11

Counting Techniques

FIRST COUNTING PRINCIPLE

If an event can occur in r different steps, and

Step 1 can occur in n_1 ways.

Step 2 can occur in n_2 ways.

....

....

Step r can occur in n_r ways.

Then the number of possible events that can occur is $= n_1 . n_2 . n_3 ... n_r$.

This is the fundamental principle of counting.

Example 1. *A child has four hats, three pair of gloves and five pair of socks. Determine different possible triplets he can wear ?*

Sol. A hat can be selected in four ways.

A pair of gloves can be selected in three ways.

A pair of socks can be selected in five ways.

$\therefore$ By principle of counting.

Total number of possible triplets the child can wear are $= 4 \times 3 \times 5 = 60$ ways.

Example 2. *A person has to arrange five books on a shelf. In how many ways can he do so ?*

Sol. The first book can be arranged in 5 ways.

The second book can be arranged in 4 ways.

The third book can be arranged in 3 ways.

The fourth book can be arranged in 2 ways.

The fifth book can be arranged in 1 way.

Thus, by principle of counting,

Total number of ways five books can be arranged is $= 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.

Theorem I. *Prove that a set containing n elements has 2^n subsets. Use the first principle of counting.*

Proof. As we have n elements in the set, a subset can be constructed in n different steps i.e., Take or do not take first element.

Take or do not take second element.

....

....

Take or do not take nth element.

So, each step can be done in two different ways.

Hence the possible number of subsets is $= 2 . 2 . 2 . 2 \dots n$ times $= 2^n$.

Example 3. *How many different 8-bit strings are there that begin and end with one.*

Sol. A 8-bit string that begins and end with 1 can be constructed in 6 steps *i.e.,*

By selecting IInd bit, IIIrd bit, IVth bit, Vth bit, VIth bit and VIIth bit and each bit can be selected in 2 ways.

Hence, the total number of 8-bit strings that begins and end with 1 is

$$= 2 . 2 . 2 . 2 . 2 . 2 = 2^6.$$

Example 4. *How many different 2-digit numbers can be made from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, 0 ? When repetition is allowed ? When repetition is not allowed ?*

Sol. When repetition is allowed

The tens place can be filled by 10 ways and the units place can be filled by 10 ways.

$\therefore$ The total number of 2 digit numbers $= 10 \times 10 = 100$.

When repetition is not allowed

The tens place can be filled by 10 ways and the units place can be filled by 9 ways.

$\therefore$ The total number of 2-digit numbers $= 10 \times 9 = 90$.

Example 5. *A five person committee having members Ankit, Arjit, Sonu, Monu and Nonu is to select a president, vice-president and secretary.*

(*a*) *How many selections exclude Nonu ?*

(*b*) *How many selections include Sonu and Monu ?*

(*c*) *How many selections exclude Sonu and Monu ?*

(*d*) *How many selections are there is which Ankit is president ?*

Sol. (*a*) After excluding Nonu, we have to select three persons from the remaining four. Therefore, president can be selected in 4 ways, vice-president can be selected in 3 ways, and secretary can be selected in 2 ways.

Hence, the total number of selections that exclude Nonu is $= 4 \times 3 \times 2 = 24$.

(*b*) We have 3 ways to assign any post to Sonu. After selecting Sonu, there are 2 ways to assign any post to Monu. After selecting Sonu and Monu, we can assign the remaining post to any of the three persons.

Hence, the total number of selections that include Sonu and Monu is $= 3 \times 2 \times 3 = 18$.

(*c*) After excluding Sonu and Monu, we have to select three persons from the remaining three. Therefore, president can be selected in 3 ways, vice-president can be selected in 2 ways, and secretary can be selected in 1 way.

Hence, the total number of selections that exclude Sonu and Monu is $= 3 \times 2 \times 1 = 6$.

(*d*) When Ankit is selected as president, then we have to select vice-president and secretary from the remaining four. Therefore,

Vice-president can be selected in 4 ways and secretary can be selected in 3 ways.

Hence, the total number of selection in which Ankit is president is $= 4 \times 3 = 12$.

Example 6. *Ram has five different 'Data Structure Books', four different 'Discrete Structure Books' and five different 'Programming Language Books'.*

(*a*) *In how many ways Ram can arrange these books on a shelf ?*

(*b*) *In how many ways can these books be arranged on a shelf if all five programming language books are on the right ?*

(*c*) *In how many ways can these books be arranged on a shelf if all five data structure books are on the left and all five programming language books are on the right ?*

Sol. We have total $= 5 + 4 + 5 = 14$ books.

(*a*) The first book can be arranged in 14 ways.

The second book can be arranged in 13 ways.

The third book can be arranged in 12 ways.

....

....

The fourteenth book can be arranged in 1 way.

Hence, the total number of ways the book can be placed on a shelf

$$= 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$= 8.717 \times 10^{10}.$$

(*b*) We have to place all the five programming language books on the right. Thus, the remaining books to be arranged are $= 14 - 5 = 9$.

The first book can be arranged in 9 ways.

The second book can be arranged in 8 ways.

....

....

The ninth book can be arranged in 1 way.

Hence, the total number of ways arranging remaining books is $= 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362880$.

The five programming language books can be arranged in following ways :

The first one can be arranged in 5 ways.

The second one can be arranged in 4 ways.

....

....

The fifth one can be arranged in 1 way.

The total number of ways arranged programming language books is

$$= 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

The total number of ways to arrange the books on a shelf when all five programming language books are on right is

$$= 362880 \times 120 = 43545600.$$

4 3 7 2

(c) As explained earlier, we have 120 ways to place five programming language books on shelf. Also we have 120 ways to arrange five data structure books on the shelf. Similarly, have 24 ways to arrange four discrete structure books on the shelf.

Hence, the total number of ways the book can be arranged on a shelf if all five data cture books are on the left and all five programming language books are on the right is

$$= 120 \times 120 \times 24 = 345600.$$

OND COUNTING PRINCIPLE

Consider that $\{D_1, D_2, D_3, \ldots, D_r\}$ is a pairwise disjoint family of sets and some set D_i has lements. Then the number of possible selection of elements from the sets D_1 or D_2 or D_3 or r ... D_r is

$$n_1 + n_2 + n_3 + n_4 + \ldots + n_r.$$

We can also define this principle in another way, consider an event A_1 can occur in n_1 s and another event A_2 can occur in n_2 ways and A_1 and A_2 are mutually exclusive, then A_1 A_2 can occur in $(n_1 + n_2)$ ways. It is applicable for any number of events.

Example 7. *A five person committee having members Ankit, Arjit, Sonu, Monu and u is to select a president, vice-president and secretary.*

(a) *In how many ways can this occur if either Sonu or Monu must be president ?*

(b) *How many selections are there in which either Nonu is a secretary or he is excluded ?*

(c) *How many selections exclude Ankit or Arjit ?*

Sol. (a) If Sonu is president, then vice-president can be selected in 4 ways and secretary be selected in 3 ways.

Hence, the total number of ways to select the remaining is $= 4 \times 3 = 12$.

Similarly, if Monu is president, then the remaining persons can be selected in 12 ways shown above.

As these are mutually exclusive events, hence the total number of ways if either Sonu or nu must be president is

$$= 12 + 12 = 24.$$

(b) If Nonu is a secretary, then the remaining two posts can be filled in 12 ways as cussed earlier. The number of selections in which Nonu is excluded is 24.

As these are mutually exclusive events, hence the total number of selections in which her Nonu is a secretary or he is excluded at all is

$$= 12 + 24 = 36.$$

(c) The number of selections in which Ankit is excluded :

Since president can be selected in from the remaining 4 persons.

Vice-president can be selected from remaining 3 persons after setting first.

Secretary can be selected from remaining 2 persons after setting second.

Hence, total number of selections $= 4 \times 3 \times 2 = 24$.

Similarly, the number of selections in which Arjit is excluded is 24.

These two sets of selections are disjoint, hence the total number of selections in which kit or Arjit are excluded is

$$= 24 + 24 = 48.$$

DEFINE FACTORIAL *N*

The product of first n natural number is called factorial n. It is denoted by $n\,!$ or $\underline{|n}$. The factorial n can also be written as

$$n\,! = n(n-1)(n-2)(n-3) \,\ldots\ldots\ldots\, 1.$$

We have, $1\,! = 1$ and $0\,! = 1$.

Example 8. *Find the value of 5 !.*

Sol. $5\,! = 5 \times (5-1)(5-2)(5-3)(5-4) = 5 \times 4 \times 3 \times 2 \times 1 = 120.$

Example 9. *Find the value of* $\dfrac{10\,!}{8\,!}$.

Sol. $\dfrac{10!}{8!} = \dfrac{10 \times 9 \times 8!}{8!} = 10 \times 9 = 90.$

Example 10. *Determine the value of* $\dfrac{n\,!}{(n-1)\,!}$.

Sol. $\dfrac{n!}{(n-1)!} = \dfrac{n(n-1)!}{(n-1)!} = n.$

Example 11. *Find the value of* $\dfrac{n\,!}{r\,!(n-r)\,!}$, *when* $n = 6$, $r = 4$.

Sol. $\dfrac{n!}{r!(n-r)!}$. Substitute the value of n and r.

We have $\dfrac{6!}{4!(6-4)!} = \dfrac{6!}{4!2!} = \dfrac{6 \times 5 \times 4!}{4! \times 2} = 15.$

Example 12. *Find the value of z, if* $\dfrac{1}{4\,!} + \dfrac{1}{5\,!} = \dfrac{z}{6\,!}$.

Sol. We have $\dfrac{1}{4!} + \dfrac{1}{5!} = \dfrac{z}{6!} = \dfrac{5+1}{5!} = \dfrac{z}{6!}$

$$\frac{6}{5!} = \frac{z}{6!}\ ;\ z = \frac{6 \times 6!}{5!}\ ;\ z = \frac{6 \times 6 \times 5!}{5!}\ ;\ z = 36.$$

Example 13. *Show that* $3\,! + 4\,! \neq (3+4)\,!$.

Sol. $3\,! + 4\,! = (3 \times 2 \times 1) + (4 \times 3 \times 2 \times 1)$

$= 6 + 24 = 30$

and $(3+4)\,! = 7\,! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

Hence $3\,! + 4\,! \neq (3+4)\,!$.

Example 14. *Show that* $10\,! - 8\,! \neq (10-8)\,!$.

Sol. $10\,! - 8\,! = 3628800 - 40320 = 3588480$

and $(10-8)\,! = (2)\,! = 2$

Hence $10\,! - 8\,! \neq (10-8)\,!$.

PERMUTATION

A permutation is an arrangement of a no. of objects in some definite order taken some or all at a time. The total number of permutations of n distinct objects taken r at a time is denoted by ${}^nP_r$ or $P(n, r)$, where $1 \le r \le n$.

Theorem II. *Prove that the number of different permutations of n distinct objects taken r at a time, $r \le n$ is given by*

$$^nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)\ldots(n-r+1).$$

Proof. The number of permutations of n distinct objects taken r at a time is like filling of r places with n objects.

The first place can be filled in by any one of the n objects. So, this can be done in n ways $= {}^nP_1 = n$.

The second place can be filled in by any one of the $n-1$ objects because after filling first place. We are left with $(n-1)$ objects. Thus, the first two places can be filled in $n(n-1)$ ways.

$$\therefore \qquad {}^nP_2 = n(n-1)$$

Similarly, the third place can be filled in by any one of the remaining $(n-2)$ objects.

Therefore, the first three places can be filled in $n(n-1)(n-2)$ ways.

Proceeding in this way, we have the number of permutations of n different objects taken r at a time

$$= n(n-1)(n-2)\ldots r$$

$$= n(n-1)(n-2)\ldots(n-\overline{r-1})$$

$$^nP_r = n(n-1)(n-2)\ldots(n-r+1).$$

Theorem III. *Prove that the number of permutations of n things taken all at a time is n!.*

Proof. We know that

$$^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} \qquad [\because 0! = 1]$$

$$= n!$$

Hence proved.

Example 15. *Determine the value of the following*

(i) ${}^4P_2$ (ii) ${}^9P_3$ (iii) ${}^{20}P_2$ (iv) ${}^{52}P_4$.

Sol. (i) $^4P_2 = \frac{4!}{(4-2)!} = \frac{4\times3\times2!}{2!} = 12$

(ii) $^9P_3 = \frac{9!}{(9-3)!} = \frac{9\times8\times7\times6!}{6!} = 504$

(iii) $^{20}P_2 = \frac{20!}{(20-2)!} = \frac{20\times19\times18!}{18!} = 380$

(iv) $^{52}P_4 = \frac{52!}{(52-4)!} = \frac{52\times51\times50\times49\times48!}{48!} = 6497400.$

Example 16. *Determine the value of n if*

(*i*) $4 \times {}^nP_3 = {}^{n+1}P_3$. (*ii*) $6 \times {}^nP_3 = 3 \times {}^{n+1}P_3$. (*iii*) $3 \times {}^nP_4 = 7 \times {}^{n-1}P_4$.

Sol. (*i*)
$$4 \times \frac{n!}{(n-3)!} = \frac{(n+1)!}{(n+1-3)!} \qquad \left[\because {}^nP_r = \frac{n!}{(n-r)!}\right]$$
$$\frac{4 \times n!}{(n-3)!} = \frac{(n+1) \times n!}{(n-2)(n-3)!}$$
$$4(n-2) = (n+1)$$
$$4n - 8 = n + 1$$
$$3n = 9$$
$$n = 3.$$

(*ii*)
$$6 \times {}^nP_3 = 3 \times {}^{n+1}P_3$$
$$6 \times \frac{n!}{(n-3)!} = 3 \times \frac{(n+1)!}{(n+1-3)!} \qquad \left[\because {}^nP_r = \frac{n!}{(n-r)!}\right]$$
$$\frac{6 \times n!}{(n-3)!} = \frac{3(n+1)(n!)}{(n-2)(n-3)!}$$
$$6(n-2) = 3 \times (n+1)$$
$$6n - 12 = 3n + 3$$
$$6n - 3n = 12 + 3$$
$$3n = 15$$
$$n = 5.$$

(*iii*)
$$3 \times {}^nP_4 = 7 \times {}^{n-1}P_4$$
$$3 \times \frac{n!}{(n-4)!} = 7 \times \frac{(n-1)!}{(n-1-4)!} \qquad \left[\because {}^nP_r = \frac{n!}{(n-r)!}\right]$$
$$\frac{3 \times n \times (n-1)!}{(n-4)(n-5)!} = \frac{7 \times (n-1)!}{(n-5)!}$$
$$3n = 7(n-4)$$
$$3n = 7n - 28$$
$$3n - 7n = -28$$
$$-4n = -28$$
$$n = 7.$$

Example 17. *How many variable names of 8 letters can be formed from the letters a, b, c, d, e, f, g, h, i if no letter is repeated.*

Sol. There are 9 letters and 8 are to be selected.

$\therefore$ Total number of variable names of 8 letters is $= {}^9P_8 = \frac{9!}{(9-8)!} = \frac{9!}{1!} = 9!$.

Example 18. *There are 10 persons called on an interview. Each one is capable to be selected for the job. How many permutation are there to select 4 from the 10.*

Sol. There are 10 persons and 4 are to be selected.

$\therefore$ Total number of permutations to select 4 persons is $= {}^{10}P_4$

$$= \frac{10!}{(10-4)!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{6!} = 5040.$$

Example 19. *How many 6-digit numbers can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, if no digit is repeated.*

Sol. There are 8 numbers are 6 are to be selected.

$$\therefore \quad \text{Total number of 6-digit numbers} = {}^8P_6 = \frac{8!}{(8-6)!} = \frac{8!}{2!}$$

$$= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2!}{2!} = 22560.$$

Permutation with Restrictions. *The number of permutations of n different objects taken r at a time in which p particular objects do not occur is

$${}^{n-p}P_r$$

*The number of permutations of n different objects taken r at a time in which p particular objects are present is

$${}^{n-p}P_{r-p} \times {}^rP_p.$$

Example 20. *How many 6-digit numbers can be formed by using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 if every number is to start with '30' with no digit repeated.*

Sol. All the numbers begin with '30'. So, we have to choose 4-digits from the remaining 7-digits.

$\therefore$ Total number of numbers that begins with '30' is

6 30 —

$${}^7P_4 = \frac{7!}{(7-4)!} = \frac{7 \times 6 \times 5 \times 4 \times 3!}{3!} = 840.$$

Example 21. *In how many ways 5 different microprocessor books and 4 different digital electronics books be arranged in a shelf so that all the four digital electronics books are together ?*

Sol. Consider the four digital electronics books as one unit. Thus, we have 6 units that can be arranged in 6 ! ways.

For each of these arrangements, 4 digital electronics books can be arranged among themselves in 4 ! ways.

$\therefore$ Total number of arrangements in which all the four digital electronics books are together is

$$= 6! \times 4! = 720 \times 24 = 17280.$$

Example 22. *How many permutations can be made out of the letter of word "COMPUTER" ? How many of these*

C 7!

(*i*) *begin with C ?* (*ii*) *end with R ?*

(*iii*) *begin with C and end with R ?* (*iv*) *C and R occupy the end places ?*

Sol. There are 8 letters in the word 'COMPUTER' and all are distinct.

$\therefore$ The total number of permutations of these letters is 8 ! = 40320.

(*i*) Permutations which begin with C.

The first position can be filled in only one way *i.e.*, C and the remaining 7 letters can be arranged in 7 ! ways.

∴ Total number of permutations starting with C are

$= 1 \times 7! = 5040.$

(*ii*) **Permutations which end with R.**

The last position can be filled in only one way *i.e.*, R and the remaining 7 letters can be arranged in 7 ! ways.

∴ The total number of permutations ending with R are $= 7! \times 1 = 5040.$

(*iii*) **Permutations begin with C and end with R.**

The first position can be filled in only one way *i.e.*, C and the last place can also be filled in only one way *i.e.*, R and the remaining 6 letters can be arranged in 6 ! ways.

∴ The total number of permutations begin with C and end with R is

$= 1 \times 6! \times 1 = 720.$

(*iv*) **Permutations is which C and R occupy end places.**

C and R occupy end positions in 2 ! ways *i.e.*, C, R and R, C and the remaining 6 letters can be arranged in 6 ! ways.

∴ The total number of permutations in which C and R occupy end places is

$= 2! \times 6! = 1440.$

PERMUTATIONS WHEN ALL OF THE OBJECTS ARE NOT DISTINCT

Theorem IV. *The number of permutations of n objects, of which n_1 objects are of one kind and n_2 objects of another kind, when all are taken at a time is* $\dfrac{n!}{n_1!\,n_2!}$.

Proof. Let us assume that the number of required permutations be K. Now consider a single particular permutation of these K permutations, in which n_1 objects of one kind is followed by n_2 objects of other kind.

Also, assume that all n_1 object are distinct from all n_2 objects.

So, number of permutations of n_1 objects taken all at a time is $= {}^{n_1}P_{n_1} = n_1!$

Also, the number of permutations of n_2 objects taken all at a time is $= {}^{n_2}P_{n_2} = n_2!$

By the fundamental principle of counting, these K permutations will give rise to $n_1!\,n_2!$ permutations by arranging the objects of one kind within the places occupied by them.

Therefore, K permutations will give rise to $K.\,n_1!\,n_2!$ permutations.

For n distinct objects, the number of permutations is $= {}^nP_n = n!$

Therefore, $K \times n_1!\,n_2! = n!$

$$K = \frac{n!}{n_1!\,n_2!}$$

This result can be generalised as follows :

If n_1 objects are of one kind, n_2 objects are of second kind, n_3 objects are of third kind, and so on upto n_1 objects are of tth type is given by

$$\frac{n!}{n_1!\,n_2!\,n_3!\ldots n_t!} \qquad [\text{Here } n_1 + n_2 + n_3 + \ldots + n_t = n]$$

Example 23. *Determine the number of permutations that can be made out of the letters of the word 'PROGRAMMING'.*

Sol. There are 11 letters in the word 'PROGRAMMING' out of which G's and M's and R's are two each.

$\therefore$ The total number of permutations is

$$= \frac{11!}{2! \times 2! \times 2!} = \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2!}{2 \times 1 \times 2 \times 1 \times 2!}$$

$$= 4989600.$$

Example 24. *There are 4 blue, 3 red and 2 black pens in a box. These are drawn one by one. Determine all the different permutations.*

Sol. There are total 9 pens in the box out of which 4 are blue, 3 are red and 2 are black.

$\therefore$ The total number of permutations is

$$= \frac{9!}{4! \times 3! \times 2!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4! \times 3 \times 2 \times 1 \times 2 \times 1} = 1080.$$

Example 25. *How many different variable names can be formed by using the letters a, a, a, b, b, b, b, c, c, c ?*

Sol. There are total 10 letters out of which 3 are a's, 4 are b's and there are 3 c's.

$\therefore$ Total number of permutations is

$$= \frac{10!}{3! \times 4! \times 3!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{3 \times 2 \times 1 \times 4! \times 3 \times 2 \times 1}$$

$$= 10 \times 3 \times 4 \times 7 \times 5 = 4200.$$

Example 26. *How many 7-digits numbers can be formed using digits 1, 7, 2, 7, 6, 7, 6 ?*

Sol. There are total 7-digits out of which 3 are 7's and 2 are 6's.

$\therefore$ Total number of permutations is $= \dfrac{7!}{3! \times 2!} = 420.$

PERMUTATIONS WITH REPEATED OBJECTS

Theorem V. *Prove that the number of different permutations of n distinct objects taken r at a time when every object is allowed to repeat any number of times is given by n^r.*

Proof. Assume that with n objects we have to fill r place when repetition of objects is allowed.

Therefore, the number of ways of filling the first place is $= n$

The number of ways of filling second place $= n$

....

....

The number of ways of filling rth place $= n$

Thus, the total number of ways of filling r places with n elements is

$$= n.\ n.\ n.\ n.\ \ldots\ldots\ r \text{ times} = n^r.$$

Example 27. *How many 4-digits numbers can be formed by using the digits 2, 4, 6, 8 when repetition of digits is allowed.*

Sol. We have 4-digits.

So, number of ways of filling unit's place = 4.

Number of ways of filling ten's place = 4.

Number of ways of filling hundred's place = 4.

Number of ways of filling thousand's place = 4.

Therefore, the total number of 4-digits numbers is

$$= 4 \times 4 \times 4 \times 4 = 256.$$

Example 28. *How many 2-digits even numbers can be formed by using the digits 1, 3, 4, 6, 8 when repetition of digits is allowed.*

Sol. We have three even numbers and two odd number.

Thus, number of ways of filling unit's place = 3.

Number of ways of filling ten's place = 5.

$\therefore$ Total number of two digits even numbers = $3 \times 5 = 15$.

Example 29. *In how many ways can 5 software projects be allotted to 6 final year students when all the 5 projects are not allotted to the same student.*

Sol. We have 5 projects and 6 students.

Each projects can be allotted in 6 ways.

Thus, the number of ways of alloting 5 projects is = $6 \times 6 \times 6 \times 6 \times 6 = 6^5$.

Number of ways in which all projects allotted to same student is = 6.

Therefore, total number of ways to allocate 5 projects to 6 students is = $6^5 - 6 = 7770$.

CIRCULAR PERMUTATIONS

The circular permutations are the permutations of the objects placed in a circle. Consider the letters *k, l, m, n, o* placed along the circle as shown in Fig. 1.

If we place letters linearly, there are five different permutations *i.e., k, l, m, n, o ; l, m, n, o, k ; m, n, o, k, l ; n, o, k, l, m ; o, k, l, m, n,* but there is only one circular permutation *k, l, m, n, o*. Therefore, there is no starting and ending in circular permutation. We only consider the relative positions.

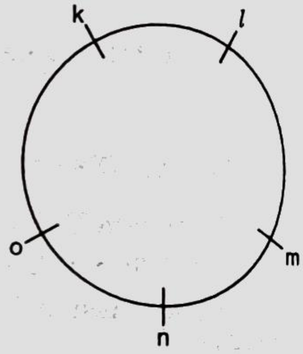

Fig. 1.

Theorem VI. *Prove that the number of circular permutations of n different objects is $(n-1)!$.*

Proof. Let us consider that K be number of permutations required.

For each such circular permutation of K, there are n corresponding linear permutations. As shown earlier, we start from every object of n object in the circular permutation. Thus, for K circular permutations, we have K. n linear permutations.

Therefore, $\quad K.\, n = n!$ or $K = \dfrac{n!}{n}$

$$K = \frac{n \times (n-1)!}{n}$$

$$K = (n-1)!$$

Hence proved.

Example 30. *In how many ways can these letters a, b, c, d, e, f be arranged in a circle ?*

Sol. There are 6 letters and hence the number of ways to arrange these 6 letters in a circle is

$$= (6-1)! = 5! = 120.$$

Example 31. *In how many ways 10 programmers can sit on a round table to discuss the project so that project leader and a particular programmer always sit together.*

Sol. There are total 10 programmers but project leader and a particular programmer always sit together. So, both become a single unit and hence there are $(10 - 2 + 1) = 9$ remains.

Thus, these 9 units can be arranged on round table in $(9-1)!$ ways.

The two programmers *i.e.*, project leader and a particular programmer can be arranged in $2!$ ways.

Therefore, the total number of ways in which 10 programmers can sit on a round table is

$$= (9-1)! \times 2! = 8! \times 2! = 80640.$$

Example 32. *Determine the number of ways in which 5 software engineers and 6 electronics engineers can be sitted at a round table so that no two software engineers can sit together.*

Sol. There are 6 electronics engineers that can be arranged round a table in $(6-1)!$ ways. There are 5 software engineers and they are not to sit together so we have six places for software engineers and can be placed in $6!$ ways as shown in Fig. 2.

Therefore, total number of ways to arrange the engineers on a round table is

$$= (6-1)! \times 6! = 5! \times 6!$$
$$= 120 \times 720$$
$$= 86400.$$

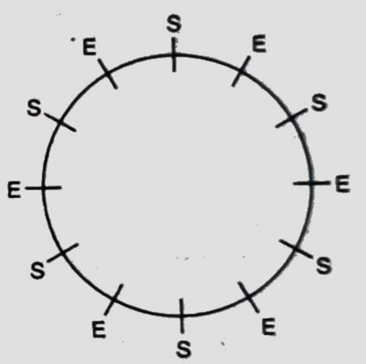

Fig. 2.

COMBINATION

A combination is a selection of some or all, objects from a set of given objects, where order of the objects does not matter. The number of combinations of n objects, taken r at a time is represented by ${}^nC_r$ or $C(n, r)$.

Theorem VII. *The number of combinations of n different things, taken r at a time is given by*

$${}^nC_r = \frac{n!}{r!\,(n-r)!},\ n \ge r \ge 1.$$

Proof. The number of permutations of n different things, taken r at a time is given by

$$^{n}P_r = \frac{n!}{(n-r)!}$$

As there is no matter about the order of arrangement of the objects, therefore, to every combination of r things, there are r ! arrangements *i.e.*,

$$^{n}P_r = r!\,^{n}C_r \quad \text{or} \quad ^{n}C_r = \frac{^{n}P_r}{r!} = \frac{n!}{(n-r)!\,r!}, \; n \geq r.$$

Thus, $$^{n}C_r = \frac{n!}{(n-r)!\,r!}.$$

Theorem VIII. *Prove that the number of combinations of n things taken all at a time is one.*

Proof. We know that

$$^{n}C_n = \frac{n!}{(n-n)!\,n!} = \frac{n!}{0!\,n!} = 1 \qquad [\because 0! = 1]$$

Theorem IX. *Prove that the number of combinations of n things taken none at a time is one.*

Proof. We know that

$$^{n}C_0 = \frac{n!}{(n-0)!\,0!} = \frac{n!}{n!\,0!} = \frac{n!}{n!} = 1 \qquad [\because 0! = 1]$$

Theorem X. *Prove that* $^{n}C_{n-r} = {}^{n}C_r\, n \geq r \geq 1$.

Proof. We know that

$$^{n}C_{n-r} = \frac{n!}{n-(n-r)!\,(n-r)!} = \frac{n!}{(n-n+r)!\,(n-r)!}$$

$$= \frac{n!}{r!\,(n-r)!} = {}^{n}C_r.$$

Example 33. *Determine the value of following*

(*i*) $^{10}C_6$ (*ii*) $^{50}C_{45}$ (*iii*) $^{52}C_4$ (*iv*) $^{20}C_{10}$.

Sol. (*i*) $$^{10}C_6 = \frac{10!}{(6)!\times(10-6)!} = \frac{10\times9\times8\times7\times6!}{6!\times4\times3\times2\times1} = 10 \times 3 \times 7 = 210.$$

(*ii*) $$^{50}C_{45} = \frac{50!}{45!\times(50-45)!} = \frac{50\times49\times48\times47\times46\times45!}{45!\times5\times4\times3\times2\times1} = 2118760.$$

(*iii*) $$^{52}C_4 = \frac{52!}{4!\times(52-4)!} = \frac{52\times51\times50\times49\times48!}{4\times3\times2\times1\times48!} = 270725.$$

(*iv*) $$^{20}C_{10} = \frac{20!}{10!\times(20-10)!} = \frac{20\times19\times18\times17\times16\times15\times14\times13\times12\times11\times10!}{10!\times10!}$$

$= 184756.$

Example 34. *Determine the value of n if*

(*i*) $^{n}C_4 = {}^{n}C_3$ (*ii*) $^{n}C_{n-2} = 10$ (*iii*) $^{20}C_{n+2} = {}^{20}C_{2n-1}$

Sol. *(i)*

$${}^nC_4 = {}^nC_3$$

$$\frac{n!}{4!\times(n-4)!} = \frac{n!}{3!(n-3)!}$$

$$\frac{n!}{n!} = \frac{4!\times(n-4)!}{3!\times(n-3)!} = \frac{4\times 3!\times(n-4)!}{3!\times(n-3)\times(n-4)!}$$

$$1 = \frac{4}{n-3}$$

$$n - 3 = 4$$

$$n = 7.$$

(ii)

$${}^nC_{n-2} = 10$$

Thus $$\frac{n!}{(n-2)!\,[n-(n-2)]!} = 10 \quad \text{or} \quad \frac{n!}{(n-2)!\times 2!} = 10$$

$$\frac{n\times(n-1)\times(n-2)!}{(n-2)!\times 2!} = 10$$

$$n \times (n-1) = 10 \times 2!$$

$$n^2 - n - 20 = 0$$

$n = -4, 5$. Since -4 is not possible, hence $n = 5$.

(iii)

$${}^{20}C_{n+2} = {}^{20}C_{2n-1}$$

Therefore, we have either

$$n + 2 = 2n - 1 \quad \text{or} \quad (n + 2) + (2n - 1) = 25$$

$$-n = -3 \quad \text{or} \quad 3n = 24$$

$$n = 3 \qquad n = 8$$

So $$n = 3, 8.$$

Example 35. *How many 16-bit strings are there containing exactly five 0's ?*

Sol. A 16-bit string having exactly five 0's is determined if we tell which bits are 0's. This can be done in ${}^{16}C_5$ ways.

Therefore, the total number of 16-bit strings is

$$= {}^{16}C_5 = \frac{16!}{5!\times(16-5)!}$$

$$= \frac{16\times 15\times 14\times 13\times 12\times 11!}{5\times 4\times 3\times 2\times 1\times 11!} = 4368.$$

Example 36. *How many ways can we select a software development group of 1 project leader, 5 programmers and 6 data entry operators from a group of 5 project leaders, 20 programmers and 25 data entry operators ?*

Sol. There are 5 project leaders out of which one can be selected in ${}^5C_1$ ways.

There are 20 programmers out of which five can be selected in ${}^{20}C_5$ ways.

There are 25 data entry operators out of which six can be selected in ${}^{25}C_6$ ways.

Therefore, the total number of ways to select the software development group is

$$= {}^5C_1 \times {}^{20}C_5 \times {}^{25}C_6 = 96101544000.$$

Example 37. *From 10 programmers in how many ways can 5 be selected when*

(a) A particular programmer is included every time.

(b) A particular programmer is not included at all.

Sol. We have to select 5 programmers from the 10 programmers. So, the number of ways to select them in $^{10}C_5$

$$= \frac{10!}{5! \times (10-5)!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5 \times 4 \times 3 \times 2 \times 1 \times 5!}$$

$= 252.$

(*a*) When a particular programmer is included every time then the remaining $= 5 - 1 = 4$ programmers can be selected from the remaining $= 10 - 1 = 9$ programmers. This can be done in $^9C_4$ ways

$$= \frac{9!}{4!(9-4)!} = \frac{9 \times 8 \times 7 \times 6 \times 5!}{4 \times 3 \times 2 \times 1 \times 5!} = 126.$$

(*b*) When a particular programmer is not included at all, then the five programmers can be selected from the remaining $= 10 - 1 = 9$ programmers.

This can be done in $^{11}C_5$ ways

$$= \frac{9!}{5!(9-5)!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4!}{5 \times 4 \times 3 \times 2 \times 1 \times 4!} = 126.$$

THE PIGEONHOLE PRINCIPLE

Theorem XI. *Show that if n pigeons are assigned to m pigeonholes and $m < n$, then there is at least one pigeonhole that contains two or more pigeons.*

Proof. Let us label the n pigeons with the numbers 1 through n and the m pigeonholes with the numbers 1 through m. Now starting with pigeon 1 and Pigeonhole 1, assign each pigeon in order to the pigeonhole with the same number. So we can assign as many pigeons as possible to distinct pigeonholes, but as we know that pigeonholes are less than pigeons *i.e.*, $m < n$. Thus, there remains $n - m$ pigeons that have not yet been assigned to a pigeonhole. Hence, there is at least one pigeonhole that will be assigned a second pigeon.

Example 38. *Show that if any four numbers from 1 to 6 are chosen, then two of them will add to 7.*

Sol. Make three sets containing two numbers whose sum is 7.

A = {1, 6}, B = {2, 5}, C = {3, 4}. The four numbers that will be chosen assigned to the set that contains it.

As there are only three sets, two numbers that are chosen is from the same set whose sum is 7.

Example 39. *Show that at least two people must have their birthday in the same month if 13 people are assembled in a room.*

Sol. We assigned each person the month of the year on which he was born. Since there are 12 months in a year.

So, according to pigeonhole principle, there must be at least two people assigned to the same month.

Example 40. *Show that if any eight +ve integers are chosen, two of them will have same remainder when divided by 7.*

Sol. Take any eight +ve integers. When these are divided by 7 each have some remainder. Since there are eight integers and only seven distinct remainders because number 7 can generate only 7 remainders, so two +ve integers must have same remainder.

EXTENDED PIGEONHOLE PRINCIPLE

It states that if n pigeons are assigned to m pigeonholes (The number of pigeons is very large than the number of pigeonholes), then one of the pigeonholes must contain at least $[(n - 1)/m] + 1$ pigeons.

Theorem XII. *Prove that extended pigeonhole principle.*

Proof. We can prove this by the method of contradiction. Assume that each pigeonhole does not contain more than $[(n - 1)/ m]$ pigeons. Then, there will be at most $m[(n - 1)/m]$ $\leq m(n - 1)/m = n - 1$ pigeons in all. This is in contradiction to our assumptions. Hence, for given m pigeonholes, one of these must contain at least $[(n - 1)/m] + 1$ pigeons.

Example 41. *Show that if 9 colours are used to paint 100 houses, at least 12 houses will be of the same colour.*

Sol. Let us assume the colours be the pigeonholes and the houses the pigeons. Now 100 pigeons are to be assigned to 9 pigeonholes. Using the extended pigeonhole principle, $[(n - 1)/m] + 1$, where $n = 100$ and $m = 9$, we have $[(100 - 1)/9] + 1 = 12$. Thus, there are 12 houses of the same colour.

SOLVED PROBLEMS

Problem 1. *How many different 8-bit strings are there that end with 0111 ?*

Sol. An 8-bit strings that end with 0111 can be constructed in 4 steps *i.e.,*

By selecting Ist bit, IInd bit, IIIrd bit and IVth bit and each bit can be selected in 2 ways.

Hence, the total no. of 8-bit strings that end with 0111 is

$$= 2 . 2 . 2 . 2 = 2^4.$$

Problem 2. *How many 2-digits numbers greater than 40 can be formed by using the digits 1, 2, 3, 4, 6, 7*

(a) When repetition is allowed (b) When repetition is not allowed.

Sol. (*a*) When repetition is allowed

We have to find the numbers greater than 40. Therefore,

Ten's place can be filled up by 3 ways.

Unit's place can be filled up by 6 ways.

$\therefore$ The total number of 2-digits numbers greater than 40 is $= 3 \times 6 = 18$.

(*b*) When repetition is not allowed

Ten's place can be filled up by 3 ways.

Unit's place can be filled up by 5 ways.

$\therefore$ The total number of 2-digits numbers greater than 40 is $= 3 \times 5 = 15$.

Problem 3. *How many words can be constructed of three English alphabets ?*

(*a*) *When repetition of alphabets is allowed*

(*b*) *When repetition is not allowed.*

Sol. There are 26 alphabets in English. Therefore,

(*a*) When repetition is allowed

First alphabet of word can be selected in 26 ways.

Second alphabet of word can be selected in 26 ways.

Third alphabet of word can be selected in 26 ways.

Hence, total number of words of three alphabets constructed is

$$= 26 \times 26 \times 26 = 17576.$$

(*b*) When repetition is not allowed

First alphabet of word can be selected in 26 ways.

Second alphabet of word can be selected in 25 ways.

Third alphabet of word can be selected in 24 ways.

Hence, the total number of words of three distinct alphabets is $= 26 \times 25 \times 24 = 15600$.

Problem 4. *Show that $0\,! = 1$.*

Sol. We have

$$^nP_r = \frac{n!}{(n-r)!} \qquad \ldots(i)$$

Now put $r = n$ in equation (*i*), we have

$$^nP_n = \frac{n!}{(n-n)!}$$

$$n\,! = \frac{n!}{0!}$$

$$0\,! = \frac{n!}{n!}$$

Hence $0\,! = 1$.

Problem 5. *There are n objects out of which r objects are to be arranged. Find the total number of permutations when*

(*a*) *four particular objects always occur.*

(*b*) *four particular objects never occur.*

Sol. (*a*) Number of ways to arrange first object $= r$

Number of ways to arrange second object $= r - 1$

Number of ways to arrange third object $= r - 2$

Number of ways to arrange fourth object $= r - 3$

Number of ways to arrange remaining $n - 4$ objects taking $r - 4$ at a time $= {}^{n-4}P_{r-4}$.

Therefore, the total number of permutation when four particular objects always occur is

$$= r(r-1)(r-2)(r-3)\,{}^{n-4}P_{r-4}. \qquad \text{[Using first principle of counting]}$$

(*b*) There are four particular objects which never occur in any arrangement. Hence set aside these four particular objects.

Thus, we have to find the number of arrangements of $n - 4$ objects taking r at a time.

$\therefore$ The total number of arrangements is $= {}^{n-4}P_r$.

Problem 6. *How many permutations can be made out of the letters of the word "Basic" ? How many of these*

(i) begin with B ? *(ii) end with C ?*

(iii) B and C occupy the end places ?

Sol. There are 5 letters in the word 'Basic' and all are distinct.

$\therefore$ The number of permutations of these letters is

$$= 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

(*i*) Permutations which begin with B

The first position can be filled in only one way *i.e.*, B and the remaining 4 letters can be arranged in 4 ! ways.

$\therefore$ Total number of permutations starting with B is $= 1 \times 4! = 24$.

(*ii*) Permutations which end with C

The first position can be filled in only one way *i.e.*, C and the remaining 4 letters can be arranged in 4 ! ways.

$\therefore$ Total number of permutations ending with C is

$$= 4! \times 1 = 24.$$

(*iii*) Permutations in which B and C occupy end places

B and C occupy end positions in 2 ! ways *i.e.*, B, C and C, B and the remaining 3 letters can be arranged in 3 ! ways.

$\therefore$ Total number of permutations in which B and C occupy end places in

$$= 2! \times 3! = 12.$$

Problem 7. *Show that* ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$, *where* $n \geq r \geq 1$ *and n and r are natural numbers.*

Sol. Take L.H.S. of equation *i.e.*,

$$
\begin{aligned}
{}^nC_r + {}^nC_{r-1} &= \frac{n!}{r! \times (n-r)!} + \frac{n!}{(r-1)! \times (n-r+1)!} \\
&= \frac{n!}{r(r-1)! \times (n-r)!} + \frac{n!}{(r-1)! \times (n-r+1)(n-r)!} \\
&= \frac{n! \times (n-r+1) + n! \times r}{r(r-1)! \times (n-r)! \times (n-r+1)} = \frac{n! \times (n-r+1+r)}{r! \times (n-r+1)!} \\
&= \frac{n! \times (n+1)}{r! \times (n-r+1)!} = {}^{n+1}C_r
\end{aligned}
$$

Hence proved.

Problem 8. *In the 'Discrete Structures Paper' there are 8 questions. In how many ways can an examiner select five questions in all if first question is compulsory.*

Sol. Since the first question is compulsory, the examiner has to select 4 questions from the remaining 7 questions.

Therefore, the number of ways to select 5 questions is $= {}^7C_4$

$$= \frac{7!}{4! \times (7-4)!} = \frac{7 \times 6 \times 5 \times 4!}{4! \times 3 \times 2 \times 1} = 35.$$

Problem 9. *Determine the number of triangles that are formed by selecting p a set of 15 points out of which 8 are collinear.*

Sol. When we take all the 15 points, the number of triangles formed is ${}^{15}C_3$. As 8 points lie on the same line, they do not form any triangle. Thus, ${}^8C_3$ tri lost.

$\therefore$ The total number of triangles produced is

$$ {}^{15}C_3 - {}^8C_3 = \frac{15!}{3 \times (15-3)!} - \frac{8!}{3!(8-3)!} = \frac{15 \times 14 \times 13 \times 12!}{3 \times 12!} - \frac{8 \times 7}{3 \times 2}$$

$$= \frac{15 \times 14 \times 13}{3} - \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 910 - 56 = 854.$$

Problem 10. *How many lines can be drawn through 10 points on a circle ?*

Sol. As all the points on the circle are not collinear. Thus, no lines will lost.

$\therefore$ The total number of lines drawn through a circle is $= {}^{10}C_2$

$$= \frac{10!}{2! \times (10-2)!} = \frac{10 \times 9 \times 8!}{2 \times 1 \times 8!} = 45.$$

Problem 11. *Determine the number of diagonals that can be drawn by joining th of octagon.*

Sol. The number of lines that can be formed by joining 2 out of 8 points is $= {}^8C_2$

$$= \frac{8 \times 7}{2} = 28$$

Out of these 28 lines, the 8 are sides of the octagon.

$\therefore$ The number of diagonals is $= 28 - 8 = 20$.

Problem 12. *In a shipment, there are 40 floppy disks of which 5 are defective. Dete*

(a) in how many ways we can select five floppy disks ?

(b) in how many ways we can select five non-defective floppy disks ?

(c) in how many ways we can select five floppy disks containing exactly three def floppy disks ?

(d) in how many ways we can select five floppy disks containing at least 1 defective f disks ?

Sol. (*a*) There are 40 floppy disks out of which we have to select 5 floppy disks. T can be done in ${}^{40}C_5$ ways *i.e.*,

$$= \frac{40!}{5!(40-5)!} = \frac{40 \times 39 \times 38 \times 37 \times 36 \times 35!}{5! \times 35!} = 658008.$$

(*b*) There are $40 - 5 = 35$ non-defective floppy disks out of which we have to select 5. T can be done in ${}^{35}C_5$ ways.

$$= \frac{35!}{5!(35-5)!} = \frac{35 \times 34 \times 33 \times 32 \times 31 \times 30!}{5 \times 4 \times 3 \times 2 \times 1 \times 30!}$$

$= 324632.$

(*c*) To select exactly three defective floppy disks out of total 5 we have ${}^5C_3$ ways and the remaining 2 floppy disks can be selected in ${}^{35}C_2$ ways.

Therefore, the total number of ways to select 5 floppy disks out of which exactly 3 are defective is $= {}^5C_3 \times {}^{35}C_2$

$$= \frac{5!}{3!(5-3)!} \times \frac{35!}{2!\times(35-2)!} = \frac{5\times4\times3!}{3!\times2\times1} \times \frac{35\times34\times33!}{2\times1\times33!}$$
$$= 5950.$$

(*d*) There are five defective floppy disks out of which at least 1 must be selected. We know that the total number of ways to select 5 floppy disks out of total 40 disks is $= {}^{40}C_5$.

Also, the number of ways to select 5 floppy disks with number one defective is

$$= {}^{35}C_5.$$

Therefore, the total number of ways to select 5 floppy disks out of which at least one is defective is

$$= {}^{40}C_5 - {}^{35}C_5 = 611625.$$

Problem 13. *Seven members of a family have total Rs. 2886 in their pockets. Show that at least one of them must have at least Rs. 416 in his pocket.*

Sol. Let us assume the members be the pigeonholes and the Rupees the pigeons. Now 2886 pigeons are to be assigned to 7 pigeonholes. Using the extended pigeonhole principle, where $n = 2886$ and $m = 7$, we have $[(2886 - 1)/7] + 1 = 416$. Hence, there are 416 Rupees in one member's pocket.

Problem 14. *How many people must you have to guarantee that at least 9 of them will have birthdays in the same day of the week.*

Sol. Let us assume the days of week the pigeonholes and the people the pigeons. Now we have 7 pigeonholes and we have to find pigeons. Using the extended pigeonhole principle, we have

$$[(n-1)/7] + 1 = 9$$
$$[(n-1)/7] = 9 - 1 = 8$$
$$n - 1 = 8 \times 7$$
$$n = 56 + 1 = 57.$$

Thus, there must be 57 people to guarantee that at least 9 of them will have birthdays in the same day of the week.

12

Recurrence Relations and Generating Functions

DEFINITION

A recurrence relation is a functional relation between the independent variable x, dependent variable $f(x)$ and the differences of various order of $f(x)$. A recurrence relation is also called a difference equation and we will use these two terms interchangeably.

For example : The equation $f(x + 3h) + 3f(x + 2h) + 6f(x + h) + 9f(x) = 0$ is a recurrence relation.

$y = f(x+3h) = a$

It can also be written as

$$a_{r+3} + 3a_{r+2} + 6a_{r+1} + 9a_r = 0$$

or

$$y_{k+3} + 3y_{k+2} + 6y_{k+1} + 9y_k = 0.$$

For example : The Fibonacci sequence is defined by the recurrence relation $a_r = a_{r-2} + a_{r-1}$, $r \geq 2$, with the initial conditions $a_0 = 1$ and $a_1 = 1$.

ORDER OF THE RECURRENCE RELATION

The order of the recurrence relation or difference equation is defined to be the difference between the highest and lowest subscripts of $f(x)$ or a_r or y_k.

For example : The equation $13a_r + 20a_{r-1} = 0$ is a first order recurrence relation.

For example : The equation $8f(x) + 4f(x + 1) + 8f(x + 2) = k(x)$ is a second order difference equation.

DEGREE OF THE DIFFERENCE EQUATION

The degree of a difference equation is defined to be the highest power of $f(x)$ or a_r or y_k.

For example : The equation $y_{k+3}^3 + 2y_{k+2}^2 + 2y_{k+1} = 0$ has the degree 3, as the highest power of y_k is 3.

For example : The equation $a_r^4 + 3a_{r-1}^3 + 6a_{r-2}^2 + 4a_{r-3} = 0$ has the degree 4, as the highest power of a_r is 4.

For example : The equation $y_{k+3} + 2y_{k+2} + 4y_{k+1} + 2y_k = k(x)$ has the degree 1, because the highest power of y_k is 1 and its order is 3.

For example : The equation $f(x + 2h) - 4f(x + h) + 2f(x) = 0$ has the degree 1 and its order is 2.

LINEAR RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

A recurrence relation is called linear if its degree is one.

The general form of linear recurrence relation with constant coefficient is

$$C_0 y_{n+r} + C_1 y_{n+r-1} + C_2 y_{n+r-2} + \ldots + C_r y_n = R(n)$$

where $C_0, C_1, C_2, \ldots, C_n$ are constants and $R(n)$ is same function of independent variable n.

A solution of a recurrence relation in any function which satisfies the given equation.

LINEAR HOMOGENEOUS RECURRENCE RELATION WITH CONSTANT COEFFICIENTS

The equation is said to be linear homogeneous difference equation if and only if $R(n) = 0$ and it will be of order n.

The equation is said to be linear non-homogeneous difference equation if $R(n) \neq 0$.

For example : The equation $a_{r+3} + 6a_{r+2} + 12a_{r+1} + 8a_r = 0$ is a linear homogeneous equation of order 3. $2, 2^2, 2^3$

For example : The equation $a_{r+2} - 4a_{r+1} + 4a_r = 3r + 2^r$ is a linear non-homogeneous equation of order 2.

A linear homogeneous difference equation with constant coefficients is given by

$$C_0 y_n + C_1 y_{n-1} + C_2 y_{n-2} + \ldots + C_r y_{n-r} = 0 \qquad \ldots(i)$$

where $C_0, C_1, C_2, \ldots, C_r$ are constants.

The solution of the equation (i) is of the form $A\alpha_1^K$, where α_1 is the characteristic root and A is constant.

Substitute the values of $A\alpha^K$ for y_n in equation (i), we have

$$C_0 A\alpha^K + C_1 A\alpha^{K-1} + C_2 A\alpha^{K-2} + \ldots + C_r A\alpha^{K-r} = 0 \qquad \ldots(ii)$$

After simplifying equation (ii), we have

$$C_0 \alpha^r + C_1 \alpha^{r-1} + C_2 \alpha^{r-2} + \ldots + C_r = 0 \qquad \ldots(iii)$$

The equation (iii) is called the characteristic equation of the difference equation.

If α_1 is one of the roots of the characteristic equation, then $A\alpha_1^K$ is a homogeneous solution to the difference equation.

To find the solution of the linear homogeneous difference equations, we have the four cases, that are discussed as follows.

Case I. If the characteristic equation has n distinct real roots $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$.

Thus, $\alpha_1^K, \alpha_2^K, \ldots\ldots, \alpha_n^K$ are all solutions of equation (i).

Also, we have, $A_1 \alpha_1^K, A_2 \alpha_1^K, \ldots\ldots, A_n \alpha_n^K$ are all solutions of equation (i). The sums of solutions are also solutions.

Hence, the homogeneous solution of the difference equation is

$$y_K = A_1\alpha_1^K + A_2\alpha_2^K + \ldots + A_n \alpha_n^K.$$

Case II. If the characteristic equation has repeated real roots.

If $\alpha_1 = \alpha_2$, then $(A_1 + A_2 K)\,\alpha_1^K$ is also a solution.

If $\alpha_1 = \alpha_2 = \alpha_3$, then $(A_1 + A_2 K + A_3 K^2)\,\alpha_1^K$ is also a solution.

Similarly, if root α_1 is repeated n times, then

$$(A_1 + A_2 K + A_3 K^2 + \ldots + A_n K^{n-1})\,\alpha_1^K$$

the solution of the homogeneous equation.

Case III. If the characteristic equation has one imaginary root.

If $\alpha + i\beta$ is the root of the characteristic equation, then $\alpha - i\beta$ is also the root, where α and β are real.

Thus, $(\alpha + i\beta)^K$ and $(\alpha - i\beta)^K$ are solutions of the equations. This implies

$$(\alpha + i\beta)^K A_1 + (\alpha - i\beta)^K A_2$$

is also a solution of the characteristic equation, where A_1 and A_2 are constants which are to be determined.

Case IV. If the characteristic equation has repeated imaginary roots.

When the characteristic equation has repeated imaginary roots,

$$(C_1 + C_2 K)(\alpha + i\beta)^K + (C_3 + C_4 K)(\alpha - i\beta)^K$$

is the solution of the homogeneous equation.

Example 1. *Solve the difference equation $a_r - 3a_{r-1} + 2a_{r-2} = 0$.*

Sol. The characteristic equation is given by

$$s^2 - 3s + 2 = 0 \quad \text{or} \quad (s - 1)(s - 2) = 0$$

$\Rightarrow \quad s = 1, 2$

Therefore, the homogeneous solution of the equation is

$$a_r = C_1 + C_2 \cdot 2^r.$$

Example 2. *Solve the difference equation $a_r - 6a_{r-1} + 8a_{r-2} = 0$.*

Sol. The characteristic equation is

$$s^2 - 6s + 8 = 0 \quad \text{or} \quad (s - 2)(s - 4) = 0$$

$\Rightarrow \quad s = 2, 4$

Therefore, the homogeneous solution of the equation is

$$a_r = C_1 \cdot 2^r + C_2 \cdot 4^r.$$

Example 3. *Solve the difference equation $9y_{K+2} - 6y_{K+1} + y_K = 0$.*

Sol. The characteristic equation is

$$9s^2 - 6s + 1 = 0 \quad \text{or} \quad (3s - 1)^2 = 0$$

$\Rightarrow \quad s = \frac{1}{3}$ and $\frac{1}{3}$

Therefore, the homogeneous solution is given by

$$y_K = (C_1 + C_2 K) \cdot \left(\frac{1}{3}\right)^K.$$

$(C_1 + C_2 t)\left(\frac{1}{3}\right)^K$

Example 4. *Solve the difference equation $a_r - 4a_{r-1} + 4a_{r-2} = 0$.*

Sol. The characteristic equation is given by

$$s^2 - 4s + 4 = 0 \quad \text{or} \quad (s - 2)^2 = 0$$

$\Rightarrow \quad s = 2$ and 2

Therefore, the homogeneous solution of equation is $a_r = (C_1 + C_2 r) \cdot 2^r$.

Example 5. *Solve the difference equation $a_r + a_{r-1} + a_{r-2} = 0$.*

Sol. The characteristic equation is $s^2 + s + 1 = 0$

The roots of this characteristic equation are imaginary, *i.e.*, $s = \dfrac{-1+i\sqrt{3}}{2}$ and $\dfrac{-1-i\sqrt{3}}{2}$.

Therefore, the homogeneous solution of the equation is

$$a_r = \left[\frac{-1+i\sqrt{3}}{2}\right]^r C_1 + \left[\frac{-1-i\sqrt{3}}{2}\right]^r C_2.$$

Example 6. *Solve the difference equation* $y_K - y_{K-1} - y_{K-2} = 0$.

Sol. The characteristic equation is $s^2 - s - 1 = 0$

$$s = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Therefore, the homogeneous solution of the equation is

$$y_K = C_1 \left[\frac{1+\sqrt{5}}{2}\right]^K + C_2 \left[\frac{1-\sqrt{5}}{2}\right]^K.$$

Example 7. *Solve the difference equation* $a_{r+4} + 2a_{r+3} + 3a_{r+2} + 2a_{r+1} + a_r = 0$.

Sol. The characteristic equation is $s^4 + 2s^3 + 3s^2 + 2s + 1 = 0$

$$(s^2 + s + 1)(s^2 + s + 1) = 0$$

$$s = \frac{-1 \pm i\sqrt{3}}{2}, \frac{-1 \pm i\sqrt{3}}{2}$$

Therefore, the homogeneous solution is given by

$$a_r = (C_1 + C_2 r)\left(\frac{-1+i\sqrt{3}}{2}\right)^r + (C_3 + C_4 r)\left(\frac{-1-i\sqrt{3}}{2}\right)^r.$$

Example 8. *Solve the difference equation* $y_{K+4} + 4y_{K+3} + 8y_{K+2} + 8y_{K+1} + 4y_K = 0$.

Sol. The characteristic equation is $s^4 + 4s^3 + 8s^2 + 8s + 4 = 0$

$$(s^2 + 2s + 2)(s^2 + 2s + 2) = 0$$

$$s = -1 \pm i, -1 \pm i$$

Therefore, the homogeneous solution is given by

$$y_K = (C_1 + C_2 K)(-1+i)^K + (C_3 + C_4 K)(-1-i)^K.$$

PARTICULAR SOLUTION

(*a*) Homogeneous Linear Difference Equations

We can find the particular solution of the difference equation, when the equation is of homogeneous linear type by putting the values of the initial conditions in the homogeneous solution.

Example 9. *Solve the difference equation* $2a_r - 5a_{r-1} + 2a_{r-2} = 0$ *and find particular solution such that* $a_0 = 0$ *and* $a_1 = 1$.

Sol. The characteristic equation is $2s^2 - 5s + 2 = 0$

$$(2s - 1)(s - 2) = 0$$

or $$s = \frac{1}{2} \text{ and } 2.$$

Therefore, the homogeneous solution is given by

$$a_{r(h)} = C_1 \left(\frac{1}{2}\right)^r + C_2 \cdot 2^r \quad \text{...}(i)$$

Putting $r = 0$ and $r = 1$ in equation (i), we get

$$a_0 = C_1 + C_2 = 0 \quad \text{...}(a)$$

$$a_1 = \frac{1}{2} C_1 + 2C_2 = 1 \quad \text{...}(b)$$

Solving eq. (a) and (b), we have

$$C_1 = -\frac{2}{3} \text{ and } C_2 = \frac{2}{3}$$

Hence, the particular solution is

$$a_{r(P)} = -\frac{2}{3} \cdot \left(\frac{1}{2}\right)^r + \frac{2}{3} \cdot (2)^r.$$

Example 10. *Solve the difference equation $a_r - 4a_{r-1} + 4a_{r-2} = 0$ and find the particular solution, given that $a_0 = 1$ and $a_1 = 6$.*

Sol. The characteristic equation is

$$s^2 - 4s + 4 = 0 \text{ or } (s-2)^2 = 0$$

or $$s = 2, 2$$

Therefore, the homogeneous solution is given by

$$a_{r(n)} = (C_1 + C_2 r) \cdot 2^r \quad \text{...}(i)$$

Putting $r = 0$ and $r = 1$ in equation (i), we get

$$a_0 = (C_1 + 0) \cdot 2^0 = 1 \quad \therefore \quad C_1 = 1$$

$$a_1 = (C_1 + C_2) \cdot 2 = 6 \quad \therefore \quad C_1 + C_2 = 3 \Rightarrow C_2 = 2$$

Hence, the particular solution is $a_{r(P)} = (1 + 2r) \cdot 2^r$.

Example 11. *Solve the difference equation $9a_r - 6a_{r-1} + a_{r-2} = 0$ satisfying the conditions $a_0 = 0$ and $a_1 = 2$.*

Sol. The characteristic equation is

$$9s^2 - 6s + 1 = 0 \text{ or } (3s-1)^2 = 0$$

or $$s = \frac{1}{3}, \frac{1}{3}$$

Therefore, the homogeneous solution is given by

$$a_{r(h)} = (C_1 + C_2 r) \cdot \left(\frac{1}{3}\right)^r \quad \text{...}(i)$$

Putting $r = 0$ and $r = 1$ in equation (i), we get

$$a_0 = C_1 = 0 \quad \text{...}(a)$$

$$a_1 = (C_1 + C_2) \cdot \frac{1}{3} = 2. \quad \therefore \quad C_1 + C_2 = 6 \Rightarrow C_2 = 6$$

Hence, the particular solution is

504165

$$a_{r(P)} = 6r \cdot \left(\frac{1}{3}\right)^r.$$

Example 12. *Solve the difference equation $a_r - 7a_{r-1} + 10a_{r-2} = 0$ satisfying the conditions $a_0 = 0$ and $a_1 = 6$.*

Sol. The characteristic equation is

$$s^2 - 7s + 10 = 0 \quad \text{or} \quad (s-5)(s-2) = 0 \quad \text{or } s = 2, 5$$

Therefore, the homogeneous solution is given by

$$a_{r(h)} = C_1 \cdot 2^r + C_2 \cdot 5^r. \qquad ...(i)$$

Putting $r = 0$ and $r = 1$ in equation (i), we get

$$a_0 = C_1 + C_2 = 0 \qquad ...(a)$$

$$a_1 = 2C_1 + 5C_2 = 6 \qquad ...(b)$$

From eq. (a) $\quad C_1 = -C_2$

From eq. (b) $\quad 3C_2 = 6 \quad \therefore \quad C_2 = 2$ and $C_1 = -2$.

Hence, the particular solution is $a_{r(P)} = -2 \cdot 2^r + 2 \cdot 5^r$.

(*b*) **Non-Homogeneous Linear Difference Equations.** There are two methods to find the particular solution of non-homogeneous linear difference equations. These are as follows :

1. Undetermined coefficients method
2. E and Δ operator method.

1. Undetermined Coefficients Method. This method is used to find particular solution of non-homogeneous linear difference equations, whose R.H.S. term R(n) consists of terms of special forms.

In this method, firstly we assume the general form of the particular solution according to the form of R(n) containing a number of unknown constant coefficients, which have to be determined. Then according to the difference equation, we will determine the exact solution.

The general form of particular solution to be assumed for the special forms of R(n), to find the exact solution is shown in the table.

Form of R(n)	General form to be assumed
Z, here z is a constant	A
Z^r, here Z is a constant	$A \cdot Z^r$
P(r), a polynomial of degree n	$A_0 r^n + A_1 r^{n-1} + ... + A_n$
Z^r. P(r), here P(r) is a polynomial of nth degree in r. Z is a constant.	$[A_0 r^n + A_1 r^{n-1} + ... + A_n] \cdot Z^r$

Example 13. *Find the particular solution of the difference equation*

$$a_{r+2} - 3a_{r+1} + 2a_r = Z^r, \qquad ...(i)$$

where Z is some constant.

Sol. The general form of solution is $= A \cdot Z^r$

Now putting this solution on L.H.S. of equation (i), we get

$$= AZ^{r+2} - 3AZ^{r+1} + 2AZ^r = (Z^2 - 3Z + 2)AZ^r \qquad ...(ii)$$

Equating equation (*ii*) with R.H.S. of equation (*i*), we get

$$(Z^2 - 3Z + 2)A = 1$$

or

$$A = \frac{1}{Z^2 - 3Z + 2} = \frac{1}{(Z-1)(Z-2)} \quad (Z \neq 1, Z \neq 2)$$

Therefore, the particular solution is $\frac{Z^r}{(Z-1)(Z-2)}$.

Example 14. *Find the particular solution of the difference equation*

$$a_{r+2} - 5a_{r+1} + 6a_r = 5^r.$$

Sol. Let us assume the general form of the solution = $A \,.\, 5^r$. ...(*i*)

Now to find the value of A, put this solution on L.H.S. of the equation (*i*), then this becomes

$$= A.\, 5^{r+2} - 5 \,.\, A5^{r+1} + 6 \,.\, A5^r$$
$$= 25A \,.\, 5^r - 25A \,.\, 5^r + 6A \,.\, 5^r$$

ar = ar(h) + ar(p)

$$= 6A \,.\, 5^r \qquad \text{...(ii)}$$

= 5^r

Equating equation (*ii*) to R.H.S. of equation (*i*), we get

6A = 1

$$A = \frac{1}{6}$$

Therefore, the particular solution of the difference equation is = $\frac{1}{6} . 5^r$.

Example 15. *Find the particular solution of the difference equation*

$$a_{r+2} - 4a_r = r^2 + r - 1. \qquad \text{...(i)}$$

Sol. The homogeneous solution of the difference equation is given by

$$a_{r(h)} = C_1(2)^r + C_2(-2)^r$$

To find the particular solution, let us assume the general form of the solution is

$$= A_1 r^2 + A_2 r + A_3.$$

Putting this solution in L.H.S. of equation (*i*), we get

$$= A_1(r+2)^2 + A_2(r+2) + A_3 - 4A_1 r^2 - 4A_2 r - 4A_3$$
$$= -3A_1 r^2 + (4A_1 - 3A_2)r + (4A_1 + 2A_2 - 3A_3) \qquad \text{...(ii)}$$

Equating equation (*ii*) with R.H.S. of equation (*i*), we get

$$-3A_1 = 1$$
$$4A_1 - 3A_2 = 1$$
$$4A_1 + 2A_2 - 3A_3 = -1$$

After solving these three equations, we get

$$A_1 = -\frac{1}{3},\ A_2 = -\frac{7}{9},\ A_3 = -\frac{17}{27}$$

Therefore, the particular solution is $= -\frac{r^2}{3} - \frac{7}{9}r - \frac{17}{27}$.

Example 16. *Find the particular solution of the difference equation*

$$a_{r+2} - 2a_{r+1} + a_r = 3r + 5. \qquad \text{...(i)}$$

Sol. The homogeneous solution of the difference equation is given by

$$a_{r(h)} = C_1 + C_2 r \qquad \text{...(ii)}$$

Corresponding to the term $3r + 5$, we assume the general form of the solution as $A_1 r + A_2$ but due to occurrence of these terms in equation (*ii*), we multiply this by suitable power of r so that none of the term will occur in equation (*ii*). Thus multiply by r^2.

Hence, the general form of the solution becomes

$$= A_1 r^3 + A_2 r^2 \ldots\ldots$$

Putting this solution in L.H.S. of equation (*i*), we get

$$= A_1(r+2)^3 + A_2(r+2)^2 - 2A_1(r+1)^3 - 2A_2(r+1)^2 + A_1 r^3 + A_2 r^2$$

$$= A_1(r^3 + 8 + 6r^2 + 12r) + A_2(r^2 + 4 + 4r) - 2A_1(r^3 + 1 + 3r^2 + 3r)$$
$$- 2A_2(r^2 + 1 + 2r) + A_1 r^3 + A_2 r^2$$

$$= (12A_1 + 4A_2 - 6A_1 - 4A_2)r + (8A_1 + 4A_2 - 2A_1 - 2A_2)$$

$$= (6A_1)r + (6A_1 + 2A_2) \qquad \ldots(iii)$$

Equating equation (*iii*) with R.H.S. of equation (*i*), we get

$$6A_1 = 3 \quad \therefore \quad A_1 = \frac{1}{2}$$

$$6A_1 + 2A_2 = 5 \quad \therefore \quad A_2 = 1$$

Therefore, the particular solution is $\frac{1}{2}r^3 + r^2$.

Example 17. *Find the particular solution of the difference equation*

$$a_{r+2} + a_{r+1} + a_r = r \,.\, 2^r. \qquad \ldots(i)$$

Sol. Let us assume the general form of the solution $= (A_0 + A_1 r) \,.\, 2^r$

Now, put this solution in the L.H.S. of equation (*i*), we get

$$= 2^{r+2}\,[A_0 + A_1(r+2)] + 2^{r+1}\,[A_0 + A_1(r+1)] + 2^r(A_0 + A_1 r)$$

$$= 4 .\, 2^r(A_0 + A_1 r + 2A_1) + 2 \,.\, 2^r(A_0 + A_1 r + A_1) + 2^r(A_0 + A_1 r)$$

$$= r .\, 2^r(7A_1) + 2^r(7A_0 + 10A_1) \qquad \ldots(ii)$$

Equating equation (*ii*) with R.H.S. of equation (*i*), we get

$$7A_1 = 1 \quad \therefore \quad A_1 = \frac{1}{7}$$

$$7A_0 + 10A_1 = 0 \quad \therefore \quad A_0 = \frac{-10}{49}$$

Therefore, the particular solution is $2^r\left(\frac{-10}{49} + \frac{1}{7}r\right)$.

Example 18. *Find the particular solution of the difference equation*

$$a_r - 4a_{r-1} + 4a_{r-2} = (r+1) \,.\, 2^r. \qquad \ldots(i)$$

Sol. The homogeneous solution of the difference equation is given by

$$a_{r(h)} = (C_1 + C_2 r) \,.\, 2^r \qquad \ldots(ii)$$

because it has two real and equal roots *i.e.*, 2 and 2.

To find the particular solution, let us assume the general form of the solution is $= 2^r(A_1 r + A_0)$, but due to occurrence of there terms in equation (*ii*), we multiply this by suitable power of r so that none of the terms will occur in equation (*ii*). Thus multiply by r^2.

Hence, the general form of the solution becomes $= 2^r(A_1 r + A_0) \,.\, r^2$

Putting this solution in L.H.S. of equation (*i*), we get

$= 2^r.(A_1r + A_0).r^2 - 4.2^{r-1}[A_1(r-2) + A_0].(r-1)^2 + 4.2^{r-2}[A_1(r-2) + A_0].(r-2)^2$

$= 2^r.(A_1r + A_0).r^2 - 2(r^2 + 1 - 2r).2^r(A_1r - A_1 + A_0) + (r^2 + 4 - 4r).2^r.(A_1r - 2A_1 + A_0)$

$= r.2^r(6A_1) + 2^r(-6A_1 + 2A_0)$ ...(*iii*)

Equating equation (*iii*) with R.H.S. of equation (*i*), we get

$$6A_1 = 1 \quad \therefore \quad A_1 = \frac{1}{6}$$

$$-6A_1 + 2A_0 = 1 \quad \therefore \quad A_0 = 1$$

Therefore, the particular solution is $= r^2.2^r\left(\frac{r}{6} + 1\right)$.

2. E and Δ Operator Method. Before discussing this method to find the particular solution of the non-homogeneous linear difference equation, we will discuss the meaning of the operator E and Δ.

Meaning of Operator E. The operation of E on $f(x)$ means that give an increment to the value of x in the function. The operation of E is, put $(x + h)$ in the function wherever there is x. Here h is increment quantity. So

$$Ef(x) = f(x + h)$$

$E^2 f(n)$

Here, E is operated on $f(x)$, therefore, E is a symbol known as shift operator.

Meaning of Operator Δ. The operator Δ is an operation of two steps.

Firstly, x in the function is incremented by a constant and then former is subtracted from the later *i.e.*,

$$\Delta f(x) = f(x + h) - f(x).$$

$E f(n) - f(n)$

$(E-1) f(n)$

Theorem I. *Prove that* $E \equiv 1 + \Delta$.

Proof. The operation of Δ on $f(x)$ is of two steps. First, increment the value of x in the function. So, whenever, there is x in $f(x)$ put $x + h$ (here h is constant increment), which means operation of E on $f(x)$ *i.e.*,

$$f(x + h) = Ef(x).$$

Second, subtract the original function from the value obtained in first step, hence

$$\Delta f(x) = Ef(x) - 1 f(x) = (E - 1) f(x)$$

[Where E and I are operators and 1 is an identity operator]

So, the operation of Δ on $f(x)$ is equivalent to the operation of $(E - 1)$ on $f(x)$.

Therefore, we have

$$E \equiv 1 + \Delta.$$

Theorem II. *Show that* $E^n f(x) = f(x + nh)$.

Proof. We know that $Ef(x) = f(x + h)$

Now $E^n f(x) = E.E.E.E. \ldots n \text{ times } f(x)$

$= E^{n-1}[Ef(x)] = E^{n-1} f(x + h)$

$= E^{n-2}[E f(x + h)] = E^{n-2} f(x + 2h)$

....

....

$= E f[x + (n-1)h] = f(x + nh)$.

Theorem III. *Show that* $E\,Cf(x) = CE\,f(x)$.

Proof. We know that $E\,Cf(x) = Cf(x+h) = CE\,f(x+h)$. Hence proved.

There is no effect of operation of E on any constant. Therefore, the operation of E on any constant will be equal to the constant itself.

By E and Δ operator method, we will find the solution of

$$C_0 y_{n+r} + C_1 y_{n+r-1} + C_2 y_{n+r-2} + \ldots\ldots + C_n y_n = R(n) \qquad \ldots(i)$$

Equation (*i*) can be written as

$$C_0 E^r y_n + C_1 E^{r-1} y_n + C_2 E^{r-2} y_n + \ldots + C_n y_n = R(n)$$

$$(C_0 E^r + C_1 E^{r-1} + C_2 E^{r-2} + \ldots\ldots + C_n) y_n = R(n)$$

Putting $\quad C_0 E^r + C_1 E^{r-1} + C_2 E^{r-2} + \ldots + C_n = P(E)$

So $\quad P(E)\, y_n = R(n)$

$$\therefore \quad y_n = \frac{R(n)}{P(E)} \qquad \ldots(ii)$$

To find the particular solution of (*ii*) for different forms of R(*n*), we have the following cases.

Case I. When R(*n*) is some constant A

We know that, the operation of E on any constant will be equal to the constant itself *i.e.*,

$$EA = A$$

Therefore,
$$P(E)\,A = (C_0 E^r + C_1 E^{r-1} + C_2 E^{r-2} + \ldots + C_n)A$$
$$= (C_0 + C_1 + C_2 + \ldots + C_n)A$$
$$= P(1)A$$

$$\frac{A}{P(E)} = \frac{1}{P(1)}\,.\,A$$

Therefore, using equation (*ii*), the particular solution of (*i*) is

$$y_n = \frac{A}{P(1)},\ P(1) \neq 0$$

P(1) is obtained by putting E = 1 in P(E).

Case II. When R(*n*) is of the form $A\,.\,Z^n$, where A, and Z are constants

We have, $\quad P(E)\,(A\,.\,Z^n) = \{C_0 E^r + C_1 E^{r-1} + \ldots + C_n\}(A\,.\,Z^n)$

$$= A\,\{C_0 Z^{r+n} + C_1 Z^{r+n-1} + \ldots + C_n Z^n\}$$
$$= A\,\{C_0 Z^r + C_1 Z^{r-1} + \ldots + C_n\}.\ Z^n$$
$$= AP(Z)\,.\,Z^n$$

To get, P(Z) put E = Z in P(E)

Therefore, $\dfrac{A\,.\,Z^n}{P(E)} = \dfrac{A\,.\,Z^n}{P(Z)}$, provided $P(Z) \neq 0$

Thus, $$y_n = \frac{A\,.\,Z^n}{P(Z)},\ P(Z) \neq 0$$

If A = 1, then $$y_n = \frac{Z^n}{P(Z)}$$

When P(Z) = 0 then for equation (*i*)

$$(E - Z)\, y_n = A\,.\,Z^n$$

For this, the particular solution becomes $= A.\ \dfrac{1}{(E-Z)}.\ Z^n = A\,.\,n\,Z^{n-1}$

(*ii*) $\quad (E - Z)^2 y_n = A \,.\, Z^n$

For this, the particular solution becomes $= A.\dfrac{1}{(E-Z)^2}.Z^n = \dfrac{A\,.\,n(n-1)}{2!}.Z^{n-2}$

(*iii*) $\quad (E - Z)^3 y_n = A \,.\, Z^n$

For this, the particular solution becomes $= A.\dfrac{1}{(E-Z)^3}.Z^n = \dfrac{A\,.\,n(n-1)(n-2)}{3!}.Z^{n-3}$ and so on.

Case III. When $R(n)$ be a polynomial of degree m in n

We know that $\quad E \equiv 1 + \Delta$

So, $\quad P(E) = P(1 + \Delta)$

or $$\frac{1}{P(E)} = \frac{1}{P(1+\Delta)},$$

which can be expanded in ascending power of Δ as far as upto Δ^m

$$\Rightarrow \quad \frac{1}{P(E)} = \frac{1}{P(1+\Delta)} = (b_0 + b_1\Delta + b_2\Delta + \ldots + b_m\,\Delta^m + \ldots)$$

$$\Rightarrow \quad \frac{1}{P(E)}.\,R(n) = (b_0 + b_1\Delta + b_2\Delta + \ldots + b_m\Delta^m + \ldots)\,.\,R(n)$$

$$= b_0\,R(n) + b_1\Delta\,R(n) + \ldots + b_m\,\Delta^m\,R(n)$$

All other higher terms will be zero because $R(n)$ is a polynomial of degree m.

Thus, the particular solution of equation (*i*), in this case will be

$$y_n = b_0\,R(n) + b_1\,\Delta\,R(n) + \ldots + b_m\,\Delta^m\,R(n).$$

Case IV. When $R(n)$ is of the form $R(n)\,.\,Z^n$, where $R(n)$ is a polynomial of degree m and Z is some constant

We have $\quad E^r\,[Z^n\,R(n)] = Z^{r+n}\,R(n+r) = Z^r.\,Z^n.\,E^r.\,R(n) = Z^n\,(ZE)^r\,R(n)$

Similarly, we have

$$\frac{1}{P(E)}\,[Z^n\,R(n)] = Z^n\,\frac{1}{P(ZE)}.\,(R(n)) = Z^n\,[P(Z + Z\Delta)]^{-1}.\,R(n)$$

Thus, the particular solution of equation (*i*), in this case will be

$$y_n = Z^n\,[P(Z + Z\Delta)]^{-1}\,R(n).$$

Example 19. *Find the particular solution of the difference equation*

$$2a_{r+1} - a_r = 12.$$

Sol. The above equation can be written as

$$(2E - 1)\,a_r = 12$$

The particular solution is given by

$$a_r = \frac{1}{(2E-1)}.\,12$$

Put $E = 1$, in the equation. The particular solution is $a_r = 12$.

Example 20. *Find the particular solution of difference equation* $a_r - 3a_{r-1} + 2a_{r-2} = 2^r$

Sol. The above equation can be written as

$$(E^2 - 3E + 2)\,a_r = 2^r$$

Therefore, $\quad P(E) = E^2 - 3E + 2$ or $P(E) = (E - 2)(E - 1)$

Thus, the particular solution is given by

$$a_r = \frac{1}{(E-2)(E-1)} \cdot 2^r$$

$$= r.\ \frac{1}{(E-1)} \cdot 2^{r-1} = r.\ \frac{1}{(2-1)} \cdot 2^{r-1}$$

$$a_r = r.\ 2^{r-1}.$$

Example 21. *Find the particular solution of difference equation*

$$a_r - 4a_{r-1} + 4a_{r-2} = 2^r.$$

Sol. The above equation can be written as

$$(E^2 - 4E + 4)\ a_r = 2^r$$

Therefore, $P(E) = E^2 - 4E + 4 = (E-2)^2$

Thus, the particular solution is given by

$$a_r = \frac{1}{(E-2)^2} \cdot 2^r = \frac{r(r-1)}{2!} \cdot 2^{r-2}$$

$$a_r = r(r-1).\ 2^{r-3}.$$

Example 22. *Find the particular solution of the difference equation*

$$a_r - 2a_{r-1} = 7r^2.$$

Sol. The above equation can be written as $(E-2)\ a_r = 7r^2$

Thus, the particular solution is given by

$$a_r = \frac{1}{(E-2)} \cdot 7r^2 = \frac{1}{(\Delta-1)} \cdot 7r^2 = -7 \cdot \frac{1}{(1-\Delta)} \cdot r^2$$

$$= -7(1-\Delta)^{-1}.\ (r^2) = -7(1+\Delta+\Delta^2)\ (r^2)$$

By factorial notation, $[r]^n = r(r-1)(r-2) \ldots (r-n+1)$.

Hence $(r^2) = [r]^2 + [r]$

$$a_r = -7(1+\Delta+\Delta^2)\ ([r]^2 + [r]) = -7([r]^2 + [r] + 2[r] + 1 + 2 + 0)$$

The operation of Δ on $[r]^n$ is analogous to its differentiation

$$a_r = -7[r]^2 - 21[r] - 21 = -7r(r-1) - 21r - 21$$

$$a_r = -7r^2 - 14r - 21.$$

Example 23. *Find the particular solution of the difference equation*

$$a_r - 2a_{r-1} = 7r.$$

Sol. The above equation can be written as $(E-2)\ a_r = 7r$

Thus, the particular solution is given by

$$a_r = \frac{1}{(E-2)} \cdot 7r = 7.\ \frac{1}{(\Delta-1)} \cdot r = -7\ \frac{1}{(1-\Delta)}\ [r]$$

(By factorial notation $r = [r]$)

$$= -7(1-\Delta)^{-1}\ [r] = -7(1+\Delta+\Delta^2+\ldots)\ [r]$$

By neglecting higher powers

$$a_r = -7(1+\Delta)\ [r] = -7([r]+1)$$

$$a_r = -7(r+1).$$

Example 24. *Find the particular solution of the difference equation*

$a_r + a_{r-1} + a_{r-2} = r.\, 2^r.$

Sol. The above equation can be written as $(E^2 + E + 1)\, a_r = r \,.\, 2^r$

Thus, the particular solution is given by

$$a_r = \frac{1}{(E^2 + E + 1)} \,.\, r.\, 2^r = 2^r \frac{1}{(2E)^2 + 2E + 1} \,.\, r$$

$$= 2^r\, (4(1+\Delta)^2 + 2(1+\Delta) + 1)^{-1}.\, r = 2^r\, (7 + 10\Delta + 4\Delta^2)^{-1}.\, r$$

$$= \frac{2^r}{7}\left(1 + \frac{10\Delta + 4\Delta^2}{7}\right)^{-1} .\, [r] = \frac{2^r}{7}\left(1 - \frac{10\Delta + 4\Delta^2}{7}\right).\, [r] = \frac{2^r}{7}\left([r] - \frac{10}{7}\right)$$

$$a_r = \frac{2^r}{7}\left(r - \frac{10}{7}\right).$$

TOTAL SOLUTION

The total solution or the general solution of a non-homogeneous linear difference equation with constant coefficients is the sum of the homogeneous solution and particular solution. If no initial conditions are given, then our work is finished. If n initial conditions are given, obtain n linear equations in n unknowns and solve them, if possible to get a total solution.

As we already know that the homogeneous solution of the recurrence relation is obtained by putting the right hand side of the equation equal to zero and the particular solution is obtained with $R(n)$ on the right hand side of the equation.

If $y_{(h)}$ denotes the homogeneous solution of the recurrence relation and $y_{(p)}$ denotes the particular solution of the recurrence relation then, the total solution or the general solution y of the recurrence relation is given by

$$y = y_{(h)} + y_{(p)}.$$

Example 25. *Solve the difference equation*

$$a_r - 4a_{r-1} + 4a_{r-2} = 3r + 2^r. \qquad ...(i)$$

Sol. The homogeneous solution of this equation is obtained by putting R.H.S. equal to zero *i.e.,*

$$a_r - 4a_{r-1} + 4a_{r-2} = 0$$

The homogeneous solution is $a_{r(h)} = (C_1 + C_2\, r) \,.\, 2^r$ as shown in example 4.

The equation (i) can be written as $(E^2 - 4E + 4)\, a_r = 3r + 2^r$

The particular solution is given as

$$a_{r(p)} = \frac{1}{(E^2 - 4E + 4)} .\, (3r + 2^r) = \frac{1}{(E-2)^2} \,.\, (3r) + \frac{1}{(E-2)^2} \,.\, 2^r$$

$$= 3 \,.\, \frac{1}{(1-\Delta)^2}\,(r) + \frac{r(r-1)}{2!} .\, 2^{r-2} = 3(1-\Delta)^{-2}\,(r) + \frac{r(r-1)}{2!} .\, 2^{r-2}$$

$$= 3(1 + 2\Delta) \,.\, [r] + \frac{r(r-1)}{2!} .\, 2^{r-2} = 3(r+2) + r(r-1) \,.\, 2^{r-3}$$

$$a_{r(p)} = 3(r+2) + r(r-1) \,.\, 2^{r-3}$$

Therefore, the total solution is $a_r = (C_1 + C_2\, r) \,.\, 2^r + 3(r+2) + r(r-1) \,.\, 2^{r-3}$.

Example 26. *Solve the difference equation*

$$a_r + 4a_{r-1} + 4a_{r-2} = r^2 - 3r + 5. \qquad ...(i)$$

Sol. Put the R.H.S. of the equation equal to zero *i.e.*,

$$a_r + 4a_{r-1} + 4a_{r-2} = 0$$

This can be written as

$$(E^2 + 4E + 4)\, a_r = 0$$

$$(E + 2)^2 = 0 \quad \text{or} \quad E = -2, -2$$

The homogeneous solution is given by

$$a_{r(h)} = (C_1 + C_2 r) \,.\, (-2)^r$$

The equation (*i*) can be written as

$$(E^2 + 4E + 4)\, a_r = r^2 - 3r + 5$$

The particular solution is given by

$$a_{r(p)} = \frac{1}{E^2 + 4E + 4} \,.\, [(r)^2 - 3(r) + 5] = \frac{1}{(E+2)^2}\,(r^2 - 3r + 5)$$

$$= \frac{1}{(3+\Delta)^2} \,.\, ([r]^2 - 2[r] + 5)$$

{By factorial notation, $[r]^2 = r(r-1)$ and $[r] = r$}

$$= \frac{1}{9} \cdot \frac{1}{\left(1+\frac{\Delta}{3}\right)^2} \,.\, ([r]^2 - 2[r] + 5)$$

$$= \frac{1}{9}\left(1+\frac{\Delta}{3}\right)^{-2} \,.\, ([r]^2 - 2[r] + 5)$$

$$= \frac{1}{9}\left(1-\frac{2\Delta}{3}+\frac{3\Delta^2}{9}\right) .\, ([r]^2 - 2[r] + 5)$$

$$= \frac{1}{9}\left([r]^2 - \frac{10}{3}[r] + 7\right)$$

$$a_{r(p)} = \frac{1}{9}\left(r^2 - \frac{13}{3} r + 7\right)$$

Therefore, the total solution is

$$a_r = a_{r(h)} + a_{r(p)}$$

$$a_r = (C_1 + C_2 r) \,.\, (-2)^r + \frac{1}{9}\left(r^2 - \frac{13}{3} r + 7\right).$$

Example 27. *Solve the difference equation*

$$a_r + a_{r-1} + a_{r-2} = r.\, 2^r. \qquad ...(i)$$

Sol. Put the R.H.S. of the equation to zero *i.e.*,

$$a_r + a_{r-1} + a_{r-2} = 0$$

The homogeneous solution is

$$a_{r(h)} = \left[\frac{-1+i\sqrt{3}}{2}\right]^r C_1 + \left[\frac{-1-i\sqrt{3}}{2}\right]^r C_2$$

shown in example 5.

The equation (i) can be written as $(E^2 + E + 1)\, a_r = r.\, 2^r$

The particular solution is given $a_{r(p)} = \dfrac{2^r}{7}\left(r - \dfrac{10}{7}\right)$

as shown in example.

Therefore, the total solution is

$$a_r = a_{r(h)} + a_{r(p)}$$

$$a_r = \left(\frac{-1+i\sqrt{3}}{2}\right)^r C_1 + \left(\frac{-1-i\sqrt{3}}{2}\right)^r C_2 + \frac{2^r}{7}\left(r - \frac{10}{7}\right).$$

Example 28. *Solve the difference equation*

$$a_r + 6a_{r-1} + 9a_{r-2} = 3 \qquad ...(i)$$

with initial conditions $a_0 = 0$ and $a_1 = 1$.

Sol. Put the R.H.S. of the equation (i) equal to zero *i.e.*,

$$a_r + 6a_{r-1} + 9a_{r-2} = 0$$

This can be written as $E^2 + 6E + 9 = 0$

or $(E + 3)(E + 3) = 0$

or $E = -3, -3$

The homogeneous solution is $a_{r(h)} = (C_1 + C_2\, r)\,.\,(-3)^r$

Now, we have $P(E) = (E + 3)(E + 3)$

Putting $E = 1$,we get $P(1) = 16$

Hence, the particular solution is $a_{r(p)} = \dfrac{3}{16}$

Therefore, the total solution is $a_r = (C_1 + C_2\, r)(-3)^r + \dfrac{3}{16}$

Now, putting $r = 0$, we have $a_0 = (C_1 + 0)(1) + \dfrac{3}{16} = 0$

$\therefore \qquad C_1 = -\dfrac{3}{16}$

Now, putting $r = 1$, we have $a_1 = (C_1 + C_2)(-3) + \dfrac{3}{16} = 1$

Substitute the value of $C_1 = -\dfrac{3}{16}$ in the equation, we have

$$\left(-\frac{3}{16} + C_2\right)(-3) + \frac{3}{16} = 1$$

$$\left(-\frac{3}{16} + C_2\right)(-3) = \frac{13}{16}$$

$$C_2 = -\frac{1}{12}$$

Therefore, the total solution satisfying initial conditions is

$$a_r = \left(-\frac{3}{16} - \frac{r}{12}\right)(-3)^r + \frac{3}{16}.$$

GENERATING FUNCTIONS

Generating functions is a method to solve the recurrence relations.

Let us consider, the sequence $a_0, a_1, a_2, \ldots, a_r, \ldots$ of real numbers. For some interval of real numbers containing zero whose value at t is given, the function $G(t)$ is defined by the series

$$G(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_r t^r + \ldots \qquad \ldots(i)$$

This function $G(t)$ is called the generating function of the sequence a_r.

Now, for the constant sequence 1, 1, 1, 1, ... the generating function is $1+t+t^2+t^3$

$$G(t) = \frac{1}{(1-t)}$$

because it can be expressed as

$$G(t) = (1-t)^{-1} = 1 + t + t^2 + t^3 + t^4 + \ldots \qquad \text{[By Binomial expansion]}$$

Comparing, this with equation (i), we get

$$a_0 = 1, a_1 = 1, a_2 = 1 \text{ and so on.}$$

For, the constant sequence 1, 2, 3, 4, 5, ... the generating function is

$$G(t) = \frac{1}{(1-t)^2} \text{ because it can be expressed as}$$

$$G(t) = (1-t)^{-2} = 1 + 2t + 3t^2 + 4t^3 + \ldots + (r+1)\, t^r$$

Comparing this with equation (i), we get

$$a_0 = 1, a_1 = 2, a_2, = 3, a_3 = 4 \text{ and so on.}$$

The generating function of Z^r, ($Z \neq 0$ and Z is a constant) is given by

$$G(t) = 1 + Zt + Z^2 t^2 + Z^3 t^3 + \ldots + Z^r t^r$$

$$G(t) = \frac{1}{(1-Zt)} \qquad \text{[Assume } |Zt| < 1]$$

So $$G(t) = \frac{1}{(1-Zt)} \text{ generates } Z^r, Z \neq 0.$$

Also, If $a_r^{(1)}$ has the generating function $G_1(t)$ and $a_r^{(2)}$ has the generating function $G_2(t)$, then $\lambda_1 a_r^{(1)} + \lambda_2 a_r^{(2)}$ has the generating function $\lambda_1 G_1(t) + \lambda_2 G_2(t)$. Here λ_1 and λ_2 are constants.

Theorem IV. *If we know the generating function for the sequence a_r, then the generating function of sequences a_{r+1}, a_{r+2} can be find easily.*

Proof. Let us consider, that $G(t)$ be the generating function for the sequence, having the general term a_r, then we have

$$G(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_r t^r + \ldots$$

$$\frac{G(t) - a_0}{t} = a_1 + a_2 t + a_3 t^2 + \ldots + a_r t^{r-1} + a_{r+1} t^r + \ldots$$

Thus $\dfrac{G(t) - a_0}{t}$ is the generating function for the sequence a_{r+1}.

In the same way,

$$\frac{G(t) - a_0 - a_1 t}{t^2} = a_2 + a_3 t + a_4 t^2 + \ldots$$

the generating function for a_{r+2}.

The summary of results, which we have deduced from the above discussion, is shown in Fig. 1.

S. No.	General term of sequence a_r	Generating function $G(t)$
1.	1	$1/(1-t)$
2.	$r+1$	$1/(1-t)^2$
3.	r	$t/(1-t)^2$
4.	$r(r+1)$	$2t/(1-t)^3$
5.	Z^r, here z is a constant	$1/(1-Zt)$
6.	${}^nC_r A^{n-r}. Z^r$ $(r = 0, 1, 2, 3, \ldots, n)$ $r > n$	$(A + Zt)^n$
7.	a_r	$G(t)$
8.	a_{r+1}	$\dfrac{G(t) - a_0}{t}$
9.	a_{r+2}	$\dfrac{G(t) - a_0 - a_1 . t}{t^2}$
10.	a_{r+3}	$\dfrac{G(t) - a_0 - a_1 . t - a_2 t^2}{t^3}$
11.	a_{r+n}	$\dfrac{G(t) - a_0 - a_1 t - \ldots\ldots - a_{n-1} t^{n-1}}{t^n}$
12.	$\lambda_1 a_r^{(1)} + \lambda_2 a_r^{(2)}$	$\lambda_1 G_1(t) + \lambda_2 G_2(t)$

Fig.1.

Example 29. *Solve the recurrence relation*

$$a_{r+2} - 3a_{r+1} + 2a_r = 0 \qquad \ldots(i)$$

by the method of generating functions with the initial conditions $a_0 = 2$ and $a_1 = 3$.

Sol. Let us assume that

$$G(t) = \sum_0^{\infty} a_r t^r$$

Multiply equation (*i*) by t^r and summing from $r = 0$ to ∞, we have

$$\sum_0^{\infty} a_{r+2} t^r - 3\sum_0^{\infty} a_{r+1} t^r + 2\sum_0^{\infty} a_r t^r = 0$$

or $$(a_2 + a_3 t + a_4 t^2 + \ldots) - 3(a_1 + a_2 t + a_3 t^2 + \ldots) + 2(a_0 + a_1 t + a_2 t^2 + \ldots) = 0$$

$$[\because \; G(t) = a_0 + a_1 t + a_2 t^2 + \ldots]$$

$\therefore$ $$\frac{G(t) - a_0 - a_1 t}{t^2} - 3\left(\frac{G(t) - a_0}{t}\right) + 2G(t) = 0 \qquad \ldots(ii)$$

Now, put $a_0 = 2$ and $a_1 = 3$ in equation (*ii*) and solving, we get

$$G(t) = \frac{2 - 3t}{1 - 3t + 2t^2} \quad \text{or} \quad G(t) = \frac{2 - 3t}{(1-t)(1-2t)}$$

Now, Let $$\frac{2-3t}{(1-t)(1-2t)} = \frac{A}{1-t} + \frac{B}{1-2t}$$

i.e., $$2 - 3t = A(1 - 2t) + B(1 - t) \qquad \ldots(iii)$$

Put $t = 1$ on both sides of equation (*iii*) to find A. Hence

$$-1 = -A \quad \therefore \quad A = 1$$

Put $t = \frac{1}{2}$ on both sides of equation (*iii*) to find B. Hence

$$\frac{1}{2} = \frac{1}{2}B \quad \therefore \quad B = 1$$

Thus $G(t) = \frac{1}{1-t} + \frac{1}{1-2t}$. Hence, $a_r = 1 + 2^r$.

Example 30. *Solve the recurrence relation*

$$a_r - 7a_{r-1} + 10a_{r-2} = 0 \quad \text{...(i)}$$

by the method of generating functions with the initial conditions $a_0 = 3$ and $a_1 = 3$.

Sol. Let us assume that

$$G(t) = \sum_{0}^{\infty} a_r t^r$$

Multiply equation (*i*) by t^r and summing from $r = 2$ to ∞, we have

$$\sum_{2}^{\infty} a_r t^r - 7\sum_{2}^{\infty} a_{r-1} t^r + 10 \sum_{2}^{\infty} a_{r-2} t^r = 0$$

or

$$(a_2 t^2 + a_3 t^3 + ...) - 7(a_1 t^2 + a_2 t^3 + ...) + 10(a_0 t^2 + a_1 t^3 + ...) = 0$$

$$G(t) - a_0 - a_1 t - 7t[G(t) - a_0] + 10t^2 G(t) = 0 \quad \text{...(ii)}$$

Now, put $a_0 = 3$ and $a_1 = 3$ in equation (*ii*) and solving, we get

$$G(t) = \frac{3 + 24t}{10t^2 - 7t + 1} = \frac{3 + 24t}{(5t - 1)(2t - 1)}$$

By partial fractions

$$G(t) = \frac{10}{(2t - 1)} - \frac{13}{(5t - 1)} \text{ or } G(t) = \frac{13}{(1 - 5t)} - \frac{10}{(1 - 2t)}$$

Therefore, $\quad a_r = 13(5)^r - 10(2)^r.$

Example 31. *Solve the recurrence relation*

$$a_{r+2} - 2a_{r+1} + a_r = 2^r \quad \text{...(i)}$$

by the method of generating functions with the initial conditions $a_0 = 2$ and $a_1 = 1$.

Sol. By taking generating functions of equation (*i*) both the sides, we have

$$\frac{G(t) - a_0 - a_1 t}{t^2} - 2\left(\frac{G(t) - a_0}{t}\right) + G(t) = \frac{1}{1 - 2t} \quad \text{...(ii)}$$

Now, put $a_0 = 2$ and $a_1 = 1$ in equation (*ii*) and solving, we get

$$G(t) - 2 - t - 2t\, G(t) - 2t + t^2 G(t) = \frac{t^2}{1 - 2t}$$

$$(t^2 - 2t + 1)\, G(t) = 2 + 3t + \frac{t^2}{1 - 2t}$$

$$(1 - t)^2 G(t) = 2 + 3t + \frac{t^2}{1 - 2t}$$

$$G(t) = \frac{2}{(1-t)^2} + \frac{3t}{(1-t)^2} + \frac{t^2}{(1-2t)(1-t)^2}$$

By partial fraction $\quad \dfrac{t^2}{(1-2t)(1-t)^2} = \dfrac{1}{(1-2t)} - \dfrac{1}{(1-t)^2}$

Hence, $\quad G(t) = \dfrac{1}{(1-t)^2} + \dfrac{3t}{(1-t)^2} + \dfrac{1}{(1-2t)}$

Therefore, $\quad a_r = (r+1) + 3r + 2^r$

$$a_r = 1 + 4r + 2^r.$$

Example 32. *Solve the recurrence relation*

$$a_{r+2} - 5a_{r+1} + 6a_r = 2$$

by the method of generating functions satisfying the initial conditions $a_0 = 1$ and $a_1 = 2$. ...(i)

Sol. Let us assume that

$$G(t) = \sum_{0}^{\infty} a_r\, t^r$$

Now, by taking generating functions of equation (*i*), we have

$$\frac{G(t) - a_0 - a_1 t}{t^2} - 5\left(\frac{G(t) - a_0}{t}\right) + 6G(t) = \frac{2}{1-t}$$

Put $a_0 = 1$ and $a_1 = 3$ in the above equation and solving, we get

$$G(t) = \frac{5t^2 - 4t + 1}{(1-t)(1-2t)(1-3t)}$$

By partial fractions

$$G(t) = \frac{1}{(1-t)} + \frac{-1}{(1-2t)} + \frac{1}{(1-3t)}$$

Therefore, the solution after applying inverse transformations

$$a_r = 1 - 2^r + 3^r.$$

SOLVED PROBLEMS

Problem 1. *Solve the following difference equation*

$$y_{K+3} + y_{K+2} - 8y_{K+1} - 12y_K = 0.$$

Sol. The characteristic equation is given by

$$s^3 + s^2 - 8s - 12 = 0$$

or $\quad (s-3)(s+2)(s+2) = 0$

or $\quad s = 3, -2, -2$

Therefore, the homogeneous solution of the equation is

$$y_K = C_1 . 3^K + (C_2 + C_3 K) . (-2)^K.$$

Problem 2. *Solve the difference equation*

$$y_{K+3} + 6y_{K+2} + 12y_{K+1} + 8y_K = 0.$$

Sol. The characteristic equation is

$$s^3 + 6s^2 + 12s + 8 = 0$$
$$(s + 2)(s + 2)(s + 2) = 0$$
$$s = -2, -2, -2.$$

Therefore, the homogeneous solution of the equation is

$$y_K = (C_1 K^2 + C_2 K + C_3) \cdot (-2)^K.$$

Problem 3. *Solve the difference equation* $a_r - 4a_{r-1} + 6a_{r-2} - 4a_{r-3} + a_{r-4} = 0.$

Sol. The characteristic equation is given by

$$s^4 - 4s^3 + 6s^2 - 4s + 1 = 0$$
$$(s - 1)(s^3 - 3s^2 + 2s - 1) = 0$$
$$(s - 1)(s - 1)(s^2 - 2s + 1) = 0$$
$$(s - 1)(s - 1)(s - 1)(s - 1) = 0$$
$$s = 1, 1, 1, 1$$

Therefore, the homogeneous solution of the equation is

$$y_K = (C_1 r^3 + C_2 r^2 + C_3 r + C_4) \cdot (1)^r.$$

Problem 4. *Solve the difference equation*

$$a_r - 2a_{r-1} + 2a_{r-2} - a_{r-3} = 0.$$

Sol. The characteristic equation is given by

$$s^3 - 2s^2 + 2s - 1 = 0$$
$$(s - 1)(s^2 - s + 1) = 0$$
$$s = 1, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}$$

Therefore, the homogeneous solution of the equation is

$$a_r = C_1 + C_2\left[\frac{1+i\sqrt{3}}{2}\right]^r + C_3\left[\frac{1-i\sqrt{3}}{2}\right]^r.$$

Problem 5. *Solve the difference equation* $a_{r+2} - 2a_{r+1} + a_r = 3r + 5.$ ...(*i*)

Sol. Put R.H.S. of the equation equal to zero *i.e.*,

$$a_{r+2} - 2a_{r+1} + a_r = 0$$

This can be written as $(E^2 - 2E + 1)\, a_r = 0$

$$(E - 1)^2 = 0 \quad \Rightarrow \quad E = +1, 1$$

Therefore, the homogeneous solution is

$$a_{r(h)} = C_1 + C_2 r$$

The particular solution of equation (*i*) is

$$a_{r(p)} = \frac{1}{2} r^3 + r^2$$

as shown in example.

Therefore, the total solution of the difference equation is

$$a_r = C_1 + C_2 r + \frac{1}{2} r^3 + r^2.$$

Problem 6. *Solve the difference equation*

$$y_K - y_{K-1} - 6y_{K-2} = -30 \quad ...(i)$$

Given that $y_0 = 20, y_1 = -5$.

Sol. The equation (*i*) can be written as

$$(s^2 - s - 6)\, y_K = -30$$

Its homogeneous solution is given as

$$(s^2 - s - 6) y_K = 0$$

or $$(s - 3)(s + 2) = 0$$

or $$s = 3, -2$$

Thus, the homogeneous solution is

$$y_{K(h)} = C_1 . 3^K + C_2 . (-2)^K$$

For the particular solution of equation (*i*), consider the general form of solution = A

Put in the general form in L.H.S. of equation (*i*), we get

$$A - A - 6A = -30 \quad \Rightarrow \quad A = 5$$

Therefore, the particular solution = 5

The total solution is $y_K = C_1 . 3^K + C_2(-2)^K + 5$

Now, putting K = 0, we have

$$y_0 = C_1 + C_2 + 5 = 20$$

$\therefore$ $$C_1 + C_2 = 15 \quad ...(ii)$$

Now, putting K = 1, we have

$$y_1 = 3C_1 - 2C_2 + 5$$

$$-5 = 3C_1 - 2C_2 + 5 = 3C_1 - 2C_2 = -10 \quad ...(iii)$$

Solving (*ii*) and (*iii*), we have

$$C_1 = 4, C_2 = 11$$

Therefore, the total solution satisfying the initial conditions is

$$y_K = 4 . 3^K + 11(-2)^K + 5.$$

Problem 7. *Solve the recurrence relation*

$$a_{r+2} - 2a_{r+1} + a_r = r^2 . 2^r. \quad ...(i)$$

Sol. Put R.H.S. of the equation (*i*) equal to zero *i.e.*,

$$a_{r+2} - 2a_{r+1} + a_r = 0$$

This can be written as $(s^2 - 2s + 1)\, a_r = 0$

or $$(s - 1)^2 = 0$$

or $$s = 1, 1$$

Therefore, the homogeneous solution is

$$a_{r(h)} = C_1 + C_2\, r$$

For, finding particular solution of equation (*i*), consider its general form

$$a_r = 2^r (A_0 + A_1 r + A_2 r^2)$$

Put this solution in the L.H.S. of equation (*i*), we get

$$= 2^{r+2} [A_2 (r + 2)^2 + A_1 (r + 2) + A_0] - 2 . 2^{r+1} [A_2 (r + 1)^2 + A_1(r + 1) + A_0] + 2^r (A_2 r^2 + A_1 r + A_0)$$

$$= r^2 . 2^r (A_2) + r. 2^r (8A_2 + A_1) + 2^r (12A_2 + 4A_1 + A_0) \quad ...(ii)$$

Equating equation (*ii*) with R.H.S. of equation (*i*), we get

$$A_2 = 1$$
$$8A_2 + A_1 = 0$$
$$12A_2 + 4A_1 + A_0 = 0$$

On solving these equations, we get

$$A_0 = 20, \ A_1 = -8, \ A_2 = 1$$

Therefore, the particular solution is

$$a_{r(p)} = 2^r (r^2 - 8r + 20)$$

Hence, the total solution is

$$a_r = C_1 + C_2 r + 2^r (r^2 - 8r + 20).$$

Problem 8. *Solve the recurrence relation*

$$a_r + a_{r-1} + a_{r-2} = 0 \qquad \text{...(i)}$$

Satisfying the initial conditions $a_0 = 0$ and $a_1 = 2$.

Sol. This equation can be written as $(s^2 + s + 1)\, a_r = 0$

$$s = \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$$

Therefore, the solution is given by

$$a_r = C_1 \left[\frac{-1 + i\sqrt{3}}{2}\right]^r + C_2 \left[\frac{-1 - i\sqrt{3}}{2}\right]^r \qquad \text{...(ii)}$$

Put $r = 0$, in equation (*ii*), we have

$$a_0 = C_1 + C_2 = 0 \qquad \text{...(iii)}$$

Put $r = 1$, in equation (*ii*), we have

$$a_1 = C_1 \left[\frac{-1 + i\sqrt{3}}{2}\right] + C_2 \left[\frac{-1 - i\sqrt{3}}{2}\right] = 2 \qquad \text{...(iv)}$$

On solving equations (*iii*) and (*iv*), we have

$$C_1 = \frac{2}{i\sqrt{3}}$$

$$C_2 = \frac{-2}{i\sqrt{3}}$$

Therefore, the solution satisfying initial conditions is

$$a_r = \frac{2}{i\sqrt{3}} \left[\frac{-1 + i\sqrt{3}}{2}\right]^r - \frac{2}{i\sqrt{3}} \left[\frac{-1 - i\sqrt{3}}{2}\right]^r.$$

Problem 9. *Solve the recurrence relation*

$$a_r - a_{r-1} - a_{r-2} = 0 \qquad \text{...(i)}$$

Satisfying the initial conditions $a_0 = 1$ and $a_1 = 1$.

Sol. The equation (*i*) can also be written as

$$(s^2 - s - 1)\, a_r = 0 \quad \text{or} \quad s = \frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}$$

Therefore, the solution is given by

$$a_r = C_1\left[\frac{1+\sqrt{5}}{2}\right]^r + C_2\left[\frac{1-\sqrt{5}}{2}\right]^r \quad ...(ii)$$

Put $r = 0$, in equation (*ii*), we have

$$a_0 = C_1 + C_2 = 1 \quad ...(iii)$$

Put $r = 1$, in equation (*ii*), we have

$$a_1 = C_1\left[\frac{1+\sqrt{5}}{2}\right] + C_2\left[\frac{1-\sqrt{5}}{2}\right] = 1 \quad ...(iv)$$

On solving equations (*iii*) and (*iv*), we have

$$C_1 = \frac{\sqrt{5}+1}{2\sqrt{5}}$$

$$C_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

Therefore, the solution satisfying initial conditions is

$$a_r = \left(\frac{\sqrt{5}+1}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^r + \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^r.$$

Problem 10. *Solve the recurrence relation*

$$a_{r+1} - 3a_r = 2^r. \quad ...(i)$$

Sol. Put R.H.S. of the equation (*i*) equal to zero *i.e.*,

$$a_{r+1} - 3a_r = 0$$

This can be written as

$$(s - 3)a_r = 0 \quad \text{or} \quad s = 3.$$

Therefore, the homogeneous solution is

$$a_{r(h)} = C_1 \cdot 3^r$$

For the particular solution of the equation (*i*), consider the general form of the solution

$$= A \cdot 2^r$$

Put this solution on L.H.S. of equation (*i*), we have

$$= A \cdot 2^{r+1} - 3A2^r = (2A - 3A) \cdot 2^r$$

$$= -A \cdot 2^r \quad ...(ii)$$

Equating equation (*ii*) with R.H.S. of equation (*i*), we get

$$A = -1$$

Therefore, the particular solution is

$$a_{r(p)} = -(2^r)$$

Hence, the total solution is $a_r = C_1 \cdot 3^r - (2^r)$.

Problem 11. *Solve the recurrence relation*

$$a_{r+2} - 5a_{r+1} + 6a_r = 5^r. \quad ...(i)$$

Sol. Put the R.H.S. of equation equal to zero *i.e.*,

$$a_{r+2} - 5a_{r+1} + 6a_r = 0$$

This can also be written as

$$(s^2 - 5s + 6)a_r = 0$$

$$(s - 3)(s - 2) = 0$$

or

$$s = 3, 2$$

or Therefore, the homogeneous solution is

$$a_{r(h)} = C_1 \cdot 2^r + C_2 \cdot 3^r$$

For, finding the particular solution consider the general form of the solution $= A \cdot 5^r$.

Now, put this solution on L.H.S. of equation (*i*), we get

$$= A \cdot 5^{r+2} - 5 \cdot A \cdot 5^{r+1} + 6 \cdot A \cdot 5^r$$

$$= 25A \cdot 5^r - 25A \cdot 5^r + 6A \cdot 5^r$$

$$= 6A \cdot 5^r \qquad ...(ii)$$

Equating equation (*ii*) to R.H.S. of equation (*i*), we get

$$A = \frac{1}{6}$$

Therefore, the particular solution is

$$a_{r(p)} = \frac{1}{6} \cdot 5^r$$

Hence, the total solution is

$$a_r = C_1 \cdot 2^r + C_2 \cdot 3^r + \frac{1}{6} \cdot 5^r.$$

Problem 12. *Solve the recurrence relation*

$$a_r - a_{r-1} = 7r. \qquad ...(i)$$

Sol. Put the R.H.S. of equation (*i*) equal to zero *i.e.*,

$$a_r - a_{r-1} = 0$$

This equation can also be written as

$$(s - 1)a_r = 0 \quad \text{or} \quad s = 1$$

Therefore, the homogeneous solution is

$$a_{r(h)} = C_1$$

For finding, the particular solution, consider the general form of the solution $= A_0 r + A_1$. Since one of the terms of the solution occurs in the homogeneous solution, so multiply the solution by r, we get $A_0 r^2 + A_1 \cdot r$.

Now, put this solution in L.H.S. of equation (*i*), we have

$$= A_0 r^2 + A_1 r - A_0(r-1)^2 - A_1(r-1) = 2A_0 r + A_1 - A_0 \qquad ...(ii)$$

Equating equation (*ii*) with R.H.S. of equation (*i*), we get

$$A_0 = \frac{7}{2}$$

$$A_1 - A_0 = 0 \Rightarrow A_1 = \frac{7}{2}$$

Therefore, the particular solution is $a_{r(p)} = \frac{7}{2}r^2 + \frac{7}{2}r$

Hence, the total solution is

$$a_r = C_1 + \frac{7}{2}r^2 + \frac{7}{2}r.$$

The particular solution of equation (*ii*), is given by

$$y_{K(p)} = \frac{1}{(E-r)} \cdot z$$

Put E = 1, because z is a constant. Hence

$$y_{K(p)} = \frac{z}{(1-r)}$$

Hence, the total solution is

$$y_K = C_1 \cdot r^K + \frac{z}{(1-r)} \quad \text{...(iii)}$$

Put K = 0, in equation (*iii*), we have

$$y_0 = C_1 + \frac{z}{(1-r)} = 0$$

$$\therefore \quad C_1 = -\frac{z}{1-r}$$

The total solution satisfying the initial condition is

$$y_K = -\frac{z}{1-r} \cdot r^K + \frac{z}{(1-r)} \quad \text{or} \quad y_K = \frac{z(1-r^K)}{(1-r)}.$$

Problem 17. *Solve the recurrence relation*

$$y_{K+3} + y_{K+2} - 8y_{K+1} - 12y_K = 2K^2 + 5. \quad \text{...(i)}$$

Sol. Put the R.H.S. of equation (*i*) equal to zero *i.e.*,

$$y_{K+3} + y_{K+2} - 8y_{K+1} - 12y_K = 0$$

i.e., $$(E^3 + E^2 - 8E - 12)y_K = 0$$

or $$E = 3, -2, -2$$

Therefore, the homogeneous solution is

$$y_{K(h)} = C_1 \cdot 3^K + (C_2 + C_3 K)(-2)^K$$

The particular solution of equation (*i*) is given as

$$y_{K(p)} = \frac{1}{(E^3 + E^2 - 8E - 12)} \cdot (2K^2 + 5)$$

$$= \frac{1}{(E-3)(E+2)^2} \cdot (2K^2 + 5) \quad \text{[Put } E = 1 + \Delta]$$

$$= \frac{1}{(\Delta - 2)(\Delta + 3)^2} \cdot (2K^2 + 5)$$

$$= (\Delta^3 + 4\Delta^2 - 3\Delta - 18)^{-1} \cdot (2K^2 + 5)$$

$$= -\frac{1}{18}\left(1 + \frac{3\Delta - 4\Delta^2 - \Delta^3}{18}\right)^{-1} \cdot (2K^2 + 5)$$

$$y_{K(p)} = -\frac{1}{18}\left(1 - \frac{3\Delta - 4\Delta^2 - \Delta^3}{18} + \frac{(3\Delta - 4\Delta^2 - \Delta^3)^2}{(18)^2} + \ldots\right) \cdot (2K^2 + 5)$$

$$= -\frac{1}{18}\left(1 - \frac{\Delta}{6} + \frac{\Delta^2}{4}\right) \cdot (2K^2 + 5) \quad \text{[By neglecting higher degre}$$

By factorial notation $2K^2 + 5 = 2[K]^2 + 2[K] + 5$.

Hence,
$$y_{K(p)} = -\frac{1}{18}\left(1-\frac{\Delta}{6}+\frac{\Delta^2}{4}\right).(2[K]^2 + 2[K] + 5)$$
$$= -\frac{1}{18}\left(2[K]^2 + 2[K] + 5 - \frac{1}{6}(4[K] + 2) + \frac{1}{4}(4)\right)$$
$$= -\frac{1}{18}\left(2[K]^2 - \frac{4}{3}[K] + \frac{17}{3}\right)$$
$$= -\frac{1}{18}\left(2K(K-1) - \frac{4}{3}K + \frac{17}{3}\right)$$
$$y_{K(p)} = -\frac{K^2}{9} + \frac{1}{27}K - \frac{17}{54}$$

Hence, the total solution is
$$y_K = C_1 . 3^K + (C_2 + C_3 K)(-2)^K + \left(\frac{-K^2}{9} + \frac{1}{27}K - \frac{17}{54}\right).$$

Problem 18. *Solve the recurrence relation*
$$y_{K+4} + 4y_{K+3} + 8Y_{K+2} + 8y_{K+1} + 4y_K = 10. \quad ...(i)$$

Sol. Put the R.H.S. of equation (*i*) equal to zero *i.e.*,
$$y_{K+4} + 4y_{K+3} + 8y_{K+2} + 8y_{K+1} + 4y_K = 0$$
$$(E^4 + 4E^3 + 8E^2 + 8E + 4)\, y_K = 0$$

Its homogeneous solution is
$$y_{K(h)} = (C_1 + C_2 K)(-1+i)^K + (C_3 + C_4 K)(-1-i)^K$$

shown before.

The particular solution of equation (*i*) is given as
$$y_{K(p)} = \frac{1}{(E^4 + 4E^3 + 8E^2 + 8E + 4)} . 10$$

By putting E = 1, we have
$$y_{K(p)} = \frac{10}{1+4+8+8+4} = \frac{10}{25} = \frac{2}{5}$$

Hence, the total solution is
$$y_K = (C_1 + C_2 K)(-1+i)^K + (C_3 + C_4 K)(-1-i)^K + \frac{2}{5}.$$