1 EXACT DIFFERENTIAL EQUATIONS

1.1. INTRODUCTION TO DIFFERENTIAL EQUATIONS

1.1.1. Differential Equation.

An equation involving independent variable, dependent variable and the derivatives of dependent variables with respect to independent variables is called a **differential equation.**

Mathematically, a differential equation can be expressed as

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots\ldots, \frac{d^ny}{dx^n}\right) = 0$$

Thus,

$$\frac{dy}{dx} = x \qquad \ldots(1)$$

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2} \qquad \ldots(2)$$

$$\frac{dy}{dx} = \tan x \qquad \ldots(3)$$

$$\left[1+\left(\frac{dy}{dx}\right)^3\right]^{4/3} = p \cdot \frac{d^2y}{dx^2} \qquad \ldots(4)$$

$$\frac{d^3y}{dx^3} + 7\frac{d^2y}{dx^2} + 8\frac{dy}{dx} - 9y = \log x \qquad \ldots(5)$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \qquad \ldots(6)$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \qquad \ldots(7)$$

are all examples of differential equations.

1.1.2. Types of Differential Equations

Differential equations are of two types :

(*i*) *Ordinary differential equations*

(*ii*) *Partial differential equations*

(*i*) **Ordinary Differential Equations.** The differential equations which involve a single independent variable are called ordinary differential equations. In Art. 1.1.1., eqns. (1), (2), (3), (4), (5) are all ordinary differential equations.

(*ii*) **Partial Differential Equations.** The differential equations which involve partial differential co-efficients with respect to more than one independent variable are called partial differential equations. In Art. 1.1.1., equations (6) and (7) are partial differential equations.

In the present book, we shall be dealing only with ordinary differential equations.

1.1.3. Order and Degree of Differential Equations [M.D.U. 2013]

The order of a differential equation is the order of the highest differential co-efficient which occurs in it.

For example, equations (1), (2) and (3) of Art. 1.1.1. are of first order, equation (4) is of second order and equation (5) is of third order.

*The **degree** of a differential equation is the degree (or power) of the highest differential co-efficient which occurs in it after the differential equation has been cleared of radicals and fractions.*

For example, the equation $\left[1+\left(\dfrac{dy}{dx}\right)^3\right]^{4/3} = p\,\dfrac{d^2y}{dx^2}$, when freed from radicals and fractions takes the form

$$\left[1+\left(\frac{dy}{dx}\right)^3\right]^4 = p^3\left(\frac{d^2y}{dx^2}\right)^3.$$

Hence, the equation is of second order and third degree. In Art. 1.1.1., equations (1), (2), (3), (5), (6) and (7) all are of first degree.

1.1.4. Solution of a Differential Equation

A relation $y = f(x)$ between the dependent and independent variables, which when substituted in the equation, satisfies the equation is called a **solution** or **integral** of the given differential equation.

Generally, solution of differential equations are of three types :

(*a*) **General Solution.** The solution of differential equation in which the number of independent arbitrary constants (which cannot be reduced to a fewer number of equivalent constants) is the same as the order of the differential equation is called the general solution. It is also called *complete primitive or complete integral*.

$\frac{d^2y}{dx^2} = x$

$\frac{dy}{dx} = \frac{x^2}{2} + c_1$

$y = \frac{x^3}{6} + c_1x + c_2$

(*b*) **Particular Solution.** The solution obtained by giving particular values to independent arbitrary constants in the general solution is called the *particular solution*.

(*c*) **Singular Solution.** A solution which does not contain any arbitrary constant and also, is not obtainable from the complete solution by giving particular values to the arbitrary constants is called a *singular solution*.

1.1.5. Formation of Differential Equations.

Formation of differential equations by elimination of arbitrary constants.

(*i*) Let the primitive be $\qquad f(x, y, a) = 0 \qquad$ ...(1)

where '*a*' is an arbitrary constant.

Differentiating (1), we get $\qquad F\left(x, y, \dfrac{dy}{dx}, a\right) = 0 \qquad$ ...(2)

Eliminating '*a*' between (1) and (2), we shall obtain a relation between x, y and $\dfrac{dy}{dx}$.

Let the relation obtained be $\psi\left(x, y, \dfrac{dy}{dx}\right) = 0$.

This is a differential equation of the first order. So we should expect one arbitrary constant in the solution of equations of first order.

(*ii*) Again consider the relation

$$f(x, y, a, b) = 0 \qquad \text{...(1)}$$

By differentiating (1), suppose we get the relation

$$\phi_1\left(x, y, \frac{dy}{dx}, a, b\right) = 0 \qquad \text{...(2)}$$

Differentiating (2) again, suppose we get

$$\phi_2\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, a, b\right) = 0 \qquad \text{...(3)}$$

Eliminating '*a*', '*b*' between (1), (2) and (3), we shall get a relation of the form

$$\psi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0,$$

which is a differential equation of second order. So we should expect two arbitrary constants in the solution of equations of second order.

1.2. GEOMETRICAL MEANING OF THE DIFFERENTIAL EQUATION

The explicit form of a differential equation of the first order and first degree is

$$\frac{dy}{dx} = f(x, y) \qquad ...(1)$$

$\frac{dy}{dx}$ is the slope of the tangent to the curve at point (x, y). Let $A_1(x_1, y_1)$ be any point in the xy-plane. Let m_1 be the value of $\frac{dy}{dx}$ at point (x_1, y_1) such that m_1 is the slope of the tangent at point (x_1, y_1). Let the point move from (x_1, y_1), in the direction m_1 for an infinitesimal distance to the point $A_2(x_2, y_2)$. Let m_2 be the slope of the tangent at (x_2, y_2). Let the point moves in the direction m_2 for an infinitesimal distance to a point $A_3(x_3, y_3)$.

Let m_3 be the slope of the tangent at point $A_3(x_3, y_3)$. Let the point moves in the direction m_3 for an infinitesimal distance to a point $A_4(x_4, y_4)$.

If the successive points $A_1, A_2, A_3, A_4, ...$ are chosen very near to one another, then the broken curve $A_1 A_2 A_3 A_4....$ approximates to a smooth curve $y = \phi(x)$, which is the solution of (1) associated with the initial condition $y_1 = \phi(x_1)$.

The slope of the tangent at any point of the curve and the co-ordinates of that point satisfy equation (1). Also different choice of the initial point will, in general, give a different curve with the same property.

Hence differential equation (1) represents a family of curves such that through every point of the xy-plane, there passes one curve of the family.

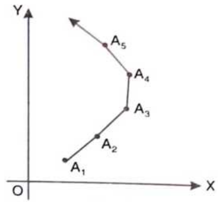

Fig. 1.1

1.3. EXACT DIFFERENTIAL EQUATIONS

Definition. *A differential equation is said to be exact, if it is obtained from its general solution directly by differentiation without any subsequent multiplication, elimination etc.*

Another form. The differential equation $M\,dx + N\,dy = 0$, where M and N are functions of x and y is said to be exact, if $M\,dx + N\,dy = du$, where u is a function of x and y.

Consider the differential equation $\frac{dy}{dx} = x$ ...(1)

General solution of (1) is

$$y = \frac{x^2}{2} + c \qquad ...(2)$$

Now, differentiating (2) w.r.t. x, we get $\frac{dy}{dx} = x$

Thus, we see that the differential equation (1) is derived from its general solution directly by differentiation.

Hence equation (1) is exact differential equation.

Some other examples of exact differential equations are

$$x\,dy + y\,dx = 0; \ \sin x \cos y\,dy + \cos x \sin y\,dx = 0 \ \text{etc.}$$

1.4. THEOREM

To find the necessary and sufficient conditions that the equation M dx + N dy = 0 may be exact.

[K.U. 2017, 16, 15, 14, 07, 04; M.D.U. 2014, 05]

Necessary Condition :

Let the equation $M\,dx + N\,dy = 0$ be exact.

$\therefore$ By definition, it must have been derived by directly differentiating some function u of x and y, performing no other operation *i.e.*,

$$M\,dx + N\,dy = du \qquad ...(1)$$

As u is a function of x and y, so by partial differentiation, we have

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad ...(2)$$

Comparing the coefficients of dx and dy in (1) and (2), we have

$$M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y}$$

so that

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\,\partial x} \qquad ...(3)$$

and

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x\,\partial y} \qquad ...(4)$$

But

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, which is the required necessary condition.

Sufficient Condition :

Given $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. *We have to prove that* $M\,dx + N\,dy = 0$ *is exact.*

Let $$\int M\,dx = u$$

where integration has been performed by treating y as constant. ...(1)

Differentiating (1) partially w.r.t. x, we get

$$\frac{\partial}{\partial x}\left(\int M\,dx\right) = \frac{\partial u}{\partial x} \Rightarrow M = \frac{\partial u}{\partial x} \quad ...(2)$$

Differentiating (2) partially both sides w.r.t. y, we get

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y\,\partial x} \quad ...(3)$$

Since $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ (Given)}$$

and $$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}$$

$\therefore$ Equation (3) becomes $$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}$$

Integrating both sides w.r.t. x, treating y as constant, we get

$$N = \frac{\partial u}{\partial y} + f(y), \text{ where } f(y) \text{ is a function of } y \text{ only.} \quad ...(4)$$

From equations (2) and (4), we have

$$M\,dx + N\,dy = \frac{\partial u}{\partial x}.\,dx + \left(\frac{\partial u}{\partial y} + f(y)\right)dy$$

$$= \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + f(y)\,dy = du + f(y)\,dy$$

$$= d\left[u + \int f(y)\,dy\right] \quad ...(5)$$

Thus $M\,dx + N\,dy$ is exact derivative of $u + \int f(y)\,dy$

Hence, $M\,dx + N\,dy = 0$ is an exact differential equation.

1.5. SOLUTION OF AN EXACT DIFFERENTIAL EQUATION

[*K.U. 2016, 14, 10; M.D.U. 2014, 13*]

If the equation $M\,dx + N\,dy = 0$ is exact, then

$$M\,dx + N\,dy = d\left[u + \int f(y)\,dy\right]$$

But $$M\,dx + N\,dy = 0$$

$$\Rightarrow \quad d\left[u + \int f(y)\,dy\right] = 0$$

Integrating both sides, we get

$$u + \int f(y)\,dy = c, \text{ where } c \text{ is an arbitrary constant.}$$

Putting $u = \int_{y\ \text{constant}} M\,dx$, we get

$$\int_{y\ \text{constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c, \text{ which is the required solution.}$$

Working Rule :

(1) In an equation of the form $M\,dx + N\,dy = 0$, first check the condition of exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(2) Its solution is $\displaystyle\int_{y\ \text{constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c$

SOLVED EXAMPLES

Example 1.

Find the value of α, assuming that the differential equation

$$(1 + x^2y^3 + \alpha x^2y^2)\,dx + (2 + x^3y^2 + x^3y)\,dy = 0 \text{ is exact.}$$

Solution. The given differential equation is

$$(1 + x^2y^3 + \alpha x^2y^2)\,dx + (2 + x^3y^2 + x^3y)\,dy = 0 \qquad ...(1)$$

Comparing equation (1) with $M\,dx + N\,dy = 0$, we have

$$M = 1 + x^2y^3 + \alpha x^2y^2 \text{ and } N = 2 + x^3y^2 + x^3y$$

$$\therefore \quad \frac{\partial M}{\partial y} = 3x^2y^2 + 2\alpha x^2 y \text{ and } \frac{\partial N}{\partial x} = 3x^2y^2 + 3x^2y$$

As the given differential equation is exact, so $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\therefore \quad 3x^2y^2 + 2\alpha x^2y = 3x^2y^2 + 3x^2y$$

$$\Rightarrow \quad 2\alpha = 3 \Rightarrow \alpha = \frac{3}{2}.$$

Example 2.

Solve the differential equation $\boldsymbol{(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0.}$

Solution. The given differential equation is

$$(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0 \qquad \ldots(1)$$

Comparing (1) with $M\,dx + N\,dy = 0$, we observe that

$$M = 1 + e^{x/y},$$

and

$$N = e^{x/y}\left(1 - \frac{x}{y}\right)$$

$$\therefore \quad \frac{\partial M}{\partial y} = e^{x/y}\left(-\frac{x}{y^2}\right) = -\frac{x}{y^2}e^{x/y}$$

and

$$\frac{\partial N}{\partial x} = e^{x/y} \cdot \frac{1}{y}\left(1 - \frac{x}{y}\right) + e^{x/y} \cdot \left(-\frac{1}{y}\right)$$

$$= e^{x/y}\left(\frac{1}{y} - \frac{x}{y^2} - \frac{1}{y}\right) = -\frac{x}{y^2}e^{x/y}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore equation (1) is exact.

Hence the solution is $\int_{y \text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\, dy = c$

i.e.,

$$\int_{y \text{ constant}} (1 + e^{x/y})\, dx = c$$

or

$$x + \frac{e^{x/y}}{1/y} = c \quad \Rightarrow \quad x + y\,e^{x/y} = c.$$

Example 3.

Solve the differential equation

$$(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0. \quad \text{[M.D.U. 2013, 12, 2000]}$$

Solution. The given differential equation is

$$(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0 \qquad ...(1)$$

Comparing equation (1) with $M\,dx + N\,dy = 0$, we observe that

$$M = x^4 - 2xy^2 + y^4$$

and
$$N = (-2x^2y + 4xy^3 - \sin y)$$

$\therefore$
$$\frac{\partial M}{\partial y} = -4xy + 4y^3$$

and
$$\frac{\partial N}{\partial x} = -4xy + 4y^3$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore the given equation (1) is exact.

Hence the solution is

$$\int_{y\text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c$$

i.e.,
$$\int_{y\text{ constant}} (x^4 - 2xy^2 + y^4)\,dx + \int -\sin y\,dy = c$$

or
$$\frac{x^5}{5} - 2y^2 \cdot \frac{x^2}{2} + y^4 \cdot x + \cos y = c$$

or
$$\frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = c.$$

Example 4.

Solve the differential equation

$$\frac{2x}{y^3}\,dx + \left(\frac{y^2 - 3x^2}{y^4}\right)dy = 0.$$

Solution. The given differential equation is

$$\frac{2x}{y^3}\,dx + \left(\frac{y^2 - 3x^2}{y^4}\right)dy = 0 \qquad ...(1)$$

Comparing equation (1) with $M\,dx + N\,dy = 0$, we observe that

$$M = \frac{2x}{y^3} \quad \text{and} \quad N = \frac{1}{y^2} - \frac{3x^2}{y^4}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{-6x}{y^4} \text{ and } \frac{\partial N}{\partial x} = -\frac{6x}{y^4}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore the given equation (1) is exact.

Hence the solution is

$$\int_{y \text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\, dy = c$$

i.e.,
$$\int_{y \text{ constant}} \frac{2x}{y^3}\,dx + \int \left(\frac{1}{y^2}\right) dy = c$$

or
$$\frac{2}{y^3} \int_{y \text{ constant}} x\,dx + \int \frac{1}{y^2}\,dy = c$$

or
$$\frac{x^2}{y^3} - \frac{1}{y} = c$$

or
$$x^2 - y^2 = cy^3.$$

Example 5.

Solve the differential equation $\left[\frac{y^2}{(y-x)^2}\,dx - \frac{x^2}{(x-y)^2}\,dy\right] = 0$.

Solution. The given differential equation is

$$\frac{y^2}{(y-x)^2}\,dx - \frac{x^2}{(x-y)^2}\,dy = 0 \qquad \text{...(1)}$$

Comparing equation (1) with $M\,dx + N\,dy = 0$, we observe that

$$M = \frac{y^2}{(y-x)^2}$$

and
$$N = -\frac{x^2}{(x-y)^2}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{(y-x)^2 \,.\, 2y - y^2 \,.\, 2\,(y-x)}{(y-x)^4}$$

$$= \frac{2\,y(y-x) - 2y^2}{(y-x)^3}$$

$$= \frac{-2xy}{(y-x)^3} = \frac{2xy}{(x-y)^3}$$

and
$$\frac{\partial N}{\partial x} = -\left[\frac{(x-y)^2 \cdot 2x - x^2 \cdot 2(x-y)}{(x-y)^4}\right]$$

$$= -\left[\frac{2x(x-y) - 2x^2}{(x-y)^3}\right] = \frac{2xy}{(x-y)^3}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore the given equation (1) is exact.

Hence the solution is

$$\int_{y \text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c$$

i.e.,
$$\int_{y \text{ constant}} \frac{y^2}{(y-x)^2}\,dx - \int 0\,dy = c$$

or
$$y^2 \int_{y \text{ constant}} \frac{1}{(y-x)^2}\,dx = c$$

or
$$y^2 \int_{y \text{ constant}} (y-x)^{-2}\,dx = c$$

or
$$y^2 \cdot \frac{(y-x)^{-1}}{(-1)\times(-1)} = c$$

or
$$\frac{y^2}{y-x} = c \;\Rightarrow\; y^2 = c\,(y-x).$$

Example 6.

Verify that the differential equation $x\,dx + y\,dy = \frac{a^2\,(x\,dy - y\,dx)}{x^2+y^2}$ ***is exact and solve it.***

Solution. The given differential equation is

$$x\,dx + y\,dy = \frac{a^2\,(x\,dy - y\,dx)}{x^2+y^2}$$

or
$$x\,dx + y\,dy = \frac{a^2 x}{x^2+y^2}\,dy - \frac{a^2 y}{x^2+y^2}\,dx$$

or
$$\left(x + \frac{a^2 y}{x^2+y^2}\right)dx + \left(y - \frac{a^2 x}{x^2+y^2}\right)dy = 0 \qquad \ldots(1)$$

Comparing equation (1) with $M\,dx + N\,dy = 0$, we observe that

$$M = x + \frac{a^2 y}{x^2 + y^2} \quad \text{and} \quad N = y - \frac{a^2 x}{x^2 + y^2}$$

$$\therefore \quad \frac{\partial M}{\partial y} = 0 + a^2\left[\frac{(x^2 + y^2) - y(2y)}{(x^2 + y^2)^2}\right] = \frac{a^2(x^2 - y^2)}{(x^2 + y^2)^2}$$

and
$$\frac{\partial N}{\partial x} = 0 - a^2\left[\frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2}\right] = \frac{a^2(x^2 - y^2)}{(x^2 + y^2)^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore equation (1) is exact.

Hence the solution is $\displaystyle\int_{y\text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\, dy = c$

or
$$\int_{y\text{ constant}}\left(x + \frac{a^2 y}{x^2 + y^2}\right) dx + \int y\, dy = c$$

i.e.,
$$\frac{x^2}{2} + a^2 y \int_{y\text{ constant}} \frac{dx}{x^2 + y^2} + \frac{y^2}{2} = c$$

or
$$\frac{x^2}{2} + a^2 y \cdot \frac{1}{y}\tan^{-1}\frac{x}{y} + \frac{y^2}{2} = c \qquad \left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

or
$$x^2 + y^2 + 2a^2 \tan^{-1}\frac{x}{y} = 2c$$

or
$$x^2 + y^2 + 2a^2 \tan^{-1}\left(\frac{x}{y}\right) = K,$$

where $2c = K$ is any arbitrary constant.

Example 7.

Give geometrical meaning of the solution of the following differential equations:

(i) $\dfrac{dy}{dx} = m$

(ii) $\dfrac{dy}{dx} = -\dfrac{y}{x}$ [*M.D.U. 2011*]

Solution. (*i*) The given differential equation is [*M.D.U. 2010*]

$$\frac{dy}{dx} = m$$

which can be written as $\qquad dy = m\,dx$

Integrating, we get $\qquad y = mx + c,$

where c is a constant of integration which can take any constant value.

Thus the equation $y = mx + c$ represents a family of straight lines such that slope of each line is m and c is the intercept on y-axis whose value is different for different lines.

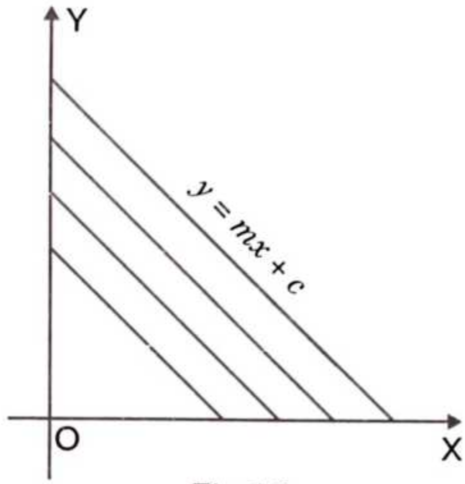

Fig. 1.2

(*ii*) The given differential equation is

$$\frac{dy}{dx} = -\frac{y}{x}$$

In variables separable form, we have

$$\frac{dy}{y} = -\frac{dx}{x}$$

or $$\frac{dx}{x} + \frac{dy}{y} = 0 \qquad ...(1)$$

Integrating (1), we have

$$\log x + \log y = \log c, \text{ where } \log c \text{ is constant of integration.}$$

or $$\log xy = \log c$$

or $$xy = c$$

As c can take any constant value, therefore $xy = c$ represents a family of hyperbolas such that co-ordinates axes are its asymptotes.

Example 8.

Give geometrical meaning of the solution of the differential equation

$$\frac{dy}{dx} + \frac{ax + hy + g}{hx + by + f} = 0.$$

Solution. The given differential equation is

$$\frac{dy}{dx} + \frac{ax + hy + g}{hx + by + f} = 0$$

or $$(ax + hy + g)\,dx + (hx + by + f)\,dy = 0 \qquad ...(1)$$

Comparing equation (1) with $M\,dx + N\,dy = 0$, we have

$$M = ax + hy + g \quad \text{and} \quad N = hx + by + f$$

$$\therefore \qquad \frac{\partial M}{\partial y} = h \quad \text{and} \quad \frac{\partial N}{\partial x} = h$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, therefore equation (1) is exact.

Hence the solution is $\displaystyle\int_{y\text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c_1$

or $$\int_{y\text{ constant}} (ax + hy + g)\,dx + \int (by + f)\,dy = c_1$$

or $$\frac{ax^2}{2} + hyx + gx + \frac{by^2}{2} + fy = c_1$$

or $$ax^2 + 2hxy + 2gx + by^2 + 2fy = 2c_1$$

or $$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

which is the solution of the given equation. It is a second degree equation in x and y and hence represents a family of conics.

EXERCISE 1.1

***Solve the following differential equations* [Q. *1 – 5*] :**

1. (*i*) $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$

(*ii*) $y \sin 2x\,dx - (y^2 + \cos^2 x)\,dy = 0$

(*iii*) $[\cos x \tan y + \cos (x + y)]\,dx + [\sin x \sec^2 y + \cos (x + y)]\,dy = 0.$

2. (*i*) $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$

(*ii*) $\left[y\left(1 + \dfrac{1}{x}\right) + \cos y\right]dx + [x + \log x - x \sin y]\,dy = 0$

(*iii*) $\cos x\,(\cos x - \sin a \sin y)\,dx + \cos y\,(\cos y - \sin a \sin x)\,dy = 0.$

3. (*i*) $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0$ [*M.D.U. 2012, 10*]

(*ii*) $(x^2 - 2xy + 3y^2)\,dx + (4y^3 + 6xy - x^2)\,dy = 0$

(*iii*) $(x^2 + y^2 + e^x)\,dx + 2xy\,dy = 0$

4. (i) $(2x^2y^3 + xy^2 + 3y)\,dx + (2x^3y^2 + x^2y + 3x)\,dy = 0$

(ii) $x\,(1 + y^2)\,dx + y\,(1 + x^2)\,dy = 0.$

(iii) $(a^2 - 2xy - y^2)\,dx - (x + y)^2\,dy = 0$ [M.D.U. 2001]

5. $(x^3 + 3xy^2)\,dx + (3x^2y + y^3)\,dy = 0$ [M.D.U. 2006]

ANSWERS

1. (i) $(e^y + 1)\sin x = c$ (ii) $\dfrac{y\cos 2x}{2} + \dfrac{y^3}{3} + c = 0$

(iii) $\sin x \tan y + \sin(x + y) = c.$

2. (i) $\tan y \tan x = c$ (ii) $xy + y\log x + x\cos y = c$

(iii) $2(x + y) + \sin 2x + \sin 2y - 4\sin a \sin y \sin x = \mathrm{K}.$

3. (i) $x^2 + y^2 - 2\tan^{-1}\dfrac{x}{y} = \mathrm{K}$ (ii) $\dfrac{1}{3}x^3 - x^2y + 3y^2x + y^4 = c$

(iii) $\dfrac{x^3}{3} + xy^2 + e^x = c.$

4. (i) $4x^3y^3 + 3x^2y^2 + 18xy = 6c$ (ii) $x^2 + (x^2 + 1)\,y^2 = 2c$

(iii) $a^2x - xy^2 - x^2y - \dfrac{y^3}{3} = c$ 5. $x^4 + 6x^2y^2 + y^4 = c$

1.6. INTEGRATING FACTOR

Definition. *If a differential equation is not exact and it becomes exact after it has been multiplied by some suitable function of x and y, then such a function is called an* ***integrating factor****.*

Integrating factor is denoted by **I.F.**

1.6.1. Theorem. Number of Integrating Factors.

To show that there is an infinite number of integrating factors for an equation $M\,dx + N\,dy = 0$.

Proof. Let μ be an integrating factor of $\mathrm{M}\,dx + \mathrm{N}\,dy = 0$ ...(1)

Then, for suitable function u of x and y, we have

$$\mu\,(\mathrm{M}\,dx + \mathrm{N}\,dy) = du \qquad ...(2)$$

$\therefore$ From (1) and (2), we have $du = 0.$

On integration, we have $u = c$, which is a solution of (2).

Multiplying (2) throughout by $f(u)$, a function of u, we get

$$\mu f(u) [M\, dx + N\, dy] = f(u)\, du$$

which is an exact differential equation.

$\therefore$ $\mu f(u)$ is an I.F. of equation (2).

Since $f(u)$ is an arbitrary function of u, hence the number of integrating factors is infinite.

1.7. INTEGRATING FACTOR BY INSPECTION

Sometimes an integrating factor can be found by inspection. For this, the reader should study the following results :

	Group of Terms	*I.F.*	*Exact Differential*
1.	$x\, dy - y\, dx$	$\frac{1}{x^2}$	$\frac{x\, dy - y\, dx}{x^2} = d\left(\frac{y}{x}\right)$
2.	$y\, dx - x\, dy$	$\frac{1}{y^2}$	$\frac{y\, dx - x\, dy}{y^2} = d\left(\frac{x}{y}\right)$
3.	$x\, dy - y\, dx$	$\frac{1}{xy}$	$\frac{dy}{y} - \frac{dx}{x} = d\left(\log \frac{y}{x}\right)$
4.	$x\, dy - y\, dx$	$\frac{1}{x^2 + y^2}$	$\frac{x\, dy - y\, dx}{x^2 + y^2} = \frac{\frac{x\, dy - y\, dx}{x^2}}{1 + \left(\frac{y}{x}\right)^2} = d\left(\tan^{-1} \frac{y}{x}\right)$
5.	$x\, dy + y\, dx$	$\frac{1}{(xy)^n}$	$\frac{x\, dy + y\, dx}{xy} = d\,[\log (xy)]$ for $n = 1$
6.	$x\, dx + y\, dy$	$\frac{1}{(x^2 + y^2)^n}$	$\frac{x\, dx + y\, dy}{(x^2 + y^2)^n} = d\left[\frac{-1}{2\,(n-1)\,(x^2 + y^2)^{n-1}}\right]$ or $\frac{x\, dx + y\, dy}{x^2 + y^2} = d\left[\frac{1}{2} \log (x^2 + y^2)\right]$ if $n = 1$
7.	$ye^x\, dx - e^x\, dy$	$\frac{1}{y^2}$	$\frac{y\, e^x\, dx - e^x\, dy}{y^2} = d\left[\frac{e^x}{y}\right]$

SOLVED EXAMPLES

Example 1.

Prove that $\frac{1}{x^2y^2}$ ***is the integrating factor of the equation*** [M.D.U. 2016]

$$(1 + xy)\, y\, dx + (1 - xy)\, x\, dy = 0$$

Solution. The given differential equation is

$$(1 + xy)\, y\, dx + (1 - xy)\, x\, dy = 0 \qquad \text{...(1)}$$

Comparing it with M dx + N dy = 0, we get

$$M = (1 + xy)\, y \quad \text{and} \quad N = (1 - xy)\, x$$

$$\therefore \quad \frac{\partial M}{\partial y} = 1 + 2xy \quad \text{and} \quad \frac{\partial N}{\partial x} = 1 - 2xy$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

∴ Equation (1) is not an exact equation.

Multiplying (1) by $\frac{1}{x^2y^2}$, we get

$$\frac{1 + xy}{x^2y}\, dx + \frac{1 - xy}{xy^2}\, dy = 0$$

or

$$\left(\frac{1}{x^2y} + \frac{1}{x}\right) dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right) dy = 0 \qquad \text{...(2)}$$

Comparing (2) with M dx + N dy = 0, we have

$$M = \frac{1}{x^2y} + \frac{1}{x} \quad \text{and} \quad N = \frac{1}{xy^2} - \frac{1}{y}$$

$$\therefore \quad \frac{\partial M}{\partial y} = -\frac{1}{x^2y^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = -\frac{1}{x^2y^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

∴ Equation (2) is an exact equation.

Hence, $\frac{1}{x^2y^2}$ is an integrating factor of equation (1).

Example 2.

Find the value of α, if y^α is an integrating factor of the differential equation $2xy\,dx - (3x^2 - y^2)\,dy = 0$.

Solution. The given differential equation is

$$2xy\,dx - (3x^2 - y^2)\,dy = 0 \quad \text{...(1)}$$

It is given that y^α is an I.F. of (1), so multiplying (1) by y^α, we get

$$2xy\,y^\alpha\,dx - (3x^2 - y^2)\,y^\alpha\,dy = 0 \quad \text{...(2)}$$

Comparing (2) with M dx + N dy = 0, we get

$$M = 2xy^{\alpha+1} \text{ and } N = -3x^2y^\alpha + y^{\alpha+2}$$

$$\therefore \quad \frac{\partial M}{\partial y} = (\alpha + 1)\,2xy^\alpha \text{ and } \frac{\partial N}{\partial x} = -6xy^\alpha$$

As the given differential equation (2) is exact, so $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$\therefore \quad 2(\alpha + 1)xy^\alpha = -6xy^\alpha$$

$$\Rightarrow \quad 2(\alpha + 1) = -6 \Rightarrow \alpha + 1 = -3 \Rightarrow \alpha = -4.$$

Example 3.

Show that $\dfrac{1}{x^2}$ is the integrating factor of the equation $(x^2 + y^2)\,dx - 2xy\,dy = 0$. Also find its solution.

[K.U. 2013]

Solution. The given differential equation is $(x^2 + y^2)\,dx - 2xy\,dy = 0$ ...(1)

Comparing it with Mdx + Ndy = 0, we get

$$M = x^2 + y^2 \quad \text{and} \quad N = -2xy$$

$$\therefore \quad \frac{\partial M}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = -2y$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, therefore equation (1) is not an exact equation.

Multiplying (1) by $\dfrac{1}{x^2}$, we have

$$\left(1 + \frac{y^2}{x^2}\right) dx - 2\,\frac{y}{x}\,dy = 0 \quad \text{...(2)}$$

Comparing equation (2) with M dx + N dy = 0, we have

$$M = 1 + \frac{y^2}{x^2} \quad \text{and} \quad N = \frac{-2y}{x}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{2y}{x^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{2y}{x^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore equation (2) is an exact equation.

Hence the solution is given by

$$\int_{y \text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c$$

i.e.,
$$\int_{y \text{ constant}} \left(1 + \frac{y^2}{x^2}\right) dx + 0 = c$$

i.e.,
$$x - \frac{y^2}{x} = c \quad \Rightarrow \quad x^2 - y^2 = cx.$$

Hence, I.F. $= \frac{1}{x^2}$ and the solution is $x^2 - y^2 = cx$.

Example 4.

Find the integrating factor and solve the differential equation

$$\boldsymbol{y\,(2xy + e^x)\,dx - e^x\,dy = 0.}$$

Solution. The given differential equation is $y\,(2xy + e^x)\,dx - e^x\,dy = 0$ ...(1)

Comparing it with $Mdx + Ndy = 0$, we have

$$M = 2xy^2 + ye^x \quad \text{and} \quad N = -e^x$$

$$\therefore \quad \frac{\partial M}{\partial y} = 4xy + e^x \quad \text{and} \quad \frac{\partial N}{\partial x} = -e^x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore equation (1) is not an exact equation.

Multiplying (1) by $\frac{1}{y^2}$, we get

$$\left(\frac{2xy + e^x}{y}\right) dx - \frac{e^x}{y^2}\,dy = 0$$

or
$$\left(2x + \frac{e^x}{y}\right) dx - \frac{e^x}{y^2}\,dy = 0 \qquad \text{...(2)}$$

Comparing (2) with $Mdx + Ndy = 0$, we have

$$M = 2x + \frac{e^x}{y} \quad \text{and} \quad N = -\frac{e^x}{y^2}$$

$$\therefore \quad \frac{\partial M}{\partial y} = 0 - \frac{e^x}{y^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = -\frac{e^x}{y^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore equation (2) is an exact equation.

Hence the solution is $\int_{y \text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c$

i.e., $$\int \left(2x + \frac{e^x}{y}\right) dx = c$$

or $$x^2 + \frac{e^x}{y} = c$$

Hence I.F. $= \dfrac{1}{y^2}$ and the solution is $x^2 + \dfrac{e^x}{y} = c$.

Aliter : From equation (2) above, we have

$$2x\,dx + \frac{y e^x dx - e^x dy}{y^2} = 0$$

or $$2x\,dx + d\left(\frac{e^x}{y}\right) = 0$$

Integrating, we have $x^2 + \dfrac{e^x}{y} = c$, which is the required solution.

EXERCISE 1.2

Find the integrating factors and solve the following differential equations :

1. $y\,dx - x\,dy = 0.$
2. $y\,dx - x\,dy + \log x\,dx = 0$ [K.U. 2010]
3. $a\,(x\,dy + 2y\,dx) = xy\,dy.$
4. $(y^2 e^x + 2xy)\,dx - x^2\,dy = 0.$

$dx\,(2ay) + dy\,(2x - xy) = 0.$

$2a - a + 2$

ANSWERS

1. $\dfrac{1}{xy}$ or $\dfrac{1}{y^2}$ or $\dfrac{1}{x^2}$; $y = cx$.
2. $\dfrac{1}{x^2}$; $cx + y + \log x + 1 = 0.$
3. $\dfrac{1}{xy}$; $2a \log x + a \log y - y = c.$
4. $\dfrac{1}{y^2}$; $e^x + \dfrac{x^2}{y} = c.$

1.8. RULES FOR FINDING THE INTEGRATING FACTORS

1.8.1. Rule 1. ***If M (x, y) and N (x, y) are homogeneous functions in x, y and the equation M dx + N dy = 0 is not exact, then*** $\dfrac{1}{Mx + Ny}$ ***is an I.F., provided*** $Mx + Ny \neq 0$.

Proof. The given equation is $Mdx + Ndy = 0$ ...(1)

As equation (1) is not exact, therefore we have to find the integrating factor.

Suppose that $\dfrac{1}{Mx + Ny}$ is an integrating factor.

Multiplying equation (1) by I.F., we get

$$\frac{M}{Mx + Ny}\,dx + \frac{N}{Mx + Ny}\,dy = 0 \qquad \text{...(2)}$$

Now, $\dfrac{\partial}{\partial y}\left(\dfrac{M}{Mx + Ny}\right) - \dfrac{\partial}{\partial x}\left(\dfrac{N}{Mx + Ny}\right)$

$$= \frac{(Mx + Ny)\left(\dfrac{\partial M}{\partial y}\right) - M\left[x\dfrac{\partial M}{\partial y} + N + y\dfrac{\partial N}{\partial y}\right]}{(Mx + Ny)^2} - \frac{(Mx + Ny)\dfrac{\partial N}{\partial x} - N\left[x\dfrac{\partial M}{\partial x} + M + y\dfrac{\partial N}{\partial x}\right]}{(Mx + Ny)^2}$$

$$\therefore\ \frac{\partial}{\partial y}\left(\frac{M}{Mx + Ny}\right) - \frac{\partial}{\partial x}\left(\frac{N}{Mx + Ny}\right) = \frac{N\left(x\dfrac{\partial M}{\partial x} + y\dfrac{\partial M}{\partial y}\right) - M\left(x\dfrac{\partial N}{\partial x} + y\dfrac{\partial N}{\partial y}\right)}{(Mx + Ny)^2} \qquad \text{...(3)}$$

It is given that M and N are homogeneous functions in x, y of degree k (*say*).

$\therefore$ By Euler's thoerem, we have

$$x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y} = kM \ \text{ and } \ x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y} = kN$$

$\therefore$ Equation (3) becomes

$$\frac{\partial}{\partial y}\left(\frac{M}{Mx + Ny}\right) - \frac{\partial}{\partial x}\left(\frac{M}{Mx + Ny}\right) = \frac{MNk - MNk}{(Mx + Ny)^2} = 0$$

$$\Rightarrow \qquad \frac{\partial}{\partial y}\left(\frac{M}{Mx + Ny}\right) = \frac{\partial}{\partial x}\left(\frac{M}{Mx + Ny}\right)$$

$\therefore$ Equation $\dfrac{M}{Mx + Ny}\,dx + \dfrac{N}{Mx + Ny}\,dy = 0$ is an exact equation.

Hence, $\dfrac{1}{Mx + Ny}$ is an I.F. of equation (1).

SOLVED EXAMPLES

Example 1.

Find the integraing factor of the differential equation

$$(y^2 - 3xy)\,dx + (x^2 - xy)\,dy = 0.$$

Solution. The given differential equation is

$$(y^2 - 3xy)\,dx + (x^2 - xy)\,dy = 0 \qquad ...(1)$$

Comparing equation (1) with $M\,dx + N\,dy = 0$, we have

$$M = y^2 - 3xy \text{ and } N = x^2 - xy$$

$$\therefore \qquad \frac{\partial M}{\partial y} = 2y - 3x \text{ and } \frac{\partial N}{\partial x} = 2x - y$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, so the given equation (1) is not exact.

Now, $Mx + Ny = xy^2 - 3x^2 y + x^2y - xy^2 = -2x^2y \neq 0$

As equation (1) is homogeneous in x, y, therefore

$$\text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{-2x^2y}. \qquad \text{[Using Rule 1]}$$

Example 2.

Solve $x^2y\,dx - (x^3 + y^3)\,dy = 0.$

Solution. The given differential equation is $x^2y\,dx - (x^3 + y^3)\,dy = 0$ ...(1)

Comparing it with $M\,dx + N\,dy = 0$, we have

$$M = x^2y \qquad \text{and} \qquad N = -x^3 - y^3$$

$$\therefore \qquad \frac{\partial M}{\partial y} = x^2 \qquad \text{and} \qquad \frac{\partial N}{\partial x} = -3x^2$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore equation (1) is not exact.

As equation (1) is homogeneous in x and y, so

$$\text{I.F.} = \frac{1}{Mx + Ny}$$

Now, $\qquad Mx + Ny = (x^2y)\,x + (-x^3 - y^3)\,y = -y^4 \neq 0$

$$\therefore \qquad \text{I.F.} = -\frac{1}{y^4}$$

Multiplying equation (1) by $-\dfrac{1}{y^4}$, we get

$$-\frac{x^2}{y^3}\,dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right) dy = 0 \qquad \text{...(2)}$$

Now, equation (2) is an exact equation $\left[\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = \dfrac{3x^2}{y^4}\right]$

Hence the solution is $\displaystyle\int_{y\text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c,$

where c is any arbitrary constant.

i.e., $$\int_{y\text{ constant}} -\frac{x^2}{y^3}\,dx + \int \frac{1}{y}\,dy = c$$

or $$-\frac{1}{y^3}\int x^2\,dx + \int \frac{1}{y}\,dy = c$$

or $$-\frac{x^3}{3y^3} + \log y = c.$$

EXERCISE 1.3

Solve the following differential equations :

1. $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$ [K.U. 2014]

2. $(3xy^2 - y^3)\,dx - (2x^2y - xy^2)\,dy = 0.$ [K.U. 2015]

3. $(x^4 + y^4)\,dx - xy^3\,dy = 0.$

4. $y^2\,dx + (x^2 - xy - y^2)\,dy = 0.$

ANSWERS

1. $\dfrac{x}{y} - 2\log x + 3\log y = c.$

2. $3\log x + \dfrac{y}{x} - 2\log y = c.$

3. $y^4 = 4x^4 \log x + cx^4.$

4. $(x - y)\,y^2 = c\,(x + y).$

1.8.2. Rule 2. ***If the equation $M\,dx + N\,dy = 0$ is not exact and is of the form $f(xy)\,y\,dx + g(xy)\,x\,dy = 0$, then $\dfrac{1}{Mx - Ny}$ is an integrating factor provided $Mx - Ny \neq 0$.***

Proof. The given equation is $M\,dx + N\,dy = 0$ ...(1)

As equation (1) is not exact, therefore we have to find the I.F.

Suppose that $\dfrac{1}{Mx - Ny}$ is an I.F.

Multiplying eqn. (1) by I.F., we get

$$\frac{M}{Mx - Ny}\,dx + \frac{N}{Mx - Ny}\,dy = 0$$

or
$$\frac{y\,f(xy)}{xy\,[f(xy) - g(xy)]}\,dx + \frac{x\,g(xy)}{xy\,[f(xy) - g(xy)]}\,dy = 0 \quad ...(2)$$

Now, $$\frac{\partial}{\partial y}\left(\frac{y\,f(xy)}{xy\,[f(xy) - g(xy)]}\right) - \frac{\partial}{\partial x}\left(\frac{x\,g(xy)}{xy\,[f(xy) - g(xy)]}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{f(xy)}{x\,[f(xy) - g(xy)]}\right) - \frac{\partial}{\partial x}\left(\frac{g(xy)}{y\,[f(xy) - g(xy)]}\right)$$

$$= \left(\frac{[f(xy) - g(xy)]\,f'(xy)\,x - f(xy)\,[f'(xy)\,x - g'(xy)\,x]}{x\,[f(xy) - g(xy)]^2}\right)$$

$$- \left(\frac{[f(xy) - g(xy)]\,g'(xy)\,y - g(xy)\,[f'(xy)\,y - g'(xy)\,y]}{y\,[f(xy) - g(xy)]^2}\right)$$

$$= \frac{x\,[f(xy)\,g'(xy) - g(xy)\,f'(xy)]}{x\,[f(xy) - g(xy)]^2} - \frac{y\,[f(xy)\,g'(xy) - g(xy)\,f'(xy)]}{y\,[f(xy) - g(xy)]^2}$$

$$\therefore \quad \frac{\partial}{\partial y}\left(\frac{y\,f(xy)}{xy\,[f(xy) - g(xy)]}\right) - \frac{\partial}{\partial x}\left(\frac{x\,g(xy)}{xy\,[f(xy) - g(xy)]}\right) = 0$$

Thus equation (2) is exact.

Hence to make equation (1) exact, $\dfrac{1}{Mx - Ny}$ is the I.F.

Example 1.

Solve $(xy^2 + 2x^2y^3)\,dx + (x^2y - x^3y^2)\,dy = 0.$ *[K.U. 2015, 12, 06; M.D.U. 2008, 07]*

Solution. The given differential equation is

$$(xy^2 + 2x^2y^3)\,dx + (x^2y - x^3y^2)\,dy = 0 \quad ...(1)$$

Comparing equation (1) with $M\,dx + N\,dy = 0$, we observe that

$$M = xy^2 + 2x^2y^3 \quad \text{and} \quad N = x^2y - x^3y^2$$

$$\therefore \quad \frac{\partial M}{\partial y} = 2xy + 6x^2y^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = 2xy - 3x^2y^2$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, therefore the given equation (1) is not exact.

Also $\quad Mx - Ny = (xy^2 + 2x^2y^3)\,x - (x^2y - x^3y^2)\,y = 3x^3y^3 \neq 0.$

Equation (1) can be written as

$$(xy + 2x^2y^2)\,y\,dx + (xy - x^2y^2)\,x\,dy = 0 \qquad ...(2)$$

$\therefore$ Equation (2) is of the form

$$y\,f(xy)\,dx + x\,g(xy)\,dy = 0$$

$$\therefore \qquad \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$$

Multiplying equation (1) by $\dfrac{1}{3x^3y^3}$, we have

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0,$$

which is an exact equation. $\left[\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = -\dfrac{1}{3x^2y^2}\right]$

Hence the solution is $\displaystyle\int_{y\text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c_1,$

where c_1 is any arbitrary constant.

or $$\int_{y\text{ constant}} \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int -\frac{1}{3y}\,dy = c_1$$

or $$\frac{1}{3y}\int \frac{1}{x^2}\,dx + \frac{2}{3}\int \frac{1}{x}\,dx - \frac{1}{3}\int \frac{1}{y}\,dy = c_1$$

or $$-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c_1$$

or $$-\frac{1}{xy} + 2\log x - \log y = 3c_1$$

or $$-\frac{1}{xy} + 2\log x - \log y = c.$$

EXERCISE 1.4

Solve the following differential equations :

1. $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0.$
2. $(x^2y^2 + xy + 1)\,y\,dx + (x^2y^2 - xy + 1)\,x\,dy = 0.$ [M.D.U. 2009, 2000; K.U. 2005]
3. $(x^4y^4 + x^2y^2 + xy)\,y\,dx + (x^4y^4 - x^2y^2 + xy)\,x\,dy = 0.$ [M.D.U. 2015]
4. $(xy \sin xy + \cos xy)\,y\,dx + (xy \sin xy - \cos xy)\,x\,dy = 0.$ [M.D.U. 2016]

ANSWERS

1. $-\dfrac{1}{xy}+\log\dfrac{x}{y}=c$
2. $xy-\dfrac{1}{xy}+\log\dfrac{x}{y}=c$
3. $\dfrac{1}{2}x^2y^2-\dfrac{1}{xy}+\log\dfrac{x}{y}=c$
4. $x\sec(xy)=cy$ or $x=cy\cos(xy)$

1.8.3. Rule 3. ***If the equation $M(x, y)dx + N(x, y)\,dy = 0$ is not exact and $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}$ is a function of x only $= f(x)$ (say), then $e^{\int f(x)\,dx}$ is an integrating factor.***

Proof. The given equation is

$$M(x, y)\,dx + N(x, y)\,dy = 0 \qquad ...(1)$$

As equation (1) is not exact, therefore we have to find the integrating factor.

Suppose that $\mu(x)$ is an integrating factor, where $\mu(x)$ depends upon x only.

Multiplying equation (1) by $\mu(x)$, we get

$$\mu(x)\,M(x, y)\,dx + \mu(x)\,N(x, y)\,dy = 0$$

This equation is exact if and only if

$$\frac{\partial}{\partial y}[\mu(x)\,M(x, y)] = \frac{\partial}{\partial x}[\mu(x)N(x, y)]$$

or
$$\mu(x)\frac{\partial}{\partial y}[M(x, y)] = \mu(x)\frac{\partial}{\partial x}[N(x, y)] + N(x, y)\frac{\partial \mu(x)}{\partial x}$$

or
$$\frac{d}{dx}\mu(x) = \mu(x)\left[\frac{\dfrac{\partial}{\partial y}M(x, y)-\dfrac{\partial}{\partial x}N(x, y)}{N(x, y)}\right] \qquad ...(2)$$

$$\left[\frac{\partial \mu(x)}{\partial x}=\frac{d\mu(x)}{dx}\ \textit{as } \mu(x) \textit{ is a function of } x \textit{ only}\right]$$

Integrating (2), we get

$$\int\frac{d\mu(x)}{\mu(x)} = \int\frac{\dfrac{\partial}{\partial y}M-\dfrac{\partial}{\partial x}N}{N}\,dx$$

i.e.,
$$\log\mu(x) = \int\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}\,dx$$

$$\Rightarrow \qquad \mu(x) = e^{\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx}$$

$$\therefore \qquad \text{I.F.} = e^{\int \frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N} dx} = e^{\int f(x)\, dx}. \qquad \left[\because \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)\right]$$

SOLVED EXAMPLES

Example 1.

Find the integrating factor of the differential equation

$$\boldsymbol{2 \cos (y^2)\, dx - xy \sin (y^2)\, dy = 0.}$$

Solution. The given differential equation is

$$2 \cos (y^2)\, dx - xy \sin (y^2)\, dy = 0 \qquad \text{...(1)}$$

Comparing equation (1) with M dx + N dy = 0, we have

$$M = 2 \cos y^2 \quad \text{and} \quad N = -xy \sin y^2$$

$$\therefore \qquad \frac{\partial M}{\partial y} = -4y \sin y^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = -y \sin y^2$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore equation (1) is not exact.

$$\text{Now,} \qquad \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-4y \sin y^2 + y \sin y^2}{-xy \sin y^2} = \frac{3}{x} = f(x),$$

which is a function of x only.

$$\therefore \qquad \text{I.F.} = e^{\int f(x)\, dx} = e^{\int \frac{3}{x} dx} = e^{3 \log x} = x^3. \qquad [\textit{Using Rule (3)}]$$

Example 2.

Solve $\boldsymbol{(x^2 + y^2 + 2x)\, dx + 2y\, dy = 0.}$ [*K.U. 2013, 08; M.D.U. 2009, 07*]

Solution. The given differential equation is

$$(x^2 + y^2 + 2x)\, dx + 2y\, dy = 0. \qquad \text{...(1)}$$

Comparing (1) with M dx + N dy = 0, we have

$$M = x^2 + y^2 + 2x \quad \text{and} \quad N = 2y$$

$$\therefore \qquad \frac{\partial M}{\partial y} = 2y \qquad \text{and} \qquad \frac{\partial N}{\partial x} = 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore equation (1) is not exact.

Now, $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y - 0}{2y} = 1 = f(x)$$

$\therefore$ $$\text{I.F.} = e^{\int f(x)\,dx} = e^{\int 1 . dx} = e^x$$

Multiplying equation (1) by e^x, we have

$$(x^2e^x + y^2e^x + 2xe^x)\,dx + 2ye^x\,dy = 0 \quad ...(2$$

Now equation (2) is exact. $\left[\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2ye^x\right]$

Hence the solution is $\int\limits_{y \text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c,$

where c is any arbitrary constant.

i.e., $$\int\limits_{y \text{ constant}} (x^2e^x + y^2e^x + 2xe^x)\,dx = c$$

or $$\int (x^2 + 2x)\,e^x\,dx + y^2 \int e^x\,dx = c$$

or $$x^2e^x + y^2e^x = c. \qquad \left[\because \int [f(x) + f'(x)]\,e^x\,dx = f(x)\,e^x\right]$$

Example 3.

Solve the differential equation

$$\boldsymbol{(2y \sin x + 3y^4 \sin x \cos x)\,dx - (4y^3 \cos^2 x + \cos x)\,dy = 0.} \qquad \text{[M.D.U. 2015]}$$

Solution. The given differential equation is

$$(2y \sin x + 3y^4 \sin x \cos x)\,dx - (4y^3 \cos^2 x + \cos x)\,dy = 0 \quad ...(1)$$

Comparing equation (1) with $M\,dx + N\,dy = 0$, we have

$$M = 2y \sin x + 3y^4 \sin x \cos x \quad \text{and} \quad N = -4y^3 \cos^2 x - \cos x$$

$$\therefore \quad \frac{\partial M}{\partial y} = 2 \sin x + 12y^3 \sin x \cos x \quad \text{and} \quad \frac{\partial N}{\partial x} = 8y^3 \cos x \sin x + \sin x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, so equation (1) is not exact.

Now, $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2 \sin x + 12y^3 \sin x \cos x - 8y^3 \cos x \sin x + \sin x}{-4y^3 \cos^2 x - \cos x}$$

$$= \frac{\sin x + 4y^3 \sin x \cos x}{-4y^3 \cos^2 x - \cos x}$$

$$= \frac{\sin x\,(1 + 4y^3 \cos x)}{-\cos x\,(1 + 4y^3 \cos x)} = -\tan x = f(x)$$

which is a function of x only.

$$\therefore \qquad \text{I.F.} = e^{\int f(x)\,dx}$$

$$= e^{-\int \tan x\,dx} = e^{\log \cos x} = \cos x$$

Multiplying equation (1) by $\cos x$, we get

$$(2y \sin x \cos x + 3y^4 \sin x \cos^2 x)\,dx - (4y^3 \cos^3 x + \cos^2 x)\,dy = 0 \qquad \text{...(2)}$$

Comparing (2) with $M\,dx + N\,dy = 0$, we have

$$M = 2y \sin x \cos x + 3y^4 \sin x \cos^2 x \quad \text{and} \quad N = -4y^3 \cos^3 x - \cos^2 x$$

$$\therefore \quad \frac{\partial M}{\partial y} = 2 \sin x \cos x + 12y^3 \sin x \cos^2 x \quad \text{and} \quad \frac{\partial N}{\partial x} = 12y^3 \sin x \cos^2 x + 2 \sin x \cos x$$

Here $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore equation (2) is exact.

Hence the solution is

$$\int_{y \text{ constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c$$

i.e.,

$$\int_{y \text{ constant}} (2y \sin x \cos x + 3y^4 \sin x \cos^2 x)\,dx + \int 0\,dy = c$$

or

$$\frac{-2y \cos^2 x}{2} - \frac{3y^4 \cos^3 x}{3} = c$$

or

$$y \cos^2 x + y^4 \cos^3 x = c_1.$$

EXERCISE 1.5

Solve the following differential equations :

1. $(x^2 + y^2 + x)\,dx + xy\,dy = 0.$
2. $(xy^2 - e^{x^{1/3}})\,dx - x^2 y\,dy = 0.$
3. $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right) dx + \frac{1}{4}(x + xy^2)\,dy = 0$ [K.U. 2017]
4. $(x^2 + y^2 + 1)\,dx - 2xy\,dy = 0.$ [M.D.U. 2017, 14, 05]

ANSWERS

1. $3x^4 + 4x^3 + 6x^2y^2 = c$
2. $\frac{-y^2}{2x^2} + \frac{1}{3} e^{x^{1/3}} = c$
3. $3x^4y + x^4y^3 + x^6 = c$
4. $x^2 - y^2 = cx + 1.$

1.8.4. Rule 4. ***If the equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is not exact and $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ is a function of y only $= f(y)$ (say), then $e^{\int f(y)\,dy}$ is an integrating factor.***

Proof. The given equation is $M(x, y)\,dx + N(x, y)\,dy = 0$ ...(1)

As equation (1) is not exact, therefore we have to find the I.F.

Suppose that $\mu(y)$ is an integrating factor, where $\mu(y)$ depends upon y only.

Multiplying eqn. (1) by $\mu(y)$, we get $\mu(y)\,M\,dx + \mu(y)\,N\,dy = 0$ ...(2)

As eqn. (2) is exact, therefore

$$\frac{\partial}{\partial y}[\mu(y)\,M] = \frac{\partial}{\partial x}[\mu(y)\,N]$$

or
$$M\frac{\partial}{\partial y}\mu(y) + \mu(y)\frac{\partial M}{\partial y} = \mu(y)\frac{\partial N}{\partial x}$$

or
$$M\frac{d}{dy}\mu(y) + \mu(y)\frac{\partial M}{\partial y} = \mu(y)\frac{\partial N}{\partial x}$$

$$\left[\textit{Here } \frac{\partial}{\partial y}\mu(y) = \frac{d}{dy}\mu(y) \textit{ as } \mu(y) \textit{ is a function of } y \textit{ only}\right]$$

$$\Rightarrow \quad M\frac{d}{dy}\mu(y) = \mu(y)\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\Rightarrow \quad \frac{\frac{d}{dy}\mu(y)}{\mu(y)} = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \qquad ...(3)$$

Integrating (3), we get $\displaystyle\int \frac{d\mu(y)}{\mu(y)} = \int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dy$

$$\Rightarrow \quad \log \mu(y) = \int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dy$$

$\therefore \quad \mu(y) = e^{\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dy} = e^{\int f(y)\,dy}$, which is the required I.F.

SOLVED EXAMPLES

Example 1.

Solve $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$. [M.D.U. 2013, 03; K.U. 2011, 09]

Solution. The given differential equation is

$$(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0 \qquad ...(1)$$

Comparing equation (1) with $M\,dx + N\,dy = 0$, we observe that

$$M = 3x^2y^4 + 2xy \quad \text{and} \quad N = 2x^3y^3 - x^2$$

$$\therefore \quad \frac{\partial M}{\partial y} = 12x^2y^3 + 2x \quad \text{and} \quad \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore equation (1) is not exact.

Now,
$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{6x^2y^3 - 2x - 12x^2y^3 - 2x}{xy\,(3xy^3 + 2)}$$

$$= -\frac{2x\,(3xy^3 + 2)}{xy\,(3xy^3 + 2)} = \frac{-2}{y} = f(y)$$

which is a function of y only.

$$\therefore \quad \text{I.F.} = e^{\int f(y)\,dy} = e^{\int -\frac{2}{y}\,dy} = e^{-2\log y} = e^{\log y^{-2}}$$

$$= e^{\log \frac{1}{y^2}} = \frac{1}{y^2}$$

Multiplying both sides of equation (1) by $\frac{1}{y^2}$, we get

$$\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0 \qquad \text{...(2)}$$

Now equation (2) is exact. $\left[\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 6x^2y - \frac{2x}{y^2}\right]$

Hence the solution is $\displaystyle\int_{y\text{ constant}} \left(3x^2y^2 + \frac{2x}{y}\right)dx + 0 = c$, where c is any arbitrary constant.

or
$$x^3y^2 + \frac{x^2}{y} = c$$

or
$$x^3y^3 + x^2 = cy.$$

EXERCISE 1.6

Solve the following differential equations :

1. $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$
2. $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4\,e^y - x^2y^2 - 3x)\,dy = 0$ [M.D.U. 2008]
3. $(xy^3 + y)\,dx + 2\,(x^2y^2 + x + y^4)\,dy = 0$ [M.D.U. 2012]
4. $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3 + y^2)\,dy = 0.$ [M.D.U. 2013, 07; K.U. 2011]

ANSWERS

1. $xy + y^2 + \frac{2x}{y^2} = c$

2. $x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$

3. $3x^2y^4 + 6xy^2 + 2y^6 = c$

4. $e^{6y}\left(\frac{x^2y^2}{2} - \frac{x^3}{3} + \frac{y^2}{6} - \frac{y}{18} + \frac{1}{108}\right) = c.$

1.8.5. Rule 5. ***If the equation $Mdx + Ndy = 0$ can be expressed as $x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0$, where a, b, c, d, m, n, p, q are constants and $\frac{m}{n} \neq \frac{p}{q}$, then $x^\alpha y^\beta$ is an integrating factor, where α and β are so chosen that $\frac{a+\alpha+1}{m} = \frac{b+\beta+1}{n}$ and $\frac{c+\alpha+1}{p} = \frac{d+\beta+1}{q}$.***

Proof. The given differential equation is

$$x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0 \quad ...(1)$$

Suppose $x^\alpha y^\beta$ is an integrating factor of (1).

Multiplying eqn. (1) by I.F., we get

$$(mx^{a+\alpha} y^{b+\beta+1}\,dx + nx^{a+\alpha+1} y^{b+\beta}\,dy) + (px^{c+\alpha} y^{d+\beta+1}\,dx + qx^{c+\alpha+1} y^{d+\beta}\,dy) = 0$$

or $$(mx^{a+\alpha} y^{b+\beta+1} + px^{c+\alpha} y^{d+\beta+1})\,dx + (nx^{a+\alpha+1} y^{b+\beta} + qx^{c+\alpha+1} y^{d+\beta})\,dy = 0$$

which is of the form $M\,dx + N\,dy = 0$.

Now,
$$\begin{aligned}
\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} &= \frac{\partial}{\partial y}(mx^{a+\alpha} y^{b+\beta+1} + px^{c+\alpha} y^{d+\beta+1}) \\
&\quad - \frac{\partial}{\partial x}(nx^{a+\alpha+1} y^{b+\beta} + qx^{c+\alpha+1} y^{d+\beta}) \\
&= [(b+\beta+1)\,mx^{a+\alpha} y^{b+\beta} + (d+\beta+1)\,px^{c+\alpha} y^{d+\beta}] \\
&\quad - [(a+\alpha+1)\,ny^{b+\beta} x^{a+\alpha} + (c+\alpha+1)qy^{d+\beta} x^{c+\alpha}] \\
&= [(b+\beta+1)\,m - (a+\alpha+1)n]\, y^{b+\beta} x^{a+\alpha} \\
&\quad + [(d+\beta+1)\,p - (c+\alpha+1)q]\, y^{d+\beta} x^{c+\alpha}
\end{aligned}$$

$$= mn\left[\frac{(b+\beta+1)}{n} - \frac{(a+\alpha+1)}{m}\right] x^{a+\alpha} y^{b+\beta} + pq\left[\frac{(d+\beta+1)}{q} - \frac{(c+\alpha+1)}{p}\right] x^{c+\alpha} y^{d+\beta}$$

$$= 0 + 0 = 0 \qquad \left[\because \frac{a+\alpha+1}{m} = \frac{b+\beta+1}{n} \text{ and } \frac{c+\alpha+1}{p} = \frac{d+\beta+1}{q}\right]$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$$

Thus, equation (2) is an exact differential equation.

Hence $x^\alpha y^\beta$ is an I.F., provided $\frac{a+\alpha+1}{m} = \frac{b+\beta+1}{n}$ and $\frac{c+\alpha+1}{p} = \frac{d+\beta+1}{q}$.

SOLVED EXAMPLES

Example 1.

***Solve* : $(2x^2y - 3y^4)\,dx + (3x^3 + 2xy^3)\,dy = 0.$** [K.U. 2016; M.D.U. 2014, 11]

Solution. The given equation is

$$(2x^2y - 3y^4)\,dx + (3x^3 + 2xy^3)\,dy = 0. \qquad ...(1)$$

Comparing equation (1) with $M\,dx + N\,dy = 0$, we observe that

$$M = 2x^2y - 3y^4 \quad \text{and} \quad N = 3x^3 + 2xy^3$$

$$\therefore \quad \frac{\partial M}{\partial y} = 2x^2 - 12y^3 \quad \text{and} \quad \frac{\partial N}{\partial x} = 9x^2 + 2y^3$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore equation (1) is not exact.

Now equation (1) can be written as

$$x^2(2y\,dx + 3x\,dy) + y^3(-3y\,dx + 2x\,dy) = 0$$

which is of the form

$$x^ay^b(my\,dx + nx\,dy) + x^cy^d(py\,dx + qx\,dy) = 0,$$

where $a = 2,\ b = 0,\ m = 2,\ n = 3,\ c = 0,\ d = 3,\ p = -3$ and $q = 2.$

Let the integrating factor be $x^\alpha y^\beta$.

Multiplying equation (1) on both sides by the I.F. *i.e.*, $x^\alpha y^\beta$, we have

$$(2x^{\alpha+2}y^{\beta+1} - 3x^\alpha y^{\beta+4})\,dx + (3x^{\alpha+3}y^\beta + 2x^{\alpha+1}y^{\beta+3})\,dy = 0 \qquad ...(2)$$

Now comparing equation (2) with $M\,dx + N\,dy = 0$, we have

$$M = 2x^{\alpha+2}y^{\beta+1} - 3x^\alpha y^{\beta+4}$$

and

$$N = 3x^{\alpha+3}y^\beta + 2x^{\alpha+1}y^{\beta+3}$$

Then

$$\frac{\partial M}{\partial y} = 2(\beta+1)x^{\alpha+2}y^\beta - 3(\beta+4)x^\alpha y^{\beta+3}$$

and

$$\frac{\partial N}{\partial x} = 3(\alpha+3)x^{\alpha+2}y^\beta + 2(\alpha+1)x^\alpha y^{\beta+3}$$

Now equation (2) will be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

i.e., if $\quad 2(\beta+1)x^{\alpha+2}y^\beta - 3(\beta+4)x^\alpha y^{\beta+3} = 3(\alpha+3)x^{\alpha+2}y^\beta + 2(\alpha+1)x^\alpha y^{\beta+3}$

or if $\quad 2(\beta+1)=3(\alpha+3)$ [Comparing co-effs. of $x^{\alpha+2}y^{\beta}$]

and $\quad -3(\beta+4)=2(\alpha+1)$ [Comparing co-effs. of $x^{\alpha}y^{\beta+3}$]

or if $\quad \alpha=-\frac{49}{13}$ and $\beta=-\frac{28}{13}$

$\therefore$ $x^{-\frac{49}{13}}y^{-\frac{28}{13}}$ is the required integrating factor.

Mutliplying equation (1) by I.F. *i.e.*, $x^{-\frac{49}{13}}y^{-\frac{28}{13}}$, we have

$$\left(2x^{-\frac{23}{13}}y^{-\frac{15}{13}}-3x^{-\frac{49}{13}}y^{\frac{24}{13}}\right)dx+\left(3x^{-\frac{10}{13}}y^{-\frac{28}{13}}+2x^{-\frac{36}{13}}y^{\frac{11}{13}}\right)dy=0 \quad ...(3)$$

Now equation (3) is exact. $\left[\because \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=-\frac{1}{13}\times\left(30x^{-\frac{23}{13}}y^{-\frac{28}{13}}+72x^{-\frac{49}{13}}y^{\frac{11}{13}}\right)\right]$

Hence the solution is

$$\int_{y\text{ constant}}\left(2x^{-\frac{23}{13}}y^{-\frac{15}{13}}-3x^{-\frac{49}{13}}y^{\frac{24}{13}}\right)dx+0=c_1,$$

where c_1 is any arbitrary constant.

or $$\frac{-13}{10}.2x^{-\frac{10}{13}}y^{-\frac{15}{13}}+3.\frac{13}{36}x^{-\frac{36}{13}}y^{\frac{24}{13}}=c_1$$

or $$5x^{-\frac{36}{13}}y^{\frac{24}{13}}-12x^{-\frac{10}{3}}y^{-\frac{15}{13}}=c.$$

EXERCISE 1.7

Solve the following differential equations :

1. $(y^3-2yx^2)\,dx+(2xy^2-x^3)\,dy=0.$ [M.D.U. 2013, 04, 01]
2. $(2x^2y^2+y)\,dx-(x^3y-3x)\,dy=0.$
3. $(y^2+2x^2y)\,dx+(2x^3-xy)\,dy=0.$
4. $(20x^2+8xy+4y^2+3y^3)\,y\,dx+4\,(x^2+xy+y^2+y^3)\,x\,dy=0.$

ANSWERS

1. $x^2y^2(y^2-x^2)=c$
2. $\frac{7}{5}.x^{\frac{10}{7}}y^{-\frac{5}{7}}-\frac{7}{4}.x^{-\frac{4}{7}}y^{-\frac{12}{7}}=c$
3. $6\sqrt{xy}-\left(\frac{y}{x}\right)^{3/2}=c$
4. $\left(4x^5+2x^4y+\frac{4}{3}x^3y^2+x^3y^3\right)y=c.$

2 EQUATIONS OF FIRST ORDER BUT NOT OF FIRST DEGREE

2.1. INTRODUCTION

In this chapter we shall consider differential equations which are of first order but not of first degree. These equations do not contain differential co-efficients higher than $\frac{dy}{dx}$. If $\frac{dy}{dx}$ be denoted by p, then the most general form of a differential equation of first order and nth degree is

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + ... + A_{n-1} p + A_n = 0$$

where $A_0, A_1, A_2, ..., A_n$ are functions of x and y.

2.2. EQUATIONS SOLVABLE FOR *p*

Suppose the equation

$p^n + A_1 p^{n-1} + A_2 p^{n-2} + ... + A_{n-1} p + A_n = 0$ can be put in the form

$$[p - F_1(x, y)]\,[p - F_2(x, y)] \,.....\, [p - F_n(x, y)] = 0$$

To obtain the solution of given equation, equate each factor to zero, which gives equations of first order and first degree namely

$$p = F_1(x, y), \quad p = F_2(x, y), \,........, \; p = F_n(x, y)$$

or

$$\frac{dy}{dx} = F_1(x, y), \; \frac{dy}{dx} = F_2(x, y), \,....., \; \frac{dy}{dx} = F_n(x, y)$$

Let their solutions be

$$f_1(x, y, c_1) = 0$$
$$f_2(x, y, c_2) = 0$$
$$f_3(x, y, c_3) = 0$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$f_n(x, y, c_n) = 0$$

where $c_1, c_2, c_3, ..., c_n$ are arbitrary constants. The most general solution of the original differential equation is

$$[f_1(x, y, c_1)] \,.\, [f_2(x, y, c_2)] \,.\, [f_3(x, y, c_3)] \,.....\, [f_n(x, y, c_n)] = 0$$

Since the given equation is of first order therefore, the general solution cannot have more than one arbitrary constant. Hence there is no loss of generality if we take

$$c_1 = c_2 = c_3 = \ldots.. = c_n = c \text{ (say)}$$

Hence the general solution of the given equation can be put as

$$[f_1(x, y, c)]\ [f_2(x, y, c)] \ldots..\ [f_n(x, y, c)] = 0.$$

Working Rule :

(1) Put $\frac{dy}{dx} = p$ in the given differential equation.

(2) Make R.H.S. zero and factorize L.H.S. into linear factors of p.

(3) Equate each linear factor to zero and find their solutions taking constant c same in all solutions.

(4) Multiply all the solutions obtained in the above step and equate to zero for obtaining general solution of the given differential equation.

SOLVED EXAMPLES

Example 1.

Solve the differential equation $\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 6 = 0.$ [M.D.U. 2015]

Solution. The given equation is

$$\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 6 = 0$$

Putting $\frac{dy}{dx} = p$, we get $\quad p^2 - 5p + 6 = 0$

or $\quad (p-3)(p-2) = 0$

$\therefore$ either $\quad p - 3 = 0$

$\Rightarrow \quad p = 3$

$\therefore \quad \frac{dy}{dx} = 3$

Integrating,

$$y = 3x + c$$

or $\quad y - 3x - c = 0$

or $\quad p - 2 = 0$

$\Rightarrow \quad p = 2$

$\therefore \quad \frac{dy}{dx} = 2$

Integrating,

$$y = 2x + c$$

or $\quad y - 2x - c = 0$

Therefore, the most general solution is

$$(y - 3x - c)(y - 2x - c) = 0.$$

Example 2.

Solve the differential equation $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$.

Solution. The given equation is

$$p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$$

or
$$p(p^2 + 2xp - y^2p - 2xy^2) = 0$$

or
$$p(p + 2x)(p - y^2) = 0$$

$\therefore$
$$p = 0, -2x, y^2$$

i.e.,
$$\frac{dy}{dx} = 0; \quad \frac{dy}{dx} = -2x; \quad \frac{dy}{dx} = y^2$$

Integrating, we have

$$\int dy = c \qquad \Bigg| \qquad \int dy = \int -2x\,dx + c \qquad \Bigg| \qquad \int \frac{dy}{y^2} = \int dx + c$$

$$\therefore \quad y = c \qquad \Bigg| \qquad \therefore \quad y = -x^2 + c \qquad \Bigg| \qquad \therefore \quad -\frac{1}{y} = x + c \Rightarrow -1 = xy + cy$$

$$\Rightarrow \quad y - c = 0 \qquad \Bigg| \qquad \Rightarrow \quad y + x^2 - c = 0 \qquad \Bigg| \qquad \Rightarrow \quad xy + cy + 1 = 0$$

Hence the most general solution is

$$(y - c)(y + x^2 - c)(xy + cy + 1) = 0.$$

Example 3.

Solve the differential equation $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$.

Solution. The given equation is

$$x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$$

Putting $\frac{dy}{dx} = p$, we get $x^2p^2 + xyp - 6y^2 = 0$

or
$$(px + 3y)(px - 2y) = 0$$

$$px + 3y = 0 \text{ gives} \qquad \Bigg| \qquad px - 2y = 0 \text{ gives}$$

$$px = -3y \qquad \Bigg| \qquad px = 2y$$

$$\text{or} \quad \frac{dy}{dx}x = -3y \qquad \Bigg| \qquad \text{or} \quad \frac{dy}{dx}x = 2y$$

or $$\frac{dy}{y} = -3\frac{dx}{x}$$

Integrating,

$$\log y = -3\log x + c_1$$
$$= \log x^{-3} + \log c$$
$$= \log x^{-3} c$$

$\therefore$ $$y = \frac{c}{x^3}$$

or $$yx^3 = c$$

or $$yx^3 - c = 0$$

or $$\frac{dy}{y} = 2\frac{dx}{x}$$

Integrating,

$$\log y = 2\log x + \log c$$
$$= \log x^2 + \log c$$
$$= \log x^2 c$$

$\therefore$ $$y = x^2 c$$

or $$\frac{y}{x^2} = c$$

or $$\left(\frac{y}{x^2} - c\right) = 0$$

Hence, the required solution is $(yx^3 - c)\left(\frac{y}{x^2} - c\right) = 0$.

Example 4.

Solve the differential equation

$$p^3(x + 2y) + 3p^2(x + y) + (y + 2x)p = 0$$

Solution. The given equation is

$$p\,[p^2(x + 2y) + 3p(x + y) + (y + 2x)] = 0$$

$\therefore$ either $p = 0$

$\Rightarrow$ $$\frac{dy}{dx} = 0$$

Integrating,

$$y = c.$$

or $p^2(x + 2y) + 3p(x + y) + (y + 2x) = 0$

which is a quadratic in p

$$\therefore p = \frac{-3(x+y) \pm \sqrt{9(x+y)^2 - 4(x+2y)(2x+y)}}{2(x+2y)}$$

$$= \frac{-3(x+y) \pm \sqrt{9(x^2+y^2+2xy) - 4(2x^2+5xy+2y^2)}}{2(x+2y)}$$

$$= \frac{(-3x-3y) \pm \sqrt{x^2+y^2-2xy}}{2(x+2y)}$$

$$= \frac{(-3x-3y) \pm (x-y)}{2(x+2y)}$$

$$= \frac{-3x-3y+x-y}{2x+4y}, \frac{-3x-3y-x+y}{2x+4y}$$

$$= \frac{-2x-4y}{2x+4y}, \frac{-4x-2y}{2x+4y}$$

$$\therefore \qquad p = -1, -\frac{2(2x+y)}{2(x+2y)}$$

$$\therefore \qquad p = -1, -\frac{(2x+y)}{x+2y}$$

When $\dfrac{dy}{dx} = -1$, then

$$dy = -dx$$

Integrating, $x + y - c = 0$

When $\dfrac{dy}{dx} = -\left(\dfrac{2x+y}{x+2y}\right)$, then

$$(x+2y)\,dy = -(2x+y)\,dx$$

or $\qquad (2x+y)\,dx + (x+2y)\,dy = 0$

or $\qquad d[xy + x^2 + y^2] = 0$

Integrating, $\qquad xy + x^2 + y^2 = c$

Hence the general solution is

$$(y-c)(y+x-c)(xy+x^2+y^2-c) = 0.$$

Example 5.

Solve the differential equation

$$\boldsymbol{x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + 2y^2 - x^2 = 0.}$$ *[K.U. 2015, 13; M.D.U. 2014, 07]*

Solution. The given equation is

$$x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + 2y^2 - x^2 = 0$$

Putting $\dfrac{dy}{dx} = p$, we have $x^2p^2 - 2xyp + 2y^2 - x^2 = 0$

which is a quadratic in p.

Solving,
$$p = \frac{2xy \pm \sqrt{4x^2y^2 - 4x^2(2y^2 - x^2)}}{2x^2}$$

$$= \frac{2xy \pm \sqrt{4x^2y^2 - 8x^2y^2 + 4x^4}}{2x^2}$$

$$= \frac{2xy \pm \sqrt{4x^4 - 4x^2y^2}}{2x^2}$$

$$= \frac{2xy \pm \sqrt{4x^2(x^2 - y^2)}}{2x^2}$$

$$= \frac{2xy \pm 2x\sqrt{x^2 - y^2}}{2x^2} = \frac{y \pm \sqrt{x^2 - y^2}}{x}$$

or $$\frac{dy}{dx} = \frac{y \pm \sqrt{x^2 - y^2}}{x}$$

which is a homogeneous equation in x and y. ...(2)

Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

∴ Equation (2) becomes

$$v + x\frac{dv}{dx} = \frac{vx \pm \sqrt{x^2 - v^2x^2}}{x}$$

or $$v + x\frac{dv}{dx} = v \pm \sqrt{1 - v^2}$$

or $$x\frac{dv}{dx} = \pm\sqrt{1 - v^2}$$

Separating the variables, we have

$$\frac{dv}{\sqrt{1 - v^2}} = \pm\frac{dx}{x}$$

or $$\int\frac{dv}{\sqrt{1 - v^2}} = \pm\int\frac{dx}{x} + c$$

or $$\sin^{-1} v = \pm \log x \pm \log c$$

or $$\sin^{-1} v = \pm \log xc$$

or $$\sin^{-1}\frac{y}{x} = \pm \log cx,$$

which is the required solution.

Example 6.

Solve the differential equation $p^2 + 2py \cot x = y^2$.

Solution. The given equation is $p^2 + 2py \cot x - y^2 = 0$, which is a quadratic in p.

Solving, $$p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

or $$p = \frac{-2y\cot x \pm 2y\,\mathrm{cosec}\, x}{2}$$

or $$p = -y\cot x \pm y\,\mathrm{cosec}\, x$$

or $$\frac{dy}{dx} = -y\cot x \pm y\,\mathrm{cosec}\, x$$

or $$\frac{dy}{dx} = y\,[-\cot x \pm \operatorname{cosec} x]$$

or $$\frac{dy}{y} = (-\cot x \pm \operatorname{cosec} x)\,dx$$

Integrating, $$\log y = -\log \sin x \pm [\log (\operatorname{cosec} x - \cot x)] + \log c$$

Taking upper sign, we have

$$\log y = -\log \sin x + \log (\operatorname{cosec} x - \cot x) + \log c$$

$$= \log \frac{c\,(\operatorname{cosec} x - \cot x)}{\sin x}$$

or $$y = \frac{c\,(1 - \cos x)}{\sin^2 x}$$

or $$y = \frac{c\,(1 - \cos x)}{1 - \cos^2 x}$$

or $$y = \frac{c}{1 + \cos x} \Rightarrow y\,(1 + \cos x) = c$$

Taking lower sign, we have

$$\log y = -\log \sin x - \log (\operatorname{cosec} x - \cot x) + \log c$$

$$= \log \frac{c}{\sin x\,(\operatorname{cosec} x - \cot x)}$$

or $$y = \frac{c}{\sin x \left(\dfrac{1 - \cos x}{\sin x}\right)}$$

or $$y = \frac{c}{1 - \cos x} \Rightarrow y\,(1 - \cos x) = c$$

Hence, the general solution is

$$[y\,(1 - \cos x) - c]\,[y\,(1 + \cos x) - c] = 0.$$

Example 7.

Solve the differential equation

$$\boldsymbol{p^3 - p\,(x^2 + xy + y^2) + xy\,(x + y) = 0.}$$ *[K.U. 2010]*

Solution. The given equation is

$$p^3 - p\,(x^2 + xy + y^2) + xy\,(x + y) = 0 \quad \text{...(1)}$$

For $p = x$, L.H.S. $= x^3 - x(x^2 + xy + y^2) + xy(x + y)$

$$= x^3 - x^3 - x^2y - xy^2 + x^2y + xy^2$$

$$= 0$$

$\therefore$ $p = x$ is a root of (1)

Hence (1) becomes $(p - x)[p^2 + px - xy - y^2] = 0$

$\therefore$ either $p - x = 0$

$\Rightarrow$ $p = x$

or $\dfrac{dy}{dx} = x$

or $dy = x.dx$

Integrating,

$$y = \frac{x^2}{2} + c_1$$

or $2y = x^2 + c$

or $(2y - x^2 - c) = 0$

or $p^2 + px - xy - y^2 = 0$

$$\therefore \quad p = \frac{-x \pm \sqrt{x^2 + 4(xy + y^2)}}{2}$$

$$= \frac{-x \pm \sqrt{(x + 2y)^2}}{2}$$

$$= \frac{-x \pm (x + 2y)}{2}$$

$$= \frac{2y}{2}, \frac{-2x - 2y}{2}$$

$$\therefore \quad p = y, -x - y$$

Now when $p = y$, we have

$$\frac{dy}{dx} = y$$

or $\dfrac{dy}{y} = dx$

Integrating, $\log y = x + \log c$

or $\log y = \log e^x + \log c$

or $\log y = \log e^x . c$

or $y = e^x . c.$

Taking $p = -x - y$, we have

$$\frac{dy}{dx} = -(x + y)$$

Put $x + y = t$

$$\therefore \quad 1 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1$$

From (2), $$\frac{dt}{dx} - 1 = -t$$

or $$\frac{dt}{dx} = 1 - t \quad \Rightarrow \quad \frac{dt}{t-1} = -dx$$

Integrating, $$\log (t-1) = -x + \log c$$

or $$\log (t-1) = \log e^{-x} + \log c$$

or $$t - 1 = ce^{-x}$$

or $$x + y - 1 = ce^{-x}$$

Hence the complete solution is

$$(2y - x^2 - c)(y - e^x c)(x + y - 1 - ce^{-x}) = 0$$

Example 8.

Solve the differential equation :

$$\boldsymbol{p^4 - (x + 1 + 2y) p^3 + (x + 2y + 2xy) p^2 - 2xyp = 0}$$

Solution. The given equation is

$$p^4 - (x + 1 + 2y) p^3 + (x + 2y + 2xy) p^2 - 2xyp = 0$$

or $$p [p^3 - (x + 1 + 2y) p^2 + (x + 2y + 2xy) p - 2xy] = 0$$

or $$p (p-1)[p^2 - (x + 2y) p + 2xy] = 0$$

Taking $p = 0$, we have

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad dy = 0$$

Integrating, $$y = c$$

Taking $p - 1 = 0$, we have $$p = 1$$

or $$\frac{dy}{dx} = 1 \quad \Rightarrow \quad dy = dx$$

Integrating, $$y = x + c$$

Taking $p^2 - (x + 2y) p + 2xy = 0$, we have

$$p = \frac{(x + 2y) \pm \sqrt{(x + 2y)^2 - 8xy}}{2}$$

$$= \frac{(x + 2y) \pm \sqrt{x^2 + 4y^2 + 4xy - 8xy}}{2}$$

$$= \frac{(x + 2y) \pm (x - 2y)}{2} = \frac{2x}{2}, \frac{4y}{2}$$

$$\therefore \qquad p = x,\ 2y$$

Taking $p = x$, we have $\dfrac{dy}{dx} = x$

or $dy = x\,dx$

Integrating, $y = \dfrac{x^2}{2} + c \Rightarrow 2y = x^2 + c$

Taking $p = 2y$, we have $\dfrac{dy}{dx} = 2y \Rightarrow \dfrac{dy}{y} = 2dx$

Integrating, $\log y = 2x + \log c$

or $\log y = \log e^{2x} + \log c$

or $\log y = \log ce^{2x} \Rightarrow y = ce^{2x}$

Hence, the complete solution is $(y-c)(y-x-c)(2y-x^2-c)(y-ce^{2x}) = 0$.

EXERCISE 2.1

Solve the following differential equations :

1. $p^2 - 7p + 12 = 0$
2. $xp^2 - (y-x)p - y = 0$ [M.D.U. 2013]
3. $(px-y)(py-x) = 0$
4. $p^2 - 9p + 18 = 0$
5. $4p^2x - (3x-a)^2 = 0$
6. $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$
7. $\left(1 - y^2 + \dfrac{y^4}{x^2}\right)p^2 - \dfrac{2y}{x}p + \dfrac{y^2}{x^2} = 0$ [M.D.U. 2016, 12]
8. $xp^2 + (y-x)p - y = 0$ [M.D.U. 2013]
9. $yp^2 + (x-y)p - x = 0$ [M.D.U. 2015]
10. $p^2 - 2p\cosh x + 1 = 0$ [M.D.U. 2016, 08]
11. $(x^2+x)p^2 + (x^2+x-2xy-y)p + y^2 - xy = 0$
12. $p^2 + \left(x + y - \dfrac{2y}{x}\right)p + xy + \dfrac{y^2}{x^2} - y - \dfrac{y^2}{x} = 0$
13. $4y^2p^2 + 2pxy(3x+1) + 3x^3 = 0$
14. $\left(1 - y^2 - \dfrac{y^4}{x^2}\right)p^2 - \dfrac{2y}{x}p + \dfrac{y^2}{x^2} = 0$
15. $p^3 - (x^2+xy+y^2)p^2 + (x^3y + x^2y^2 + xy^3)p - x^3y^3 = 0$
16. $p^2 + 2py\cot x = y^2$.

ANSWERS

1. $(y-4x-c)(y-3x-c)=0$
2. $(y+x-c)\left(\dfrac{y}{x}-c\right)=0$
3. $(y^2-x^2-c)(y-cx)=0$ or $(y^2-x^2-c)\left(\dfrac{y}{x}-c\right)=0$
4. $(y-6x-c)(y-3x-c)=0$
5. $(y-c)^2=x(x-a)^2$
6. $(y-cx^2)(y^2+3x^2-c)=0$
7. $\log\dfrac{x+\sqrt{x^2-y^2}}{y}=\pm y+c$
8. $(y-x-c)(yx-c)=0$
9. $(y-x-c)(x^2+y^2-c)=0$
10. $(y-e^x-c)(y+e^{-x}-c)=0$
11. $[y+x\log cx][y-c(x+1)]=0$
12. $(y+x^2-cx)(y-cx\,e^{-x})=0$
13. $(y^2+x^3-c)(2y^2+x^2-c)=0$
14. $\left(x+\sqrt{x^2+y^2}\right)=cy\,e^{\pm y}$
15. $(x^3-3y+c)\ (y-ce^{x^2/2})(xy+cy+1)=0$
16. $[y(1+\cos x)-c]\,[y(1-\cos x)-c]=0$

2.3. EQUATIONS SOLVABLE FOR *y*

If y can be expressed explicitly in terms of x and p, then the equation is solvable for y.

Consider
$$y=f(x,p) \quad \text{...(1)}$$

$f(x,y,p)=0$ This is implicit form of gen diff eq^n.

Differentiating w.r.t x, we get
$$\frac{dy}{dx}=F\left(x,p,\frac{dp}{dx}\right) \quad \text{...(2)}$$

or
$$p=F\left(x,p,\frac{dp}{dx}\right)$$

The equation is in two variables x and p and can be solved. Let its solution be
$$\phi(x,p,c)=0 \quad \text{...(3)}$$

Eliminating p between (1) and (3), we will get a relation between x, y and c which will be the required solution.

If p cannot be eliminated easily, then express the values of x and y in terms of parameter p in the form
$$x=\phi_1(p,c),\quad y=\phi_2(p,c)$$

These two relations together give the complete solution of the given equation.

Working Rule :

(1) Express the given equation in the form $y = f(x, p)$.

(2) Differentiate w.r.t. x and replace $\frac{dy}{dx}$ by p so that the equation has two variables p and x.

(3) Solve the equation obtained in step 2.

(4) Let the solution of equation be $\phi(x, p, c) = 0$.

(5) Eliminate p from equations of steps 1 and 4.

SOLVED EXAMPLES

Example 1.

Solve the differential equation $y = 2px + p^4x^2$.

Solution. The given equation is $y = 2px + p^4x^2$ ...(1)

Differentiating w.r.t. x, we have

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 2xp^4 + 4p^3x^2\frac{dp}{dx}$$

or
$$p = 2p + 2x\frac{dp}{dx} + 2xp^4 + 4p^3x^2\frac{dp}{dx}$$

or
$$p + 2x\frac{dp}{dx} + 2xp^4 + 4p^3x^2\frac{dp}{dx} = 0$$

or
$$\left(p + 2x\frac{dp}{dx}\right)(1 + 2p^3x) = 0$$

The factor $1 + 2p^3x = 0$ is discarded as it does not contain $\frac{dp}{dx}$.

Thus
$$p + 2x\frac{dp}{dx} = 0$$

or
$$\frac{2dp}{p} = -\frac{dx}{x}$$

Integrating,
$$2\log p = -\log x + \log c$$

or
$$\log p^2 = \log\frac{c}{x}$$

or $$p^2 = \frac{c}{x} \Rightarrow p = \sqrt{\frac{c}{x}}$$

Putting this value of p in (1), we have

$$y = 2\sqrt{\frac{c}{x}}\,x + \frac{c^2}{x^2}\cdot x^2$$

or $$y = 2\sqrt{cx} + c^2$$

which is the complete solution.

Example 2.

Solve the differential equation $y = -px + x^4p^2$ [*K.U. 2014, 12, 11; M.D.U. 2014*]

Solution. The given equation is $y = -px + x^4p^2$ ...(1)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -p - x\frac{dp}{dx} + 4x^3p^2 + 2x^4p\frac{dp}{dx}$$

or $$p = -p - x\frac{dp}{dx} + 4x^3p^2 + 2x^4p\frac{dp}{dx}$$

or $$2p + x\frac{dp}{dx} - 4x^3p^2 - 2x^4p\frac{dp}{dx} = 0$$

or $$2p + x\frac{dp}{dx} - 2px^3\left(2p + x\frac{dp}{dx}\right) = 0$$

or $$\left(2p + x\frac{dp}{dx}\right)(1 - 2px^3) = 0$$

Rejecting the factor $(1 - 2px^3)$, we get

$$2p + x\frac{dp}{dx} = 0$$

or $$\frac{dp}{p} + \frac{2dx}{x} = 0$$

Integrating, we get $$\log p + 2\log x = \log c$$

or $$\log px^2 = \log c$$

or $$px^2 = c \Rightarrow p = \frac{c}{x^2}$$

$\therefore$ Equation (1) becomes, $$y = -\frac{c}{x^2}\cdot x + x^4\cdot\frac{c^2}{x^4}$$

or $$y = -\frac{c}{x} + c^2$$

which is the required solution.

Example 3.

Solve the differential equation $y = 3x + \log p$. [M.D.U. 2006]

Solution. The given equation is $y = 3x + \log p$ ...(1)

Differentiating w.r.t. x, we have

$$\frac{dy}{dx} = 3 + \frac{1}{p}\frac{dp}{dx}$$

or $$p = 3 + \frac{1}{p}\frac{dp}{dx}$$

or $$dx = \frac{dp}{p(p-3)}$$

or $$dx = \frac{1}{3}\left[\frac{1}{p-3} - \frac{1}{p}\right] dp$$

Integrating, $$x = \frac{1}{3}[\log(p-3) - \log p] + \log c_1$$

or $$3x = \log\frac{p-3}{p} + \log c_2$$

or $$3x = \log\left(\frac{p-3}{p}\right) . c_2$$

or $$e^{3x} = \frac{p-3}{p} . c_2 \quad \Rightarrow \quad \frac{p-3}{p} = ce^{3x}$$

or $$1 - \frac{3}{p} = ce^{3x} \quad \Rightarrow \quad p = \frac{3}{1 - ce^{3x}}$$

Putting this value of p in (1), we have

$$y = 3x + \log\frac{3}{1 - ce^{3x}}$$

which is the required solution.

Example 4.

Solve the differential equation $p^3 + p = e^y$. [K.U. 2008]

Solution. The given equation is $p^3 + p = e^y$

Taking log of both sides, we have

$$\log p(p^2 + 1) = \log e^y$$

or $$\log p + \log (p^2 + 1) = y$$

or $$y = \log p + \log (1 + p^2)$$

Differentiating w.r.t. x, we have

$$\frac{dy}{dx} = \frac{1}{p}\frac{dp}{dx} + \frac{2p}{1+p^2}\frac{dp}{dx}$$

or $$p = \frac{1}{p}\frac{dp}{dx} + \frac{2p}{1+p^2}\frac{dp}{dx}$$

or $$dx = \left(\frac{1}{p^2} + \frac{2}{1+p^2}\right) dp$$

Integrating, $$x = -\frac{1}{p} + 2\tan^{-1} p + c \quad \text{...(1)}$$

Also, $$y = \log p + \log (1 + p^2) \quad \text{...(2)}$$

These two equations (1) and (2) form the required solution.

• EXERCISE 2.2

Solve the following differential equations :

1. $y = 2px - p^2$ *[M.D.U. 2013]*

2. $x - yp = ap^2$ *[M.D.U. 2007]*

3. $p^2 - py + x = 0$

4. $y = 2px - xp^2$

5. $y - 2px = f(xp^2)$

6. $xp^2 - 2yp - ax = 0$

7. $y = (1 + p)x + p^2$ *[M.D.U. 2011]*

8. $y = p \tan p + \log \cos p$ *[M.D.U. 2009]*

9. $y = x + a \tan^{-1} p$

10. $y = p \sin p + \cos p$

11. $y = 2p + \sqrt{1 + p^2}$ *[K.U. 2015]*

12. $p - y = \log (p^2 - 1)$

13. $16x^2 + 2p^2y - p^3x = 0$

14. $p^3 + mp^2 = a(y + mx)$

15. $4y = x^2 + p^2$

16. $y = p + p^2x$

17. $x^2 + p^2x = yp$

18. $y = 2xp + p^2$

19. $y = 1 + xp^3$

20. $y + px = p^2 x^4$

ANSWERS

1. $x = \frac{c}{p^2} + \frac{2}{3}p,\ y = \frac{1}{3}p^2 + \frac{2c}{p}$

2. $x = \frac{ap\sin^{-1}p}{\sqrt{1-p^2}} + \frac{cp}{\sqrt{1-p^2}},\ y = \frac{1}{\sqrt{1-p^2}}[c + a\sin^{-1}p] - ap$

3. $x = \frac{p}{\sqrt{1-p^2}}[c - \sin^{-1}p],\ y = \frac{1}{\sqrt{1-p^2}}[c - \sin^{-1}p] + p$

4. $y = 2\sqrt{cx} - c$

5. $y = 2\sqrt{cx} + f(c)$

6. $2y = cx^2 - \frac{a}{c}$

7. $x = 2(1-p) + ce^{-p},\ y = (1+p)\,[2(1-p) + ce^{-p}] + p^2$

8. $x = \tan p + c,\ y = p\tan p + \log\cos p$

9. $x + c = \frac{a}{2}[\log(1+p^2)^{-1/2}(p-1) - \tan^{-1}p],\ y = x + a\tan^{-1}p$

10. $x = c + \sin p$, given relation

11. $x = 2\log p + \log[p + \sqrt{1+p^2}\,] + c$, given relation

12. $x = \log\frac{p(p+1)}{p-1} + c$, given relation

13. $2c^2y + 16 - x^2c^3 = 0$

14. $p^3 + mp^2 = a(y + mx),\ ax + c = \frac{3}{2}p^2 - mp + m^2\log(m+p)$

15. $\log(p - x) = \frac{x}{p-x} + c$, given relation

16. $(p-1)^2x = c - p + \log p,\ (p-1)^2y = p^2(c - 2 + \log p) + p$

17. $x = c\sqrt{p} - \frac{1}{3}p^2$, given relation

18. $xp^2 = -\frac{2}{3}p^3 + c$

19. $x^{2/3} = (y-1)^{2/3} + c$

20. $y = -\frac{c}{x} + c^2$

2.4. EQUATIONS SOLVABLE FOR x

The equation is said to be solvable for x if x can be expressed explicitly in terms of y and p. Such an equation can be put in the form

$$x = f(y, p) \qquad ...(1)$$

Differentiating (1) w.r.t. y, we get

$$\frac{dx}{dy} = F\left(y, p, \frac{dp}{dy}\right)$$

or
$$\frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right) \qquad ...(2)$$

The equation involves two variables y and p. Suppose the solution is

$$\phi(y, p, c) = 0 \qquad ...(3)$$

where c is any arbitrary constant.

Eliminating p from (1) and (3), we get the solution of the equation (1). If elimination of p is not possible, then values of x and y expressed in terms of parameter p together form the solution of the equation.

SOLVED EXAMPLES

Example 1.

[K.U. 2016]

Solve the differential equation $x = y + p^2$.

Solution. The given equation is $x = y + p^2$ ...(1)

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = 1 + 2p\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{1}{p} = 1 + 2p\frac{dp}{dy} \quad \Rightarrow \quad 2p\frac{dp}{dy} = \frac{1}{p} - 1$$

$$\Rightarrow \quad \frac{dp}{dy} = \frac{1-p}{2p^2} \quad \Rightarrow \quad \frac{2p^2}{1-p}\,dp = dy$$

$$\Rightarrow \quad -2\left[p + 1 + \frac{1}{p-1}\right]dp = dy$$

Integrating, we get
$$c - 2\left[\frac{p^2}{2} + p + \log(p-1)\right] = y$$

or
$$y = c - [p^2 + 2p + 2\log(p-1)]$$

Putting the value of y in (1), we get

$$x = c - [2p + 2\log(p-1)]$$

which is the required solution.

Example 2.

Solve the differential equation $y = 2px + y^2p^3$.

Solution. The given equation is $y = 2px + y^2p^3$ ...(1)

or
$$x = \frac{y}{2p} - \frac{y^2p^2}{2}$$

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - yp^2 - p^2y^2\frac{dp}{dy}$$

or
$$\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - yp^2 - p^2y^2\frac{dp}{dy}$$

or
$$\left(\frac{1}{2p} + yp^2\right) = -\left(\frac{y}{2p^2} + py^2\right)\frac{dp}{dy}$$

or
$$\left(\frac{1}{2p} + yp^2\right) = -\frac{y}{p}\left(\frac{1}{2p} + yp^2\right)\frac{dp}{dy}$$

or
$$\left(\frac{1}{2p} + yp^2\right) + \frac{y}{p}\left(\frac{1}{2p} + yp^2\right)\frac{dp}{dy} = 0$$

or
$$\left(\frac{1}{2p} + yp^2\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

Neglecting the factor $\frac{1}{2p} + yp^2$, we get

$$1 + \frac{y}{p}\frac{dp}{dy} = 0$$

or
$$\frac{dy}{y} + \frac{dp}{p} = 0$$

Integrating, we get $\quad \log y + \log p = \log c$

or
$$\log yp = \log c$$

or
$$yp = c \Rightarrow p = \frac{c}{y} \qquad ...(2)$$

Eliminating p from (1) and (2), we get

$$y = \frac{2c}{y}x + y^2 \cdot \frac{c^3}{y^3}$$

or $$y^2 = 2cx + c^3$$

which is the required solution.

Example 3.

Solve the differential equation $x = y + a \log p$.

Solution. The given equation is $x = y + a \log p$ ...(1)

Differentiating w.r.t. y, we get $\frac{dx}{dy} = 1 + \frac{a}{p}\frac{dp}{dy} \Rightarrow \frac{1}{p} = 1 + \frac{a}{p}\frac{dp}{dy}$

$$\Rightarrow \quad (1-p) = a\frac{dp}{dy} \quad \Rightarrow \quad dy = a\frac{dp}{1-p}$$

Integrating, we get $$y = c - a\log(1-p) \quad ...(2)$$

Equation (1) and (2) constitute the required solution.

EXERCISE 2.3

Solve the following differential equations :

1. $y = 3px + 6p^2y^2$
2. $p = \tan\left(x - \frac{p}{1+p^2}\right)$
3. $y^2 \log y = xyp + p^2$
4. $p^3 - 4xyp + 8y^2 = 0$ [K.U. 2011]
5. $y = 2px + p^2y$
6. $yp^2 - 2xp + y = 0$

ANSWERS

1. $y^3 = 3cx + 6c^2$
2. $x = \tan^{-1} p + \frac{p}{1+p^2}, y = c - \frac{1}{1+p^2}$
3. $\log y = c^2 + cx$
4. $64y = c\,(c - 4x)^2$
5. $y^2 = 2cx + c^2$
6. $y^2 = 2cx - c^2$

DO

2.5. LAGRANGE'S EQUATION

[M.D.U. 2017; K.U. 2017]

The differential equation of the form $\boxed{y = x\phi(p) + f(p)}$, *where* $\phi(p)$ *and* $f(p)$ *are functions of* p *only is called* **Lagrange's equation**.

For example, the equation $y = xp^2 + p$ is Lagrange's equation.

2.5.1. To solve the differential equation $y = x\,\phi(p) + f(p)$ (Lagrange's Equation).

The given equation is $\quad y = x\phi(p) + f(p) \qquad$ ...(1)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \phi(p) + x\,\phi'(p)\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

Putting $\frac{dy}{dx} = p$, we get

$$p = \phi(p) + x\,\phi'(p)\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

or
$$p - \phi(p) = [x\,\phi'(p) + f'(p)]\frac{dp}{dx}$$

or
$$[p - \phi(p)]\frac{dx}{dp} = x\phi'(p) + f'(p)$$

or
$$\frac{dx}{dp} = \frac{x\phi'(p) + f'(p)}{p - \phi(p)}$$

$$= \frac{x\phi'(p)}{p - \phi(p)} + \frac{f'(p)}{p - \phi(p)}$$

or
$$\frac{dx}{dp} + \left[\frac{\phi'(p)}{\phi(p) - p}\right]x = \frac{f'(p)}{p - \phi(p)}$$

which is a linear differential equation in terms of x and p.

Let the solution of this equation be $F(x, p, c) = 0 \qquad$ ...(2)

Equations (1) and (2) taken together give the required solution.

Working Rule :

(1) Express the given equation in the form $y = x\,\phi(p) + f(p)$, where $\phi(p)$ and $f(p)$ are functions of p only.

(2) Differentiate w.r.t. x and put $\frac{dy}{dx} = p$.

(3) The equation reduces to a linear differential equation in variables x and p which can be easily solved.

(4) The solution obtained in step (3) and the given equation together form the required solution.

The following solved example will further illustrate the method.

Example 1.

Solve ***the differential equation*** $x - yp = ap^2$

Solution. The given equation is $x - yp = ap^2$

or $$yp = x - ap^2$$

or $$y = \frac{x}{p} - ap \qquad \ldots(1)$$

which being of the form $y = x\phi(p) + f(p)$ is a Lagrange's equation.

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{p} - \frac{x}{p^2}\frac{dp}{dx} - a\frac{dp}{dx}$$

Putting $\frac{dy}{dx} = p$, we have

$$p = \frac{1}{p} - \frac{x}{p^2}\frac{dp}{dx} - a\frac{dp}{dx}$$

or $$p^3 = p - x\frac{dp}{dx} - ap^2\frac{dp}{dx}$$

or $$(x + ap^2)\frac{dp}{dx} = p - p^3$$

or $$\frac{dx}{dp} = \frac{x + ap^2}{p - p^3}$$

or $$\frac{dx}{dp} = \frac{x}{p - p^3} + \frac{ap^2}{p - p^3}$$

or $$\frac{dx}{dp} - \frac{x}{p - p^3} = \frac{ap}{1 - p^2}$$

or $$\frac{dx}{dp} + \left(\frac{1}{p^3 - p}\right)x = \frac{ap}{1 - p^2} \qquad \ldots(2)$$

which is a linear differential equation of the form

$$\frac{dx}{dp} + \text{P}x = \text{Q}, \text{ where } \text{P} = \frac{1}{p^3 - p} \text{ and } \text{Q} = \frac{ap}{1 - p^2}$$

$$\text{I.F.} = e^{\int P\,dp} = e^{\int \frac{dp}{p^3 - p}} = e^{\int -\frac{dp}{p - p^3}}$$

$$= e^{-\int\left(\frac{A}{p} + \frac{B}{1-p} + \frac{C}{1+p}\right)dp}$$ [Using partial fractions]

$$= e^{-\int\left[\frac{1}{p} + \frac{1}{2(1-p)} - \frac{1}{2(1+p)}\right]dp}$$

$$= e^{-\left[\log p - \frac{1}{2}\log(1-p) - \frac{1}{2}\log(1+p)\right]}$$

$$= e^{-\left[\log p - \frac{1}{2}\log(1-p^2)\right]}$$

$$= e^{\left[-\log p + \log(1-p^2)^{\frac{1}{2}}\right]}$$

$$= e^{\log\frac{\sqrt{1-p^2}}{p}} = \frac{\sqrt{1-p^2}}{p}$$

The solution of equation (2) is given by

$$x\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dp + c$$

i.e.,

$$\frac{x\sqrt{1-p^2}}{p} = \int \frac{ap}{1-p^2}\cdot\frac{\sqrt{1-p^2}}{p}\,dp + c$$

or

$$\frac{x\sqrt{1-p^2}}{p} = a\int \frac{1}{\sqrt{1-p^2}}\,dp + c$$

or

$$x\,\frac{\sqrt{1-p^2}}{p} = a\sin^{-1}p + c$$

$$\therefore \quad x = \frac{ap\sin^{-1}p}{\sqrt{1-p^2}} + \frac{cp}{\sqrt{1-p^2}} \qquad \ldots(3)$$

Putting the value of x in (1), we get

$$y = \frac{a\sin^{-1}p}{\sqrt{1-p^2}} + \frac{c}{\sqrt{1-p^2}} - ap \qquad \ldots(4)$$

Equations (3) and (4) taken together form the required solution.

2.6. CLAIRAUT'S EQUATION [K.U. 2011]

Clairaut's equation is a particular case of Lagrange's equation which is obtained by taking $\phi(p) = p$.

Def. *The differential equation of the type $y = px + f(p)$ is known as Clairaut's equation.*

2.6.1. To solve the differential equation $y = px + f(p)$ (Clairaut's Equation)

[M.D.U. 2016]

The given equation is $$y = px + f(p) \quad ...(1)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = p + x \cdot \frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

or $$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

or $$[x + f'(p)]\frac{dp}{dx} = 0$$

Neglecting $x + f'(p)$, we get $\dfrac{dp}{dx} = 0$

Integrating both sides, we obtain

$$p = c$$

Putting $p = c$ in (1), we get $y = cx + f(c)$, which is the required solution.

If however we eliminate p between the given equation (1) and the relation $x + f'(p) = 0$, we get another solution called the **singular solution**.

SOLVED EXAMPLES

Example 1. *Solve the following differential equations :*

(*i*) $(y - px)^2 = 1 + p^2$ (*ii*) $(y - px)(p + 1) = p^2$. [M.D.U. 2016]

Solution. (*i*) The given equation is $(y - px)^2 = 1 + p^2$

$$\therefore \quad y - px = \sqrt{1 + p^2}$$

or $$y = px + \sqrt{1 + p^2} \quad ...(1)$$

which is a Clairaut's equation of the form $y = px + f(p)$

Replacing p by constant c in (1), we get

$$y = cx + \sqrt{1+c^2}$$, which is the required solution.

(*ii*) The given equation is $(y - px)(p+1) = p^2$

or $$y - px = \frac{p^2}{p+1}$$

or $$y = px + \frac{p^2}{p+1} \quad ...(1)$$

which is a Clairaut's equation of the form $y = px + f(p)$

Replacing p by constant c in (1), we get

$$y = cx + \frac{c^2}{c+1}$$, which is the required solution.

Example 2. ***Solve*** $\boldsymbol{p = \log(px - y)}$ [M.D.U. 2014]

Solution. The given equation is

$$p = \log(px - y)$$

or $$px - y = e^p$$

or $$y = px - e^p, \quad ...(1)$$

which is a Clairaut's equation of the form $y = px + f(p)$.

Replacing p by constant c in (1), we get

$$y = cx - e^c$$, which is the required solution.

Example 3. ***Solve the differential equation*** $\boldsymbol{\sin px \cos y = \cos px \sin y + p}$. [M.D.U. 2015]

Solution. The given differential equation is

$$\sin px \cos y = \cos px \sin y + p$$

or $$\sin px \cos y - \cos px \sin y = p$$

or $$\sin(px - y) = p$$

or $$px - y = \sin^{-1} p \quad ...(1)$$

or $$y = px - \sin^{-1} p$$

which is a Clairaut's equation of the form $y = px + f(p)$.

Replacing p by constant c in (1), we get

$$y = cx - \sin^{-1} c$$, which is the required solution.

EXERCISE 2.4

Solve the following differential equations (Lagrange's Equation) :

1. $y = 2px + p^2$

2. $y = (1 + p)\,x + p^2$

3. $y = p\,(x - b) + \dfrac{a}{p}$

4. $y = xp^2 - \dfrac{1}{p}$ [M.D.U. 2012]

Solve the following differential equations (Clairaut's Equation) :

5. $y = px + p - p^2$

6. $p = \tan\,(px - y)$ [K.U. 2016, 15; M.D.U. 2012]

7. $y = px + p^2$

8. $y = px + \sqrt{a^2\,p^2 + b^2}$. [M.D.U. 2015]

9. $p^2(x^2 - a^2) - 2pxy + y^2 - b^2 = 0.$ [M.D.U. 2016, 15]

10. $(y - px)\,(p - 1) = p.$

11. $p^2x\,(x - 2) + p\,(2y - 2xy - x + 2) + y^2 + y = 0.$

ANSWERS

1. $y = 2px + p^2,\;\; xp^2 = -\dfrac{2}{3}\,p^3 + c$

2. $y = 2 - p^2 + ce^{-p}\,(1 + p),\;\; x = -\,2p + 2 + ce^{-p}$

3. $y = c\,(x - b) + \dfrac{a}{c}$

4. $x = \dfrac{2p - 1}{2p^2\,(p - 1)^2} + \dfrac{c}{(p - 1)^2},\;\; y = \dfrac{2p - 1}{2(p - 1)^2} + \dfrac{cp^2}{(p - 1)^2} - \dfrac{1}{p}$

5. $y = cx + c - c^2$

6. $y = cx - \tan^{-1} c$

7. $y = cx + c^2$

8. $y = cx + \sqrt{a^2c^2 + b^2}$

9. $y = cx \pm \sqrt{a^2c^2 + b^2}$

10. $y = cx + \dfrac{c}{c - 1}$

11. $(y - cx + 2c)\,(y - cx + 1) = 0$

2.7. EQUATIONS REDUCIBLE TO CLAIRAUT'S FORM

By making some suitable substitutions the differential equations of first order and higher degree can be reduced to Clairaut's form. However there are no general rules for making such substitutions.

SOLVED EXAMPLES

Example 1.

Solve the differential equation $(px - y)(py + x) = a^2p$.

Solution. The given equation is

$$(px - y)(py + x) = a^2p \quad \text{...(1)}$$

The given equation can be reduced to Clairaut's form by putting

$$x^2 = X \quad \text{and} \quad y^2 = Y$$

$$\therefore \quad 2x\,dx = dX \quad \text{and} \quad 2y\,dy = dY$$

$$\therefore \quad p = \frac{dy}{dx} = \frac{\frac{dY}{2y}}{\frac{dX}{2x}} = \frac{x}{y}\frac{dY}{dX} = \frac{x}{y}P, \text{ where } P = \frac{dY}{dX}$$

Putting these values in (1), we get

$$\left(\frac{x^2}{y}P - y\right)\left(\frac{x}{y}Py + x\right) = a^2\frac{x}{y}P$$

or $$\frac{(x^2P - y^2)}{y}x(P+1) = a^2\frac{x}{y}P$$

or $$(x^2P - y^2)(P+1) = a^2P$$

or $$(XP - Y) = \frac{a^2P}{P+1}$$

or $$-Y = -XP + \frac{a^2P}{P+1}$$

or $$Y = PX - \frac{a^2P}{P+1}$$

which is Clairaut's equation of type $Y = XP + f(P)$

Thus the solution is $$Y = CX - \frac{a^2C}{C+1}$$ *[Replacing P by constant C]*

or $$y^2 = cx^2 - \frac{a^2c}{c+1}.$$

Example 2.

Solve the differential equation $e^{3x}(p-1)+p^3e^{2y}=0$.

Solution. The given equation is

$$e^{3x}(p-1)+p^3e^{2y}=0 \qquad ...(1)$$

Put $X=e^{kx}$ and $Y=e^{ky}$, where k is H.C.F. of 3 and 2.

Hence $X=e^x$, $Y=e^y$

$\Rightarrow$ $dX=e^x\,dx$, $dY=e^y\,dy$

$$\therefore \quad p=\frac{dy}{dx}=\frac{\frac{dY}{e^y}}{\frac{dX}{e^x}}=\frac{e^x}{e^y}\frac{dY}{dX}=\frac{e^x}{e^y}P, \text{ where } P=\frac{dY}{dX}$$

From (1),

$$X^3\left[\frac{e^x}{e^y}P-1\right]+\frac{e^{3x}}{e^{3y}}P^3Y^2=0$$

or

$$X^3\left[\frac{X}{Y}P-1\right]+\frac{X^3}{Y^3}P^3Y^2=0$$

or

$$X^3\left(\frac{PX-Y}{Y}\right)+\frac{X^3P^3}{Y}=0$$

or

$$PX-Y+P^3=0$$

or

$$Y=PX+P^3, \text{ which is of Clairaut's form.}$$

Thus the solution is

$$Y=CX+C^3 \qquad \textit{[Replacing P by C]}$$

or

$$e^y=ce^x+c^3.$$

EXERCISE 2.5

1. Solve the equation $x^2(y-px)=yp^2$.
2. Solve the equation $(y+px)^2=x^2p$. X = x, Y = xy find P then p in terms of P.
3. Solve $e^{5x}(p-1)+p^5e^{4y}=0$.
4. Solve $(x^2+y^2)(1+p)^2-2(x+y)(1+p)(x+yp)+(x+yp)^2=0$. X = x+y, Y = x²+y² find P then p in terms of P
5. Solve $e^{4x}(p-1)+e^{2y}p^2=0$ by using the substitution $X=e^{2x}$ and $Y=e^{2y}$.
6. Solve $(px^2+y^2)(px+y)=(p+1)^2$ by using the substitution $X=x+y$, $Y=xy$. *[M.D.U. 2014, 13]*

   $p = \frac{y+xp}{1+p}$ + ... xy & xyp
7. Solve the differential equation $y^2(y-px)=x^4p^2$ by making substitution $x=\frac{1}{X}$ and $y=\frac{1}{Y}$.

   find $p = \frac{dy}{dx}$ in terms of P.

ANSWERS

1. $y^2 = cx^2 + c^2$
2. $xy = cx - c^2$
3. $e^y = ce^x + c^5$
4. $x^2 + y^2 = c\,(x + y) - \dfrac{c^2}{4}$
5. $e^{2y} = ce^{2x} + c^2$
6. $xy = c\,(x + y) - \dfrac{1}{c}$
7. $x = c^2xy + cy$

2.8. SINGULAR SOLUTION

Sometimes a particular solution satisfies a differential equation of the first order but this solution is not obtained from the general solution by giving some values to the arbitrary constants. Such a solution is called **singular solution**. In other words, *a solution which does not contain arbitrary constants and is not obtained from the general solution by giving some values to the arbitrary constants is called a singular solution*.

2.9. DISCRIMINANT

The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $b^2 - 4ac$.

If $b^2 - 4ac = 0$, then the equation has two equal roots. But if equation is of higher degree than two, then the condition of two equal roots is obtained by eliminating x between $f(x) = 0$ and $f'(x) = 0$.

2.10. p-DISCRIMINANT AND c-DISCRIMINANT $\left(p = \dfrac{dy}{dx}\right)$

Let

$$f(x, y, p) = 0 \quad ...(1)$$

be the differential equation whose solution is

$$\phi\,(x, y, c) = 0 \quad ...(2)$$

where c is any arbitrary constant.

Differentiating (2) partially w.r.t. c, we have

$$\frac{\partial \phi}{\partial c} = 0 \quad ...(3)$$

Differentiating (1) partially w.r.t. p, we have

$$\frac{\partial f}{\partial p} = 0 \qquad \text{...(4)}$$

Then, p-discriminant is obtained by eliminating p between (1) and (4)

c-discriminant is obtained by eliminating c between (2) and (3).

Note.

If $F(x, y) = 0$ is a singular solution of the differential equation $f(x, y, p) = 0$ whose primitive is $\phi(x, y, c) = 0$, then $F(x, y)$ is a factor of both the discriminants.

Working Rule for Obtaining a Singular Solution :

(1) Let the differential equation be $f(x, y, p) = 0$ ...(1)

(2) Let the complete solution of (1) be $\phi(x, y, c) = 0$ ...(2)

(3) Find p-discriminant by eliminating p between $f(x, y, p) = 0$ and $\dfrac{\partial f}{\partial p} = 0$.

(4) Find c-discriminant by eliminating c between $\phi(x, y, c) = 0$ and $\dfrac{\partial \phi}{\partial c} = 0$.

(5) Obtain a common factor of p-discriminant and c-discriminant which occurs once. This common factor equated to zero is the required singular solution.

Remarks :

(*i*) In Clairaut's equation, p-discriminant = c-discriminant.

(*ii*) If the given equation is a quadratic in p and is of the form $Ap^2 + Bp + C = 0$, then p-discriminant is $B^2 - 4AC = 0$.

(*iii*) If the general solution is a quadratic in c and is of the form $Ac^2 + Bc + D = 0$, then c-discriminant is $B^2 - 4AD = 0$.

SOLVED EXAMPLES

Example 1.

Solve the differential equation $y = px + \dfrac{a}{p}$ and obtain the singular solution.

[*K.U. 2013; M.D.U. 2003*]

Solution. The given equation is

$$y = px + \frac{a}{p} \qquad ...(1)$$

which is a Clairaut's equation of the form $y = px + f(p)$ and so its solution is

$$y = cx + \frac{a}{c} \qquad ...(2) \quad [\text{Replacing } p \text{ by } c]$$

Also from (1), we have

$$py = p^2x + a$$

or

$$p^2x - py + a = 0$$

which is a quadratic equation in p

$\therefore$ p-discriminant is given by $y^2 - 4ax = 0$

Hence, the general solution is $y = cx + \frac{a}{c}$ and singular solution is $y^2 - 4ax = 0$.

Example 2.

Solve the differential equation $\sin px \cos y = \cos px \sin y + p$ and obtain the singular solution. [K.U. 2017; M.D.U. 2011]

Solution. The given equation is

$$\sin px \cos y = \cos px \sin y + p$$

or

$$\sin px \cos y - \cos px \sin y = p$$

or

$$\sin (px - y) = p$$

or

$$px - y = \sin^{-1} p$$

or

$$y = px - \sin^{-1} p \qquad ...(1)$$

which is a Clairaut's equation of the form $y = px + f(p)$ and so its solution is

$$y = cx - \sin^{-1} c \qquad ...(2) \quad [\text{Replacing } p \text{ by } c]$$

Differentiating (2) partially w.r.t. c, we get

$$0 = x - \frac{1}{\sqrt{1 - c^2}}$$

or

$$x^2 (1 - c^2) = 1$$

or

$$x^2 - x^2c^2 = 1$$

or $$c^2 = \frac{x^2-1}{x^2} \Rightarrow c = \frac{\sqrt{x^2-1}}{x}$$

Eliminating c between (2) and (3), we get

$$y = \sqrt{x^2-1} - \sin^{-1}\frac{\sqrt{x^2-1}}{x}$$

which is the c-discriminant and is the required singular solution.

Example 3.

Solve and find the complete primitive and singular solution of the equation

$$3y = 2px - \frac{2p^2}{x}.$$

Solution. The given equation is

$$3y = 2px - \frac{2p^2}{x} \quad ...(1)$$

or $$y = \frac{2}{3}px - \frac{2}{3}\frac{p^2}{x}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{2}{3}x.\frac{dp}{dx} + \frac{2}{3}p - \frac{2}{3}.\frac{1}{x}.2p\frac{dp}{dx} + \frac{2}{3}\frac{p^2}{x^2}$$

or $$p = \frac{2}{3}x\frac{dp}{dx} + \frac{2}{3}p - \frac{4p}{3x}\frac{dp}{dx} + \frac{2}{3}\frac{p^2}{x^2}$$

or $$\frac{2}{3}x\frac{dp}{dx} - \frac{1}{3}p - \frac{4}{3}\frac{p}{x}\frac{dp}{dx} + \frac{2p^2}{3x^2} = 0$$

or $$2x\frac{dp}{dx} - p - \frac{2p}{x^2}\left(2x\frac{dp}{dx} - p\right) = 0$$

or $$\left(2x.\frac{dp}{dx} - p\right)\left(1 - \frac{2p}{x^2}\right) = 0$$

$\therefore$ Either $2x\dfrac{dp}{dx} - p = 0$ or $1 - \dfrac{2p}{x^2} = 0$

Rejecting $1 - \frac{2p}{x^2} = 0$, we get

$$2x\frac{dp}{dx} - p = 0$$

or $$2\frac{1}{p}\,dp = \frac{1}{x}\,dx \qquad \text{[Separating the variables]}$$

Integrating both sides, we get

$$2\log p = \log x + \log c$$

where c is any arbitrary constant.

or $$\log p^2 = \log cx \;\Rightarrow\; p^2 = cx$$

Putting the value of p^2 in (1), we get

$$3y = 2px - 2c$$

or $$3y + 2c = 2px$$

Squaring both sides, we have

$$(3y + 2c)^2 = 4p^2x^2$$

or $$(3y + 2c)^2 = 4cx^3, \qquad ...(2)$$

which is the complete primitive.

Now, given differential equation (1) can be written as

$$2p^2 - 2x^2p + 3xy = 0 \qquad ...(3)$$

Equation (2) can be written as

$$4c^2 + 4c\,(3y - x^3) + 9y^2 = 0 \qquad ...(4)$$

From (4), c-discriminant is

$$16(x^3 - 3y)^2 - 144y^2 = 0$$

i.e., $$x^6 - 6x^3y = 0$$

or $$x^3(x^3 - 6y) = 0$$

Also from (3), p-discriminant is

$$4x^4 - 24xy = 0$$

i.e., $$x(x^3 - 6y) = 0$$

We find that factor $x^3 - 6y$ occurs only once in both p and c-discriminants.

Hence $x^3 - 6y = 0$ is the required singular solution.

Example 4.

Obtain the complete primitive and the singular solution of

$$x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} + 4x = 0. \qquad \text{[K.U. 2016,13; M.D.U. 2009, 08, 04]}$$

Solution. The given differential equation is

$$x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} + 4x = 0$$

or $$xp^2 - 2yp + 4x = 0 \qquad ...(1) \qquad \left[\because \; p = \frac{dy}{dx}\right]$$

or $$y = \frac{xp}{2} + \frac{2x}{p}$$

Differentiating w.r.t. x, we get $\dfrac{dy}{dx} = \dfrac{x}{2}\dfrac{dp}{dx} + \dfrac{p}{2} + \dfrac{2}{p} - \dfrac{2x}{p^2}\cdot\dfrac{dp}{dx}$

or $$p = \frac{x}{2}\cdot\frac{dp}{dx} + \frac{p}{2} + \frac{2}{p} - \frac{2x}{p^2}\cdot\frac{dp}{dx}$$

or $$\frac{p}{2} - \frac{2}{p} = \left(\frac{x}{2} - \frac{2x}{p^2}\right)\frac{dp}{dx}$$

or $$\frac{p^2 - 4}{2p} = \frac{x}{2p^2}(p^2 - 4)\frac{dp}{dx}$$

or $$(p^2 - 4)\left(1 - \frac{x}{p}\frac{dp}{dx}\right) = 0$$

$\therefore$ Either $p^2 - 4 = 0$ or $1 - \dfrac{x}{p}\cdot\dfrac{dp}{dx} = 0$

$\therefore$ Either $p = 2$ or $\dfrac{x}{p}\cdot\dfrac{dp}{dx} = 1$ or $\dfrac{dp}{p} = \dfrac{dx}{x}$ [*Separating the variables*]

Integrating both sides, we get

$$\log p = \log x + \log c$$

where c is any arbitrary constant.

or $$\log p = \log cx \;\Rightarrow\; p = cx$$

Putting this value of p in (1), we get

$$c^2x^3 - 2ycx + 4x = 0$$

or $$c^2x^2 - 2yc + 4 = 0$$

which is the complete primitive of (1). ...(2)

From (2), c-discriminant is

$$4y^2 - 16x^2 = 0 \quad i.e., \quad y^2 - 4x^2 = 0$$

Also from (1), p-discriminant is $4y^2 - 16x^2 = 0$

Since $y^2 - 4x^2$ is a non-repeated common factor in p and c- discriminants and it satisfies the differential equation (1), therefore the singular solution of (1) is

$$y^2 - 4x^2 = 0 \quad i.e., \quad y = \pm 2x.$$

Example 5.

Obtain the singular solution of the equation

$$p^2y^2 \cos^2 \alpha - 2pxy \sin^2 \alpha + y^2 - x^2 \sin^2 \alpha = 0$$

directly from the equation and from the complete primitive explaining the geometrical siginificance of the irrelevant factors that present themselves.

Solution. The given differential equation is

$$p^2y^2 \cos^2 \alpha - 2pxy \sin^2 \alpha + y^2 - x^2 \sin^2 \alpha = 0, \quad ...(1)$$

which is a quadratic in p.

$\therefore$ p-discriminant is

$$4x^2y^2 \sin^4 \alpha - 4y^2 \cos^2 \alpha\, (y^2 - x^2 \sin^2 \alpha) = 0$$

or $$4y^2\, [x^2 \sin^4 \alpha - y^2 \cos^2 \alpha + x^2 \sin^2 \alpha \cos^2 \alpha] = 0$$

or $$4y^2\, [x^2 \sin^2 \alpha\, (\sin^2 \alpha + \cos^2 \alpha) - y^2 \cos^2 \alpha] = 0$$

or $$4y^2\, [x^2 \sin^2 \alpha - y^2 \cos^2 \alpha] = 0$$

or $$4y^2 \cos^2 \alpha\, [x^2 \tan^2 \alpha - y^2] = 0 \quad ...(2)$$

Dividing (1) by $\cos^2 \alpha$, we have

$$p^2y^2 - 2pxy \tan^2 \alpha + y^2 \sec^2 \alpha - x^2 \tan^2 \alpha = 0$$

$$\therefore \quad p = \frac{2xy \tan^2 \alpha \pm \sqrt{4x^2y^2 \tan^4 \alpha - 4y^4 \sec^2 \alpha + 4x^2y^2 \tan^2 \alpha}}{2y^2}$$

$$= \frac{2xy \tan^2 \alpha \pm 2y\sqrt{x^2 \tan^4 \alpha - y^2 \sec^2 \alpha + x^2 \tan^2 \alpha}}{2y^2}$$

$$= \frac{x \tan^2 \alpha \pm \sqrt{x^2 \tan^2 \alpha (\tan^2 \alpha + 1) - y^2 \sec^2 \alpha}}{y}$$

$$= \frac{x \tan^2 \alpha \pm \sec \alpha \sqrt{x^2 \tan^2 \alpha - y^2}}{y}$$

or $$py = x \tan^2 \alpha \pm \sec \alpha \sqrt{x^2 \tan^2 \alpha - y^2}$$

or $$y \frac{dy}{dx} = x \tan^2 \alpha \pm \sec \alpha \sqrt{x^2 \tan^2 \alpha - y^2}$$

or $$y\, dy = x \tan^2 \alpha\, dx \pm \sec \alpha \sqrt{(x^2 \tan^2 \alpha - y^2)}\, dx$$ [*Separating the variables*]

or $$\frac{x \tan^2 \alpha\, dx - y\, dy}{\sqrt{x^2 \tan^2 \alpha - y^2}} = \pm (\sec \alpha)\, dx$$

Integrating both sides, we have

$$(x^2 \tan^2 \alpha - y^2)^{1/2} = c \pm x \sec \alpha,$$

where c is an arbitrary constant.

Squaring both sides, we get

$$x^2 \tan^2 \alpha - y^2 = c^2 \pm 2cx \sec \alpha + x^2 \sec^2 \alpha$$

or $$c^2 \pm 2cx \sec \alpha + x^2 + y^2 = 0 \quad ...(3)$$

which clearly represents a family of circles for all values of c.

Also, equation (3) is the complete primitive.

$\therefore$ c-discriminant from (3) is

$$4x^2 \sec^2 \alpha - 4(x^2 + y^2) = 0$$

or $$x^2 \tan^2 \alpha - y^2 = 0 \quad ...(4)$$

From (2) and (4), we observe that the common factor of p and c-discriminants occurring once in them is

$$x^2 \tan^2 \alpha - y^2 = 0$$

Thus the required singular solution is

$$y = \pm x \tan \alpha$$

Equation (1) represent a family of circles given by eqn. (3), whose envelope is given by

$$y = \pm x \tan \alpha.$$

Example 6.

Reduce the differential equation (px − y) (x − py) = 2p to Clairaut's form by substitution $x^2 = u$ and $y^2 = v$ and find its complete primitive and its singular solution, if any.

[M.D.U. 2014]

Solution. The given differential equation is

$$(px - y)(x - py) = 2p$$

or
$$px^2 - xy - p^2xy + py^2 = 2p$$

or
$$p^2xy - (x^2 + y^2 - 2)\,p + xy = 0 \qquad \text{...(1)}$$

Since $x^2 = u$ and $y^2 = v$

$\therefore$
$$2x\,dx = du \quad \text{and} \quad 2y\,dy = dv$$

Dividing, we have $\dfrac{2y}{2x}\dfrac{dy}{dx} = \dfrac{dv}{du} \quad \Rightarrow \quad \dfrac{y}{x}\cdot\dfrac{dy}{dx} = \dfrac{dv}{du}$

or
$$\frac{y}{x}\cdot p = P \quad \text{where} \quad \frac{dy}{dx} = p \quad \text{and} \quad \frac{dv}{du} = P$$

or
$$p = \frac{xP}{y}$$

Substituting this value of p in (1), we get

$$xy\cdot\frac{x^2P^2}{y^2} - (x^2 + y^2 - 2)\frac{xP}{y} + xy = 0$$

or
$$x^2P^2 - (x^2 + y^2 - 2)\,P + y^2 = 0$$

or
$$(x^2P - y^2)(1 - P) = 2P$$

or
$$(uP - v)(1 - P) = 2P$$

or
$$uP - v = \frac{2P}{1-P} \quad \Rightarrow \quad v = Pu - \frac{2P}{1-P},$$

which is of Clairaut's form, $v = u.p + f(p)$

Thus the complete primitive is $v = cu - \dfrac{2c}{1-c}$ [Replacing P by c]

i.e.,
$$y^2 = cx^2 - \frac{2c}{1-c}$$

or
$$c^2x^2 - c\,(x^2 + y^2 - 2) + y^2 = 0 \qquad \text{...(2)}$$

Now from (1), p-discriminant is

$$(x^2 + y^2 - 2)^2 - 4x^2y^2 = 0$$

or $$[x^2 + y^2 - 2 - 2xy]\,[x^2 + y^2 - 2 + 2xy] = 0$$

or $$[(x - y)^2 - 2]\,[(x + y)^2 - 2] = 0$$

or $$(x - y + \sqrt{2})(x - y - \sqrt{2})(x + y + \sqrt{2})(x + y - \sqrt{2}) = 0 \qquad ...(3)$$

The c-discriminant from (2) is

$$(x^2 + y^2 - 2)^2 - 4x^2y^2 = 0 \qquad ...(4)$$

which is same as p-discriminant.

Hence, the common factors of p-discriminant and c-discriminant occurring once in them give singular solutions

i.e., $$(x - y + \sqrt{2})(x - y - \sqrt{2})(x + y + \sqrt{2})(x + y - \sqrt{2}) = 0$$

or $$x - y + \sqrt{2} = 0,\quad x - y - \sqrt{2} = 0,\quad x + y + \sqrt{2} = 0,\quad x + y - \sqrt{2} = 0$$

are the singular solutions

Example 7.

Reduce the equation $xyp^2 - (x^2 + y^2 - 1)\,p + xy = 0$ to Clairaut's form by substituting $x^2 = u$ and $y^2 = v$. Hence show that the equation represents a family of conics touching the four sides of a square. [K.U. 2014; M.D.U. 2011]

Solution. The given differential equation is

$$xyp^2 - (x^2 + y^2 - 1)\,p + xy = 0 \qquad ...(1)$$

Since $x^2 = u$ and $y^2 = v$

$\therefore$ $2x\,dx = du$ and $2y\,dy = dv$

Dividing, we have $$\frac{y}{x}\cdot\frac{dy}{dx} = \frac{dv}{du}$$

i.e., $$\frac{y}{x}\,p = \text{P} \text{ where } p = \frac{dy}{dx} \text{ and } \text{P} = \frac{dv}{du}$$

or $$p = \frac{\text{P}x}{y}$$

Substituting this value of p in (1), we get

$$xy\,\frac{x^2\text{P}^2}{y^2} - (x^2 + y^2 - 1)\,\frac{x\text{P}}{y} + xy = 0$$

or $$\frac{x^2\text{P}^2}{y} - (x^2 + y^2 - 1)\,\frac{\text{P}}{y} + y = 0$$

or $$x^2\text{P}^2 - (x^2 + y^2 - 1)\,\text{P} + y^2 = 0$$

or $$u\text{P}^2 - (u + v - 1)\,\text{P} + v = 0$$

or $$u(P^2 - p) - v(P - 1) + P = 0$$

or $$v(P - 1) = uP(P - 1) + P$$

or $$v = Pu + \frac{P}{P - 1}$$

which is of Clairaut's form.

Hence replacing P by c, the complete primitive is

$$v = cu + \frac{c}{c - 1}$$

or $$y^2 = cx^2 + \frac{c}{c - 1}$$

or $$c^2x^2 - (x^2 + y^2 - 1)\,c + y^2 = 0 \qquad ...(2)$$

From (1), p-discriminant is

$$(x^2 + y^2 - 1)^2 - 4x^2y^2 = 0$$

or $$(x^2 + y^2 - 2xy - 1)(x^2 + y^2 + 2xy - 1) = 0$$

or $$[(x - y)^2 - 1]\,[(x + y)^2 - 1] = 0$$

or $$(x - y - 1)(x - y + 1)(x + y - 1)(x + y + 1) = 0 \qquad ...(3)$$

From (2), c-discriminant is

$$(x^2 + y^2 - 1)^2 - 4x^2y^2 = 0, \qquad ...(4)$$

which is same as p-discriminant.

The singular solutions *i.e.*, envelope of the family of conics given by eq. (2) is

$$(x - y - 1)(x - y + 1)(x + y - 1)(x + y + 1) = 0$$

i.e., $$x - y - 1 = 0,\quad x - y + 1 = 0,\quad x + y - 1 = 0,\quad x + y + 1 = 0$$

These four lines clearly form a square.

Hence the differential equation (1) represents conic (2) touching the four sides of a square.

EXERCISE 2.6

1. How would you find the singular solution of a differential equation whose complete primitive is $\phi(x, y, c) = 0$.

Find the complete primitive and singular solution of the following :

2. $(xp - y)^2 = p^2 - 1$ [K.U. 2012]

3. $y^2 - 2pxy + p^2(x^2 - 1) = m^2$

4. $4xp^2 = (3x - a)^2$ [K.U. 2017, 12]

5. $p^2(x^2 - a^2) - 2pxy + y^2 - b^2 = 0$

6. $y = px + \sqrt{b^2 + a^2p^2}$

7. $xp^2 - 2py + x + 2y = 0$

8. $x^2(y - xp) = yp^2$

9. $4x(x-1)(x-2)p^2 = (3x^2 - 6x + 2)^2$

10. $8p^3x = y(12p^2 - 9)$

11. $y^2(y - px) = x^4p^2$.

12. $p^2 + y^2 = 1$

13. $4p^2(x - 2) = 1$ [M.D.U. 2017, 13]

14. $xp^2 - 2yp + 4x = 0$

15. Find the singular solution of the following :

(i) $y = px - 2p^2$ [K.U. 2005]

(ii) $xp^2 - (x - a)^2 = 0$ [M.D.U. 2015]

(iii) $yp^2 - 2xp + y = 0$ [M.D.U. 2017; K.U. 2015, 06, 04]

(iv) $(8p^3 - 27)x = 12p^2y$

16. Find the differential equation of the family of circles $x^2 + y^2 + 2cx + 2c^2 - 1 = 0$ (c is arbitrary constant). Determine the singular solution of the differential equation.

17. Reduce the equation $x^2p^2 + py(2x + y) + y^2 = 0$ where $p = \frac{dy}{dx}$ to Clairaut's form by putting $u = y$ and $v = xy$ and find its complete primitive and also its singular solution. [M.D.U. 2015; K.U. 2003]

18. Reduce the equation $xp^2 - 2py + x + 2y = 0$ to Clairaut's form by putting $y - x = v$ and $x^2 = u$. Hence obtain and interpret the primitive and singular solution of the equation. [M.D.U. 2016, 15]

ANSWERS

2. $(y - cx)^2 = c^2 - 1; x^2 - y^2 = 1$ is singular solution.

3. $(y - cx)^2 = m^2 + c^2; y^2 + m^2x^2 = m^2$ is singular solution.

4. $(y - c)^2 = x(x - a)^2 ; x = 0$ is singular solution.

5. $(y - cx)^2 = a^2c^2 + b^2; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is singular solution.

6. $(y - cx)^2 = a^2c^2 + b^2; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is singular solution.

7. $y - x = x^2c + \frac{1}{2c}; y - x = \pm\sqrt{2}x$ is singular solution.

8. $y^2 = cx^2 + c^2 ; x^4 + 4y^2 = 0$ is singular solution.

9. $(y - c)^2 = x^3 - 3x^2 + 2x;\ x(x - 1)(x - 2) = 0$ is singular solution.

10. $3cy^2 = (x + c)^3$; $y(9x^2 - 4y^2) = 0$ is singular solution.

11. $c^2xy + cy - x = 0$; $y(y + 4x^2) = 0$ is singular solution.

12. $y = \sin(x + c)$; $y^2 - 1 = 0$ is singular solution.

13. $(y - c)^2 - (x - 2) = 0$, $x - 2 = 0$ is singular solution.

14. $2cy - c^2x^2 - 4 = 0$, $y^2 - 4x^2 = 0$ is singular solution.

15. (i) $x^2 - 8y = 0$ (ii) $x = 0$

(iii) $x^2 - y^2 = 0$ (iv) $4y^3 + 27x^3 = 0$

16. $2y^2p^2 + 2xyp + (x^2 + y^2 - 1) = 0$; $x^2 + 2y^2 - 2 = 0$ is singular solution.

17. $xy = cy + c^2$: $y = 0$ and $y + 4x = 0$ are singular solution.

18. $2c^2x^2 - 2c(y - x) + 1 = 0$; $(y - x)^2 - 2x^2 = 0$ is singular solution.

3 ORTHOGONAL TRAJECTORIES

3.1. TRAJECTORIES

Trajectory : *A curve is called* ***trajectory*** *if it cuts every member of a given family of curves according to a certain law.*

Oblique trajectory : *A curve is called* ***oblique trajectory*** *if it cuts every member of a given family of curves at certain angle other than right angles.*

Orthogonal trajectory : *A curve is called* ***orthogonal trajectory*** *if it cuts every member of a given family of curves at right angles.* [M.D.U. 2013]

Illustration. For a parameter a, $x = a$ is a family of straight lines parallel to y-axis. For parameter b, $y = b$ is another family of straight lines parallel to x-axis. As each member of family $x = a$ cuts every member of family $y = b$ at right angles, therefore these two curves are said to be orthogonal trajectories.

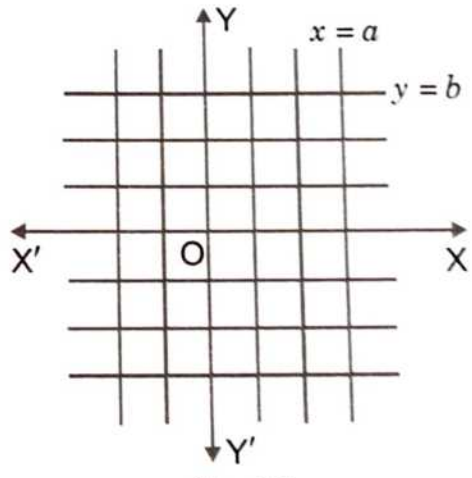

Fig. 3.1

Self orthogonal : *If $f(x, y, c) = 0$ is a family of curves and its orthogonal trajectories are also the same family of curves $f(x, y, c) = 0$, then such a family of curves is called* **self orthogonal**.

3.2. ORTHOGONAL TRAJECTORIES IN CARTESIAN CO-ORDINATES

To find the equation of orthogonal trajectories in cartesian co-ordinates the following steps are employed :

Step I : Let the equation of the family of given curves be

$$f(x, y, c) = 0 \qquad ...(1)$$

Differentiate equation (1) w.r.t. x and then eliminate the arbitrary constant between this derived equation and given equation (1). The resulting equation is the differential equation of the given family.

Step II : Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in the equation obtained in step I and the resulting equation is the differential equation of the orthogonal trajectory.

Step III : Integrate the equation obtained in step II and the resulting equation is the equation of the orthogonal trajectories of the given family.

3.3. ORTHOGONAL TRAJECTORIES IN POLAR CO-ORDINATES

Let the given family of polar curves be $\phi(r, \theta, c) = 0$ ...(1)

Differentiating (1) w.r.t. θ and eliminating parameter c between (1) and the resulting equation, we have

$$F\left(r, \theta, \frac{dr}{d\theta}\right) = 0 \quad ...(2)$$

which is a differential equation of the family (1).

We know that the tangent of the angle between the radius vector and tangent to a curve of the given system at any point (r, θ) is $r\frac{d\theta}{dr}$.

Since the tangents of the curve and its trajectory are perpendicular to each other

$$m = -\frac{1}{r}\frac{dr}{d\theta}$$

where m is the tangent of the angle between the radius-vector and the tangent to the trajectory through (r, θ).

The differential equation of the required trajectory is obtained by putting $-\frac{1}{r}\frac{dr}{d\theta}$ for $r\frac{d\theta}{dr}$ or $-r^2\frac{d\theta}{dr}$ for $\frac{dr}{d\theta}$ in (2).

Thus, the required differential equation is

$$F\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0 \quad ...(3)$$

Integrating both sides of (3), we get the equation of orthogonal trajectories.

SOLVED EXAMPLES

Example 1.

Find the orthogonal trajectories of the family of parabolas $y = ax^2$.

[*M.D.U. 2014; K.U. 2013*]

Solution. The given equation is $y = ax^2$ ...(1)

Differentiating (1) w.r.t. x, we have

$$\frac{dy}{dx} = 2ax \qquad ...(2)$$

Eliminating arbitrary constant a between (1) and (2), we get

$$y = x^2 \cdot \frac{1}{2x}\frac{dy}{dx}$$

or
$$y = \frac{x}{2}\frac{dy}{dx} \qquad ...(3)$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (3), we have

$$y = -\frac{x}{2}\frac{dx}{dy}$$

or
$$2y\,dy = -x\,dx \qquad \text{[Separating the variables]}$$

Integrating both sides, we get

$$\int 2y\,dy = -\int x\,dx + c_1$$

where c_1 is any arbitrary constant.

i.e.,
$$y^2 = -\frac{x^2}{2} + c_1$$

or
$$2y^2 = -x^2 + 2c_1$$

or
$$x^2 + 2y^2 = 2c_1$$

or
$$x^2 + 2y^2 = c^2 \qquad [\because 2c_1 = c^2]$$

which is the equation of orthogonal trajectories of the given family of parabolas.

Example 2.

Find the orthogonal trajectories of the family of co-axal circles $x^2 + y^2 + 2gx + c = 0$, where g is a parameter and c is a constant.

[K.U. 2014]

Solution. The given equation is $x^2 + y^2 + 2gx + c = 0$ ...(1)

Differentiating (1), we have $2x + 2y\dfrac{dy}{dx} + 2g = 0$

or $$x + y\frac{dy}{dx} + g = 0 \quad \text{...(2)}$$

Eliminating arbitrary constant g between (1) and (2), we get

$$x^2 + y^2 - 2x\left(x + y\frac{dy}{dx}\right) + c = 0$$

or $$y^2 - x^2 - 2xy\frac{dy}{dx} + c = 0 \quad \text{...(3)}$$

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ in (3), we get

$$y^2 - x^2 + 2xy\frac{dx}{dy} + c = 0$$

or $$2xy\frac{dx}{dy} - x^2 = -c - y^2 \quad \text{...(4)}$$

Put $x^2 = t$, so that $2x\dfrac{dx}{dy} = \dfrac{dt}{dy}$

$\therefore$ From (4), $$y\frac{dt}{dy} - t = -c - y^2$$

or $$\frac{dt}{dy} - \frac{1}{y}t = -\frac{c}{y} - y$$

which is a linear differential equation in t.

$$\therefore \quad \text{I.F.} = e^{\int -\frac{1}{y}dy} = e^{-\log y} = e^{\log y^{-1}} = e^{\log \frac{1}{y}} = \frac{1}{y}$$

Thus, the solution is

$$t\frac{1}{y} = \int -\left(\frac{c}{y} + y\right)\frac{1}{y}\,dy = \frac{c}{y} - y - f, \text{ where } f \text{ is an arbitrary constant.}$$

or $$\frac{t}{y} = \frac{c}{y} - y - f$$

or $$t = c - fy - y^2$$

or $$x^2 + y^2 + fy - c = 0, \qquad [\because t = x^2]$$

which is the equation of the orthogonal trajectories of the given family of curves.

Example 3.

Show that the system of confocal conics $\dfrac{x^2}{a^2 + \lambda} + \dfrac{y^2}{b^2 + \lambda} = 1$ ***is self-orthogonal.***

[K.U. 2016, 10; M.D.U. 2015]

Solution. The given equation is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \qquad ...(1)$$

Differentiating (1) w.r.t. x, we get

$$\frac{2x}{a^2+\lambda}+\frac{2y\dfrac{dy}{dx}}{b^2+\lambda}=0$$

or
$$\frac{x}{a^2+\lambda}+\frac{y\dfrac{dy}{dx}}{b^2+\lambda}=0$$

or
$$\lambda=-\frac{b^2x+a^2y\dfrac{dy}{dx}}{x+y\dfrac{dy}{dx}}$$

$\therefore$
$$a^2+\lambda=\frac{(a^2-b^2)\,x}{x+y\dfrac{dy}{dx}} \qquad ...(2)$$

and
$$b^2+\lambda=\frac{-(a^2-b^2)\,y\dfrac{dy}{dx}}{x+y\dfrac{dy}{dx}}$$

Eliminating λ between (1) and (2), we have

$$\frac{x^2\left(x+y\dfrac{dy}{dx}\right)}{(a^2-b^2)x}-\frac{y^2\left(x+y\dfrac{dy}{dx}\right)}{(a^2-b^2)\,y\dfrac{dy}{dx}}=1$$

or
$$\left(x+y\frac{dy}{dx}\right)\left(x-\frac{y}{\dfrac{dy}{dx}}\right)=a^2-b^2 \qquad ...(3)$$

Now replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ in (3), we get

$$\left(x-y\frac{dx}{dy}\right)\left(x+y\frac{dy}{dx}\right)=a^2-b^2 \qquad ...(4)$$

Now (4) is same as (3). Thus the system of confocal conics is self orthogonal.

Example 4.

Find the orthogonal trajectories of the cardioid $r = a(1 - \cos\theta)$, where a is the parameter. *[K.U. 2012, 09; M.D.U. 2008, 07, 06]*

Solution. The given equation is $r = a(1 - \cos\theta)$ ...(1)

Differentiating, we get $$\frac{dr}{d\theta} = a\sin\theta \quad ...(2)$$

Eliminating a between (1) and (2), we have

$$r = \frac{1}{\sin\theta}\frac{dr}{d\theta}(1-\cos\theta)$$

or $$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$, we get

$$-r\frac{d\theta}{dr} = \cot\left(\frac{\theta}{2}\right)$$

or $$\frac{1}{r}dr = -\tan\frac{\theta}{2}d\theta$$

Integrating both sides, we get

$$\int\frac{1}{r}dr = -\int\tan\frac{\theta}{2}d\theta + c_1$$

where c_1 is any arbitrary constant.

or $$\log r = 2\log\cos\frac{\theta}{2} + \log 2c \qquad [\text{Putting } c_1 = \log 2c]$$

or $$\log r = \log\cos^2\frac{\theta}{2} + \log 2c$$

or $$\log r = \log 2c\cos^2\frac{\theta}{2}$$

or $$r = 2c\cos^2\frac{\theta}{2} = c\left(2\cos^2\frac{\theta}{2}\right)$$

or $$r = c(1+\cos\theta),$$

which is the equation of the orthogonal trajectories of the given cardioid.

Example 5.

Find the orthogonal trajectory for the family of curves $r = a(1 + \sin\theta)$.

[M.D.U. 2017, 15, 11; K.U. 2016]

Solution. The given equation of the family of curves is

$$r = a(1 + \sin\theta) \qquad \text{...(1)}$$

Differentiating (1) w.r.t. θ, we get $\dfrac{dr}{d\theta} = a\cos\theta$ ...(2)

Eliminating a between (1) and (2), we have

$$\frac{dr}{d\theta} = \frac{r}{1+\sin\theta}\cdot\cos\theta \Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{1+\sin\theta} \qquad \text{...(3)}$$

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ in (3), we get

$$-r\frac{d\theta}{dr} = \frac{\cos\theta}{1+\sin\theta}$$

or
$$\frac{1+\sin\theta}{\cos\theta}d\theta = -\frac{dr}{r} \qquad \text{...(4)}$$

which is the differential equation of the orthogonal trajectory.

Integrating both sides of (4), we have

$$\int\frac{1+\sin\theta}{\cos\theta}d\theta = -\int\frac{dr}{r} + \log c$$

or
$$\int\frac{1-\sin^2\theta}{\cos\theta(1-\sin\theta)}d\theta = -\log r + \log c$$

or
$$\int\frac{\cos\theta}{1-\sin\theta}d\theta = -\log r + \log c$$

or
$$-\log(1-\sin\theta) = -\log\frac{r}{c}$$

or
$$1-\sin\theta = \frac{r}{c}$$

or
$$r = c(1-\sin\theta),$$

which is the equation of the orthogonal trajectories of the given family of curves.

Example 6.

Find the orthogonal trajectories of the curves $r^n \sin n\theta = a^n$.

[K.U. 2015, 13; M.D.U. 2004]

Solution. The given equation of the system of curves is

$$r^n \sin n\theta = a^n \quad \text{...(1)}$$

Applying log on both sides, we have

$$n \log r + \log \sin n\theta = n \log a$$

Differentiating both sides w.r.t. θ, we get

$$n \frac{1}{r} \cdot \frac{dr}{d\theta} + \frac{1}{\sin n\theta} \cdot n \cos n\theta = 0$$

or $$\frac{1}{r} \frac{dr}{d\theta} + \cot n\theta = 0$$

or $$r \frac{d\theta}{dr} = -\tan n\theta \quad \text{...(2)}$$

Replacing $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ in (2), we have

$$-\frac{1}{r} \frac{dr}{d\theta} = -\tan n\theta$$

or $$\frac{dr}{r} = \tan n\theta \, d\theta \quad \text{[Separating the variables]}$$

Integrating both sides, we get $\int \frac{dr}{r} = \int \tan n\theta \, d\theta + c_1$,

which c_1 is any arbitrary constant.

or $$\log r = \frac{1}{n} \log \sec n\theta + \log c \quad \text{[Putting } c_1 = \log c\text{]}$$

or $$\log r = \log (\sec n\theta)^{1/n} + \log c$$

or $$\log r = \log c \, (\sec n\theta)^{1/n}$$

or $$r = c \, (\sec n\theta)^{1/n}$$

or $$r^n = c^n \sec n\theta$$

or $$r^n \cos n\theta = c^n,$$

which is equation of the orthogonal trajectories of the given curves.

EXERCISE 3.1

1. Find the orthogonal trajectories of the following family of curves :

(*i*) $y = ax^3$

(*ii*) $y = ax^n$ [*M.D.U. 2016*]

(*iii*) $ax^2 + y^2 = 1$

(*iv*) $x^2 + y^2 = a^2$ [*M.D.U. 2016*]

(*v*) $y^2 = 4ax$

(*vi*) $px^2 + qy^2 = a^2$, where p and q are constants
[*K.U. 2015, 03; M.D.U. 2010, 08, 05*]

2. Find the orthogonal trajectories of the parabola whose equation is $y = 2ax^2$.

3. Find the orthogonal trajectories of the family of circles $x^2 + y^2 + 2fy + 1 = 0$, f being a parameter.
[*M.D.U. 2009*]

4. Find the orthogonal trajectories of $x^2 + y^2 + 2gx + c = 0$, where c is a parameter.
[*K.U. 2015, 07; M.D.U. 2009*]

5. Find the orthogonal trajectories of the family of curves $x^2 + y^2 + 2gx + c = 0$, g being a parameter.

6. Show that the orthogonal trajectories of $x^2 + y^2 + 2gx + 1 = 0$ is $x^2 + y^2 + 2fy - 1 = 0$.

7. Find the orthogonal trajectories of the series of hypocycloids $x^{2/3} + y^{2/3} = a^{2/3}$. [*M.D.U. 2013*]

8. Find the orthogonal trajectories of the family of semi-cubical parabolas $ay^2 = x^3$, where a is a variable parameter.

9. Find the orthogonal trajectories of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2 + \lambda} = 1$, where λ is a parameter. [*M.D.U. 2012, 05*]

10. Show that the system of co-focal and co-axal parabolas $y^2 = 4a(x + a)$ is self orthogonal.
[*M.D.U. 2014*]

Or

Find the differential equation satisfied by the system of parabolas $y^2 = 4a(x + a)$ and find the orthogonal trajectories of the system.

11. Find the orthogonal trajectories of the series of curves $r = a + \sin 5\theta$.

12. Determine the orthogonal trajectories of the system of curves $r^n = a^n \cos n\theta$.

13. Find the orthogonal trajectories of the family of cardioids :

(*i*) $r = a(1 + \cos\theta)$, a being a parameter.

(*ii*) $r = b(1 - \cos\theta)$, b being a parameter.

14. Find orthogonal trajectories of the series of logarithmic spirals $r = a^\theta$, where a varies.

15. A family of parabolas has a common focus and common axis. Find the orthogonal family. [M.D.U. 2016]

16. Find the orthogonal trajectories of $\left(r+\dfrac{k^2}{r}\right)\cos\theta=\alpha$ where α is a parameter. [K.U. 2011]

17. Show that the family of confocal conics $\dfrac{x^2}{a}+\dfrac{y^2}{a-b}=1$ is self orthogonal, where a is the parameter and b is constant. [M.D.U. 2017]

[**Hint** : Proceed as in Example 3 on page 3.4]

ANSWERS

1. (i) $x^2+3y^2=c$ (ii) $x^2+ny^2=c$

(iii) $y=ce^{(x^2+y^2)/2}$ (iv) $y=cx$

(v) $2x^2+y^2=c^2$ (vi) $y^p=cx^q$

2. $x^2+2y^2=c^2$

3. $x^2+y^2+cx-1=0$

4. $x+g=ay$

5. $x^2+y^2-c'y-c=0$, where c' is an arbitrary constant.

7. $x^{4/3}=y^{4/3}+c^{4/3}$

8. $3y^2+2x^2=c^2$

9. $x^2+y^2-2a^2\log x=k$

11. $25=r\log(\tan 5\theta+\sec 5\theta)+cr$

12. $r^n=c^n\sin n\theta$

13. (i) $r=c(1-\cos\theta)$ (ii) $r=c(1+\cos\theta)$

14. $r=e^{\sqrt{c^2-\theta^2}}$

15. $r=\dfrac{c}{1-\cos\theta}$

16. $r^2-k^2=cr\operatorname{cosec}\theta$

4 LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT CO-EFFICIENTS

4.1. DEFINITION

A linear differential equation with constant co-efficients is the one in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together and the coefficients are constants.

A differential equation of the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + ... + P_n y = X \qquad .....(1)$$

where P_0, P_1, P_2, ..., P_n and X are functions of x or constants, is called a *linear differential equation of nth order.*

And if P_0, P_1, P_2, ..., P_n are all constants (not functions of x) and X is some function of x, then the equation is a *linear differential equation with constant co-efficients of nth order.*

4.2. THE DIFFERENTIAL OPERATOR 'D'

The part $\frac{d}{dx}$ of the symbol $\frac{dy}{dx}$ may be regarded as a differential operator, such that when it operates on y, the result is the derivative of y.

It is usual to write D for $\frac{d}{dx}$, D^2 for $\frac{d^2}{dx^2}$, D^3 for $\frac{d^3}{dx^3}$, ..., D^n for $\frac{d^n}{dx^n}$

In terms of the operator D, the differential equation (1) can be written as

$$(P_0 D^n + P_1 D^{n-1} + P_2 D^{n-2} + ... + P^n)\, y = X$$

4.3. COMPLETE SOLUTION OF THE LINEAR DIFFERENTIAL EQUATION

Theorem. ***If $y = y_1, y = y_2, \ldots\ldots, y = y_n$ are linear independent solutions of $(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n)\, y = 0$ then $y = c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n$ is the general or complete solution of the differential equation (1), where $c_1, c_2, c_3, \ldots\ldots, c_n$ are n arbitrary constants.***

Proof. The given equation is

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \ldots\ldots + a_n y = 0 \qquad \ldots(1)$$

Since $y = y_1, y = y_2, \ldots\ldots, y = y_n$ are solutions of equation (1), therefore $y_1, y_2, \ldots\ldots, y_n$ must satisfy (1).

$$\begin{aligned} \therefore \quad & a_0 D^n y_1 + a_1 D^{n-1} y_1 + a_2 D^{n-2} y_1 + \ldots\ldots + a_n y_1 = 0 \\ & a_0 D^n y_2 + a_1 D^{n-1} y_2 + a_2 D^{n-2} y_2 + \ldots\ldots + a_n y_2 = 0 \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots\ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots\ldots \\ & a_0 D^n y_n + a_1 D^{n-1} y_n + a_2 D^{n-2} y_n + \ldots\ldots + a_n y_n = 0 \end{aligned} \qquad \ldots(2)$$

Multiplying above equations successively by $c_1, c_2, \ldots\ldots, c_n$ and adding vertically, we get

$$\begin{aligned} a_0 D^n [c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n] + a_1 D^{n-1} [c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n] \\ + a_n D^{n-2} [c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n] + \ldots\ldots \\ \ldots\ldots\ldots + a_n [c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n] = 0 \end{aligned}$$

This shows that $y = c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n$ is also a solution of equation (1). As this solution contains n arbitrary constants, so this *solution is the general solution or complete solution of the differential equation (1).*

4.4. AUXILIARY EQUATION (A.E.)

Definition. *The equation obtained by equating to zero the symbolic coefficient of y is called the auxiliary equation, provided D is taken as an algebraic quantity.*

Consider the differential equation

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a^n)\, y = 0 \qquad \ldots(1)$$

where $a_0, a_1, a_2, \ldots\ldots, a_n$ are all constants.

Let $y = e^{mx}$ be a solution of this equation (1).

Then putting $y = e^{mx}$, $Dy = me^{mx}$, $D^2y = m^2e^{mx}$,......, $D^ny = m^n e^{mx}$ in equation (1), we have

$$(a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + ... + a_n) e^{mx} = 0$$

Cancelling e^{mx} ($\because e^{mx} \neq 0$ for any m), we have

$$a_0m^n + a_1m^{n-1} + a_2m^{n-2} + ... + a_n = 0 \qquad ...(2)$$

Hence e^{mx} will be root of equation (1) if m is a root of the algebraic equation (2).

Equation (2) is called the auxiliary equation for the differential equation (1).

Remark.

It is observed that the values of D obtained on solving the equation

$$a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n = 0$$

are same as the values of m obtained on solving equation (2).

Hence $a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n = 0$ can be regarded as the auxiliary equation in place of equation $a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + ... + a_n = 0$.

Therefore in practice we do not replace D by m to form the auxiliary equation. The equation in D may be regarded as auxiliary equation which is obtained by equating to zero the symbolic co-efficient of y in equation (1).

4.5. TO FIND THE COMPLETE SOLUTION OF THE DIFFERENTIAL EQUATION

$$\mathbf{(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n) y = 0.}$$

The given equation is

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n) y = 0. \qquad ...(1)$$

The auxiliary equation for (1) is

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + ... + a_n = 0. \qquad ...(2)$$

Now the following different cases arise :

Case I. ***When all the roots of auxiliary equation (2) are real and different.***

Let $m_1, m_2, m_3, ..., m_n$ be the n different and real roots of equation (2).

Then $y = e^{m_1 x}$, $y = e^{m_2 x}$, ..., $y = e^{m_n x}$ are n independent solutions of (1).

Therefore the general or complete solution of (1) is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \dots + c_ne^{m_nx}$$

Case II. ***When two roots of the auxiliary equation (2) are equal and all others are different.***

Let $m_1, m_2, m_3, \dots, m_n$ be the different roots of (2), then general solution is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \dots + c_ne^{m_nx}$$

But if two roots are equal say $m_1 = m_2$ then the solution becomes

$$y = (c_1 + c_2)e^{m_1x} + c_3e^{m_3x} + \dots + c_ne^{m_nx}$$

which clearly contains only $n - 1$ arbitrary constants (since $c_1 + c_2$ is equivalent to only one arbitrary constant.)

Therefore this is no longer a general solution.

The part of the differential equation (1) corresponding to the equal roots $m_1 = m_2$ is

$$(D - m_1)(D - m_1)y = 0$$

Putting $(D - m_1)y = v$, we get

$$(D - m_1)v = 0 \quad \text{or} \quad \frac{dv}{dx} = m_1 v$$

Separating the variables, we have

$$\frac{dv}{v} = m_1\,dx$$

Integrating,

$$\log v = m_1x + \log c$$
$$= \log e^{m_1x} + \log c$$
$$= \log ce^{m_1x}$$

$$\therefore \quad v = ce^{m_1x}$$

or $$(D - m_1)y = ce^{m_1x} \qquad [\because v = (D - m_1)y]$$

or $$\frac{dy}{dx} - m_1y = ce^{m_1x}$$

which is a linear equation of the first order.

$$\therefore \quad \text{I.F.} = e^{\int -m_1\,dx} = e^{-m_1x}$$

Hence its solution is $$y \cdot e^{-m_1x} = \int ce^{m_1x} \cdot e^{-m_1x}\,dx + c_1$$

or $$y \cdot e^{-m_1x} = \int c\,dx + c_1 = cx + c_1$$

or $$y = (cx + c_1)e^{m_1x}$$

or $$y = (c_1 + c_2x)e^{m_1x} \qquad [\textit{Taking } c = c_2 \textit{ as constant}]$$

Therefore the most general or complete solution of equation (1), when two roots of the auxiliary equation are equal, is

$$y = (c_1 + c_2 x)\, e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

Cor. In case three roots are equal, *i.e.*, $m_1 = m_2 = m_3$, then the general solution is

$$y = (c_1 + c_2 x + c_3 x^2)\, e^{m_1 x} + c_4 e^{m_4 x} + \ldots\ldots + c_n e^{m_n x}$$

Case III. *When two roots of the auxiliary equation (2) are imaginary and the rest are all real and different.*

Let $\alpha \pm i\beta$ be the imaginary roots of equation (2). Let other real and different roots be $m_3, m_4, \ldots\ldots, m_n$.

Then the complete solution is

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= e^{\alpha x} (c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$\Rightarrow \quad y = e^{\alpha x} [c_1(\cos \beta x + i \sin \beta x) + c_2(\cos \beta x - i \sin \beta x)] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$[\because e^{i\theta} = \cos\theta + i \sin\theta]$$

$$= e^{\alpha x} [(c_1 + c_2) \cos \beta x + i\,(c_1 - c_2) \sin \beta x] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= e^{\alpha x} [\mathrm{A} \cos \beta x + \mathrm{B} \sin \beta x] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}.$$

Cor. 1. Imaginary roots repeated. If the auxiliary equation has two equal pairs of imaginary roots *i.e.*, if $\alpha + i\beta$ and $\alpha - i\beta$ occur twice, then the complete solution is

$$y = e^{\alpha x} [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + \ldots\ldots + c_n e^{m_n x}.$$

Cor. 2. If a pair of roots of the auxiliary equation occur in the form of quadratic surd $\alpha \pm \sqrt{\beta}$ where β is +ve, then the corresponding term in solution may be written as

$$e^{\alpha x} [c_1 \cosh x\sqrt{\beta} + c_2 \sinh x\sqrt{\beta}]$$

or

$$c_1 e^{\alpha x} \cosh (x\sqrt{\beta} + c_2) \quad \text{or} \quad c_1 e^{\alpha x} \sinh (x\sqrt{\beta} + c_2).$$

4.6. SYNOPSIS OF THE FORMS OF SOLUTION OR RULE TO SOLVE AN EQUATION

To solve an equation of the form

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n)\, y = 0$$

1. Find the roots of the auxiliary equation

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n = 0$$

2. Find the complete solution as shown in the following table.

Roots of Auxiliary equation	***Complete solution***
Case I. All roots, $m_1, m_2, m_3, \ldots, m_n$ are real and different.	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$.
Case II. $m_1 = m_2$ are two real roots and other roots are real and different.	$y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$.
Case III. (Imaginary Roots) $\alpha \pm i\beta$, are two imaginary roots and other roots are real.	Corresponding part of the general solution is $e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$ or $c_1 e^{\alpha x} \cos(\beta x + c_2)$ or $c_1 e^{\alpha x} \sin(\beta x + c_2)$
Case IV. $\alpha \pm i\beta, \alpha \pm i\beta$ are four imaginary roots and other roots are real.	Corresponding part of complete solution is $e^{\alpha x}[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x]$

SOLVED EXAMPLES

Example 1.

If the two roots of the auxiliary equation $\lambda^2 + a_1\lambda + a_2 = 0$ are real and equal, then prove that $y = (c_1 + c_2 x) e^{\lambda x}$ is a solution of the equation $y_2 + a_1 y_1 + a_2 y = 0$, where c_1, c_2 are arbitrary constants.

Solution. The given equation is

$$y_2 + a_1 y_1 + a_2 y = 0$$

Equation in symbolic form is

$$D^2 y + a_1 D y + a_2 y = 0$$

or

$$(D^2 + a_1 D + a_2) y = 0 \quad \ldots(1)$$

Auxiliary equation (A.E.) is

$$\lambda^2 + a_1 \lambda + a_2 = 0 \quad \ldots(2)$$

The roots of (2) are given to be real and equal.

Let each root $= \lambda$

$\therefore$ Equation (1) becomes $$(D - \lambda)^2 y = 0$$

or $$(D - \lambda)(D - \lambda) y = 0 \quad ...(3)$$

Putting $(D - \lambda) y = v$, we get

$$(D - \lambda) v = 0$$

or $$\frac{dv}{dx} = \lambda v$$

or $$\frac{dv}{v} = \lambda\, dx$$

Integrating, $$\log v = \lambda x + \log c$$

or $$\log v - \log c = \lambda x$$

or $$\log \frac{v}{c} = \lambda x$$

$\therefore$ $$\frac{v}{c} = e^{\lambda x} \Rightarrow v = ce^{\lambda x}$$

or $$(D - \lambda) y = ce^{\lambda x} \qquad [\because v = (D - \lambda) y]$$

or $$\frac{dy}{dx} - \lambda y = c.e^{\lambda x},$$

which is a linear differential equation of the first order.

$$\text{I.F.} = e^{\int -\lambda dx} = e^{-\lambda x}$$

Hence, the complete solution is

$$y \cdot e^{-\lambda x} = \int c \cdot e^{\lambda x} \cdot e^{-\lambda x}\, dx + c'$$

$$= \int c\, dx + c' = cx + c'$$

or $$y = (cx + c')\, e^{\lambda x}$$

or $$y = (c_1 + c_2 x)\, e^{\lambda x}.$$

Example 2.

If λ_1, λ_2 are real and distinct roots of the auxiliary equation $\lambda^2 + a_1\lambda + a_2 = 0$ and $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$ are its solutions, then prove that $y = c_1 y_1 + c_2 y_2$ is a solution of

$$\frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$$

Is this solution general ? [M.D.U. 2001]

Solution. The auxiliary equation is $\lambda^2 + a_1\lambda + a_2 = 0$ ...(1)

Since its roots are λ_1 and λ_2 $(\lambda_1 \neq \lambda_2)$

$\therefore$ $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are two independent solutions of

$$D^2y + a_1 Dy + a_2 y = 0 \quad \ldots(2)$$

$$\therefore \quad D^2 y_1 + a_1 D y_1 + a_2 y_1 = 0 \quad \ldots(3)$$

and
$$D^2 y_2 + a_1 D y_2 + a_2 y_2 = 0 \quad \ldots(4)$$

Multiplying (3) by c_1, (4) by c_2 and adding, we get

$$D^2 [c_1 y_1 + c_2 y_2] + a_1 D [c_1 y_1 + c_2 y_2] + a_2 [c_1 y_1 + c_2 y_2] = 0$$

This shows that $y = c_1 y_1 + c_2 y_2$ is a solution of equation

$$D^2 y + a_1 Dy + a_2 y = 0$$

i.e., $y = c_1 y_1 + c_2 y_2$ is a solution of equation

$$\frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$$

Since the solution involves two arbitrary constants and the order of the given equation is also 2, therefore this solution is general.

Example 3.

Solve the differential equation

$$2\frac{d^3 y}{dx^3} - 7\frac{d^2 y}{dx^2} + 7\frac{dy}{dx} - 2y = 0.$$

Solution. The given equation is

$$2\frac{d^3 y}{dx^3} - 7\frac{d^2 y}{dx^2} + 7\frac{dy}{dx} - 2y = 0$$

Equation in symbolic form is

$$(2D^3 - 7D^2 + 7D - 2)\, y = 0$$

Auxiliary equation is $2D^3 - 7D^2 + 7D - 2 = 0$*

i.e.,
$$(D - 1)(D - 2)(2D - 1) = 0$$

$\therefore$ $D = 1, 2, \frac{1}{2}$

Here, we have three real and different roots.

Hence, the complete solution is $y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{x/2}$.

* As we have already mentioned in Remark on page 4.3, we shall be taking the auxiliary equation in D instead of m throughout this book for the sake of convenience.

Example 4.

Solve the differential equation

$$\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0.$$

Solution. The given equation is

$$\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0$$

The equation in symbolic form is

$$(D^4 - D^3 - 9D^2 - 11D - 4)\,y = 0$$

Auxiliary equation is $\quad D^4 - D^3 - 9D^2 - 11D - 4 = 0$

i.e., $\quad (D+1)^3(D-4) = 0$

$\therefore\ D = -1, -1, -1, 4$

Here, we have three equal roots and one different root.

Hence the complete solution is $y = (c_1 + c_2x + c_3x^2)\,e^{-x} + c_4e^{4x}$.

Example 5.

Solve the differential equation $(D^4 + 5D^2 + 6)\,y = 0$. *[K.U. 2016]*

Solution. The given equation is

$$(D^4 + 5D^2 + 6)\,y = 0$$

The auxiliary equation is $\quad D^4 + 5D^2 + 6 = 0$

i.e., $\quad (D^2+3)(D^2+2) = 0$

$\therefore\ D = \pm\sqrt{3}\,i, \pm\sqrt{2}\,i$

So, we have two pairs of complex roots, $\quad D = \pm\sqrt{3}\,i, \pm\sqrt{2}\,i$

Hence, the complete solution is

$$y = c_1\cos\sqrt{3}\,x + c_2\sin\sqrt{3}\,x + c_3\cos\sqrt{2}\,x + c_4\sin\sqrt{2}\,x.$$

Example 6.

Solve the differential equation $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0$.

Solution. The given equation is $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0$

Writing the equation in symbolic form, we have

$$(D^2 - 4D + 1)y = 0$$

The auxiliary equation is $\quad D^2 - 4D + 1 = 0$

$$\therefore \quad D = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

Hence the complete solution is $y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$

Another form :

$$y = e^{2x}\,[c_1(\cosh\sqrt{3}\,x + \sinh\sqrt{3}\,x) + c_2(\cosh\sqrt{3}\,x - \sinh\sqrt{3}\,x)]$$

or
$$y = e^{2x}\,[(c_1 + c_2)\cosh\sqrt{3}x + (c_1 - c_2)\sinh\sqrt{3}x]$$

or
$$y = e^{2x}\,[A\cosh\sqrt{3}x + B\sinh\sqrt{3}x].$$

Example 7.

Solve the differential equation $\boldsymbol{\dfrac{d^4y}{dx^4} + a^4 y = 0.}$ [M.D.U. 2017]

Solution. The given equation is $\quad \dfrac{d^4y}{dx^4} + a^4 y = 0$

The symbolic form of the equation is

$$(D^4 + a^4)y = 0$$

Auxiliary equation is $\quad D^4 + a^4 = 0$

or
$$(D^2 + a^2)^2 - 2D^2a^2 = 0$$

or
$$(D^2 + a^2)^2 - (\sqrt{2}\,Da)^2 = 0$$

or
$$(D^2 - \sqrt{2}\,Da + a^2)(D^2 + \sqrt{2}\,Da + a^2) = 0$$

$$\therefore \quad D = \frac{-a}{\sqrt{2}} \pm i\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}} \pm i\frac{a}{\sqrt{2}}$$

Hence, we have two pairs of complex roots, *viz.,* $\quad \dfrac{-a}{\sqrt{2}} \pm i\dfrac{a}{\sqrt{2}}, \dfrac{a}{\sqrt{2}} \pm i\dfrac{a}{\sqrt{2}}$

Hence the general solution is

$$y = e^{-\frac{a}{\sqrt{2}}x}\left(c_1\cos\frac{a}{\sqrt{2}}x + c_2\sin\frac{a}{\sqrt{2}}x\right) + e^{\frac{a}{\sqrt{2}}x}\left(c_3\cos\frac{a}{\sqrt{2}}x + c_4\sin\frac{a}{\sqrt{2}}x\right)$$

or
$$y = e^{\frac{-a}{\sqrt{2}}x}c_1\cos\left(\frac{a}{\sqrt{2}}x + c_2\right) + e^{\frac{a}{\sqrt{2}}x}c_3\cos\left(\frac{a}{\sqrt{2}}x + c_4\right).$$

EXERCISE 4.1

***Solve the following differential equations* [Q. 1 – 10] :**

1. $\dfrac{d^2y}{dx^2} + (a+b)\dfrac{dy}{dx} + aby = 0$

2. $\dfrac{d^3y}{dx^3} - 13\dfrac{dy}{dx} - 12y = 0$

3. $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = 0$

4. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$

5. $\dfrac{d^4y}{dx^4} + 5\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 8y = 0$

6. $(D^3 - 2D^2 - 4D + 8)\,y = 0$

$D^2(D-2) - 4(D-2)$

7. $\dfrac{d^2y}{dx^2} + \mu^2 y = 0$

8. $\dfrac{d^4y}{dx^4} + 4y = 0$

9. $\dfrac{d^4y}{dx^4} + 13\dfrac{d^2y}{dx^2} + 36y = 0.$

10. $\dfrac{d^6y}{dx^6} + 6\dfrac{d^4y}{dx^4} + 12\dfrac{d^2y}{dx^2} + 8y = 0.$

$(D^2+2)^3 = 0$

11. If $\dfrac{d^4y}{dx^4} - a^4 y = 0$, show that $y = c_1 \cos ax + c_2 \sin ax + c_3 \cosh ax + c_4 \sinh ax$.

$D^4 - a^4 = 0 \quad (D^2-a^2)(D^2+a^2) = 0$

12. Solve $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 3x = 0$, given that for $t = 0, x = 0$ and $\dfrac{dx}{dt} = 12$.

13. Solve $\dfrac{d^2y}{dx^2} + y = 0$, given that $y = 2$ for $x = 0$ and $y = -2$ for $x = \dfrac{\pi}{2}$.

ANSWERS

1. $y = c_1 e^{-ax} + c_2 e^{-bx}$.

2. $y = c_1 e^{-x} + c_2 e^{-3x} + c_3 e^{4x}$

3. $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$

4. $y = (c_1 + c_2 x)\,e^{2x}$

5. $y = c_1 e^{x} + (c_2 + c_3 x + c_4 x^2)\,e^{-2x}$

6. $y = (c_1 + c_2 x)\,e^{2x} + c_3 e^{-2x}$

7. $y = c_1 \cos \mu x + c_2 \sin \mu x$

8. $y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^{x}(c_3 \cos x + c_4 \sin x)$

9. $y = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos 3x + c_4 \sin 3x$

10. $y = (c_1 + c_2 x + c_3 x^2) \cos \sqrt{2}x + (c_4 + c_5 x + c_6 x^2) \sin \sqrt{2}x$

12. $x = -6e^{-3t} + 6e^{-t}$

13. $y = 2 \cos x - 2 \sin x$.

4.7. THEOREM

If $y = Y$ is the complete solution of the equation

$$(P_0 D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots\ldots + P_n)\, y = 0 \qquad \ldots(1)$$

and $y = u$ be particular solution (containing no arbitrary constants) of the equation

$$(P_0 D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots\ldots + P_n)\, y = X \qquad \ldots(2)$$

where X is a function of x, then the complete solution of equation (2) is $y = Y + u$.

Proof. Since $y = Y$ is the complete solution of (1), so we have

$$(P_0 D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots\ldots + P_n)\, Y = 0 \qquad \ldots(3)$$

Also, since $y = u$ is a solution of equation (2), we have

$$(P_0 D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots\ldots + P_n)\, u = X \qquad \ldots(4)$$

Putting $y = Y + u$ in equation (2), we get

$$P_0 D^n (Y + u) + P_1 D^{n-1} (Y + u) + P_2 D^{n-2} (Y + u) + \ldots\ldots + P_n (Y + u) = X$$

or $\quad (P_0 D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots\ldots + P^n)\, Y + (P_0 D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots\ldots + P_n)\, u = X$

or $\quad 0 + X = X \qquad$ [*Using (3) and (4)*]

or $\quad X = X$, which is true.

$\therefore \quad y = Y + u$ is a solution of (2).

Now Y being a general solution of (1), contains n arbitrary constants and is of nth order and as such $Y + u$ also contains n arbitrary constants. Therefore $y = Y + u$ is a general or complete solution of equation (2).

4.7.1. Complementary Function and Particular Integral

C.F ⇒ The general soln which is obtained by solving the A.E

Consider the following two linear differential equations of order n

$$[a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots + a_n]\, y = X \qquad \ldots(5)$$

$$[\underbrace{a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots + a_n}_{f(D)}]\, y = 0 \qquad \ldots(6)$$

where AE is $f(D) = 0$

The general solution of equation (6) containing n arbitrary constants is called **complementary function**.

Any solution of equation (5) containing no arbitrary constant is called **particular integral**. Hence complete solution of equation (1) is given by

Complete Solution = Complementary Function + Particular Integral

briefly written as $\qquad$ **C.S. = C.F. + P.I.**

4.8. MEANING OF THE SYMBOL $\frac{1}{f(D)}$ OR THE INVERSE OPERATOR $\frac{1}{f(D)}$

Definition. $\frac{1}{f(D)}$ *X is the function of x, free from arbitrary constants, which when operated by f (D) gives X.*

Thus
$$f(D).\frac{1}{f(D)}X = X.$$

Therefore $f(D)$ and $\frac{1}{f(D)}$ are inverse operators (*i.e.*, they cancel each others effect when both act on any function, leaving the function intact).

4.8.1. Prove that $\frac{1}{f(D)}$ X is the particular integral of $f(D)\,y = X$ [M.D.U. 2014]

or particular soln

It is called particular integral or particular solution since it does not contain any arb. constt.

Proof. The given equation is $f(D)\,y = X$ ...(1)

Let $y = u$ be a solution of (1)

$$\therefore \quad f(D)\,u = X$$

Applying operator $\frac{1}{f(D)}$ on both sides, we have

$$\frac{1}{f(D)}f(D)\,u = \frac{1}{f(D)}X$$

$$\Rightarrow \quad u = \frac{1}{f(D)}X$$

Hence $\frac{1}{f(D)}$ X is a solution of (1) containing no arbitrary constant.

$\therefore$ Particular integral $= \frac{1}{f(D)}X$.

4.9. THEOREM

Prove that $\boldsymbol{\frac{1}{D} X = \int X\, dx}$***, no arbitrary constant being added.***

Proof. Let $\frac{1}{D} X = Z$ ...(1)

Operating on both sides of (1) by D, we have

$$D.\frac{1}{D} X = DZ$$

or $$X = DZ$$

or $$DZ = X$$

or $$\frac{dZ}{dx} = X.$$

Integrating both sides w.r.t. x, we get

$$Z = \int X\, dx, \text{ no arbitrary constant is being added.}$$

$$\left[\because\ Z = \frac{1}{D} X \text{ contains no arbitrary constant}\right]$$

$$\frac{1}{D} X = \int X\, dx, \text{ no arbitrary constant being added.}$$

Note.

As proved above the symbol $\frac{1}{D}$ stands for integration.

$e^{\alpha x}\int e^{-\alpha x} X\, dx,$

$\frac{1}{D-\alpha} X =$

4.10. THEOREM

To show that $\boldsymbol{\frac{1}{D-\alpha} X = e^{\alpha x}\int (e^{-\alpha x} X)\, dx}$***, no arbitrary constant being added.***

[M.D.U. 2007; K.U. 2001]

Proof. Let $\frac{1}{D-\alpha} X = y$ ...(1)

Operating on both sides by $(D-\alpha)$, we get

$$D - \alpha \cdot \frac{1}{D - \alpha} \cdot X = (D - \alpha)\, y$$

or $$X = (D - \alpha)\, y$$

or $$X = Dy - \alpha\, y$$

or $$\frac{dy}{dx} - \alpha\, y = X, \text{ which is a linear equation in } y. \qquad \left[\because\ D = \frac{d}{dx}\right]$$

Comparing it with $\frac{dy}{dx} + Py = Q$, we have $P = -\alpha$ and $Q = X$

$$\therefore \qquad \text{I.F.} = e^{\int P\,dx} = e^{\int -\alpha\,.\,dx} = e^{-\alpha x}$$

Hence the solution is $y \,.\, e^{-\alpha x} = \int (e^{-\alpha x} \,.\, X)\, dx$, no arbitrary constant being added.

$$\left[\because\ y\left(= \frac{1}{D - \alpha} X\right) \textit{contains no arbitrary constant.}\right]$$

or $$y = e^{\alpha x} \int (e^{-\alpha x}.X)\,dx$$

or $$\frac{1}{D - a} X = e^{\alpha x} \int (e^{-\alpha x}.X)\,dx, \qquad [\textit{Using (1)}]$$

no arbitrary constant being added.

Cor. I. ***Prove that*** $\boldsymbol{\frac{1}{D - a} X = e^{ax} \int X e^{-ax}\, dx}$ ***and hence show that***

$$\frac{1}{(D - a)^2} \cdot e^{ax} = \frac{x^2}{2} e^{ax}. \qquad [\textit{K.U. 2010}]$$

$\frac{1}{(D-a)^n} e^{ax},\ \frac{x^n}{n!} e^{a}$

Proof. For the first part, see previous theorem.

Second Part :

Here $\qquad X = e^{ax}$

$$\therefore \qquad \frac{1}{D - a} e^{ax} = e^{ax} \int (e^{-ax} \,.\, e^{ax})\, dx \qquad [\textit{Using theorem, Art. 4.10}]$$

$$= e^{ax} \int 1\, dx$$

$$= e^{ax} \,.\, x = x \,.\, e^{ax} \qquad ...(1)$$

Now $$\frac{1}{(D-a)^2}\cdot e^{ax} = \frac{1}{(D-a)}\left[\frac{1}{(D-a)}e^{ax}\right] = \frac{1}{(D-a)}(xe^{ax})$$ [Using (1)]

$$= e^{ax}\int (e^{-ax}\,.\,xe^{ax})\,dx$$

$$= e^{ax}\int x\,dx = e^{ax}\,.\,\frac{x^2}{2} = \frac{x^2}{2}\,.\,e^{ax}$$

$\therefore$ $$\frac{1}{(D-a)^2}e^{ax} = \frac{x^2}{2}e^{ax}.$$

Cor. 2. ***Show that*** $\dfrac{1}{(D-a)^n}\,.\,e^{ax} = \dfrac{x^n}{n!}\,.\,e^{ax}$.

Proof. By Cor. I, we know that

$$\frac{1}{(D-a)^2}e^{ax} = \frac{x^2}{2}e^{ax} = \frac{x^2}{2!}e^{ax} \quad ...(1)$$

Now, $$\frac{1}{(D-a)^3}e^{ax} = \frac{1}{D-a}\left[\frac{1}{(D-a)^2}\cdot e^{ax}\right] = \frac{1}{D-a}\cdot\frac{x^2}{2}\cdot e^{ax}$$ [By (1)]

$$= e^{ax}\int\left(\frac{x^2}{2}\,.\,e^{ax}\,.\,e^{-ax}\right)dx$$ [*Using theorem, Art. 4.10*]

$$= e^{ax}\int\frac{x^2}{2}\,.\,dx = e^{ax}\,.\,\frac{x^3}{3\,.\,2} = \frac{x^3}{3!}\,.\,e^{ax}$$

$\therefore$ $$\frac{1}{(D-a)^3}\cdot e^{ax} = \frac{x^3}{3!}\cdot e^{ax} \quad ...(2)$$

Similarly, $$\frac{1}{(D-a)^4}\cdot e^{ax} = \frac{x^4}{4!}\cdot e^{ax} \quad ...(3)$$

Proceeding in the same way, we have

$$\frac{1}{(D-a)^n}\cdot e^{ax} = \frac{x^n}{n!}e^{ax}$$

4.11. THEOREM

To determine the particular integral of $f(D)\,y = X$ where

$$f(D) = (D - \alpha_1)(D - \alpha_2) \ldots\ldots (D - \alpha_n).$$

Proof. We have $f(\mathrm{D}) = (\mathrm{D} - \alpha_1)(\mathrm{D} - \alpha_2) \ldots\ldots (\mathrm{D} - \alpha_n)$

$$\therefore \qquad \frac{1}{f(\mathrm{D})} = \frac{1}{(\mathrm{D} - \alpha_1)(\mathrm{D} - \alpha_2) \ldots\ldots (\mathrm{D} - \alpha_n)}$$

Resolving into partial fractions, we have

$$\frac{1}{f(\mathrm{D})} = \frac{\mathrm{A}_1}{\mathrm{D} - \alpha_1} + \frac{\mathrm{A}_2}{\mathrm{D} - \alpha_2} + \ldots\ldots + \frac{\mathrm{A}_n}{\mathrm{D} - \alpha_n} \text{ (say)}$$

Now,
$$\text{P.I.} = \frac{1}{f(\mathrm{D})}\mathrm{X} = \left\{\frac{\mathrm{A}_1}{\mathrm{D} - \alpha_1} + \frac{\mathrm{A}_2}{\mathrm{D} - \alpha_2} + \ldots\ldots + \frac{\mathrm{A}_n}{\mathrm{D} - \alpha_n}\right\}\mathrm{X}$$

$$= \mathrm{A}_1 \frac{1}{\mathrm{D} - \alpha_1}\mathrm{X} + \mathrm{A}_2 \frac{1}{\mathrm{D} - \alpha_2}\mathrm{X} + \ldots\ldots + \mathrm{A}_n \frac{1}{\mathrm{D} - \alpha_n}\mathrm{X}$$

$$= \mathrm{A}_1 \,.\, e^{\alpha_1 x} \int e^{-\alpha_1 x} \,.\, \mathrm{X}\, dx + \mathrm{A}_2 \,.\, e^{\alpha_2 x} \int e^{-\alpha_2 x} \,.\, \mathrm{X}\, dx + \ldots\ldots + \mathrm{A}_n \,.\, e^{\alpha_n x} \int e^{-\alpha_n x} \,.\, \mathrm{X}\, dx$$

which can in general be evaluated and thus the particular integral can be found.

4.12. PARTICULAR INTEGRAL IN SOME SPECIAL CASES

4.12.1. Case I. Prove that $\dfrac{1}{f(D)} \,.\, e^{ax} = \dfrac{1}{f(a)} \,.\, e^{ax}$ provided $f(a) \neq 0$. [*M.D.U. 2001*]

$\frac{1}{f(D)} e^{ax}$, $\frac{1}{f(\cdot)}$

Proof. By successive differentiation, we find that

$\frac{x}{\frac{d}{dD}(f(D))} e^{x}$

$$e^{ax} = e^{ax} \qquad \ldots(1)$$

$$\mathrm{D}e^{ax} = ae^{ax} \qquad \ldots(2)$$

$$\mathrm{D}^2 e^{ax} = a^2 e^{ax} \qquad \ldots(3)$$

....

....

$$\mathrm{D}^n e^{ax} = a^n e^{ax} \qquad \ldots(n)$$

Let $\quad f(\mathrm{D}) = a_0\mathrm{D}^n + a_1\mathrm{D}^{n-1} + a_2\mathrm{D}^{n-2} + \ldots\ldots + a_{n-1}\mathrm{D} + a_n$

$\therefore \quad f(a) = a_0 a^n + a_1 a^{n-1} + a_2 a^{n-2} + \ldots\ldots + a_{n-1} a + a_n$

Now,
$$\begin{aligned} f(\mathrm{D})\, e^{ax} &= [a_0\, \mathrm{D}^n + a_1\, \mathrm{D}^{n-1} + a_2\, \mathrm{D}^{n-2} + \ldots\ldots + a_{n-1}\, \mathrm{D} + a_n]\, e^{ax} \\ &= a_0\, \mathrm{D}^n e^{ax} + a_1\, \mathrm{D}^{n-1} e^{ax} + a_2\, \mathrm{D}^{n-2}\, e^{ax} + \ldots\ldots + a_{n-1}\, \mathrm{D}e^{ax} + a_n\, e^{ax} \\ &= a_0\, a^n\, e^{ax} + a_1\, a^{n-1}\, e^{ax} + a_2\, a^{n-2}\, e^{ax} + \ldots\ldots + a_{n-1}\, ae^{ax} + a_n\, e^{ax} \\ &= [a_0\, a^n + a_1\, a^{n-1} + a_2\, a^{n-2} + \ldots\ldots + a_{n-1}\, a + a_n]\, e^{ax} \\ &= f(a)\, e^{ax} \end{aligned}$$

$\therefore \quad f(\mathrm{D})\, e^{ax} = f(a)\, e^{ax}$

Now operating on both sides by $\dfrac{1}{f(\mathrm{D})}$, we have

$$\frac{1}{f(\mathrm{D})} \cdot f(\mathrm{D})\, e^{ax} = \frac{1}{f(\mathrm{D})} \cdot f(a)\, e^{ax}$$

or
$$e^{ax} = f(a) \frac{1}{f(\mathrm{D})} \cdot e^{ax}$$

or
$$\frac{1}{f(a)} \cdot e^{ax} = \frac{1}{f(\mathrm{D})}\, e^{ax} \qquad \text{[Dividing by } f(a) \neq 0]$$

or
$$\frac{1}{f(\mathrm{D})} \cdot e^{ax} = \frac{1}{f(a)} \cdot e^{ax}, \text{ where } f(a) \neq 0$$

Case of Failure. If $f(a) = 0$, then

$$\frac{1}{f(\mathrm{D})} \cdot e^{ax} = x \cdot \frac{1}{\dfrac{d}{d\mathrm{D}}[f(\mathrm{D})]}\, e^{ax}.$$

Proof. If $f(a) = 0$, then $\mathrm{D} - a$ must be a factor of $f(\mathrm{D})$. *[Factor Theorem]*

So, let $f(\mathrm{D}) = (\mathrm{D} - a)\, \phi(\mathrm{D})$ ...(1)

$$\begin{aligned} \therefore \quad \frac{1}{f(\mathrm{D})}\, e^{ax} &= \frac{1}{(\mathrm{D}-a)\, \phi(\mathrm{D})}\, e^{ax} = \frac{1}{\mathrm{D}-a}\left(\frac{1}{\phi(\mathrm{D})}\, e^{ax}\right) \\ &= \frac{1}{\mathrm{D}-a}\left(\frac{1}{\phi(a)}\, e^{ax}\right) \qquad \text{[Assuming } \phi(a) \neq 0] \\ &= \frac{1}{\phi(a)} \cdot \frac{1}{\mathrm{D}-a}\, e^{ax} \\ &= \frac{1}{\phi(a)} \cdot e^{ax} \int e^{-ax} \cdot e^{ax}\, dx \qquad \text{[By theorem 4]} \end{aligned}$$

$$= \frac{1}{\phi(a)} e^{ax} \int 1\, dx = \frac{1}{\phi(a)} . e^{ax} . x$$

$$\therefore \quad \frac{1}{f(D)} e^{ax} = x . \frac{1}{\phi(a)} . e^{ax} \qquad ...(2)$$

Differentiating (1) w.r.t. D, we get

$$f'(D) = \phi(D) + (D - a)\phi'(D)$$

$$\therefore \quad f'(a) = \phi(a)$$

From (2),

$$\frac{1}{f(D)} e^{ax} = \frac{xe^{ax}}{f'(a)}$$

$$= x . \frac{1}{f'(D)} e^{ax} = x . \frac{1}{\frac{d}{dD}[f(D)]} e^{ax}$$

$$\therefore \quad \frac{1}{f(D)} . e^{ax} = x . \frac{1}{\frac{d}{dD}[f(D)]} . e^{ax} \text{ if } f(a) = 0.$$

Note.

If by using the above rule the denominator vanishes, repeat the rule again and so on.

SOLVED EXAMPLES

Example 1.

Solve the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 104e^{3x}.$$ **[K.U. 2013, 03; M.D.U. 2007]**

Solution. The given equation is $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 104e^{3x}$

The symbolic form of the equation is

$$(D^2 + 6D + 25)y = 104e^{3x}$$

Auxiliary equation is $D^2 + 6D + 25 = 0$

$$\therefore \quad D = \frac{-6 \pm \sqrt{36 - 100}}{2}$$

or $$D = \frac{-6 \pm \sqrt{-64}}{2} = \frac{-6 \pm 8i}{2} = -3 \pm 4i$$

$\therefore$ $$\text{C.F.} = e^{-3x}\,[c_1 \cos 4x + c_2 \sin 4x]$$

and $$\text{P.I.} = \frac{1}{D^2 + 6D + 25} \cdot 104\, e^{3x} = \frac{1}{9 + 18 + 25} \cdot 104\, e^{3x} \qquad \text{[Putting } D = 3]$$

$$= \frac{104\, e^{3x}}{52} = 2e^{3x}$$

Thus, the complete solution is $y = \text{C.F.} + \text{P.I.}$

or $$y = e^{-3x}\,(c_1 \cos 4x + c_2 \sin 4x) + 2e^{3x}.$$

Example 2.

Solve the differential equation $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = e^{3x}$.

Solution. The given equation is $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = e^{3x}$ ...(1)

The symbolic form of the equation is

$$(D^2 - 6D + 9)\, y = e^{3x}$$

Auxiliary equation is $D^2 - 6D + 9 = 0$

or $$(D - 3)^2 = 0 \;\Rightarrow\; D = 3, 3$$

$\therefore$ $$\text{C.F.} = (c_1 + c_2 x)\, e^{3x}$$

and $$\text{P.I.} = \frac{1}{(D-3)^2} \cdot e^{3x} = \frac{1}{(3-3)^2} \cdot e^{3x} \qquad \text{[Putting } D = 3]$$

$$= \frac{1}{0} \cdot e^{3x}, \text{ which is a case of failure}$$

$\therefore$ $$\text{P.I.} = \frac{1}{(D-3)^2}\, e^{3x} = x \cdot \frac{1}{\frac{d}{dD}(D-3)^2} \cdot e^{3x} = \frac{x}{2} \cdot \frac{1}{D-3} \cdot e^{3x}$$

$$= \frac{x}{2}\left[\frac{1}{3-3}\, e^{3x}\right] \qquad \text{[Putting } D = 3]$$

$$= \frac{x}{2}\left[\frac{1}{0}\, e^{3x}\right] \qquad \text{[Again a case of failure]}$$

$$\therefore \qquad \text{P.I.} = \frac{x}{2} \cdot x \cdot \frac{1}{\frac{d}{dD}(D-3)} e^{3x} = \frac{x^2}{2} \cdot e^{3x}$$

Hence the complete solution is $y = \text{C.F.} + \text{P.I.}$

i.e., $$y = (c_1 + c_2 x) e^{3x} + \frac{x^2}{2} e^{3x}.$$

Example 3. ***Solve the differential equation*** $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^{4x}$

Solution. The given equation is $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^{4x}$

The symbolic form of the equation is $(D^2 - 5D + 6)\, y = e^{4x}$

$\therefore$ Auxiliary equation is $D^2 - 5D + 6 = 0$

or $$(D-2)(D-3) = 0 \Rightarrow D = 2, 3$$

$$\therefore \qquad \text{C.F.} = c_1 e^{2x} + c_2 e^{3x}$$

and $$\text{P.I.} = \frac{1}{D^2 - 5D + 6} e^{4x} = \frac{1}{(D-2)(D-3)} e^{4x}$$

$$= \left[\frac{1}{D-3} - \frac{1}{D-2}\right] e^{4x} \qquad \text{[Resolving into partial fractions]}$$

$$= \frac{1}{D-3} e^{4x} - \frac{1}{D-2} e^{4x}$$

$$= e^{3x} \int e^{-3x} \cdot e^{4x}\, dx - e^{2x} \int e^{-2x} \cdot e^{4x}\, dx$$

$$= e^{3x} \int e^{x}\, dx - e^{2x} \int e^{2x}\, dx = e^{3x} \cdot e^{x} - e^{2x} \cdot \frac{e^{2x}}{2}$$

$$\therefore \qquad \text{P.I.} = e^{4x} - \frac{1}{2} e^{4x} = \frac{1}{2} e^{4x}$$

Hence, the complete solution is $y = \text{C.F.} + \text{P.I.}$

i.e., $$y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{2} e^{4x}.$$

Example 4. ***Solve the differential equation*** $\dfrac{d^2y}{dx^2} + a^2 y = \sec ax$. [K.U. 2012, 09, 05]

Solution. The given equation is $\dfrac{d^2y}{dx^2} + a^2 y = \sec ax$

The symbolic form of the equation is

$$(D^2 + a^2)\, y = \sec ax$$

$\therefore$ Auxiliary equation is $\quad D^2 + a^2 = 0$

i.e., $\quad D^2 = -a^2$

i.e., $\quad D = \pm ia = 0 \pm ia$

$\therefore \quad$ C.F. $= e^{0x}(c_1 \cos ax + c_2 \sin ax)$

$= c_1 \cos ax + c_2 \sin ax$

Now, $\quad$ P.I. $= \dfrac{1}{D^2 + a^2} \cdot \sec ax = \dfrac{1}{(D + ai)(D - ai)} \sec ax$

i.e., $\quad$ P.I. $= \dfrac{1}{2ai}\left[\dfrac{1}{D - ai} - \dfrac{1}{D + ai}\right] \sec ax$ $\quad$ [*Resolving into partial fractions*]

$$= \frac{1}{2ai}\left[\frac{1}{D - ai} \sec ax - \frac{1}{D + ai} \sec ax\right]$$

$$= \frac{1}{2ai}\left[e^{iax} \int \frac{e^{-iax}}{\cos ax}\, dx - e^{-iax} \int \frac{e^{iax}}{\cos ax}\, dx\right]$$ [*Using theorem, Art 4.10*]

$$= \frac{1}{2ai} e^{iax} \int \frac{\cos ax - i \sin ax}{\cos ax}\, dx - \frac{1}{2ai} e^{-iax} \int \frac{\cos ax + i \sin ax}{\cos ax}\, dx$$

$\left[\because \quad e^{\pm i\theta} = \cos\theta \pm i \sin\theta \text{ (Euler's theorem)}\right]$

$$= \frac{1}{2ai} e^{iax} \int (1 - i \tan ax)\, dx - \frac{e^{-iax}}{2ai} \int (1 + i \tan ax)\, dx$$

$$= \frac{1}{2ai} \cdot e^{iax}\left[x + \frac{i}{a} \log \cos ax\right] - \frac{e^{-iax}}{2ia}\left[x - \frac{i}{a} \log \cos ax\right]$$

$$= \frac{1}{2ai}[x(e^{iax} - e^{-iax})] + \frac{1}{2a^2} \log \cos ax\,(e^{iax} + e^{-iax})$$

$$= \frac{x}{a}\left[\frac{e^{iax} - e^{-iax}}{2i}\right] + \frac{1}{a^2} \log \cos ax \left[\frac{e^{iax} + e^{-iax}}{2}\right]$$

$$= \frac{x}{a} \sin ax + \frac{1}{a^2} \log(\cos ax) \cos ax$$

$\left[\because \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} \text{ and } \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}\right]$

$$= \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax \,.\, \log \cos ax$$

Hence, the complete solution is

$$y = c_1 \cos ax + c_2 \sin ax + \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax \log(\cos ax).$$

Example 5. ***Solve the differential equation***

$$\frac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$$ [M.D.U. 2013; K.U. 2013]

Solution. The given equation is $\dfrac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$

The symbolic form of the equation is

$$(D^3 + 1)\, y = 3 + e^{-x} + 5e^{2x}$$

$\therefore$ Auxiliary equation is $\quad D^3 + 1 = 0$

or $\quad (D + 1)(D^2 - D + 1) = 0$

$\therefore$ $$D = -1 ; \frac{1 \pm \sqrt{1-4}}{2} = -1, \frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}$$

$\therefore$ $$\text{C.F.} = c_1 e^{-x} + e^{x/2}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^3+1}(3 + e^{-x} + 5e^{2x}) \\
&= 3 \cdot \frac{1}{D^3+1} \cdot e^{0x} + \frac{1}{D^3+1} \cdot e^{-x} + 5 \cdot \frac{1}{D^3+1} \cdot e^{2x} \\
&= 3 \cdot \frac{1}{0+1} \cdot e^{0x} + \frac{1}{(-1)^3+1} \cdot e^{-x} + 5 \cdot \frac{1}{2^3+1} e^{2x} \\
&= 3 + \frac{1}{0} e^{-x} + \frac{5}{9} e^{2x} \qquad \text{[Case of failure]} \\
&= 3 + x \cdot \frac{1}{\frac{d}{dD}(D^3+1)} e^{-x} + \frac{5}{9} e^{2x} = 3 + x \cdot \frac{1}{3D^2} \cdot e^{-x} + \frac{5}{9} e^{2x} \\
&= 3 + x \cdot \frac{1}{3(-1)^2} \cdot e^{-x} + \frac{5}{9} e^{2x} = 3 + \frac{xe^{-x}}{3} + \frac{5}{9} e^{2x}
\end{aligned}$$

Hence the complete solution is

$$y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right) + 3 + \frac{xe^{-x}}{3} + \frac{5}{9} e^{2x}.$$

Example 6. ***Solve the differential equation*** $\dfrac{d^2y}{dx^2} - y = \cosh x$. [M.D.U. 2017]

Solution. The given equation is $\dfrac{d^2y}{dx^2} - y = \cosh x$

or $$(D^2 - 1)\, y = \cosh x$$

$\therefore$ Auxiliary equation is $D^2 - 1 = 0 \;\Rightarrow\; D^2 = 1 \;\Rightarrow\; D = \pm 1$

$\therefore$ $$\text{C.F} = c_1 e^x + c_2 e^{-x}$$

and $$\text{P.I.} = \frac{1}{D^2-1}\cosh x = \frac{1}{D^2-1}\left(\frac{e^x+e^{-x}}{2}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{D^2-1}e^x + \frac{1}{2}\cdot\frac{1}{D^2-1}e^{-x}$$

$$= \frac{1}{2}\cdot\frac{1}{0}e^x + \frac{1}{2}\cdot\frac{1}{0}e^{-x}, \text{ which is a case of failure.}$$

$\therefore$ $$\text{P.I.} = \frac{x}{2}\cdot\frac{1}{\frac{d}{dD}(D^2-1)}e^x + \frac{x}{2}\frac{1}{\frac{d}{dD}(D^2-1)}e^{-x}$$

$$= \frac{x}{2}\cdot\frac{1}{2D}e^x + \frac{x}{2}\cdot\frac{1}{2D}e^{-x}$$

$$= \frac{x}{4}e^x - \frac{x}{4}e^{-x}$$

Hence, the complete solution is y = C.F. + P.I.

i.e., $$y = c_1 e^x + c_2 e^{-x} + \frac{x}{4}(e^x - e^{-x}).$$

Example 7. ***If*** $\dfrac{d^2x}{dt^2} + \dfrac{g}{b}(x-a) = 0$, $(a > 0, b > 0, g > 0)$ ***and*** $x = \alpha$, $\dfrac{dx}{dt} = 0$, ***when*** $t = 0$; ***show that*** $x = a + (\alpha - a)\cos\sqrt{\dfrac{g}{b}}\,.t$.

Solution. The given equation is $\dfrac{d^2x}{dt^2} + \dfrac{g}{b}(x-a) = 0$

or $$\frac{d^2x}{dt^2} + \frac{g}{b}x - \frac{ga}{b} = 0$$

The symbolic form of the equation is

$$\left(D^2 + \frac{g}{b}\right)x = \frac{ga}{b}, \text{ where } D = \frac{d}{dt}$$

Auxiliary equation is $D^2 + \dfrac{g}{b} = 0$

$\therefore$ $$D = \pm i\sqrt{\frac{g}{b}} = 0 \pm i\sqrt{\frac{g}{b}}$$

$\therefore$ $$\text{C.F.} = e^{0.t}\left(c_1\cos\sqrt{\frac{g}{b}}\,.t + c_2\sin\sqrt{\frac{g}{b}}\,.t\right)$$

$$= c_1 \cos \sqrt{\frac{g}{b}}\,.\,t + c_2 \sin \sqrt{\frac{g}{b}}\,.\,t$$

$$\text{P.I.} = \frac{1}{D^2 + \frac{g}{b}} \cdot \frac{ga}{b} = \frac{ga}{b} \cdot \frac{1}{D^2 + \frac{g}{b}} \cdot e^{0.t}$$

$$= \frac{ga}{b} \cdot \frac{1}{0 + \frac{g}{b}} \cdot e^{0.t} = a$$

Hence the complete solution is

$$x = c_1 \cos \sqrt{\frac{g}{b}}\,.\,t + c_2 \sin \sqrt{\frac{g}{b}}\,.\,t + a \qquad ...(1)$$

When $t = 0, x = \alpha$ [*Given*]

$\therefore$ From (1), $\alpha = c_1 \cos 0 + c_2 \sin 0 + a$

i.e., $\alpha = c_1 + a \Rightarrow c_1 = \alpha - a$

Differentiating (1) w.r.t. 't', we have

$$\frac{dx}{dt} = -c_1 \sqrt{\frac{g}{b}} \sin \sqrt{\frac{g}{b}}\,.\,t + c_2 \sqrt{\frac{g}{b}} \cos \sqrt{\frac{g}{b}}\,.\,t \qquad ...(2)$$

When $t = 0$, $\dfrac{dx}{dt} = 0$ [*Given*]

$\therefore$ From (2), $0 = c_2 \sqrt{\dfrac{g}{b}} \Rightarrow c_2 = 0$

Substituting the values of c_1 and c_2 in (1), we have

$$x = a + (\alpha - a) \cos\left(\sqrt{\frac{g}{b}}\,.\,t\right),$$ which is the required result.

EXERCISE 4.2

***Solve the following differential equations* [Q. *1 – 12*] :**

1. $4\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 3y = e^{2x}$

2. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^{2x}$

3. $\dfrac{d^2y}{dx^2} - 2k\dfrac{dy}{dx} + k^2y = e^x$

4. $2\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + y = e^x + 1$

5. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = e^{2x} + e^{-2x}$ [K.U. 2000]

6. (i) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 2\sinh 2x$ (ii) $(D^2 - 3D + 2)\,y = \cosh x$

7. $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x}$, given that $y = 0$ when $x = 0$. [M.D.U. 2014]

8. $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 3y = e^{2x}\cosh x$ [K.U. 2016]

9. $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = e^{2x}$

10. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{3x}$

11. $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ [K.U. 2017, 15; M.D.U. 2015, 05]

12. $\frac{d^2y}{dx^2} + y = \sec x$ [M.D.U. 2011; K.U. 2004]

13. $\frac{d^2y}{dx^2} + 16y = \sec 4x$ [M.D.U. 2014, 03]

14. $(D^2 - 3D + 2)\,y = \cos e^{-x}$ [M.D.U. 2007] Put $e^{-x} = t$ for integration

15. Prove that $\frac{1}{D+a}X = e^{-ax}\int (e^{ax}.\ X)\,dx$, where X is function of x. [M.D.U. 2016]

[**Hint** : Proceed as in Art. 4.10, page 4.14].

ANSWERS

1. $y = c_1e^{\frac{x}{2}} + c_2e^{-\frac{3}{2}x} + \frac{1}{21}e^{2x}$

2. $y = c_1e^{x} + c_2e^{2x} + x\,e^{2x}$

3. $y = (c_1 + c_2x)\,e^{kx} + \frac{1}{(1-k)^2}.e^x$

4. $y = (c_1 + c_2x)\,e^x + c_3e^{-\frac{x}{2}} + \frac{1}{6}x^2e^x + 1$

5. $y = (c_1 + c_2x)\,e^{-2x} + \frac{e^{2x}}{16} + \frac{1}{2}x^2e^{-2x}$

6. (i) $y = (c_1 + c_2x)\,e^{-2x} + \frac{e^{2x}}{16} - \frac{x^2}{2}e^{-2x}$ (ii) $y = c_1e^x + c_2e^{2x} - \frac{x}{2}e^x + \frac{1}{12}e^{-x}$

7. $y = c_1(e^x - e^{6x}) + \frac{1}{4}(e^{6x} - e^{2x})$

8. $y = e^x(c_1 + c_2x) + c_3e^{3x} + \frac{x}{8}e^{3x} - \frac{1}{8}x^2e^x$

9. $y = c_1e^x + c_2e^{2x} + c_3e^{3x} - x\,e^{2x}$

10. $y = c_1e^{-2x} + c_2e^x + c_3e^{3x} + \frac{x}{10}e^{3x}$

11. $y = c_1\cos x + c_2\sin x + \sin x\log\sin x - x\cos x$

12. $y = c_1\cos x + c_2\sin x + x\sin x + \cos x\log\cos x$

13. $y = c_1\cos 4x + c_2\sin 4x + \frac{x}{4}\sin 4x + \frac{1}{16}\cos 4x\log(\cos 4x)$.

14. $y = c_1e^x + c_2e^{2x} - e^{2x}\cos e^{-x}$.

4.12.2. Case II. (a) (i) Show that $\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$, **if** $f(-a^2) \neq 0$.

[*M.D.U. 2005*]

Proof. We have $e^{iax} = \cos ax + i \sin ax$

$\therefore$ $\sin ax$ = Imaginary part of e^{iax}

Now,
$$\frac{1}{f(D^2)} \sin ax = \frac{1}{f(D^2)} \text{ I.P. of } e^{iax}$$
$$= \text{I.P. of } \frac{1}{f(i^2 a^2)} e^{iax}$$
$$= \text{I.P. of } \frac{1}{f(-a^2)} e^{iax}$$
$$= \frac{\text{I.P. of } (\cos ax + i \sin ax)}{f(-a^2)}$$
$$= \frac{\sin ax}{f(-a^2)}$$

$$\therefore \quad \frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax, \text{ if } f(-a^2) \neq 0.$$

(ii) Show that $\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} . \cos ax$, **if** $f(-a)^2 \neq 0$.

Proof. Similar as in part (*i*).

(b) CASE OF FAILURE. If $f(-a^2) = 0$, **then**

(*i*) $\frac{1}{f(D^2)} \sin ax = x . \frac{1}{\frac{d}{dD} f(D^2)} \sin ax.$

(*ii*) $\frac{1}{f(D^2)} \cos ax = x . \frac{1}{\frac{d}{dD} f(D^2)} \cos ax.$

Proof. We have $\frac{1}{f(D^2)} (\cos ax + i \sin ax) = \frac{1}{f(D^2)} . e^{iax}$ [*Euler's Theorem*]

[*If we replace D by ia*, $f(D^2) = f[(ia)^2] = f(-a^2) = 0$]

$$= x \cdot \frac{1}{\frac{d}{dD}[f(D^2)]} e^{iax} \qquad [\because f(-a^2) = 0]$$

$$= x \cdot \frac{1}{\frac{d}{dD}[f(D^2)]} (\cos ax + i \sin ax)$$

(*i*) Equating the imaginary parts, we have

$$\frac{1}{f(D^2)} \cdot \sin ax = x \cdot \frac{1}{\frac{d}{dD}[f(D^2)]} \cdot \sin ax$$

(*ii*) Equating the real parts, we have

$$\frac{1}{f(D^2)} \cdot \cos ax = x \cdot \frac{1}{\frac{d}{dD}[f(D^2)]} \cdot \cos ax$$

Note.

If by using the above rule, the denominator vanishes, repeat the rule again and so on.

An Important Remark (Working Rule).

When the particular integral is of the form $\frac{1}{f(D)} \sin ax$ or $\frac{1}{f(D)} \cos ax$:

Put $D^2 = -a^2$

$D^3 = D^2 . D = -a^2 D$

$D^4 = (D^2)^2 = (-a^2)^2$ and so on.

Note that we cannot put anything in place of D.

Thus, ultimately $f(D)$ becomes linear in D say of the form $(D + \beta)$. Then we proceed as follows :

$$\frac{1}{D+\beta} \sin ax = \frac{D-\beta}{(D+\beta)(D-\beta)} \sin ax$$

$$= \frac{D-\beta}{D^2-\beta^2} \sin ax = \frac{D-\beta}{-a^2-\beta^2} \sin ax$$

[*Putting* $-a^2$ *for* D^2 *in the denominator*]

$$= \frac{1}{-a^2-\beta^2} \left(\frac{d}{dx} \sin ax - \beta \sin ax \right) \qquad \left[\because D = \frac{d}{dx}\right]$$

$$= \frac{1}{-a^2 - \beta^2}(a \cos ax - \beta \sin ax)$$

Thus, the particular integral in case of $\sin ax$ and $\cos ax$ can be completely evaluated.

SOLVED EXAMPLES

Example 1.

Solve the differential equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = \sin 2x.$ [K.U. 2001]

Solution. The given equation is $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = \sin 2x$

The symbolic form of the equation is

$$(D^2 + D + 1)\, y = \sin 2x.$$

$\therefore$ Auxiliary equation is $D^2 + D + 1 = 0$

$$\therefore \quad D = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\therefore \quad \text{C.F.} = e^{-x/2}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$$

$$\text{P.I.} = \frac{1}{D^2 + D + 1}\sin 2x$$

$$= \frac{1}{-2^2 + D + 1}\sin 2x \qquad [\text{Put } D^2 = -2^2]$$

$$= \frac{1}{D-3}\sin 2x$$

$$= \frac{D+3}{(D-3)(D+3)}\sin 2x = \frac{D+3}{D^2-9}\sin 2x$$

$$= \frac{D+3}{-2^2-9}\sin 2x \qquad [\because D^2 = -2^2]$$

$$= -\frac{1}{13}(D+3)\sin 2x = -\frac{1}{13}[D \sin 2x + 3\sin 2x]$$

$$= -\frac{1}{13}\left[\frac{d}{dx}\sin 2x + 3\sin 2x\right].$$

$$= -\frac{1}{13}[2\cos 2x + 3\sin 2x]$$

Hence the complete solution is

$$y = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) - \frac{1}{13}(2\cos 2x + 3\sin 2x).$$

Example 2.

Solve the differential equation

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x.$$ [K.U. 2012]

Solution. The given equation is $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x.$

The symbolic form of the equation is $(D^3 + D^2 + D + 1)\,y = \sin 2x$

Auxiliary equation is $D^3 + D^2 + D + 1 = 0$

or $D^2(D+1) + (D+1) = 0$

or $(D^2+1)(D+1) = 0 \Rightarrow D = \pm i, -1$

$\therefore$ C.F. $= c_1 e^{-x} + c_2 \cos x + c_3 \sin x$

Now, P.I. $= \dfrac{1}{D^3 + D^2 + D + 1}\sin 2x = \dfrac{1}{(D+1)(D^2+1)}.\sin 2x$

$$= \frac{1}{(D+1)(-2^2+1)}.\sin 2x \quad \text{[Put } D^2 = -2^2]$$

$$= -\frac{1}{3}.\frac{1}{D+1}.\sin 2x = -\frac{1}{3}.\frac{D-1}{(D+1)(D-1)}.\sin 2x$$

$$= -\frac{1}{3}.\frac{D-1}{D^2-1}.\sin 2x = -\frac{1}{3}.\frac{D-1}{-4-1}.\sin 2x \quad \text{[Put } D^2 = -2^2]$$

$$= \frac{1}{15}(D-1)\sin 2x = \frac{1}{15}[D\sin 2x - \sin 2x]$$

$$= \frac{1}{15}[2\cos 2x - \sin 2x]$$

Hence the complete solution is

$$y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x + \frac{1}{15}[2\cos 2x - \sin 2x].$$

Example 3.

Solve the differential equation

$$\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}-\frac{dy}{dx}-y=\cos 2x.$$

Solution. The given equation is $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}-\frac{dy}{dx}-y=\cos 2x$

The symbolic form of the equation is

$$(D^3+D^2-D-1)\,y=\cos 2x$$

Auxiliary equation is $\quad D^3+D^2-D-1=0$

or $\quad D^2(D+1)-(D+1)=0$

or $\quad (D+1)(D^2-1)=0$

or $\quad (D+1)^2(D-1)=0$

i.e., $\quad D=-1,-1,1$

$\therefore \quad$ C.F. $=c_1e^x+(c_2+c_3x)\,e^{-x}$

Now,

$$\text{P.I.}=\frac{1}{D^3+D^2-D-1}\cdot\cos 2x=\frac{1}{(D+1)(D^2-1)}\cdot\cos 2x$$

$$=\frac{1}{(D+1)(-4-1)}\cdot\cos 2x \qquad [Put\ D^2=-2^2=-4]$$

$$=-\frac{1}{5}\cdot\frac{1}{D+1}\cdot\cos 2x=-\frac{1}{5}\cdot\frac{D-1}{(D+1)(D-1)}\cdot\cos 2x$$

$$=-\frac{1}{5}\cdot\frac{D-1}{D^2-1}\cdot\cos 2x$$

$$=-\frac{1}{5}\cdot\frac{D-1}{-4-1}\cdot\cos 2x=\frac{1}{25}(D-1)\cos 2x$$

$$=\frac{1}{25}(D\cos 2x-\cos 2x)$$

$$=\frac{1}{25}(-2\sin 2x-\cos 2x) \qquad \left[\because D=\frac{d}{dx}\right]$$

$$=-\frac{1}{25}(2\sin 2x+\cos 2x)$$

Hence the complete solution is

$$y=c_1e^x+(c_2+c_3x)\,e^{-x}-\frac{1}{25}(2\sin 2x+\cos 2x).$$

Example 4.

Solve the differential equation $\frac{d^2y}{dx^2} - 4y = e^x + \sin 2x$.

Solution. The given equation is $\frac{d^2y}{dx^2} - 4y = e^x + \sin 2x$

The symbolic form of the equation is $(D^2 - 4)\, y = e^x + \sin 2x$

Auxiliary equation is $D^2 - 4 = 0$

$$D = \pm 2$$

$$\therefore \quad \text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}$$

Now,
$$\text{P.I.} = \frac{1}{D^2 - 4} \cdot (e^x + \sin 2x) = \frac{1}{D^2 - 4} \cdot e^x + \frac{1}{D^2 - 4} \sin 2x$$

$$= \frac{1}{1^2 - 4} \cdot e^x + \frac{1}{-4-4} \cdot \sin 2x \quad \left[\because \frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax\right]$$

$$= -\frac{1}{3} e^x - \frac{1}{8} \sin 2x$$

Hence, the complete or general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3} e^x - \frac{1}{8} \sin 2x.$$

Example 5.

Solve the differential equation $\frac{d^2y}{dx^2} + y = \sin x \sin 2x.$ [K.U. 2016]

Solution. The given equation is $\frac{d^2y}{dx^2} + y = \sin x \sin 2x.$

The symbolic form of the equation is $(D^2 + 1)\, y = \sin x \sin 2x$

Auxiliary equation is $D^2 + 1 = 0$

i.e.,
$$D = \pm i = 0 \pm i$$

$$\therefore \quad \text{C.F.} = e^{0x}(c_1 \cos x + c_2 \sin x) = c_1 \cos x + c_2 \sin x$$

Now,
$$\text{P.I.} = \frac{1}{D^2 + 1} \cdot \sin x \sin 2x = \frac{1}{D^2 + 1}\left[\frac{1}{2}(2 \sin 2x \sin x)\right]$$

$$= \frac{1}{2} \cdot \frac{1}{D^2 + 1} (\cos x - \cos 3x) \quad [\because 2 \sin A \sin B = \cos (A - B) - \cos (A + B)]$$

$$= \frac{1}{2}\left[\frac{1}{D^2+1}\cdot\cos x - \frac{1}{D^2+1}\cdot\cos 3x\right]$$

$$= \frac{1}{2}\left[x\cdot\frac{1}{\frac{d}{dD}(D^2+1)}\cdot\cos x - \frac{1}{-9+1}\cos 3x\right]$$

$$\left[\because \frac{1}{f(D^2)}\cos ax = x\cdot\frac{1}{\frac{d}{dD}[f(D^2)]}\cdot\cos ax \text{ if } f(-a^2)=0 \text{ (case of failure)}\right]$$

$$= \frac{1}{2}\left[x\cdot\frac{1}{2D}\cdot\cos x + \frac{1}{8}\cos 3x\right] = \frac{1}{2}\left[\frac{x}{2}\cdot\frac{1}{D}\cos x + \frac{1}{8}\cos 3x\right]$$

$$= \frac{1}{2}\left[\frac{x}{2}\int\cos x\,dx + \frac{1}{8}\cos 3x\right] = \frac{x}{4}\sin x + \frac{1}{16}\cos 3x$$

Hence, the complete solution is

$$y = c_1\cos x + c_2\sin x + \frac{x}{4}\sin x + \frac{1}{16}\cos 3x.$$

Example 6.

Solve the differential equation $\frac{d^2y}{dx^2} + a^2y = \sin ax$ [M.D.U. 2009]

Solution. The given equation is $\frac{d^2y}{dx^2} + a^2y = \sin ax$

The symbolic form of the equation is $(D^2 + a^2)\,y = \sin ax$

Auxiliary equation is $D^2 + a^2 = 0$

i.e., $D = \pm ia = 0 \pm ia$

$\therefore$ C.F. $= e^{0x}(c_1\cos ax + c_2\sin ax) = c_1\cos ax + c_2\sin ax$

Now, P.I. $= \frac{1}{D^2+a^2}\sin ax$

$$= x.\frac{1}{\frac{d}{dD}(D^2+a^2)}.\sin ax \qquad \text{[Case of failure]}$$

$$= x\cdot\frac{1}{2D}\cdot\sin ax = \frac{x}{2}\cdot\frac{1}{D}\sin ax$$

$$= \frac{x}{2}\int\sin ax\,dx$$

$$= \frac{x}{2}\left(-\frac{\cos ax}{a}\right) = \frac{-x\cos ax}{2a}$$

Hence the complete solution is $y = c_1 \cos ax + c_2 \sin ax - \frac{x}{2a} \cdot \cos ax.$

EXERCISE 4.3

***Solve the following differential equations* [Q. 1 – 5] :**

1. (i) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin 3x$ (ii) $(D^3 + 6D^2 + 11D + 6)\,y = 2\sin x$

   (iii) $(D^3 - 2D^2 + 3)\,y = \cos x$ (iv) $(D^4 - 1)\,y = \sin 2x$ [K.U. 2014; M.D.U. 2006]

2. (i) $\frac{d^4y}{dx^4} + y = \cos x$ (ii) $\frac{d^4y}{dx^4} + 2n^2\frac{d^2y}{dx^2} + n^4y = \cos mx.$

3. (i) $\frac{d^4y}{dx^4} - a^4y = \sin ax$ (ii) $\frac{d^4y}{dx^4} - m^4y = \sin mx$ [K.U. 2016]

   (iii) $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = \cos x$

4. (i) $(D^4 + 1)\,y = \sin^2 x$ $\cos 2x = 1 - 2\sin^2 x$ (ii) $(D^2 + 1)(D^2 + 4)\,y = \cos\left(\frac{x}{2}\right)\cos\left(\frac{3x}{2}\right)$

   (iii) $(D^2 + 1)(D^2 + 4)\,y = \cos\frac{x}{2}\sin\frac{x}{2}$ [M.D.U. 2017] (iv) $(D^2 - 4D + 3)\,y = \sin 3x \cos 2x$ [K.U. 2000]

   (v) $(D^2 - 4D + 4)\,y = e^{-4x} + 5\cos 3x$

5. (i) $\frac{d^3y}{dx^3} + y = \sin 3x - \cos^2\frac{x}{2}$ (ii) $\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x$

   (iii) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x.$

6. Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y + 37\sin 3x = 0$ and find the value of y when $x = \frac{\pi}{2}$ if it is given that $y = 3$ and $\frac{dy}{dx} = 0$ when $x = 0$.

7. Solve $\frac{d^2x}{dt^2} + b^2x = k\cos bt$, given that $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.

ANSWERS

1. (i) $y = c_1e^{x} + c_2e^{2x} - \frac{7}{130}\sin 3x + \frac{9}{130}\cos 3x$

(ii) $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x} - \frac{\cos x}{5}$

(iii) $y = c_1e^{-x} + e^{\frac{3}{2}x}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right] + \frac{1}{26}(5\cos x - \sin x).$

(iv) $y = c_1e^{x} + c_2e^{-x} + c_3\cos x + c_4\sin x + \frac{\sin 2x}{15}$

2. (i) $y = e^{x/\sqrt{2}}\left[c_1\cos\frac{x}{\sqrt{2}} + c_2\sin\frac{x}{\sqrt{2}}\right] + e^{-x/\sqrt{2}}\left[c_3\cos\frac{x}{\sqrt{2}} + c_4\sin\frac{x}{\sqrt{2}}\right] + \frac{1}{2}\cos x$

(ii) $y = (c_1 + c_2x)\cos nx + (c_3 + c_4x)\sin nx + \frac{\cos mx}{(m^2 - n^2)^2}.$

3. (i) $y = c_1e^{ax} + c_2e^{-ax} + c_3\cos ax + c_4\sin ax + \frac{x}{4a^3}\cos ax$

(ii) $y = c_1e^{mx} + c_2e^{-mx} + c_3\cos mx + c_4\sin mx + \frac{x}{4m^3}\cos mx$

(iii) $y = (c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x - \frac{x^2}{8}\cos x$

4. (i) $y = e^{\frac{1}{\sqrt{2}}x}\left[c_1\cos\frac{x}{\sqrt{2}} + c_2\sin\frac{x}{\sqrt{2}}\right] + e^{-\frac{1}{\sqrt{2}}x}\left[c_3\cos\frac{x}{\sqrt{2}} + c_4\sin\frac{x}{\sqrt{2}}\right] + \frac{1}{2} - \frac{1}{34}\cos 2x$

(ii) $y = (c_1\cos x + c_2\sin x) + (c_3\cos 2x + c_4\sin 2x) + \frac{x}{12}\left[\sin x - \frac{1}{2}\sin 2x\right]$

(iii) $y = (c_1\cos x + c_2\sin x) + (c_3\cos 2x + c_4\sin 2x) - \frac{x}{12}\cos x$

(iv) $y = c_1e^{x} + c_2e^{3x} + \frac{1}{884}(-11\sin 5x + 10\cos 5x) + \frac{1}{20}(\sin x + 2\cos x)$

(v) $y = (c_1 + c_2x)e^{2x} + \frac{1}{36}e^{-4x} - \frac{5}{169}(12\sin 3x + 5\cos 3x).$

5. (i) $y = c_1e^{-x} + e^{x/2}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right] + \frac{1}{730}[\sin 3x + 27\cos 3x] - \frac{1}{2} - \frac{1}{4}(\cos x - \sin x)$

(*ii*) $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} x \cos 2x + \frac{1}{5} e^x.$

(*iii*) $y = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x) + x\, e^x + \frac{3}{10} \sin x + \frac{1}{10} \cos x.$

6. $y = e^{-x} (c_1 \cos 3x + c_2 \sin 3x) + 6 \cos 3x - \sin 3x \,;\, y = 1.$

7. $x = \frac{kt}{2b} \sin bt.$

4.12.3. Case III. To evaluate $\frac{1}{f(D)} x^m$, where m is a positive integer.

To evaluate the particular integral $\frac{1}{f(D)} . x^m$, where m is a +ve integer; from f (D) take out the common lowest degree term and the remaining factor will be of the form $[1 \pm \phi(D)]$.

Take $[1 \pm \phi(D)]$ to the numerator with negative index and expand it by Binomial theorem. The expansion will be made only upto the terms where the index of D is m, because D^{m+1} when multiplied by x^m (*i.e.*, $D^{m+1}. x^m$) will be zero. Operate on x^m with each term.

Important Expansions. The following binomial expansions will be commonly used in these type of questions :

1. $(1 - D)^{-1} = 1 + D + D^2 + D^3 + ...$ to ∞.
2. $(1 + D)^{-1} = 1 - D + D^2 - D^3 + ...$ to ∞.
3. $(1 - D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + ... + (r + 1) D^r + ...$ to ∞.
4. $(1 - D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + ... + \frac{(r+1)(r+2)}{2} D^r + ...$ to ∞.

By Binomial expansion $(1+x)^n$

$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$

SOLVED EXAMPLES

Example 1.

Solve the differential equation $\frac{d^2y}{dx^2} - 4y = x^2.$

Solution. The given equation is $\frac{d^2y}{dx^2} - 4y = x^2$

The symbolic form of the equation is $(D^2 - 4)\, y = x^2$

$\therefore$ Auxiliary equation is $D^2 - 4 = 0 \;\Rightarrow\; D = \pm 2$

$\therefore$ $$\text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}$$

Now.
$$\text{P.I.} = \frac{1}{D^2 - 4} \cdot x^2$$

$$= \frac{1}{-4\left(1 - \frac{D^2}{4}\right)} x^2 = -\frac{1}{4}\left(1 - \frac{D^2}{4}\right)^{-1} \cdot x^2$$

$$= -\frac{1}{4}\left[1 + \frac{D^2}{4} + \ldots\ldots\right] x^2 \qquad \textit{[By Binomial theorem]}$$

$$= -\frac{1}{4}\left[x^2 + \frac{1}{4} D^2 (x^2)\right] \qquad \textit{[Retaining terms upto } D^2\textit{]}$$

$$= -\frac{1}{4}\left[x^2 + \frac{1}{4} \cdot 2\right] = -\frac{1}{4}\left(x^2 + \frac{1}{2}\right)$$

Hence, the complete solution is $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4}\left(x^2 + \frac{1}{2}\right)$.

Example 2. ***Solve the differential equation*** $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$. [*K.U. 2012*]

Solution. The given equation is $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$

The symbolic form of the equation is
$$(D^2 + D - 2)\, y = x + \sin x$$

$\therefore$ Auxiliary equation is $\quad D^2 + D - 2 = 0$

or $\quad (D + 2)(D - 1) = 0 \quad \Rightarrow \quad D = 1, -2$

$\therefore \quad$ C.F. $= c_1 e^{x} + c_2 e^{-2x}$

Now,
$$\text{P.I.} = \frac{1}{D^2 + D - 2}(x + \sin x)$$

$$= \frac{1}{D^2 + D - 2} x + \frac{1}{D^2 + D - 2} \sin x$$

$$= \frac{1}{-2\left(1 - \frac{D}{2} - \frac{D^2}{2}\right)} \cdot x + \frac{1}{-1^2 + D - 2} \cdot \sin x$$

$$= -\frac{1}{2}\left(1 - \frac{D}{2} - \frac{D^2}{2}\right)^{-1} \cdot x + \frac{1}{D - 3} \cdot \sin x$$

$$= -\frac{1}{2}\left[1 - \left(\frac{D}{2} + \frac{D^2}{2}\right)\right]^{-1} \cdot x + \frac{D + 3}{(D - 3)(D + 3)} \cdot \sin x$$

$$= -\frac{1}{2}\left[1 + \left(\frac{D}{2} + \frac{D^2}{2}\right) + \ldots\ldots\right] x + \frac{D + 3}{D^2 - 9} \cdot \sin x$$

$$= -\frac{1}{2}\left[x + \frac{1}{2}D(x)\right] + \frac{D+3}{-1-9}.\sin x$$

$$= -\frac{1}{2}\left(x + \frac{1}{2}\right) - \frac{1}{10}(D+3).\sin x$$

$$= -\frac{1}{2}\left(x + \frac{1}{2}\right) - \frac{1}{10}(D\sin x + 3\sin x)$$

$$= -\frac{1}{2}\left(x + \frac{1}{2}\right) - \frac{1}{10}(\cos x + 3\sin x)$$

Hence, the complete solution is $y = c_1e^x + c_2e^{-2x} - \frac{1}{2}\left(x + \frac{1}{2}\right) - \frac{1}{10}(\cos x + 3\sin x)$.

Example 3.

Solve the differential equation $\boldsymbol{\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 + e^x + \cos 2x}$. [K.U. 2014]

Solution. The given equation is $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 + e^x + \cos 2x$

The symbolic form of the equation is

$$(D^2 - 4D + 4)\,y = x^2 + e^x + \cos 2x$$

Auxiliary equation is

$$D^2 - 4D + 4 = 0$$

i.e., $(D-2)^2 = 0 \quad \Rightarrow \quad D = 2, 2$

$\therefore$ $\text{C.F.} = (c_1 + c_2x)\,e^{2x}$

Now, $\text{P.I.} = \frac{1}{D^2 - 4D + 4}(x^2 + e^x + \cos 2x)$

$$= \frac{1}{D^2 - 4D + 4}x^2 + \frac{1}{D^2 - 4D + 4}.e^x + \frac{1}{D^2 - 4D + 4}\cos 2x$$

$$= \frac{1}{4\left(1 - D + \frac{D^2}{4}\right)}x^2 + \frac{1}{1^2 - 4.1 + 4}e^x + \frac{1}{-2^2 - 4D + 4}\cos 2x$$

$$= \frac{1}{4}\left[1 - \left(D - \frac{D^2}{4}\right)\right]^{-1}.x^2 + e^x - \frac{1}{4}\left(\frac{1}{D}\cos 2x\right)$$

$$= \frac{1}{4}\left[1 + D - \frac{D^2}{4} + \left(D - \frac{D^2}{2}\right)^2 + ...\right]x^2 + e^x - \frac{1}{4}\int \cos 2x\,dx$$

$$= \frac{1}{4}\left[1 + D - \frac{D^2}{4} + D^2 + ...\right]x^2 + e^x - \frac{1}{4}.\frac{\sin 2x}{2}$$

$$= \frac{1}{4}\left[1 + D + \frac{3}{4}D^2 + \ldots\right]x^2 + e^x - \frac{1}{8}\sin 2x$$

$$= \frac{1}{4}\left[x^2 + D(x^2) + \frac{3}{4}D^2(x^2)\right] + e^x - \frac{1}{8}\sin 2x$$

$$= \frac{1}{4}\left[x^2 + 2x + \frac{3}{2}\right] + e^x - \frac{1}{8}\sin 2x$$

Hence the complete solution is

$$y = (c_1 + c_2 x)e^{2x} + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right) + e^x - \frac{1}{8}\sin 2x.$$

Example 4. ***Solve the differential equation***

$$\boldsymbol{(D^2 + 1)(D^2 + 4)\,y = \cos\frac{x}{2}\cos\frac{3x}{2} + x.}$$ [*M.D.U. 2016*]

Solution. The given equation is $(D^2 + 1)(D^2 + 4)\,y = \cos\left(\frac{x}{2}\right)\cos\left(\frac{3x}{2}\right) + x$

$\therefore$ Auxiliary equation is $(D^2 + 1)(D^2 + 4) = 0 \Rightarrow D = \pm i, \pm 2i$

$\therefore$ C.F. $= (c_1 \cos x + c_2 \sin x) + (c_3 \cos 2x + c_4 \sin 2x)$

Now, P.I. $= \frac{1}{(D^2 + 1)(D^2 + 4)}\left(\cos\frac{x}{2}\cos\frac{3x}{2} + x\right)$

$$= \frac{1}{(D^2 + 1)(D^2 + 4)}\left[\frac{1}{2}.2\cos\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right) + x\right]$$

$$= \frac{1}{(D^2 + 1)(D^2 + 4)}\left[\frac{1}{2}(\cos 2x + \cos x) + x\right]$$

$$= \frac{1}{2}\left[\frac{1}{(D^2 + 1)(D^2 + 4)}\cos 2x + \frac{1}{(D^2 + 1)(D^2 + 4)}\cos x\right] + \frac{1}{(D^2 + 1)(D^2 + 4)}x$$

$$= \frac{1}{0}\cos 2x + \frac{1}{0}\cos x + \frac{1}{D^4 + 5D^2 + 4}.x$$ [*Case of failure*]

$$= \frac{x}{2}\left[\frac{1}{\frac{d}{dD}(D^4 + 5D^2 + 4)}\cos 2x + \frac{1}{\frac{d}{dD}(D^4 + 5D^2 + 4)}\cos x\right] + \frac{1}{4\left(1 + \frac{5}{4}D^2 + \frac{D^4}{4}\right)}.x$$

$$= \frac{x}{2}\left[\underset{[\text{Put } D^2 = -4]}{\frac{1}{4D^3 + 10D}}\cos 2x + \underset{[\text{Put } D^2 = -1]}{\frac{1}{4D^3 + 10D}}\cos x\right] + \frac{1}{4\left[1 + \left(\frac{5}{4}D^2 + \frac{D^4}{4}\right)\right]}.x$$

$$= \frac{x}{2}\left[\frac{1}{-6D}\cos 2x + \frac{1}{6D}\cos x\right] + \frac{1}{4}\left[1 + \left(\frac{5}{4}D^2 + \frac{D^4}{4}\right)\right]^{-1}x$$

$$= \frac{x}{2}\left[-\frac{1}{6}\int \cos 2x\, dx + \frac{1}{6}\int \cos x\, dx\right] + \frac{1}{4}\left[1 - \left(\frac{5}{4}D^2 + \frac{D^4}{4}\right) + \ldots\right]x$$

$$= \frac{x}{2}\left[-\frac{1}{12}\sin 2x + \frac{1}{6}\sin x\right] + \frac{1}{4}\left[1 - \frac{5}{4}D^2 + \ldots\right]x$$

$$= \frac{x}{12}\left[\sin x - \frac{1}{2}\sin 2x\right] + \frac{1}{4}[x - 0]$$

$$= \frac{x}{12}\left[\sin x - \frac{1}{2}\sin 2x\right] + \frac{x}{4}$$

Hence, the complete solution is

$$y = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x + \frac{x}{12}\left(\sin x - \frac{1}{2}\sin 2x\right) + \frac{x}{4}.$$

EXERCISE 4.4

Solve the following differential equations [Q. 1 – 6] :

1. (*i*) $\dfrac{d^3y}{dx^3} - 13\dfrac{dy}{dx} + 12y = x.$ (*ii*) $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 8y = x$

 (*iii*) $2\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 2y = 5 + 2x$

2. (*i*) $(D^3 - D^2 - 6D)y = x^2$ (*ii*) $(D^4 - 2D^3 + D^2)y = x^3$ (*iii*) $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} = 1 + x^2.$

3. (*i*) $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = e^{2x} + x^2 + x$ (*ii*) $(D^4 + 2D^3 - 3D^2)y = x^2 + 3e^{2x} + 4\sin x$ [K.U. 2017]

 (*iii*) $(D^4 - a^4)y = x^4 + \sin bx$

4. (*i*) $(D-1)^2(D+1)^2 y = \sin^2\dfrac{x}{2} + e^x + x$ (*ii*) $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 2e^{-x} + x^2$

5. $\dfrac{d^2y}{dx^2} + 4y = e^x + \sin 3x + x^2$

6. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 8(x^2 + e^{2x} + \sin 2x)$ [M.D.U. 2009]

7. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 2y = x - \sin x$ [M.D.U. 2007]

ANSWERS

1. (*i*) $y = c_1e^x + c_2e^{3x} + c_3e^{-4x} + \dfrac{1}{144}(12x + 13).$ (*ii*) $y = c_1e^x + c_2e^{-2x} + c_3e^{4x} + \dfrac{1}{8}\left(x + \dfrac{3}{4}\right).$

 (*iii*) $y = c_1e^{-x/2} + c_2e^{-2x} + x$

2. (*i*) $y = c_1 + c_2e^{3x} + c_3e^{-2x} - \dfrac{1}{18}\left(x^3 - \dfrac{1}{2}x^2 + \dfrac{7}{6}x\right)$

(ii) $y = c_1 + c_2 x + (c_3 + c_4 x)\, e^x + \dfrac{x^5}{20} + \dfrac{x^4}{2} + 3x^3 + 12x^2.$

(iii) $y = c_1 + c_2\, e^{3x} + c_3\, e^{-2x} - \dfrac{1}{108}(6x^3 - 3x^2 + 25x).$

3. (i) $y = c_1 + (c_2 + c_3 x)\, e^{-x} + \dfrac{1}{18} e^{2x} + \dfrac{x^3}{3} - \dfrac{3x^2}{2} + 4x.$

(ii) $y = (c_1 + c_2\, x) + c_3\, e^{x} + c_4 e^{-3x} - \left(\dfrac{1}{36} x^4 + \dfrac{2}{27} x^3 + \dfrac{7}{27} x^2\right) + \dfrac{3}{20} e^{2x} + \dfrac{2}{5}(\cos x + 2\sin x).$

(iii) $y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax - \dfrac{1}{a^4}\left(x^4 + \dfrac{24}{a^4}\right) + \dfrac{\sin bx}{b^4 - a^4}.$

4. (i) $y = e^x (c_1 + c_2\, x) + e^{-x}(c_3 + c_4 x) + \dfrac{1}{2}(1 + 2x) - \dfrac{1}{8}\cos x + \dfrac{x^2 e^x}{8}.$

(ii) $y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x + x e^{-x} + (x^2 - 2x).$

5. $y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{e^x}{5} - \dfrac{\sin 3x}{5} + \dfrac{x^2}{4} - \dfrac{1}{8}.$

6. $y = e^{2x}(c_1 + c_2 x) + 2x^2 + 4x + 3 + 4x^2 e^{2x} + \cos 2x.$

7. $y = c_1\, e^{(-2+\sqrt{2})x} + c_2\, e^{(-2-\sqrt{2})x} + \dfrac{1}{2}(x - 2) + \dfrac{1}{17}(4\cos x - \sin x)\,.$

4.12.4. Case IV. To prove that $\dfrac{1}{f(D)}(e^{ax}\, V) = e^{ax}\, \dfrac{1}{f(D+a)}\, V$, where V is a function of x.

[*M.D.U. 2014, 2000*]

Proof. By successive differentiation, we have

$$D\,(e^{ax}\, V) = e^{ax}\, DV + a e^{ax}\, V = e^{ax}\,(D + a)\, V$$

$$D^2 (e^{ax}\, V) = e^{ax}\, D^2 V + a e^{ax}\, DV + a^2 e^{ax}\, V + a e^{ax}\, DV$$

$$= e^{ax}\,(D^2 + 2aD + a^2)\, V = e^{ax}\,(D + a)^2\, V$$

Similarly, $\quad D^3 (e^{ax}\, V) = e^{ax}\,(D + a)^3\, V$

....

....

$$D^n (e^{ax}\, V) = e^{ax}\,(D + a)^n\, V$$

$$\therefore \quad f(D)\,(e^{ax}\, V) = e^{ax}\, f(D + a)\, V \qquad \ldots(1)$$

Putting $f(D + a)\, V = X$, *i.e.*, $V = \dfrac{1}{f(D + a)}\, X$ in (1), we have

$$f(D)\left[e^{ax} \cdot \frac{1}{f(D+a)} \cdot X\right] = e^{ax} \,.\, X$$

$\frac{1}{f(D)} e^{ax} \cdot V = e^{ax} \frac{1}{f(D+a)} V$

$\frac{1}{f(D)}(xV) = x \frac{1}{f(D)} \cdot V + \frac{d}{dD}\left(\frac{1}{f(D)}\right) V$

Operating on both sides by $\dfrac{1}{f(D)}$, we have

$$\frac{1}{f(D)} \cdot f(D)\left[e^{ax} \cdot \frac{1}{f(D+a)} \cdot X\right] = \frac{1}{f(D)}\,(e^{ax}\; X)$$

$$\Rightarrow \qquad e^{ax}\frac{1}{f(D+a)}X = \frac{1}{f(D)}(e^{ax}X)$$

$$\Rightarrow \qquad \frac{1}{f(D)}(e^{ax}X) = e^{ax}\frac{1}{f(D+a)}.X$$

In general, $\quad \dfrac{1}{f(D)}(e^{ax}V) = e^{ax}\dfrac{1}{f(D+a)}.V$

Hence the required result.

Working Rule :

In general practice, take out e^{ax} and put $D + a$ for every D in $f(D)$, so that $f(D)$ becomes $f(D + a)$; then operate $\dfrac{1}{f(D+a)}$ with V alone by previous methods as discussed in case II and III.

Note.

Here V shall be of type $x^m \sin ax$ or $\cos ax$.

SOLVED EXAMPLES

Example 1.

Solve the differential equation $\dfrac{d^2y}{dx^2} + y = x\,.\,e^{2x}$

Solution. The given equation is $\dfrac{d^2y}{dx^2} + y = x\,.\,e^{2x}$

The symbolic form of the equation is $(D^2 + 1)\,y = x.e^{2x}$

Auxiliary equation is

$$D^2 + 1 = 0 \quad \Rightarrow \quad D = \pm i = 0 \pm i$$

$$\therefore \qquad \text{C.F.} = e^{0x}(c_1 \cos x + c_2 \sin x)$$

$$= c_1 \cos x + c_2 \sin x$$

Now, $\qquad \text{P.I.} = \dfrac{1}{D^2+1}(xe^{2x}) = \dfrac{1}{D^2+1}(e^{2x}\,.\,x)$

$$= e^{2x} \frac{1}{(D+2)^2+1} . x \qquad \text{[Note this step]}$$

$$= e^{2x} . \frac{1}{D^2+4D+4+1} . x = e^{2x} . \frac{1}{D^2+4D+5} . x$$

$$= e^{2x} . \frac{1}{5\left(1+\frac{4D}{5}+\frac{D^2}{5}\right)} . x$$

$$= \frac{1}{5} . e^{2x}\left[1+\left(\frac{4D}{5}+\frac{D^2}{5}\right)\right]^{-1} . x$$

$$= \frac{1}{5} e^{2x}\left[1-\left(\frac{4D}{5}+\frac{D^2}{5}\right)+\ldots\ldots\right]x$$

$$= \frac{1}{5} . e^{2x}\left[x-\frac{4}{5}D(x)\right] = \frac{1}{5} e^{2x}\left(x-\frac{4}{5}\right)$$

Hence, the complete solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{5} e^{2x}\left(x-\frac{4}{5}\right).$$

Example 2.

Solve the differential equation $\dfrac{d^3y}{dx^3} - 3\dfrac{dy}{dx} + 2y = x^2 . e^x.$ [K.U. 2016, 15, 08, 06]

Solution. The given equation is $\dfrac{d^3y}{dx^3} - 3\dfrac{dy}{dx} + 2y = x^2 . e^x$

The symbolic form of the equation is

$$(D^3-3D+2)\,y = x^2.e^x$$

Auxiliary equation is

$$D^3-3D+2 = 0$$

or

$$(D-1)(D^2+D-2) = 0$$

or

$$(D-1)(D+2)(D-1) = 0$$

or

$$(D-1)^2(D+2) = 0 \quad \Rightarrow \quad D = 1, 1, -2$$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^x + c_3 e^{-2x}$

Now, $$\text{P.I.} = \frac{1}{D^3 - 3D + 2}(x^2 . e^x) = \frac{1}{D^3 - 3D + 2}(e^x . x^2)$$

$$= e^x \cdot \frac{1}{(D+1)^3 - 3(D+1) + 2} \cdot x^2 = e^x \cdot \frac{1}{D^3 + 3D^2} \cdot x^2$$

$$= e^x \cdot \frac{1}{3D^2\left(1 + \frac{D}{3}\right)} \cdot x^2$$

$$= \frac{1}{3} e^x \frac{1}{D^2}\left(1 + \frac{D}{3}\right)^{-1} . x^2$$

$$= \frac{1}{3} e^x \frac{1}{D^2}\left[1 - \frac{D}{3} + \frac{D^2}{9} .....\right] x^2 \qquad \text{[By Binomial theorem]}$$

$$= \frac{1}{3} e^x \frac{1}{D^2}\left[x^2 - \frac{1}{3} D(x^2) + \frac{1}{9} D^2(x^2)\right]$$

$$= \frac{1}{3} e^x \left[\frac{1}{D^2} x^2 - \frac{1}{3} \cdot \frac{1}{D}(x^2) + \frac{1}{9} x^2\right]$$

$$= \frac{1}{3} e^x \left[\frac{1}{D} \int x^2\, dx - \frac{1}{3} \int x^2\, dx + \frac{1}{9} x^2\right]$$

$$= \frac{1}{3} e^x \left[\frac{1}{D}\left(\frac{x^3}{3}\right) - \frac{x^3}{9} + \frac{x^2}{9}\right]$$

$$= \frac{1}{3} e^x \left[\frac{1}{3} \int x^3\, dx - \frac{x^3}{9} + \frac{x^2}{9}\right] = \frac{1}{3} e^x \left[\frac{x^4}{12} - \frac{x^3}{9} + \frac{x^2}{9}\right]$$

$$= \frac{1}{108} x^2 e^x [3x^2 - 4x + 4]$$

Hence, the complete solution is

$$y = (c_1 + c_2 x)e^x + c_3 e^{-2x} + \frac{1}{108} x^2 e^x (3x^2 - 4x + 4).$$

Example 3.

Solve the differential equation

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = e^x \sin 2x.$$

Solution. The given equation is $\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = e^x \sin 2x$

The symbolic form of the equation is

$$(D^2 + 2D + 1)\, y = e^x \sin 2x$$

Auxiliary equation is $D^2 + 2D + 1 = 0$

i.e., $(D + 1)^2 = 0 \Rightarrow D = -1, -1$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{-x}$

Now, P.I. $= \dfrac{1}{D^2 + 2D + 1} \cdot e^x \sin 2x$

$$= \frac{1}{(D+1)^2} e^x \sin 2x = e^x \cdot \frac{1}{(D+1+1)^2} \sin 2x$$

$$= e^x \cdot \frac{1}{(D+2)^2} \sin 2x = e^x \cdot \frac{1}{D^2 + 4D + 4} \sin 2x$$

$$= e^x \cdot \frac{1}{-2^2 + 4D + 4} \cdot \sin 2x$$

$$= e^x \cdot \frac{1}{4D} \cdot \sin 2x = \frac{1}{4} e^x \int \sin 2x \, dx$$

$$= \frac{1}{4} e^x \cdot \frac{-\cos 2x}{2} = -\frac{1}{8} e^x \cos 2x$$

Hence, the complete solution is $y = (c_1 + c_2 x)\, e^{-x} - \dfrac{1}{8} e^x \cos 2x.$

Example 4.

Solve the differential equation

$$\frac{d^2y}{dx^2} + 2y = x^2 \cdot e^{3x} + e^x \cdot \cos 2x \qquad \text{[M.D.U. 2015, 13, 11; K.U. 2007]}$$

Solution. The given equation is $\dfrac{d^2y}{dx^2} + 2y = x^2 \cdot e^{3x} + e^x \cdot \cos 2x$

The symbolic form of the equation is

$$(D^2 + 2)\, y = x^2 e^{3x} + e^x \cos 2x$$

Auxiliary equation is $D^2 + 2 = 0$

$\Rightarrow$ $D = \pm i\sqrt{2} = 0 \pm i\sqrt{2}$

$\therefore$ C.F. $= e^{0x} (c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x) = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x$

Now, P.I. $= \dfrac{1}{D^2 + 2} (x^2 e^{3x} + e^x \cos 2x)$

$$= \frac{1}{D^2+2}(e^{3x}x^2) + \frac{1}{D^2+2}(e^x \cos 2x)$$

$$= e^{3x} \cdot \frac{1}{(D+3)^2+2} \cdot x^2 + e^x \cdot \frac{1}{(D+1)^2+2} \cos 2x$$

$$= e^{3x} \cdot \frac{1}{D^2+6D+11} x^2 + e^x \cdot \frac{1}{D^2+2D+3} \cos 2x$$

$$= e^{3x} \frac{1}{11\left(1+\frac{6}{11}D+\frac{D^2}{11}\right)} \cdot x^2 + e^x \frac{1}{-4+2D+3} \cos 2x$$

$$= \frac{1}{11} e^{3x} \left[1+\frac{6}{11}D+\frac{D^2}{11}\right]^{-1} \cdot x^2 + e^x \frac{1}{2D-1} \cos 2x$$

$$= \frac{1}{11} e^{3x} \left[1-\left(\frac{6}{11}D+\frac{D^2}{11}\right)+\left(\frac{6}{11}D+\frac{D^2}{11}\right)^2 \ldots\ldots\right] x^2 + e^x \cdot \frac{2D+1}{(2D-1)(2D+1)} \cdot \cos 2x$$

$$= \frac{1}{11} e^{3x} \left[1-\frac{6}{11}D-\frac{D^2}{11}+\frac{36}{121}D^2+\ldots\ldots\right] x^2 + e^x \cdot \frac{2D+1}{4D^2-1} \cdot \cos 2x$$

$$= \frac{1}{11} e^{3x} \left[1-\frac{6}{11}D+\frac{25}{121}D^2+\ldots\ldots\right] x^2 + e^x \cdot \frac{(2D+1)}{4(-4)-1} \cdot \cos 2x$$

$$= \frac{1}{11} e^{3x} \left[x^2-\frac{6}{11}D(x^2)+\frac{25}{121}D^2(x^2)\right] - \frac{e^x}{17}(2D+1) \cos 2x$$

$$= \frac{1}{11} e^{3x} \left[x^2-\frac{6}{11}.2x+\frac{25}{121}.2\right] - \frac{e^x}{17}[2D(\cos 2x)+\cos 2x]$$

$$= \frac{1}{11} e^{3x} \left[x^2-\frac{12}{11}x+\frac{50}{121}\right] - \frac{1.e^x}{17}[2(-2\sin 2x)+\cos 2x]$$

$$= \frac{1}{11} e^{3x} \left[x^2-\frac{12}{11}x+\frac{50}{121}\right] - \frac{1}{17}e^x[-4\sin 2x+\cos 2x]$$

$$= \frac{1}{11} e^{3x} \left[x^2-\frac{12}{11}x+\frac{50}{121}\right] + \frac{1}{17}e^x[4\sin 2x-\cos 2x]$$

Hence, the complete solution is

$$y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x + \frac{1}{11} e^{3x}\left(x^2-\frac{12}{11}x+\frac{50}{121}\right) + \frac{1}{17}e^x(4\sin 2x - \cos 2x).$$

$(1+x)^n = {}^nC_0(1)\,x + {}^nC_1(1)\,x + \dots + {}^nC_n(1)\,x^n$

EXERCISE 4.5

***Solve the following differential equations* [Q. 1 – 6] :**

1. (i) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x^2 . e^{3x}$ [M.D.U. 2012; K.U. 2000]

 (ii) $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = x^2 e^{2x}$

 (iii) $(D^2 - 5D + 6)\, y = x\, e^{4x}$.

2. (i) $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = x\,(x + e^x)$

 (ii) $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = e^{2x}\,(1 + x)$.

3. (i) $\dfrac{d^3y}{dx^3} - 2\dfrac{dy}{dx} + 4y = e^x \cos x$

 (ii) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 4y = e^x \cos x$

 (iii) $\dfrac{d^4y}{dx^4} - y = e^x \cos x$

 $(D^2-1)(D^2+1)$

 (iv) $(D^2 + 3D + 2)\, y = e^{2x} \sin x$.

4. (i) $(D^2 - 4D + 3)\, y = e^x \cos 2x + \cos 3x$

 (ii) $(D^2 - 4D + 4)\, y = e^{2x} \cos^2 x$. [M.D.U. 2001]

5. (i) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = e^x \sin x$

 (ii) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^x \sin 2x + x^2$ [M.D.U. 2008]

6. $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = e^{2x} \cos x + k$, where k is a constant.

ANSWERS

1. (i) $y = (c_1 + c_2 x)\, e^x + \dfrac{e^{3x}}{8}(2x^2 - 4x + 3)$

 (ii) $y = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{-2x} - \dfrac{e^{2x}}{12}\left(x^2 + \dfrac{5}{6}x + \dfrac{97}{72}\right)$

 (iii) $y = c_1 e^{2x} + c_2 e^{3x} + \dfrac{1}{2} e^{4x}\left(x - \dfrac{3}{2}\right)$.

2. (i) $y = c_1 e^{2x} + c_2 e^{3x} + \dfrac{1}{6}\left(x^2 + \dfrac{5}{3}x + \dfrac{19}{18}\right) + \dfrac{1}{2} e^x \left(x + \dfrac{3}{2}\right)$

 (ii) $y = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{-2x} - \dfrac{1}{12} e^{2x}\left[x + \dfrac{17}{12}\right]$

3. (*i*) $y = c_1 e^{-2x} + e^x (c_2 \cos x + c_3 \sin x) + \dfrac{1}{20} xe^x (3 \sin x - \cos x)$

(*ii*) $y = e^x (c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x) + \dfrac{1}{2} e^x \cos x$

(*iii*) $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \dfrac{1}{5} e^x \cos x$

(*iv*) $y = c_1 e^{-x} + c_2 e^{-2x} - \dfrac{1}{170} e^{2x} (7 \cos x - 11 \sin x)$.

4. (*i*) $y = c_1 e^x + c_2 e^{3x} - \dfrac{1}{8} e^x (\sin 2x + \cos 2x) - \dfrac{1}{30} (2 \sin 3x + \cos 3x)$

(*ii*) $y = (c_1 + c_2 x) e^{2x} + \dfrac{1}{8} e^{2x} (2x^2 - \cos 2x)$.

5. (*i*) $y = e^x (c_1 \cos x + c_2 \sin x) - \dfrac{x}{2} e^x \cos x$.

(*ii*) $y = (c_1 + c_2 x) e^{-x} - \dfrac{1}{8} e^x \cos 2x + (x^2 - 4x + 6)$.

6. $y = c_1 e^{2x} + c_2 \cos x + c_3 \sin x - \dfrac{e^{2x}}{8} (\cos x - \sin x) - \dfrac{k}{2}$.

4.12.5. Case V. To evaluate $\dfrac{1}{f(D)} \cdot (x V)$, where V is any function of x.

***i.e.*, Show that** $\dfrac{1}{f(D)} (x V) = x \cdot \dfrac{1}{f(D)} \cdot V + \dfrac{d}{dD} \left[\dfrac{1}{f(D)} \right] \cdot V$

Proof. By Leibnitz's Theorem on successive differentiation, we have

$$\begin{aligned} D^n (xV) &= D^n (Vx) = (D^n V) x + {}^nC_1 (D^{n-1} V) \\ &= x . D^n V + n . D^{n-1} V \\ &= x D^n V + \frac{d}{dD} (D^n) V \qquad \left[\because \frac{d}{dD} D^n = nD^{n-1} \right] \end{aligned}$$

$$\therefore \quad f(D) (xV) = x f(D) V + f'(D) V \qquad \ldots(1)$$

$$[\because f(D) = P_0 + P_1 D + P_2 D^2 + \ldots\ldots + P_n . D^n]$$

Putting $f(D) V = X$, we have

$$\frac{1}{f(D)} [f(D) V] = \frac{1}{f(D)} X \quad \Rightarrow \quad V = \frac{1}{f(D)} . X$$

Also since V is a function of x, so is X

$\therefore$ From (1), $\quad f(\mathrm{D})\left[x\,\dfrac{1}{f(\mathrm{D})}\,.\,\mathrm{X}\right] = x.\mathrm{X} + f'(\mathrm{D}).\dfrac{1}{f(\mathrm{D})}\,\mathrm{X}$

Operating on both sides by $\dfrac{1}{f(\mathrm{D})}$, we have

$$x\,\frac{1}{f(\mathrm{D})}\,\mathrm{X} = \frac{1}{f(\mathrm{D})}\,(x\,\mathrm{X}) + \frac{f'(\mathrm{D})}{[f(\mathrm{D})]^2}\,.\,\mathrm{X}$$

or
$$\frac{1}{f(\mathrm{D})}\,(x\,\mathrm{X}) = x\,.\,\frac{1}{f(\mathrm{D})}\,.\,\mathrm{X} - \frac{f'(\mathrm{D})}{[f(\mathrm{D})]^2}\,.\,\mathrm{X}$$

or
$$\frac{1}{f(\mathrm{D})}\,(x\,\mathrm{X}) = x\,.\,\frac{1}{f(\mathrm{D})}\,\mathrm{X} + \frac{d}{d\mathrm{D}}\left[\frac{1}{f(\mathrm{D})}\right].\,\mathrm{X} \qquad \left[\because \frac{d}{dD}\left(\frac{1}{f(\mathrm{D})}\right) = -\frac{f'(D)}{[f(D)]^2}\right]$$

In general, replacing X by V, we have

$$\frac{1}{f(\mathrm{D})}\,(x\mathrm{V}) = x\,.\,\frac{1}{f(\mathrm{D})}\,.\,\mathrm{V} + \frac{d}{d\mathrm{D}}\left[\frac{1}{f(\mathrm{D})}\right].\,\mathrm{V}$$

[$\because$ *X and V are both functions of x*]

Hence the result.

SOLVED EXAMPLES

Example 1.

Solve the differential equation $\boldsymbol{\dfrac{d^2y}{dx^2} + 4y = x\sin x}$ **[*M.D.U. 2015*]**

Solution. The given equation is $\dfrac{d^2y}{dx^2} + 4y = x\sin x.$

The symbolic form of the equation is

$$(\mathrm{D}^2 + 4)\,y = x\sin x$$

Auxiliary equation is $\quad \mathrm{D}^2 + 4 = 0 \;\Rightarrow\; \mathrm{D} = \pm\, 2i = 0 \pm 2i$

$$\therefore \qquad \text{C.F.} = e^{0.x}\,(c_1\cos 2x + c_2\sin 2x)$$
$$= c_1\cos 2x + c_2\sin 2x$$

$$\frac{d}{dD}\left(\frac{1}{D^2+4}\right) = \frac{d}{dD}\left((D^2+4)^{-1}\right)$$
$$= (-1)\cdot(D^2+4)^{-2}$$

Now, $\qquad \text{P.I.} = \dfrac{1}{\mathrm{D}^2 + 4}\,(x\sin x)$

$$= x \cdot \frac{1}{D^2+4} \sin x - \frac{2D}{(D^2+4)^2} \sin x$$

$$\left[\because \quad \frac{1}{f(D)}(xV) = x.\frac{1}{f(D)}V + \frac{d}{dD}\left(\frac{1}{f(D)}\right).V\right]$$

$$= x \cdot \frac{1}{-1+4} \sin x - \frac{2D}{(-1+4)^2} . \sin x$$

$$= \frac{1}{3} x \sin x - \frac{2}{9} D(\sin x)$$

$$= \frac{1}{3} x \sin x - \frac{2}{9} \cdot \frac{d}{dx}(\sin x) = \frac{1}{3} x \sin x - \frac{2}{9} \cos x$$

Hence, the complete solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3} x \sin x - \frac{2}{9} \cos x.$$

Example 2.

Solve the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x. \qquad \text{[K.U. 2016, 01; M.D.U. 2006]}$$

Solution. The given equation is $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$

The symbolic form of the given equation is

$$(D^2 - 2D + 1)y = x e^x \sin x$$

Auxiliary equation is

$$D^2 - 2D + 1 = 0$$

or $$(D-1)^2 = 0 \Rightarrow D = 1, 1$$

$\therefore$ $$\text{C.F.} = (c_1 + c_2 x) e^x$$

Now, $$\text{P.I.} = \frac{1}{D^2 - 2D + 1}(xe^x \sin x) = \frac{1}{(D-1)^2}(xe^x \sin x)$$

$$= e^x \cdot \frac{1}{(D+1-1)^2}(x \sin x) = e^x \cdot \frac{1}{D^2}(x \sin x)$$

$$= e^x\left[x \cdot \frac{1}{D^2} \sin x + \frac{d}{dD}\left(\frac{1}{D^2}\right) \sin x\right]$$

$$= e^x\left[x \cdot \frac{1}{D^2}\sin x - \frac{2}{D^3} \cdot \sin x\right]$$

$$= e^x\left[x \cdot \frac{1}{-1}\sin x - \frac{2}{D}\left(\frac{1}{D^2} \cdot \sin x\right)\right]$$

$$= e^x\left[-x\sin x - \frac{2}{D} \cdot \frac{1}{-1}\sin x\right]$$

$$= e^x\left[-x\sin x + \frac{2}{D}\sin x\right] = e^x\left[-x\sin x + 2\int \sin x\,dx\right]$$

$$= e^x\left[-x\sin x + 2(-\cos x)\right] = -e^x\left[x\sin x + 2\cos x\right]$$

Hence, the complete solution is $y = (c_1 + c_2 x)\, e^x - e^x\,(x \sin x + 2\cos x)$.

Example 3.

Solve the differential equation

$$\frac{d^2y}{dx^2} - y = x^2 \cos x. \qquad \text{[K.U. 2015, 11; M.D.U. 2014, 04]}$$

Solution. The given differential equation is

$$\frac{d^2y}{dx^2} - y = x^2 \cos x.$$

The symbolic form of the equation is $(D^2 - 1)\,y = x^2 \cos x$

Auxiliary equation is $D^2 - 1 = 0 \;\Rightarrow\; D = \pm 1$

$\therefore$ C.F. $= c_1 e^x + c_2 e^{-x}$

Now, P.I. $= \dfrac{1}{D^2 - 1}(x^2 \cos x)$

$$= \text{Real part of } \frac{1}{D^2 - 1}(x^2 e^{ix}) \qquad [\because e^{ix} = \cos x + i \sin x]$$

$$= \text{Real part of } e^{ix}\,\frac{1}{(D+i)^2 - 1}\,x^2 \qquad \left[\because \frac{1}{f(D)}\,X e^{ax} = e^{ax} \cdot \frac{1}{f(D+a)}\,X\right]$$

$$= \text{R.P. of } e^{ix}\,\frac{1}{D^2 + 2iD - 2}\,x^2$$

$$= \text{R.P. of } e^{ix}\,\frac{1}{-2\left(1 - iD - \dfrac{D^2}{2}\right)}\,x^2$$

$$= \text{R.P. of } \frac{-e^{ix}}{2}\left[1-\left(iD+\frac{D^2}{2}\right)\right]^{-1} x^2$$

$$= \text{R.P. of } \frac{-e^{ix}}{2}\left[1+\left(iD+\frac{D^2}{2}\right)+\left(iD+\frac{D^2}{2}\right)^2+\ldots..\right]x^2$$

$$= \text{R.P. of } -\frac{1}{2}e^{ix}\left[1+iD+\frac{D^2}{2}+i^2D^2+\ldots..\right]x^2$$

$$= \text{R.P. of } -\frac{1}{2}e^{ix}\left[1+iD-\frac{D^2}{2}+\ldots..\right]x^2$$

$$= \text{R.P. of } -\frac{1}{2}e^{ix}\left[x^2+iDx^2-\frac{1}{2}D^2x^2+\ldots..\right]$$

$$= \text{R.P. of } -\frac{1}{2}(\cos x+i\sin x)(x^2+2ix-1)$$

$$= -\frac{1}{2}x^2\cos x+\frac{1}{2}\cos x+x\sin x$$

$$= -\frac{1}{2}(x^2-1)\cos x+x\sin x$$

Hence, the complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

i.e.,

$$y = c_1e^x+c_2e^{-x}-\frac{1}{2}(x^2-1)\cos x+x\sin x.$$

Example 4.

Solve the differential equation $\boldsymbol{\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=3x^2e^{2x}\sin 2x}$.

Solution. The given differential equation is

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=3x^2e^{2x}\sin 2x$$

The symbolic form of the equation is

$$(D^2-4D+4)y = 3x^2e^{2x}\sin 2x$$

Auxiliary equation is

$$D^2-4D+4 = 0$$

or

$$(D-2)^2 = 0 \Rightarrow D = 2, 2$$

$\therefore \quad \text{C.F.} = (c_1 + c_2 x)\, e^{2x}$

Now, $\quad \text{P.I.} = \dfrac{1}{(D-2)^2}(3x^2 e^{2x} \sin 2x)$

$$= 3e^{2x} \cdot \frac{1}{(D+2-2)^2}(x^2 \sin 2x) = 3e^{2x} \frac{1}{D^2}(x^2 \sin 2x)$$

$$= \text{I.P. of } 3e^{2x} \cdot \frac{1}{D^2} x^2 e^{i2x} \qquad [\because e^{i2x} = \cos 2x + i \sin 2x]$$

$$= \text{I.P. of } 3e^{2x} \cdot e^{i2x} \frac{1}{(D+2i)^2} x^2$$

$$= \text{I.P. of } 3e^{2x} \cdot e^{i2x} \frac{1}{\left[2i\left(1+\frac{D}{2i}\right)\right]^2}(x^2)$$

$$= \text{I.P. of } 3e^{2x} \cdot e^{i2x} \cdot \frac{-1}{4}\left(1 - \frac{iD}{2}\right)^{-2}(x^2) \qquad [\because i^2 = -1]$$

$$= \text{I.P. of } \frac{-3}{4} e^{2x} \cdot e^{i2x}\left[1 + 2 \cdot \frac{iD}{2} + 3\left(\frac{iD}{2}\right)^2 + \ldots\right] x^2$$

$$= \text{I.P. of } \frac{-3}{4} e^{2x} \cdot e^{i2x}\left[1 + iD - \frac{3}{4} D^2 + \ldots\right] x^2 \qquad [\textit{By binomial theorem}]$$

$$= \text{I.P. of } \frac{-3}{4} e^{2x} \cdot e^{i2x}\left[x^2 + 2ix - \frac{3}{2}\right]$$

$$= \text{I.P. of } \frac{-3}{4} e^{2x} (\cos 2x + i \sin 2x)\left(x^2 + 2ix - \frac{3}{2}\right)$$

$$= \frac{-3}{4} e^{2x}\left[2x \cos 2x + \left(x^2 - \frac{3}{2}\right)\sin 2x\right]$$

$$= \frac{-3}{8} e^{2x}\left[4x \cos 2x + (2x^2 - 3)\sin 2x\right]$$

Hence, the complete solution is given by

$$y = \text{C.F.} + \text{P.I.}$$

i.e.,

$$y = (c_1 + c_2 x)\, e^{2x} - \frac{3}{8} e^{2x}\, [4x \cos 2x + (2x^2 - 3) \sin 2x].$$

EXERCISE 4.6

Solve the following differential equations [Q. 1 – 8]:

Q1

1. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x \sin x.$ [M.D.U. 2008]

2. $\dfrac{d^2y}{dx^2} + 4y = x \cos x.$

3. $\dfrac{d^2y}{dx^2} - 4y = x \cos 2x.$

4. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = xe^x \sin x$ [K.U. 2011, 09, 04]

5. (i) $\dfrac{d^4y}{dx^4} - y = x \sin x$ [M.D.U. 2003] (ii) $\dfrac{d^4y}{dx^4} - y = x \sin x + e^x.$ [M.D.U. 2016]

$\left[\textbf{Hint.} \text{ (i) P.I.} = \dfrac{1}{D^4 - 1}(x \sin x) = \text{Imaginary part of } \dfrac{1}{D^4 - 1} xe^{ix} = \text{I.P. of } e^{ix} \dfrac{1}{(D+i)^4 - 1} x.\right]$

Q16

6. $\dfrac{d^2y}{dx^2} - y = x \sin x + (1 + x^2)\, e^x.$

7. $\dfrac{d^2y}{dx^2} - y = x \sin x + x^2 e^x.$

8. $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = x^2 \sin x.$ [M.D.U. 2015]

ANSWERS

1. $y = (c_1 + c_2 x)\, e^x + \dfrac{x}{2} \cos x + \dfrac{1}{2}(\cos x - \sin x)$

2. $y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{1}{3} x \cos x + \dfrac{2}{9} \sin x$

3. $y = c_1 e^{2x} + c_2 e^{-2x} - \dfrac{1}{8} x \cos 2x + \dfrac{1}{16} \sin 2x.$

4. $y = c_1 e^{-x} + c_2 e^{-2x} + e^x \left[\dfrac{x}{10}(\sin x - \cos x) + \dfrac{1}{10} \cos x - \dfrac{1}{25} \sin x\right]$

5. (i) $y = c_1 \cos x + c_2 \sin x + c_3 e^x + c_4 e^{-x} + \dfrac{1}{8} x^2 \cos x - \dfrac{3}{8} x \sin x$

(ii) $y = c_1 \cos x + c_2 \sin x + c_3 e^x + c_4 e^{-x} + \dfrac{1}{8} x^2 \cos x - \dfrac{3}{8} x \sin x + \dfrac{xe^x}{4}$

6. $y = c_1 e^x + c_2 e^{-x} - \dfrac{1}{2}(x \sin x + \cos x) + \dfrac{1}{12} xe^x (2x^2 - 3x + 9).$

7. $y = c_1 e^x + c_2 e^{-x} - \dfrac{1}{2}(x \sin x + \cos x) + \dfrac{1}{12} xe^x (2x^2 - 3x + 3).$

8. $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x - \dfrac{x^3}{12} \cos x - \left(\dfrac{x^4}{48} - \dfrac{3}{16} x^2\right) \sin x.$

5 HOMOGENEOUS LINEAR EQUATIONS

of Cauchy-Euler's Eq^ns.

5.1. HOMOGENEOUS LINEAR EQUATION

Definition. *An equation of the form :*

$$P_0 x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + P_n y = X \qquad \ldots(1)$$

where $P_0, P_1, P_2, \ldots, P_n$ are constants and X is a function of x, is called a homogeneous linear differential equation. of Cauchy-Euler Eqn.

5.2. METHOD OF SOLUTION

$x^n \frac{d^n y}{dx^n} = D(D-1)(D-2) \ldots (D-(n-1))y$

$x \frac{d}{dx} = D$

$x^2 \frac{d^2}{dx^2} = D(D-1)$

To reduce the homogeneous linear equation

$$\mathbf{P_0 x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + P_n y = X}$$

into linear equation with constant coefficients.

Also prove that

$$\mathbf{x^n \frac{d^n y}{dx^n} = D (D-1) (D-2) \ldots (D-n+1) y.}$$

Soln of Homog. linear diff eqn is obtained by converting it into linear diff eqn with constant coeffs.

Proof. The given homogeneous equation is

$$P_0 x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + P_{n-1} x \frac{dy}{dx} + P_n y = X \qquad \ldots(1)$$

$x = e^z$

$\log x = z$

Put $x = e^z$ so that $z = \log x$ and $\frac{dz}{dx} = \frac{1}{x}$ $\qquad \ldots(2)$

$\frac{dz}{dx} = \frac{1}{x}$

Now, $\quad \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x}$ [*Using* (2)]

$\Rightarrow \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$

$= \frac{1}{x} \frac{dy}{dz}$

$\Rightarrow x \frac{dy}{}$

or $$x\frac{dy}{dx} = \frac{dy}{dz} \quad \text{...(3)}$$

Also, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) \quad \text{[From (3)]}$$

$$= \frac{1}{x}\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx} - \frac{1}{x^2}\cdot\frac{dy}{dz}$$

$$= \frac{1}{x}\frac{d^2y}{dz^2}\cdot\frac{1}{x} - \frac{1}{x^2}\cdot\frac{dy}{dz} \quad \text{[By (2)]}$$

or $$\frac{d^2y}{dx^2} = \frac{1}{x^2}\cdot\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

or $$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} \quad \text{...(4)}$$

Similarly $$x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz} \quad \text{...(5)}$$

Put $\frac{d}{dz} = \text{D}$. Then $$x\frac{dy}{dx} = \text{D}y \quad \text{[From (3)]}$$

and $$x^2\frac{d^2y}{dx^2} = \text{D}(\text{D}-1)y$$

....

....

Thus, $$x^n\frac{d^ny}{dx^n} = \text{D}(\text{D}-1)(\text{D}-2)\,.....\,(\text{D}-n+1)y$$

Making these substitutions, the given equation (1) reduces to

$$[\text{P}_0\{\text{D}(\text{D}-1)\,....(\text{D}-n+1)\} + \text{P}_1\{\text{D}(\text{D}-1)\,...\,(\text{D}-n+2)\} + \,.....\, + \text{P}_{n-1}\text{D} + \text{P}_n]\,y = \text{Z} \quad \text{...(6)}$$

or $$f(\text{D})\,y = \text{Z}$$

where Z is the function of z into which X is changed.

This is now a linear differential equation with constant co-efficients and hence is solvable for y in terms of z by the methods discussed in previous chapter.

If the solution of this differential equation (6) is $y = \psi(z)$, then putting $z = \log x$, the required solution is $y = \psi(\log x)$.

SOLVED EXAMPLES

Example 1.

Solve the differential equation

$$x^2\frac{d^2y}{dx^2}-2y=x^2+\frac{1}{x}$$ [M.D.U. 2017; K.U. 2006]

Solution. The given equation is

$$x^2\frac{d^2y}{dx^2}-2y=x^2+\frac{1}{x} \quad \text{...(1)}$$

Put $x=e^z$ so that $z=\log x$ and $D=x\dfrac{d}{dx}=\dfrac{d}{dz}$. Then, given equation (1) becomes

$$[D(D-1)-2]\,y=e^{2z}+e^{-z}$$

or $$[D^2-D-2]\,y=e^{2z}+e^{-z}$$

Auxiliary equation is $D^2-D-2=0$

or $$(D-2)(D+1)=0 \quad \text{i.e.,} \quad D=2,-1$$

$\therefore$ $$\text{C.F.}=c_1e^{2z}+c_2e^{-z}$$

$$\text{P.I.}=\frac{1}{D^2-D-2}(e^{2z}+e^{-z})$$

$$=\frac{1}{D^2-D-2}e^{2z}+\frac{1}{D^2-D-2}e^{-z}$$

$$=z\cdot\frac{1}{2D-1}e^{2z}+z\cdot\frac{1}{2D-1}\cdot e^{-z} \quad \text{[Cases of failure]}$$

$$=z\cdot\frac{1}{2(2)-1}\cdot e^{2z}+z\frac{1}{2(-1)-1}\cdot e^{-z}$$

$$=\frac{1}{3}ze^{2z}-\frac{1}{3}ze^{-z}=\frac{1}{3}\cdot z(e^{2z}-e^{-z})$$

Hence the complete solution is

$$y=c_1e^{2z}+c_2e^{-z}+\frac{1}{3}(e^{2z}-e^{-z})z$$

or $$y=c_1x^2+\frac{c_2}{x}+\frac{1}{3}\left(x^2-\frac{1}{x}\right)\log x. \quad [\because x=e^z]$$

Example 2.

Solve the differential equation $x\dfrac{d^3y}{dx^3}+\dfrac{d^2y}{dx^2}=\dfrac{1}{x}$.

Solution. The given equation is

$$x\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}=\frac{1}{x}$$

Multiplying both sides by x^2, we have

$$x^3\frac{d^3y}{dx^3}+x^2\frac{d^2y}{dx^2}=x \qquad ...(1)$$

Put $x=e^z$ so that $z=\log x$

Let $x\dfrac{d}{dx}=\dfrac{d}{dz}=\text{D}$. Then the given equation (1), becomes

$$[\text{D}(\text{D}-1)(\text{D}-2)+\text{D}(\text{D}-1)]\,y=e^z$$

or
$$(\text{D}^3-2\text{D}^2+\text{D})\,y=e^z$$

$\therefore$ Auxiliary equation is $\text{D}^3-2\text{D}^2+\text{D}=0$

or
$$\text{D}(\text{D}-1)^2=0 \quad\Rightarrow\quad \text{D}=0,1,1$$

$\therefore$
$$\text{C.F.}=c_1e^{0z}+(c_2+c_3z)\,e^z=c_1+(c_2+c_3z)\,e^z$$

Now,
$$\text{P.I.}=\frac{1}{\text{D}^3-2\text{D}^2+\text{D}}\cdot e^z \qquad \textit{[Case of failure]}$$

$$=z\cdot\frac{1}{3\text{D}^2-4\text{D}+1}\cdot e^z \qquad \textit{[Case of failure]}$$

$$=z^2\cdot\frac{1}{6\text{D}-4}\cdot e^z$$

$$=z^2\cdot\frac{1}{6(1)-4}\cdot e^z=\frac{z^2}{2}\cdot e^z$$

Hence the complete solution is

$$y=c_1+(c_2+c_3z)\,e^z+\frac{z^2}{2}\cdot e^z$$

or
$$y=c_1+(c_2+c_3\log x)\,x+\frac{x}{2}(\log x)^2 \qquad [\because e^z=x]$$

Example 3.

Solve the differential equation $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^4$.

Solution. The given equation is $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^4$ ...(1)

Put $x = e^z$ so that $z = \log x$ and let $D = \dfrac{d}{dz} = x\dfrac{d}{dx}$

Then the given equation (1) becomes

$$[D(D-1) - 2D - 4]\,y = e^{4z}$$

or

$$(D^2 - 3D - 4)\,y = e^{4z}$$

$\therefore$ Auxiliary equation is $D^2 - 3D - 4 = 0$

$$(D-4)(D+1) = 0 \Rightarrow D = 4, -1$$

$\therefore$

$$\text{C.F.} = c_1 e^{4z} + c_2 e^{-z}$$

$$= c_1 x^4 + \frac{c_2}{x} \qquad [\because x = e^z]$$

Now,

$$\text{P.I.} = \frac{1}{D^2 - 3D - 4} \cdot e^{4z}$$

$$= \frac{1}{16 - 12 - 4} \cdot e^{4z} = \frac{1}{0} e^{4z} \qquad [\textit{Case of failure}]$$

$$= z \cdot \frac{1}{2D - 3} e^{4z}$$

[*Differentiating denominator w.r.t. D and multiplying by z*]

$$= z \cdot \frac{1}{2.4 - 3} \cdot e^{4z}$$

$$= \frac{z \cdot e^{4z}}{5} = \frac{1}{5} (\log x)\, x^4 \qquad [\because e^z = x]$$

Hence the complete or general solution is

$$y = c_1 x^4 + \frac{c_2}{x} + \frac{x^4 . \log x}{5} \qquad [\because \textit{C.S.} = \textit{C.F.} + \textit{P.I.}]$$

Example 4.

Solve the differential equation

$$(x^2 D^2 - 3xD + 4)\,y = x^m.$$

Solution. The given equation is $(x^2D^2 - 3xD + 4)\,y = x^m$

Put $x = e^z$ so that $z = \log x$

and let $D' = x\dfrac{d}{dx} = \dfrac{d}{dz}$. Then the given equation becomes

$$[D'(D'-1) - 3D' + 4]\, y = e^{mz}$$

or $$[(D'^2 - 4D' + 4]\, y = e^{mz} \Rightarrow (D'-2)^2 y = e^{mz}$$

∴ Auxiliary equation is $(D'-2)^2 = 0 \Rightarrow D' = 2, 2$

$$\text{C.F.} = (c_1 + c_2 z)\, e^{2z}$$
$$= (c_1 + c_2 \log x)\, x^2 \qquad [\because e^z = x]$$

Now, $$\text{P.I.} = \frac{1}{(D'-2)^2} \cdot e^{mz} = \frac{1}{(m-2)^2}\, e^{mz} = \frac{1}{(m-2)^2}\, x^m \qquad [\because e^z = x]$$

Hence the complete solution is

$$y = (c_1 + c_2 \log x)\, x^2 + \frac{1}{(m-2)^2} \cdot x^m$$

Example 5.

Solve the differential equation

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right).$$

Solution. The given equation is

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right) \qquad ...(1)$$

Put $x = e^z$ so that $z = \log x$ and $D = x\dfrac{d}{dx} = \dfrac{d}{dz}$.

Then given equation (1) becomes

$$[D(D-1)(D-2) + 2D(D-1) + 2]y = 10\,(e^z + e^{-z})$$

or $$(D^3 - D^2 + 2)\, y = 10\,(e^z + e^{-z})$$

Auxiliary equation is $D^3 - D^2 + 2 = 0$

or $$(D+1)(D^2 - 2D + 2) = 0$$

$$\therefore \quad D = -1, \frac{2 \pm \sqrt{4-8}}{2} \Rightarrow D = -1, 1 \pm i$$

$$\therefore \quad \text{C.F.} = c_1 e^{-z} + e^z (c_2 \cos z + c_3 \sin z)$$
$$= c_1 x^{-1} + x\,[c_2 \cos(\log x) + c_3 \sin(\log x)]$$

Now,

$$\text{P.I.} = \frac{1}{D^3 - D^2 + 2} \cdot 10(e^z + e^{-z})$$

$$= 10 \cdot \frac{1}{D^3 - D^2 + 2} e^z + 10 \frac{1}{D^3 - D^2 + 2} e^{-z}$$

[*Second term is a case of failure*]

$$= 10 \cdot \frac{1}{1 - 1 + 2} \cdot e^z + 10 \cdot z \frac{1}{3D^2 - 2D} \cdot e^{-z}$$

[*Multiplying second term by z and differentiating its den. w.r.t. D*]

$$= 5e^z + 10z \cdot \frac{1}{3.1 - 2(-1)} \cdot e^{-z} = 5\,e^z + 2\,z e^{-z}$$

$$= \left(5x + 2\log x \cdot \frac{1}{x}\right) \qquad [\because x = e^z]$$

Hence the complete solution is

$$y = c_1 x^{-1} + x\,[c_2 \cos(\log x) + c_3 \sin(\log x)] + 5x + \frac{2}{x}\log x.$$

Example 6.

Solve the differential equation

$$x^2 \frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + 13y = \log x \qquad \text{[K.U. 2007, 05, 2000]}$$

Solution. The given equation is

$$x^2 \frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + 13y = \log x$$

Put $x = e^z$ so that $z = \log x$ and $D = x\frac{d}{dx} = \frac{d}{dz}$. Then the given equation becomes

$$[D(D-1) + 8D + 13]\,y = z$$

$$(D^2 + 7D + 13)\,y = z$$

Auxiliary equation is $\quad D^2 + 7D + 13 = 0$

$$\therefore \quad D = \frac{-7 \pm \sqrt{49 - 52}}{2} = -\frac{7}{2} \pm \frac{i\sqrt{3}}{2}$$

$$\therefore \quad \text{C.F.} = e^{-\frac{7}{2}z}\left(c_1 \cos\frac{\sqrt{3}}{2}.z + c_2 \sin\frac{\sqrt{3}}{2}.z\right)$$

or $$\text{C.F.} = x^{-7/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right] \qquad [\because e^z = x]$$

Now, $$\text{P.I.} = \frac{1}{D^2 + 7D + 13} \cdot z = \frac{1}{13\left[1 + \frac{7}{13}D + \frac{D^2}{13}\right]} \cdot z$$

$$= \frac{1}{13} \cdot \left[1 + \frac{7}{13}D + \frac{D^2}{13}\right]^{-1} \cdot z$$

$$= \frac{1}{13}\left[1 - \left(\frac{7}{13}D + \frac{D^2}{13}\right) + \ldots\ldots\right] \cdot z \qquad \text{[By Binomial theorem]}$$

$$= \frac{1}{13}\left[1 - \frac{7}{13}D \ldots\ldots\right] z = \frac{1}{13}\left[z - \frac{7}{13}D(z)\right]$$

$$= \frac{1}{13}\left[z - \frac{7}{13}\frac{d}{dz}(z)\right] \qquad \left[\because D = \frac{d}{dz}\right]$$

$$= \frac{1}{13}\left(z - \frac{7}{13}\right) = \frac{1}{169}(13z - 7)$$

$$= \frac{1}{169}(13 \log x - 7) \qquad [\because e^z = x]$$

Hence the complete the solution is

$$y = x^{-7/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right] + \frac{1}{169}(13 \log x - 7).$$

Example 7.

Solve the differential equation

$$x^2 . \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2} \qquad \text{[M.D.U. 2013, 12; K.U. 2011]}$$

Solution. The given equation is

$$x^2 . \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$$

Put $x = e^z$ so that $z = \log x$ and let $D = x\frac{d}{dx} = \frac{d}{dz}$. Then the given equation reduces to

$$[D(D-1)+3D+1]\,y = \frac{1}{(1-e^z)^2}$$

or

$$(D^2+2D+1)\,y = \frac{1}{(1-e^z)^2}$$

Auxiliary equation is $D^2 + 2D + 1 = 0$

i.e.,

$$(D+1)^2 = 0 \quad \Rightarrow \quad D = -1, -1$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 z)\,e^{-z} = (c_1 + c_2 \log x)\,\frac{1}{x} \qquad [\because e^z = x]$$

Now,

$$\text{P.I.} = \frac{1}{D^2+2D+1}\cdot\frac{1}{(1-e^z)^2} = \frac{1}{(D+1)^2}\cdot\frac{1}{(1-e^z)^2}$$

$$= \frac{1}{(D+1)(D+1)}\cdot\frac{1}{(1-e^z)^2} = \frac{1}{D+1}\left[\frac{1}{D+1}\cdot\frac{1}{(1-e^z)^2}\right]$$

$$= \frac{1}{D+1}\cdot e^{-z}\int \frac{1}{(1-e^z)^2}\cdot e^z\,dz \qquad \left[\because \frac{1}{D-a}X = e^{ax}\int X\,e^{-ax}\,dx\right]$$

$$= \frac{1}{D+1}\cdot e^{-z}\int (1-t)^{-2}\,dt, \text{ where } e^z = t$$

$e^z dz = dt$

$$= \frac{1}{D+1}\cdot e^{-z}\left[-\int (1-t)^{-2}\,(-dt)\right]$$

$$= \frac{1}{D+1}\cdot e^{-z}\,\frac{-(1-t)^{-1}}{-1} \qquad \left[\because \int [f(x)]^n\cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= \frac{1}{D+1}\cdot e^{-z}\cdot\frac{1}{1-t} = \frac{1}{D+1}\cdot e^{-z}\,\frac{1}{1-e^z} \qquad [\because t = e^z]$$

$$= \frac{1}{D+1}\cdot\frac{e^{-z}}{1-e^z}$$

$$= e^{-z}\int \frac{e^{-z}}{1-e^z}\cdot e^z\,dz \qquad \left[\because \frac{1}{D-a}X = e^{ax}\int Xe^{-ax}\,dx\right]$$

$$= e^{-z}\int \frac{dz}{1-e^z} = e^{-z}\int \frac{e^{-z}}{e^{-z}-1}\,dz \qquad [\textit{Multiplying num. and den. by } e^{-z}]$$

$$= -e^{-z}\int \frac{-e^{-z}}{e^{-z}-1}\,dz$$

$$= -e^{-z}.\log(e^{-z}-1) \qquad \left[\because \int \frac{f'(z)}{f(z)}\,dz = \log[f(z)]\right]$$

$$= -\frac{1}{x}\log\left(\frac{1}{x}-1\right) \qquad [\because e^z = x]$$

$$= -\frac{1}{x}\log\left(\frac{1-x}{x}\right) = \frac{1}{x}\log\left(\frac{1-x}{x}\right)^{-1}$$

$$= \frac{1}{x}\log\frac{x}{1-x}$$

Hence the complete solution is $y = (c_1 + c_2 \log x)\frac{1}{x} + \frac{1}{x}\log\frac{x}{1-x}$.

Example 8.

Solve the differential equation :

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x \qquad \text{[M.D.U. 2016, 06]}$$

e^{2z}

Solution. The given equation is

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$$

Put $x = e^z$ so that $z = \log x$

Let $D = x\frac{d}{dx}$. Then the given equation becomes

$$[D(D-1) - D - 3]\,y = z\,e^{2z}$$

or

$$[D^2 - 2D - 3]\,y = z\,e^{2z}$$

Auxiliary equation is $D^2 - 2D - 3 = 0$

i.e.,

$$(D-3)(D+1) = 0 \;\Rightarrow\; D = 3, -1$$

$\therefore$ C.F. $= c_1 e^{3z} + c_2 e^{-z}$

Now, P.I. $= \dfrac{1}{D^2 - 2D - 3}(ze^{2z})$

$$= e^{2z}\frac{1}{(D+2)^2 - 2(D+2) - 3}\cdot z \qquad \left[\because \frac{1}{f(D)}(Xe^{ax}) = e^{ax}\frac{1}{f(D+a)}X\right]$$

$$= e^{2z}\frac{1}{D^2 + 2D - 3}z = e^{2z}\frac{1}{-3\left(1 - \frac{2D}{3} - \frac{D^2}{3}\right)}z$$

$$= \frac{e^{2z}}{-3}\left[1 - \left(\frac{2D}{3} + \frac{D^2}{3}\right)\right]^{-1} z$$

$$= -\frac{1}{3}e^{2z}\left[1+\frac{2D}{3}\right]z$$

$$= -\frac{1}{3}e^{2z}\left[z+\frac{2}{3}\right] \qquad \text{[By Binomial theorem]}$$

Hence the complete solution is

$$y = c_1e^{3z} + c_2e^{-z} - \frac{1}{3}e^{2z}\left(z+\frac{2}{3}\right)$$

or

$$y = c_1x^3 + c_2x^{-1} - \frac{1}{3}x^2\left(\log x + \frac{2}{3}\right). \qquad [\because e^z = x \Rightarrow z = \log x]$$

Example 9.

Solve the differential equation $(x^2D^2 - 3xD + 5)y = \sin(\log x)$. [M.D.U. 2015]

Solution. The given equation is

$$(x^2D^2 - 3xD + 5)y = \sin(\log x)$$

Put $x = e^z$ so that $z = \log x$

Let $D' \equiv xD \equiv x.\frac{d}{dx}$. Then the given equation becomes

$$[D'(D'-1) - 3D' + 5]y = \sin z$$

or

$$(D'^2 - 4D' + 5)y = \sin z$$

Auxiliary equation is $D'^2 - 4D' + 5 = 0$

$$\therefore \qquad D' = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

$$\therefore \qquad \text{C.F.} = e^{2z}(c_1\cos z + c_2\sin z)$$

Now,

$$\text{P.I.} = \frac{1}{D'^2 - 4D' + 5}(\sin z) = \frac{1}{-1-4D'+5}\sin z$$

$$= \frac{1}{4}.\frac{1}{1-D'} \times \frac{1+D'}{1+D'}\sin z = \frac{1}{4}.\frac{(1+D')}{1-D'^2}\sin z$$

$$= \frac{1}{4}.\frac{(1+D')}{[1-(-1)]}\sin z$$

$$= \frac{1}{8}[\sin z + D'(\sin z)] = \frac{1}{8}[\sin z + \cos z]$$

Hence the complete solution is y = C.F. + P.I.

i.e., $$y = e^{2z}(c_1 \cos z + c_2 \sin z) + \frac{1}{8}(\sin z + \cos z)$$

or $$y = x^2[c_1 \cos(\log x) + c_2 \sin(\log x)] + \frac{1}{8}[\sin(\log x) + \cos(\log x)]$$

$$[\because e^z = x \Rightarrow z = \log x]$$

Example 10.

Solve $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x^2 e^x$.

Solution. The given equation is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2 e^x$$

Put $x = e^z$ so that $z = \log x$

Let $D \equiv x\dfrac{d}{dx} = \dfrac{d}{dz}$. Then the given equation becomes

$$[D(D-1) + D - 1]\,y = e^{2z}e^{e^z} \quad i.e., \quad (D^2 - 1)\,y = e^{2z}e^{e^z}$$

Auxiliary equation is

$$D^2 - 1 = 0 \Rightarrow D = \pm 1$$

$\therefore$ $$\text{C.F.} = c_1 e^z + c_2 e^{-z}$$

Now, $$\text{P.I.} = \frac{1}{D^2-1}(e^{2z}e^{e^z}) = \frac{1}{(D+1)(D-1)}(e^{2z}e^{e^z})$$

$$= \frac{1}{2}\left[\frac{1}{D-1} - \frac{1}{D+1}\right](e^{2z}e^{e^z}) \qquad [\textit{Using partial fractions}]$$

$$= \frac{1}{2}\cdot\frac{1}{D-1}(e^{2z}e^{e^z}) - \frac{1}{2}\cdot\frac{1}{D+1}(e^{2z}e^{e^z})$$

$$= \frac{1}{2}e^z \int e^{2z}e^{e^z}e^{-z}\,dz - \frac{1}{2}e^{-z}\int e^{2z}e^{e^z}e^z\,dz$$

$$\left[\because \frac{1}{D-a}X = e^{ax}\int Xe^{-ax}\,dx\right]$$

$$= \frac{1}{2}e^z \int e^t\,dt - \frac{1}{2}e^{-z}\int t^2 e^t\,dt, \quad \text{where } e^z = t$$

$$= \frac{1}{2} e^z e^t - \frac{1}{2} e^{-z} (t^2 - 2t + 2)e^t$$

$$= \frac{1}{2} e^z e^{e^z} - \frac{1}{2} e^{-z} (e^{2z} - 2e^z + 2)e^{e^z}$$

$$= \frac{1}{2} e^{e^z} [e^z - e^z + 2 - 2e^{-z}] = e^{e^z} - e^{-z} e^{e^z}$$

Hence the complete solution is $y = \text{C.F.} + \text{P.I.}$

i.e.,
$$y = c_1 e^z + c_2 e^{-z} + e^{e^z} - e^{-z} e^{e^z}$$

or
$$y = c_1 x + c_2 x^{-1} + e^x (1 - x^{-1})$$
$[\because e^z = x]$

EXERCISE 5.1

Solve the following differential equations [Q. 1 – 7] :

1. (*i*) $(x^2 D^2 + 2xD - 2) y = 0$ (*ii*) $x^3 \frac{d^3y}{dx^3} + 6x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} - 4y = 0$ [*M.D.U. 2015, 07*]

(*iii*) $x^2 \frac{d^2y}{dx^2} + 9x \frac{dy}{dx} + 25y = 50$ (*iv*) $\frac{d^3y}{dx^3} - \frac{4}{x} \cdot \frac{d^2y}{dx^2} + \frac{5}{x^2} \frac{dy}{dx} - \frac{2y}{x^3} = 1.$ [*K.U. 2015*]

2. (*i*) $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = x^m$ (*ii*) $x^2 \frac{d^3y}{dx^3} - 4x \frac{d^2y}{dx^2} + 6 \frac{dy}{dx} = 4$

(*iii*) $x^4 \frac{d^3y}{dx^3} + 2x^3 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1$

3. (*i*) $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 4x^3$ (*ii*) $x^3 \frac{d^3y}{dx^3} - x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 2y = x^3 + 3x$

(*iii*) $x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 20y = (x + 1)^2.$

4. (*i*) $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + 2y = x \log x$ (*ii*) $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = 2 \log x$ [*K.U. 2001; M.D.U. 2001*]

(*iii*) $\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} = \frac{12 \log x}{x^2}.$

5. (*i*) $[x^4 D^4 + 6x^3 D^3 + 9x^2 D^2 + 3x D + 1] y = (1 + \log x)^2$

(*ii*) $[x^4 D^4 + 2x^3 D^3 + x^2 D^2 - x D + 1] y = x + \log x$ [*M.D.U. 2011*]

(*iii*) $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + y = \frac{\log x \sin (\log x) + 1}{x}.$

[K.U. 2014; M.D.U. 2000]

6. (*i*) $x^3 \dfrac{d^3y}{dx^3} + 3x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 8y = 65\cos(\log x)$ [M.D.U. 2008]

(*ii*) $(x^2 D^2 - x D + 4) y = \cos(\log x) + x \sin(\log x)$ [M.D.U. 2016]

7. (*i*) $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = e^x$

(*ii*) $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = x + \sin x.$

8. Solve $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = x$, given that $y = 0$ when $x = 1$ and $y = e^2$ when $x = e$.

HINTS

1. (*iv*) Multiplying by x^3, the given equation is $x^3 \dfrac{d^3y}{dx^3} - 4x^2\dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} - 2y = x^3$

2. (*iii*) Dividing by x, the given equation is $x^3 \dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = x^{-1}$

7. (*i*) $\text{P.I.} = \dfrac{1}{(D+1)(D+2)} e^{e^z} = \left[\dfrac{1}{D+1} - \dfrac{1}{D+2}\right] e^{e^z}$

Use $\dfrac{1}{D-\alpha} X = e^{\alpha x} \displaystyle\int X e^{-\alpha x}\, dx.$

ANSWERS

1. (*i*) $y = c_1 x^{-2} + c_2 x$ (*ii*) $y = c_1 x + [c_2 + c_3 (\log x)]\, x^{-2}.$

(*iii*) $y = x^{-4}[c_1 \cos(3\log x) + c_2 \sin(3\log x)] + 2$

(*iv*) $y = c_1 x^2 + x^{5/2}\left[c_2 x^{\sqrt{21}/2} + c_3 x^{-\sqrt{21}/2}\right] - \dfrac{x^3}{5}.$

2. (*i*) $y = c_1 x + c_2 x^{-1} + \dfrac{x^m}{m^2 - 1}$ (*ii*) $y = c_1 + c_2 x^3 + c_3 x^4 + \dfrac{2}{3}x$

(*iii*) $y = (c_1 + c_2 \log x)\, x + c_3 x^{-1} + \dfrac{1}{4x}\log x.$

3. (*i*) $y = c_1 x + c_2 x^2 + 2x^3$

(*ii*) $y = (c_1 + c_2 \log x) . x + c_3 x^2 + \dfrac{1}{4}x^3 - \dfrac{3}{2} . x(\log x)^2$ (*iii*) $y = c_1 x^{-5} + c_2 x^4 - \dfrac{1}{14}x^2 - \dfrac{1}{9}x - \dfrac{1}{20}$

4. (i) $y = x[c_1 \cos(\log x) + c_2 \sin(\log x)] + x \log x$
(ii) $y = (c_1 + c_2 \log x)x + 2 \log x + 4$
(iii) $y = c_1 + c_2 \log x + 2(\log x)^3$.

5. (i) $y = [c_1 + c_2(\log x)] \cos(\log x) + [c_3 + c_4(\log x)] \sin(\log x) + (\log x)^2 + 2 \log x - 3$.
(ii) $y = [c_1 + c_2 \log x + c_3(\log x)^2 + c_4(\log x)^3]x + \dfrac{x(\log x)^4}{24} + \log x + 4$
(iii) $y = x^2(c_1 x^{\sqrt{3}} + c_2 x^{-\sqrt{3}}) + \dfrac{x^{-1}\log x}{61}[6\cos(\log x) + 5\sin(\log x)]$

$$+ \frac{2x^{-1}}{3721}[191\cos(\log x) + 27\sin(\log x)] + \frac{1}{6}x^{-1}.$$

6. (i) $y = c_1 x^{-2} + x[c_2 \cos(\sqrt{3}\log x) + c_3 \sin(\sqrt{3}\log x)] + 8\cos(\log x) - \sin(\log x)$
(ii) $y = x[c_1 \cos(\sqrt{3}\log x) + c_2 \sin(\sqrt{3}\log x)] + \dfrac{1}{13}[3\cos(\log x) - 2\sin(\log x)] + \dfrac{1}{2}\cdot x \sin(\log x)$.

7. (i) $y = \dfrac{c_1}{x} + \dfrac{c_2}{x^2} + \dfrac{e^x}{x^2}$ (ii) $y = \dfrac{c_1}{x} + \dfrac{c_2}{x^2} + \dfrac{x}{6} - \dfrac{1}{x^2}\cdot \sin x$

8. $y = \left[-1 + \left(2 - \dfrac{1}{e}\right)\log x\right]x^2 + x.$

$x\frac{d}{dx} = D$
$x^2\frac{d^2}{dx^2} = D$
$(a+bx)\frac{d}{dx} = b\frac{dy}{dz}$

x^2
$P_0(a+bx)$

5.3. EQUATIONS REDUCIBLE TO HOMOGENEOUS LINEAR FORM

To explain the method of solution of linear differential equations reducible to homogeneous linear form.

[M.D.U. 2013]

Consider an equation of the type

$$P_0(a+bx)^n \frac{d^n y}{dx^n} + P_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + P_{n-1}(a+bx)\frac{dy}{dx} + P_n y = f(x) \qquad \ldots(1)$$

where $P_0, P_1, P_2, \ldots\ldots, P_{n-1}, P_n$ are constants.

Put $a + bx = e^z$ so that $z = \log(a+bx)$

$a+bx = e^z$
$z = \log(a+bx)$
$\frac{dz}{dx} = \frac{b}{a+bx}$

$$\therefore \qquad \frac{dz}{dx} = \frac{1}{a+bx}\cdot b = \frac{b}{a+bx} \qquad \ldots(2)$$

Now, $$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{b}{a+bx}\cdot\frac{dy}{dz} \qquad \ldots(3)\ [By\ (2)]$$

or $$(a+bx)\frac{dy}{dx} = b\frac{dy}{dz} \qquad \ldots(4)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{b}{a+bx}\cdot\frac{dy}{dz}\right) \qquad [\text{By }(3)]$$

$$\left(\frac{d}{dx}\left(\frac{dy}{dz}\right) = \frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}\right.$$

$$= \frac{-b^2}{(a+bx)^2}\cdot\frac{dy}{dz} + \frac{b}{a+bx}\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

$$= \frac{-b^2}{(a+bx)^2}\cdot\frac{dy}{dz} + \frac{b}{a+bx}\cdot\frac{d^2y}{dz^2}\cdot\frac{b}{a+bx} \qquad [\text{By }(2)]$$

$$= \frac{-b^2}{(a+bx)^2}\cdot\frac{dy}{dz} + \frac{b^2}{(a+bx)^2}\cdot\frac{d^2y}{dz^2}$$

$$\Rightarrow \qquad (a+bx)^2\frac{d^2y}{dx^2} = b^2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) \qquad \ldots(5)$$

Put $\frac{d}{dz} = D$.

Then from (4), $(a+bx)\frac{dy}{dx} = bD\,y$

and from (5), $(a+bx)^2\frac{d^2y}{dx^2} = b^2\,[D^2y - D\,y] = b^2 D\,(D-1)\,y$

Similarly, $(a+bx)^3\frac{d^3y}{dx^3} = b^3 D\,(D-1)\,(D-2)\,y$

....

and $$(a+bx)^n\frac{d^ny}{dx^n} = b^n D\,(D-1)\,(D-2)\;\ldots\ldots\;(D-n+1)\,y$$

Making the use of these substitutions in equation (1), we have

$$[P_0 b^n D(D-1)\ldots\ldots(D-n+1) + P_1 b^{n-1} D\,(D-1)\ldots\ldots(D-n+2) + \ldots\ldots$$

$$\ldots. + P_{n-1}\,b\,D + P_n]\,y = f\left(\frac{e^z - a}{b}\right).$$

This is a linear equation with constant coefficients and hence solvable for y in terms of z by methods discussed in previous chapter.

Let its solution be $y = F(z)$

Putting $z = \log(a+bx)$, the required solution is $y = F[\log(a+bx)]$.

SOLVED EXAMPLES

Example 1.

Solve the differential equation

$$(x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x. \qquad \text{[M.D.U. 2013; 01]}$$

Solution. The given equation is

$$(x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x$$

Put $x + a = e^z$ so that $z = \log(x+a)$

Let $D = \frac{d}{dz}$. Then the given equation becomes

$$[D(D-1) - 4D + 6]\, y = e^z - a$$

or

$$[D^2 - 5D + 6]\, y = e^z - a$$

$\therefore$ Auxiliary equation is

$$D^2 - 5D + 6 = 0$$

or

$$(D-3)(D-2) = 0 \Rightarrow D = 2, 3$$

$$\therefore \quad \text{C.F.} = c_1 e^{2z} + c_2 e^{3z} = c_1 (x+a)^2 + c_2 (x+a)^3 \qquad [\because e^z = x + a]$$

Now,

$$\text{P.I.} = \frac{1}{D^2 - 5D + 6}(e^z - a)$$

$$= \frac{1}{D^2 - 5D + 6} \cdot e^z - a \cdot \frac{1}{D^2 - 5D + 6} \cdot e^{0.z}$$

$$= \frac{1}{1^2 - 5.1 + 6} \cdot e^z - a \cdot \frac{1}{0 - 5.0 + 6}$$

$$= \frac{1}{2} \cdot e^z - \frac{1}{6} \cdot a = \frac{1}{2}(x+a) - \frac{a}{6} \qquad [\because e^z = x + a]$$

$$= \frac{1}{6}(3x + 2a)$$

Hence the complete solution is

$$y = c_1 (x+a)^2 + c_2 (x+a)^3 + \frac{1}{6}(3x + 2a).$$

Example 2. *Solve the differential equation*

$$(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} = (2x+3)(2x+4).$$ [M.D.U. 2000]

Solution. The given equation is

$$(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} = (2x+3)(2x+4)$$

Put $x+1 = e^z$ so that $z = \log(x+1)$ and $D = \frac{d}{dz}$. Then the given equation becomes

$$[D(D-1)+D]\,y = (2e^z+1)(2e^z+2)$$

or $$D^2 y = (2e^z+1)(2e^z+2)$$

Auxiliary equation is $D^2 = 0 \Rightarrow D = 0, 0$

$$\text{C.F.} = (c_1 + c_2 z)\,.\,e^{0.z} = c_1 + c_2 z$$

Now. $$\text{P.I.} = \frac{1}{D^2}(2e^z+1)(2e^z+2) = \frac{1}{D^2}(4e^{2z}+6e^z+2)$$

$$= \frac{1}{D}\int (4e^{2z}+6e^z+2)\,dz \qquad \left[\because \frac{1}{D} \text{ mean integration w.r.t.}\right]$$

$$= \frac{1}{D}\left[\frac{4e^{2z}}{2} + 6e^z + 2z\right] = \frac{1}{D}[2e^{2z}+6e^z+2z]$$

$$= \int (2e^{2z}+6e^z+2z)\,dz = 2\,.\,\frac{e^{2z}}{2} + 6e^z + 2\,.\,\frac{z^2}{2}$$

$$= e^{2z} + 6e^z + z^2$$

Hence the complete or general solution is

$$y = c_1 + c_2 z + e^{2z} + 6e^z + z^2 \qquad [\because C.S. = C.F. + P.I.]$$

i.e., $$y = c_1 + c_2 \log(x+1) + (x+1)^2 + 6(x+1) + [\log(x+1)]^2$$

i.e., $$y = c_1 + c_2.\log(x+1) + x^2 + 8x + [\log(x+1)]^2$$

leaving the constant term 7 which can be considered to be included in part c_1.

Example 3. ***Solve the differential equation***

$$(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1.$$ [K.U. 2014; M.D.U. 2014, 11]

Solution. The given equation is

$$(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$$

Put $3x+2 = e^z$ so that $z = \log(3x+2)$. Denoting $\frac{d}{dx}$ by D, the given equation becomes

$$[3^2\,.\,D(D-1) + 3.3D - 36]\,y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1 \qquad [\because \text{Here } b = 3]$$

$$9(D^2-4)y = \frac{1}{3}(e^{2z}-4e^z+4)+\frac{4}{3}\cdot(e^z-2)+1$$

or

$$9(D^2-4)y = \frac{1}{3}e^{2z}-\frac{1}{3}$$

$\therefore$ Auxiliary equation is $9(D^2-4)=0$

i.e. $$D^2-4=0 \Rightarrow D=\pm 2$$

$\therefore$ $$\text{C.F.} = c_1 e^{2z}+c_2 e^{-2z}$$

Now, $$\text{P.I.} = \frac{1}{9(D^2-4)}\left(\frac{1}{3}e^{2z}-\frac{1}{3}\right)$$

$$=\frac{1}{27}\cdot\frac{1}{D^2-4}\cdot e^{2z}-\frac{1}{27}\cdot\frac{1}{D^2-4}\cdot e^{0.z}$$

$$=\frac{1}{27}\cdot z\cdot\frac{1}{2D}e^{2z}-\frac{1}{27}\cdot\frac{1}{0-4}\cdot e^{0.z}$$ [$\because$ *First term is a case of failure*]

$$=\frac{z}{54}\cdot\frac{1}{D}e^{2z}+\frac{1}{108}$$

$$=\frac{z}{54}\int e^{2z}\,dz+\frac{1}{108}$$

$$=\frac{z}{54}\cdot\frac{e^{2z}}{2}+\frac{1}{108}=\frac{1}{108}(ze^{2z}+1)$$

Hence the complete solution is $y = \text{C.F.} + \text{P.I.}$

i.e., $$y = c_1e^{2z}+c_2e^{-2z}+\frac{1}{108}(ze^{2z}+1)$$

i.e., $$y = c_1(3x+2)^2+c_2(3x+2)^{-2}+\frac{1}{108}[(3x+2)^2\log(3x+2)+1].$$

Example 4. ***Solve the differential equation***

$$(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=2\sin\log(1+x)$$ [M.D.U. 2017]

Solution. The given equation is $(1+x)^2\dfrac{d^2y}{dx^2}+(1+x)\dfrac{dy}{dx}+y=2\sin\log(1+x)$ ...(1)

Put $1+x=e^z$ so that $\log(1+x)=z$

Also, $(1+x)\dfrac{d}{dx}=\dfrac{d}{dz}=D$ and $(1+x)^2\dfrac{d^2}{dx^2}=D(D-1)$. Then the given equation becomes

$$[D(D-1)+D+1]\,y = 2\sin z$$

or $$(D^2+1)\,y = 2\sin z$$

$\therefore$ Auxiliary equation is $D^2+1=0$

$$D^2 = -1 \Rightarrow D = \pm i$$

or $\quad$ C.F. $= e^{0.z}[c_1 \cos z + c_2 \sin z]$

$\therefore \quad = c_1 \cos [\log(1+x)] + c_2 \sin [\log(1+x)]$

Now, $\quad$ P.I $= \dfrac{1}{D^2+1}.2\sin z$

Putting $D^2 = -1$, $\quad$ P.I. $= \dfrac{1}{0}.2\sin z$ $\quad$ [*Case of failure*]

$\therefore \quad$ P.I. $= 2z.\dfrac{1}{2D}\sin z = z.\dfrac{1}{D}\sin z$

$= z(-\cos z) = -z\cos z = -\log(1+x)\cos[\log(1+x)]$

Hence the complete solution is $y =$ C.F. + P.I.

i.e., $\quad y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] - \log(1+x)\cos[\log(1+x)]$.

EXERCISE 5.2

Solve the following differential equations [Q. 1 – 6] :

1. $(5+2x)^2 \dfrac{d^2y}{dx^2} - 6(5+2x)\dfrac{dy}{dx} + 8y = 0.$ [*K.U. 2017, 16; M.D.U. 2007*]
2. $(2x-1)^3 \dfrac{d^3y}{dx^3} + (2x-1)\dfrac{dy}{dx} - 2y = 0.$ [*K.U. 2017; M.D.U. 2014*]
3. $(1+2x)^2 \dfrac{d^2y}{dx^2} - 6(1+2x)\dfrac{dy}{dx} + 16y = 8(1+2x)^2.$ [*K.U. 2016,15, 08; M.D.U. 2012, 08*]
4. $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4\cos\log(1+x).$ [*M.D.U. 2009*]
5. $(3x+2)^2 \dfrac{d^2y}{dx^2} + 5(3x+2)\dfrac{dy}{dx} - 3y = x^2 + x + 1.$
6. $16(x+1)^4 \dfrac{d^4y}{dx^4} + 96(x+1)^3 \dfrac{d^3y}{dx^3} + 104(x+1)^2 \dfrac{d^2y}{dx^2} + 8(x+1)\dfrac{dy}{dx} + y = x^2 + 4x + 3.$

ANSWERS

1. $y = (5+2x)^2 [c_1 (5+2x)^{\sqrt{2}} + c_2 (5+2x)^{-\sqrt{2}}]$
2. $y = (2x-1)[c_1 + c_2 (2x-1)^{\sqrt{3}/2} + c_3 (2x-1)^{-\sqrt{3}/2}]$
3. $y = (1+2x)^2 [c_1 + c_2 \log(1+2x) + \{\log(1+2x)\}^2].$
4. $y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] + 2[\log(1+x)]\sin[\log(1+x)]$
5. $y = c_1 (3x+2)^{1/3} + c_2 (3x+2)^{-1} + \dfrac{(3x+2)^2}{405} - \dfrac{(3x+2)}{108} - \dfrac{7}{27}.$
6. $y = [c_1 + c_2 \log(1+x)](x+1)^{1/2} + [c_3 + c_4 \log(1+x)](x+1)^{-1/2} + \dfrac{(x+1)^2}{225} + \dfrac{2(x+1)}{9}.$