# 1

# Sets

## DEFINITION

A set is defined as a collection of distinct objects of same type or class of objects.

The objects of a set are called elements or members of the set. Objects can be numbers, alphabets, names etc.

*e.g.*, $$A = \{1, 2, 3, 4, 5\}$$

A is a set of numbers containing elements 1, 2, 3, 4 and 5.

* A set is usually denoted by capital letters A, B, C, D, P, Q, R, S, T etc.

* Elements of the set are defined by $p, q, r, t$ or $p_1, q_1, r_1, t_1$ etc.

## SET FORMATION

The set can be formed by two ways :

(*i*) Tabular form of a set

(*ii*) Builder form of a set.

(*i*) **Tabular Form of a Set.** If a set is defined by actually listing its members, *e.g.*, if set P contains elements $a, b, c, d$ then it is expressed as $P = \{a, b, c, d\}$.

This is called tabular form of a set.

(*ii*) **Builder Form of a Set.** If a set is defined by the properties which its elements must satisfy *e.g.*,

$$P = \{x : x \in N, x \text{ is a multiple of } 3\}.$$

$$R = \{x : x > 1 \text{ and } x < 10 \text{ and } x \text{ is an odd integer}\}.$$

$$T = \{x : x \text{ is even less than } 9\}.$$

## STANDARD NOTATIONS

| | |
|---|---|
| $x \in A$ | $x$ belongs to A or $x$ is an element of set A. |
| $x \notin A$ | $x$ does not belong to set A. |
| $\phi$ | Empty set. |
| U | Universal set. |
| N | The set of all natural numbers. |
| I | The set of all integers. |
| $I_0$ | The set of all non-zero integers. |

| | |
|---|---|
| $I_+$ | The set of all + ve integers. |
| $C, C_0$ | The sets of all complex, non-zero complex numbers respectively. |
| $Q, Q_0, Q_+$ | The sets of rational, non-zero rational, + ve rational numbers respectively. |
| $R, R_0, R_+$ | The sets of real, non-zero real, +ve real number respectively. |

## FINITE SET

If a set consists of specific number of different elements then that set is called finite set. *e.g.*,

$$P = \{x : x \in N, 3 < x < 9\}.$$
$$Q = \{2, 4, 6, 8\}.$$
$$R = \{\text{months of year}\}.$$

## INFINITE SET

If a set consists of infinite number of different elements or if the counting of different elements of the set does not come to an end, the set is called infinite set. *e.g.*,

$$I = \{\text{The set of all integers}\}$$
$$E = \{x : x \in N, x \text{ is a multiple of } 2\}.$$

## EQUALITY OF SETS

Two sets A and B are said to be equal and written as A = B if both have the same elements. Therefore, every element which belongs to A is also an element of the set B and every element which belongs to the set B is also an element of the set A.

$$A = B \quad \Leftrightarrow \quad (x \in A \ \Leftrightarrow \ x \in B).$$

If there is some element in a set A that does not belong to set B or vice-versa then $A \neq B$ *i.e.*, A is not equal to B.

* A set does not change if one or more of its elements are repeated.

* A set does not change if we change the order in which its elements are tabulated. *e.g.*,

(*a*) $A = \{x : x < 10 \text{ and } x \text{ is even}\}$

$B = \{2, 4, 6, 8\}$

$C = \{x : x > 1 \text{ and } x < 10 \text{ and } x \text{ is even}\}$

All the three sets are equal.

(*b*) $P = \{r, s, t\}$

$Q = \{r, r, s, t\}$

$R = \{s, r, t\}$

$S = \{r, s, t, t\}$

All the four sets P, Q, R, S are equal.

(*c*) $A = \{a, b, c\}$

$B = \{b, a, c\}$

$C = \{b, a\}$

The sets A and B are equal but C is not equal to either A or B.

## DISJOINT SETS

Two sets A and B are said to be disjoint if no element of A is in B and no element of B is in A. *e.g.*, $R = \{a, b, c\}$, $S = \{k, p, m\}$

R and S are disjoint sets.

## FAMILY OF SETS

If a set A contains elements which are itself sets then it is called family of sets or a set of sets. *e.g.*, $A = \{\{1, 2\}, \{3, 4\}, \phi\}$

A is set of sets.

## SUBSET OF A SET

If every element of a set A is also an element of a set B then A is called subset of B and is written as $A \subseteq B$. B is called superset of A.

$$A \subseteq B = \{x : x \in A \Rightarrow x \in B\}$$

(*a*) **Proper Subset.** If A is subset of B and $A \neq B$ then A is said to be proper subset of B. If A is a proper subset of B then B is not subset of A *i.e.*, there is atleast one element in B which is not in A. *e.g.*,

(*i*) Let $A = \{2, 3, 4\}$

$B = \{2, 3, 4, 5\}$

A is a proper subset of B.

(*ii*) The null set $\phi$ is a proper subset of every set.

(*b*) **Improper Subset.** If A is subset of B and A = B, then A is said to be an improper subset of B. *e.g.*,

(*i*) $A = \{2, 3, 4\}$, $B = \{2, 3, 4\}$

A is improper subset of B.

(*ii*) Every set is improper subset of itself.

## NULL SET OR EMPTY SET

The set that contains no element is called the null set or the empty set and is denoted by $\phi$. *e.g.*,

(*a*) $P = \{x : x^2 = 4, x \text{ is odd}\}$.

(*b*) $Q = \{x : x^2 = 9, x \text{ is even}\}$.

(*c*) $R = \{x : x^2 = 9, 2x = 4\}$.

The set $\phi = \{0\}$ is not a null set because 0 is the element of the set.

The set $A = \{\phi\}$ is not a null set because set $\phi$ is the element of the set.

## POWER SET

The power set of any given set A is the set of all subsets of A and is denoted by P(A). If A has $n$ elements then P(A) has $2^n$ elements.

If $A = \{r, s\}$ then its subsets are $\phi$, $\{r\}$, $\{s\}$, $\{r, s\}$

$P(A) = \{\phi, \{r\}, \{s\}, \{r, s\}\}$ which has $2^2 = 4$ elements.

## UNIVERSAL SET

If all the sets under investigation are subsets of a fixed set U, then the set U is called universal set. *e.g.*, in plane geometry, the universal set consists of all the points in the plane.

## COMPARABILITY

Two sets A and B are comparable if one of them is subset of other, *i.e.*, $A \subseteq B$ or $B \subseteq A$. If $A \subseteq B$ and $B \subseteq A$, then $A = B$. Set $\phi$ is comparable to every set. Every set is comparable to the universal set U.

Two sets A and B are said to be incomparable if $A \not\subseteq B$ and also $B \not\subseteq A$ *i.e.*, there is atleast one element in A not in B and vice-versa.

**Theorem I.** *Prove that, the null set $\phi$ is a subset of every set.*

**Proof.** Suppose that $\phi$ is not a subset of A *i.e.*, $\phi \not\subset A$.

Then there exists an element $x \in \phi$ such that $x \notin A$.

But $\phi$ is the null set and hence for every $x$, $x \notin \phi$ because null set does not contain any element.

From above, $x \in \phi$ and $x \notin \phi$ which is contradiction. Hence, $\phi \not\subset A$ is wrong supposition.

Therefore, $\phi$ is a subset of A. Hence proved.

**Theorem II.** *Prove that, every set is a subset of itself i.e., $A \subseteq A$.*

**Proof.** Let $x \in A \;\Rightarrow\; x \in A \;\therefore\; A \subseteq A$

As every element belonging to set A is also an element of set A. Therefore, A is subset of A or itself. Hence proved.

**Theorem III.** *Prove that, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

**Proof.** Let $x$ be any element of the set A.

Since, $A \subseteq B \quad \therefore\; x \in A \;\Rightarrow\; x \in B$

$B \subseteq C \quad \therefore\; x \in B \;\Rightarrow\; x \in C \qquad \therefore\; x \in A \;\Rightarrow\; x \in C$

$\therefore\quad A \subseteq C$. Hence proved.

**Theorem IV.** *Prove that, if $A \subseteq B$, $B \subseteq C$, $C \subseteq A$, then $A = C$.*

**Proof.** If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

$C \subseteq A$ (Given)

$\therefore\quad A \subseteq C$ and $C \subseteq A$

$\therefore\quad A = C$. Hence proved.

**Theorem V.** *Prove that, if $A \subset \phi$, then $A = \phi$.*

**Proof.** $\phi \subset A$ (As $\phi$ is subset of every set).

But $A \subset \phi$ (Given)

$\therefore\quad A = \phi$. Hence proved.

**Example 1.** *How many subsets can be formed from a set of n elements ? How many of these will be proper and how many improper ?*

**Sol.** There are $^nC_1$ subsets each consisting of one of the elements of the given set.

There will be $^nC_2$ subsets each consisting of any two of the $n$ elements of the given set.

Also, there will be $^nC_3$ subsets each consisting of any three of the $n$ elements of given set.

At last, there will be $^nC_n$ subset consisting of all the $n$ elements of the given set.

Also, there will be one set $\phi$.

Hence, the total number of subsets will be

$$(^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n) + 1 \qquad \text{(For the null set)}$$

or
$$^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n = 2^n \qquad \because \ {}^nC_0 = 1.$$

Therefore, $2^n$ subsets formed from $n$ elements of a set. Out of $2^n$ subsets $2^n - 1$ subsets will be proper and 1 (one) subset improper *i.e.*, the set itself.

## OPERATIONS ON SETS

The basic set operations are :

**1. Union of Sets.** Union of the sets A and B is defined to be the set of all those elements which belong to A or B or both and is denoted by $A \cup B$.

$$A \cup B = \{x : x \in A \quad \text{or} \quad x \in B\}$$

*e.g.*, Let
$$A = \{1, 2, 3\}, B = \{3, 4, 5, 6\}$$
$$A \cup B = \{1, 2, 3, 4, 5, 6\}.$$

**2. Intersection of Sets.** Intersection of two sets A and B is the set of all those elements which belong to both A and B and is denoted by $A \cap B$.

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

*e.g.*, Let
$$A = \{a, b, c, d\}, B = \{a, b, l, m\}$$
$$A \cap B = \{a, b\}.$$

**3. Difference of Sets.** The difference of two sets A and B is a set of all those elements which belong to A but do not belong to B and is denoted by $A - B$.

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

*e.g.*, Let
$$A = \{a, b, c, d\}, B = \{d, l, m, n\}$$
$$A - B = \{a, b, c\}.$$

**4. Complement of a Set w.r.t. a Universal Set.** The complement of a set A is a set of all those elements of the universal set which do not belong to A and is denoted by $A^c$.

$$A^c = U - A = \{x : x \in U \text{ and } x \notin A\} = \{x : x \notin A\}$$

*e.g.*, Let U is the set of all natural numbers.

$$A = \{1, 2, 3\}$$
$$A^c = \{\text{all natural numbers except } 1, 2, 3\}.$$

**5. Symmetric Difference of Sets.** The symmetric difference of two sets A and B is the set containing all the elements that are in A or in B but not in both and is denoted by $A \oplus B$ *i.e.*,

$$A \oplus B = (A \cup B) - (A \cap B)$$

e.g., (i) Let $A = \{a, b, c, d\}$

$B = \{a, b, l, m\}$

$A \oplus B = \{c, d, l, m\}$

(ii) Let $A = \{1, 2, 3, 4\}$

$A \oplus \phi = \{1, 2, 3, 4\}$

(iii) Let $A = \{l, m, n, m\}$

$A \oplus A = \phi$.

## ALGEBRA OF SETS

A set is obtained from a set formula by replacing the variables by definite sets. When the set variables appearing in two set formulas are replaced by any sets and both the set formulas are equal as sets, then we call that both the set formula are equal. The equality of set formulas are called set identities. Some of the identities describe certain properties of the operations involved. These properties describe an algebra called algebra of sets.

**Table I**

| | |
|---|---|
| (i) **Idempotent Laws**<br>(a) $A \cup A = A$<br>(b) $A \cap A = A$<br>(ii) **Associative Laws**<br>(a) $(A \cup B) \cup C = A \cup (B \cup C)$<br>(b) $(A \cap B) \cap C = A \cap (B \cap C)$<br>(iii) **Commutative Laws**<br>(a) $A \cup B = B \cup A$<br>(b) $A \cap B = B \cap A$<br>(iv) **Distributive Laws**<br>(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$<br>(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$<br>(v) **De Morgan's Laws**<br>(a) $(A \cup B)^c = A^c \cap B^c$<br>(b) $(A \cap B)^c = A^c \cup B^c$ | (vi) **Identity Laws**<br>(a) $A \cup \phi = A$<br>(b) $A \cap U = A$<br>(c) $A \cup U = U$<br>(d) $A \cap \phi = \phi$<br>(vii) **Complement Laws**<br>(a) $A \cup A^c = U$<br>(b) $A \cap A^c = \phi$<br>(c) $U^c = \phi$<br>(d) $\phi^c = U$<br>(viii) **Involution Law**<br>(a) $(A^c)^c = A$ |

The Table I shows the laws of algebra of sets.

**Example 2.** *Prove (a) $A \cup A = A$ (b) $A \cap A = A$.*

**Sol.** (a) To prove $A \cup A = A$

Since, $B \subset A \cup B$, therefore $A \subset A \cup A$

Let $x \in A \cup A \Rightarrow x \in A$ or $x \in A \Rightarrow x \in A$

$\therefore \quad A \cup A \subset A$

As $A \cup A \subset A$ and $A \subset A \cup A \Rightarrow A = A \cup A$. Hence proved.

(b) To prove $A \cap A = A$

Since, $A \cap B \subset B$, therefore $A \cap A \subset A$

Let $x \in A \Rightarrow x \in A$ and $x \in A$

A∩A

$\Rightarrow \quad x \in A \cap A \qquad \therefore \; A \subset A \cap A$

As $A \cap A \subset A$ and $A \subset A \cap A \;\Rightarrow\; A = A \cap A$. Hence proved.

**Example 3.** *Prove (a)* $(A \cup B) \cup C = A \cup (B \cup C)$
*(b)* $(A \cap B) \cap C = A \cap (B \cap C)$.

**Sol.** (*a*) To prove $(A \cup B) \cup C = A \cup (B \cup C)$

Let some $x \in (A \cup B) \cup C$

$\Rightarrow \quad (x \in A \text{ or } x \in B) \text{ or } x \in C$

$\Rightarrow \quad x \in A \text{ or } x \in B \text{ or } x \in C$

$\Rightarrow \quad x \in A \text{ or } (x \in B \text{ or } x \in C)$

$\Rightarrow \quad x \in A \text{ or } x \in B \cup C$

$\Rightarrow \quad x \in A \cup (B \cup C)$.

Similarly, if some $x \in A \cup (B \cup C)$, then $x \in (A \cup B) \cup C$.

Thus, any $x \in A \cup (B \cup C) \;\Leftrightarrow\; x \in (A \cup B) \cup C$. Hence proved.

(*b*) To prove $(A \cap B) \cap C = A \cap (B \cap C)$

Let some $x \in A \cap (B \cap C) \;\Rightarrow\; x \in A \text{ and } x \in B \cap C$

$\Rightarrow \quad x \in A \text{ and } (x \in B \text{ and } x \in C) \;\Rightarrow\; x \in A \text{ and } x \in B \text{ and } x \in C$

$\Rightarrow \quad (x \in A \text{ and } x \in B) \text{ and } x \in C \;\Rightarrow\; x \in A \cap B \text{ and } x \in C$

$\Rightarrow \quad x \in (A \cap B) \cap C$.

Similarly, if some $x \in A \cap (B \cap C)$, then $x \in (A \cap B) \cap C$

Thus, any $x \in (A \cap B) \cap C \;\Leftrightarrow\; x \in A \cap (B \cap C)$. Hence proved.

**Example 4.** *Prove (a)* $A \cup B = B \cup A$   *(b)* $A \cap B = B \cap A$.

**Sol.** (*a*) To prove

$A \cup B = B \cup A$

3, 4,

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

$= \{x : x \in B \text{ or } x \in A\}$   ($\because$ Order is not preserved in case of sets)

$= B \cup A$. Hence proved.

(*b*) To prove   $A \cap B = B \cap A$

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

$= \{x : x \in B \text{ and } x \in A\}$

($\because$ Order is not preserved in case of sets)

$= B \cap A$. Hence proved.

**Example 5.** *Prove (a) Intersection of sets is distributive w.r.t. union of sets i.e.,*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

*(b) Union of sets is distributive w.r.t. intersection of sets i.e.,*

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

**Sol.** (*a*) To prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let $\quad x \in A \cap (B \cup C) \;\Rightarrow\; x \in A \text{ and } x \in B \cup C$

$\Rightarrow \quad (x \in A \text{ } \cancel{\text{and}} \text{ or } x \in B) \text{ and } (x \in B \text{ or } x \in C)$

$\Rightarrow \quad (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)$

$\Rightarrow \quad x \in A \cap B \text{ or } x \in A \cap C$

$\Rightarrow \quad x \in (A \cap B) \cup (A \cap C)$

Therefore, $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ ...(*i*)

Again, let $y \in (A \cap B) \cup (A \cap C) \Rightarrow y \in A \cap B$ or $y \in A \cap C$

$\Rightarrow \quad (y \in A \text{ and } y \in B) \text{ or } (y \in A \text{ and } y \in C)$

$\Rightarrow \quad (y \in A \text{ or } y \in A) \text{ and } (y \in B \text{ or } y \in C)$

$\Rightarrow \quad y \in A \text{ and } y \in B \cup C$

$\Rightarrow \quad y \in A \cap (B \cup C)$

Therefore, $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$ ...(*ii*)

Combining (*i*) and (*ii*), we get $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Hence proved.

(*b*) To prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let $x \in A \cup (B \cap C) \Rightarrow x \in A$ or $x \in B \cap C$

$\Rightarrow \quad (x \in A \text{ or } x \in A) \text{ or } (x \in B \text{ and } x \in C)$

$\Rightarrow \quad (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)$

$\Rightarrow \quad x \in A \cup B \text{ and } x \in A \cup C$

$\Rightarrow \quad x \in (A \cup B) \cap (A \cup C)$

Therefore, $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ ...(*i*)

Again, let $y \in (A \cup B) \cap (A \cup C) \Rightarrow y \in A \cup B$ and $y \in A \cup C$

$\Rightarrow \quad (y \in A \text{ or } y \in B) \text{ and } (y \in A \text{ or } y \in C)$

$\Rightarrow \quad (y \in A \text{ and } y \in A) \text{ or } (y \in B \text{ and } y \in C)$

$\Rightarrow \quad y \in A \text{ or } y \in B \cap C$

$\Rightarrow \quad y \in A \cup (B \cap C)$

Therefore, $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$ ...(*ii*)

Combining (*i*) and (*ii*), we get $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Hence proved.

**Example 6.** *Prove De Morgan's Laws*

(*a*) $(A \cup B)^c = A^c \cap B^c$ (*b*) $(A \cap B)^c = A^c \cup B^c$.

**Sol.** (*a*) To prove $(A \cup B)^c = A^c \cap B^c$

Let $x \in (A \cup B)^c \Rightarrow x \notin A \cup B$ $(\because a \in A \Leftrightarrow a \notin A^c)$

$\Rightarrow x \notin A \text{ and } x \notin B$

$\Rightarrow x \in A^c \text{ and } x \in B^c$

$\Rightarrow x \in A^c \cap B^c$

Therefore, $(A \cup B)^c \subset A^c \cap B^c$ ...(*i*)

Again, let $x \in A^c \cap B^c \Rightarrow x \in A^c \text{ and } x \in B^c$

$\Rightarrow x \notin A \text{ and } x \notin B$

$\Rightarrow x \notin A \cup B$

$\Rightarrow x \in (A \cup B)^c$

Therefore, $A^c \cap B^c \subset (A \cup B)^c$ ...(*ii*)

Combining (*i*) and (*ii*), we get $A^c \cap B^c = (A \cup B)^c$. Hence proved.

(*b*) To prove $(A \cap B)^c = A^c \cup B^c$

Let $x \in (A \cap B)^c \Rightarrow x \notin A \cap B$ $(\because a \in A \Leftrightarrow a \notin A^c)$

$\Rightarrow x \notin A \text{ or } x \notin B$

$\Rightarrow x \in A^c \text{ or } x \in B^c$

$\Rightarrow x \in A^c \cup B^c$

$\therefore \quad (A \cap B)^c \subset A^c \cup B^c$ ...(*i*)

Again, let $x \in A^c \cup B^c \Rightarrow x \in A^c$ or $x \in B^c$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \notin A \cap B$

$\Rightarrow x \in (A \cap B)^c$

$\therefore \quad A^c \cup B^c \subset (A \cap B)^c$ ....(*ii*)

Combining (*i*) and (*ii*), we get $(A \cap B)^c = A^c \cup B^c$. Hence proved.

**Example 7.** *Prove*

(*a*) $A \cup \phi = A$ (*b*) $A \cap \phi = \phi$ (*c*) $A \cup U = U$ (*d*) $A \cap U = A$.

**Sol.** (*a*) To prove $A \cup \phi = A$

Let $x \in A \cup \phi \Rightarrow x \in A$ or $x \in \phi$

$\Rightarrow x \in A$ ($\because x \notin \phi$, as $\phi$ is the null set)

Therefore, $x \in A \cup \phi \Rightarrow x \in A$ $A \cup B$

Hence $A \cup \phi \subset A$.

We know that $A \subset A \cup B$ for any set B.

But for $B = \phi$, we have $A \subset A \cup \phi$

From above, $A \subset A \cup \phi$, $A \cup \phi \subset A \Rightarrow A = A \cup \phi$. Hence proved.

(*b*) To prove $A \cap \phi = \phi$

If $x \in A$, then $x \notin \phi$ ($\because \phi$ is null set)

Therefore, $x \in A, x \notin \phi \Rightarrow A \cap \phi = \phi$. Hence proved.

(*c*) To prove $A \cup U = U$

Every set is a subset of universal set

$\therefore \quad A \cup U \subseteq U$

Also, $U \subseteq A \cup U$

Therefore, $A \cup U = U$. Hence proved.

(*d*) To prove $A \cap U = A$

We know $A \cap U \subset A$ ...(*i*)

So we have to show that $A \subset A \cap U$

Let $x \in A \Rightarrow x \in A$ and $x \in U$ ($\because A \subset U$ so $x \in A \Rightarrow x \in U$)

$\therefore \quad x \in A \Rightarrow x \in A \cap U$

$\therefore \quad A \subset A \cap U$ ...(*ii*)

From (*i*) and (*ii*), we get $A \cap U = A$. Hence proved.

**Example 8.** *Prove* (*a*) $A \cup A^c = U$ (*b*) $A \cap A^c = \phi$

(*c*) $U^c = \phi$ (*d*) $\phi^c = U$.

**Sol.** (*a*) To prove $A \cup A^c = U$

Every set is a subset of U

$\therefore \quad A \cup A^c \subset U$ ...(*i*)

We have to show that $U \subseteq A \cup A^c$

Let $x \in U \Rightarrow x \in A$ or $x \notin A$

$\Rightarrow x \in A$ or $x \in A^c \Rightarrow x \in A \cup A^c$

$\therefore \quad U \subseteq A \cup A^c$

From (*i*) and (*ii*), we get $A \cup A^c = U$. Hence proved.

(*b*) To prove $A \cap A^c = \phi$

As $\phi$ is a subset of every set

$\therefore \quad \phi \subseteq A \cap A^c$

We have to show that $A \cap A^c \subseteq \phi$

Let $x \in A \cap A^c \Rightarrow x \in A$ and $x \in A^c$

$\Rightarrow x \in A$ and $x \notin A$ (A contradictory stateme

$\Rightarrow x \in \phi$

$\therefore \quad A \cap A^c \subset \phi$

From (*i*) and (*ii*), we get $A \cap A^c = \phi$. Hence proved.

(*c*) To prove $U^c = \phi$

Let $x \in U^c \Leftrightarrow x \notin U \Leftrightarrow x \in \phi$ (As U is a universal s

$\therefore \quad U^c = \phi$. Hence proved.

(*d*) To prove $\phi^c = U$

Let $x \in \phi^c \Leftrightarrow x \notin \phi \Leftrightarrow x \in U$ (As $\phi$ is an empty s

$\therefore \quad \phi^c = U$. Hence proved.

**Example 9.** *Prove* $(A^c)^c = A$.

**Sol.** Let $x \in (A^c)^c \Leftrightarrow x \notin A^c \Leftrightarrow x \in A$

$\therefore \quad (A^c)^c = A$. Hence proved.

## CARDINALITY OF A SET

The total numbers of unique elements in the set is called the cardinality of the set. T cardinality of the countably infinite set is countably infinite *e.g.*,

**1.** Let $P = \{k, l, m, n\}$

The cardinality of the set P is 4.

**2.** Let A is the set of all non-negative even integers *i.e.*,

$$A = \{0, 2, 4, 6, 8, 10, .........\}$$

As A is countably infinite set hence the cardinality.

## VENN DIAGRAMS

The Venn diagram represents sets as regions in a plane. These are used to understa the relationships between sets. Results from a Venn diagram usually are not considered constitute a proof, but the results in the Venn diagram can be converted into a proof.

**Venn Diagrams and Set Operations.** Venn diagrams are used to represent set o erations. In Venn diagrams, shadings are used to indicate which regions represent which set The results are obtained by determining which regions in the diagram are shaded in whi fashion.

To draw Venn diagrams following points are to be taken into consideration :

(*i*) All the sets are usually represented by circles.

(*ii*) The universal set is represented by a rectangle.

(*iii*) The complement of the set is represented by that portion of the universal set which is not in the set.

**Example 10.** *Draw Venn diagram to represent the sets A, $A^c$ and U.*

**Sol.** The shaded portion represents the set A. (Fig. 1).

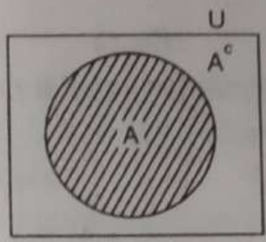



Fig. 1

The unshaded portion represents the set $A^c$ w.r.t. the universal set represented by the rectangle.

**Example 11.** *Draw the Venn diagrams for the following :*

(*i*) $A \cup B$ (*ii*) $A \cap B$.

**Sol.** (*i*) The Venn diagram for $A \cup B$ is shown in Fig. 2.

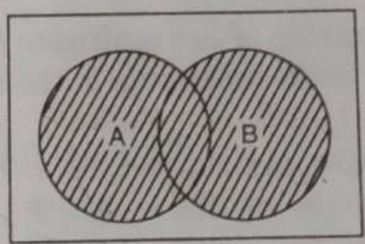



Fig. 2

(*ii*) The Venn diagram for $A \cap B$ is shown in Fig. 3.

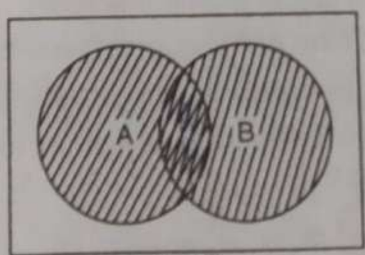
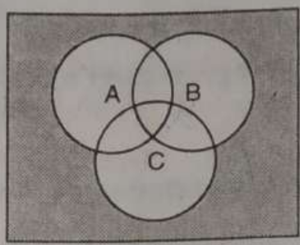



Fig. 3

The region shaded in both directions represents $A \cap B$.

**Example 12.** *Draw the Venn diagram for* $A^c \cap B^c \cap C^c$.

**Sol.** The Venn diagram for $A^c \cap B^c \cap C^c$ is shown in Fig. 4.




$A^c \cap B^c \cap C^c$

Fig. 4

The shaded portion represents $A^c \cap B^c \cap C^c$.

## MULTISETS

**Definition.** A multiset is an unordered collection of elements, in which the multiplicity of an element may be one or more than one or zero. The multiplicity of an element is the number of times the element repeated in the multiset. In other words, we can say that an element can appear any number of times in a set *e.g.*,

$$A = \{l, l, m, m, n, n, n, n\}$$
$$B = \{a, a, a, a, a, c\}.$$

## OPERATIONS ON MULTISETS

**1. Union of Multisets.** The union of two multisets A and B, is a multiset such that the multiplicity of an element is equal to the maximum of the multiplicity of an element in A and B and is denoted by $A \cup B$, *e.g.*,

Let $A = \{l, l, m, m, n, n, n, n\}$

$B = \{l, m, m, m, n\}$, then $A \cup B = \{l, l, m, m, m, n, n, n, n\}$.

**2. Intersection of Multisets.** The intersection of two multisets A and B, is a multiset such that the multiplicity of an element is equal to the minimum of the multiplicity of the element in A and B, and is denoted by $A \cap B$, *e.g.*,

Let $A = \{l, l, m, n, p, q, q, r\}$

$B = \{l, m, m, p, q, r, r, r, r\}$, then $A \cap B = \{l, m, p, q, r\}$.

**3. Difference of Multisets.** The difference of two multisets A and B, is a multiset such that the multiplicity of an element is equal to the multiplicity of the element in A minus the multiplicity of the element in B if the difference is +ve, and is equal to 0 if the difference is 0 or negative *e.g.*,

Let $A = \{l, m, m, m, n, n, n, p, p, p\}$

$B = \{l, m, m, m, m, n, r, r, r\}$, then $A - B = \{n, n, p, p, p\}$.

**4. Sum of Multisets.** The sum of two multisets A and B, is a multiset such that the multiplicity of an element is equal to the sum of the multiplicities of the element in A and B, *e.g.*, Let $\quad A = \{l, m, n, p, r\}$

$B = \{l, l, m, n, n, n, p, r, r\}$

then $\quad A + B = \{l, l, l, m, m, n, n, n, n, p, p, r, r, r\}$

**5. Cardinality of a Multiset.** The cardinality of a multiset is the number of distinct elements in a multiset without considering the multiplicity of an element *e.g.*,

Let $\quad A = \{l, l, m, m, n, n, n, p, p, p, p, q, q, q\}$

The cardinality of the multiset A is 5.

## ORDERED PAIRS

An ordered pair consists of two elements such that one of them is designated as first member and other as second member. If $p$ is the first element and $q$ is the second element, then the ordered pair is written as $(p, q)$.

*e.g.*, Let $\quad A = \{1, 2\}$, then possible ordered pairs of $A \times A$

$$A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

## CARTESIAN PRODUCT OF TWO SETS

The cartesian product of two sets P and Q in that order is the set of all ordered pairs whose first member belongs to the set P and second member belongs to set Q and is denoted by $P \times Q$ *i.e.*,

$$P \times Q = \{(x, y) : x \in P, y \in Q\}.$$

**Example 13.** *Let $P = \{a, b, c\}$ and $Q = \{k, l, m, n\}$. Determine the cartesian product of P and Q.*

**Sol.** The cartesian product of P and Q is

$$P \times Q = \begin{Bmatrix} (a,k),(a,l),(a,m),(a,n) \\ (b,k),(b,l),(b,m),(b,n) \\ (c,k),(c,l),(c,m),(c,n) \end{Bmatrix}.$$

**Example 14.** *Let $R = \{1, 2, 3\}$ and $S = \{4, 5, 6\}$. Determine the cartesian product.*

**Sol.** The cartesian product of R and S is

$$R \times S = \begin{Bmatrix} (1,4),(1,5),(1,6) \\ (2,4),(2,5),(2,6) \\ (3,4),(3,5),(3,6) \end{Bmatrix}.$$

**Example 15.** *Prove that $(A \times B) \cap (P \times Q) = (A \cap P) \times (B \cap Q)$.*

**Sol.** Let $\quad (x, y) \in (A \times B) \cap (P \times Q)$

$\Rightarrow \quad (x, y) \in (A \times B)$ and $(x, y) \in (P \times Q)$

$\Rightarrow \quad (x \in A$ and $y \in B)$ and $(x \in P$ and $y \in Q)$

$\Rightarrow \quad (x \in A$ and $x \in P)$ and $(y \in B$ and $y \in Q)$

$\Rightarrow \quad x \in (A \cap P)$ and $y \in (B \cap Q)$

$\Rightarrow \quad (x, y) \in (A \cap P) \times (B \cap Q)$

Therefore, $(A \times B) \cap (P \times Q) \subset (A \cap P) \times (B \cap Q)$ ...(i)

Now, conversely let $(x, y) \in (A \cap P) \times (B \cap Q)$

$\Rightarrow \quad x \in (A \cap P)$ and $y \in (B \cap Q)$

$\Rightarrow \quad (x \in A$ and $x \in P)$ and $(y \in B$ and $y \in Q)$

$\Rightarrow \quad (x \in A$ and $y \in B)$ and $(x \in P$ and $y \in Q)$

$\Rightarrow \quad (x, y) \in (A \times B)$ and $(x, y) \in (P \times Q)$

$\Rightarrow \quad (x, y) \in (A \times B) \cap (P \times Q)$

Therefore, $(A \cap P) \times (B \cap Q) \subset (A \times B) \cap (P \times Q)$ ...(ii)

From (*i*) and (*ii*), we have

$(A \times B) \cap (P \times Q) = (A \cap P) \times (B \cap Q)$. Hence proved.

**Example 16.** *Prove that* $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

**Sol.** Let $(x, y) \in A \times (B \cap C) \quad \Rightarrow \quad x \in A$ and $y \in B \cap C$

$\Rightarrow \quad x \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow \quad x \in A$ and $y \in B$ and $x \in A$ and $y \in C$

$\Rightarrow \quad (x, y) \in A \times B$ and $(x, y) \in A \times C$

$\Rightarrow \quad (x, y) \in (A \times B) \cap (A \times C)$

Therefore, $A \times (B \cap C) \subset (A \times B) \cap (A \times C)$ ...(i)

Now, conversely let $(x, y) \in (A \times B) \cap (A \times C)$

$\Rightarrow \quad (x, y) \in A \times B$ and $(x, y) \in A \times C$

$\Rightarrow \quad (x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C)$

$\Rightarrow \quad x \in A$ and $y \in B$ and $y \in C$

$\Rightarrow \quad x \in A$ and $y \in B \cap C$

$\Rightarrow \quad (x, y) \in A \times (B \cap C)$

Therefore, $(A \times B) \cap (A \times C) \subset A \times (B \cap C)$ ...(ii)

From (*i*) and (*ii*), we have $A \times (B \cap C) = (A \times B) \cap (A \times C)$. Hence proved.

**Example 17.** *Prove that* $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

**Sol.** Let $(x, y) \in A \times (B \cup C) \quad \Rightarrow \quad x \in A$ and $y \in B \cup C$

$\Rightarrow \quad x \in A$ and $(y \in B$ or $y \in C) \quad \Rightarrow \quad (x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$

$\Rightarrow \quad (x, y) \in A \times B$ or $(x, y) \in (A \times C) \quad \Rightarrow \quad (x, y) \in (A \times B) \cup (A \times C)$

Therefore, $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$ ...(i)

Now, conversely let $(x, y) \in (A \times B) \cup (A \times C)$

$\Rightarrow \quad (x, y) \in (A \times B)$ or $(x, y) \in (A \times C) \quad \Rightarrow \quad (x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$

$\Rightarrow \quad x \in A$ and $y \in B$ or $y \in C \quad \Rightarrow \quad x \in A$ and $y \in (B \cup C)$

$\Rightarrow \quad (x, y) \in A \times (B \cup C)$

Therefore, $(A \times B) \cup (A \times C) \subset A \times (B \cup C)$ ...(ii)

From (*i*) and (*ii*), we have $A \times (B \cup C) = (A \times B) \cup (A \times C)$. Hence proved.

**Example 18.** *Prove that if* $A \subset B$, *then* $A \times C \subset B \times C$.

**Sol.** Let $(x, y) \in A \times C \quad \Rightarrow \quad x \in A$ and $y \in C$

$\Rightarrow \quad x \in B$ and $y \in C$ $\quad (\because \; A \subset B)$

$$\Rightarrow \quad (x, y) \in B \times C$$

Therefore, $A \times C \subset B \times C$. Hence proved.

**Example 19.** *If S and T have n elements in common. Show that* $S \times T$ *and* $T \times S$ *have* $n^2$ *elements in common.*

**Sol.** Suppose, a set R, consisting of $n$ common elements of S and T.

Therefore, $R \subset S$ and $R \subset T$

Let $(x, y) \in (R \times R) \quad \Leftrightarrow x \in R$ and $y \in R$

$\Leftrightarrow (x \in R \text{ and } y \in R)$ and $(x \in R \text{ and } y \in R)$

$\Leftrightarrow (x \in S \text{ and } y \in T)$ and $(x \in T \text{ and } y \in S) \quad (\because \; R \subset S ; R \subset T)$

$\Leftrightarrow (x, y) \in (S \times T)$ and $(x, y) \in (T \times S)$

$\Leftrightarrow (x, y) \in (S \times T) \cap (T \times S)$

Therefore, $\quad (R \times R) = (S \times T) \cap (T \times S)$.

The right hand side contains ordered pairs common to both $S \times T$ and $T \times S$.

The left hand side $R \times R$ has $n^2$ elements $\quad (\because \;$ R has $n$ elements)

Since, the two sets are equal, both have the same number of elements.

Hence, $S \times T$ and $T \times S$ have $n^2$ common elements.

## SOLVED PROBLEMS

**Problem 1.** *Write the following sets in builder form :*

(*a*) $A = \{1, 3, 5, 7, 9, ...\}$ (*b*) $B = \{1, 8, 27, 64, ...\}$

(*c*) $R = \{a, e, i, o, u\}$ (*d*) $S = \{2, 3, 5, 7, 11, 13, 17, 19, ...\}$

(*e*) *T = {JAWAHAR LAL NEHRU, INDIRA GANDHI, ..., ATAL BEHARI VAJPAI}*.

**Sol.** (*a*) $\{x : x$ is an odd +ve integer$\}$.

(*b*) $\{x : x$ is a +ve integer, $x$ is a perfect cube$\}$.

(*c*) $\{x : x$ is a letter of alphabet and $x$ is a vowel$\}$.

(*d*) $\{x : x$ is a +ve integer and $x$ is a prime number$\}$.

(*e*) $\{x : x$ was prime minister of India$\}$.

**Problem 2.** *Write the following sets in builder from :*

(*a*) $A = \{2, 4, 6, 8, 10, 12, 14\}$ (*b*) $K = \{3, 6, 9, 12, 15, 18, ...\}$

(*c*) *L = {PUNJAB, HARYANA, DELHI, UP, ..., BIHAR}*.

**Sol.** (*a*) $\{x : x$ is a +ve integer divisible by 2 and less than 15$\}$.

(*b*) $\{x : x$ is +ve integer and is multiple of 3$\}$.

(*c*) $\{x : x$ is the stage of India$\}$.

**Problem 3.** *Write the following sets in tabular form :*

(*a*) $A = \{x : x^2 = 9\}$ (*b*) $B = \{x : x$ *is a multiple of* 3 *and* $0 < x < 20\}$

(*c*) $C = \{x : x$ *is a +ve even integer*$\}$ (*d*) $D = \{x : x$ *is a multiple of* 5$\}$.

**Sol.** (*a*) $A = \{3\}$ (*b*) $B = \{3, 6, 9, 12, 15, 18\}$

(*c*) $C = \{2, 4, 6, 8, 10, ...\}$ (*d*) $D = \{5, 10, 15, 20, 25, ...\}$

**Problem 4.** *Write the following sets in tabular form :*

(*a*) $A = \{x : x$ *is a +ve integer and a perfect square*$\}$

(*b*) $B = \{x : x$ *is a divisor of* $24\}$

(*c*) $C = \{x : x$ *is a multiple of 3 and* $5\}$

(*d*) $D = \{x : x$ *is a multiple of 3 or* $5\}$

(*e*) $T = \{x : x$ *is a letter in the alphabet*$\}$.

**Sol.** (*a*) $A = \{1, 4, 9, 16, 25, 36, ...\}$ (*b*) $B = \{1, 2, 3, 4, 6, 8, 12, 24\}$

(*c*) $C = \{15, 30, 45, 60, ...\}$ (*d*) $D = \{3, 5, 6, 10, 15, ...\}$

(*e*) $T = \{a, b, c, d, e, ...\}$.

**Problem 5.** *Determine finite or infinite sets among the sets in Example 4.*

**Sol.** (*a*) Infinite. There are infinite integers that are perfect square.

(*b*) Finite. Divisor of 24 are finite.

(*c*) Infinite. There are infinite numbers that are multiple of 3 and 5.

(*d*) Infinite. There are infinite numbers that are multiple of 3 or 5.

(*e*) Finite. There are 26 letters in the alphabet.

**Problem 6.** *Determine finite or infinite sets among the following :*

(*i*) $R = \{$*month in a year*$\}$ (*ii*) $S = \{$*lines through the origin*$\}$

(*iii*) $T = \{$*positive odd integer*$\}$ (*iv*) $K = \{$*cities in India*$\}$

(*v*) $L = \{$*members of the Parliament*$\}$ (*vi*) $M = \{$*positive integer between 1 and* $-1\}$

**Sol.** (*i*) Finite. Twelve months in a year.

(*ii*) Infinite. There are infinite number of lines passing through origin.

(*iii*) Infinite. There are infinite number of +ve odd integers.

(*iv*) Finite. There are finite number of cities in India.

(*v*) Finite. The members of the parliament are finite.

(*vi*) Finite. The set is empty. There is no +ve integer between 1 and $-1$.

**Problem 7.** *Which of the following sets are equal :*

(*i*) $A = \{a, b, c, d\}$ (*ii*) $B = \{a, a, b, b, c, d, d, d\}$

(*iii*) $K = \{d, a, c, b\}$ (*iv*) $L = \{x : x$ *is the letter of alphabet before*

(*v*) $M = \{$*First four letters of the alphabet*$\}$.

**Sol.** All the sets are equal. In set B multiple number of same elements does not af the set. In set K, the order does not change the set.

**Problem 8.** *Which of the following sets are equal :*

(*a*) $R = \{x : x^2 = 7\}$ (*b*) $S = \{x : x + 4 = 4\}$

(*c*) $T = \{x : x^2 + 2 = 8\}$.

**Sol.** The sets R and T are equal as they are empty sets. The set S is not, because elem a belongs to set S.

**Problem 9.** *Determine the power set P(A) of the set* $A = \{1, 2, 3\}$.

**Sol.** $P(A) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \phi\}$.



Pr
(a)
Sc
P
false.

**Problem 10.** *Determine the power sets of the following sets :*

(*a*) $\{a\}$ (*b*) $\{\{a\}\}$ (*c*) $\{\phi, \{\phi\}\}$.

**Sol.** (*a*) $\{\{a\}, \phi\}$ (*b*) $\{\{\{a\}\}, \phi\}$ (*c*) $\{\{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}, \phi\}$.

**Problem 11.** *Let* $A = \{\phi, \{\phi\}\}$. *Determine whether the following statements are true or false.*

(*a*) $\phi \in P(A)$ (*b*) $\phi \subseteq P(A)$ (*c*) $\{\phi\} \subseteq P(A)$

(*d*) $\{\phi\} \subseteq A$ (*e*) $\{\phi\} \in P(A)$ (*f*) $\{\phi\} \in A$

(*g*) $\{\{\phi\}\} \subseteq P(A)$ (*h*) $\{\{\phi\}\} \subseteq A$ (*i*) $\{\{\phi\}\} \in P(A)$

(*j*) $\{\{\phi\}\} \in A$.

**Sol.** The power $P(A) = \{\{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}, \phi\}$

(*a*) True (*b*) True (*c*) True (*d*) True

(*e*) True (*f*) True (*g*) True (*h*) True

(*i*) True (*j*) False.

**Problem 12.** *Let* $A = \{a, \{a\}\}$. *Determine whether each of the following is true or false :*

(*a*) $\phi \in P(A)$ (*b*) $\phi \subseteq P(A)$ (*c*) $\{a\} \in P(A)$ (*d*) $\{a\} \subseteq P(A)$

(*e*) $\{\{a\}\} \in P(A)$ (*f*) $\{\{a\}\} \subseteq P(A)$ (*g*) $\{a, \{a\}\} \in P(A)$ (*h*) $\{a, \{a\}\} \subseteq P(A)$

(*i*) $\{\{\{a\}\}\} \in P(A)$ (*j*) $\{\{\{a\}\}\} \subseteq P(A)$.

**Sol.** The power set $P(A) = \{\{a\}, \{\{a\}\}, \{a, \{a\}\}, \phi\}$

(*a*) True (*b*) True (*c*) True (*d*) False

(*e*) True (*f*) True (*g*) True (*h*) False

(*i*) False (*j*) True.

**Problem 13.** *Let* $A = \{\phi\}$. *Let* $B = P(P(A))$.

(*a*) *Is* $\phi \in B$ ? $\phi \subseteq B$ ? (*b*) *Is* $\{\phi\} \in B$ ? $\{\phi\} \subseteq B$ ?

(*c*) *Is* $\{\{\phi\}\} \in B$ ? $\{\{\phi\}\} \subseteq B$ ?

**Sol.** The power set of A *i.e.*, $P(A) = \{\{\phi\}, \phi\}$

$$B = P(P(A)) = \{\{\phi\}, \{\{\phi\}\}, \{\{\phi\}, \phi\}, \phi\}$$

(*a*) Yes, $\phi \in B$ and $\phi \subseteq B$ as $\phi$ is a subset of every set.

(*b*) Yes, $\{\phi\} \in B$. Also, $\{\phi\} \subseteq B$ as $\phi$ is an element in B.

(*c*) Yes, $\{\{\phi\}\} \in B$ and also $\{\{\phi\}\} \subseteq B$ since $\{\phi\}$ is an element of set B.

**Problem 14.** *Suppose*

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$A = \{1, 4, 9\}$

$P = \{x : x \in U \text{ and } x \text{ is a perfect square}\}$

$R = \{1, 2, 3, 5, 7, 9\}$

$D = \{2, 3, 5, 7\}$

$N = \{x : x \in U \text{ and } x \text{ is a prime number}\}$

$\phi$

(*a*) *Determine which sets are subsets of others.*

(*b*) *Determine which sets are proper subsets of others.*

(*c*) *Determine pair of sets which are disjoint.*

(*d*) *Determine pair of sets which are comparable.*

(*e*) *Determine pair of sets which are incomparable.*

**Sol.** (*a*) All the sets are subsets of U since the elements of every set belong to U. Set $\phi$ is subset of all other sets. Set A is subset of P. Set D is subset of R and N.

(*b*) All the sets are proper subsets of U since they are not equal to U. Set D is proper subset of R. Set N is proper subset of R. Set $\phi$ is proper subset of all the other sets.

(*c*) The pairs A and D, A and N, P and D, and P and N are disjoint sets.

(*d*) All the sets are comparable with the set U.

Set D is comparable with R as $D \subseteq R$

Set N is comparable with R as $N \subseteq R$

All sets comparable with $\phi$ as $\phi \subseteq$ Every set.

(*e*) The pairs A and R, A and D, A and N, P and R, P and D, P and N, are incomparable

**Problem 15.** *Determine the cardinalities of the sets :*

(*a*) $P = \{n^7 : n \text{ is a positive integer}\}$ (*b*) $Q = \{n^{109} : n \text{ is a positive integer}\}$

(*c*) $P \cup Q$ (*d*) $P \cap Q$.

**Sol.** (*a*) The cardinality of the set P is infinite as the number of +ve integers are infinite.

(*b*) The cardinality of the set Q is infinite as the number of positive integers are infinite

(*c*) Since sets P and Q are both infinite, their union is also infinite and hence cardinality

(*d*) The cardinality of $P \cap Q$ is one because for $n = 1$

$$1^7 = 1^{109}$$

But for $n = 2$ ; $2^7 \neq 2^{109}$ and so on.

Hence, $P \cap Q$ contains only one element.

**Problem 16.** *Let N is the set of all natural numbers. Let P denotes all finite subsets of N What is the cardinality of set P ? Give reason.*

**Sol.** The number of subsets of any set is given by $2^n$, where $n$ is number of elements in the set. As the number of subsets of set N is $2^n$, hence there are $2^n$ number of subsets in set P Therefore, the cardinality of set S is $2^n$.

**Problem 17.** *Let* $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *and suppose that*

$A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7\}$, $C = \{5, 6, 7, 8, 9\}$, $D = \{1, 3, 5, 7, 9\}$.

*Find the following subsets of U :*

(*a*) $A \cup B$ (*b*) $A \cap B$ (*c*) $A'$

(*d*) $C - D$ (*e*) $(C \cup D)'$.

**Sol.** (*a*) $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$ (*b*) $A \cap B = \{4, 5\}$

(*c*) $A' = \{6, 7, 8, 9\}$ (*d*) $C - D = \{6, 8\}$

(*e*) $(C \cup D)' = \{2, 4\}$.

**Problem 18.** *Determine the following sets :*

(*a*) $\phi \cup \{\phi\}$ (*b*) $\phi \cap \{\phi\}$ (*c*) $\{\phi\} \cup \{a, \phi, \{\phi\}\}$ (*d*) $\{\phi\} \cap \{a, \phi, \{\phi\}\}$

(*e*) $\phi \oplus \{a, \phi, \{\phi\}\}$ (*f*) $\{\phi\} \oplus \{a, \phi, \{\phi\}\}$.

**Sol.** (*a*) $\{\phi\}$ (*b*) $\phi$

(*c*) $\{a, \phi, \{\phi\}\}$ (*d*) $\{\phi\}$

(*e*) $\{a, \phi, \{\phi\}\}$ (*f*) $\{a, \{\phi\}\}$.

**Problem 19.** *Determine whether each of the following statements is true or false. Explain your answer.*

(*a*) $A \cup P(A) = P(A)$ (*b*) $A \cap P(A) = A$

(*c*) $\{A\} \cup P(A) = P(A)$ (*d*) $\{A\} \cap P(A) = A$

(*e*) $A - P(A) = A$ (*f*) $P(A) - \{A\}$.

**Sol.** (*a*) $A \cup P(A) = P(A)$.

**False.** P(A) contains all subsets of A but does not contain all elements of A. Therefore, $A \cup P(A) \neq P(A)$.

(*b*) $A \cap P(A) = A$.

**False.** Since power set of A contains all subsets of A but no elements of A, hence there is no element common to both the sets.

(*c*) $\{A\} \cup P(A) = P(A)$

**True.** P(A) contains all subsets of A and {A} is also an element of P(A).

(*d*) $\{A\} \cap P(A) = A$.

**False.** {A} is common to both {A} and P(A) because P(A) contains all subsets of A but their intersection is not A but {A}.

(*e*) $A - P(A) = A$ **False.**

(*f*) $P(A) - \{A\} = P(A)$.

**False.** {A} is also an element of P(A). Therefore, $P(A) - \{A\} \neq P(A)$.

**Problem 20.** *Let* $A = \{\phi, a\}$. *Construct the following sets :*

(*a*) $A - \phi$ (*b*) $\{\phi\} - A$ (*c*) $A \cup P\{A\}$ (*d*) $A \cap P\{A\}$.

**Sol.** (*a*) $\{\phi, a\}$ (*b*) $\phi$

(*c*) $\{\phi, a\} \cup \{\{\phi\}, \{a\}, \{\phi, a\}, \phi\} = \{\phi, a, \{\phi\}, \{a\}, \{\phi, a\}\}$

(*d*) $\{\phi, a\} \cap \{\{\phi\}, \{a\}, \{\phi, a\}, \phi\} = \{\phi\}$.

**Problem 21.** *Let A, B, C be sets. Under what conditions is each of the following true ?*

(*a*) $(A - B) \cup (A - C) = \phi$ (*b*) $(A - B) \cup (A - C) = \phi$

(*c*) $(A - B) \cap (A - C) = \phi$ (*d*) $(A - B) \oplus (A - C) = \phi$.

**Sol.** (*a*) This is true if B and C are null sets or A and B are disjoint sets and A and C are disjoint sets.

(*b*) This is true if A = B = C or A, B and C are null sets.

(*c*) This is true if A = B = C or all three sets are null sets.

(*d*) This is true if $A \subseteq B$ and $A \subseteq C$.

It is also true if $A \subseteq B$ and $A \not\subseteq C$ but B = C.

**Problem 22.** *What can you say about P and Q if*

(*a*) $P \cap Q = P$ (*b*) $P \cup Q = P$ (*c*) $P \oplus Q = P$ (*d*) $P \cap Q = P \cap Q$.

**Sol.** (*a*) This tells that $P \subseteq Q$. (*b*) This tells that P = Q or $Q \subseteq P$.

(*c*) This tells that Q is a null set. (*d*) This tells that P = Q.

**Problem 23.** *Let A, B, C be arbitrary sets :*

(*a*) *Show that* $(A - B) - C = A - (B \cup C)$.

(*b*) *Show that* $(A - B) - C = (A - C) - B$.

(*c*) *Show that* $(A - B) - C = (A - C) - (B - C)$.

**Sol.** (*a*) $(A - B) - C = A - (B \cup C)$

Let $x \in (A - B) - C \Rightarrow x \in (A - B), x \notin C$

$\Rightarrow x \in A, x \notin B, x \notin C$

$\Rightarrow x \in A, x \notin B$ or $C$

$\Rightarrow x \in A, x \notin B \cup C$

$\Rightarrow x \in A - (B \cup C)$.

$\therefore \quad A - (B \cup C) \subseteq (A - B) - C$ ...(*i*)

Conversely, let $x \in A - (B \cup C)$

$\Rightarrow x \in A, x \notin B \cup C$

$\Rightarrow x \in A, x \notin B, x \notin C$

$\Rightarrow x \in A - B, x \notin C$

$\Rightarrow x \in (A - B) - C$

$\therefore \quad (A - B) - C \subseteq A - (B \cup C)$ ...(*ii*)

From (*i*) and (*ii*), we get $(A - B) - C = A - (B \cup C)$.

(*b*) $(A - B) - C = (A - C) - B$

Let $x \in (A - B) - C$

$\Rightarrow x \in A - B, x \notin C \quad \Rightarrow x \in A, x \notin B, x \notin C$

$\Rightarrow x \in A - C, x \notin B \quad \Rightarrow x \in (A - C) - B$

$\therefore \quad (A - C) - B \subseteq (A - B) - C$ ...(*i*)

Conversely, let $x \in (A - C) - B$

$\Rightarrow x \in (A - C), x \notin B \quad \Rightarrow x \in A, x \notin C, x \notin B$

$\Rightarrow x \in A, x \notin B, x \notin C \quad \Rightarrow x \in (A - B), x \notin C$

$\Rightarrow x \in (A - B) - C$

$\therefore \quad (A - B) - C \subseteq (A - C) - B$ ...(*ii*)

From (*i*) and (*ii*), we get $(A - B) - C = (A - C) - B$.

(*c*) $(A - B) - C = (A - C) - (B - C)$

Let $x \in (A - B) - C \quad \Rightarrow x \in (A - B), x \notin C$

$\Rightarrow x \in A, x \notin B, x \notin C \quad \Rightarrow x \in A, x \notin C, x \notin B, x \notin C$

$\Rightarrow x \in A - C, x \notin B - C \quad \Rightarrow x \in (A - C) - (B - C)$

$\therefore \quad (A - C) - (B - C) \subseteq (A - B) - C$ ...(*i*)

Conversely, let $x \in (A - C) - (B - C)$

$\Rightarrow x \in (A - C), x \notin (B - C) \quad \Rightarrow x \in A, x \notin C, x \notin B, x \notin C$

$\Rightarrow x \in A, x \notin B, x \notin C \quad \Rightarrow x \in A - B, x \notin C$

$\Rightarrow x \in (A - B) - C$

$\therefore \quad (A - B) - C \subseteq (A - C) - (B - C)$ ...(*ii*)

From (*i*) and (*ii*), we get $(A - B) - C = (A - C) - (B - C)$.

**Problem 24.** *Given that $P \cup Q = P \cup R$, is it necessary that $Q = R$ ? Justify your answer.*

**Sol.** This is not necessary.

$P \cup Q \quad \Rightarrow$ all elements of P or Q are in this set.

$P \cup R \quad \Rightarrow$ all elements of P or R are in this set.

If $\quad Q \subseteq P$ and also $R \subseteq P$. Then $P \cup Q = P \cup R = P$.

Therefore, it is not necessary that $Q = R$.

*e.g.*, Let $P = \{1, 2, 3, 4\}$, $Q = \{1, 2\}$, $R = \{3, 4\}$, then $P \cup Q = P \cup R = P$
But here $Q = R$.

**Problem 25.** *Given that $P \cap Q = P \cap R$, is it necessary that $Q = R$ ? Justify your answer.*

**Sol.** This is not necessary

$P \cap Q$ contains elements common to both P and Q.

$P \cap R$ contains elements common to both P and R.

But it is not necessary that $Q = R$ because set R can have elements of set Q which are elements of set P as well, but it can also have elements other than those in set Q.

*e.g.*, Let $P = \{1, 2, 5, 6\}$, $Q = \{1, 2, 8\}$, $R = \{1, 2, 9\}$
Then $P \cap Q = P \cap R = \{1, 2\}$ ; but $Q \neq R$.

**Problem 26.** *Given that $P \oplus Q = P \oplus R$, is it necessary that $Q = R$ ? Justify your answer.*

**Sol.** It is necessary that $Q = R$.

Because $P \oplus Q = (P \cup Q) - (P \cap Q)$ *i.e.*, this set contains elements which are in sets P or Q but does not contain elements common to both sets.

Similarly, $P \oplus R = (P \cup R) - (P \cap R)$ *i.e.*, this set contains elements which are in sets P or R but does not contain elements common to both sets.

As $P \oplus Q = P \oplus R$

$\therefore \quad (P \cup Q) - (P \cap Q) = (P \cup R) - (P \cap R)$

$\Rightarrow \quad P \cup Q = P \cup R$ and $P \cap Q = P \cap R$

$\Rightarrow \quad Q = R$

*e.g.*, Let $P = \{1, 2, 3, 4\}$, $Q = \{1, 5\}$, $R = \{1, 5\}$

$P \oplus Q = (P \cup Q) - (P \cap Q) = \{1, 2, 3, 4, 5\} - \{1\} = \{2, 3, 4, 5\}$

$P \oplus R = (P \cup R) - (P \cap R) = \{1, 2, 3, 4, 5\} - \{1\} = \{2, 3, 4, 5\}$

$\Rightarrow \quad Q = R$.

**Problem 27.** *Prove that $(A - B) \cap B = \phi$.*

**Sol.** $(A - B) \cap B = \phi$

If $x \in B$, then $x \notin A - B$ due to definition of $A - B$

$x \in B, x \notin A - B \quad \Rightarrow \quad (A - B) \cap B = \phi$. Hence proved.

**Problem 28.** *Prove that $A \cup B = \phi \quad \Rightarrow \quad A = \phi, B = \phi$.*

**Sol.** Let $A \cup B = \phi$

Now $\phi$ is a subset of every set $\Rightarrow \quad \phi \subset A, \phi \subset B$

Since, $\quad A \subset A \cup B, B \subset A \cup B$

Hence, $\quad A \subset \phi, B \subset \phi$

So $\quad A \subset \phi, \phi \subset A \quad \Rightarrow \quad A = \phi$

$\quad B \subset \phi, \phi \subset B \quad \Rightarrow \quad B = \phi$. Hence proved.

**Problem 29.** *Prove that $A - B \subset B'$.*

**Sol.** Let $\quad x \in A - B \quad \Rightarrow \quad x \in A$ and $x \notin B$

$\Rightarrow \quad x \in A$ and $x \in B' \quad \Rightarrow \quad x \in A \cap B'$ as $A \cap B' \subset B'$

$\Rightarrow \quad x \in B'$

Therefore, any $x \in A - B \Rightarrow x \in B'$

So $A - B \subset B'$. Hence proved.

**Problem 30.** *Given that*

$$(A \cap C) \subseteq (B \cap C)$$
$$(A \cap \overline{C}) \subseteq (B \cap \overline{C})$$

*Show that* $A \subseteq B$.

**Sol.** For $(A \cap C) \subseteq (B \cap C)$, all elements of set A which are also elements of set C are contained in a set which contains elements common to both B and C. It implies $A \subseteq B$ for elements which are common in A and C, and B and C.

For $(A \cap \overline{C}) \subseteq (B \cap \overline{C})$, elements common to sets A and $\overline{C}$ are contained in a set containing elements common to both B and C. It implies $A \subseteq B$ for elements common in A and $\overline{C}$ and also in B and $\overline{C}$.

Hence, from both conditions it is shown that $A \subseteq B$.

# 2

# Principle of Inclusion and Exclusion

As we know the cardinality of the set P is the number of unique elements in set P. It is denoted as $|P|$ and read as cardinality of set P.

## FIRST PRINCIPLE

If P and Q are disjoint sets, then

$$|P \cup Q| = |P| + |Q|.$$

**Theorem I.** *Let P and Q be any two non-disjoint sets. Then*

$$|P \cup Q| = |P| + |Q| - |P \cap Q|.$$

**Proof.** Draw Venn diagram for the above as shown in Fig. 1.

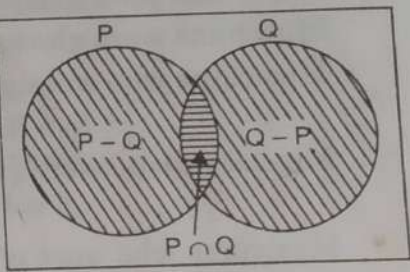



Fig. 1

From figure, we see that $P \cup Q$ can be seen to be the union of three disjoint sets $P - Q$, $Q - P$ and $P \cap Q$.

So $|P \cup Q| = |P - Q| + |Q - P| + |P \cap Q|$ ...(*i*)

Also, $|P| = |P - Q| + |P \cap Q|$ ...(*ii*)

and $|Q| = |Q - P| + |P \cap Q|$ ...(*iii*)

Combining (*ii*) and (*iii*)

$$|P| + |Q| = |P - Q| + |Q - P| + 2|P \cap Q|$$

$$|P| + |Q| = |P \cup Q| + |P \cap Q|$$

(As $|P \cup Q| = |P - Q| + |Q - P| + |P \cap Q|$)

$$|P \cup Q| = |P| + |Q| - |P \cap Q|$$

Hence proved.

**Theorem II.** *Let P, Q and R are three finite sets. Then*

$$|P \cup Q \cup R| = |P| + |Q| + |R| - |P \cap Q| - |P \cap R| - |Q \cap R| + |P \cap Q \cap R|.$$

**Proof.** Using theorem I, we have

$$|P \cup (Q \cup R)| = |P| + |Q \cup R| - |P \cap (Q \cup R)|$$
$$= |P| + |Q| + |R| - |Q \cap R| - |P \cap (Q \cup R)| \quad ...(i)$$

As $P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$

So
$$|P \cap (Q \cup R)| = |P \cap Q| + |P \cap R| - |(P \cap Q) \cap (P \cap R)|$$
$$= |P \cap Q| + |P \cap R| - |P \cap Q \cap R| \quad ...(ii)$$

Putting (*ii*) in (*i*), we get

$$|P \cup Q \cup R| = |P| + |Q| + |R| - |P \cap Q| - |P \cap R| - |Q \cap R| + |P \cap Q \cap R|$$

Hence proved.

## INCLUSION-EXCLUSION PRINCIPLE IN GENERAL

Let $P_1, P_2, \ldots, P_n$ are finite sets. Then $|P_1 \cup P_2 \cup \ldots P_n| = \sum_{1 \le i \le n} |P_i| - \sum_{1 \le i < j \le n} |P_i \cap P_j|$

$$+ \sum_{1 \le i < j < k \le n} |P_i \cap P_j \cap P_k| \ldots + (-1)^{n-1} |P_1 \cap P_2 \cap \ldots P_n|$$

## SOLVED PROBLEMS

**Problem 1.** *In a survey of 200 musicians, it was found that 40 wore gloves on the left hand and 39 wore gloves on the right hand. If 160 wore no gloves at all, how many wore a glove on only the right hand ? Only the left hand ? On both hands ?* $L \cap R$

**Sol.** Total number of musicians wore gloves on left, right or both hands *i.e.*,

$$|L \cup R| = 200 - 160 = 40$$

Musicians wore gloves on left hand $|L| = 40$

Musicians wore gloves on right hand $|R| = 39$

Musicians who wore gloves on both hands

$$|L \cap R| = |L| + |R| - |L \cup R| = 40 + 39 - 40 = 39$$

Musicians who wore gloves only on right hand

$$= 40 - 39 = 1$$

Musicians who wore gloves only on left hand

$$= 39 - 39 = 0.$$

**Problem 2.** *Out of 1200 students at a college*

*582 took Economics*
*627 took English*
*543 took Mathematics*
*217 took both Economics and English*
*307 took both Economics and Mathematics*
*250 took both Mathematics and English*
*222 took all three courses.*

*How many took none of the three ?*

**Sol.** Suppose $|A| = 582$, $|B| = 627$, $|C| = 543$, $|A \cap B| = 217$, $|A \cap C| = 307$, $|B \cap C| = 250$, $|A \cap B \cap C| = 222$

The total number of students who took any of three subjects

$$|A \cup B \cup C| = 582 + 627 + 543 - 217 - 307 - 250 + 222 = 1200$$

Students who took none of three

= (total students in the college) – (total students who took any of three subjects)

= 1200 – 1200 = 0.

**Problem 3.** *40 computer programmers interviewed for a job. 25 knew JAVA, 28 knew ORACLE, and 7 knew neither language. How many knew both languages ?*

**Sol.** Now $|J| = 25$

$|O| = 28$

$|J \cup O| = 40 - 7 = 33$

Sorry

Sorry

Computer programmers who knew both languages are

$|J \cap O| = 25 + 28 - 33 = 20.$

**Problem 4.** *Among 100 students, 32 study Mathematics, 20 study Physics, 45 study Biology, 15 study Mathematics and Biology, 7 study Mathematics and Physics, 10 study Physics and Biology and 30 do not study any of three subjects.*

(*a*) *Find the number of students studying all three subjects.*

(*b*) *Find the number of students studying exactly one of the three subjects.*

**Sol.** $|M| = 32$ $|P| = 20$ $|B| = 45$

$|M \cap B| = 15$ $|M \cap P| = 7$ $|P \cap B| = 10$

$|M \cup P \cup B| = 100 - 30 = 70.$

(*a*) Number of students studying all three subjects

$|M \cap P \cap B| = 70 - 32 - 20 - 45 + 15 + 7 + 10 = 5.$

(*b*) 5 study all three subjects.

15 – 5 = 10 study Mathematics and Biology but not all the three.

7 – 5 = 2 study Mathematics and Biology but not all the three.

10 – 5 = 5 study Physics and Biology but not all the three.

32 – 10 – 2 – 5 = 15 study only Mathematics.

20 – 2 – 5 – 5 = 8 study only Physics.

45 – 10 – 5 – 5 = 25 study only Biology.

Number of students studying exactly one of three subjects

= 15 + 8 + 25 = 48.

**Problem 5.** *A survey of 550 television watchers produced the following information :*

*285 watch football games*

*195 watch hockey games*

*115 watch baseball games*

*45 watch football and baseball games*

*70 watch football and hockey games*

*50 watch hockey and baseball games*

*100 do not watch any of the three games.*

(*a*) *How many people in the survey watch all three games ?*

(*b*) *How many people watch exactly one of the three games ?*

**Sol.** $|F| = 285$ ; $|H| = 195$ ; $|B| = 115$

$|F \cap B| = 45$ ; $|F \cap H| = 70$ ; $|H \cap B| = 50$

$$| F \cup H \cup B | = 550 - 100 = 450$$

(*a*) The number of people watch all three games

$$| F \cap H \cap B | = 450 - 285 - 195 - 115 + 45 + 70 + 50 = 20.$$

(*b*) 20 watch all three games.

$45 - 20 = 25$ watch football and baseball but not all three.

$70 - 20 = 50$ watch football and hockey but not all three.

$50 - 20 = 30$ watch hockey and baseball but not all three.

$285 - 25 - 50 - 20 = 190$ watch only football.

$195 - 50 - 30 - 20 = 95$ watch only hockey.

$115 - 25 - 30 - 20 = 40$ watch only baseball.

Number of people exactly watch one of the three games

$$= 190 + 95 + 40 = 325.$$

**Problem 6.** *In a survey of 300 students,*

*64 had taken a Mathematics course*
*94 had taken a English course*
*58 had taken a Computer course*
*28 had taken both a Mathematics and a Computer course*
*26 had taken both a English and a Mathematics course*
*22 had taken both a English and a Computer course*
*14 had taken all three courses.*

(*a*) *How many students were surveyed who had taken non of the three courses ?*

(*b*) *How many had taken only a Computer course ?*

28-14 = 14

14

22-14 = 8

**Sol.** $| M | = 64 \,;\, | E | = 94 \,;\, | C | = 58$

$| M \cap C | = 28 \,;\, | M \cap E | = 26 \,;\, | E \cap C | = 22$

$| M \cap E \cap C | = 14$

(*a*) $| M \cup E \cup C | = | M | + | E | + | C | - | M \cap C | - | M \cap E | - | E \cap C | + | M \cap E \cap C |$

$$= 64 + 94 + 58 - 28 - 26 - 22 + 14 = 154$$

Students who had taken none of the courses

$$= 300 - 154 = 146.$$

(*b*) 14 had taken all three courses.

$28 - 14 = 14$ had taken both a Mathematics and a Computer but not all thr

$22 - 14 = 8$ had taken both a English and a Computer courses but not all thr

$58 - 14 - 8 - 14 = 22$ had taken only Computer course.

**Problem 7.** *A survey was conducted among 1000 people. Of these 595 like Metro Cha nel, 595 like Star Movies and 550 like Zee TV ; 395 of them like Metro Channel and Star Movi 350 of them like Metro Channel and Zee TV and 400 of them like Star Movies and Zee TV ; 2 of them like Metro Channel, Star Movies and Zee TV.*

(*a*) *How many of them who do not like Metro Channel, do not like Star Movies and do n like Zee TV ?*

(*b*) *How many of them who like Metro Channel, do not like Star Movies and do not li Zee TV ?*

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**Sol.** $|M| = 595$ ; $|S| = 595$ ; $|Z| = 550$
$|M \cap S| = 395$ ; $|M \cap Z| = 350$ ; $|S \cap Z| = 400$
$|M \cap S \cap Z| = 250$

($a$) Number of people like at least one of the channels
$$|M \cup S \cup Z| = 595 + 595 + 550 - 395 - 350 - 400 + 250 = 845$$

The number of people who do not like Metro Channel, do not like Star Movies and do not like Zee TV

= total number of people surveyed – number of people like at least one of the channels
= 1000 – 845 = 155.

($b$) 250 people like all the three channels

395 – 250 = 145 like Metro and Star Movies but not all the three

350 – 250 = 100 like Metro and Zee TV but not all the three

595 – 145 – 100 – 250 = 100 people like Metro Channel but do not like Star Movies and do not like Zee TV.

**Problem 8.** *Among the first 500 positive integers :*

($a$) *Determine the integers which are not divisible by 2, nor by 3, nor by 5.*

($b$) *Determine the integers which are exactly divisible by one of them.*

**Sol.** Let A is the number of integers divisible by 2

B is the number of integers divisible by 3

C is the number of integers divisible by 5.

$$|A| = \left[\frac{500}{2}\right] = 250 \; ; \qquad |B| = \left[\frac{500}{3}\right] = 166 \; ; \qquad |C| = \left[\frac{500}{5}\right] = 100$$

$$|A \cap B| = \left[\frac{500}{2 \times 3}\right] = 83 \; ; \qquad |A \cap C| = \left[\frac{500}{2 \times 5}\right] = 50$$

$$|B \cap C| = \left[\frac{500}{3 \times 5}\right] = 33 \; ; \qquad |A \cap B \cap C| = \left[\frac{500}{2 \times 3 \times 5}\right] = 16.$$

($a$) $|A \cup B \cup C| = 250 + 166 + 50 - 83 - 100 - 33 + 16 = 366$

The integers not divisible by 2, 3 and 5 = 500 – 366 = 134.

($b$) The integers divisible by all the three = 16

83 – 16 = 67 integers are divisible by 2 and 3 but not all the three

50 – 16 = 34 integers are divisible by 2 and 5 but not by all the three

33 – 16 = 17 integers are divisible by 3 and 5 but not by all the three

250 – 67 – 34 – 16 = 133 integers are only divisible by 2

166 – 67 – 17 – 16 = 66 integers are only divisible by 3

100 – 34 – 17 – 16 = 33 integers are only divisible by 5

Total number of integers only divisible by 2, 3 and 5

= 133 + 33 + 66 = 232.

**Problem 9.** *Among the first 1000 positive integers :*

($a$) *Determine the integers which are not divisible by 5, nor by 7, nor by 9.*

($b$) *Determine the integers divisible by 5, but not by 7, not by 9.*

**Sol.** Let A is the number of integers divisible by 5

B is the number of integers divisible by 7

C is the number of integers divisible by 9.

So $\quad |A| = \left|\frac{1000}{5}\right| = 200;$ $\qquad |B| = \left[\frac{1000}{7}\right] = 142$

$$|C| = \left[\frac{1000}{9}\right] = 111; \qquad |A \cap B| = \left[\frac{1000}{5 \times 7}\right] = 28$$

$$|A \cap C| = \left[\frac{1000}{5 \times 9}\right] = 22; \qquad |B \cap C| = \left[\frac{1000}{7 \times 9}\right] = 15$$

$$|A \cap B \cap C| = \left[\frac{1000}{5 \times 7 \times 9}\right] = 3.$$

(*a*) The number of integers divisible by 5, 7 and 9

$$|A \cup B \cup C| = 200 + 142 + 111 - 28 - 22 - 15 + 3$$
$$= 391.$$

The number of integers not divisible by 5, nor by 7, nor by 9

= Total number of integers – integers divisible by 5, 7 and 9

$= 1000 - 391 = 609.$

(*b*) The integers divisible by all the three integers = 3

$28 - 3 = 25$ integers divisible by 5 and 7 but not by all the three

$22 - 3 = 19$ integers divisible by 5 and 9 but not by all the three

$\therefore$ $200 - 25 - 19 - 3 = 153$ integers divisible by 5 but not by 7, not by 9.

3

# Mathematical Induction

## PRINCIPLE

The process to establish the validity of a general result involving natural numbers is the principle of mathematical induction.

## WORKING RULE

Let $n_0$ be a fixed integer. Suppose P($n$) is a statement involving the natural number $n$ and we wish to prove that P($n$) is true for all $n \geq n_0$.

**1. Basis of Induction.** P($n_0$) is true *i.e.,* P($n$) is true for $n = n_0$.

**2. Induction Step.** Assume that the P($k$) is true for $n = k$. $(k \geq n_0)$

Then P($k + 1$) must also be true.

Then P($n$) is true for all $n \geq n_0$.

## PEANO'S AXIOMS

Peano, a mathematician defined the natural number in the following way. According to him, the member of set N satisfying the following properties are called natural number.

**1.** $1 \in N$.

**2.** For all $n \in N$, there exists a unique $n' \in N$, such that

($a$) $m' = n' \iff m = n$

($b$) There exists no element $p$ in N such that $p' = 1$.

**3.** If $S \subset N$ and

($a$) $1 \in S$

($b$) $x \in S \Rightarrow x' \in S$ then $S = N$.

$x'$ is called the successor of $x$ and $x$ is known as predecessor of $x'$.

These properties are called Peano's Axioms.

The number $x'$ and $x + 1$.

Therefore $2 = 1'$, $3 = 2' = (1')'$, $4 = 3' = (2')' = ((1')')'$ and so on.

## SOLVED PROBLEMS

**Prove the following by Mathematical Induction :**

**Problem 1.** $1 + 3 + 5 + ... + 2n - 1 = n^2$.

**Sol. Basis of Induction.** Let us assume that

$$P(n) = 1 + 3 + 5 + ... + (2n - 1) = n^2.$$

For $n = 1$, $\quad P(1) = 1 = 1^2 = 1$

It is true for $n = 1$.

**Induction Step.** For $n = r$,

$$P(r) = 1 + 3 + 5 + ... + 2r - 1 = r^2 \text{ is true}$$

Adding $2r + 1$ in both sides

$$P(r + 1) = 1 + 3 + 5 + ... + 2r - 1 + 2r + 1$$
$$= r^2 + (2r + 1) = r^2 + 2r + 1 = (r + 1)^2$$

As $P(r)$ is true. Hence $P(r + 1)$ is also true.

From (*i*), (*ii*) and (*iii*), we conclude that

$$1 + 3 + 5 + ... + 2n - 1 = n^2$$

is true for $n = 1, 2, 3, 4, 5, ...$ . Hence proved.

**Problem 2.** $1^2 + 2^2 + 3^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

**Sol. Basis of Induction.** For $n = 1$,

$$P(1) = 1^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{6}{6} = 1$$

It is true for $n = 1$.

**Induction Step.** For $n = r$,

$$P(r) = 1^2 + 2^2 + 3^2 + ... + r^2 = \frac{r(r+1)(2r+1)}{6} \text{ is true}$$

Adding $(r + 1)^2$ on both sides, we get

$$P(r + 1) = 1^2 + 2^2 + 3^2 + ... + r^2 + (r + 1)^2 = \frac{r(r+1)(2r+1)}{6} + (r + 1)^2$$
$$= \frac{r(r+1)(2r+1) + 6(r+1)^2}{6} = (r+1)\left[\frac{r(2r+1) + 6(r+1)}{6}\right]$$
$$= \frac{(r+1)}{6}[r(2r + 1) + 6(r + 1)] = \frac{(r+1)}{6}[2r^2 + 7r + 6]$$
$$= \frac{(r+1)(r+2)(2r+3)}{6}$$

As $P(r)$ is true, hence $P(r + 1)$ is true.

From (*i*), (*ii*) and (*iii*), we conclude that

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

is true for $n = 1, 2, 3, 4, 5, ...$ . Hence proved.

**Problem 3.** $1^2 - 2^2 + 3^2 + ... + (-1)^{n+1} n^2 = \frac{(-1)^{n+1} n(n+1)}{2}$.

**Sol.** Let $\quad P(n) = 1^2 - 2^2 + 3^2 + ... + (-1)^{n+1} n^2 = \frac{(-1)^{n+1} n(n+1)}{2}$.

**Basic Step.** For $n = 1$,

$$P(1) = (-1)^2 \cdot 1^2 = 1 = \frac{(-1)^{1+1} \cdot 1 \cdot (1+1)}{2} = 1$$

It is true for $n = 1$. ...(*i*)

**Induction Step.** For $n = r$,

$$P(r) = 1^2 - 2^2 + 3^2 + ... + (-1)^{r+1} r^2 = \frac{(-1)^{r+1} r(r+1)}{2} \text{ is true} \qquad ...(ii)$$

Adding $(-1)^{r+2} \cdot (r+1)^2$ in both sides, we get

$$P(r+1) = 1^2 - 2^2 + 3^2 + ... + (-1)^{r+1} \cdot r^2 + (-1)^{r+2} (r+1)^2$$

$$= \frac{(-1)^{r+1} r(r+1)}{2} + (-1)^{r+2} (r+1)^2$$

$$= \frac{(-1)^{r+1} r(r+1) + 2\{(-1)^{r+2} (r+1)^2\}}{2}$$

$$= \frac{(-1)^{r+2} (r+1)\left\{(-1)^{-1} r + 2(r+1)\right\}}{2}$$

$$= \frac{(-1)^{r+2} (r+1)(r+2)}{2} \qquad ...(iii)$$

As $P(r)$ is true, hence $P(r+1)$ is also true. From (*i*), (*ii*) and (*iii*), we conclude that

$$1^2 - 2^2 + 3^2 + ... + (-1)^{n+1} n^2 = \frac{(-1)^{n+1} n(n+1)}{2}$$

s true for $n = 1, 2, 3, 4, 5, ....$ Hence proved.

**Problem 4.** $1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

**Sol.** Let $\quad P(n) = 1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

**Basic Step.** For $n = 1$,

$$P(1) = 1^3 = \left[\frac{1(1+1)}{2}\right]^2 = (1)^2 = 1$$

Hence $P(n)$ is true for $n = 1$. ...(*i*)

**Induction Step.** For $n = r$,

$$P(r) = 1^3 + 2^3 + 3^3 + ... + r^3 = \left[\frac{r(r+1)}{2}\right]^2 \text{ is true} \qquad ...(ii)$$

Adding $(r+1)^3$ on both sides

$$P(r+1) = 1^3 + 2^3 + 3^3 + ... + r^3 + (r+1)^3 = \left[\frac{r(r+1)}{2}\right]^2 + (r+1)^3$$

$$= \frac{r^2(r+1)^2 + 4(r+1)(r+1)^2}{4} = \frac{(r+1)^2\{r^2 + 4(r+1)\}}{4}$$

$$= \frac{(r+1)^2\,[r^2 + 4r + 4]}{4} = \frac{(r+1)^2 (r+2)^2}{4}$$

$$= \left[\frac{(r+1)(r+2)}{2}\right]^2$$

As P($r$) is true, hence P($r$ + 1) is also true.

From ($i$), ($ii$) and ($iii$), we conclude that

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

is true for $n = 1, 2, 3, 4, 5, \ldots$. Hence proved.

**Problem 5.** $1\,.\,2 + 2\,.\,3 + 3\,.\,4 + \ldots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$.

**Sol.** Let $\quad P(n) = 1\,.\,2 + 2\,.\,3 + 3\,.\,4 + \ldots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$.

**Basis of induction.** For $n = 1$,

$$P(1) = 1\,.\,2 = \frac{1(1+1)(1+2)}{3} = 1\,.\,2$$

It is true for $n = 1$.

**Induction Step.** For $n = r$,

$$P(r) = 1\,.\,2 + 2\,.\,3 + 3\,.\,4 + \ldots + r(r+1) = \frac{r(r+1)(r+2)}{3} \text{ is true}$$

Adding $(r + 1)(r + 2)$ on both sides,

$$P(r+1) = 1\,.\,2 + 2\,.\,3 + 3\,.\,4 + \ldots + r(r+1) + (r+1)(r+2)$$

$$= \frac{r(r+1)(r+2)}{3} + (r+1)(r+2)$$

$$= \frac{r(r+1)(r+2) + 3(r+1)(r+2)}{3} = \frac{(r+1)(r+2)(r+3)}{3}$$

As P($r$) is true, hence P($r$ + 1) is also true.

From ($i$), ($ii$) and ($iii$), we conclude that

$$1\,.\,2 + 2\,.\,3 + 3\,.\,4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

is true for $n = 1, 2, 3, 4, \ldots$. Hence proved.

**Problem 6.** $\dfrac{1}{1\,.\,3} + \dfrac{1}{3\,.\,5} + \dfrac{1}{5\,.\,7} + \ldots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$.

**Sol. Basic Step.** For $n = 1$,

$$\frac{1}{(2-1)(2+1)} = \frac{1}{3} = \frac{1}{2(1)+1} = \frac{1}{3}$$

Hence P($n$) is true for $n = 1$.

**Induction Step.** For $n = r$,

$$P(r) = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2r-1)(2r+1)} = \frac{r}{2r+1} \text{ is true} \quad \ldots(ii)$$

Adding $\frac{1}{(2r+1)(2r+3)}$ on both sides

$$P(r+1) = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2r-1)(2r+1)} + \frac{1}{(2r+1)(2r+3)}$$

$$= \frac{r}{2r+1} + \frac{1}{(2r+1)(2r+3)} = \frac{r(2r+3)+1}{(2r+1)(2r+3)}$$

$$= \frac{2r^2+3r+1}{(2r+1)(2r+3)} = \frac{(2r+1)(r+1)}{(2r+1)(2r+3)} = \frac{(r+1)}{(2r+3)} \quad \ldots(iii)$$

As P($r$) is true, hence P($r$ + 1) is also true.

From ($i$), ($ii$) and ($iii$), we conclude that

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

is true for $n$ = 1, 2, 3, 4, … . Hence proved.

**Problem 7.** *1(1 !) + 2(2 !) + … + n(n !) = (n + 1) ! – 1.*

**Sol.** Let P($n$) = 1(1 !) + 2(2 !) + … + $n$($n$ !) = ($n$ + 1) ! – 1.

**Basis of Induction.** For $n$ = 1,

$$P(1) = 1(1\,!) = (1+1)\,! - 1 \;\Rightarrow\; 2\,! - 1 \;\Rightarrow\; 1 = 1$$

It is true for $n$ = 1. …($i$)

**Induction Step.** For $n = r$,

$$P(r) = 1(1\,!) + 2(2\,!) + \ldots + r(r\,!) = (r+1)\,! - 1 \text{ is true.} \quad \ldots(ii)$$

Adding ($r$ + 1) [($r$ + 1) !] on both sides,

$$P(r+1) = 1(1\,!) + 2(2\,!) + \ldots + r(r\,!) + (r+1)(r+1)\,!$$

$$= (r+1)\,! - 1 + (r+1)(r+1)\,!$$

$$= (r+1)\,!\,(r+1+1) - 1 = (r+1)\,!\,(r+2) - 1$$

$$= (r+2)\,.\,(r+1)\,! - 1 = (r+2)\,! - 1 \quad \ldots(iii)$$

As P($r$) is true, hence P($r$ + 1) is also true.

From ($i$), ($ii$) and ($iii$), we conclude that

$$1(1\,!) + 2(2\,!) + \ldots + n(n\,!) = (n+1)\,! - 1$$

true for $n$ = 1, 2, 3, 4, 5, … . Hence proved.

**Problem 8.** *Prove 1 + 3 + 5 + … + (2n – 1) = $n^2$ by in induction (n ≥ 1).*

**Sol.** Consider P($n$) = 1 + 3 + 5 + … + (2$n$ – 1) = $n^2$

**Basis of Induction.** For $n$ = 1,

$$P(1) = 2 \times 1 - 1 = 1 = 1^2 = 1$$

It is true for $n$ = 1. …($i$)

**Induction Step.** For $n = r$,

$$P(r) = 1 + 3 + 5 + \ldots + (2r-1) = r^2 \text{ is true} \quad \ldots(ii)$$

Adding $(2r+1)$ to both sides,

$$P(r+1) = 1 + 3 + 5 + \dots + (2r-1) + (2r+1)$$
$$= r^2 + (2r+1) = (r+1)^2$$

As $P(r)$ is true, hence $P(r+1)$ is also true.

From (*i*), (*ii*) and (*iii*), we conclude that

$$1 + 3 + 5 + \dots + (2n-1) = n^2$$

is true for $n = 1, 2, 3, \dots$. Hence proved.

**Problem 9.** *Show that* $1 + 2 + 2^2 + 2^3 + \dots + 2^n = 2^{n+1} - 1$ *by induction (for* $n \geq 0$*).*

**Sol.** Consider $P(n) = 1 + 2 + 2^2 + 2^3 + \dots + 2^n = 2^{n+1} - 1$.

**Basis of Induction.** For $n = 0$,

$$P(0) = 1 = 2^1 - 1 = 1$$

It is true for $n = 0$.

**Induction Step.** For $n = r$,

$$P(r) = 1 + 2 + 2^2 + 2^3 + \dots + 2^r = 2^{r+1} - 1 \text{ is true.}$$

Adding $2^{r+1}$ to both sides,

$$P(r+1) = 1 + 2 + 2^2 + 2^3 + \dots + 2^r + 2^{r+1} = 2^{r+1} - 1 + 2^{r+1}$$
$$= 2(2^{r+1}) - 1 = 2^{r+2} - 1$$

As $P(r)$ is true, hence $P(r+1)$ is also true.

From (*i*), (*ii*) and (*iii*), we conclude that

$$1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1.$$

is true for $n = 1, 2, 3, \dots$.

**Problem 10.** *Prove by induction that for* $n \geq 0$ *and* $a \neq 1$ *;* $1 + a + a^2 + \dots + a^n = \dfrac{1-a^{n+1}}{1-a}$

**Sol. Basis of Induction.** For $n = 1$,

$$1 + a^1 = \frac{1-a^{1+1}}{1-a} = \frac{1-a^2}{1-a} = 1 + a$$

It is true for $n = 1$.

**Induction Step.** For $n = r$,

$$P(r) = 1 + a + a^2 + \dots + a^r = \frac{1-a^{r+1}}{1-a} \text{ is true}$$

Adding $a^{r+1}$ to both sides,

$$P(r+1) = 1 + a + a^2 + \dots + a^r + a^{r+1} = \frac{1-a^{r+2}}{1-a}$$

$$= \frac{1-a^{r+1}}{1-a} + a^{r+1} = \frac{1-a^{r+1}+a^{r+1}-a^{r+2}}{1-a} = \frac{1-a^{r+2}}{1-a}$$

As $P(r)$ is true, hence $P(r+1)$ is also true.

From (*i*), (*ii*) and (*iii*), we conclude that

$$1 + a + a^2 + \dots + a^n = \frac{1-a^{n+1}}{1-a} \text{ is true for } n \geq 0. \text{ Hence proved.}$$

**Problem 11.** *Prove*

$$1 . 2 . 3 + 2 . 3 . 4 + 3 . 4 . 5 + ... + n(n + 1)(n + 2) = \frac{1}{4} n(n + 1)(n + 2)(n + 3).$$

**Sol. Basis of Induction.** For $n = 1$,

$$\text{L.H.S.} = 1(1 + 1)(1 + 2) = 1 . 2 . 3 = 6$$

$$\text{R.H.S.} = \frac{1}{4} . 1(1 + 1)(1 + 2)(1 + 3) = \frac{1}{4} . 1 . 2 . 3 . 4 = 6$$

It is true for $n = 1$.

**Induction Step.** For $n = r$,

$$1 . 2 . 3 + 2 . 3 . 4 + 3 . 4 . 5 + ... + r(r + 1)(r + 2) = \frac{1}{4} r(r + 1)(r + 2)(r + 3) \text{ is true.}$$

For $n = r + 1$,

$$\begin{aligned} \text{L.H.S.} &= 1 . 2 . 3 + 2 . 3 . 4 + ... + r(r + 1)(r + 2) + (r + 1)(r + 2)(r + 3) \\ &= \frac{1}{4} r(r + 1)(r + 2)(r + 3) + (r + 1)(r + 2)(r + 3) \\ &= \frac{1}{4} (r + 1)(r + 2)(r + 3)(r + 4) \\ &= \text{R.H.S., for } n = r + 1. \end{aligned}$$

Hence, $1 . 2 . 3 + 2 . 3 . 4 + 3 . 4 . 5 + ... + n(n + 1)(n + 2) = \frac{1}{4} n(n + 1)(n + 2)(n + 3)$. Hence proved.

**Problem 12.** *Show that for any integer n*

$11^{n+2} + 12^{2n+1}$ *is divisible by 133.*

**Sol.** Let $\quad P(n) = 11^{n+2} + 12^{2n+1}$

**Basis of Induction.** For $n = 1$,

$$P(1) = 11^3 + 12^3 = 3059 = 133 \times 23$$

So, 133 divides P(1). ...(*i*)

**Induction Step.** For $n = r$,

$$P(r) = 11^{r+2} + 12^{2r+1} = 133 \times s \qquad ...(ii)$$

Now, for $n = r + 1$,

$$\begin{aligned} P(r + 1) &= 11^{r+2+1} + 12^{2(r)+3} = 11[133s - 12^{2r+1}] + 144 . 12^{2r+1} \\ &= 11 \times 133s + 12^{2r+1} . 133 = 133[11s + 12^{2r+1}] = 133 \times t \qquad ...(iii) \end{aligned}$$

As (*i*), (*ii*) and (*iii*) all are true, hence P(*n*) is divisible by 133.

**Problem 13.** *Prove by induction that the sum of the cubes of three consecutive integers is divisible by 9.*

**Sol.** Let $P(n) = n^3 + (n + 1)^3 + (n + 2)^3$

P(*n*) is divisible by 9

$$P(1) = 1 + 8 + 27 = 36$$

which is divisible by 9. ...(*i*)

For $n = r$,

$$P(r) = r^3 + (r + 1)^3 + (r + 2)^3 = 9 . q \qquad ...(ii)$$

For $n = r + 1$, $P(r + 1) = (r + 1)^3 + (r + 2)^3 + (r + 3)^3$

$$= r^3 + (r + 1)^3 + (r + 2)^3 + [9r^2 + 27r + 27]$$
$$= 9q + 9(r^2 + 3r + 3) = 9[q + r^2 + 3r + 3]$$
$$= 9 \cdot q$$

From (*i*), (*ii*) and (*iii*), we have the required result by induction. Hence proved.

**Problem 14.** *Prove* $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{13}{24}$, *for* $n \geq 2$.

**Sol.** For $n = 2$,

$$\text{L.H.S.} = \frac{1}{2+1} + \frac{1}{2+2} = \frac{7}{12} > \frac{13}{24} = \text{R.H.S.}$$

It is true for $n = 2$.

Now, for $n = r$, where $r > 2$

$$\frac{1}{r+1} + \frac{1}{r+2} + \ldots + \frac{1}{2r} > \frac{13}{24}$$

For $n = r + 1$,

$$\text{L.H.S.} = \frac{1}{r+2} + \frac{1}{r+3} + \ldots + \frac{1}{2r} + \frac{1}{2r+1} + \frac{1}{2r+2} > \frac{13}{24} + \frac{1}{2r+1} + \frac{1}{2r+2} - \frac{1}{r+1}$$ [Using

$$= \frac{13}{24} + \frac{2r+2+2r+1-2(2r+1)}{(2r+1)(2r+2)} = \frac{13}{24} + \frac{1}{(2r+1)(2r+2)} > \frac{13}{24}$$

As $n = r$ is true. Thus, the result is also true for $n = r + 1$. Hence, we have the resul
$n \geq 2$ by induction. Hence proved.

**Problem 15.** *Prove* $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

**Sol. Basis of Induction.** For $n = 1$,

$$\frac{1}{1(1+1)} = \frac{1}{2} = \frac{1}{1+1} = \frac{1}{2}$$

It is true for $n = 1$.

**Induction Step.** For $n = r$,

$$P(r) = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{r(r+1)} = \frac{r}{r+1} \text{ is true.}$$

For $n = r + 1$,

$$\text{L.H.S.} = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{(r+1)(r+1+1)}$$
$$= \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \ldots + \frac{1}{r(r+1)} + \frac{1}{(r+1)(r+2)}$$
$$= \frac{r}{r+1} + \frac{1}{(r+1)(r+2)} = \frac{1}{r+1}\left(r + \frac{1}{r+2}\right)$$
$$= \frac{r^2 + 2r + 1}{(r+1)(r+2)} = \frac{r+1}{r+2}$$

$$\text{R.H.S.} = \frac{r+1}{r+1+1} = \frac{r+1}{r+2}$$

L.H.S. = R.H.S.

Therefore, $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ is true for $n = r + 1$.

Hence proved.

**Problem 16.** *Show that $n^3 + 2n$ is divisible by 3 for all $n \geq 1$ by induction.*

**Sol. Basis of Induction.** For $n = 1$,

$$P(1) = 1^3 + 2 \times 1 = 3. \text{ It is divisible by 3.}$$

**Induction Step.** For $n = r$,

$$P(r) = r^3 + 2r = 3r$$

For $n = r + 1$,

$$\begin{aligned} P(r+1) &= (r+1)^3 + 2(r+1) = r^3 + 1 + 3r(r+1) + 2r + 2 \\ &= r^3 + 1 + 3r^2 + 3r + 2r + 2 \\ &= r^3 + 3 + 5r + 3r^2 = r^3 + 2r + 3 + 3r + 3r^2 \qquad (\because\ r^3 + 2r = 3r) \\ &= 3r + 3 + 3r + 3r^2 = 3r^2 + 6r + 3 \\ &= 3(r^2 + 2r + 1) = 3(r+1)^2. \end{aligned}$$

It is divisible by 3.

Hence, we have the required result by induction.

**Problem 17.** *Show that $2^n \times 2^n - 1$ is divisible by 3 for all $n \geq 1$ by induction.*

**Sol. Basis of Induction.** For $n = 1$,

$$2^1 \times 2^1 - 1 = 3 \text{ divisible by 3.}$$

It is true for $n = 1$.

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**Induction Step.** For $n = r$,

$$2^r\, 2^r - 1 = 3r \quad \textit{i.e.,} \quad 2^{r+1} - 1 = 3r$$

For $n = r + 1$,

$$\begin{aligned} 2^{r+1} \,.\, 2^{r+1} - 1 &= 2^{r+1}(3r + 1) - 1 \\ &= 3 \,.\, 2^{r+2} + 2^{r+1} - 1 = 3 \,.\, 2^{r+2} + 3r \\ &= 3(2^{r+2} + r). \end{aligned}$$

It is divisible by 3.

Hence, we have the required result by induction.

**Problem 18.** *Show that $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$.*

**Sol. Basis of Induction.** For $n = 1$,

$$(2 \times 1 - 1)^2 = 1 = \frac{1(2\times 1 - 1)(2 \times 1 + 1)}{3} = 1$$

It is true of $n = 1$.

**Induction Step.** For $n = r$,

$$1^2 + 3^2 + 5^2 + \ldots + (2r-1)^2 = \frac{r(2r-1)(2r+1)}{3} \text{ is true.}$$

For $n = r + 1$,

$$P(r+1) = 1^2 + 3^2 + 5^2 + \dots + (2r-1)^2 + [2(r+1)-1]^2 = \frac{(r+1)(2r+1)(2r+3)}{3}$$

$$\text{L.H.S.} = \frac{r(2r-1)(2r+1)}{3} + (2r+1)^2 = \frac{r(2r-1)(2r+1)+3(2r+1)^2}{3}$$

$$= \frac{(2r+1)[2r^2 - r + 6r + 3]}{3} = \frac{(2r+1)[2r^2+5r+3]}{3}$$

$$= \frac{(2r+1)[2r^2+3r+2r+3]}{3} = \frac{(2r+1)(2r+3)(r+1)}{3}$$

L.H.S. = R.H.S.

So, $\quad 1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}.$

**Problem 19.** *Show that* $\dfrac{1^2}{1.3} + \dfrac{2^2}{3.5} + \dots + \dfrac{n^2}{(2n-1)(2n+1)} = \dfrac{n(n+1)}{2(2n+1)}.$

**Sol. Basis of Induction.** For $n = 1$,

$$\frac{1}{1.3} = \frac{1.2}{2.3} = \frac{1}{3}$$

It is true for $n = 1$.

**Induction Step.** For $n = r$,

$$\frac{1^2}{1.3} + \frac{2^2}{3.5} + \dots + \frac{r^2}{(2r-1)(2r+1)} = \frac{r(r+1)}{2(2r+1)}$$

Now, for $n = r + 1$,

$$\frac{\text{[redacted]}}{1.3} + \frac{2^2}{3.5} + \dots + \frac{r^2}{(2r-1)(2r+1)} + \frac{(r+1)^2}{(2r+1)(2r+3)} = \frac{(r+1)(r+2)}{2(2r+3)}$$

$$\text{L.H.S.} = \frac{r(r+1)}{2(2r+1)} + \frac{(r+1)^2}{(2r+1)(2r+3)} = \frac{r+1}{2r+1}\left[\frac{r}{2} + \frac{(r+1)}{(2r+3)}\right]$$

$$= \frac{r+1}{2r+1}\left[\frac{2r^2+3r+2r+2}{2(2r+3)}\right]$$

$$= \frac{(r+1)(2r^2+5r+2)}{2(2r+1)(2r+3)} = \frac{(r+1)(r+2)}{2(2r+3)}$$

L.H.S. = R.H.S.

So, $$\frac{1^2}{1.3} + \frac{2^2}{3.5} + \dots + \frac{n^2}{(2n-1)(2n+1)} = \frac{n(n+1)}{2(2n+1)}.$$

**Problem 20.** *Prove by mathematical induction*

$$1 + 2 + 3 + 4 + \dots + n = \frac{n(n+1)}{2}.$$

**Sol.** Let $\quad P(n) = 1 + 2 + 3 + \dots + n = \dfrac{n(n+1)}{2}$

**Basis of Induction.** For $n = 1$,

$$P(1) = 1 = \frac{1(1+1)}{2} = 1 \qquad ...(i)$$

It is true for $n = 1$.

**Induction Step.** For $n = r$,

$$P(r) = 1 + 2 + 3 + ... + r = \frac{r(r+1)}{2} \qquad ...(ii)$$

Now, for $n = r + 1$,

$$P(r+1) = 1 + 2 + 3 + 4 + ... + r + (r+1) = \frac{(r+1)(r+2)}{2}$$

Consider, $\text{L.H.S.} = 1 + 2 + 3 + ... + r + (r+1)$

$$= \frac{r(r+1)}{2} + (r+1) = \frac{r(r+1) + 2(r+1)}{2}$$

$$= \frac{r+1}{2}[r+2] = \frac{(r+1)(r+2)}{2} = \text{R.H.S.} \qquad ...(iii)$$

As $P(r)$ is true. Hence, $P(r + 1)$ is also true.

From $(i)$, $(ii)$ and $(iii)$, we conclude that

$$1 + 2 + 3 + 4 + ... + n = \frac{n(n+1)}{2}. \text{ Hence proved.}$$

**Problem 21.** *Prove that $n(n + 1)(n + 2)$ is a multiple of 6.*

**Sol.** $P(n) = n(n + 1)(n + 2)$

**Basis of Induction.** For $n = 1$,

$$P(1) = 1 \,.\, (1 + 1)(1 + 2) = 1 \,.\, 2 \,.\, 3 = 6$$

which is true.

**Induction Step.** For $n = r$,

$$P(r) = r(r + 1)(r + 2) \text{ be a multiple of } 6$$

For $n = r + 1$,

$$P(r + 1) = (r + 1)(r + 2)(r + 3) \text{ is also a multiple of } 6.$$

Now, $\quad P(r + 1) = (r + 1)(r + 2)(r + 3) = (r + 1)(r + 2)\, r + 3(r + 1)(r + 2)$

$$= r(r + 1)(r + 2) + 3(2r')$$

where $(r + 1)(r + 2) = \text{even} = 2r' = r(r + 1)(r + 2) + 6r'$

which is a multiple of 6.

Thus, $P(r + 1)$ is true as $P(r)$ is true.

$\therefore \quad P(n)$ is true for all natural numbers $n$

*i.e.*, $n(n + 1)(n + 2)$ is a multiple of 6. Hence proved.

6

**Problem 22.** *Prove that $\forall n \in N$*

$$\frac{1}{5}n^5 + \frac{1}{3}n^3 + \frac{7}{15}n \text{ is a natural number.}$$

**Sol.** Let $P(n) = \frac{1}{5}n^5 + \frac{1}{3}n^3 + \frac{7}{15}n$ is a natural number.

**Basis of Induction.** For $n = 1$,

$$\frac{1}{5} + \frac{1}{3} + \frac{7}{15} = 1, \text{ which is a natural number.}$$

It is true for $n = 1$.

**Induction Step.** For $n = r$,

$$\left(\frac{1}{5}r^5 + \frac{1}{3}r^3 + \frac{7}{15}r\right) \text{ is a natural number}$$

For $n = r + 1$,

$$\frac{1}{5}(r+1)^5 + \frac{1}{3}(r+1)^3 + \frac{7}{15}(r+1)$$

$$= \frac{1}{5}(r^5 + 5r^4 + 10r^3 + 10r^2 + 5r + 1) + \frac{1}{3}(r^3 + 3r^2 + 3r + 1) + \frac{7}{15}(r$$

$$= \left(\frac{1}{5}r^5 + \frac{1}{3}r^3 + \frac{7}{15}r\right) + (r^4 + 2r^3 + 3r^2 + 2r) + 1$$

= (a natural number) + (a natural number) + 1 [Using

= a natural number

As (P($r$)) is true, hence P($r$ + 1) is also true.

Hence, by induction, P($n$) is true $\forall n \in$ N. Hence proved.

**Problem 23.** *Prove that $n(n + 1)(2n + 1)$ is divisible by 6.*

**Sol.** Let $P(n) = n(n + 1)(2n + 1)$ is divisible by 6.

**Basis of Induction.** For $n = 1$,

$$P(1) = 1(1 + 1)(2 + 1) = 1 . 2 . 3 = 6 \text{ which is true for } n = 1$$

**Induction Step.** For $n = r$,

$$P(r) = r(r + 1)(2r + 1) \text{ be divisible by 6}$$

For $n = r + 1$,

$$P(r + 1) = (r + 1)(r + 2)(2r + 3)$$

$$\text{L.H.S.} = (r + 1)(r + 2)(2r + 3) = 2r^3 + 9r^2 + 13r + 6$$

$$= (2r^3 + 3r^2 + r) + (6r^2 + 12r + 6)$$

$$= r(r + 1)(2r + 1) + 6(r^2 + 2r + 1)$$

which is divisible by 6. [From

As (P($r$)) is true, hence P($r$ + 1) is also true.

$\therefore$ P($n$) is true for all natural numbers $n$

*i.e.,* $n(n + 1)(2n + 1)$ is divisible by 6. Hence proved.

# Propositional Calculus

## PROPOSITION

A proposition is a statement which is either true or false. It is a declarative sente

**For example :** The following statements are all propositions :

(*i*) Jawahar Lal Nehru is the first prime minister of India.

(*ii*) It rained yesterday.

(*iii*) If $x$ is an integer, then $x^2$ is a + ve integer.

**For example :** The following statements are not propositions :

(*i*) Please report at 11 a.m. sharp

(*ii*) What is your name ?

(*iii*) $x^2 = 13$.

## PROPOSITIONAL VARIABLES

The lower case letters starting from P onwards are used to represent propositions

$p$ : India is in Asia

$q : 2 + 2 = 4$.

**Example 1.** *Classify the following statements as propositions or non-propositions.*

(*i*) *The population of India goes upto 100 million in year 2000.*

(*ii*) $x + y = 30$

(*iii*) *Come here*

(*iv*) *The Intel Pentium-III is a 64-bit computer.*

**Sol.** (*i*) Proposition (*ii*) Not a proposition

(*iii*) Not a proposition (*iv*) Proposition.

## COMBINATION OF PROPOSITIONS

We can combine the propositions to produce new propositions. There are t fundamental and three derived connectors to combine the propositions. These are expl as follows one by one.

## (a) Fundamental Connectors

1. **Conjunction.** It means ANDing of two statements. Assume $p$ and $q$ be two propositions. Conjunction of $p$ and $q$ to be a proposition which is true when both $p$ and $q$ are true, otherwise false. It is denoted by $p \wedge q$. (Fig. 1)

Truth tables are used to determine the truth or falsity of the combined proposition.

$(p \wedge q)$

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

Fig. 1. Truth Table of $p \wedge q$.

2. **Disjunction.** It means ORing of two statements. Assume $p$ and $q$ be two propositions. Disjunction of $p$ and $q$ to be a proposition which is true when either one or both $p$ and $q$ are true and is false when both $p$ and $q$ are false. It is denoted by $p \vee q$. (Fig. 2)

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

Fig. 2. Truth Table of $p \vee q$.

3. **Negation.** It means opposite of original statement. Assume $p$ be a proposition. Negation of $p$ to be a proposition which is true when $p$ is false, and is false when $p$ is true. It is denoted by $\sim p$. (Fig. 3)

| $p$ | $\sim p$ |
|---|---|
| T | F |
| F | T |

Fig. 3. Truth Table of $\sim p$.

**Example 2.** *Consider the following :*

*$p$ : He is rich*

*$q$ : He is Generous.*

*Write the proposition which combines the proposition $p$ and $q$ using conjunction ($\wedge$), disjunction ($\vee$), and negation ($\sim$).*

**Sol. Conjunction.** He is rich and generous *i.e.*, $p \wedge q$.

**Disjunction.** He is rich or generous *i.e.*, $p \vee q$.

**Negation.** He is not rich *i.e.*, $\sim p$

He is not generous *i.e.*, $\sim q$.

It is false that he is rich or generous *i.e.*, $\sim (p \vee q)$.

He is neither rich nor generous *i.e.*, $\sim p \wedge \sim q$.

It is false that he is not rich *i.e.*, $\sim (\sim p)$.

**Example 3.** *Let p be "It is hot day" and q be "The temperature is 45°C". Write in simple sentences the meaning of following :*

(i) $\sim p$ (ii) $\sim (p \vee q)$ (iii) $\sim (p \wedge q)$

(iv) $\sim (\sim p)$ (v) $p \vee q$ (vi) $p \wedge q$

(vii) $\sim p \wedge \sim q$ (viii) $\sim (\sim p \vee \sim q)$.

**Sol.** (i) It is not a hot day.

(ii) It is false that it is hot day or temperature is 45°C.

(iii) It is not true that it is hot day and temperature is 45°C.

(iv) It is false that it is not a hot day.

(v) It is hot day or temperature is 45°C.

(vi) It is hot day and temperature is 45°C.

(vii) It is neither a hot day nor temperature is 45°C.

(viii) It is false that it is not a hot day or temperature is not 45°C.

**Example 4.** *Consider the following statements*

*p : He is coward.*

*q : He is lazy.*

*r : He is rich.*

*Write the following compound statements in the symbolic form.*

(i) *He is either coward or poor.*

(ii) *He is neither coward nor lazy.* $\sim p \wedge \sim q$

(iii) *It is false that he is coward but not lazy.*

(iv) *He is coward or lazy but not rich.* $\sim (p \wedge \sim q)$

(v) *It is false that he is coward or lazy but not rich.*

(vi) *It is not true that he is not rich.*

(vii) *He is rich or else he is both coward and lazy.*

**Sol.** (i) $p \wedge \sim r$ (ii) $\sim p \wedge \sim q$ (iii) $\sim (p \wedge \sim q)$

(iv) $(p \vee q) \wedge \sim r$ (v) $\sim ((p \vee q) \wedge \sim r)$ (vi) $\sim (\sim r)$

(vii) $r \vee (p \wedge q)$.

**(b) Derived Connectors**

1. **NAND.** It means negation after ANDing of two statements. Assume $p$ and $q$ be two propositions. Nanding of $p$ and $q$ to be a proposition which is false when both $p$ and $q$ are true otherwise true. It is denoted by $p \uparrow q$. (Fig. 4)

$p \vee \sim q$

| $p$ | $q$ | $p \uparrow q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | T |

Fig. 4.

2. **NOR or Joint Denial.** It means negation after ORing of two statements. Assume $p$ and $q$ be two propositions. NOring of $p$ and $q$ to be a proposition which is true when both $p$ and $q$ are false, otherwise false. It is denoted by $p \downarrow q$. (Fig. 5)

| $p$ | $q$ | $p \downarrow q$ |
|---|---|---|
| T | T | F |
| T | F | F |
| F | T | F |
| F | F | T |

Fig. 5.

3. **XOR.** Assume $p$ and $q$ be two propositions. XORing of $p$ and $q$ is true if $p$ is true or if $q$ is true but not both and vice-versa. It is denoted by $p \oplus q$. (Fig. 6)

| $p$ | $q$ | $p \oplus q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

Fig. 6.

**Example 5.** *Generate the truth table for following :*

(*i*) $A \oplus B \oplus C$ (*ii*) $A \uparrow B \uparrow C$.

**Sol.** The truth table for above formulas are as shown in Figs. 7 and 8.

(*i*)

| A | B | C | $A \oplus B$ | $A \oplus B \oplus C$ |
|---|---|---|---|---|
| T | T | T | F | T |
| T | T | F | F | F |
| T | F | T | T | F |
| T | F | F | T | T |
| F | T | T | T | F |
| F | T | F | T | T |
| F | F | T | F | T |
| F | F | F | F | F |

Fig. 7.

(*ii*) Truth table for (*ii*) is

| A | B | C | $A \uparrow B$ | $A \uparrow B \uparrow C$ |
|---|---|---|---|---|
| T | T | T | F | T |
| T | T | F | F | T |
| T | F | T | T | F |
| T | F | F | T | T |
| F | T | T | T | F |
| F | T | F | T | T |
| F | F | T | T | F |
| F | F | F | T | T |

Fig. 8.

**Example 6.** *Prove that* $X \oplus Y \equiv (X \wedge \sim Y) \vee (\sim X \wedge Y)$.

**Sol.** Construct the truth table for both the propositions. (Fig. 9)

| X | Y | $X \oplus Y$ | $\sim Y$ | $\sim X$ | $X \wedge \sim Y$ | $\sim X \wedge Y$ | $(X \wedge \sim Y) \vee (\sim X \wedge Y)$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | F | F | F | F |
| T | F | T | T | F | T | F | T |
| F | T | T | F | T | F | T | T |
| F | F | F | T | T | F | F | F |

Fig. 9.

As the truth table for both the proposition are same.

$$X \oplus Y \equiv (X \wedge \sim Y) \vee (\sim X \wedge Y). \text{ Hence proved.}$$

**Example 7.** *Show that* $(p \oplus q) \vee (p \downarrow q)$ *is equivalent to* $p \uparrow q$.

**Sol.** Construct the truth table for both the propositions.

| $p$ | $q$ | $(p \oplus q)$ | $(p \downarrow q)$ | $(p \oplus q) \vee (p \downarrow q)$ | $p \uparrow q$ |
|---|---|---|---|---|---|
| T | T | F | F | F | F |
| T | F | T | F | T | T |
| F | T | T | F | T | T |
| F | F | F | T | T | T. |

Fig. 10.

Since the values of $(p \oplus q) \vee (p \downarrow q)$ is same as $p \uparrow q$ as in Fig. 10. Hence, they equivalent.

**Example 8.** *Show that* $(p \uparrow q) \oplus (p \uparrow q)$ *is equivalent to* $(p \vee q) \wedge (p \downarrow q)$.

**Sol.** Construct the truth table for both the propositions

| $p$ | $q$ | $p \uparrow q$ | $(p \uparrow q) \oplus (p \uparrow q)$ | $p \vee q$ | $p \downarrow q$ | $(p \vee q) \wedge (p \downarrow q)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F |
| T | F | T | F | T | F | F |
| F | T | T | F | T | F | F |
| F | F | T | F | F | T | F |

Fig. 11.

Since, the values of $(p \uparrow q) \oplus (p \uparrow q)$ and $(p \vee q) \wedge (p \downarrow q)$ are same as in Fig. 11. Hen they are equivalent.

(*c*) **Some Other Connectors**

1. **Conditional.** Statements of the form "If $p$ then $q$" are called conditional stateme It is denoted as $p \rightarrow q$ and read as "$p$ implies $q$" or "$q$ is necessary for $p$" or "$p$ is suffici for $q$".

Conditional statement is true if both $p$ and $q$ are true or if $p$ is false. It is false if $p$ is t and $q$ is false. The proposition $p$ is called hypothesis and the proposition $q$ is called conclus The truth table of conditional statement is

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

Fig. 12. Truth Table of $p \rightarrow q$.

**For example :** The followings are conditional statements :

1. If $a = b$ and $b = c$, then $a = c$.
2. If I will get money, then I will purchase computer.

## RIATIONS IN CONDITIONAL STATEMENT

**Contrapositive.** The proposition $\sim q \to \sim p$ is called contrapositive of $p \to q$.
**Converse.** The proposition $q \to p$ is called the converse of $p \to q$.
**Inverse.** The proposition $\sim p \to \sim q$ is called the inverse of $p \to q$.

**Example 9.** *Show that $p \to q$ and its contrapositive $\sim q \to \sim p$ are logically equivalent.*

**Sol.** Construct truth table for both the propositions. (as in Fig. 13)

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \to q$ | $\sim q \to \sim p$ |
|---|---|---|---|---|---|
| T | T | F | F | T | T |
| T | F | F | T | F | F |
| F | T | T | F | T | T |
| F | F | T | T | T | T |

Fig. 13.

*As, the values in both cases are same, hence both propositions are equivalent.*

**Example 10.** *Show that proposition $q \to p$ and $\sim p \to \sim q$ is not equivalent to $p \to q$.*

**Sol.** Construct truth table for all the above propositions :

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \to q$ | $q \to p$ | $\sim p \to \sim q$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T |
| T | F | F | T | F | T | T |
| F | T | T | F | T | F | F |
| F | F | T | T | T | T | T |

Fig. 14.

As the values of $p \to q$ in table is not equal to $q \to p$ and $\sim p \to \sim q$ as in Fig. 14. So both f them are not equal to $p \to q$ but they are themselves logically equivalent.

**Example 11.** *Prove that the following propositions are equivalent to $p \to q$.*

(*i*) $\sim (p \wedge \sim q)$ (*ii*) $\sim p \vee q$ (*iii*) $\sim q \to \sim p$.

**Sol.** Construct the truth table for all the above propositions :

| $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim p \vee q$ | $\sim q \to \sim p$ | $(p \wedge \sim q)$ | $\sim (p \wedge \sim q)$ | $p \to q$ |
|---|---|---|---|---|---|---|---|---|
| T | T | F | F | T | T | F | T | T |
| T | F | F | T | F | F | T | F | F |
| F | T | T | F | T | T | F | T | T |
| F | F | T | T | T | T | F | T | T |

Fig. 15.

In the above table (Fig. 15) the values of $p \to q$ is equivalent to (*i*), (*ii*) and (*iii*), hence hey are equivalent to $p \to q$. Hence proved.

2. **Biconditional.** Statements of the form "if and only if" are called biconditional statements.

It is denoted as $p \leftrightarrow q$ and read as "$p$ if and only if $q$". The proposition $p \leftrightarrow q$ is true if $p$ and $q$ have the same truth values and is false if $p$ and $q$ do not have the same truth values. The name of biconditional comes from the fact that $p \leftrightarrow q$ is equivalent to $(p \to q) \wedge (q \to p)$.

The truth table of $p \leftrightarrow q$ is

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

Fig. 16. Truth Table of $p \leftrightarrow q$.

**For example :** (*i*) Two lines are parallel if and only if they have same slope.

(*ii*) You will pass the exam if and only if you will work hard.

**Example 12.** *Prove that $p \leftrightarrow q$ is equivalent to $(p \to q) \wedge (q \to p)$.*

**Sol.** Construct the truth tables of both propositions :

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

Fig. 17.

| $p$ | $q$ | $p \to q$ | $q \to p$ | $(p \to q) \wedge (q \to p)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | T |

Fig. 18.

Since, the truth tables are same, hence they are logically equivalent. (Fig. 17 and Fig. 18). Hence proved.

## PRINCIPLE OF DUALITY

Two formulas $A_1$ and $A_2$ are said to be duals of each other if either one can be obtained from the other by replacing $\wedge$ (AND) by $\vee$ (OR) and $\vee$ (OR) by $\wedge$ (AND). Also if the formula contains T (True) or F (False), then we replace T by F and F by T to obtain the dual.

**Note 1.** The two connectives $\wedge$ and $\vee$ are called dual of each other.

**2.** Like AND and OR, $\uparrow$ (NAND) and $\downarrow$ (NOR) are dual of each other.

**3.** If any formula of proposition is valid, then its dual is also a valid formula.

**Example 13.** *Determine the dual of each of the following :*

(*a*) $p \wedge (q \wedge r)$ (*b*) $\sim p \vee \sim q$ (*c*) $(p \wedge \sim q) \vee (\sim p \wedge q)$

(*d*) $(p \uparrow q) \uparrow (p \uparrow q)$ (*e*) $((\sim p \vee q) \wedge (q \wedge \sim s)) \vee (p \vee F)$, *here F means false.*

**Sol.** To obtain the dual of all the above formulas, replace $\wedge$ by $\vee$ and $\vee$ by $\wedge$, and also replace T by F and F by T.

Also replace $\uparrow$ by $\downarrow$ and vice-versa.

(*a*) $p \wedge (q \wedge r) = p \vee (q \vee r)$ (*b*) $\sim p \vee \sim q = \sim p \wedge \sim q$

(*c*) $(p \wedge \sim q) \vee (\sim p \wedge q) = (p \vee \sim q) \wedge (\sim p \vee q)$

(*d*) $(p \downarrow q) \downarrow (p \downarrow q)$

(*e*) $((\sim p \vee q) \wedge (q \wedge \sim s)) \vee (p \vee F) = ((\sim p \wedge q) \vee (q \vee \sim s)) \wedge (p \wedge T)$

## EQUIVALENCE OF PROPOSITIONS

Two propositions are said to be logically equivalent if they have exactly the same truth values under all circumstances. The table 1 contains the fundamental logical equivalent expressions :

**Table 1.**

1. **De Morgan's Laws**
   $\sim(p \wedge q) \equiv \sim p \vee \sim q$
   $\sim(p \vee q) \equiv \sim p \wedge \sim q$
2. **Commutative Properties**
   $p \vee q \equiv q \vee p$; $p \wedge q \equiv q \wedge p$.
3. **Associative Properties**
   $(p \vee q) \vee r \equiv p \vee (q \vee r)$
   $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
4. **Distributive Properties**
   $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
   $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
5. **Impotent Laws**
   $p \vee p \equiv p$ and $p \wedge p \equiv p$
6. **Complement Properties**
   $p \equiv \sim\sim p$
7. **Transposition**
   $(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$
8. **Material Implication**
   $(p \rightarrow q) \equiv (\sim p \vee q)$
9. **Material Equivalence**
   $(p \leftrightarrow q) \equiv [(p \rightarrow q) \wedge (q \rightarrow p)]$
   $(p \leftrightarrow q) \equiv [(p \rightarrow q) \vee (\sim p \wedge \sim q)]$
10. **Exportation**
    $[(p \wedge q) \rightarrow r] \equiv [p \rightarrow (q \rightarrow r)]$

$\sim p \rightarrow \sim q$
$(p \vee q)$

$p \vee q = q \vee p$

**Example 14.** *Consider the following propositions*

$$\sim p \vee \sim q \text{ and } \sim (p \wedge q).$$

*Are they equivalent ?*

**Sol.** Construct the truth table for both (as in Fig. 19).

| $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim p \vee \sim q$ | $p \wedge q$ | $\sim (p \wedge q)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | F | T | F |
| T | F | F | T | T | F | T |
| F | T | T | F | T | F | T |
| F | F | T | T | T | F | T |

Fig. 19.

Since, the final values of both the propositions are same, hence the two propositions are equivalent.

## TAUTOLOGIES

A proposition P is a tautology if it is true under all circumstances. It means it contains only T in the final column of its truth table.

**Example 15.** *Prove that the statement $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$ is a tautology.*

**Sol.** Make the truth table of above statement :

| $p$ | $q$ | $p \rightarrow q$ | $\sim q$ | $\sim p$ | $\sim q \rightarrow \sim p$ | $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | T | T |
| F | F | T | T | T | T | T |

Fig. 20.

As the final column contains all T's, so it is a tautology. (Fig. 20)

**Example 16.** *Prove that $p \vee p \leftrightarrow p$ is a tautology.*

**Sol.** Construct the truth table of the given statement :

| $p$ | $p \vee p$ | $p \vee p \leftrightarrow p$ |
|---|---|---|
| T | T | T |
| F | F | T |

Fig. 21.

As the last column contains all T's, so it is a tautology. (Fig. 21)

## CONTRADICTION

A statement that is always false is called a contradiction.

**Example 17.** *Show that the statement $p \wedge \sim p$ is a contradiction.*

| $p$ | $\sim p$ | $p \wedge \sim p$ |
|---|---|---|
| T | F | F |
| F | T | F |

Fig. 22.

**Sol.** Construct the truth table of above statement.

Since, the last column contains all F's, so it is a contradiction. (Fig. 22)

## CONTINGENCY

A statement that can be either true or false depending on the truth values of its variables, is called a contingency.

| $p$ | $q$ | $p \rightarrow q$ | $p \wedge q$ | $(p \rightarrow q) \rightarrow (p \wedge q)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | T | F | F |
| F | F | T | F | F |

Fig. 23.

**Example 18.** *Prove that the statement $(p \rightarrow q) \rightarrow (p \wedge q)$ is a contingency.*

**Sol.** Construct the truth table of above statement.

As, the value of final column depends on the truth value of the variables, so it is a contingency. (Fig. 23)

**Example 19.** *From the following formulae, find out tautology, contingency and contradiction.*

*(i)* $A \equiv A \wedge (A \vee B)$ *(ii)* $(p \wedge \sim q) \vee (\sim p \wedge q)$ *(iii)* $\sim (p \vee q) \vee (\sim p \vee \sim q)$

**Sol.** (*i*) Construct the truth table for $A \to A \wedge (A \vee B)$.

| $A$ | $B$ | $A \vee B$ | $A \wedge (A \vee B)$ | $A \to A \wedge (A \vee B)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | T | T | T |
| F | T | T | F | T |
| F | F | F | F | T |

Fig. 24.

Since, the last column of the table contains all T's, hence it is a tautology. (Fig. 24)

(*ii*) Construct the truth table for $(p \wedge \sim q) \vee (\sim p \wedge q)$ as in Fig. 25.

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \wedge \sim q$ | $\sim p \wedge q$ | $(p \wedge \sim q) \vee (\sim p \wedge q)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | F | F | F |
| T | F | F | T | T | F | T |
| F | T | T | F | F | T | T |
| F | F | T | T | F | F | F |

Fig. 25.

Since, the value of the final column depends on the value of the different variables, hence it is a contingency.

(*iii*) Construct the truth table of the proposition $\sim (p \vee q) \vee (\sim p \vee \sim q)$ as in Fig. 26.

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \wedge q$ | $\sim (p \wedge q)$ | $\sim p \vee \sim q$ | $\sim (p \vee q) \vee (\sim p \vee \sim q)$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F | F |
| T | F | F | T | F | T | T | T |
| F | T | T | F | F | T | T | T |
| F | F | T | T | F | T | T | T |

Fig. 26.

Since, the value of final column depends upon the value of different variables, hence it is a contingency.

**Example 20.** *Verify that proposition $p \vee \sim (p \wedge q)$ is tautology.*

**Sol.** Construct the truth table for above proposition. (Fig. 27)

| $p$ | $q$ | $p \wedge q$ | $\sim(p \wedge q)$ | $p \vee \sim (p \wedge q)$ |
|---|---|---|---|---|
| T | T | T | F | T |
| T | F | F | T | T |
| F | T | F | T | T |
| F | F | F | T | T |

Fig. 27.

Since, the last column contains all T' s, hence it is a tautology.

**Example 21.** *Determine whether the following is a tautology, contingency and a contradiction :*

(*i*) $p \to (p \to q)$ (*ii*) $p \to (q \to p)$ (*iii*) $p \wedge \sim p$.

**Sol.** (*i*) Construct truth table for $p \to (p \to q)$ as in Fig. 28.

| $p$ | $q$ | $p \to q$ | $p \to (p \to q)$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | F |
| F | T | T | T |
| F | F | T | T |

Fig. 28.

Since, the value of last column depends on the value of different variables, hence it is a contingency.

(*ii*) Construct truth table for $p \to (q \to p)$ as in Fig. 29.

| $p$ | $q$ | $q \to p$ | $p \to (q \to p)$ |
|---|---|---|---|
| T | T | T | T |
| T | F | T | T |
| F | T | F | T |
| F | F | T | T |

Fig. 29.

Since, the last column contains all T's, hence it is a tautology.

(*iii*) Construct truth table for $p \wedge \sim p$ as in Fig. 30.

Since, the last column contain all F's, hence it is a contradiction.

| $p$ | $\sim p$ | $p \wedge \sim p$ |
|---|---|---|
| T | F | F |
| F | T | F |

Fig. 30.

## FUNCTIONALLY COMPLETE SETS OF CONNECTIVES

We have three basic and two conditional connectives *i.e.*, $\wedge$, $\vee$, $\sim$ $\Rightarrow$ and $\Leftrightarrow$. If we have given any formula containing all these connectives, we can write an equivalent formula with certain proper subsets of these connectives.

A set of connectives is called functionally complete if every formula can be expresses on terms of an equivalent formula containing the connectives from this set.

**Example 22.** *Write an equivalent formula for $P \wedge (R \Leftrightarrow S) \vee (S \Leftrightarrow P)$ which does not involve biconditional.*

**Sol.** We know that $P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$ ...(*i*)

So, apply eqn. (*i*) to formula to obtain the formula equivalent to

$$[P \wedge (R \Leftrightarrow S) \vee (S \Leftrightarrow P)],$$

which does not involve biconditional.

$$= [P \wedge ((R \Rightarrow S) \wedge (S \Rightarrow R)) \vee ((S \Rightarrow P) \wedge (P \Rightarrow S))].$$

**Example 23.** *Write an equivalent formula for $R \vee (S \Leftrightarrow T)$, which does not involve biconditional as well as conditional.*

**Sol.** We know that

$$(P \Leftrightarrow Q) = (P \Rightarrow Q) \wedge (Q \Rightarrow P) \qquad ...(i)$$

$$(P \Rightarrow Q) = \sim P \wedge Q \qquad ...(ii)$$

So, applying the eqn. $(i)$ and $(ii)$ on the above formula, we can obtain an equivalent nula, which does not involve biconditional as well as conditional.

$$[R \vee (S \Leftrightarrow T)] = [R \vee ((S \Rightarrow T) \vee (T \Rightarrow S))]$$
$$= [R \vee ((\sim S \vee T) \vee (\sim T \vee S))].$$

**Example 24.** *Show that* $\{\sim, \wedge\}$ *is functionally complete.*

**Sol.** Take any formula which involve all the five connectives $\wedge, \vee, \sim, \Rightarrow$ and $\Leftrightarrow$. We can ain an equivalence formula by first replacing biconditional and then replacing conditional finally we can replace $\vee$.

As $\quad P \Leftrightarrow Q = (P \Rightarrow Q) \wedge (Q \Rightarrow P)$

$$= (\sim P \vee Q) \wedge (\sim Q \vee P) = (\sim (\sim\sim P \wedge \sim Q)) \wedge (\sim (\sim \sim Q \wedge \sim P))$$

Hence, $\{\sim, \wedge\}$ is functionally complete.

Similarly, we can show that $(\sim, \vee\}$ is functionally complete.

**Example 25.** *Show that* $\{\sim, \rightarrow\}$ *is functionally complete.*

**Sol.** We know that $\quad P \Rightarrow Q \equiv \sim P \vee Q.$

So, we have $\quad P \vee Q \equiv \sim P \Rightarrow Q.$

Since, $\{\sim, \vee\}$ is functionally complete. Hence from above, $\{\sim, \Rightarrow\}$ is also functionally nplete.

given any formula which involve all the five connectives, we can obtain an equivalence mula using $\{\sim, \rightarrow\}$ by first replacing biconditional and then replacing $(\wedge)$ ANDing and filly replacing $\vee$.

**Example 26.** *Express* $P \Leftrightarrow Q$ *in terms of* $\{\sim, \wedge\}$ *only.*

**Sol.** $(P \Leftrightarrow Q) = (P \Rightarrow Q) \wedge (Q \Rightarrow P) \qquad \because (P \Leftrightarrow Q) = (P \Rightarrow Q) \wedge (Q \Rightarrow P)$

$$= (\sim P \vee Q) \wedge (\sim Q \vee P) \qquad \because P \Rightarrow Q = \sim P \vee Q$$
$$= (\sim (\sim \sim P \wedge \sim Q)) \wedge (\sim (\sim \sim Q \wedge \sim P)) \qquad \because P \vee Q = \sim (\sim P \wedge \sim Q).$$

**Example 27.** *Express* $(P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$ *in terms of* $(\sim, \vee)$ *only.*

**Sol.** $(P \wedge \sim Q) \vee (\sim P \wedge \sim Q) = (\sim (\sim P \vee \sim \sim Q)) \vee (\sim (\sim \sim P \vee \sim \sim Q)) \because P \wedge Q = \sim (\sim P \vee \sim Q)$

## RGUMENT

An argument is an assertion ; that a group of propositions called premises, yields another proposition, called the conclusion. Let $P_1, P_2, P_3, ..., P_n$ is the group of propositions that lds the conclusion Q. Then, it is denoted as $P_1, P_2, P_n, ..., P_n \mid\!- Q$.

**Conclusion.** The conclusion of an argument is the proposition that is asserted on the sis of other proposition of the argument.

**Premises.** The propositions, which are assumed for accepting the conclusion, are called e premises of that argument.

**(a) Valid Argument**

An argument is called valid argument if the conclusion is true whenever all the pre[...] are true.

The argument is also valid if and only if the ANDing of the group of propositions im[...] conclusion is a tautology *i.e.*, $P(p_1, p_2, p_3, ..., p_n) \rightarrow Q$ is a tautology. Where $P(p_1, p_2, p_3, ..., p_n)$ [...] the group of propositions and Q is the conclusion.

Some common valid argument forms are given in Table 2.

**(b) Falacy Argument**

An argument is called falacy or an invalid argument if it is not a valid argument.

**Elementary Valid Argument Forms**

**Table 2.**

**1. Modus ponens**

$p \rightarrow q$

$p$

$\therefore\ q$

**2. Modus tollens**

$p \rightarrow q$

$\sim q$

$\therefore\ \sim p$

**3. Hypothetical syllogism**

$p \rightarrow q$

$q \rightarrow r$

$\therefore\ p \rightarrow r$

**4. Disjunctive syllogism**

$p \vee q$

$\sim p$

$\therefore\ q$

**5. Constructive dilemma**

$(p \rightarrow q) \wedge (r \rightarrow s)$

$p \vee r$

$\therefore\ q \vee s$

**6. Absorption**

$p \rightarrow q$

$\therefore\ p \rightarrow (p \wedge q)$

**7. Simplification**

$p \wedge q$

$\therefore\ p$

**8. Conjunction**

$p$

$q$

$\therefore\ p \wedge q$

**9. Addition**

$p$

$\therefore\ p \vee q.$

**Example 28.** *Show that the following rule is valid :*

$$p \mid\!- p \vee q \quad or \quad p \quad \therefore \quad p \vee q.$$

**Sol.** We can prove this rule from the truth table (Fig. 31)

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

Fig. 31.

$p$ is true in line 1 and 2 and $p \vee q$ is also true in line 1 and 2. Hence, argument i[...]

**Example 29.** *Show that the rule modus ponens is valid.*

$$p \to q$$
$$p$$
$$\therefore \quad q$$

**Sol.** The truth table of this rule is as follows : (Fig. 32)

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

Fig. 32.

The $p$ is the true in line 1 and 2 and $p \to q$ and $p$ both are true in line 1 and $p, p \to q$ and l are true in line 1. Hence, argument is valid.

**Example 30.** *Show that the rule of hypothetical syllogism is valid*

$$p \to q$$
$$q \to r$$
$$\therefore \quad p \to r.$$

**Sol.** The truth table of above rule is as in Fig. 33.

| $p$ | $q$ | $r$ | $p \to q$ | $q \to r$ | $p \to r$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | T | F | T | F | F |
| T | F | F | F | T | F |
| T | F | T | F | T | T |
| F | T | T | T | T | T |
| F | T | F | T | F | T |
| F | F | T | T | T | T |
| F | F | F | T | T | T |

Fig. 33.

$p \to q$ is true in lines 1, 2, 5, 6, 7, 8.

$q \to r$ is true in lines 1, 3, 4, 5, 7, 8.

Both $p \to q$ and $q \to r$ is true in lines 1, 5, 7, 8 $p \to r$ is also true in lines 1, 5, 7, 8. Hence, ırgument is valid.

**Example 31.** *Show that the rule of Modus Tollens is valid*

$$p \to q$$
$$\sim q$$
$$\therefore \quad \sim p.$$

**Sol.** The truth table of above propositions are (Fig. 34)

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \to q$ |
|---|---|---|---|---|
| T | T | F | F | T |
| T | F | F | T | F |
| F | T | T | F | T |
| F | F | T | T | T |

Fig. 34.

$p \to q$ is true in line 1, 3 and 4. $\sim q$ is true in line 2 and 4. Both $p \to q$ and $\sim q$ are true in line 4. $\sim p$ is also true in line 4. Hence, the argument is valid.

**Example 32.** *Show that the rule of disjunctive syllogism is valid*

$$p \vee q$$
$$\sim p$$
$$\therefore \quad q.$$

**Sol.** The truth table of above rule is as follows : (Fig. 35)

| $p$ | $q$ | $\sim p$ | $p \vee q$ |
|---|---|---|---|
| T | T | F | T |
| T | F | F | T |
| F | T | T | T |
| F | F | T | F |

Fig. 35.

$p \vee q$ is true in line 1, 2 and 3. $\sim p$ is true in line 3. Both $p \vee q$ and $\sim p$ is true in line 3. As $q$ is also true in line 3. Hence, argument is valid.

**Example 33.** *Show that the rule of simplification is valid*

$$p \wedge q$$
$$\therefore \quad p.$$

**Sol.** The truth table of above argument is as follows : (Fig. 36)

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

Fig. 36.

$p \wedge q$ is true in line 1 and $p$ is also true in line 1. Hence, the argument is valid.

**Example 34.** *Show that the rule of conjunction is valid.*

$$p$$
$$q$$
$$\therefore \quad p \wedge q.$$

**Sol.** The argument is valid if $p \wedge q \to p \wedge q$ is a tautology. The truth table for the above proposition is as follows : (Fig. 37)

| $p$ | $q$ | $p \wedge q$ | $p \wedge q \to p \wedge q$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | F | T |
| F | F | F | T |

Fig. 37.

As the proposition is a tautology. Hence, the argument is valid.

**Example 35.** *Show that the rule of absorption is valid*

$$p \to q$$
$$\therefore \quad p \to (p \wedge q).$$

**Sol.** We have to show that $(p \to q) \to [p \to (p \wedge q)]$ is tautology. The truth table of the above argument is as follows : (Fig. 38)

| $p$ | $q$ | $p \wedge q$ | $p \to q$ | $p \to (p \wedge q)$ | $(p \to q) \to [p \to (p \wedge q)]$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | F | F | T |
| F | T | F | T | T | T |
| F | F | F | T | T | T |

Fig. 38.

Since, the argument is a tautology. Hence, it is a valid argument.

## PROOF OF VALIDITY

We can test the validity of any argument by constructing the truth tables. But as the no. of variable statements increases, the truth tables grow unwieldly. So, a more efficient method to test the validity of the argument is to deduce its conclusion from its premises by a sequence of elementary arguments each of which is known to be valid.

**Example 36.** *Prove that the argument $p \to \sim q, r \to q, r \vdash \sim p$ is valid without using truth tables.*

| | | |
|---|---|---|
| **Sol.** (*i*) | $p \to \sim q$ | (Given) |
| (*ii*) | $r \to q$ | (Given) |
| (*iii*) | $\sim q \to \sim r$ | Contrapositive of (*ii*) |
| (*iv*) | $p \to \sim r$ | Hypothetical syllogism using (*i*) and (*iii*) |
| (*v*) | $r \to \sim p$ | Contrapositive of (*iv*) |

| | | |
|---|---|---|
| | | (Given) |
| (vi) | $r$ is true | Modus ponens using ($v$) and (...) |
| (vii) | $\sim p$ is true | |

**Example 37.** *Prove that the argument $p \to q, q \to r, r \to s, \sim s, p \vee t \vdash t$ is valid without using truth tables.*

| | | |
|---|---|---|
| **Sol.** ($i$) | $p \to q$ | (Given) |
| ($ii$) | $q \to r$ | (Given) |
| ($iii$) | $r \to s$ | (Given) |
| ($iv$) | $\sim s$ | (Given) |
| ($v$) | $p \vee t$ | (Given) |
| ($vi$) | $p \to r$ | Hypothetical syllogism using ($i$) and ($ii$) |
| ($vii$) | $p \to s$ | Hypothetical syllogism using ($vi$) and ($iii$) |
| ($viii$) | $\sim p$ | Modus tollens using ($vii$) and ($iv$) |
| ($ix$) | $t$ | Disjunctive syllogism using ($v$) and ($viii$) |

**Example 38.** *Prove that the argument $p, q \vdash (p \vee r) \wedge q$ is valid without using truth tables.*

| | | |
|---|---|---|
| **Sol.** ($i$) | $p$ | (Given) |
| ($ii$) | $p \vee r$ | Rule of addition using ($i$) |
| ($iii$) | $q$ | (Given) |
| ($iv$) | $(p \vee r) \wedge q$ | Rule of conjunction using ($ii$) and ($iii$) |

**Example 39.** *Prove that the argument $p \to q, p \wedge r \vdash q$ is valid without using truth table.*

| | | |
|---|---|---|
| **Sol.** ($i$) | $p \to q$ | (Given) |
| ($ii$) | $p \wedge r$ | (Given) |
| ($iii$) | $p$ | Rule of simplification using ($ii$) |
| ($iv$) | $q$ | Modus ponens using ($i$) and ($iii$) |

**Example 40.** *Prove that the argument $(p \to q) \wedge (r \to s), (p \vee r) \wedge (q \vee r) \vdash q \vee s$*

| | | |
|---|---|---|
| **Sol.** ($i$) | $(p \to q) \wedge (r \to s)$ | (Given) |
| ($ii$) | $(p \vee r) \wedge (q \vee r)$ | (Given) |
| ($iii$) | $(p \vee r)$ | Simplification using ($ii$) |
| ($iv$) | $q \vee s$ | Constructive dilemma using ($i$) and ($iii$) |

**Example 41.** *Prove that the argument $(p \wedge q) \vee (r \to s), t \to r, \sim (p \wedge q) \vdash t \to s$ is valid without using truth tables.*

| | | |
|---|---|---|
| **Sol.** ($i$) | $(p \wedge q) \vee (r \to s)$ | (Given) |
| ($ii$) | $t \to r$ | (Given) |
| ($iii$) | $\sim (p \wedge q)$ | (Given) |
| ($iv$) | $r \to s$ | Disjunctive syllogism using ($i$) and ($iii$) |
| ($v$) | $t \to s$ | Hypothetical syllogism using ($ii$) and ($iv$) |

**Example 42.** *Prove that the argument $(p \to q) \wedge (r \to s), q \to s, (q \to s) \to (p \vee r) \vdash q \vee s$ is valid using deduction method.*

**Sol.** (*i*) $(p \to q) \wedge (r \to s)$ (Given)

(*ii*) $q \to s$ (Given)

(*iii*) $(q \to s) \to (p \vee r)$ (Given)

(*iv*) $p \vee r$ Modus ponens using (*iii*) and (*ii*)

(*v*) $q \vee s$ Constructive dilemma using (*i*) and (*iv*).

**Example 43.** *Prove that the argument $p \to (q \vee r)$, $(s \wedge t) \to q$, $(q \vee r) \to (s \wedge t) \vdash p \to q$ is d without using truth table.*

**Sol.** (*i*) $p \to (q \vee r)$ (Given)

(*ii*) $(s \wedge t) \to q$ (Given)

(*iii*) $(q \vee r) \to (s \wedge t)$ (Given)

(*iv*) $p \to (s \wedge t)$ Hypothetical syllogism using (*i*) and (*iii*)

(*v*) $p \to q$ Hypothetical syllogism using (*ii*) and (*iv*).

**Example 44.** *Prove that the argument $p \vee (q \to p)$, $\sim p \wedge r \vdash \sim q$ is valid without using th tables.*

**Sol.** (*i*) $p \vee (q \to p)$ (Given)

(*ii*) $\sim p \wedge r$ (Given)

(*iii*) $\sim p$ Rule of simplification using (*ii*)

(*iv*) $q \to p$ Disjunctive syllogism using (*i*) and (*iii*)

(*v*) $\sim q$ Modus tollens using (*iv*) and (*iii*).

**Example 45.** *Test the validity of following argument. If I will select in IAS examination, en I will not be able to go to London. Since, I am going to London, I will not select in IAS amination.*

**Sol.** Let $p$ be "I will select in IAS examination" and $q$ be "I am going to London". Then e above argument can be written in symbolic form as follows :

$$p \to \sim q$$
$$q$$
$$\therefore \quad \sim p$$

Construct the truth table for above argument (Fig. 39)

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \to \sim q$ |
|---|---|---|---|---|
| T | T | F | F | F |
| T | F | F | T | T |
| F | T | T | F | T |
| F | F | T | T | T |

Fig. 39.

$p \to \sim q$ is true in line 2, 3 and 4. $q$ is true in line 1 and 4 and $\sim p$ is true in line 3 and 4. Hence, all three are true in line 4. So it is a valid statement.

**Example 46.** *Consider the following argument and determine whether it is valid.*

*Either I will get good marks or I will not graduate. If I did not graduate I will go to Canada. I get good marks. Thus, I would not go to Canada.*

**Sol.** Let $p$ be "I will get good marks" and $q$ be "I will graduate" and $r$ be "I will go to Canada". Thus, the above argument can be written in symbolic form as follows :

$$p \vee \sim q$$
$$\sim q \rightarrow r$$
$$p$$
$$\therefore \quad \sim r$$

The truth table of above proposition is (Fig. 40)

| $p$ | $q$ | $r$ | $\sim q$ | $\sim r$ | $p \vee \sim q$ | $\sim q \rightarrow r$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | T | T |
| T | T | F | F | T | T | T |
| T | F | F | T | T | T | F |
| T | F | T | T | F | T | T |
| F | T | T | F | F | F | T |
| F | T | F | F | T | F | T |
| F | F | T | T | F | T | T |
| F | F | F | T | T | T | F |

Fig. 40.

$p \vee \sim q$ is true in line 1, 2, 3, 4, 7, 8 and $\sim q \rightarrow r$ is true in line 1, 2, 4, 5, 6, 7 and $p$ is true in line 1, 2, 3 and 4. $\sim r$ is true in 2, 3, 6 and 8. All the above are true in line 2. Hence, the argument is valid.

**Example 47.** *Determine the validity of the following argument without using truth tables.*

*Either I will pass the examination, or, I will not graduate. If I do not graduate, I will go to Canada. I failed : Thus, I will go to Canada.*

**Sol.** Let $p$ be "I will pass the examination" and $q$ be "I will graduate" and $t$ be "I will go to Canada". Thus the above argument, in symbolic form can be written as

$$p \vee \sim q$$
$$\sim q \rightarrow t$$
$$\sim p$$
$$\therefore \quad t$$

Thus to prove the validity of the argument, use the standard results as follows :

| | | |
|---|---|---|
| (*i*) | $p \vee \sim q$ | (Given) |
| (*ii*) | $\sim q \rightarrow t$ | (Given) |
| (*iii*) | $\sim p$ | (Given) |
| (*iv*) | $\sim q$ | Disjunctive syllogism using (*i*) and (*iii*) |
| (*v*) | $t$ | Modus ponens using (*ii*) and (*iv*) |

Hence proved.

**Example 48.** *Determine the validity of the following argument using deduction method.*

*If I study, then I will pass examination. If I do not go to picnic, then I will study. But I d examination. Therefore, I went to picnic.*

**Sol.** Let $p$ be "I study" and $q$ be "I will pass examination" and $t$ be "I go to picnic". Then ıbove argument is written in symbolic form as follows :

$$p \to q$$
$$\sim t \to p$$
$$\sim p$$
$$\therefore \quad t$$

Thus to prove the validity of the argument use the rules of inference.

| | | |
|---|---|---|
| (*i*) | $p \to q$ | (Given) |
| (*ii*) | $\sim t \to p$ | (Given) |
| (*iii*) | $\sim p$ | (Given) |
| (*iv*) | $\sim \neg t$ | Modus tollens using (*ii*) and (*iii*) |
| (*v*) | $t$ | Complement property using (*iv*) |

Hence proved.

**Example 49.** *Prove the validity of the following argument using truth tables as well as uction method.*

*"If the market is free then there is no inflation. If there is no inflation then there are price trols. Since there are price controls, therefore, the market is free".*

**Sol.** Let $p$ be "The market is free" and $q$ be "There is inflation" and $r$ be "There are price trols". Then the above argument can be written in symbolic form as follows :

$$p \to \sim q$$
$$\sim q \to r$$
$$r$$
$$\therefore \quad p$$

**Ist Method.** By using truth tables

Construct the truth table of above argument (Fig. 41)

| $p$ | $q$ | $r$ | $\sim q$ | $p \to \sim q$ | $\sim q \to r$ |
|---|---|---|---|---|---|
| T | T | T | F | F | T |
| T | T | F | F | F | T |
| T | F | F | T | T | F |
| T | F | T | T | T | T |
| F | T | T | F | T | T |
| F | T | F | F | T | T |
| F | F | T | T | T | T |
| F | F | F | T | T | F |

Fig. 41.

$p \to \sim q$ is true in line 3, 4, 5, 6, 7 and 8 $\sim q \to r$ is true in line 1, 2, 4, 5, 6, 7 $r$ is true in line 1, 5, 7. All the above three are true in line 4 and 5. Also $p$ is true in line 4. Hence the ument is valid.

**IInd Method.** Using deduction method

| | | |
|---|---|---|
| (*i*) | $p \to \sim q$ | (Given) |
| (*ii*) | $\sim q \to r$ | (Given) |
| (*iii*) | $p \to r$ | Hypothetical syllogism using (*i*) and (*ii*) |
| (*iv*) | $\sim p \to \sim r$ | Transposition using (*iii*) |
| (*v*) | $r$ | (Given) |
| (*vi*) | $\sim \sim p$ | Modus tollens using (*iv*) and (*v*) |
| (*vii*) | $p$ | Complement of (*vi*) |

## EXISTENTIAL QUANTIFIER

If $p(x)$ is a proposition over the universe U. Then it is denoted as $\exists x\, p(x)$ and read as "There exists at least one value in the universe of variable $x$ such that $p(x)$ is true. The quantifier $\exists$ is called the existential quantifier.

There are several ways to write a proposition, with an existential quantifier *i.e.*,

$(\exists x \in A)\, p(x)$ or $\exists x \in A$ such that $p(x)$ or $(\exists x)\, p(x)$ or $p(x)$ is true for some $x \in A$.

## UNIVERSAL QUANTIFIER

If $p(x)$ is a proposition over the universe U. Then it is denoted as $\forall\, x, p(x)$ and read as "For every $x \in U, p(x)$ is true". The quantifier $\forall$ is called the universal quantifier.

There are several ways to write a proposition, with a universal quantifier.

$\forall x \in A, p(x)$ or $p(x), \forall x \in A$

or $\forall x, p(x)$ or $p(x)$ is true for all $x \in A$.

**Example 50.** *Let $A(x)$ : $x$ has a white colour, $B(x)$ : $x$ is a polar bear, $C(x)$ : $x$ is found in cold regions, over the universe of animals. Translate the following into simple sentences :*

(*i*) $\exists x\, (B(x) \wedge \sim A(x))$ (*ii*) $(\exists x)\, (\sim C(x))$

(*iii*) $(\forall x)\, (B(x) \wedge C(x) \to A(x))$.

**Sol.** (*i*) There exists a polar bear whose colour is not white.

(*ii*) There exists an animal that is not found in cold regions.

(*iii*) Every polar bear that is found in cold regions has a white colour.

**Example 51.** *Let $K(x)$ : $x$ is a two-wheeler, $L(x)$ : $x$ is a scooter. $M(x)$ : $x$ is manufactured by Bajaj. Express the following using quantifiers.*

(*i*) *Every two wheeler is a scooter.*

(*ii*) *There is a two wheeler that is not manufactured by Bajaj.*

(*iii*) *There is a two wheeler manufactured by Bajaj that is not a scooter.*

(*iv*) *Every two wheeler that is a scooter is manufactured by Bajaj.*

**Sol.** (*i*) $(\forall\, x)\, (K(x) \to L(x))$

(*ii*) $(\exists\, x)\, (K(x) \wedge M(x))$

(*iii*) $(\exists\, x)\, (K(x) \wedge M(x) \to \sim L(x))$

(*iv*) $(\forall\, x)\, (K(x) \wedge L(x) \to M(x))$

## NEGATION OF QUANTIFIED PROPOSITIONS

When we negate a quantified proposition *i.e.*, when a universally quantified proposition is negated, we obtain an existentially quantified proposition and when an existentially quantified proposition is negated, we obtain a universally quantified proposition.

The two rules for negation of quantified proposition are as follows. These are also called De Morgan's law.

(*i*) $\sim \exists x\, p(x) \equiv \forall x \sim p(x)$ (*ii*) $\sim \forall x\, p(x) \equiv \exists x \sim p(x)$.

**Example 52.** *Negate each of the following propositions :*

(*i*) *All boys can run faster than all girls.*

(*ii*) *Some girls are more intelligent than all boys.*

(*iii*) *Some students do not live in hostel.*

(*iv*) *All students pass the semester exams.*

(*v*) *Some of the students are absent and the classroom's empty.*

**Sol.** (*i*) Some boys can run faster than some girls.

(*ii*) All girls are more intelligent than some boys.

(*iii*) All students live in hostel.

(*iv*) Some students do not pass the semester exams.

(*v*) All students are present and the class-room is full.

**Example 53.** *Negate each of the following propositions :*

(*i*) $\forall x\, P(x) \wedge \exists y\, q(y)$ (*ii*) $\forall x\, p(x) \wedge \forall y\, q(y)$

(*iii*) $\exists x\, p(x) \vee \forall y\, q(y)$ (*iv*) $\exists x\, p(x) \vee \exists y\, q(y)$

(*v*) $(\exists x \in U)\,(x + 6 = 25)$ (*vi*) $(\forall x \in U)\,(x < 25)$.

**Sol.** (*i*) $\sim (\forall x\, p(x) \wedge \exists y\, q(y))$

$\equiv \sim \forall x\, p(x) \vee \sim \exists y\, q(y)$ $(\because \sim (p \wedge q) = \sim p \vee \sim q)$

$\equiv \exists x \sim p(x) \vee \forall y \sim q(y)$

(*ii*) $\sim (\forall x\, p(x) \wedge \forall y\, q(y))$

$\equiv \sim \forall x\, p(x) \vee \sim \forall y\, q(y)$ $(\because \sim (p \wedge q) = \sim p \sim q)$

$\equiv \exists x \sim p(x) \vee \exists y \sim q(y)$

(*iii*) $\sim (\exists x\, p(x) \vee \forall y\, q(y))$

$\equiv \sim \exists x\, p(x) \wedge \sim \forall y\, q(y)$ $(\because \sim (p \vee q) = \sim p \wedge \sim q)$

$\equiv \forall x \sim p(x) \wedge \exists y \sim q(y)$

(*iv*) $\sim (\exists x\, p(x) \vee \exists y\, q(y))$

$\equiv \sim \exists x\, p(x) \wedge \sim \exists y\, q(y)$ $(\because \sim (p \vee q) = \sim p \wedge \sim q)$

$\equiv \forall x \sim p(x) \wedge \forall y \sim q(y)$

(*v*) $\sim (\exists x \in U)\,(x + 6) = 25$

$\equiv \forall x \in U \sim (x + 6) = 25$

$\equiv (\forall x \in U)\,(x + 6) \neq 25$

(*vi*) $\sim (\forall x \in U)\,(x < 25)$

$\equiv \exists x \in U \sim (x < 25)$

$\equiv (\exists x \in U)\,(x \geq 25)$

## PROPOSITIONS WITH MULTIPLE QUANTIFIERS

The proposition having more than one variable can be quantified with multiple quantifiers. The multiple universal quantifiers can be arranged in any order without altering the meaning of the resulting proposition. Also the multiple existential quantifiers can be arranged in any order without altering the meaning of proposition.

The proposition which contains both universal and existential quantifiers, the order of these quantifiers can't be exchanged without altering the meaning of proposition *e.g.*, the proposition $\exists x \forall y\, p(x, y)$ means " There exists some $x$ such that $p(x, y)$ is true for every $y$".

**Example 54.** *Write the negation for each of the following. Determine whether the resulting statement is true or false. Assume $U = R$.*

(*i*) $\forall x \exists m(x^2 < m)$ (*ii*) $\exists m \forall x(x^2 < m)$.

**Sol.** (*i*) Negation of $\forall x \exists m(x^2 < m)$ is $\exists x \forall m\, (x^2 \geq m)$. The meaning of $\exists x \forall m(x^2 \geq m)$ is that there exists some $x$ such that $x^2 \geq m$, for every $m$. The statement is true as there is some greatest $x$ such that $x^2 \geq m$, for every $m$.

(*ii*) Negation of $\exists m \forall x\, (x^2 < m)$ is $\forall m \exists x\, (x^2 \geq m)$. The meaning of $\forall m \exists x\, (x^2 \geq m)$ is that for every $m$, there exists some $x$ such that $(x^2 \geq m)$. The statement is true as for every $m$, there exists some greatest $x$ such that $x^2 \geq m$.

**Example 55.** *Check the validity of following formula under given interpretation.*

*(a) $\forall x \exists y\, P(x, y)$ under interpretation domain = {1, 2} and $P(1, 1) = T$, $P(1, 2) = F$, $P(2, 1) = T$, $P(2, 2) = T$ where T and F refer to true and false respectively.*

*(b) $\forall x\, (P(x) \to Q(a, f(x))$ under interpretation domain = {1, 2} and $a = 1$, $f(1) = 2$, $f(2) = 1$, $P(1) = F$, $P(2) = T$, $Q(1, 1) = T$, $Q(1, 2) = T$, $Q(2, 1) = F$ and $Q(2, 2) = T$. Here T and F refer to true and false respectively.*

**Sol.** (*a*) $\forall x \exists y\, P(x, y)$ means for every $x$, there exists some $y$ such that $P(x, y)$ is true. Since $P(1, 1) = T$ and $P(2, 1) = T$. Hence, the formula is valid under the domain = {1, 2}.

(*b*) $\forall x\, (P(x) \to Q(a, F(x))$ means for every $x$, whenever $P(x)$ is true implies $Q(a, f(x))$ is also true. So, under the domain {1, 2}, the different values of $\forall x\, (P(x) \to Q(a, f(x))$ are as follows :

| $x$ | $P(x)$ | $Q(a, f(x))$ |
|---|---|---|
| 1 | F | T |
| 2 | T | T |

So, when $P(x)$ is true, $Q(a, f(x))$ is also true. Hence, the formula is valid.

## SOLVED PROBLEMS

**Problem 1.** *Translate the following statements in proposition logic :*

(*i*) *If you study, you will get good marks. If you do not study, you will enjoy. Therefore, either you will get good marks or you will enjoy.*

(*ii*) *If the catalogue is correct, then if the seeds are planted on April, flowers will bloom in July.*

(*iii*) *If John is elected class representative, then either Mary is elected treasurer or Alice is* *ted vice-treasurer.*

(*iv*) *Either taxes are increased or if expenditures rise, then the debt ceiling is raised.*

**Sol.** (*i*) Let $p$ be "You study" and $q$ be "You will get good marks" and $r$ be "You will ...y". Then the proposition logic is

$$p \to q$$
$$\sim p \to r$$
$$\therefore \quad q \vee r$$

(*ii*) Let $p$ be "The catalogue is correct" and $q$ be "Seeds are planted in April" and $r$ be ...e flowers bloom in July". Then, proposition logic is

$$p \to (q \to r)$$

(*iii*) Let $p$ be "John is elected class representative" and $q$ be "Mary is elected treasurer" ...l $r$ be "Alice is elected vice-treasurer". Then, proposition logic is

$$p \to q \vee r$$

(*iv*) Let $p$ be "Taxes are increased" and $q$ be "Expenditures rise" and $r$ be "The debt ...ling is raised". Then proposition logic is

$$p \vee (q \to r).$$

**Problem 2.** *The meaning of proposition $p \to q$ is "If p then q" or "q is a necessary condi-...n for p" or "p only if q" or "p is a sufficient condition for q".*

*Write the following statements in terms of above.*

*1. $p$ : New Delhi is capital of India     $q$ : India is in Asia.*

*2. $p : x^2 = 4$     $q : x = 2$*

*3. $p : x^2 = y^2$     $q : x = y$.*

*4. $p$ = he works hard $q$ : he is a Gold Medalist.*

**Sol.** 1. (*i*) If New Delhi is capital of India, then India is in Asia.

(*ii*) India is in Asia is a necessary condition for New Delhi to be capital of India.

(*iii*) New Delhi is capital of India only if India is in Asia.

(*iv*) The fact that New Delhi is capital of India is a sufficient condition that India is in ...ia.

2. (*i*) If $x^2 = 4$, then $x = 2$

(*ii*) $x = 2$ is necessary for $x^2 = 4$

(*iii*) $x^2 = 4$ only if $x = 2$

(*iv*) $x^2 = 4$ is sufficient for $x = 2$.

3. (*i*) If $x^2 = y^2$ then $x = y$

(*ii*) $x = y$ is necessary for $x^2 = y^2$

(*iii*) $x^2 = y^2$ only if $x = y$

(*iv*) $x^2 = y^2$ is sufficient for $x = y$.

4. (*i*) If he works hard then he is a Gold Medalist.

(*ii*) Gold medal is necessary for hard work.

(*iii*) He works hard only if he is a Gold Medalist.

(*iv*) Hard work is sufficient condition for Gold Medalist.

**Problem 3.** *Construct the truth table for the following statements*

(*i*) $(p \to p) \to (p \to \sim p)$     (*ii*) $(\sim q \to \sim p) \to (p \to q)$.

**Sol.** (*i*) The truth table for $(p \to p) \to (p \to \sim p)$ is as follows. (Fig. 42)

| $p$ | $\sim p$ | $p \to p$ | $p \to \sim p$ | $(p \to p) \to (p \to \sim p)$ |
|---|---|---|---|---|
| T | F | T | F | F |
| F | T | T | T | T |

Fig. 42.

(*ii*) The truth table of statement $(\sim q \to \sim p) \to (p \to q)$ is as follows. (Fig. 43)

| $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim q \to \sim p$ | $p \to q$ | $(\sim q \to \sim p) \to (p \to q)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | T | T |
| F | F | T | T | T | T | T |

Fig. 43.

**Problem 4.** *Construct the truth table for following statements*

(*i*) $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$

(*ii*) $p \leftrightarrow (\sim p \vee \sim q)$

(*iii*) $(p \to p) \vee (p \to \sim p)$.

**Sol.** (*i*) Truth table for $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ is as follows. (Fig. 44)

| $p$ | $q$ | $r$ | $q \to r$ | $p \to q$ | $p \to r$ | $\sim(p \to (q \to r))$ | $((p \to q) \to (p \to r))$ | $K \to$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T | T |
| T | T | F | F | T | F | F | F | T |
| T | F | F | T | F | F | T | T | T |
| T | F | T | T | F | T | T | T | T |
| F | T | T | T | T | T | T | T | T |
| F | T | F | F | T | T | T | T | T |
| F | F | T | T | T | T | T | T | T |
| F | F | F | T | T | T | T | T | T |

Fig. 44.

(*ii*) Truth table for $p \leftrightarrow (\sim p \vee \sim q)$ is as following. (Fig. 45)

| $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim p \vee \sim q$ | $p \leftrightarrow (\sim p \vee \sim q)$ |
|---|---|---|---|---|---|
| T | T | F | F | F | F |
| T | F | F | T | T | T |
| F | T | T | F | T | F |
| F | F | T | T | T | F |

Fig. 45.

(*iii*) The truth table for $(p \to p) \vee (p \to \sim p)$ is as follows. (Fig. 46)

| $p$ | $\sim p$ | $p \to p$ | $p \to \sim p$ | $(p \to p) \vee (p \to \sim p)$ |
|---|---|---|---|---|
| T | F | T | F | T |
| F | T | T | T | T |

Fig. 46.

**Problem 5.** *Assume the value of $p \to q$ is false. Determine the value of $(\sim p \vee \sim q) \to q$.*

**Sol.** Construct the truth table for both the statements and determine the value of $(\sim p \vee \sim q) \to q$ against the false values of $p \to q$. (Fig. 47)

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \to q$ | $\sim p \vee \sim q$ | $(\sim p \vee \sim q) \to q$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | T |
| T | F | F | T | F | T | F |
| F | T | T | F | T | T | T |
| F | F | T | T | T | T | F |

Fig. 47.

So, when $p \to q$ is false, the value of $(\sim p \vee \sim q) \to q$ is also false.

**Problem 6.** *Given the value of $p \to q$ is true. Determine the value of $\sim p \vee (p \leftrightarrow q)$.*

**Sol.** Construct truth table for both statements. (Fig. 48)

| $p$ | $q$ | $\sim p$ | $p \to q$ | $p \leftrightarrow q$ | $\sim p \vee (p \leftrightarrow q)$ |
|---|---|---|---|---|---|
| T | T | F | T | T | T |
| T | F | F | F | F | F |
| F | T | T | T | F | T |
| F | F | T | T | T | T |

Fig. 48.

So, when the value of $p \to q$ is true *i.e.*, in line 1, 3 and 4. The value of $\sim p \vee (p \leftrightarrow q)$ is also true.

**Problem 7.** *Prove that the negation of conditional statement $\sim (p \to q)$ is equivalent of $p \wedge \sim q$.*

**Sol.** The truth table of the above propositions are as follows. (Fig. 49)

| $p$ | $q$ | $\sim q$ | $p \to q$ | $\sim (p \to q)$ | $p \wedge \sim q$ |
|---|---|---|---|---|---|
| T | T | F | T | F | F |
| T | F | T | F | T | T |
| F | T | F | T | F | F |
| F | F | T | T | F | F |

Fig. 49.

As the values of both the propositions are same, hence $\sim (p \to q) \equiv p \wedge \sim q$.

**Problem 8.** *Prove that the negation of biconditional statement $\sim (p \leftrightarrow q)$ is equivalent to $p \leftrightarrow \sim q$ or $\sim p \leftrightarrow q$.*

**Sol.** The truth table of all the above propositions is as follows. (Fig. 50)

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \leftrightarrow q$ | $\sim (p \leftrightarrow q)$ | $p \leftrightarrow \sim q$ | $\sim p \leftrightarrow q$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F | F |
| T | F | F | T | F | T | T | T |
| F | T | T | F | F | T | T | T |
| F | F | T | T | T | F | F | F |

Fig. 50.

As the values of $\sim (p \leftrightarrow q)$ and $p \leftrightarrow \sim q$ and $\sim p \leftrightarrow q$ are same, hence they all are equivalent.

**Problem 9.** *From the following formulae find out tautology, contingency and contradiction.*

(*i*) $\sim (A \rightarrow B) \vee (\sim A \vee (A \wedge B))$ (*ii*) $(H \rightarrow (I \wedge J)) \rightarrow \sim (H \rightarrow I)$

(*iii*) $(p \leftrightarrow Q) \equiv (p \wedge Q) \vee (\sim p \wedge Q)$.

**Sol.** (*i*) Construct the truth table for $\sim (A \rightarrow B) \vee (\sim A \vee (A \wedge B))$ as in Fig. 51.

| A | B | $\sim$ A | $\sim A \vee (A \wedge B)$ | $A \rightarrow B$ | $\sim (A \rightarrow B)$ | $\sim (A \rightarrow B) \vee (\sim A \vee (A \wedge B))$ |
|---|---|---|---|---|---|---|
| T | T | F | T | T | F | T |
| T | F | F | F | F | T | T |
| F | T | T | T | T | F | T |
| F | F | T | T | T | F | T |

Fig. 51.

As the last column of the table contains all T's, hence it is a tautology.

(*ii*) Construct truth table for $(H \rightarrow I \wedge J) \rightarrow \sim (H \rightarrow I))$ as in (Fig. 52)

| H | I | J | $(I \wedge J)$ | $H \rightarrow (I \wedge J)$ | $H \rightarrow I$ | $\sim (H \rightarrow I)$ | $(H \rightarrow I \wedge J) \rightarrow \sim (H \rightarrow I)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | F | F |
| T | T | F | F | F | T | F | T |
| T | F | F | F | F | F | T | T |
| T | F | T | F | F | F | T | T |
| F | T | T | T | T | T | F | F |
| F | F | T | F | T | T | F | F |
| F | T | F | F | T | T | F | F |
| F | F | F | F | T | T | F | F |

Fig. 52.

As the value of last column depends upon the value of the variables, hence it is a contingency.

(*iii*) Construct the truth table for $(p \leftrightarrow q) \rightarrow (p \wedge q) \vee (\sim p \wedge q)$ as in (Fig. 53)

| $p$ | $q$ | $\sim p$ | $p \leftrightarrow q$ | $p \wedge q$ | $\sim p \wedge q$ | $(p \wedge q) \vee (\sim p \wedge q)$ | $(p \leftrightarrow q) \rightarrow (p \wedge q) \vee (\sim p \wedge q)$ |
|---|---|---|---|---|---|---|---|
| T | T | F | T | T | F | T | T |
| T | F | F | F | F | T | T | T |
| F | T | T | F | F | T | T | T |
| F | F | T | T | F | F | F | T |

Fig. 53.

As the last column contains all T's, hence the above formulae is a tautology.

**Problem 10.** *Prove that following is a tautology*

$$A \vee (\overline{B \wedge C}) \equiv (A \vee \overline{B}) \vee \overline{C}.$$

**Sol.** Construct the truth table for $A \vee (\overline{B \wedge C}) \rightarrow (A \vee \overline{B}) \vee \overline{C}$ as in (Fig. 54)

| A | B | C | $\overline{B}$ | $\overline{C}$ | $\overline{B \wedge C}$ | $A \vee (\overline{B \wedge C})$ | $(A \vee \overline{B})$ | $(A \vee \overline{B}) \vee \overline{C}$ | $A \vee (\overline{B \wedge C}) \rightarrow (A \vee \overline{B}) \vee \overline{C}$ |
|---|---|---|---|---|---|---|---|---|---|
| T | T | T | F | F | F | T | T | T | T |
| T | T | F | F | T | T | T | T | T | T |
| T | F | F | T | T | T | T | T | T | T |
| T | F | T | T | F | T | T | T | T | T |
| F | T | T | F | F | F | F | F | F | T |
| F | T | F | F | T | T | T | F | T | T |
| F | F | T | T | F | T | T | T | T | T |
| F | F | F | T | T | T | T | T | T | T |

Fig. 54.

As the last column of the table contains all T's, hence it is a tautology.

**Problem 11.** *Determine whether the following are equivalent, using biconditional statement.*

(*i*) $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

(*ii*) $(p \rightarrow q) \rightarrow t \equiv (p \wedge \sim q) \rightarrow t.$

**Sol.** To prove that the above pairs are equivalent. Prove that $(p \leftrightarrow q) \leftrightarrow ((p \wedge q) \vee (\sim p \wedge \sim q))$ and $((p \rightarrow q) \rightarrow t) \leftrightarrow ((p \wedge \sim q) \rightarrow t)$ are tautologies.

(*i*) Construct truth table for $(p \leftrightarrow q) \leftrightarrow ((p \wedge q) \vee (\sim p \wedge \sim q))$ as in (Fig. 55)

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \leftrightarrow q$ | $p \wedge q$ | $\sim p \wedge \sim q$ | $(p \wedge q) \vee (\sim p \wedge \sim q)$ | $(p \leftrightarrow q) \leftrightarrow (p \wedge q) \vee (\sim p \wedge \sim q)$ |
|---|---|---|---|---|---|---|---|---|
| T | T | F | F | T | T | F | T | T |
| T | F | F | T | F | F | F | F | T |
| F | T | T | F | F | F | F | F | T |
| F | F | T | T | T | F | T | T | T |

Fig. 55.

As the proposition $(p \leftrightarrow q) \leftrightarrow ((p \wedge q) \vee (\sim p \wedge \sim q))$ is a tautology, hence they are equivalent.

(*ii*) Construct the truth table for $((p \rightarrow q) \rightarrow t) \leftrightarrow ((p \wedge \sim q) \rightarrow t)$ as in Fig. 56.

| $p$ | $q$ | $t$ | $\sim q$ | $p \rightarrow q$ | $(p \rightarrow q) \rightarrow t$ | $p \wedge \sim q$ | $p \wedge \sim q \rightarrow t$ | $((p \rightarrow q) \rightarrow t) \leftrightarrow ((p \wedge \sim q) \rightarrow t))$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | F | T | T | F | T | T |
| T | T | F | F | T | F | F | T | F |
| T | F | F | T | F | T | T | F | F |
| T | F | T | T | F | T | T | T | T |
| F | T | T | T | T | T | F | T | T |
| F | T | F | F | T | F | F | T | F |
| F | F | T | T | T | T | F | T | T |
| F | F | F | F | T | F | F | T | F |

Fig. 56.

Since, the proposition is not a tautology, hence they are not equivalent.

**Problem 12.** *Determine the dual of each of the following :*

(a) $(P \wedge Q) \vee (\sim P \vee (\sim P \vee Q))$

(b) $\sim P \wedge (\sim Q \wedge R) \vee (Q \wedge R) \vee (P \wedge R)$

(c) $(P \downarrow Q) \uparrow R$.

**Sol.** The dual of above propositions are as follows :

(a) $(P \wedge Q) \vee (\sim \phi \vee (\sim P \vee Q)) = (P \vee Q) \wedge (\sim P \wedge (\sim P \wedge Q))$

(b) $\sim P \wedge (\sim Q \wedge R) \vee (Q \wedge R) \vee (P \wedge R) = \sim P \vee (\sim Q \vee R) \wedge (Q \vee R) \wedge (P \vee R)$

(c) $(P \downarrow Q) \uparrow R = (P \uparrow Q) \downarrow R$.

**Problem 13.** *Express $P \Rightarrow Q$ using Nand $\uparrow$ only.*

**Sol.**
$$\begin{aligned}
P \Rightarrow Q &\equiv \sim P \vee Q \equiv \sim(\sim\sim P \wedge \sim Q) && \because\ P \vee Q \equiv \sim(\sim P \wedge \sim Q)\\
&\equiv \sim(P \wedge \sim Q) && \because\ \sim\sim P \equiv P\\
&\equiv P \uparrow \sim Q && \because\ P \uparrow Q \equiv \sim(P \wedge Q)\\
&\equiv P \uparrow Q \uparrow Q && \because\ \sim P = P \uparrow P.
\end{aligned}$$

**Problem 14.** *Express $P \downarrow Q$ using $\uparrow$ only.*

**Sol.**
$$\begin{aligned}
P \downarrow Q &\equiv \sim(P \vee Q) \equiv (P \vee Q) \uparrow (P \vee Q)\\
&\equiv [(P \uparrow P) \uparrow (Q \uparrow Q)] \uparrow [(P \uparrow P) \uparrow (Q \uparrow Q)] \quad \because\ P \vee Q \equiv (P \uparrow Q) \uparrow (P \uparrow Q)
\end{aligned}$$

**Problem 15.** *Express the following formula using only $\sim$ and $\wedge$.*

$$(P \downarrow Q) \uparrow R$$

*where $\downarrow$ denotes NOR and $\uparrow$ denotes NAND.*

**Sol.**
$$\begin{aligned}
(P \downarrow Q) \uparrow R &\equiv \sim(P \vee Q) \uparrow R && \because\ P \downarrow Q \equiv \sim(P \vee Q)\\
&\equiv \sim((\sim P \vee Q) \wedge R) && \because\ P \uparrow Q \equiv \sim(P \wedge Q)\\
&\equiv \sim((\sim\sim\sim(P \wedge \sim Q)) \wedge R) && \because\ P \vee Q \equiv \sim(\sim P \wedge \sim Q)\\
&\equiv \sim((\sim(P \wedge \sim Q)) \wedge R) && \because\ \sim\sim P = P
\end{aligned}$$

**Note.** $\uparrow$ (NAND) and $\downarrow$ (NOR) are dual of each other.

**Problem 16.** *Show that the connective $\uparrow$ (Nand) is functionally complete.*

**Sol.** To show that the connective $\uparrow$ is functionally complete. We have to show that the set of connectives $(\wedge, \sim)$ and $(\vee, \sim)$ can be expressed in terms of $\uparrow$ alone which is shown as follows :

$$\sim P \equiv \sim P \vee \sim P \equiv \sim(P \wedge P) \equiv P \uparrow P$$
$$P \vee Q \equiv \sim(\sim P \wedge \sim Q) \equiv \sim P \uparrow \sim Q \equiv (P \uparrow P) \uparrow (Q \uparrow Q)$$

and
$$P \wedge Q \equiv \sim(P \uparrow Q) \equiv (P \uparrow Q) \uparrow (P \uparrow Q)$$

Since the above expresses $\wedge$, $\vee$ and $\sim$ in terms of $\uparrow$ alone. Hence $\uparrow$ is functionally complete.

**Problem 17.** *Prove the validity of following arguments without using truth tables.*

1. $p \vee q, \sim p \mid\!- q$   2. $p, p \to q, q \to r \mid\!- r$
3. $p, q, (p \wedge q) \to r \mid\!- r$   4. $p, (p \wedge \sim q) \to \sim p \mid\!- p \to q$.

**Sol.** 1. (i)   $p \vee q$   (Given)

(ii)   $\sim p$   (Given)

(iii)   $q$   Disjunctive syllogism

| | | |
|---|---|---|
| 2. (*i*) | $p$ | (Given) |
| (*ii*) | $p \to q$ | (Given) |
| (*iii*) | $q \to r$ | (Given) |
| (*iv*) | $p \to r$ | Hypothetical syllogism using (*ii*) and (*iii*) |
| (*v*) | $r$ | Modus ponens using (*iv*) and (*i*) |
| 3. (*i*) | $p$ | (Given) |
| (*ii*) | $q$ | (Given) |
| (*iii*) | $(p \wedge q) \to r$ | (Given) |
| (*iv*) | $p \wedge q$ | Rule of conjunction using (*i*) and (*ii*) |
| (*v*) | $r$ | Modus ponens using (*iii*) and (*iv*) |
| 4. (*i*) | $p$ | (Given) |
| (*ii*) | $(p \wedge \sim q) \to \sim p$ | (Given) |
| (*iii*) | $\sim (p \wedge \sim q)$ | Modus tollens using (*ii*) and (*i*) |
| (*iv*) | $\sim\sim (p \to q)$ | As $\sim (p \to q) \equiv (p \wedge \sim q)$ |
| (*v*) | $p \to q$ | Complement property using (*iv*). |

**Problem 18.** *Prove that the argument* $(p \wedge q) \to r,\ p \to q \ |\!-\ p \to ((p \wedge q) \wedge r)$ *is valid without using truth table.*

| | | |
|---|---|---|
| **Sol.** (*i*) | $(p \wedge q) \to r$ | (Given) |
| (*ii*) | $p \to q$ | (Given) |
| (*iii*) | $p \to (p \wedge q)$ | Rule of absorption using (*ii*) |
| (*iv*) | $(p \wedge q) \to ((p \wedge q) \wedge r)$ | Rule of absorption using (*i*) |
| (*v*) | $p \to ((p \wedge q) \wedge r)$ | Hypothetical syllogism using (*iii*) and (*iv*). |

**Problem 19.** *Prove that the following arguments are valid without using truth tables.*

1. $(p \vee q) \to \sim r,\ r \vee t,\ p \ |\!-\ t$
2. $(p \wedge q) \to r,\ (r \to q),\ (r \to q) \to (q \wedge r)\ |\!-\ (p \wedge q) \to (q \wedge r)$

| | | |
|---|---|---|
| **Sol.** 1. (*i*) | $(p \vee q) \to \sim r$ | (Given) |
| (*ii*) | $r \vee t$ | (Given) |
| (*iii*) | $p$ | (Given) |
| (*iv*) | $p \vee q$ | Rule of addition using (*iii*) |
| (*v*) | $\sim r$ | Modus ponens using (*i*) and (*iv*) |
| (*vi*) | $t$ | Disjunctive syllogism using (*ii*) and (*v*) |
| 2. (*i*) | $p \wedge q \to r$ | (Given) |
| (*ii*) | $r \to q$ | (Given) |
| (*iii*) | $(r \wedge q) \to (q \wedge r)$ | (Given) |
| (*iv*) | $r \to (r \wedge q)$ | Rule of absorption using (*ii*) |
| (*v*) | $(p \wedge q) \to (r \wedge q)$ | Hypothetical syllogism using (*i*) and (*iv*) |
| (*vi*) | $(p \wedge q) \to (q \wedge r)$ | Hypothetical syllogism using (*v*) and (*iii*) |

**Problem 20.** *Prove the validity of following argument using deduction system.*

(1) $A \to B$ (2) $B \to \sim C$

(3) $\sim C \wedge D$ (4) $A \to D$

$\therefore B$.

| | | |
|---|---|---|
| **Sol.** (*i*) | $A \to B$ | (Given) |
| (*ii*) | $B \to \sim C$ | (Given) |
| (*iii*) | $\sim C \wedge D$ | (Given) |
| (*iv*) | $A \to D$ | (Given) |
| (*v*) | $A \to \sim C$ | Hypothetical syllogism using (*i*) and (*ii*) |
| (*vi*) | $\sim C$ | Simplification of (*iii*) |
| (*vii*) | $\sim A \to C$ | Transposition of (*v*) |
| (*viii*) | $A$ | Modus tollens using (*vii*) and (*vi*) |
| (*ix*) | $B$ | Modus ponens using (*i*) and (*viii*) |

**Problem 21.** *Prove the validity of following by deduction method.*

(A) (*i*) $P \to Q$
(*ii*) $\sim Q \vee R$
(*iii*) $\sim (R \wedge \sim S)$
(*iv*) $P$
$\therefore S$

(B) (*i*) $P \vee Q$
(*ii*) $Q \to R$
(*iii*) $R \wedge S$
(*iv*) $P \to S$
(*v*) $P$
$\therefore S$

(C) (*i*) $(Q \to R) \wedge (S \to T)$
(*ii*) $(U \to V) \wedge (W \to X)$
(*iii*) $Q \vee U$
$\therefore R \vee V$

| | | |
|---|---|---|
| **Sol.** (A) (*i*) | $P \to Q$ | (Given) |
| (*ii*) | $\sim Q \vee R$ | (Given) |
| (*iii*) | $\sim (R \wedge \sim S)$ | (Given) |
| (*iv*) | $P$ | (Given) |
| (*v*) | $Q$ | Modus ponens using (*i*) and (*iv*) |
| (*vi*) | $R \vee \sim Q$ | Commutative property using (*ii*) |
| (*vii*) | $\sim R$ | Rule of Modus tollens using (*vi*) and (*v*) |
| (*viii*) | $\sim R \vee S$ | De Morgan's Law using (*iii*) |
| (*ix*) | $S$ | Modus ponens using (*viii*) and (*vii*) |
| (B) (*i*) | $P \vee Q$ | (Given) |
| (*ii*) | $Q \to R$ | (Given) |
| (*iii*) | $R \wedge S$ | (Given) |
| (*iv*) | $P \to S$ | (Given) |
| (*v*) | $P$ | (Given) |
| (*vi*) | $S$ | Modus ponens using (*iv*) and (*v*) |
| (C) (*i*) | $(Q \to R) \wedge (S \to T)$ | (Given) |
| (*ii*) | $(U \to V) \wedge (W \to X)$ | (Given) |
| (*iii*) | $Q \vee U$ | (Given) |
| (*iv*) | $(Q \to R)$ | Simplification using (*i*) |
| (*v*) | $(U \to V)$ | Simplification using (*ii*) |
| (*vi*) | $(Q \to R) \wedge (U \to V)$ | Conjunction using (*iv*) and (*v*) |
| (*vii*) | $R \vee V$ | Constructive dilemma using (*vi*) and (*iii*) |

**Problem 22.** *Prove the validity of following argument using truth tables.*

*"If it rains then it will be cold. If it is cold then I shall stay at home. Since it rains therefore, I shall stay at home".*

**Sol.** Let $p$ be "It rains" and $q$ be "It will be cold" and $r$ be "I shall stay at home". Then the above argument in symbolic form is

$$p \to q$$
$$q \to r$$
$$p$$
$$\therefore \quad r$$

Construct the truth table of $(p \to q) \wedge (q \to r) \wedge p \to r$ as in (Fig. 57).

| $p$ | $q$ | $r$ | $p \to q$ | $q \to r$ | $(p \to q) \wedge (q \to r)$ | $(p \to q) \wedge (q \to r) \wedge p$ | $(p \to q) \wedge (q \to r) \wedge p \to r$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | F | F | F | T |
| T | F | F | F | T | F | F | T |
| T | F | T | F | T | F | F | T |
| F | T | T | T | T | T | F | T |
| F | T | F | T | F | F | F | T |
| F | F | T | T | T | T | F | T |
| F | F | F | T | T | T | F | T |

Fig. 57.

As the last column contain all T's, hence the argument is valid.

**Problem 23.** *Translate the following into symbolic form and test the validity of the argument.*

*If 6 is even then 2 does not divide 7. Either 5 is not prime or 2 divides 7. But 5 is prime, therefore, 6 is odd.*

**Sol.** Let $p$ be "6 is even" and $q$ be "2 divide 7" and $r$ be "5 is primed". Thus, the above argument in symbolic form can be written as

$$p \to \sim q$$
$$\sim r \vee q$$
$$r$$
$$\therefore \quad \sim p$$

Construct the truth table to test the validity. (Fig. 58)

| $p$ | $q$ | $r$ | $\sim p$ | $\sim q$ | $\sim r$ | $p \to \sim q$ | $\sim r \vee q$ |
|---|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F | T |
| T | T | F | F | F | T | F | T |
| T | F | F | F | T | T | T | F |
| T | F | T | F | T | F | T | T |
| F | T | T | T | F | F | T | T |
| F | T | F | T | F | T | T | T |
| F | F | T | T | T | F | T | F |
| F | F | F | T | T | T | T | F |

Fig. 58.

$p \to \sim q$ is true in line 3, 4, 5, 6, 7 and 8, $\sim r \vee q$ is true in line 1, 2, 3, 4, 5 and $r$ is t... line 1, 4, 5 and 7. All the three are true in line 4 and 5 and $\sim p$ is true in line 5. Hence, arg... is valid.

**Problem 24.** *Determine the negation of the following statements*

(*i*) $\forall x\ \forall y\ \forall z,\ p(x, y, z)$ (*ii*) $\forall x\ \exists_{y'}\ p(x, y)$

(*iii*) $\exists_y\ \forall_x\ \forall_{z'}\ p(x, y, z)$.

**Sol.** (*i*) $\sim (\forall_x \forall_y \forall_{z'}\ p(x, y, z)) \equiv \exists_x \exists_y \exists_z \sim p(x, y, z)$

(*ii*) $\sim (\forall_x \exists_{y'}\ p(x, y)) \equiv \exists_x \forall_{y'} \sim p(x, y)$

(*iii*) $\sim (\exists_y \forall_x \forall_{z'}\ p(x, y, z)) \equiv \forall_y \exists_x \exists_{z'} \sim p(x, y, z)$.

**Problem 25.** *Determine the negation of the following statements*

(*i*) $\exists_x \forall_y\ (p(x) \vee q(y))$ (*ii*) $\forall_x \exists_y\ (p(x, y) \to q(x, y))$

(*iii*) $\forall_x \forall_y\ (p(x) \wedge q(y))$.

**Sol.** (*i*) $\sim (\exists x\ \forall_{y'}\ (p(x) \vee q(y)) \equiv \forall_x \exists_y \sim (p(x) \vee q(y))$

$\equiv \forall_x \exists_y (\sim p(x) \wedge \sim q(y))$

(*ii*) $\sim \forall_x \exists_y\ (p(x, y) \to q(x, y)) \equiv \exists_x \forall_y \sim (p(x, y) \to q(x, y))$

(*iii*) $\equiv \forall_x \forall_y\ (p\ (x) \wedge p(y)) \equiv \exists_x \exists_y \sim (p(x) \wedge p(y)) \equiv \exists_x \exists_y (\sim p(x) \vee \sim p(y))$.

**Problem 26.** *Let* $U = Q$. *Use quantifiers to express the following statements*

(*i*) $\sqrt{5}$ *is not rational.*

(*ii*) *Subtraction of any two rational numbers is rational.*

**Sol.** (*i*) $\sim(\exists\, x)\ (x^2 = 5)$

(*ii*) $(\forall x)\ (\forall y)\ (x - y$ is a rational$)$.