Ques
$$f(x) = \begin{cases} x^2, & 0 \leq x \leq \pi \\ -x^2, & -\pi \leq x \leq 0 \end{cases}$$

write its fourier series expansion.

Sol^n

$$f(-x) = \begin{cases} (-x)^2, & 0 \leq -x \leq +\pi \\ -(-x)^2, & +\pi \leq -x \leq 0 \end{cases}$$

$$= \begin{cases} x^2, & 0 \geq x \geq -\pi \\ -x^2, & \pi \geq x \geq 0 \end{cases}$$

$$= \begin{cases} x^2, & -\pi \leq x \leq 0 \\ -x^2, & 0 \leq x \leq \pi \end{cases} = -f(x)$$

hence $f(x)$ is an odd function from $-\pi$ to π.

Since function is odd

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x)}_{odd} \underbrace{\cos nx}_{even} \, dx = 0$$

odd function

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x)}_{odd} \underbrace{\sin nx}_{odd} \, dx$$

Even function

$$= \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} \underbrace{x^2}_{I} \underbrace{\sin nx}_{II} \, dx$$

$$= \frac{2}{\pi} \left[x^2 \left(\frac{-\cos nx}{n} \right) - \int \underbrace{2x}_{I} \underbrace{\left(\frac{-\cos nx}{-n} \right)}_{II} \, dx \right]_{0}^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{-x^2 \cos nx}{n} + \frac{2}{n} \left(x \frac{\sin nx}{n} - \int 1 \cdot \frac{\sin nx}{n} \, dx \right) \right]_{0}^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{-x^2 \cos nx}{n} + \frac{2}{n^2} x \sin nx + \frac{2}{n^3} \cos nx \right]_{0}^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{-\pi^2 \cos n\pi}{n} + \frac{2}{n^3} \cos n\pi - \frac{2}{n^3} \right]$$

$$= 2 \left[\frac{-\pi (-1)^n}{n} + \frac{2}{\pi n^3} \left((-1)^n - 1 \right) \right]$$

Now fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 \text{ and } a_n = 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\sum_{n=1}^{\infty} 2 \left(\frac{-\pi(-1)^n}{n} + \frac{2}{n^3 \pi} ((-1)^n - 1) \right) \sin nx$$

$$= 2\left(\pi - \frac{4}{\pi}\right) \sin x - \pi \sin 2x + 2\left(\pi - \frac{4}{3^3}\right) \sin 3x$$

$$- \frac{\pi}{2} \sin 4x - \underline{\hspace{3cm}} \infty.$$

Ques: $\quad f(x) = |\cos x| \quad, \quad -\pi < x < \pi$

fourier series expansion?

Sol^n: $\qquad f(x) = |\cos x| , \quad -\pi < x < \pi$

fourier series function expansion is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$|\cos x|$ is an even function.

when function is even i's $-\pi < x < \pi$

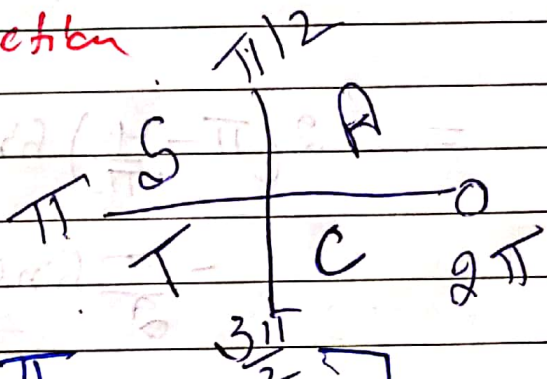$$b_n = 0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

Now $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \underline{f(x)}\,dx = \frac{2}{\pi} \int_{0}^{\pi} f(x)\,dx$

Even function

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underline{f(x)} \cdot \underline{\cos nx}\,dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx\,dx$$

Even   Even

Even function

Now $a_0 = \frac{2}{\pi} \int_{0}^{\pi} |\cos x|\,dx$

$$= \frac{2}{\pi} \left[\int_{0}^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx \right]$$

$$= \frac{2}{\pi} \left[\left(+\sin x \right)_{0}^{\pi/2} - \left(\sin x \right)_{\pi/2}^{\pi} \right]$$

$$= \frac{2}{\pi} \left[1 - 0 - (0 - 1) \right] = \boxed{\frac{4}{\pi}}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} |\cos x| \cos nx \, dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x \cos nx \, dx + \int_{\pi/2}^{\pi} -\cos x \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\int_0^{\pi/2} \left(\cos(n+1)x + \cos(n-1)x \right) dx - \int_{\pi/2}^{\pi} \left(\cos(n+1)x + \cos(n-1)x \right) dx \right]$$

$$= \frac{1}{\pi} \left[\left(\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right)_0^{\pi/2} - \left(\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right)_{\pi/2}^{\pi} \right]$$

$$(n \neq 1)$$

$$= \frac{1}{\pi} \left[\frac{\sin(n+1)\frac{\pi}{2}}{(n+1)} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1} - \left(0+0 - \frac{\sin(n+1)\frac{\pi}{2}}{n+1} - \frac{\sin(n-1)\frac{\pi}{2}}{n-1} \right) \right]$$

$$= \frac{1}{\pi} \left[\frac{2 \cdot \sin(n+1)\frac{\pi}{2}}{n+1} + \frac{2 \cdot \sin(n-1)\frac{\pi}{2}}{n-1} \right] \qquad (n \neq 1)$$

for $n=1$, we will find a_1

$$a_1 = \frac{2}{\pi} \int_0^{\pi} |\cos x| \cdot \cos x \, dx$$

$$a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^3 x\, dx - \int_{\pi/2}^{\pi} \cos^2 x\, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\left(\frac{1+\cos 2x}{2}\right)dx - \int_{\pi/2}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx\right]$$

$$= \frac{2}{2\pi}\left[\left(x + \frac{\sin 2x}{2}\right)_0^{\pi/2} - \left(x + \frac{\sin 2x}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \left(\pi - \frac{\pi}{2}\right)\right] = 0 \checkmark$$

Now Fourier Series is

$$\cos x = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{2}{\pi} + \sum_{n=2}^{\infty} \cdot \frac{2}{\pi}\left[\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right]\cos nx$$

$$= \frac{2}{\pi} + \frac{2}{\pi}\left[\cos 2x - \frac{1}{3}\cos 2x + \left(\frac{1}{5} - \frac{1}{3}\right)\cos 4x\right.$$
$$\left. + \left(-\frac{1}{7} + \frac{1}{5}\right)\cos 6x - - - - \infty\right]$$

$$= \frac{2}{\pi} + \frac{2}{\pi}\left(\frac{2}{3}\cos 2x - \frac{2}{15}\cos 4x + \frac{2}{35}\cos 6x - \right.$$

$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{\cos 2x}{1\cdot 3} - \frac{\cos 4x}{3\cdot 5} + \frac{\cos 6x}{5\cdot 7} - \cdots \infty\right]$$

$\underline{n=2}$

$$\frac{\sin \frac{3\pi}{2}}{3} + \frac{\sin \frac{\pi}{2}}{1} = \frac{\sin\left(\pi + \frac{\pi}{2}\right)}{3} + \frac{1}{1}$$

$$= \frac{-\sin\frac{\pi}{2}}{3} + 1 = \boxed{1 - \frac{1}{3}} \quad \checkmark$$

$\underline{n=4}$

$$\frac{\sin \frac{5\pi}{2}}{5} + \frac{\sin \frac{3\pi}{2}}{3} = \frac{\sin\left(2\pi + \frac{\pi}{2}\right)}{5} + \frac{\sin\left(\pi + \frac{\pi}{2}\right)}{3}$$

$$= \boxed{\frac{1}{5} - \frac{1}{3}} \quad \checkmark$$

hence evaluate

$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \cdots \infty = \frac{?}{0}$$