Date :
Page No.

Change of Interval :-

Consider a periodic function $f(x)$ in the limit $\alpha < x < \alpha + 2l$

here Fourier series expansion is written as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Ques $f(x) = x - x^2$ in the interval $-1 < x < 1$

Solⁿ Fourier series expansion is written as

$$f(x) = x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

here $\alpha = -1$

$\alpha + 2l = 1 \Rightarrow -1 + 2l = 1 \Rightarrow l = 1$

Date :
Page No.

- $a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \frac{1}{1}\int_{-1}^{1}(x-x^2)\,dx$

$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_{-1}^{1} = \left(\frac{1}{2} - \left(+\frac{1}{3}\right)\right) - \left(\frac{1}{2} - \left(-\frac{1}{3}\right)\right)$

$= \frac{1}{2} - \frac{1}{3} - \frac{1}{2} - \frac{1}{3} = -\frac{2}{3}$

$a_n = \frac{1}{1}\int_{-1}^{1}(x-x^2)\cos\frac{n\pi x}{1}\,dx$

$= \int_{-1}^{1}\underbrace{(x-x^2)}_{\text{I}}\ \underbrace{\cos n\pi x}_{\text{II}}\,dx$

$= \left((x-x^2)\frac{\sin n\pi x}{n\pi} - \int \underbrace{(1-2x)}_{\text{I}}\cdot\underbrace{\frac{\sin n\pi x}{n\pi}}_{\text{II}}\,dx\right)_{-1}^{1}$

$= (x-x^2)\frac{\sin n\pi x}{n\pi} - \frac{1}{n\pi}\left((1-2x)\left(-\frac{\cos n\pi x}{n\pi}\right) - \int(-2)\left(-\frac{\cos n\pi x}{n\pi}\right)dx\right)_{-1}^{1}$

$= \left((x-x^2)\frac{\sin n\pi x}{n\pi} + \frac{(1-2x)\cos n\pi x}{n^2\pi^2} + \frac{2}{n^2\pi^2}\int \cos n\pi x\,dx\right)_{-1}^{1}$

Scanned with CamScanner

$$a_n = \left(\frac{(x-x^2)\sin n\pi x}{n\pi} + \frac{(1-2x)\cos n\pi x}{n^2\pi^2} + \frac{2}{n^3\pi^3}\sin n\pi x \right)_{-1}^{1}$$

$$= -\frac{\cos n\pi}{n^2\pi^2} - \left(\frac{3\cos n\pi}{n^2\pi^2} \right) = -\frac{4\cos n\pi}{n^2\pi^2}$$

$$a_n = \frac{-4(-1)^n}{n^2\pi^2}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} \underbrace{(x-x^2)}_{\mathrm{I}} \underbrace{\sin \frac{n\pi x}{1}}_{\mathrm{II}}\, dx$$

$$= \left((x-x^2)\left(-\frac{\cos n\pi x}{n\pi}\right) - \int (1-2x)\left(-\frac{\cos n\pi x}{n\pi}\right) dx \right)_{-1}^{1}$$

$$= \left(-\frac{(x-x^2)\cos n\pi x}{n\pi} + \frac{1}{n\pi}\int \underbrace{(1-2x)}_{\mathrm{I}} \underbrace{\cos n\pi x}_{\mathrm{II}}\, dx \right)_{-1}^{1}$$

$$= \left(-\frac{(x-x^2)\cos n\pi x}{n\pi} + \frac{1}{n\pi}\left((1-2x)\frac{\sin n\pi x}{n\pi} - \int (-2)\frac{\sin n\pi x}{n\pi}\, dx \right) \right)_{-1}^{1}$$

Page No.

$$b_n = \left(-\frac{(x-x^2)\cos n\pi x}{n\pi} + \frac{(1-2x)\sin n\pi x}{n^2\pi^2} - \frac{2}{n^3\pi^3}\cos n\pi x\right)_{-1}^{1}$$

$$b_n = -\frac{2}{n^3\pi^3}\cos n\pi - \left(-\frac{(-1-(-1)^2)\cos n\pi}{n\pi} - \frac{2}{n^3\pi^3}\cos n\pi\right)$$

$$= \frac{-2}{n^3\pi^3}(-1)^n + \left(\frac{-2(-1)^n}{n\pi}\right) + \frac{2}{n^3\pi^3}(-1)^n$$

$$b_n = \frac{-2(-1)^n}{n\pi}$$

Fourier series is

$$x - x^2 = \frac{-1}{3} + \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n^2\pi^2}\cos n\pi x + \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n\pi}\sin n\pi x$$

$$x - x^2 = -\frac{1}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x \quad — A$$

hence evaluate

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \text{———} \infty$$

Expand eq^n A

$$x - x^2 = \frac{-1}{3} - \frac{4}{\pi^2}\left(-\frac{\cos \pi x}{1^2} + \frac{\cos 2\pi x}{2^2} - \frac{\cos 3\pi x}{3^2} \text{——} \infty\right)$$

$$-\frac{2}{\pi}\left(-\sin \pi x + \frac{\sin 2\pi x}{2} - \frac{\sin 3\pi x}{3} - \infty\right)$$

Date :

Page No.

Put $x=0$

$$0 = -\frac{1}{3} - \frac{4}{\pi^2}\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - - \infty\right)$$

$$- \frac{2}{\pi}(0)$$

$$\frac{1}{3} = \frac{4}{\pi^2}\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - - \infty\right)$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} ——— \infty = \frac{\pi^2}{12}$$

Ques $f(x) = 2x - x^2$ in $(0,3)$

hence deduce that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} —— \infty = \frac{\pi^2}{12}$$

Sol^n Fourier series expansion is written as.

$$2x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

here $\alpha = 0$

$\alpha + 2l = 3 \Rightarrow 2l = 3 \Rightarrow l = 3/2$

Date :
Page No.

$$2x-x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{3/2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{3/2}$$

$$2x-x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{3}$$

$$a_0 = \frac{1}{3/2}\int_0^3 (2x-x^2)\,dx = \frac{2}{3}\left(x^2-\frac{x^3}{3}\right)_0^3$$

$$a_0 = \frac{2}{3}\left(9-\frac{27}{3}\right) = 0$$

$$a_n = \frac{1}{3/2}\int_0^3 \underbrace{(2x-x^2)}_{\mathrm{I}} \underbrace{\cos\frac{2n\pi x}{3}}_{\mathrm{II}}\,dx$$

$$= \frac{2}{3}\left[(2x-x^2)\frac{\sin\frac{2n\pi x}{3}}{2n\pi/3} - \int (2-2x)\frac{\sin\frac{2n\pi x}{3}}{2n\pi/3}\,dx\right]_0^3$$

$$= \frac{2}{3}\left[\frac{3}{2n\pi}(2x-x^2)\sin\frac{2n\pi x}{3} - \frac{2\times 3}{2n\pi}\int \underbrace{(1-x)}_{\mathrm{I}} \underbrace{\sin\frac{2n\pi x}{3}}_{\mathrm{II}}\,dx\right]_0^3$$

$$= \frac{2}{3}\left[\frac{3}{2n\pi}(2x-x^2)\sin\frac{2n\pi x}{3} - \frac{3}{n\pi}\left((1-x)\left(\frac{-\cos\frac{2n\pi x}{3}}{2n\pi/3}\right) - \int(-1)\left(\frac{-\cos\frac{2n\pi x}{3}}{2n\pi/3}\right)\right)\right]_0^3$$

$$= \frac{2}{3}\left[\frac{3}{2n\pi}(2x-x^2)\sin\frac{2n\pi x}{3} - \frac{3}{n\pi}\left(-(1-x)\cdot\frac{3}{2n\pi}\cos\frac{2n\pi x}{3} - \frac{3}{2n\pi}\frac{\sin\frac{2n\pi x}{3}}{2n\pi/3}\right)\right]_0^3$$

Scanned with CamScanner

Page No. 3

$$a_n = \frac{2}{3}\left[\frac{3}{2n\pi}(2x-x^2)\sin\frac{2n\pi x}{3} + \frac{9(1-x)}{2n^2\pi^2}\cos\frac{2n\pi x}{3} - \frac{9}{4n^2\pi^2}\sin\frac{2n\pi x}{3}\cdot\frac{3}{n\pi}\right]_0^3$$

$$= \frac{2}{3}\left[-\frac{9}{n^2\pi^2} - \frac{9}{2n^2\pi^2}\right] = \frac{2}{3}\left(-\frac{27}{2}\right)\cdot\frac{1}{n^2\pi^2}$$

$$= -\frac{9}{n^2\pi^2}$$

Similarly $b_n = \frac{3}{n\pi}$

Now Fourier series is

$$2x - x^2 = \sum_{n=1}^{\infty} \frac{-9}{n^2\pi^2}\cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty}\frac{3}{n\pi}\sin\frac{2n\pi x}{3}$$

$$= \frac{-9}{\pi^2}\left[\frac{\cos\frac{2\pi x}{3}}{1^2} + \frac{\cos\frac{4\pi x}{3}}{2^2} + \frac{\cos\frac{6\pi x}{3}}{3^2} \cdots \infty\right]$$

$$+\frac{3}{\pi}\left[\frac{\sin\frac{2\pi x}{3}}{1} + \frac{\sin\frac{4\pi x}{3}}{2} + \frac{\sin\frac{6\pi x}{3}}{3} \cdots \infty\right]$$

Put $x = 3/2$

$$\left(2\times\frac{3}{2}\right)-\left(\frac{3}{2}\right)^2 = \frac{-9}{\pi^2}\left[\frac{\cos\pi}{1^2}+\frac{\cos 2\pi}{2^2}+\frac{\cos 3\pi}{3^2}-\infty\right]$$

$$+\frac{3}{\pi}\left[\frac{\sin\pi}{1^2}^{=0}+\frac{\sin 2\pi}{2}^{=0}+\frac{\sin 3\pi}{3}^{=0}-\infty\right]$$

$$3-\frac{9}{4} = \frac{-9}{\pi^2}\left[-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\infty\right]$$

$$\frac{3}{4} = \frac{9}{\pi^2}\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}--\infty\right]$$

$$\Rightarrow \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}-\infty = \frac{\pi^2}{12}$$