8·6. LINEAR EQUATIONS

A differential equation is said to be linear if the dependent variable and its differential coefficients occur only in the first degree and not multiplied together.

Thus **the standard form of a linear equation of the first order,** *commonly known as* **Leibnitz's linear equation,** *is*

$$\frac{dy}{dx} + Py = Q \quad \text{where } P, Q \text{ are the functions of } x. \qquad ...(1)$$

To solve the equation, multiply both sides by $e^{\int P dx}$ so that we get

$$\frac{dy}{dx} \cdot e^{\int P dx} + y\,(e^{\int P dx} P) = Qe^{\int P dx} \quad i.e. \quad \frac{d}{dx}(ye^{\int P dx}) = Qe^{\int P dx}$$

Integrating both sides, we get $ye^{\int P dx} = \int Qe^{\int P dx}\, dx + c$ as the required solution.

Obs. The factor $e^{\int P dx}$ on multiplying by which the left-hand side of (1) becomes the differential coefficient of a single function, is called the **integrating factor (I.F.)** of the linear equation (1).

It is important to remember that **I.F.** $= e^{\int P dx}$

and the solution is **y (I.F.)** $= \int$ **Q (I.F.) dx + c.**

Example 8·15. *Solve* $(x + 1)\dfrac{dy}{dx} - y = e^{3x}(x + 1)^2$.

Sol. Dividing throughout by $(x + 1)$, given equation becomes

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x + 1) \text{ which is Leibnitz's equation.} \qquad ...(i)$$

Here $P = -\dfrac{1}{x+1}$ and $\int P dx = -\int \dfrac{dx}{x+1} = -\log(x + 1) = \log(x + 1)^{-1}$

$\therefore$
$$\text{I.F.} = e^{\int P dx} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$$

Thus the solution of (1) is $y\,(\text{I.F.}) = \int [e^{3x}(x + 1)](\text{I.F.})\, dx + c$

$$\frac{y}{x+1} = \int e^{3x} dx + c = \frac{1}{3}e^{3x} + c \quad \text{or} \quad y = (\tfrac{1}{3}e^{3x} + c)(x + 1).$$

Example 8·16. *Solve* $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}}\right)\dfrac{dx}{dy} = 1$

Sol. Given equation can be written as $\dfrac{dy}{dx} + \dfrac{y}{\sqrt{x}} = \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}$ ...(*i*)

$\therefore$
$$I.F. = e^{\int x^{1/2} dx} = e^{2\sqrt{x}}$$

Thus solution of (*i*) is $y\,(I.F.) = \int \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}(I.F.)\, dx + c$

$$ye^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \cdot e^{2\sqrt{x}} dx + c$$

$$ye^{2\sqrt{x}} = \int x^{-1/2} dx + c \quad \text{or} \quad ye^{2\sqrt{x}} = 2\sqrt{x} + c.$$

Example 8·17. *Solve* $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$.

(Bhopal, 2008 ; V.T.U., 2008 ; U.P.T.U., 2005)

Sol. This equation contains y^2 and $\tan^{-1} y$ and is, therefore, not a linear in y ; but since only x occurs, it can be written as

$$(1 + y^2)\frac{dx}{dy} = \tan^{-1} y - x \quad \text{or} \quad \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$$

which is a Leibnitz's equation in x.

$$\therefore \qquad \text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Thus the solution is $x\,(\text{I.F.}) = \int \frac{\tan^{-1} y}{1+y^2}\,(\text{I.F.})\, dy + c$

$\left[\text{Put } \tan^{-1} y = t \quad \therefore \quad \frac{dy}{1+y^2} = dt\right]$

or
$$x e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} \cdot e^{\tan^{-1} y}\, dy + c$$

$$= \int t e^t\, dt + c = t \cdot e^t - \int 1 \cdot e^t\, dt + c$$ (Integrating by parts)

$$= t \cdot e^t - e^t + c = (\tan^{-1} y - 1)\, e^{\tan^{-1} y} + c$$

or
$$x = \tan^{-1} y - 1 + c e^{-\tan^{-1} y}.$$

Example 8·18. *Solve* $r \sin \theta \, d\theta + (r^3 - 2r^2 \cos \theta + \cos \theta)\, dr = 0$.

Sol. Given equation can be rewritten as

$$\sin \theta \frac{d\theta}{dr} + \frac{1}{r}(1 - 2r^2) \cos \theta = -r^2 \qquad \text{...(i)}$$

Put $\cos \theta = y$ so that $-\sin \theta \, d\theta/dr = dy/dr$

Then (*i*) becomes $-\frac{dy}{dr} + \left(\frac{1}{r} - 2r\right) y = -r^2$ or $\frac{dy}{dr} + \left(2r - \frac{1}{r}\right) y = r^2$

which is a Leibnitz's equation. $\therefore$ I.F. $= e^{\int (2r - 1/r)\, dr} = e^{r^2 - \log r} = \frac{1}{r} e^{r^2}$

Thus its solution is $y\left(\frac{1}{r} e^{r^2}\right) = \int r^2 \cdot e^{r^2} \cdot \frac{1}{r}\, dr + c$

or
$$y e^{r^2}/r = \frac{1}{2} \int e^{r^2}\, 2r\, dr + c = \frac{1}{2} e^{r^2} + c$$

or
$$2 e^{r^2} \cos \theta = r e^{r^2} + 2 c r \quad \text{or} \quad r\,(1 + 2 c e^{-r^2}) = 2 \cos \theta.$$

Problems

Solve the following differential equations :

1. $\cos^2 x \frac{dy}{dx} + y = \tan x.$
2. $x \log x \frac{dy}{dx} + y = \log x^2.$ (*V.T.U., 2009*)
3. $2y' \cos x + 4y \sin x = \sin 2x$, given $y = 0$ when $x = \pi/3$. (*V.T.U., 2003*)
4. $\cosh x \frac{dy}{dx} + y \sinh x = 2 \cosh^2 x \sinh x$ (*J.N.T.U., 2003*)
5. $(1 - x^2) \frac{dy}{dx} - xy = 1.$ (*V.T.U., 2010*)
6. $(1 - x^2) \frac{dy}{dx} + 2xy = x \sqrt{(1 - x^2)}$ (*Nagpur, 2009*)
7. $\frac{dy}{dx} + 2xy = 2e^{-x^2}$ (*P.T.U., 2005*)
8. $\frac{dy}{dx} = -\frac{x + y \cos x}{1 + \sin x}.$
9. $dr + (2r \cot \theta + \sin 2\theta)\, d\theta = 0.$ (*J.N.T.U., 2003*)
10. $3x(1 - x^2) y^3 \frac{dy}{dx} + (2x^2 - 1) y^3 = ax^3$ (*Rajasthan, 2006*)
11. $(x + 2y^3) \frac{dy}{dx} = y.$ (*Marathwada, 2008*)
12. $\sqrt{(1 - y^2)}\, dx = (\sin^{-1} y - x)\, dy.$
13. $y e^y\, dx = (y^3 + 2xe^y)\, dy.$
14. $y \log y\, dx + (x - \log y)\, dy = 0.$ (*U.P.T.U., 2000*)
15. $(1 + y^2)\, dx + (x - e^{-\tan^{-1} y})\, dy = 0.$ (*V.T.U., 2006*)

8.7. BERNOULLI'S EQUATION

The equation $\frac{dy}{dx} + Py = Qy^n$ ...(1)

where P, Q are functions of x, is reducible to the Leibnitz's linear equation and is usually called the Bernoulli's equation.

To solve (1), divide both sides by y^n, so that $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$ ...(2)

Put $y^{1-n} = z$ so that $(1-n)\,y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$.

$\therefore$ (2) becomes $\quad \frac{1}{1-n}\frac{dz}{dx} + Pz = Q \quad$ or $\quad \frac{dz}{dx} + P(1-n)z = Q(1-n),$

which is Leibnitz's linear in z and can be solved easily.

Example 8·19. *Solve* $x\frac{dy}{dx} + y = x^3y^6$.

Sol. Dividing throughout by xy^6, $y^{-6}\frac{dy}{dx} + \frac{y^{-5}}{x} = x^2$ ...(*i*)

Put $y^{-5} = z$, so that $-5y^{-6}\frac{dy}{dx} = \frac{dz}{dx}$ $\quad\therefore$ (*i*) becomes $-\frac{1}{5}\frac{dz}{dx} + \frac{z}{x} = x^2$

or $$\frac{dz}{dx} - \frac{5}{x}z = -5x^2 \text{ which is Leibnitz's linear in } z. \quad ...(ii)$$

I.F. $= e^{-\int (5/x)dx} = e^{-5\log x} = e^{\log x^{-5}} = x^{-5}$

$\therefore$ The solution of (*ii*) is $z\,(\text{I.F.}) = \int (-5x^2)\,(\text{I.F.})\,dx + c \quad$ or $\quad zx^{-5} = \int (-5x^2)\,x^{-5}\,dx + c$

or $$y^{-5}x^{-5} = -5\cdot\frac{x^{-2}}{-2} + c \qquad [\because z = y^{-5}]$$

Dividing throughout by $y^{-5}x^{-5}$, $1 = (2{\cdot}5 + cx^2)x^3y^5$ which is the required solution.

Example 8·20. *Solve* $xy\,(1 + xy^2)\,\frac{dy}{dx} = 1.$ *(Nagpur, 2009)*

Sol. Rewriting the given equation as

$$\frac{dx}{dy} - yx = y^3x^2$$

and dividing by x^2, we have

$$x^{-2}\frac{dx}{dy} - yx^{-1} = y^3 \quad ...(i)$$

Putting $x^{-1} = z$ so that $-x^{-2}\frac{dx}{dy} = \frac{dz}{dy}$, $\quad$ (*i*) becomes

$$\frac{dz}{dy} + yz = -y^3 \text{ which is Leibnitz's linear in } z.$$

Here I.F. $= e^{\int y\,dy} = e^{y^2/2}$

$\therefore$ The solution is $z\,(\text{I.F.}) = \int (-y^3)\,(\text{I.F.})\,dy + c$

or $$ze^{y^2/2} = -\int y^2\cdot e^{\frac{1}{2}y^2}\cdot y\,dy + c \qquad \left|\begin{array}{l}\text{Put } \frac{1}{2}y^2 = t\\ \text{so that } y\,dy = dt\end{array}\right.$$

$$= -2\int t\cdot e^t\,dt + c \qquad \text{[Integrate by parts]}$$

$$= -2\,[t \cdot e^t - \int 1 \cdot e^t\, dt] + c$$

$$= -2\,[te^t - e^t] + c = (2 - y^2)\, e^{y^2/2} + c$$

or $$z = (2 - y^2) + ce^{-\frac{1}{2}y^2}$$

or $$1/x = (2 - y^2) + ce^{-\frac{1}{2}y^2}.$$

Note. *General equation reducible to Leibnitz's linear is* $f'(y)\dfrac{dy}{dx} + Pf(y) = Q$ ...(A)

where P, Q are functions of x. To solve it, put $f(y) = z$.

Example 8·21. *Solve* $\dfrac{dy}{dx} + x \sin 2y = x^3 \cos^2 y.$ (Marathwada, 2008 ; J.N.T.U., 2005)

Sol. Dividing throughout by $\cos^2 y$, $\sec^2 y \dfrac{dy}{dx} + 2x \dfrac{\sin y \cos y}{\cos^2 y} = x^3$

or $$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \text{ which is of the form (A) above.} \quad ...(i)$$

$\therefore$ Put $\tan y = z$ so that $\sec^2 y \dfrac{dy}{dx} = \dfrac{dz}{dx}$ $\therefore$ (*i*) becomes $\dfrac{dz}{dx} + 2xz = x^3$.

This is Leibnitz's linear equation in z. $\therefore$ I.F. $= e^{\int 2x dx} = e^{x^2}$

$\therefore$ The solution is $ze^{x^2} = \int e^{x^2} x^3 dx + c = \dfrac{1}{2}(x^2 - 1)\, e^{x^2} + c.$

Replacing z by $\tan y$, we get $\tan y = \dfrac{1}{2}(x^2 - 1) + ce^{-x^2}$ which is the required solution.

Example 8·22. *Solve* $\dfrac{dz}{dx} + \left(\dfrac{z}{x}\right) \log z = \dfrac{z}{x}(\log z)^2.$

Sol. Dividing by z, the given equation becomes

$$\frac{1}{z}\frac{dz}{dx} + \frac{1}{x}\log z = \frac{1}{x}(\log z)^2 \quad ...(i)$$

Put $\log z = t$ so that $\dfrac{1}{z}\dfrac{dz}{dx} = \dfrac{dt}{dx}$. $\therefore$ (*i*) becomes

$$\frac{dt}{dx} + \frac{t}{x} = \frac{t^2}{x} \quad \text{or} \quad \frac{1}{t^2}\frac{dt}{dx} + \frac{1}{x}\cdot\frac{1}{t} = \frac{1}{x} \quad ...(ii)$$

This being Bernoulli's equation, put $1/t = v$ so that (*ii*) reduces to

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x} \quad \text{or} \quad \frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x}$$

This is Leibnitz's linear in v. $\therefore$ I.F. $= e^{-\int 1/x\, dx} = 1/x$.

$\therefore$ The solution is $\quad v \cdot \dfrac{1}{x} = -\int \dfrac{1}{x}\cdot\dfrac{1}{x}\, dx + c = \dfrac{1}{x} + c$

Replacing v by $1/\log z$, we get $(x \log z)^{-1} = x^{-1} + c$ or $(\log z)^{-1} = 1 + cx$ which is the required solution.

Problems

Solve the following equations :

1. $\dfrac{dy}{dx} + y \tan x = y^2 \sec x.$ (P.T.U., 2005)
2. $r \sin\theta - \cos\theta \dfrac{dr}{d\theta} = r^2.$ (V.T.U., 2005)
3. $(x^3y^2 + xy)\, dx = dy.$

4. $\dfrac{dy}{dx}=\dfrac{x^2+y^2+1}{2xy}$. (Bhilai, 2005)

5. $x(x-y)\,dy+y^2\,dx=0$ (I.S.M., 2001)

6. $\dfrac{dy}{dx}-\dfrac{\tan y}{1+x}=(1+x)\,e^x\sec y$. (Bhopal, 2009)

7. $e^y\left(\dfrac{dy}{dx}+1\right)=e^x$. (V.T.U., 2000)

8. $\sec^2 y\,\dfrac{dy}{dx}+x\tan y=x^3$

9. $\tan y\,\dfrac{dy}{dx}+\tan x=\cos y\cos^2 x$. (Sambalpur, 2002)

10. $\dfrac{dy}{dx}=\dfrac{y}{x+\sqrt{(xy)}}$. (V.T.U., 2004)

11. $(y\log x-2)\,ydx-xdy=0$ (V.T.U., 2006)

8·8. EXACT DIFFERENTIAL EQUATIONS

(1) **Def.** *A differential equation of the form* $\mathbf{M}(x,y)\,\mathbf{dx}+\mathbf{N}(x,y)\,\mathbf{dy}=\mathbf{0}$ *is said to be* **exact** *if its left hand member is the exact differential of some function* $u(x,y)$ *i.e.* $du\equiv Mdx+Ndy=0$. Its solution, therefore, is $u(x,y)=c$.

(2) **Theorem.** *The necessary and sufficient condition for the differential equation* $Mdx+Ndy=0$ *to be exact is*

$$\frac{\partial \mathbf{M}}{\partial \mathbf{y}}=\frac{\partial \mathbf{N}}{\partial \mathbf{x}}$$

Condition is necessary :

The equation $Mdx+Ndy=0$ will be exact, if

$$Mdx+Ndy\equiv du \quad \text{...(1)}$$

where u is some function of x and y.

But $$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy \quad \text{...(2)}$$

$\therefore$ Equating coefficients of dx and dy in (1) and (2), we get $M=\dfrac{\partial u}{\partial x}$ and $N=\dfrac{\partial u}{\partial y}$

$$\therefore \quad \frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y\partial x} \text{ and } \frac{\partial N}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}.$$

But $$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y} \quad \text{(Assumption)}$$

$\therefore$ $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$ which is the necessary condition for exactness.

Condition is sufficient : i.e. if $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$, then $Mdx+Ndy=0$ is exact.

Let $\int Mdx=u$, where y is supposed constant while performing integration.

Then $$\frac{\partial}{\partial x}\left(\int Mdx\right)=\frac{\partial u}{\partial x},\ i.e.\ M=\frac{\partial u}{\partial x} \quad \text{...(3)}$$

$$\therefore \quad \frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y\partial x} \text{ or } \frac{\partial N}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \quad \left\{\begin{array}{l}\because \dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}\ \text{(given)}\\ \text{and } \dfrac{\partial^2 u}{\partial y\partial x}=\dfrac{\partial^2 u}{\partial x\partial y}\end{array}\right.$$

Integrating both sides w.r.t. x (taking y as constant).

$$N=\frac{\partial u}{\partial y}+f(y), \text{ where } f(y) \text{ is a function of } y \text{ alone.} \quad \text{...(4)}$$

$$\therefore \quad Mdx+Ndy=\frac{\partial u}{\partial x}dx+\left\{\frac{\partial u}{\partial y}+f(y)\right\}dy \quad \text{[by (3) and (4)]}$$

$$=\left\{\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy\right\}+f(y)\,dy=du+f(y)\,dy=d\left[u+\int f(y)\,dy\right] \quad \text{...(5)}$$

which shows that $Mdx+Ndy=0$ is exact.

(3) Method of solution. By (5), the equation $Mdx + Ndy = 0$ becomes $d\,[u + \int f(y)\,dy] = 0$

Integrating $\quad u + \int f(y)\,dy = 0.$

But $\quad u = \int_{y \text{ constant}} Mdx$ and $f(y)$ = terms of N not containing x.

$\therefore$ *The solution of $Mdx + Ndy = 0$ is*

$$\int_{(y \text{ cons.})} \mathbf{Mdx} + \int (\textbf{terms of N not containing x})\,\mathbf{dy = c}$$

provided $\quad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}.$

Example 8·23. *Solve* $(y^2 e^{xy^2} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0.$

Sol. Here $\quad M = y^2 e^{xy^2} + 4x^3$ and $N = 2xy\,e^{xy^2} - 3y^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 2ye^{xy^2} + y^2 e^{xy^2} \cdot 2xy = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const.})} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

i.e. $$\int_{(y \text{ const.})} (y^2 e^{y^2 x} + 4x^3)\,dx + \int (-3y^2)\,dy = c \quad \text{or} \quad e^{xy^2} + x^4 - y^3 = c.$$

Example 8·24. *Solve* $\left\{ y\left(1 + \dfrac{1}{x}\right) + \cos y \right\} dx + (x + \log x - x \sin y)\,dy = 0$

(Marathwada, 2008 ; V.T.U., 2006)

Sol. Here $M = y\,(1 + 1/x) + \cos y$ and $N = x + \log x - x \sin y$

$$\therefore \quad \frac{\partial M}{\partial y} = 1 + 1/x - \sin y = \frac{\partial N}{\partial x}$$

Then the equation is exact and its solution is

$$\int_{(y\ const)} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

$$\int_{(y\ const)} \left\{ \left(1 + \frac{1}{x}\right) y + \cos y \right\} dx = c \quad \text{or} \quad (x + \log x)\,y + x \cos y = c.$$

Example 8·25. *Solve* $(1 + 2xy \cos x^2 - 2xy)\,dx + (\sin x^2 - x^2)\,dy = 0.$

Sol. Here $M = 1 + 2xy \cos x^2 - 2xy$ and $N = \sin x^2 - x^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 2x \cos x^2 - 2x = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y\ const)} M\,dx + \int (\text{terms of } N \text{ not containing } x) = c$$

i.e., $$\int_{(y\ const)} (1 + 2xy \cos x^2 - 2xy)\,dx = c \quad \text{or} \quad x + y\left[\int \cos x^2 \cdot 2x dx - \int 2\,x\,dx\right] = c$$

or $$x + y \sin x^2 - y\,x^2 = c.$$

Example 8·26. *Solve* $\dfrac{dy}{dx} + \dfrac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0.$ *(Kurukshetra, 2002)*

Sol. Given equation can be written as

$$(y \cos x + \sin y + y)\, dx + (\sin x + x \cos y + x)\, dy = 0.$$

Here $M = y \cos x + \sin y + y$ and $N = \sin x + x \cos y + x$.

$$\therefore \quad \frac{\partial M}{\partial y} = \cos x + \cos y + 1 = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

i.e. $$\int_{(y \text{ const.})} (y \cos x + \sin y + y)\, dx + \int (0)\, dx = c \quad \text{or} \quad y \sin x + (\sin y + y)\, x = c.$$

Example 8·27. *Solve* $(2x^2 + 3y^2 - 7)\, x\, dx - (3x^2 + 2y^2 - 8)\, y dy = 0.$ *(U.P.T.U., 2005)*

Sol. Rewriting the given equation, we have

$$\frac{x\, dx}{y\, dy} = \frac{3x^2 + 2y^2 - 8}{2x^2 + 3y^2 - 7}$$

or $$\frac{x\, dx + y\, dy}{x\, dx - y\, dy} = \frac{5x^2 + 5y^2 - 15}{x^2 - y^2 - 1}$$ [by Componendo-dividendo]

or $$\frac{x\, dx + y\, dy}{x^2 + y^2 - 3} = 5\, \frac{x\, dx - y\, dy}{x^2 - y^2 - 1}$$

or $$\frac{2x\, dx + 2y\, dy}{x^2 + y^2 - 3} = 5\, \frac{2x\, dx - 2y\, dy}{x^2 - y^2 - 1}$$ [Multiplying both sides by 2.

Integrating both sides,

$$\log (x^2 + y^2 - 3) = 5 \log (x^2 - y^2 - 1) + \log c$$

or $$x^2 + y^2 - 3 = c\, (x^2 + y^2 - 1)^5$$

Problems

Solve the following equations :

1. $(x^2 - ay)\, dx = (ax - y^2)\, dy.$
2. $(x^2 + y^2 - a^2)\, x dx + (x^2 - y^2 - b^2)\, y dy = 0.$ *(Kurukshetra, 2005)*
3. $(x^4 - 2xy^2 + y^4)\, dx - (2x^2 y - 4xy^3 + \sin y)\, dy = 0.$
4. $y e^{xy} dx + (x e^{xy} + 2y)\, dy = 0.$
5. $(5x^4 + 3x^2 y^2 - 2xy^3)\, dx + (2x^3 y - 3x^2 y^2 - 5y^4)\, dy = 0.$ *(V.T.U., 2008)*
6. $(3x^2 + 6xy^2)\, dx + (6x^2 y + 4y^3)\, dy = 0.$
7. $\dfrac{2x}{y^3}\, dx + \dfrac{y^2 - 3x^2}{y^4}\, dy = 0.$
8. $y \sin 2x\, dx - (1 + y^2 + \cos^2 x)\, dy = 0.$
9. $(\sec x \tan x \tan y - e^x)\, dx + \sec x \sec^2 y\, dy = 0.$ *(Nagpur, 2009)*
10. $(2xy + y - \tan y)\, dx + (x^2 - x \tan^2 y + \sec^2 y)\, dy.$

8·9. EQUATIONS REDUCIBLE TO EXACT EQUATIONS

Sometimes a differential equation which is not exact, can be made so on multiplication by a suitable factor called an *integrating factor*. The rules for finding integrating factors of the equation $Mdx + Ndy = 0$ are as follows :

(1) **I.F. found by inspection.** In a number of cases, the integrating factor can be found after regrouping the terms of the equation and recognizing each group as being a part of an exact differential. In this connection the following integrable combinations prove quite useful :

$$xdy + ydx = d\,(xy)$$

$$\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right); \frac{xdy - ydx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$$

$$\frac{xdy - ydx}{y^2} = -d\left(\frac{x}{y}\right); \frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$$

$$\frac{xdy - ydx}{x^2 - y^2} = d\left(\frac{1}{2}\log\frac{x+y}{x-y}\right).$$

Example 8·28. *Solve* $y(2xy + e^x)\,dx = e^x\,dy.$ *(Kurukshetra, 2005 ; J.N.T.U., 2002)*

Sol. It is easy to note that the terms $ye^x dx$ and $e^x dy$ should be put together.

$$\therefore \quad (ye^x dx - e^x dy) + 2xy^2\,dx = 0$$

Now we observe that the term $2xy^2\,dx$ should not involve y^2. This suggests that $1/y^2$ may be I.F. Multiplying throughout by $1/y^2$, it follows

$$\frac{ye^x\,dx - e^x\,dy}{y^2} + 2xdx = 0 \text{ or } d\left(\frac{e^x}{y}\right) + 2xdx = 0$$

Integrating, we get $\frac{e^x}{y} + x^2 = c$ which is the required solution.

(2) I.F. of a homogeneous equation. *If $Mdx + Ndy = 0$ be a homogeneous equation in x and y, then $1/(Mx + Ny)$ is an integrating factor ($Mx + Ny \neq 0$).*

Example 8·29. *Solve* $(x^2y - xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$ *(Osmania, 2003 S)*

Sol. This equation is homogeneous in x and y.

$$\therefore \quad \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{(x^2y - 2xy^2)\,x - (x^3 - 3x^2y)y} = \frac{1}{x^2y^2}$$

Multiplying throughout by $1/x^2y^2$, the equation becomes

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \text{ which is exact.}$$

$\therefore$ The solution is $\frac{x}{y} - 2\log x + 3\log y = c.$

(3) I.F. for an equation of the type $\mathbf{f_1(xy)ydx + f_2(xy)xdy = 0}$.

If the equation $Mdx + Ndy = 0$ be of this form, then $1/(Mx - Ny)$ is an integrating factor ($Mx - Ny \neq 0$).

Example 8·30. *Solve* $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0.$ *(S.V.T.U., 2008)*

Sol. The given equation is of the form $f_1(xy)\,ydx + f_2(xy)\,xdy = 0$

Here $\quad M = (1 + xy)\,y, N = (1 - xy)\,x.$

$$\therefore \quad \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{(1 + xy)\,yx - (1 - xy)\,xy} = \frac{1}{2x^2y^2}$$

Multiplying throughout by $1/2x^2y^2$, it becomes

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0, \text{ which is an exact equation.}$$

$\therefore$ The solution is $\frac{1}{2y}\left(-\frac{1}{x}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = c$

or $$\log\frac{x}{y} - \frac{1}{xy} = c'.$$

(4) In the equation Mdx + Ndy = 0,

(a) if $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}$ *be a function of x only = f (x) say, then* $e^{\int f(x)\,dx}$ *is an integrating factor.*

(b) if $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}$ *be a function of y only = F (y) say, then* $e^{\int F(y)\,dy}$ *is an integrating factor.*

Example 8·31. *Solve* $(xy^2 - e^{1/x^3})\,dx - x^2 y\,dy = 0.$ *(S.V.T.U., 2009 ; Mumbai, 2007)*

Sol. Here $M = xy^2 - e^{1/x^3}$ and $N = -x^2y$

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N} = \frac{2xy-(-2xy)}{-x^2y} = -\frac{4}{x} \text{ which is a function of } x \text{ only.}$$

$$\therefore \quad \text{I.F.} = e^{\int \frac{-4}{x}dx} = e^{-4\log x} = x^{-4}$$

Multiplying throughout by x^{-4}, we get $\left(\dfrac{y^2}{x^3}-\dfrac{1}{x^4}e^{1/x^3}\right)dx - \dfrac{y}{x^2}\,dy = 0$

which is an exact equation.

$$\therefore \text{ The solution is } \int \left(\frac{y^2}{x^3}-\frac{1}{x^4}e^{1/x^3}\right)dx + 0 = c$$

or
$$-\frac{y^2x^{-2}}{2}+\frac{1}{3}\int e^{x^{-3}}(-3x^{-4})\,dx = c \qquad \text{or} \quad \frac{1}{3}e^{x^{-3}} - \frac{1}{2}\frac{y^2}{x^2} = c.$$

Otherwise it can be solved as a Bernoulli's equation. (§ 8·7)

Example 8·32. *Solve* $(y\log y)\,dx + (x-\log y)\,dy = 0$ *(U.P.T.U., 2004)*

Sol. Here $M = y\log y$ and $N = x - \log y$

$$\therefore \quad \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) = \frac{1}{y\log y}(1-\log y-1) = -\frac{1}{y}, \quad \text{which is a function of } y \text{ alone.}$$

$$\therefore \quad \text{I.F.} = e^{-\int \frac{1}{y}dy} = e^{-\log y} = \frac{1}{y}$$

Multiplying the given equation throughout by $1/y$, it becomes

$$\log y\,dx + \frac{1}{y}(x-\log y)\,dy = 0$$

which is an exact equation $\left[\because \dfrac{\partial}{\partial y}(\log y) = \dfrac{\partial}{\partial x}\left(\dfrac{x-\log y}{y}\right)\right]$

$$\therefore \text{ Its solution is } \int_{(y\text{ const.})} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

or
$$\log y\int dx + \int\left(\frac{-\log y}{y}\right)dy = c \quad \text{or} \quad x\log y - \frac{1}{2}(\log y)^2 = c.$$

(5) For the equation of the type

$$x^a y^b (my\,dx + nx\,dy) + x^{a'}y^{b'}(m'y\,dx + n'x\,dy) = 0,$$

an integrating factor is $x^h y^k$

where $\dfrac{a+h+1}{m} = \dfrac{b+k+1}{n}, \quad \dfrac{a'+h+1}{m'} = \dfrac{b'+k+1}{n'}.$

(Hissar, 2005)

Example 8·33. *Solve* $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0.$

Sol. Rewriting the equation as $xy\,(ydx + xdy) + x^2y^2\,(2ydx - xdy) = 0$ and comparing with

$$x^a y^b\,(mydx + nxdy) + x^{a'} y^{b'}\,(m'ydx + n'xdy) = 0,$$

we have $a = b = 1, m = n = 1\,;\, a' = b' = 2, m' = 2, n' = -1.$

$\therefore$ $\quad$ I.F. $= x^h y^k$.

where $\quad \dfrac{a+h+1}{m} = \dfrac{b+k+1}{n}, \quad \dfrac{a'+h+1}{m'} = \dfrac{b'+k+1}{n'}$

i.e. $\quad \dfrac{1+h+1}{1} = \dfrac{1+k+1}{1}, \quad \dfrac{2+h+1}{2} = \dfrac{2+k+1}{-1}$

or $\quad h - k = 0,\ h + 2k + 9 = 0$

Solving these, we get $h = k = -3$. $\therefore$ I.F. $= 1/x^3y^3$.

Multiplying throughout by $1/x^3y^3$, it becomes

$$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dx = 0, \text{ which is an exact equation.}$$

$\therefore$ The solution is $\dfrac{1}{y}\left(-\dfrac{1}{x}\right) + 2\log x - \log y = c$ or $2\log x - \log y - 1/xy = c$.

Problems

Solve the following equations :

1. $xdy - ydx + a\,(x^2 + y^2)\,dx = 0.$
2. $xdx + ydy = \dfrac{a^2\,(xdy - ydx)}{x^2 + y^2}$. (U.P.T.U., 2005)
3. $ydx - xdy + \log x\,dx = 0.$
4. $2ydx + x\,(2\log x - y)\,dy = 0$ (P.T.U., 2005)
5. $(x^3y^2 + x)\,dy + (x^2y^3 - y)\,dx = 0.$
6. $(x^2y^2 + xy + 1)\,ydx + (x^2y^2 - xy + 1)\,xdy = 0.$
7. $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$
8. $(4xy + 3y^2 - x)\,dx + x\,(x + 2y)\,dy = 0.$ (Mumbai, 2006)
9. $x^4\dfrac{dy}{dx} + x^3y + \operatorname{cosec}(xy) = 0.$
10. $(y - xy^2)\,dx - (x + x^2y)\,dy = 0.$ (Mumbai, 2006)
11. $ydx - xdy + 3x^2y^2e^{x^3}\,dx = 0.$ (Kurukshetra, 2006)
12. $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0.$ (Rajasthan, 2005)

8·10. EQUATIONS OF THE FIRST ORDER AND HIGHER DEGREE

As dy/dx will occur in higher degrees, it is convenient to denote dy/dx by p. Such equations are of the form $f(x, y, p) = 0$. Three cases arise for discussion :

Case I. **Equation solvable for p.** A differential equation of the first order but of the nth degree is of the form

$$p^n + P_1p^{n-1} + P_2p^{n-2} + \ldots\ldots + P_n = 0 \quad \ldots(1)$$

where $P_1, P_2, \ldots\ldots, P_n$ are functions of x and y.

Splitting up the left hand side of (1) into n linear factors, we have

$$[p - f_1(x, y)]\,[p - f_2(x, y)] \ldots\ldots [p - f_n(x, y)] = 0.$$

Equating each of the factors to zero,

$$p = f_1(x, y),\ p = f_2(x, y), \ldots\ldots, p = f_n(x, y)$$

Solving each of these equations of the first order and first degree, we get the solutions

$$F_1(x, y, c) = 0,\ F_2(x, y, c) = 0, \ldots\ldots, F_n(x, y, c) = 0.$$

These n solutions constitute the general solution of (1).

Otherwise, the general solution of (1) may be written as

$$F_1(x, y, c) \cdot F_2(x, y, c) \ldots\ldots F_n(x, y, c) = 0.$$

Example 8·34. *Solve* $\dfrac{dy}{dx} - \dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y}{x}$.

Sol. Given equation is $p - \dfrac{1}{p} = \dfrac{x}{y} - \dfrac{y}{x}$ where $p = \dfrac{dy}{dx}$ or $p^2 + p\left(\dfrac{y}{x} - \dfrac{x}{y}\right) - 1 = 0.$

Factorising $(p + y/x)(p - x/y) = 0.$

Thus we have $\quad p + y/x = 0 \quad \ldots(i) \quad$ and $\quad p - x/y = 0 \quad \ldots(ii)$

From (*i*), $\quad \dfrac{dy}{dx} + \dfrac{y}{x} = 0$ or $xdy + ydx = 0$

i.e., $\quad d(xy) = 0$. Integrating, $xy = c$.

From (*ii*), $\quad \dfrac{dy}{dx} - \dfrac{x}{y} = 0$ or $xdx - ydy = 0$

Integrating, $\quad x^2 - y^2 = c$. Thus $xy = c$ or $x^2 - y^2 = c$, constitute the required solution.

Otherwise, combining these into one, the required solution can be written as

$$(xy - c)(x^2 - y^2 - c) = 0.$$

Example 8·35. *Solve* $p^2 + 2py \cot x = y^2$. *(Bhopal, 2008 ; Kerala, 2005)*

Sol. We have $p^2 + 2py \cot x + (y \cot x)^2 = y^2 + y^2 \cot^2 x$

or $\quad p + y \cot x = \pm y \operatorname{cosec} x$

i.e., $\quad p = y(-\cot x + \operatorname{cosec} x) \quad \ldots(i)$

or $\quad p = y(-\cot x - \operatorname{cosec} x) \quad \ldots(ii)$

From (*i*), $\dfrac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$ or $\dfrac{dy}{y} = (\operatorname{cosec} x - \cot x)\, dx$

Integrating, $\quad \log y = \log \tan \dfrac{x}{2} - \log \sin x + \log c = \log \dfrac{c \tan x/2}{\sin x}$

or $\quad y = \dfrac{c}{2 \cos x^2/2}$ or $y(1 + \cos x) = c \quad \ldots(iii)$

From (*ii*), $\dfrac{dy}{dx} = -y(\cot x + \operatorname{cosec} x)$ or $\dfrac{dy}{y} = -(\cot x + \operatorname{cosec} x)\, dx$

Integrating, $\log y = -\log \sin x - \log \tan \dfrac{x}{2} + \log c = \log \dfrac{c}{\sin x \tan \dfrac{x}{2}}$

or $\quad y = \dfrac{c}{2 \sin^2 \dfrac{x}{2}}$ or $y(1 - \cos x) = c \quad \ldots(iv)$

Thus combining (*iii*) and (*iv*), the required general solution is

$$y(1 \pm \cos x) = c.$$

Case II. **Equations solvable for y.** If the given equation, on solving for y, takes the form

$$y = f(x, p), \quad \ldots(1)$$

then differentiation with respect to x gives as equation of the form

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right).$$

Now it may be possible to solve this new differential equation in x and p.

Let its solution be $F(x, p, c) = 0. \quad \ldots(2)$

The elimination of p from (1) and (2) gives the required solution.

In case elimination of p is not possible, then we may solve (1) and (2) for x and y and obtain

$$x = F_1(p, c),\ y = F_2(p, c)$$

as the required solution, where p is the parameter.

Obs. This method is especially useful for equations which do not contain x.

Example 8·36. *Solve* $y - 2px = \tan^{-1}(xp^2)$.

Sol. Given equation is $y = 2px + \tan^{-1}(xp^2)$ ...(*i*)

Differentiating both sides with respect to x, $\dfrac{dy}{dx} = p = 2\left(p + x\dfrac{dp}{dx}\right) + \dfrac{p^2 + 2xp\dfrac{dp}{dx}}{1 + x^2p^4}$

or $$p + 2x\frac{dp}{dx} + \left(p + 2x\frac{dp}{dx}\right)\cdot\frac{p}{1 + x^2p^4} = 0 \text{ or } \left(p + 2x\frac{dp}{dx}\right)\left(1 + \frac{p}{1 + x^2p^4}\right) = 0$$

This gives $p + 2x\,dp/dx = 0$.

Separating the variables and integrating, we have $\int \dfrac{dx}{x} + 2\int \dfrac{dp}{p}$ = a constant

or $$\log x + 2\log p = \log c \text{ or } \log xp^2 = \log c$$

whence $$xp^2 = c \text{ or } p = \sqrt{(c/x)} \qquad ...(ii)$$

Eliminating p from (*i*) and (*ii*), we get $y = 2\sqrt{(c/x)}x + \tan^{-1} c$

or $y = 2\sqrt{(cx)} + \tan^{-1} c$ which is the general solution of (*i*).

Obs. The significance of the factor $1 + p/(1 + x^2p^4) = 0$ which we didn't consider, will not be considered here as it concerns 'singular solution' of (*i*) whereas *we are interested only in finding general solution.*

Caution. Sometimes one is tempted to write (*ii*) as

$$\frac{dy}{dx} = \sqrt{\left(\frac{c}{x}\right)}$$

and integrating it to say that the required solution is $y = 2\sqrt{(cx)} + c'$. Such a reasoning is *incorrect.*

(*Bhopal, 2009*)

Example 8·37. *Solve* $y = 2px + p^n$.

Sol. Given equation is $y = 2px + p^n$ ...(*i*)

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = p = 2p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx} \text{ or } p\frac{dx}{dp} + 2x = -np^{n-1}$$

or $$\frac{dx}{dp} + \frac{2x}{p} = -np^{n-2} \qquad ...(ii)$$

This is Leibnitz's linear equation in x and p. Here I.F. $= e^{\int \frac{2}{p}dp} = e^{\log p^2} = p^2$

∴ The solution of (*ii*) is

$$x(\text{I.F.}) = \int(-np^{n-2})\cdot(\text{I.F.})\,dp + c \text{ or } xp^2 = -n\int p^n\,dp + c = -\frac{np^{n+1}}{n+1} + c$$

or $$x = cp^{-2} - \frac{np^{n-1}}{n+1} \qquad ...(iii)$$

Substituting this value of x in (*i*), we get $y = \dfrac{2c}{p} + \dfrac{1-n}{1+n}p^n$ ...(*iv*)

The equations (*iii*) and (*iv*) taken together, with parameter p, constitute the general solution (*i*).

Obs. In general, the equations of the form $y = xf(p) + \phi(p)$, known as *Lagrange's* equation, are solvable for y and lead to Leibnitz's equation in dx/dp.

Case III. **Equations solvable for x.** If the given equation on solving for x, takes the form

$$x = f(y, p) \quad \ldots(1)$$

then differentiation with respect to y gives an equation of the form

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right)$$

Now it may be possible to solve the new differential equation in y and p. Let its solution be $f(y, p, c) = 0$. ...(2)

The elimination of p from (1) and (2) gives the required solution. In case the elimination is not feasible, (1) and (2) may be expressed in terms of p and p may be regarded as a parameter.

Obs. This method is especially useful for equations which do not contain y.

Example 8.38. *Solve* $y = 2px + y^2p^3$. (*Bhopal, 2008*)

Sol. Given equation, on solving for x, takes the form $x = \dfrac{y - y^2p^3}{2p}$

Differentiating with respect to y, $\dfrac{dx}{dy}\left(=\dfrac{1}{p}\right) = \dfrac{1}{2}\cdot\dfrac{p\left(1 - 2y\cdot p^3 - y^2 3p^2\dfrac{dp}{dy}\right) - (y - y^2p^3)\dfrac{dp}{dy}}{p^2}$

or $$2p = p - 2yp^4 - 3y^2p^3\frac{dp}{dy} - y\frac{dp}{dy} + y^2p^3\frac{dp}{dy}$$

or $$p + 2yp^4 + 2y^2p^3\frac{dp}{dy} + y\frac{dp}{dy} = 0 \text{ or } p(1 + 2yp^3) + y\frac{dp}{dy}(1 + 2yp^3) = 0$$

or $$\left(p + y\frac{dp}{dy}\right)(1 + 2py^3) = 0. \text{ This gives } p + y\frac{dp}{dy} = 0 \text{ or } \frac{d}{dy}(py) = 0.$$

Integrating $\quad py = c.$ ...(*i*)

Thus eliminating p from the given equation and (*i*), we get $y = 2\dfrac{c}{y}x + \dfrac{c^3}{y^3}y^2$ or $y^2 = 2cx + c^3$

which is the required solution.

Case IV. **Clairaut's Equation**

An equation of the form $y = px + f(p)$ is known as Clairaut's equation ...(1)

Differentiating with respect to x, we have $p = p + x\dfrac{dp}{dx} + f'(p)\dfrac{dp}{dx}$

or $$[x + f'(p)]\frac{dp}{dx} = 0 \qquad \therefore \frac{dp}{dx} = 0, \text{ or } x + f'(p) = 0$$

$$\frac{dp}{dx} = 0, \text{ gives } p = c \quad \ldots(2)$$

Thus eliminating p from (1) and (2), we get $y = cx + f(c)$ ...(3)

as the general solution of (1).

Hence the solution of the Clairaut's equation is obtained on replacing p by c.

Obs. If we eliminate p from $x + f'(p) = 0$ and (1), we get an equation involving no constant. *This is the* **singular solution** of (1) *which gives the envelope of the family of straight lines (3).*

To obtain the singular solution, we proceed as follows :

(*i*) Find the general solution by replacing p by c *i.e.* (3)

(*ii*) Differentiate this w.r.t. c giving $x + f'(c) = 0$. ...(4)

(*iii*) Eliminate c from (3) and (4) which will be the *singular solution.*

Example 8.39. *Solve $p = \sin(y - xp)$. Also find its singular solutions.*

Sol. Given equation can be written as

$$\sin^{-1} p = y - xp \quad \text{or} \quad y = px + \sin^{-1} p \text{ which is the Clairaut's equation.}$$

$\therefore$ Its solution is $y = cx + \sin^{-1} c$. ...(i)

To find the singular solution, differentiate (i) w.r.t. c giving

$$0 = x + \frac{1}{\sqrt{1-c^2}} \quad \text{...(ii)}$$

To eliminate c from (i) and (ii), we rewrite (ii) as

$$c = N(x^2 - 1)/x$$

Now substituting this value of c in (i), we get

$$y = N(x^2 - 1) + \sin^{-1}\{N(x^2 - 1)/x\}$$

which is the desired singular solution.

Obs. Equations reducible to Clairaut's form. Many equations of the first order but of higher degree can be easily reduced to the Clairaut's form by making suitable substitutions.

Example 8.40. *Solve $(px - y)(py + x) = a^2 p$.* (J.N.T.U., 2006)

Sol. Put $x^2 = u$ and $y^2 = v$ so that $2xdx = du$ and $2ydy = dv$

$\therefore$
$$p = \frac{dy}{dx} = \frac{dv}{y} \bigg/ \frac{du}{x} = \frac{x}{y} P, \text{ where } P = \frac{dv}{du}$$

Then the given equation becomes $\left(\frac{xP}{y} \cdot x - y\right)\left(\frac{xP}{y} \cdot y + x\right) = a^2 \frac{xP}{y}$

or
$$(uP - v)(P + 1) = a^2 P \quad \text{or} \quad uP - v = \frac{a^2 P}{P+1}$$

or
$$v = uP - a^2 P/(P + 1), \text{ which is Clairaut's form.}$$

$\therefore$ Its solution is $v = uc - a^2 c/(c + 1)$, *i.e.* $y^2 = cx^2 - a^2 c/(c + 1)$.

Problems

Solve the following equations :

1. (i) $y\left(\frac{dy}{dx}\right)^2 + (x - y)\left(\frac{dy}{dx}\right) - x = 0$. (ii) $p(p + y) = x(x + y)$.
   (iii) $y = x[p + \sqrt{(1 + p^2)}]$. (iv) $xy\left(\frac{dy}{dx}\right)^2 - (x^2 + y^2)\frac{dy}{dx} + xy = 0$. (Madras, 2003)
   (v) $p^3 + 2xp^2 - y^2 p^2 - 2xy^2 p = 0$. (S.V.T.U., 2007)
2. (i) $y = x + a \tan^{-1} p$. (i) $y + px = x^4 p^2$.
   (iii) $x^2 p^4 + 2xp - y = 0$. (iv) $xp^2 + x = 2yp$.
   (v) $y = xp^2 + p$. (vi) $y = p \sin p + \cos p$.
3. (i) $p^3 - 4xyp + 8y^2 = 0$. (ii) $p^3 y + 2px = y$. (S.V.T.U., 2008)
   (iii) $x - yp = ap^2$. (iv) $p = \tan\left(x - \frac{p}{1+p^2}\right)$.
4. Find the general and singular solution of the equations :
   (i) $xp^2 - yp + a = 0$. (J.N.T.U., 2006) (ii) $p = \log(px - y)$. (P.T.U., 2006)
   (iii) $y = px + \sqrt{(a^2p^2 + b^2)}$ (W.B.T.U., 2005) (iv) $\sin px \cos y = \cos px \sin y + p$

5. **Solve the following equations :**

(i) $y + 2\left(\dfrac{dy}{dx}\right)^2 = (x+1)\dfrac{dy}{dx}$.

(ii) $(y - px)(p - 1) = p$.

(iii) $(x - a)\left(\dfrac{dy}{dx}\right)^2 + (x - y)\dfrac{dy}{dx} - y = 0$.

(iv) $(px - y)(x + py) = 2p$