Star

Solutions to Linear Algebra

B.A./B.Sc. III

Sixth Semester

Paper-II

RISING STAR PUBLICATIONS

Contents

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Vector Spaces

Exercise 1.1

Example 1 : Prove that the set of all vectors in a plane over the field of real numbers is a vector space with respect to vector addition and scalar multiplication.

Solution : Let V be the set of all vectors in a plane and R be the field of real numbers i.e. $V = \{(x,y) : x,y \in R\}$ [Since V is a set of vectors in a plane, so elements are ordered pairs]

Let us define addition and scalar multiplication in V(R) as

$$x + y = (x_1 + y_1 , x_2 + y_2)$$

$$\alpha.x = (\alpha x_1 , \alpha x_2) \text{ for all } x = (x_1 , y_1) , y = (y_1 , y_2) \in V(R) \text{ and for } \alpha \in R$$

Let us prove all the axioms of vector space.

I. We prove that V is an additive abelian group.

(i) **Closure property :** Let $x = (x_1 , x_2), y = (y_1 , y_2)$ be two elements of V(R), then $x + y = (x_1 + y_1, x_2 + y_2)$. Clearly $x_1 + y_1 , x_2 + y_2 \in R \Rightarrow x + y \in V(R)$.

(ii) **Associativity :** Let $x = (x_1 , x_2)$, $y = (y_1 , y_2)$ and $z = (z_1 , z_2)$ be elements of V(R) then

$$\begin{aligned}(x+y)+z &= [(x_1, x_2)+(y_1, y_2)]+(z_1, z_2)\\ &= (x_1 + y_1 , x_2 + y_2)+(z_1 , z_2)\\ &= ((x_1 + y_1)+z_1 , (x_2 + y_2)+z_2)\\ &= (x_1 + (y_1 + z_1) , x_2 + (y_2 + z_2)) \quad \text{[Since associativity holds in R]}\\ &= (x_1 + x_2)+(y_1 + z_1 , y_2 + z_2)\\ &= (x_1 + x_2)+[(y_1 , y_2)+(z_1 , z_2)] = x+(y+z).\end{aligned}$$

(iii) **Existence of additive identity :** Consider $\bar{0} = (0 , 0) \in V(R)$, where 0 is the additive identity in the field F. Then

$$x + \bar{0} = (x_1 , x_2) + (0 , 0) = (x_1 , x_2) = x$$

Similarly, $\bar{0} + x = x$ for all $x \in V(R)$. Hence $\bar{0}$ is additive identity of V(R).

(iv) **Existence of additive inverse :** Let $x = (x_1 , x_2) \in V(R)$ be any element, then

$$-x = (-x_1 , -x_2) \in V(R) \text{ and}$$

$$\begin{aligned}x + (-x) &= (x_1 , x_2) + (-x_1 , -x_2)\\ &= (x_1 - x_1)(x_2 - x_2) = (0 , 0) = \bar{0}.\end{aligned}$$

Similarly, $(-x)+x=\overline{0}$, so $-x$ is inverse of x.

(v) Commutativity : Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be two elements of V

Then
$$\begin{aligned} x+y &= (x_1, x_2)+(y_1, y_2) \\ &= (x_1+y_1, x_2+y_2) \\ &= (y_1+x_1, y_2+x_2) \\ &= (y_1, y_2)+(x_1, x_2) \\ &= y+x \qquad \text{[Since commutative law holds]} \end{aligned}$$

II. Scalar multiplication is closed :- Let $x=(x_1, x_2)\in V(R)$ and let $\alpha\in R$. Then

$\alpha x=(\alpha x_1, \alpha x_2)$ but $\alpha\in R, x_i\in R \Rightarrow \alpha x_i \in R$ for all i

$\Rightarrow \alpha x\in V(R)$.

III. Properties of Scalar multiplication :-

(i) We have for $x, y\in V(R)$, $\alpha\in R$

$$\begin{aligned} \alpha(x+y) &= \alpha(x_1+y_1, x_2+y_2) \\ &= (\alpha x_1+\alpha y_1, \alpha x_2+\alpha y_2) \\ &= (\alpha x_1, \alpha x_2)+(\alpha y_1, \alpha y_2) \\ &= \alpha(x_1, x_2)+\alpha(y_1, y_2) = \alpha x+\alpha y. \end{aligned}$$

(ii) We have for $x\in V(R)$, $\alpha, \beta\in R$

$$\begin{aligned} (\alpha+\beta)x &= (\alpha+\beta)(x_1, x_2) = ((\alpha+\beta)x_1, (\alpha+\beta)x_2) \\ &= (\alpha x_1+\beta x_1, \alpha x_2+\beta x_2) = (\alpha x_1, \alpha x_2)+(\beta x_1, \beta x_2) \\ &= \alpha(x_1, x_2)+\beta(x_1, x_2) = \alpha x+\beta y. \end{aligned}$$

(iii)
$$\begin{aligned} (\alpha\beta)x &= (\alpha\beta)(x_1, x_2) = ((\alpha\beta)x_1, (\alpha\beta)x_2) \\ &= (\alpha(\beta x_1), \alpha(\beta x_2)) = \alpha(\beta x_1, \beta x_2) \\ &= \alpha(\beta(x_1, x_2)) = \alpha(\beta x). \end{aligned}$$

(iv) Let $1\in R$ be unity of field R, then $1.x_i=x_i$ for all $x_i\in R$.

Now, let $x=(x_1, x_2)\in V(R)$ then

$$1.x = 1.(x_1, x_2) = (1.x_1, 1.x_2) = (x_1, x_2) = x$$

Hence V(R) is a vector space.

Example 2 : Prove that the set C of all complex numbers (i.e. the set of all orde[red] pairs of real numbers) is a vector space over the field R of all real numbers wh[en] vector addition is defined by $(x_1, x_2)+(y_1, y_2)=(x_1+y_1, x_2+y_2)$ for (x_1, x_2), $(y_1, y_2)\in C$ and scalar multiplication is defined by $a(x_1, x_2)=(ax_1, ax_2)$ for all $a\in$

Solution : Similar to example (1) as $C = \{(x, y) : x, y\in R\}$.

Example 3 : If $P_n(x)$ denotes the set of all polynomials of degree at the most n o[ver] a field F, then prove that $P_n(x)$ is a vector space w.r.t. addition of polynomials a[nd] scalar multiplication defined as the product of polynomial by an element of F.

Solution : We have

$P_n(x) = \{f(x) : f(x)$ is a polynomial of degree $\leq n$ or $f(x)$ is a zero polynomial$\}$.

We define addition and scalar multiplication as follows :

Let $f(x) = \alpha_0 + \alpha_1 x + \dots\dots\dots + \alpha_k x^k$ and $g(x) = \beta_0 + \beta_1 x + \dots\dots\dots + \beta_t x^t$, be two elements of $P_n(x)$. Then $f(x) + g(x) = (\alpha_0 + \beta_0) + (\alpha_1 + \beta_1)x + \dots\dots\dots$

and $af(x) = (a\alpha_0) + (a\alpha_1)x_1 + \dots\dots\dots + (a\alpha_k)x^k$, $a \in F$

Let us prove all the axioms of vector space.

I. We prove that V is an additive abelian group

(i) **Closure :** For any $f(x), g(x) \in P_n(x)$

$$f(x) + g(x) = (\alpha_0 + \beta_0) + (\alpha_1 + \beta_1)x + \dots\dots\dots\dots \in P_n(x)$$

Since $f(x) + g(x)$ is either a zero polynomial or a polynomial in x over F of degree $\le n$

(ii) **Commutativity :** Addition of polynomials is always commutative.

(iii) **Associativity :** Addition of polynomials is always associative

$$\text{i.e.} \quad (f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$$

(iv) **Additive Identity :** Zero polynomial is the additive identity of V(F).

(v) **Existence of additive inverse :** Additive inverse of

$$f(x) = \alpha_0 + \alpha_1 x + \dots.. + \alpha_k x^k \text{ is } -f(x) = (-\alpha_0) + (-\alpha_1)x + \dots\dots + (-\alpha_k)x^k$$

II. **Scalar multiplication is closed :** For all $\alpha \in F$ and for all $f(x) \in P_n(x)$

We have

$$\alpha f(x) = \alpha \sum a_n x^n = \sum (\alpha a_n) x^n \in P_n(x)$$

[Since $\alpha \in F$, $a_n \in F \Rightarrow \alpha a_n \in F$ because F is closed for multiplication]

III. **Properties of scalar multiplication :** Let $f(x), g(x) \in P_n(x)$ be arbitrary element and $\alpha, \beta \in F$ be scalars then

(i)
$$\begin{aligned} [\alpha(f+g)](x) = \alpha[(f+g)(x)] &= \alpha[f(x) + g(x)] \\ &= \alpha f(x) + \alpha g(x) = (\alpha f)(x) + (\alpha g)(x) \\ &= (\alpha f + \alpha g)(x) \text{ for all } x \end{aligned}$$

$$\Rightarrow \quad \alpha(f+g) = \alpha f + \alpha g.$$

(ii)
$$\begin{aligned} [(\alpha+\beta)f](x) = (\alpha+\beta)f(x) &= \alpha f(x) + \beta f(x) \\ &= (\alpha f)(x) + (\beta f)(x) = (\alpha f + \beta f)(x) \end{aligned}$$

$$\Rightarrow \quad (\alpha+\beta)f = \alpha f + \beta f.$$

(iii)
$$\begin{aligned} [(\alpha\beta)f](x) = (\alpha\beta)f(x) &= \alpha(\beta f(x)) = \alpha[(\beta f)(x)] \\ &= [\alpha(\beta f)](x) \text{ for all } x \end{aligned}$$

$$\Rightarrow \quad (\alpha\beta)f = \alpha(\beta f).$$

(iv) $1.f(x) = f(x)$

Hence $P_n(x)$ is a vector space.

Example 4 : Prove the set M_{nn} of all square matrices of order n over a field F is a vector space over F w.r.t. matrix addition and scalar multiplication.

Solution : Let us prove all the axioms of vector space.

I We prove that M_{nn} is an additive abelian group.

(i) Closure : If $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$ are two matrices with entries over F, then clearly $A + B = [a_{ij} + b_{ij}]_{n\times n}$ is also a matrix of order $n\times n$ with entries over F. Thus $A + B \in M_{n\times n}$ for all $A, B \in M_{n\times n}$.

(ii) Associativity : As addition of elements in a field is always associative so matrix addition in $M_{n\times n}$ is also associative because all the matrices in $M_{n\times n}$ are with entries over a field F.

(iii) Existence of additive identity : Let $A \in M_{n\times n}$ be any arbitrary matrix. The zero matrix 0 of order $n\times n$ acts as the additive identity because we know that

$$A + 0 = 0 + A = A$$

(iv) Existence of additive inverse : For any arbitrary matrix $A \in M_{n\times n}$, the matrix $-A$ also belongs to $M_{n\times n}$ and

$$A + (-A) = 0 = (-A) + A$$

Thus, $-A$ is the additive inverse of A.

(v) Commutativity of addition : As addition of a field is always commutative so matrix addition in $M_{n\times n}$ is also commutative because all the matrices in $M_{n\times n}$ are with entries over the field F.

Hence $M_{n\times n}$ is an additive abelian group.

II. Scalar multiplication is closed : Let $A \in M_{n\times n}$, and $k \in F$ be arbitrary.

Here A is $n\times n$ matrix over F and $k \in F$. So clearly kA is also $n\times n$ matrix over F. $k \in M_{n\times n}$.

III. Properties of scalar multiplication : For any $a, b \in F$ and $A = [a_{ij}]$, $B = [b_{ij}]$ in $M_{n\times n}$, we see that

(i) $$(a+b)A = (a+b)[a_{ij}] = \left[(a+b)a_{ij}\right]$$
$$= \left[aa_{ij} + ba_{ij}\right] \quad \text{[multiplication is distributive over F]}$$
$$= a[a_{ij}] + b[a_{ij}] = aA + bA.$$

(ii) $$a(A+B) = a\left[[a_{ij}] + [b_{ij}]\right] = a[a_{ij} + b_{ij}]$$
$$= \left[a\left[a_{ij}\right] + a\left[b_{ij}\right]\right] = [aa_{ij} + ab_{ij}]$$
$$= [aa_{ij}] + [ab_{ij}] = a[a_{ij}] + a[b_{ij}] = aA + aB.$$

(iii) $$a(bA) = a(b[a_{ij}]) = a[ba_{ij}] = [aba_{ij}] = (ab)[a_{ij}] = abA.$$

(iv) If 1 is unit of F, then for any matrix $A \in M_{n\times n}$, we have $1.A = A$.

Hence $M_{n\times n}$ is a vector space.

Example 5 : Prove that the set of all diagonal matrices of order $n \times n$ forms a vector space over the field of real numbers R with respect to matrix addition and scalar multiplication.

Solution : Let $V = \left\{[a_{ij}]_{n\times n} : a_{ij} = 0 \text{ for } i \neq j,\ a_{ij} \in R\right\}$

Let us all the axioms of vector space.

I We prove that V(R) is an additive abelian group

(i) Closure : If $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$ are two matrices in V.

Then $a_{ij} = 0$ for $i \neq j$ and $b_{ij} = 0$ for $i \neq j$

Clearly $A + B = [a_{ij} + b_{ij}]_{n\times n} = [c_{ij}]$ is also a matrix of order $n \times n$ with real entries and $c_{ij} = 0$ for $i \neq j$. Then $A+B \in V$ for all $A, B \in V$.

(ii) **Associativity :** As addition of real numbers is always associative so matrix addition in V is also associative because all the matrices in V are with real entries.

(iii) **Existence of additive identity :** Let $A \in V$ be an arbitrary matrix.
The zero matrix O of order $n \times n$ clearly belong to V and acts as the additive identity because we know that

$$A + 0 = 0 = 0 + A.$$

(iv) **Existence of additive inverse :** For any arbitrary matrix $A \in V$, the matrix $-A \in V$ and $A + (-A) = 0 = (-A)+A$.
Thus $-A$ is the additive inverse of A.

(v) **Commutativity of addition :** As addition of real numbers is always commutative so matrix addition in V is also commutative because all the matrices in V are with real entries. Hence V is an additive abelian group.

II. Scalar multiplication is closed : Let $A = [a_{ij}] \in V$ and $k \in R$ then $a_{ij} = 0$ for $i \neq j$. So, clearly $kA = [ka_{ij}] = [c_{ij}]$ is also a $n \times n$ matrix with real entries and $c_{ij} = 0$ for $i \neq j$. Thus $kA \in V$.

III. Properties of scalar multiplication : For any $a, b \in R$ and $A = [a_{ij}]$, $B = [b_{ij}]$ in V, we have

$$a_{ij} = 0 \text{ for } i \neq j \quad \text{and} \quad b_{ij} = 0 \text{ for } i \neq j$$

(i) $(a+b)A = (a+b)[a_{ij}] = [(a+b)a_{ij}]$

$= [aa_{ij} + ba_{ij}]$ [Since multiplication is distributive in R]

$= a[a_{ij}] + b[a_{ij}] = aA + bA$

(ii) $a(A+B) = a\left([a_{ij}] + [b_{ij}]\right) = a[a_{ij} + b_{ij}] = [a(a_{ij} + b_{ij})]$

$= [aa_{ij} + ab_{ij}]$ [Since multiplication is distributive in R]

$= [aa_{ij}] + [ab_{ij}] = a[a_{ij}] + a[b_{ij}] = aA + aB.$

(iii) $a(bA) = a(b[a_{ij}]) = a[ba_{ij}]$

$= [ab\, a_{ij}] = (ab)[a_{ij}] = (ab)A$

(iv) If 1 is unity of R then for any matrix $A \in V$, we have $1.A = A$

Hence V(R) is a vector space.

Example 6 : Prove that the set of all matrices of the form $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}$, where $x, y \in C$ the set of complex numbers, is a vector space over C with respect to matrix addition and scalar multiplication.

Solution : Let $V = \left\{\begin{bmatrix} x & y \\ -y & x \end{bmatrix} : x, y \in C\right\}$. Let us prove all the axioms of vector space.

I. We prove that V is an additive abelian group.

(i) **Closure :** Let $A = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}$, $B = \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix} \in V$

So, $A + B = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 \\ -(y_1 + y_2) & x_1 + x_2 \end{bmatrix} \in V$.

Thus addition is closed in V.

(ii) Associativity : Let $A = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}$, $B = \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix}$, $C = \begin{bmatrix} x_3 & y_3 \\ -y_3 & x_3 \end{bmatrix} \in V$

So, $(A + B) + C = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 \\ -y_1 - y_2 & x_1 + x_2 \end{bmatrix} + \begin{bmatrix} x_3 & y_3 \\ -y_3 & x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 & y_1 + y_2 + y_3 \\ -y_1 - y_2 - y_3 & x_1 + x_2 + x_3 \end{bmatrix}$

Similarly, $A + (B + C) = \begin{bmatrix} x_1 + x_2 + x_3 & y_1 + y_2 + y_3 \\ -y_1 - y_2 - y_3 & x_1 + x_2 + x_3 \end{bmatrix}$

So, $(A + B) + C = A + (B + C)$.

(iii) Existence of additive Identity : Let $A \in V$ be any arbitrary matrix. The zero m
$0 \in V$ act as the additive identity because we know that

$A + 0 = A = 0 + A$.

(iv) Existence of additive inverse : For any arbitrary matrix $A \in V$, the matrix $-A$
belong to V and $A + (-A) = 0 = (-A) + A$

Thus $-A$ is the additive inverse of A.

(v) Commutativity of addition : As addition of complex numbers is always commu
so matrix addition in V is also commutative because all the matrices in V are
complex entries.

Hence V(C) is an additive abelian group.

II. Scalar multiplication is closed : Let $A \in V$ and $k \in C$ be arbitrary, then

$$A = \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \Rightarrow kA = \begin{bmatrix} kx & ky \\ -ky & kx \end{bmatrix}$$

So, $kA \in V$, since product of two complex numbers is again a complex number.

III. Properties of Scalar multiplication : Let $a, b \in C$ and $A, B \in V$

(i) $$(a+b)A = (a+b)\begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} = \begin{bmatrix} (a+b)x_1 & (a+b)y_1 \\ -(a+b)y_1 & (a+b)x_1 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + bx_1 & ay_1 + by_1 \\ -ay_1 - by_1 & ax_1 + bx_1 \end{bmatrix} = \begin{bmatrix} ax_1 & ay_1 \\ -ay_1 & ax_1 \end{bmatrix} + \begin{bmatrix} bx_1 & by_1 \\ -by_1 & bx_1 \end{bmatrix}$$

$$= a\begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} + b\begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} = aA + bB.$$

(ii) $$a(A+B) = a\begin{bmatrix} x_1 + x_2 & y_1 + y_2 \\ -(y_1 + y_2) & x_1 + x_2 \end{bmatrix} = \begin{bmatrix} a(x_1 + x_2) & a(y_1 + y_2) \\ -a(y_1 + y_2) & a(x_1 + x_2) \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + ax_2 & ay_1 + ay_2 \\ -ay_1 - ay_2 & ax_1 + ax_2 \end{bmatrix} = \begin{bmatrix} ax_1 & ay_1 \\ -ay_1 & ax_1 \end{bmatrix} + \begin{bmatrix} ax_2 & ay_2 \\ -ay_2 & ax_2 \end{bmatrix}$$

$$= aA + aB.$$

(iii) Similarly, one can show that $a(bA) = (ab)A$

(iv) $1.A = A$

Hence V(C) is a vector space.

Example 7 : Show that the set $Q(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in Q\}$ is a vector space over Q with respect to the vector addition and scalar multiplication defined as

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c + (b + d)\sqrt{2})$$

and $\alpha(a + b\sqrt{2}) = a\alpha + b\alpha\sqrt{2}$ where a, b, c, d and α are all rational numbers.

Solution : Let us prove all axioms of vector space

I We prove that $Q(\sqrt{2})$ is an additive abelian group.

(i) Closure : Let $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$ be two arbitrary elements of $Q(\sqrt{2})$. Then

$$x + y = (a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2} \in Q\sqrt{2}$$

Since $a + c \in Q$, $b + d \in Q$

(ii) Associativity : Let $x = (a + b\sqrt{2})$, $y = (c + d\sqrt{2})$ and $z = (e + f\sqrt{2})$ be an arbitrary elements of $Q(\sqrt{2})$.

Then $[x + y] + z = [(a + b\sqrt{2}) + (c + d\sqrt{2})] + (e + f\sqrt{2})$

$$= (a + c + (b + d)\sqrt{2}) + (e + f\sqrt{2})$$

$$= a + c + e + (b + d + f)\sqrt{2}$$

Again, $x + [y + z] = (a + b\sqrt{2}) + [(c + d\sqrt{2}) + (e + f\sqrt{2})]$

$$= \left[(a + b\sqrt{2}) + c + e + (d + f)\sqrt{2}\right]$$

$$= a + c + e + (b + d + f)\sqrt{2}.$$

Hence $(x + y) + z = x + (y + z)$.

(iii) Existence of additive identity :

There exists $\bar{0} = 0 + 0\sqrt{2} \in Q(\sqrt{2})$ such that

$$(a + b\sqrt{2}) + (0 + 0\sqrt{2}) = a + b\sqrt{2}.$$

(iv) Existence of additive inverse : For each $x = a + b\sqrt{2} \in Q(\sqrt{2})$ there exists $-x = -a - b\sqrt{2} \in Q(\sqrt{2})$ such that

$$x + (-x) = (a + b\sqrt{2}) + (-a - b\sqrt{2}) = 0 + 0\sqrt{2} = \bar{0}.$$

(v) Commutativity of addition :

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c + (b + d)\sqrt{2}) = \left(c + a + (d + b)\sqrt{2}\right)$$

$$= (c + d\sqrt{2}) + (a + b\sqrt{2})$$

Hence $Q\sqrt{2}$ is an additive abelian group.

II. Scalar multiplication is closed :- Clearly for each $a + b\sqrt{2} \in Q(\sqrt{2})$ and $k \in Q$

$$k(a + b\sqrt{2}) = ak + bk\sqrt{2} \in Q(\sqrt{2}).$$

III. Properties of scalar multiplication : For any $\alpha, \beta \in Q$ and $x = a + b\sqrt{2}$, $y = c + d\sqrt{2} \in Q(\sqrt{2})$ we see that

(i) $(\alpha + \beta)x = (\alpha + \beta)(a + b\sqrt{2})$

$$= (\alpha + \beta)a + (\alpha + \beta)b\sqrt{2}$$

$$= \alpha a + \beta a + \alpha b\sqrt{2} + \beta b\sqrt{2}$$

$$= \alpha a + \alpha b\sqrt{2} + \beta a + \beta b\sqrt{2}$$
$$= \alpha(a + b\sqrt{2}) + \beta(a + b\sqrt{2})$$
$$= \alpha x + \beta x.$$

(ii) $\alpha(x + y) = \alpha\left[(a + b\sqrt{2}) + (c + d\sqrt{2})\right]$
$$= \alpha\left[(a + c) + (b + d)\sqrt{2}\right]$$
$$= \alpha(a + c) + \alpha(b + d)\sqrt{2}$$
$$= \alpha a + \alpha b\sqrt{2} + \alpha c + \alpha d\sqrt{2}$$
$$= \alpha(a + b\sqrt{2}) + \alpha(c + d\sqrt{2})$$
$$= \alpha x + \alpha y$$

Similarly one can show that

(iii) $\alpha(\beta x) = (\alpha\beta)x$

(iv) $1.x = x$, 1 is the unity of R.

Hence $Q\sqrt{2}(\sqrt{2})$ is a vector space.

Example 8 : Show that the set V of all polynomials with constant term 2 over R is not a vector space. Further tell which fixed value can be assigned to the constant term so that V can become a vector space.

Solution : Let V = Set of all polynomials over R with constant term 2. Then V is not closed w.r.t. addition.

For example, if $u = 3x + 2$, $v = 2x + 2 \in V$

Then $u + v = 5x + 4 \notin V$

Hence V cannot be a vector space over R.

Now, let V = set of all polynomials over R with constant term zero. Since sum of two polynomials over R with constant term zero is always a polynomial over R with constant term zero. Therefore V is closed w.r.t. addition. Since multiplication of such a polynomial by a scalar will produce a polynomial over R with constant term zero, therefore V is also closed w.r.t. scalar multiplication and all other properties for a vector space are obviously satisfied.

Exercise 1.2

Example 1 : Let V be a vector space given by $V = R^3 = \{(x, y, z) : x, y, z \in R\}$ then which of the following are subspaces of V over R ?

(i) $W = \{(x, y, z) : 3x + y - z = 0, x, y, z \in R\}$

(ii) $W = \{(x, y, z) : x + y - z = 0, 2x + 3y - z = 0, x, y, z \in R\}$

(iii) $W = \{(x, y, z) : x + y \geq 0, x, y, z \in R\}$

(iv) $W = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1, x, y, z \in R\}$

Solution : (i) Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ be two elements of W. Then we have $3x_1 + y_1 - z_1 = 0$ and $3x_2 + y_2 - z_2 = 0$

Let $a, b \in R$ be any two scalars. To prove that W is a subspace. We have to show that $au + bv \in W$.

Now, $au + bv = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$

$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

Now, $3(ax_1 + bx_2) + (ay_1 + by_2) - (az_1 + bz_2)$

$= 3ax_1 + 3bx_2 + ay_1 + by_2 - az_1 - bz_2$

$= (3ax_1 + ay_1 - az_1) + (3bx_2 + by_2 - bz_2)$

$= a(3x_1 + y_1 - z_1) + b(3x_2 + y_2 - z_2)$

$= a.0 + b.0 = 0$

$\Rightarrow$ $au + bv \in W$ for all $u, v \in W$; $a, b \in F$

Hence W is a subspace of V.

(ii) Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ be two elements of W so that

$x_1 + y_1 - z_1 = 0$, $2x_1 + 3y_1 - z_1 = 0$

and $x_2 + y_2 - z_2 = 0$, $2x_2 + 3y_2 - z_2 = 0$

Let $a, b \in R$ be any two scalars. To prove that W is a subspace we have to show that $au + bv \in W$

Now, $au + bv = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$

$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

Now; $ax_1 + bx_2 + ay_1 + by_2 - az_1 + bz_2$

$= a(x_1 + y_1 - z_1) + b(x_2 + y_2 - z_2) = 0$

$\Rightarrow$ $a.0 + b.a = 0$

Again, $2(ax_1 + bx_2) + 3(ay_1 + by_2) - (az_1 + bz_2)$

$= a(2x_1 + 3y_1 - z_1) + b(2x_2 + 3y_2 - z_2)$

$= a.0 + b.0 = 0$

$\Rightarrow$ $au + bv \in W$ for all $u, v \in W$; $a, b \in F$.

(iii) W is not a vector space since for $(1, 2, 3) \in W$ and $-1 \in R$

$-1(1, 2, 3) = (-1, -2, -3) \notin W$

So W is not closed under scalar multiplication.

(iv) W is not a vector space since for $(a, b, c) \in W$, we have

$a^2 + b^2 + c^2 \le 1$ and $-2 \in R$ then, $(-2)(a, b, c) = (-2a, -2a, -c) \notin W$

Thus W is not closed under scalar multiplication.

Example 2 : Let V be the vector space of all n – square matrices over reals. Which of the following sets of matrices are subspaces of V.

(i) all scalar matrices.

(ii) all singular matrices

(iii) all diagonal matrices

(iv) all upper triangular matrices

(v) all skew symmetric matrices.

Solution : (i) Let W = collection of all scalar matrices

$$= \left\{ [a_{ij}]_{n\times n} : \begin{matrix} a_{ij} = k \text{ for } i = j \\ a_{ij} = 0 \text{ for } i \pm j \end{matrix} ; k \in R \right\}$$

Let $x, y \in W$ where $x = [a_{ij}]$ for which $a_{ij} = k_1$ for $i = j$ and $a_{ij} = 0$ for $i \neq j$

and $y = [b_{ij}]$ for which $b_{ij} = k_2$ for $i = j$ and $a_{ij} = 0$ for $i \neq j$.

Then $\alpha x + \beta y = \alpha[a_{ij}] + \beta[b_{ij}]$, for $\alpha, \beta \in R$

$= [\alpha a_{ij} + \beta b_{ij}] = [c_{ij}]$ where $c_{ij} = \alpha a_{ij} + \beta b_{ij}$

Now, $c_{ij} = k_1 + k_2$ for $i = j$ and $a_{ij} = 0$ for $i \neq j$.

So, $c_{ij} \in W$. Hence W is a subspace of V.

(ii) Let W = collection of all singular matrices. Clearly W is not a subspace of because W is not closed under addition.

For example, Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \in W$

Then $A + B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \notin W$ as $\det(A + B) = 6 \neq 0$.

(iii) Let W = collection of all diagonal matrices

$= \{[a_{ij}]_{n \times n} : a_{ij} = 0 \text{ for } i \neq j,\ a_{ij} \in R\}$

Clearly $W \subseteq V$.

Let $x, y \in W$ where $x = [a_{ij}]$ for which $a_{ij} = 0$ for $i \neq j$. and $y = [b_{ij}]$ for which $b_{ij} = 0$ for $i \neq j$

Then $\alpha x + \beta y = \alpha[a_{ij}] + \beta[b_{ij}] = [\alpha a_{ij} + \beta b_{ij}] = [c_{ij}]$ where $c_{ij} = \alpha a_{ij} + \beta b_{ij}$

Clearly $c_{ij} = 0$ for $i \neq j$.

Thus $[c_{ij}]$ is also a diagonal matrix and therefore $[c_{ij}] \in W$.

Hence W is a subspace of V.

(iv) Let W = collection of all upper triangular matrices

i.e. $W = \{A : A = [a_{ij}]_{n \times n},\ a_{ij} = 0 \text{ for } i > j\}$

Let $A, B \in W$ be any two matrices then A and B are two upper triangular matrices i.e.

$A = [a_{ij}]_{n \times n},\ a_{ij} = 0$ for $i > j$

and $B = [b_{ij}]_{n \times n},\ b_{ij} = 0$ for $i > j$

Let $\alpha, \beta \in R$ be any two scalars. To prove that W is a subspace we have to show that

$\alpha A + \beta B \in W$

Now $\alpha A + \beta B = \alpha[a_{ij}]_{n \times n} + \beta[b_{ij}]_{n \times n}$

$= [\alpha a_{ij} + \beta b_{ij}]$ where $\alpha a_{ij} + \beta b_{ij} = \alpha.0 + \beta.0$ for $i > j$

$= 0$ for $i > j$

Thus $\alpha a + \beta b$ is a upper triangular matrix so $\alpha a + \beta b \in W$ and hence W is a subspace of V.

(v) Here W = collection of all skew-symmetric matrices i.e. $W = \{A : A^T = -A\}$

Let $A, B \in W$ be any two matrices. Then A and B are skew-symmetric

i.e. $A^T = -A$ and $B^T = -B$

Let $\alpha, \beta \in R$ be any two scalars. To prove that W is a subspace we have to show that

$\alpha A + \beta B \in W$

Now, $(\alpha A + \beta B)^T = (\alpha A)^T + (\beta B)^T = \alpha A^T + \beta B^T = \alpha(-A) + \beta(-B) = -[\alpha A + \beta B]$

So, $\alpha A+\beta B$ is skew symmetric matrix and therefore $\alpha A+\beta B \in W$ and hence W is a subspace of V.

Example 3 : Prove that the intersection of any family of subspaces of a vector space is a subspace .

Solution : Let $\{W_\alpha : \alpha \in V\}$ be an arbitrary collection of subspaces of a vector space V.

Let $W = \bigcap W_\alpha \quad \alpha = 1, 2, ..., n$

Since $0 \in W_\alpha$. So, $0 \in \cap W_\alpha \Rightarrow 0 \in W$

$\Rightarrow$ W is non – empty.

Let $a, b \in F$ and $x, y \in W$ then $x, y \in$ each W_α

$\Rightarrow \quad ax+by \in$ each W_α, which is a subspace.

$\Rightarrow \quad ax+by \in \cap W_\alpha \quad \Rightarrow \quad ax+by \in W$

Hence $W = \bigcap W_\alpha$ is a subspace.

Example 4 : Let $V = R^3$ be the vector space over R. Prove that W is a vector sub space over R where $W = \{(x, y, z) : ax+by+cz=0 ; x, y, z \in R\}$.

OR

Show that any plane passing through the origin is a subspace of R^3.

Solution : Here $W = \{(x, y, z) : ax+by+cz=0 : x, y, z \in R\}$

Let $u=(x_1, y_1, z_1)$ and $v=(x_2, y_2, z_2)$ be two elements of W so that we have

$$ax_1+by_1+cz_1=0$$
$$ax_2+by_2+cz_2=0$$

Let $\alpha, \beta \in R$ be any two scalars. To prove that W is a subspace we have to show that

$$\alpha u+\beta v \in W$$

Now $\quad \alpha u+\beta v = \alpha(x_1, y_1, z_1)+\beta(x_2, y_2, z_2) = (\alpha x_1+\beta x_2, \alpha y_1+\beta y_2, \alpha z_1+\beta z_2)$

Now, $\quad a(\alpha x_1+\beta x_2)+b(\alpha y_1+\beta y_2)+c(\alpha z_1+\beta z_2) = \alpha(ax_1+by_1+cz_1)+\beta(ax_2+by_2+cz_2)$

$$= \alpha.0+\beta.0 = 0+0 = 0$$

$\Rightarrow \quad \alpha u+\beta v \in W$ for all $u, v \in W ; \alpha, \beta \in F$

Hence W is a subspace of V.

Example 5 : Let V(R) be the vector space of all function from R to R (reals). Prove that the following subsets are the subspaces of V(R)

(a) set of all odd functions **(b) set of all even functions**

(c) set of all continuous functions.

Solution : (a) Let W = set of all odd functions

i.e. $W = \{f : f \in V \text{ and } f(-x) = -f(x)\}$

Let $f(x) = x^3+x$ then $f(-x) = (-x)^3+(-x) = -[x^3+x] = f(x)$. Hence $f \in W$ and $W \neq \phi$

Now, let $f, g \in W$ be any two functions then f and g are odd functions.

$$f(-x) = -f(x) \quad \text{and} \quad g(-x) = -g(x)$$

Let $\alpha, \beta \in R$ be any two scalars. To prove that W is a subspace we have to show that

$$\alpha f + \beta g \in W$$

Now, $(\alpha f + \beta g)(-x) = (\alpha f)(-x) + (\beta g)(-x) = \alpha f(-x) + \beta g(-x)$

$$= \alpha(-f(x)) + \beta(-g(x))$$
$$= -[\alpha f(x) + \beta g(x)]$$
$$= -(\alpha f + \beta g)(x)$$

$\Rightarrow$ $\alpha f + \beta g$ is an odd function so $\alpha f + \beta g \in W$.

Hence W is a subspace of V(R).

(b) We have $W = \{f : f \in V \text{ and } f(-x) = f(x)\}$

Let $f(x) = x^2 + x^4$ then $f(-x) = (-x)^2 + (-x)^4 = x^2 + x^4 = f(x)$

Hence $f \in W$ and $W \neq \phi$

Now, let $f, g \in W$ be any two functions then f and g are even functions i.e.

$$f(-x) = f(x) \quad , \quad g(-x) = g(x)$$

Let $\alpha, \beta \in R$ be any two scalars. To prove that W is a subspace we have to show that

$$\alpha f + \beta g \in W$$

Now, $(\alpha f + \beta g)(-x) = (\alpha f)(-x) + (\beta g)(-x)$

$$= \alpha f(-x) + \beta g(-x) = \alpha f(x) + \beta g(x) = (\alpha f + \beta g)(x)$$

$\Rightarrow$ $\alpha f + \beta g$ is a even function so $\alpha f + \beta g \in W$

Hence W is a subspace of $V(R)$.

(c) Let W = collection of all continuous functions. Clearly $W \neq \phi$

Now, let $f, g \in W$ be any two functions. Then f and g are two continuous functions.

Let $\alpha, \beta \in R$ be any two scalars. To prove that W is a subspace we have to show that

$$\alpha f + \beta g \in W$$

Clearly $\alpha f + \beta g$ is also a continuous function because sum of two continuous functions is a continuous function and product of a continuous function by a real number is also a continuous function.

So, $\alpha f + \beta g \in W$. Thus W is a subspace of $V(R)$.

Example 6 : Let V be the vector space of all functions from R to R (reals). *Let* W_1 and W_2 denotes the subspaces of all even functions and all odd functions *respectively.* Prove that V is the direct sum of W_1 and W_2.

Solution : To prove that V is the direct sum of W_1 and W_2 i.e. $V = W_1 \oplus W_2$, we have to show that (i) $V = W_1 + W_2$ and (ii) $W_1 \cap W_2 = 0$

(i) Let $f(x) = \frac{1}{2}[f(x) + f(-x)]$ for all $x \in R$

$= F(x) + G(x)$ where $F(x) = \frac{1}{2}[f(x) + f(-x)]$ and $G(x) = \frac{1}{2}(f(x) - f(-x))$

Clearly $F(-x) = F(x)$ and $G(-x) = -G(x)$

$\Rightarrow$ $F(x) \in W_1$ and $G(x) \in W_2$

Thus $f = F + G$ where $F \in W_1$ and $G \in W_2$. So, $V = W_1 + W_2$.

(ii) Let $f \in W_1 \cap W_2$

$\Rightarrow$ $f \in W_1$ and $f \in W_2$ $\Rightarrow$ $f(-x) = f(x)$ and $f(-x) = -f(x)$

which is true only when $f(x) = 0$ for all x.

i.e. $f = 0$ $\Rightarrow$ $W_1 \cap W_2 = 0$.

So, from (i) and (ii), we get $V = W_1 \oplus W_2$.

Example 7 : Let $M_{22}(R)$ denotes the vector space of all 2×2 matrices over reals. Determine which of the following are subspaces of $M_{22}(R)$.

(i) $W = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} : a, b, c \in R \right\}$ **(ii)** $W = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in R \right\}$

(iii) $W = \left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} : a \in R \right\}$ **(iv)** $W = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in R \right\}$

(v) $W = \{ A \in M_{22}(R) : A^2 = A \}$ **(vi)** $W = \{ A \in M_{22}(R) : A^2 = I \}$

(vii) $W = \{ A \in M_{22}(R) : A \text{ is singular} \}$

Solution : (i) $W = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} : a, b \in R \right\}$

Let $A, B \in W$, so that

$$A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}, \quad B = \begin{bmatrix} d & e \\ f & 0 \end{bmatrix}$$

Let $\alpha, \beta \in R$ be any two scalars. To prove that W is a subspace we have to show that

$$\alpha A + \beta B \in W$$

Now $\alpha A + \beta B = \alpha \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} + \beta \begin{bmatrix} d & e \\ f & 0 \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & 0 \end{bmatrix} + \begin{bmatrix} \beta d & \beta e \\ \beta f & 0 \end{bmatrix} = \begin{bmatrix} \alpha a + \beta d & \alpha b + \beta e \\ \alpha c + \beta f & 0 \end{bmatrix} \in W$

Thus W is a subspace of $M_{22}(R)$.

(ii) Let $A, B \in W$ so that $A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}$

Let $\alpha, \beta \in R$ be any two scalars. To prove that W is a subspace we have to show that

$$\alpha A + \beta B \in W$$

$$\alpha A + \beta B = \alpha \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \beta b & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha a + \beta b & 0 \\ 0 & 0 \end{bmatrix} \in W.$$

Thus W is a subspace of $M_{22}(R)$.

(iii) Let $A, B \in W$, so that $A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$.

Let $\alpha, \beta \in R$ be any two scalars. To prove that W is a subspace we have to show that

$$\alpha A + \beta B \in W$$

$$\alpha A+\beta B=\alpha\begin{bmatrix}0 & a\\ 0 & 0\end{bmatrix}+\beta\begin{bmatrix}0 & b\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & \alpha a+\beta b\\ 0 & 0\end{bmatrix}\in W.$$

Thus W is a subspace of $M_{22}(R)$.

(iv) $W=\left\{\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix} : a, b\in R\right\}$ let $A, B\in W$ so that

$$A=\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}, \quad B=\begin{bmatrix}a' & 0\\ 0 & b'\end{bmatrix}.$$

Let $\alpha, \beta\in R$ be any two scalars. To prove that W is a subspace we have to show that

$$\alpha A+\beta B\in W$$

$$\alpha A+\beta B=\alpha\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}+\beta\begin{bmatrix}a' & 0\\ 0 & b'\end{bmatrix}=\begin{bmatrix}\alpha a+\beta a' & 0\\ 0 & \alpha b+\beta b'\end{bmatrix}\in W.$$

Thus W is a subspace of $M_{22}(R)$.

(v) $W=\{A\in M_{22}(R): A^2=A\}$. Here W is not a subspace of $M_{22}(R)$ as

$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\in W$ but $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\notin W$ i.e., W is not closed under addition.

(vi) $W=\{A\in M_{22}(R) : A^2=I\}$, W is not subspace of M_{22} since $A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\in W$

but, $A+A=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\notin W$.

(vii) W is not a subspace of M_{22} since $\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 3\end{bmatrix}\in W$

but, $\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 3\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}\notin W$.

Example 8 : Let $P_4(x)$ denotes the vector space of all polynomials of degree at the most 4 over the field of real numbers. Determine which of the following are subspaces of $P_4(x)$.

(i) **W = $\{f(x):$ constant term of $f(x)=1\}$**

(ii) **W = $\{f(x):$ deg $f(x)=2\}$**

(iii) **W = $\{f(x):$ deg $f(x)\le 2\}$**

(iv) **W = $\{f(x):$ coeff. of x in $f(x)$ is zero$\}$**

(v) **W = $\{f(x):$ $f(2)=f(4)=0\}$**

(vi) **W = $\{f(x):$ constant term of $f(x)=0\}$**

Solution : (i) $W=\{f(x)$: constsnt term of $f(x)=1\}$, W is not a subspace of $P_4(x)$ since $f(x)=x^2+1$, $g(x)=x+1\in W$ but $f(x)+g(x)=x^2+x+2\notin W$.

(ii) $W = \{f(x) : \deg f(x) = 2\}$. W is not a subspace of $P_4(x)$ as $f(x) = 3x^2 + 2$, $g(x) = -3x^2 + 3 \in W$ but $f(x) + g(x) = 5 \notin W$.

(iii) $W = \{f(x) : \deg f(x) \le 2\}$. Let $f(x), g(x) \in W$ so that $\deg f(x) \le 2$, $\deg g(x) \le 2$.

Let $\alpha, \beta \in R$ be any two scalars. To prove that W is a subspace we have to prove that $\alpha f(x) + \beta g(x) \in W$.

Clearly, $\deg[\alpha f(x) + \beta g(x)] \le 2 \Rightarrow \alpha f(x) + \beta g(x) \in W$. Hence W is a subspace of $P_4(x)$.

(iv) $W = \{f(x)$: co-efficient of x in $f(x)$ is zero$\}$

Let $f(x), g(x) \in W$ so that co-efficient of x in $f(x)$ and $g(x)$ is zero.

Let $\alpha, \beta \in R$ be any two scalars. To prove that W is a subspace of $P_4(x)$ we have to prove that $\alpha f(x) + \beta g(x) \in W$

Clearly co – efficient of x in $\alpha f(x) + \beta g(x)$ is zero.

So, $\alpha f(x) + \beta g(x) \in W$. Thus W is a subspace of $P_4(x)$.

(v) $W = \{f(x) : f(2) = f(4) = 0\}$

Let $f(x), g(x) \in W$. So that $f(2) = f(4) = 0$, $g(2) = g(4) = 0$

Let $\alpha, \beta \in R$ be arbitrary constants. To prove that W is a subspace of $P_4(x)$ we have to prove that $\alpha f(x) + \beta g(x) \in W$

Now $(\alpha f + \beta g)(2) = \alpha f(2) + \beta g(2) = \alpha.0 + \beta.0 = 0$

Again, $(\alpha f + \beta g)(4) = \alpha f(4) + \beta g(4) = \alpha.0 + \beta.0 = 0 + 0 = 0$

So, $\alpha f + \beta g \in W$. Hence W is a subspace of $P_4(x)$.

(vi) $W = \{f(x)$; constant term of $f(x) = 0\}$.

Since sum of two polynomials over R with constant term zero is always a polynomial over R with constant term zero. Therefore W is closed w.r.t. addition.

Since multiplication of such a polynomial by a scalar will produce a polynomial over R with constant term zero, therefore W is closed w.r.t. scalar multiplication. Hence W is a subspace of $P_4(x)$.

Example 9 : In the vector space $V = R^2(R)$. Consider $W_1 = \{(x, 0) : x \in R\}$, $W_2 = \{(0, x) : x \in R\}$, $W_3 = \{(x, x) : x \in R\}$.

Prove that W_1, W_2, W_3 are subspaces of $R^2(R)$ and prove that

(i) $V = W_1 \oplus W_2$ (ii) $V = W_1 \oplus W_3$

Solution : $W_1 = \{(x, 0) : x \in R\}$, $W_2 = \{(0, x) : x \in R\}$, $W_3 = \{(x, x) : x \in R\}$

Firstly we show that W_1 is subspace of $R^2(R)$.

Let $X = (x, 0)$, $Y = (y, 0)$ be two arbitrary elements of W_1.

Let $\alpha, \beta \in R$ be any two scalars. To prove that W_1 is a subspace of $R^2(R)$ we have to prove that $\alpha X + \beta Y \in W$

Now $\alpha X + \beta Y = \alpha(x, 0) + \beta(y, 0) = (\alpha x + \beta y, 0) \in W_1$

So, W_1 is a subspace of $R^2(R)$.

Similarly we can prove that W_2 and W_3 are subspaces of $R^2(R)$.
Firstly we shall prove that V is the direct sum of W_1 and W_2

(i) i.e. $V = W_1 \oplus W_2$

Let $(x, y) \in V \Rightarrow (x, y) = (x, 0) + (0, y)$ where $(x, 0) \in W_1$ and $(0, y) \in W_2$

$\Rightarrow (x, y) \in W_1 + W_2 \Rightarrow V = W_1 + W_2$

Again, let $(x, y) \in W_1 \cap W_2$

$\Rightarrow (x, y) \in W_1$ and $(x, y) \in W_2 \Rightarrow y = 0$ and $x = 0$.

$\Rightarrow (x, y) = 0$ i.e. $W_1 \cap W_2 = \{0\}$.

So, we get $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$

Hence $V = W_1 \oplus W_2$.

(ii) Now, we shall prove $V = W_1 \oplus W_3$. Let $(x, y) \in V$ be arbitrary element.

Then $(x, y) = (x - y, 0) + (y, y)$ where $(x - y, 0) \in W_1$ and $(y, y) \in W_2$

So, $(x, y) \in W_1 + W_2$ i.e. $V = W_1 + W_2$

Again, let $(x, y) \in W_1 \cap W_3 \Rightarrow y = 0$ and $x = y$ i.e. $x = 0$, $y = 0$

i.e. $(x, y) = 0$ i.e. $W_1 \cap W_2 = \{0\}$

So, we get $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$. Hence $V = W_1 \oplus W_2$.

Exercise 1.3

Example 1 : Express the vector $v = (4, -5, 9, -7)$ as a linear combination vectors $v_1 = (1, 1, -2, 1)$, $v_2 = (3, 0, 4, -1)$, $v_3 = (-1, 2, 5, 2)$.

Solution : Let there exist scalars a, b and c such that $v = av_1 + bv_2 + cv_3$.

$$(4, -5, 9, -7) = a(1, 1, -2, 1) + b(3, 0, 4, -1) + c(-1, 2, 5, 2) \quad \text{......(}$$

Equating coefficients of like term on both sides, we get

$$a + 3b - c = 4, \quad a + 2c = -5, \quad -2a + 4b + 5c = 9, \quad a - b + 2c = -7$$

From 2nd equation, we get $a = -5 - 2c$.

Using this value in the remaining equations and solving, we get $a = -3$, $b = 2$, $c = -1$.

Using these values in (1), we obtain

Hence
$$(4, -5, 9, -7) = (-3)(1, 1, -2, 1) + 2(3, 0, 4, -1) + (-1)(-1, 2, 5, 2)$$
$$v = -3v_1 + 2v_2 - v_3$$

Example 2 : Is it possible to express the vector $v = (2, -5, 4)$ as a linear combinati of $v_1 = (1, -3, 2)$ and $v_2 = (2, -1, 1)$.

Solution : Let a and $b \in R$ be two scalars such that

$$(2, -5, 4) = a(1, -3, 2) + b(2, -1, 1)$$
$$\Rightarrow (2, -5, 4) = (a + 2b, 3a - b, 2a + b)$$
$$\Rightarrow a + 2b = 2$$
$$-3a - b = -5$$
$$2a + b = 4$$

On solving first two equations, we get, $a = \frac{8}{5}$, $b = \frac{1}{5}$

Putting these values in third equation, we get $2\left(\frac{8}{5}\right)+\left(\frac{1}{5}\right) = \frac{16}{5}+\frac{1}{5} = \frac{17}{5} \neq 4$

Hence the given vector v can not be expressed as a linear combination of v_1 and v_2.

Example 3 : Consider the vector $v = (1, -2, k)$ in $R^3(R)$. For what value of k (if any) the vector v can be expressed as a linear combination of vectors $v_1 = (3, 0, -2)$ and $v_2 = (2, -1, -5)$?

Solution : Let a and b are any two scalars such that $v = av_1 + bv_2$.

i.e. $(1, -2, k) = a(3, 0, -2) + b(2, -1, -5)$

$\Rightarrow$ $(1, -2, k) = (3a + 2b, -b, -2a - 5b)$

$\Rightarrow$ $3a + 2b = 1$

$-b = -2$

$-2a - 5b = k$

Second equation gives $b = 2$

Using this value of b in first equation, we get

$3a + 4 = 1 \quad \Rightarrow \quad 3a = -3$

$\Rightarrow$ $a = -1$ i.e. $a = -1$, $b = 2$

Using these values in last equation we get

$-2.(-1) - 5(2) = k$ i.e. $k = 2 - 10 = -8$

Example 4 : Express the vector $v = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ in the vector space of 2×2 matrices as a linear combination of $v_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

Solution : Let a , b , c be any scalars such that

$$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = a\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + c\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} a & a \\ 0 & -a \end{bmatrix} + \begin{bmatrix} b & b \\ -b & 0 \end{bmatrix} + \begin{bmatrix} c & -c \\ 0 & 0 \end{bmatrix}$$

$$\text{i.e.} \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} a+b+c & a+b-c \\ -b & -a \end{bmatrix}$$

$\Rightarrow$ $a + b + c = 3$

$a + b - c = -1$

$-b = 1$

$-a = -2$ i.e. $a = 2$, $b = -1$

Using these values of a and b in first two equations we get $c = 2$.

So, $v = 2v_1 - v_2 + 2v_3$.

Example 5 : Find the condition on a, b, c such that the matrix $\begin{bmatrix} a & -b \\ b & c \end{bmatrix}$ is a li combination of $v_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

Solution : Consider the scalars a_1, a_2, a_3 such that

$$\begin{bmatrix} a & -b \\ b & c \end{bmatrix} = a_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & -b \\ b & c \end{bmatrix} = \begin{bmatrix} a_1 + a_2 + a_3 & a_1 + a_2 - a_3 \\ -a_2 & -a_1 \end{bmatrix}$$

$$\Rightarrow \quad a_1 + a_2 + a_3 = a$$

$$a_1 + a_2 - a_3 = -b$$

$$-a_2 = b \;, \; -a_1 = c$$

From last two equations, we get $a_1 = -c$, $a_2 = -b$

From first two equations, we get $a_3 = -c$

So , $a_1 + a_2 + a_3 = a \Rightarrow -c - b - c = a \Rightarrow a + b + 2c = 0$

Example 6 : If S and T are two non-empty subsets of a vector space V(F), then s that (a) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$

(b) $L(S \cup T) = L(S) + L(T)$ (c) $L(L(S)) = L(S)$

Solution : (a) We prove that $S \subseteq T \Rightarrow L(S) \subseteq L(T)$

Let $u \in L(S)$, so u must be linear combination of finite elements of S but $S \subseteq T$

$\Rightarrow$ u must be linear combination of finite elements of T i.e. $u \in L(T)$.

So, $u \in L(S) \Rightarrow u \in L(T)$ i.e. $L(S) \subseteq L(T)$

(b) We prove that $L(S \cup T) = L(S) + L(T)$. Since $S \subseteq S \cup T \Rightarrow L(S) \subseteq L(S \cup T)$ ....

Again $T \subseteq S \cup T \Rightarrow L(T) \subseteq L(S \cup T)$ ....

From (1) and (2), we get

$$L(S) + L(T) \subseteq L(S \cup T)$$ ....

Now, let $u \in L(S \cup T)$

$\Rightarrow$ there exists $x_1, x_2, \ldots\ldots\ldots, x_n \in S \cup T$ and scalars $\alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_n \in F$ such t

$u = \sum_{i=1}^{n} \alpha_i x_i \Rightarrow u = \sum \alpha_j x_j + \sum \alpha_k x_k$, where $x_j's \in S$ and remaining $x_k's \in T$

[Since each x_i is either an element of S or an element of T or an element of bot dividing the elements x_j into elements belonging to S and belonging to T]

$\Rightarrow$ $u \in L(S) + L(T)$ i.e. $u \in L(S \cup T) \Rightarrow u \in L(S) + L(T)$

$\Rightarrow$ $L(S \cup T) \subseteq L(S) + L(T)$ ....

From (3) and (4), we get $L(S \cup T) = L(S) + L(T)$

(c) Now we prove that $L(L(S)) = L(S)$

Since $L(S)$ is a subspace of V(F). So $L(L(S)) = L(S)$.

Example 7 : Is the vector $(3,-1,0,-1)$ in the subspace of R^4 spanned by the vectors $v_1 = (2,-1,3,2)$, $v_2 = (-1,1,1,-3)$ and $v_3 = (1,1,9,-5)$?

Solution : Let a , b , c are any scalars so that

$$(3\,,\,-1\,,\,0\,,\,-1) = a(2\,,\,-1\,,\,3\,,\,2) + b(-1\,,\,1\,,\,1\,,\,-3) + c(1\,,\,1\,,\,9\,,\,-5)$$

$\Rightarrow$ $(3,\,-1\,,\,0\,,\,-1) = (2a-b+c\,,\ -a+b+c\,,\ 3a+b+9c\,,\ 2a-3b-5c)$

$\Rightarrow$
$$2a-b+c=3$$
$$-a+b+c=-1$$
$$3a+b+9c=0$$
$$2a-3b-5c=-1$$

Solving first two equations, we get $a+2c=2 \Rightarrow a=2-2c$

Using this value in third equation, we get $b+3c=2 \Rightarrow a=-6-3c$

Now using these values of a and b in the last equation, we get

$$22=-1, \text{ which is not possible.}$$

So, the given vector $(3\,,-1\,,0\,,-1)$ is not in the subspace of R^4 spanned by v_1, v_2, v_3.

Example 8 : Let $V_3 = R^3\,(R)$ and $S = \{v_1 = (1,1,0),\ v_2 = (0,-1,1),\ v_3 = (1,0,1)\}$. Prove that $(a\,,b\,,c) \in L(S)$ iff $a = b + c$.

Solution : By definition of $L(S)$, $(a,b,c) \in L(S)$

$$\Leftrightarrow \quad (a,b,c) = p(1,1,0) + q(0,-1,1) + r(1,0,1)$$
$$\Leftrightarrow \quad (a,b,c) = (p+r,\ p-q,\ q+r)$$
$$\Leftrightarrow \quad a=p+r,\quad b=p-q,\quad c=q+r$$
$$\Leftrightarrow \quad a=b+c$$

Example 9 : In the complex vector space $V_2\,(C)$ does $(1+i, 1-i)$ belongs to $<(1+i,1), (1,1-i)>$

Solution : By definition of linear span, $(1+i,1-i) \in <(1+i,1),(1,1-i)>$ if there exist scalars a, b such that $(1+i,1-i) = a(1+i,1)+b(1,1-i)$.

Equating the coefficients of like term on both sides, we get

$$a(1+i)+b=1+i \quad \text{and} \quad a+b(1-i)=1-i$$

On solving these equations, we obtain $a=1+i,\ b=1-i$.

Hence, $$(1+i,1-i) \in <(1+i,1),(1,1-i)>$$

Example 10 : Which of the following polynomials belong to the vector space generated by $\{x^3,\ x^2+2x\ ,\ x^2+2\ ,\ 1-x\}$

(i) $3x^2+x+5$ **(ii)** $3x^2+x+5$ **(iii)** x^4+7x+2

Solution : (i) Let $3x^2+x+5 = ax^3+b(x^2+2x)+c(x^2+2)+d(1-x)$

Equating the coefficients of like term on both sides, we get

$$a=0,\quad b+c=3,\quad 2b-d=1,\quad 2c+d=5$$

On solving these equations, we get

$$a=0,\quad b=3-c,\quad d=5-2c$$

Let us take $c = 1$, then $a = 0$, $b = 2$, $d = 3$.

Thus $3x^2+x+5$ can be expressed as a linear combination of $\{x^3, x^2+2x, x^2+2, 1-x$ and therefore belongs to the vector space generated by them.

(ii) Let $2x^3+3x^2+3x+7 = ax^3+b(x^2+2x)+c(x^2+2)+d(1-x)$

Equating the coefficients of like term on both sides, we get

$$a=2, \quad b+c=3, \quad 2b-d=3, \quad 2c+d=7$$

On solving these equations, we get

$$a=2, \quad b=3-c, \quad d=7-2c$$

Using these values in $2b-d=3$, we obtain

$$2(3-c)-(7-2c)=3$$

$\Rightarrow$ $6-2c-7+2c=3$

$\Rightarrow$ $6-7=3$

$\Rightarrow$ $-1=3$, which is not possible.

Thus $2x^3+3x^2+3x+7$ cannot be expressed as a linear combination of $\{x^3, x^2+2x, x^2+2, 1-x\}$ and therefore does not belong to the vector space generated b them.

(iii) Let $x^4+7x+2 = ax^3+b(x^2+2x)+c(x^2+2)+d(1-x)$.

Equating the coefficients of like terms on both sides, we get

$$a=0, \quad b+c=0, \quad 2b-d=7, \quad 2c+d=2$$

On solving these equations, we get $a=0, \quad b=-c, \quad d=-2c-7$

Using these values in $2c+d=2$, we obtain $2c-2c-7=2$

$\Rightarrow$ $-7=2$, which is not possible.

Thus x^4+7x+2 cannot be expressed as a linear combination of $\{x^3, x^2+2x, x^2+2, 1-$ and therefore does not belong to the vector space generated by them.

Exercise 1.4

Example 1 : Determine whether the following set of vectors are linearly dependent or independent

(i) $\{(1,1,1),(1,2,3),(0,1,2)\}$ in $\mathbf{R}^3(\mathbf{R})$

(ii) $\{(1,-2,1),(2,1,-1),(7,-4,1)\}$ in $\mathbf{R}^3(\mathbf{R})$

(iii) $\{(1,-2,3),(2,3,4),(0,1,2)\}$ in $\mathbf{R}^3(\mathbf{R})$

(iv) $\{(2,3,-1,-1),(1,-1,-2,-4),(3,1,3,-2),(6,3,0,-7)\}$ in $\mathbf{R}^4(\mathbf{R})$

(v) $\{(1,2,3,4),(0,1,0,0),(0,0,1,1),(0,1,0,0)\}$ in $\mathbf{R}^4(\mathbf{R})$

(vi) $\{(1,1,0,0),(0,1,-1,0),(0,0,0,3)\}$ in $\mathbf{R}^4(\mathbf{R})$

(vii) $\{(0,1,0,1,1),(1,0,1,0,1),(0,1,0,1,1),(1,1,1,1,1)\}$ in $\mathbf{R}^5(\mathbf{Q})$

(viii) $\{(1+i, 2i),(1, 1+i)\}$ in $\mathbf{C}^2(\mathbf{R})$

(ix) $\{(1+i, 2i),(1, 1+i)\}$ in $\mathbf{C}^2(\mathbf{C})$

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Solution : (i) We consider a matrix A whose columns are given vectors

i.e., $$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \quad [\text{Operating } R_2 \to R_2 - R_1 \,;\, R_3 \to R_3 - R_1]$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad [\text{Operating } R_3 \to R_3 - 2R_2]$$

which is in row echelon form. Here $\rho(A) = 2 <$ number of columns. So the given Vectors are L.D.

(ii) We construct a matrix A whose columns are given vectors

$$A = \begin{bmatrix} 1 & 2 & 7 \\ -2 & 1 & -4 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 5 & 10 \\ 0 & -3 & -6 \end{bmatrix} \quad [\text{Operating } R_2 \to R_2 + 2R_1 \,;\, R_3 \to R_3 - R_1]$$

$$\sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 5 & 10 \\ 0 & 0 & 0 \end{bmatrix} \quad [\text{Operating } R_3 \to R_3 + \frac{3R_2}{5}]$$

which is in row echelon form. Here $\rho(A) = 2 <$ number of columns. So the given vectors are L.D.

(iii) We construct a matrix A whose columns are given vectors

i.e. $$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 7 & 1 \\ 0 & -2 & 2 \end{bmatrix} \quad [\text{Operating } R_2 \to R_2 + 2R_1 \,;\, R_3 \to R_3 - 3R_1]$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & \frac{16}{7} \end{bmatrix} \quad [\text{Operating } R_3 \to R_3 + \frac{2}{7} R_2]$$

which is in row echelon form. Here $\rho(A) = 3 =$ number of columns. So the given vectors are linearly independent.

(iv) We construct a matrix A whose columns are given vectors.

i.e. $$A = \begin{bmatrix} 2 & 1 & 3 & 6 \\ 3 & -1 & 1 & 3 \\ -1 & -2 & 3 & 0 \\ -1 & -4 & -2 & -7 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 & 3 & 0 \\ 3 & -1 & 1 & 3 \\ 2 & 1 & 3 & 6 \\ -1 & -4 & -2 & -7 \end{bmatrix} \quad [\text{Operating } R_1 \leftrightarrow R_3]$$

$$\sim \begin{bmatrix} -1 & -2 & 3 & 0 \\ 0 & -7 & 10 & 3 \\ 0 & -3 & 9 & 6 \\ 0 & -2 & -5 & -7 \end{bmatrix} \quad [\text{Operating } R_2 \to R_2 + 3R_1 \,;\, R_3 \to R_3 + 2R_1 ;\, R_4 \to R_4 - R_1]$$

$$\sim \begin{bmatrix} -1 & -2 & 3 & 0 \\ 0 & 1 & -\frac{10}{7} & -\frac{3}{7} \\ 0 & -3 & 9 & 6 \\ 0 & -2 & -5 & -7 \end{bmatrix} \quad \text{[Operating } R_2 \to -\frac{R_2}{7}\text{]}$$

$$\sim \begin{bmatrix} -1 & -2 & 3 & 0 \\ 0 & 1 & -\frac{10}{7} & -\frac{3}{7} \\ 0 & 0 & \frac{33}{7} & \frac{33}{7} \\ 0 & 0 & -\frac{55}{7} & -\frac{55}{7} \end{bmatrix} \quad \text{[Operating } R_3 \to R_3 + 3R_2\text{ ; } R_4 \to R_4 + 2R_2$$

$$\sim \begin{bmatrix} -1 & -2 & 3 & 0 \\ 0 & 1 & -\frac{10}{7} & -\frac{3}{7} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \quad \text{[Operating } R_3 \to \frac{7}{33}R_3\text{ ; } R_4 \to \frac{7}{55}R_4\text{]}$$

$$\sim \begin{bmatrix} -1 & -2 & 3 & 0 \\ 0 & 1 & -\frac{10}{7} & -\frac{3}{7} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{[Operating } R_4 \to R_4 + R_3\text{]}$$

which is in row echelon form. Here $\rho(A) = 3 <$ number of columns. So, given vectors are L.D.

(v) We construct a matrix A whose columns are given vectors

$$\text{i.e. } A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{[Operating } R_2 \to R_2 - 2R_1\text{ ; } R_3 \to R_3 - 3R_1\text{ ; } R_4 \to R_4 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{[Operating } R_4 \to R_4 - R_3\text{]}$$

which is in row echelon form. Here ρ (A) = 3 < number of columns. So, given vectors are L.D.

(vi) We construct a matrix A whose columns are given vectors

$$\text{i.e.}\quad A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad [\text{Operating } R_2 \to R_2 - R_1]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad [\text{Operating } R_3 \to R_3 + R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad [\text{Operating } R_3 \leftrightarrow R_4]$$

which is in row echelon form. Here $\rho(A) = 3 =$ number of columns. So, given vectors are L.I.

(vii) We construct a matrix A whose columns are given vectors

$$\text{i.e.}\quad A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad [\text{Operating } R_1 \leftrightarrow R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad [\text{Operating } R_4 \to R_4 - R_1\ ;\ R_5 \to R_5 - R_1]$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad [\text{Operating } R_3 \to R_3 - R_2;\ R_5 \to R_5 - R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [\text{Operating } R_3 \leftrightarrow R_5]$$

which is in row echelon form. Here $\rho(A) = 3 <$ number of columns. So, given vectors are L.D.

(viii) We construct a matrix A whose columns are given vectors,

i.e.,
$$A = \begin{bmatrix} 1+i & 1 \\ 2i & 1+i \end{bmatrix} \sim \begin{bmatrix} 1+i & 1 \\ -2 & i-1 \end{bmatrix} \quad \text{(operating } R_2 \to R_2 - 2R_1\text{)}$$

Now, the matrix A cannot be reduced further to row echelon form as we have to choo scalars from the field of real numbers i.e., R. So, $\rho(A) = 2 =$ number of columns. Thus the given vectors are L.I.

(ix) We construct a matrix A whose column are given vectors

$$\text{i.e. } A = \begin{bmatrix} 1+i & 1 \\ 2i & 1+i \end{bmatrix} \sim \begin{bmatrix} 1+i & 1 \\ 0 & 0 \end{bmatrix} \quad \left[\text{Operating } R_2 \to R_2 - \frac{2i}{1+i} R_1\right]$$

which is in row echelon form. Here ρ (A) = 1 < number of columns. So given vector L.D.

Example 2 : In the vector space of polynomials of degree ≤ 4, which of the follo vectors are L.I.

(i) $x^2 + x + 1$, x , 1 **(ii) $x^3 + 2x + 1$, $x^3 - x + 1$, $x+1$**

(iii) $x^4 + x^3$, $x^4 - 1$, $x^3 + x^2$, $x^2 + x$, $x + 1$ **(iv) $x^4 - x$, $x^3 + 1$, $x^3 - 1$**

Solution : (i) Let a_1 , a_2 , a_3 be the scalars such that. $a_1(x^2 + x + 1) + a_2 x + a_3 = 0$

Comparing co-efficient of same power of x , we get

$$\left.\begin{aligned} a_1 + 0.a_2 + 0.a_3 &= 0 \\ a_1 + a_2 + 0.a_3 &= 0 \\ a_1 + 0.a_2 + a_3 &= 0 \end{aligned}\right] \quad \ldots\ldots(1)$$

$$\text{Then , determinant of co-efficient matrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

Thus the system of equations (1) has zero solution as the only solution i.e. $a_1 = a_2 = a_3$
Hence the given vectors are linearly independent.

(ii) Let a_1 , a_2 , a_3 be the scalars such that $a_1(x^3 + 2x + 1) + a_2(x^3 - x + 1) + a_3(x + 1) = 0$

Comparing coefficient of same power of x , we get

$$\begin{aligned} a_1 + a_2 + 0.a_3 &= 0 \\ 0.a_1 + 0.a_2 + 0.a_3 &= 0 \\ 2a_1 - a_2 + a_3 &= 0 \\ a_1 + a_2 + a_3 &= 0 \end{aligned}$$

i.e.

$$\left.\begin{aligned} a_1 + a_2 + 0.a_3 &= 0 \\ 2a_1 - a_2 + a_3 &= 0 \\ a_1 + a_2 + a_3 &= 0 \end{aligned}\right] \quad \ldots\ldots(1)$$

From first and last equation we get , $a_3 = 0$
So last two equations become

$$\begin{aligned} 2a_1 - a_2 &= 0 \\ a_1 + a_2 &= 0 \end{aligned}$$

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$$\Rightarrow \qquad a_1 = a_2 = 0$$

So, system (1) has only one solution which is $a_1 = a_2 = a_3 = 0$
Hence the given vectors are linearly independent.

(iii) Since $x^4 - 1 = (x^4 + x^3) - (x^3 + x^2) + (x + x^2) - (1 + x)$ which shows that one of the vector $x^4 - 1$ has been expressed as linear combination of the remaining vectors.
Hence the given vectors are L.D.

(iv) Consider scalars a_1, a_2, a_3 such that $a_1(x^4 - x) + a_2(x^3 + 1) + a_3(x^3 - 1) = 0$.
Comparing coefficient of like powers, we get

$$a_1 + 0.a_2 + 0.a_3 = 0 \qquad \text{......(i)}$$
$$0.a_1 + a_2 + a_3 = 0 \qquad \text{......(ii)}$$
$$-a_1 + 0.a_2 + 0.a_3 = 0 \qquad \text{......(iii)}$$
$$0.a_1 + a_2 - a_3 = 0 \qquad \text{......(iv)}$$

From (i) and (iii), we get $a_1 = 0$ and from equation (ii) and (iv), we get

$$a_2 + a_3 = 0$$
$$a_2 - a_3 = 0$$
$$\Rightarrow \qquad a_2 = a_3 = 0$$

So, we get $a_1 = a_2 = a_3 = 0$. Hence the given vectors are L.I.

Example 3 : Show that the three row vectors as well as the three column vectors of the matrix $\begin{bmatrix} 2 & 3 & 6 \\ 0 & 4 & 2 \\ 5 & 2 & 2 \end{bmatrix}$ **are linearly independent.**

Solution : The three row vectors in the given matrix are $(2, 3, 6)$, $(0, 4, 2)$ and $(5, 2, 2)$

Consider, $\qquad a_1(2, 3, 6) + a_2(0, 4, 2) + a_3(5, 2, 2) = 0$

$$\Rightarrow \qquad \left.\begin{aligned} 2a_1 + 5a_3 &= 0 \\ 3a_2 + 4a_2 + 2a_3 &= 0 \\ 6a_1 + 2a_2 + 2a_3 &= 0 \end{aligned}\right\} \qquad \text{......(1)}$$

Also, $\qquad \begin{vmatrix} 2 & 0 & 5 \\ 3 & 4 & 2 \\ 6 & 2 & 2 \end{vmatrix} \neq 0.$

Thus, the system of equations given by (1) has only the zero solution i.e.,
$a_1 = 0, a_2 = 0, a_3 = 0$.
Hence, the row vectors of the given matrix are linearly independent.
Similarly, the three column vectors $(2, 0, 5)$, $(3, 4, 2)$ and $(6, 2, 2)$ are linearly independent.

Example 4 : Prove that the four vectors (1, 0, 0), (0, 1, 0), (0, 0, 1) and (1, 1, 1) are linearly independent but any three of them are linearly independent .

Solution : Let $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, $v_3 = (0, 0, 1)$ and $v_4 = (1, 1, 1)$

Then clearly $v_4 = v_1 + v_2 + v_3$. So, $\{v_1, v_2, v_3, v_4\}$ are L.D.

Now consider the vectors v_2, v_3, v_4.

Consider, $$av_2 + bv_3 + cv_4 = 0 = (0, 0, 0)$$

$$\Rightarrow \quad a(0, 1, 0) + b(0, 0, 1) + c(1, 1, 1) = (0, 0, 0)$$

$$\Rightarrow \quad c = 0, a + c = 0, b + c = 0$$

$$\Rightarrow \quad a = 0, b = 0, c = 0$$

Thus the set $\{v_2, v_3, v_4\}$ is L.I.

Similarly, we can show that any three of v_1, v_2, v_3, v_4 are L.I.

Example 5 : Prove that the non – zero rows in an echelon matrix form a linearly independent set.

Solution : If the echelon matrix is a null matrix, then the set of non-zero rows in A is ϕ and hence 0 is linearly independent.

Let us consider the echelon matrix A in which not all the rows are zero

i.e. $$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_l \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Suppose $\quad a_1R_1 + a_2R_2 + \dots\dots\dots + a_lR_l = 0$ for some sclars $a_1, a_2, \dots\dots\dots, a_l$ ......(1)

Now, $R_1, R_2, \dots\dots\dots, R_l$ are all the non – zero rows having a non – zero entry (leading element) in each row; in such a manner that all other elements below the leading element of a particular row are all zero.

If λ_1 is leading element of R_1, then from (1), we have :

$$a_1\lambda_1 = 0 \Rightarrow a_1 = 0$$

Proceeding in the same manner, we get

$$a_2 = a_3 = \dots\dots\dots = a_l = 0$$

So, all the coefficient a_i's are zero. Hence $\{R_1, R_2, \dots\dots\dots, R_l\}$ are L.I.

xample 6 : Find α if the vectors $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} \alpha \\ 0 \\ 1 \end{bmatrix}$ **are linearly dependent.**

Solution : Consider $a_1\begin{bmatrix}1\\-1\\3\end{bmatrix}+a_2\begin{bmatrix}1\\2\\-3\end{bmatrix}+a_3\begin{bmatrix}\alpha\\0\\1\end{bmatrix}=0$, where a_1, a_2, a_3 are scalars not all zero.

$$\Rightarrow \quad \begin{aligned} a_1+a_2+\alpha a_3 &= 0 \\ -a_1+2a_2+0.a_3 &= 0 \\ 3a_1-3a_2+a_3 &= 0 \end{aligned}$$

$$\Rightarrow \quad \begin{vmatrix}1 & 1 & \alpha\\ -1 & 2 & 0\\ 3 & -3 & 1\end{vmatrix}=0 \qquad \text{[Since } a_1, a_2, a_3 \text{ are not all zero]}$$

i.e. $\quad 1(2-0)-1(-1-0)+\alpha(3-6)=0$

$\Rightarrow \quad 2+1-3\alpha=0 \quad \Rightarrow \quad \alpha=1$

Example 7 : If u, v, w are linearly independent in V(F) where F is any subfield of C, then show that the vectors $u+v$, $v-w$, $u-2v+w$ are linearly independent.

Solution : Let $a, b, c \in F$ be the scalars such that

$$a(u+v)+b(v-w)+c(u-2v+w)=0$$

$\Rightarrow \quad (a+c)u+(a+b-2c)v+(-b+c)w=0$

$\Rightarrow \quad a+c=0, \; a+b-2c=0$

$\quad -b+c=0 \qquad$ [Since u,v,w are L.I.]

$\quad (a+c)-(-b+c)=0$

$\Rightarrow \quad a+b=0$

Now, using $a+b-2c=0$ and $a+b=0$, we get $c=0$

Also, $-b+c=0$, $c=0$, $a+c=0 \Rightarrow a=0, b=0$

So, $a=b=c=0$ and hence the vectors $u+v, v-w, u-2v+w$ are linearly independent.

Example 8 : Prove that the vectors (a_1, a_2) and (b_1, b_2) in $V_2(F)$ are linearly dependent if $a_1b_2-a_2b_1=0$.

Solution : We know that two vectors (a_1, a_2) and (b_1, b_2) are linearly dependent iff one is a scalar multiple of other.

i.e. iff $(a_1, a_2) = \lambda(b_1, b_2)$ for some $\lambda(\neq 0)\in F$

i.e. iff $a_1=\lambda b_1$, $a_2=\lambda b_2$, i.e. iff $\frac{a_1}{b_1}=\frac{a_2}{b_2}$, i.e. iff $a_1b_2-a_2b_1=0$.

Example 9 : If v_1 and v_2 are vectors in V and $a, b \in F$, show that the set $\{v_1, v_2, av_1+bv_2\}$ is linearly dependent.

Solution : Clearly the vector av_1+bv_2 is a linear combination of the vectors v_1 and v_2. Hence the vectors $\{v_1, v_2, av_1+bv_2\}$ are linearly dependent.

Example 10 : Show that if $\{v_1, v_2, \ldots\ldots, v_n\}$ is a linearly independent set of vecto and $a_1, a_2, \ldots, a_{n-1}$ are any scalars, then the set $\{v_1 - a_1 v_n, v_2 - a_2 v_n, \ldots, v_{n-1} - a_{n-1} v_n$ is also linearly independent.

Solution : Let $b_1, b_2, \ldots\ldots, b_{n-1}$ be any scalars such that

$$b_1(v_1 - a_1 v_n) + b_2(v_2 - a_2 v_n) + \ldots\ldots + b_{n-1}(v_{n-1} - a_{n-1} v_n) = 0 \quad \ldots\ldots(1)$$

$$\Rightarrow \quad b_1 v_1 + b_2 v_2 + \ldots\ldots + b_{n-1} v_{n-1} - (a_1 b_1 + a_2 b_2 + \ldots\ldots + a_{n-1} b_{n-1}) v_n = 0$$

$$\Rightarrow \quad b_1 = 0,\ b_2 = 0, \ldots\ldots, b_{n-1} = 0 \quad \text{and} \quad a_1 b_1 + a_2 b_2 + \ldots\ldots + a_{n-1} b_{n-1} = 0,$$ sinc

$v_1, v_2, \ldots, v_n$ are linearly independent vectors.

As $b_1 = b_2 = \ldots\ldots = b_{n-1} = 0$, therefore from (1), it follows that the set

$\{v_1 - a_1 v_n, v_2 - a_2 v_n, \ldots\ldots, v_{n-1} - a_{n-1} v_n\}$ is linearly independent.

Example 11 : If $v_1, v_2, \ldots\ldots\ldots, v_n$ are linearly independent vectors of a vector spac V, then show that none of them can be a zero vector.

Solution : Let, if possible, $v_i = 0$ for some i, $1 \le i \le n$, then

$$0v_1 + 0v_2 + \ldots\ldots + 0v_{i-1} + 1v_i + 0v_i + 0v_{i+1} + \ldots. + 0v_n = 0$$

But $1 \ne 0$. This shows that $v_1, v_2, \ldots.., v_n$ are linearly dependent, which is a contradiction a $v_1, v_2, \ldots.., v_n$ are given to be linearly independent vectors. Hence none of $v_1, v_2, \ldots.., v_n$ ca be a zero vector.

Example 12 : Let S be a set of four vectors such that any three of them are linearly independent. Does it follow that the four vectors are linearly independent ?

Solution : In the set of vectors $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$ any three vectors are linearly independent but these four vectors are L.D.

Exercise 1.5

Example 1 : Determine whether or not the following sets form a basis of $R^3(R)$.

(i) $\{(4, 3, 2), (2, 1, 0), (-1, 1, -1)\}$

(ii) $\{(1, -3, 2), (2, 4, 1), (1, 1, 1)\}$

(iii) $\{(1, 0, 0), (1, 1, 0), (4, 5, 0)\}$

(iv) $\{(2, 4, -3), (0, 1, 1), (0, 1, -1)\}$

(v) $\{(1, 2, 1), (5, 0, -1), (2, -1, 0), (5, 2, 1)\}$

Solution : (i) The given set of vectors is $\{(4, 3, 2), (2, 1, 0), (-1, 1, -1)\}$

Let there exists scalars a_1, a_2, a_3 such that

$$a_1(4, 3, 2) + a_2(2, 1, 0) + a_3(-1, 1, -1) = 0$$

$$\Rightarrow \left.\begin{aligned} 4a_1 + 2a_2 - a_3 &= 0 \\ 3a_1 + a_2 + a_3 &= 0 \\ 2a_1 + 0.a_2 - a_3 &= 0 \end{aligned}\right\} \quad \ldots\ldots(1)$$

Also ,
$$\begin{vmatrix} 4 & 2 & -1 \\ 3 & 1 & 1 \\ 2 & 0 & -1 \end{vmatrix} = 8 \neq 0$$

which shows that the system of equations in (1) has only zero solution.

$\Rightarrow$ The given set of vectors is linearly independent.

Also , if $(x , y , z) \in R^3$ be arbitrary ,

then $(x , y , z) = a_1(4 , 3 , 2) + a_2(2 , 1 , 0) + a_3(-1 , 1 , -1)$

$$\Rightarrow \quad 4a_1 + 2a_2 - a_3 = x \qquad \text{......(2)}$$

$$3a_1 + a_2 + a_3 = y \qquad \text{......(3)}$$

$$2a_1 - a_3 = z \qquad \text{......(4)}$$

From (2) and (3) , we get $7a_1 + 3a_2 = x + y$

From (3) and (4) , we get $5a_1 + a_2 = y + z$

$$\Rightarrow \quad a_1 = -\frac{1}{8}(x - 2y - 3z)$$

$$a_2 = \frac{1}{8}(5x - 2y - 7z)$$

$$a_3 = -\frac{1}{4}(x - 2y + z)$$

which shows the existence of a_1 , a_2 , a_3 such that any $(x , y , z) \in R^3$ can be expressed as a linear combination of the given vectors. Hence the given vectors form a basis of R^3.

(ii) Consider $a_1(1, -3, 2) + a_2(2, 4, 1) + a_3(1, 1, 1) = 0$

$$\Rightarrow \quad \left.\begin{aligned} a_1 + 2a_2 + a_3 &= 0 \\ -3a_1 + 4a_2 + a_3 &= 0 \\ 2a_1 + a_2 + a_3 &= 0 \end{aligned}\right\} \qquad \text{......(1)}$$

Then ,
$$\begin{vmatrix} 1 & 2 & 1 \\ -3 & 4 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 1(4-1)\_2(-3-2) + 1(-3-8)$$

$$= 3 - 2(-5) + (-11)$$

$$= 3 + 10 - 11 = 2 \neq 0$$

which shows that system of equations (1) has only the zero solution.

$\Rightarrow$ The given vectors are linearly independent.

Also, if $(x, y, z) \in R^3(R)$ be arbitrary, then

$$(x, y, z) = a_1(1, -3, 2) + a_2(2, 4, 1) + a_3(1, 1, 1)$$

$$\Rightarrow \quad a_1 + 2a_2 + a_3 = x \qquad \text{......(2)}$$

$$-3a_1 + 4a_2 + a_3 = y \qquad \text{......(3)}$$

$$2a_1 + a_2 + a_3 = z \qquad \text{......(4)}$$

From (2) and (3), we get $4a_1 - 2a_2 = x - y$

From (3) and (4), we get $-5a_1 + 3a_2 = y - z$

$$\Rightarrow \quad a_1 = \frac{1}{2}(3x - y - 2z)$$

$$a_2 = \frac{1}{2}(5x - y - 4z)$$

$$a_3 = \frac{1}{2}(-11x + 3y + 10z)$$

which shows the existence of a_1, a_2, a_3 such that any $(x, y, z) \in R^3$ can be expressed linear combination of the given vectors. Hence the given vectors form a basis of R^3.

(iii) Consider $\quad a_1(1, 0, 0) + a_2(1, 1, 0) + a_3(4, 5, 0) = 0$

$$\Rightarrow \quad \left.\begin{aligned} a_1 + a_2 + 4a_3 &= 0 \\ a_2 + 5a_3 &= 0 \end{aligned}\right] \quad \ldots\ldots$$

Then $\quad \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{vmatrix} = 0$

which shows that system of equations (1) has non zero solution. So, given vectors L.D., and therefore cannot form a basis for R^3.

(iv) Consider $\quad a_1(2, 4, -3) + a_2(0, 1, 1) + a_3(0, 1, -1) = 0$

$$\Rightarrow \quad \left.\begin{aligned} 2a_1 + 0.a_2 + 0.a_3 &= 0 \\ 4a_1 + a_2 + a_3 &= 0 \\ -3a_1 + a_2 - a_3 &= 0 \end{aligned}\right] \quad \ldots\ldots(1)$$

Then $\quad \begin{vmatrix} 2 & 0 & 0 \\ 4 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 2(-1-1) = -4 \neq 0$

which shows that the system of equations in (1) has only zero solution.

$\Rightarrow$ The given set of vectors is L.I.

Now let $(x, y, z) \in R^3$ be arbitrary.

Then $(x, y, z) = a(2, 4, -3) + b(0, 1, 1) + c(0, 1, -1)$

$$\Rightarrow \quad 2a = x$$

$$4a + b + c = y$$

$$-3a + b - c = z$$

Now first equation gives $\quad a = \frac{x}{2}$

Using $a = \frac{x}{2}$ in second and third equation, we get

$$b + c = y - 2x \quad , \quad b - c = z + \frac{3}{2}x$$

$$\Rightarrow \quad b = \frac{1}{2}\left[y + z - \frac{1}{2}x\right] \quad , \quad c = \frac{1}{2}\left[y - 2x - z - \frac{3}{2}x\right] = \frac{1}{2}\left[y - z - \frac{7}{2}x\right]$$

So, $\quad (x, y, z) = \frac{x}{2}(2, 4, -3) + \frac{1}{2}(y + z - \frac{1}{2}x)(0, 1, 1) + \frac{1}{2}(y - z - \frac{7}{2}x)(0, 1, -1)$

Hence the given vectors form a basis of R^3.

(v) Clearly, any basis of R^3 has three vectors. But the given set contains four vectors. So the given set cannot form a basis of $R^3(R)$.

Example 2 : Determine whether or not the following sets form a basis of $R^4(R)$.

(i) $\{(1,0,0,0), (1,2,0,0), (1,2,3,0), (1,2,0,4)\}$

(ii) $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

(iii) $\{(1,-2,5,-3), (0,7,-9,2), (0,0,1,0), (0,0,0,1)\}$.

Solution : (i) Firstly, we shall show that given vectors are L.I.

Consider $a_1(1,0,0,0)+a_2(1,2,0,0)+a_3(1,2,3,0)+a_4(1,2,0,4)=0$

$$\Rightarrow \quad a_1+a_2+a_3+a_4=0$$
$$2a_2+2a_3+2a_4=0$$
$$3a_3=0$$
$$4a_4=0$$
$$\Rightarrow \quad a_1=0,\ a_2=0,\ a_3=0,\ a_4=0$$

Hence the given vectors are L.I.

Also if $(x, y, z, t) \in R^4$ be arbitrary, then

$$(x, y, z, t)=a_1(1,0,0,0)+a_2(1,2,0,0)+a_3(1,2,3,0)+a_4(1,2,0,4) \quad \ldots\ldots(1)$$
$$\Rightarrow \quad a_1+a_2+a_3+a_4=x$$
$$2a_2+2a_3+2a_4=y$$
$$3a_3=z, \qquad 4a_4=t$$
$$\Rightarrow \quad a_3=\frac{z}{3},\ a_4=\frac{t}{4},\ a_2=\frac{1}{12}(6y-4z-3t),\ a_1=\frac{2x-y}{2}$$

which shows that existence of a_1, a_2, a_3, a_4 such that (1) holds

Thus any vector (x, y, z, t) can be expressed as linear combination of given vectors. Hence the given set of vectors form a basis of R^4.

(ii) Consider, $a_1(1,0,0,0)+a_2(0,1,0,0)+a_3(0,0,1,0)+a_4(0,0,0,1)=0$

$$\Rightarrow \quad a_1=0,\ a_2=0,\ a_3=0,\ a_4=0$$

which shows that the given vectors are linearly independent.

Also, if $(x, y, z, t) \in R^4(R)$ be arbitrary, then

$$(x, y, z, t)=a_1(1,0,0,0)+a_2(0,1,0,0)+a_3(0,0,1,0)+a_4(0,0,0,1)$$

Then, clearly $a_1=x,\ a_2=y,\ a_3=z,\ a_4=t$.

which shows the existence of a_1, a_2, a_3, a_4 such that any $(x, y, z, t) \in R^4(R)$ can be expressed as a linear combination of the given vectors. Hence the given vectors form a basis of $R^4(R)$.

(iii) Consider, $a_1(1,-2,5,-3)+a_2(0,7,-9,2)+a_3(0,0,1,0)+a_4(0,0,0,1)=0$

$$\Rightarrow \quad a_1=0,\ -2a_1+7a_2=0,\ 5a_1-9a_2+a_3=0,\ -3a_1+2a_2+a_4=0.$$

Solving these equations, we get

$$a_1=0,\ a_2=0,\ a_3=0,\ a_4=0$$

which shows that the given vectors are linearly independent.

Also, if $(x, y, z, t) \in R^4(R)$ be arbitrary, then

$$(x, y, z, t) = a_1(1, -2, 5, -3) + a_2(0, 7, -9, 2) + a_3(0, 0, 1, 0) + a_4(0, 0, 0, 1)$$

$$\Rightarrow \quad a_1 = x, \quad -2a_1 + 7a_2 = y, \quad 5a_1 - 9a_2 + a_3 = z, \quad -3a_1 + 2a_2 + a_4 = t$$

Solving these equations, we get

$$a_1 = x, \ a_2 = \frac{y+2x}{7}, \ a_3 = -17x + \frac{9y}{7} + z, \ a_4 = \frac{17x}{7} - \frac{2y}{7} + t$$

which shows the existence of a_1, a_2, a_3, a_4 such that any $(x, y, z, t) \in R^4(R)$ can be expressed as a linear combination of the given vectors. Hence the given vectors form a basis of $R^4(R)$.

Example 3 : Extend the following sets of vectors to form a basis of R^3.

(i) {(1, 2, 3), (2, −2, 0)} (ii) {(0, 1, 2), (2, −1, 4)} (iii) {(1, 1, 1), (1, 0, 0)}

Solution : (i) Let $v_1 = (1, 2, 3)$, $v_2 = (2, -2, 0)$

Now, we know that standard basis of R^3 is $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$

Consider the set $S = \{v_1, v_2, e_1, e_2, e_3\}$

Any basis of R^3 has three vectors, so only one vector out of $\{e_1, e_2, e_3\}$ is required to form a basis together with v_1 and v_2.

First we check v_1, v_2, e_1 for linear independence. Construct a matrix whose rows are v_1, v_2 and e_1

$$\text{i.e.} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & -6 \\ 0 & -2 & -3 \end{bmatrix} \quad [\text{Operating } R_2 \to R_2 - 2R_1;\ R_3 \to R_3 - R_1]$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & -6 \\ 0 & 0 & -1 \end{bmatrix} \quad [\text{Operating } R_3 \to R_3 - \frac{1}{3}R_2]$$

which is in row echelon form. Thus $\rho(A) = 3$ = number of vectors. Hence first three vectors are L.I. and hence form a basis of R^3.

(ii) We know that standard basis of R^3 is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

We consider the set $S = \{(0, 1, 2), (2, -1, 4), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

Clearly, any basis of R^3 has three vectors. So we check the first three vectors of S for linear independence. Consider a matrix whose rows are these three vectors.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & -1 & 4 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 4 \\ 0 & 1 & 2 \end{bmatrix} \quad (\text{operating } R_1 \leftrightarrow R_3)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 1 & 2 \end{bmatrix} \quad (\text{operating } R_2 \to R_2 - 2R_1)$$

$$\sim\begin{bmatrix}1 & 0 & 0\\0 & -1 & 4\\0 & 0 & 6\end{bmatrix} \quad (\text{operating } R_3 \to R_3 + R_2)$$

which is row echelon form.
Here, $\rho(A) = 3 =$ number of vectors

Thus the set $\{(0, 1, 2), (2, -1, 4), (1, 0, 0)\}$ is a linearly independent set and hence it is a basis of R^3.

(iii) Let $v_1 = (1, 1, 1)$, $v_2 = (1, 0, 0)$

Now, we know that standard basis of R^3 is $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$

Consider the set $S = \{v_1, v_2, e_1, e_2, e_3\}$

Any basis of R^3 has three vectors. So only one vector out of $\{e_1, e_2, e_3\}$ is required to form a basis together with v_1 and v_2. First we choose v_1, v_2, e_1 for linear independence. Construct a matrix whose rows are v_1, v_2 and e_1

i.e. $$A = \begin{bmatrix}1 & 1 & 1\\1 & 0 & 0\\1 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 1\\1 & 0 & 0\\0 & 0 & 0\end{bmatrix} \quad [\text{Operating } R_3 \to R_3 - R_2]$$

$\rho(a) = 2 <$ number of vectors. Hence first three vectors of S are L.D. Now, we check the vectors v_1, v_2, e_2 for linear independence.

Construct a matrix whose rows are v_1, v_2, e_2.

$$B = \begin{bmatrix}1 & 1 & 1\\1 & 0 & 0\\0 & 1 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 1\\0 & -1 & -1\\0 & 1 & 0\end{bmatrix} \quad [\text{Operating } R_2 \to R_2 - R_1]$$

$$\sim \begin{bmatrix}1 & 1 & 1\\0 & -1 & -1\\0 & 0 & -1\end{bmatrix} \quad [\text{Operating } R_3 \to R_3 + R_2]$$

which is in row echelon form. Then $\rho(B) = 3 =$ number of vectors. Thus the three vectors v_1, v_2, e_2 i.e. $(1, 1, 1), (1, 0, 0), (0, 1, 0)$ are linearly independent and hence form a basis of R^3.

Example 4 : Extend the set $\{(0, 0, 0, 3), (1, 1, 0, 0), (0, 1, -1, 0)\}$ to form a basis of R^4.

Solution : Let $v_1 = (0, 0, 0, 3)$, $v_2 = (1, 1, 0, 0)$; $v_3 = (0, 1, -1, 0)$

Now, we know that standard basis of R^4 is

$$e_1 = (1, 0, 0, 0), \quad e_2 = (0, 1, 0, 0), \quad e_3 = (0, 0, 1, 0), \ e_4 = (0, 0, 0, 1)$$

Consider the set $S = \{v_1, v_2, v_3, e_1, e_2, e_3, e_4\}$

Any basis of R^4 has four vectors, so one vector out of $\{e_1, e_2, e_3, e_4\}$ is required to form a basis of R^4 together with v_1, v_2, v_3.

First we check vectors v_1, v_2, v_3, e_1 for linear independence. Construct a matrix whose rows are v_1, v_2, v_3, e_1

i.e. $A = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ [Operating $R_1 \leftrightarrow R_2$]

$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$ [Operating $R_3 \to R_3 - R_1$]

$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$ [Operating $R_4 \to R_4 + R_3$]

$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -1 & 0 \end{bmatrix}$ [Operating $R_2 \leftrightarrow R_3$]

$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ [Operating $R_3 \leftrightarrow R_4$]

which is in row echelon form. Here $\rho(A) = 4$ = number of vectors. Hence v_1, v_2, v_3, e_1 i.e. $(0, 0, 0, 3)$, $(1, 1, 0, 0)$, $(0, 1, -1, 0)$, $(1, 0, 0, 0)$ are linearly independent and hence form a basis of R^4.

Example 5 : Let $\{u, v, w\}$ be a basis for the vector space R^3. Prove that the sets $\{u+v, v+w, w+u\}$ and $\{u, u+v, u+v+w\}$ are also bases of R^3.

Solution : (i) Consider $a(u+v)+b(v+w)+c(w+u)=0$ where $a, b, c \in R$ ......(1)

$\Rightarrow$ $u(a+c)+v(a+b)+w(b+c)=0$ ......(2)

Since $\{u, v, w\}$ being basis of R^3 is linearly independent.

So, from (2), it follows that $a+c=0$, $a+b=0$, $b+c=0$ $\Rightarrow$ $a=b=c=0$

Thus from (1), we have that $\{u+v, v+w, u+w\}$ is linearly independent.

Also, $\dim R^3(R) = 3$. Hence $\{u+v, v+w, u+w\}$ is also a basis of $R^3(R)$.

(ii) Consider $a(u)+b(u+v)+c(u+v+w)=0$ where $a, b, c \in R$

$\Rightarrow$ $(a+b+c)u+(b+c)v+cw=0$ but since $\{u, v, w\}$ being basis of R^3 is linearly independent.

So, we have : $\begin{aligned} a+b+c&=0 \\ b+c&=0 \\ c&=0 \end{aligned}$ $\Rightarrow$ $a=b=c=0$.

Hence $\{u, u+v, u+v+w\}$ is linearly independent.

Also $\dim R^3(R) = 3$. $\{u, u+v, u+v+w\}$ is a basis of $R^3(R)$.

Example 6 : Extend the set $S = \{(1, 1, 0)\}$ to form two different bases of $R^3(R)$.

Solution : As we know that $(1, 1, 0) \neq (0, 0, 0)$. So, S is a linearly independent set and $L(S) = \{a(1, 1, 0) = (a, a, 0) : a \in R\}$.

Clearly, $(1, 0, 0) \notin L(S)$ and thus $S_1 = \{(1, 1, 0), (1, 0, 0)\}$ is a linearly independent set.

Now,
$$L(S_1) = \{a(1,1, 0) + b(1, 0, 0) : a, b \in R\}$$
$$= \{(a + b, a, 0) : a, b \in R\}$$

Clearly $(0, 1, 1) \notin L(S_1)$. Thus $S_2 = \{(1, 1, 0), (1, 0, 0), (0, 1, 1)\}$ is a linearly independent set and hence a basis of R^3.

Similarly, we can show that $\{(1, 1, 0), (0,1, 0), (0, 0, 1)\}$ is also a basis of R^3.

Example 7 : Show that the set $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ is a basis of $C^3(C)$ but not a basis of $C^3(R)$ or $C^3(Q)$.

Solution : First we check these vectors for linear independence. Let $a, b, c \in C$ be any arbitrary scalars such that
$$a(1,0, 0) + b(1, 1, 0) + c(1, 1, 1) = 0$$
$$\Rightarrow \quad (a+b+c, b+c, c) = (0, 0, 0)$$
$$\Rightarrow \quad a+b+c = 0$$
$$b+c = 0$$
$$c = 0$$

Solving these, we get $a = 0$, $b = 0$, $c = 0$.

Thus, the given vectors are linearly independent. Now, we show that every vector $(z_1, z_2, z_3) \in C^3$ can be written as a linear combination of these vectors.

Let $(z_1, z_2, z_3) \in C^3$ be any vector. Suppose $a, b, c \in C$ be any three scalars such that
$$(z_1, z_2, z_3) = a(1, 0, 0) + b(1, 1, 0) + c(1, 1, 1) \quad \text{......(1)}$$
$$\Rightarrow \quad (z_1, z_2, z_3) = (a+b+c, b+c, c)$$
$$\Rightarrow \quad z_1 = a+b+c$$
$$z_2 = b+c$$
$$z_3 = c$$

Solving these, we get $c = z_3$, $b = z_2 - z_3$, $a = z_1 - z_2$. Using these values in (1), we observe that every vector of C^3 can be written as a linear combination of the given vectors. Thus, the given set forms a basis for $C^3(C)$.

The given set of vectors is not a basis for $C^3(R)$ because if we take $(2i, i, i) \in C^3$ then clearly there exist no real numbers a, b, c such that
$$(2i, i, i) = a(1, 0, 0) + b(1, 1, 0) + c(1, 1, 1).$$

Similarly, the given set of vectors is not a basis for $C^3(Q)$.

Example 8 : In $R^3(R)$, find the coordinates of vector u w.r.t. a basis (u_1, u_2, u_3) where

(i) $u = (4, 5, 6)$; $u_1 = (-1, 1, 1)$, $u_2 = (-1, 1, 0)$, $u_3 = (1, 0, -1)$

(ii) $u = (1, 0, -1)$; $u_1 = (0, 1, -1)$, $u_2 = (1, 1, 0)$, $u_3 = (1, 0, 2)$.

Solution : (i) Consider $(4, 5, 6) = a_1(-1, 1, 1) + a_2(-1, 1, 0) + a_3(1, 0, -1)$

$$= (-a_1 - a_2 + a_3, a_1 + a_2, a_1 - a_3)$$

$$\Rightarrow \quad -a_1 - a_2 + a_3 = 4 \quad \text{...... (i)}$$

$$a_1 + a_2 = 5 \quad \text{...... (ii)}$$

$$a_1 - a_3 = 6 \quad \text{......(iii)}$$

From (iii) we get, $a_3 = a_1 - 6$. Putting this value in (i), we have

$$-a_1 - a_2 + a_1 - 6 = 4 \quad \Rightarrow \quad -a_2 - 6 = 4$$

$$\Rightarrow \quad -a_2 = +10 \quad \Rightarrow \quad a_2 = -10$$

Putting $a_2 = -10$ in (ii), we get $a_1 = 15$

So, from (i) we get, $a_3 = 9$

Hence the co-ordinates of $(4, 5, 6)$ relative to the basis (u_1, u_2, u_3) are $15, -10, 9$.

(ii) Consider, $u = a_1u_1 + a_2u_2 + a_3u_3$

$$\Rightarrow \quad (1, 0, -1) = a_1(0, 1, -1) + a_2(1, 1, 0) + a_3(1, 0, 2)$$

$$= (a_2 + a_3, a_1 + a_2, -a_1 + 2a_3)$$

$$\Rightarrow \quad a_2 + a_3 = 1, \quad a_1 + a_2 = 0, \quad -a_1 + 2a_3 = -1$$

On solving, we have $a_3 = -2$, $a_2 = 3$, $a_1 = -3$.

Hence, the coordinates of $u = (1, 0, -1)$ relative to the basis $\{u_1, u_2, u_3\}$ are $(-3, 3, -2)$.

Example 9 : In $C^3(C)$, find the coordinates of $(2i, 3+4i, 5)$ w.r.t. the basis $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$.

Solution : Consider, $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, $z_3 = a_3 + ib_3 \in C$

Then, $(2i, 3+4i, 5) = z_1(1, 0, 0) + z_2(1, 1, 0) + z_3(1, 1, 1)$

$$= (z_1 + z_2 + z_3, z_2 + z_3, z_3)$$

$$= ((a_1 + a_2 + a_3) + i(b_1 + b_2 + b_3), (a_2 + a_3) + i(b_2 + b_3), a_3 + ib_3)$$

$$\Rightarrow \quad a_1 + a_2 + a_3 = 0, \quad a_2 + a_3 = 3, \quad a_3 = 5$$

$$\Rightarrow \quad a_2 = -2 \text{ and } a_1 = -a_2 - a_3 = 2 - 5 = -3$$

$$\Rightarrow \quad a_1 = -3, \quad a_2 = -2, \quad a_3 = 5$$

Again $\quad b_1 + b_2 + b_3 = 2, \quad b_2 + b_3 = 4, \quad b_3 = 0$

$$\Rightarrow \quad b_2 = 4 \text{ and } b_1 = 2 - b_2 - b_3 = 2 - 4 = -2$$

i.e. $\quad b_1 = -2, \; b_2 = 4, \; b_3 = 0$

So, $z_1 = -3 - 2i$, $z_2 = -2 + 4i$, $z_3 = 5$

Hence the co-ordinates of $(2i, 3+4i, 5)$ relative to the basis $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ are $-3 - 2i, -2 + 4i, 5$.

Example 10 : Let V be the vector space of square matrices over R and let W be the sub-space generated by $\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix}, \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix}$. **Show that** $\left\{\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\right\}$ **form a basis and dim W = 2.**

Solution : Let S = $\{A_1, A_2, A_3, A_4\}$ where $A_1 = \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}$; $A_2 = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix}$, $A_4 = \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix}$.

Consider
$$a\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix} + b\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\Rightarrow \quad a = 0 \ , \ -5a + 2b = 0, \ -4a + b = 0 \ ; \ 2a + b = 0$$
$$\Rightarrow \quad a = 0 \ , \ b = 0$$

So, set $\left\{\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\right\}$ is linearly independent.

Now, let $u = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be arbitrary and let $\alpha, \beta \in R$ such that
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix} + \beta\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & -5\alpha \\ -4\alpha & 2\alpha \end{bmatrix} + \begin{bmatrix} 0 & 2\beta \\ \beta & \beta \end{bmatrix} = \begin{bmatrix} \alpha & -5\alpha+\beta \\ -4\alpha+\beta & 2\alpha+\beta \end{bmatrix}$$
$$\Rightarrow \quad \alpha = a, \ -5\alpha + \beta = b, \ -4\alpha + \beta = c, \ 2\alpha + \beta = d$$
$$\Rightarrow \quad \alpha = a, \ \beta = \frac{b + 5\alpha}{2} = \frac{b + 5a}{2}$$

which shows the existence of $\alpha, \beta \in R$ such that W is generated by these two vectors. Thus it forms a basis and dim W = 2.

Example 11 : Let V be the vector space of solutions of the differential equation $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$. **Show that V(R) is a 3 – dimensional real vector space. Find a basis of this vector space.**

Solution : The auxiliary equation of the given differential equation is
$$m^3 - 7m - 6 = 0$$
$$\Rightarrow \quad (m+1)(m^2 - m - 6) = 0$$
$$\Rightarrow \quad (m+1)(m-3)(m+2) = 0$$
$$\Rightarrow \quad m = -1, -2, 3$$

Thus any solution of the given differential equation is
$$y = ae^{-x} + be^{-2x} + ce^{3x} \quad \text{......(1)}$$
where a, b, c are arbitrary real constants.

It can be verified that the set

$$V = \left\{ y = ae^{-x} + be^{-2x} + ce^{3x} : a, b, c \in R \right\}$$ is a vector space over

Let $$S = \left\{ y_1 = e^{-x}, y_2 = e^{-2x}, y_3 = e^{3x} \right\}$$

The wronskian of y_1, y_2, y_3 is

$$w(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} & e^{3x} \\ -e^{-x} & -2e^{-2x} & 3e^{3x} \\ e^{-x} & 4e^{-2x} & 9e^{3x} \end{vmatrix}$$

$$= e^{-x} \cdot e^{-2x} \cdot e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ -1 & -2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= e^{-3x+3x} \begin{vmatrix} 1 & 0 & 0 \\ -1 & -1 & 4 \\ 1 & 3 & 8 \end{vmatrix} \quad \text{(operating } C_2 \to C_2 - C_1, C_3 \to C_3 - C_1)$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ -1 & -1 & 4 \\ 1 & 3 & 8 \end{vmatrix} = -20 \neq 0$$

Thus S is a linearly independent subset of V and from (1), $L(S) = V$.

Hence $S = \left\{ e^{-x}, e^{-2x}, e^{-3x} \right\}$ is a basis of V and dim $V = 3$.

Exercise 1.6

Example 1 : Determine a basis of the subspace spanned by the vectors

(i) (–3 , 1 , 2) (0 , 1 , 3) , (2 , 1 , 0) , (1 , 1 , 1)

(ii) (1 , 1 , 1) , (1 , 0 , –1) , (3 , –1 , 0) , (2 , 1 , – 2).

Solution : (i) We construct a matrix A whose rows are the given vectors i.e.

$$A = \begin{bmatrix} -3 & 1 & 2 \\ 0 & 1 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & 0 \\ -3 & 1 & 2 \end{bmatrix} \quad [\text{Operating } R_1 \leftrightarrow R_4]$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -2 \\ 0 & 4 & 5 \end{bmatrix} \quad [\text{Operating } R_3 \to R_3 - 2R_1 : R_4 \to R_4 + 3R_1]$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 4 & 5 \end{bmatrix} \quad [\text{Operating } R_3 \to R_3 + R_2]$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -7 \end{bmatrix} \text{[Operating } R_4 \to R_4 - 4R_2 \text{]}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{[Operating } R_4 \to R_4 + 7R_1 \text{]}$$

which is in row echelon form. Hence basis is $\{(1, 1, 1), (0, 1, 3), (0, 0, 1)\}$.

(ii) We construct a matrix A whose rows are the given vectors i.e.,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 3 & -1 & 0 \\ 2 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -4 & -3 \\ 0 & -1 & -4 \end{bmatrix} \text{(operating } R_2 \to R_2 - R_1,\ R_3 \to R_3 - 3R_1,\ R_4 \to R_4 - 2R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 5 \\ 0 & 0 & -2 \end{bmatrix} \text{(operating } R_3 \to R_3 - 4R_2,\ R_4 \to R_4 - R_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \left(\text{operating } R_4 \to R_4 + \frac{2}{5} R_3\right)$$

which is row echelon form. Hence basis is $\{(1, 1, 1), (1, 0, -1), (3, -1, 0)\}$.

Example 2 : (i) Find the dimension of subspace generated by (0, 0, 1) and (1, 1, 1).
(ii) Find the dimension of sub – space W of R^4 generated by (1, 4, –1, 3), (2, 1, –3, –1), (0, 2, 1, –5).

Solution : (i) Construct a matrix A whose rows are given vectors

i.e. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ which is in row echelon form.

So the two vectors $(1, 1, 1), (0, 0, 1)$ are linearly independent. Hence the dimension of subspace generated by $(0, 0, 1), (1, 1, 1)$ is two.

(ii) Construct a matrix with rows equal to given vectors.

$$\text{i.e. } A = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 2 & 1 & -3 & -1 \\ 0 & 2 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & -7 & -1 & -7 \\ 0 & 2 & 1 & -5 \end{bmatrix} \text{[Operating } R_2 \to R_2 - 2R_1\text{]}$$

$$\sim \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & -7 & -1 & -7 \\ 0 & 0 & \frac{5}{7} & -7 \end{bmatrix} \qquad [\text{Operating } R_3 \to R_3 + \frac{2}{7}R_2]$$

which is in row echelon form. Here $\rho(A) = 3$ = number of vectors. So these vectors are linearly independent and hence the dimension of the subspace generated by these vectors is 3.

Example 3 : Determine a basis and dimension of the following sub-space W of $R^3(R)$

(i) $W = \{(u_1, u_2, u_3) : u_2 = 2u_1, u_3 = u_1 + u_2\}$

(ii) $W = \{(u_1, u_2, u_3) : 2u_1 + u_2 - 3u_3 = 0\}$

Solution : (i) Here $W = \{(u_1, u_2, u_3) : u_i \in R, u_2 = 2u_1, u_3 = u_1 + u_2\}$

Let $(u_1, u_2, u_3) \in W$ be arbitrary. Note that $u_2 = 2u_1$, $u_3 = u_1 + u_2 = u_1 + 2u_1$

So, $u_3 = 3u_1$

Thus $(u_1, u_2, u_3) = (u_1, 2u_1, 3u_1) = (1, 2, 3)u$

Thus basis of W is $\{(1, 2, 3)\}$ and dim W = 1.

(ii) Here $W = \{(u_1, u_2, u_3) : u_i \in R, 2u_1 + u_2 - 3u_3 = 0\}$

Let $(u_1, u_2, u_3) \in W$ be arbitrary. Note that $2u_1 + u_2 - 3u_3 = 0$.

If we take $u_1 = 1$, $u_3 = 0$ then $u_2 = -2$.

Also if we set $u_1 = 0$, $u_3 = 1$, then $u_2 = 3$

Moreover, there exist no scalar λ for which $(1, -2, 0) = \lambda(0, 3, 1)$

i.e. $\{(1, -2, 0), (0, 3, 1)\}$ is linearly independent set and it spans W subject to condition $2u_1 + u_2 - 3u_3 = 0$

Further $u_1 = 1$, $u_2 = 1$ and $u_3 = 1$, satisfies the condition $2u_1 + u_2 - 3u_3 = 0$ but $(1, -2, 0)$, $(0, 3, 1)$ and $(1, 1, 1)$ are not linearly independent since

$$(1, 1, 1) = 1.(1, -2, 0) + 1.(0, 3, 1)$$

So, dim W cannot be 3. So, basis = $(1, -2, 0), (0, 3, 1)$ and dim W = 2.

Example 4 : Show that the subspace spanned by the vectors $v_1 = (1, 1, -1)$, $v_2 = (2, 3, -1)$, $v_3 = (3, 1, -5)$ and the subspace spanned by the vectors $w_1 = (1, -1, -3)$, $w_2 = (3, -2, -8)$, $w_3 = (2, 1, -3)$ are identical.

Solution : Let us express v_1, v_2 and v_3 as linear combination of w_1, w_2 and w_3.

Let $\quad (1, 1, -1) = a(1, -1, -3) + b(3, -2, -8) + c(2, 1, -3)$

$\Rightarrow \quad a + 3b + 2c = 1, \quad -a - 2b + c = 1, \quad -3a - 8b - 3c = -1$

Solving first two equations, we get

$$b + 3c = 2 \quad \Rightarrow \quad b = 2 - 3c$$

Using this value in third equation, we get

$$a = 7c - 5$$

Let us take $c = 1$, then we have $b = -1, a = 2$

$\Rightarrow \qquad v_1 = 2w_1 - w_2 + w_3$

Now, let $(2, 3, -1) = a(1, -1, -3) + b(3, -2, -8) + c(2, 1, -3)$

$\Rightarrow \qquad a + 3b + 2c = 2, \quad -a - 2b + c = 3, \quad -3a - 8b - 3c = -1$

Solving first two equations, we get

$b + 3c = 5 \quad \Rightarrow \quad b = 5 - 3c$

Using this value in third equation, we get $a = 7c - 13$

Let us take $c = 1$, then we have $a = -6$, $b = 2$

$\Rightarrow \qquad v_2 = -6w_1 + 2w_2 + w_3$

Now, let $(3, 1, -5) = a(1, -1, -3) + b(3, -2, -8) + c(2, 1, -3)$

$\Rightarrow \qquad a + 3b + 2c = 3, \quad -a - 2b + c = 1, \quad -3a - 8b - 3c = -5$

Solving first two equations, we get $b + 3c = 4 \quad \Rightarrow \quad b = 4 - 3c$

Using this value in third equation, we get $a = 7c - 9$

Let us take $c = 1$, then we have $a = -2$, $b = 1$. $\quad \Rightarrow \quad v_3 = -2w_1 + w_2 + w_3$

Again, we express w_1, w_2 and w_3 as linear combinations of v_1, v_2 and v_3.

Let $(1, -1, -3) = a(1, 1, -1) + b(2, 3, -1) + c(3, 1, -5)$

$\Rightarrow \qquad a + 2b + 3c = 1, \quad a + 3b + c = -1, \quad -a - b - 5c = -3$

Solving first and third equation, we get $b = 2c - 2$

Using this value in second equation, we get $a = 5 - 7c$

If $c = 1$, then we have $a = -2$, $b = 0$ $\quad \Rightarrow \quad w_1 = -2v_1 + 0v_2 + v_3$

Now, let $(3, -2, -8) = a(1, 1, -1) + b(2, 3, -1) + c(3, 1, -5)$

$a + 2b + 3c = 3, \quad a + 3b + c = -2, \quad -a - b - 5c = -8$

Solving first two equations, we get $= 2c - 5$.

Using this value in third equation, we get $a = 13 - 7c$.

If $c = 1$, then we have $a = 6$, $b = -3$ $\quad \Rightarrow \quad w_2 = 6v_1 - 3v_2 + v_3$

Now let $(2, 1, -3) = a(1, 1, -1) + b(2, 3, -1) + c(3, 1, -5)$

$\Rightarrow \qquad a + 2b + 3c = 2, \quad a + 3b + c = 1, \quad -a - b - 5c = -3$

Solving first two equations, we get $b = 2c - 1$.

Using this value in third equation, we get $a = 4 - 7c$.

If $c = 1$, then we have $a = -3$, $b = 1$. $\quad \Rightarrow \quad w_3 = -3v_1 + v_2 + v_3$

Thus, v_1, v_2 and v_3 are linear combinations of w_1, w_2, w_3 and w_1, w_2, w_3 are linear combination of v_1, v_2, v_3. Therefore the subspace generated by v_1, v_2, v_3 is identical with the subspace generated by w_1, w_2, w_3.

Example 5 : Find a complement of a subspace W generated by (1, − 2, 5, − 3), (2, 3, 1, − 4) and (3, 8, − 3, − 5) in V = R^4(R).

Solution : Here $w = \langle (1, -2, 5, -3), (3, 8, -3, -5) \rangle$

Clearly $(1, -2, 5, -3)$ and $(3, 8, -3, -5)$ are linearly independent as neither of them is a multiple of other. So, the set $\{(1, -2, 5, -3), (3, 8, -3, -5)\}$ is a basis of W.

Now, we extend this to form a basis of R^4. For this we have to choose two vectors out of standard basis $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$. Let us choose

(0, 0, 1, 0) and (0, 0, 0, 1) for this purpose . To check the independence of these four vectors, we see that
$$\begin{vmatrix} 1 & -2 & 5 & -3 \\ 3 & 8 & -3 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 14 \neq 0$$
and so $\{(1,-2,5,-3),(3,8,-3,-5),(0,0,1,0),(0,0,0,1)\}$ is a *L.I.* set and hence it is a basis of R^4. Now, we know that complement of a subspace W is generated by additional vectors which are used to extend the basis of W to a basis of V. Here the additional vectors used for extension of basis are (0, 0, 1, 0) and (0, 0, 0, 1).

So, complement of $W = W' = \langle (0,0,1,0),(0,0,0,1) \rangle$.

Exercise 1.7

Example 1 : Determine dim V/W, where $V = C(R)$, $W = R(R)$.

Solution : As we know that $\{1, i\}$ is a basis of $C(R)$ and $\{1\}$ is a basis of $R(R)$.

Therefore dim $V = 2$ and dim $W = 1$.

Hence $\dim V/W = \dim V - \dim W = 2 - 1 = 1$.

Example 2 : Determine $\dim V/W$, where $V = C^2(R)$, $W = R^2(R)$

Solution : As we know that $\{(1,0),(0,1),(i,0),(0,i)\}$ is a basis of $C^2(R)$ and $\{(1,0),(0,1)\}$ is a basis of $R^2(R)$.

Therefore dim $V = 4$ and dim. $W = 2$

Hence dim V/W = dim V – dim W = 4 – 2 = 2.

Example 3 : Let $A = \{(x, y, 0), x, y \in R\}$ and $B = \{(0, y, z) : y, z \in R\}$ be two subspaces of $R^3(R)$. Verify that $\dim((A+B)/A) = \dim(B/A \cap B)$.

Solution : Any element $(x, y, 0) \in A$ can be expressed as $(x,y,0) = x(1,0,0) + y(0,1,0)$

$\Rightarrow \quad A = L(\{e_1, e_2\})$, where $e_1 = (1,0,0)$ and $e_2 = (0,1,0)$ are L.I.

Thus, $\{e_1, e_2\}$ is a basis of A and therefore dim $A = 2$.

Similarly dim $B = 2$. Now, it is clear that
$$A \cap B = \{(0, y, 0) : y \in R\}$$
$$= \{y(0,1,0) : y \in R\}$$
Thus, $\{(0,1,0)\}$ is a basis of $A \cap B$.
$$\Rightarrow \quad \dim(A \cap B) = 1$$
Therefore $\dim(A+B) = \dim A + \dim B - \dim(A \cap B)$
$$= 2 + 2 - 1 = 3$$
Now, $\dim\left(\frac{(A+B)}{A}\right) = \dim(A+B) - \dim A = 3 - 2 = 1$

and $\dim\left(\dfrac{B}{(A\cap B)}\right) = \dim B - \dim(A\cap B) = 2-1=1$

Hence $\dim\left(\dfrac{(A+B)}{A}\right) = \dim\left(\dfrac{B}{(A\cap B)}\right)$

Example 4 : If W is a subspace of $R^3(R)$ generated by {(1 , 0 , 0) , (1 , 1 , 0)} , find basis and dimension of V/W .

Solution : Given that the subspace W is generated by $\{(1, 0, 0), (1, 1, 0)\}$.

Clearly, this set is linearly independent , so we can extend it to get the basis of $R^3(R)$. We have to take one vector from the standard basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ of $R^3(R)$.

Let us take $(0, 0, 1)$ for this purpose. Thus the set, $\{(1, 0, 0), (1, 1, 0), (0, 0, 1)\}$ forms a basis of $R^3(R)$.

Therefore,
$$\begin{aligned} V/W &= \{W+v : v\in V\} \qquad \text{[Here } V = R^3(R)\text{]}\\ &= \{W+(0, 0, a) : a\in R\}\\ &= \{W+a(0, 0, 1) : a\in R\}\end{aligned}$$

Hence basis of V/W is $\{W+(0, 0, 1)\}$ and so dim. $V/W = 1$.

Example 5 : IF V is a vector space of all 2×2 matrices over R and W is the set of all 2×2 diagonal matrices over R , then show that W is a subspace of V. Also find dim (V/W).

Solution : Let W = collection of all 2×2 diagonal matrices.

i.e.,
$$W = \left\{\begin{bmatrix} a & 0 \\ 0 & b\end{bmatrix} : a, b\in R\right\}$$

Clearly $W \subseteq V$.

Let $A = \begin{bmatrix} a & 0 \\ 0 & b\end{bmatrix}$ and $B = \begin{bmatrix} c & 0 \\ 0 & d\end{bmatrix}$ be two arbitrary matrices of W. Then

$$A+B = \begin{bmatrix} a+c & 0 \\ 0 & b+d\end{bmatrix} \in W$$

Thus, W is closed under addition. Also, for any scalar k, we have

$$kA = k\begin{bmatrix} a & 0 \\ 0 & b\end{bmatrix} = \begin{bmatrix} ka & 0 \\ 0 & kb\end{bmatrix} \in W.$$

Thus, W is closed under scalar multiplication.
So, W is a subspace of V.

Now, the set $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix}\right\}$ is L.I. and spans W, so it is a basis of W.

Now, we extend this basis of W to get a basis of V by choosing two vectors from the standard basis $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ of V.

Let us take $\begin{bmatrix}0&1\\0&0\end{bmatrix}$ and $\begin{bmatrix}0&0\\1&0\end{bmatrix}$ for this purpose.

Now, the set $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix}\right\}$ is L.I.

Let $\begin{bmatrix}a&b\\c&d\end{bmatrix}$ be any arbitrary matrix, then it can be expressed as

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}=a\begin{bmatrix}1&0\\0&0\end{bmatrix}+b\begin{bmatrix}0&1\\0&0\end{bmatrix}+c\begin{bmatrix}0&0\\1&0\end{bmatrix}+d\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

We have $$V/W=\{W+v : v\in V\}$$

We claim that $W+\begin{bmatrix}0&1\\0&0\end{bmatrix}$ and $W+\begin{bmatrix}0&0\\1&0\end{bmatrix}$ form a basis of V/W.

First we prove that these two vectors are L.I.

Let a, b be any two scalars such that $a\left(W+\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)+b\left(W+\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)=W$

$\Rightarrow$ $$W+a\begin{bmatrix}0&1\\0&0\end{bmatrix}+W+b\begin{bmatrix}0&0\\1&0\end{bmatrix}=W$$

$\Rightarrow$ $$W+\begin{bmatrix}0&a\\0&0\end{bmatrix}+\begin{bmatrix}0&0\\b&a\end{bmatrix}=W$$

$\Rightarrow$ $$W+\begin{bmatrix}0&a\\b&0\end{bmatrix}=W$$

$\Rightarrow$ $$\begin{bmatrix}0&a\\b&0\end{bmatrix}\in W$$

$\Rightarrow$ $\begin{bmatrix}0&a\\b&0\end{bmatrix}$ is a diagonal matrix

$\Rightarrow$ non-diagonal elements are zero i.e., $a=b=0$.

Thus $W+\begin{bmatrix}0&1\\0&0\end{bmatrix}$ and $W+\begin{bmatrix}0&0\\1&0\end{bmatrix}$ are L.I.

Now, we prove that these two vectors span V/W

Let $W+v\in V/W$ be arbitrary. Then $W+v=W+\begin{bmatrix}a&b\\c&d\end{bmatrix}$; $a,b,c,d\in V$

Now $$W+v=W+\begin{bmatrix}a&b\\c&d\end{bmatrix}=W+\begin{bmatrix}a&0\\0&0\end{bmatrix}+\begin{bmatrix}0&b\\0&0\end{bmatrix}+\begin{bmatrix}0&0\\c&0\end{bmatrix}+\begin{bmatrix}0&0\\0&d\end{bmatrix}$$

$$=\left(W+\begin{bmatrix}a&0\\0&0\end{bmatrix}\right)+\left(W+\begin{bmatrix}0&b\\0&0\end{bmatrix}\right)+\left(W+\begin{bmatrix}0&0\\c&0\end{bmatrix}\right)+\left(W+\begin{bmatrix}0&0\\0&d\end{bmatrix}\right)$$

$$= W + \left(W + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) + \left(W + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}\right) + W \qquad \left[\because \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \in W\right]$$

$$= \left(W + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) + \left(W + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}\right)$$

$$= \left(W + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) + \left(W + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)$$

$$= b\left(W + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) + c\left(W + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)$$

which shows that $W + v \in V/W$ is a linear combination of $W + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $W + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Hence the set $\left\{W + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, W + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$ is a basis of V/W and so $\dim(V/W) = 2$.

Example 6 : Let V be the vector space of all 2×2 matrices over C. Let $W_1 = \left\{\begin{bmatrix} a & -a \\ b & c \end{bmatrix} : a, b, c \in C\right\}$, $W_2 = \left\{\begin{bmatrix} a & b \\ -a & c \end{bmatrix} : a, b, c \in C\right\}$ be two subsets of V. Verify that W_1 and W_2 are subspaces of V and $\dim\left(\dfrac{W_1+W_2}{W_2}\right) = \dim\left(\dfrac{W_1}{W_1 \cap W_2}\right)$.

Solution : Here $W_1 = \left\{\begin{bmatrix} a & -a \\ b & c \end{bmatrix} : a, b, c \in C\right\}$ and $W_2 = \left\{\begin{bmatrix} a & b \\ -a & c \end{bmatrix} : a, b, c \in C\right\}$

Clearly $W_1 \subseteq V$ and $W_2 \subseteq V$. Firstly, we prove that W_1 and W_2 are subspaces of V.

Let $A = \begin{bmatrix} a_1 & -a_1 \\ b_1 & c_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & -a_2 \\ b_2 & c_2 \end{bmatrix}$ be two arbitrary matrices of W_1. Then

$$A + B = \begin{bmatrix} a_1 & -a_1 \\ b_1 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & -a_2 \\ b_2 & c_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & -(a_1 + a_2) \\ b_1 + b_2 & c_1 + c_2 \end{bmatrix} \in W_1$$

Thus, W_1 is closed under addition. Also, for any scalar k, we have

$$kA = k\begin{bmatrix} a_1 & -a_1 \\ b_1 & c_1 \end{bmatrix} = \begin{bmatrix} ka_1 & -ka_1 \\ kb_1 & kc_1 \end{bmatrix} \in W_1$$

Thus, W_1 is closed under scalar multiplication. So, W_1 is a subspace of V.

Now, let $A = \begin{bmatrix} a_1 & b_1 \\ -a_1 & c_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ -a_2 & c_2 \end{bmatrix}$ be two arbitrary matrices of W_2. Then

$$A + B = \begin{bmatrix} a_1 & b_1 \\ -a_1 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ -a_2 & c_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ -(a_1 + a_2) & c_1 + c_2 \end{bmatrix} \in W_2$$

Thus W_2 is closed under addition. Also, for any scalar k, we have

$$kA = k\begin{bmatrix} a_1 & b_1 \\ -a_1 & c_1 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 \\ -ka_1 & kc_1 \end{bmatrix} \in W_2$$

Thus W_2 is closed under scalar multiplication. So, W_2 is a subspace of V.

We see that , $\begin{bmatrix} a & -a \\ b & c \end{bmatrix} + \begin{bmatrix} a & b \\ -a & c \end{bmatrix} = \begin{bmatrix} 2a & b-a \\ b-a & 2c \end{bmatrix} \in V$

So, we get $V = W_1 + W_2$.

We know dim V= 4 and so $\dim (W_1 + W_2) = 4$.

Now, $W_1 \cap W_2 = \left\{ \begin{bmatrix} a & -a \\ -a & c \end{bmatrix} : a, c \in C \right\}$. So dim $(W_1 \cap W_2) = 2$, since $\left\{ \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis of $W_1 \cap W_2$.

Now, $\dim W_1 = 3$, since $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis of W_1.

Similarly, $\dim W_2 = 3$, since $\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis of W_2.

So, $$\dim.\left(\frac{W_1 + W_2}{W_2}\right) = \dim.(W_1 + W_2) - \dim.W_2 = 4 - 3 = 1$$

and $$\dim.\left(\frac{W_1}{W_1 \cap W_2}\right) = \dim W_1 - \dim(W_1 \cap W_2) = 3 - 2 = 1$$

Hence $$\dim\left(\frac{W_1 + W_2}{W_2}\right) = \dim.\left(\frac{W_1}{W_1 \cap W_2}\right).$$