Def. Sequence : A sequence is a function whose domain is the set of natural numbers and range can be any set.

Def. Real Sequence : A real sequence is a function whose domain is the set of natural numbers and range is any subset of real numbers.

Note : In this chapter, we shall study only real sequences. So by a 'sequence' we shall always mean a 'real sequence'.

Representation of a Sequence : A sequence is a function and functions are usually denoted by symbols f, g, h etc. However, in case of sequences, these symbols are not preferred by most of the authors, instead the symbols a, b, u, v, x, y are used for distinction of the concept of the sequences from the concept of calculus of function. We shall also use the symbols a, b, u, v to denote the sequences.

Symbolically, a sequence is a function $a : N \to R$ (or $x : N \to R$) where N denotes the set of natural numbers and R denotes the set of real numbers. If $a : N \to R$ be a sequence, the image of $n \in N$ is denoted by a_n instead of denoting it by $a(n)$. Thus a_1, a_2, a_3 are real numbers associated with 1, 2, 3 by the function a and are called first, second, third terms of the sequence.

The sequence $a : N \to R$ is denoted by $< a_n >$ which when represented in expanded form is written as $< a_1, a_2, ..., a_n, ... >$

Note : Some authors also denote the sequence $< a_n >$ by $\{a_n\}$. But we shall reserve the notation $\{a_n\}$ to denote the range set of a sequence.

Methods to describe a Sequence : Some commonly used methods to describe a sequence are as follows :

(i) A sequence may be described by writing first few terms of the sequence till the rule or pattern of writing down the terms becomes clear. For example, the sequence $<1, 4, 9, 16, 25, ...>$ is the sequence whose n^{th} term is n^2.

Similarly the sequence, $< 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... >$ is the sequence whose n^{th} term is $\frac{1}{n}$.

(ii) Most often, a sequence is described by giving a formula for its n^{th} term. For example, the sequences $< n^2 >, < \frac{1}{n} >, < (-1)^n >$ can be written in expanded form as follows :

$$< n^2 > = < 1, 4, 9, 16, ... >, \quad < \frac{1}{n} > = < 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... >, \quad < (-1)^n > = < -1, 1, -1, 1, -1, 1, ... >$$

(iii) Recurrence Relation : A sequence is sometimes described by giving its first few terms and a relatio (formula) to determine the other terms of the sequence.

For example, $a_1 = 2$ and $a_{n+1} = 7 + a_n \quad \forall\, n \in N$

By this relation, we get $a_2 = 7 + a_1 = 7 + 2 = 9$, $\quad a_3 = 7 + 9 = 16$, $a_4 = 7 + 16 = 23$ etc.

Thus the sequence in expanded form is $< 2, 9, 16, 23, ... >$, where each term is obtained by adding 7 its preceeding term.

Def. Range set of a sequence : The set of all distinct terms of a sequence is called its range. The range o a sequence $< a_n >$ is denoted by $\{a_n\}$.

Note : The number of terms in a sequence is always infinite but range set of a sequence may be a finite set since it contain only distinct terms of the sequence.

For example, if $a_n = <(-1)^n>$, then $<a_n> = <-1, 1, -1, 1, \ldots\ldots>$
The range set of this sequence is $\{-1, 1\}$, which is a finite set.

Def. Bounded above sequence : A sequence $<a_n>$ is said to be bounded above if there exists a real number K such that $a_n \leq K \;\; \forall \;\; n \in N$.

Def. Bounded below sequence : A sequence $<a_n>$ is said to be bounded below if there exists a real number k such that $k \leq a_n \;\; \forall \;\; n \in N$

Def. Bounded sequence : A sequence is said to be bounded if it is bounded above as well as bounded below. Thus, a sequence $<a_n>$ is bounded if there exist two real numbers k and K such that $k \leq a_n \leq K \quad \forall n \in N$.

Def. A sequence $<a_n>$ is said to be bounded if there exists a positive number K such that $|a_n| \leq K \; \forall \; n \in N$

Def. Unbounded sequence : A sequence is said to be unbounded if it is not bounded i.e. either it is unbounded above or unbounded below or both. To be precise let us define unbounded above and below sequences.

Def. Unbounded above sequence : A sequence $<a_n>$ is said to unbounded above if for every real number K, there exists a $m \in N$ such that $a_m > K$.

Def. Unbounded below sequence : A sequence $<a_n>$ is said to be unbounded below if for every real number k, there exists a $m \in N$ such that $a_m < k$.

Def. Constant sequence : A sequence $<a_n>$ defined by $a_n = c \in R$ for all $n \in N$ is called a constant sequence. Thus $<a_n> = <c, c, c, \ldots\ldots>$ is a constant sequence with range $= <c>$ a singleton.

Def. Eventually constant sequence : A sequence $<a_n>$ is said to be eventually constant if there exists a positive integer m such that $a_n = c$ for $n \geq m$.

Def. Convergent sequence : A sequence $<a_n>$ is said to converge to a real number l if for given $\varepsilon > 0$, however small, there exists a positive integer m such that $|a_n - l| < \varepsilon \;\; \forall \;\; n \geq m$. The real number l is called the limit of the sequence $<a_n>$ and symbolically it is expressed as $a_n \to l$ as $n \to \infty$ or $\lim\limits_{n\to\infty} a_n = l$

Explanation of the definition : The above definition says that after first $m-1$ terms of the sequence, the difference between any term of the sequence and the number l is less than ε.
This definition can be understood by an another efficient way as follows :

We have $\quad |a_n - l| < \varepsilon \;\; \forall \;\; n \geq m \;\; \Rightarrow \;\; -\varepsilon < a_n - l < \varepsilon \;\; \forall \;\; n \geq m$

$$\Rightarrow \; l - \varepsilon < a_n < l + \varepsilon \;\; \forall \;\; n \geq m$$

$\Rightarrow$ all terms of the sequence, except the first $m-1$ terms, lie in the interval $(l-\varepsilon, l+\varepsilon)$ i.e., only finitely many terms of the sequence lie outside the interval $(l-\varepsilon, l+\varepsilon)$.

Another definition of convergent sequence : A sequence $<a_n>$ is said to converge to a real number l if for given $\varepsilon > 0$, however small, the interval $(l-\varepsilon, l+\varepsilon)$ contains all terms of the sequence except some finitely many terms. If there are infinitely many terms of the sequence outside the interval $(l-\varepsilon, l+\varepsilon)$, then the sequence can not converge to l.

Def. Divergent sequence :

(i) A sequence $<a_n>$ is said to diverge to $+\infty$ if for any positive real number K, however large, there exists a positive integer m such that $a_n > K \ \forall \ n \geq m$ i.e., all the terms of the sequence lie on the right of K, on the real line, except some finitely many terms and then symbolically, we write $\lim_{n\to\infty} a_n = \infty$ or $a_n \to \infty$ as $n \to \infty$.

(ii) A sequence $<a_n>$ is said to diverge to $-\infty$ if for any positive real number K, however large, there exists a positive integer m such that $a_n < -K \ \forall \ n \geq m$ i.e., all the terms of the sequence lie on the left of K, on the real line, except some finitely many terms and then symbolically we write $\lim_{n\to\infty} a_n = -\infty$ or $a_n \to -\infty$ as $n \to \infty$

(iii) A sequence $<a_n>$ is said to be divergent sequence if it either diverges to $+\infty$ or diverges to $-\infty$.

Def. Oscillating sequence : A sequence is said to be oscillating sequence if it is neither convergent nor divergent.

Oscillating sequence are of two types :

(i) A sequence is said to be finitely oscillating if it is bounded as well as oscillating.

(ii) A sequence is said to be infinitely oscillating if it is unbounded as well as oscillating.

Def. Null sequence : A sequence $<a_n>$ is said to be a null sequence if it converges to 0 i.e., $\lim_{n\to\infty} a_n = 0$

For example, the sequences $<\frac{1}{n}>, <\frac{1}{n^2}>, <\frac{1}{2^n}>, <\frac{(-1)^n}{n}>$ are null sequences.

Def. Monotonic sequence :

(i) A sequence $<a_n>$ is said to be monotonically increasing if $a_n \leq a_{n+1} \ \forall \ n$. i.e., $a_1 \leq a_2 \leq a_3 \leq a_4 \leq$.........

(ii) A sequence $<a_n>$ is said to be monotonically decreasing if $a_n \geq a_{n+1} \ \forall \ n$ i.e., $a_1 \geq a_2 \geq a_3 \geq a_4 \geq$........

(iii) A sequence is said to be monotonic if either it is monotonically decreasing or monotonically increasing.

(iv) A sequence $<a_n>$ is said to be strictly monotonically increasing if $a_n < a_{n+1} \ \forall \ n$. i.e. $a_1 < a_2 < a_3$.......

(v) A sequence $<a_n>$ is said to be strictly monotonically decreasing if $a_n > a_{n+1} \ \forall \ n$. i.e., $a_1 > a_2 > a_3$ ......

Remark : Every strictly monotonically increasing (decreasing) sequence is also monotonically increasing (decreasing), according to the definition. But converse may not be true, because any constant sequence say, $<7, 7, 7, ..., 7, ...>$ falls under the definition of monotonically increasing as well as decreasing but do not fall under the definition of strictly monotonically increasing or decreasing. In fact constant sequence is the only sequence which is both monotonically increasing as well as decreasing.

Def. Least Upper Bound of a sequence : A real number u is said to be least upper bound of a bounded above sequence if

(i) $a_n \leq u \ \forall \ n \in N$ i.e., u is an upper bound of $< a_n >$.

(ii) If u' is any other real number such that $a_n \leq u' \ \forall \ n \in N$, then $u \leq u'$ i.e., no other upper bound is less than u.

Condition (ii) can be expressed in an another way as : For every $\varepsilon > 0$, there exists a positive integer m such that $u - \varepsilon < a_m$.

Def. Greatest Lower Bound of a sequence : A real number l is called a greatest lower bound of a bounded below sequence if

(i) $l \leq a_n \ \forall \ n \in N$ i.e., l is an lower bound of $< a_n >$.

(ii) If is any other real number such that $l' \leq a_n \ \forall \ n \in N$, then $l \geq l'$ i.e., no other lower bound is greater than l.

Condition (ii), as before, can be expressed in an another way as

For every $\varepsilon > 0$, there exists a positive integer m such that $a_m < l + \varepsilon$.

Def. 1. Limit Point : A real number l is said to be a limit point or cluster point of a sequence if every neighbourhood of l contains infinitely many terms of the sequence i.e. for every $\varepsilon > 0$, the interval $(l - \varepsilon, l + \varepsilon)$ contains infinitely many terms of the sequence.

Def. 2. Limit Point : A real number l is said to be a limit point of a sequence $< a_n >$ if for a given $\varepsilon > 0$ and for a given positive integer m, there exists a positive integer $k > m$ such that $|a_k - l| < \varepsilon$ i.e., $a_k \in (l - \varepsilon , l + \varepsilon)$.

Difference between limit and limit point : We know that a real number l is the limit of a sequence $< a_n >$ if for every $\varepsilon > 0$, there exists a positive integer m such that $|a_n - l| < \varepsilon \ \forall \ n \geq m$.

$\Rightarrow$ $a_n \in (l - \varepsilon, \ l + \varepsilon)$ for all $n \geq m$

$\Rightarrow$ the interval $(l - \varepsilon , l + \varepsilon)$ contains all terms of the sequence except the first m–1 terms.

$\Rightarrow$ infinitely many terms of the sequence lie in the interval $(l - \varepsilon , l + \varepsilon)$ and only finitely many terms lie outside this interval. Thus the main difference between limit and limit point is that, in case of limit, the number of terms that lie inside the interval $(l - \varepsilon , l + \varepsilon)$ must be infinite and the number of terms that lie outside this interval must be finite, whereas in case of limit point, the number of terms that lie inside the interval $(l - \varepsilon , l + \varepsilon)$ must be infinite and there is no condition on the number of terms that lie outside this interval i.e. they may be finite or infinite.

Remarks :

(i) By the above discussion, it is clear that the limit of a sequence is also the limit point of the sequence. In fact, if l is the limit of a sequence then it is the only limit point of that sequence.

(ii) Limit point may not be the limit of the sequence.

(iii) Limit point of a sequence may or may not be term of the sequence.

(iv) If any term of the sequence is repeated infinitely many times then that term is a limit point of the sequence. e.g. the sequence $< 1,2,3, 1,2,3, 1,2,3, 1,2,3, ... >$ has three limit points, namely, 1, 2, 3.

Results (A) :

1. Every convergent sequence has a unique limit.

2. Every convergent sequence is bounded.

3. A bounded sequence may or may not be convergent.

4. A unbounded sequence can not be convergent.

5. (i) If a sequence diverges to $+\infty$ then it is unbounded above but bounded below.
(ii) If a sequence diverges to $-\infty$, then it is unbounded below but bounded above.

6. A sequence $<a_n>$ converges to l iff the sequence $<a_n - l>$ is a null sequence.

7. If $<a_n>$ is a sequence then prove that $a_n \to 0$ iff $|a_n| \to 0$.

8. If a sequence $<a_n>$ converges to a and $a_n \geq 0\ \forall\ n$ then $a \geq 0$.

OR

A convergent sequence of non-negative terms converges to a non-negative number.

Remark (i) : If $a_n \geq 0$ is replaced by $a_n > 0$ in the above theorem, then student should not conclude that $a > 0$. For example, $a_n = \frac{1}{n} > 0\ \forall n$. But $\lim_{n\to\infty} a_n = 0$ i.e., $a = 0$.

Remark (ii) : The above result can also be given for non-positive terms as :

If a sequence $<a_n>$ converges to a and $a_n \leq 0\ \forall\ n$ then $a \leq 0$.

- or -

A convergent sequence of non-positive terms always converges to a non-positive number.

9. If $<a_n>$ converges to a, $<b_n>$ converges to b and $a_n \geq b_n\ \forall\ n$, then $a \geq b$.

10. If $<a_n>$ converges to a and $a_n \geq k\ \forall n$, then $a \geq k$.

11. If $<a_n>$ and $<b_n>$ both are null sequences, then prove that $<a_n + b_n>$ is a null sequence.

12. If $<a_n>$ is bounded sequence and $<b_n>$ is a null sequence, then prove that $<a_n \cdot b_n>$ is a null sequence.

13. If $<a_n>$ is a null sequence and k is a constant, then $<k\,a_n>$ is a null sequence.

14. (i) If $<a_n>$ diverges to ∞ and $b_n \geq a_n\ \forall\ n$ then $<b_n>$ also diverges to ∞.
(ii) If $<a_n>$ diverges to $-\infty$ and $b_n \leq a_n$ then $<b_n>$ also diverges to $-\infty$.

Results (B) : Algebra of limits :

If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then

(i) $\lim_{n\to\infty} ka_n = ka$, where k is any real number.

(ii) $\lim_{n\to\infty} |a_n| = |a|$

Remark : Converse of above result is not true in general. i.e., existence of $\lim_{n\to\infty} |a_n|$ does not necessarily imply the existence of $\lim_{n\to\infty} a_n$. For example , the sequence $< a_n > = < (-1)^n >$ is not convergent but $< |a_n| > = < 1 >$ is convergent.

(iii) $\lim_{n\to\infty} (a_n + b_n) = a + b$

Remark : Converse of above theorem is not true in general i.e., existence of $\lim_{n\to\infty} (a_n + b_n)$ does not necessarily imply the existence of two separate limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$.

(iv) $\lim_{n\to\infty} (a_n - b_n) = a - b$

Remark : The converse of above theorem is not true i.e., existence of $\lim_{n\to\infty} (a_n - b_n)$ does not necessarily imply the existence of two separate limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$.

(v) $\lim_{n\to\infty} (a_n b_n) = ab$

Remark : The converse of above theorem is not true in general i.e., the existence of $\lim_{n\to\infty} (a_n b_n)$ the does not necessarily imply the existence of two separate limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$. For example, $a_n = (-1)^n$ and $b_n = (-1)^n$, then both the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ do not exist but $\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} (-1)^{2n} = 1$ exists.

(vi) $\lim_{n\to\infty} \left(\frac{1}{a_n}\right) = \frac{1}{a}$ provided $a_n \neq 0\ \forall n$ and $a \neq 0$

(vii) $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$ provided $b_n \neq 0\ \forall n$ and $b \neq 0$

Remark : The converse of above theorem is not true in general i.e., the existence of $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right)$ does not necessarily imply the existence of two separate limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$. For example, if we consider $a_n = n$ and $b_n = n$, then both the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ do not exist. But $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = 1$ exists. $\frac{a}{b}$

Definition : Let $< a_n >$ and $< b_n >$ be two sequences, then the sequences $< a_n + b_n >$, $< a_n - b_n >$, $< a_n b_n >$, $< \frac{a_n}{b_n} >$ (provided $b_n \neq 0$) are called respectively the sum, difference, product and quotient of sequences $< a_n >$ and $< b_n >$.

Keeping the definition made above in mind, we can say, by the above result that sum, difference, product and quotient of two convergent sequences is convergent.
Now if we extend our thought towards two divergent sequences or one convergent and one divergent sequence, then their sum, difference, product and quotient may not be so well-behaved as in the case of two convergent sequence.
Considering sum and product (as difference and quotient can be treated similarly) of two divergent sequences or one convergent and one divergent, all the possibilities are gathered in the following tables :

	If		then	
Sr. No.	$< a_n >$	$< b_n >$	$< a_n + b_n >$	Example/Proof
1.	converges	diverges to ∞	diverges to ∞	requires proof
2.	converges	diverges to $-\infty$	diverges to $-\infty$	requires proof
3.	diverges to ∞	diverges to ∞	diverges to ∞	requires proof
4.	diverges to ∞	diverges to $-\infty$	may converge	$a_n = n,\ b_n = -n$
			may diverge to ∞	$a_n = 2n,\ b_n = -n$
			may diverge to $-\infty$	$a_n = n,\ b_n = -2n$
			may oscillate	$a_n = \begin{cases} n^2 & \text{if } n \text{ is odd} \\ n+1 & \text{if } n \text{ is even} \end{cases}$ $b_n = -n$

	If		then	
Sr. No.	$< a_n >$	$< b_n >$	$< a_n . b_n >$	Example/Proof
1.	converges	diverges to ∞	may converge	$a_n = \dfrac{(-1)^n}{n^2},\ b_n = n$
			may diverge to ∞	$a_n = 1, b_n = n$
			may diverge to $-\infty$	$a_n = -1,\ b_n = n$
			may oscillate	$a_n = \dfrac{(-1)^n}{n},\ b_n = n$
2.	converges	diverges to $-\infty$	may converge	$a_n = \dfrac{(-1)^n}{n^2},\ b_n = -n$
			may diverge to ∞	$a_n = -1, b_n = -n$
			may diverge to $-\infty$	$a_n = 1,\ b_n = -n$
			may oscillate	$a_n = \dfrac{(-1)^n}{n},\ b_n = -n$
3.	diverges to ∞	diverges to ∞	diverges to ∞	requires proof
4.	diverges to ∞	diverges to $-\infty$	diverges to $-\infty$	requires proof