9. A particle moves along the curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time. Determine its velocity and acceleration vectors and also the magnitudes of velocity and acceleration at $t = 0$.

10. The position vector of a particle at time t is $\vec{r} = \cos(t-1)\,\hat{i} + \sinh\,(t-1)\,\hat{j} + \alpha t^3 \hat{k}$. Find the condition imposed on α by requiring that at time $t = 1$, the acceleration is normal to the position vector.

11. A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 5$ where t is the time. Find the components of its velocity and acceleration at $t = 1$ in the direction $2\hat{i} + 3\hat{j} + 6\hat{k}$. (*P.T.U. Jan. 2009*)

12. A particle moves so that its position vector is given by $\vec{r} = \cos \omega t\hat{i} + \sin \omega t\hat{j}$. Show that the velocity $\vec{v}$ of the particle is perpendicular to $\vec{r}$ and $\vec{r} \times \vec{v}$ is a constant vector.

13. The position vector of a point at time t is given by $\vec{r} = e^t(\cos t\hat{i} + \sin t\hat{j})$. Show that

    (*i*) $\vec{a} = 2(\vec{v} - \vec{r})$, where $\vec{a}, \vec{v}$ are acceleration and velocity of the particle.

    (*ii*) the angle between the radius vector and the acceleration is constant.

14. A particle moves along the curve $\vec{r} = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)\hat{k}$ where t denotes time. Find the magnitudes of acceleration along the tangent and normal at time $t = 2$. (*P.T.U. 2011*)

15. For the curve $x = \cos t + t\sin t$, $y = \sin t - t\cos t$, find the tangential and the normal components of the acceleration at any time t. (*M.D.U. May 2011*)

    [**Hint.** $\vec{r} = x\hat{i} + y\hat{j} = (\cos t + t\sin t)\,\hat{i} + (\sin t - t\cos t)\hat{j}$]

16. A particle (position vector $\vec{r}$) is moving in a circle with constant angular velocity ω. Show by vector methods, that the acceleration is equal to $-\omega^2\vec{r}$.

Answers

1. 1

4. (*i*) $\dfrac{d\vec{r}}{dt} \cdot \vec{a}$ (*ii*) $\dfrac{d\vec{r}}{dt} \times \vec{a}$ (*iii*) $\vec{r} \times \dfrac{d^2\vec{r}}{dt^2}$ (*iv*) $\left(\dfrac{d\vec{r}}{dt}\right)^2 + \vec{r} \cdot \dfrac{d^2\vec{r}}{dt^2}$

6. (*b*) -11 (*c*) $7\hat{i} + 6\hat{j} - 6\hat{k}$

7. $\dfrac{(-a\sin t)\,\hat{i} + (a\cos t)\,\hat{j} + b\hat{k}}{\sqrt{a^2 + b^2}}$

8. $\cos^{-1}\left(\dfrac{3}{7}\right)$

9. $\sqrt{37}, 5\sqrt{13}$

10. $\pm\dfrac{1}{\sqrt{6}}$

11. $\dfrac{24}{7}, \dfrac{18}{7}$

14. $16, 2\sqrt{73}$

15. $1, t$.

13.10. SCALAR AND VECTOR FIELDS

A variable quantity whose value at any point in a region of space depends upon the position of the point, is called a *point function*. There are two types of point functions.

(*a*) *Scalar Point Function*. Let R be a region of space at each point of which a scalar $\phi = \phi(x, y, z)$ is given, then ϕ is called a *scalar function* and R is called a *scalar field*.

The temperature distribution in a medium, the distribution of atmospheric pressure in space are examples of scalar point functions.

(*b*) *Vector Point Function*. Let R be a region of space at each point of which a vector $\vec{v} = \vec{v}\,(x, y, z)$ is given, then $\vec{v}$ is called a *vector point function* and R is called a *vector field*.

Each vector $\vec{v}$ of the field is regarded as a localised vector attached to the corresponding point (x, y, z).

The velocity of a moving fluid at any instant, the gravitational force are examples of vector point functions.

13.11. GRADIENT OF A SCALAR FIELD

(K.U.K. 2009, Dec. 2013, Dec. 2014; U.P.T.U. 2008; P.T.U. 2009)

Let $\phi(x, y, z)$ be a function defining a scalar field, then the vector $\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$ is called the **gradient** of the scalar field ϕ and is denoted by grad ϕ.

Thus, $$\text{grad } \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

The gradient of scalar field ϕ is obtained by operating on ϕ by the vector operator

$$\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

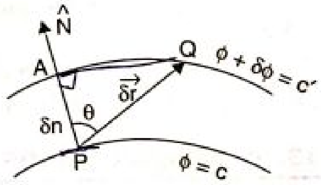
This operator is denoted by the symbol ∇, read as **del** (also called nabla).

Thus, grad $\phi = \nabla\phi$.

13.12. GEOMETRICAL INTERPRETATION OF GRADIENT

(M.D.U. 2013; K.U.K. 2009, Dec. 2013, Dec. 2014; P.T.U. May 2010)

If a surface $\phi(x, y, z) = c$ is drawn through any point P such that at each point on the surface, the function has the same value as at P, then such a surface is called a *level surface* through P. For example, if $\phi(x, y, z)$ represents potential at the point (x, y, z), the **equipotential surface** $\phi(x, y, z) = c$ is a level surface.



Through any point passes one and only one level surface. Moreover, no two level surfaces can intersect.

Consider the level surface through P at which the function has value ϕ and another level surface through a neighbouring point Q where the value is $\phi + \delta\phi$.

Let $\vec{r}$ and $\vec{r} + \delta\vec{r}$ be the position vectors of P and Q respectively, then $\overrightarrow{PQ} = \delta\vec{r}$.

Now $$\nabla\phi . \delta\vec{r} = \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)\cdot(\hat{i}\,\delta x + \hat{j}\delta y + \hat{k}\delta z)$$

$$= \frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial z}\delta z = \delta\phi \qquad \text{...(1)}$$

If Q lies on the same level surface as P, then $\delta\phi = 0$,

$\therefore$ (1) reduces to $\nabla\phi \cdot \delta\vec{r} = 0$.

Thus, $\nabla\phi$ is perpendicular to every $\delta\vec{r}$ lying in the surface.

Hence **$\nabla\phi$ is normal to the surface** $\phi(x, y, z) = c$.

Let $\nabla\phi = |\nabla\phi|\,\hat{N}$, where $\hat{N}$ is a unit vector normal to the surface. Let PA $= \delta n$ be the perpendicular distance between the two level surfaces through P and Q. Then the rate of change of ϕ in the direction of normal to the surface through P is

$$\frac{\partial \phi}{\partial n} = \lim_{\delta n \to 0} \frac{\partial \phi}{\partial n} = \lim_{\delta n \to 0} \frac{\nabla \phi . \delta \vec{r}}{\delta n} \qquad \text{[by (1)]}$$

$$= \lim_{\delta n \to 0} \frac{|\nabla \phi| \hat{N} . \delta \vec{r}}{\delta n} = |\nabla \phi| \qquad (\because \hat{N} \cdot \delta \vec{r} = |\hat{N}||\delta \vec{r}| \cos \theta = |\delta \vec{r}| \cos \theta = \delta n)$$

$$\therefore \quad |\nabla \phi| = \frac{\partial \phi}{\partial n}.$$

Hence the gradient of a scalar field ϕ is a vector normal to the surface $\phi = c$ and has a magnitude equal to the rate of change of ϕ along this normal.

13.13. DIRECTIONAL DERIVATIVE

(P.T.U. Jan. 2010)

Let $PQ = \delta r$, then $\lim_{\delta r \to 0} \frac{\delta \phi}{\delta r} = \frac{\partial \phi}{\partial r}$ is called the directional derivative of ϕ at P in the direction PQ.

Let $\hat{N}'$ be a unit vector in the direction PQ, then $\delta r = \frac{\delta n}{\cos \theta} = \frac{\delta n}{\hat{N} \cdot \hat{N}'}$

$$\therefore \quad \frac{\partial \phi}{\partial r} = \lim_{\delta r \to 0} \left[\hat{N} \cdot \hat{N}' \frac{\delta \phi}{\delta n} \right] = \hat{N} \cdot \hat{N}' \frac{\partial \phi}{\partial n}$$

$$= \hat{N}' \cdot \hat{N} \frac{\partial \phi}{\partial n} = \hat{N}' \cdot \hat{N} |\nabla \phi| = \hat{N}' \cdot \nabla \phi \qquad \left(\because |\nabla \phi| = \frac{\partial \phi}{\partial n} \text{ and } \hat{N} |\nabla \phi| = \nabla \phi \right)$$

Thus, the directional derivative $\frac{\partial \phi}{\partial r}$ is the resolved part of $\nabla \phi$ in the direction $\hat{N}'$.

Since $\quad \frac{\partial \phi}{\partial r} = \hat{N}' \cdot \nabla \phi = |\nabla \phi| \cos \theta \leq |\nabla \phi|$.

$\therefore$ $\nabla \phi$ *gives the maximum rate of change of ϕ and the magnitude of this maximum is* $|\nabla \phi|$.

Hence the directional derivative of a scalar field ϕ at a point (x, y, z) in the direction of unit vector $\hat{a}$ is given by $(\nabla \phi) . \hat{a}$.

13.14. PROPERTIES OF GRADIENT

(a) If ϕ is a constant scalar point function, then $\nabla \phi = \vec{0}$

(b) If ϕ_1 and ϕ_2 are two scalar point functions, then

(i) $\nabla (\phi_1 \pm \phi_2) = \nabla \phi_1 \pm \nabla \phi_2$

(ii) $\nabla (c_1 \phi_1 + c_2 \phi_2) = c_1 \nabla \phi_1 + c_2 \nabla \phi_2$, where c_1, c_2 are constant

(iii) $\nabla (\phi_1 \phi_2) = \phi_1 \nabla \phi_2 + \phi_2 \nabla \phi_1$

(iv) $\nabla \left(\frac{\phi_1}{\phi_2} \right) = \frac{\phi_2 \nabla \phi_1 - \phi_1 \nabla \phi_2}{{\phi_2}^2}, \phi_2 \neq 0.$

All the above results can be easily proved. For example

(iii) $$\nabla (\phi_1 \phi_2) = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (\phi_1 \phi_2) = \hat{i} \frac{\partial}{\partial x} (\phi_1 \phi_2) + \hat{j} \frac{\partial}{\partial y} (\phi_1 \phi_2) + \hat{k} \frac{\partial}{\partial z} (\phi_1 \phi_2)$$

$$= \hat{i} \left(\phi_1 \frac{\partial \phi_2}{\partial x} + \phi_2 \frac{\partial \phi_1}{\partial x} \right) + \hat{j} \left(\phi_1 \frac{\partial \phi_2}{\partial y} + \phi_2 \frac{\partial \phi_1}{\partial y} \right) + \hat{k} \left(\phi_1 \frac{\partial \phi_2}{\partial z} + \phi_2 \frac{\partial \phi_1}{\partial z} \right)$$

$$= \phi_1\left(\hat{i}\frac{\partial\phi_2}{\partial x}+\hat{j}\frac{\partial\phi_2}{\partial y}+\hat{k}\frac{\partial\phi_2}{\partial z}\right)+\phi_2\left(\hat{i}\frac{\partial\phi_1}{\partial x}+\hat{j}\frac{\partial\phi_1}{\partial y}+\hat{k}\frac{\partial\phi_1}{\partial z}\right)$$

$$= \phi_1\nabla\phi_2+\phi_2\nabla\phi_1.$$

(iv) $$\nabla\left(\frac{\phi_1}{\phi_2}\right)=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(\frac{\phi_1}{\phi_2}\right)=\hat{i}\frac{\partial}{\partial x}\left(\frac{\phi_1}{\phi_2}\right)+\hat{j}\frac{\partial}{\partial y}\left(\frac{\phi_1}{\phi_2}\right)+\hat{k}\frac{\partial}{\partial z}\left(\frac{\phi_1}{\phi_2}\right)$$

$$=\hat{i}\,\frac{\phi_2\frac{\partial\phi_1}{\partial x}-\phi_1\frac{\partial\phi_2}{\partial x}}{\phi_2^2}+\hat{j}\,\frac{\phi_2\frac{\partial\phi_1}{\partial y}-\phi_1\frac{\partial\phi_2}{\partial y}}{\phi_2^2}+\hat{k}\,\frac{\phi_2\frac{\partial\phi_1}{\partial z}-\phi_1\frac{\partial\phi_2}{\partial z}}{\phi_2^2}$$

$$=\frac{1}{{\phi_2}^2}\left[\phi_2\left(\hat{i}\frac{\partial\phi_1}{\partial x}+\hat{j}\frac{\partial\phi_1}{\partial y}+\hat{k}\frac{\partial\phi_1}{\partial z}\right)-\phi_1\left(\hat{i}\frac{\partial\phi_2}{\partial x}+\hat{j}\frac{\partial\phi_2}{\partial y}+\hat{k}\frac{\partial\phi_2}{\partial z}\right)\right]$$

$$=\frac{\phi_2\nabla\phi_1-\phi_1\nabla\phi_2}{\phi_2^2}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find grad ϕ when ϕ is given by $\phi = 3x^2y - y^3z^2$ at the point (1, –2, –1).*

Sol. $$\text{Grad }\phi=\nabla\phi=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(3x^2y-y^3z^2)$$

$$=\hat{i}\frac{\partial}{\partial x}(3x^2y-y^3z^2)+\hat{j}\frac{\partial}{\partial y}(3x^2y-y^3z^2)+\hat{k}\frac{\partial}{\partial z}(3x^2y-y^3z^2)$$

$$=\hat{i}\,(6xy)+\hat{j}(3x^2-3y^2z^2)+\hat{k}(-2y^3z)$$

$$=-12\hat{i}-9\hat{j}-16\hat{k}\ \text{at the point } (1,-2,-1).$$

Example 2. *If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, show that*

(i) *grad* $r=\dfrac{\vec{r}}{r}$ (G.B.T.U. 2011) (ii) *grad* $\left(\dfrac{1}{r}\right)=-\dfrac{\vec{r}}{r^3}$ (G.B.T.U. 2011)

(iii) $\nabla r^n = nr^{n-2}\,\vec{r}$ (G.B.T.U. 2011) (iv) $\nabla(\vec{a}\cdot\vec{r})=\vec{a}$, *where $\vec{a}$ is a constant vector.* (U.P.T.U. 2008)

Sol. $r=|\,\vec{r}\,|=\sqrt{x^2+y^2+z^2}$, or $r^2=x^2+y^2+z^2$

Differentiating partially w.r.t. x, we have $2r\dfrac{\partial r}{\partial x}=2x$ or $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}$

Similarly, $\dfrac{\partial r}{\partial y}=\dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z}=\dfrac{z}{r}$

(i) $$\text{grad } r=\nabla r=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)r=\hat{i}\frac{\partial r}{\partial x}+\hat{j}\frac{\partial r}{\partial y}+\hat{k}\frac{\partial r}{\partial z}$$

$$=\hat{i}\left(\frac{x}{r}\right)+\hat{j}\left(\frac{y}{r}\right)+\hat{k}\left(\frac{z}{r}\right)=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{r}=\frac{\vec{r}}{r}.$$

(ii) $$\text{grad}\left(\frac{1}{r}\right)=\nabla\left(\frac{1}{r}\right)=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\cdot\frac{\partial r}{\partial x}\right) + \hat{j}\left(-\frac{1}{r^2}\cdot\frac{\partial r}{\partial y}\right) + \hat{k}\left(-\frac{1}{r^2}\cdot\frac{\partial r}{\partial z}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\cdot\frac{x}{r}\right) + \hat{j}\left(-\frac{1}{r^2}\cdot\frac{y}{r}\right) + \hat{k}\left(-\frac{1}{r^2}\cdot\frac{z}{r}\right)$$

$$= -\frac{1}{r^3}(x\hat{i} + y\hat{j} + z\hat{k}) = -\frac{\vec{r}}{r^3}.$$

(iii) $$\nabla r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r^n = \hat{i}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) + \hat{j}\left(nr^{n-1}\frac{\partial r}{\partial y}\right) + \hat{k}\left(nr^{n-1}\frac{\partial r}{\partial z}\right)$$

$$= \hat{i}\left(nr^{n-1}\cdot\frac{x}{r}\right) + \hat{j}\left(nr^{n-1}\cdot\frac{y}{r}\right) + \hat{k}\left(nr^{n-1}\cdot\frac{z}{r}\right) = nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k}) = nr^{n-2}\vec{r}.$$

(iv) Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, where a_1, a_2, a_3 are constants.

$$\vec{a}\cdot\vec{r} = a_1x + a_2y + a_3z$$

$$\therefore \quad \nabla(\vec{a}\cdot\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$$

$$= \hat{i}\frac{\partial}{\partial x}(a_1x + a_2y + a_3z) + \hat{j}\frac{\partial}{\partial y}(a_1x + a_2y + a_3z) + \hat{k}\frac{\partial}{\partial z}(a_1x + a_2y + a_3z)$$

$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}.$$

Example 3. *Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point (1, 2, –1).*

Sol. Let $\phi = x^3 + y^3 + 3xyz = 3$, then $\frac{\partial\phi}{\partial x} = 3x^2 + 3yz$, $\frac{\partial\phi}{\partial y} = 3y^2 + 3xz$, $\frac{\partial\phi}{\partial z} = 3xy$

$$\therefore \quad \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = (3x^2 + 3yz)\hat{i} + (3y^2 + 3xz)\hat{j} + (3xy)\hat{k}$$

At $(1, 2, -1)$, $\nabla\phi = -3\hat{i} + 9\hat{j} + 6\hat{k}$

Which is a vector normal to the given surface at (1, 2, –1).

Hence a unit vector normal to the given surface at (1, 2, –1)

$$= \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{\sqrt{[(-3)^2 + (9)^2 + (6)^2]}} = \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{3\sqrt{14}} = \frac{1}{\sqrt{14}}(-\hat{i} + 3\hat{j} + 2\hat{k}).$$

Example 4. *Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point P(1, 2, 3) in the direction of the line PQ, where Q is the point (5, 0, 4). In what direction it will be maximum ? Find also the magnitude of this maximum.* (U.K.T.U. 2011)

Sol. We have $\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = 2x\hat{i} - 2y\hat{j} + 4z\hat{k} = 2\hat{i} - 4\hat{j} + 12\hat{k}$ at P(1, 2, 3)

Also, $$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (5\hat{i} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 4\hat{i} - 2\hat{j} + \hat{k}$$

If $\hat{n}$ is a unit vector in the direction $\overrightarrow{PQ}$, then $\hat{n} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16 + 4 + 1}} = \frac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} + \hat{k})$

$\therefore$ Directional derivative of f in the direction $\overrightarrow{PQ} = (\nabla f)\cdot\hat{n}$

$$= (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} + \hat{k}) = \frac{1}{\sqrt{21}}[2(4) - 4(-2) + 12(1)]$$

$$= \frac{28}{\sqrt{21}} = \frac{4}{3}\sqrt{21}$$

The directional derivative of f is maximum in the direction of the normal to the given surface *i.e.*, in the direction of $\nabla f = 2\hat{i} - 4\hat{j} + 12\hat{k}$

The maximum value of this directional derivative $= |\nabla f|$

$$= \sqrt{(2)^2 + (-4)^2 + (12)^2} = \sqrt{164} = 2\sqrt{41}.$$

Example 5. *Find the directional derivative of* $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ *at the point* $P(1, 1, 1)$ *in the direction of the line* $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$. (G.B.T.U. 2010)

Sol. Here,
$$\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$$

$\therefore$
$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \left(10xy + \frac{5}{2}z^2\right)\hat{i} + (5x^2 - 10yz)\hat{j} + (-5y^2 + 5zx)\hat{k}$$

$$= \frac{25}{2}\hat{i} - 5\hat{j} \qquad \text{at P(1, 1, 1)}$$

The direction of the given line is $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$

$\Rightarrow$
$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}$$

$\therefore$ The required directional derivative

$$= (\nabla\phi)\cdot\hat{a} = \left(\frac{25}{2}\hat{i} - 5\hat{j}\right)\cdot\left(\frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}\right)$$

$$= \left(\frac{25}{2}\right)\left(\frac{2}{3}\right) + (-5)\left(-\frac{2}{3}\right) + (0)\left(\frac{1}{3}\right) = \frac{35}{3}.$$

Example 6. *Find the directional derivative of* $f(x, y, z) = e^{2x}\cos(yz)$ *at* $(0, 0, 0)$ *in the direction of the tangent to the curve* $x = a\sin t$, $y = a\cos t$, $z = at$ *at* $t = \pi/4$.

Sol.
$$f(x, y, z) = e^{2x}\cos(yz)$$

$\therefore$
$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= (2e^{2x}\cos(yz))\,\hat{i} - (ze^{2x}\sin(yz))\,\hat{j} - (ye^{2x}\sin(yz))\,\hat{k}$$

$$= 2\hat{i} \quad \text{at } (0, 0, 0)$$

The given curve is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

i.e., $$\vec{r} = (a \sin t)\hat{i} + (a \cos t)\hat{j} + (at)\hat{k}$$

Tangent to the given curve is

$$\frac{d\vec{r}}{dt} = (a \cos t)\hat{i} - (a \sin t)\hat{j} + a\hat{k}$$

$$= a\left(\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} + \hat{k}\right) \quad \text{at} \quad t = \frac{\pi}{4}$$

$$\therefore \qquad \left|\frac{d\vec{r}}{dt}\right| = \sqrt{\frac{a^2}{2} + \frac{a^2}{2} + a^2} = a\sqrt{2}$$

$\Rightarrow$ Unit vector along the tangent at $t = \dfrac{\pi}{4}$ is

$$\hat{n} = \frac{\dfrac{d\vec{r}}{dt}}{\left|\dfrac{d\vec{r}}{dt}\right|} = \frac{a\left(\dfrac{1}{\sqrt{2}}\hat{i} - \dfrac{1}{\sqrt{2}}\hat{j} + \hat{k}\right)}{a\sqrt{2}} = \frac{\hat{i}}{2} - \frac{\hat{j}}{2} + \frac{\hat{k}}{\sqrt{2}}$$

$\therefore$ The required directional derivative

$$= (\nabla f) \,.\, \hat{n} = 2\hat{i} \,.\left(\frac{\hat{i}}{2} - \frac{\hat{j}}{2} + \frac{\hat{k}}{\sqrt{2}}\right) = 2\left(\frac{1}{2}\right) = 1$$

Example 7. *If $\vec{a}, \vec{b}$ are constant vectors, prove that $\nabla\, [\vec{r}\ \vec{a}\ \vec{b}] = \vec{a} \times \vec{b}$.*

Sol. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

where $a_1, a_2, a_3, b_1, b_2, b_3$ are constants.

Also $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ so that

$$\left[\vec{r}\ \ \vec{a}\ \ \vec{b}\right] = \begin{vmatrix} x & y & z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)x + (a_3b_1 - a_1b_3)\,y + (a_1b_2 - a_2\,b_1)\,z$$

$$\therefore \quad \nabla\left[\vec{r}\ \ \vec{a}\ \ \vec{b}\right] = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)[(a_2b_3 - a_3b_2)x + (a_3b_1 - a_1b_3)y + (a_1b_2 - a_2b_1)z]$$

$$= (a_2\,b_3 - a_3\,b_2)\hat{i} + (a_3b_1 - a_1b_3)\,\hat{j} + (a_1b_2 - a_2\,b_1)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{a} \times \vec{b}$$

Hence $\nabla\left[\vec{r}\ \ \vec{a}\ \ \vec{b}\right] = \vec{a} \times \vec{b}$.

Example 8. *If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at the point (1, 1, 1) has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$, find the values of a, b and c.* (M.D.U. Dec. 2010)

Sol. Here,
$$\phi = ax^2y + by^2z + cz^2x$$

$\therefore$
$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$
$$= (2axy + cz^2)\hat{i} + (ax^2 + 2byz)\hat{j} + (by^2 + 2czx)\hat{k}$$
$$= (2a + c)\hat{i} + (a + 2b)\hat{j} + (b + 2c)\hat{k} \text{ at } (1, 1, 1)$$

Now, the directional derivative of ϕ is maximum in the direction of the normal to the given surface *i.e.*, in the direction of $\nabla\phi$.

But we are given that the directional derivative of ϕ is maximum in the direction parallel to the line.
$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1} \quad i.e., \text{ parallel to the vector } 2\hat{i} - 2\hat{j} + \hat{k}.$$

$\therefore$
$$\frac{2a+c}{2} = \frac{a+2b}{-2} = \frac{b+2c}{1}$$

[Two vectors are parallel if the corresponding scalar components are proportional].

$\Rightarrow$
$$\frac{2a+c}{2} = \frac{a+2b}{-2} \quad \text{and} \quad \frac{a+2b}{-2} = \frac{b+2c}{1}$$

$\Rightarrow$
$$2a + c = -a - 2b \quad \text{and} \quad a + 2b = -2b - 4c$$

$\Rightarrow$
$$3a + 2b + c = 0 \quad \text{and} \quad a + 4b + 4c = 0$$

By cross-multiplication, we have
$$\frac{a}{8-4} = \frac{b}{1-12} = \frac{c}{12-2}$$

or
$$\frac{a}{4} = \frac{b}{-11} = \frac{c}{10} = \lambda \quad \text{(say)}$$

$\Rightarrow$
$$a = 4\lambda,\ b = -11\lambda,\ c = 10\lambda$$

The maximum value of directional derivative of ϕ
$$= |\nabla\phi| = \sqrt{(2a+c)^2 + (a+2b)^2 + (b+2c)^2}$$

Since it is given to be 15, we have
$$\sqrt{(2a+c)^2 + (a+2b)^2 + (b+2c)^2} = 15$$

$\Rightarrow$
$$(8\lambda + 10\lambda)^2 + (4\lambda - 22\lambda)^2 + (-11\lambda + 20\lambda)^2 = 225$$

$\Rightarrow$
$$(324 + 324 + 81)\lambda^2 = 225 \Rightarrow \lambda^2 = \frac{225}{729} = \frac{25}{81}$$

$\Rightarrow$
$$\lambda = \pm 5/9$$

$\therefore$
$$a = \pm\frac{20}{9},\ b = \mp\frac{55}{9},\ c = \pm\frac{50}{9}$$

Example 9. *Find the values of constants a, b and c so that the maximum value of the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at (1, 2, −1) has a magnitude 64 in the direction parallel to z-axis.* (M.D.U. Dec. 20..)

Sol. Here,

$$\phi = axy^2 + byz + cz^2x^3$$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= (ay^2 + 3cx^2z^2)\hat{i} + (2axy + bz)\hat{j} + (by + 2cx^3z)\hat{k}$$

$$= (4a + 3c)\hat{i} + (4a - b)\hat{j} + (2b - 2c)\hat{k} \text{ at } (1, 2, -1)$$

Now, the directional derivative of ϕ is maximum in the direction of the normal to the given surface i.e., in the direction of $\nabla\phi$. But we are given that the directional derivative is maximum in the direction parallel to z-axis i.e., parallel to $\hat{k}$.

Hence co-efficients of $\hat{i}$ and $\hat{j}$ in $\nabla\phi$ should be zero and the co-efficient of $\hat{k}$ positive.

Thus,

$$4a + 3c = 0 \quad ...(1)$$

$$4a - b = 0 \quad ...(2)$$

and

$$2b - 2c > 0 \quad i.e., \quad b > c \quad ...(3)$$

Then,

$$\nabla\phi = 2(b - c)\hat{k}$$

Also maximum value of directional derivative $= |\nabla\phi|$ (Given)

$\therefore$

$$|2(b - c)\hat{k}| = 64$$

or

$$2(b - c) = 64 \quad \text{or} \quad b - c = 32 \quad ...(4)$$

Solving (1), (2) and (4), we have

$$a = 6,\ b = 24,\ c = -8.$$

Example 10. *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2, −1, 2).*

Sol. Angle between two surfaces at a point is the angle between the normals to the surfaces at that point.

Let

$$\phi_1 = x^2 + y^2 + z^2 = 9 \quad \text{and} \quad \phi_2 = x^2 + y^2 - z = 3$$

Then

$$\text{grad } \phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k} \quad \text{and} \quad \text{grad } \phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

Let $\vec{n_1}$ = grad ϕ_1 at the point (2, −1, 2) and $\vec{n_2}$ = grad ϕ_2 at the point (2, −1, 2).

Then

$$\vec{n_1} = 4\hat{i} - 2\hat{j} + 4\hat{k} \quad \text{and} \quad \vec{n_2} = 4\hat{i} - 2\hat{j} - \hat{k}$$

The vectors $\vec{n_1}$ and $\vec{n_2}$ are along normals to the two surfaces at the point (2, −1, 2). If θ is the angle between these vectors, then

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{4(4) - 2(-2) + 4(-1)}{\sqrt{16 + 4 + 16}\ \sqrt{16 + 4 + 1}} = \frac{16}{6\sqrt{21}}$$

$\therefore$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

EXERCISE 13.2

1. Find grad ϕ when ϕ is given by
   (i) $\phi = x^2 + yz$ (ii) $\phi = x^3 + y^3 + 3xyz$ (iii) $\phi = \log (x^2 + y^2 + z^2)$.

2. If $r = |\vec{r}|$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that
   (i) $\nabla f(r) = f'(r)\, \nabla r$ (ii) $\nabla \log r = \dfrac{\vec{r}}{r^2}$
   (iii) $\nabla (e^{r^2}) = 2e^{r^2}\, \vec{r}$ (iv) grad $|\vec{r}|^2 = 2\vec{r}$
   (v) $\text{grad}\left(\dfrac{1}{r^2}\right) = -\dfrac{2\vec{r}}{r^4}$ (M.D.U. 2013)
   (vi) $\nabla\phi(r) = \dfrac{\phi'(r)}{r}\vec{r}$ and hence show that $\nabla\left(\int r^n\, dr\right) = r^{n-1}\vec{r}$ (M.D.U. May 2011)

3. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, prove that:
   (i) (grad u) . [(grad v) × (grad w)] = 0
   (ii) grad u, grad v and grad w are coplanar vectors. (M.T.U. 2013; U.K.T.U. 2010)
   [**Hint.** Three vectors are coplanar if their scalar triple product is zero].

4. Find a unit vector normal to the surface
   (i) $xy^3z^2 = 4$ at the point (–1, –1, 2) (U.P.T.U. 2008)
   (ii) $x^2y + 2xz = 4$ at the point (2, –2, 3).

5. Find the directional derivative of the function
   (i) $f(x, y, z) = xy^2 + yz^3$ at the point (2, –1, 1) in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$. (P.T.U. 2011)
   (ii) $f(x, y, z) = 2xy + z^2$ at the point (1, –1, 3) in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$. (Anna 2009)
   (iii) $\phi = x^2yz + 4xz^2$ at the point (1, –2, –1) in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$. (V.T.U. 2012)
   (iv) $\phi = 4xz^3 - 3x^2yz^2$ at (2, –1, 2) along z-axis.
   (v) $\phi = xy^2 + yz^3$ at the point (2, –1, 1) in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at (–1, 2, 1). ((Anna 2009; G.B.T.U. 2011)
   (vi) $\phi = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ at the point P(3, 1, 2) in the direction of the vector $yz\hat{i} + zx\hat{j} + xy\hat{k}$.
   (vii) $\psi(x, y, z) = 4e^{x + 5y - 13z}$ at the point (1, 2, 3) in the direction towards the point (–3, 5, 7). (U.P.T.U. 2009)

6. Find the directional derivative of the function $\phi = \dfrac{y}{x^2 + y^2}$ at the point (0, 1) making an angle 30° with the positive x-axis.
   [**Hint.** Here $\hat{a} = \cos 30°\,\hat{i} + \sin 30°\,\hat{j}$]

7. In what direction from (3, 1, –2) is the directional derivative of $\phi = x^2y^2z^4$ maximum and what is its magnitude ? (K.U.K. 2010)

8. What is the greatest rate of increase of $u = x^2 + yz^2$ at the point (1, – 1, 3)?

9. The temperature at a point (x, y, z) in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at (1, 1, 2) desires to fly in such a direction that it will get warm as soon as possible. In what direction should it fly?

10. Calculate the angle between the normals to the surface $xy = z^2$ at the points (4, 1, 2) and (3, 3, –3). (U.K.T.U. 2012)

11. If θ is the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$, show that $\cos\theta = \dfrac{3}{7\sqrt{6}}$.

12. Find the constants a and b so that the surface $ax^2 - byz = (a + 2)x$ is orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$.
[**Hint.** The point P$(1, -1, 2)$ lies on both the surfaces and $(\text{grad } \phi_1)_P \cdot (\text{grad } \phi_2)_P = 0$]

13. Find the angle between the tangent planes to the surfaces $x \log z = y^2 - 1$ and $x^2y = 2 - z$ at the point $(1, 1, 1)$.

14. Find the directional derivative of $\vec{V}^2$ where $\vec{V} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ at the point $(2, 0, 3)$ in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 14$ at the point $(3, 2, 1)$.
(M.T.U. 2013)

Answers

1. (*i*) $2x\hat{i} + z\hat{j} + y\hat{k}$ (*ii*) $3(x^2 + yz)\hat{i} + 3(y^2 + xz)\hat{j} + 3xy\hat{k}$ (*iii*) $\dfrac{2(x\hat{i} + y\hat{j} + z\hat{k})}{x^2 + y^2 + z^2}$

4. (*i*) $-\dfrac{1}{\sqrt{11}}(\hat{i} + 3\hat{j} - \hat{k})$ (*ii*) $\dfrac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$

5. (*i*) $-\dfrac{11}{3}$ (*ii*) $\dfrac{14}{3}$ (*iii*) $\dfrac{37}{3}$ (*iv*) 144 (*v*) $\dfrac{15}{\sqrt{17}}$ (*vi*) $\dfrac{-9}{49\sqrt{14}}$ (*vii*) $-4\sqrt{41}\, e^{-28}$

6. $-\dfrac{1}{2}$ 7. $96(\hat{i} + 3\hat{j} - 3\hat{k})$; $96\sqrt{19}$

8. 11 9. $\dfrac{1}{3}(2\hat{i} + 2\hat{j} - \hat{k})$ 10. $\cos^{-1}\left(\dfrac{1}{\sqrt{22}}\right)$ 12. $a = 2.5, b = 1$

13. $\cos^{-1}\left(\dfrac{1}{\sqrt{30}}\right)$ 14. $\dfrac{1404}{\sqrt{14}}$.

13.15. DIVERGENCE OF A VECTOR POINT FUNCTION *(U.P.T.U. 2009)*

The divergence of a differentiable vector point function $\vec{V}$ is denoted by div $\vec{V}$ and is defined as

$$\text{div } \vec{V} = \nabla \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot \vec{V} = \hat{i}\cdot\frac{\partial \vec{V}}{\partial x} + \hat{j}\cdot\frac{\partial \vec{V}}{\partial y} + \hat{k}\cdot\frac{\partial \vec{V}}{\partial z}.$$

Obviously, the divergence of a vector point function is a scalar point function.

If
$$\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$$
then
$$\text{div } \vec{V} = \nabla\cdot\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot[V_1\hat{i} + V_2\hat{j} + V_3\hat{k}] = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

Since $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ and $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$.

13.16. CURL OF A VECTOR POINT FUNCTION *(G.B.T.U. 2012; U.P.T.U. 2008)*

The curl (or rotation) of a differentiable vector point function $\vec{V}$ is denoted by curl $\vec{V}$ and is defined as

$$\text{curl } \vec{V} = \nabla\times\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\vec{V} = \hat{i}\times\frac{\partial \vec{V}}{\partial x} + \hat{j}\times\frac{\partial \vec{V}}{\partial y} + \hat{k}\times\frac{\partial \vec{V}}{\partial z}.$$

Obviously, the curl of a vector point function is a vector point function.

If $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

$$\text{curl } \vec{V} = \nabla \times \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times [V_1\hat{i} + V_2\hat{j} + V_3\hat{k}]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right).$$

ILLUSTRATIVE EXAMPLES

Example 1. *If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that*

(i) div $\vec{r} = 3$ *(ii) curl $\vec{r} = \vec{0}$.*

Sol. (i) div $\vec{r} = \nabla \cdot \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3.$

(ii) curl $\vec{r} = \nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] + \hat{j}\left[\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]$$

$$= \hat{i}(0) + \hat{j}(0) + \hat{k}(0) = \vec{0}.$$

Example 2. *Find the divergence and curl of the vector $\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ at the point $(2, -1, 1)$.*

Sol.
$$\text{div } \vec{V} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$
$$= yz + 3x^2 + 2xz - y^2 = -1 + 12 + 4 - 1 = 14 \text{ at } (2, -1, 1)$$

$$\text{curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix} = \hat{i}(-2yz - 0) + \hat{j}(xy - z^2) + \hat{k}(6xy - xz)$$

$$= 2\hat{i} - 3\hat{j} - 14\hat{k} \text{ at } (2, -1, 1).$$

Example 3. *Find div $\vec{F}$ and curl $\vec{F}$ where $\vec{F} = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz)$.* (M.D.U. 2013)

Sol. Let $\phi = x^3 + y^3 + z^3 - 3xyz$, then

$$\vec{F} = \text{grad } \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^3 + y^3 + z^3 - 3xyz)$$
$$= (3x^2 - 3yz)\hat{i} + (3y^2 - 3zx)\hat{j} + (3z^2 - 3xy)\hat{k}$$

$$\therefore \qquad \text{div } \vec{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3zx) + \frac{\partial}{\partial x}(3z^2 - 3xy)$$

$$= 6x + 6y + 6z = 6(x + y + z)$$

and

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3zx & 3z^2 - 3xy \end{vmatrix}$$

$$= \hat{i}(-3x + 3x) + \hat{j}(-3y + 3y) + \hat{k}(-3z + 3z) = \vec{0}.$$

Example 4. *Find curl (curl $\vec{V}$) where $\vec{V} = (2xz^2)\hat{i} - yz\hat{j} + 3xz^3\hat{k}$, at (1, 1, 1).*

Sol. Here, $\vec{V} = (2xz^2)\hat{i} - yz\hat{j} + 3xz^3\hat{k}$

$$\therefore \qquad \text{curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(3xz^3) - \frac{\partial}{\partial z}(-yz)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(3xz^3) - \frac{\partial}{\partial z}(2xz^2)\right\}$$

$$+ \hat{k}\left\{\frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial y}(2xz^2)\right\}$$

$$= \hat{i}(0 + y) - \hat{j}(3z^3 - 4xz) + \hat{k}(0 - 0) = y\hat{i} + (4xz - 3z^3)\hat{j}$$

$$\text{curl (curl } \vec{V}) = \text{curl } \{y\hat{i} + (4xz - 3z^3)\hat{j}\}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & 4xz - 3z^3 & 0 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(4xz - 3z^3)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(y)\right\}$$

$$+ \hat{k}\left\{\frac{\partial}{\partial x}(4xz - 3z^3) - \frac{\partial}{\partial y}(y)\right\}$$

$$= \hat{i}\left(0 - (4x - 9z^2)\right) - \hat{j}(0 - 0) + \hat{k}(4z - 1)$$

$$= (9z^2 - 4x)\hat{i} + (4z - 1)\hat{k} = 5\hat{i} + 3\hat{k} \text{ at } (1, 1, 1).$$

Example 5. *Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a}$ be a constant vector, find the value of* $\text{div}\left(\dfrac{\vec{a} \times \vec{r}}{r^n}\right)$.

Sol. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}.$

$$\therefore \quad \text{div } \vec{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3zx) + \frac{\partial}{\partial x}(3z^2 - 3xy)$$

$$= 6x + 6y + 6z = 6(x + y + z)$$

and
$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3zx & 3z^2 - 3xy \end{vmatrix}$$

$$= \hat{i}(-3x + 3x) + \hat{j}(-3y + 3y) + \hat{k}(-3z + 3z) = \vec{0}.$$

Example 4. *Find curl (curl $\vec{V}$) where $\vec{V} = (2xz^2)\hat{i} - yz\hat{j} + 3xz^3\hat{k}$, at (1, 1, 1).*

Sol. Here, $\vec{V} = (2xz^2)\hat{i} - yz\hat{j} + 3xz^3\hat{k}$

$$\therefore \quad \text{curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(3xz^3) - \frac{\partial}{\partial z}(-yz)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(3xz^3) - \frac{\partial}{\partial z}(2xz^2)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial y}(2xz^2)\right\}$$

$$= \hat{i}(0 + y) - \hat{j}(3z^3 - 4xz) + \hat{k}(0 - 0) = y\hat{i} + (4xz - 3z^3)\hat{j}$$

$$\text{curl (curl } \vec{V}) = \text{curl } \{y\hat{i} + (4xz - 3z^3)\hat{j}\}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 4xz - 3z^3 & 0 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(4xz - 3z^3)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(y)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(4xz - 3z^3) - \frac{\partial}{\partial y}(y)\right\}$$

$$= \hat{i}\left(0 - (4x - 9z^2)\right) - \hat{j}(0 - 0) + \hat{k}(4z - 1)$$

$$= (9z^2 - 4x)\hat{i} + (4z - 1)\hat{k} = 5\hat{i} + 3\hat{k} \text{ at } (1, 1, 1).$$

Example 5. *Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a}$ be a constant vector, find the value of div $\left(\frac{\vec{a} \times \vec{r}}{r^n}\right)$.*

Sol. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

$$\vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - a_3 y)\ \hat{i} + (a_3 x - a_1 z)\ \hat{j} + (a_1 y - a_2 x)\ \hat{k}$$

$$\frac{\vec{a} \times \vec{r}}{r^n} = \frac{(a_2 z - a_3 y)\hat{i} + (a_3 x - a_1 z)\hat{j} + (a_1 y - a_2 x)\hat{k}}{(x^2 + y^2 + z^2)^{n/2}}$$

$$\therefore \quad \text{div}\left(\frac{\vec{a} \times \vec{r}}{r^n}\right) = \nabla \cdot \frac{\vec{a} \times \vec{r}}{r^n}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \frac{(a_2 z - a_3 y)\hat{i} + (a_3 x - a_1 z)\hat{j} + (a_1 y - a_2 x)\hat{k}}{(x^2 + y^2 + z^2)^{n/2}}$$

$$= \frac{\partial}{\partial x}\left\{\frac{a_2 z - a_3 y}{(x^2 + y^2 + z^2)^{n/2}}\right\} + \frac{\partial}{\partial y}\left\{\frac{a_3 x - a_1 z}{(x^2 + y^2 + z^2)^{n/2}}\right\} + \frac{\partial}{\partial z}\left\{\frac{a_1 y - a_2 x}{(x^2 + y^2 + z^2)^{n/2}}\right\}$$

$$= (a_2 z - a_3 y) \cdot \left(-\frac{n}{2}\right)(x^2 + y^2 + z^2)^{-\frac{n}{2} - 1} \cdot 2x$$

$$+ (a_3 x - a_1 z)\left(-\frac{n}{2}\right)(x^2 + y^2 + z^2)^{-\frac{n}{2} - 1} \cdot 2y$$

$$+ (a_1 y - a_2 x)\left(-\frac{n}{2}\right)(x^2 + y^2 + z^2)^{-\frac{n}{2} - 1} \cdot 2z$$

$$= \frac{-n}{(x^2 + y^2 + z^2)^{\frac{n}{2} + 1}}\,[(a_2 z - a_3 y)\, x + (a_3 x - a_1 z)\, y + (a_1 y - a_2 x)\, z]$$

$$= \frac{-n}{(x^2 + y^2 + z^2)^{\frac{n}{2} + 1}}\,[0] = 0$$

Hence, $\text{div}\left(\frac{\vec{a} \times \vec{r}}{r^n}\right) = 0.$

Example 6. *Find the directional derivative of div $(\vec{u})$ at the point (1, 2, 2) in the direction of the outer normal to the sphere $x^2 + y^2 + z^2 = 9$ for $\vec{u} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$.*

Sol. Here, $\quad \vec{u} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$

$$\therefore \quad \text{div}\,(\vec{u}) = \nabla \cdot \vec{u} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (x^4\hat{i} + y^4\hat{j} + z^4\hat{k})$$

$$= \frac{\partial}{\partial x}(x^4) + \frac{\partial}{\partial y}(y^4) + \frac{\partial}{\partial z}(z^4)$$

$$= 4\,(x^3 + y^3 + z^3)$$

Directional derivative of div $\vec{u} = \nabla\,(4x^3 + 4y^3 + 4z^3)$

$$= \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)(4x^3 + 4y^3 + 4z^3)$$

$$= 12\,(x^2\hat{i} + y^2\hat{j} + z^2\hat{k})$$

$$= 12\,(\hat{i} + 4\hat{j} + 4\hat{k}) \text{ at } (1, 2, 2)$$

Outer normal to the sphere $= \nabla\,(x^2 + y^2 + z^2 - 9)$

$$= \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9)$$

$$= \hat{i}\,(2x) + \hat{j}\,(2y) + \hat{k}\,(2z)$$

$$= 2(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= 2(\hat{i} + 2\hat{j} + 2\hat{k}) \text{ at } (1, 2, 2)$$

$$= 2\hat{i} + 4\hat{j} + 4\hat{k}$$

Unit outer normal to the sphere at (1, 2, 2) is

$$\hat{n} = \frac{2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{4 + 16 + 16}} = \frac{2\hat{i} + 4\hat{j} + 4\hat{k}}{6}$$

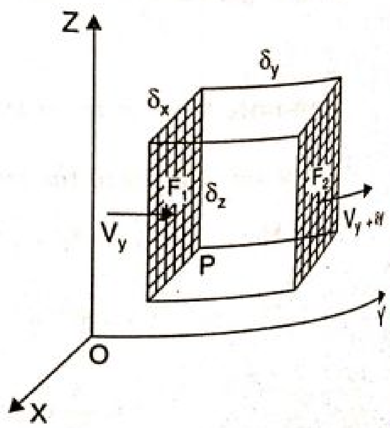
$\therefore$ Directional derivative of div $\vec{u}$ at (1, 2, 2) in the direction of outer normal

$$= 12\,(\hat{i} + 4\hat{j} + 4\hat{k})\cdot\frac{2\hat{i} + 4\hat{j} + 4\hat{k}}{6} = 2\,(2 + 16 + 16) = 68$$

13.17. PHYSICAL INTERPRETATION OF DIVERGENCE

(G.B.T.U. 2010; K.U.K. 2012; P.T.U. May 2010)

Consider a fluid having density $\rho = \rho\,(x, y, z, t)$ and velocity $\vec{v} = \vec{v}\,(x, y, z, t)$ at a point (x, y, z) at time t. Let $\vec{V} = \rho\vec{v}$, then $\vec{V}$ is a vector having the same direction as $\vec{v}$ and magnitude $\rho\,|\,\vec{v}\,|$. It is known as *flux*. Its direction gives the direction of the fluid flow, and its magnitude gives the mass of the fluid crossing per unit time a unit area placed perpendicular to the direction of flow.



Consider the motion of the fluid having velocity $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$ at a point P(x, y, z). Consider a small parallelopiped with edges δx, δy, δz parallel to the axes with one of its corners at P.

The mass of the fluid entering through the face F_1 per unit time is $V_y\,\delta x\,\delta z$ and that flowing out through the opposite face F_2 is $V_{y+\delta y}\,\delta x\,\delta z = \left(V_y + \dfrac{\partial V_y}{\partial y}\,\partial y\right)\delta x\,\delta z$ by using Taylor's series.

$\therefore$ The net decrease in the mass of fluid flowing across these two faces

$$= \left(V_y + \frac{\partial V_y}{\partial y}\,\partial y\right)\delta x\,\delta z - V_y\,\delta x\,\delta z = \frac{\partial V_y}{\partial y}\,\delta x\,\delta y\,\delta z$$

Similarly, considering the other two pairs of faces, we get the total decrease in the mass of fluid inside the parallelopiped per unit time $= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)\delta x\,\delta y\,\delta z.$

Dividing this by the volume $\delta x\,\delta y\,\delta z$ of the parallelopiped, we have the rate of loss of fluid per unit time $= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \text{div}\,\vec{V}$

Hence div $\vec{V}$ gives the rate of outflow per unit volume at a point of the fluid.

Note. If div $\vec{V} = 0$ everywhere in some region R of space, then $\vec{V}$ is called *solenoidal* vector point function.

13.18. PHYSICAL INTERPRETATION OF CURL

(P.T.U. May 2010, Dec. 2011; G.B.T.U. 2010; M.T.U. 2011)

Consider a rigid body rotating about a fixed axis through O with uniform angular velocity

$$\vec{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$$

The velocity $\vec{V}$ of any point P(x, y, z) on the body is given by $\vec{V} = \vec{\omega} \times \vec{r}$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector of P.

$$\therefore \quad \vec{V} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= (\omega_2 z - \omega_3 y)\hat{i} + (\omega_3 x - \omega_1 z)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}$$

$$\text{curl}\,\vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= (\omega_1 + \omega_1)\hat{i} + (\omega_2 + \omega_2)\hat{j} + (\omega_3 + \omega_3)\hat{k} \qquad [\because \ \omega_1, \omega_2, \omega_3 \text{ are constants}]$$

$$= 2\,(\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}) = 2\vec{\omega} \quad \Rightarrow \quad \vec{\omega} = \tfrac{1}{2}\,\text{curl}\,\vec{V}.$$

Thus, the angular velocity at any point is equal to half the curl of the linear velocity at that point of the body.

Note. If curl $\vec{V} = \vec{0}$, then $\vec{V}$ is said to be an **irrotational vector,** otherwise **rotational.**

13.19. PROPERTIES OF DIVERGENCE AND CURL

1. For a constant vector $\vec{a}$, div $\vec{a} = 0$, curl $\vec{a} = \vec{0}$

2. $\textbf{div}\,(\vec{\mathbf{A}}+\vec{\mathbf{B}}) = \textbf{div}\,\vec{\mathbf{A}} + \textbf{div}\,\vec{\mathbf{B}}$ **or** $\nabla\cdot(\vec{\mathbf{A}}+\vec{\mathbf{B}}) = \nabla\cdot\vec{\mathbf{A}} + \nabla\cdot\vec{\mathbf{B}}$

Proof. $\text{div}\,(\vec{A}+\vec{B}) = \nabla\cdot(\vec{A}+\vec{B}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(\vec{A}+\vec{B})$

$$= \hat{i}\cdot\frac{\partial}{\partial x}(\vec{A}+\vec{B}) + \hat{j}\cdot\frac{\partial}{\partial y}(\vec{A}+\vec{B}) + \hat{k}\cdot\frac{\partial}{\partial z}(\vec{A}+\vec{B})$$

$$= \hat{i}\cdot\left(\frac{\partial\vec{A}}{\partial x}+\frac{\partial\vec{B}}{\partial x}\right) + \hat{j}\cdot\left(\frac{\partial\vec{A}}{\partial y}+\frac{\partial\vec{B}}{\partial y}\right) + \hat{k}\cdot\left(\frac{\partial\vec{A}}{\partial z}+\frac{\partial\vec{B}}{\partial z}\right)$$

$$= \left(\hat{i}\cdot\frac{\partial\vec{A}}{\partial x}+\hat{j}\cdot\frac{\partial\vec{A}}{\partial y}+\hat{k}\cdot\frac{\partial\vec{A}}{\partial z}\right) + \left(\hat{i}\cdot\frac{\partial\vec{B}}{\partial x}+\hat{j}\cdot\frac{\partial\vec{B}}{\partial y}+\hat{k}\cdot\frac{\partial\vec{B}}{\partial z}\right)$$

$$= \nabla\cdot\vec{A} + \nabla\cdot\vec{B} = \text{div}\,\vec{A} + \text{div}\,\vec{B}.$$

3. $\textbf{curl}\,(\vec{\mathbf{A}}+\vec{\mathbf{B}}) = \textbf{curl}\,\vec{\mathbf{A}} + \textbf{curl}\,\vec{\mathbf{B}}$ **or** $\nabla\times(\vec{\mathbf{A}}+\vec{\mathbf{B}}) = \nabla\times\vec{\mathbf{A}} + \nabla\times\vec{\mathbf{B}}$

Proof. $\text{curl}\,(\vec{A}+\vec{B}) = \nabla\times(\vec{A}+\vec{B}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times(\vec{A}+\vec{B})$

$$= \sum \hat{i}\times\frac{\partial}{\partial x}(\vec{A}+\vec{B}) = \sum \hat{i}\times\left(\frac{\partial\vec{A}}{\partial x}+\frac{\partial\vec{B}}{\partial x}\right)$$

$$= \sum \hat{i}\times\frac{\partial\vec{A}}{\partial x} + \sum \hat{i}\times\frac{\partial\vec{B}}{\partial x} = \nabla\times\vec{A}+\nabla\times\vec{B} = \text{curl}\,\vec{A}+\text{curl}\,\vec{B}.$$

4. If $\vec{A}$ is a vector function and ϕ is a scalar function, then

$\textbf{div}\,(\phi\vec{A}) = \phi\,\textbf{div}\,\vec{\mathbf{A}} + (\textbf{grad}\,\phi)\cdot\vec{A}$ **or** $\nabla\cdot(\phi\vec{A}) = \phi\,(\nabla\cdot\vec{\mathbf{A}}) + (\nabla\phi)\cdot\vec{A}$

Proof. $\text{div}\,(\phi\vec{A}) = \nabla\cdot(\phi\vec{A}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(\phi\vec{A})$

$$= \hat{i}\cdot\frac{\partial}{\partial x}(\phi\vec{A}) + \hat{j}\cdot\frac{\partial}{\partial y}(\phi\vec{A}) + \hat{k}\cdot\frac{\partial}{\partial z}(\phi\vec{A})$$

$$= \sum \hat{i}\cdot\frac{\partial}{\partial x}(\phi\vec{A}) = \sum \hat{i}\cdot\left(\phi\frac{\partial\vec{A}}{\partial x}+\frac{\partial\phi}{\partial x}\vec{A}\right) = \phi\sum\left(\hat{i}\cdot\frac{\partial\vec{A}}{\partial x}\right) + \sum\left(\hat{i}\frac{\partial\phi}{\partial x}\right)\cdot\vec{A}$$

$$= \phi\,(\nabla\cdot\vec{A}) + (\nabla\phi)\cdot\vec{A} = \phi\,\text{div}\,\vec{A} + (\text{grad}\,\phi)\cdot\vec{A}.$$

5. If A is a vector function and ϕ is a scalar function, then

$\textbf{curl}\,(\phi\vec{\mathbf{A}}) = (\textbf{grad}\,\phi)\times\vec{\mathbf{A}} + \phi\,\textbf{curl}\,\vec{\mathbf{A}}$ **or** $\nabla\times(\phi\vec{\mathbf{A}}) = (\nabla\phi)\times\vec{\mathbf{A}} + \phi\,(\nabla\times\vec{\mathbf{A}})$

(P.T.U. Jan. 2010)

Proof. $\text{curl}\,(\phi\vec{A}) = \nabla\times(\phi\vec{A}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times(\phi\vec{A})$

$$= \sum \hat{i} \times \frac{\partial}{\partial x}(\phi \vec{A}) = \sum \hat{i} \times \left(\frac{\partial \phi}{\partial x} \vec{A} + \phi \frac{\partial \vec{A}}{\partial x} \right) = \sum \hat{i} \times \frac{\partial \phi}{\partial x} \vec{A} + \phi \sum \hat{i} \times \frac{\partial \vec{A}}{\partial x}$$

$$= \sum \frac{\partial \phi}{\partial x} \hat{i} \times \vec{A} + \phi \sum \hat{i} \times \frac{\partial \vec{A}}{\partial x} \qquad [\because \vec{a} \times (m\vec{b}) = (m\vec{a}) \times \vec{b} = m(\vec{a} \times \vec{b})]$$

$$= (\nabla \phi) \times \vec{A} + \phi (\nabla \times \vec{A}) = (\text{grad } \phi) \times \vec{A} + \phi \text{ curl } \vec{A}.$$

$(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}) = (\vec{\mathbf{A}} \cdot \nabla) \vec{\mathbf{B}} + (\vec{\mathbf{B}} \cdot \nabla) \vec{\mathbf{A}} + \vec{\mathbf{A}} \times (\nabla \times \vec{\mathbf{B}}) + \vec{\mathbf{B}} \times (\nabla \times \vec{\mathbf{A}})$

$$\nabla(\vec{A} \cdot \vec{B}) = \sum \hat{i} \frac{\partial}{\partial x}(\vec{A} \cdot \vec{B}) = \sum \hat{i} \left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x} + \frac{\partial \vec{A}}{\partial x} \cdot \vec{B} \right)$$

$$= \sum \left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x} \right) \hat{i} + \sum \left(\vec{B} \cdot \frac{\partial \vec{A}}{\partial x} \right) \hat{i} \qquad \ldots(1)$$

ve know that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$

$$(\vec{a} \cdot \vec{b}) \vec{c} = (\vec{a} \cdot \vec{c}) \vec{b} - \vec{a} \times (\vec{b} \times \vec{c})$$

$$\left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x} \right) \hat{i} = (\vec{A} \cdot \hat{i}) \frac{\partial \vec{B}}{\partial x} - \vec{A} \times \left(\frac{\partial \vec{B}}{\partial x} \times \hat{i} \right) = (\vec{A} \cdot \hat{i}) \frac{\partial \vec{B}}{\partial x} + \vec{A} \times \left(\hat{i} \times \frac{\partial \vec{B}}{\partial x} \right)$$

$$\sum \left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x} \right) \hat{i} = \left(\vec{A} \cdot \sum \hat{i} \frac{\partial}{\partial x} \right) \vec{B} + \vec{A} \times \sum \left(\hat{i} \times \frac{\partial \vec{B}}{\partial x} \right)$$

$$= (\vec{A} \cdot \nabla) \vec{B} + \vec{A} \times (\nabla \times \vec{B}) \qquad \ldots(2)$$

terchanging $\vec{A}$ and $\vec{B}$, $\sum \left(\vec{B} \cdot \frac{\partial \vec{A}}{\partial x} \right) \hat{i} = (\vec{B} \cdot \nabla) \vec{A} + \vec{B} \times (\nabla \times \vec{A})$ ...(3)

ubstituting the values from (2) and (3) in (1), we get

$$\nabla(\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \nabla) \vec{B} + (\vec{B} \cdot \nabla) \vec{A} + \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}).$$

7. $\nabla \cdot (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \vec{\mathbf{B}} \cdot (\nabla \times \vec{\mathbf{A}}) - \vec{\mathbf{A}} \cdot (\nabla \times \vec{\mathbf{B}})$

Or

$\text{div}(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \vec{\mathbf{B}} \cdot \text{curl } \vec{\mathbf{A}} - \vec{\mathbf{A}} \cdot \text{curl } \vec{\mathbf{B}}$ *(G.B.T.U. 2012; P.T.U. Jan. 2010)*

Proof. $\nabla \cdot (\vec{A} \times \vec{B}) = \sum \hat{i} \cdot \frac{\partial}{\partial x}(\vec{A} \times \vec{B}) = \sum \hat{i} \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} + \vec{A} \times \frac{\partial \vec{B}}{\partial x} \right)$

$$= \sum \hat{i} \cdot \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} \right) + \sum \hat{i} \cdot \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x} \right) = \sum \hat{i} \cdot \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} \right) - \sum \hat{i} \cdot \left(\frac{\partial \vec{B}}{\partial x} \times \vec{A} \right)$$

$$= \sum \left(\hat{i} \times \frac{\partial \vec{A}}{\partial x} \right) \cdot \vec{B} - \sum \left(\hat{i} \times \frac{\partial \vec{B}}{\partial x} \right) \cdot \vec{A} \qquad [\text{Since } \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}]$$

$$= (\nabla \times \vec{A}) \cdot \vec{B} - (\nabla \times \vec{B}) \cdot \vec{A} = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}).$$

8. $\nabla \times (\vec{A} \times \vec{B}) = (\nabla \cdot \vec{B})\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$

(G.B.T.U. 2013; P.T.U. May 2010; M.D.U. May 2012)

Proof. $\nabla \times (\vec{A} \times \vec{B}) = \sum \hat{i} \times \frac{\partial}{\partial x}(\vec{A} \times \vec{B}) = \sum \hat{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} + \vec{A} \times \frac{\partial \vec{B}}{\partial x}\right)$

$$= \sum \hat{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B}\right) + \sum \hat{i} \times \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x}\right)$$

$$= \sum \left[(\hat{i} \cdot \vec{B})\frac{\partial \vec{A}}{\partial x} - \left(\hat{i} \cdot \frac{\partial \vec{A}}{\partial x}\right)\vec{B}\right] + \sum \left[\left(\hat{i} \cdot \frac{\partial \vec{B}}{\partial x}\right)\vec{A} - (\hat{i} \cdot \vec{A})\frac{\partial \vec{B}}{\partial x}\right]$$

[Since $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$]

$$= \sum (\vec{B} \cdot \hat{i})\frac{\partial \vec{A}}{\partial x} - \left(\sum \hat{i} \cdot \frac{\partial \vec{A}}{\partial x}\right)\vec{B} + \left(\sum \hat{i} \cdot \frac{\partial \vec{B}}{\partial x}\right)\vec{A} - \sum (\vec{A} \cdot \hat{i})\frac{\partial \vec{B}}{\partial x}$$

$$= \left(\vec{B} \cdot \sum \hat{i}\frac{\partial}{\partial x}\right)\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\nabla \cdot \vec{B})\vec{A} - \left(A \cdot \sum \hat{i}\frac{\partial}{\partial x}\right)\vec{B}$$

$$= (\vec{B} \cdot \nabla)\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\nabla \cdot \vec{B})\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$

$$= (\nabla \cdot \vec{B})\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}.$$

13.20. REPEATED OPERATIONS BY ∇

Let $\phi(x, y, z)$ and $\vec{V}(x, y, z)$ be scalar and vector point functions respectively.

Since grad ϕ and curl $\vec{V}$ are also vector point functions, we can find their divergence as well as curl, whereas div $\vec{V}$ being a scalar point function, we can find its gradient only.

1. **div (grad ϕ) = $\nabla^2\phi$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$**

Proof. div (grad ϕ) = $\nabla \cdot (\nabla\phi) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$$= \nabla^2\phi \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

∇^2 is called the *Laplacian operator* and $\nabla^2\phi = 0$ is called *Laplace's equation*.

2. **curl (grad ϕ) = $\nabla \times \nabla\phi = \vec{0}$** (*G.B.T.U. 2011*)

Proof. curl (grad ϕ) $= \nabla \times \nabla\phi = \nabla \times \left(\hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}\right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\phi}{\partial z} \end{vmatrix} = \sum \hat{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) = \vec{0}$$

3. **div (curl $\vec{V}$) = $\nabla \cdot (\nabla \times \vec{V}) = 0$.** (*G.B.T.U. 2011; M.D.U. May 2011*)

Proof. Let $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$, then curl $\vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$\therefore$ div (curl $\vec{V}$) $= \nabla \cdot (\nabla \times \vec{V})$

$$= \frac{\partial}{\partial x}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$= \left(\frac{\partial^2 V_3}{\partial x\partial y} - \frac{\partial^2 V_2}{\partial x\partial z}\right) + \left(\frac{\partial^2 V_1}{\partial y\partial z} - \frac{\partial^2 V_3}{\partial y\partial x}\right) + \left(\frac{\partial^2 V_2}{\partial z\partial x} - \frac{\partial^2 V_1}{\partial z\partial y}\right) = 0.$$

(*M.D.U. 2013*)

4. **curl (curl $\vec{V}$) = grad div $\vec{V} - \nabla^2\vec{V}$**

or $\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2\vec{V}$.

Proof. Let $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

then curl $\vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$\therefore$ curl (curl $\vec{V}$) $= \nabla \times (\nabla \times \vec{V}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial V_3}{\partial y} - \dfrac{\partial V_2}{\partial z} & \dfrac{\partial V_1}{\partial z} - \dfrac{\partial V_3}{\partial x} & \dfrac{\partial V_2}{\partial x} - \dfrac{\partial V_1}{\partial y} \end{vmatrix}$

$$= \sum \hat{i}\left\{\frac{\partial}{\partial y}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\right\}$$

$$= \sum \hat{i} \left\{ \frac{\partial}{\partial x} \left(\frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \right) - \left(\frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2} \right) \right\}$$

$$= \sum \hat{i} \left\{ \frac{\partial}{\partial x} \left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \right) - \left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2} \right) \right\}$$

$$\left[\because \frac{\partial}{\partial x} \left(\frac{\partial V_1}{\partial x} \right) = \frac{\partial^2 V_1}{\partial x^2} \right]$$

$$= \sum \hat{i} \left\{ \frac{\partial}{\partial x} (\nabla \cdot \vec{V}) - (\nabla^2 V_1) \right\} = \sum \hat{i} \frac{\partial}{\partial x} (\nabla \cdot \vec{V}) - \nabla^2 \Sigma \hat{i} V_1$$

$$= \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V} = \text{grad (div } \vec{V}) - \nabla^2 \vec{V}.$$

Note 1. The above result can also be written as grad (div $\vec{V}$) = curl (curl $\vec{V}$) + $\nabla^2 \vec{V}$

or $$\nabla(\nabla \cdot \vec{V}) = \nabla \times (\nabla \times \vec{V}) + \nabla^2 \vec{V}.$$

Note 2. Treating ∇ as a vector, the results of repeated application of ∇ can be easily written down. Thus

$$\nabla \cdot \nabla \phi = \nabla^2 \phi \qquad (\because \vec{a} \cdot \vec{a} = a^2)$$

$$\nabla \times \nabla \phi = \vec{0} \qquad (\because \vec{a} \times \vec{a} = \vec{0})$$

$$\nabla \cdot (\nabla \times \vec{V}) = 0 \qquad (\because \vec{a} \cdot (\vec{a} \times \vec{b}) = [\vec{a}\ \vec{a}\ \vec{b}] = 0)$$

$$\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V}$$ (By expanding as a vector triple product)

ILLUSTRATIVE EXAMPLES

Example 1. *A vector field is given by* $\vec{A} = (x^2 + xy^2)\hat{i} + (y^2 + x^2 y)\hat{j}$. *Show that the field is irrotational and find the scalar potential.*

Sol. Field $\vec{A}$ is irrotational if curl $\vec{A} = \vec{0}$

Now, $$\text{curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2 y & 0 \end{vmatrix}$$

$$= \hat{i}(0 - 0) - \hat{j}(0 - 0) + \hat{k}(2xy - 2xy) = \vec{0}.$$

$\therefore$ Field $\vec{A}$ is irrotational.

If ϕ is the scalar potential, then $\vec{A}$ = grad ϕ $\qquad [\because$ curl (grad $\phi) = \vec{0}]$

$$\Rightarrow \quad (x^2 + xy^2)\hat{i} + (y^2 + x^2 y)\hat{j} = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial x} = x^2 + xy^2, \quad \frac{\partial \phi}{\partial y} = y^2 + x^2 y, \quad \frac{\partial \phi}{\partial z} = 0$$

$$\therefore \quad d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}\,dz$$

$$= (x^2 + xy^2)\,dx + (y^2 + x^2y)dy$$

$$= x^2\,dx + y^2\,dy + xy(y\,dx + x\,dy)$$

$$= d\left(\frac{x^3}{3}\right) + d\left(\frac{y^3}{3}\right) + d\left(\frac{x^2y^2}{2}\right)$$

Integrating, $$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2} + c.$$

Example 2. *A fluid motion is given by* $\vec{V} = (y \sin z - \sin x)\,\hat{i} + (x \sin z + 2yz)\hat{j} + (xy \cos z + y^2)\hat{k}$ *is the motion irrotational ? If so, find the velocity potential.*

(P.T.U. May 2010; U.K.T.U. 2011)

Sol. The motion is irrotational if curl $\vec{V} = \vec{0}$

Now, $$\text{curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y \sin z - \sin x & x \sin z + 2yz & xy \cos z + y^2 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(xy \cos z + y^2) - \frac{\partial}{\partial z}(x \sin z + 2yz)\right\} + \hat{j}\left\{\frac{\partial}{\partial z}(y \sin z - \sin x) - \frac{\partial}{\partial x}(xy \cos z + y^2)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(x \sin z + 2yz) - \frac{\partial}{\partial y}(y \sin z - \sin x)\right\}$$

$$= \hat{i}\,(x \cos z + 2y - x \cos z - 2y) + \hat{j}(y \cos z - y \cos z) + \hat{k}\,(\sin z - \sin z)$$

$$= \hat{i}\,(0) + \hat{j}(0) + \hat{k}\,(0) = \vec{0}$$

$\therefore$ The motion is irrotational.

Let u be the velocity potential, then $\vec{V} = \text{grad } u$ $\quad [\because \text{ curl (grad } u) = \vec{0}]$

$$\Rightarrow \quad (y \sin z - \sin x)\,\hat{i} + (x \sin z + 2yz)\,\hat{j} + (xy \cos z + y^2)\,\hat{k} = \hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = y \sin z - \sin x, \quad \frac{\partial u}{\partial y} = x \sin z + 2yz, \quad \frac{\partial u}{\partial z} = xy \cos z + y^2$$

$$\therefore \quad du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz$$

$$= (y \sin z - \sin x)\, dx + (x \sin z + 2yz)\, dy + (xy \cos z + y^2)\, dz$$
$$= (y \sin z\, dx + x \sin z\, dy + xy \cos z\, dz) + (-\sin x\, dx) + (2yz\, dy + y^2\, dz)$$
$$= d(xy \sin z) + d(\cos x) + d(y^2 z)$$

Integrating, $\quad u = xy \sin z + \cos x + y^2 z + c.$

Example 3. *If the vector* $\vec{F} = (ax^2y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}$ *is solenoidal, find the value of a. Find also the curl of this solenoidal vector.*

Sol. Here $\vec{F} = (ax^2y + yz)\hat{i} + (xy^2 - xz^2)\,\hat{j} + (2xyz - 2x^2y^2)\,\hat{k}$

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[(ax^2y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}\right]$$
$$= \frac{\partial}{\partial x}(ax^2y + yz) + \frac{\partial}{\partial y}(xy^2 - xz^2) + \frac{\partial}{\partial z}(2xyz - 2x^2y^2)$$
$$= 2axy + 2xy + 2xy = 2(a + 2)xy$$

Since $\vec{F}$ is solenoidal, div $\vec{F} = 0$

$\Rightarrow \quad 2(a + 2)xy = 0 \qquad \therefore \quad a = -2$

Now, $\quad \vec{F} = (-2x^2y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}$

$$\therefore \quad \text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -2x^2y + yz & xy^2 - xz^2 & 2xyz - 2x^2y^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2xyz - 2x^2y^2) - \frac{\partial}{\partial z}(xy^2 - xz^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(2xyz - 2x^2y^2) - \frac{\partial}{\partial z}(-2x^2y + yz)\right]$$
$$+ \hat{k}\left[\frac{\partial}{\partial x}(xy^2 - xz^2) - \frac{\partial}{\partial y}(-2x^2y + yz)\right]$$
$$= \hat{i}\,(2xz - 4x^2y + 2xz) - \hat{j}(2yz - 4xy^2 - y) + \hat{k}\,(y^2 - z^2 + 2x^2 - z)$$
$$= 4x(z - xy)\hat{i} + (y + 4xy^2 - 2yz)\,\hat{j} + (2x^2 + y^2 - z^2 - z)\hat{k}.$$

Example 4. *Show that* $r^\alpha \vec{R}$ *is an irrotational vector for any value of* α *but it is solenoidal if* $\alpha + 3 = 0$ *where* $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ *and r is the magnitude of* $\vec{R}$. (M.D.U. 2010)

Sol. Let $\quad \vec{V} = r^\alpha \vec{R} = (x^2 + y^2 + z^2)^{\frac{\alpha}{2}}\,(x\hat{i} + y\hat{j} + z\hat{k})$

$$= x(x^2 + y^2 + z^2)^{\alpha/2}\,\hat{i} + y(x^2 + y^2 + z^2)^{\alpha/2}\,\hat{j} + z(x^2 + y^2 + z^2)^{\alpha/2}\,\hat{k}$$

$$\therefore \quad \text{curl}\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x\,(x^2 + y^2 + z^2)^{\alpha/2} & y\,(x^2 + y^2 + z^2)^{\alpha/2} & z\,(x^2 + y^2 + z^2)^{\alpha/2} \end{vmatrix}$$

$$= \sum \hat{i}\left\{\frac{\alpha z}{2}(x^2 + y^2 + z^2)^{\frac{\alpha}{2} - 1} . 2y - \frac{\alpha y}{2}(x^2 + y^2 + z^2)^{\frac{\alpha}{2} - 1} . 2z\right\}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

$\Rightarrow$ $\vec{V} = r^\alpha \vec{R}$ is irrotational for any value of α.

Now, $\text{div } \vec{V} = \nabla \cdot (r^\alpha \vec{R})$

$$= r^\alpha (\text{div } \vec{R}) + \text{grad } r^\alpha \cdot \vec{R} \quad \text{...(1)}$$

$$\left[\because \text{ div } (\phi \vec{A}) = \phi (\text{div } \vec{A}) + \text{grad } \phi \cdot \vec{A}\right]$$

and $\text{div } (\vec{R}) = \nabla \cdot (x\hat{i} + y\hat{j} + z\hat{k})$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$$

Also, $r^2 = x^2 + y^2 + z^2$ so that $\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\text{grad } r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\vec{R}}{r}$$

$\therefore$ $\text{grad } r^\alpha = \alpha\, r^{\alpha-1} \text{ grad } r = \alpha r^{\alpha-1}\frac{\vec{R}}{r} = \alpha r^{\alpha-2}\vec{R}$

$\therefore$ From (1), we have

$$\text{div } \vec{V} = r^\alpha (3) + \alpha r^{\alpha-2}\vec{R}\cdot\vec{R} = 3r^\alpha + \alpha\, r^{\alpha-2}(x^2 + y^2 + z^2)$$
$$= 3r^\alpha + \alpha\, r^{\alpha-2}(r^2) = (3 + \alpha)\, r^\alpha$$

Now, $\vec{V}$ is solenoidal if $\text{div } \vec{V} = 0$ *i.e.*, $(3 + \alpha)\, r^\alpha = 0$

$\Rightarrow$ $r^\alpha \vec{R}$ is solenoidal if $\alpha + 3 = 0$.

Note. The first part of the above example may be done using properties of curl of a vector. Thus

$$\text{curl } \vec{V} = \text{curl } (r^\alpha \vec{R})$$

$$= (\text{grad } r^\alpha) \times \vec{R} + r^\alpha (\text{curl } \vec{R}) \quad [\because \text{ curl } (\phi\vec{A}) = (\text{grad } \phi) \times \vec{A} + \phi(\text{curl } \vec{A})]$$

$$= (\alpha r^{\alpha-1} \text{ grad } r) \times \vec{R} + r^\alpha(\vec{0}) \quad [\because \text{ curl } \vec{R} = \vec{0}]$$

$$= \alpha r^{\alpha-1}\frac{\vec{R}}{r} \times \vec{R} + \vec{0}$$

$$= \alpha r^{\alpha-2}(\vec{R} \times \vec{R}) = \alpha r^{\alpha-2}(\vec{0})$$

$$= \vec{0}$$

Example 5. *If $\vec{a}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that curl $(\vec{a} \times \vec{r}) = 2\vec{a}$.*

(Calicut 2009; K.U.K. 2009; P.T.U. May 2010; M.D.U. May 2010)

Sol. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, where a_1, a_2, a_3 are constants.

$$\vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - a_3 y)\hat{i} + (a_3 x - a_1 z)\hat{j} + (a_1 y - a_2 x)\hat{k}$$

$$\text{curl } (\vec{a} \times \vec{r}) = \nabla \times (\vec{a} \times \vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= (a_1 + a_1)\hat{i} + (a_2 + a_2)\hat{j} + (a_3 + a_3)\hat{k} = 2(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = 2\vec{a}.$$

Example 6. *Prove that*

(*i*) $\nabla(\vec{a} \cdot \vec{u}) = (\vec{a} \cdot \nabla)\vec{u} + \vec{a} \times (\nabla \times \vec{u})$ (*ii*) $\nabla \times (\vec{a} \times \vec{u}) = (\nabla \cdot \vec{u})\vec{a} - (\vec{a} \cdot \nabla)\vec{u}$

where $\vec{a}$ *is a constant vector.*

Sol. (*i*) $$\nabla(\vec{a} \cdot \vec{u}) = \sum \hat{i} \frac{\partial}{\partial x}(\vec{a} \cdot \vec{u}) = \sum \hat{i} \left(\vec{a} \cdot \frac{\partial \vec{u}}{\partial x}\right) \qquad \text{...(1)}$$

Now $$\vec{a} \times \left(\hat{i} \times \frac{\partial \vec{u}}{\partial x}\right) = \left(\vec{a} \cdot \frac{\partial \vec{u}}{\partial x}\right)\hat{i} - (\vec{a} \cdot \hat{i})\frac{\partial \vec{u}}{\partial x}$$

$$\Rightarrow \quad \left(\vec{a} \cdot \frac{\partial \vec{u}}{\partial x}\right)\hat{i} = \vec{a} \times \left(\hat{i} \times \frac{\partial \vec{u}}{\partial x}\right) + (\vec{a} \cdot \hat{i})\frac{\partial \vec{u}}{\partial x}$$

$\therefore$ From (1), we have $$\nabla(\vec{a} \cdot \vec{u}) = \sum \vec{a} \times \left(\hat{i} \times \frac{\partial \vec{u}}{\partial x}\right) + \sum (\vec{a} \cdot \hat{i})\frac{\partial \vec{u}}{\partial x}$$

$$= \vec{a} \times (\nabla \times \vec{u}) + (\vec{a} \cdot \nabla)\vec{u} = (\vec{a} \cdot \nabla)\vec{u} + \vec{a} \times (\nabla \times \vec{u}).$$

(*ii*) $$\nabla \times (\vec{a} \times \vec{u}) = \sum \hat{i} \frac{\partial}{\partial x} \times (\vec{a} \times \vec{u}) = \sum \hat{i} \times \left(\vec{a} \times \frac{\partial \vec{u}}{\partial x}\right)$$

$$= \sum \left(\hat{i} \cdot \frac{\partial \vec{u}}{\partial x}\right)\vec{a} - \sum (\hat{i} \cdot \vec{a})\frac{\partial \vec{u}}{\partial x} = (\nabla \cdot \vec{u})\vec{a} - (\vec{a} \cdot \nabla)\vec{u}.$$

Example 7. *Prove that* $\nabla \cdot (\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$.

Sol. For a scalar function f and a vector function $\vec{G}$, we know that

$$\nabla \cdot (f\vec{G}) = f(\nabla \cdot \vec{G}) + (\nabla f) \cdot \vec{G}$$

Also $$\nabla \cdot (\vec{F} - \vec{G}) = \nabla \cdot \vec{F} - \nabla \cdot \vec{G}$$

$$\therefore \quad \nabla \cdot (\phi\nabla\psi - \psi\nabla\phi) = \nabla \cdot (\phi\nabla\psi) - \nabla \cdot (\psi\nabla\phi)$$
$$= [\phi(\nabla \cdot \nabla\psi) + \nabla\phi \cdot \nabla\psi] - [\psi(\nabla \cdot \nabla\phi) + \nabla\psi \cdot \nabla\phi]$$
$$= \phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi - \psi\nabla^2\phi - \nabla\psi \cdot \nabla\phi$$
$$= \phi\nabla^2\psi - \psi\nabla^2\phi \qquad [\because \text{ dot product is commutative}]$$

Example 8. *Prove that*

(*i*) $div\left(\frac{\vec{r}}{r^3}\right) = 0$ (Calicut 2009) (*ii*) $\nabla^2 (r^n) = n(n+1)\, r^{n-2}$

(U.K.T.U. 2012; K.U.K. Dec. 2013; P.T.U. 2008)

(*iii*) $\nabla^2\left(\frac{x}{r^3}\right) = 0$, *where r is the magnitude of* $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Sol. Here $r^2 = x^2 + y^2 + z^2$ so that $\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\text{grad } r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z} = \frac{\vec{r}}{r}$$

(*i*) Since $\text{div }(\phi\vec{A}) = \phi(\text{div }\vec{A}) + \text{grad }\phi \cdot \vec{A}$

$$\therefore \quad \text{div}\left(\frac{\vec{r}}{r^3}\right) = \text{div }(r^{-3}\vec{r}) = r^{-3}(\text{div }\vec{r}) + (\text{grad } r^{-3}) \cdot \vec{r}$$

$$[\because \text{ div } \vec{r} = 3]$$

$$= 3r^{-3} + (-3r^{-4} \text{ grad } r) \cdot \vec{r}$$
$$= 3r^{-3} + \left(-3r^{-4}\frac{\vec{r}}{r}\right) \cdot \vec{r} = 3r^{-3} - 3r^{-5}(\vec{r} \cdot \vec{r}) = 3r^{-3} - 3r^{-5}(r^2) = 0.$$

(*ii*)
$$\nabla^2(r^n) = \nabla\cdot(\nabla r^n) = \nabla\cdot\left(nr^{n-1}\frac{\vec{r}}{r}\right) = n\nabla\cdot(r^{n-2}\vec{r})$$
$$= n\,[(\nabla r^{n-2}) \cdot \vec{r} + r^{n-2}(\nabla \cdot \vec{r})] \quad [\because \nabla \cdot (\phi\vec{A}) = (\nabla\phi) \cdot \vec{A} + \phi(\nabla \cdot \vec{A})]$$

$$[\because \nabla \cdot \vec{r} = 3]$$

$$= n\left[(n-2)\,r^{n-3}\frac{\vec{r}}{r} \cdot \vec{r} + r^{n-2}(3)\right]$$

$$[\because \vec{r} \cdot \vec{r} = r^2]$$

$$= n\,[(n-2)\,r^{n-4}(r^2) + 3r^{n-2}]$$
$$= n(n+1)r^{n-2}.$$

Second Method

$$\nabla^2(r^n) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^n$$
$$= \sum \frac{\partial^2}{\partial x^2}(r^n) = \sum \frac{\partial}{\partial x}\left(\frac{\partial r^n}{\partial x}\right)$$
$$= \sum \frac{\partial}{\partial x}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) = \sum \frac{\partial}{\partial x}\left(nr^{n-1}\frac{x}{r}\right) = \sum n\frac{\partial}{\partial x}(r^{n-2}x)$$

$$= n \sum \left[(n-2) r^{n-3} \frac{\partial r}{\partial x} \cdot x + r^{n-2} \right] = n \sum \left[(n-2) r^{n-3} \frac{x}{r} \cdot x + r^{n-2} \right]$$

$$= n \sum [(n-2) r^{n-4} x^2 + r^{n-2}] = n[(n-2) r^{n-4} (x^2 + y^2 + z^2) + 3r^{n-2}]$$

$$= n [(n-2) r^{n-4}(r^2) + 3r^{n-2}] = n(n+1)r^{n-2}.$$

(Second Statement)

Prove that div (grad r^n) = $n(n+1)r^{n-2}$.

(*iii*)
$$\nabla^2 \left(\frac{x}{r^3}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{x}{r^3}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right) \quad ...(1)$$

Now
$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{1}{r^3} - \frac{3x}{r^4}\frac{\partial r}{\partial x} = \frac{1}{r^3} - \frac{3x^2}{r^5} \qquad \Big| \because \frac{\partial r}{\partial x} = \frac{x}{r}$$

$\therefore$
$$\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right) = -\frac{3}{r^4}\frac{\partial r}{\partial x} - \frac{6x}{r^5} + \frac{15x^2}{r^6}\frac{\partial r}{\partial x}$$

$$= -\frac{3x}{r^5} - \frac{6x}{r^5} + \frac{15r^3}{r^7} = -\frac{9x}{r^5} + \frac{15x^3}{r^7}$$

$$\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right) = -\frac{3x}{r^4}\frac{\partial r}{\partial y} = -\frac{3xy}{r^5} \qquad \Big| \because \frac{\partial r}{\partial y} = \frac{y}{r}$$

$\therefore$
$$\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right) = \frac{\partial}{\partial y}\left(-\frac{3xy}{r^5}\right) = -\frac{3x}{r^5} + \frac{15xy}{r^6}\frac{\partial r}{\partial y} = -\frac{3x}{r^5} + \frac{15xy^2}{r^7}$$

Similarly,
$$\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right) = -\frac{3x}{r^5} + \frac{15xz^2}{r^7}$$

$\therefore$ From (1), we have

$$\nabla^2\left(\frac{x}{r^3}\right) = \left(-\frac{9x}{r^5} + \frac{15x^3}{r^7}\right) + \left(-\frac{3x}{r^5} + \frac{15xy^2}{r^7}\right) + \left(-\frac{3x}{r^5} + \frac{15xz^2}{r^7}\right)$$

$$= -\frac{15x}{r^5} + \frac{15x}{r^7}(x^2 + y^2 + z^2) = -\frac{15x}{r^5} + \frac{15x}{r^7}(r^2) \quad \Big| \because x^2 + y^2 + z^2 = r^2$$

$$= -\frac{15x}{r^5} + \frac{15x}{r^5} = 0$$

Example 9. *Prove that the vector $f(r)\,\vec{r}$ is irrotational.* (Anna 2009)

Sol. The vector $f(r)\,\vec{r}$ will be irrotational if curl $[f(r)\,\vec{r}\,] = \vec{0}$

Since
$$\text{curl}\,(\phi\vec{A}) = (\text{grad}\,\phi) \times \vec{A} + \phi\,\text{curl}\,\vec{A}$$

$\therefore$
$$\text{curl}\,[f(r)\,\vec{r}\,] = [\text{grad}\,f(r)] \times \vec{r} + f(r)\,\text{curl}\,\vec{r}$$

$$= [f'(r)\,\text{grad}\,r] \times \vec{r} + f(r)\,\vec{0} \qquad [\because \text{curl}\,\vec{r} = \vec{0}]$$

$$= \left[f'(r)\frac{\vec{r}}{r}\right] \times \vec{r} = \frac{f'(r)}{r}(\vec{r} \times \vec{r}) = \vec{0}, \text{ since } \vec{r} \times \vec{r} = \vec{0}.$$

$\therefore$ The vector $f(r)\,\vec{r}$ is irrotational.

Example 10. *Prove that* $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$. *Hence evaluate* $\nabla^2 (\log r)$

(M.D.U. May 2011, May 2012; U.K.T.U. 2010; G.B.T.U. 2012)

Sol. $\nabla^2 f(r) = \nabla \cdot \{\nabla f(r)\} = \text{div}\,\{\text{grad } f(r)\} = \text{div}\,\{f'(r) \text{ grad } r\} = \text{div}\left\{f'(r)\frac{\vec{r}}{r}\right\}$

$$= \text{div}\left\{\frac{1}{r} f'(r)\,\vec{r}\right\} = \frac{1}{r} f'(r)\,\text{div}\,\vec{r} + \vec{r} \cdot \text{grad}\left\{\frac{1}{r} f'(r)\right\}$$

$$= \frac{3}{r} f'(r) + \vec{r} \cdot \left[\frac{d}{dr}\left(\frac{1}{r} f'(r)\right) \text{grad } r\right] = \frac{3}{r} f'(r) + \vec{r} \cdot \left[\left\{-\frac{1}{r^2} f'(r) + \frac{1}{r} f''(r)\right\} \frac{\vec{r}}{r}\right]$$

$$= \frac{3}{r} f'(r) + \left[-\frac{1}{r^3} f'(r) + \frac{1}{r^2} f''(r)\right](\vec{r} \cdot \vec{r}) = \frac{3}{r} f'(r) + \left[-\frac{1}{r^3} f'(r) + \frac{1}{r^2} f''(r)\right] r^2$$

$$= \frac{3}{r} f'(r) - \frac{1}{r} f'(r) + f''(r) = f''(r) + \frac{2}{r} f'(r).$$

If $\quad f(r) = \log r$, then $f'(r) = \frac{1}{r}$ and $f''(r) = -\frac{1}{r^2}$

$$\therefore \; \nabla^2 (\log r) = -\frac{1}{r^2} + \frac{2}{r}\left(\frac{1}{r}\right) = \frac{1}{r^2} = \frac{1}{x^2 + y^2 + z^2}.$$

Example 11. *If* $u\vec{F} = \nabla v$, *where* u, v *are scalar fields and* $\vec{F}$ *is a vector, field, show that* $\vec{F} \cdot \text{curl}\,\vec{F} = 0$.

$\left[\because \; \vec{F} = \frac{1}{u} \nabla v\right]$

Sol. $\quad \text{curl}\,\vec{F} = \nabla \times \left(\frac{1}{u} \nabla v\right)$ $\quad [\because \; \nabla \times (\phi \vec{A}) = \nabla\phi \times \vec{A} + \phi \nabla \times \vec{A}]$

$$= \nabla \frac{1}{u} \times \nabla v + \frac{1}{u} \nabla \times (\nabla v)$$ $[\because \; \nabla \times \nabla v = \vec{0}]$

$$= \nabla \frac{1}{u} \times \nabla v$$

$$\therefore \quad \vec{F} \cdot \text{curl}\,\vec{F} = \frac{1}{u} \nabla v \cdot \left(\nabla \frac{1}{u} \times \nabla v\right) \frac{1}{u}\left[\nabla v \cdot \left(\nabla \frac{1}{u} \times \nabla v\right)\right] = 0$$

being the scalar triple product in which two factors are equal.

Example 12. *Prove that* $\vec{a} \cdot \nabla\left(\vec{b} \cdot \nabla \frac{1}{r}\right) = \frac{3(\vec{a} \cdot \vec{r})(\vec{b} \cdot \vec{r})}{r^5} - \frac{\vec{a} \cdot \vec{b}}{r^3}$ *where* $\vec{a}$ *and* $\vec{b}$ *are constant vectors.*

Sol. We know that $\quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$$\therefore \quad \nabla \frac{1}{r} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

$$= \frac{-1}{2(x^2 + y^2 + z^2)^{\frac{3}{2}}} (2x\hat{i} + 2y\hat{j} + 2z\hat{k}) = -\frac{\vec{r}}{r^3}$$

$$\Rightarrow \quad \nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right)=\nabla\left[\vec{b}\cdot\left(-\frac{\vec{r}}{r^3}\right)\right]=-\nabla\left[\frac{1}{r^3}(\vec{b}\cdot\vec{r})\right] \quad \text{[Form } \nabla(fg)]$$

$$=-\left[\frac{1}{r^3}\nabla(\vec{b}\cdot\vec{r})+(\vec{b}\cdot\vec{r})\nabla\frac{1}{r^3}\right]$$

$$=-\left[\frac{1}{r^3}\{(\vec{b}\cdot\nabla)\vec{r}+(\vec{r}\cdot\nabla)\vec{b}+\vec{b}\times(\nabla\times\vec{r})+\vec{r}\times(\nabla\times\vec{b})\}\right.$$

$$\left.+(\vec{b}\cdot\vec{r})\left\{\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right\}(x^2+y^2+z^2)^{-\frac{3}{2}}\right] \quad \ldots(1)$$

Let $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ where b_1, b_2, b_3 are independent of x, y, z (since $\vec{b}$ is a constant vector)

$$(\vec{b}\cdot\nabla)\vec{r}=(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})\cdot\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\vec{r}$$

$$=\left(b_1\frac{\partial}{\partial x}+b_2\frac{\partial}{\partial y}+b_3\frac{\partial}{\partial z}\right)(x\hat{i}+y\hat{j}+z\hat{k})$$

$$=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}=\vec{b}$$

$$(\vec{r}\cdot\nabla)\vec{b}=(x\hat{i}+y\hat{j}+z\hat{k})\cdot\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\vec{b}$$

$$=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})=\vec{0}$$

$$\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-\frac{3}{2}}=\frac{-3}{2(x^2+y^2+z^2)^{\frac{5}{2}}}(2x\hat{i}+2y\hat{j}+2z\hat{k})$$

$$=-\frac{3\vec{r}}{r^5}$$

Also $\quad \nabla\times\vec{r}=\vec{0},\ \nabla\times\vec{b}=\vec{0}$

$\therefore$ From (1), we have

$$\nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right)=-\left[\frac{1}{r^3}\left\{\vec{b}+\vec{0}+\vec{0}+\vec{0}\right\}+(\vec{b}\cdot\vec{r})\left(\frac{-3\vec{r}}{r^5}\right)\right]=-\frac{\vec{b}}{r^3}+\frac{3(\vec{b}\cdot\vec{r})\vec{r}}{r^5}$$

$$\Rightarrow \quad \vec{a}\cdot\nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right)=\vec{a}\cdot\left[-\frac{\vec{b}}{r^3}+\frac{3(\vec{b}\cdot\vec{r})\vec{r}}{r^5}\right]=-\frac{\vec{a}\cdot\vec{b}}{r^3}+\frac{3(\vec{b}\cdot\vec{r})(\vec{a}\cdot\vec{r})}{r^5}$$

$$=\frac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{r^5}-\frac{\vec{a}\cdot\vec{b}}{r^3}$$

Example 13. *If r is the distance of a point (x, y, z) from the origin, prove that curl* $\left(\hat{k}\times \text{grad}\,\frac{1}{r}\right) + \text{grad}\left(\hat{k}\cdot \text{grad}\,\frac{1}{r}\right) = \vec{0}$, *where* $\hat{k}$ *is the unit vector in the direction of OZ.*

Sol. Here, $r = \sqrt{x^2 + y^2 + z^2}$ so that $r^2 = x^2 + y^2 + z^2$

and
$$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \quad \text{grad}\, r = \hat{i}\,\frac{\partial r}{\partial x} + \hat{j}\,\frac{\partial r}{\partial y} + \hat{k}\,\frac{\partial r}{\partial z} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}$$

$$\text{grad}\,\frac{1}{r} = -\frac{1}{r^2}\ \text{grad}\, r = -\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^3}$$

$$\Rightarrow \quad \hat{k}\times \text{grad}\,\frac{1}{r} = -\frac{x\,(\hat{k}\times\hat{i}) + y\,(\hat{k}\times\hat{j}) + z\,(\hat{k}\times\hat{k})}{r^3}$$

$$= -\frac{x\hat{j} - y\hat{i}}{r^3} = \frac{y\hat{i} - x\hat{j}}{(x^2 + y^2 + z^2)^{3/2}}$$

and
$$\hat{k}\cdot \text{grad}\,\frac{1}{r} = -\frac{\hat{k}\cdot(x\hat{i} + y\hat{j} + z\hat{k})}{r^3} = -\frac{z}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\therefore \quad \text{curl}\left(\hat{k}\times \text{grad}\,\frac{1}{r}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{(x^2 + y^2 + z^2)^{3/2}} & \frac{-x}{(x^2 + y^2 + z^2)^{3/2}} & 0 \end{vmatrix}$$

$$= \hat{i}\,\frac{\partial}{\partial z}\left\{\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right\} + \hat{j}\,\frac{\partial}{\partial z}\left\{\frac{y}{(x^2 + y^2 + z^2)^{3/2}}\right\}$$
$$+ \hat{k}\left[\frac{\partial}{\partial x}\left\{\frac{-x}{(x^2 + y^2 + z^2)^{3/2}}\right\} - \frac{\partial}{\partial y}\left\{\frac{y}{(x^2 + y^2 + z^2)^{3/2}}\right\}\right]$$

$$= \hat{i}\left[-\frac{3}{2}\,x\,(x^2 + y^2 + z^2)^{-\frac{5}{2}}\cdot 2z\right] + \hat{j}\left[-\frac{3}{2}\,y\,(x^2 + y^2 + z^2)^{-\frac{5}{2}}\cdot 2z\right]$$
$$+ \hat{k}\left[\left\{-(x^2 + y^2 + z^2)^{-\frac{3}{2}} + \frac{3}{2}\,x\,(x^2 + y^2 + z^2)^{-\frac{5}{2}}\cdot 2x\right\}\right.$$
$$\left. - \left\{(x^2 + y^2 + z^2)^{-\frac{3}{2}} - \frac{3}{2}\,y\,(x^2 + y^2 + z^2)^{-\frac{5}{2}}\cdot 2y\right\}\right]$$

$$= \frac{-3xz\hat{i}}{(x^2 + y^2 + z^2)^{5/2}} - \frac{3yz\hat{j}}{(x^2 + y^2 + z^2)^{5/2}}$$
$$+ \hat{k}\left[\frac{-(x^2 + y^2 + z^2) + 3x^2 - (x^2 + y^2 + z^2) + 3y^2}{(x^2 + y^2 + z^2)^{5/2}}\right]$$

$$= \frac{-3xz\hat{i} - 3yz\hat{j} + (x^2 + y^2 - 2z^2)\,\hat{k}}{r^5} \qquad \ldots(1)$$

Also, grad $\left(\hat{k}\cdot \text{grad}\,\frac{1}{r}\right) = \nabla\left[\frac{-z}{(x^2+y^2+z^2)^{3/2}}\right]$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left[\frac{-z}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \hat{i}\left[\frac{3}{2}z(x^2+y^2+z^2)^{-\frac{5}{2}}\cdot 2x\right]+\hat{j}\left[\frac{3}{2}z(x^2+y^2+z^2)^{-\frac{5}{2}}\cdot 2y\right]$$

$$+\hat{k}\left[-(x^2+y^2+z^2)^{-\frac{3}{2}}+\frac{3}{2}z(x^2+y^2+z^2)^{-\frac{5}{2}}\cdot 2z\right]$$

$$= \frac{3xz\hat{i}}{(x^2+y^2+z^2)^{5/2}}+\frac{3yz\hat{j}}{(x^2+y^2+z^2)^{5/2}}+\frac{\{-(x^2+y^2+z^2)+3z^2\}\hat{k}}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{3xz\hat{i}+3yz\hat{j}+(2z^2-x^2-y^2)\hat{k}}{r^5} \quad \text{...(2)}$$

Adding (1) and (2), we have

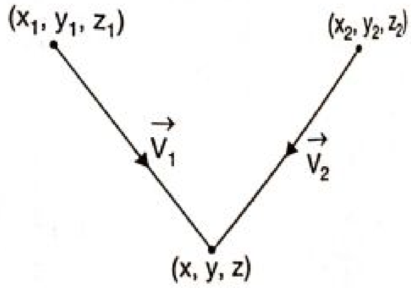
$$\text{curl}\left(\hat{k}\times\text{grad}\,\frac{1}{r}\right)+\text{grad}\left(\hat{k}\cdot\text{grad}\,\frac{1}{r}\right)=\vec{0}.$$

Example 14. *If $\vec{V}_1$ and $\vec{V}_2$ be the vectors joining the fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively to a variable point (x, y, z), prove that curl $(\vec{V}_1\times\vec{V}_2)=2(\vec{V}_1-\vec{V}_2)$.*

Sol. Given

$$\vec{V}_1 = (x-x_1)\hat{i}+(y-y_1)\hat{j}+(z-z_1)\hat{k}$$

$$\vec{V}_2 = (x-x_2)\hat{i}+(y-y_2)\hat{j}+(z-z_2)\hat{k}$$



$\therefore$ Curl $(\vec{V}_1\times\vec{V}_2) = \nabla\times(\vec{V}_1\times\vec{V}_2)$

$$= \sum\hat{i}\times\frac{\partial}{\partial x}(\vec{V}_1\times\vec{V}_2)$$

$$= \sum\hat{i}\times\left(\frac{\partial\vec{V}_1}{\partial x}\times\vec{V}_2+\vec{V}_1\times\frac{\partial\vec{V}_2}{\partial x}\right)$$

$$= \sum\hat{i}\times\left(\frac{\partial\vec{V}_1}{\partial x}\times\vec{V}_2\right)+\sum\hat{i}\times\left(\vec{V}_1\times\frac{\partial\vec{V}_2}{\partial x}\right)$$

$$= \sum\left[\left(\hat{i}\cdot\vec{V}_2\right)\frac{\partial\vec{V}_1}{\partial x}-\left(\hat{i}\cdot\frac{\partial\vec{V}_1}{\partial x}\right)\vec{V}_2\right]+\sum\left[\left(\hat{i}\cdot\frac{\partial\vec{V}_2}{\partial x}\right)\vec{V}_1-\left(\hat{i}\cdot\vec{V}_1\right)\frac{\partial\vec{V}_2}{\partial x}\right]$$

$[\because\ \vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}]$

$$= \sum [(x - x_2)\,\hat{i} - (\hat{i} \cdot \hat{i})\,\vec{V}_2] + \sum [(\hat{i} \cdot \hat{i})\,\vec{V}_1 - (x - x_1)\,\hat{i}]$$

$$= \sum [(x - x_2)\,\hat{i} - \vec{V}_2] + \sum [\vec{V}_1 - (x - x_1)\,\hat{i}]$$

$$= [(x - x_2)\,\hat{i} - \vec{V}_2 + (y - y_2)\hat{j} - \vec{V}_2 + (z - z_2)\,\hat{k} - \vec{V}_2]$$
$$+ [\vec{V}_1 - (x - x_1)\,\hat{i} + \vec{V}_1 - (y - y_1)\hat{j} + \vec{V}_1 - (z - z_1)\,\hat{k}]$$

$$= (\vec{V}_2 - 3\,\vec{V}_2) + (3\vec{V}_1 - \vec{V}_1)$$

$$= -2\,\vec{V}_2 + 2\,\vec{V}_1 = 2(\vec{V}_1 - \vec{V}_2)$$

EXERCISE 13.3

1. Evaluate

   (i) div $(3x^2\hat{i} + 5xy^2\hat{j} + xyz^3\hat{k})$ at the point (1, 2, 3).

   (ii) div $[(xy \sin z)\hat{i} + (y^2 \sin x)\hat{j} + (z^2 \sin xy)\hat{k}]$ at the point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$.

   (iii) curl $[e^{xyz}\,(\hat{i} + \hat{j} + \hat{k})]$

2. Find the divergence and curl of the vectors

   (i) $\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ at the point (2, –1, 1).

   (ii) $\vec{R} = (x^2 + yz)\hat{i} + (y^2 + zx)\hat{j} + (z^2 + xy)\hat{k}$.

   (iii) $\vec{F}\,(x, y, z) = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$

   (iv) $\vec{F}\,(x, y, z) = e^{xyz}\,(xy^2\hat{i} + yz^2\hat{j} + zx^2\hat{k})$ at the point (1, 2, 3).

3. If $\vec{F} = (x + y + 1)\hat{i} + \hat{j} - (x + y)\hat{k}$, show that $\vec{F} \cdot \text{Curl}\,\vec{F} = 0$.

4. If $\vec{V} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$, show that $\nabla \cdot \vec{V} = \dfrac{2}{\sqrt{x^2 + y^2 + z^2}}$ and $\nabla \times \vec{V} = \vec{0}$. *(M.T.U. 2011; Calicut 2009)*

5. (i) If $\vec{A} = (3xz^2)\hat{i} - (yz)\hat{j} + (x + 2z)\hat{k}$, find curl (curl $\vec{A}$).

   (ii) Evaluate curl curl of $\vec{V} = (2xz^2)\hat{i} - yz\hat{j} + (3xz^3)\hat{k}$ at (1, 1, 1). *(P.T.U. Dec. 2010)*

6. Show that each of the following vectors are solenoidal: *(P.T.U. May 2010, May 2011)*

   (i) $(x + 3y)\hat{i} + (y - 3z)\hat{j} + (x - 2z)\hat{k}$

   (ii) $(-x^2 + yz)\hat{i} + (4y - z^2x)\hat{j} + (2xz - 4z)\hat{k}$

   (iii) $3y^4z^2\hat{i} + 4x^3z^2\hat{j} + 3x^2y^2\hat{k}$. *(Calicut 2009)*

7. If $u = x^2 + y^2 + z^2$ and $\vec{V} = x\hat{i} + y\hat{j} + z\hat{k}$, show that div $(u\vec{V}) = 5u$.

8. (a) Show that the vector field $\vec{V} = (\sin y + z)\hat{i} + (x \cos y - z)\hat{j} + (x - y)\hat{k}$ is irrotational.

   (b) Find the value of constant 'a' such that $\vec{A} = (ax + 4y^2z)\hat{i} + (x^3 \sin z - 3y)\hat{j} - (e^x + 4 \cos x^2y)\hat{k}$ is solenoidal. *(M.D.U. May 2012)*

(c) Show that $\vec{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrational. Find the velocity potential ϕ such that $\vec{A} = \nabla\phi$. *(Anna 2012; U.K.T.U. 2011)*

9. (i) Find the values of a, b, c for which the vector $\vec{V} = (x + y + az)\hat{i} + (bx + 3y - z)\hat{j} + (3x + cy + z)\hat{k}$ is irrotational. *(P.T.U. Dec. 2011; M.D.U. May 2010)*

(ii) Prove that $(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidal and irrotational. *(U.P.T.U. 2009)*

(iii) A fluid motion is given by $\vec{V} = (y + z)\hat{i} + (z + x)\hat{j} + (x + y)\hat{k}$. Is this motion irrotational? If so, find the velocity potential. *(P.T.U. Dec. 2011)*

(iv) Show that the vector field $\vec{A}$, where $\vec{A} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ is irrotational and find its scalar potential. *(Anna 2010; M.T.U. 2013; P.T.U. May 2010)*

(v) Show that the vector field defined by $\vec{F} = 2xyz^3\,\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$ is irrotational and find its scalar potential.

(vi) Show that the vector field $\vec{F} = 2x(y^2 + z^3)\,\hat{i} + 2x^2y\hat{j} + 3x^2z^2\hat{k}$ is irrotational and find its scalar potential.

(vii) Prove that $\vec{F} = (y^2 \cos x + z^3)\,\hat{i} + (2y \sin x - 4)\hat{j} + 3xz^2\hat{k}$ is irrotational and find its scalar potential.

10. (a) Show that $\vec{E} = \dfrac{\vec{r}}{r^2}$ is irrotational.

(b) Show that the vector field $\vec{F} = \dfrac{\vec{r}}{r^3}$ is irrotational as well as solenoidal.

11. If $\vec{E}$ and $\vec{H}$ are irrotational, prove that $\vec{E} \times \vec{H}$ is solenoidal.

12. For a solenoidal vector $\vec{F}$, prove that curl curl curl curl $\vec{F} = \nabla^4\vec{F}$. *(G.B.T.U. 2010; Calicut 2008)*

13. Find the directional derivative of $\nabla \cdot (\nabla\phi)$ at the point $(1, -2, 1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $\phi = 2x^3y^2z^4$. *(U.P.T.U. 2009)*

14. If $\vec{V}_1$ and $\vec{V}_2$ be the vectors joining the fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively to a variable point (x, y, z), prove that

(i) div $(\vec{V}_1 \times \vec{V}_2) = 0$ (ii) grad $(\vec{V}_1 \cdot \vec{V}_2) = \vec{V}_1 + \vec{V}_2$

15. Prove that $\nabla \times (\vec{r} \times \vec{u}) = \vec{r}\,(\nabla \cdot \vec{u}) - 2\vec{u} - (\vec{r} \cdot \nabla)\,\vec{u}$.

16. (a) Find curl (curl $\vec{V}$) given $\vec{V} = x^2y\hat{i} + y^2z\hat{j} + z^2y\hat{k}$

(b) If $f = (x^2 + y^2 + z^2)^{-n}$, find div grad f and determine n if div grad $f = 0$

(c) If $u = 3x^2y$, $v = xz^2 - 2y$, find

(i) $\nabla(\nabla u \cdot \nabla v)$ (ii) $\nabla \cdot (\nabla u \times \nabla v)$.

17. If $\vec{a}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

(i) div $(\vec{a} \times \vec{r}) = 0$ *(K.U.K. 2009)* (ii) curl $[(\vec{a} \cdot \vec{r})\,\vec{r}] = \vec{a} \times \vec{r}$ *(M.T.U. 2012)*

(iii) $\nabla(\vec{a}\cdot\vec{a}) = 2(\vec{a}\cdot\nabla)\vec{a} + 2\vec{a}\times(\nabla\times\vec{a})$

(iv) $\vec{a}\times(\nabla\times\vec{r}) = \nabla(\vec{a}\cdot\vec{r}) - (\vec{a}\cdot\nabla)\vec{r}$. (U.P.T.U. 2008)

18. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

(i) $\text{div}\,(r^n\,\vec{r}) = (n+3)\,r^n$ (M.D.U. 2013)

(ii) $\text{curl}\,(r^n\,\vec{r}) = \vec{0}$

(iii) $\dfrac{\vec{r}}{r^3}$ is solenoidal as well as irrotational

(iv) $\nabla^2\left(\dfrac{1}{r}\right) = 0$

(v) $\nabla^2(r^n\,\vec{r}) = n(n+3)r^{n-2}\,\vec{r}$

(vi) $\nabla\cdot\left\{r\nabla\left(\dfrac{1}{r^3}\right)\right\} = \dfrac{3}{r^4}$

(vii) $\nabla^2\left[\nabla\cdot\left(\dfrac{\vec{r}}{r^2}\right)\right] = 2r^{-4}$

(viii) $\nabla\left[\nabla\cdot\dfrac{\vec{r}}{r}\right] = -\dfrac{2}{r^3}\vec{r}$

(ix) $\nabla^2(r\,\vec{r}) = 4\hat{r}$

(x) $\nabla^2(xy\hat{i} + yz\hat{j} + zx\hat{k}) = 0$.

19. Show that the vector $\nabla\phi\times\nabla\psi$ is solenoidal.

20. (i) Show that $\text{div}\,(\text{grad}\,r^n) = n(n+1)r^{n-2}$, where $r^2 = x^2 + y^2 + z^2$. (K.U.K. 2012; U.K.T.U. 2012)

(ii) Find the value of n, if $f = (x^2 + y^2 + z^2)^{-n}$ and div. grad $f = 0$ (K.U.K. Dec. 2014)

21. If r and $\vec{r}$ have their usual meanings and $\vec{a}$ is a constant vector, prove that

$$\nabla\times\left(\frac{\vec{a}\times\vec{r}}{r^n}\right) = \frac{2-n}{r^n}\vec{a} + \frac{n(\vec{a}\cdot\vec{r})}{r^{n+2}}\vec{r}.$$

[Hint. $\nabla\times[r^{-n}(\vec{a}\times\vec{r})] = (\nabla r^{-n})\times(\vec{a}\times\vec{r}) + r^{-n}[\nabla\times(\vec{a}\times\vec{r})]$

{Since $\nabla\times(\phi\vec{A}) = (\nabla\phi)\times\vec{A} + \phi(\nabla\times\vec{A})$ by Art. 13.19}]

Answers

1. (i) 80 (ii) $\dfrac{\pi}{2}$ (iii) $e^{xyz}[x(z-y)\hat{i} + y(x-z)\hat{j} + z(y-x)\hat{k}]$

2. (i) 14; $2\hat{i} - 3\hat{j} - 14\hat{k}$ (ii) $2(x+y+z)$; $\vec{0}$

(iii) $z^3 - 2x^2z + 8yz^3$; $2(x^2y + z^4)\hat{i} + 3xz^2\hat{j} - 4xyz\hat{k}$

(iv) $98e^6$; $-39e^6\hat{i} - 16e^6\hat{j} + 92e^6\hat{k}$

5. (i) $-6x\hat{i} + (6z-1)\hat{k}$ (ii) $5\hat{i} + 3\hat{k}$

(c) $\phi = 3x^2y + xz^3 - zy + c$

8. (b) 3

9. (i) $a = 3,\ b = 1,\ c = -1$ (iii) $xy + yz + zx + c$

(iv) $\dfrac{1}{3}x^3 - xy^2 + \dfrac{1}{2}(x^2 - y^2) + c$ (v) $x^2yz^3 + c$

(vi) $x^2(y^2 + z^3) + c$ (vii) $y^2\sin x + xz^3 - 4y + c$

13. $\dfrac{1724}{\sqrt{21}}$

16. (a) $2(x+z)\hat{j} + 2y\hat{k}$ (b) $\dfrac{2n(2n-1)}{(x^2+y^2+z^2)^{n+1}}$; $n = \dfrac{1}{2}$

(c) (i) $(6yz^2 - 12x)\hat{i} + 6xz^2\hat{j} + 12xyz\hat{k}$ (ii) 0.

20. (ii) 0, $\dfrac{1}{2}$

Answers

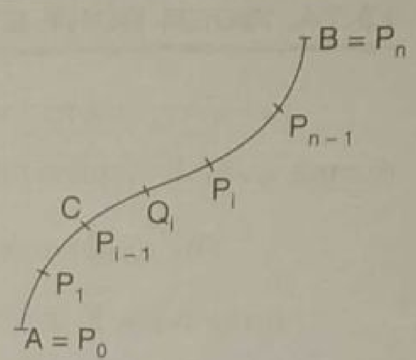
1. $\frac{1}{2}\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{4}\hat{k}$ 2. (i) 12 (ii) $-24\hat{i} - \frac{40}{3}\hat{j} + \frac{64}{5}\hat{k}$

3. $\vec{r} = (t^3 - t + 2)\hat{i} + (1 - t^4)\hat{j} + (4 - 4\cos t - 3t)\hat{k}$

4. $\vec{v} = 6\sin 2t\hat{i} + 4(\cos 2t - 1)\hat{j} + 8t^2\hat{k}$, $\vec{r} = 3(1 - \cos 2t)\hat{i} + 2(\sin 2t - 2t)\hat{j} + \frac{8}{3}t^3\hat{k}$.

13.22. LINE INTEGRALS

Any integral which is to be evaluated along a curve is called a line integral.



Let $\vec{F}(P)$ be a continuous vector point function defined at every point of a curve C is space. Divide the curve C into n parts by the points

$$A = P_0, P_1, P_2, ..., P_n = B$$

and let $\vec{R}_0, \vec{R}_1, \vec{R}_2, \ldots, \vec{R}_n$ be the position vectors of these points. Let Q_i be any point on the arc $P_{i-1} P_i$. Then the limit of the sum

$$\sum_{i=1}^{n} \vec{F}(Q_i) \cdot \delta\vec{R}_i, \quad \text{where} \quad \delta\vec{R}_i = \vec{R}_i - \vec{R}_{i-1} \qquad ...(1)$$

as $n \to \infty$ and every $|\delta\vec{R}_i| \to 0$, if it exists, is called a line integral of $\vec{F}$ along C and is denoted by

$$\int_C \vec{F} \cdot d\vec{R} \quad \text{or} \quad \int_C \vec{F} \cdot \frac{d\vec{R}}{dt}\, dt.$$

Clearly, it is a scalar. It is called the *tangential line integral* of $\vec{F}$ along C.

If the scalar products in (1) are replaced by vector products, then the corresponding line integral is defined as $\int_C \vec{F} \times d\vec{R}$ which is a vector.

If the vector function $\vec{F}$ is replaced by a scalar function ϕ, then the corresponding line integral is defined as $\int_C \phi\, d\vec{R}$, which is a vector.

If $\quad \vec{F}(x, y, z) = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ and $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ then $d\vec{R} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

$$\therefore \quad \int_C \vec{F} \cdot d\vec{R} = \int_C (f_1 dx + f_2 dy + f_3 dz)$$

If the parametric equations of the curve C are $x = x(t)$, $y = y(t)$, $z = z(t)$ and $t = t_1$ at A, $t = t_2$ at B, then

$$\int_C \vec{F} \cdot d\vec{R} = \int_{t_1}^{t_2} \left(f_1 \frac{dx}{dt} + f_2 \frac{dy}{dt} + f_3 \frac{dz}{dt} \right) dt$$

If C is a closed curve, then the integral sign $\int_C$ is replaced by $\oint_C$.

13.23. CIRCULATION

(G.B.T.U. 2010)

In fluid dynamics, if $\vec{V}$ represents the velocity of a fluid particle and C is a closed curve, then the integral $\oint_C \vec{V}\cdot d\vec{R}$ is called the *circulation* of $\vec{V}$ around the curve C.

If the circulation of $\vec{V}$ around every closed curve in a region D vanishes, then $\vec{V}$ is said to be *irrotational* in D.

13.24. WORK DONE BY A FORCE

Let $\vec{F}$ represent the force acting on a particle moving along an arc AB. The work done during a small displacement $\delta\vec{R}$ is $\vec{F}\cdot\delta\vec{R}$.

$\therefore$ The total work done by $\vec{F}$ during displacement from A to B is given by $\int_A^B \vec{F}\cdot d\vec{R}$

If the force $\vec{F}$ is conservative, then there exists a scalar function ϕ such that

$$\vec{F} = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$\therefore$ The work done by $\vec{F}$ during displacement from A to B $= \int_A^B \vec{F}\cdot d\vec{R}$

$$= \int_A^B \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_A^B \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = \int_A^B d\phi = \Big[\phi\Big]_A^B = \phi_B - \phi_A$$

Thus, in a conservative field, the work done depends on the value of ϕ at the end points A and B, and not on the path joining A and B.

ILLUSTRATIVE EXAMPLES

Example 1. *If* $\vec{F} = 3xy\hat{i} - y^2\hat{j}$, *evaluate* $\int_C \vec{F}\cdot d\vec{r}$, *where C is the arc of the parabola* $y = 2x^2$ *from (0, 0) to (1, 2).* (Calicut 2009)

Sol. Since the integration is performed in the xy-plane ($z = 0$), we take

$$\vec{r} = x\hat{i} + y\hat{j} \quad \text{so that } d\vec{r} = dx\hat{i} + dy\hat{j}$$

$$\therefore \quad \vec{F}\cdot d\vec{r} = (3xy\hat{i} - y^2\hat{j})\cdot(dx\hat{i} + dy\hat{j}) = 3xy\,dx - y^2\,dy$$

On C : $y = 2x^2$ from (0, 0) to (1, 2)

$$\vec{F}\cdot d\vec{r} = 3x(2x^2)\,dx - (2x^2)^2\,(4x\,dx) = (6x^3 - 16x^5)\,dx$$

Also, x varies from 0 to 1.

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_0^1 (6x^3 - 16x^5)\,dx = \left[\frac{6x^4}{4} - \frac{16x^6}{6}\right]_0^1 = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}$$

Note that if the curve is traversed in the opposite sense, *i.e.*, from (1, 2) to (0, 0), the value of the integral would be $\frac{7}{6}$.

Second Method

Let $x = t$, then the parametric equations of the parabola $y = 2x^2$ are $x = t$, $y = 2t^2$.

At the point (0, 0), $\quad x = 0$ and so $t = 0$

At the point (1, 2), $\quad x = 1$ and so $t = 1$

If $\vec{r}$ is the position vector of any point (x, y) on C, then $\vec{r} = x\hat{i} + y\hat{j} = t\hat{i} + 2t^2\hat{j}$

Also in terms of t, $\vec{F} = (3t \times 2t^2)\hat{i} - (2t^2)^2\hat{j} = 6t^3\hat{i} - 4t^4\hat{j}$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_C \left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right)dt = \int_0^1 (6t^3\hat{i} - 4t^4\hat{j})\cdot(\hat{i} + 4t\hat{j})\,dt$$

$$= \int_0^1 (6t^3 - 16t^5)\,dt = \left[\frac{6t^4}{4} - \frac{16t^6}{6}\right]_0^1 = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}.$$

Example 2. *A vector field is given by* $\vec{F} = (\sin y)\hat{i} + x(1 + \cos y)\hat{j}$. *Evaluate the line integral over the circular path given by* $x^2 + y^2 = a^2$, $z = 0$.

Sol. The parametric equations of the circular path are $x = a\cos t$, $y = a\sin t$, $z = 0$, where t varies from 0 to 2π.

Since the particle moves in the xy-plane ($z = 0$), we can take $\vec{r} = x\hat{i} + y\hat{j}$ so that $d\vec{r} = dx\hat{i} + dy\hat{j}$

$$\therefore \quad \oint_C \vec{F}\cdot d\vec{r} = \oint_C [\sin y\hat{i} + x(1 + \cos y)\hat{j}]\cdot(dx\hat{i} + dy\hat{j})$$

$$= \oint_C [\sin y\,dx + x\,(1 + \cos y)\,dy] = \oint_C [(\sin y\,dx + x\cos y\,dy) + x\,dy]$$

$$= \oint_C d\,(x\sin y) + \oint_C x\,dy = \int_0^{2\pi} d\,[a\cos t\sin(a\sin t)] + \int_0^{2\pi} a\cos t \cdot a\cos t\,dt$$

$$= \Big[a\cos t\sin(a\sin t)\Big]_0^{2\pi} + a^2\int_0^{2\pi} \cos^2 t\,dt$$

$$= \frac{a^2}{2}\int_0^{2\pi} (1 + \cos 2t)\,dt = \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = \frac{a^2}{2}(2\pi) = \pi a^2.$$

Example 3. *If* $\vec{F} = 2y\hat{i} - z\hat{j} + x\hat{k}$, *evaluate* $\oint_C \vec{F}\times d\vec{r}$ *along the curve* $x = \cos t$, $y = \sin t$, $z = 2\cos t$ *from* $t = 0$ *to* $t = \frac{\pi}{2}$.

Sol. $$\vec{F}\times d\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix} = (-z\,dz - x\,dy)\hat{i} + (x\,dx - 2y\,dz)\hat{j} + (2y\,dy + z\,dx)\hat{k}$$

In terms of t,

$$\vec{F} \times d\vec{r} = [-2\cos t(-2\sin t)\,dt - \cos t(\cos t)\,dt]\hat{i}$$
$$+ [\cos t(-\sin t)\,dt - 2\sin t(-2\sin t)\,dt]\hat{j} + [2\sin t(\cos t)\,dt + 2\cos t(-\sin t)\,dt]\hat{k}$$
$$= [(4\cos t\sin t - \cos^2 t)\hat{i} + (4\sin^2 t - \cos t\sin t)\hat{j}]\,dt$$

$$\therefore \int_C \vec{F} \times d\vec{r} = \int_0^{\pi/2} [(4\cos t\sin t - \cos^2 t)\hat{i} + (4\sin^2 t - \cos t\sin t)\hat{j}]\,dt$$
$$= \left(4\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{\pi}{2}\right)\hat{i} + \left(4\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\right)\hat{j} = \left(2 - \frac{\pi}{4}\right)\hat{i} + \left(\pi - \frac{1}{2}\right)\hat{j}.$$

Example 4. *Compute* $\int_C \vec{F}\cdot d\vec{r}$, *where* $\vec{F} = \dfrac{\hat{i}y - \hat{j}x}{x^2 + y^2}$ *and C is the circle* $x^2 + y^2 = 1$ *traversed counter clockwise.*

Sol. For the circle $x^2 + y^2 = 1$ in the xy-plane, we take $\vec{r} = x\hat{i} + y\hat{j}$ so that $d\vec{r} = dx\hat{i} + dy\hat{j}$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \left(\frac{\hat{i}y - \hat{j}x}{x^2 + y^2}\right)\cdot(dx\hat{i} + dy\hat{j})$$
$$= \int_C \frac{y\,dx - x\,dy}{x^2 + y^2} = \int_C (y\,dx - x\,dy) \qquad [\because\ x^2 + y^2 = 1]$$

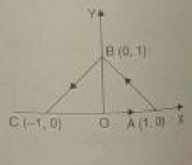
Parametric equation of the circle $C : x^2 + y^2 = 1$ is $x = \cos\theta$, $y = \sin\theta$

so that $\quad dx = -\sin\theta\,d\theta$, $dy = \cos\theta\,d\theta$ and θ varies from 0 to 2π.

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \sin\theta(-\sin\theta\,d\theta) - \cos\theta(\cos\theta\,d\theta)$$
$$= -\int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\,d\theta = -\int_0^{2\pi} d\theta$$
$$= -[\theta]_0^{2\pi} = -2\pi.$$

Example 5. *Compute the line integral* $\int_C (y^2 dx - x^2 dy)$ *about the triangle whose vertices are (1, 0), (0, 1) and (–1, 0).*

Sol. Here C is the triangle ABC.



On AB Equation of AB is

$$y - 0 = \frac{1-0}{0-1}(x-1) \quad \text{or} \quad y = 1 - x$$

$\therefore$ $\quad dy = -dx$ and x varies from 1 to 0.

On BC Equation of BC is

$$y - 1 = \frac{0-1}{-1-0}(x-0) \quad \text{or} \quad y = 1 + x$$

$\therefore$ $\quad dy = dx$ and x varies from 0 to –1.

On CA $\quad y = 0 \quad \therefore \quad dy = 0$ and x varies from –1 to 1.

$$\therefore \int_C (y^2 dx - x^2 dy) = \int_{AB} (y^2 dx - x^2 dy) + \int_{BC} (y^2 dx - x^2 dy) + \int_{CA} (y^2 dx - x^2 dy)$$

$$= \int_1^0 [(1-x)^2 dx - x^2(-dx)] + \int_0^{-1} [(1+x)^2 dx - x^2 dx] + \int_{-1}^1 0\,dx$$
$$= \int_1^0 (2x^2 - 2x + 1)\,dx + \int_0^{-1} (2x + 1)\,dx + 0$$
$$= \left[\frac{2x^3}{3} - \frac{2x^2}{2} + x\right]_1^0 + \left[\frac{2x^2}{2} + x\right]_0^{-1}$$
$$= \left(-\frac{2}{3} + 1 - 1\right) + (1 - 1) = -\frac{2}{3}.$$

Example 6. *Find the work done in moving a particle in the force field* $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ *along*

(i) *the straight line from (0, 0, 0) to (2, 1, 3)* (P.T.U. May 2010)

(ii) *the curve defined by* $x^2 = 4y$, $3x^3 = 8z$ *from* $x = 0$ *to* $x = 2$. (M.T.U. 2012)

Sol. Work done $= \int_C \vec{F}\cdot d\vec{r} = \int_C [3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}]\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$

$$= \int_C [3x^2 dx + (2xz - y)dy + z\,dz] \qquad \ldots(1)$$

(i) Equation of straight line from (0, 0, 0) to (2, 1, 3) are

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} \quad \text{or} \quad \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \quad \text{(say)}$$

$\therefore$ C : $x = 2t$, $y = t$, $z = 3t$

so that $\quad dx = 2dt$, $dy = dt$, $dz = 3dt$

The points (0, 0, 0) and (2, 1, 3) correspond to $t = 0$ and $t = 1$ respectively.

$\therefore$ From (1), we have

$$\text{Work done} = \int_0^1 [3(2t)^2\, 2dt + \{2(2t)(3t) - t\}\,dt + (3t)\,3dt]$$
$$= \int_0^1 (24t^2 + 12t^2 - t + 9t)\,dt = \int_0^1 (36t^2 + 8t)\,dt$$
$$= \left[12t^3 + 4t^2\right]_0^1 = 16$$

(ii) Let $x = t$, then C: $x = t$, $y = \dfrac{t^2}{4}$, $z = \dfrac{3t^3}{8}$

so that $\quad dx = dt$, $dy = \dfrac{t}{2}\,dt$, $dz = \dfrac{9t^2}{8}\,dt$

From $x = 0$ to $x = 2$, t varies from 0 to 2. $\qquad (\because\ t = x)$

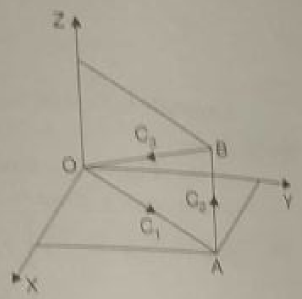
$\therefore$ From (1), we have

$$\text{Work done} = \int_0^2 \left[3t^2 dt + \left\{2(t)\left(\frac{3t^3}{8}\right) - \frac{t^2}{4}\right\}\left(\frac{t}{2}\,dt\right) + \frac{3t^3}{8}\left(\frac{9t^2}{8}\,dt\right)\right]$$
$$= \int_0^2 \left(3t^2 + \frac{3}{8}t^5 - \frac{1}{8}t^3 + \frac{27}{64}t^5\right)dt = \int_0^2 \left(\frac{51}{64}t^5 - \frac{1}{8}t^3 + 3t^2\right)dt$$

$$= \left[\frac{51}{64} - \frac{t^6}{6} - \frac{t^4}{32} + t^3\right]_0^2 = \frac{17}{2} - \frac{1}{2} + 8 = 16.$$

Example 7. *Find the work done by the force* $\vec{F} = x\hat{i} - z\hat{j} + 2y\hat{k}$ *in the displacement along the closed path C consisting of the segments* C_1, C_2 *and* C_3 *where*

on C_1,	$0 \le x \le 1$,	$y = x$,	$z = 0$
on C_2,	$0 \le z \le 1$,	$x = 1$,	$y = 1$
on C_3,	$1 \ge x \ge 0$,	$y = z = x$.	



Sol. Total work done

$$= \oint_C \vec{F}\cdot d\vec{r} = \oint_C (x\hat{i} - z\hat{j} + 2y\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \oint_C (xdx - zdy + 2ydz)$$

$$= \oint_{C_1} (xdx - zdy + 2ydz) + \oint_{C_2} (xdx - zdy + 2ydz) + \oint_{C_3} (xdx - zdy + 2ydz)$$

$$= W_1 + W_2 + W_3$$

where W_1, W_2, W_3 denote the work done in displacement along C_1, C_2 and C_3 respectively.

On C_1, $\quad 0 \le x \le 1, \quad y = x, \quad z = 0, \quad dy = dx, \quad dz = 0$

$$\therefore \quad W_1 = \int_{C_1} x\,dx = \int_0^1 x\,dx = \frac{1}{2}$$

On C_2, $\quad 0 \le z \le 1, \quad x = 1, \quad y = 1, \quad dx = 0, \quad dy = 0$

$$\therefore \quad W_2 = \int_{C_2} 2dz = 2\int_0^1 dz = 2$$

On C_3, $\quad 1 \ge x \ge 0, \quad y = z = x, \quad dy = dz = dx$

$$\therefore \quad W_3 = \int_{C_3} (x\,dx - x\,dx + 2\,xdx) = 2\int_1^0 x\,dx = -1.$$

Total work done $\quad = W_1 + W_2 + W_3 = \frac{1}{2} + 2 - 1 = \frac{3}{2}.$

EXERCISE 13.5

1. (*a*) If $\phi = 2xyz^2$, $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$, and C is the curve $x = t^2, y = 2t, z = t^3$ from $t = 0$ to $t = 1$, evaluate the line integrals

 (*i*) $\int_C \phi\, d\vec{r}$ (*ii*) $\int_C \vec{F}\times d\vec{r}$.

 (*b*) Evaluate $\int_C (x^2 + yz)\,dz$, where C is the curve defined by $x = t, y = t^2, z = 3t$ for t lying in the interval $1 \le t \le 2$. (*P.T.U. May 2008*)

2. (*a*) Evaluate the line integral $\int_C [(x^2 + xy)dx + (x^2 + y^2)dy]$, where C is the square formed by the lines $y = \pm 1$ and $x \pm 1$.

 (*b*) If $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the curve C in the xy-plane, $y = x^3$ from the point (1, 1) to (2, 8).

3. If $\vec{F} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$ from (0, 0, 0) to (1, 1, 1) along the following paths C:

 (*i*) $x = t, y = t^2, z = t^3$ (*G.B.T.U. 2010*) (*ii*) the straight line joining (0, 0, 0) to (1, 1, 1)

 (*iii*) the straight line from (0, 0, 0) to (1, 0, 0), then to (1, 1, 0) and then to (1, 1, 1).

4. (*i*) Show that $\int_C \vec{F}\cdot d\vec{r} = 3\pi$, given that $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ and C being the arc of the curve $\vec{r} = \cos t\hat{i} + \sin t\hat{j} + t\hat{k}$ from $t = 0$ to $t = 2\pi$.

 (*ii*) Evaluate $\int_C \vec{F}\cdot d\vec{R}$, where $\vec{F} = (2z, x, -y)$ and C is $\vec{R} = (\cos t, \sin t, 2t)$ from (1, 0, 0) to (1, 0, 4π). (*K.U.K. Dec. 2012*)

5. Evaluate $\int_C [(2x^2y + y + z^2)\hat{i} + 2(1 + yz^2)\hat{j} + (2z + 3y^2z^2)\hat{k}]\cdot d\vec{r}$ along the curve C: $y^2 + z^2 = a^2$, $x = 0$.

6. Evaluate $\int_C [(y + 3z)\,dx + (2z + x)\,dy + (3x + 2y)\,dz]$, where C is the arc of helix $x = a\cos\theta$, $y = a\sin\theta$, $z = \frac{2a\theta}{\pi}$ between the points $(a, 0, 0)$ and $(0, a, a)$.

7. (*a*) Find the circulation of $\vec{F}$ round the curve C, where $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ and C is the circle $x^2 + y^2 = 1, z = 0$.

 (*b*) If $\vec{A} = (x - y)\hat{i} + (x + y)\hat{j}$, evaluate $\oint_C \vec{A}\cdot d\vec{r}$ around the curve C consisting of $y = x^2$ and $y^2 = x$. (*G.B.T.U. 2012*)

8. Find the circulation of $\vec{F}$ round the curve C, where $\vec{F} = e^x\sin y\hat{i} + e^x\cos y\hat{j}$ and C is the rectangle whose vertices are (0, 0), (1, 0), $\left(1, \frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right)$.

9. Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ moves a particle in the xy-plane from (0, 0) to (1, 1) along the parabola $y^2 = x$. Is the work done different when the path is the straight line $y = x$?

10. Find the work done in moving a particle once round the circle $x^2 + y^2 = 9$ in the xy-plane if the field of force is $\vec{F} = (2x - y - z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$. If possible, find its scalar potential. (*M.D.U. May 2010*)

11. Compute the work done by the force $\vec{F} = (2y + 3)\hat{i} + xz\hat{j} + (yz - x)\hat{k}$ when it moves a particle from the point (0, 0, 0) to the point (2, 1, 1) along the curve $x = 2t^2, y = t, z = t^3$. (*G.B.T.U. 2011*)

12. (*a*) Suppose $\vec{F}(x, y, z) = x^3\hat{i} + y\hat{j} + z\hat{k}$ is the force field. Find the work done by $\vec{F}$ along the line from (1, 2, 3) to (3, 5, 7).

 (*b*) Find the total work done by a force $\vec{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$ in moving a point from (0, 0) to (a, b) along the rectangle bounded by the lines $x = 0, x = a, y = 0$ and $y = b$.

Answers

1. (a) (i) $\frac{8}{11}\hat{i}+\frac{4}{5}\hat{j}+\hat{k}$ (ii) $-\frac{9}{10}\hat{i}-\frac{2}{3}\hat{j}+\frac{7}{5}\hat{k}$ (b) $\frac{163}{4}$
2. (a) 0 (b) 35
3. (i) 5 (ii) $\frac{13}{3}$ (iii) $\frac{23}{3}$
5. 0
6. $2a^3$
7. (a) $-\pi$ (b) $\frac{2}{3}$
8. 0
9. $-\frac{2}{3}$; No, it is same
10. 18π ; not possible
11. $8\frac{8}{35}$
12. (a) $\frac{101}{2}$ (b) $\frac{a^3}{3}-ab^2$

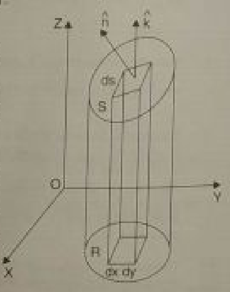
13.25. SURFACE INTEGRALS

Any integral which is to be evaluated over a surface is called a surface integral.

Let $\vec{F}(P)$ be a continuous vector point function and S a two sided surface. Divide S into a finite number of sub-surfaces $\delta S_1, \delta S_2, ..., \delta S_k$. Let P_i be any point in δS_i and $\hat{n}_i$ be the unit vector at P in the direction of outward drawn normal to the surface at P_i. Then the limit of the sum

$$\sum_{i=1}^{k} \vec{F}(P_i)\cdot\hat{n}_i\,\delta S_i, \text{ as } k \to \infty \text{ and each } \delta S_i \to 0$$ is called the *normal surface integral* of $\vec{F}(P)$ over S and is denoted by $\iint_S \vec{F}\cdot\hat{n}\,dS$.

The surface element $\delta\vec{S}$ surrounding any point P can be regarded as a vector whose magnitude is area δS and the direction that of the outward drawn normal $\hat{n}$ *i.e.* $\delta\vec{S} = \hat{n}\,\delta S$. The surface integral may alternatively be written as $\iint_S \vec{F}\cdot d\vec{S}$.



If $\vec{F}$ represents the velocity of a fluid at any point P on a closed surface S, then $\vec{F}\cdot\hat{n}$ is the normal component of $\vec{F}$ at P and $\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_S \vec{F}\cdot d\vec{S}$ is a measure of volume emerging from S per unit time, *i.e.* it measures the *flux of* $\vec{F}$ *over S*.

Other types of surface integrals are $\iint_S \vec{F}\times d\vec{S}$, $\iint_S \phi\,d\vec{S}$.

Note. In order to evaluate surface integrals, it is convenient to express them as double integrals taken over the orthogonal projection of S on one of the coordinate planes. But this is possible only when any line perpendicular to the coordinate plane chosen meets the surface S in not more than one point.

Let R be the orthogonal projection of S on the *xy*-plane.

Let $\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$ where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of $\hat{n}$.

Now, $dxdy$ = projection of dS on the *xy*-plane $= dS\cos\gamma \Rightarrow dS = \frac{dx\,dy}{\cos\gamma}$

Also $|\hat{k}\cdot\hat{n}| = \cos\gamma \quad \therefore \quad dS = \frac{dxdy}{|\hat{k}\cdot\hat{n}|}$

Hence $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{F}\cdot\hat{n}\,\frac{dxdy}{|\hat{k}\cdot\hat{n}|}$.

13.26. VOLUME INTEGRALS

Any integral which is to be evaluated over a volume is called a volume integral.

If V is a volume bounded by a surface S, then the triple integrals

$$\iiint_V \phi\,dV \quad \text{and} \quad \iiint_V \vec{F}\,dV$$

are called volume integrals. The first of these is a scalar and the second is a vector.

If we sub-divide the volume V into small cuboids by drawing planes parallel to the coordinate planes, then $dV = dx\,dy\,dz$.

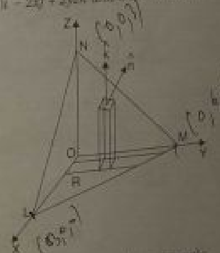
$$\therefore \quad \iiint_V \phi\,dV = \iiint_V \phi(x, y, z)\,dx\,dy\,dz$$

If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, then

$$\iiint_V \vec{F}\,dV = \hat{i}\iiint_V F_1(x, y, z)\,dx\,dy\,dz + \hat{j}\iiint_V F_2(x, y, z)\,dx\,dy\,dz + \hat{k}\iiint_V F_3(x, y, z)\,dx\,dy\,dz.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $\iint_S \vec{A}\cdot\hat{n}\,dS$, *where* $\vec{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ *and S is the surface of the plane* $2x + y + 2z = 6$ *in the first octant.*



Sol. A vector normal to the surface S is given by

$$\nabla(2x + y + 2z) = 2\hat{i} + \hat{j} + 2\hat{k}$$

$\therefore$ $\hat{n}$ = a unit vector normal to surface S

$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{(2)^2 + 1^2 + (2)^2}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

$$\hat{k}\cdot\hat{n} = \hat{k}\cdot\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = \frac{2}{3}$$

$$\therefore \quad \iint_S \vec{A}\cdot\hat{n}\,dS = \iint_R \vec{A}\cdot\hat{n}\,\frac{dxdy}{|\hat{k}\cdot\hat{n}|},$$

where R is the projection of S, *i.e.* triangle LMN on the *xy*-plane. The region R *i.e.* triangle OLM is bounded by *x*-axis, *y*-axis and the line $2x + y = 6$, $z = 0$.

Now, $\quad \vec{A}\cdot\hat{n} = [(x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}]\cdot\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$

$$= \frac{2}{3}(x+y^2) - \frac{2}{3}x + \frac{4}{3}yz = \frac{2}{3}y^2 + \frac{4}{3}yz$$

$$= \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6-2x-y}{2}\right)$$

$$\left(\because \text{ on the plane } 2x+y+2z=6,\ z=\frac{6-2x-y}{2}\right)$$

$$= \frac{2}{3}y(y+6-2x-y) = \frac{4}{3}y(3-x)$$

Hence, $\iint_S \vec{A}\cdot\hat{n}\,dS = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{k}\cdot\hat{n}|}$

$$= \iint_R \frac{4}{3}y(3-x)\cdot\frac{3}{2}\,dx\,dy = \int_0^3\int_0^{6-2x} 2y(3-x)\,dy\,dx$$

$$= \int_0^3 2(3-x)\cdot\left[\frac{y^2}{2}\right]_0^{6-2x} dx = \int_0^3 (3-x)(6-2x)^2\,dx$$

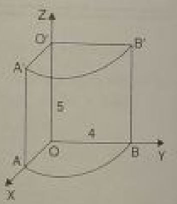
$$= 4\int_0^3 (3-x)^3\,dx = 4\cdot\left[\frac{(3-x)^4}{4(-1)}\right]_0^3 = -(0-81) = 81.$$

Example 2. *Evaluate* $\iint_S \vec{A}\cdot\hat{n}\,dS$, *where* $\vec{A} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ *and S is the curved surface of the cylinder* $x^2+y^2=16$ *included in the first octant between* $z=0$ *and* $z=5$.

Sol. A vector normal to the surface S is given by $\nabla(x^2+y^2) = 2x\hat{i} + 2y\hat{j}$

$\therefore \quad \hat{n}$ = a unit vector normal to surface S

$$= \frac{2x\hat{i}+2y\hat{j}}{\sqrt{(2x)^2+(2y)^2}} = \frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}} = \frac{x\hat{i}+y\hat{j}}{4}$$



($\because$ on the surface of cylinder, $x^2+y^2=16$)

Let R be the projection of S on yz-plane, then

$$\iint_S \vec{A}\cdot\hat{n}\,dS = \iint_R \vec{A}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{i}\cdot\hat{n}|}$$

The region R is OBB'O' enclosed by $y=0$ to $y=4$ and $z=0$ to $z=5$.

Now, $\quad \hat{i}\cdot\hat{n} = \hat{i}\cdot\left(\frac{1}{4}x\hat{i} + \frac{1}{4}y\hat{j}\right) = \frac{1}{4}x$

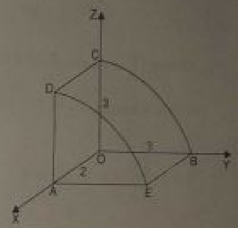
$$\vec{A}\cdot\hat{n} = (z\hat{i} + x\hat{j} - 3y^2z\hat{k})\cdot\left(\frac{1}{4}x\hat{i} + \frac{1}{4}y\hat{j}\right)$$

$$= \frac{1}{4}zx + \frac{1}{4}xy = \frac{1}{4}x(y+z).$$

Hence $\iint_S \vec{A}\cdot\hat{n}\,dS = \iint_R \vec{A}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{i}\cdot\hat{n}|} = \iint_R \frac{1}{4}x(y+z)\frac{dy\,dz}{\frac{1}{4}x} = \int_0^5\int_0^4 (y+z)\,dy\,dz$

$$= \int_0^5\left[\frac{y^2}{2} + zy\right]_0^4 dz = \int_0^5 (8+4z)\,dz = \left[8z+2z^2\right]_0^5 = 40+50 = 90.$$

Example 3. *Evaluate* $\iint_S \vec{F}\cdot\hat{n}\,dS$, *where* $\vec{F} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ *and S is the closed surface of the region in the first octant bounded by the cylinder* $y^2+z^2=9$ *and the planes* $x=0$, $x=2$, $y=0$ *and* $z=0$.



Sol. The given surface is comprised of the following five surfaces:

Surface	*Equation*	*ds*	*Outward normal*
$S_1 \to$ rectangle OAEB	$z=0$	$dx\,dy$	$-\hat{k}$
$S_2 \to$ rectangle OADC	$y=0$	$dx\,dz$	$-\hat{j}$
$S_3 \to$ circular quadrant OBC	$x=0$	$dy\,dz$	$-\hat{i}$
$S_4 \to$ circular quadrant ADE	$x=2$	$dy\,dz$	$\hat{i}$
$S_5 \to$ curved surface BCDE	$y^2+z^2=9$, $0 \le x \le 2$	$\frac{dx\,dy}{\lvert\hat{n}\cdot\hat{k}\rvert}$	$\hat{n}$

$$\therefore \quad \iint_S \vec{F}\cdot\hat{n}\,dS = \iint_{S_1}\vec{F}\cdot\hat{n}\,dS + \iint_{S_2}\vec{F}\cdot\hat{n}\,dS + \iint_{S_3}\vec{F}\cdot\hat{n}\,dS$$

$$+ \iint_{S_4}\vec{F}\cdot\hat{n}\,dS + \iint_{S_5}\vec{F}\cdot\hat{n}\,dS \quad \ldots(1)$$

Now, $\quad \iint_{S_1}\vec{F}\cdot\hat{n}\,dS = \iint_{S_1}(2x^2y\hat{i} - y^2\hat{j})\cdot(-\hat{k})\,dx\,dy \quad (\because z=0)$

$$= 0$$

$$\iint_{S_2}\vec{F}\cdot\hat{n}\,dS = \iint_{S_2}(4xz^2\hat{k})\cdot(-\hat{j})\,dx\,dz \quad (\because y=0)$$

$$= 0$$

$$\iint_{S_3}\vec{F}\cdot\hat{n}\,dS = \iint_{S_3}(-y^2\hat{j})\cdot(-\hat{i})\,dy\,dz \quad (\because x=0)$$

$$= 0$$

$$\iint_{S_4}\vec{F}\cdot\hat{n}\,dS = \iint_{S_4}(8y\hat{i} - y^2\hat{j} + 8z^2\hat{k})\cdot(\hat{i})\,dy\,dz \quad (\because x=2)$$

$$= \iint_{S_4} 8y\,dy\,dz = \int_0^3 \int_0^{\sqrt{9-z^2}} 8y\,dy\,dz$$

$$= \int_0^3 [4y^2]_0^{\sqrt{9-z^2}}\, dz = \int_0^3 4(9-z^2)\,dz$$

$$= 4\left[9z - \frac{z^3}{3}\right]_0^3 = 4(27-9) = 72$$

To find $\iint_{S_5} \vec{F}\cdot\hat{n}\, dS$, we first find $\hat{n}$.

A vector normal to $y^2+z^2=9$ is $\vec{n} = \nabla(y^2+z^2) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(y^2+z^2)$

$$= 2(y\hat{j} + z\hat{k})$$

$\hat{n}$ = unit vector normal to S_5

$$= \frac{\vec{n}}{|\vec{n}|} = \frac{2(y\hat{j}+z\hat{k})}{2\sqrt{y^2+z^2}} = \frac{y\hat{j}+z\hat{k}}{3} \qquad (\because y^2+z^2=9)$$

$$|\hat{n}\cdot\hat{k}| = \left|\frac{y\hat{j}+z\hat{k}}{3}\cdot\hat{k}\right| = \frac{z}{3}$$

$$\therefore \quad ds = \frac{dx\,dy}{\frac{z}{3}} = \frac{3\,dx\,dy}{z}$$

$$\vec{F}\cdot\hat{n} = (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot\frac{y\hat{j}+z\hat{k}}{3} = \frac{1}{3}(-y^3+4xz^3)$$

$$\therefore \quad \iint_{S_5} \vec{F}\cdot\hat{n}\,dS = \iint_{S_5} \frac{1}{3}(-y^3+4xz^3)\frac{3\,dx\,dy}{z} = \iint_{S_5}\left(-\frac{y^3}{z} + 4xz^2\right)dx\,dy$$

$$= \int_0^3\int_0^2\left[-\frac{y^3}{\sqrt{9-y^2}} + 4x(9-y^2)\right]dx\,dy$$

$$\left[\because \text{ On } S_5,\ y^2+z^2=9 \Rightarrow z=\sqrt{9-y^2}\right]$$

$$= \int_0^3\left[-\frac{xy^3}{\sqrt{9-y^2}} + 2x^2(9-y^2)\right]_0^2 dy = \int_0^3\left[-\frac{2y^3}{\sqrt{9-y^2}} + 8(9-y^2)\right]dy$$

$$\left[\text{Put } y = 3\sin\theta, \text{ then } dy = 3\cos\theta\, d\theta \right.$$

$$\left.\text{when } y=0,\ \theta=0;\ \text{when } y=3,\ \theta=\frac{\pi}{2}\right]$$

$$= \int_0^{\frac{\pi}{2}}\left[\frac{-2\times 27\sin^3\theta}{3\cos\theta} + 8\cdot 9\cos^2\theta\right]3\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}(-54\sin^3\theta + 216\cos^3\theta)\,d\theta$$

$$= -54\times\frac{2}{3} + 216\times\frac{2}{3} = \frac{2}{3}(162) = 108$$

$\therefore$ From (1),

$$\iint_S \vec{F}\cdot\hat{n}\,dS = 0+0+0+72+108 = 180$$

Example 4. *If* $\vec{F} = (2x^2-3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, *then evaluate* $\iiint_V \nabla\cdot\vec{F}\,dV$, *where V is bounded by the planes* $x=0,\ y=0,\ z=0$ *and* $2x+2y+z=4$.

Sol. $$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(2x^2-3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x) = 4x - 2x = 2x$$

$$\therefore \quad \iiint_V \nabla\cdot\vec{F}\,dV = \iiint_V 2x\,dx\,dy\,dz$$

$$= \int_0^2\int_0^{2-x}\int_0^{4-2x-2y} 2x\,dz\,dy\,dx = \int_0^2\int_0^{2-x} 2x\,[z]_0^{4-2x-2y}\,dy\,dx$$

$$= \int_0^2\int_0^{2-x} 2x(4-2x-2y)\,dy\,dx = \int_0^2\int_0^{2-x}[4x(2-x)-4xy]\,dy\,dx$$

$$= \int_0^2\left[4x(2-x)y - 2xy^2\right]_0^{2-x} dx = \int_0^2 [4x(2-x)^2 - 2x(2-x)^2]\,dx$$

$$= \int_0^2 2x(2-x)^2\,dx = 2\int_0^2(4x-4x^2+x^3)\,dx$$

$$= 2\left[2x^2 - 4\frac{x^3}{3} + \frac{x^4}{4}\right]_0^2 = 2\left(8 - \frac{32}{3} + 4\right) = \frac{8}{3}$$

Example 5. *If* $\vec{A} = 2xz\hat{i} - x\hat{j} + y^2\hat{k}$, *evaluate* $\iiint_V \vec{A}\,dV$, *where V is the region bounded by the surface* $x=0,\ y=0,\ x=2,\ y=6,\ z=x^2,\ z=4$.

Sol. $$\iiint_V \vec{A}\,dV = \iiint_V (2xz\hat{i} - x\hat{j} + y^2\hat{k})\,dx\,dy\,dz$$

$$= \int_0^2\int_0^6\int_{x^2}^4 (2xz\hat{i} - x\hat{j} + y^2\hat{k})\,dz\,dy\,dx$$

$$= \hat{i}\int_0^2\int_0^6\int_{x^2}^4 2xz\,dz\,dy\,dx - \hat{j}\int_0^2\int_0^6\int_{x^2}^4 x\,dz\,dy\,dx + \hat{k}\int_0^2\int_0^6\int_{x^2}^4 y^2\,dz\,dy\,dx$$

$$= \hat{i}\int_0^2\int_0^6\left[xz^2\right]_{x^2}^4 dy\,dx - \hat{j}\int_0^2\int_0^6\left[xz\right]_{x^2}^4 dy\,dx + \hat{k}\int_0^2\int_0^6\left[y^2z\right]_{x^2}^4 dy\,dx$$

$$= \hat{i}\int_0^2\int_0^6(16x - x^5)\,dy\,dx - \hat{j}\int_0^2\int_0^6(4x-x^3)\,dy\,dx + \hat{k}\int_0^2\int_0^6 y^2(4-x^2)\,dy\,dx$$

$$= \hat{i}\int_0^2(16x-x^5)\left[y\right]_0^6 dx - \hat{j}\int_0^2(4x-x^3)\left[y\right]_0^6 dx + \hat{k}\int_0^2(4-x^2)\left[\frac{y^3}{3}\right]_0^6 dx$$

$$= 6\hat{i}\int_0^2 (16x - x^5)\,dx - 6\hat{j}\int_0^2 (4x - x^3)\,dx + 72\hat{k}\int_0^2 (4 - x^2)\,dx$$

$$= 6\hat{i}\left[8x^2 - \frac{x^6}{6}\right]_0^2 - 6\hat{j}\left[2x^2 - \frac{x^4}{4}\right]_0^2 + 72\hat{k}\left[4x - \frac{x^3}{3}\right]_0^2$$

$$= 128\hat{i} - 24\hat{j} + 384\hat{k}.$$

EXERCISE 13.6

1. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,dS$, where $\vec{F} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is the surface of the plane $2x + 3y + 6z = 12$ in the first octant. *(M.T.U. 2012; P.T.U. May 2010)*
2. (*a*) Evaluate $\iint_S \vec{A}\cdot\hat{n}\,dS$, where $\vec{A} = 12x^2y\hat{i} - 3yz\hat{j} + 2z\hat{k}$ and S is the portion of the plane $x + y + z = 1$ included in the first octant.

   (*b*) Evaluate $\iint_S \vec{F}\cdot\hat{n}\,dS$, where $\vec{F} = 6z\hat{i} - 4\hat{j} + y\hat{k}$ and S is the portion of the plane $2x + 3y + 6z = 12$ in the first octant.
3. Show that $\iint_S \vec{F}\cdot\hat{n}\,dS = \frac{3}{2}$, where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.
4. Evaluate $\iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and S is that part of the surface $x^2 + y^2 + z^2 = 1$ which lies in the first octant.
5. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,dS$, where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the surface bounding the region
$$x^2 + y^2 = 4,\ z = 0,\ z = 3.$$
6. Evaluate $\iiint_V \phi\,dV$, where $\phi = 45x^2y$ and V is the closed region bounded by the planes *(Calicut 2009; K.U.K. 2012)*
$$4x + 2y + z = 8,\ x = 0,\ y = 0,\ z = 0.$$
7. If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, then evaluate $\iiint_V \nabla\times\vec{F}\,dV$ and $\iiint_V \nabla\cdot\vec{F}\,dV$, where V is the closed region bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.
8. If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$, evaluate $\iiint_V \vec{F}\,dV$, where V is the region bounded by the surface $x = 0, y = 0, x = 2, y = 4, z = x^2, z = 2$.

Answers

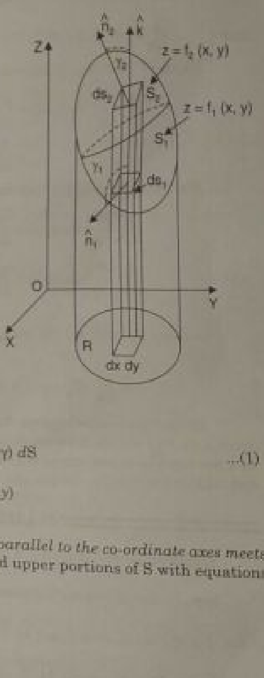
1. 24 2. (*a*) $\frac{49}{120}$ (*b*) 8 4. $\frac{3}{8}$
5. 84π 6. 128 7. $\frac{8}{3}(\hat{j} - \hat{k});\ \frac{8}{3}$ 8. $\frac{32}{15}(3\hat{i} + 5\hat{k})$.

13.27. DIVERGENCE THEOREM OF GAUSS

(Relation between surface and volume integrals) *(G.B.T.U. 2012; Calicut 2010)*

Statement. *If $\vec{F}$ is a vector point function having continuous first order partial derivatives in the* **region V bounded by a closed surface S,** *then* $\iiint_V \nabla\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{n}\,dS$, *where $\hat{n}$ is the outwards drawn unit normal vector to the surface S.*

[*i.e.*, the volume integral of the divergence of a vector point function $\vec{F}$ taken over the volume V enclosed by a surface S, is equal to the surface integral of the normal component of $\vec{F}$ taken over the closed surface S].



Proof. Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, then
$$\nabla\cdot\vec{F} = \operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Let α, β, γ be the angles which the outwards drawn unit normal vector $\hat{n}$ makes with the positive directions of x, y, z-axes respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of $\hat{n}$ and $\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$.

$$\therefore\ \vec{F}\cdot\hat{n} = (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\cdot(\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k})$$
$$= F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma$$

Therefore, the cartesian equivalent of divergence theorem is

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$
$$= \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,dS \qquad \ldots(1)$$
$$= \iint_S (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy)$$

since $\cos\alpha\,dS = dy\,dz$, etc.

Suppose that S is such a closed surface that a line parallel to the co-ordinate axes meets it in two points only. Let S_1 and S_2 denote the lower and upper portions of S with equations $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively.

Let R be the projection of S on the xy-plane, then

$$\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_R \left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\partial F_3}{\partial z}\,dz\right] dx\,dy$$
$$= \iint_R \Big[F_3(x, y, z)\Big]_{f_1}^{f_2} dx\,dy = \iint_R [F_3(x, y, f_2) - F_3(x, y, f_1)]\,dx\,dy$$
$$= \iint_R F_3(x, y, f_2)\,dx\,dy - \iint_R F_3(x, y, f_1)\,dx\,dy \qquad \ldots(2)$$

Now for the upper portion S_2 of S, the normal $\hat{n}_2$ to S_2 makes an acute angle γ_2 with $\hat{k}$.

$$\therefore \quad dxdy = \cos\gamma_2\, dS_2 = \hat{k}\cdot\hat{n}_2\, dS_2$$

For the lower portion S_1 of S, the normal $\hat{n}_1$ to S_1 makes an obtuse angle γ_1 with $\hat{k}$.

$$\therefore \quad dxdy = -\cos\gamma_1\, dS_1 = -\hat{k}\cdot\hat{n}_1\, dS_1$$

$$\therefore \quad \iint_R F_3(x, y, f_2)\, dxdy = \iint_{S_2} F_3\, \hat{k}\cdot\hat{n}_2\, dS_2 \qquad \text{...(3)}$$

and
$$\iint_R F_3(x, y, f_1)\, dxdy = -\iint_{S_1} F_3\, \hat{k}\cdot\hat{n}_1\, dS_1 \qquad \text{...(4)}$$

Using (3) and (4), (2) becomes

$$\iiint_V \frac{\partial F_3}{\partial z}\, dx\,dy\,dz = \iint_{S_2} F_3\hat{k}\cdot\hat{n}_2\, dS_2 + \iint_{S_1} F_3\hat{k}\cdot\hat{n}_1\, dS_1$$

$$= \iint_S F_3\hat{k}\cdot\hat{n}\, dS = \iint_S F_3 \cos\gamma\, dS \qquad \text{...(5)}$$

Similarly, by considering the projection of S on yz and zx-planes, we have

$$\iiint_V \frac{\partial F_1}{\partial x}\, dx\,dy\,dz = \iint_S F_1\hat{i}\cdot\hat{n}\, dS = \iint_S F_1 \cos\alpha\, dS \qquad \text{...(6)}$$

and
$$\iiint_V \frac{\partial F_2}{\partial y}\, dx\,dy\,dz = \iint_S F_2\hat{j}\cdot\hat{n}\, dS = \iint_S F_2 \cos\beta\, dS \qquad \text{...(7)}$$

Adding (5), (6) and (7), we get (1) *i.e.*, $\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz$

$$= \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\, dS \quad \text{or} \quad \iiint_V \nabla\cdot\vec{F}\, dV = \iint_S \vec{F}\cdot\hat{n}\, dS$$

In case the region be such that the lines drawn parallel to the coordinate axes meet it in more than two points, then we divide the region into various sub-regions each of which is met by a line parallel to any axis in only two points. Applying the theorem to each of these sub-regions and adding the results, we get the volume integral over the whole region.

ILLUSTRATIVE EXAMPLES

Example 1. *For any closed surface S, prove that* $\iint_S \text{curl}\, \vec{F}\cdot\hat{n}\, dS = 0$.

Sol. By the divergence theorem, we have $\iint_S \text{curl}\, \vec{F}\cdot\hat{n}\, dS = \iiint_V (\text{div curl}\, \vec{F})\, dV$,

where V is the volume enclosed by S = 0. Since div curl $\vec{F} = 0$, therefore, $\iint_S \text{curl}\, \vec{F}\cdot\hat{n}\, dS = 0$

Example 2. *Evaluate* $\iint_S \vec{r}\cdot\hat{n}\, dS$, *where S is a closed surface.* (M.D.U. May 2010)

Sol. By the divergence theorem, we have $\iint_S \vec{r}\cdot\hat{n}\, dS = \iiint_V \nabla\cdot\vec{r}\, dV$,

where V is the volume enclosed by S

$$= \iiint_V 3dV, \text{ since } \nabla\cdot\vec{r} = \text{div}\,\vec{r} = 3$$
$$= 3V.$$

Example 3. *Use divergence theorem to show that* $\oint_S \nabla r^2\cdot d\vec{S} = 6V$, *where S is any closed surface enclosing a volume V.*

Sol. By the divergence theorem, we have $\oint_S \nabla r^2\cdot d\vec{S} = \int_V \text{div}\,(\nabla r^2)\, dV$

$$= \int_V \nabla\cdot(\nabla r^2)\, dV = \int_V \nabla^2 r^2\, dV$$

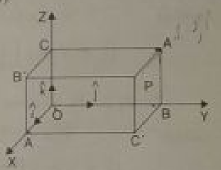
$$= \int_V \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(x^2+y^2+z^2)\, dV = \int_V 6dV = 6V.$$

Example 4. *Verify divergence theorem for* $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ *taken over the rectangular parallelopiped* $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$.

(M.T.U. 2013; M.D.U. 2013; K.U.K. Dec. 2010; P.T.U. Dec. 2010, May 2011)

Sol. For verification of divergence theorem, we shall evaluate the volume and surface integrals separately and show that they are equal.

Now,
$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$
$$= 2(x + y + z)$$



$$\therefore \quad \iiint_V \text{div}\,\vec{F}\, dV$$

$$= \int_0^c\int_0^b\int_0^a 2(x + y + z)\, dxdydz$$

$$= \int_0^c\int_0^b 2\left[\left(\frac{x^2}{2} + yx + zx\right)\right]_0^a dydz$$

$$= \int_0^c\int_0^b 2\left(\frac{a^2}{2} + ya + za\right) dydz = \int_0^c 2\left[\left(\frac{a^2}{2}y + \frac{ay^2}{2} + azy\right)\right]_0^b dz$$

$$= 2\int_0^c \left(\frac{a^2b}{2} + \frac{ab^2}{2} + abz\right) dz = 2\left[\frac{a^2b}{2}z + \frac{ab^2}{2}z + ab\frac{z^2}{2}\right]_0^c$$

$$= a^2bc + ab^2c + abc^2 = abc\,(a + b + c) \qquad \text{...(1)}$$

To evaluate the surface integral, divide the closed surface S of the rectangular parallelopiped into 6 parts

S_1 : the face OAC'B, S_2 : the face CB'PA', S_3 : the face OBA'C,
S_4 : the face AC'PB', S_5 : the face OCB'A, S_6 : the face BA'PC'.

Also
$$\iint_S \vec{F}\cdot\hat{n}\, dS = \iint_{S_1} \vec{F}\cdot\hat{n}\, dS + \iint_{S_2} \vec{F}\cdot\hat{n}\, dS + \iint_{S_3} \vec{F}\cdot\hat{n}\, dS$$
$$+ \iint_{S_4} \vec{F}\cdot\hat{n}\, dS + \iint_{S_5} \vec{F}\cdot\hat{n}\, dS + \iint_{S_6} \vec{F}\cdot\hat{n}\, dS$$

On S_1 $(z = 0)$, we have $\hat{n} = -\hat{k}$, $\vec{F} = x^2\hat{i} + y^2\hat{j} - xy\hat{k}$

so that $\vec{F}\cdot\hat{n} = (x^2\hat{i} + y^2\hat{j} - xy\hat{k}).(-\hat{k}) = xy$

$$\therefore \quad \iint_{S_1} \vec{F}\cdot\hat{n}\, dS = \int_0^b \int_0^a xy\, dx\, dy = \int_0^b \left[y\frac{x^2}{2} \right]_0^a dy = \frac{a^2}{2}\int_0^b y\, dy = \frac{a^2b^2}{4}$$

On S_2 $(z = c)$, we have $\hat{n} = \hat{k}$, $\vec{F} = (x^2 - cy)\hat{i} + (y^2 - cx)\hat{j} + (c^2 - xy)\hat{k}$

so that $\vec{F}\cdot\hat{n} = [(x^2 - cy)\hat{i} + (y^2 - cx)\hat{j} + (c^2 - xy)\hat{k}]\cdot\hat{k} = c^2 - xy$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\, dS = \int_0^b \int_0^a (c^2 - xy)\, dx\, dy = \int_0^b \left(c^2a - \frac{a^2}{2}y \right) dy = abc^2 - \frac{a^2b^2}{4}$$

On S_3 $(x = 0)$, we have $\hat{n} = -\hat{i}$, $\vec{F} = -yz\hat{i} + y^2\hat{j} + z^2\hat{k}$

so that $\vec{F}\cdot\hat{n} = (-yz\hat{i} + y^2\hat{j} + z^2\hat{k})\cdot(-\hat{i}) = yz$

$$\therefore \quad \iint_{S_3} \vec{F}\cdot\hat{n}\, dS = \int_0^c \int_0^b yz\, dy\, dz = \int_0^c \frac{b^2}{2} z\, dz = \frac{b^2c^2}{4}$$

On S_4 $(x = a)$, we have $\hat{n} = \hat{i}$, $\vec{F} = (a^2 - yz)\hat{i} + (y^2 - az)\hat{j} + (z^2 - ay)\hat{k}$

so that $\vec{F}\cdot\hat{n} = [(a^2 - yz)\hat{i} + (y^2 - az)\hat{j} + (z^2 - ay)\hat{k}]\cdot\hat{i} = a^2 - yz$

$$\therefore \quad \iint_{S_4} \vec{F}\cdot\hat{n}\, dS = \int_0^c \int_0^b (a^2 - yz)\, dy\, dz = \int_0^c \left(a^2b - \frac{b^2}{2}z \right) dz = a^2bc - \frac{b^2c^2}{4}$$

On S_5 $(y = 0)$, we have $\hat{n} = -\hat{j}$, $\vec{F} = x^2\hat{i} - zx\hat{j} + z^2\hat{k}$

so that $\vec{F}\cdot\hat{n} = (x^2\hat{i} - zx\hat{j} + z^2\hat{k})\cdot(-\hat{j}) = zx$

$$\therefore \quad \iint_{S_5} \vec{F}\cdot\hat{n}\, dS = \int_0^a \int_0^c zx\, dz\, dx = \int_0^a \frac{c^2}{2} x\, dx = \frac{a^2c^2}{4}$$

On S_6 $(y = b)$, we have $\hat{n} = \hat{j}$, $\vec{F} = (x^2 - bz)\hat{i} + (b^2 - zx)\hat{j} + (z^2 - bx)\hat{k}$

so that $\vec{F}\cdot\hat{n} = [(x^2 - bz)\hat{i} + (b^2 - zx)\hat{j} + (z^2 - bx)\hat{k}]\cdot\hat{j} = b^2 - zx$

$$\therefore \quad \iint_{S_6} \vec{F}\cdot\hat{n}\, dS = \int_0^a \int_0^c (b^2 - zx)\, dz\, dx = \int_0^a \left(b^2c - \frac{c^2}{2}x \right) dx = ab^2c - \frac{a^2c^2}{4}$$

$$\therefore \quad \iint_S \vec{F}\cdot\hat{n}\, dS = \frac{a^2b^2}{4} + abc^2 - \frac{a^2b^2}{4} + \frac{b^2c^2}{4} + a^2bc - \frac{b^2c^2}{4} + \frac{a^2c^2}{4} + ab^2c - \frac{a^2c^2}{4} \quad ...(2)$$

$$= abc\,(a + b + c)$$

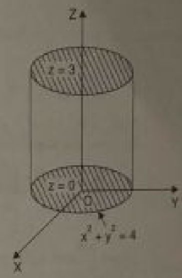
The equality of (1) and (2) verifies divergence theorem.

Example 5. *Verify divergence theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by the cylinder $x^2 + y^2 = 4$, $z = 0$, $z = 3$.*

(M.D.U. Dec. 2010, May 2010; K.U.K. Dec. 2013; Calicut 2010)

Sol. Since div $\vec{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$

$$\therefore \quad \iiint_V \text{div}\,\vec{F}\, dV = \iiint_V (4 - 4y + 2z)\, dx dy dz$$



$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 (4 - 4y + 2z)\, dz dy dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + z^2 \right]_0^3 dy dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9)\, dy dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21\, dy dx.$$

(Since 12y is an odd function, $\int_{-a}^{a} 12y\, dy = 0$)

$$= \int_{-2}^{2} 42\sqrt{4 - x^2}\, dx = 84\int_0^2 \sqrt{4 - x^2}\, dx = 84\left[\frac{x\sqrt{4 - x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2} \right]_0^2 = 84[2\sin^{-1} 1]$$

$$= 84\left[2 \times \frac{\pi}{2} \right] = 84\pi \quad ...(1)$$

To evaluate the surface integral, divide the closed surface S of the cylinder into 3 parts.

S_1 : the circular base in the plane $z = 0$

S_2 : the circular top in the plane $z = 3$

S_3 : the curved surface of the cylinder, given by the equation $x^2 + y^2 = 4$.

Also, $\iint_S \vec{F}\cdot\hat{n}\, dS = \iint_{S_1} \vec{F}\cdot\hat{n}\, dS + \iint_{S_2} \vec{F}\cdot\hat{n}\, dS + \iint_{S_3} \vec{F}\cdot\hat{n}\, dS$

On S_1 $(z = 0)$, we have $\hat{n} = -\hat{k}$, $\vec{F} = 4x\hat{i} - 2y^2\hat{j}$

so that $\vec{F}\cdot\hat{n} = (4x\hat{i} - 2y^2\hat{j})\cdot(-\hat{k}) = 0$

$$\therefore \quad \iint_{S_1} \vec{F}\cdot\hat{n}\, dS = 0$$

On S_2 $(z = 3)$, we have $\hat{n} = \hat{k}$, $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + 9\hat{k}$

so that $\vec{F}\cdot\hat{n} = (4x\hat{i} - 2y^2\hat{j} + 9\hat{k})\cdot\hat{k} = 9$

$$\therefore \quad \iint_{S_2} \vec{F}\cdot\hat{n}\, dS = \iint_{S_2} 9\,dx\,dy = 9\iint_{S_2} dx\,dy$$

$$= 9 \times \text{area of surface } S_2 = 9\,(\pi \cdot 2^2) = 36\pi$$

On S_3, $x^2 + y^2 = 4$

A vector normal to the surface S_3 is given by $\nabla(x^2 + y^2) = 2x\hat{i} + 2y\hat{j}$

$\therefore$ $\hat{n}$ = a unit vector normal to surface S_3

$$= \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2 + 4y^2}} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4 \times 4}}, \text{ since } x^2 + y^2 = 4$$

$$= \frac{x\hat{i} + y\hat{j}}{2}$$

$$\vec{F} \cdot \hat{n} = (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}) \cdot \left(\frac{x\hat{i} + y\hat{j}}{2}\right) = 2x^2 - y^3$$

Also, on S_3, *i.e.*, $x^2 + y^2 = 4$, $x = 2\cos\theta$, $y = 2\sin\theta$ and $dS = 2d\theta\, dz$.
To cover the whole surface S_3, z varies from 0 to 3 and θ varies from 0 to 2π.

$$\therefore \quad \iint_{S_3} \vec{F} \cdot \hat{n}\, dS = \int_0^{2\pi} \int_0^3 [2(2\cos\theta)^2 - (2\sin\theta)^3]\, 2dz\, d\theta$$

$$= \int_0^{2\pi} 16(\cos^2\theta - \sin^3\theta) \times 3\, d\theta = 48 \int_0^{2\pi} (\cos^2\theta - \sin^3\theta)\, d\theta = 48\pi$$

$$\left(\text{since } \int_0^{2\pi} \cos^2\theta\, d\theta = 2\int_0^{\pi} \cos^2\theta\, d\theta = 4\int_0^{\pi/2} \cos^2 d\theta = 4 \times \frac{1}{2} \times \frac{\pi}{2} = \pi, \int_0^{2\pi} \sin^3\theta\, d\theta = 0\right)$$

$$\therefore \quad \iint_S \vec{F} \cdot \hat{n}\, dS = 0 + 36\pi + 48\pi = 84\pi \qquad ...(2)$$

The equality of (1) and (2) verifies divergence theorem.

Example 6. *Find* $\iint_S \vec{F} \cdot \hat{n}\, dS$, *where* $\vec{F} = (2x + 3z)\hat{i} - (xz + y)\hat{j} + (y^2 + 2z)\hat{k}$ *and S is the surface of the sphere having centre at (3, –1, 2) and radius 3.*

Sol. Let V be the volume enclosed by the surface S. Then by Gauss divergence theorem, we have

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V \text{div}\, \vec{F}\, dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(2x + 3z) + \frac{\partial}{\partial y}(-xz - y) + \frac{\partial}{\partial z}(y^2 + 2z)\right] dV$$

$$= \iiint_V (2 - 1 + 2)\, dV = 3\iiint_V dV = 3V$$

But V is the volume of a sphere of radius 3.

$$\therefore \quad V = \frac{4}{3}\pi(3)^3 = 36\pi.$$

Hence $\iint_S \vec{F} \cdot d\vec{S} = 3 \times 36\pi = 108\pi.$

Example 7. *Evaluate* $\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}) \cdot \hat{n}\, dS$, *where S is the part of the sphere* $x^2 + y^2 + z^2 = 1$ *above the xy-plane and bounded by this plane.*

Sol. Let V be the volume enclosed by the surface S. Then by divergence theorem, we have

$$\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}) \cdot \hat{n}\, dS = \iiint_V \text{div}\,(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\, dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(y^2z^2) + \frac{\partial}{\partial y}(z^2x^2) + \frac{\partial}{\partial z}(z^2y^2)\right] dV$$

$$= \iiint_V 2zy^2\, dV = 2\iiint_V zy^2\, dV$$

Changing to spherical polar coordinates by putting

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$
$$dV = r^2\sin\theta\, dr\, d\theta\, d\phi$$

To cover V, the limits of r will be 0 to 1, those of θ will be 0 to $\frac{\pi}{2}$ and those of ϕ will be 0 to 2π.

$$\therefore \quad 2\iiint_V zy^2\, dV = 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (r\cos\theta)(r^2\sin^2\theta\sin^2\phi)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 r^5\sin^3\theta\cos\theta\sin^2\phi\, dr\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2} \sin^3\theta\cos\theta\sin^2\phi \left[\frac{r^6}{6}\right]_0^1 d\theta\, d\phi$$

$$= \frac{2}{6}\int_0^{2\pi} \sin^2\phi \cdot \frac{2}{4.2}\, d\phi = \frac{1}{12}\int_0^{2\pi} \sin^2\phi\, d\phi = \frac{\pi}{12}.$$

Example 8. *Evaluate* $\iint_S \vec{F} \cdot \hat{n}\, dS$ *over the entire surface of the region above the xy-plane bounded by the cone* $z^2 = x^2 + y^2$ *and the plane* $z = 4$, *if* $\vec{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$.

Sol. If V is the volume enclosed by S, then V is bounded by the surfaces $z = 0$, $z = 4$, $z^2 = x^2 + y^2$.

By divergence theorem, we have $\iint_S \vec{F} \cdot \hat{n}\, dS = \iiint_V \text{div}\, \vec{F}\, dV$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(3z)\right] dV = \iiint_V (4z + xz^2 + 3)\, dV$$

$$= \int_0^4 \int_{-z}^{z} \int_{-\sqrt{z^2 - y^2}}^{\sqrt{z^2 - y^2}} (4z + xz^2 + 3)\, dx\, dy\, dz$$

$$= 2\int_0^4 \int_{-z}^{z} \int_0^{\sqrt{z^2 - y^2}} (4z + 3)\, dx\, dy\, dz, \text{ since } \int_{-\sqrt{z^2 - y^2}}^{\sqrt{z^2 - y^2}} x\, dx = 0$$

$$= 2\int_0^4 \int_{-z}^{z} (4z + 3)\sqrt{z^2 - y^2}\, dy\, dz = 4\int_0^4 \int_0^z (4z + 3)\sqrt{z^2 - y^2}\, dy\, dz$$

$$= 4\int_0^4 (4z + 3)\left[\frac{y\sqrt{z^2 - y^2}}{2} + \frac{z^2}{2}\sin^{-1}\frac{y}{z}\right]_0^z dz$$

$$= 4\int_0^4 (4z + 3)\left[\frac{z^2}{2}\sin^{-1} 1\right] dz = 4 \times \frac{\pi}{4}\int_0^4 (4z^3 + 3z^2)\, dz$$

$$= \pi\left[z^4 + z^3\right]_0^4 = \pi(256 + 64) = 320\pi.$$

Example 9. *Evaluate* $\iint_S (\nabla \times \vec{A}) \cdot \hat{n}\, dS$, *where S is the surface of the cone* $z = 2 - \sqrt{x^2 + y^2}$ *above the xy-plane and* $\vec{A} = (x - z)\hat{i} + (x^3 + yz)\hat{j} - 3xy^2\hat{k}$.

Sol. Here S is not a closed surface. The surface $z = 2 - \sqrt{x^2 + y^2}$ meets the xy-plane in a circle C given by $x^2 + y^2 = 4$, $z = 0$. Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then *S' is a closed surface*. By divergence theorem, we have

$$\iint_{S'} \text{curl } \vec{A} \cdot \hat{n} \, dS = 0 \qquad \text{[See Example 1]}$$

$$\Rightarrow \iint_S \text{curl } \vec{A} \cdot \hat{n} \, dS + \iint_{S_1} \text{curl } \vec{A} \cdot \hat{n} \, dS = 0 \qquad [\because \text{ S' consists of S and } S_1]$$

$$\Rightarrow \iint_S \text{curl } \vec{A} \cdot \hat{n} \, dS - \iint_{S_1} \text{curl } \vec{A} \cdot \hat{k} \, dS = 0 \qquad [\because \text{ on } S_1, \hat{n} = -\hat{k}]$$

$$\Rightarrow \int_S \text{curl } \vec{A} \cdot \hat{n} \, dS = \iint_{S_1} \text{curl } \vec{A} \cdot \hat{k} \, dS$$

Now,
$$\text{curl } \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - z & x^3 + yz & -3xy^2 \end{vmatrix} = \hat{i}(-6xy - y) + \hat{j}(-1 + 3y^2) + \hat{k}(3x^2 - 0)$$

$$\therefore \quad \text{curl } \vec{A} \cdot \hat{k} = 3x^2$$

$$\therefore \iint_S \text{curl } \vec{A} \cdot \hat{n} \, dS = \iint_S 3x^2 \, dS = \int_0^{2\pi} \int_0^2 3(r^2 \cos^2 \theta) \, r \, dr \, d\theta, \text{ changing to polars}$$

$$= 3 \int_0^{2\pi} \int_0^2 r^3 \cos^2 \theta \, dr \, d\theta = 3 \int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^2 \cos^2 \theta \, d\theta = 12 \int_0^{2\pi} \cos^2 \theta \, d\theta = 12\pi.$$

Example 10. *Use divergence theorem to evaluate* $\iint_S (xdydz + ydzdx + zdxdy)$, *where S is the portion of the plane* $x + 2y + 3z = 6$ *in the first octant.*

Sol. We know that

$$\iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy) = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz$$

where V is the region bounded by closed surface S.

Comparing L.H.S. with $\iint_S (xdydz + ydzdx + zdxdy)$, we have

$F_1 = x, F_2 = y, F_3 = z$

$$\therefore \iint_S (xdydz + ydzdx + zdxdy)$$

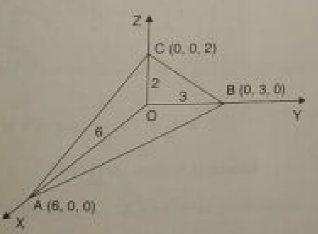
$$= \iiint_V (1 + 1 + 1) \, dxdydz$$

$$= 3 \iiint_V dxdydz$$

$= 3 \times$ volume of tetrahedron OABC

$$= 3 \left\{\frac{1}{3} \Delta AOB \times \text{hight OC}\right\}$$

$$= \frac{1}{2} (OA)(OB)(OC) = \frac{1}{2} (6)(3)(2) = 18.$$



EXERCISE 13.7

1. Use divergence theorem to show that $\iint_S r^n \vec{r} \cdot d\vec{S} = (n+3) \int_V r^n \, dV \, (n \neq -3)$, where S is any closed surface enclosing a volume V.

2. Verify divergence theorem for $\vec{F} = xz\hat{i} + y^2\hat{j} + yz\hat{k}$ taken over the cube bounded by $x = 0, x = 1; y = 0; y = 1; z = 0, z = 1$. *(U.P.T.U. 2009; G.B.T.U. 2010)*

3. (i) Verify divergence theorem for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ taken over the cube bounded by $x = 0, x = 1; y = 0, y = 1; z = 0, z = 1$. *(K.U.K. Dec. 2014; Anna 2010; Calicut 2010; U.P.T.U. 2009)*

   (ii) Verify Gauss divergence theorem for $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ taken over the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

4. Verify divergence theorem for $\vec{F} = (x^3 - yz)\hat{i} - 2x^2y\hat{j} + 2\hat{k}$ taken over the cube bounded by the planes $x = 0, x = a; y = 0, y = a; z = 0, z = a$.

5. (a) Verify divergence theorem for $\vec{F} = y\hat{i} + x\hat{j} + z^2\hat{k}$ over the cylindrical region bounded by $x^2 + y^2 = a^2$, $z = 0$ and $z = h$.

   (b) Verify Gauss divergence theorem for the function $\vec{F} = y\hat{i} + x\hat{j} + z^2\hat{k}$ over the cylindrical region bounded by $x^2 + y^2 = 9$, $z = 0$ and $z = 2$.

6. Verify divergence theorem for $\vec{F} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ taken over the region in the first octant bounded by $y^2 + z^2 = 9$ and $x = 2$.

7. If S is any closed surface enclosing a volume V and $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$, prove that $\iint_S \vec{F} \cdot \hat{n} \, dS = 6V$.

8. If $\vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$, a, b, c are constants, show that $\iint_S \vec{F} \cdot d\vec{S} = \frac{4}{3}\pi(a + b + c)$, where S is the surface of a unit sphere.

9. Use divergence theorem to evaluate $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$. *(P.T.U. May 2009, Jan. 2010)*

10. Use divergence theorem to evaluate $\iint_S \vec{F} \cdot \hat{n} dS$, where $\vec{F} = x^3\hat{i} + x^2y\hat{j} + x^2z\hat{k}$ and S is the surface bounding the region $x^2 + y^2 = a^2$, $z = 0$, $z = b$. *(P.T.U. Dec. 2010)*

11. Evaluate $\iint_S (x\hat{i} + y\hat{j} + z^2\hat{k}) \cdot \hat{n} dS$, where S is the closed surface bounded by the cone $x^2 + y^2 = z^2$ and the plane $z = 1$.

12. If $\vec{F} = (x^2 + y - 4)\hat{i} + 3xy\hat{j} + (2xz + z^2)\hat{k}$, evaluate $\iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS$, where S is the surface of the sphere $x^2 + y^2 + z^2 = 16$ above the xy-plane.

13. Evaluate $\iint_S [x^3\, dydz + y^3\, dzdx + 2z\,(xy - x - y)\, dxdy]$, where S is the surface of the cube $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$.

14. Use the divergence theorem to evaluate $\iint_S [(x^3 - yz)\, dydz - 2x^2y\, dzdx + z\, dxdy]$ over the surface of a cube bounded by the coordinate planes and the planes $x = y = z = a$.

15. Show that $\iint_S (x^2\hat{i} + y^2\hat{j} + z^2\hat{k}) \cdot \hat{n}\, dS = 0$, where Su denotes the surface of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

16. Prove that $\iiint_V \frac{1}{r^2}\, dV = \iint_S \frac{\vec{r} \cdot \hat{n}}{r^2}\, dS$.

17. Use divergence theorem to evaluate $\iint (x\, dy\, dz + y\, dz\, dx + z dx\, dy)$ over the surface of a sphere of radius a. (K.U.K. 2009)

Answers

9. $\frac{12}{5}\pi a^5$	10. $\frac{5}{4}\pi a^4 b$	11. $\frac{7}{6}\pi$
12. -16π	13. $\frac{1}{2}$	14. $\frac{1}{3}a^6 + a^3$
17. $4\pi a^3$		

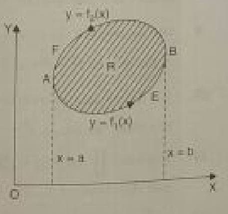
13.28. GREEN'S THEOREM IN THE PLANE

(M.T.U. 2012; P.T.U. Jan. 2009, May 2011)

Statement. *If $M(x, y)$ and $N(x, y)$ be continuous functions of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in a region R of the xy-plane bounded by a closed curve C, then*

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

where C is traversed in the counterclockwise direction.



Proof. Let us assume that the region R is such that any line parallel to either axes meets the boundary curve C in at most two points.

[The proof can be easily extended to other cases.]

Suppose the region R is bounded between the lines $x = a$, $x = b$ and two arcs AEB and BFA whose equations are $y = f_1(x)$ and $y = f_2(x)$ respectively such that $f_2(x) > f_1(x)$.

Now,
$$\iint_R \frac{\partial M}{\partial y}\, dxdy = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\, dy\right] dx$$
$$= \int_a^b \Big[M(x, y)\Big]_{f_1(x)}^{f_2(x)} dx = \int_a^b [M(x, f_2) - M(x, f_1)]\, dx$$
$$= \int_a^b M(x, f_2)\, dx - \int_a^b M(x, f_1)\, dx = -\int_b^a M(x, f_2)\, dx - \int_a^b M(x, f_1)\, dx$$

$$= -\left[\int_a^b M(x, f_1)\, dx + \int_b^a M(x, f_2)\, dx\right] = -\oint_C Mdx$$

$$\oint_C Mdx = -\iint_R \frac{\partial M}{\partial y}\, dxdy \qquad ...(1)$$

Similarly, we can show that $\oint_C Ndy = \iint_R \frac{\partial N}{\partial x}\, dxdy$ ...(2)

Adding (1) and (2), we have $\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

This theorem is useful for changing a line integral around a closed curve C into a double integral over the region R enclosed by C.

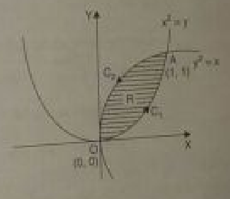
ILLUSTRATIVE EXAMPLES

Example 1. *Verify Green's theorem in the plane for $\oint_C (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$ where C is the boundary of the region defined by*

(a) $y = \sqrt{x}$, $y = x^2$ (U.K.T.U. 2011) (b) $x = 0, y = 0, x + y = 1$.

(K.U.K. Dec. 2013, Dec. 2014; Anna 2011)

Sol. (a) $y = \sqrt{x}$ *i.e.*, $y^2 = x$ and $y = x^2$ are two parabolas intersecting at O(0, 0) and A(1, 1).



Here $M = 3x^2 - 8y^2$, $N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y, \qquad \frac{\partial N}{\partial x} = -6y$$

$$\therefore \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 10y$$

If R is the region bounded by C, then

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$
$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\, dydx = \int_0^1 5\Big[y^2\Big]_{x^2}^{\sqrt{x}} dx$$
$$= 5\int_0^1 (x - x^4)\, dx = 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 \qquad ...(1)$$
$$= 5\left(\frac{1}{2} - \frac{1}{5}\right) = 5\left(\frac{3}{10}\right) = \frac{3}{2}$$

Also, $\oint_C (Mdx + Ndy) = \int_{C_1} (Mdx + Ndy) + \int_{C_2} (Mdx + Ndy)$

Along C_1, $x^2 = y$, $\therefore$ $2xdx = dy$ and the limits of x are from 0 to 1.

$\therefore$ Line integral along $C_1 = \int_{C_1} (Mdx + Ndy)$

$$= \int_0^1 (3x^2 - 8x^4)\, dx + (4x^2 - 6x \cdot x^2)\, 2x\, dx = \int_0^1 (3x^2 + 8x^3 - 20x^4)\, dx$$
$$= \Big[x^3 + 2x^4 - 4x^5\Big]_0^1 = -1$$

Along C_2, $y^2 = x$, $\therefore$ $2y\,dy = dx$ and the limits of y are from 1 to 0.

$\therefore$ Line integral along $C_2 = \int_{C_2} (M\,dx + N\,dy)$

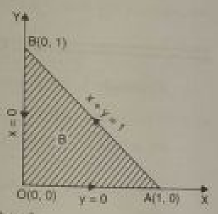
$$= \int_1^0 (3y^4 - 8y^2)\, 2y\,dy + (4y - 6y^2 \cdot y)\, dy$$

$$= \int_1^0 (4y - 22y^3 + 6y^5)\, dy = \left[2y^2 - \frac{11}{2}y^4 + y^6\right]_1^0 = \frac{5}{2}$$

$\therefore$ Line integral along $C = -1 + \frac{5}{2} = \frac{3}{2}$ *i.e.*, $\oint_C (M\,dx + N\,dy) = \frac{3}{2}$ ...(2)

The equality of (1) and (2) verifies Green's theorem in the plane.

(*b*) Here $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1 \int_0^{1-x} 10y\, dy\,dx$



$$= \int_0^1 5\left[y^2\right]_0^{1-x} dx$$

$$= 5\int_0^1 (1-x)^2\, dx = 5\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$= -\frac{5}{3}(0-1) = \frac{5}{3} \qquad ...(1)$$

Along OA, $y = 0$ $\therefore$ $dy = 0$ and the limits of x are from 0 to 1.

$\therefore$ Line integral along OA $= \int_0^1 3x^2\, dx = \left[x^3\right]_0^1$

Along AB, $y = 1 - x$ $\therefore$ $dy = -dx$ and the limits of x are from 1 to 0.

$\therefore$ Line integral along AB $= \int_1^0 [3x^2 - 8(1-x)^2]\, dx + [4(1-x) - 6x(1-x)](-dx)$

$$= \int_1^0 (3x^2 - 8 + 16x - 8x^2 - 4 + 4x + 6x - 6x^2)\, dx = \int_1^0 (-12 + 26x - 11x^2)\, dx$$

$$= \left[-12x + 13x^2 - \frac{11}{3}x^3\right]_1^0 = -\left[-12 + 13 - \frac{11}{3}\right] = \frac{8}{3}$$

Along BO, $x = 0$ $\therefore$ $dx = 0$ and the limits of y are from 1 to 0.

$\therefore$ Line integral along BO $= \int_1^0 4y\, dy = \left[2y^2\right]_1^0 = -2$

$\therefore$ Line integral along C (*i.e.*, along OABO) $= 1 + \frac{8}{3} - 2 = \frac{5}{3}$

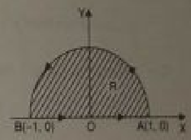
i.e.,
$$\oint_C (M\,dx + N\,dy) = \frac{5}{3} \qquad ...(2)$$

The equality of (1) and (2) verifies Green's theorem in the plane.

Example 2. *Use Green's theorem in a plane to evaluate the integral* $\oint_C [(2x^2 - y^2)\, dx + (x^2 + y^2)\, dy]$, *where C is the boundary in the xy-plane of the area enclosed by the x-axis and the semi-circle* $x^2 + y^2 = 1$ *in the upper half xy-plane.* (K.U.K. Dec. 2010; P.T.U. Jan. 2009)

Sol. If R is the region bounded by the closed curve C, then by Green's theorem in the plane, we have

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$



Here $M = 2x^2 - y^2$, $N = x^2 + y^2$

$$\frac{\partial M}{\partial y} = -2y, \frac{\partial N}{\partial x} = 2x$$

$$\therefore \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2(x + y)$$

The region R is bounded by

$$x = -1, x = 1, y = 0, y = \sqrt{1 - x^2}$$

$$\therefore \oint_C [(2x^2 - y^2)\, dx + (x^2 + y^2)\, dy] = \iint_R 2(x + y)\, dx\,dy$$

$$= \int_{-1}^1 \int_0^{\sqrt{1-x^2}} 2(x + y)\, dy\,dx = 2\int_{-1}^1 \left[xy + \frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= 2\int_{-1}^1 \left[x\sqrt{1-x^2} + \frac{1}{2}(1 - x^2)\right] dx$$

$$= \int_{-1}^1 (1 - x^2)\, dx, \text{ since } \int_{-1}^1 x\sqrt{1-x^2}\, dx = 0$$

$$= 2\int_0^1 (1 - x^2)\, dx = 2\left[x - \frac{x^3}{3}\right]_0^1 = 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}.$$

EXERCISE 13.8

1. Verify Green's theorem in the plane for $\oint_C (xy + y^2)\, dx + x^2\, dy$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$. (U.P.T.U. 2008)
2. Verify Green's theorem in the plane for $\oint_C (2xy - x^2)\, dx + (x^2 + y^2)\, dy$, where C is the boundary of the region enclosed by $y = x^2$ and $y^2 = x$.
3. Verify Green's theorem in the xy-plane for $\oint_C (xy^2 - 2xy)\, dx + (x^2y + 3)\, dy$, around the boundary C of the region enclosed by $y^2 = 8x$ and $x = 2$.
4. Verify Green's theorem in the plane for $\int_C (x^3 - xy^3)\, dx + (y^2 - 2xy)\, dy$, where C is the square having vertices at the points (0, 0), (2, 0), (2, 2) and (0, 2). (U.P.T.U. 2009)
5. Evaluate by Green's theorem $\int_C (\cos x \sin y - xy)\, dx + \sin x \cos y\, dy$, where C is the circle $x^2 + y^2 = 1$.

6. Apply Green's theorem to evaluate $\oint_C [(y - \sin x)\, dx + \cos x\, dy]$, where C is the plane triangle enclosed by the lines $y = 0$, $x = \frac{\pi}{2}$ and $y = \frac{2}{\pi} x$. *(P.T.U. Dec. 2011; M.D.U. 2012)*

7. Evaluate by Green's theorem $\oint_C e^{-x} (\sin y\, dx + \cos y\, dy)$, where C is the rectangle with vertices $(0, 0)$, $(\pi, 0)$, $\left(\pi, \frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right)$. *(M.D.U. 2013)*

8. Apply Green's theorem to prove that the area enclosed by a plane curve is $\frac{1}{2}\int_C (x\,dy - y\,dx)$. Hence find the area of an ellipse whose semi major and minor axes are of lengths a and b.

9. Find the area of a circle of radius 'a' using Green's theorem.

10. A vector field $\vec{F}$ is given by $\vec{F} = \sin y \hat{i} + x(1 + \cos y)\hat{j}$. Using Green's theorem, evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the circular path given by $x^2 + y^2 = a^2$.

11. Use Green's theorem to evaluate $\int_C (x^2 + xy)\, dx + (x^2 + y^2)\, dy$, where C is the square formed by the lines $x = \pm 1$, $y = \pm 1$. *(G.B.T.U. 2010; M.T.U. 2011; M.D.U. Dec. 2010)*

12. Using Green's theorem evaluate $\int_C (x^2 y\, dx + x^2\, dy)$, where C is the boundary described counter clockwise of the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$. *(U.K.T.U. 2010)*

Answers

5. 0	6. $-\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$	7. $2(e^{-\pi} - 1)$	8. πab
9. πa^2	10. πa^2	11. 0	12. $\frac{5}{12}$

13.29. STOKE'S THEOREM *(Relation between line and surface integrals)*

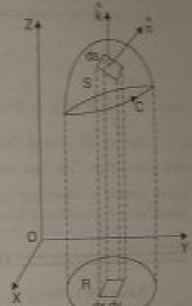
(U.P.T.U. 2009; P.T.U. Jan. 2009, Dec. 2010; U.K.T.U. 2012)

Statement. *If S be an open surface bounded by a closed curve C and $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be any vector point function having continuous first order partial derivatives, then $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n}\, dS$ where $\hat{n}$ is a unit normal vector at any point of S drawn in the sense in which a right handed screw would advance when rotated in the sense of description of C.*

Proof. Let $\hat{n}$ make angles α, β, γ with positive directions of x, y, z axes, then

$$\hat{n} = \cos\alpha\, \hat{i} + \cos\beta\, \hat{j} + \cos\gamma\, \hat{k}.$$

Also, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ so that $d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$



$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$$\therefore \quad \text{curl } \vec{F} \cdot \hat{n} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma$$

Also, $\vec{F} \cdot d\vec{r} = (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$

$$= F_1\, dx + F_2\, dy + F_3\, dz$$

$\therefore$ Stoke's theorem can be written as $\oint_C (F_1 dx + F_2 dy + F_3 dz)$

$$= \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right] dS \quad \text{...(1)}$$

Let $z = \phi(x, y)$ be the equation of the surface S whose projection on the xy-plane is R. Then the projection of C on the xy-plane is the curve C' which bounds the region R.

$$\therefore \quad \oint_C F_1(x, y, z)\, dx = \oint_{C'} F_1(x, y, \phi)\, dx = \oint_{C'} [F_1(x, y, \phi)\, dx + 0\, dy]$$

$$= -\iint_R \frac{\partial}{\partial y} F_1(x, y, \phi)\, dx dy \qquad \text{By Green's theorem in plane for the region R}$$

$$= -\iint_R \left[\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial \phi}{\partial y}\right] dx dy \quad \text{...(2)}$$

Now the direction ratios of the normal $\hat{n}$ to the surface $z = \phi(x, y)$ are $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, -1$

$$\Rightarrow \quad \frac{\cos\alpha}{\frac{\partial \phi}{\partial x}} = \frac{\cos\beta}{\frac{\partial \phi}{\partial y}} = \frac{\cos\gamma}{-1} \Rightarrow \frac{\partial \phi}{\partial y} = -\frac{\cos\beta}{\cos\gamma} \qquad [\because z = \phi(x, y) \Rightarrow \phi(x, y) - z = 0]$$

Since $dx\, dy$ is the projection of dS on the xy-plane

$$\therefore \quad dx dy = \cos\gamma\, dS$$

From (2), we have $\oint_C F_1(x, y, z)\, dx = -\iint_S \left[\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z}\frac{\cos\beta}{\cos\gamma}\right]\cos\gamma\, dS$

$$= -\iint_S \left[\frac{\partial F_1}{\partial y}\cos\gamma - \frac{\partial F_1}{\partial z}\cos\beta\right] dS \quad \text{...(3)}$$

$$= \iint_S \left[\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right] dS$$

Similarly, we can prove that $\oint_C F_2(x, y, z)\, dy$

$$= \iint_S \left[\frac{\partial F_2}{\partial x}\cos\gamma - \frac{\partial F_2}{\partial z}\cos\alpha\right] dS \qquad \ldots(4)$$

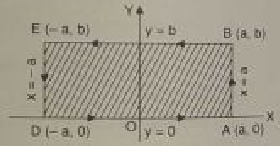
and $$\oint_C F_3(x, y, z)\, dz = \iint_S \left[\frac{\partial F_3}{\partial y}\cos\alpha - \frac{\partial F_3}{\partial x}\cos\beta\right] dS \qquad \ldots(5)$$

Adding (3), (4) and (5), we get (1). Hence the theorem is proved. In words, Stoke's theorem states that "The line integral of the tangential component of a vector point function $\vec{F}$ taken around a closed curve C is equal to the surface integral of the normal component of $\vec{F}$ taken over any surface S having C as its boundary".

ILLUSTRATIVE EXAMPLES

Example 1. *Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\hat{i} - 2xy\,\hat{j}$ taken round the rectangle bounded by the lines $x = \pm a$, $y = 0$, $y = b$.*

Sol. Let C denote the boundary of the rectangle ABED, then



$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C [(x^2 + y^2)\hat{i} - 2xy\hat{j}]\cdot(\hat{i}\,dx + \hat{j}\,dy)$$

$$= \oint_C [(x^2 + y^2)dx - 2xy\,dy]$$

The curve C consists of four lines AB, BE, ED and DA.

Along AB, $x = a$, $dx = 0$ and y varies from 0 to b.

$$\therefore \int_{AB} [(x^2 + y^2)\,dx - 2xydy] = \int_0^b -2ay\,dy = -a\left[y^2\right]_0^b = -ab^2 \qquad \ldots(1)$$

Along BE, $y = b$, $dy = 0$ and x varies from a to $-a$.

$$\therefore \int_{BE} [(x^2 + y^2)\,dx - 2xydy] = \int_a^{-a} (x^2 + b^2)\,dx = \left[\frac{x^3}{3} + b^2x\right]_a^{-a} = -\frac{2a^3}{3} - 2ab^2 \qquad \ldots(2)$$

Along ED, $x = -a$, $dx = 0$ and y varies from b to 0.

$$\therefore \int_{ED} [(x^2 + y^2)\,dx - 2xydy] = \int_b^0 2ay\,dy = a\left[y^2\right]_b^0 = -ab^2 \qquad \ldots(3)$$

Along DA, $y = 0$, $dy = 0$ and x varies from $-a$ to a.

$$\therefore \int_{DA} [(x^2 + y^2)\,dx - 2dxdy] = \int_{-a}^{a} x^2\,dx = \frac{2a^3}{3} \qquad \ldots(4)$$

Adding (1), (2), (3) and (4), we get

$$\oint_C \vec{F}\cdot d\vec{r} = -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2 \qquad \ldots(5)$$

Now, $$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} = (-2y - 2y)\hat{k} = -4y\hat{k}$$

For the surface S, $\hat{n} = \hat{k}$

$$\text{curl}\,\vec{F}\cdot\hat{n} = -4y\hat{k}\cdot\hat{k} = -4y$$

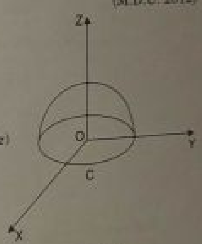
$$\therefore \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS = \int_0^b\int_{-a}^{a} -4y\,dxdy = \int_0^b -4y\left[x\right]_{-a}^{a} dy$$

$$= -8a\int_0^b y\,dy = -8a\left[\frac{y^2}{2}\right]_0^b = -4ab^2 \qquad \ldots(6)$$

The equality of (5) and (6) verifies Stoke's Theorem.

Example 2. *Verify Stoke's Theorem for the vector field $\vec{F} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ over the upper half surface of $x^2 + y^2 + z^2 = 1$, bounded by its projection on the xy-plane.* (M.D.U. 2012)

Sol. Let S be the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$. The boundary C of S is a circle in the xy-plane of radius unity and centre O. The equations of C are $x^2 + y^2 = 1$, $z = 0$ whose parametric form is $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t < 2\pi$.



$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C [(2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}]\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \oint_C [(2x - y)\,dx - yz^2\,dy - y^2z\,dz]$$

$$= \oint_C (2x - y)\,dx, \text{ since on C, } z = 0,\ dz = 0$$

$$= \int_0^{2\pi} (2\cos t - \sin t)\frac{dx}{dt}\,dt = \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\,dt$$

$$= \int_0^{2\pi} (-\sin 2t + \sin^2 t)\,dt = \int_0^{2\pi}\left(-\sin 2t + \frac{1 - \cos 2t}{2}\right)dt$$

$$= \left[\frac{\cos 2t}{2} + \frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = \frac{1}{2} + \pi - \frac{1}{2} = \pi$$

Also, $$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix} = (-2yz + 2yz)\hat{i} + (0 - 0)\hat{j} + (0 + 1)\hat{k} = \hat{k}$$

$$\text{Curl}\,\vec{f}\cdot\hat{n} = \hat{k}\cdot\hat{n} = \hat{n}\cdot\hat{k}$$

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS = \iint_S \hat{k}\cdot\hat{n}\,dS = \iint_R \hat{k}\cdot\hat{n}\frac{dxdy}{|\hat{k}\cdot\hat{n}|} = \iint_R dxdy$$

where R is the projection of S on xy-plane.

= area bounded by R

= area of a circle with radius 1

$= \pi$

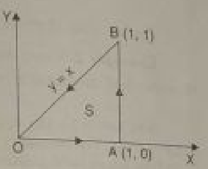
Since $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS$, Stoke's theorem is verified.

Example 3. *Evaluate* $\oint_C \vec{F}\cdot d\vec{r}$ *by Stoke's Theorem, where* $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ *and C is the boundary of the triangle with vertices at (0, 0, 0), (1, 0, 0) and (1, 1, 0).*

(P.T.U. May 2010; U.K.T.U. 2012)

Sol. Since z-coordinates of each vertex of the triangle is zero, therefore, the triangle lies in the xy-plane and $\hat{n} = \hat{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = \hat{j} + 2(x-y)\hat{k}$$



$\therefore$ $\text{curl } \vec{F}\cdot\hat{n} = [\hat{j} + 2(x-y)\hat{k}]\cdot\hat{k} = 2(x-y)$

The equation of line OB is $y = x$.

By Stoke's theorem, $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\hat{n}\, dS$

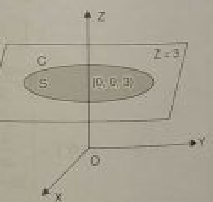
$$= \int_0^1\int_0^x 2(x-y)\, dy dx$$

$$= \int_0^1 2\left[xy - \frac{y^2}{2}\right]_0^x dx = 2\int_0^1\left(x^2 - \frac{x^2}{2}\right)dx = \int_0^1 x^2\, dx = \frac{1}{3}$$

Example 4. *Using Stoke's theorem for the vector function* $\vec{F} = (2x+y-2z)\hat{i} + (2x-4y+z^2)\hat{j} + (x-2y+z^2)\hat{k}$ *evaluate the integral* $\oint_C \vec{F}\cdot d\vec{R}$, *where C is the circle with centre at (0, 0, 3) and radius 5 in the plane z = 3.*

Sol. Here $\vec{F} = (2x+y-2z)\hat{i} + (2x-4y+z^2)\hat{j} + (x-2y+z^2)dz$ and C is the curve given by $(x-0)^2 + (y-0)^2 + (z-3)^2 = 5^2$, $z = 3$ i.e., $x^2 + y^2 + (z-3)^2 = 25$, $z = 3$

[It is a great circle with centre at (0, 0, 3) and radius 5]



By Stoke's theorem

$$\oint_C \vec{F}\cdot d\vec{R} = \iint_S \text{curl } \vec{F}\cdot\hat{n}\, dS \qquad ...(1)$$

Now,
$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x+y-2z & 2x-4y+z^2 & x-2y+z^2 \end{vmatrix}$$

$$= \hat{i}\,[-2-2z] - \hat{j}\,[1-(-2)] + \hat{k}\,[2-1]$$

$$= (-2-2z)\hat{i} - 3\hat{j} + \hat{k}$$

$\hat{n}$ = a unit vector normal to the plane $z = 3$

i.e., $\hat{n} = \hat{k}$

$\therefore$ From (1), $\oint_C \vec{F}\cdot d\vec{R} = \iint_S [(-2-2z)\hat{i} - 3\hat{j} + \hat{k}]\cdot\hat{k}\, dS = \iint_S dS = S$

$$= \text{area of circle} = \pi\times 5^2 = 25\pi$$

Example 5. *Evaluate the surface integral* $\iint_S \text{curl } \vec{F}\cdot\hat{n}\, dS$ *by transforming it into a line integral, S being that part of the surface of the paraboloid* $z = 1 - x^2 - y^2$, *for which* $z \geq 0$ *and* $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$.

Sol. The boundary C of the surface S is the circle $x^2 + y^2 = 1$, $z = 0$ whose parametric equations are

$$x = \cos t,\ y = \sin t,\ z = 0,\ 0 \leq t < 2\pi.$$

By Stoke's theorem, we have $\iint_S \text{curl } \vec{F}\cdot\hat{n}\, dS = \oint_C \vec{F}\cdot d\vec{r}$

$$= \oint_C (y\hat{i} + z\hat{j} + x\hat{k})\cdot(\hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz)$$

$$= \oint_C (y\, dx + z\, dy + x\, dz) = \oint_C y\, dx, \text{ since on C, } z = 0,\ dz = 0$$

$$= \oint_C y\frac{dx}{dt}\, dt = \int_0^{2\pi} \sin t(-\sin t)\, dt = -\int_0^{2\pi} \sin^2 t\, dt$$

$$= -2\int_0^{\pi} \sin^2 t\, dt = -4\int_0^{\pi/2} \sin^2 t\, dt = -4\times\frac{1}{2}\times\frac{\pi}{2} = -\pi.$$

EXERCISE 13.9

1. Verify Stoke's theorem for the vector field $\vec{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j}$ integrated round the rectangle in the plane $z = 0$ and bounded by the lines $x = 0$, $y = 0$, $x = a$, $y = b$.
2. Verify Stoke's theorem for the function $\vec{F} = x^2\hat{i} + xy\hat{j}$ integrated round the square whose sides are $x = 0$, $y = 0$, $x = a$ and $y = a$ in the plane $z = 0$.
3. Verify Stoke's theorem for $\vec{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j}$ over the box bounded by the planes $x = 0$, $x = a$; $y = 0$; $y = b$, $z = 0$, $z = c$; if the face $z = 0$ is cut.
4. Verify Stoke's theorem for $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.
5. Evaluate by Stoke's theorem $\oint_C (yz\, dx + zx\, dy + xy\, dz)$, where C is the curve $x^2 + y^2 = 1$, $z = y^2$.
6. Use Stoke's theorem to evaluate $\iint_S (\nabla\times\vec{F})\cdot\hat{n}\, dS$, where $\vec{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane.
7. Evaluate by Stoke's theorem $\oint_C (\sin z\, dx - \cos x\, dy + \sin y\, dz)$, where C is the boundary of the rectangle $0 \leq x \leq \pi$, $0 \leq y \leq 1$, $z = 3$. (Calicut 2010)

8. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = y\hat{i} + xz^3\hat{j} - zy^3\hat{k}$, C is the circle $x^2 + y^2 = 4$, $z = 1.5$.

9. Apply Stoke's theorem to evaluate $\int_C [(x + y)dx + (2x - z)dy + (y + z)dz]$, where C is the boundary of the triangle with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6).

(K.U.K. 2009; P.T.U. May 2010; Calicut 2010; M.T.U. 2013)

10. Apply Stoke's theorem to evaluate $\int_C (ydx + zdy + xdz)$, where C is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$. *(M.D.U. May 2010)*

Answers

5. 0 6. 0 7. 2 8. $\frac{19\pi}{2}$

9. 21 10. $-\frac{\pi a^2}{\sqrt{2}}$.

APPENDIX—I

SHORT ANSWER TYPE QUESTIONS

According to the latest syllabus of MDU, Q. 1 will be compulsory. It will cover all the four sections and will be of short answer type.

Answer the following short answer type questions by choosing the correct option or by filling up the blanks.

Infinite Series:

1. The geometric series $1 + x + x^2 + \cdots \infty$ converges in the interval ....

2. The hyper-harmonic series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \infty$ converges if

(a) $p < 1$ (b) $p = 1$ (c) $p > 1$ (d) $p \le 1$

(M.D.U. Dec. 2013)

3. If $\sum u_n$ is convergent, then $\lim_{n\to\infty} u_n = \cdots$

4. If $\sum u_n$ is a series of positive terms and $\lim_{n\to\infty} u_n \neq 0$, then $\sum u_n$ must be ...

5. A series of positive terms either ... or ... but never ...

6. The series $\sum_{n=1}^{\infty} (-1)^{n-1}$

(a) converges (b) diverges

(c) oscillates finitely (d) oscillates infinitely

7. The series $1 + \frac{1}{4^{2/3}} + \frac{1}{9^{2/3}} + \frac{1}{16^{2/3}} + \cdots \infty$ is

(a) convergent (b) divergent

8. The series $\sum_{n=0}^{\infty} (2x)^n$ converges if

(a) $-2 < x < 2$ (b) $-1 < x < 1$ (c) $-\frac{1}{2} \le x \le \frac{1}{2}$ (d) $-\frac{1}{2} < x < \frac{1}{2}$

9. The series $\frac{1}{\sqrt{1.2}} + \frac{1}{\sqrt{2.3}} + \frac{1}{\sqrt{3.4}} + \cdots \infty$ is

(a) convergent (b) divergent

10. The series $\sum \frac{\sqrt[3]{3n^2+1}}{\sqrt[4]{4n^3+2n+7}}$ is

(a) convergent (b) divergent