Half Range Series

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If function is required to expand in limit $(0,\pi)$ or $(0,l)$ in fourier series of period 2π or $2l$ than it does not matter what is the function outside the range $(0,\pi)$ or $(0,l)$. We can assume any function in $(-\pi,0)$ or $(-l,0)$

* If we extend function in such a way that $f(-x) = f(x)$ (even function) in $(-\pi,\pi)$ or $(-l,l)$ then Half range cosine series is written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \quad , \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

or

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx \quad , \quad a_n = \frac{2}{l}\int_0^{l} f(x)\cos \frac{n\pi x}{l}\,dx$$

*
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \underbrace{f(x)\,dx}_{\text{even } f^n}$$

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* If we extend function $f(x)$ in interval $(-l,0)$ or $(-\pi,0)$ such that $f(-x)=-f(x)$ in $(-l,l)$ or $(-\pi,\pi)$. function is odd,

then Half range Sine series is written as:

$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$	$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$
$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx$	$b_n = \frac{2}{l}\int_0^{l} f(x) \sin \frac{n\pi x}{l}\, dx$

Practice Questions:

① Half range cosine series of $(x-1)^2$ in the interval $0<x<1$

hence show that (i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2} --- \infty = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2} --- \infty = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+ --- \infty = \frac{\pi^2}{8}$

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Solution $f(x) = (x-1)^2 \quad 0 < x < 1$

Half range Cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

here $l = 1$

$$(x-1)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

$$a_0 = \frac{2}{1}\int_0^1 (x-1)^2\,dx = 2\left(\frac{(x-1)^3}{3}\right)_0^1 = \frac{2}{3}$$

$$a_n = \frac{2}{1}\int_0^1 \underbrace{(x-1)^2}_{\text{I}}\ \underbrace{\cos n\pi x}_{\text{II}}\,dx$$

$$= 2\left(\frac{(x-1)^2 \sin n\pi x}{n\pi} - \int \underbrace{2(x-1)}_{\text{I}}\ \underbrace{\frac{\sin n\pi x}{n\pi}}_{\text{II}}\,dx\right)_0^1$$

$$= 2\left[\frac{(x-1)^2 \sin n\pi x}{n\pi} - \frac{2}{n\pi}\left((x-1)\left(-\frac{\cos n\pi x}{n\pi}\right) - \int\left(-\frac{\cos n\pi x}{n\pi}\right)dx\right)\right]_0^1$$

$$= 2\left[\frac{(x-1)^2 \sin n\pi x}{n\pi} + \frac{2}{n^2\pi^2}(x-1)\cos n\pi x - \frac{2}{n^3\pi^3}\sin n\pi x\right]_0^1$$

$$= +\frac{2\times 2}{n^2\pi^2} = \frac{+4}{n^2\pi^2}$$

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Half range cosine series is

$$(x-1)^2 = \frac{1}{3} + \sum_{n=1}^{\infty}\left(+\frac{4}{n^2\pi^2}\right)\cos n\pi x$$

$$(x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} \text{---} \infty\right)$$

— A

Now (i) for $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \text{---} \infty = ?$

Put $x = 0$

$$1 = \frac{1}{3} + \frac{4}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \text{---} \infty\right)$$

$$\frac{2}{3} = \frac{4}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \text{---} \infty\right)$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \text{---} \infty = \frac{\pi^2}{6}$$

— B

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \text{---} = ?$

Put $x = 1$

$$0 = \frac{1}{3} + \frac{4}{\pi^2}\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} \text{---} \infty\right)$$

$$-\frac{1}{3} = -\frac{4}{\pi^2}\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \text{---} \infty\right)$$

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$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - - - \infty = \frac{\pi^2}{12} \quad — (C)$$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + - - - - = ?$

Adding (B) and (C)

$$\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} - - \infty\right) + \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - - \infty\right)$$

$$= \frac{\pi^2}{6} + \frac{\pi^2}{12} = \frac{3\pi^2}{12} = \frac{\pi^2}{4}$$

$$2\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} - - \infty\right) = \frac{\pi^2}{4}$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} - - - - \infty = \frac{\pi^2}{8}$$

Ques 2

$$f(x) = \begin{cases} \sin x, & 0 \le x \le \frac{\pi}{4} \\ \cos x, & \frac{\pi}{4} \le x \le \frac{\pi}{2} \end{cases}$$

Expand as Fourier sine series.

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$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

here $l = \frac{\pi}{2}$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\pi/2} = \sum_{n=1}^{\infty} b_n \sin 2nx$$

$$b_n = \frac{2}{\pi/2} \int_0^{\pi/2} f(x) \sin 2nx \, dx$$

$$= \frac{4}{\pi} \left[\int_0^{\pi/4} \sin x \sin 2nx \, dx + \int_{\pi/4}^{\pi/2} \cos x \sin 2nx \, dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/4} 2 \sin 2nx \sin x \, dx + \int_{\pi/4}^{\pi/2} 2 \sin 2nx \cos x \, dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/4} \left(\cos(2n-1)x - \cos(2n+1)x \right) dx + \int_{\pi/4}^{\pi/2} \left(\sin(2n+1)x + \sin(2n-1)x \right) dx \right]$$

$$= \frac{2}{\pi} \left[\left(\frac{\sin(2n-1)x}{2n-1} - \frac{\sin(2n+1)x}{2n+1} \right)_0^{\pi/4} + \left(\frac{-\cos(2n+1)x}{2n+1} - \frac{\cos(2n-1)x}{2n-1} \right)_{\pi/4}^{\pi/2} \right]$$

$$= \frac{2}{\pi} \left[\frac{\sin(2n-1)\frac{\pi}{4}}{2n-1} - \frac{\sin(2n+1)\frac{\pi}{4}}{2n+1} - \underbrace{\frac{\cos(2n+1)\frac{\pi}{2}}{2n+1}}_{=0} - \underbrace{\frac{\cos(2n-1)\frac{\pi}{2}}{2n-1}}_{=0} + \frac{\cos(2n+1)\frac{\pi}{4}}{2n+1} + \frac{\cos(2n-1)\frac{\pi}{4}}{2n-1} \right]$$

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$$b_n = \frac{2}{\pi}\left[\frac{\sin(2n-1)\frac{\pi}{4}}{2n-1} - \frac{\sin(2n+1)\frac{\pi}{4}}{2n+1} + \frac{\cos(2n+1)\frac{\pi}{4}}{2n+1} + \frac{\cos(2n-1)\frac{\pi}{4}}{2n-1}\right]$$

Fourier sine Series is

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{\sin(2n-1)\frac{\pi}{4}}{2n-1} - \frac{\sin(2n+1)\frac{\pi}{4}}{2n+1} + \frac{\cos(2n+1)\frac{\pi}{4}}{2n+1} + \frac{\cos(2n-1)\frac{\pi}{4}}{2n-1}\right)\sin 2nx$$

$$= \frac{2}{\pi}\left[\left(\frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}} - \frac{1}{3\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\sin 2x + \left(\frac{1}{3\sqrt{2}} + \frac{1}{5\sqrt{2}} - \frac{1}{5\sqrt{2}} - \frac{1}{3\sqrt{2}}\right)\sin 4x + \left(\frac{-1}{5\sqrt{2}} + \frac{1}{7\sqrt{2}} - \frac{1}{5\sqrt{2}} + \frac{1}{7\sqrt{2}}\right)\sin 6x \text{ ------ } \infty\right]$$

$$= \frac{2}{\sqrt{2}\pi}\left(\left(1 - \frac{1}{3}\right)\sin 2x + \left(\frac{1}{7} - \frac{1}{5}\right)\sin 6x + \left(\frac{1}{9} - \frac{1}{11}\right)\sin 10x \text{ --- } \infty\right)$$

$$\left(\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}},\ \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}},\ \sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}},\ \sin\frac{7\pi}{4} = -\frac{1}{\sqrt{2}},\ \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},\ \cos\frac{3\pi}{4} = \frac{-1}{\sqrt{2}}\right)$$

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$$= \frac{2}{\pi\sqrt{2}}\left(\frac{2}{3}\sin 2x + \frac{2}{5\cdot 7}\sin 6x + \frac{2}{9\cdot 11}\sin 10x \text{ --- } \infty\right)$$

$$= \frac{4}{\pi\sqrt{2}}\left(\frac{\sin 2x}{1\cdot 3} - \frac{\sin 6x}{5\cdot 7} + \frac{\sin 10x}{9\cdot 11} \text{ --- } \infty\right)$$