# 4

# Relations

## BINARY RELATION

Let P and Q be two non-empty sets. A binary relation R is defined to be a subset of P × Q from a set P to Q. If $(a, b) \in R$ and $R \subseteq P \times Q$ then $a$ is related to $b$ by R *i.e.*, $aRb$. If sets P and Q are equal, then we say $R \subseteq P \times P$ is a relation on P *e.g.*,

(*i*) Let $A = \{a, b, c\}$

$B = \{r, s, t\}$

Then $R = \{(a, r), (b, r), (b, t), (c, s)\}$

is a relation from A to B.

(*ii*) Let $A = \{1, 2, 3\}$ and $B = A$

$R = \{(1, 1), (2, 2), (3, 3)\}$

is a relation (equal) on A.

**Example 1.** *If a set A has n elements, how many relations are there from A to A.*

**Sol.** If a set A has $n$ elements, A × A has $n^2$ elements. So, there are $2^{n^2}$ relations from A to A.

**Example 2.** *If A has m elements and B has n elements. How many relations are there from A to B and vice-versa ?*

**Sol.** There are $m \times n$ elements, hence there are $2^{m \times n}$ relations from A to B.

**Example 3.** *If a set A = {1, 2}. Determine all relations from A to A.*

**Sol.** There are $2^2 = 4$ elements *i.e.*, {(1, 2), (2, 1), (1, 1), (2, 2)} in A × A. So, there are $2^4 = 16$ relations from A to A. *i.e.*,

{(1, 2), (2, 1), (1, 1), (2, 2)}, {(1, 2), (2, 1)}, {(1, 2), (1, 1)}, {(1, 2), (2, 2)}, {(2, 1), (1, 1)}, {(2, 1), (2, 2)}, {(1, 1), (2, 2)}, {(1, 2), (2, 1), (1, 1)}, {(1, 2), (1, 1), (2, 2)}, {(2, 1), (1, 1), (2, 2)}, {(1, 2), (2, 1), (2, 2)}, {(1, 2), (2, 1), (1, 1), (2, 2)} and $\phi$.

## DOMAIN OF RELATION

The domain of relation R is the set of elements in P which are related to some element in Q or it is the set of all first entries of the ordered pairs in R. It is denoted by DOM (R).

## RANGE OF RELATION

The range of a relation R is the set of elements in Q which are related to some elem in P or it is the set of all second entries of the ordered pairs in R. It is denoted by RAN (R).

**For example** : Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$ and $R = \{(1, a), (1, b), (1, c), (2, b), (2, d)\}$.

Then $\quad$ $DOM(R) = \{1, 2\}$

$RAN(R) = \{a, b, c, d\}$

## COMPLEMENT OF A RELATION

Consider a relation R from a set A to B. The complement of relation R denoted by $\overline{R}$ i relation from A to B such that

$$\overline{R} = \{(a, b) : (a, b) \notin R\}.$$

**Example 4.** *Consider the relation R from X to Y*

$$X = \{1, 2, 3\} ; Y = \{8, 9\} \text{ and } R = \{(1, 8), (2, 8), (1, 9), (3, 9)\}.$$

*Find the complement of relation R.*

**Sol.** First we find the universal relation X × Y *i.e.,*

$$X \times Y = \{(1, 8), (2, 8), (3, 8), (1, 9), (2, 9), (3, 9)\}$$

Now, we find the complement relation $\overline{R}$ w.r.t. X × Y

$$\overline{R} = \{(3, 8), (2, 9)\}.$$

**Example 5.** *Let $A = \{7, 8, 9\}$ and $B = \{k, l, m, n\}$ and R is relation from A to B.*

$$R = \{(7, k), (8, k), (8, l), (8, m), (9, m), (9, n)\}.$$

*Determine its complement.*

**Sol.** The relation A × B from the set A to B is

$$A \times B = \{(7, k), (7, l), (7, m), (7, n), (8, k), (8, l), (8, m), (8, n), (9, k), (9, l), (9, m), (9, n\}$$

The complement of relation R w.r.t. A × B is

$$\overline{R} = \{(7, l), (7, m), (7, n), (8, n), (9, k), (9, l)\}.$$

## INVERSE OF A RELATION

Consider a relation R from a set A to B. The inverse of a relation R, denoted by $R^{-1}$, is relation from B to A such that $bR^{-1}a$ iff $aRb$ *i.e.,*

$$R^{-1} = \{(b, a) : (a, b) \in R\}.$$

**Example 6.** *Consider the relation '≤' on the set $A = \{2, 3, 4, 5\}$. Determine its inverse.*

**Sol.** The relation '≤' is defined by $(a,b) \in R$ if $a \le b$, $a, b \in A$. Then

$$R = \{(2, 2), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5), (4, 4), (4, 5), (5, 5)\}$$

and $\quad R^{-1} = \{(2, 2), (3, 2), (4, 2), (5, 2), (3, 3), (4, 3), (5, 3), (4, 4), (5, 4), (5, 5)\}.$

**Example 7.** *Consider the following relation R on the set of +ve integers. Find its inverse*

$$R = \{(1, 1), (1, 2), (1, 3), (2, 1), (3, 1) (3, 2), (2, 3)\}.$$

**Sol.** The inverse of relation R is

$$R^{-1} = \{(1, 1), (2, 1), (3, 1), (1, 2), (1, 3), (2, 3), (3, 2)\}.$$

## REPRESENTATION OF RELATIONS

Relations can be represented in many ways. Some of which are as follows :

**1. Relation as a Matrix.** Let $P = \{a_1, a_2, ...., a_m\}$ and $Q = \{b_1, b_2, ...., b_n\}$ are finite sets, containing $m$ and $n$ number of elements respectively. R is a relation from P to Q. The relation R can be represented by $m \times n$ matrix $M = [M_{ij}]$, defined as

$$M_{ij} = \begin{cases} 0 & \text{if } (a_i, b_j) \notin R \\ 1 & \text{if } (a_i, b_j) \in R \end{cases}$$

*e.g.,* Let $P = \{1, 2, 3, 4\}$, $Q = \{a, b, c, d\}$

and $R = \{(1, a), (1, b), (1, c), (2, b), (2, c), (2, d)\}$.

The matrix of the relation R is shown in Fig. 1.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Fig. 1.

**2. Relation as a Directed Graph.** If P is a finite set and R is a relation on P, R can be represented as a directed graph as follows :

Draw a small circle for each element of P and label the circle with corresponding element of P. These circles are called vertices. Draw a directed line from vertex $a_i$ to vertex $a_j$ if $a_i$ R $a_j$. These directed lines are called edges. The obtained picture is the directed graph.

**For example :** Let $P = \{1, 2, 3, 4\}$

and $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 2), (3, 4), (4, 3), (4, 4)\}$

which is a relation on P.

Then the directed graph of relation R is shown in Fig. 2.

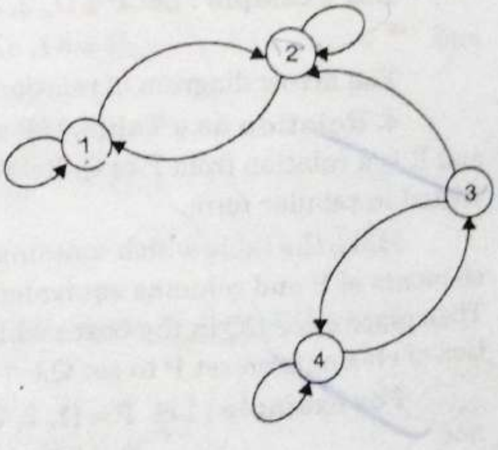


Fig. 2.

From Fig. 2, we see that the edges in the directed graph of R corresponds to the pairs in R, and the vertices correspond to the elements in the set P.

**Example 8.** *Find the relations determined by the following figures.*

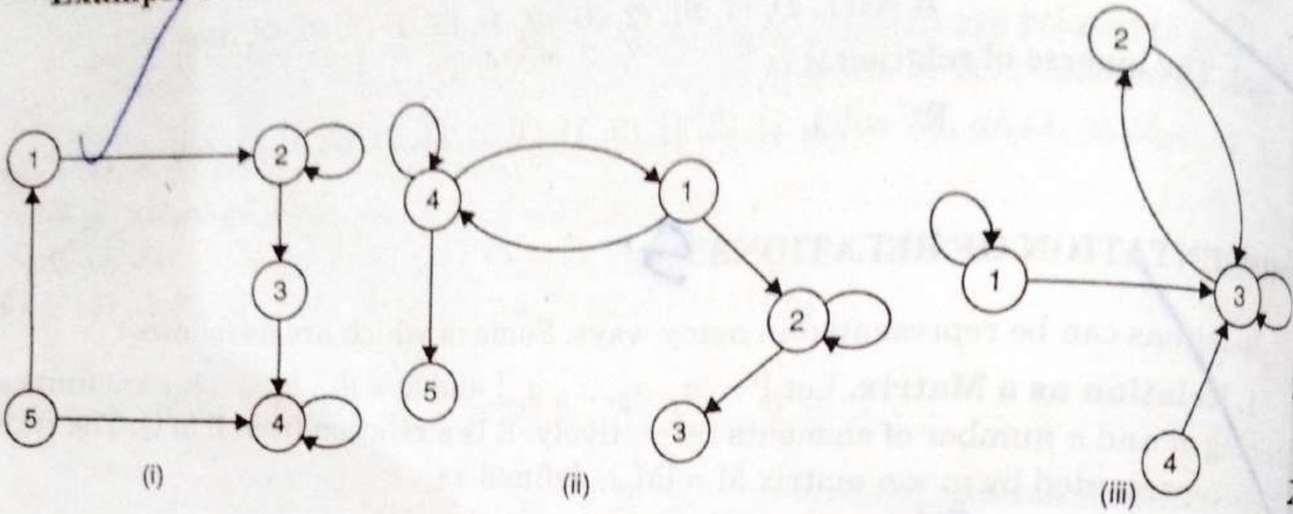



Fig. 3.

**Sol.** The relations determined by above figures are as follows :

(*i*) R = {(1, 2), (2, 2), (2, 3), (3, 4), (4, 4), (5, 4), (5, 1)}

(*ii*) R = {(1, 2), (1, 4), (2, 2), (2, 3), (4, 1), (4, 4), (4, 5)}

(*iii*) R = {(1, 1), (1, 3), (2, 3), (3, 2), (3, 3), (4, 3)}.

**3. Relation as an Arrow Diagram.** If P and Q are finite sets and R is a relation from P to Q. Relation R can be represented as an arrow diagram as follows.

Draw two ellipses for the sets P and Q. Write down the elements of P and elements of Q columnwise in three ellipses. Then draw an arrow from first ellipse to second ellipse if $a$ is related to $b$ and $a \in$ P and $b \in$ Q.

**For example :** Let P = {1, 2, 3, 4}, Q = {$a$, $b$, $c$, $d$}

and R = {(1, $a$), (2, $a$), (3, $a$), (1, $b$), (4, $b$), (4, $c$), (4, $d$)}

The arrow diagram of relation R is shown in Fig. 4.

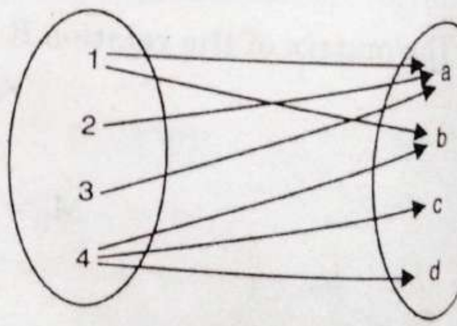


Fig. 4.

**4. Relation as a Table.** If P and Q are finite sets and R is a relation from P to Q. Relation R can be represented in tabular form.

Make the table which contains rows equivalent to elements of P and columns equivalent to elements of Q. Then place cross (X) in the boxes which represents relation of elements on set P to set Q.

**For example :** Let P = {1, 2, 3, 4}, Q = {$x$, $y$, $z$, $k$}

and R = {(1, $x$), (1, $y$), (2, $z$), (3, $z$), (4, $k$)}.

The tabular form of relation is as shown in Fig. 5.

| | x | y | z | k |
|---|---|---|---|---|
| 1 | × | × | | |
| 2 | | | × | |
| 3 | | | × | |
| 4 | | | | × |

Fig. 5.

## COMPOSITION OF RELATIONS

Consider a relation R ...

**Example 9.** *Let P and Q be the relations on set A = {1, 2, 3, 4} defined by*

$P = \{(1, 2), (2, 2), (2, 3), (2, 4), (3, 2), (4, 2), (4, 3)\}$

*and* $Q = \{(2, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2)\}$

*Find (i) P o P (ii) P o Q (iii) P o P o Q.*

**Sol.** (i) P o P = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (2, 2), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)}

(ii) P o Q = {(1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (2, 1), (3, 2), (3, 3), (4, 2), (4, 3), (4, 4)}

(iii) P o P = {(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (4, 2), (3, 2), (3, 4), (4, 3), (4, 4)}

P o P o Q = {(1, 2), (1, 3), (1, 4), (1, 1), (2, 2), (2, 3), (2, 4), (2, 1), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3)}.

**Example 10.** *Let X = {4, 5, 6}, Y = {a, b, c} and Z = {l, m, n}. Consider the relation $R_1$ from X to Y and $R_2$ from Y to Z.*

$$R_1 = \{(4, a), (4, b), (5, c), (6, a), (6, c)\}$$
$$R_2 = \{(a, l), (a, n), (b, l), (b, m), (c, l), (c, m), (c, n)\}$$

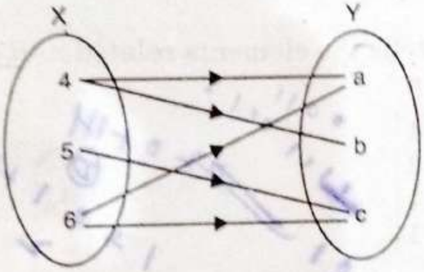



Fig. 6.

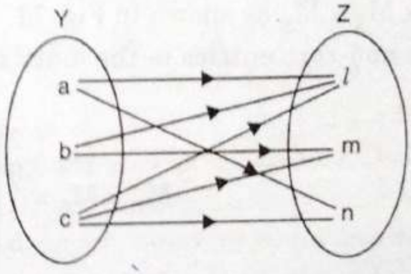



Fig. 7.

*Find the composition relation (i) $R_1$ o $R_2$ (ii) $R_1$ o $R_1^{-1}$.*

**Sol.** (i) The composition relation $R_1$ o $R_2$ is as shown in Fig. 8.

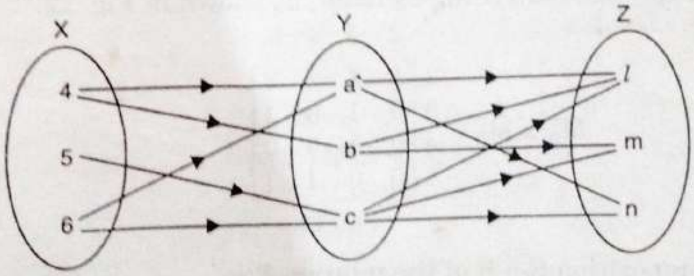



Fig. 8. $R_1$ o $R_2$.

$$R_1 \text{ o } R_2 = \{(4, l), (4, n), (4, m), (5, l), (5, m), (5, n), (6, l), (6, m), (6, n)\}$$

(ii) The composition relation $R_1$ o $R_1^{-1}$ is shown in Fig. 9.

4a

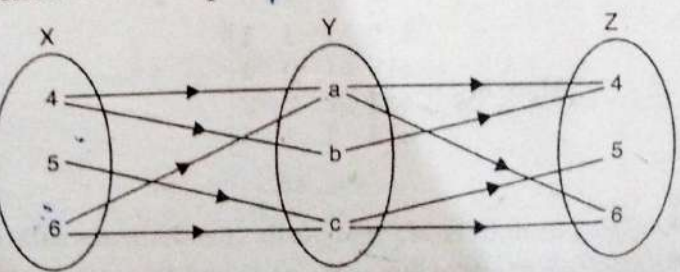



Fig. 9. $R_1$ o $R_1^{-1}$.

$$R_1 \text{ o } R_1^{-1} = \{(4, 4), (5, 5), (5, 6), (6, 4), (6, 5), (4, 6), (6, 6)\}.$$

**Example 11.** *Let* $P = \{2, 3, 4, 5\}$. *Consider the relation R and S on P defined by*

$$R = \{(2, 2), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5), (5, 3)\}$$

*and* $$S = \{(2, 3), (2, 5), (3, 4), (3, 5), (4, 2), (4, 3), (4, 5), (5, 2), (5, 5)\}.$$

*Find the matrices of the above relations.*

*Use matrices to find the following compositions of the relations R and S.*

(*i*) $R \circ S$ (*ii*) $R \circ R$ (*iii*) $S \circ R$.

**Sol.** The matrices of the relations R and S are as shown in Fig. 10.

$$M_R = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad M_S = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Fig. 10.

(*i*) To obtain the composition of relations R and S. First multiply $M_R$ with $M_S$ to obtain the matrix $M_R \times M_S$ as shown in Fig. 11.

The non-zero entries in the matrix $M_R \times M_S$ tells the elements related in R o S. So,

$$M_R \times M_S = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 2 & 2 & 1 & 4 \\ 2 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Fig. 11.

Hence, the composition R o S of the relation R and S is

$$R \circ S = \{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (4, 2), (4, 5), (5, 2), (5, 3), (5, 4), (5, 5)\}.$$

(*ii*) First, multiply the matrix $M_R$ by itself, as shown in Fig. 12.

$$M_R \times M_S = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Fig. 12.

Hence, the composition R o R of the relation R is

$$R \circ R = \{(2, 2), (3, 2), (3, 3), (3, 4), (4, 2), (4, 5), (5, 2), (5, 3), (5, 5)\}.$$

(*iii*) Multiply the matrix $M_S$ with $M_R$ to obtain the matrix $M_S \times M_R$ as shown in Fig. 13.

$$M_S \times M_R = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Fig. 13.

The non-zero entries in matrix $M_S \times M_R$ tells the elements related in S o R.

Hence, the composition S o R of the relation S and R is

$$S \circ R = \{(2, 4), (2, 5), (3, 3), (3, 4), (3, 5), (4, 2), (4, 4), (4, 5) \ldots$$

## PATHS IN RELATIONS

Consider a relation R on set A. A path of length $n$ in R from $a$ to $b$, is a finite sequence $a, Y_1, Y_2, Y_3, \ldots\ldots, Y_{n-1}, b$ which begins with $a$ and ends with $b$ such that

$$aRY_1, Y_1RY_2, Y_2RY_3, \ldots\ldots, Y_{n-1}Rb$$

The path which has length $n$ must have $n + 1$ elements of A. The elements may be distinct or same.

A path that begins and ends at the same vertex is called a **cycle.**

As we have seen earlier that a relation can be uniquely represented by a directed graph. The path in a directed graph is a sequence of edges in the indicated direction. Hence, the path length is the number of edges in the path.

**Example.** *Consider the graph shown in Fig. 13(i).*

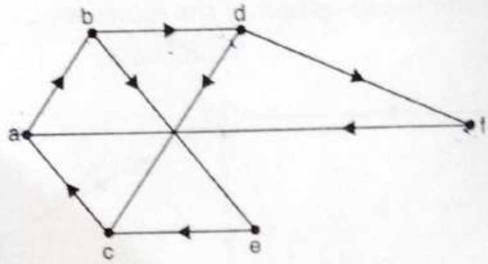



Fig. 13(*i*).

The graph has a path $P_1 = a, b, d, f$ of length 3.
The graph has a path $P_2 = a, b, e, c, d, f$ of length 5.
The graph has a cycle $P_3 = a, b, d, f, a$ of length 4.
The graph has a cycle $P_4 = a, b, e, c, d, f, a$ of length 6.

**Example 12.** *Consider the directed graph as shown in Fig. 13 (ii).*

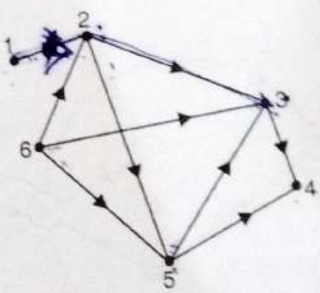



Fig. 13(*ii*).

*Determine all the paths of length 1 and 2.*

**Sol.** Paths that have length 1 are as follows :

$P_1 = 1, 2$ $\quad P_2 = 2, 3$ $\quad P_3 = 3, 4$ $\quad P_4 = 5, 4$
$P_5 = 6, 5$ $\quad P_6 = 6, 3$ $\quad P_7 = 2, 5$ $\quad P_8 = 5, 3$ $\quad P_9 = 6, 2.$

Paths that have length 2 are as follows :

$P_{10} = 1, 2, 3$ $\quad P_{11} = 6, 2, 3$ $\quad P_{12} = 6, 3, 4$ $\quad P_{13} = 6, 5, 3$
$P_{14} = 6, 5, 4$ $\quad P_{15} = 2, 5, 4$ $\quad P_{16} = 2, 5, 3$ $\quad P_{17} = 2, 3, 4$ $\quad P_{18} = 6, 2, 5.$

Paths in a relation R can be used to define following relation on the set A.

$\mathbf{R}^n$. The relation $R^n$ on the set A is defined as a path of length $n$ from $a$ to $b$ in R denoted by $R^n\, b$.

$\mathbf{R}^\infty$. The relation $R^\infty$ on the set A is defined as some path from $a$ to $b$ in R. The le the path will depend on $a$ and $b$. It is also called the connectivity relation for R.

$\mathbf{R}^n(x)$. It is defined as a set consisting of all the vertices that can be reached fro means of a path in R of length $n$.

$\mathbf{R}^\infty(x)$. It is defined as a set consisting of all the vertices that can be reached fro some path in R.

$\mathbf{R}^*$. Consider a set A having $n$ elements and let R is a relation on A. Then relatio R is defined as a relation such that $aR^*b$ if $a = b$ or $a\ R^\infty b$. It is also called the reach relation. The meaning of this is that $b$ is reachable from $a$ if either $b = a$ or there is som from $a$ to $b$.

$R^* = \{(a,b) : a = b \text{ or } aR^\infty b\}$

**Example 13.** *Consider the set A = {a, b, c, d, e}. Let R be the relation whose diagra shown in Fig. 13(iii). Determine the diagraph of the relation*

*(i)* $R^2$ *(ii)* $R^3$ *on A.*

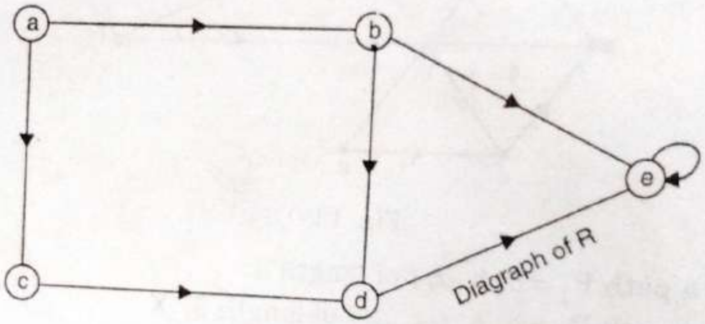



Fig. 13(*iii*).

**Sol.** (*i*) To obtain the diagraph of the relation $R^2$, we will find all lines that connects t vertices in $R^2$, when there is a path of length two connecting those vertices in R. The diagra of $R^2$ is shown in Fig. 13(*iv*).

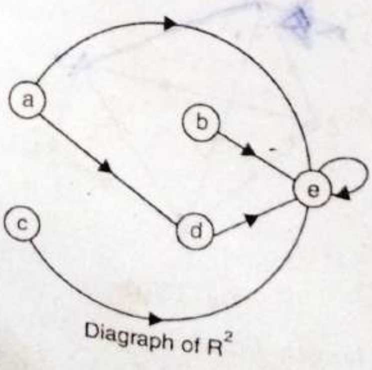
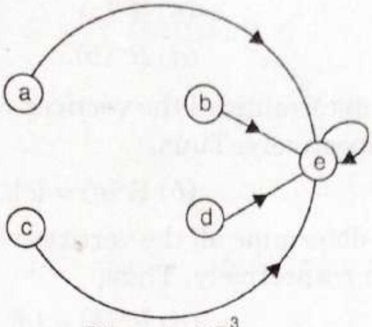



Fig. 13(*iv*).

(*ii*) To obtain the diagraph of the relation $R^3$, we will find all paths of length 3 in R from any vertex to any other vertex and then join those vertices in $R^3$ by a direct line. The diagraph of $R^3$ is shown in Fig. 13(*v*).




Diagraph of $R^3$

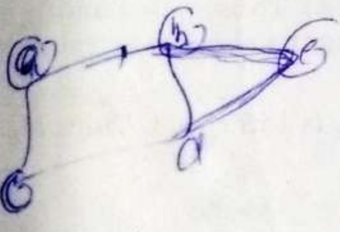
Fig. 13($v$).

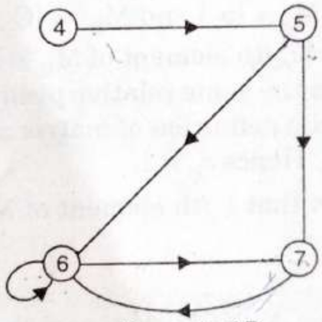
**Example 14.** *Consider the set A = {4, 5, 6, 7} and the relation R on A is given by*

$$R = \{(4, 5), (5, 6), (5, 7), (6, 6), (6, 7), (7, 6), (7, 7)\}.$$

*Determine* ($a$) $R^3$ ($b$) $R^\infty$.

**Sol.** The diagraph of R is shown in Fig. 13($vi$).






6, 6, 7, 6

Diagraph of R

Fig. 13($vi$)

($a$) Now find all the paths of length 3 from any vertex to any other vertex in R and write these vertices by pair vertices in $R^3$. Thus, the $R^3$ is given by

$$R^3 = \{(4, 6), (4, 7), (5, 6), (5, 7), (6, 6), (6, 7), (6, 7), (7, 7)\}.$$

($b$) To determine $R^\infty$, find all the ordered pairs of vertices for which there is a path of any length from any vertex to any other vertex of the set A. Thus, $R^\infty$ is given by

$$R^\infty = \{(4, 5), (4, 6), (4, 7), (5, 6), (5, 7), (6, 6), (6, 7), (7, 6), (7, 7)\}.$$

**Example 15.** *Consider the diagraph of the relation R on set A = {a, b, c, d, e, f, g} as shown in Fig. 13(vii).*

c, d, f, g

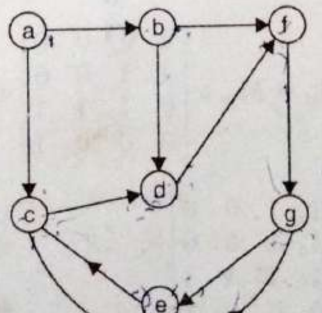



Fig. 13($vii$).

Determine (a) $R^2(d)$ (b) $R^2(g)$

(c) $R^\infty(e)$ (d) $R^\infty(b)$.

**Sol.** To find $R^2(d)$ and $R^2(g)$ determine all the vertices which can be reached from $d$ by means of a path of length 2 respectively. Thus,

(a) $R^2(d) = \{g\}$ (b) $R^2(g) = \{c\}$

Now to find $R^\infty(e)$ and $R^\infty(b)$, determine all the vertices which can be reached from $e$ by means of a path of some length respectively. Thus,

(c) $R^\infty(e) = \{c, d, f, g\}$ (d) $R^\infty(b) = \{d, f, g, e, c\}$.

**Computation of $M_R{}^2$ using $M_R$.** Consider a relation R on the set $A = \{a_1, a_2, ...$ Also let $M_R$ be the matrix representation of R, we can compute the matrix $M_{R^2}$ from $M_R$.

**Theory.** *Show that if R is a relation on the set $A = \{a_1, a_2, \ldots, a_n\}$ and $M_R$ is the corresponding matrix of R, Then $M_{R^2} = M_R \otimes M_R$ {The symbol $\otimes$ means not an ordinary product}.*

**Proof.** Let us consider that $M_R = [b_{ij}]$ and $M_R{}^2 = [C_{ij}]$

Now from definition, we have $i, j$th element of $M_R \otimes M_R$ is equal to 1 iff row $i$ of first and column $j$ of second $M_R$ have 1 in the same relative position (say $k$). Thus, $b_{ik} = 1$ and $b_{kj} = 1$ for some $k$, where $1 \le k \le n$. From the definition of matrix and using the above constraint we have $a_i R a_k$ and $a_k R a_j$. Thus, $a_i R^2 a_j$. Hence $c_{ij} = 1$.

In the above, we have shown that $i, j$th element of $M_R \otimes M_R$ is 1 iff $c_{ij} = 1$. Thus $M_R = M_{R^2}$.

**Example 16.** *Consider the set $A = \{1, 2, 3, 4\}$ and the relation*

*$R = \{(1, 2), (2, 1), (2, 2), (3, 2), (3, 3), (3, 4), (4, 4)\}$ on set A. Determine $M_{R^2}$ from $M_R$.*

**Sol.** The matrix representation of R is given by

$$M_R = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$M_{ij}$ of $R^2$ is 1 iff $a_{ij} = 1$ of $M_R$ & $a_{jk} = 1$ of $M_R$

Now, compute $M_{R^2}$ using $M_R$, which is given as follows :

$a_{12}$

$$M_{R^2} = M_R \otimes M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{R^2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, the result $M_{R^2}$ obtained by using the matrix is same as if we obtain it by using diagraph of R.

**Computation of the Matrix of $R^n$.** Consider a relation R on a finite set A. Also let $n \geq 2$. The matrix $M_{R^n}$ of R is given by

$$M_{R^n} = M_R \circledast M_R \circledast M_R \ldots \circledast M_R \ (n \text{ times})$$

**Computation of the Matrix of $R^\infty$.** Consider a relation R on a finite set A. A vertex is $aR^\infty b$ iff both the vertices are connected by a path in R of length $n$. Thus, $aR^\infty b$ iff $aRb$ or $aR^2b$ or $aR^3b$ or...... Hence,

$$M_{R^\infty} = M_R \vee M_R^2 \vee M_R^3 \vee \ldots\ldots$$

**Note 1.** If R and S are two relations on A, then $R \cup S$ is a relation such that $a(R \cup S)b$ iff $aRb$ or $aSb$.

**2.** The relation $R^\infty$ can be defined also as $R^\infty = R \cup R^2 \cup R^3 \cup R^4 \ldots\ldots = \bigcup_{n=1}^{\infty} R^n$.

**3.** The matrix of $R \cup S$ denoted by $(M_{R \cup S}$ can be defined as $M_{R \cup S} = M_R \vee M_S$

**Computation of the Matrix of $R^*$.** As discussed earlier, we know $R^*$ is a reachability relation, which mean $b$ is reachable from $a$ if either $b = a$ or there is a path from $a$ to $b$. Thus, $M_{R^*}$ is defined as :

$R^* = I_R \cup R^\infty$

$$M_{R^*} = I_n \vee M_{R^\infty}$$

so, $M_{R^*} = I_n \vee M_{R^\infty}$

[Here $I_n$ is the identity matrix of $n \times n$ and $n$ is the number of elements in A]

or putting the value of $M_{R^\infty}$, we have

$$M_{R^*} = I_n \vee M_R \vee M_R^2 \vee M_R^3 \vee \ldots\ldots$$

**Example 17.** *Consider the diagraph of the relation R on set A = {1, 2, 3, 4, 5} as shown in Fig. 13(viii).*

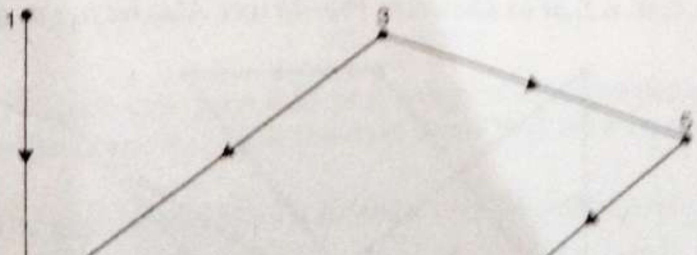



Fig. 13(viii).

$R^\infty = \{$

*Determine the following (a) $R^*$ (b) $M_{R^*}$.*

**Sol.** (*a*) The relation $R^*$ of R on set A is given by

$R^* = \{(1, 1), (1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 3), (3, 4), (3, 5), (4, 4), (5, 4), (5, 5)\}$

The matrix $M_{R^*}$ is obtained by finding the value of $M_{R^-}$ and then ORing with $I_n$ (Identity matrix), which is as follows :

$$M_{R^-_y} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Now,

$$M_{R^*} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

which is same as we obtained from $R^*$.

## COMPOSITION OF PATHS

Consider a relation R on a set A. Let $p_1 : x, a_1, a_2, a_3, \ldots, a_{n-1}, Y$ be a path in the R whose length is $n$ from $x$ to $y$. Also let $p_2 : y, b_1, b_2, b_3, b_4, \ldots, b_{m-1}, z$ be a path in the R whose length is $m$ from $y$ to $z$.

Then, the composition of $p_1$ and $p_2$ denoted $p_2 \circ p_1$ is a path $x, a_1, a_2, a_3, \ldots$ $b_3, \ldots, b_{m-1}, z$ of length $n + m$, which is a path from $x$ to $z$.

**Example 18.** *Consider the diagraph of the relation R on the set* $A = \{a, b, c, d, e, f, g\}$ *as shown in Fig. 13 (ix). Also let* $p_1 : a, d, b, c$ *and* $p_2$ :

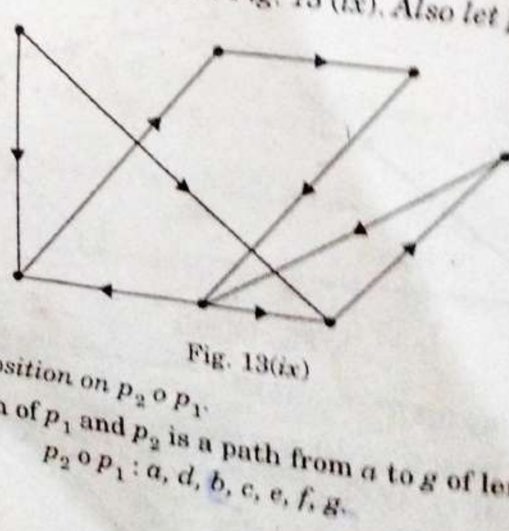


Fig. 13(*ix*)

*Determine the composition on* $p_2 \circ p_1$.

**Sol.** The composition of $p_1$ and $p_2$ is a path from $a$ to $g$ of length 6 given by

$$p_2 \circ p_1 : a, d, b, c, e, f, g.$$

## COMPUTER REPRESENTATION OF RELATIONS

There are two most common methods for starting data items. These are :

1. Array representation
2. Linked list representation.

### 1. Array Representation

It refers to storing consecutive homogeneous data items in sequential storage locations in the memory.

**Example.** *The Fig. 13(x) shows the 10 integers stored in memory in consecutive memory locations.*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $I_0$ | $I_1$ | $I_2$ | $I_3$ | $I_4$ | $I_5$ | $I_6$ | $I_7$ | $I_8$ | $I_9$ |

Fig. 13(*x*).

If A is the name of the array then the first, second, third, ......., data items can be stored and accessed in locations A[0], A[1], A[2], A[3],...., and so on.

**Advantages.** (*i*) We can access any data items $I_i$ by simply supplying its index *i*. So direct access to any item.

(*ii*) Sorting and searching problems are very efficient.

**Disadvantages.** (*i*) Insertions are difficult as we have to move a large number of data items to new positions to accommodate the new data item.

(*ii*) Deletions are also difficult as we have to move a large number of data items to take the position vacated by the deleted item.

(*iii*) Once an array is declared and data is stored we cannot change the size of the array.

### 2. Linked List Representation

Here the data is represented by a linear list of nodes, each one is connected by a pointer or link with another node.

Every element consists of two parts called info and linked.

node

| | |
|---|---|
| info | link |

The info part contains the data item and link part contains the address of the next node. So, the nodes are in linear sequence but the data items that they represent may not be in same sequence.

**Example.** *The Fig. 13(xi) shows a linked list of 10 data items :*

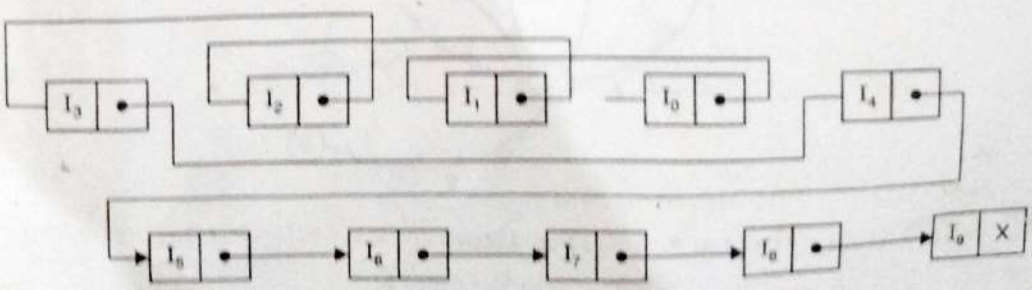



Fig. 13(*xi*)

*The starting node is $I_0$ and the last node contain X in the link part, which tells that linked list ends here.*

**Advantages.** The insertions and deletions are very easy. No movement of records.

**Disadvantages.** (*i*) Sorting is very difficult.

(*ii*) No direct access to any element of the list.

## REPRESENTATION OF RELATION IN COMPUTER

There are two possible ways to store a relation :

1. Two dimensional array representation.
2. Linked list representation.

### 1. Two Dimensional Array Representation

Consider a set S and relation R on the set S. The relation R can be represented by $n \times n$ matrix M, where $n$ is the number of elements in S. The matrix M is a boolean matrix containing only 0 and 1.

**Example.** *Let $S = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$.*

*Then the relation R on S can be represented by an $n \times n$ array.*

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

The value in matrix is 1 if the corresponding pair is in R otherwise zero.

### 2. Linked List

We can create a linked list that contains all the edges of the diagraph of a relation. The data can be represented by three arrays : TAIL, HEAD and NEXT. The Tail and Head arrays contains the beginning vertex and end vertex, for all rows, respectively. The array NEXT is array of pointers from each edge to the next edge, which make the edge data into a linked list.

**Example :** *Consider the relation represented by diagraph as shown in Fig. 13(xii).*

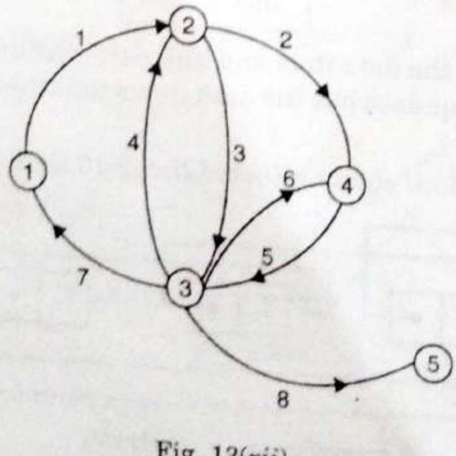



Fig. 13(*xii*).

*The diagraph shown in Fig. 13(xii) is stored in linked list form as shown in Fig. 13 (xiii). The logical ordering coincides with the numbering of edges.*

START

| 1 |
|---|

| Index Tail | HEAD | NEXT |
|---|---|---|
| 1 | 2 | 6 |
| 3 | 2 | 4 |
| 3 | 4 | 5 |
| 4 | 3 | 3 |
| 3 | 1 | 8 |
| 2 | 4 | 7 |
| 2 | 3 | 2 |
| 3 | 5 | 0 |

Fig. 13(*xiii*).

In the Fig. 13(*xiii*), the START contains the index 1 *i.e.,* the address of the first edge which is started at index 1. The NEXT (1) contains 6, so we locate the next edge in position 6. The NEXT [6] contains 7, so we locate the next edge at index 7. The NULL pointer is represented by 0, which indicates no further data.

**Disadvantages.** If we want to investigate the edges that begin or end with a particular vertex, it is not possible in this type of storage scheme.

**Modified linked list.** The modified linked list representation consists of an additional array VERT having one position for each vertex in the diagraph. Here, VERT[J] is the index for each vertex $j$, in TAIL and HEAD. The VERT contains pointers to the edges.

For every vertex $j$, the pointers are arranged in the NEXT in such a way that they link together all edges leaving $j$, starting with the edge pointed to by VERT[$j$]. The last of these edges point to zero in each case. Thus, several linked lists of edges (one for each vertex) contained in the data array TAIL and HEAD.

**Example.** *Consider the relation represented by diagraph as shown in Fig. 13(xiv).*

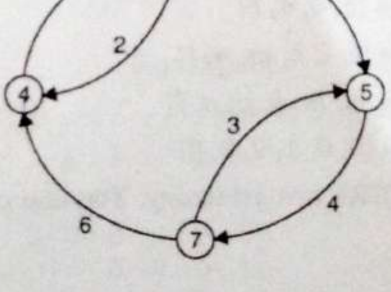



Fig. 13(*xiv*).

The diagraph shown in Fig.13(*xiv*) is stored in modified linked list represent... shown in Fig.13(*xv*).

| | VERT |
|---|---|
| 1 | 5 |
| 2 | 1 |
| 3 | 3 |
| 4 | 2 |

| | TAIL | HEAD | NEXT |
|---|---|---|---|
| 1 | 5 | 7 | 4 |
| 2 | 7 | 5 | 0 |
| 3 | 6 | 4 | 0 |
| 4 | 5 | 6 | 0 |
| 5 | 4 | 6 | 6 |
| 6 | 4 | 7 | 0 |
| 7 | | | |
| 8 | | | |

Fig. 13(*xv*).

In the modified linked list representation, VERT[1] contains 5, thus the first edg... ing vertex 1 must be stored in the 5th data position *i.e.*, the edge (4, 6). Again NEXT... tains 6, so the next edge leaving vertex 1 is (4, 7) in the 6th data position. Again NE... contains 0. Thus, we have come to the end of all those edges that begin at vertex 1.

Proceeding similarly we can trace through the edges coming from each vertex.

**Example 19.** *Consider a set* A = {*k*, *l*, *m*, *n*} *and let R be a relation on A defined... matrix* $M_R$ *as shown :*

$$M_R = \begin{array}{c} \\ k \\ l \\ m \\ n \end{array} \begin{array}{c} \begin{array}{cccc} k & l & m & n \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

*Construct a modified linked list representation for the relation R.*

**Sol.** The modified linked list representation has four arrays *i.e.*, VERT, TAIL, ... and NEXT. We have four vertices *k*, *l*, *m* and *n* and six edges. The four arrays are as sh...

VERT = [3, 6, 4, 5]

TAIL = [*k*, *l*, *k*, *m*, *n*, *l*]

HEAD = [*n*, *m*, *k*, *n*, *n*, *l*]

NEXT = [0, 0, 1, 0, 0, 2]

The indexes defined in VERT are arbitrary. You can choose any index ordering

**Example 20.** *Consider the set A = {a, b, c, d} and let R be the relation whose diagraph is shown in Fig. 13(xvi).*

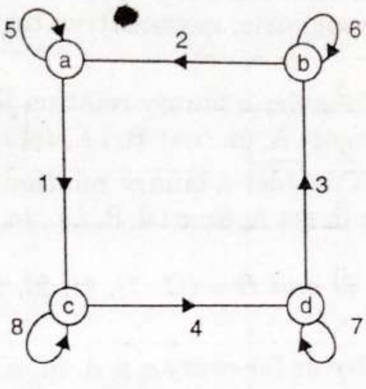



Fig. 13(*xvi*)

*Construct a modified linked list representation for the relation R.*

**Sol.** The modified linked list representation has four arrays *i.e.*, VERT, TAIL, HEAD and NEXT, which are as shown below :

VERT = [6, 8, 3, 5]
TAIL = [*a*, *b*, *c*, *d*, *d*, *a*, *c*, *b*]
HEAD = [*a*, *b*, *d*, *d*, *b*, *c*, *c*, *a*]
NEXT = [0, 0, 7, 0, 4, 1, 0, 2]

**Example 21.** *Consider the arrays as shown :*

*VERT = [5, 4, 3, 7, 6]*
*TAIL = [a, a, c, b, a, e, d]*
*HEAD = [c, d, e, e, b, d, e]*
*NEXT = [2, 0, 0, 0, 1, 0, 0]*

*These arrays describe a relation R on the set A = {a, b, c, d, e}. Determine the diagraph of R and the corresponding matrix* $M_R$.

**Sol.** The diagraph described by the arrays is shown in Fig. 13(*xvii*).

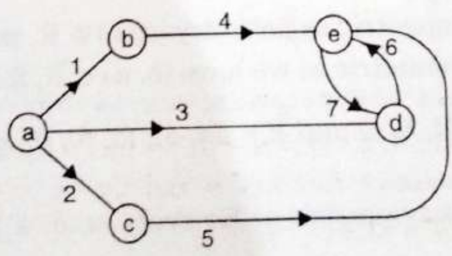



Fig. 13(*xvii*).

The corresponding matrix $M_R$ of the relation R is as shown in Fig. 13(*xviii*).

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Fig. 13(*xviii*).

## PROPERTIES OF RELATIONS

Property of the relation tells the nature or type of the relation. There are many properties of relations *i.e.*, reflexive, symmetric, asymmetric, transitive, antisymmetric etc. are as follows :

**1. Reflexive Relation.** Consider a binary relation R on a set A. Relation R is called reflexive relation if, for every $a$ in set A, $(a, a) \in$ R, *i.e.*, $(a, a) \in$ R, $\forall\, a \in$ A.

**2. Irreflexive Relation.** Consider a binary relation R on a set A. Relation R is irreflexive relation if, for every $a$ in set A, $(a, a) \notin$ R, *i.e.*, $(a, a) \notin$ R, $\forall\, a \in$ A.

**Example 22.** *Let A = {1, 2} and R = {(1, 1), (1, 2), (2, 2)}. Is the relation reflexive irreflexive ?*

**Sol.** The relation is reflexive as for every $a \in$ A. $(a, a) \in$ R, *i.e.*, (1, 1), (2, 2) $\in$ R.

The relation is not irreflexive.

**Example 23.** *Let A = {1, 2, 3, 4} and R = {(1, 2), (2, 1), (3, 1), (4, 1)}. Is the relation reflexive or irreflexive ?*

**Sol.** The relation R is not reflexive as for every $a \in$ A $(a, a) \notin$ R.

The relation is irreflexive as for every $a \in$ A, $(a, a) \notin$ R.

**Example 24.** *Let A = {1, 2, 3} and R = {(1, 2), (2, 2), (3, 1), (1, 3)}. Is the relation reflexive or irreflexive ?*

**Sol.** The relation R is not reflexive as for every $a \in a$, $(a, a) \notin$ R, *i.e.*, (1, 1) and (3, 3)

The relation is not irreflexive as $(a, a) \notin$ R, for some $a \in$ A, *i.e.*, (2, 2) $\in$ R.

**3. Symmetric Relation.** Consider a binary relation R on a set A. Relation R is called symmetric relation if for every $(a, b) \in$ R implies that $(b, a)$ also belong to R.

**4. Asymmetric Relation.** Consider a binary relation R on a set A. Relation R is called an asymmetric relation if for every $(a, b) \in$ R implies that $(b, a)$ does not belongs to R.

**Example 25.** *Let A = {1, 2, 3} and R = {(1, 2), (2, 1), (3, 2), (2, 3)}. Is the relation symmetric or asymmetric ?*

**Sol.** The relation is symmetric as for every $(a, b) \in$ R, we have $(b, a) \in$ R.

The relation is not asymmetric as we have $(b, a) \in$ R, for some $(a, b) \in$ R.

**Example 26.** *Let A = {4, 5, 6} and R = {(4, 3), (5, 6), (6, 4)}. Is the relation symmetric asymmetric ?*

**Sol.** The relation is not symmetric as for every $(a, b) \in$ R, we have $(b, a) \notin$ R *i.e.*, (4, 3) $\in$ R but (3, 4) $\notin$ R.

The relation is asymmetric as for every $(a, b) \in$ R, we have $(b, a) \notin$ R.

**Example 27.** *Let A = {7, 8, 9} and R = {(7, 8), (8, 7), (7, 9), (8, 9)}. Is the relation symmetric or asymmetric ?*

**Sol.** The relation is not symmetric as for every $(a, b) \in$ R, we have $(b, a) \notin$ R *i.e.*, (7, 9) $\in$ R but (9, 7) $\notin$ R.

The relation is not asymmetric as for some $(a, b) \in$ R, we have $(b, a) \in$ R *i.e.*, (7, 8) and (8, 7) $\in$ R.

**5. Transitive Relations.** Consider a binary relation R on a set A. Relation R is called transitive relation if whenever both $(a, b)$ and $(b, c)$ belong to R, implies that $(a, c)$ also belongs to R *i.e.*, $(a, b), (b, c) \in R \Rightarrow (a, c) \in R$.

**Example 28.** *Let* $A = \{1, 2, 3\}$ *and* $R = \{(1, 2), (2, 1), (1, 1), (2, 2)\}$. *Is the relation transitive.*

**Sol.** The relation R is transitive as for every $(a, b), (b, c)$ belong to R, we have $(a, c) \in R$ *i.e.*, $(1, 2), (2, 1) \in R \Rightarrow (1, 1) \in R$.

**Example 29.** *Let* $A = \{3, 4, 5\}$ *and* $R = \{(3, 4), (4, 3), (5, 4), (5, 3)\}$. *Is the relation R transitive ?*

**Sol.** The relation R is not transitive as for every $(a, b), (b, c) \in R$. We have $(a, c) \notin R$ *i.e.*, $(3, 4), (4, 3) \in R$ but $(3, 3) \notin R$.

**6. Antisymmetric Relations.** Consider a binary relation R on a set A. Relation R is called antisymmetric relation if $(a, b) \in R$ implies that $(b, a) \notin R$ unless $a = b$. In other words, we can say if $(a, b)$ and $(b, a)$ belong to R, then $a = b$.

**Example 30.** *Let* $A = \{4, 5, 6\}$ *and* $R = \{(4, 4), (4, 5), (5, 4), (5, 6), (4, 6)\}$. *Is the relation R antisymmetric ?*

**Sol.** The relation R is not antisymmetric as $4 \neq 5$ but $(4, 5)$ and $(5, 4)$ both belong to R.

**Example 31.** *Let* $A = \{1, 2, 3\}$ *and* $R = \{(1, 1), (2, 2)\}$. *Is the relation R antisymmetric ?*

**Sol.** The relation R is antisymmetric as $a = b$ when $(a, b)$ and $(b, a)$ both belong to R.

**Example 32.** *Let* $A = \{4, 5, 6, 7\}$. *Determine whether the following relation are reflexive, symmetric, transitive or antisymmetric.*

(*i*) $R_1 = \{(4, 4), (5, 5), (6, 6), (7, 7)\}$ (*ii*) $R_2 = \{(4, 4), (5, 5)\}$ asym

(*iii*) $R_3 = \phi$ (*iv*) $R_4 = \{(4, 5), (5, 4), (7, 6), (6, 7)\}$.

**Sol.** (*i*) **Reflexive.** The relation is reflexive as for every $a \in A$, $(a, a) \in R_1$, *i.e.*, $(4, 4), (5, 5), (6, 6), (7, 7) \in R_1$.

**Symmetric.** The relation is symmetric as there is no such pair $(a, b) \in R_1$, such that $(b, a)$ also belongs to $R_1$.

**Transitive.** The relation is transitive as there is no such pairs *i.e.*, $(a, b) \in R_1$ and $(b, c) \in R_1$ for which we have $(a, c) \in R_1$.

**Antisymmetric.** The relation is antisymmetric as if $(a, b)$ and $(b, a)$ belong to $R_1$, we have $a = b$.

(*ii*) **Reflexive.** The relation is not reflexive as for every $a$, $(a, a) \notin R_2$, *i.e.*, $(b, b) \notin R_2$.

**Symmetric.** The relation is symmetric as there is no such pair $(a, b) \in R_2$, such that $(b, a)$ also belong to $R_2$.

**Transitive.** The relation is transitive as there is no such pair $(a, b) \in R_2$ and $(b, c) \in R_2$, such that $(a, c)$ also belong to $R_2$.

**Antisymmetric.** The relation is antisymmetric because whenever $(a, b) \in R_2$ and $(b, a) \in R_2$, we have $a = b$.

(*iii*) **Reflexive.** The relation is not reflexive as for every $a \in A$, $(a, a) \notin R_3$.

**Symmetric.** The relation is symmetric as there is no such element $(a, b) \in R_3$ for which we have $(b, a) \in R_3$.

**Transitive.** The relation is transitive as there is no such elements $(a, b) \in$ $(b, c) \in R_3$ for which we have $(a, c) \in R_3$.

**Antisymmetric.** The relation is antisymmetric as there is no such elements $(a, b)$ and $(b, a) \in R_3$, such that $a = b$.

(*iv*) **Reflexive.** The relation is not reflexive as for every $a \in A$, $(a, a) \notin R_4$.

**Symmetric.** The relation is symmetric as for each $(a, b) \in R_4$, we have $(b, a) \in$

**Transitive.** The relation is not transitive as for elements $(a, b) \in R_4$ and $(b, c) \in$ have $(a, c) \notin R_4$ *i.e.*, $(4, 5) \in R_4$ and $(5, 4) \in R_4$ but $(4, 4) \notin R_4$.

**Antisymmetric.** The relation is not antisymmetric as for pairs $(a, b) \in R_4$ and $(b,$ we have $a \neq b$ *i.e.*, $(7, 6) \in R_4$ and $(6, 7) \in R_4$ but $6 \neq 7$.

**Example 33.** *Determine whether the relation R on set A = {a, b, c, d} whose graph is given in Fig. 14 is (i) reflexive (ii) symmetric (iii) transitive (iv) antisymmetric*

(a,a)(a,c)(b,a)(b,d)(c,b)(c,d)
(d,a) (a,a) (c,c)

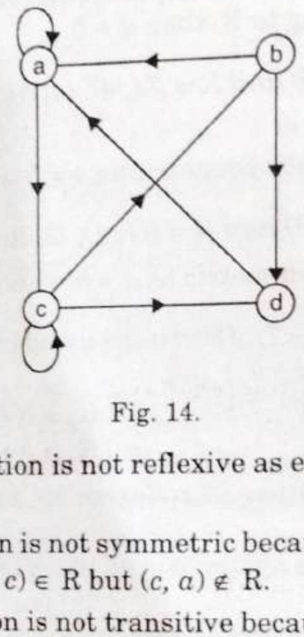



Fig. 14.

**Sol.** (*i*) **Reflexive.** The relation is not reflexive as each vertex does not contain on itself *i.e.*, a self-loop.

(*ii*) **Symmetric.** The relation is not symmetric because when there is an edge $(a$ we have no edge $(b, a) \in R$ *e.g.*, $(a, c) \in R$ but $(c, a) \notin R$.

(*iii*) **Transitive.** The relation is not transitive because when there is an edge $(a$ and $(b, c) \in R$, we have no edge $(a, c) \in R$ *e.g.*, $(a, c) \in R$ and $(c, d) \in R$ but $(a, d) \notin R$.

(*iv*) **Antisymmetric.** The relation is antisymmetric because we have no such $(a, b) \in R$ and $(b, a) \in R$ for which $a \neq b$.

## CLOSURE PROPERTIES OF RELATIONS

Consider a relation R on some set A. Suppose the relation R does not possess the property. We will add as few new pairs as possible to relation R to get a relation that the desired property.

The smallest relation S on A that contains R and the desired property is ca closure of relation R with the desired property.

**1. Reflexive Closure.** Consider a relation R on a set A. Relation $R_F$ is called closure of R if $R_F$ is the smallest relation containing R, having the reflexive prope $R_F = R \cup \Delta$ where $\Delta$ is a diagonal relation.

**Example 34.** *Consider the relation R on A = {7, 8, 9} defined by R = {(7, 8), (7, 9), (8, 8), (9, 7)}. Find the reflexive closure of R.*

**Sol.** The smallest relation containing R having the reflexive property is

$$R \cup \Delta = \{(7, 7), (7, 8), (7, 9), (8, 8), (9, 7), (9, 9)\}.$$

$R \cup I_R$

**Example 35.** *Let A = {k, l, m}. Let R is a relation on A defined by*

*R = {(k, k), (k, l), (l, m), (m, k)}. Find the reflexive closure of R.*

**Sol.** $R \cup \Delta$ is the smallest relation having reflexive property. Hence,

$$R_F = R \cup \Delta = \{(k, k), (k, l), (l, l), (l, m), (m, m), (m, k)\}.$$ smallest relation

**2. Symmetric Closure.** Consider a relation R on set A. Relation $R_s$ is called the symmetric closure of R if $R_S$ is the smallest relation containing R, having the symmetric property. The relation $\boxed{R_S = R \cup R^{-1}}$ is the smallest symmetric relation containing R.

**Example 36.** *Consider the relation R on A = {4, 5, 6, 7} defined by*

*R = {(4, 5), (5, 5), (5, 6), (6, 7), (7, 4), (7, 7)}* $\to R^{-1} = \{(5,4), (5,5), (6,5), (7,6), (4,7), (7,7)\}$

*Find the symmetric closure of R.*

**Sol.** The smallest relation containing R, having the symmetric property is $R \cup R^{-1}$ *i.e.,*

$$R_S = R \cup R^{-1} = \{(4, 5), (5, 4), (5, 5), (5, 6), (6, 5), (6, 7), (7, 6), (7, 4), (4, 7), (7, 7)\}.$$

**Example 37.** *Let A = { Maths, Physics, English, Chemistry}. Let R is relation 'combination of subjects' on the set A, defined by*

*R = {(Maths, Physics), (Physics, English), (Chemistry, English), (Chemistry, Physics), (Physics, Chemistry)}.*

*Find the symmetric closure $R_S$ of R.*

**Sol.** $R \cup R^{-1}$ is the smallest relation having the symmetric property. Thus,

$R_S = R \cup R^{-1}$ = {(Maths, Physics), (Physics, Math), (Physics, English), (English, Physics), (Chemistry, English), (English, Chemistry), (Chemistry, Physics), (Physics, Chemistry)}.

**3. Transitive Closure.** Consider a relation R on a set A. The transitive closure R of a relation R is the smallest transitive relation containing R. $R^* = \bigcup_{i=1} R^i$

**Method to find transitive closure.** Draw directed graph of the relation R. Find whether there is a path from Ist vertex to any other vertex of the graph. Include the ordered pair for that vertices R* if there is path from Ist vertex to any other vertex. Similarly, we proceed to check all the paths from all vertices to all other vertices and include the corresponding ordered pairs in R*. At the end we got the transitive closure R* of relation R. $R^+ = R \cup R^2 \cup R^3 \cup \dots \cup R^n$ if dom A = n.

**Example 38.** *Let A = {a, b, c, d}. Let R = {(a, b), (a, c), (b, a), (b, c), (c, d), (d, a)}.*

*Find the transitive closure of R.*

**Sol.** The directed graph of relation R is shown in Fig. 15.

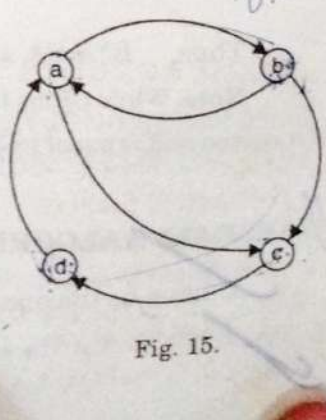


Fig. 15.

From vertex *a*, we have paths to vertices *a*, *b*, *c* and *d*. So, the ordered pairs included in R* is (*a*, *a*), (*a*, *b*), (*a*, *c*) and (*a*, *d*). Similarly, from vertex *b*, we have paths to vertices *a*, *b*, *c* and *d*. So, the ordered pairs included in R* is (*b*, *b*), (*b*, *c*), (*b*, *d*), (*b*, *a*). From vertex *c*, we have paths to vertices *c*, *b*, *d* and *a* from vertex *d*, we have paths to vertices *d*, *a*, *b* and *c*. So, the vertices included in R* is (*c*, *c*), (*c*, *d*), (*c*, *b*), (*c*, *a*) and (*d*, *d*), (*d*, *a*), (*d*, *b*), (*d*, *c*). Hence, transitive closure R* is

$R^* = \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c), (a, d), (b, c), (b, a), (b, d), (c, a), (c, b),$ $(d, a), (d, b), (d, c)\}$.

**Example 39.** *Let $A = \{4, 6, 8, 10\}$ and $R = \{(4, 4), (4, 10), (6, 6), (6, 8), (8, 10)\}$ is a on set A. Determine transitive closure of R.* $= R^*$

**Sol.** The matrix of relation R is shown in Fig. 16.

$R^* = I_4 \vee M_R^{\infty}$

$$M_R = \begin{array}{c|c} & \begin{matrix} 4 & 6 & 8 & 10 \end{matrix} \\ \begin{matrix} 4 \\ 6 \\ 8 \\ 10 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 16.

Now, find the powers of $M_R$ as in Figs. 17, 18 and 19.

$$M_{R^2} = \begin{array}{c|c} & \begin{matrix} 4 & 6 & 8 & 10 \end{matrix} \\ \begin{matrix} 4 \\ 6 \\ 8 \\ 10 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Fig. 17.

$$M_{R^3} = \begin{array}{c|c} & \begin{matrix} 4 & 6 & 8 & 10 \end{matrix} \\ \begin{matrix} 4 \\ 6 \\ 8 \\ 10 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Fig. 18.

$$M_{R^4} = \begin{array}{c|c} & \begin{matrix} 4 & 6 & 8 & 10 \end{matrix} \\ \begin{matrix} 4 \\ 6 \\ 8 \\ 10 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Fig. 19.

Hence, the transitive closure of $M_R$ is $M_{R^*}$ as shown in Fig. 20. (where $M_{R^*}$ is the of powers of $M_R$).

$$M_{R^*} = M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4}\ ; \qquad M_{R^*} = \begin{array}{c|c} & \begin{matrix} 4 & 6 & 8 & 10 \end{matrix} \\ \begin{matrix} 4 \\ 6 \\ 8 \\ 10 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Fig. 20.

Thus, $R^* = \{(4, 4), (4, 10), (6, 8), (6, 6), (6, 10), (8, 10)\}$.

**Note.** While ORing the powers of the matrix R, we can eliminate $M_{R^n}$ because it is equal to $M_{R^2}$ if $n$ is even and is equal to $M_{R^3}$ if $n$ is odd.

## WARSHALL'S ALGORITHM

1. Let $n$ be the number of elements in a given set A *i.e.*, $n \to |A|$
2. Let $w_0, w_1, w_2, \ldots\ldots, w_n$ be Warshall sets.

3. To find transitive closure of relation R on A, maximum of $n$ Warshall sets of which $w_n$ is the last need to be computed.

The procedure to compute $w_k$ from $w_{k-1}$ is as follows :

(*i*) Copy 1 to all entries in $w_k$ from $w_{k-1}$, where there is 1 in $w_{k-1}$.

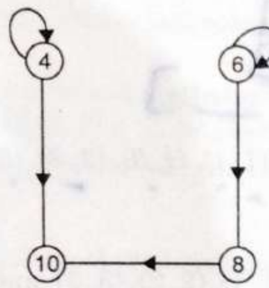
(*ii*) Find the row numbers $p_1, p_2, p_3, \ldots\ldots$ for which there is 1 in column $k$ in $w_{k-1}$ and the column numbers $q_1, q_2, q_3, \ldots\ldots\ldots$ for which there is 1 in row $k$ in $w_{k-1}$.

(*iii*) Mark entries in $w_k$ as 1 for $(p_i, q_i)$ of $w_k$ if there are not already 1.

4. Stop when $w_n$ is obtained and it is the needed result (transitive closure).

**Example 40.** *Consider the example 39, where A = {4, 6, 8, 10} and R = {(4, 4), (4, 10), (6, 6), (6, 8), (8, 10)} is a relation on set A. Determine the transitive closure of R using Warshall's algorithm.*

**Sol.** The diagraph of relation R and its corresponding matrix $M_R$ is shown in Fig. 21.




$$M_R = \begin{array}{c} \\ 4 \\ 6 \\ 8 \\ 10 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 6 & 8 & 10 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Fig. 21.

The value of $n = |A| = 4$. Thus, we have to find the Warshall's sets $w_0$, $w_1$, $w_2$, $w_3$ and $w_4$.

The first set $w_0$ is same as $M_R$, which is shown below :

$$w_0 = M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now to find $w_1$ from $w_0$ we have row number 1 for column 1 in $w_0$ and column number 1 and 4 for row 1 in $w_0$. Thus, new entries in $w_1$ are (4, 4) and (4, 10) which are already one. Thus, $w_1$ is same as $w_0$, which is as follows :

$$w_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To find $w_2$ from $w_1$, we have row number 2 for column 2 in $w_1$ and column number 2 and 3 for row 2 in $w_1$. Thus, new entries in $w_2$ are (6, 6) and (6, 8), which are already one. Thus, $w_2$ is same as $w_1$, which is as follows :

$$w_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Similarly, $w_3$ is obtained from $w_2$. Here, we have row number 2 in column 3 and ... number 4 in row 3. Thus, the new entries in $w_3$ are (6, 16). So, $w_3$ is as follows :

$$w_3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Similarly, $w_4$ is obtained from $w_3$. But there are no new entries of 1's in $w_4$. $M_{R^*} = w_4 = w_3$, which is the transitive closure of R, and is same as in example 39.

## EQUIVALENCE RELATIONS

A relation R on a set A is called an equivalence relation if it satisfies following the properties :

1. Relation R is reflexive i.e., $aRa \; \forall \; a \in A$.
2. Relation R is symmetric i.e., $aRb \Rightarrow bRa$.
3. Relation R is transitive i.e., $aRb$ and $bRc \Rightarrow aRc$.

**Example 41.** *Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 4)\}$.*

*Show that R is an equivalence relation.*

**Sol. Reflexive.** Relation R is reflexive as (1, 1), (2, 2), (3, 3), and (4, 4) $\in$ R.

**Transitive.** Relation R is transitive because whenever $(a, b)$ and $(b, c)$ belong to R, $(a, c)$ also belong to R e.g., (3, 1) $\in$ R and (1, 3) $\in$ R $\Rightarrow$ (3, 3) $\in$ R.

**Symmetric.** Relation R is symmetric because whenever $(a, b) \in$ R, $(b, a)$ also belongs to R e.g., (2, 4) $\in$ R $\Rightarrow$ (4, 2) $\in$ R.

So, as R is reflexive, symmetric and transitive. Hence, R is an equivalence relation.

**Example 42.** *Let S be the set of all points in a plane. Let R be a relation such that for any two points a and b ; $(a, b) \in R$ if b is within two centimeter from A. Show that R is an equivalence relation.*

**Sol. Reflexive.** We have $(a, a) \in$ R, for every $a \in$ S [$\because$ a lies within 2 cm from itself]

Therefore, R is reflexive.

**Symmetric.** Assume that $(a, b) \in$ R

$\Rightarrow$ $b$ lies within 2 cm from $a$ $\Rightarrow$ $a$ lies within 2 cm from $b$.

$\Rightarrow$ $(b, a) \in$ R.

Thus, $(a, b) \in R \Rightarrow (b, a) \in R$.

Therefore, R is symmetric.

**Transitive.** Assume that $(a, b)$ and $(b, c) \in$ R.

Therefore, $b$ lies within 2 cm from $a$.

$c$ lies within 2 cm from $b$.

It does not implies $c$ lies within 2 cm from $a$. Therefore, $(a, b) \in R$, $(b, c) \in R$ then $(a, c)$ may not belong to R.

Hence, relation R is not always transitive. Therefore, relation is not an equivalence relation.

## PARTIAL ORDER RELATION

A relation R on a set A is called a partial order relation if it satisfies the following three properties :

1. Relation R is reflexive *i.e.*, $a\text{R}a \ \forall \ a \in \text{A}$.
2. Relation R is antisymmetric $a\text{R}b$ and $b\text{R}a \Rightarrow a = b$.
3. Relation R is transitive $a\text{R}b$ and $b\text{R}c \Rightarrow a\text{R}c$.

**Example 43.** *Show whether the relation $(x, y) \in \text{R}$, if $x \geq y$ defined on the set of + ve integers is a partial order relation.*

**Sol.** Consider the set $\text{A} = \{1, 2, 3, 4\}$ containing four + ve integers. Find the relation for this set such as $\text{R} = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (1, 1), (2, 2), (3, 3), (4, 4)\}$.

**Reflexive.** The relation is reflexive as for every $a \in \text{A}$ $(a, a) \in \text{R}$ *i.e.*, $(1, 1), (2, 2), (3, 3)$ and $(4, 4) \in \text{R}$.

**Antisymmetric.** The relation is antisymmetric as whenever $(a, b)$ and $(b, a) \in \text{R}$, we have $a = b$.

**Transitive.** The relation is transitive as whenever $(a, b)$ and $(b, c) \in \text{R}$, we have $(a, c) \in \text{R}$ *e.g.*, $(4, 2) \in \text{R}$ and $(2, 1) \in \text{R}$, implies $(4, 1) \in \text{R}$.

As the relation is reflexive, antisymmetric and transitive. Hence, it is a partial order relation.

**Example 44.** *Show that the relation 'Divides' defined on N is a partial order relation.*

**Sol. Reflexive.** We have a divides $a$, $\forall \ a \in \text{N}$. Therefore, relation 'Divides' is reflexive.

**Antisymmetric.** Let $a, b \in \text{N}$, such that $a$ divides $b$. It implies $b$ divides $a$ iff $a = b$.

So, relation is antisymmetric.

**Transitive.** Let $a, b, c \in \text{N}$, such that $a$ divides $b$ and $b$ divides $c$.

Then $a$ divides $c$. Hence, relation is transitive. Thus, the relation being reflexive, antisymmetric and transitive, the relation 'divides' is a partial order relation.

A, R
(A, R)

## PARTIAL ORDER SET (POSET)

The set A together with a partially order relation R on the set A and is denoted by (A, R) is called a partially ordered set or POSET.

## TOTAL ORDER RELATION

Consider the relation R on the set A. If it is also the case that for all $a, b \in \text{A}$, we have either $(a, b) \in \text{R}$ or $(b, a) \in \text{R}$ or $a = b$, then the relation R is called total order relation on set A.

**Example 45.** *Show that the relation '<' (less than) defined on N, the set of + ve integers is neither an equivalence relation nor partially ordered relation but is a total order relation.*

**Sol. Reflexive.** Let $a \in \text{N}$, then $a < a$

$\Rightarrow$ '<' is not reflexive.

As, the relation '<' (less than) is not reflexive, it is neither an equivalence relation nor the partial order relation.

But, as $\forall a, b \in N$, we have either $a < b$ or $b < a$ or $a = b$. So, the relation is a total or
relation.

## PARTITION

A partition $(A_1, A_2, ..., A_i)$ of a non-empty set A is defined as a collection of non-empty subsets of A, such that

(a) Every element of A belongs to one of $A_i$ *i.e.*, the union of $A_i$ is equal to set A.

(b) If $A_i$ and $A_j$ are distinct, then $A_i \cap A_j = \phi$ *i.e.*, the partitions divides the elements of set A into disjoint subsets.

The subsets in a partition are called blocks or cells.

**Example 46.** *Let A = {7, 8, 9}. Determine all the partitions of the set A.*

**Sol.** We have three elements in set A. So, in a partition there can be 1, 2 or 3 elements at the maximum. Hence, the partitions are as follows :

(*i*) [{7}, {8}, {9}] (*ii*) [{7}, {8, 9}]
(*iii*) [{8}, {7, 9}] (*iv*) [{9}, {7, 8}] (*v*) [{7, 8, 9}].

**Example 47.** *Let A = {1, 2, 3, 4, 5, 6, 7, 8}. Determine whether the following is a partition of A or not*

(*i*) $P_1$ = [{1, 2, 3, 4}, {1, 3, 5, 6, 7, 8}]
(*ii*) $P_2$ = [{1, 3, 5, 7}, {2, 4}, {6, 8}]
(*iii*) $P_3$ = [{1, 2, 3, 4}, {6, 8}]
(*iv*) $P_4$ = [{1, 3, 5, 6, 7, 8}, {2, 4}].

**Sol.** (*i*) Not a partition because $1 \in A$ belongs to both the cells.
(*ii*) $P_2$ is a partition.
(*iii*) $P_3$ is not a partition because $7 \in A$ does not belong to any cell of $P_3$.
(*iv*) $P_4$ is a partition.

## EQUIVALENCE CLASS

Consider, an equivalence relation R on a set A. The equivalence class of an element $a \in A$, is the set of elements of A to which element $a$ is related. It is denoted by $[a]$.

**Example 48.** *Let R be an equivalence relation on the set A = {4, 5, 6, 7} defined by*

$$R = \{(4, 4), (5, 5), (6, 6), (7, 7), (4, 6), (6, 4)\}.$$

*Determine its equivalence classes.*

**Sol.** The equivalence classes are as follows :

$$[4] = [6] = \{4, 6\}$$
$$[5] = \{5\}$$
$$[7] = \{7\}.$$

**Example 49.** *Let R be an equivalence relation on set A = {1, 2, 3, 4, 5} defined by*

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 4), (4, 1), (2, 4), (4, 2), (1, 2), (2, 1)\}$$

*Determine equivalence classes of R.*

**Sol.** The equivalence classes of R is

$$[1] = [2] = [4] = \{1, 2, 4\}$$
$$[3] = \{3\}$$
$$[5] = \{5\}.$$

**Example 50.** *Let R be an equivalence relation on the set A = {6, 7, 8, 9, 10} defined by R = {(6, 6), (7, 7), (8, 8), (9, 9), (10, 10), (6, 7), (7, 6), (8, 9), (9, 8), (9, 10), (10, 9), (8, 10), (10, 8)}.*

*Find the equivalence classes of R and hence find the partition of A corresponding to R.*

**Sol.** The equivalence class of R are as follows. Element 6 is related with 6 and 7, therefore

$$[6] = [7] = \{6, 7\}.$$

Element 8 is related with 8, 9 and 10, therefore, $[8] = [9] = [10] = \{8, 9, 10\}$.

Hence, the partition corresponding to R is

$$P = \{(6, 7), (8, 9, 10)\}.$$

## CIRCULAR RELATION

Consider a binary relation R on a set A. Relation R is called circular if $(a, b) \in$ R and $(b, c) \in$ R implies $(c, a) \in$ R.

**Example 51.** *Show that relation R is reflexive and circular if and only if it is an equivalence relation.*

**Sol.** Consider, the relation is reflexive and circular.

Therefore, $(a, a) \in$ R for every $a \in$ A (∵ R is reflexive)

Also, let $(a, b) \in$ R and $(b, c) \in$ R implies $(c, a) \in$ R (∵ R is circular)

Now, as $(c, a) \in$ R and $(a, a) \in$ R implies $(a, c) \in$ R (∵ R is circular)

⇒ R is transitive.

Let $(a, b) \in$ R implies $(a, a) \in$ R and $(b, b) \in$ R (∵ R is reflexive)

It implies $(b, a) \in$ R (∵ R is circular)

Therefore, R is symmetric.

Therefore, relation R being reflexive, transitive and symmetric, hence it is an equivalence relation.

**Example 52.** *Consider R is an equivalence relation. Show that R is reflexive and circular.*

**Sol. Reflexive.** As, the relation R is an equivalence relation. So, reflexivity is the property of an equivalence relation. Hence, R is reflexive.

**Circular.** Let $(a, b) \in$ R and $(b, c) \in$ R

⇒ $(a, c) \in$ R (∵ R is transitive)

⇒ $(c, a) \in$ R (∵ R is symmetric)

Thus, R is circular.

## SOLVED PROBLEMS

**Problem 1.** *Let R be a relation on set A = {k, l, m, n} defined by*

$$R = \{(k, l), (m, l), (n, l), (l, l), (k, k), (m, k), (l, k), (n, k)\}.$$

*Find domain and range of relation R.*

**Sol.** The domain of relation R is

$$\text{Dom}(R) = \{k, l, m, n\} = A$$

The range of relation R is

$$\text{RAN}(R) = \{k, l\}.$$

**Problem 2.** *Let R be a relation on set A = {1, 2, 3, 4} defined by*

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (4, 3), (4, 2), (4, 1), (3, 2),$$

*Find the matrix and directed graph of relation R.*

**Sol.** The matrix of relation R is a 4 × 4 matrix as shown in Fig. 22.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Fig. 22.

The directed graph of relation R is as shown in Fig. 23.

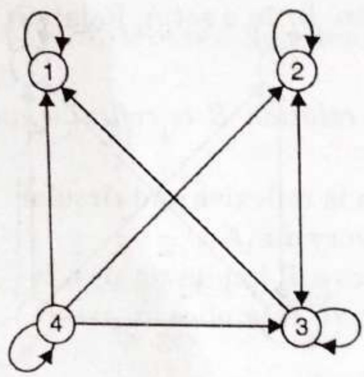



Fig. 23.

**Problem 3.** *Let R be a relation on set A = {1, 2, 3, 4, 5}, whose matrix is shown. Find the directed graph of relation R.*

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Fig. 24.

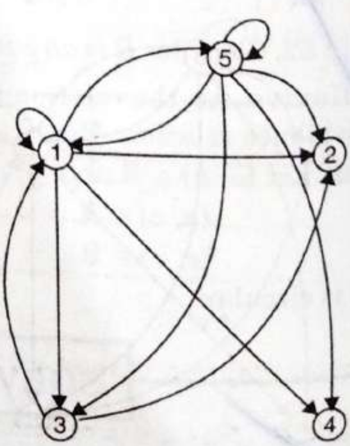
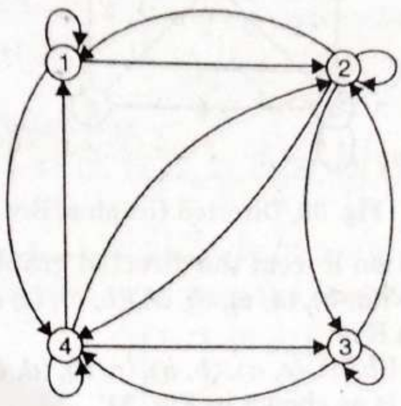



Fig. 25.

**Sol.** The directed graph of relation R whose matrix $M_R$ is given is as shown in Fig. 25.

**Problem 4.** *Let R be a relation on set A whose directed graph is as shown in Fig. 26. Determine its matrix.*




(1,1) (2,

Fig. 26.

**Sol.** The matrix $M_R$ of the relation R whose directed graph is shown above is as shown in Fig. 27.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Fig. 27.

**Problem 5.** *Let R be a relation on set A = {x, y, z} defined by*

$$R = \{(x, x), (y, y), (z, z), (x, z), (x, y), (y, z)\}.$$

*Write the relation as a table and also find its arrow diagram.*

**Sol.** Relation R as a table shown in Fig. 28.

| | *x* | *y* | *z* |
|---|---|---|---|
| *x* | × | × | × |
| *y* | | × | × |
| *z* | | | × |

Fig. 28.

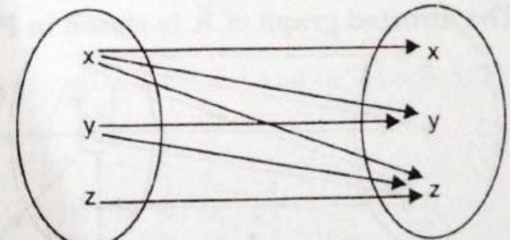



Fig. 29.

The arrow diagram of relation R is shown in Fig. 29.

**Problem 6.** *Consider a relation R whose directed graph is shown in Fig. 30. Determine its inverse $R^{-1}$ and complement $\overline{R}$.*

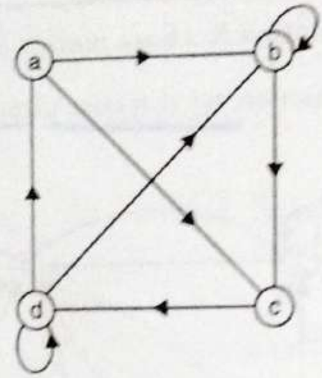



Fig. 30. Directed Graph of R.

**Sol.** The elements of relation R from the directed graph is

$$R = \{(a, b), (a, c), (b, b), (b, c), (c, d), (d, d), (d, a), (d, b)\}$$

(*i*) The inverse of relation R is

$$R^{-1} = \{(b, a), (c, a), (b, b), (c, b), (d, c), (d, d), (a, d), (b, d)\}.$$

The directed graph of $R^{-1}$ is as shown in Fig. 31.

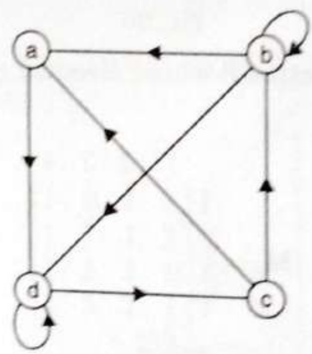



Fig. 31. Directed Graph of $R^{-1}$.

(*ii*) The universal relation R × R on set $\{a, b, c, d\}$ is

$R \times R = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (b, a), (c, c), (c, d), (c, b), (c,$ $(d, a), (d, b), (d, c)\}$.

The complement of R is

$$\overline{R} = \{(a, a), (a, d), (b, d), (b, a), (c, a), (c, b), (c, c), (d, c)\}.$$

The directed graph of $\overline{R}$ is shown in Fig. 32.

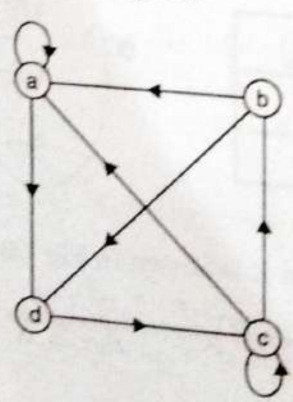



Fig. 32. Directed Graph of $\overline{R}$.

**Problem 7.** *Consider a relation R from a set A to B whose matrix is shown in Fig. 33. Determine is inverse $R^{-1}$ and complement $\overline{R}$.*

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Fig. 33.

**Sol.** The elements of relation R is

$$R = \{(1, 1), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 1), (4, 2)\}$$

(*i*) The inverse of relation R is

$$R^{-1} = \{(1, 1), (4, 1), (1, 2), (2, 2), (3, 2), (4, 2), (3, 3), (4, 3), (1, 4), (2, 4)\}.$$

For writing in the matrix form, transpose the original matrix *i.e.*, (Fig. 34.).

$$M_R^{-1} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Fig. 34.

(*ii*) The complement of relation R is taken by exchanging 0's with 1's and 1's with 0's. (Fig. 35).

$$M_{\overline{R}} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Fig. 35.

So, $\overline{R} = \{(1, 2), (1, 3), (3, 1), (3, 2), (4, 3), (4, 4)\}$.

**Problem 8.** *Let R and S be two relation from A to B. Show that*

(*i*) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ (*ii*) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.

**Sol.** (*i*) Let $(a, b) \in (R \cap S)^{-1}$. So, we have $(b, a) \in (R \cap S)$. Now $(b, a) \in R$ and $(b, a) \in S$. This means $(a, b) \in R^{-1}$ and $(a, b) \in S^{-1}$.

Hence, $(a, b) \in R^{-1} \cap S^{-1}$ ....(*i*)

Conversely, let $(a, b) \in R^{-1} \cap S^{-1}$. So, we have $(a, b) \in R^{-1}$ and $(a, b) \in S^{-1}$. This means $(b, a) \in R$ and $(b, a) \in S$. So, $(b, a) \in (R \cap S)$.

Hence, $(a, b) \in (R \cap S)^{-1}$ ....(*ii*)

Form (*i*) and (*ii*), we have

$$(R \cap S)^{-1} = R^{-1} \cap S^{-1}$$

Hence proved.

(*ii*) Let $(a, b) \in (R \cup S)^{-1}$. So, we have $(b, a) \in (R \cup S)$

Now, $(b, a) \in R$ or $(b, a) \in S$. This means $(a, b) \in R^{-1}$ or $(a, b) \in S^{-1}$

Hence, $(a, b) \in R^{-1} \cup S^{-1}$ ...(*i*)

Conversely, let $(a, b) \in R^{-1} \cup S^{-1}$. So, we have $(a, b) \in R^{-1}$ or $(a, b) \in S^{-1}$. This m $(b, a) \in R$ or $(b, a) \in S$. So, $(b, a) \in R \cup S$.

Hence, $(a, b) \in (R \cup S^{-1})$

From (*i*) and (*ii*), we have

$$(R \cup S)^{-1} = R^{-1} \cup S^{-1}.$$

Hence proved.

**Problem 9.** *Let* $A = \{k, l, m, n\}$

*Let* $R = \{(k, k), (l, l), (m, m), (k, l), (k, m), (l, m), (m, n), (n, k)\}$

*Let* $S = \{(n, k), (n, l), (n, m), (m, k), (m, l), (l, k), (k, k)\}$.

*Find the composition* (*i*) *RoR* (*ii*) *SoS*.

**Sol.** (*i*)The composition RoR is shown in Fig. 36.

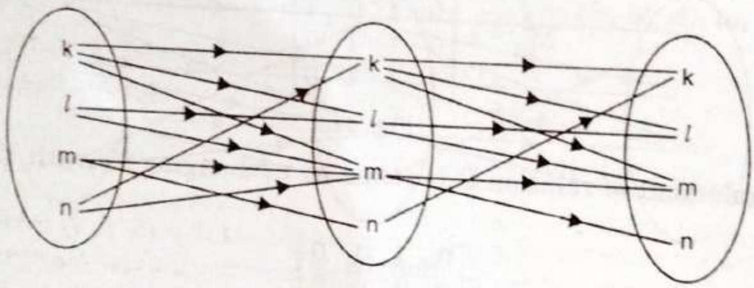


Fig. 36. RoR.

$RoR = \{(k, k), (k, l), (k, m), (k, n), (l, l), (l, m), (l, n), (m, m), (m, n), (m, k), (n, k), (n, m)\}$

(*ii*) The composition SoS is shown in Fig. 37.

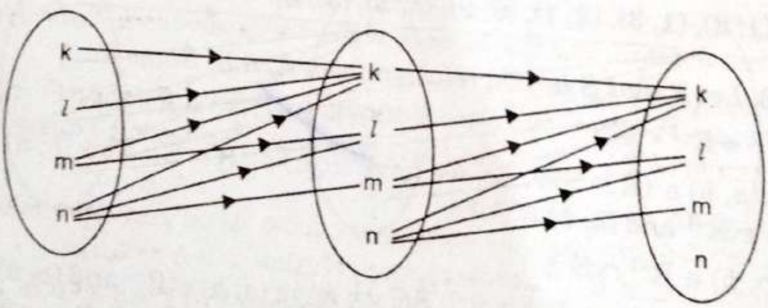


Fig. 37. SoS.

$$SoS = \{(k, k), (l, k), (m, k), (n, k), (n, l)\}$$

**Problem 10.** *Give an example of relations* $R_1$, $R_2$, $R_3$, $R_4$ *and* $R_5$ *on* $A = \{4, 5, 6,$ *having property*

(*i*) $R_1$ *is reflexive and transitive but not symmetric.*

(*ii*) $R_2$ *is symmetric and antisymmetric.*

(*iii*) $R_3$ *is antisymmetric but not reflexive.*

(*iv*) $R_4$ *is neither symmetric nor antisymmetric.*

(*v*) $R_5$ *is neither symmetric and asymmetric nor antisymmetric.*

**Sol.** (*i*) $R_1 = \{(4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (4, 5), (5, 6), (4, 6)\}$.

(*ii*) $R_2 = \{(4, 4), (5, 5), (6, 6), (7, 7)\}$.

(*iii*) $R_3 = \{(4, 4), (5, 5), (6, 6)\}$.

(*iv*) $R_4 = \{(4, 4), (4, 3), (3, 4), (4, 2)\}$.

(*v*) $R_5 = \{(4, 4), (4, 5), (5, 4), (4, 6)\}$.

**Problem 11.** *Give an example of relations $R_1$, $R_2$, $R_3$ and $R_4$ on $A = \{a, b, c, d\}$ having property*

(*i*) *$R_1$ is irreflexive and antisymmetric.*

(*ii*) *$R_2$ is asymmetric and antisymmetric.*

(*iii*) *$R_3$ is asymmetric but $R_3 \cup R_3^{-1}$ is symmetric.*

(*iv*) *$R_4$ is transitive but $R_4 \cup R_4^{-1}$ is not transitive.*

**Sol.** (*i*) $R_1 = \{(a, b), (b, c), (c, d), (d, a)\}$ (*ii*) $R_2 = \{(a, a), (b, b), (c, d), (b, d), (a, d)\}$

(*iii*) $R_3 = \{(a, b), (b, c), (c, d), (d, a)\}$ (*iv*) $R_4 = \{(a, c), (d, d)\}$.

**Problem 12.** *Give an example of a relation R on the set $A = \{4, 5, 6, 7\}$ which have no property i.e., the relation R is not reflexive, irreflexive, symmetric, asymmetric, transitive and antisymmetric.*

**Sol.** The relation R having no property is $R = \{(4, 4), (5, 5), (4, 5), (5, 4), (4, 7)\}$.

**Problem 13.** *Give an example of a relation R on the set $A = \{5, 6, 7\}$ which is reflexive, symmetric, transitive and antisymmetric i.e., a relation having all these four properties.*

**Sol.** The relation R having all the above four properties is

$$R = \{(5, 5), (6, 6), (7, 7)\}.$$

**Problem 14.** *Let A be a set of books.*

(*i*) *Let $R_1$ be a binary relation on A, such that $(a, b)$ is in $R_1$ if book 'a' costs more and contains fewer pages than book 'b'. In general, is $R_1$ reflexive ? symmetric ? antisymmetric ? and transitive ?*

(*ii*) *Let $R_2$ is a binary relation on A, such that $(a, b)$ is in $R_2$ if book 'a' costs more or contains fewer pages than book 'b'. In general is $R_2$ reflexive ? symmetric ? antisymmetric ? and transitive ?*

**Sol.** (*i*) **Reflexive.** A book does not costs more from itself and does not contain fewer pages than itself. Hence, $(a, a) \notin R_1$. Therefore, the relation is not reflexive.

**Symmetric.** If book a costs more and contains fewer pages than book $b$, it implies $(a, b) \in R_1$. But it is not possible that book $b$ also costs more and contains fewer pages than book $a$. Hence $(b, a) \notin R_1$. Therefore relation is not symmetric.

**Antisymmetric.** There is no such condition in the relation that $(a, b) \in R_1$ and $(b, a) \in R_1$ such that $a \neq b$. Hence, relation is antisymmetric.

**Transitive.** Let us assume that $(a, b) \in R_1$ and $(b, c) \in R_1$ *i.e.*, book a costs more and contains fewer pages than book $b$ and costs more and contains fewer pages than book $c$. So, it is sure that book $a$ costs more and contains fewer pages than book $c$ *i.e.*, $(a, c) \in R_1$. Therefore, the relation is transitive.

(*ii*) **Reflexive.** A book $a$ doesn't costs more or contains fewer pages than itself. Hence $(a, a) \notin R_2$. Therefore, the relation is not reflexive.

**Symmetric.** Let $(a, b) \in R_2$ *i.e.*, book $a$ costs more or contains fewer pages than book $b$. But it does not imply that book $b$ costs more or contains fewer pages than book $a$. $(b, a) \notin R_1$. Therefore, relation is not always symmetric.

**Antisymmetric.** Sometimes, there exists condition that $(a, b) \in R$ and $(b, a) \in R$ and $a \neq b$. Hence, the relation is not antisymmetric.

**Transitive.** Let $(a, b) \in R$ and $(b, c) \in R$ *i.e.*, book $a$ costs more than $b$ and fewer pages than book $c$. So, we can have that book $a$ costs less or does not contain fewer pages than book $c$. Hence, the relation is not always transitive.

**Problem 15.** *Determine whether the relation $R = \{(a, b) \in R, a - b \leq 1$ on the set of + ve integer)\} is*

(*i*) *reflexive* (*ii*) *symmetric* (*iii*) *transitive*

(*iv*) *antisymmetric* (*v*) *a partial order relation* (*vi*) *an equivalence*

**Sol.** (*i*) **Reflexive.** The relation is reflexive because $a - a \leq 1$, for every $a \in I_+$. So for every $a \in I_+$.

(*ii*) **Symmetric.** The relation is symmetric because whenever $a - b \leq 1$, we have $\in 1$. So, for $(a, b) \in I_+$, we also have $(b, a) \in I_+$.

(*iii*) **Transitive.** The relation is not transitive because when some $a - b \leq 1$ and we have $a - c > 1$ *e.g.*, $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \notin R$.

(*iv*) **Antisymmetric.** The relation is not antisymmetric because whenever $a -$ $b - a < 1$, we have $a \neq b$ *e.g.*, $(1, 2) \in R$ and $(2, 1) \in R$ but $1 \neq 2$.

(*v*) **Partial Order Relation.** The relation is not partial order relation as it is but not transitive and antisymmetric.

(*vi*) **Equivalence Relation.** The relation is not an equivalence relation as it is and symmetric but not transitive.

**Problem 16.** *Determine whether the relation*

*$R = \{(a, b) \in R : a + b$ is even. On the set $I_+$ (set of +ve integers) is (i) reflexive (ii) (iii) transitive (iv) antisymmetric (v) a partial order relation (vi) an equivalence relation.*

**Sol.** (*i*) **Reflexive.** The relation is reflexive because when a number whether or odd added to itself, it always is even. So, for every $a \in I_+$ $(a, a) \in R$.

(*ii*) **Symmetric.** The relation is symmetric because whenever $a + b$ is even, $b +$ even. So, for each $(a, b) \in R$ we have $(b, a) \in R$.

(*iii*) **Transitive.** The relation is transitive because whenever $a + b$ is even $b +$ we have $a + c$ is even. So, whenever $(a, b) \in R$ and $(b, c) \in R$ we have $(a, c) \in R$.

(*iv*) **Antisymmetric.** The relation is not antisymmetric because we have $(a, b) \in R$ and $(b, a) \in R$ such that $a \neq b$ *e.g.*, $(2, 4) \in R$ and $(4, 2) \in R$ but $2 \neq 4$.

(*v*) **Partial Order Relation.** The relation is not a partial order relation as it is and transitive but not antisymmetric.

(*vi*) **Equivalence Relation.** The relation is an equivalence relation as it is symmetric and transitive also.

**Problem 17.** *Let R be the relation on set A = \{a, b, c, d\} defined by*

$$R = [(a, b), (b, c), (d, c), (d, a), (a, d), (d, d)]$$

*Determine (i) Reflexive closure of R*

*(ii) Symmetric closure of R*

*(iii) Transitive closure of R.*

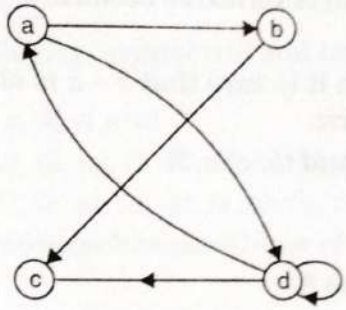
**Sol.** (*i*) The smallest relation R having the reflexive property is

$$R \cup \Delta = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, c), (d, c), (d, a), (a, d)\}$$

(*ii*) The smallest relation R having symmetric property is

$$R \cup R^{-1} = \{(a, b), (b, a), (b, c), (c, b), (d, c), (c, d), (d, a), (a, d), (d, d)\}$$

(*iii*) To find the transitive closure of R, draw the directed graph of R as shown in Fig. 38.




$H_1 \cup H_2 = G.$

$H_{1,2}$ 4-2

$H_2 = 9-6 = 3$

Fig. 38.

Now, from vertex $a$, we have paths to the vertices $a$, $b$, $c$ and $d$. So, the ordered pairs included in R* are $(a, a)$ $(a, b)$ $(a, c)$ and $(a, d)$. From vertex $b$, we have paths to vertex $c$. So, ordered pairs included in R* are $(b, c)$. From vertex $c$, we have no paths to any other vertex. Hence, no ordered pair is included in R*. Similarly, from vertex $d$, we have paths to vertices $a$, $b$, $c$ and $d$. So, the ordered pairs included in R* are $(d, a)$, $(d, b)$, $(d, c)$ and $(d, d)$.

Therefore, transitive closure of R is

$$R^* = \{(a, a), (a, b), (a, c), (a, d), (b, c), (d, a), (d, b), (d, c), (d, d)\}.$$

**Problem 18.** *Determine whether the relation R on the set A of all triangles in the plane defined by*

*R = {(a, b) : triangle a is similar to triangle b} is an equivalence relation.*

**Sol. Reflexive.** Triangle $a$ is similar to itself. Hence $(a, a) \in R$ for every $a \in A$. Therefore, R is reflexive.

**Symmetric.** If triangle $a$ is similar to $b$, then it is sure that triangle $b$ is also similar to $a$. Hence, whenever $(a, b) \in R$, we have $(b, a)$ also belong to R. So, R is symmetric.

**Transitive.** It triangle $a$ is similar to $b$ and triangle $b$ is similar to $c$.

We have, triangle $a$ similar to $c$.

Hence, whenever $(a, b) \in R$ and $(b, c) \in R$, we have $(a, c) \in R$.

Therefore, R is transitive.

So, being reflexive, symmetric and transitive ; the relation R is an equivalence relation.

**Problem 19.** *Determine whether the relation S = {(a, b) : a ≥ b} on the set R of real numbers is an equivalence relation.*

**Sol. Reflexive.** We have $(a, a) \in S$, for every $a \in R$ because $a = a$ for every $a \in R$. Hence, the relation is reflexive.

**Symmetric.** Let $(a, b) \in S$. So, we have $a \geq b$ but for this it is not possible that $b \geq a$. Hence, $(b, a) \notin S$. Therefore, relation is not symmetric.

**Transitive.** Let $(a, b) \in S$ and $(b, c) \in S$. So, when we have $a \geq b$ and $b \geq c$, it is sure that $a \geq c$. Hence, $(a, c) \in S$. Therefore, the relation is transitive.

So, being the relation reflexive and transitive but not symmetric, it is not an lence relation.

**Problem 20.** *Prove that the relation R "a – b is divisible by 5"* $\forall a, b \in I_+$ *(a se integers) is an equivalence relation.*

**Sol. Reflexive.** The relation is reflexive because $a - a$ is divisible by 5 for every

**Symmetric.** Let $(a, b) \in R$.

If $a - b$ is divisible by 5 then it is sure that $b - a$ is also divisible by 5. Hence, (b Therefore, the relation is symmetric.

**Transitive.** Let $(a, b) \in R$ and $(b, c) \in R$

*i.e.,* $a - b$ is divisible by 5.

$b - c$ is divisible by 5.

It implies $a - c$ is divisible by 5.

Hence, $(a, c) \in R$.

Therefore, the relation is transitive.

So, being reflexive, symmetric and transitive, the relation "$a - b$ is divisible by 5" equivalence relation. Hence proved.

**Problem 21.** *If R and S are equivalence relations on the set A. Show that follow equivalence relations*

(*i*) $R \cap S$ (*ii*) $R \cup S$.

**Sol.** (*i*) **Reflexive.** The relations R and S are equivalence relations. We have (a and $(a, a) \in S$, for every $a \in A$. So, $(a, a) \in R \cap S$ for every $a \in A$. Therefore, $R \cap S$ is re

**Symmetric.** Let $(a, b) \in R \cap S$

So, we have $(a, b) \in R$ also $(b, a) \in R$ ($\because$ R is sym

Similarly, we have $(a, b) \in S$

Also $(b, a) \in S$ ($\because$ S is sym

Thus, from above, we conclude that $(b, a) \in R \cap S$

Therefore, $R \cap S$ is symmetric.

**Transitive.** Let $(a, b) \in R \cap S$ and $(b, c) \in R \cap S$.

So, we have $(a, b) \in R$ and $(b, c) \in R$

Also, $(a, c) \in R$ ($\because$ R is tra

Similarly, we have $(a, b) \in S$ and $(b, c) \in S$

Also, $(a, c) \in S$ ($\because$ S is tra

Thus, from above we conclude that $(a, c) \in R \cap S$.

Therefore, $R \cap S$ is transitive.

So, being reflexive, symmetric and transitive, $R \cap S$ is an equivalence relation.

(*ii*) **Reflexive.** We have $(a, a) \in R$ and $(a, a) \in S$, for every $a \in A$. Hence, $(a, a) \in$ Therefore, the $R \cup S$ is reflexive.

**Symmetric.** Let $(a, b) \in R \cup S$.

So, we have $(a, b) \in R \Rightarrow (b, a) \in R$ ($\because$ R is sym

or $(a, b) \in S \Rightarrow (b, a) \in S$ ($\because$ S is sym

Hence, from above we conclude that $(b, a) \in R \cup S$

Therefore, $R \cup S$ is symmetric.

**Transitive.** Let $(a, b) \in R \cup S$ and $(b, c) \in R \cup S$.

So, we have $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ ($\because$ R is transitive)

or $(a, b) \in S$ and $(b, c) \in S \Rightarrow (a, c) \in S$ ($\because$ S is transitive)

Hence, from above we conclude that $(a, c) \in R \cup S$

Therefore, $R \cup S$ is transitive.

So, relation $R \cup S$ being reflexive, symmetric and transitive is an equivalence relation.

**Problem 22.** *Let* $A = \{a, b, c, d, e\}$ *and*

$$R = \{(a, b), (a, a), (b, a), (b, b), (c, c), (d, d), (d, e), (e, d), (e, e)\}$$
$$S = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, c), (c, a), (d, e), (e, d)\}$$

*be equivalence relations on A. Determine the partitions corresponding to following (if it is an equivalence relation).*

*(i)* $R^{-1}$ *(ii)* $R \cup S$ *(iii)* $R \cap S$.

**Sol.** (*i*) The relation $R^{-1}$ is

$$R^{-1} = \{(b, a), (a, a), (a, b), (b, b), (c, c), (d, d), (e, d), (d, e), (e, e)\}$$

The partition corresponding $R^{-1}$ is

$$P = \{\{a, b\}, \{c\}, \{d, e\}\}.$$

(*ii*) The relation $R \cup S$ is

$$R \cup S = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (b, a), (a, c), (c, a), (e, d), (d, e)\}$$

The relation $R \cup S$ is not an equivalence relation.

(*iii*) The relation $R \cap S$ is

$$R \cap S = \{(a, a), (b, b), (c, c), (d, d), (e, e), (d, e), (e, d)\}$$

The partition corresponding to $R \cap S$ is

$$P = \{\{a\}, \{b\}, \{c\}, \{d, e\}\}.$$

**Problem 23.** *Let R is an equivalence relation on the set* $A = \{p, q, r, s\}$ *defined by partition* $P = \{\{p, s\}, \{q, r\}\}$. *Determine the elements of equivalence relation and also find the equivalence classes of R.*

**Sol.** The elements of equivalence relation defined by partition P is

$$R = \{(p, p), (s, s), (p, s), (s, p), (q, q), (r, r), (q, r), (r, q)\}$$

The equivalence classes of R are

$$[p] = [s] = \{p, s\}$$
$$[q] = [r] = \{q, r\}.$$

**Problem 24.** *Let R is an equivalence relation on the set* $A = \{7, 8, 9, 10\}$ *defined by partition* $P = \{(7), (8), (9), (10)\}$. *Determine the elements of equivalence relation and also find the equivalence classes of R.*

**Sol.** The elements of equivalence relation defined by partition P is

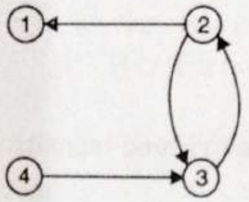
$$R = \{(7, 7), (8, 8), (9, 9), (10, 10)\}.$$

The equivalence classes of R are

$$[7] = \{7\}, \qquad [8] = \{8\}$$
$$[9] = \{9\}, \qquad [10] = \{10\}.$$

**Problem 25.** *Let A = {1, 2, 3, 4} and R = {(2, 1), (2, 3), (3, 2), (4, 3)}. Find the tr closure of R using Warshall's algo.*

**Sol.** The diagraph of relation R and its corresponding matrix $M_R$ is shown in Fi



$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Fig. 39.

The value of $n = |A| = 4$. Thus, we have to find the Warshall sets $w_0, w_1, w_2, w_3$
The first set $w_0$ is same as $M_R$ and is shown below :

$$w_0 = M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Now to find $w_1$ from $w_0$, we have row number 2 from column 1 in $w_0$. But there column numbers for row 1 in $w_0$. Thus, no new entries of 1 are added in $w_1$. So, $w_1$ is same and is shown below :

$$w_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

To find $w_2$ from $w_1$, we have row number 3 for column 2 and column numbers 1 and row 2. Thus, the new entries in $w_2$ are (3, 1) and (3, 3). Thus, $w_2$ is shown as follows :

$$w_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Similarly, $w_3$ is obtained from $w_2$. Here we have row number 2, 3 and 4 for column column numbers 1, 2 and 3 for row 3. Thus, the possible new entries in $w_3$ are (2, 1), (2, 3), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2) and (4, 3). Thus, $w_3$ is as follows :

$$w_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Similarly, $w_4$ can be obtained from $w_3$. But there are no new entries of 1's in $w_4$ as t is row number for column 4, where there is 1 in $w_3$. Hence,

$$M_{R^\infty} = w_4 = w_3$$

which is the transitive closure of R.