Relation between Beta and Gamma function.

$$\Gamma(m) = \int_0^\infty e^{-t} t^{m-1} \, dt$$

Put $t = x^2 \Rightarrow dt = 2x\,dx$.

Also when $t = 0 \Rightarrow x = 0$

& when $t = \infty \Rightarrow x = \infty$

$$\Gamma(m) = \int_0^\infty e^{-x^2} (x^2)^{m-1} 2x\,dx$$

$$= 2\int_0^\infty e^{-x^2} x^{2m-2} \cdot x\,dx.$$

$$\boxed{\Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-1} \, dx} \quad \text{--- (1)}$$

$$\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1} \, dy. \quad \text{--- (2)}$$

(1) × (2)

$$\therefore \Gamma(m)\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2m-1} \, dx \cdot 2\int_0^\infty e^{-y^2} y^{2n-1} \, dy$$

$$= 4\int_0^\infty\int_0^\infty e^{-x^2} x^{2m-1} \cdot e^{-y^2} y^{2n-1} \, dx\,dy$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} \, dx\,dy.$$

$$\Gamma(m)\,\Gamma(n) = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,x^{2m-1}\,y^{2n-1}\,dx\,dy.$$

Converting into polar co-ordinates.

Put $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow$ $dx\,dy = r\,dr\,d\theta$. , $0 \le \theta \le \pi/2$

$0 \le r \le \infty$

$$\therefore \Gamma(m)\,\Gamma(n) = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,(r\cos\theta)^{2m-1}\,(r\sin\theta)^{2n-1}\,r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\cdot r^{2m-1+2n-\cancel{1}+\cancel{1}}\,\cos^{2m-1}\theta\,\sin^{2n-1}\theta\,dr\,d\theta.$$

$$= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r^{2m+2n-1}\cdot\cos^{2m-1}\theta\,\sin^{2n-1}\theta\,dr\,d\theta.$$

$$= \boxed{2\int_0^{\pi/2}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\,d\theta}\cdot\boxed{2\int_0^\infty e^{-r^2}\,r^{2m+2n-1}\,dr.}$$

$$2\int_0^\infty e^{-r^2}\,r^{2(m+n)-1}\,dr$$

From eq. 1

$$\Gamma(m)\,\Gamma(n) = \beta(m,n)\,\Gamma(m+n).$$

$$\Rightarrow \boxed{\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}}$$

Value of $\Gamma(\frac{1}{2})$ [Second Method]

Using $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ — 1

Put $m = \frac{1}{2}$, $n = \frac{1}{2}$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})} = \frac{[\Gamma(\frac{1}{2})]^2}{\Gamma(1)} = [\Gamma(\tfrac{1}{2})]^2 \quad (\because \Gamma(1) = 1)$$

$$\Rightarrow [\Gamma(\tfrac{1}{2})]^2 = \beta\left(\frac{1}{2}, \frac{1}{2}\right) \quad \text{— 2}$$

Now we know that

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta.$$

Put $m = \frac{1}{2}$, $n = \frac{1}{2}$

$$\Rightarrow \beta(m,n) = 2\int_0^{\pi/2} \sin^{2\cdot\frac{1}{2}-1}\theta \cos^{2\cdot\frac{1}{2}-1}\theta \cdot d\theta$$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^0\theta \cos^0\theta \, d\theta = 2\int_0^{\pi/2} d\theta$$

$$= 2(\theta)_0^{\pi/2} = 2\left(\frac{\pi}{2} - 0\right) = \pi$$

$$\Rightarrow \boxed{\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi} \quad \text{— 3}$$

using 3 in 2 we've

$$[\Gamma(\tfrac{1}{2})]^2 = \pi \Rightarrow \boxed{\Gamma(\tfrac{1}{2}) = \sqrt{\pi}}$$

Questions

① $\Gamma\left(\frac{7}{2}\right) = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}$

$\left(\because \Gamma(n+1) = n\,\Gamma(n)\right]$

② $\beta\left(\frac{5}{2}, \frac{3}{2}\right) = ?$

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\therefore \beta\left(\frac{5}{2}, \frac{3}{2}\right) = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}+\frac{3}{2}\right)} = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(4)}$$

$$= \frac{\left[\frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)\right]\left[\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right]}{3!}$$

$$= \frac{\frac{3}{4}\sqrt{\pi} \times \frac{1}{2}\sqrt{\pi}}{6} = \frac{3 \times \pi}{8 \times 62} = \frac{\pi}{16} \text{ Ans}$$

Find $\beta\left(\frac{9}{2}, \frac{7}{2}\right)$ — H.W.

To Evaluate $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$.

we know that

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta.$$

Put $2m-1 = p$ & $2n-1 = q$

$\Rightarrow m = \frac{p+1}{2}$ $\Rightarrow n = \frac{q+1}{2}$

$$\Rightarrow 2\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\Rightarrow \boxed{\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)}$$

Now $\frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+1}{2} + \frac{q+1}{2}\right)}$

$$\therefore \boxed{\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}}$$