1.6 VOLUME OF SOLIDS OF REVOLUTION

1.6.1 Solid of Revolution

If a plane area (R) is revolved about a fixed line (L) lying in its own plane, then the body so generated by the revolution of the plane area is called a solid of revolution.

1.6.2 Axis of Rotation

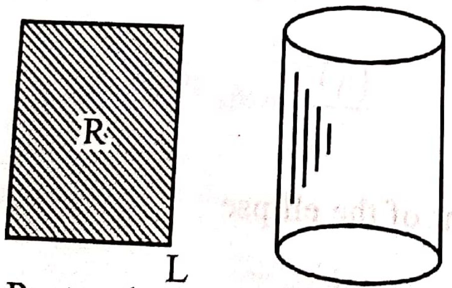
The fixed straight line (L) about which the area revolves is called the axis of revolution or axis of rotation.

Note

The straight line (L) does not intersect the plane area R but may touch the boundary of plane area (R).

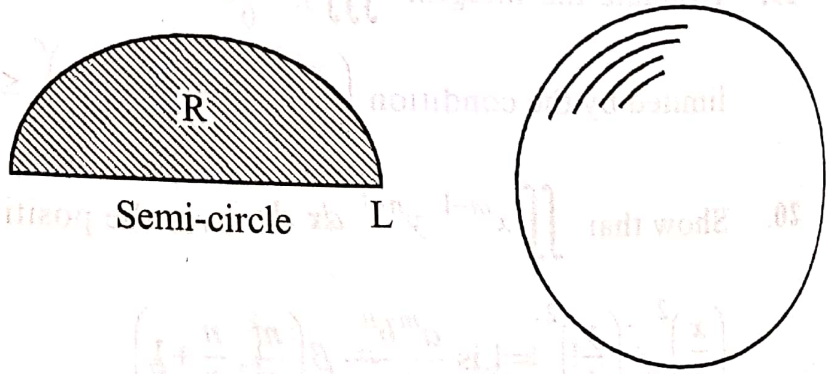
For example:

1. Right circular cylinder is a solid of revolution obtained by revolving a rectangle R about its edge (Fig. 1.1)



Rectangle Cylinder

Fig. 1.1.



Semi-circle Sphere

Fig. 1.2

2. A right angle triangle when revolved about one of its sides generated a right circular cone.

3. Sphere is a solid of revolution generated by revolving the semi circular region R about its diameter L (Fig. 1.2)

1.7 VOLUME OF SOLIDS OF REVOLUTION FOR CARTESIAN CURVES OR EQUATIONS

1.7.1 Revolution about x-axis

The volume of the solid generated by revolution about the x-axis, of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is

$$\int_a^b \pi y^2 dx$$

Proof: Let AB be the arc of the curve $y = f(x)$ included between ordinates $x = a$ and $x = b$.

Let P (x, y) and Q$(x+\delta x, y+\delta y)$ be any two points on the curve $y = f(x)$. (Fig. 4.3). Draw the $\perp$ PL and QM. Also draw PR and QS $\perp$'s to these ordinates.

Let V denote the volume of the solid generated by revolution of the area ACLP about the x-axis and $V+\delta V$ is the volume of solid generated by the revolution of the Area ACMQ.

So that volume of the solid generated by the revolution of the strip PLMQ about the x-axis is δV.

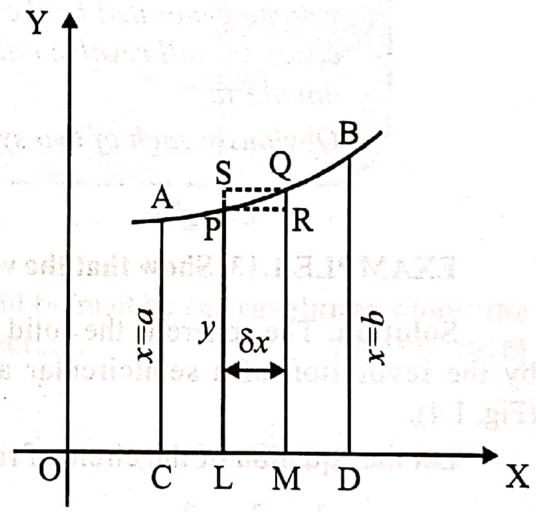


Fig. 1.3

Now, PL $= y$, QM $= y + \delta y$ and LM $= (x + \delta x) - x = \delta x$.

Then, the volume of the solid generated by revolving the area PLMR $= \pi y^2 \delta x$, and the volume of the solid generated by revolving rhe area SLMQ $= \pi (y + \delta y)^2 \delta x$.

$\therefore$ δV lies between $\pi y^2 \delta x$ and $\pi (y+\delta y)^2 \delta x$.

or $\dfrac{\delta V}{\delta x}$ lies between πy^2 and $\pi (y+\delta y)^2$.

In the limiting position as Q $\to$ P, $\delta x \to 0$ and therefore $\delta y \to 0$.

$\therefore$ $\lim\limits_{\delta x \to 0} \dfrac{\delta V}{\delta x}$ lies between πy^2 and $\lim\limits_{\delta y \to 0} \pi (y+\delta y)^2$.

$$\therefore \quad \frac{dV}{dx} = \pi y^2 \quad \Rightarrow \quad dV = \pi y^2 dx$$

$$\therefore \quad \int_a^b \pi y^2 dx = \int_a^b dV = |V|_a^b$$

$= ($Volume V when $x = b) - ($Volume V when $x = a)$

$=$ Volume generated by the area ACDB $- 0$

Hence, the volume of the solid generated by the area ACDB about the x-axis is $\int_a^b \pi y^2 dx$.

1.7.2 Revolution about y-axis

The volume of the solid generated by revolution about the y-axis of the area bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$, $y = b$ is

$$\int_a^b \pi x^2 dy$$

Note

If the given curve is symmetrical about x-axis and we have to find the volume generated by the revolution of the area about x-axis, then in such case we shall revolve only one of the two symmetrical areas and shall not double it.

Obviously each of two symmetrical parts will generate the same volume.

EXAMPLE 1.13. Show that the volume of a sphere of radius a is $\frac{4}{3}\pi a^3$.

Solution. The sphere is the solid of revolution generated by the revolution of a semicircular area about its diameter (Fig. 1.4).

Let the equation of the circle of radius a be

$$x^2 + y^2 = a^2 \qquad \text{....(i)}$$

Now x varies from $-a$ to a for the semicircle about the x-axis.

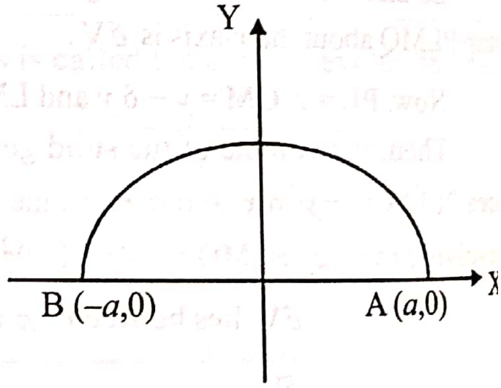


Fig. 1.4

$\therefore$ Required volume of the sphere is

$$= \int_{-a}^{a} \pi y^2 dx = \pi \int_{-a}^{a} (a^2 - x^2)\, dx \qquad [\text{from (i)}]$$

$$= \pi\left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \pi\left[a^3 - \frac{a^3}{3} - \left(-a^3 + \frac{a^3}{3}\right)\right] = \pi\left[2a^3 - 2\frac{a^3}{3}\right] = \frac{4}{3}\pi a^3$$

Note

If the curve is symmetrical about y-axis and the curve be made to revolve about x-axis, then volume generated = 2 × volume generated by the portion lying in the right of y-axis.

EXAMPLE 1.14. Find the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the major axis. **[K.U.K. 2006]**

Solution. The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \Rightarrow \quad y^2=\frac{b^2}{a^2}(a^2-x^2)$$

Since, the ellipse is symmetrical about y-axis (Fig. 1.5).

Required volume by the revolution about x-axis

= 2 × volume generated by arc in 1st quadrant about the x-axis.

$$=2\int_0^a \pi y^2\,dx=2\pi\int_0^a \frac{b^2}{a^2}(a^2-x^2)\,dx$$

$$=2\pi\frac{b^2}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a=2\pi\frac{b^2}{a^2}\left[a^3-\frac{a^3}{3}-0\right]$$

$$=2\pi\frac{b^2}{a^2}\cdot\frac{2a^3}{3}=\frac{4}{3}\pi ab^2$$

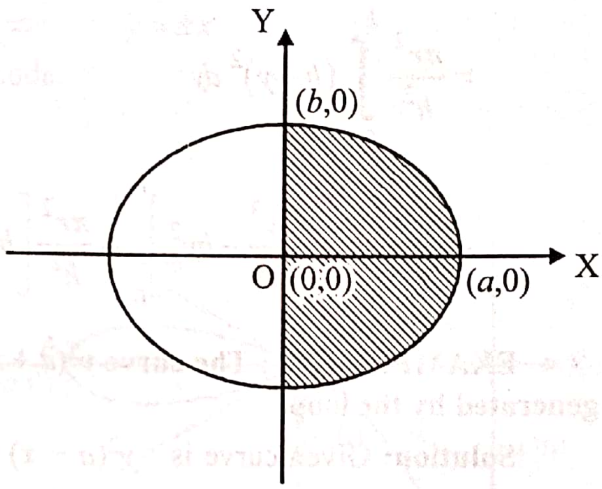


Fig. 1.5

EXAMPLE 1.15. Find the volume of the reel-shaped solid formed by the revolution about the y-axis, of the part of the parabola $y^2 = 4ax$ cut off by the latus-rectum. **[M.D.U. 2010]**

Solution. Given equation of the parabola is

$$y^2=4ax \quad \Rightarrow \quad x=\frac{y^2}{4a}$$

Let A be the vertex and L one extremity of the latus-rectum.

For the arc AL, y varies from 0 to $2a$ (Fig. 1.6).

Curve is symmetrical about x-axis.

∴ Required volume is

= 2 (volume generated by the revolution about the y-axis of the area ALC)

$$=2\int_0^{2a}\pi x^2\,dy=2\pi\int_0^{2a}\frac{y^4}{16a^2}\,dy$$

$$=\frac{\pi}{8a^2}\left[\frac{y^5}{5}\right]_0^{2a}=\frac{\pi}{40a^2}(32a^5-0)=\frac{4\pi a^3}{5}$$

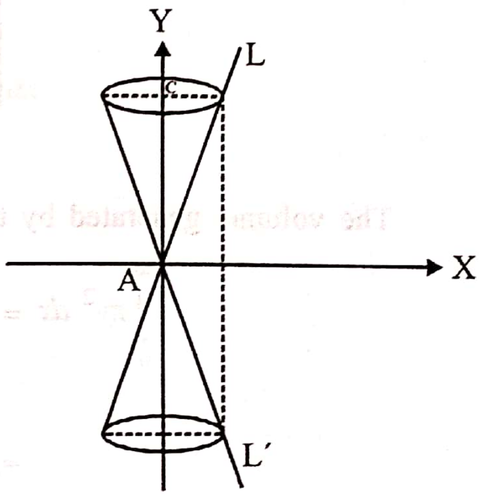


Fig. 1.6

EXAMPLE 1.16. Find the volume of a cone of height h and base radius r.

Solution. Let OX and OY be the co-ordinate axis. Let P be any point (x, y) on the cone. Let us draw a circle through P, radius x and thickness δy (Fig. 1.7).

Volume of the plate $(\delta V) = \pi x^2 \, \delta y$.

$$V = \pi \int_0^h x^2 dy$$

$$= \pi \int_0^h r^2 \left(\frac{h-y}{h}\right)^2 dy \qquad \left(\because \frac{x}{r} = \frac{h-y}{h}\right)$$

$$= \frac{\pi r^2}{h^2} \int_0^h (h-y)^2 dy$$

$$= \frac{\pi r^2}{h^2}\left[h^2 y + \frac{y^3}{3} - hy^2\right]_0^h = \frac{\pi r^2}{h^2}\left[h^3 - h^3 + \frac{h^3}{3}\right] = \frac{\pi r^2 h}{3}$$

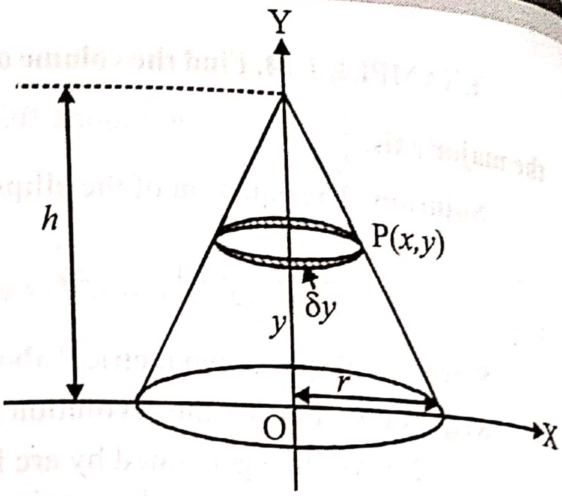


Fig. 1.7

EXAMPLE 1.17 : The curve $y^2(a + x) = x^2 (3a - x)$ revolves about the x-axis. Find the volume generated by the loop. **[KU 2015]**

Solution: Given curve is $y^2(a + x) = x^2 (3a - x)$

The given curve is symmetrical about the x-axis and cuts the x-axis at $(3a, 0)$ and passes through the origin.

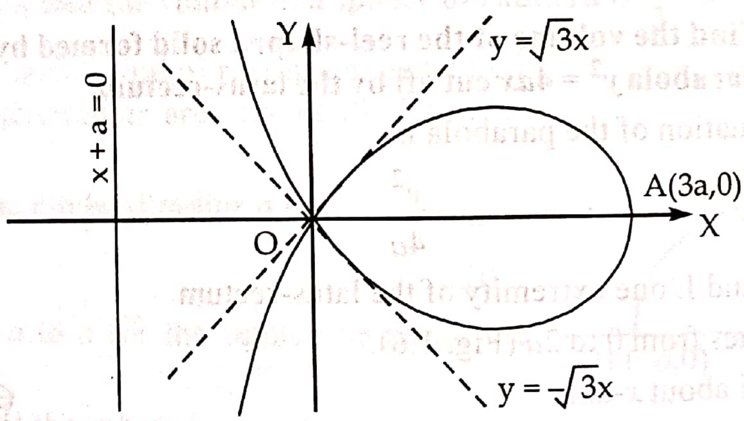


Fig. 1.8

The volume, generated by the loop is given by

$$\int_0^{3a} \pi y^2 \, dx = \pi \int_0^{3a} \frac{x^2(3a-x)}{x+a} dx = \pi \int_0^{3a} \left[-x^2 + 4ax - 4a^2 + \frac{4a^3}{x+a}\right] dx$$

$$= \pi\left[-\frac{x^3}{3} + 4a\frac{x^2}{2} - 4a^2 x + 4a^3 \log(x+a)\right]_0^{3a}$$

$$= \pi\left[-9a^3 + 18a^3 - 12a^3 + 4a^3(\log 4a - \log a)\right]$$

$$= \pi\left[-3a^3 + 4a^3 \log 4\right]$$

$$= \pi a^3 (8 \log 2 - 3)$$

EXAMPLE 1.18. Find the volume of the solid formed by the revolution about the x-axis, of the loop of the curve $y^2(a-x)=x^2(a+x)$.

Solution. The equation of the given curve is

$$y^2(a-x)=x^2(a+x) \qquad ....(i)$$

Let us trace the curve of the equation (i)

1. The curve of the given equation is symmetrical about x-axis because only even powers of y occurs.
2. Equation of tangent: Put lowest degree term equal to zero, we have

$$\cancel{a}y^2=\cancel{a}x^2 \quad\Rightarrow\quad y^2=x^2 \quad\Rightarrow\quad y=\pm x$$

   Tangents are real and distinct so double point is a node.
3. The curve passes through origin.
4. **Asymptote**
   (a) **Asymptote parallel to x-axis:** No asymptote parallel to x-axis because the co-efficient of highest degree term is x is constant.
   (b) **Asymptote parallel to y-axis:** Put co-efficient of highest degree term in y equal to zero, we have

$$a-x=0 \quad\Rightarrow\quad x=a$$

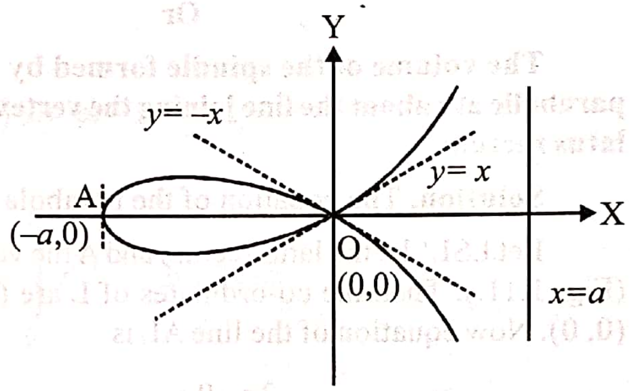


Fig. 1.9

Thus, the rough shape of the curve is shown in Fig. 1.9.

For upper half of the loop x varies from $-a$ to a.

$\therefore$ Required volume is

$$=\int_{-a}^{0}\pi y^2\,dx=\pi\int_{-a}^{0}\frac{x^2(a+x)}{a-x}\,dx \qquad \text{[from (i)]}$$

$$=\pi\int_{-a}^{0}\frac{ax^2+x^3}{a-x}\,dx=\pi\int_{-a}^{0}\left[-x^2-2ax-2a^2+\frac{2a^3}{a-x}\right]dx$$

$$=\pi\left[\frac{-x^3}{3}-ax^2-2a^2x-2a^3\log(a-x)\right]_{-a}^{0}$$

$$=\pi\left[-2a^3\log a-\left(\frac{a^3}{3}-a^3+2a^3-2a^3\log 2a\right)\right]$$

$$=\pi\left[-\frac{4}{3}a^3+2a^3(\log 2a-\log a)\right]=2\pi a^3\left(\log 2-\frac{2}{3}\right)$$

1.7.3 Revolution About Any Axis

The volume of the solid generated by the revolution about any axis LM of the area bounded by the curve AB, the axis LM and the perpendiculars AL, BM on the axis is

$$\int_{OL}^{OM} \pi (PN)^2 \, d\,(ON)$$

where O is a fixed point in LM and PN is perpendicular from any point P of the curve AB on LM (Fig. 1.10).

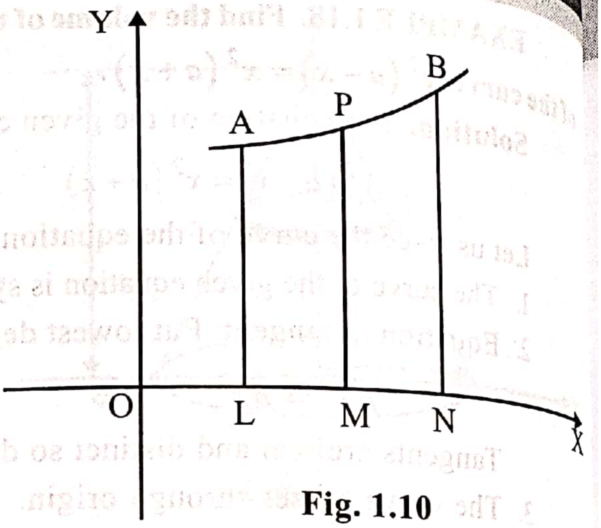


Fig. 1.10

EXAMPLE 1.19. The area cut off from the parabola $y^2 = 4ax$ by the chord joining the vertex to an end of the latus rectum is rotated through four right angles about the chord. Find the volume of the solid generated.

Or

The volume of the spindle formed by the revolution of a parabolic arc about the line joining the vertex to one extremity of latus rectum.

Solution. The equation of the parabola is $y^2 = 4ax$.

Let LSL′ be the latus rectum and A the vertex of the parabola (Fig. 1.11.). Then the co-ordinates of L are $(a, 2a)$ and of A are (0, 0). Now equation of the line AL is

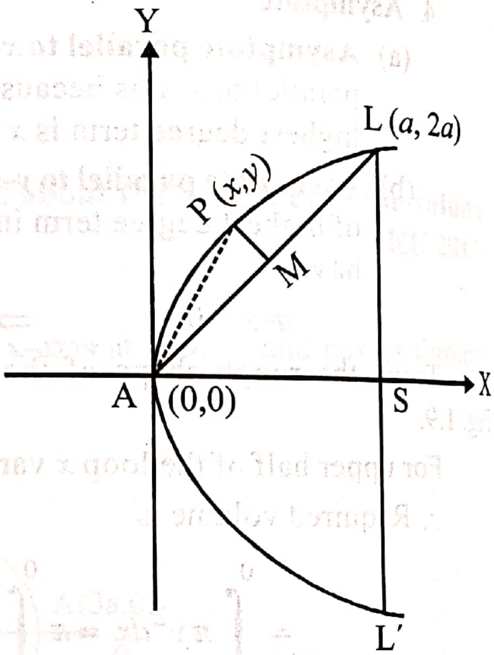


Fig. 1.11

$$y - 0 = \frac{2a-0}{a-0}(x-0)$$

or $\qquad 2x - y = 0$

Let P (x, y) be any point on the arc AL of the parabola.

Draw PM $\perp$ AL and join AP.

$\Rightarrow$ PM = $\perp$ distance of P (x, y) from the line $2x - y = 0$

$$= \frac{2x-y}{\sqrt{2^2+(-1)^2}} = \frac{2x-y}{\sqrt{4+1}} = \frac{2x-y}{\sqrt{5}} = \frac{2x-2\sqrt{ax}}{\sqrt{5}} \qquad \left[\because y^2 = 4ax\right]$$

$$\therefore \qquad PM^2 = \frac{\left(2x-2\sqrt{ax}\right)^2}{5} = \frac{4x\left(x+a-2\sqrt{ax}\right)}{5}$$

Now, $$AM^2 = AP^2 - PM^2 = x^2 + y^2 - \frac{4x\left(x+a-2\sqrt{ax}\right)}{5}$$

$$= x^2 + 4ax - \frac{4}{5}x\left(x+a-2\sqrt{ax}\right) \qquad \Big| \because y^2 = 4ax$$

$$= \frac{1}{5}\left[x^2 + 16ax + 8\sqrt{a}\,x^{3/2}\right] = \frac{1}{5}\left[x+4\sqrt{ax}\right]^2$$

$$\therefore \qquad AM = \frac{1}{\sqrt{5}}\left(x+4\sqrt{a}\sqrt{x}\right)$$

$$\therefore \qquad d(AM) = \frac{1}{\sqrt{5}}\left(1+4\sqrt{a}\cdot\frac{1}{2\sqrt{x}}\right)dx = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{x}+2\sqrt{a}}{\sqrt{x}}\right)dx$$

$\therefore$ Required volume $= \int \pi \left(PM^2 \right) \cdot d\left(AM \right)$

$$= \pi \int_0^a \frac{4x\left(\sqrt{x}-\sqrt{a}\right)^2}{5} \cdot \frac{1}{\sqrt{5}} \frac{\left(\sqrt{x}+2\sqrt{a}\right)}{\sqrt{x}} dx$$

$$= \frac{4\pi}{5\sqrt{5}} \int_0^a \sqrt{x}\left(\sqrt{x}-\sqrt{a}\right)^2 \left(\sqrt{x}+2\sqrt{a}\right) dx$$

Put $x = at^2$ $\quad \therefore \quad dx = 2at\ dt$

When $x = 0, t = 0$; when $x = a, t = 1$

$$= \frac{4\pi}{5\sqrt{5}} \int_0^1 \sqrt{a}\,t\left(\sqrt{a}\,t-\sqrt{a}\right)^2 \cdot \left(\sqrt{a}\,t+2\sqrt{a}\right) 2at\ dt$$

$$= \frac{8\pi a^3}{5\sqrt{5}} \int_0^1 t^2 (t+2)\left(t^2-2t+1\right) dt = \frac{8\pi a^3}{5\sqrt{5}} \int_0^1 t^2\left(t^3-3t+2\right) dt$$

$$= \frac{8\pi a^3}{5\sqrt{5}} \left[\frac{t^6}{6} - 3\frac{t^4}{4} + 2\frac{t^3}{3}\right]_0^1 = \frac{8\pi a^3}{5\sqrt{5}} \left[\frac{1}{6} - \frac{3}{4} + \frac{2}{3}\right] = \frac{8\pi a^3}{5\sqrt{5}} \left[\frac{2-9+8}{12}\right] = \frac{2\pi a^3}{15\sqrt{5}}$$

1.7.4 Volume of Solids of Resolution for Parametric Curves or Equations

1.7.4.1 Revolution about X-axis

The volume of the solid generated by the revolution about the x-axis, of the area bounded by the curves $x = f_1(t), y = f_2(t)$, the x-axis and the ordinates, where $t = a$, $t = b$ is

$$\int_a^b \pi y^2 \frac{dx}{dt} \cdot dt$$

1.7.4.2 Revolution about Y-axis

The volume of the solid generated by the revolution about the y-axis, of the area bounded by the curve $x = f_1(t), y = f_2(t)$, the y-axis and the abscissae at the points where $t = a$, $t = b$ is

$$\int_a^b \pi x^2 \frac{dy}{dt} \cdot dt$$

EXAMPLE 1.20. Find the volume of the solid generated by the revolution of the tractrix $x = a\cos t + \frac{1}{2} a \log\tan^2\left(\frac{t}{2}\right)$, $y = a\sin t$ about its asymptote.

Solution. The given curve is

$$x = a\cos t + \frac{1}{2}a\log\tan^2\left(\frac{t}{2}\right),\ y = a\sin t \quad \text{....(i)}$$

$$\therefore \frac{dx}{dt} = -a\sin t + \frac{1}{2}a\frac{1}{\tan^2\left(\frac{t}{2}\right)}\cdot 2\tan\left(\frac{t}{2}\right)\sec^2\left(\frac{t}{2}\right)\cdot\frac{1}{2}$$

$$= -a\sin t + \frac{a}{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)} = -a\sin t + \frac{a}{\sin t}$$

$$= \frac{a\left(1-\sin^2 t\right)}{\sin t} = a\frac{\cos^2 t}{\sin t} \quad \text{....(ii)}$$

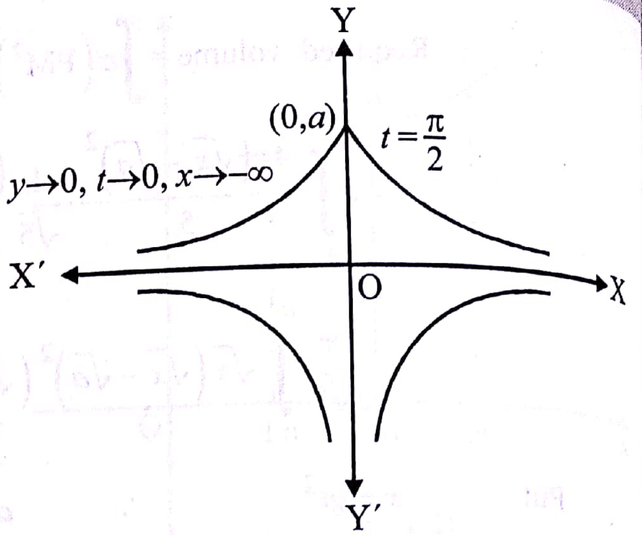


Fig. 1.12

Now, the given curve is symmetrical about both the axes and the asymptote in the line $y=0$ i.e. x-axis (Fig. 1.12)

For the portion of the curve lying in the second quadrant y varies from a to 0, t varies from $\frac{\pi}{2}$ to 0 and x varies from 0 to $-\infty$.

$\therefore$ Required volume

$$= 2\int_{-\infty}^{0}\pi y^2 dx = 2\int_{0}^{\pi/2}\pi y^2\frac{dx}{dt}\cdot dt = 2\pi\int_{0}^{\pi/2}a^2\sin^2 t\frac{a\cos^2 t}{\sin t}\cdot dt \quad \text{[From (i) and (ii)]}$$

$$= 2\pi a^3\int_{0}^{\pi/2}\cos^2 t\sin t\, dt = 2\pi a^3\cdot\frac{1}{3\cdot 1} = \frac{2}{3}\pi a^3 \quad \textbf{Ans.}$$

EXAMPLE 1.21. Prove the volume of the reel formed by the revolution of the cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$ about the tangent at the vertex is π^2a^3.

Solution. The given equation of the cycloid is

$$x = a(\theta+\sin\theta),\ y = a(1-\cos\theta) \quad \text{....(i)}$$

The cycloid is symmetrical about the y-axis and the tangent at the vertex is x-axis (Fig. 1.13)

For half of the curve, θ varies from 0 to π

$\therefore$ Required volume

$$= 2\int_{0}^{\pi}\pi y^2\frac{dx}{d\theta}\cdot d\theta$$

$$= 2\pi\int_{0}^{\pi}a^2(1-\cos\theta)^2\, a(1+\cos\theta)\,d\theta$$

$$= 2\pi a^3\int_{0}^{\pi}\left(2\sin^2\frac{\theta}{2}\right)^2\cdot\left(2\cos^2\frac{\theta}{2}\right)d\theta$$

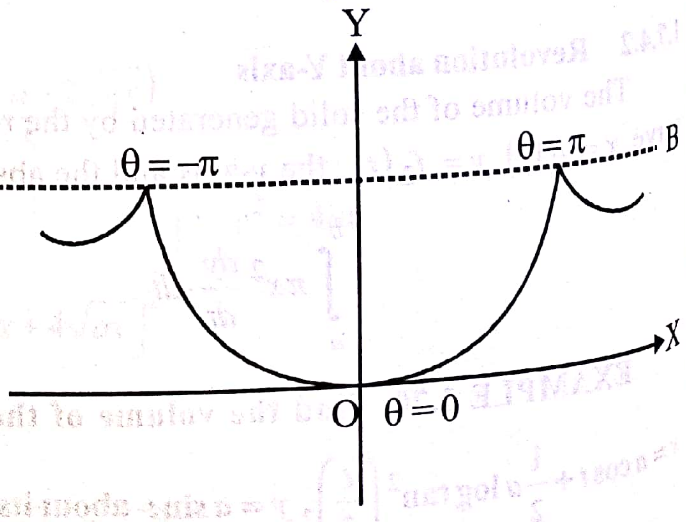


Fig. 1.13

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$$= 2\pi a^3 \int_0^{\pi/2} \left(2\sin^2 t\right)^2 \cdot \left(2\cos^2 t\right) 2dt \quad \left[\begin{array}{l} \text{Put } \frac{\theta}{2} = t,\ d\theta = 2dt \\ \text{Limit } \theta = 0, t = 0, \\ \theta = \pi, t = \frac{\pi}{2} \end{array}\right]$$

$$= 32\pi a^3 \int_0^{\pi/2} \sin^4 t \cos^2 t\, dt = 32\pi a^3 \frac{3\cdot1\cdot1}{6\cdot4\cdot2}\cdot\frac{\pi}{2} = \pi^2 a^3$$

EXAMPLE 1.22. Calculate the volume of the solid of revolution generated by revolving the hypo cycloid $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$, about x-axis and also deduce the results when Astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is revolved about x-axis.

Solution. The equation of the curve is $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$

The parametric equation of the hypo cycloid are

$$x = a\cos^3 t, \quad y = b\sin^3 t \qquad \text{....(i)}$$

The curve is symmetrical bout both the axis and for the portion of the curve in the first quadrant, t varies from $\frac{\pi}{2}$ to 0 (Fig. 1.14)

Required vlume is

$= 2\times$ volume generated by the arc in the Ist quadrant

$$= 2\int_0^a \pi y^2 dx = \int_0^a \pi y^2 \frac{dx}{dt}\cdot dt$$

$$= 2\int_{\pi/2}^{0} \pi\left(b^2 \sin^6 t\right)\left(-3a\cos^2 t \sin t\right) dt$$

$$= 6\pi ab^2 \int_0^{\pi/2} \sin^7 t \cdot \cos^2 t\, dt = 6\pi ab^2 \frac{6\cdot4\cdot2\cdot1}{9\cdot7\cdot5\cdot3} = \frac{32}{105}\pi ab^2$$

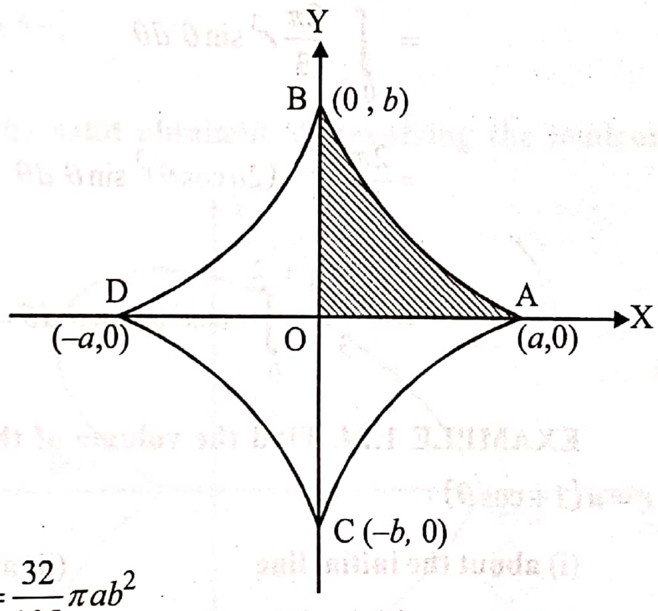

Fig. 1.14

Now, Astroid is a special case of hypocycloid for $b = a$

So, volume generated by revolving astroid about x-axis is

$$V = \frac{32}{105}\pi a \cdot a^2 = \frac{32}{105}\pi a^3$$

1.7.5 The Volume of Solids of Revolution for Polar Curves

The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$ (Fig. 1.15)

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(1) Revolution about x-axis i.e. initial line OX $(\theta = 0)$

$$\int_{\alpha}^{\beta} \frac{2\pi}{3} r^3 \sin\theta \, d\theta$$

(2) Revolution about y-axis i.e. OY $\left(\theta = \frac{\pi}{2}\right)$

$$\int_{\alpha}^{\beta} \frac{2\pi}{3} r^3 \cos\theta \, d\theta$$

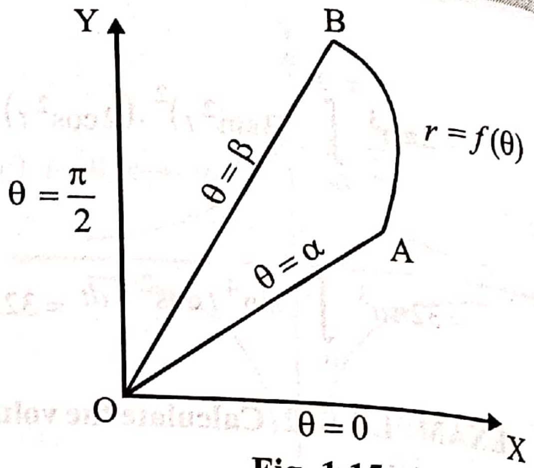

Fig. 1.15

EXAMPLE 1.23. Find the volume generated by the revolution of the curve $r = 2a\cos\theta$ about the initial line.

Solution. The equation of the curve is

$$r = 2a\cos\theta \qquad \text{....(i)}$$

The curve is symmetrical about the initial line and for the upper half of the curve θ varies from 0 to $\frac{\pi}{2}$ (Fig. 1.16)

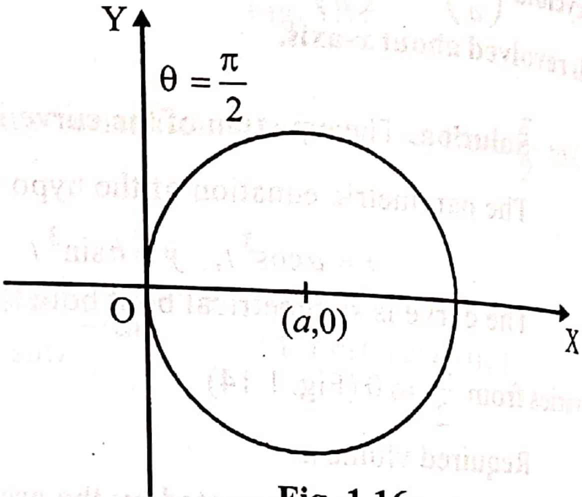

Fig. 1.16

$\therefore$ Required volume is

$$= \int_0^{\pi/2} \frac{2\pi}{3} r^3 \sin\theta \, d\theta$$

$$= \frac{2\pi}{3} \int_0^{\pi/2} (2a\cos\theta)^3 \sin\theta \, d\theta$$

$$= \frac{16\pi a^3}{3} \int_0^{\pi/2} \cos^3\theta \cdot \sin\theta \, d\theta = \frac{16\pi a^3}{3} \cdot \frac{2}{4} \cdot \frac{1}{2} = \frac{4\pi a^3}{3}$$

EXAMPLE 1.24. Find the volume of the solid generated by the revolution of the cardioid $r = a(1+\cos\theta)$.

(i) about the initial line **(ii) about the line $\theta = \frac{\pi}{2}$**

Solution. The equation of the cardiod is $r = a(1+\cos\theta)$

The cardioid is symmetrical about the initial line and for its upper half, θ increases from 0 to π (Fig. 1.17)

(i) About the initial line :

Required volume is

$$= \int_0^{\pi} \frac{2}{3}\pi r^3 \sin\theta \, d\theta = \frac{2\pi}{3} \int_0^{\pi} a^3 (1+\cos\theta)^3 \sin\theta \, d\theta$$

$$= \frac{-2\pi a^3}{3} \int_0^{\pi} (1+\cos\theta)^3 (-\sin\theta) \, d\theta$$

$$= \frac{-2\pi a^3}{3}\left|\frac{(1+\cos\theta)^4}{4}\right|_0^{\pi}$$

$$= \frac{-\pi a^3}{6}(0-16) = \frac{8}{3}\pi a^3$$

(ii) About the line $\theta = \frac{\pi}{2}$:

Required volume is

$$= \int_{-\pi/2}^{\pi/2} \frac{2}{3}\pi r^3 \cos\theta \, d\theta$$

$$= \frac{4\pi a^3}{3}\int_0^{\pi/2} (1+\cos\theta)^3 \cos\theta \, d\theta$$

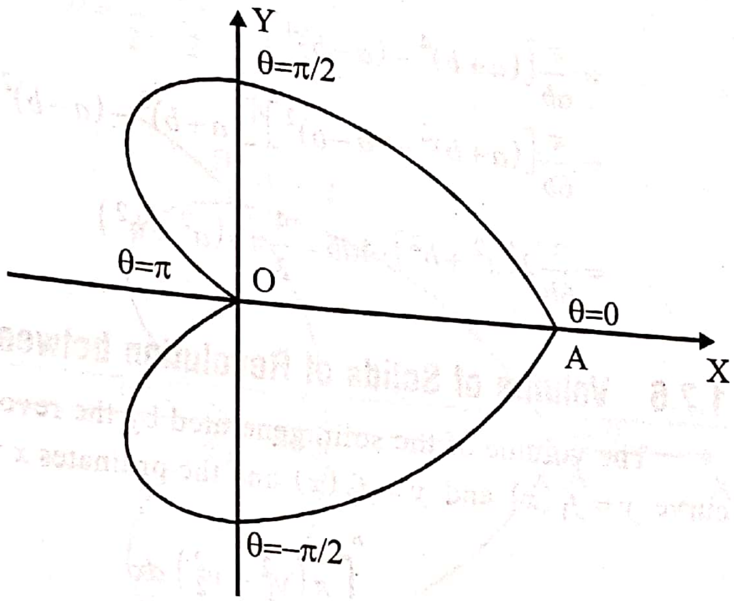

Fig. 1.17

$$= \frac{4\pi a^3}{3}\int_0^{\pi/2} \left[\cos^4\theta + 3\cos^3\theta + 3\cos^2\theta + \cos\theta\right] d\theta$$

$$= \frac{4\pi a^3}{3}\left[1 + \frac{3}{4}\cdot\frac{1}{3}\cdot\frac{\pi}{2} + 3\cdot\frac{2}{3} + 3\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{\pi a^3}{4}[16+5\pi]$$

EXAMPLE 1.25. Determine the volume of the solid obtained by revolving the lemicon $r = a + b\cos\theta$ $(a > b)$ about the initial line.

Solution. The equation of the curve is

$$r = a + b\cos\theta \qquad (a > b) \qquad ....(i)$$

The curve is symmetrical about the initial line and for the upper half of the curve θ varies from 0 to π (Fig. 1.18).

$\therefore$ Required volume is

$$= \int_0^{\pi} \frac{2}{3}\pi r^3 \sin\theta \, d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi} (a+b\cos\theta)^3 \sin\theta \, d\theta$$

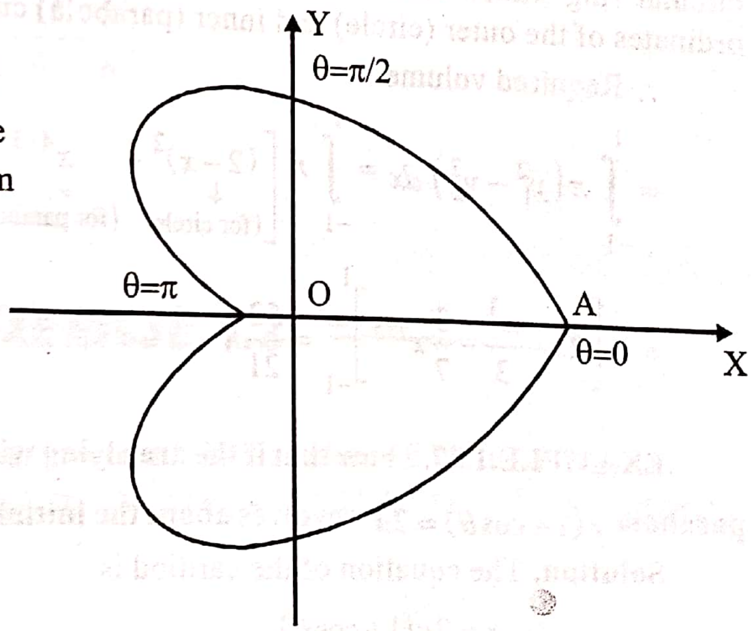

Fig. 1.18

$$= \frac{-2}{3}\frac{\pi}{b}\int_0^{\pi} (a+b\cos\theta)^3(-b\sin\theta)\, d\theta$$

$$= \frac{-2}{3}\frac{\pi}{b}\left[\frac{(a+b\cos\theta)^4}{4}\right]_0^{\pi} = \frac{-2\pi}{3b}\left[\frac{(a-b)^4}{4} - \frac{(a+b)^4}{4}\right]$$

$$=\frac{\pi}{6b}\left[(a+b)^4-(a-b)^4\right]$$

$$=\frac{\pi}{6b}\left[(a+b)^2+(a-b)^2\right]\left[(a+b)^2-(a-b)^2\right]$$

$$=\frac{\pi}{6b}2\left(a^2+b^2\right).4ab=\frac{4}{3}\pi a\left(a^2+b^2\right)$$

1.7.6 Volume of Solids of Revolution between Two Solids

The volume of the solid generated by the revolution about the x-axis of the area bounded by the curve $y=f_1(x)$ and $y=f_2(x)$ and the ordinates $x=a$, $x=b$ is

$$\int_a^b \pi\left(y_1^2-y_2^2\right)dx$$

where y_1 and y_2 are the y's of upper and lower curve respectively.

EXAMPLE 1.26. Find the volume of the solid generated by revolving about the x-axis, the smaller area bounded by the circle $x^2+y^2=2$ and the semicubical parabola $y^3=x^2$.

Solution. The smaller area OABC bounded by the circle and parabola is shown in Fig. 1.19.

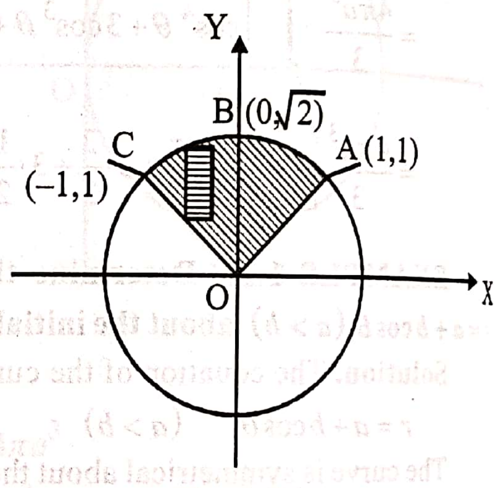

Fig. 1.19

The vertical element revolved about x-axis generates a circular ring whose outer radii and inner radii are respective ordinates of the outer (circle) and inner (parabola) curves.

$\therefore$ Required volume

$$=\int_{-1}^{1}\pi\left(y_1^2-y_2^2\right)dx=\int_{-1}^{1}\pi\left[\underset{\text{(for circle)}}{\underset{\downarrow}{(2-x)^2}}-\underset{\text{(for parabola)}}{\underset{\downarrow}{x^{4/3}}}\right]dx$$

$$=\pi\left[2x-\frac{x^3}{3}-\frac{3}{7}x^{7/3}\right]_{-1}^{1}=\frac{52}{21}\pi$$

EXAMPLE.1.27. Show that if the area lying within the cardiod $r=2a(1+\cos\theta)$ and without the parabola $r(1+\cos\theta)=2a$ revolves about the initial line, the volume generated is $18\pi a^3$.

Solution. The equation of the cardiod is

$$r=2a(1+\cos\theta) \qquad \text{....(i)}$$

and that of the parabola is

$$r=\frac{2a}{1+\cos\theta} \qquad \text{....(ii)}$$

Equating the values of r from (i) and (ii), we get

$$2a(1+\cos\theta)=\frac{2a}{(1+\cos\theta)} \Rightarrow (1+\cos\theta)^2=1 \Rightarrow \cos\theta(\cos\theta+2)=0$$

Now, $\cos \neq -2 \quad \Rightarrow \quad \cos\theta = 0$ i.e. $\theta = \frac{\pi}{2}, \frac{-\pi}{2}$

Thus, curve intersect where

$$\theta = \frac{\pi}{2} \text{ and } \theta = \frac{-\pi}{2}.$$

Also both the curves are symmetrical about the initial line (x-axis) as shown in Fig. 1.20.

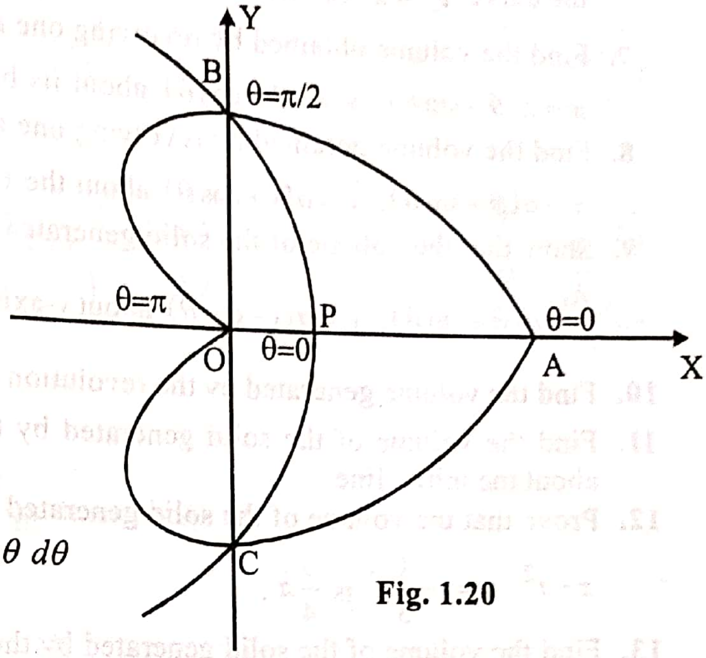

Fig. 1.20

$\therefore$ Required volume

$$= \underbrace{\frac{2\pi}{3}\int_0^{\pi/2} r^3 \sin\theta \, d\theta}_{\substack{\text{(for cardioid)} \\ \downarrow \\ \text{outer curve}}} - \underbrace{\frac{2\pi}{3}\int_0^{\pi/2} r^3 \sin\theta \, d\theta}_{\substack{\text{(for parabola)} \\ \downarrow \\ \text{inner curve}}}$$

$$= \frac{2\pi}{3}\int_0^{\pi/2}\left[8a^3(1+\cos\theta)^3 - \frac{8a^3}{(1+\cos\theta)^3}\right]\sin\theta \, d\theta$$

$$= \frac{-16\pi a^3}{3}\int_0^{\pi/2}\left[(1+\cos\theta)^3 - (1+\cos\theta)^{-3}\right](-\sin\theta)\, d\theta$$

$$= \frac{-16\pi a^3}{3}\left[\frac{(1+\cos\theta)^4}{4} - \frac{(1+\cos\theta)^{-2}}{-2}\right]_0^{\pi/2}$$

$$= \frac{-16\pi a^3}{3}\left[\frac{1}{4}(1-16) + \frac{1}{2}\left(1-\frac{1}{4}\right)\right] = \frac{-16\pi a^3}{3}\left[\frac{-15}{4} + \frac{3}{8}\right]$$

$$= \frac{-16\pi a^3}{3}\left(\frac{-27}{8}\right) = 18\pi a^3.$$

EXERICSE 1.2

1. Find the volume of a spherical cap of height h cut off from a sphere of radius a.
2. Find the volume of the solid generated by the revolution of an arc of the catenary $y = c\cosh\left(\frac{x}{c}\right)$ about the x-axis.
3. Find the volume of the solid generated by the revolution of the curve $y = \frac{a^3}{a^2+x^2}$ about its asymptote. **(K.U. 2003)**
4. Show that the volume of the solid obtained by the revolution of the curve $a^2y^2 = x^2(a^2-x^2)$ about the x-axis is $\frac{4}{15}a^3$.
5. Obtain the volume of the frustrum of a right circular cone whose lower base has radius R, upper base of radius r and altitute is h.

6. Show that the volume of the solid generated by the revolution of the upper half of the loop of the curve $y^2 = x^2(2-x)$ about x-axis is $\frac{4}{3}\pi$.

7. Find the volume obtained by revolving one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1+\cos\theta)$ about its base.

8. Find the volume obtained by revolving one arch of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1+\cos\theta)$ about the x-axis.

9. Show that the volume of the solid generated by the revolution of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ about y-axis is $\pi a^3\left(\frac{3}{2}\pi^2 - \frac{8}{3}\right)$.

10. Find the volume generated by the revolution of $r = 2a\cos\theta$ about the initial line.

11. Find the volume of the solid generated by the revolution of the cardioid $r = a(1-\cos\theta)$ about the initial line.

12. Prove that the volume of the solid generated by the revolution about the x-axis of the curve $x = t^2$, $y = t - \frac{1}{3}t^3$ is $\frac{3}{4}\pi$.

13. Find the volume of the solid generated by the revolution of the cissoid $x = 2a\sin^2 t$, $y = \dfrac{2a\sin^3 t}{\cos t}$ about its asymptote.

14. Prove that the volume of the solid generated by the revolution of an ellipse about its minor axis is $\frac{4}{3}\pi a^2 b$.

15. Show that the volume of the solid generated by revolving the area included between the curves $y^2 = x^3$ and $x^2 = y^3$ about x-axis is $\frac{5\pi}{28}$.

ANSWERS

1. $\pi h^2\left(a - \frac{h}{3}\right)$

2. $\frac{\pi c^2}{2}\left[x + \frac{c}{2}\sinh\frac{2x}{c}\right]$

3. $\frac{\pi^2 a^3}{2}$

5. $\frac{\pi h}{3}(r^2 + rR + R^2)$

7. $5\pi^2 a^3$

8. $5\pi^2 a^3$

10. $\frac{4}{3}\pi a^3$

11. $\frac{8}{3}\pi a^3$

13. $2\pi^2 a^3$

1.8 SURFACE AREA OF REVOLUTION

If a plane curve is revolved about a fixed line lying in its own plane, then the surface generated by the perimeter of the curve is called a surface of revolution.