# CHAPTER 1

# CALCULUS

## 1.1 BETA AND GAMMA FUNCTIONS

The definite Integral

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \qquad \text{for } m>0,\ n>0$$

is called the *Beta function* or *Eulerian Integral* of first kind and denoted by $\beta(m, n)$. Thus,

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \qquad m>0,\ n>0 \qquad \text{....(i)} \qquad \textbf{[KU 2013]}$$

### 1.1.1 Properties

**(*i*) Show that Beta function is symmetric or $\beta(m,n)=\beta(n,m)$.**

**Proof:** From (i), we have

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \qquad m>0,\ n>0$$

Put $\quad x=1-y$, we get $dx=-\,dy$, when $x=0$, $y=1$ and $x=1$, $y=0$

$$\beta(m,n)=-\int_1^0 (1-y)^{m-1}\,y^{n-1}\,dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\, dy = \beta(n,m)$$

Thus, $\beta(m,n) = \beta(n,m)$

**(ii) Prove that $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$**

**Proof:** $\quad$ R.H.S $= \beta(m+1,n) + \beta(m,n+1)$

$$= \int_0^1 x^m (1-x)^{n-1}\, dx + \int_0^1 x^{m-1}(1-x)^n\, dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\, dx\,[x+1-x]$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

$$= \beta(m,n) = \text{L.H.S.}$$

**(iii) Prove $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$, where $m, n > 0$**

**Proof:** We have $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$ ...(i)

The integral can be written as $\int_0^\infty \left(\frac{x}{1+x}\right)^{m-1} \left(\frac{1}{1+x}\right)^{n-1} \frac{1}{(1+x)^2}\, dx$

Put $\frac{x}{1+x} = t \quad \Rightarrow \quad x = \frac{t}{1-t}$ ...(ii)

$$\frac{1(1+x) - x.1}{(1+x)^2}\, dx = dt$$

$$\frac{1}{(1+x)^2}\, dx = dt \qquad \text{...(iii)}$$

$$1 + x = 1 + \frac{t}{1-t} = \frac{1}{1-t} \qquad \text{...(iv)}$$

or $\frac{1}{1+x} = (1-t)$

$$\lim_{x\to 0} = \frac{x}{1+x} = 0 \qquad \text{...(v)}$$

$$\lim_{x\to\infty} t = \lim_{x\to\infty} \frac{x}{1+x} = \lim_{x\to\infty} = \frac{1}{\frac{1}{x}+1} = 1 \qquad \text{...(vi)}$$

(*iii*), (*iv*), (*v*) and (*vi*), in (*ii*), integral (*i*) changes to

$$\int_0^1 t^{m-1}(1-t)^{n-1}\,dt = \beta(m,n) \text{ Hence proved.}$$

The integral $\beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$ is also considered as alternate from of beta function.

Similarly, we can prove that

$$\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\therefore \quad \beta(m,n) = \beta(n,m)$$

$$\therefore \quad \beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

**(*iv*) Prove that $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$**

**Proof:** $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Substituting $x = \sin^2\theta$ so that

$dx = 2\sin\theta\cos\theta\,d\theta$

When $x = 0, \theta = 0$ and $x = 1, \theta = \frac{\pi}{2}$

$$= \int_0^{\pi/2} \sin^{2(m-1)}\theta\cos^{2(n-1)}\theta . 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta \text{ Hence proved.}$$

**(*v*) If both $m$ and $n$ are positive integers, then $\beta(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$**

**Proof:** L.H.S. $= \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Integrating by parts, we have

$$= \left[x^{m-1}\frac{(1-x)^n}{-n}\right]_0^1 - \int_0^1 (m-1)x^{m-2}\frac{(1-x)^n}{-n}\,dx$$

$$= 0 + \frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^n\,dx$$

$$= \frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^2(1-x)\,dx$$

$$= \frac{m-1}{n}\int_0^1 \left[x^{m-2}(1-x)^{n-1} - x^{m-1}(1-x)^{n-1}\right]dx$$

$$= \frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^{n-1}\,dx - \frac{m-1}{n}\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$\therefore \quad \beta(m,n) = \frac{m-1}{n}\beta(m-1,n) - \frac{m-1}{n}\beta(m,n)$$

or $\beta(m,n) + \frac{m-1}{n}\beta(m,n) = \frac{m-1}{n}\beta(m-1,n)$

or $\left(1+\frac{m-1}{n}\right)\beta(m,n) = \frac{m-1}{n}\beta(m-1,n)$

or $\left(\frac{n+m-1}{n}\right)\beta(m,n) = \frac{m-1}{n}\beta(m-1,n)$

$$\therefore \quad \beta(m,n) = \frac{(m-1)}{(m+n-1)}\beta(m-1,n) \qquad \text{...(i)}$$

Changing $m$ by $(m-1)$, we get

$$\beta(m-1,n) = \frac{m-2}{m+n-2}\beta(m-2,n)$$

Put in equation (*i*)

$$\therefore \quad \beta(m,n) = \frac{(m-1)(m-2)}{(m+n-1)(m+n-2)}\beta(m-2,n) \qquad \text{...(ii)}$$

Continuing this way, we get

$$\beta(m,n) = \frac{(m-1)(m-2).......1}{(n+m-1)(n+m-2).....(n+1)}\beta(1,n) \qquad \text{...(iii)}$$

Now, $\beta(l,n) = \int_0^1 x^0(1-x)^{n-1}\,dx = \int_0^1 (1-x)^{n-1}\,dx = \left[\frac{(1-x)^n}{-n}\right]_0^1$

$$= \frac{1}{n}$$

From (*iii*), we get $\beta(m,n) = \frac{(m-1)(m-2).......1}{(n+m-1)(n+m-2).....(n+1)}\cdot\frac{1}{n}$

$$= \frac{(m-1)!}{(n+m-1)(n+m-2)....(n+1)n}$$

Multiplying and dividing by $(n-1)!$ we get

$$\beta(m,n) = \frac{(m-1)!(n-1)!}{(n+m-1)(n+m-2).....(n+1)\,n\,(n-1)!}$$

$$= \frac{(m-1)!(n-1)!}{(n+m-1)}$$

## 1.2 GAMMA FUNCTION

The Gamma function is defined as

$$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1} dx \quad (n > 0)$$ [KU, 2013]

This integral is also known as Euler's integral of the second kind.

In particular,

$$\Gamma(n) = \int_0^\infty e^{-x} dx = \left| -e^{-x} \right|_0^\infty = 1$$

### 1.2.1 Properties of Gamma Function

*(i)* $\Gamma(n+1) = n\Gamma n$

Since $\quad \Gamma(n+1) = \int_0^\infty e^{-x} x^n\, dx = \int_0^\infty \underset{\mathrm{I}}{x^n}\; \underset{\mathrm{II}}{e^{-x}}\, dx$

Integrating by parts, we have

$$\Gamma(n+1) = \left| -x^n e^{-x} \right|_0^\infty + n\int_0^\infty x^{n-1} e^{-x}\, dx$$

$$= n\int_0^\infty e^{-x}\, x^{n-1}\, dx \qquad \because \lim_{x\to 0} \frac{x^n}{e^x} = 0$$

$$= n\Gamma(n)$$ [By definition of Gamma function]

$\therefore \quad \boxed{\Gamma(n+1) = n\Gamma(n)}$ , is the reduction formula for $\Gamma(n)$.

***(ii)*** $\Gamma(n+1) = n!$

Using $\quad \Gamma(n+1) = n\Gamma(n)$

$$\Gamma(2) = 1\times\Gamma(1) = 1!$$

$$\Gamma(3) = 2\cdot\Gamma(2) = 2\times 1 = 2!$$

$$\Gamma(4) = 3\cdot\Gamma(3) = 3\times 2! = 3!$$

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$$\Gamma(n+1) = n!, \qquad n > 0$$

**(iii)** $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx, \qquad n > 0$$

$$\Rightarrow \qquad \Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x} x^{-\frac{1}{2}}\, dx$$

Put $x = y^2$. so that $dx = 2y\,dy$, when $x = 0, y = 0$ and $x = \infty, y = \infty$

$$= 2\int_0^\infty e^{-y^2}\, dy \qquad \text{...(1)}$$

Also $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}\, dx$ ...(2)

Multiplying (1) and (2), we get

$$\therefore \quad \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\, dx\, dy = 4\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2} r\, dr\, d\theta \qquad \begin{bmatrix} \text{Put } x = r\cos\theta \\ y = r\sin\theta \end{bmatrix}$$

[$dx\,dy = r\,dr\,d\theta$

$\theta$ varies from 0 to $\pi/2$ and $r$ varies from 0 to $\infty$]

$$= 4\int_0^{\frac{\pi}{2}} \left[-\frac{1}{2}e^{-r^2}\right]_0^\infty d\theta = 2\int_0^{\frac{\pi}{2}} d\theta = \pi$$

Hence, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

## 1.3 RELATION BETWEEN BETA AND GAMMA FUNCTIONS

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

**(M.D.U. 1999, 2000, 01, 02, 03; K.U. 2009, 2013, 2016, 2017, 2018)**

**Proof:** We know that $\Gamma(m) = \int_0^\infty e^{-t} t^{m-1} dt$ [by definition of Gamma function]

Put $t = x^2$ so that $dt = 2x\,dx$, when $t = 0, x = 0$ and $t = \infty, x = \infty$

$$\Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-1}\, dx \qquad \text{....}(i)$$

Similarly $\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy$ ....(ii)

Multiplying (*i*) and (*ii*), we get

$$\Rightarrow \quad \Gamma(m)\Gamma(n) = 4\int_0^{\infty} e^{-x^2} x^{2m-1}\,dx \cdot \int_0^{\infty} e^{-y^2} y^{2n-1}\,dy$$

$$= 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\,dx\,dy$$

Changing to polar cordinates

Put $x = r\cos\theta,\ y = r\sin\theta$, so that $dx\,dy = r\,dr\,d\theta$. Region of integration is entire first quadrant. In polar co-ordinates $r$ varies from 0 to $\infty$ and $\theta$ varies from 0 to $\frac{\pi}{2}$.

$$\Rightarrow \quad \Gamma(m)\,\Gamma(n) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2} r^{2(m+n)-1} \cos^{2m-1}\theta\ \sin^{2n-1}\theta\,dr\,d\theta$$

$$= 4\int_0^{\infty} e^{-r^2} r^{2(m+n)-1}\,dr \cdot \int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta\ \sin^{2n-1}\theta\,d\theta \qquad \text{....}(iii)$$

Now $\quad 2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1}\,dr = \Gamma(m+n)$ from (*i*)

and from $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\,d\theta$

Result (*iii*) becomes,

$$\Gamma(m)\,\Gamma(n) = \Gamma(m+n)\,\beta(m,n) \quad \Rightarrow \quad \beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

## 1.4 TO EVALUATE $\int_0^{\frac{\pi}{2}} \sin^p x\cos^q x\,dx$

$$\int_0^{\frac{\pi}{2}} \sin^p x\cos^q x\,dx = \int_0^{\frac{\pi}{2}} \left(\sin^{p-1} x\cos^{q-1} x\right)\sin x\cos x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin^2 x)^{\frac{p-1}{2}} (\cos^2 x)^{\frac{q-1}{2}}\,.2\sin x\cos x\,dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 x)^{\frac{p-1}{2}} (1-\sin^2 x)^{\frac{q-1}{2}} \sin x\cos x\,dx$$

Put $\sin^2 x = t$ so that $2\sin x\cos x\,dx = dt$, when $x = 0,\ t = 0$ and $x = \frac{\pi}{2},\ t = 1$

$$= \frac{1}{2}\int_0^1 t^{\frac{p-1}{2}}(1-t)^{\frac{q-1}{2}}\,dt$$

$$= \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

## 1.4.1 Particular Cases

**(1)** Put $q = 0$ and $p = n$, we have

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\frac{\sqrt{\pi}}{2}$$

**(2)** Put $p = 0$ and $q = n$, we have

$$\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\frac{\sqrt{\pi}}{2}$$

**(3)** Put $p = q = 0$, we have

$$\frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{2\Gamma(1)} = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2} \quad \text{or} \quad \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi$$

$$\Rightarrow \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

**EXAMPLE 1.1. Compute :**

**(i)** $\beta(2.5, 1.5)$ **[M.D.U.2001]** **(ii)** $\beta\left(\frac{9}{2}, \frac{7}{2}\right)$

**Solution. (i)** $\beta(2.5, 1.5) = \beta\left(\frac{5}{2}, \frac{3}{2}\right)$

$$= \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}+\frac{3}{2}\right)}$$

$$= \frac{\left(\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right)\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(4)}$$

$\because \beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

$\Gamma(n+1) = n\Gamma(n)$

$\Gamma(n+1) = n!$

$$=\frac{\frac{3}{8}\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{3!}=\frac{\frac{3}{8}[\sqrt{\pi}]^2}{6}=\frac{\pi}{16}$$

(ii) $$\beta\left(\frac{9}{2},\frac{7}{2}\right)=\frac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{9}{2}+\frac{7}{2}\right)}$$

$$=\frac{\left[\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right]\left[\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right]}{\Gamma(8)}$$

$$=\frac{7\cdot5\cdot3\cdot5\cdot3\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{16\times8\times7!}=\frac{7\cdot5\cdot3\cdot5\cdot3[\sqrt{\pi}]^2}{128\times7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}=\frac{5\pi}{2048}$$

**EXAMPLE 1.2. Show that** $\beta(p,q)=\int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy$ **[K.U. 2014, 2017]**

$$=\int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}\,dx$$ **[M.D.U. 2012, 2013]**

**Solution.** $\beta(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx$ [By definition of Beta function]

Put $x=\frac{1}{1+y}$ so that $dx=-\frac{1}{(1+y)^2}dy$ when $x=0, y=\infty$ and $x=1, y=0$

$$\beta(p,q)=\int_\infty^0\left(\frac{1}{1+y}\right)^{p-1}\left(\frac{y}{1+y}\right)^{q-1}\left(\frac{-1}{(1+y)^2}\right)dy$$

$$=\int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy$$

$$=\int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}}\,dy+\int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy \qquad \text{....}(i)$$

Now put $y=\frac{1}{z}$ in second integral, so that $dy=-\frac{1}{z^2}dz$, when $y=1, z=1;\quad y=\infty, z=0$

$$\int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy=\int_1^0 \frac{\left(\frac{1}{z}\right)^{q-1}}{\left(1+\frac{1}{z}\right)^{p+q}}\left(-\frac{1}{z^2}\right)dz=\int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}}\,dz$$

$\therefore$ From (i), we have

$$\beta(p,q)=\int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}}\,dy+\int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}}\,dz=\int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}\,dx$$

**EXAMPLE 1.3. Prove that** $\boldsymbol{\beta\left(m,\frac{1}{2}\right)=2^{2m-1}\beta(m,m)}$. **[K.U. 2013]**

**Solution.** We know that $$\beta(m,n)=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta \qquad ...(1)$$

Putting $n=\frac{1}{2}$, we get

$$\beta\left(m,\frac{1}{2}\right)=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2\left(\frac{1}{2}\right)-1}\theta\,d\theta=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\,d\theta \qquad ...(2)$$

Now, $$\beta(m,m)=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2m-1}\theta\,d\theta=2\int_0^{\frac{\pi}{2}}(\sin\theta\cos\theta)^{2m-1}d\theta$$

$$=2\int_0^{\frac{\pi}{2}}\left(\frac{2\sin\theta\cos\theta}{2}\right)^{2m-1}d\theta$$

$$=\frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}}\sin^{2m-1}2\theta\,d\theta=\frac{1}{2^{2m-2}}\int_0^{\frac{\pi}{2}}\sin^{2m-1}2\theta\,d\theta$$

Put $2\theta=\alpha$ so that $d\theta=\frac{d\alpha}{2}$, when $\theta=0$, $\alpha=0$ and $\theta=\frac{\pi}{2}$, $\alpha=\pi$

$$=\frac{1}{2^{2m-2}}\int_2^{\pi}\frac{\sin^{2m-1}\alpha}{2}\,d\alpha=\frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}}\sin^{2m-1}\alpha\,d\alpha$$

or $$2^{2m-1}\beta(m,m)=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\alpha\,d\alpha=\beta\left(m,\frac{1}{2}\right) \qquad [\because \text{from (2)}]$$

Hence, $\beta=\left(m,\frac{1}{2}\right)=2^{2m-1}\beta(m,m)$

**EXAMPLE 1.4.** Show that $\Gamma(n) = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy, (n > 0)$. **[M.D.U. 2012]**

**Solution.** $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \quad (n > 0)$ [By definition of Gamma function]

Put $y = e^{-x}$ i.e. $x = \log\left(\frac{1}{y}\right)$ when $x = 0, y = 1$ and $x = \infty, y = 0$

$$\Rightarrow \quad dx = -\frac{1}{y} dy$$

$$\Rightarrow \quad \Gamma(n) = \int_1^0 \left(\log \frac{1}{y}\right)^{n-1} y\left(-\frac{1}{y} dy\right) = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$$

**EXAMPLE 1.5.** Show that $\Gamma(m)\,\Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma(2m)$. **[Duplication Formula]**

**OR** **Prove Duplication formula.**

**Solution.** By definition of Beta function, we have

$$\beta(m, m) = \int_0^1 x^{m-1} (1 - x)^{m-1}\, dx \qquad \ldots(i)$$

Put $x = \sin^2\theta$, so that $dx = 2\sin\theta\cos\theta\, d\theta$

When $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \frac{\pi}{2}$

$\therefore$ From (i), we have

$$\beta(m, m) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1} (1 - \sin^2\theta)^{m-1} (2\sin\theta\cos\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-2} (\cos\theta)^{2m-2} (2\sin\theta\cos\theta)\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1} (\cos\theta)^{2m-1}\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^{2m-1}\, d\theta = 2\int_0^{\frac{\pi}{2}} \left(\frac{\sin 2\theta}{2}\right)^{2m-1} d\theta$$

$$= \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} (\sin 2\theta)^{2m-1}\, d\theta \qquad \text{....(i}$$

Put $2\theta = t$ so that $d\theta = \frac{dt}{2}$, When $\theta = 0, t = 0$ and when $\theta = \frac{\pi}{2}, t = \pi$

$\therefore$ From (ii), we have $\beta(m,m) = \frac{2}{2^{2m-1}} \int_0^{\pi} (\sin t)^{2m-1} \cdot \frac{dt}{2}$

or
$$\frac{\Gamma(m)\,\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \int_0^{\pi} (\sin t)^{2m-1}\, dt$$

$$= \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} (\sin t)^{2m-1}\, dt \qquad \left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ if } f(2a-x) = f(x)\right]$$

$$= \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} (\sin t)^{2m-1} (\cos t)^0\, dt$$

$$= \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} (\sin t)^{2m-1} (\cos t)^{2\left(\frac{1}{2}\right)-1}\, dt = \frac{2}{2^{2m-1}} \beta\left(m, \frac{1}{2}\right)$$

$$\therefore \quad \frac{\Gamma(m)\,\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \frac{\Gamma(m)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2m+1}{2}\right)} \quad \text{or} \quad \frac{\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\therefore \quad \Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma(2m)$$

**EXAMPLE 1.6. Show that :**

**(i)** $\displaystyle \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta \times \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}} = \pi$ **(ii)** $\displaystyle \int_a^b (x-a)^m (b-x)^n\, dx = (b-a)^{m+n+1} \beta(m+1, n+$

**Solution. (i)** $\displaystyle \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta \times \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}} = \int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{1}{2}}\, d\theta \cdot \int_0^{\frac{\pi}{2}} (\sin\theta)^{-\frac{1}{2}}\, d\theta$

$$= \int_0^{\frac{\pi}{2}} (\sin\theta)^{2\left(\frac{3}{4}\right)-1} (\cos\theta)^{2\left(\frac{1}{2}\right)-1} d\theta \cdot \int_0^{\frac{\pi}{2}} (\sin\theta)^{2\left(\frac{1}{4}\right)-1} (\cos\theta)^{2\left(\frac{1}{2}\right)-1} d\theta$$

$$= \frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{2}\right) \cdot \frac{1}{2}\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

$$= \left[\frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)}\right] \cdot \left[\frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}\right] = \left[\frac{1}{2}\frac{\sqrt{\pi}\ \Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}\right]\left[\frac{1}{2}\frac{\sqrt{\pi}\ \Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right]$$

$$= \left[\frac{1}{2}\frac{\sqrt{\pi}\ \cancel{\Gamma\left(\frac{3}{4}\right)}}{\frac{1}{4}\cancel{\Gamma\left(\frac{1}{4}\right)}}\right]\left[\frac{1}{2}\frac{\sqrt{\pi}\ \cancel{\Gamma\left(\frac{1}{4}\right)}}{\cancel{\Gamma\left(\frac{3}{4}\right)}}\right] = \pi$$

**(ii)** $\int_a^b (x-a)^m (b-x)^n\, dx$

Put $(x-a) = y \Rightarrow x = a+y$ so that $dx = dy$ when $x = a$, $y = 0$ and $x = b$, $y = b-a$

$$\therefore \quad \int_a^b (x-a)^m (b-x)^n\, dx = \int_0^{b-a} y^m (b-a-y)^n\, dy$$

Put $y = (b-a)\sin^2\theta$

So, that $dy = 2(b-a)\ \sin\theta\ \cos\theta\ d\theta$, when $y = 0$, $\theta = 0$ and $y = b-a$, $\theta = \frac{\pi}{2}$

$$\therefore \quad \int_a^b (x-a)^m (b-x)^n\, dx = \int_0^{\frac{\pi}{2}} (b-a)^m \sin^{2m}\theta (b-a)^n \cos^{2n}\theta\ 2(b-a)\sin\theta\cos\theta\ d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (b-a)^{m+n+1} (\sin\theta)^{2m+1} (\cos\theta)^{2n+1}\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (b-a)^{m+n+1} (\sin\theta)^{2(m+1)-1} (\cos\theta)^{2(n+1)-1}\, d\theta = (b-a)^{m+n+1}\ \beta(m+1, n+1)$$

**EXAMPLE 1.7. Express the integral $\int_0^1 x^m (1-x^n)^p\, dx$ in terms of gamma functions and evaluate $\int_0^1 x^5 (1-x^3)^{10}\, dx$.** **[K.U. 2005, 2011, 2014]**

**Solution.** $\int_0^1 x^m(1-x^n)^p\,dx$

Put $x^n = z \Rightarrow x = z^{1/n}$ so that $dx = \frac{1}{n}z^{\frac{1}{n}-1}\,dz$

When $x = 0,\ z = 0$ and $x = 1,\ z = 1$

$$\therefore \quad \int_0^1 x^m(1-x^n)^p\,dx = \int_0^1 z^{m/n}(1-z)^p \frac{1}{n} z^{\frac{1}{n}-1}\,dz = \frac{1}{n}\int_0^1 z^{\frac{m}{n}+\frac{1}{n}-1}(1-z)^p\,dz$$

$$= \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right) = \frac{1}{n}\frac{\left\lceil\left(\frac{m+1}{n}\right)\right. \left\lceil(p+1)\right.}{\left\lceil\left(\frac{m+1}{n}+p+1\right)\right.} \quad \text{....(i)}$$

Now, find the value of $\int_0^1 x^5(1-x^3)^{10}\,dx$ compare with $\int_0^1 x^m(1-x^n)^p\,dx$, we get $m = 5, n = 3,$ $p = 10$

Put the value of $m, n, p$ in equation (i), we get

$$\int_0^1 x^5(1-x^3)^{10}\,dx = \frac{1}{3}\frac{\left\lceil\left(\frac{5+1}{3}\right)\right. \left\lceil(10+1)\right.}{\left\lceil\left(\frac{5+1}{3}+10+1\right)\right.} = \frac{1}{3}\frac{\lceil 2 \lceil 11}{\lceil 13}$$

$$= \frac{1}{3}\cdot\frac{1\cdot 10!}{12!} = \frac{1}{3}\cdot\frac{10!}{12\times 11\times 10!} = \frac{1}{396}$$

**EXAMPLE 1.8. Prove that** $\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\cdot\Gamma(2n+1)}{2^{2n}\Gamma(n+1)}$. **[K.U. 2006, 2014]**

**Solution.** Since $\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

$$\therefore \quad \beta\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2}{\Gamma(2n+1)} \quad \text{....(i)}$$

Now $\quad \beta\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \int_0^1 x^{n-\frac{1}{2}}\,(1-x)^{n-\frac{1}{2}}\,dx$

Put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$, when $x = 0, \theta = 0$ and $x = 1, \theta = \frac{\pi}{2}$

$$\beta\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\,\cos^{2n-1}\theta\cdot 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2n}\theta \cos^{2n}\theta \, d\theta = 2\int_0^{\frac{\pi}{2}} \left(\frac{\sin 2\theta}{2}\right)^{2n} d\theta$$

$$= \frac{1}{2^{2n-1}}\int_0^{\frac{\pi}{2}} \sin^{2n} 2\theta \, d\theta \qquad \left[\begin{array}{l}\text{Put } \phi = 2\theta \\ d\phi = 2d\theta\end{array} \text{ when } \theta = 0, \phi = 0 \text{ and } \theta = \frac{\pi}{2}, \phi = \pi\right]$$

$$= \frac{1}{2^{2n}}\int_0^{\pi} \sin^{2n}\phi \, d\phi = \frac{2}{2^{2n}}\int_0^{\frac{\pi}{2}} \sin^{2n}\phi \, d\phi = \frac{2}{2^{2n}} \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma\left(\frac{2n+2}{2}\right)} = \frac{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}{2^{2n}\,\Gamma(n+1)}$$

$\therefore$ From equation (i), we have

$$\frac{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}{2^{2n}\,\Gamma(n+1)} = \frac{\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2}{\Gamma(2n+1)} \quad \Rightarrow \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}\,\Gamma(n+1)}$$

**EXAMPLE 1.9. Prove that** $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$. **[K.U. 2008]**

**Solution.** L.H.S. $= \displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}}$

Put $x^4 = z \;\Rightarrow\; x = z^{1/4}$, so that $dx = \frac{1}{4}z^{-3/4}dz$

When $x = 0,\ z = 0$ and $x = 1,\ z = 1$

$$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^1 \frac{1}{4}\frac{z^{-3/4}dz}{\sqrt{1-z}} = \frac{1}{4}\int_0^1 z^{-3/4}(1-z)^{-1/2}\,dz = \frac{1}{4}\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

$$= \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)} \qquad \left[\because B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{1}{4}\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \text{R.H.S.}$$

**EXAMPLE 1.10. Show that** $\displaystyle\int_0^{\pi/2} \sqrt{\cot\theta}\,d\theta = \frac{1}{2}\gamma\left(\frac{1}{4}\right)\gamma\left(\frac{3}{4}\right)$, **where $\gamma$ is a gamma rotation.**

**[K.U. 2007, 2010]**

**Solution.** L.H.S.= $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta\, d\theta$

$$=\frac{1}{2}\beta\left(\frac{1}{4},\frac{3}{4}\right)=\frac{1}{2}\frac{\gamma\left(\frac{1}{4}\right)\gamma\left(\frac{3}{4}\right)}{\gamma\left(\frac{1}{4}+\frac{3}{4}\right)}=\frac{1}{2}\gamma\left(\frac{1}{4}\right)\gamma\left(\frac{3}{4}\right)$$

**EXAMPLE 1.11.** **Prove that** (i) $\beta(m,n)=\beta(m+1,n)+\beta(m,n+1)$ **[K.U. 2003, 2006, 2012**

(ii) $\dfrac{\beta(m+1,n)}{m}=\dfrac{\beta(m,n+1)}{n}=\dfrac{\beta(m,n)}{m+n}$

**Solution. (i)** $\beta(m+1,n)+\beta(m,n+1)=\int_0^1 x^m(1-x)^{n-1}\,dx+\int_0^1 x^{m-1}(1-x)^n\,dx$

$$=\int_0^1 x^m(1-x)^{n-1}[x+1-x]\,dx=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$=\beta(m,n)$$

**(ii)** $\dfrac{\beta(m+1,n)}{m}=\dfrac{\Gamma(m+1)\cdot\Gamma(n)}{m\,\Gamma(m+1+n)}=\dfrac{\cancel{m}\,\Gamma(m)\cdot\Gamma(n)}{\cancel{m}\,\Gamma(m+1+n)}$

$$=\frac{\Gamma(m)\cdot n\,\Gamma(n)}{n\,\Gamma(m+1+n)}=\frac{\Gamma(m)\,\Gamma(n+1)}{n\Gamma(m+1+n)}=\frac{\beta(m,n+1)}{n}$$

Also $\dfrac{\beta(m+1,n)}{m}=\dfrac{1}{m}\dfrac{\Gamma(m+1)\ \Gamma n}{\Gamma(m+1+n)}=\dfrac{1}{m}\dfrac{m\,\Gamma m\,\Gamma n}{(m+n)\,\Gamma m+n}=\dfrac{\Gamma m\,\Gamma n}{(m+n)\Gamma m+n}=\dfrac{\beta(m,n)}{m+n}$

$\therefore \quad \dfrac{\beta(m+1,n)}{m}=\dfrac{\beta(m,n+1)}{n}=\dfrac{\beta(m,n)}{m+n}$

## 1.5 DIRICHLET'S INTEGRAL

The triple integral $\iiint\limits_V x^{p-1}\cdot y^{m-1}z^{n-1}\,dx\,dy\,dz=\dfrac{\Gamma(p)\,\Gamma(m)\,\Gamma(n)}{\Gamma(1+p+m+n)}$

where V is the region given by $x\ge 0,\ y\ge 0,\ z\ge 0,\ x+y+z\le 1$ is called Dirichlet's integral.

**Proof:** Here $V=\{(x,y,z):\ x\ge 0,\ y\ge 0,\ z\ge 0,\ x+y+z\le 1\}$

or $V=\{(x,y,z):\ 0\le x\le 1,\ 0\le y\le 1-x,\ 0\le z\le 1-x-y\}$

Let $I=\iiint\limits_V x^{p-1}y^{m-1}z^{n-1}\,dx\,dy\,dz=\int_0^1\int_0^{1-x}\int_0^{1-x-y} x^{p-1}y^{m-1}z^{n-1}\,dz\,dy\,dx$

$$=\int_0^1\int_0^{1-x} x^{p-1}y^{m-1}\left(\frac{z^n}{n}\right)_0^{1-x-y}dy\,dx=\int_0^1\int_0^{1-x} x^{p-1}y^{m-1}\frac{(1-x-y)^n}{n}\,dy\,dx \quad ....(i$$

Put $y=(1-x)t$, so that $dy=(1-x)\,dt$

When $y=0$, $t=0$, when $y=1-x$, $t=1$

$\therefore$ From equation (i), we have

$$I=\int_0^1\int_0^1 x^{p-1}(1-x)^{m-1}t^{m-1}\frac{[1-x+(1-x)t]^n}{n}(1-x)\,dt\,dx$$

$$=\int_0^1\int_0^1 x^{p-1}(1-x)^{m-1}t^{m-1}\frac{[(1-x)(1-t)]^n}{n}(1-x)\,dt\,dx$$

$$=\frac{1}{n}\int_0^1\int_0^1 x^{p-1}(1-x)^{m+n}t^{m-1}(1-t)^n\,dt\,dx$$

$$=\frac{1}{n}\int_0^1 x^{p-1}(1-x)^{m+n}\beta(m,n+1)\,dx$$

$$=\frac{1}{n}\beta(m,n+1)\int_0^1 x^{p-1}(1-x)^{m+n}\,dx=\frac{1}{n}\beta(m,n+1)\beta(p,m+n+1)$$

$$=\frac{1}{n}\frac{\Gamma(m)\Gamma(n+1)}{\cancel{\Gamma(m+n+1)}}\cdot\frac{\Gamma(p)\cdot\cancel{\Gamma(m+n+1)}}{\Gamma(p+m+n+1)}=\frac{1}{n}\frac{\Gamma(m)\Gamma(n+1)\Gamma(p)}{\Gamma(1+p+m+n)}$$

$$=\frac{1}{n}\frac{\Gamma(m)\cdot n\Gamma(n)\cdot\Gamma(p)}{\Gamma(1+p+m+n)}=\frac{\Gamma(p)\Gamma(m)\Gamma(n)}{\Gamma(1+p+m+n)}$$

## 1.5.1 Theorem

Prove that $\iint\limits_D x^{m-1}y^{n-1}\,dx\,dy=\frac{\Gamma(m)\Gamma(n)}{\Gamma(1+m+n)}\cdot h^{m+n}$, where D is the region $x\ge 0$, $y\ge 0$ and $x+y\le h$.

**Proof:** Let the transformation be $x = hu$, $y = hv$ so that $dx = h\,dy$, $dy = h\,dv$ and $dx\,dy = h^2 dx\,dv$

Also $x+y\le h$ becomes $hu+hv\le h$ i.e. $u+v\le 1$.

The region D in $xy$-plane becomes the region D′ in $uv$-plane where

$$D'=\{(u,v): u\ge 0,\ v\ge 0 \text{ and } u+v\le 1\}$$

$$\text{Given integral}=\iint\limits_D x^{m-1}y^{n-1}\,dx\,dy=\iint\limits_{D'}(hu)^{m-1}(hv)^{n-1}h^2\,du\,dv$$

$$=h^{m+n}\int_0^1\int_0^{1-u}u^{m-1}v^{n-1}\,dv\,du=h^{m+n}\int_0^1 u^{m-1}\left(\frac{v^n}{n}\right)_0^{1-u}du$$

$$=\frac{h^{m+n}}{n}\int_0^1 u^{m-1}(1-u)^n\,du=\frac{h^{m+n}}{n}\beta(m,n+1)$$

$$=\frac{h^{m+n}}{n}\frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}=\frac{h^{m+n}\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)}\qquad[\because \Gamma(m+1)=n\,\Gamma(n)]$$

Thus we have $\displaystyle\iint\limits_{x+y\le h} x^{m-1}y^{n-1}\,dx\,dy = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)}\cdot h^{m+n}$

**EXAMPLE 1.12. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in A, B, C. Apply Dirichlet's integral to find volume of the tetrahedron OABC.**

**Solution.** The given equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Put $x = au,\ y = bv,\ z = cw$ so that

$dx = a\,du,\ dy = b\,dv,\ dz = c\,dw$

$\therefore$ $dx\,dy\,dz = abc\,du\,dv\,dw$

The conditions are $u \ge 0,\ v \ge 0,\ w \ge 0$ and $u+v+w \le 1$

$\therefore$ Volume of tetrahedron OABC

$$= \iiint\limits_{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\le 1} 1\,dx\,dy\,dz = \iiint\limits_{u+v+w\le 1} (abc)\,du\,dv\,dw$$

$$= abc \iiint\limits_{u+v+w\le 1} u^{1-1}\,v^{1-1}\,w^{1-1}\,du\,dv\,dw$$

Applying Dirichlet's integral, we have

$$\text{Volume of tetrahedron} = abc\frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1+1)} = \frac{abc}{\Gamma(4)} = \frac{abc}{3!} = \frac{abc}{6}$$

## EXERICSE 1.1

***Show that:***

1. $\displaystyle\int_0^\infty e^{-4x}x^{3/2}\,dx = \frac{3}{128}\sqrt{\pi}$

2. $\displaystyle\int_{-\infty}^{\infty} e^{-k^2x^2}\,dx = \frac{\sqrt{\pi}}{k}$

3. $\displaystyle\int_0^\infty e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a}$

4. $\displaystyle\int_0^\infty \sqrt{x}\,e^{-x^3}\,dx = \frac{\sqrt{\pi}}{3}$

5. $\displaystyle\int_0^\infty \frac{x^c}{c^x}\,dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}},\ c>1$

6. $\displaystyle\int_0^\infty \frac{e^{-\sqrt{x}}}{x^{7/4}}\,dx = \frac{8}{3}\sqrt{\pi}$

7. $\displaystyle\int_0^\infty x^{p-1}e^{-kx}\,dx = \frac{\Gamma(p)}{k^p}$

***Prove that:***

8. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{3}\right)}{3\Gamma\left(\frac{5}{6}\right)}$

9. $\displaystyle\int_0^3 \frac{dx}{\sqrt{3x-x^2}} = \beta\left(\frac{1}{2},\frac{1}{2}\right) = \pi$

10. $\int_0^1 x^4\left(\log\frac{1}{x}\right)^3 dx=\frac{6}{625}$ [M.D.U.2004]

11. $\int_0^1 \frac{x\,dx}{\sqrt{1-x^5}}=\frac{1}{5}\,\beta\left(\frac{2}{5},\frac{1}{2}\right)$

12. $\int_0^1 x^2(1-x)^3\,dx=\frac{1}{60}$

13. $\int_0^1 x^4(1-x)^3\,dx=\frac{1}{280}$

14. Prove that $\int_0^1 x^m(\log x)^n\,dx=\frac{(-1)^n\,n!}{(m+1)^{n+1}}$, where $n$ is a positive integer and $m>-1$.

15. Show that $\Gamma(n)\,\Gamma(1-n)=\frac{\pi}{\sin n\pi}$, given $\int_0^\infty \frac{x^{n-1}}{1+x}dx=\frac{\pi}{\sin n\pi}$

16. Prove that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}dx=\frac{\beta(m,n)}{a^n(a+b)^m}$ [Hint. Put $\frac{x}{a+bx}=\frac{z}{a+b}$ ]

17. (a) Show that $\frac{\beta(m+1,n)}{\beta(m,n)}=\frac{m}{m+n}$

(b) $\frac{\beta(p,q+1)}{q}=\frac{\beta(p+1,q)}{p}=\frac{\beta(p,q)}{p+q}$

18. Prove that $\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}}\times\int_0^1 \frac{dx}{\sqrt{1+x^4}}=\frac{\pi}{4\sqrt{2}}$ (M.D.U. 2008)

[Hint : Put $x^2=\sin\theta$ in 1st integral and $x^2=\tan\theta$ in 2nd integral. ]

19. Evaluate the integral $\iiint x^{l-1}\,y^{m-1}\,z^{n-1}\,dx\,dy\,dz$, where $x$, $y$, $z$ are always positve but limited by the condition $\left(\frac{x}{a}\right)^p+\left(\frac{y}{b}\right)^q+\left(\frac{z}{c}\right)^r\le 1$

20. Show that $\iint x^{m-1}\,y^{n-1}\,dx\,dy$ over the positive octant of the ellipse $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$ is $\frac{a^m b^n}{2n}\cdot\beta\left(\frac{m}{2},\frac{n}{2}+1\right)$

21. Show that the area in the first quadrant enclosed by the curve $\left(\frac{x}{a}\right)^{\alpha}+\left(\frac{y}{b}\right)^{\beta}=1, \alpha>0, \beta>0$ i

given by $\dfrac{ab}{\alpha+\beta}\cdot\dfrac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\beta}\right)}{\Gamma\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)}$

## Answers

19. $\dfrac{a^l b^m c^n}{pqr}=\dfrac{\Gamma\left(\frac{l}{p}\right)\Gamma\left(\frac{m}{q}\right)\Gamma\left(\frac{n}{r}\right)}{\Gamma\left(1+\frac{l}{p}+\frac{m}{q}+\frac{n}{r}\right)}$