Beta Function.

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad — (1)$$

Also $\beta(m,n) = \beta(n,m)$ [Symmetry]

Now Put $x = \frac{t}{1+t}$ in eq. (1)

$\Rightarrow x(1+t) = t \Rightarrow x + xt = t \Rightarrow x = t(1-x)$

$\Rightarrow \boxed{t = \frac{x}{1-x}}$

when $x = 0$

$t = 0$

when $x = 1 \Rightarrow t = \infty$

Also $dx = \frac{(1+t)\cdot 1 - t(\cdot 1)}{(1+t)^2}\,dt = \frac{1+t-t}{(1+t)^2}\,dt$

$$\Rightarrow dx = \frac{dt}{(1+t)^2}$$

Now from eq. (1) we have

$$\beta(m,n) = \int_0^\infty \left(\frac{t}{1+t}\right)^{m-1}\left(1-\frac{t}{1+t}\right)^{n-1}\frac{dt}{(1+t)^2}$$

$$\beta(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m-1}}\left(\frac{1+\cancel{t}-\cancel{t}}{1+t}\right)^{n-1}\frac{dt}{(1+t)^2}$$

$$= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m-1}}\cdot\frac{1}{(1+t)^{n-1}}\cdot\frac{dt}{(1+t)^2}$$

$$= \int_0^\infty \frac{t^{m-1}\,dt}{(1+t)^{m-\cancel{1}+n-\cancel{1}+\cancel{2}}} = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt$$

$$\therefore \boxed{\beta(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\, dt.}$$

Also As β-function is symmetrical

$$\therefore \boxed{\beta(m,n) = \int_0^\infty \frac{t^{n-1}}{(1+t)^{m+n}}\, dt,}$$

Another form of β-function.

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx.$$

Put $x = \sin^2\theta$

$\Rightarrow dx = 2\sin\theta\cos\theta\, d\theta.$

Also when $x = 0 \Rightarrow \sin^2\theta = 0 \Rightarrow \theta = 0$
when $x = 1 \Rightarrow \sin^2\theta = 1 \Rightarrow \theta = \pi/2.$

$$\therefore \beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} (\sin\theta)^{2m-2}(\cos\theta)^{2n-2} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$\therefore \beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\, d\theta$$

$$\boxed{\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\, \cos^{2n-1}\theta\, d\theta}$$

$$\beta(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\, dt \quad \text{—— (2)}$$

$$= \int_0^1 \frac{t^{m-1}}{(1+t)^{m+n}}\, dt + \int_1^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\, dt$$

Let $t = \frac{1}{z}$ in second integral only.

when $t=1$, $z=1$ Also $dt = \frac{-1}{z^2}\, dz$

when $t=\infty$ $z=0$

$$\therefore \int_1^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\, dt = \int_1^0 \frac{\left(\frac{1}{z}\right)^{m-1}}{\left(1+\frac{1}{z}\right)^{m+n}} \left(\frac{-1}{z^2}\right) dz$$

$$= \int_1^0 \frac{1}{z^{m-1}} \times \frac{1}{\left(\frac{z+1}{z}\right)^{m+n}} \left(\frac{-1}{z^2}\right) dz$$

$$= -\int_1^0 \frac{1}{z^{m-1+2}} \cdot \frac{z^{m+n}}{(1+z)^{m+n}}\, dz$$

$$= \int_0^1 \frac{z^{m+n-m-1}}{(1+z)^{m+n}}\, dz \quad \left(\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right)$$

$$= \int_0^1 \frac{z^{n-1}}{(1+z)^{m+n}}\, dz$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\, dt \quad \text{—— (3)}$$

Using (3) in (2)

$$\beta(m,n) = \int_0^1 \frac{t^{m-1}}{(1+t)^{m+n}}\,dt + \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\,dt$$

$$\boxed{\therefore\ \beta(m,n) = \int_0^1 \frac{t^{m-1}+t^{n-1}}{(1+t)^{m+n}}\,dt}$$

So we've

(1) $$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

(2) $$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx.$$

(3) $$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta.$$

(4) $$= \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$$

(5) $$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx.$$