CHAPTER 1

Fourier Series

1.1. PERIODIC FUNCTIONS

A function $f(x)$ which satisfies the relation $f(x + T) = f(x)$ for all x is called a periodic function. The *smallest positive number* T, for which this relation holds, is called the **period** of $f(x)$.

If T is the period of $f(x)$, then $f(x) = f(x + T) = f(x + 2T) = \ldots\ldots = f(x + nT) = \ldots\ldots$

Also $f(x) = f(x - T) = f(x - 2T) = \ldots\ldots = f(x - nT) = \ldots\ldots$

$\therefore$ $f(x) = f(x \pm nT)$, where n is a positive integer.

Thus, $f(x)$ repeats itself after periods of T.

For example, $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are periodic functions with period 2π while $\tan x$ and $\cot x$ are periodic functions with period π. The functions $\sin nx$ and $\cos nx$ are periodic with period $\frac{2\pi}{n}$.

The sum of a number of periodic functions is also periodic. If T_1 and T_2 are the periods of $f(x)$ and $g(x)$, then the period of $a\, f(x) + b\, g(x)$ is the least common multiple of T_1 and T_2.

For example, $\cos x$, $\cos 2x$, $\cos 3x$ are periodic functions with periods 2π, π and $\frac{2\pi}{3}$ respectively.

$\therefore$ $f(x) = \cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x$ is also periodic with period 2π, the L.C.M. of 2π, π and $\frac{2\pi}{3}$.

1.2. FOURIER SERIES

Periodic functions are of common occurrence in many physical and engineering problems; for example, in conduction of heat and mechanical vibrations. It is useful to express these functions in a series of sines and cosines. Most of the single valued functions which occur in applied mathematics can be expressed in the form $\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots\ldots$ $+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots\ldots$

within a desired range of values of the variable. Such a series is known as *Fourier Series*. Thus, any function $f(x)$ defined in the interval $c_1 \le x \le c_2$ can be expressed in the Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where a_0, a_n, b_n ($n = 1, 2, 3, \ldots\ldots$) are constants, called the Fourier co-efficients of $f(x)$.

Note. To determine a_0, a_n and b_n, we shall need the following results: (m and n are integers)

(i) $\displaystyle\int_c^{c+2\pi} \sin nx\, dx = -\left[\frac{\cos nx}{n}\right]_c^{c+2\pi} = 0, \quad \int_c^{c+2\pi} \cos nx\, dx = \left[\frac{\sin nx}{n}\right]_c^{c+2\pi} = 0, n \neq 0$

(ii) $\displaystyle\int_c^{c+2\pi} \sin mx \cos nx\, dx = \frac{1}{2}\int_c^{c+2\pi} [\sin (m+n)x + \sin (m-n)x]\, dx$

$$= -\frac{1}{2}\left[\frac{\cos (m+n)x}{m+n} + \frac{\cos (m-n)x}{m-n}\right]_c^{c+2\pi} = 0, m \neq n$$

(iii) $\displaystyle\int_c^{c+2\pi} \cos mx \cos nx\, dx = \frac{1}{2}\int_c^{c+2\pi} [\cos (m+n)x + \cos (m-n)x]\, dx$

$$= \frac{1}{2}\left[\frac{\sin (m+n)x}{m+n} + \frac{\sin (m-n)x}{m-n}\right]_c^{c+2\pi} = 0, m \neq n$$

(iv) $\displaystyle\int_c^{c+2\pi} \sin mx \sin nx\, dx = \frac{1}{2}\int_c^{c+2\pi} [\cos (m-n)x - \cos (m+n)x]\, dx$

$$= \frac{1}{2}\left[\frac{\sin (m-n)x}{m-n} - \frac{\sin (m+n)x}{m+n}\right]_c^{c+2\pi} = 0, m \neq n$$

(v) $\displaystyle\int_c^{c+2\pi} \cos^2 nx\, dx = \left[\frac{x}{2} + \frac{\sin 2nx}{4n}\right]_c^{c+2\pi} = \pi, \quad \int_c^{c+2\pi} \sin^2 nx\, dx = \left[\frac{x}{2} - \frac{\sin 2nx}{4n}\right]_c^{c+2\pi} = \pi, n \neq 0$

(vi) $\displaystyle\int_c^{c+2\pi} \sin nx \cos nx\, dx = \frac{1}{2}\int_c^{c+2\pi} \sin 2nx\, dx = -\frac{1}{2}\left[\frac{\cos 2nx}{2n}\right]_c^{c+2\pi} = 0, n \neq 0$

(vii) To integrate the product of two functions, one of which is a positive integral power of x, we apply the *generalised rule of integration by parts.* If dashes denote differentiation and suffixes denote integration w.r.t. x, the rule can be stated as follows:

$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots$ where u and v are functions of x. *i.e.*, Integral of the product of two functions

= 1st function × integral of 2nd – go on differentiating 1st, integrating 2nd, signs alternately + ve and – ve.

[Simplification should be done only when the integration is over.]

For example, $\displaystyle\int x^3 e^{-2x}\, dx = x^3\left(\frac{e^{-2x}}{-2}\right) - 3x^2\left[\frac{e^{-2x}}{(-2)^2}\right] + 6x\left[\frac{e^{-2x}}{(-2)^3}\right] - 6\left[\frac{e^{-2x}}{(-2)^4}\right]$

$$= e^{-2x}\left[-\frac{1}{2}x^3 - \frac{3}{4}x^2 - \frac{3}{4}x - \frac{3}{8}\right] = -\frac{1}{8}e^{-2x}(4x^3 + 6x^2 + 6x + 3)$$

$$\int x^2 \cos nx\, dx = x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)$$

$$= \frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx.$$

(viii) $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$

$\sin\left(n + \frac{1}{2}\right)\pi = (-1)^n$ and $\cos\left(n + \frac{1}{2}\right)\pi = 0$, where n is an integer.

(*ix*) **Even and Odd Functions**

A function $f(x)$ is said to be *even* if $f(-x) = f(x)$ e.g., x^2, $\cos x$, $\sin^2 x$ are even functions.

The graph of an even function is symmetrical about the y-axis.

A function $f(x)$ is said to be *odd* if $f(-x) = -f(x)$ e.g., x^3, $\sin x$, $\tan^3 x$ are odd functions.

The graph of an odd function is symmetrical about the origin.

The product of two even functions or two odd functions is an even function while the product of an even function and an odd function is an odd function.

Also, $\int_{-c}^{c} f(x)\,dx = 0$, when $f(x)$ is an odd function

and $\int_{-c}^{c} f(x)\,dx = 2\int_{0}^{c} f(x)\,dx$, when $f(x)$ is an even function.

1.3. EULER'S FORMULAE

The Fourier series for the function $f(x)$ in the interval $c < x < c + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$$

In finding the co-efficients a_0, a_n and b_n, we assume that the series on the right hand side of (1) is uniformly convergent for $c < x < c + 2\pi$ and it can be integrated term by term in the given interval.

To find a_0. Integrate both sides of (1) w.r.t. x, between the limits c to $c + 2\pi$.

$$\int_c^{c+2\pi} f(x)\,dx = \frac{a_0}{2}\int_c^{c+2\pi} dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)dx$$

$$= \frac{a_0}{2}(c + 2\pi - c) + 0 + 0 \qquad \text{[By formulae } (i) \text{ above]}$$

$$= a_0\pi$$

$$\therefore \qquad \boxed{a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx}$$

To find a_n, multiply both sides of (1) by $\cos nx$ and integrate w.r.t. x, between the limits c to $c + 2\pi$.

$$\int_c^{c+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi} \cos nx\,dx$$

$$+ \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\cos nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\cos nx\,dx$$

[By formulae (*i*), (*v*) and (*vi*)]

$$= 0 + a_n\pi + 0$$

$$= a_n\pi$$

$$\therefore \qquad \boxed{a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx}$$

To find b_n, multiply both sides of (1) by sin nx and integrate w.r.t. x between the limits c to c + 2π.

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi}\sin nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\sin nx\,dx$$

$$+\int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\sin nx\,dx$$

[By formulae (i), (vi) and (v)]

$$= 0 + 0 + b_n\pi$$

$$= b_n\pi$$

$$\therefore \qquad \boxed{b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx}$$

Hence $a_0 = \dfrac{1}{\pi}\displaystyle\int_c^{c+2\pi} f(x)\,dx$; $a_n = \dfrac{1}{\pi}\displaystyle\int_c^{c+2\pi} f(x)\cos nx\,dx$; and $b_n = \dfrac{1}{\pi}\displaystyle\int_c^{c+2\pi} f(x)\sin nx\,dx$ ...(I)

These values of a_0, a_n and b_n are called Euler's formulae.

Cor. 1. *If c = 0, the interval becomes $0 < x < 2\pi$ and the formulae I reduce to*

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx;\quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx;\quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Cor. 2. *If c = − π, the interval becomes $-\pi < x < \pi$, and the formulae I reduce to*

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx;\; a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx;\; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Cor. 3. *When f(x) is an odd function* $\quad a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = 0$

Since cos nx is an even function, therefore, $f(x)$ cos nx is an odd function.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

Since sin nx is an odd function, therefore, $f(x)$ sin nx is an even function.

$$\therefore \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Hence, if a periodic function $f(x)$ is odd, its Fourier expansion contains only sine terms,

i.e., $$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad \text{where} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

When f(x) is an even function $\quad a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\,dx$

Since cos nx is an even function, therefore, $f(x)$ cos nx is an even function.

$$\therefore \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Since sin nx is an odd function, therefore, $f(x)$ sin nx is an odd function.

$$\therefore \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

Hence, if a periodic function $f(x)$ is even, its Fourier expansion contains only cosine terms.

i.e.,
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \text{ where } a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \text{ and } a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

An Important Note. If a periodic function f is defined in the open interval (a, b), then how to find $f(a)$ and $f(b)$, the values of f at the end points?

Consider, $f(x) = x + x^2, \; -\pi < x < \pi$

where f is periodic with period 2π.

$$f(-\pi) = \frac{1}{2}\,(\text{LHL} + \text{RHL}) = \frac{1}{2}\,[f(-\pi - 0) + f(-\pi + 0)]$$

$$= \frac{1}{2}\,[f(2\pi + (-\pi - 0)) + f(-\pi + 0)] \qquad \mid \because \text{ Period of } f \text{ is } 2\pi$$

$$= \frac{1}{2}\,[f(\pi - 0) + f(-\pi + 0)]$$

$$= \frac{1}{2}\,[(\pi + \pi^2) + (-\pi) + (-\pi)^2] = \pi^2$$

Clearly, $f(-\pi) \neq -\pi + \pi^2$

Similarly, $f(\pi) = \frac{1}{2}\,[\text{LHL} + \text{RHL}]$

$$= \frac{1}{2}\,[f(\pi - 0) + f(\pi + 0)] = \frac{1}{2}\,[f(\pi - 0) + f(-2\pi + (\pi + 0)]$$

$$\mid \because \text{ Period of } f \text{ is } 2\pi$$

$$= \frac{1}{2}\,[f(\pi - 0) + f(-\pi + 0)] = \frac{1}{2}\,[(\pi + \pi^2) + (-\pi + (-\pi)^2)] = \pi^2$$

Clearly, $f(\pi) \neq \pi + \pi^2$.

ILLUSTRATIVE EXAMPLES

Example 1. *Obtain the Fourier series to represent* $f(x) = \left(\frac{\pi - x}{2}\right)^2, 0 \le x \le 2\pi$.

(M.D.U. 2010, 2012, Dec. 2014; K.U.K. Jan. 2013)

Hence obtain the following relations:

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (M.D.U. Dec. 2014)

(ii) $\dfrac{1}{1^2}-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+\ldots\ldots=\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{n^2}=\dfrac{\pi^2}{12}$ (M.D.U., 2010, Dec. 2014)

(iii) $\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\ldots\ldots=\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}=\dfrac{\pi^2}{8}$

Sol. Let $f(x)=\dfrac{1}{4}(\pi-x)^2=\dfrac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

By Euler's formulae, we have

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-x)^2\,dx=\frac{1}{4\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{2\pi}=-\frac{1}{12\pi}[-\pi^3-\pi^3]=\frac{\pi^2}{6}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-x)^2\cos nx\,dx$$

$$=\frac{1}{4\pi}\left[(\pi-x)^2\frac{\sin nx}{n}-\{-2(\pi-x)\}\left(-\frac{\cos nx}{n^2}\right)+2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$=\frac{1}{4\pi}\left[\left(0+\frac{2\pi\cos 2n\pi}{n^2}+0\right)-\left(0-\frac{2\pi\cos 0}{n^2}+0\right)\right]=\frac{1}{4\pi}\left[\frac{2\pi}{n^2}+\frac{2\pi}{n^2}\right]=\frac{1}{n^2}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-x)^2\sin nx\,dx$$

$$=\frac{1}{4\pi}\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right)-\{-2(\pi-x)\}\left(-\frac{\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$=\frac{1}{4\pi}\left[\left(-\frac{\pi^2\cos 2n\pi}{n}-0+\frac{2\cos 2n\pi}{n^3}\right)-\left(-\frac{\pi^2}{n}-0+\frac{2\cos 0}{n^3}\right)\right]$$

$$=\frac{1}{4\pi}\left[\left(-\frac{\pi^2}{n}+\frac{2}{n^3}\right)-\left(-\frac{\pi^2}{n}+\frac{2}{n^3}\right)\right]=0$$

$$\therefore\quad f(x)=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}=\frac{\pi^2}{12}+\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\ldots\ldots \qquad \ldots(1)$$

Deductions

(i) Putting $x=0$ in equation (1), we get

$$f(0)=\frac{\pi^2}{12}+\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\right)$$

$$\Rightarrow\quad \frac{\pi^2}{4}=\frac{\pi^2}{12}+\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\right)$$

$$\Rightarrow\quad \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots=\frac{\pi^2}{6} \qquad \ldots(2)$$

(*ii*) Putting $x = \pi$ in equation (1), we get

$$f(\pi) = \frac{\pi^2}{12} + \left[\left(\frac{-1}{1^2}\right) + \frac{1}{2^2} + \left(\frac{-1}{3^2}\right) + \frac{1}{4^2} + \ldots\ldots\right]$$

$$\Rightarrow \quad 0 = \frac{\pi^2}{12} - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots$$

$$\Rightarrow \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12} \qquad \ldots(3)$$

(*iii*) Adding (2) and (3), we get

$$2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right) = \frac{\pi^2}{6} + \frac{\pi^2}{12}$$

$$\Rightarrow \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{1}{2}\left(\frac{\pi^2}{4}\right) = \frac{\pi^2}{8}.$$

Note. In the above example, if $f(x) = \left(\frac{\pi - x}{2}\right)^2$, $0 < x < 2\pi$, then

$$f(0) = \frac{1}{2}[f(0-0) + f(0+0)]$$

$$= \frac{1}{2}[f(2\pi - 0) + f(0+0)] \qquad [\because f \text{ is periodic with period } 2\pi]$$

$$= \frac{1}{2}\left[\left(\frac{\pi - 2\pi}{2}\right)^2 + \left(\frac{\pi - 0}{2}\right)^2\right] = \frac{1}{2}\left(\frac{\pi^2}{4} + \frac{\pi^2}{4}\right) = \frac{\pi^2}{4}.$$

Example 2. *Expand $f(x) = x \sin x$, $0 < x < 2\pi$ as a Fourier series.*
(K.U.K. 2010; M.D.U. 2010, 2013; G.B.T.U. 2010)

Sol. Let $f(x) = x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

By Euler's formulae, we have $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x\, dx$

$$= \frac{1}{\pi}\Big[x(-\cos x) - 1.(-\sin x)\Big]_0^{2\pi} = \frac{1}{\pi}[-2\pi] = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx\, dx = \frac{1}{2\pi}\int_0^{2\pi} x(2 \cos nx \sin x)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\sin (n+1)x - \sin (n-1)x]\, dx$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1}\right\} - 1.\left\{-\frac{\sin (n+1)x}{(n+1)^2} + \frac{\sin (n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right\}\right]$$

$$= -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1}, \; n \neq 1$$

When $n = 1$, we have $a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi} = \frac{1}{2\pi}[-\pi] = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2 \sin nx \sin x) \, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,[\cos(n-1)x - \cos(n+1)x] \, dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\} - 1.\left\{-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] = 0, \; n \neq 1$$

When $n = 1$, we have $b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x(1 - \cos 2x) \, dx$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1.\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi(2\pi) - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \frac{1}{2\pi}(2\pi^2) = \pi$$

$$\therefore \quad f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx + 0$$

$$= -1 + \pi \sin x - \frac{1}{2}\cos x + \frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x + \ldots\ldots$$

Example 3. *Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$. Hence show that*

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}.$$

(G.B.T.U. 2012; K.U.K. May 2013; DCRUST, Murthal May 2014)

Sol. Let $x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

By Euler's formulae, we have $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - \frac{\pi^3}{3}\right) - \left(\frac{\pi^2}{2} + \frac{\pi^3}{3}\right)\right] = -\frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1 - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1 - 2\pi)\frac{\cos n\pi}{n^2} - (1 + 2\pi)\frac{\cos n\pi}{n^2}\right] = \frac{1}{\pi}\left(-4\pi\,.\frac{\cos n\pi}{n^2}\right)$$

$$= -4\frac{(-1)^n}{n^2} \qquad [\because \cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(-\frac{\cos nx}{n}\right) - (1 - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi^2 - \pi)\frac{\cos n\pi}{n} - 2\frac{\cos n\pi}{n^3} + (-\pi - \pi^2)\frac{\cos n\pi}{n} + 2\frac{\cos n\pi}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-2\pi\,.\frac{\cos n\pi}{n}\right] = -2\frac{(-1)^n}{n}$$

$$\therefore \quad x - x^2 = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$= -\frac{\pi^2}{3} - 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \ldots\ldots\right] - 2\left[\frac{-\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \ldots\ldots\right]$$

$$= -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\ldots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\ldots\right]$$

Putting $x = 0$, we get $0 = -\frac{\pi^2}{3} + 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\right)$

$$\Rightarrow \qquad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}.$$

Example 4. *Obtain the Fourier series for the function $f(x) = x^2, -\pi \le x \le \pi$. Hence show that*

(*i*) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ (K.U.K. Jan. 2014)

(*ii*) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$ (M.D.U. 2011)

(*iii*) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

Sol. Since $f(-x) = (-x)^2 = x^2 = f(x)$.

$\therefore$ $f(x)$ is an even function and hence $b_n = 0$

Let $\quad f(x) = x^2 = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Then $\quad a_0 = \dfrac{2}{\pi}\int_0^{\pi} f(x)\,dx = \dfrac{2}{\pi}\int_0^{\pi} x^2\,dx = \dfrac{2}{\pi}\left[\dfrac{x^3}{3}\right]_0^{\pi} = \dfrac{2}{3}\pi^2$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[2\pi\,.\,\frac{\cos n\pi}{n^2}\right] = 4\frac{(-1)^n}{n^2}$$

$$\therefore \quad x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx$$

$$= \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\ldots\right) \qquad \ldots(1)$$

Putting $x = \pi$ in (1), we get

$$\pi^2 = \frac{\pi^2}{3} - 4\left(-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\right) \Rightarrow \frac{2\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\right)$$

$$\therefore \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{6} \qquad \text{[Result } (i)\text{]}$$

Putting $x = 0$ in (1), we get $0 = \dfrac{\pi^2}{3} - 4\left(\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots\ldots\right)$

$$\therefore \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12} \qquad \text{[Result } (ii)\text{]}$$

Adding (i) and (ii), we get $2\left(\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots\right) = \dfrac{\pi^2}{4}$

$$\therefore \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}. \qquad \text{[Result } (iii)\text{]}$$

Example 5. *Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.*

Sol. Let $\quad f(x) = e^{-x} = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Then $\quad a_0 = \dfrac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \dfrac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \dfrac{1}{\pi}\left[-e^{-x}\right]_0^{2\pi} = \dfrac{1-e^{-2\pi}}{\pi}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$\left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$= \frac{1-e^{-2\pi}}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\sin nx - n\cos nx)\right]_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi}\cdot\frac{n}{1+n^2}$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$\therefore \quad e^{-x} = \frac{1-e^{-2\pi}}{2\pi} + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{1+n^2} + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{n\sin nx}{1+n^2}$$

$$= \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + \ldots\ldots\right)\right.$$

$$\left. + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x + \ldots\ldots\right)\right]$$

Example 6. *Find the Fourier series to represent e^{ax} in the interval $-\pi < x < \pi$. Hence derive series for $\frac{\pi}{\sinh \pi}$.*

Sol. Let $f(x) = e^{ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Then $\quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\,dx = \frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_{-\pi}^{\pi} = \frac{1}{a\pi}(e^{a\pi} - e^{-a\pi}) = \frac{2\sinh a\pi}{a\pi}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx)\right]_{-\pi}^{\pi} = \frac{1}{\pi(a^2+n^2)}[ae^{a\pi}\cos n\pi - ae^{-a\pi}\cos n\pi]$$

$$= \frac{a\cos n\pi\,(e^{a\pi} - e^{-a\pi})}{\pi(a^2+n^2)} = \frac{2a(-1)^n\sinh a\pi}{\pi(a^2+n^2)}$$

Similarly, $b_n = \frac{-2n(-1)^n\sinh a\pi}{\pi(a^2+n^2)}$

$$\therefore \quad e^{ax} = \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty}\frac{2a(-1)^n\sinh a\pi}{\pi(a^2+n^2)}\cos nx - \sum_{n=1}^{\infty}\frac{2n(-1)^n\sinh a\pi}{\pi(a^2+n^2)}\sin nx$$

$$= \frac{2\sinh a\pi}{\pi}\left[\frac{1}{2a} - a\left(\frac{\cos x}{a^2+1^2} - \frac{\cos 2x}{a^2+2^2} + \frac{\cos 3x}{a^2+3^2} - \ldots\ldots\right)\right.$$
$$\left. + \left(\frac{\sin x}{a^2+1^2} - \frac{2\sin 2x}{a^2+2^2} + \frac{3\sin 3x}{a^2+3^2} - \ldots\ldots\right)\right]$$

Deduction. Putting $x = 0$ and $a = 1$, we get

$$1 = \frac{2\sinh \pi}{\pi}\left[\frac{1}{2} - \left(\frac{1}{1+1^2} - \frac{1}{1+2^2} + \frac{1}{1+3^2} - \frac{1}{1+4^2} + \ldots\ldots\right)\right]$$

$$\Rightarrow \qquad \frac{\pi}{\sinh \pi} = 2\left(\frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} - \ldots\ldots\right)$$

Example 7. *Express $f(x) = |x|$, $-\pi < x < \pi$, as Fourier series.* (M.D.U. 2013, Dec. 2015)

Sol. Since $\quad f(-x) = |-x| = |x| = f(x)$

$\therefore$ $f(x)$ is an even function and hence $b_n = 0$

Let $$f(x) = |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Then, $$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \frac{2}{\pi}\int_0^\pi |x|\,dx = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi = \pi$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi |x|\cos nx\,dx = \frac{2}{\pi}\int_0^\pi x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1.\left(-\frac{\cos nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}[(-1)^n - 1] = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{-4}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \qquad |x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right)$$

Note. Putting $x = 0$ in the above result, we get $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$.

Example 8. *Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $-\pi \le x \le \pi$.*

Deduce that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \ldots\ldots = \frac{\pi - 2}{4}$. (M.D.U. 2012)

Sol. Since $x \sin x$ is an even function of x, $b_n = 0$

Let $$f(x) = x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Then $$a_0 = \frac{2}{\pi}\int_0^\pi x\sin x\,dx = \frac{2}{\pi}\Big[x(-\cos x) - 1.(-\sin x)\Big]_0^\pi = \frac{2}{\pi}(-\pi\cos\pi) = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx \, dx = \frac{1}{\pi}\int_0^{\pi} x(2\cos nx \sin x)\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]\, dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1.\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\}\right], \ n \neq 1$$

$$= \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}$$

When n is odd, $n \neq 1$, $n-1$ and $n+1$ are even

$$\therefore \qquad a_n = \frac{1}{n-1} - \frac{1}{n+1} = \frac{2}{n^2-1}$$

When n is even, $n-1$ and $n+1$ are odd

$$\therefore \qquad a_n = \frac{-1}{n-1} + \frac{1}{n+1} = \frac{-2}{n^2-1}$$

When $\quad n = 1$, we have $a_1 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} x \sin x \cos x \, dx = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} x \sin 2x \, dx$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[-\frac{\pi \cos 2\pi}{2}\right] = -\frac{1}{2}$$

$$\therefore \qquad x \sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - \frac{\cos 5x}{5^2-1} + \dots\dots\right)$$

Putting $\quad x = \dfrac{\pi}{2}$, we get $\dfrac{\pi}{2} = 1 - 2\left(\dfrac{-1}{2^2-1} + \dfrac{1}{4^2-1} - \dfrac{1}{6^2-1} + \dots\dots\right)$

$$\Rightarrow \qquad \frac{\pi}{2} - 1 = 2\left(\frac{1}{2^2-1} - \frac{1}{4^2-1} + \frac{1}{6^2-1} - \dots\dots\right) \Rightarrow \frac{\pi-2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\dots$$

Example 9. *Show that for* $-\pi < x < \pi$,

$$\sin ax = \frac{2\sin a\pi}{\pi}\left(\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} - \dots\dots\right).$$

Sol. Since $\sin ax$ is an odd function of x, $a_0 = 0$ and $a_n = 0$.

Let $\quad \sin ax = \displaystyle\sum_{n=1}^{\infty} b_n \sin nx$

Then $\quad b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \sin ax \sin nx \, dx = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} [\cos(n-a)x - \cos(n+a)x]\, dx$

$$= \frac{1}{\pi}\left[\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{n+a}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\sin(n-a)\pi}{n-a} - \frac{\sin(n+a)\pi}{n+a}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(-\sin a\pi)}{n-a} - \frac{(-1)^n \sin a\pi}{n+a}\right] = -\frac{(-1)^n \sin a\pi}{\pi}\left[\frac{1}{n-a} + \frac{1}{n+a}\right]$$

$$= (-1)^{n+1} \cdot \frac{2n \sin a\pi}{\pi(n^2 - a^2)}$$

$$\therefore \quad \sin ax = \frac{2 \sin a\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 - a^2} \sin nx$$

$$= \frac{2 \sin a\pi}{\pi}\left(\frac{\sin x}{1^2 - a^2} - \frac{2 \sin 2x}{2^2 - a^2} + \frac{3 \sin 3x}{3^2 - a^2} - \ldots\ldots\right).$$

Example 10. *Obtain Fourier series for the function $f(x)$ given by*

$$f(x) = 1 + \frac{2x}{\pi}, \; -\pi \le x \le 0$$

$$= 1 - \frac{2x}{\pi}, \; 0 \le x \le \pi.$$

(U.K.T.U. 2011)

(K.U.K. Jan. 2013)

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$.

Sol. When $\quad -\pi \le x \le 0, \qquad 0 \le -x \le \pi$

$$\therefore \quad f(-x) = 1 - \frac{2(-x)}{\pi} = 1 + \frac{2x}{\pi} = f(x)$$

When $\quad 0 \le x \le \pi, \quad -\pi \le -x \le 0$

$$\therefore \quad f(-x) = 1 + \frac{2(-x)}{\pi} = 1 - \frac{2x}{\pi} = f(x)$$

$\Rightarrow$ $f(x)$ is an even function of x in $[-\pi, \pi]$. This is also clear from its graph which is symmetrical above the y-axis.

$$\therefore \quad b_n = 0$$

Let $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

then $\quad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right) dx = \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi} = 0$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\left(1 - \frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

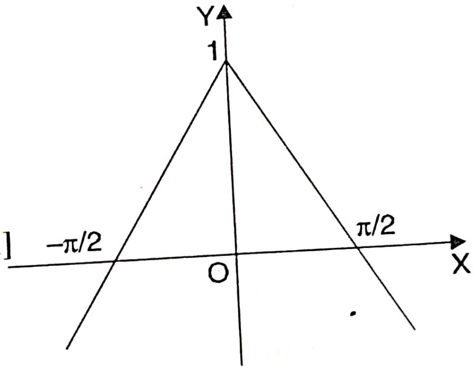
$$= \frac{2}{\pi}\left[-\frac{2\cos n\pi}{\pi n^2} + \frac{2}{\pi n^2}\right] = \frac{4}{\pi^2 n^2}[1 - (-1)^n]$$

$$\therefore \quad f(x) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} [1-(-1)^n] \frac{\cos nx}{n^2}$$

$$= \frac{4}{\pi^2}\left(\frac{2\cos x}{1^2} + \frac{2\cos 3x}{3^2} + \frac{2\cos 5x}{5^2} + \ldots\ldots\right)$$

$[\because \quad 1-(-1)^n = 0$ when n is even$]$

$$= \frac{8}{\pi^2}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right)$$



Putting $\quad x = 0$, we get $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots = \dfrac{\pi^2}{8}$, since $f(0) = 1$.

Example 11. *Show that for* $-\pi \le x \le \pi$,

$$\cos cx = \frac{\sin c\pi}{\pi}\left[\frac{1}{c} - \frac{2c\cos x}{c^2 - 1^2} + \frac{2c\cos 2x}{c^2 - 2^2} - \ldots\ldots\right]$$

where c is non-integral. Hence deduce that

$$\pi \operatorname{cosec}(c\pi) = \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{n+c} + \frac{1}{n+1-c}\right].$$

(M.D.U. 2011)

Sol. Since $\cos cx$ is an even function of x, $b_n = 0$

Let $\quad \cos cx = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx$ ...(1)

Then $\quad a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \cos cx \, dx = \dfrac{2}{\pi}\left[\dfrac{\sin cx}{c}\right]_0^{\pi}$

$$= \frac{2\sin c\pi}{c\pi}, \quad \text{since } c \text{ is non-integral, } \sin c\pi \ne 0$$

Also, $\quad a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \cos cx \cos nx \, dx = \dfrac{1}{\pi}\int_0^{\pi} [\cos(n+c)x + \cos(n-c)x]\, dx$

$$= \frac{1}{\pi}\left[\frac{\sin(n+c)x}{n+c} + \frac{\sin(n-c)x}{n-c}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+c)\pi}{n+c} + \frac{\sin(n-c)\pi}{n-c}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n\pi + c\pi)}{n+c} + \frac{\sin(n\pi - c\pi)}{n-c}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n \sin c\pi}{n+c} + \frac{(-1)^n \sin(-c\pi)}{n-c}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n \sin c\pi}{n+c} - \frac{(-1)^n \sin c\pi}{n-c}\right] = \frac{(-1)^n \sin c\pi}{\pi}\left(\frac{1}{n+c} - \frac{1}{n-c}\right)$$

$$= \frac{(-1)^n \sin c\pi}{\pi}\left[\frac{-2c}{n^2-c^2}\right] = \frac{(-1)^n \sin c\pi}{\pi} \cdot \frac{2c}{c^2-n^2}$$

$\therefore$ From (1),
$$\cos cx = \frac{\sin c\pi}{c\pi} + \frac{2c \sin c\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{c^2-n^2} \cos nx$$

$$= \frac{\sin c\pi}{\pi}\left[\frac{1}{c} + 2c\left\{-\frac{\cos x}{c^2-1^2} + \frac{\cos 2x}{c^2-2^2} - \ldots\ldots\right\}\right]$$

$$= \frac{\sin c\pi}{\pi}\left[\frac{1}{c} - \frac{2c \cos x}{c^2-1^2} + \frac{2c \cos 2x}{c^2-2^2} - \ldots\ldots\right]$$

Deduction Put $x = 0$

$$1 = \frac{\sin c\pi}{\pi}\left[\frac{1}{c} - \frac{2c}{c^2-1^2} + \frac{2c}{c^2-2^2} - \ldots\ldots\right]$$

$$\Rightarrow \quad \pi \operatorname{cosec}(c\pi) = \frac{1}{c} - \frac{(c+1)+(c-1)}{(c+1)(c-1)} + \frac{(c+2)+(c-2)}{(c+2)(c-2)} - \ldots\ldots$$

$$= \frac{1}{c} - \frac{1}{c-1} - \frac{1}{c+1} + \frac{1}{c-2} + \frac{1}{c+2} - \frac{1}{c-3} + \ldots\ldots$$

$$= \left(\frac{1}{c} + \frac{1}{1-c}\right) - \left(\frac{1}{c+1} + \frac{1}{2-c}\right) + \left(\frac{1}{c+2} + \frac{1}{3-c}\right) - \ldots\ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{n+c} + \frac{1}{n+1-c}\right]$$

Hence,
$$\pi \operatorname{cosec}(c\pi) = \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{n+c} + \frac{1}{n+1-c}\right]$$

EXERCISE 1.1

1. Expand in a Fourier series the function $f(x) = x$ in the interval $0 < x < 2\pi$.
2. Express $f(x) = \frac{1}{2}(\pi - x)$ in a Fourier series in the interval $0 < x < 2\pi$.
3. Find the Fourier series for the function $f(x) = x + x^2$ in the interval $-\pi < x < \pi$.

   Hence show that: (G.B.T.U. 2010)

   (i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}$. (U.K.T.U. 2011)

   (ii) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{6}$

4. Prove that for all values of x between $-\pi$ and π, $x = 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\ldots\right]$ *(M.D.U. 2011)*

5. Obtain the Fourier series to represent e^x in the interval $0 < x < 2\pi$.

6. Find the Fourier series to represent e^x in the interval $-\pi < x < \pi$. *(M.D.U. 2011)*

7. Find the Fourier series to represent the function $f(x) = |\sin x|, -\pi < x < \pi$.

8. Expand $f(x) = |\cos x|$ as a Fourier series in the interval $-\pi < x < \pi$.

(M.D.U. 2010, Dec. 2013; M.T.U. 2011)

9. Prove that in the interval $-\pi < x < \pi$, $x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{n(-1)^n}{n^2-1}\sin nx$.

10. Prove that for $-\pi < x < \pi$, $\frac{x(\pi^2 - x^2)}{12} = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \frac{\sin 4x}{4^3} + \ldots\ldots$

11. (*a*) Obtain a Fourier expansion for $\sqrt{1-\cos x}$ in the interval $-\pi < x < \pi$.

[**Hint.** For all integral values of n, $\cos\left(n + \frac{1}{2}\right)\pi = \cos(2n+1)\frac{\pi}{2} = 0 = \cos\left(n - \frac{1}{2}\right)\pi$.

$$\sqrt{1-\cos x} = \sqrt{2\sin^2\frac{x}{2}} = \sqrt{2}\left|\sin\frac{x}{2}\right| = \begin{cases} -\sqrt{2}\sin\frac{x}{2}, & -\pi < x \le 0 \\ \sqrt{2}\sin\frac{x}{2}, & 0 \le x < \pi \end{cases}\Bigg]$$

(*b*) Obtain a Fourier series for $\sqrt{1-\cos x}$ in the interval $(0, 2\pi)$ and hence find the value of

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots\ldots$$

12. Express $f(x) = \cos wx$, $-\pi < x < \pi$, where w is a fraction, as a Fourier series. Hence, prove that

$$\cot\theta = \frac{1}{\theta} + \frac{2\theta}{\theta^2 - \pi^2} + \frac{2\theta}{\theta^2 - 4\pi^2} + \ldots\ldots$$

13. Find the Fourier series for $f(x)$ in the interval $(-\pi, \pi)$ when

$$\begin{aligned} f(x) &= \pi + x, & -\pi < x < 0 \\ &= \pi - x, & 0 < x < \pi. \end{aligned}$$

14. Obtain a Fourier series to represent e^{-ax} from $x = -\pi$ to $x = \pi$. Hence derive series for $\frac{\pi}{\sinh \pi}$.

15. Prove that in the range $-\pi < x < \pi$, $\cosh ax = \frac{2a}{\pi}\sinh a\pi\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\cos nx\right]$.

16. Given $f(x) = \begin{cases} -x+1 & \text{for } -\pi \le x \le 0 \\ x+1 & \text{for } 0 \le x \le \pi \end{cases}$

Is the function even or odd ? Find the Fourier series for $f(x)$ and deduce the value of

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

$x+1$ $\quad -\pi \le -x \le 0$

$\pi \geq x \geq 0$

17. Find the Fourier series expansion for $f(x) = x + \frac{x^2}{4}$, $-\pi \le x \le \pi$.

18. Find the Fourier series of $f(x) = \frac{3x^2 - 6\pi x + 2\pi^2}{12}$ in the interval $(0, 2\pi)$. Hence deduce that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots.$$ *(M.T.U. 2012)*

19. Find the Fourier series to represent the function $f(x)$ given by $f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \pi \end{cases}$

and hence show that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$. *(DCRUST, Murthal May 2014)*

20. Obtain the Fourier series expansion for $f(x) = \pi^2 - x^2$ in the interval $-\pi < x < \pi$. *(M.D.U. May 2014)*

21. Find the Fourier series of period 2π for the function

$$f(x) = \begin{cases} \cos x - \sin x, & -\pi < x \le 0 \\ \cos x + \sin x, & 0 \le x < \pi \end{cases}$$

22. Find the Fourier series expansion of $f(x)$ in $(-\pi, \pi)$, where $f(x) = \sin^3 x + \cos^3 x$.

23. Find the Fourier series of $f(x) = \cos^4 x$ in $(0, 2\pi)$.

Answers

1. $f(x) = \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ **2.** $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$

3. $f(x) = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\ldots\right) + 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\ldots\right)$

5. $e^x = \frac{e^{2\pi} - 1}{2\pi} + \frac{e^{2\pi} - 1}{\pi}\sum_{n=1}^{\infty}\left(\frac{\cos nx}{1+n^2} - \frac{n}{1+n^2}\sin nx\right)$

6. $e^x = \frac{2\sinh \pi}{\pi}$

$$\left[\frac{1}{2} - \left(\frac{1}{2}\cos x - \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x - \ldots\ldots\right) - \left(\frac{1}{2}\sin x - \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x - \ldots\ldots\right)\right]$$

7. $|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \ldots\ldots + \frac{\cos 2nx}{4n^2 - 1} + \ldots\ldots\right)$

8. $|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left(\frac{\cos 2x}{3} - \frac{\cos 4x}{15} + \ldots\ldots\right)$

11. (*a*) $\sqrt{1 - \cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{4n^2 - 1}$ (*b*) Same as in part (*a*); $\frac{1}{2}$

12. $\cos wx = \frac{2w \sin w\pi}{\pi}\left(\frac{1}{2w^2} + \frac{\cos x}{1^2 - w^2} - \frac{\cos 2x}{2^2 - w^2} + \frac{\cos 3x}{3^2 - w^2} - \ldots\ldots\right)$

13. $f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\dots\right)$

14. $e^{-ax} = \frac{2\sinh a\pi}{\pi}\left[\left(\frac{1}{2a} - \frac{a\cos x}{1^2+a^2} + \frac{a\cos 2x}{2^2+a^2} - \dots\dots\right) - \left(\frac{\sin x}{1^2+a^2} - \frac{2\sin 2x}{2^2+a^2} + \frac{3\sin 3x}{3^2+a^2} - \dots\dots\right)\right]$

$\frac{\pi}{\sinh \pi} = 2\left[\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \dots\dots\right]$

16. Even. $f(x) = \frac{\pi}{2} + 1 - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\dots\right); \frac{\pi^2}{8}$.

17. $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$.

18. $f(x) = \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$.

19. $f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \dots\dots\right)$.

20. $f(x) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$.

21. $f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=2,4,6,\dots}\frac{n}{n^2-1}\cos nx$

22. $f(x) = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x + \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$.

23. $f(x) = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos 4x$.

1.4. DIRICHLET'S CONDITIONS

(M.D.U. May 2014, Dec. 2014, May 2015)

The sufficient conditions for the uniform convergence of a Fourier series are called Dirichlet's conditions (after Dirichlet, a German mathematician). All the functions that normally arise in engineering problems satisfy these conditions and hence they can be expressed as a Fourier series.

Any function $f(x)$ can be expressed as a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

where a_0, a_n, b_n are constants, provided

(i) $f(x)$ is periodic, single valued and finite.

(ii) $f(x)$ has a finite number of finite discontinuities in any one period.

(iii) $f(x)$ has a finite number of maxima and minima.

When these conditions are satisfied, the Fourier series converges to $f(x)$ at every point of continuity. At a point of discontinuity, the sum of the series is equal to the mean of the limits on the right and left.

i.e., $\frac{1}{2}[f(x+0)+f(x-0)]$

where $f(x+0)$ and $f(x-0)$ denote the limit on the right and the limit on the left respectively.

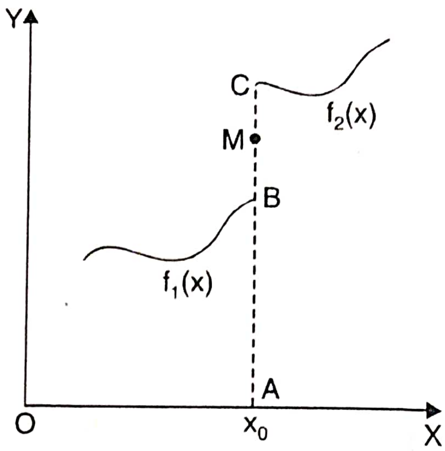
1.5. FOURIER SERIES FOR DISCONTINUOUS FUNCTIONS

In Art. 1.3, we derived Euler's formulae for a_0, a_n, b_n on the assumption that $f(x)$ is continous in $(c, c+2\pi)$. However, if $f(x)$ has finitely many points of finite discontinuity, even then it can be expressed as a Fourier series. The integrals for a_0, a_n, b_n are to be evaluated by breaking up the range of integration.

Let $f(x)$ be defined by $f(x) = f_1(x),\ c < x < x_0$

$= f_2(x),\ x_0 < x < c + 2\pi$

where x_0 is the point of finite discontinuity in the interval $(c, c+2\pi)$.



The values of a_0, a_n, b_n are given by

$$a_0 = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\,dx + \int_{x_0}^{c+2\pi} f_2(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\cos nx\,dx + \int_{x_0}^{c+2\pi} f_2(x)\cos nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\sin nx\,dx + \int_{x_0}^{c+2\pi} f_2(x)\sin nx\,dx\right]$$

At $x = x_0$, there is a finite jump in the graph of the function. Both the limits $f(x_0-0)$ and $f(x_0+0)$ exist but are unequal. The sum of the Fourier series $= \frac{1}{2}[f(x_0-0)+f(x_0+0)]$ $= \frac{1}{2}$ (AB + AC) = AM, where M is the mid-point of BC.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Fourier series to represent the function f(x) given by*

$$f(x) = x \quad \text{for} \quad 0 \le x \le \pi$$

and

$$= 2\pi - x \ \text{for} \quad \pi \le x \le 2\pi.$$

(G.B.T.U. 2010)

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$.

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Then $\quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right]$

$$= \frac{1}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\pi} + \left|2\pi x - \frac{x^2}{2}\right|_{\pi}^{2\pi}\right] = \frac{1}{\pi}\left[\frac{\pi^2}{2} + (4\pi^2 - 2\pi^2) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right] = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|x.\frac{\sin nx}{n} - 1.\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi} + \left|(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right) + \left(-\frac{\cos 2n\pi}{n^2} + \frac{\cos n\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi n^2}[(-1)^n - 1 - 1 + (-1)^n] = \frac{2}{\pi n^2}[(-1)^n - 1] = \begin{cases} -\dfrac{4}{\pi n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even}\end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|x\left(-\frac{\cos nx}{n}\right) - 1.\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi}\right.$$

$$\left. + \left|(2\pi - x)\times\left(-\frac{\cos nx}{n}\right) - 1.\left(-\frac{\sin nx}{n^2}\right)\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n} + \frac{\pi\cos n\pi}{n}\right] = 0$$

$$\therefore \qquad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right)$$

Putting $x = 0$, we get $0 = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots\right) \Rightarrow \dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots$

Example 2. *If* $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi,\end{cases}$

prove that $f(x) = \dfrac{1}{\pi} + \dfrac{1}{2}\sin x - \dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{\cos 2nx}{4n^2 - 1}$. (M.D.U. Dec. 2015; G.B.T.U. 2012)

Hence show that

(i) $\dfrac{1}{1.3} + \dfrac{1}{3.5} + \dfrac{1}{5.7} + \ldots\ldots = \dfrac{1}{2}$ (ii) $\dfrac{1}{1.3} - \dfrac{1}{3.5} + \dfrac{1}{5.7} - \ldots\ldots = \dfrac{\pi - 2}{4}$.

Sol. Let $\quad f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}\sin x\,dx\right] = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}\sin x\cos nx\,dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\cos nx\sin x\,dx = \frac{1}{2\pi}\int_0^{\pi}[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}, \; n \neq 1$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \begin{cases} \dfrac{1}{2\pi}\left(-\dfrac{1}{n+1} + \dfrac{1}{n-1} + \dfrac{1}{n+1} - \dfrac{1}{n-1}\right), & \text{when } n \text{ is odd} \\[2ex] \dfrac{1}{2\pi}\left(\dfrac{1}{n+1} - \dfrac{1}{n-1} + \dfrac{1}{n+1} - \dfrac{1}{n-1}\right), & \text{when } n \text{ is even} \end{cases}$$

$$= \begin{cases} 0, \text{ when } n \text{ is odd, } i.e., n = 3, 5, 7, \ldots.. \\ -\dfrac{2}{\pi(n^2-1)}, \text{ when } n \text{ is even} \end{cases}$$

When $n = 1$, we have $a_1 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi}\sin x\cos x\,dx = \dfrac{1}{2\pi}\int_0^{\pi}\sin 2x\,dx = \dfrac{1}{2\pi}\left[-\dfrac{\cos 2x}{2}\right]_0^{\pi} = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}\sin x\sin nx\,dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\sin nx\sin x\,dx = \frac{1}{2\pi}\int_0^{\pi}[\cos(n-1)x - \cos(n+1)x]\,dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi} = 0, \; n \neq 1$$

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_0^{\pi}\sin x\sin x\,dx = \frac{1}{2\pi}\int_0^{\pi}(1-\cos 2x)\,dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}$$

$\therefore$ $$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \ldots\ldots\right] + \frac{1}{2}\sin x$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(2n)^2-1} \qquad \ldots(1)$$

Putting $x = 0$ in (1), we have $0 = \dfrac{1}{\pi} - \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$

$$\Rightarrow \quad \frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots\ldots$$

Putting $x = \frac{\pi}{2}$ in (1), we have $1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi}{4n^2 - 1}$

$$\Rightarrow \quad \frac{1}{2} - \frac{1}{\pi} = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}$$

$$\Rightarrow \quad \frac{\pi - 2}{4} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)(2n+1)} = -\left(-\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \ldots\ldots\right)$$

$$\Rightarrow \quad \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \ldots\ldots = \frac{\pi - 2}{4}.$$

EXERCISE 1.2

1. Find the Fourier series to represent the function

$$f(x) = -k \quad \text{when} \quad -\pi < x < 0$$
$$= k \quad \text{when} \quad 0 < x < \pi$$

Also deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$ *(G.B.T.U. 2010, 2013)*

2. *(a)* Develop $f(x)$ in a Fourier series in the interval $(-\pi, \pi)$ if $f(x) = 0$ when $-\pi < x < 0$
$= 1$ when $0 < x < \pi$.

Deduce that sum of the Gregory series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$ is $\frac{\pi}{4}$.

(b) Find the Fourier series of the function defined by

$$f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ \pi, & 0 \le x < \pi \end{cases}$$ *(M.D.U. 2011)*

3. Find the Fourier series expansion for $f(x)$, if $f(x) = -\pi, \quad -\pi < x < 0$
$= x, \quad 0 < x < \pi$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$. *(M.D.U. Dec. 2013; U.K.T.U. 2012; G.B.T.U. 2011)*

[**Hint. For the deduction, put $x = 0$ in the expansion of $f(x)$.**

$$f(0-0) = -\pi \text{ and } f(0+0) = 0 \quad \therefore \quad f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = -\frac{\pi}{2}$$]

4. Find the Fourier expansion of the function defined in one period by the relations

$$f(x) = 1 \text{ for } 0 < x < \pi$$
$$= 2 \text{ for } \pi < x < 2\pi$$

and deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$

5. Find the Fourier series of $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ x^2, & 0 \le x \le \pi \end{cases}$

which is assumed to be periodic with period 2π. *(M.T.U. 2011)*

6. Find the Fourier series of the following function:

$$f(x) = x^2, \qquad 0 \le x \le \pi$$
$$= -x^2, \qquad -\pi \le x \le 0.$$

7. An alternating current after passing through a rectifier has the form

$$i = I_0 \sin x \qquad \text{for} \quad 0 \le x \le \pi$$
$$= 0 \qquad \text{for} \quad \pi \le x \le 2\pi$$

where I_0 is the maximum current and the period is 2π. Express i as a Fourier series.

8. Obtain Fourier series for the function

$$f(x) = \begin{cases} x & \text{for } -\pi < x < 0 \\ -x & \text{for } 0 < x < \pi \end{cases}$$ (M.T.U. 2013)

and hence show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}.$$ (G.B.T.U. 2010)

9. Find the Fourier series for the function

$$f(x) = \begin{cases} -1 & \text{for } -\pi < x < -\frac{\pi}{2} \\ 0 & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}. \\ 1 & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$$

Hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$. (G.B.T.U. 2012)

Answers

1. $f(x) = \frac{4k}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$

2. (a) $f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$

(b) $f(x) = \frac{\pi}{2} + 2\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$

3. $f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right) + \left(3\sin x - \frac{\sin 2x}{2} + \sin 3x - \frac{\sin 4x}{4} + \ldots\ldots\right)$

4. $f(x) = \frac{3}{2} - \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$

5. $f(x) = \frac{\pi^2}{6} - 2\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\ldots\right)$

$$-\frac{1}{\pi}\left[\left(\frac{4}{1^3} - \frac{\pi^2}{1}\right)\sin x + \frac{\pi^2}{2}\sin 2x + \left(\frac{4}{3^3} - \frac{\pi^2}{3}\right)\sin 3x + \frac{\pi^2}{4}\sin 4x + \ldots\ldots\right]$$

6. $f(x) = 2\left(\pi - \frac{4}{\pi}\right)\sin x - \pi \sin 2x + \frac{2}{3}\left(\pi - \frac{4}{9\pi}\right)\sin 3x - \frac{\pi}{2}\sin 4x + \ldots\ldots$

7. $i = \frac{I_0}{\pi} + \frac{I_0}{2}\sin x - \frac{2I_0}{\pi}\left(\frac{\cos 2x}{2^2 - 1} + \frac{\cos 4x}{4^2 - 1} + \frac{\cos 6x}{6^2 - 1} + \ldots\ldots\right)$

8. $f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right)$

9. $f(x) = \frac{2}{\pi}\left(\sin x - \sin 2x + \frac{\sin 3x}{3} - \ldots\ldots\right)$

1.6. CHANGE OF INTERVAL

In many engineering problems, it is desired to expand a function in a Fourier series over an interval of length $2l$ and not 2π. In order to apply foregoing theory, this interval must be transformed into an interval of length 2π. This can be achieved by a transformation of the variable.

Consider a periodic function $f(x)$ defined in the interval $c < x < c + 2l$. To change the interval into one of length 2π, we put

$$\frac{x}{l} = \frac{z}{\pi} \quad \text{or} \quad z = \frac{\pi x}{l} \quad \text{so that when } x = c,\ z = \frac{\pi c}{l} = d \text{ (say)}$$

and when $\quad x = c + 2l, \quad z = \frac{\pi(c + 2l)}{l} = \frac{\pi c}{l} + 2\pi = d + 2\pi.$

Thus the function $f(x)$ of period $2l$ in $(c, c + 2l)$ is transformed to the function $f\left(\frac{lz}{\pi}\right)$ $= F(z)$, say, of period 2π in $(d, d + 2\pi)$ and the latter function can be expressed as the Fourier series

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \qquad \ldots(1)$$

where $a_0 = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\, dz$; $a_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z) \cos nz\, dz$; and $b_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z) \sin nz\, dz \qquad \ldots(2)$

Now making the inverse substitution $z = \frac{\pi x}{l}$, $dz = \frac{\pi}{l}\, dx$

When $\quad z = d, \quad x = c \quad$ and when $\quad z = d + 2\pi,\ x = c + 2l.$

The expression (1) becomes $F(z) = F\left(\frac{\pi x}{l}\right) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

and the co-efficients a_0, a_n, b_n from (2) reduce to

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\, dx; \quad a_n = \frac{1}{l}\int_c^{c+2l} f(x) \cos \frac{n\pi x}{l}\, dx; \quad \text{and} \quad b_n = \frac{1}{l}\int_c^{c+2l} f(x) \sin \frac{n\pi x}{l}\, dx$$

Hence the Fourier series for f(x) in the interval c < x < c + 2l is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where $\quad a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\, dx, \quad a_n = \frac{1}{l}\int_c^{c+2l} f(x) \cos \frac{n\pi x}{l}\, dx \quad$ and $\quad b_n = \frac{1}{l}\int_c^{c+2l} f(x) \sin \frac{n\pi x}{l}\, dx.$

Cor. 1. If we put $c = 0$, the interval becomes $0 < x < 2l$ and the above results reduce to

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\, dx; \quad a_n = \frac{1}{l}\int_0^{2l} f(x) \cos\frac{n\pi x}{l}\, dx; \quad \text{and} \quad a_n = \frac{1}{l}\int_0^{2l} f(x) \sin\frac{n\pi x}{l}\, dx.$$

Cor. 2. If we put $c = -l$, the interval becomes $-l < x < l$ and the above results reduce to

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\, dx, \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\, dx \quad \text{and} \quad b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}\, dx.$$

If f(x) is an even function, we have

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\, dx, \quad a_n = \frac{2}{l}\int_0^{l} f(x) \cos\frac{n\pi x}{l}\, dx \quad \text{and} \quad b_n = 0$$

If f(x) is an odd function, we have $a_0 = 0, \quad a_n = 0$

and
$$b_n = \frac{2}{l}\int_0^{l} f(x) \sin\frac{n\pi x}{l}\, dx.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find Fourier expansion for the function $f(x) = x - x^2$, $-1 < x < 1$.*

Sol. Let $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos n\pi x + \sum\limits_{n=1}^{\infty} b_n \sin n\pi x$

Then
$$a_0 = \int_{-1}^{1} (x - x^2)\, dx = \int_{-1}^{1} x\, dx - \int_{-1}^{1} x^2\, dx = 0 - 2\int_0^1 x^2\, dx = -2\left[\frac{x^3}{3}\right]_0^1 = -\frac{2}{3}$$

$$a_n = \int_{-1}^{1} (x - x^2) \cos n\pi x\, dx = \int_{-1}^{1} x \cos n\pi x\, dx - \int_{-1}^{1} x^2 \cos n\pi x\, dx$$

$$= 0 - 2\int_0^1 x^2 \cos n\pi x\, dx = -2\left[x^2 \cdot \frac{\sin n\pi x}{n\pi} - 2x\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= -2\left[\frac{2\cos n\pi}{n^2\pi^2}\right] = \frac{-4(-1)^n}{n^2\pi^2} = \frac{4(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \int_{-1}^{1} (x - x^2) \sin n\pi x\, dx = \int_{-1}^{1} x \sin n\pi x\, dx - \int_{-1}^{1} x^2 \sin n\pi x\, dx$$

$$= 2\int_0^1 x \sin n\pi x\, dx - 0 = 2\left[x\left(-\frac{\cos n\pi x}{n\pi}\right) - 1.\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= 2\left[-\frac{\cos n\pi}{n\pi}\right] = \frac{-2(-1)^n}{n\pi} = \frac{2(-1)^{n+1}}{n\pi}$$

$$\therefore \quad x - x^2 = -\frac{1}{3} + \frac{4}{\pi^2}\left(\frac{\cos \pi x}{1^2} - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - \ldots\ldots\right)$$
$$+ \frac{2}{\pi}\left(\frac{\sin \pi x}{1} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \ldots\ldots\right)$$

Example 2. *Find the Fourier series to represent $f(x) = x^2 - 2$, when $-2 \le x \le 2$.*

(M.D.U. 2011)

Sol. Since $f(x)$ is an even function, $b_n = 0$.

Let $\qquad f(x) = x^2 - 2 = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \dfrac{n\pi x}{2}$

Then $\qquad a_0 = \dfrac{2}{2}\int_0^2 (x^2 - 2)\, dx = \left[\dfrac{x^3}{3} - 2x\right]_0^2 = \dfrac{8}{3} - 4 = -\dfrac{4}{3}$

$$a_n = \frac{2}{2}\int_0^2 (x^2 - 2) \cos \frac{n\pi x}{2}\, dx$$

$$= \left[(x^2 - 2) \cdot \frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} - 2x\left(-\frac{\cos \frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right) + 2\left(-\frac{\sin \frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}}\right)\right]_0^2$$

$$= \frac{16 \cos n\pi}{n^2\pi^2} = \frac{16(-1)^n}{n^2\pi^2}$$

$\therefore \qquad x^2 - 2 = -\dfrac{2}{3} - \dfrac{16}{\pi^2}\left(\cos \dfrac{\pi x}{2} - \dfrac{1}{4}\cos \pi x + \dfrac{1}{9}\cos \dfrac{3\pi x}{2} - \ldots\ldots\right).$

Example 3. *Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$.*

Sol. Let $f(x) = e^{-x} = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \dfrac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{l}$

Then $\quad a_0 = \dfrac{1}{l}\int_{-l}^{l} e^{-x}\, dx = \dfrac{1}{l}\left[-e^{-x}\right]_{-l}^{l} = \dfrac{1}{l}(e^l - e^{-l}) = \dfrac{2 \sinh l}{l}$

$$a_n = \frac{1}{l}\int_{-l}^{l} e^{-x} \cos \frac{n\pi x}{l}\, dx = \frac{1}{l}\left[\frac{e^{-x}}{1 + \left(\frac{n\pi}{l}\right)^2}\left(-\cos \frac{n\pi x}{l} + \frac{n\pi}{l} \sin \frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$\left[\because \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)\right]$$

$$= \frac{l}{l^2 + (n\pi)^2}[-e^{-l} \cos n\pi + e^l \cos n\pi] = -\frac{2l \cos n\pi}{l^2 + (n\pi)^2}\left(\frac{e^l - e^{-l}}{2}\right) = \frac{2l(-1)^n \sinh l}{l^2 + (n\pi)^2}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} e^{-x} \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1 + \left(\frac{n\pi}{l}\right)^2}\left(-\sin \frac{n\pi x}{l} - \frac{n\pi}{l} \cos \frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$\left[\because \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)\right]$$

$$= -\frac{1}{l^2+(n\pi)^2}\left[\frac{n\pi}{l}(e^{-l}-e^{l})\cos n\pi\right] = \frac{2n\pi\cos n\pi}{l^2+(n\pi)^2}\left(\frac{e^{l}-e^{-l}}{2}\right) = \frac{2n\pi(-1)^n\sinh l}{l^2+(n\pi)^2}$$

$$\therefore \quad e^{-x} = \sinh l\left[\frac{1}{l} - 2l\left(\frac{1}{l^2+\pi^2}\cos\frac{\pi x}{l} - \frac{1}{l^2+2^2\pi^2}\cos\frac{2\pi x}{l} + \frac{1}{l^2+3^2\pi^2}\cos\frac{3\pi x}{l} - \ldots\ldots\right)\right.$$

$$\left. - 2\pi\left(\frac{1}{l^2+\pi^2}\sin\frac{\pi x}{l} - \frac{2}{l^2+2^2\pi^2}\sin\frac{2\pi x}{l} + \frac{3}{l^2+3^2\pi^2}\sin\frac{3\pi x}{l} - \ldots\ldots\right)\right].$$

Example 4. *Obtain Fourier series for the function* $f(x) = \pi x, \quad 0 \le x \le 1$
$= \pi(2-x), \quad 1 \le x \le 2.$

(M.D.U. May 2013)

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$

Then $a_0 = \int_0^2 f(x)\,dx = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx = \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$

$$= \pi\left(\frac{1}{2}\right) + \pi\left[(4-2) - \left(2-\frac{1}{2}\right)\right] = \pi$$

$$a_n = \int_0^2 f(x)\cos n\pi x\,dx = \int_0^1 \pi x\cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \left[\pi x\,.\frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\,.\frac{\sin n\pi x}{n\pi} - (-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[\frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi}\right] + \left[-\frac{\cos 2n\pi}{n^2\pi} + \frac{\cos n\pi}{n^2\pi}\right] = \frac{2}{n^2\pi}(\cos n\pi - 1) = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$= 0$ or $-\frac{4}{n^2\pi}$ according as n is even or odd.

$$b_n = \int_0^2 f(x)\sin n\pi x\,dx = \int_0^1 \pi x\sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \left[\pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[-\frac{\cos n\pi}{n}\right] + \left[\frac{\cos n\pi}{n}\right] = 0$$

$$\therefore \quad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\ldots\right).$$

Note. Putting $x = 0$, we have $f(0) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right)$

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right)$$

$$\therefore \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}.$$

EXERCISE 1.3

1. Find a Fourier series for $f(t) = 1 - t^2$ when $-1 \le t \le 1$.
2. Expand $f(x)$ in Fourier series in the interval $(-2, 2)$ when $f(x) = 0, \quad -2 < x < 0$
$= 1, \quad 0 < x < 2.$
3. Develop $f(x)$ in a Fourier series in the interval $(0, 2)$ if $f(x) = x, \quad 0 < x < 1$
$= 0, \quad 1 < x < 2.$
4. Find the Fourier expansion for $f(x) = \pi x$ from $x = -c$ to $x = c$.
5. Find the Fourier expansion for the function $f(x) = x - x^3$ in the interval $-1 < x < 1$.
6. (a) Find the Fourier series for the function given by $f(t) = t, \quad 0 < t < 1$
$= 1 - t, \quad 1 < t < 2.$

Hence deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$ *(G.B.T.U. 2011)*

(b) Find the Fourier series for the function

$$f(x) = \begin{cases} x, & -1 < x \le 0 \\ x + 2, & 0 < x < 1 \end{cases}$$

where $f(x) = f(x + 2)$

(c) Find the Fourier series expansion of $f(x) = 2x - x^2$ in $(0, 3)$ and hence deduce that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}.$$ *(K.U.K. Dec. 2015)*

7. Find a Fourier series to represent x^2 in the interval $(-l, l)$.
8. Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-2, 2)$.
9. Expand: $$f(x) = \begin{cases} \frac{1}{4} - x, & 0 < x < \frac{1}{2} \\ x - \frac{3}{4}, & \frac{1}{2} < x < 1 \\ 1, & 1 < x < \frac{3}{2} \\ x - 1, & \frac{3}{2} < x < 2 \end{cases}$$

as a Fourier series.

10. Find the Fourier series for the function $f(x) = \begin{cases} 0 & \text{when } -2 < x < -1 \\ k & \text{when } -1 < x < 1 \\ 0 & \text{when } 1 < x < 2. \end{cases}$
11. A sinusoidal voltage E sin ωt is passed through a half-wave rectifier which clips the negative portion of the wave.

Expand the resulting periodic function $u(t) = \begin{cases} 0 & \text{when } -\frac{T}{2} < t < 0 \\ E \sin \omega t & \text{when } 0 < t < \frac{T}{2} \end{cases}$

and $T = \dfrac{2\pi}{\omega}$, in a Fourier series. (G.B.T.U. 2011)

Hence deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

12. Obtain the Fourier series expansion of (M.T.U. 2011)

$f(x) = \left(\dfrac{\pi - x}{2}\right)$ for $0 < x < 2$.

13. Find the Fourier series expansion of $f(x) = 1 + |x|$ defined in $-3 < x < 3$. (M.T.U. 2011)

14. Obtain the Fourier series for $f(x) = \begin{cases} -1 & \text{in} \quad -1 < x < 0 \\ 2x & \text{in} \quad 0 < x < 1 \end{cases}$

Hence show that $\quad 1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots = \dfrac{\pi^2}{8}$

and $\quad 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots = \dfrac{\pi}{4}$.

[**Hint:** For deductions, put $x = 0$ and $x = \dfrac{1}{2}$]

15. Obtain the Fourier series expansion of $f(x) = \begin{cases} 1 & \text{in} \quad 0 < x < 1 \\ 2 & \text{in} \quad 1 < x < 3 \end{cases}$.

16. Obtain the Fourier series expansion of $f(x) = x^2$ defined in the interval $0 < x < 2l$. Hence deduce that

(*i*) $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots = \dfrac{\pi^2}{6}$ (*ii*) $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \cdots = \dfrac{\pi^2}{12}$

(*iii*) $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots = \dfrac{\pi^2}{8}$

[**Hint:** (*i*) Put $x = 0$ (*ii*) Put $x = l$ (*iii*) Add]

Answers

1. $1 - t^2 = \dfrac{2}{3} + \dfrac{4}{\pi^2}\left(\cos \pi t - \dfrac{\cos 2\pi t}{2^2} + \dfrac{\cos 3\pi t}{3^2} - \ldots\ldots\right)$

2. $f(x) = \dfrac{1}{2} + \dfrac{2}{\pi}\left(\sin \dfrac{\pi x}{2} + \dfrac{1}{3}\sin \dfrac{3\pi x}{2} + \dfrac{1}{5}\sin \dfrac{5\pi x}{2} + \ldots\ldots\right)$

3. $f(x) = \dfrac{1}{4} - \dfrac{2}{\pi^2}\left(\cos \pi x + \dfrac{\cos 3\pi x}{3^2} + \dfrac{\cos 5\pi x}{5^2} + \ldots\ldots\right) + \dfrac{1}{\pi}\left(\sin \pi x - \dfrac{\sin 2\pi x}{2} + \dfrac{\sin 3\pi x}{3} - \ldots\ldots\right)$

4. $f(x) = 2c\left[\sin\left(\dfrac{\pi x}{c}\right) - \dfrac{1}{2}\sin\left(\dfrac{2\pi x}{c}\right) + \dfrac{1}{3}\sin\left(\dfrac{3\pi x}{c}\right) - \ldots\ldots\right]$

5. $f(x) = \dfrac{12}{\pi^3}\left(\sin \pi x - \dfrac{\sin 2\pi x}{2^3} + \dfrac{\sin 3\pi x}{3^3} - \ldots\ldots\right)$

6. (a) $f(t) = -\dfrac{4}{\pi^2}\left(\cos \pi t + \dfrac{\cos 3\pi t}{3^2} + \dfrac{\cos 5\pi t}{5^3} + \ldots\ldots\right) + \dfrac{2}{\pi}\left(\sin \pi t + \dfrac{\sin 3\pi t}{3} + \ldots\ldots\right)$

(b) $f(x) = 1 + \dfrac{2}{\pi}\sum_{n=1}^{\infty} \dfrac{1-2(-1)^n}{n} \sin n\pi x = 1 + \dfrac{2}{\pi}\left(3 \sin \pi x - \dfrac{\sin 2\pi x}{2} + \sin 3\pi x - \dfrac{\sin 4\pi x}{4} + \ldots\ldots\right)$

(c) $2x - x^2 = -\sum_{n=1}^{\infty} \dfrac{9}{n^2\pi^2} \cos \dfrac{2n\pi x}{3} + \sum_{n=1}^{\infty} \dfrac{3}{n\pi} \sin \dfrac{2n\pi x}{3}$

(For deduction, put $x = \dfrac{3}{2}$)

7. $x^2 = \dfrac{l^2}{3} - \dfrac{4l^2}{\pi^2}\left(\dfrac{\cos \pi x/l}{1^2} - \dfrac{\cos 2\pi x/l}{2^2} + \dfrac{\cos 3\pi x/l}{3^2} - \dfrac{\cos 4\pi x/l}{4^2} + \ldots\ldots\right)$

8. $$e^{-x} = \sinh 2\left[\frac{1}{2} - 4\left(\frac{1}{2^2+\pi^2}\cos\frac{\pi x}{2} - \frac{1}{2^2+2^2\pi^2}\cos \pi x + \frac{1}{2^2+3^2\pi^2}\cos\frac{3\pi x}{2} - \ldots\ldots\right)\right.$$
$$\left. - 2\pi\left(\frac{1}{2^2+\pi^2}\sin\frac{\pi x}{2} - \frac{2}{2^2+2^2\pi^2}\sin \pi x + \frac{3}{2^2+3^2\pi^2}\sin\frac{3\pi x}{2} - \ldots\ldots\right)\right]$$

9. $f(x) = \dfrac{7}{16} + \dfrac{1}{\pi}\left(\dfrac{1}{\pi} - \dfrac{1}{2}\right)\cos \pi x - \dfrac{1}{\pi}\left(\dfrac{1}{\pi} + \dfrac{3}{2}\right)\sin \pi x + \dfrac{3}{2\pi^2}\cos 2\pi x + \dfrac{1}{4\pi}\sin 2\pi x + \ldots\ldots$

10. $f(x) = \dfrac{k}{2} + \dfrac{2k}{\pi}\left(\cos\dfrac{\pi x}{2} - \dfrac{1}{3}\cos\dfrac{3\pi x}{2} + \dfrac{1}{5}\cos\dfrac{5\pi x}{2} - \ldots\ldots\right)$

11. $u(t) = \dfrac{E}{\pi} + \dfrac{E}{2}\sin \omega t - \dfrac{2E}{\pi}\left(\dfrac{\cos 2\omega t}{1.3} + \dfrac{\cos 4\omega t}{3.5} + \dfrac{\cos 6\omega t}{5.7} + \ldots\ldots\right)$

12. $f(x) = \dfrac{\pi - 1}{2} + \dfrac{1}{\pi}\sum_{n=1}^{\infty} \dfrac{\sin n\pi x}{n}$.

13. $f(x) = \dfrac{5}{2} - \dfrac{12}{\pi^2}\left(\dfrac{\cos x}{1^2} + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \ldots\right)$

14. $$f(x) = -\frac{4}{\pi^2}\left(\cos \pi x + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\right)$$
$$+ \frac{2}{\pi}\left(2\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{2\sin 3\pi x}{3} - \frac{\sin 4\pi x}{4} + \frac{2\sin 5\pi x}{5} - \ldots\right)$$

15. $f(x) = \dfrac{5}{3} - \dfrac{1}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{n}\sin\dfrac{2n\pi}{3}\cos\dfrac{2n\pi x}{3} + \dfrac{1}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\left(\cos\dfrac{2n\pi}{3} - 1\right)\sin\dfrac{2n\pi x}{3}$

16. $f(x) = \dfrac{4l^2}{3} + \dfrac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{n^2}\cos\dfrac{n\pi x}{l} - \dfrac{4l^2}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\sin\dfrac{n\pi x}{l}$

1.7. HALF RANGE SERIES

Sometimes it is required to expand a function $f(x)$ in the range $(0, \pi)$ in a Fourier series of period 2π or more generally in the range $(0, l)$ in a Fourier series of period $2l$.

If it is required to expand $f(x)$ in the interval $(0, l)$, then it is immaterial what the function may be outside the range $0 < x < l$. We are free to choose it arbitrarily in the interval $(-l, 0)$.

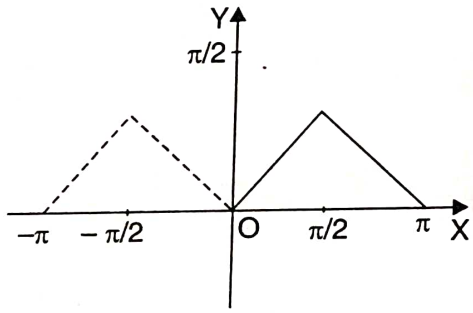
If we extend the function $f(x)$ by reflecting it in the y-axis so that $f(-x) = f(x)$, then the extended function is even for which $b_n = 0$. The Fourier expansion of $f(x)$ will contain only cosine terms.

If we extend the function $f(x)$ by reflecting it in the origin so that $f(-x) = -f(x)$, then the extended function is odd for which $a_0 = a_n = 0$. The Fourier expansion of $f(x)$ will contain only sine terms.

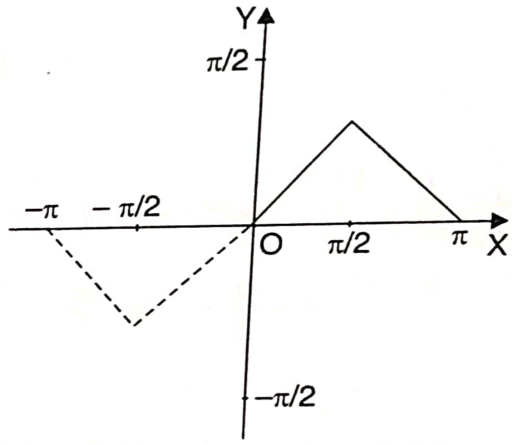
For example, consider the function

$$f(x) = x, \qquad 0 < x < \frac{\pi}{2}$$

$$= \pi - x \qquad \frac{\pi}{2} < x < \pi$$



(Reflection in the y-axis)



(Reflection in the origin)

Hence a function $f(x)$ defined over the interval $0 < x < l$ is capable of two distinct half-range series.

The half-range cosine series is $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos \dfrac{n\pi x}{l}$

where $a_0 = \dfrac{2}{l} \displaystyle\int_0^l f(x)\, dx$; $a_n = \dfrac{2}{l} \displaystyle\int_0^l f(x) \cos \dfrac{n\pi x}{l}\, dx$.

The half-range sine series is $f(x) = \sum\limits_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{l}$, where $b_n = \dfrac{2}{l} \displaystyle\int_0^l f(x) \sin \dfrac{n\pi x}{l}\, dx$.

Cor. If the range is $0 < x < \pi$, then

(*i*) The half-range cosine series is $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \dfrac{2}{\pi} \displaystyle\int_0^\pi f(x)\, dx$; $a_n = \dfrac{2}{\pi} \displaystyle\int_0^\pi f(x) \cos nx\, dx$.

(*ii*) The half-range sine series is $f(x) = \sum\limits_{n=1}^{\infty} b_n \sin nx$, where $b_n = \dfrac{2}{\pi} \displaystyle\int_0^\pi f(x) \sin nx\, dx$.

ILLUSTRATIVE EXAMPLES

Example 1. *Expand $\pi x - x^2$ in a half-range sine series in the interval $(0, \pi)$ upto the first three terms.*

Sol. Let $\pi x - x^2 = \sum_{n=1}^{\infty} b_n \sin nx$, then

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin nx \, dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2\cos n\pi}{n^3} + \frac{2}{n^3}\right] = \frac{4}{\pi n^3}[1 - (-1)^n]$$

$$= 0 \quad \text{or} \quad \frac{8}{\pi n^3} \text{ according as } n \text{ is even or odd.}$$

$$\therefore \qquad \pi x - x^2 = \frac{8}{\pi}\left(\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots\ldots\right).$$

Example 2. *If* $\quad f(x) = x, \quad 0 < x < \frac{\pi}{2}$

$$= \pi - x, \quad \frac{\pi}{2} < x < \pi$$

show that (i) $f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\ldots\right]$ (M.D.U. 2012)

(ii) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \ldots\ldots\right].$

Sol. (*i*) **For the half-range sine series**

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

Then $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\, dx\right]$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1.\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right] + \frac{2}{\pi}\left[\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\sin\frac{n\pi}{2}\right] = \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$$

When n is even, $b_n = 0$.

$$\therefore \qquad f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\ldots\right]$$

(*ii*) **For the half-range cosine series**

Let $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Then $a_0 = \dfrac{2}{\pi}\int_0^{\pi} f(x)\,dx = \dfrac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right]$

$$= \frac{2}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\pi/2} + \left|\pi x - \frac{x^2}{2}\right|_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right] = \frac{2}{\pi}\left[\frac{\pi^2}{4}\right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[x\,.\,\frac{\sin nx}{n} - 1.\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\,.\,\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2}\right] + \frac{2}{\pi}\left[-\frac{\cos n\pi}{n^2} - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - \cos n\pi - 1\right]$$

$\therefore \quad a_1 = 0,\ a_2 = \dfrac{2}{\pi\,.\,2^2}(2\cos\pi - \cos 2\pi - 1) = \dfrac{-2}{\pi\,.\,1^2},$

$a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ a_6 = \dfrac{2}{\pi\,.\,6^2}(2\cos 3\pi - \cos 6\pi - 1) = \dfrac{-2}{\pi\,.\,3^2},$

$a_7 = 0,\ a_8 = 0,\ a_9 = 0,\ a_{10} = \dfrac{2}{\pi\,.\,10^2}(2\cos 5\pi - \cos 10\pi - 1) = \dfrac{-2}{\pi\,.\,5^2},\ \ldots\ldots$

Hence $f(x) = \dfrac{\pi}{4} - \dfrac{2}{\pi}\left[\dfrac{\cos 2x}{1^2} + \dfrac{\cos 6x}{3^2} + \dfrac{\cos 10x}{5^2} + \ldots\ldots\right].$

Example 3. *Find a series of cosines of multiples of x which will represent x sin x in the interval (0, π) and show that* $\dfrac{1}{1\cdot3} - \dfrac{1}{3\cdot5} + \dfrac{1}{5\cdot7} - \ldots\ldots = \dfrac{\pi - 2}{4}.$ (M.T.U. 2013)

Sol. Let $x \sin x = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Then $a_0 = \dfrac{2}{\pi}\int_0^{\pi} x\sin x\,dx = \dfrac{2}{\pi}\Big[x(-\cos x) - 1.(-\sin x)\Big]_0^{\pi} = \dfrac{2}{\pi}[-\pi\cos\pi] = 2$

$a_n = \dfrac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx = \dfrac{1}{\pi}\int_0^{\pi} x\,(2\cos nx\sin x)\,dx$

$$= \frac{1}{\pi}\int_0^{\pi} x\,[\sin (n+1)\,x - \sin (n-1)\,x]\,dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1}\right\} - 1.\left\{-\frac{\sin (n+1)x}{(n+1)^2} - \frac{\sin (n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos (n+1)\pi}{n+1} + \frac{\pi \cos (n-1)\pi}{n-1}\right], \quad \text{when } n \neq 1$$

$$= -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} = (-1)^{n-1}\left[\frac{1}{n-1} - \frac{1}{n+1}\right] = \frac{2(-1)^{n-1}}{(n-1)(n+1)}$$

When $n = 1$, we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x\,dx = \frac{1}{\pi}\int_0^{\pi} x \sin 2x\,dx = \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{2^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi \cos 2\pi}{2}\right] = -\frac{1}{2}$$

$$\therefore \quad x \sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{1.3} - \frac{\cos 3x}{2.4} + \frac{\cos 4x}{3.5} - \dots\dots\right).$$

Putting $x = \frac{\pi}{2}$, we get $\frac{\pi}{2} = 1 - 2\left(\frac{-1}{1.3} + \frac{1}{3.5} + \frac{-1}{5.7} - \dots\dots\right)$

$$\Rightarrow \qquad 1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \dots\dots = \frac{\pi}{2}$$

$$\Rightarrow \qquad \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \dots\dots = \frac{\pi}{2} - 1$$

Hence $$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots\dots = \frac{\pi - 2}{4}.$$

Example 4. *Obtain the half-range sine series for e^x in $0 < x < 1$.*

Sol. Let $e^x = \sum_{n=1}^{\infty} b_n \sin n\pi x$, (since $l = 1$)

Then $$b_n = 2\int_0^1 e^x \sin n\pi x\,dx = 2\left[\frac{e^x}{1+(n\pi)^2}(\sin n\pi x - n\pi \cos n\pi x)\right]_0^1$$

$$\left[\because \int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)\right]$$

$$= 2\left[\frac{e}{1+(n\pi)^2}(-n\pi \cos n\pi) - \frac{1}{1+(n\pi)^2}(-n\pi)\right]$$

$$= \frac{2}{1+n^2\pi^2}[-en\pi(-1)^n + n\pi] = \frac{2n\pi}{1+n^2\pi^2}[1-e(-1)^n]$$

Hence $e^x = 2\pi \sum_{n=1}^{\infty} \frac{n[1-e(-1)^n]}{1+n^2\pi^2}$

$$= 2\pi\left[\frac{1+e}{1+\pi^2}\sin \pi x + \frac{2(1-e)}{1+4\pi^2}\sin 2\pi x + \frac{3(1+e)}{1+9\pi^2}\sin 3\pi x + \ldots\ldots\right].$$

Example 5. *Develop* $\sin\left(\frac{\pi x}{l}\right)$ *in half-range cosine series in the range* $0 < x < l$.

Sol. Let $\sin\left(\frac{\pi x}{l}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

then $$a_0 = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\,dx = \frac{2}{l}\left[-\frac{\cos\frac{\pi x}{l}}{\frac{\pi}{l}}\right]_0^l = -\frac{2}{\pi}[\cos\pi - 1] = \frac{4}{\pi}$$

$$a_n = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^l \left[\sin(n+1)\frac{\pi x}{l} - \sin(n-1)\frac{\pi x}{l}\right]dx$$

$$= \frac{1}{l}\left[-\frac{\cos(n+1)\frac{\pi x}{l}}{(n+1)\frac{\pi}{l}} + \frac{\cos(n-1)\frac{\pi x}{l}}{(n-1)\frac{\pi}{l}}\right]_0^l, \quad \text{when } n \neq 1$$

$$= \frac{1}{\pi}\left[\left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\} - \left\{-\frac{1}{n+1} + \frac{1}{n-1}\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

When $n = 1$, we have

$$a_1 = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{\pi x}{l}\,dx = \frac{1}{l}\int_0^l \sin\frac{2\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\frac{-\cos\frac{2\pi x}{l}}{\frac{2\pi}{l}}\right]_0^l = -\frac{1}{2\pi}(\cos 2\pi - \cos 0)$$

$$= -\frac{1}{2\pi}(1-1) = 0.$$

When n is odd, $n \neq 1$, $a_n = \frac{1}{\pi}\left[-\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] = 0$, also $a_1 = 0$

When n is even, $a_n = \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$

$$= \frac{2}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right] = -\frac{4}{\pi(n+1)(n-1)}$$

$$\therefore \qquad \sin\left(\frac{\pi x}{l}\right) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos\frac{2\pi x}{l}}{1\cdot 3} + \frac{\cos\frac{4\pi x}{l}}{3\cdot 5} + \frac{\cos\frac{6\pi x}{l}}{5\cdot 7} + \ldots\ldots\right]$$

Example 6. *Obtain a half-range cosine series for*

$$f(x) = kx \qquad \textit{for} \quad 0 \le x \le \frac{l}{2}$$

$$= k(l - x) \qquad \textit{for} \quad \frac{l}{2} \le x \le l.$$

Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$ (M.D.U. Dec. 2014)

Sol. Let $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

then
$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\left[\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right]$$

$$= \frac{2}{l}\left[\left|\frac{kx^2}{2}\right|_0^{l/2} + \left|k\left(lx - \frac{x^2}{2}\right)\right|_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\frac{kl^2}{8} + k\left(l^2 - \frac{l^2}{2}\right) - k\left(\frac{l^2}{2} - \frac{l^2}{8}\right)\right] = \frac{2}{l}\left(\frac{kl^2}{4}\right) = \frac{kl}{2}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\left[\int_0^{l/2} kx\cdot\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} k(l-x)\cdot\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\left|kx\cdot\frac{1}{n\pi}\sin\frac{n\pi x}{l} + k\cdot\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right|_0^{l/2}\right.$$
$$\left. + \left|k(l-x)\,.\frac{1}{n\pi}\sin\frac{n\pi x}{l} - k\,.\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right|_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\left|\frac{kl^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{kl^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\right|\right.$$
$$\left. + \left|\frac{-kl^2}{n^2\pi^2}\cos n\pi - \frac{kl^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{kl^2}{n^2\pi^2}\cos\frac{n\pi}{2}\right|\right]$$

$$= \frac{2}{l}\left[\frac{2kl^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{kl^2}{n^2\pi^2} - \frac{kl^2}{n^2\pi^2}\cos n\pi\right] = \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$

When n is odd, $\cos\frac{n\pi}{2} = 0$ and $\cos n\pi = -1$ $\quad \therefore \quad a_n = 0 \;\Rightarrow\; a_1 = a_3 = a_5 = \ldots\ldots = 0$

When n is even, $a_2 = \dfrac{2kl}{2^2\pi^2}\,[2\cos\pi - 1 - \cos 2\pi] = -\dfrac{8kl}{2^2\pi^2}$;

$$a_4 = \frac{2kl}{4^2\pi^2}\,[2\cos 2\pi - 1 - \cos 4\pi] = 0$$

$$a_6 = \frac{2kl}{6^2\pi^2}\,[2\cos 3\pi - 1 - \cos 6\pi]$$

$$= \frac{2kl}{6^2\pi^2}(-2 - 1 - 1) = -\frac{8kl}{6^2\pi^2} \text{ and so on.}$$

$$\therefore \qquad f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \ldots\ldots\right) \qquad \ldots(1)$$

Putting $\quad x = l,\ f(l) = 0$

$\therefore$ From (1), we have $\quad 0 = \dfrac{kl}{4} - \dfrac{8kl}{\pi^2}\left(\dfrac{1}{2^2} + \dfrac{1}{6^2} + \ldots\ldots\right)$

$$\Rightarrow \qquad \frac{1}{2^2} + \frac{1}{6^2} + \ldots\ldots = \frac{\pi^2}{32} \quad \Rightarrow \quad \frac{1}{2^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \ldots\ldots\right] = \frac{\pi^2}{32}$$

Hence $\quad \dfrac{1}{1^2} + \dfrac{1}{3^2} + \ldots\ldots = \dfrac{\pi^2}{8}$.

EXERCISE 1.4

1. (*a*) Obtain cosine and sine series for $f(x) = x$ in the interval $0 \le x \le \pi$. Hence show that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots = \dfrac{\pi^2}{8}$.

   (*b*) Prove that for $0 < x < l$

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \ldots\ldots\right)$$

2. Find the half-range cosine series for the function $f(x) = x^2$ in the range $0 \le x \le \pi$.

   (M.D.U. Dec. 2015; K.U.K. Jan. 2014)

3. Find the half-range cosine series for the function $f(x) = (x - 1)^2$ in the interval $0 < x < 1$. Hence show that

   (*i*) $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \ldots\ldots = \dfrac{\pi^2}{6}$ $\qquad$ (*ii*) $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots\ldots = \dfrac{\pi^2}{12}$

   (*iii*) $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \ldots\ldots = \dfrac{\pi^2}{8}$.

When n is odd, $\cos \frac{n\pi}{2} = 0$ and $\cos n\pi = -1$ $\quad \therefore \quad a_n = 0 \Rightarrow a_1 = a_3 = a_5 = \ldots\ldots = 0$

When n is even, $a_2 = \frac{2kl}{2^2\pi^2}[2\cos\pi - 1 - \cos 2\pi] = -\frac{8kl}{2^2\pi^2}$;

$$a_4 = \frac{2kl}{4^2\pi^2}[2\cos 2\pi - 1 - \cos 4\pi] = 0$$

$$a_6 = \frac{2kl}{6^2\pi^2}[2\cos 3\pi - 1 - \cos 6\pi]$$

$$= \frac{2kl}{6^2\pi^2}(-2-1-1) = -\frac{8kl}{6^2\pi^2} \text{ and so on.}$$

$$\therefore \quad f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \ldots\ldots\right) \qquad \ldots(1)$$

Putting $\quad x = l,\ f(l) = 0$

$\therefore$ From (1), we have $0 = \frac{kl}{4} - \frac{8kl}{\pi^2}\left(\frac{1}{2^2} + \frac{1}{6^2} + \ldots\ldots\right)$

$$\Rightarrow \quad \frac{1}{2^2} + \frac{1}{6^2} + \ldots\ldots = \frac{\pi^2}{32} \quad \Rightarrow \quad \frac{1}{2^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \ldots\ldots\right] = \frac{\pi^2}{32}$$

Hence $\quad \frac{1}{1^2} + \frac{1}{3^2} + \ldots\ldots = \frac{\pi^2}{8}$.

EXERCISE 1.4

1. (a) Obtain cosine and sine series for $f(x) = x$ in the interval $0 \le x \le \pi$. Hence show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}.$$

(b) Prove that for $0 < x < l$

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \ldots\ldots\right)$$

2. Find the half-range cosine series for the function $f(x) = x^2$ in the range $0 \le x \le \pi$.

(M.D.U. Dec. 2015; K.U.K. Jan. 2014)

3. Find the half-range cosine series for the function $f(x) = (x-1)^2$ in the interval $0 < x < 1$. Hence show that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{6}$ $\qquad$ (ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots = \frac{\pi^2}{8}$.

4. (a) Express $\sin x$ as a cosine series in $0 < x < \pi$.

(b) Show that a constant function c can be expanded in an infinite series $\frac{4c}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$ in the range $0 < x < \pi$.

(c) Find the Fourier half-range sine series of $f(x) = 1,\ 0 \le x \le 2$.

5. If

$$f(x) = \begin{cases} \dfrac{\pi}{3}, & 0 \le x \le \dfrac{\pi}{3} \\ 0, & \dfrac{\pi}{3} < x < \dfrac{2\pi}{3} \\ -\dfrac{\pi}{3}, & \dfrac{2\pi}{3} \le x \le \pi \end{cases}$$

then show that $f(x) = \frac{2}{\sqrt{3}}\left[\cos x - \frac{\cos 5x}{5} + \frac{\cos 7x}{7} - \ldots\ldots\right]$.

6. If

$$f(x) = \begin{cases} mx, & 0 \le x \le \dfrac{\pi}{2} \\ m(\pi - x), & \dfrac{\pi}{2} \le x \le \pi \end{cases}$$

then show that $f(x) = \frac{4m}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\ldots\right]$.

7. Express $f(x) = x$ as a half-range.

(i) sine series in $0 < x < 2$. *(M.D.U. Dec. 2013, Dec. 2014, May 2015; U.K.T.U. 2*

(ii) cosine series in $0 < x < 2$. *(K.U.K. 2010, May 2013; M.D.U. 2*

8. Find the Fourier sine and cosine series of

$$f(x) = \begin{cases} x, & 0 < x < \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} < x < \pi \end{cases}$$

9. Show that the series $\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{l}{n} \sin \frac{2n\pi x}{l}$ represents $\frac{1}{2}(l - x)$ when $0 < x < l$

10. Find the half-range sine series for

$$f(x) = \begin{cases} \dfrac{1}{4} - x, & 0 < x < \dfrac{1}{2} \\ x - \dfrac{3}{4}, & \dfrac{1}{2} < x < 1 \end{cases}$$

11. Represent the following function by Fourier sine series

$$f(x) = \begin{cases} 1 & \text{when } 0 < x < \dfrac{l}{2} \\ 0 & \text{when } \dfrac{l}{2} < x < l \end{cases}$$

12. Find the half-range sine series for the function [illegible]

4. (*a*) Express $\sin x$ as a cosine series in $0 < x < \pi$.

(*b*) Show that a constant function c can be expanded in an infinite series

$$\frac{4c}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right) \text{ in the range } 0 < x < \pi.$$

(*c*) Find the Fourier half-range sine series of $f(x) = 1,\ 0 \le x \le 2$.

5. If

$$f(x) = \begin{cases} \frac{\pi}{3}, & 0 \le x \le \frac{\pi}{3} \\ 0, & \frac{\pi}{3} < x < \frac{2\pi}{3} \\ -\frac{\pi}{3}, & \frac{2\pi}{3} \le x \le \pi \end{cases}$$

then show that $f(x) = \frac{2}{\sqrt{3}}\left[\cos x - \frac{\cos 5x}{5} + \frac{\cos 7x}{7} - \ldots\ldots\right]$.

6. If

$$f(x) = \begin{cases} mx, & 0 \le x \le \frac{\pi}{2} \\ m(\pi - x), & \frac{\pi}{2} \le x \le \pi \end{cases}$$

then show that $f(x) = \frac{4m}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\ldots\right]$.

7. Express $f(x) = x$ as a half-range. *(M.D.U. Dec. 2013, Dec. 2014, May 2015; U.K.T.U. 2011)*

(*i*) sine series in $0 < x < 2$. *(K.U.K. 2010, May 2013; M.D.U. 2010)*

(*ii*) cosine series in $0 < x < 2$.

8. Find the Fourier sine and cosine series of

$$f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi \end{cases}$$

9. Show that the series $\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{l}{n}\sin\frac{2n\pi x}{l}$ represents $\frac{1}{2}l - x$ when $0 < x < l$.

10. Find the half-range sine series for

$$f(x) = \begin{cases} \frac{1}{4} - x, & 0 < x < \frac{1}{2} \\ x - \frac{3}{4}, & \frac{1}{2} < x < 1. \end{cases}$$

(U.K.T.U. 2012)

11. Represent the following function by Fourier sine series

$$f(x) = \begin{cases} 1 \text{ when } 0 < x < \frac{l}{2} \\ 0 \text{ when } \frac{l}{2} < x < l. \end{cases}$$

(M.T.U. 2011)

12. Find the half-range sine series for the function $f(t) = t - t^2,\ 0 < t < 1$.

13. Prove that for $0 < x < \pi$,

$$x(\pi - x) = \frac{\pi^2}{6} - \left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \ldots\ldots\right)$$

14. Let $f(x) = \begin{cases} \omega x, & \text{when } 0 \le x \le \dfrac{l}{2} \\ \omega(l - x), & \text{when } \dfrac{l}{2} \le x \le l \end{cases}$

Show that $f(x) = \dfrac{4\omega l}{\pi^2} \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)^2} \sin \dfrac{(2n+1)\pi x}{l}$.

Hence obtain the sum of the series

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$ (G.B.T.U. 2011)

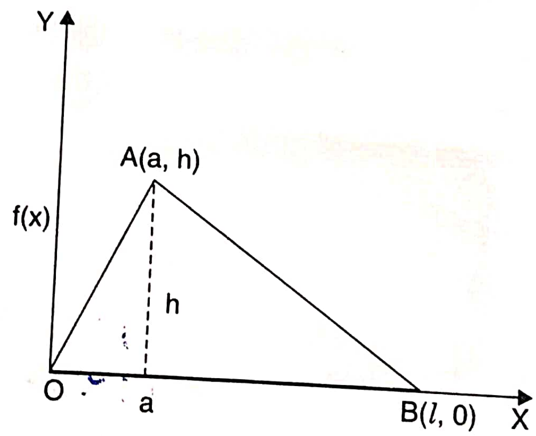
15. If $f(x) = \begin{cases} \sin x, & \text{for } 0 \le x < \dfrac{\pi}{4} \\ \cos x, & \text{for } \dfrac{\pi}{4} \le x \le \dfrac{\pi}{2} \end{cases}$ expand $f(x)$ in a series of sines. (M.T.U. 2011)

16. Find the half range sine series of $f(x) = lx - x^2$ in the interval $(0, l)$. Hence, deduce that

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$ (M.T.U. 2012)

17. Obtain the half-range sine series of the function $f(x) = x \sin x$ in $0 < x < \pi$. (K.U.K. Dec. 2015)

18. For the function defined by the graph OAB, find the half-range Fourier sine series.



Answers

1. (a) $\dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\dfrac{\cos x}{1^2} + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \ldots\ldots\right)$; $2\left(\sin x - \dfrac{\sin 2x}{2} + \dfrac{\sin 3x}{3} - \dfrac{\sin 4x}{4} + \ldots\ldots\right)$

2. $\dfrac{\pi^2}{3} - 4\left[\dfrac{\cos x}{1^2} - \dfrac{\cos 2x}{2^2} + \dfrac{\cos 3x}{3^2} - \dfrac{\cos 4x}{4^2} + \ldots\ldots\right]$

3. $\dfrac{1}{3} + \dfrac{4}{\pi^2}\left(\dfrac{\cos \pi x}{1^2} + \dfrac{\cos 2\pi x}{2^2} + \dfrac{\cos 3\pi x}{3^2} + \ldots\ldots\right)$ **4.** (a) $\dfrac{2}{\pi} - \dfrac{4}{\pi}\left[\dfrac{\cos 2x}{1 \cdot 3} + \dfrac{\cos 4x}{3 \cdot 5} + \dfrac{\cos 6x}{5 \cdot 7} + \ldots\ldots\right]$

(c) $1 = \dfrac{4}{\pi}\left(\sin \dfrac{\pi x}{2} + \dfrac{1}{3}\sin \dfrac{3\pi x}{2} + \dfrac{1}{5}\sin \dfrac{5\pi x}{2} + \ldots\ldots\right)$

7. (i) $\dfrac{4}{\pi}\left(\sin\dfrac{\pi x}{2}-\dfrac{1}{2}\sin\dfrac{2\pi x}{2}+\dfrac{1}{3}\sin\dfrac{3\pi x}{2}-\dfrac{1}{4}\sin\dfrac{4\pi x}{2}+\ldots\ldots\right)$

(ii) $1-\dfrac{8}{\pi^2}\left[\cos\dfrac{\pi x}{2}+\dfrac{1}{3^2}\cos\dfrac{3\pi x}{2}+\dfrac{1}{5^2}\cos\dfrac{5\pi x}{2}+\ldots\ldots\right]$

8. (i) $f(x)=\dfrac{2}{\pi}\left(\dfrac{\sin x}{1^2}-\dfrac{\sin 3x}{3^2}+\dfrac{\sin 5x}{5^2}-\ldots\ldots\right)+\left(\dfrac{\sin 2x}{2}-\dfrac{\sin 4x}{4}+\dfrac{\sin 6x}{6}-\ldots\ldots\right)$

(ii) $f(x)=\dfrac{\pi}{8}+\dfrac{2}{\pi}\left[\left(\dfrac{\pi}{2}-1\right)\cos x-\dfrac{1}{2}\cos 2x-\left(\dfrac{\pi}{6}+\dfrac{1}{3^2}\right)\cos 3x+\left(\dfrac{\pi}{10}-\dfrac{1}{5^2}\right)\cos 5x\ldots\ldots\right]$

10. $f(x)=\left(\dfrac{1}{\pi}-\dfrac{4}{\pi^2}\right)\sin\pi x+\left(\dfrac{1}{3\pi}+\dfrac{4}{3^2\pi^2}\right)\sin 3\pi x+\left(\dfrac{1}{5\pi}-\dfrac{4}{5^2\pi^2}\right)\sin 5\pi x+\cdots$

11. $f(x)=\dfrac{2}{\pi}\left[\sin\dfrac{\pi x}{l}+\sin\dfrac{2\pi x}{l}+\dfrac{1}{3}\sin\dfrac{3\pi x}{l}+\dfrac{1}{5}\sin\dfrac{5\pi x}{l}+\ldots\ldots\right]$

12. $\dfrac{8}{\pi^3}\left(\dfrac{\sin\pi t}{1^3}+\dfrac{\sin 3\pi t}{3^3}+\dfrac{\sin 5\pi t}{5^3}+\ldots\ldots\right)$ 14. $\dfrac{\pi^2}{8}$

15. $\dfrac{4\sqrt{2}}{\pi}\left(\dfrac{\sin 2x}{1\cdot 3}-\dfrac{\sin 6x}{5\cdot 7}+\dfrac{\sin 10x}{9\cdot 11}\ldots\ldots\right)$

16. $f(x)=\dfrac{8l^2}{\pi^3}\left(\dfrac{1}{1^3}\sin\dfrac{\pi x}{l}+\dfrac{1}{3^3}\sin\dfrac{3\pi x}{l}+\dfrac{1}{5^3}\sin\dfrac{5\pi x}{l}+\ldots\right)$.

17. $f(x)=\left(\dfrac{\pi}{2}+\dfrac{2}{\pi}\right)\sin x-\dfrac{8}{\pi}\left[\dfrac{2\sin 2x}{(3\times 1)^2}+\dfrac{4\sin 4x}{(5\times 3)^2}+\dfrac{6\sin 6x}{(7\times 5)^2}+\ldots\ldots\right]$.

18. $\dfrac{2l^2h}{a(l-a)\pi^2}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n^2}\sin\dfrac{n\pi a}{l}\sin\dfrac{n\pi x}{l}$.

1.8. PARSEVAL'S THEOREM ON FOURIER CONSTANTS

If the Fourier series of f(x) over an interval $c<x<c+2l$ is given as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right\}$$

then
$$\frac{1}{2l}\int_c^{c+2l}[f(x)]^2\,dx=\frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

Proof. The Fourier series of $f(x)$ in $c<x<c+2l$ is given as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right\} \qquad \ldots(1)$$

where $a_0=\dfrac{1}{l}\displaystyle\int_c^{c+2l}f(x)\,dx$; $a_n=\dfrac{1}{l}\displaystyle\int_c^{c+2l}f(x)\cos\dfrac{n\pi x}{l}\,dx$; $b_n=\dfrac{1}{l}\displaystyle\int_c^{c+2l}f(x)\sin\dfrac{n\pi x}{l}\,dx$ $\qquad\ldots(2)$

Multiplying both sides of (1) by $f(x)$, we have

$$[f(x)]^2 = \frac{a_0}{2} f(x) + \sum_{n=1}^{\infty} a_n f(x) \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n f(x) \sin \frac{n\pi x}{l}$$

Integrating both sides w.r.t. x, between the limits c to $c + 2l$, we have

$$\int_c^{c+2l} [f(x)]^2 \, dx = \frac{a_0}{2} \int_c^{c+2l} f(x) \, dx + \sum_{n=1}^{\infty} a_n \int_c^{c+2l} f(x) \cos \frac{n\pi x}{l} \, dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{a_0}{2} . la_0 + \sum_{n=1}^{\infty} a_n (la_n) + \sum_{n=1}^{\infty} b_n (lb_n) \qquad \text{[Using (2)]}$$

$$\Rightarrow \quad \int_c^{c+2l} [f(x)]^2 \, dx = \frac{la_0^2}{2} + l \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\therefore \quad \frac{1}{2l} \int_c^{c+2l} [f(x)]^2 \, dx = \frac{1}{2l} \left\{ \frac{la_0^2}{2} + l \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right\}$$

or

$$\mathbf{\frac{1}{2l} \int_c^{c+2l} [f(x)]^2 \, dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)} \qquad \text{(Parseval's identity)}$$

Hence the proof.

Note. *Parseval's identities in different cases:*

(*i*) If $c = 0$, the interval becomes $0 < x < 2l$ and Parseval's identity reduces to

$$\frac{1}{2l} \int_0^{2l} [f(x)]^2 \, dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

If $c = -l$, the interval becomes $-l < x < l$ and Parseval's identity reduces to

$$\frac{1}{2l} \int_{-l}^{l} [f(x)]^2 \, dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

(*ii*) If $f(x)$ is an even function in $(-l, l)$ then $\frac{2}{l} \int_0^l [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$

(*iii*) If $f(x)$ is an odd function in $(-l, l)$ then $\frac{2}{l} \int_0^l [f(x)]^2 \, dx = \sum_{n=1}^{\infty} b_n^2$

(*iv*) If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$ in $(0, l)$ then $\frac{2}{l} \int_0^l [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$

(*v*) If $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ in $(0, l)$ then $\frac{2}{l} \int_0^l [f(x)]^2 \, dx = \sum_{n=1}^{\infty} b_n^2$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Fourier sine series for unity in $0 < x < \pi$ and hence show that*

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots = \frac{\pi^2}{8}.$$

Sol. We require half-range Fourier sine series for 1 in $(0, \pi)$

Let $$1 = \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots(1)$$

Then $$b_n = \frac{2}{\pi}\int_0^{\pi} (1) \sin nx \, dx = \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = -\frac{2}{n\pi}(\cos n\pi - 1)$$

$$= \frac{2}{n\pi}[1-(-1)^n] \qquad [\because \cos n\pi = (-1)^n]$$

Now $b_n = 0$ when n is even; and $b_n = \dfrac{4}{n\pi}$ when n is odd.

Substituting in (1), we get

$$\therefore \quad 1 = \sum_{m=1}^{\infty} \frac{4}{(2m-1)\pi} \sin (2m-1)x \quad \text{or} \quad 1 = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right) \quad \ldots(2)$$

Now from Parseval's theorem on Fourier constants

$$\int_c^{c+2l} [f(x)]^2 \, dx = 2l\left[\frac{{a_0}^2}{4} + \frac{l}{2}\sum_{n=1}^{\infty}({a_n}^2 + {b_n}^2)\right] \quad \ldots(3)$$

Applying (3) to half-range sine series for 1 in $(0, \pi)$

$$c = 0,\ 2l = \pi,\ f(x) = 1,\ a_0 = 0,\ a_n = 0, \text{ and } b_n = \frac{4}{(2m-1)\pi},\ m = 1, 2 \ldots\ldots$$

We get, $$\int_0^{\pi} (1)^2 \, dx = \pi \cdot \frac{1}{2}\sum_{m=1}^{\infty} \frac{16}{(2m-1)^2 \cdot \pi^2}$$

$$\Rightarrow \quad \left[x\right]_0^{\pi} = \frac{8}{\pi}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right\} \quad \text{or} \quad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

Hence the result.

Example 2. *Find Fourier series of x^2 in $(-\pi, \pi)$. Use Parseval's identity to prove that*

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots$$

Sol. The Fourier series of x^2 in $(-\pi, \pi)$ is

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx \qquad \ldots(1)$$

Here $\qquad a_0 = \frac{2\pi^2}{3},\ a_n = \frac{4(-1)^n}{n^2},\ b_n = 0,\ f(x) = x^2$

Now by Parseval's identity from (1), we get

$$\int_{-\pi}^{\pi} (x^2)^2\, dx = 2\pi\left[\frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{16}{n^4}\right]$$

$$\Rightarrow \qquad \left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \frac{2\pi^5}{9} + \pi\sum_{n=1}^{\infty}\frac{16}{n^4} \quad \text{or} \quad \frac{2\pi^5}{5} - \frac{2\pi^5}{9} = \pi\sum_{n=1}^{\infty}\frac{16}{n^4}$$

or $$\frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4} \quad \text{or} \quad 1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots = \frac{\pi^4}{90}.$$

EXERCISE 1.5

1. If $f(x)$ has the Fourier series expansion

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \text{ in } a \le x \le a + 2l$$

show that $\displaystyle\int_a^{a+2l} [f(x)]^2\, dx = 2l\left[\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right].$

2. If $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ in $0 < x < l$, then show that

$$\int_0^l [f(x)]^2\, dx = \frac{l}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right].$$

3. If $f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ in $(0, l)$, then show that $\displaystyle\int_0^l [f(x)]^2\, dx = \frac{l}{2}\sum_{n=1}^{\infty} b_n^2.$

4. Prove that in the range $(0, l)$, $x = \dfrac{l}{2} - \dfrac{4l}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2m-1)^2}\cos\frac{(2m-1)\pi x}{l}$ and deduce that

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots = \frac{\pi^4}{96}.$$

5. Show that for $0 < x < \pi$,

$$x(\pi - x) = \frac{\pi^2}{6} - \left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \dots\dots\right)$$

and hence evaluate $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

1.9. TYPICAL WAVEFORMS

A periodic waveform is a waveform that repeats a basic pattern. It is a single-valued periodic function. Therefore it can be developed as a Fourier series.

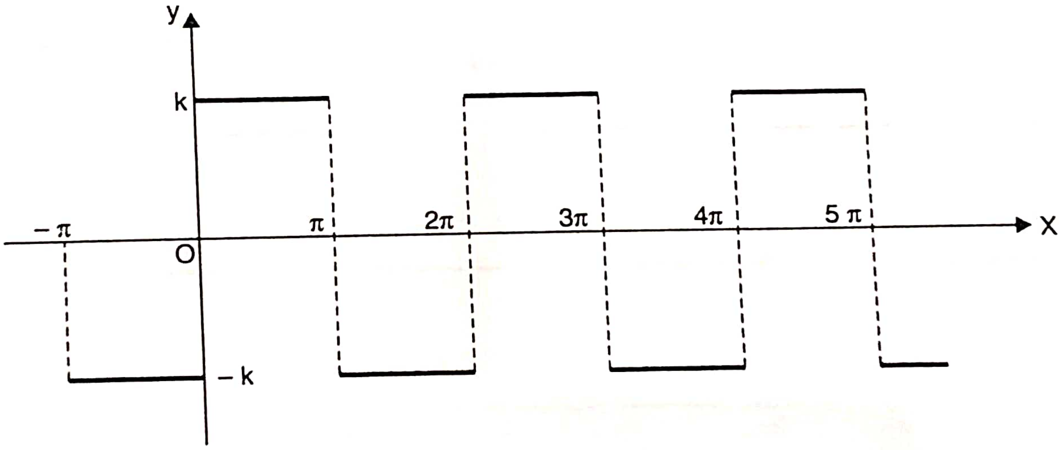
We give below some typical waveforms usually met with in communication engineering and also the corresponding Fourier series. The student is urged to construct the Fourier series in each case.

I. Square (or Rectangular) Waveform

(*M.D.U. May 2015*)

It is a periodic function of the form given below.

(*i*) $$f(x) = \begin{cases} -k & \text{for } -\pi < x < 0 \\ k & \text{for } \quad 0 < x < \pi \end{cases}, \quad f(x + 2\pi) = f(x)$$

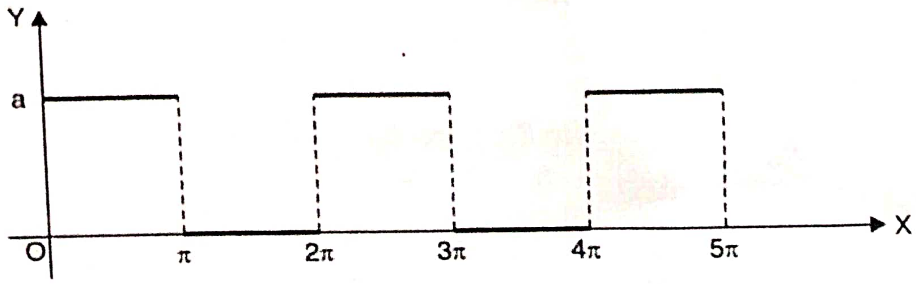


Its Fourier expansion is

$$f(x) = \frac{4k}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\dots\right]$$

[See Question 1 in Exercise 1.2]

(*ii*) $$f(x) = \begin{cases} a & \text{when } \; 0 < x < \pi \\ 0 & \text{when } \pi < x < 2\pi \end{cases}, \quad f(x + 2\pi) = f(x)$$



Its Fourier expansion is

$$f(x) = \frac{a}{2} + \frac{2a}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$$

Exercise. Draw and write the name of the wave form for the following function:

$$f(x) = \begin{cases} a & \text{for } 0 < x < \pi \\ 0 & \text{for } \pi < x < 2\pi \end{cases}$$

(M.D.U. May 2014)

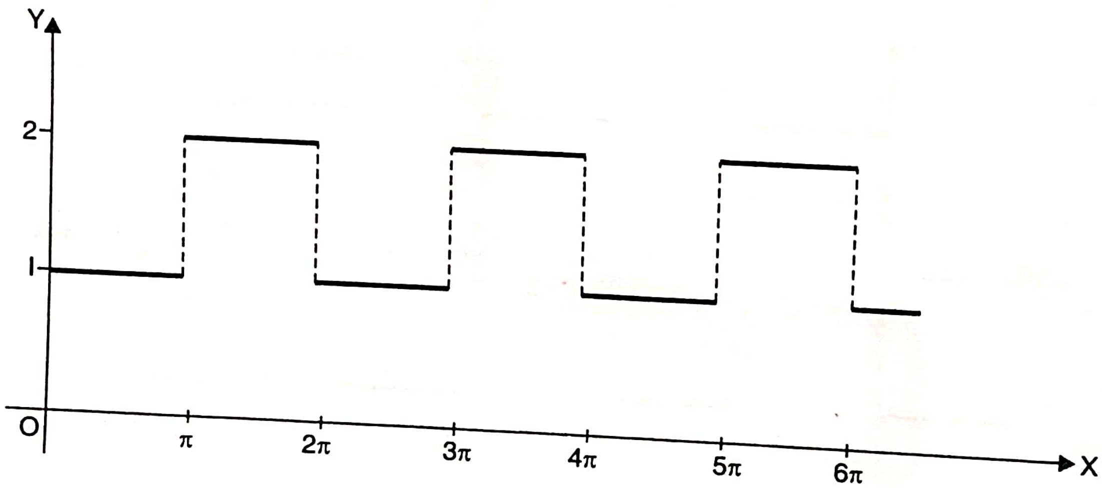
(*iii*)
$$f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 1 & \text{for } \quad 0 < x < \pi \end{cases}, f(x + 2\pi) = f(x)$$

Its Fourier expansion is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$$

[See Question 2 in Exercise 1.2]

(*iv*)
$$f(x) = \begin{cases} 1 & \text{for } 0 < x < \pi \\ 2 & \text{for } \pi < x < 2\pi \end{cases}, f(x + 2\pi) = f(x)$$



Its Fourier expansion is

$$f(x) = \frac{3}{2} - \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$$

(*v*)
$$f(x) = \begin{cases} -\dfrac{\pi}{4} & \text{for } -\pi < x < 0 \\ \dfrac{\pi}{4} & \text{for } 0 < x < \pi \end{cases}, f(x + 2\pi) = f(x)$$

Its Fourier expansion is

$$f(x) = \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots$$

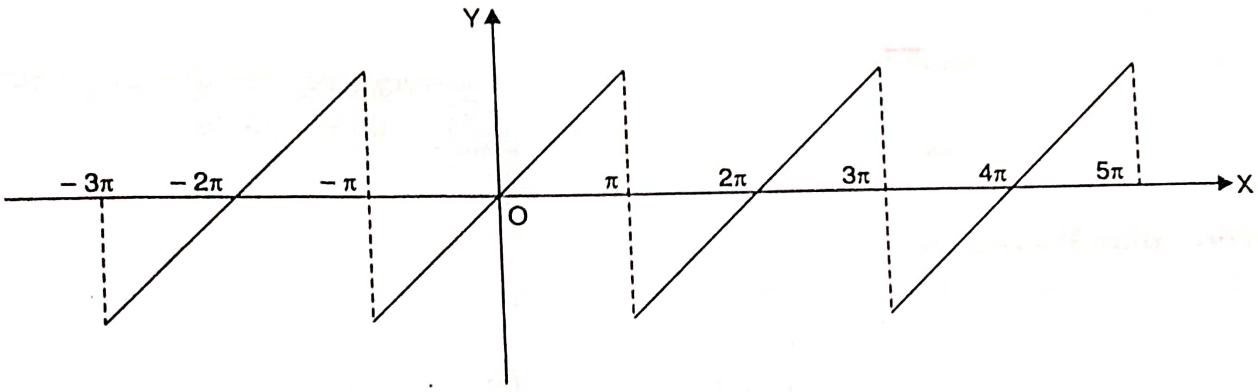
II. Saw-toothed Waveform

It is a periodic function of the form given below.

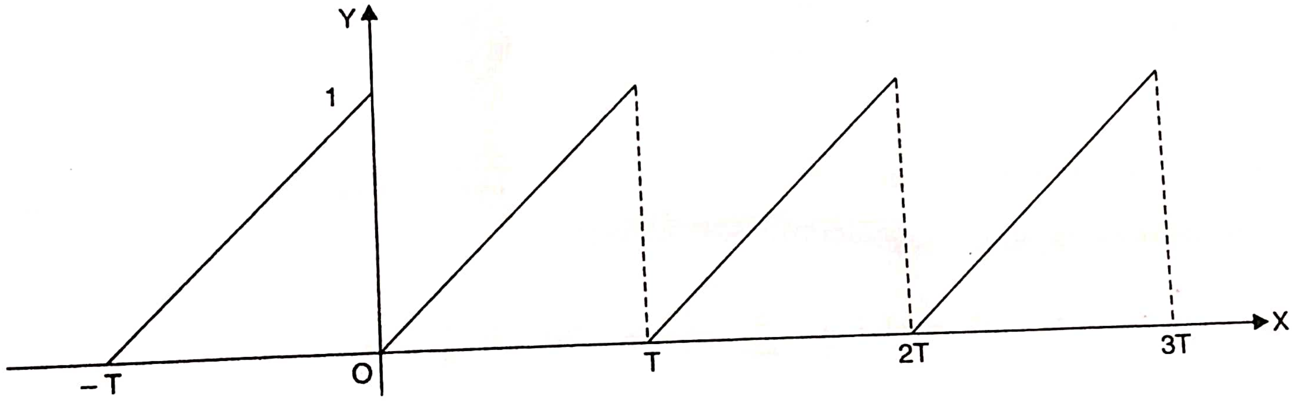
(*i*) $f(x) = x, \ -\pi < x < \pi$ and $f(x + 2\pi) = f(x)$

Its Fourier expansion is

$$f(x) = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\ldots\right)$$



(*ii*) $f(x) = \frac{1}{T}x$ when $0 < x < T$ and $f(x + T) = f(x)$



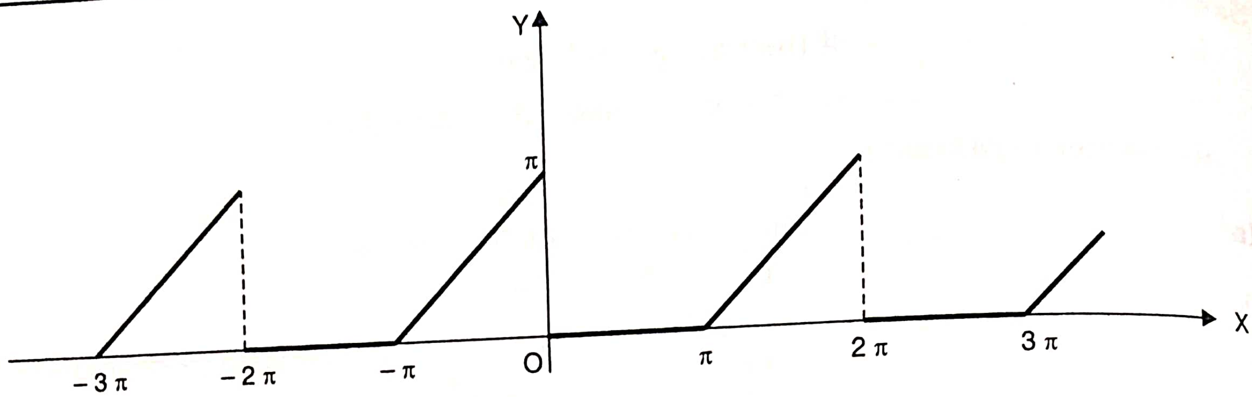
Its Fourier expansion is

$$f(x) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\omega x}{n}, \qquad \text{where } \omega = \frac{2\pi}{T}.$$

III. Modified Saw-toothed Waveform

It is a periodic function of the form given below.

$$f(x) = \begin{cases} \pi + x & \text{for } -\pi < x < 0 \\ 0 & \text{for } \ 0 \le x < \pi \end{cases}, \ f(x + 2\pi) = f(x)$$



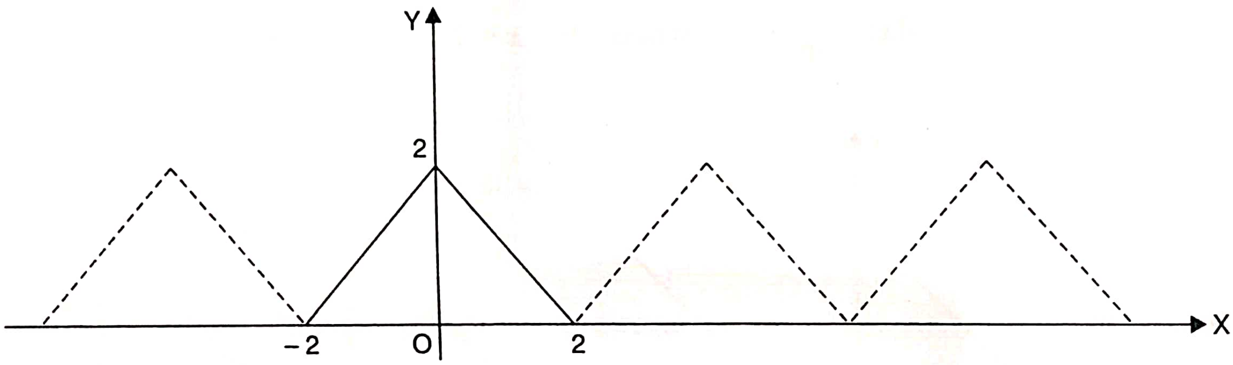
Its Fourier expansion is

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\ldots\right) - \left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \ldots\ldots\right)$$

IV. Triangular Waveform

It is a periodic function of the form given below.

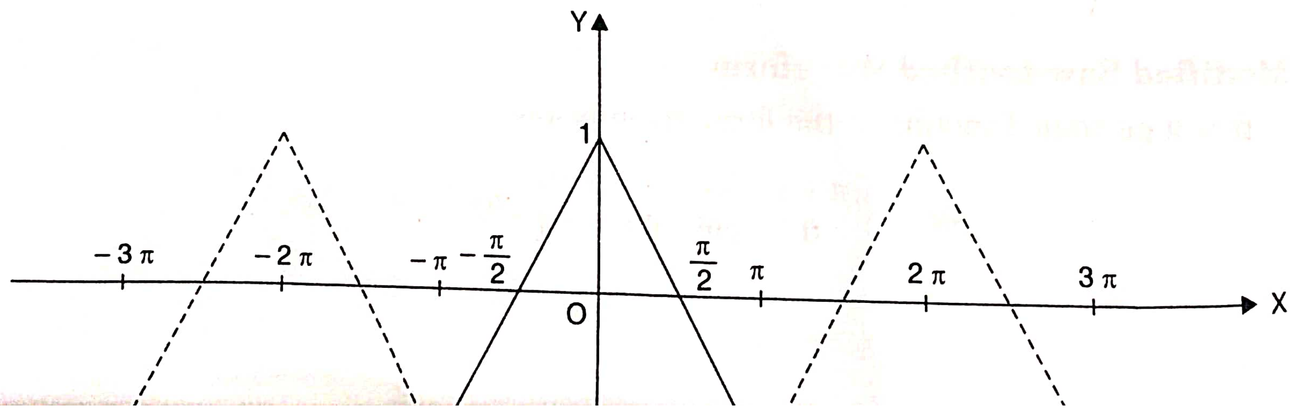
(*i*) $$f(x) = \begin{cases} 2 + x & \text{for } -2 \le x \le 0 \\ 2 - x & \text{for } \ \ 0 < x \le 2 \end{cases}, \ f(x+4) = f(x)$$



Its Fourier expansion is

$$f(x) = 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left[(2n-1)\frac{\pi x}{2}\right]$$

(*ii*) $$f(x) = \begin{cases} 1 + \dfrac{2x}{\pi} & \text{for } -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi} & \text{for } \ \ 0 < x \le \pi \end{cases}, \ f(x + 2\pi) = f(x)$$



Its Fourier expansion is

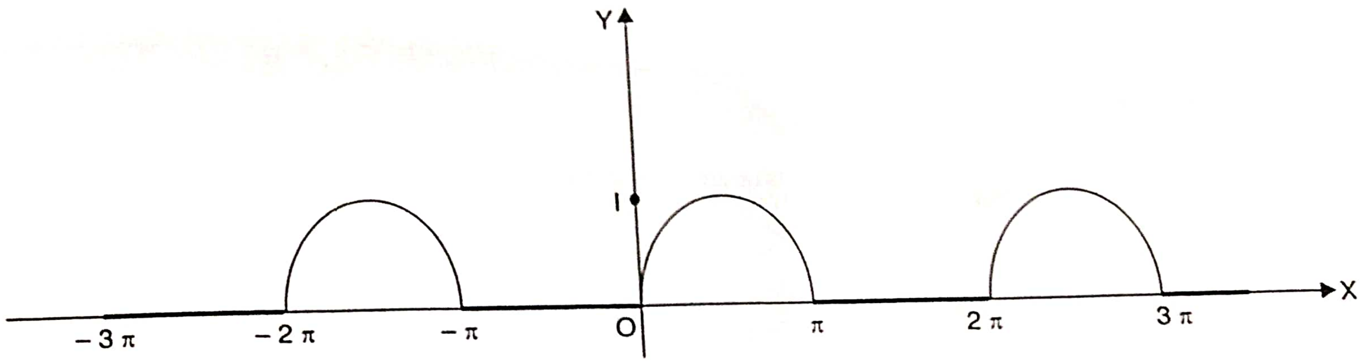
$$f(x) = \frac{8}{\pi^2}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\dots\right)$$

[See Example 10 before Exercise 1.1]

V. Half Rectified Waveform

It is a periodic function of the form given below.

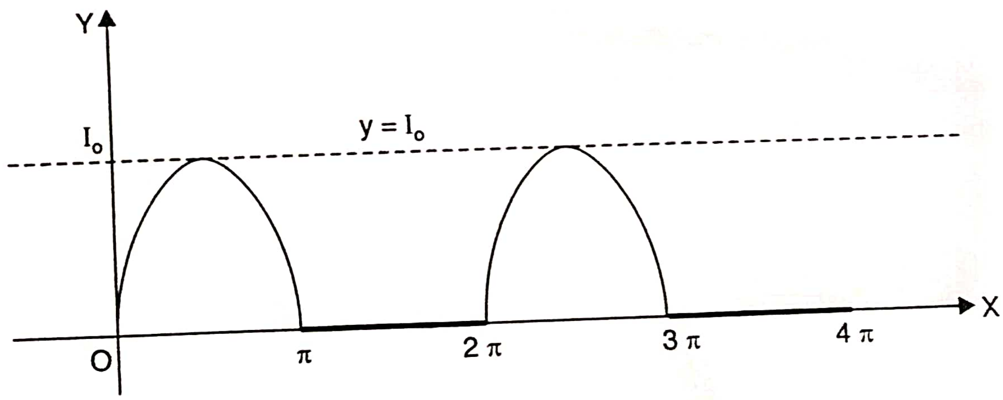
(i) $$f(x) = \begin{cases} 0 & \text{for } -\pi \le x \le 0 \\ \sin x & \text{for } 0 \le x \le \pi \end{cases}$$



Its Fourier expansion is

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2 - 1}$$

(ii) $$f(x) = \begin{cases} I_0 \sin x & \text{for } 0 \le x \le \pi \\ 0 & \text{for } \pi \le x \le 2\pi \end{cases}, \; f(x + 2\pi) = f(x)$$



Its Fourier expansion is

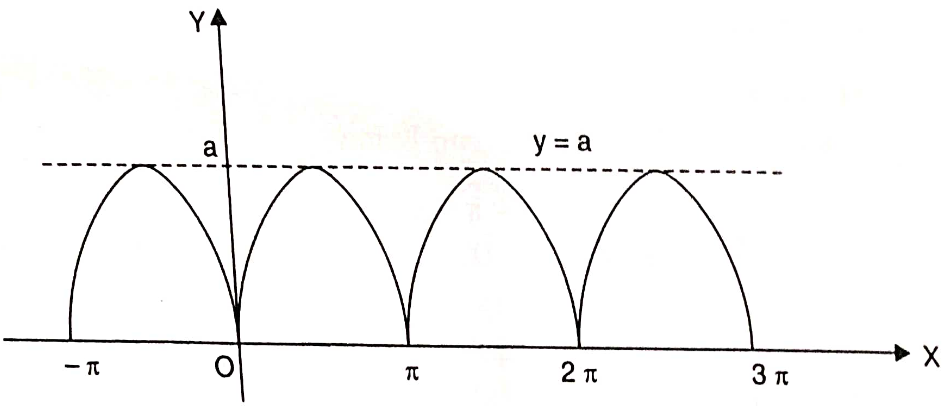
$$f(x) = \frac{I_0}{\pi} + \frac{1}{2}I_0 \sin x - \frac{2I_0}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2n\,x}{4n^2 - 1}$$

[See Question 6 in Exercise 1.2]

VI. Full Rectified Waveform

It is a periodic function of the form given below.

$$f(x) = a \sin x \text{ for } 0 \le x \le \pi,\ f(x + \pi) = f(x)$$



Its Fourier expansion is

$$f(x) = \frac{4a}{\pi}\left[\frac{1}{2} - \frac{\cos 2x}{1.3} - \frac{\cos 4x}{3.5} - \frac{\cos 6x}{5.7} - \ldots\ldots\right]$$

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