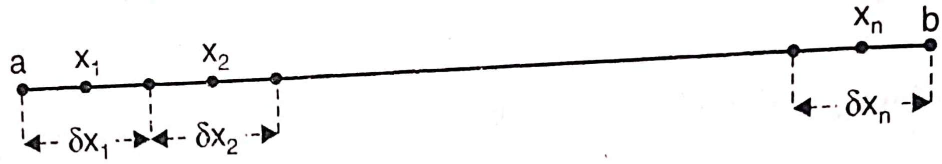
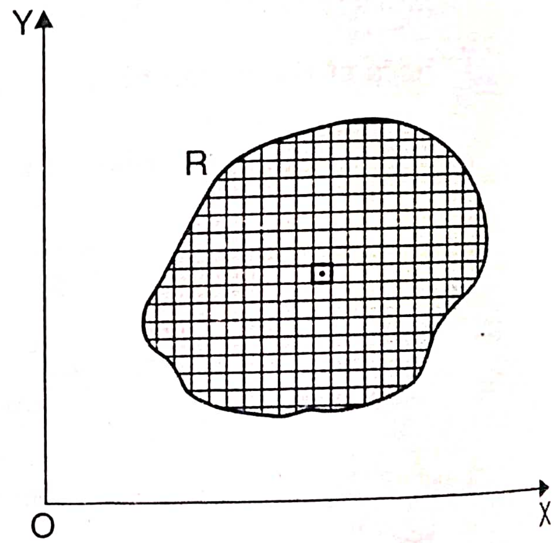
Multiple Integrals

12.1. DOUBLE INTEGRALS

The definite integral $\int_a^b f(x)\,dx$ is defined as the limits of the sum $f(x_1)\delta x_1 + f(x_2)\delta x_2 + \dots + f(x_n)\delta x_n$ when $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \dots, \delta x_n$ tends to zero. Here $\delta x_1, \delta x_2, \dots, \delta x_n$ are n sub-intervals into which the range $b - a$ has been divided and $x_1, x_2, \dots, x_n$ are values of x lying respectively in the first, second, ..., nth sub-interval.



A double integral is its counterpart in two dimensions. Let a single-valued and bounded function $f(x, y)$ of two independent variables x, y be defined in a closed region R of the xy-plane. Divide the region R into sub-regions by drawing lines parallel to co-ordinate axes. Number the rectangles which lie entirely inside the region R, from 1 to n. Let (x_r, y_r) be any point inside the rth rectangle whose area is δA_r.



Consider the sum

$$f(x_1, y_1)\delta A_1 + f(x_2, y_2)\delta A_2 + \dots + f(x_n, y_n)\delta A_n$$

$$= \sum_{r=1}^{n} f(x_r, y_r)\delta A_r \qquad \dots(1)$$

Let the number of these sub-regions increase indefinitely, such that the largest linear dimension (*i.e.*, diagonal) of δA_r approaches zero. The limit of the sum (1), if it exists, irrespective of the mode of sub-division, is called the *double integral* of $f(x, y)$ over the region R and is denoted by $\iint_R f(x, y)\, dA$

In other words,

$$\lim_{\substack{n \to \infty \\ \delta A_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r = \iint_R f(x, y)\, dA$$

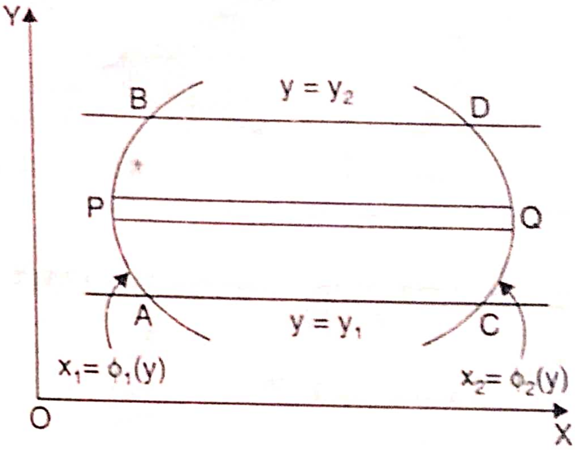
which is also expressed as

$$\iint_R f(x, y)\, dxdy \quad \text{or} \quad \iint_R f(x, y)\, dy\, dx$$

12.2. EVALUATION OF DOUBLE INTEGRALS

The methods of evaluating the double integrals depend upon the nature of the curves bounding the region R. Let the region R be bounded by the curves $x = x_1$, $x = x_2$ and $y = y_1$, $y = y_2$.

(i) When x_1, x_2 are functions of y and y_1, y_2 are constants. Let AB and CD be the curves $x_1 = \phi_1(y)$ and $x_2 = \phi_2(y)$.

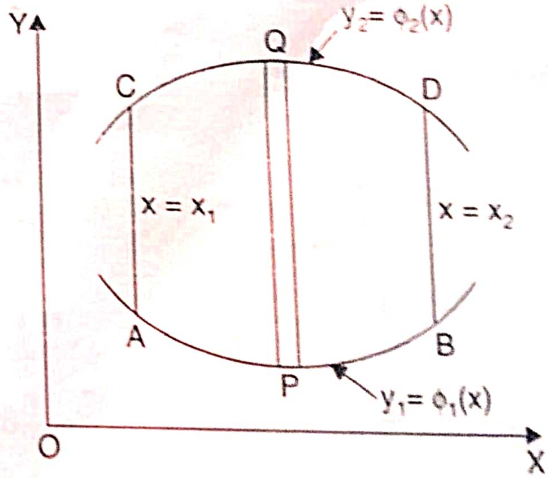


Take a horizontal strip PQ of width δy. Here the double integral is evaluated first w.r.t. x (treating y as a constant). The resulting expression which is a function of y is integrated w.r.t. y between the limits $y = y_1$ and $y = y_2$. Thus

$$\iint_R f(x, y)\, dx\, dy = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1 = \phi_1(y)}^{x_2 = \phi_2(y)} f(x, y)\, dx}\; dy}$$

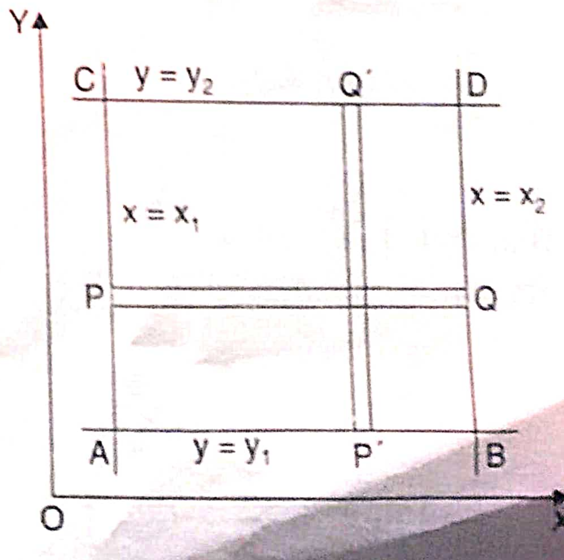
the integration being carried from the inner to the outer rectangle. Geometrically, the integral in the inner rectangle indicates that the integration is performed along the horizontal strip PQ (keeping y constant) while the outer rectangle corresponds to the sliding of the strip PQ from AC to BD thus covering the entire region ABDC of integration.

(ii) When y_1, y_2 are functions of x and x_1, x_2 are constants. Let AB and CD be the curves $y_1 = \phi_1(x)$ and $y_2 = \phi_2(x)$. Take a vertical strip PQ of width δx. Here the double integral is evaluated first w.r.t. y (treating x as constant). The resulting expression which is a function of x is integrated w.r.t. x between the limits $x = x_1$ and $x = x_2$. Thus,



$$\iint_R f(x, y)\, dx\, dy = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1 = \phi_1(x)}^{y_2 = \phi_2(x)} f(x, y)\, dy}\; dx}$$

the integration being carried from the inner to the outer rectangle. Geometrically, the integral in the inner rectanlge indicates that the integration is performed along the vertical strip PQ (keeping x constant) while the outer rectangle corresponds to the sliding of the strip PQ from AC to BD thus covering the entire region ABDC of integration.



(iii) When x_1, x_2, y_1, y_2 are constants. Here the region of integration R is the rectangle ABDC. It is immaterial whether we integrate first along the horizontal strip PQ and then slide it from AB to CD; or we integrate first along the vertical strip P′Q′ and then slide it from AC to BD. Thus the order of integration is immaterial, provided the limits of integration are changed accordingly.

$$\iint_R f(x, y)\,dx\,dy = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x, y)\,dx}\,dy} = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x, y)\,dy}\,dx}$$

Note 1. From cases (*i*) and (*ii*) above, we observe that *integration is to be performed w.r.t. the variable having variable limits first and then w.r.t. the variable with constant limits.*

Note 2. If $f(x, y)$ has discontinuities within or on the boundary of the region of integration, then the change of the order of integration does not result into the same integrals.

ILLUSTRATIVE EXAMPLES

Example 1. *Prove that* $\int_1^2 \int_3^4 (xy + e^y)\,dy\,dx = \int_3^4 \int_1^2 (xy + e^y)\,dx\,dy.$

Sol. $$\int_1^2 \int_3^4 (xy + e^y)\,dy\,dx = \int_1^2 \left[\int_3^4 (xy + e^y)\,dy\right] dx$$

$$= \int_1^2 \left[\frac{xy^2}{2} + e^y\right]_3^4 dx = \int_1^2 \left(8x + e^4 - \frac{9}{2}x - e^3\right) dx$$

$$= \int_1^2 \left(\frac{7}{2}x + e^4 - e^3\right) dx = \left[\frac{7x^2}{4} + (e^4 - e^3)x\right]_1^2$$

$$= 7 + 2(e^4 - e^3) - \frac{7}{4} - (e^4 - e^3) = \frac{21}{4} + e^4 - e^3$$

$$\int_3^4 \int_1^2 (xy + e^y)\,dx\,dy = \int_3^4 \left[\int_1^2 (xy + e^y)\,dx\right] dy = \int_3^4 \left[\frac{yx^2}{2} + xe^y\right]_1^2 dy$$

$$= \int_3^4 \left(2y + 2e^y - \frac{y}{2} - e^y\right) dy = \int_3^4 \left(\frac{3y}{2} + e^y\right) dy$$

$$= \left[\frac{3y^2}{4} + e^y\right]_3^4 = 12 + e^4 - \frac{27}{4} - e^3 = \frac{21}{4} + e^4 - e^3$$

Hence the result.

Example 2. *Evaluate* $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1 - x^2)(1 - y^2)}}$.

Sol. $$\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}} = \int_0^1 \left[\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-y^2)}}\right] dy = \int_0^1 \frac{1}{\sqrt{1-y^2}} \left[\sin^{-1} x\right]_0^1 dy$$

$$= \int_0^1 \frac{1}{\sqrt{1-y^2}} \cdot \frac{\pi}{2}\,dy = \frac{\pi}{2}\left[\sin^{-1} y\right]_0^1 = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}.$$

Example 3. *Show that* $\int_0^1 dx \int_0^1 \frac{x - y}{(x + y)^3}\,dy \neq \int_0^1 dy \int_0^1 \frac{x - y}{(x + y)^3}\,dx.$

Sol. $\text{LHS} = \int_0^1 dx \int_0^1 \frac{2x - (x+y)}{(x+y)^3}\, dy = \int_0^1 dx \int_0^1 \left[\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2}\right] dy$

$$= \int_0^1 \left[2x \cdot \frac{(x+y)^{-2}}{-2} - \frac{(x+y)^{-1}}{-1}\right]_0^1 dx = \int_0^1 \left[\frac{-x}{(x+y)^2} + \frac{1}{x+y}\right]_0^1 dx$$

$$= \int_0^1 \left[\frac{-x}{(x+1)^2} + \frac{1}{x+1} + \frac{1}{x} - \frac{1}{x}\right] dx = \int_0^1 \frac{-x+x+1}{(x+1)^2}\, dx = \int_0^1 \frac{1}{(x+1)^2}\, dx$$

$$= \left[-\frac{1}{x+1}\right]_0^1 = -\frac{1}{2} + 1 = \frac{1}{2}$$

$$\text{RHS} = \int_0^1 dy \int_0^1 \frac{(x+y) - 2y}{(x+y)^3}\, dx = \int_0^1 dy \int_0^1 \left[\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3}\right] dx$$

$$= \int_0^1 \left[\frac{(x+y)^{-1}}{-1} - 2y \cdot \frac{(x+y)^{-2}}{-2}\right]_0^1 dy = \int_0^1 \left[-\frac{1}{x+y} + \frac{y}{(x+y)^2}\right]_0^1 dy$$

$$= \int_0^1 \left[-\frac{1}{1+y} + \frac{y}{(1+y)^2} + \frac{1}{y} - \frac{1}{y}\right] dy = \int_0^1 \frac{-1-y+y}{(1+y)^2}\, dy = -\int_0^1 \frac{1}{(1+y)^2}\, dy$$

$$= \left[\frac{1}{1+y}\right]_0^1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$\therefore$ The two integrals are not equal.

Example 4. *Evaluate* $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{1+x^2+y^2}$. (Kerala 2012)

Sol. $I = \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{(1+x^2)+y^2}\, dy\right] dx$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1} \frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} [\tan^{-1} 1 - \tan^{-1} 0]\, dx = \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}}$$

$$= \frac{\pi}{4} \left[\log (x + \sqrt{1+x^2})\right]_0^1 = \frac{\pi}{4} [\log (1+\sqrt{2}) - \log 1] = \frac{\pi}{4} \log (\sqrt{2}+1).$$

Example 5. *Evaluate* $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$.

Sol. $I = \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\, dx\right] dy.$

$$= \int_0^a \left[\frac{x\sqrt{a^2 - y^2 - x^2}}{2} + \frac{a^2 - y^2}{2} \sin^{-1} \frac{x}{\sqrt{a^2 - y^2}} \right]_0^{\sqrt{a^2 - y^2}} dy$$

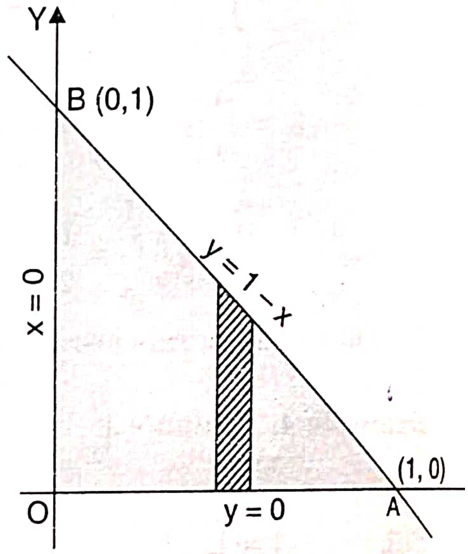
$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \right.$$

$$= \int_0^a \frac{a^2 - y^2}{2} \sin^{-1} 1\, dy = \frac{\pi}{4} \int_0^a (a^2 - y^2)\, dy$$

$$= \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3} \right]_0^a = \frac{\pi}{4} \left[a^3 - \frac{a^3}{3} \right] = \frac{\pi a^3}{6}.$$

Example 6. *Evaluate* $\iint e^{2x+3y}\, dx\, dy$ *over the triangle bounded by* $x = 0$, $y = 0$ *and* $x + y = 1$.

Sol. The region R of integration is the triangle OAB. Here x varies from 0 to 1 and y varies from x-axis upto the line $x + y = 1$ *i.e.*, from 0 to $1 - x$.



$\therefore$ The region R can be expressed as

$$0 \le x \le 1,\ 0 \le y \le 1 - x$$

$$\therefore \quad \iint_R e^{2x+3y}\, dx\, dy$$

$$= \int_0^1 \int_0^{1-x} e^{2x+3y}\, dy\, dx$$

$$= \int_0^1 \left[\frac{1}{3} e^{2x+3y} \right]_0^{1-x} dx$$

$$= \frac{1}{3} \int_0^1 (e^{3-x} - e^{2x})\, dx$$

$$= \frac{1}{3} \left[-e^{3-x} - \frac{1}{2} e^{2x} \right]_0^1 = -\frac{1}{3} \left[\left(e^2 + \frac{1}{2} e^2 \right) - \left(e^3 + \frac{1}{2} \right) \right]$$

$$= -\frac{1}{3} \left[-e^2(e - 1) + \frac{1}{2}(e^2 - 1) \right]$$

$$= \frac{1}{6} (e - 1)\, [2e^2 - (e + 1)] = \frac{1}{6} (e - 1)(2e^2 - e - 1)$$

$$= \frac{1}{6} (e - 1)(e - 1)(2e + 1) = \frac{1}{6} (e - 1)^2 (2e + 1).$$

Example 7. *Evaluate* $\iint_R y\, dx\, dy$, *where R is the region bounded by the parabolas* $y^2 = 4x$ *and* $x^2 = 4y$.

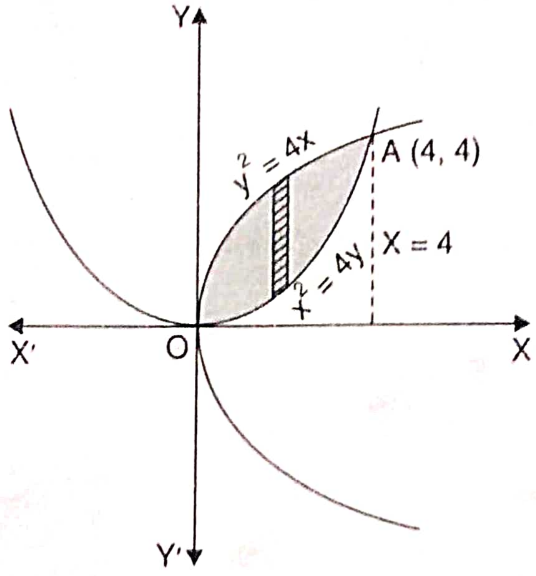
Sol. Solving $y^2 = 4x$ and $x^2 = 4y$, we have

$$\left(\frac{x^2}{4}\right)^2 = 4x \quad \text{or} \quad x(x^3 - 64) = 0$$

$\therefore$ $\quad x = 0, 4$

When $\quad x = 4, \ y = 4$

$\therefore$ Co-ordinates of A are (4, 4)



The region R can be expressed as

$$0 \le x \le 4, \ \frac{x^2}{4} \le y \le 2\sqrt{x}$$

$$\therefore \quad \iint_R y \, dx \, dy = \int_0^4 \int_{x^2/4}^{2\sqrt{x}} y \, dy \, dx$$

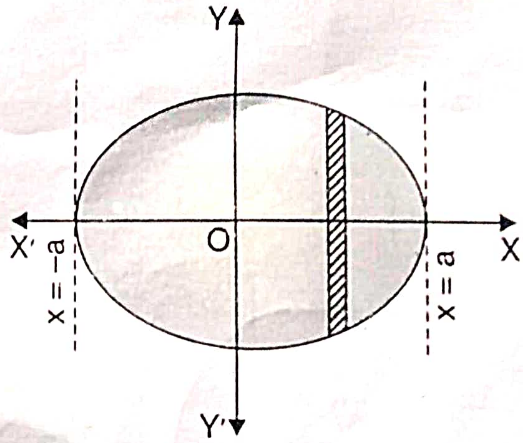
$$= \int_0^4 \frac{1}{2}\left[y^2\right]_{x^2/4}^{2\sqrt{x}} dx = \frac{1}{2}\int_0^4 \left(4x - \frac{x^4}{16}\right) dx$$

$$= \frac{1}{2}\left[2x^2 - \frac{x^5}{80}\right]_0^4 = \frac{1}{2}\left[32 - \frac{1024}{80}\right] = \frac{48}{5}.$$

Example 8. *Evaluate* $\iint (x + y)^2 \, dx \, dy$ *over the area bounded by the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ (K.U.K. 2009; U.K.T.U. 2011)

Sol. For the ellipse

$$\frac{y}{b} = \pm\sqrt{1 - \frac{x^2}{a^2}} \quad \text{or} \quad y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$



$\therefore$ The region of integration R can be expressed as

$$-a \le x \le a, \ -\frac{b}{a}\sqrt{a^2 - x^2} \le y \le \frac{b}{a}\sqrt{a^2 - x^2}$$

$$\therefore \quad \iint_R (x + y)^2 \, dx \, dy = \iint_R (x^2 + y^2 + 2xy) \, dx \, dy$$

$\int_{-a}^{a}$ f(x) - = 0

$$= \int_{-a}^{a} \int_{-b/a\sqrt{a^2 - x^2}}^{b/a\sqrt{a^2 - x^2}} (x^2 + y^2 + 2xy) \, dy \, dx$$

$$= \int_{-a}^{a} \int_{-b/a\sqrt{a^2 - x^2}}^{b/a\sqrt{a^2 - x^2}} (x^2 + y^2) \, dy \, dx + \int_{-a}^{a} \int_{-b/a\sqrt{a^2 - x^2}}^{b/a\sqrt{a^2 - x^2}} 2xy \, dy \, dx$$

0

$$= \int_{-a}^{a} \int_{0}^{b/a\sqrt{a^2 - x^2}} 2(x^2 + y^2) \, dy \, dx + 0$$

[Since $(x^2 + y^2)$ is an even function of y and $2xy$ is an odd function of y]

$$= \int_{-a}^{a} \left[2\left(x^2 y + \frac{y^3}{3}\right)\right]_0^{b/a\sqrt{a^2 - x^2}} dx$$

$$= 2\int_{-a}^{a} \left[x^2 \cdot \frac{b}{a}\sqrt{a^2 - x^2} + \frac{1}{3} \cdot \frac{b^3}{a^3}(a^2 - x^2)^{3/2}\right] dx$$

$$= 4\int_0^a \left[\frac{b}{a}x^2\sqrt{a^2 - x^2} + \frac{b^3}{3a^3}(a^2 - x^2)^{3/2}\right] dx$$

$$= 4\int_0^{\pi/2}\left(\frac{b}{a}\cdot a^2 \sin^2\theta \cdot a\cos\theta + \frac{b^3}{3a^3}\cdot a^3\cos^3\theta\right)\times a\cos\theta\, d\theta$$

on putting $x = a \sin\theta$ and $dx = a\cos\theta\, d\theta$

$$= 4\int_0^{\pi/2}\left(a^3 b\sin^2\theta\cos^2\theta + \frac{ab^3}{3}\cos^4\theta\right) d\theta$$

$$= 4\left[a^3 b\cdot\frac{1\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} + \frac{ab^3}{3}\cdot\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}\right] = \frac{\pi}{4}(a^3 b + ab^3) = \frac{\pi}{4}ab(a^2 + b^2).$$

EXERCISE 12.1

Evaluate the following integrals (1–10):

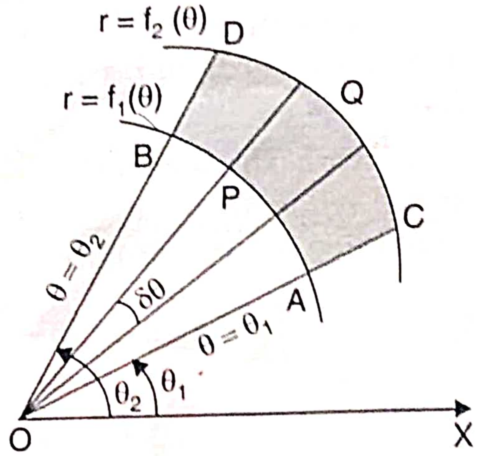
1. $\int_0^3\int_0^1 (x^2 + 3y^2)\, dy\, dx$.
2. $\int_0^3\int_1^2 xy\,(1 + x + y)\, dy\, dx$.
3. $\int_1^a\int_1^b \frac{dy\, dx}{xy}$.
4. $\int_1^2\int_0^x \frac{dy\, dx}{x^2 + y^2}$. (P.T.U. May 20
5. $\int_0^1 dx\int_0^x e^{y/x}\, dy$.
6. $\int_0^1\int_{x^2}^x (x^2 + 3y + 2)\, dy\, dx$.
7. $\int_0^1\int_0^{x^2} e^{y/x}\, dy\, dx$. (Kerala 2009)
8. $\int_0^1\int_y^{y^2+1} x^2 y\, dx\, dy$.
9. $\int_0^{2a}\int_0^{\sqrt{2ax - x^2}} (x^2 + y^2)\, dy\, dx$.
10. $\int_0^1\int_x^{\sqrt{x}} (x^2 + y^2)\, dy\, dx$.
11. Evaluate $\iint (x^2 + y^2)\, dx\, dy$ over the region in the positive quadrant for which $x + y \le 1$.
12. Evaluate $\iint x^2y^2\, dx\, dy$ over the circle $x^2 + y^2 = 1$.
13. Evaluate $\iint xy\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$.
    (U.P.T.U. 2009; M.D.U. Dec. 201
14. Compute the value of $\iint_R y\, dx\, dy$, where R is the region in the first quadrant bounded by t
    ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
15. Evaluate $\iint_S \sqrt{xy - y^2}\, dx\, dy$, where S is a triangle with vertices (0, 0), (10, 1) and (1, 1).
16. Evaluate $\iint xy\,(x + y)\, dx\, dy$ over the area between $y = x^2$ and $y = x$. (U.K.T.U. 2011, 201
17. Evaluate $\iint_A xy\, dx\, dy$, where A is the domain bounded by x-axis, ordinate $x = 2a$ and the cur
    $x^2 = 4ay$. (M.T.U. 2012; G.B.T.U. 201
18. Evaluate $\iint_R xy\, dx\, dy$, where R is the region inside the square $|x| + |y| = 1$.

Answers

1. 12	2. $30\frac{3}{4}$	3. $\log a \log b$	4. $\frac{\pi}{4}\log 2$
5. $\frac{1}{2}(e-1)$	6. $\frac{7}{12}$	7. $\frac{1}{2}$	8. $\frac{67}{120}$
9. $\frac{3\pi a^4}{4}$	10. $\frac{3}{35}$	11. $\frac{1}{6}$	12. $\frac{\pi}{24}$
13. $\frac{a^4}{8}$	14. $\frac{ab^2}{3}$	15. 6	16. $\frac{3}{56}$
17. $\frac{a^4}{3}$	18. $\frac{1}{6}$.		

12.3. EVALUATION OF DOUBLE INTEGRALS IN POLAR CO-ORDINATES

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\, dr\, d\theta$ over the region bounded by the straight lines $\theta=\theta_1$, $\theta=\theta_2$ and the curves $r=r_1$, $r=r_2$, we first integrate w.r.t. r between the limits $r=r_1$ and $r=r_2$ (treating θ as a constant). The resulting expression is then integrated w.r.t. θ between the limits $\theta=\theta_1$ and $\theta=\theta_2$.



Geometrically, AB and CD are the curves $r=f_1(\theta)$ and $r=f_2(\theta)$ bounded by the lines $\theta=\theta_1$ and $\theta=\theta_2$ so that ACDB is the region of integration. PQ is a wedge of angular thickness $\delta\theta$. Then $\int_{r=r_1}^{r=r_2} f(r,\theta)\,dr$ indicates that the integration is performed along PQ (*i.e.*, r varies, θ is constant) and the integration w.r.t. θ means rotation of this strip PQ from AC to BD. dr indicates that the integration is performed along PQ (*i.e.*, r varies, θ is constant) and the integration w.r.t. θ means rotation of this strip PQ from AC to BD.

ILLUSTRATIVE EXAMPLES

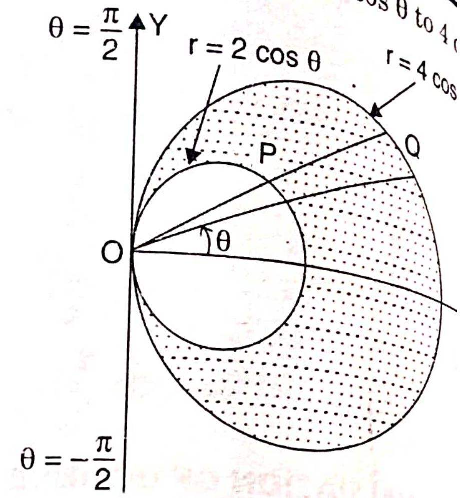
Example 1. *Evaluate* $\int_0^{\pi/2}\left[\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\right]d\theta.$

Sol.
$$I=\int_0^{\pi/2}\left[\int_0^{a\cos\theta} -\frac{1}{2}(a^2-r^2)^{1/2}(-2r)\,dr\right]d\theta$$
$$=\int_0^{\pi/2}\left[-\frac{1}{2}\cdot\frac{(a^2-r^2)^{3/2}}{3/2}\right]_0^{a\cos\theta} d\theta$$
$$=-\frac{1}{3}\int_0^{\pi/2}(a^3\sin^3\theta-a^3)\,d\theta=-\frac{a^3}{3}\left[\frac{2}{3}-\frac{\pi}{2}\right]=\frac{a^3}{18}(3\pi-4).$$

Example 2. *Evaluate* $\iint r^3\, dr\, d\theta$, *over the area bounded between the circles* $r=2\cos\theta$ *and* $r=4\cos\theta$. (M.D.U. Dec. 2012; Kerala 2009)

Sol. The region of integration R is shown shaded. Here r varies from $2 \cos \theta$ to 4

while θ varies from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$.



$$\therefore \quad \iint_R r^3\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \int_{2\cos\theta}^{4\cos\theta} r^3 dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4}\right]_{2\cos\theta}^{4\cos\theta} d\theta$$

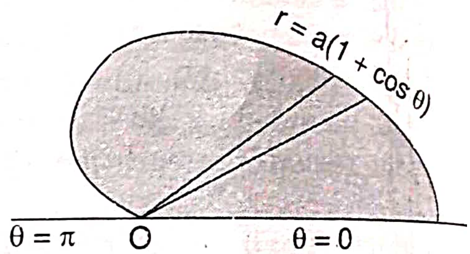
$$= \int_{-\pi/2}^{\pi/2} \frac{1}{4}(256 \cos^4 \theta - 16 \cos^4 \theta)\, d\theta$$

$$= 60 \int_{-\pi/2}^{\pi/2} \cos^4 \theta\, d\theta$$

$$= 120 \int_0^{\pi/2} \cos^4 \theta\, d\theta \qquad \text{[Since } \cos^4\theta \text{ is an even function}$$

$$= 120 \times \frac{3\times 1}{4\times 2}\cdot\frac{\pi}{2} = \frac{45}{2}\pi.$$

Example 3. *Evaluate* $\iint r \sin \theta\, dr\, d\theta$ *over the area of the cardioid* $r = a\,(1 + \cos \theta)$ *above the initial line.*



Sol. The region of integration R is covered by radial strips whose ends are $r = 0$ and $r = a\,(1 + \cos \theta)$, the strips starting from $\theta = 0$ and ending at $\theta = \pi$.

$$\therefore \quad \iint_R r \sin \theta\, dr\, d\theta = \int_0^{\pi} \int_0^{a(1+\cos\theta)} r \sin \theta\, dr\, d\theta\cdot$$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta = \frac{1}{2}\int_0^{\pi} \sin\theta\,.\,a^2(1+\cos\theta)^2\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot\left(2\cos^2\frac{\theta}{2}\right)^2 d\theta = 4a^2 \int_0^{\pi} \sin\frac{\theta}{2}\cos^5\frac{\theta}{2}\, d\theta$$

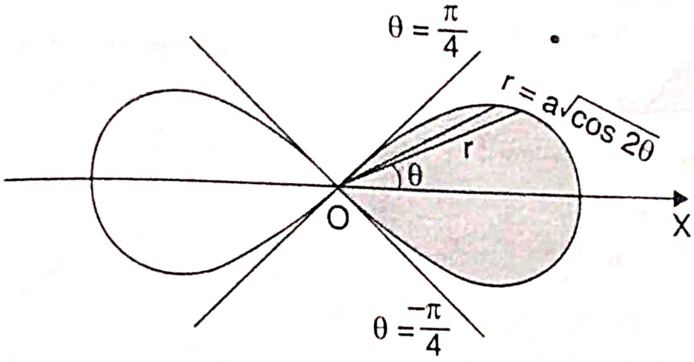
Putting $\cos\dfrac{\theta}{2} = t$ so that $\sin\dfrac{\theta}{2}\, d\theta = -\,2dt$

$$\iint_R r \sin\theta\, dr\, d\theta = 4a^2 \int_1^0 t^5\,(-\,2dt) = 8a^2 \int_0^1 t^5 dt = 8a^2 \left[\frac{t^6}{6}\right]_0^1 = 8a^2\left(\frac{1}{6}\right) = \frac{4a^2}{3}$$

Example 4. *Evaluate* $\displaystyle\iint \frac{r\, dr\, d\theta}{\sqrt{a^2 + r^2}}$ *over one loop of the lemniscate* $r^2 = a^2 \cos 2\theta$.

Sol. The region of integration R is covered by radial strips whose ends are $r = 0$ a

$r = a\sqrt{\cos 2\theta}$, the strips starting from $\theta = -\dfrac{\pi}{4}$ and ending at $\theta = \dfrac{\pi}{4}$.



$$\therefore \quad \iint_R \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}} = \int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \frac{1}{2}(a^2+r^2)^{-1/2}\cdot 2r\,dr\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\left[\frac{1}{2}\cdot\frac{(a^2+r^2)^{1/2}}{1/2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta = \int_{-\pi/4}^{\pi/4}[(a^2+a^2\cos 2\theta)^{1/2} - a]\,d\theta$$

$$= a\int_{-\pi/4}^{\pi/4}[(1+\cos 2\theta)^{1/2} - 1]\,d\theta = a\int_{-\pi/4}^{\pi/4}[(2\cos^2\theta)^{1/2} - 1]\,d\theta$$

$$= a\int_{-\pi/4}^{\pi/4}(\sqrt{2}\cos\theta - 1)\,d\theta = 2a\int_0^{\pi/4}(\sqrt{2}\cos\theta - 1)\,d\theta = 2a\left[\sqrt{2}\sin\theta - \theta\right]_0^{\pi/4}$$

$$= 2a\left[\sqrt{2}\,.\,\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right] = 2a\left(1 - \frac{\pi}{4}\right).$$

EXERCISE 12.2

Evaluate the following integrals (1–4):

1. $\displaystyle\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta.$
2. $\displaystyle\int_0^{\pi/2}\int_0^{a\cos\theta} r\sin\theta\,dr\,d\theta.$
3. $\displaystyle\int_0^{\pi/2}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta.$
4. $\displaystyle\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta\,dr\,d\theta.$
5. Show that $\displaystyle\iint_R r^2\sin\theta\,dr\,d\theta = \frac{2a^3}{3}$, where R is the region bounded by the semi-circle $r = 2a\cos\theta$, above the initial line.
6. Evaluate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.
7. Evaluate $\iint r\sin\theta\,dr\,d\theta$ over the cardioid $r = a(1-\cos\theta)$ above the initial line. *(Kerala 2010)*
8. Evaluate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r = 2a\cos\theta$, $r = 2b\cos\theta$, $(b < a)$.

Answers

1. $\dfrac{\pi a^2}{4}$
2. $\dfrac{a^2}{6}$
3. $a^2\left(1+\dfrac{\pi}{8}\right)$
4. $\dfrac{5}{8}\pi a^3$
6. $\dfrac{45\pi}{2}$
7. $\dfrac{4a^2}{3}$
8. $\dfrac{3\pi}{2}(a^4 - b^4)$.

12.4. CHANGE OF ORDER OF INTEGRATION

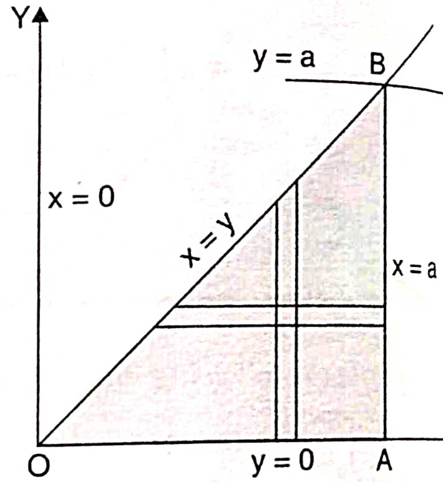
In a double integral, if the limits of integration are constant, then the order of integration is immaterial, provided the limits of integration are changed accordingly. Thus

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx .$$

But if the limits of integration are variable, a change in the order of integration necessitates change in the limits of integration. A rough sketch of the region of integration helps in fixing the new limits of integration.

ILLUSTRATIVE EXAMPLES

Example 1. *Change the order of integration in* $\int_0^a \int_y^a \frac{x\, dx\, dy}{x^2 + y^2}$ *and hence evaluate the same.* (M.D.U. Dec. 2012; Kerala 200



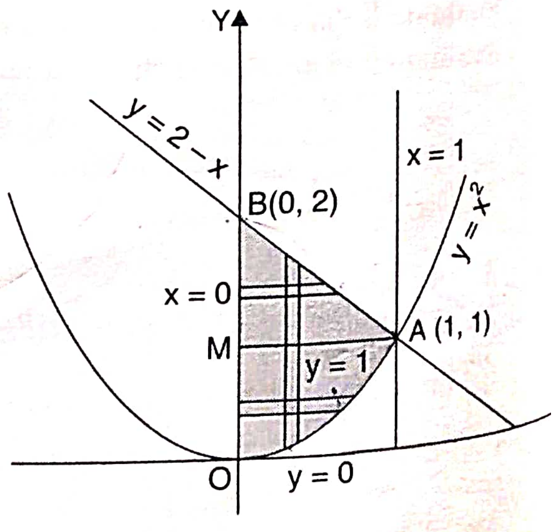
Sol. From the limits of integration, it is clear that the region of integration is bounded by $x = y$, $x = a$, $y = 0$ and $y = a$. Thus the region of integration is the ΔOAB and is divided into horizontal strips. For changing the order of integration, we divide the region of integration into vertical strips. The new limits of integration become: y varies from 0 to x and x varies from 0 to a.

$$\int_0^a \int_y^a \frac{x\, dx\, dy}{x^2 + y^2} = \int_0^a \int_0^x \frac{x\, dy\, dx}{x^2 + y^2}$$

$$= \int_0^a x \cdot \left[\frac{1}{x} \tan^{-1} \frac{y}{x} \right]_0^x dx$$

$$= \int_0^a \frac{\pi}{4}\, dx = \frac{\pi}{4} \cdot \left[x \right]_0^a = \frac{\pi a}{4}.$$

Example 2. *Change the order of integration in* $I = \int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$ *and hence evaluate the same.* (Kerala 2010, 2012; M.D.U. Dec. 2010; P.T.U. 2009, 201



Sol. From the limits of integration, it is clear that we have to integrate first with respect to y which varies from $y = x^2$ to $y = 2 - x$ and then with respect to x which varies from $x = 0$ to $x = 1$. The region of integration (shown shaded) is divided into vertical strips. For changing the order of integration, we divide the region of integration into horizontal strips.

Solving $y = x^2$ and $y = 2 - x$, the co-ordinates of A are (1, 1). Draw AM $\perp$ OY. The region of integration is divided into two parts, OAM and MAB.

For the region OAM, x varies from 0 to $\sqrt{y}$ and y varies from 0 to 1. For the region MAB, x varies from 0 to $2-y$ and y varies from 1 to 2.

$$\therefore \int_0^1\int_{x^2}^{2-x} xy\,dx\,dy = \int_0^1\int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2\int_0^{2-y} xy\,dx\,dy$$

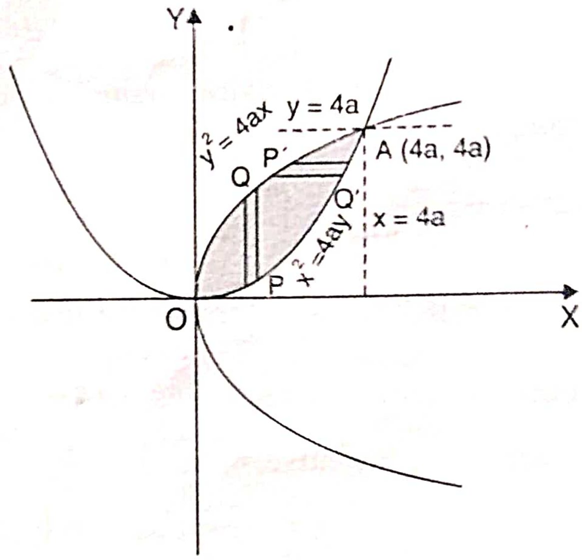
$$= \int_0^1 y\cdot\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y\cdot\left[\frac{x^2}{2}\right]_0^{2-y} dy = \frac{1}{2}\int_0^1 y^2dy + \frac{1}{2}\int_1^2 y(2-y)^2\,dy$$

$$= \frac{1}{2}\cdot\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2(4y-4y^2+y^3)\,dy = \frac{1}{6}+\frac{1}{2}\left[2y^2-\frac{4y^3}{3}+\frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{6}+\frac{1}{2}\left[\left(8-\frac{32}{3}+4\right)-\left(2-\frac{4}{3}+\frac{1}{4}\right)\right] = \frac{3}{8}.$$

Example 3. *Change the order of integration in the following integral and evaluate:*

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx.$$ (P.T.U. Jan. 2010; M.D.U. May 2009)



Sol. From the limits of integration, it is clear that we have to integrate first w.r.t. y which varies from $y=\dfrac{x^2}{4a}$ to $y=2\sqrt{ax}$ and then w.r.t. x which varies from $x=0$ to $x=4a$. Thus integration is first performed along the vertical strip PQ which extends from a point P on the parabola $y=\dfrac{x^2}{4a}$ (*i.e.,* $x^2=4ay$) to the point Q on the parabola $y=2\sqrt{ax}$ (*i.e.,* $y^2=4ax$). Then the strip slides from O to A $(4a, 4a)$, the point of intersection of the two parabolas.

For changing the order of integration, we divide the region of integration OPAQO into horizontal strips P′Q′ which extend from P′ on the parabola $y^2=4ax$ *i.e.,* $x=\dfrac{y^2}{4a}$ to Q′ on the parabola $x^2=4ay$ *i.e.,* $x=2\sqrt{ay}$. Then this strip slides from O to A$(4a, 4a)$, *i.e.,* varies from 0 to $4a$.

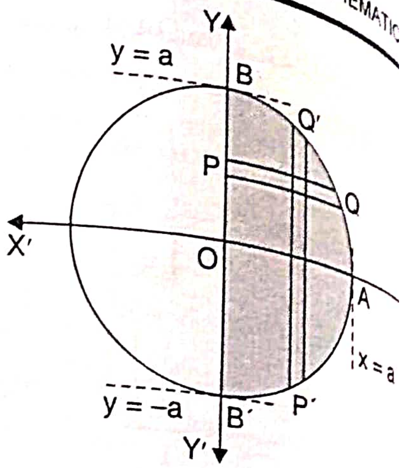
$$\therefore \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy$$

$$= \int_0^{4a}\left[x\right]_{y^2/4a}^{2\sqrt{ay}} dy = \int_0^{4a}\left(2\sqrt{ay}-\frac{y^2}{4a}\right)dy$$

$$= \left[2\sqrt{a}\cdot\frac{y^{3/2}}{3/2}-\frac{y^3}{12a}\right]_0^{4a} = \frac{4}{3}\sqrt{a}\cdot(4a)^{3/2}-\frac{64a^3}{12a}$$

$$= \frac{4}{3}\sqrt{a}\cdot 8a^{3/2}-\frac{16a^2}{3} = \frac{32a^2}{3}-\frac{16a^2}{3} = \frac{16a^2}{3}.$$

Example 4. *Change the order of integration in the integral* $\displaystyle\int_{-a}^{a}\int_0^{\sqrt{a^2-y^2}} f(x, y)\,dx\,dy.$



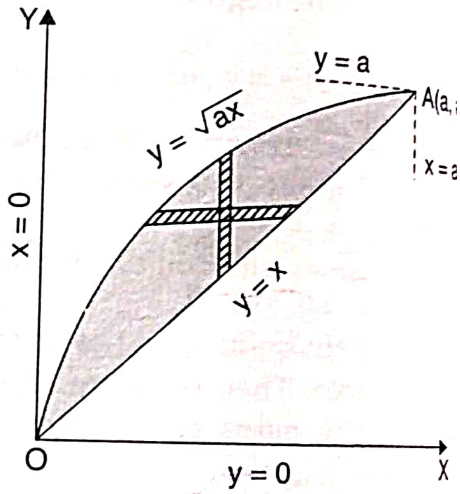
Sol. From the limits of integration, it is clear that we have to integrate first w.r.t. x which varies from $x = 0$ to $x = \sqrt{a^2 - y^2}$ and then w.r.t. y which varies from $y = -a$ to $y = a$. Thus integration is first performed along the horizontal strip PQ which extends from a point P on $x = 0$ (*i.e.*, y-axis) to the point Q on the circle $x = \sqrt{a^2 - y^2}$ (*i.e.*, $x^2 = a^2 - y^2$ or $x^2 + y^2 = a^2$). Then the strip slides from B′ to B.

For changing the order of integration, we divide the region of integration B′AQBPB′ into vertical strips P′Q′ which extend from P′ on the circle $y = -\sqrt{a^2 - x^2}$ to Q′ on the circle $y = \sqrt{a^2 - x^2}$. [$x^2 + y^2 = a^2 \Rightarrow y = \pm\sqrt{a^2 - x^2}$; for points in the 4th quadran $y = -\sqrt{a^2 - x^2}$ and for points in the first quadrant, $y = \sqrt{a^2 - x^2}$] . Then this strip slides fro y-axis ($x = 0$) to A, where $x = a$.

$$\therefore \quad \int_{-a}^{a}\int_{0}^{\sqrt{a^2 - x^2}} f(x, y)\, dx\, dy = \int_{0}^{a}\int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} f(x, y)\, dy\, dx .$$

Example 5. *By changing the order of integration, evaluate*

$$\int_0^a \int_{y^2/a}^{y} \frac{y}{(a - x)\sqrt{ax - y^2}}\, dx\, dy .$$



Sol. The given limits show that the area of integration (shown shaded) lies between $x = \dfrac{y^2}{a}$, $x = y$, $y = 0$, $y = a$. We can consider it as lying between

$$y = x,\ y = \sqrt{ax}\,,\ x = 0,\ x = a.$$

$$\therefore \quad \int_0^a \int_{\frac{y^2}{a}}^{y} \frac{y}{(a - x)\sqrt{ax - y^2}}\, dx\, dy = \int_0^a \int_x^{\sqrt{ax}} \frac{y}{(a - x)\sqrt{ax - y^2}}\, dy\, dx$$

$$= \int_0^a \int_x^{\sqrt{ax}} -\frac{1}{2}\cdot\frac{1}{a - x}\cdot(ax - y^2)^{-1/2}\cdot(-2y)\, dy dx$$

$$= \int_0^a \frac{-1}{2(a - x)}\cdot\left[\frac{(ax - y^2)^{1/2}}{\frac{1}{2}}\right]_x^{\sqrt{ax}} dx = \int_0^a \frac{-1}{a - x}\left[0 - (ax - x^2)^{1/2}\right] dx$$

$$= \int_0^a \frac{(ax - x^2)^{1/2}}{a - x}\, dx = \int_0^a \frac{[x(a - x)]^{1/2}}{a - x}\, dx$$

$$= \int_0^a \left(\frac{x}{a - x}\right)^{1/2} dx \qquad \text{Put } x = a\sin^2\theta,\ dx = 2a\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{a\sin^2\theta}{a\cos^2\theta}\right)^{1/2} \cdot 2a\sin\theta\cos\theta\, d\theta = 2a\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta = 2a\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi a}{2}.$$

EXERCISE 12.3

Evaluate the following integrals by changing the order of integration:

1. $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dy \, dx$. (M.D.U. Dec. 2010; K.U.K. Dec. 2014)

2. $\int_0^1 \int_{4y}^4 e^{x^2} dx \, dy$. (Kerala 2012)

3. $\int_0^4 \int_y^4 \frac{x \, dx \, dy}{x^2 + y^2}$.

4. $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2 - y^2}} x \, dx \, dy$.

5. $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x \, dy \, dx}{\sqrt{x^2 + y^2}}$. (K.U.K. 2012, 2013; M.D.U. Dec. 2013)

6. $\int_0^a \int_{x^2/a}^{2a-x} xy \, dy \, dx$. (U.P.T.U. 2008)

7. $\int_0^b \int_0^{a/b\sqrt{b^2 - y^2}} xy \, dx \, dy$.

8. $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2 \, dy \, dx}{\sqrt{y^4 - a^2x^2}}$.

9. $\int_0^2 \int_{\sqrt{2y}}^2 \frac{x^2 \, dx \, dy}{\sqrt{x^4 - 4y^2}}$. (M.D.U. Dec. 2011)

10. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy \, dx$. (M.D.U. Dec. 2009; K.U.K. Dec. 2010; M.T.U. 2013; P.T.U. 2011)

11. $\int_0^\infty \int_0^x xe^{-\frac{x^2}{y}} dy \, dx$. (P.T.U. 2010; M.T.U. 2013; M.D.U. Dec. 2014)

12. $\int_0^3 \int_1^{\sqrt{4-y}} (x + y) \, dx \, dy$.

13. $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2) \, dy \, dx$.

Answers

1. $\frac{\pi}{16}$	2. $\frac{1}{8}(e^{16} - 1)$	3. π	4. $\frac{a^3\sqrt{2}}{6}$
5. $1 - \frac{1}{\sqrt{2}}$	6. $\frac{3a^4}{8}$	7. $\frac{1}{8}a^2b^2$	8. $\frac{\pi a^2}{6}$
9. $\frac{2\pi}{3}$	10. 1	11. $\frac{1}{2}$	12. $\frac{241}{60}$
13. $\frac{a^3}{28} + \frac{a}{20}$.			

12.5. TRIPLE INTEGRALS

Consider a function $f(x, y, z)$ which is continuous at every point of a finite region V of three dimensional space. Divide the region V into n sub-regions of respective volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$. Let (x_r, y_r, z_r) be an arbitrary point in the rth sub-region. Consider the sum

$$\sum_{r=1}^{n} f(x_r, y_r, z_r)\, \delta V_r .$$

The limit of this sum as $n \to \infty$ and $\delta V_r \to 0$, if it exists, is called the *triple integral* of $f(x, y, z)$ over the region V and is denoted by $\iiint_V f(x, y, z)\, dV$.

For purposes of evaluation, it can be expressed as the repeated integral

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y, z)\, dx\, dy\, dz \quad ...(i)$$

the order of integration depending upon the limits.

Let x_1, x_2 be function of y, z; y_1, y_2 be function of z and z_1, z_2 be constants, *i.e.*,

Let $x_1 = f_1(y, z), x_2 = f_2(y, z), y_1 = \phi_1(z), y_2 = \phi_2(z)$ and $z_1 = a, z_2 = b$.

Then the integral (*i*) is evaluated as follows:

$$\boxed{\int_{z_1=a}^{z_2=b} \boxed{\int_{y_1=\phi_1(z)}^{y_2=\phi_2(z)} \boxed{\int_{x_1=f_1(y,z)}^{x_2=f_2(y,z)} f(x,y,z)\, dx}\; dy}\; dz}$$

First $f(x, y, z)$ is integrated w.r.t. x (keeping y and z constant) between the limits x_1 and x_2. The resulting expression, which is a function of y and z is then integrated w.r.t. y (keeping z constant) between the limits y_1 and y_2. The resulting expression, which is a function of z only is then integrated w.r.t. z between the limits z_1 and z_2. The order of integration is from the innermost rectangle to the outermost rectangle.

Limits involving two variables are kept innermost, then the limits involving one variable and finally the constant limits.

If $x_1, x_2; y_1, y_2$ and z_1, z_2 are all constants, then the order of integration is immaterial provided the limits are changed accordingly. Thus

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\, dx\, dy\, dz = \int_{x_1}^{x_2}\int_{z_1}^{z_2}\int_{y_1}^{y_2} f(x,y,z)\, dy\, dz\, dx$$

$$= \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\, dz\, dy\, dx, \text{ etc.}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx.$ (Kerala 2009)

Sol.

$$I = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{(1-x^2-y^2)-z^2}}\, dz\, dy\, dx$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} (\sin^{-1}1 - \sin^{-1}0)\, dy\, dx$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} \frac{\pi}{2}\, dy\, dx = \int_0^1 \frac{\pi}{2}\Big[y\Big]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{2}\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right]_0^1 = \frac{\pi}{4}[\sin^{-1}1] = \frac{\pi}{4}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}.$$

Example 2. *Evaluate* $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy$. (K.U.K. Dec. 2014)

Sol.
$$I = \int_1^e \int_1^{\log y} \left[\int_1^{e^x} \log z \, dz \right] dx \, dy$$

Since $\int_1^{e^x} \log z \, dz = \int_1^{e^x} \log z \, . \, 1 \, dz$

Integrating by parts $= \left[\log z \cdot z \right]_1^{e^x} - \int_1^{e^x} \frac{1}{z} \cdot z \, dz$

$$= e^x \log e^x - 0 - \left[z \right]_1^{e^x} = xe^x - e^x + 1 = (x-1) e^x + 1$$

$$\therefore \quad I = \int_1^e \int_1^{\log y} [(x-1) e^x + 1] \, dx \, dy$$

Now $\int_1^{\log y} [(x-1) e^x + 1] \, dx = \int_1^{\log y} (x-1) e^x \, dx + \left[x \right]_1^{\log y}$

$$= \left[(x-1) e^x \right]_1^{\log y} - \int_1^{\log y} 1 \cdot e^x \, dx + \log y - 1$$

$$= (\log y - 1) e^{\log y} - \left[e^x \right]_1^{\log y} + \log y - 1$$

$$= y(\log y - 1) - (e^{\log y} - e) + \log y - 1 \qquad [\because \; e^{\log y} = y]$$

$$= y(\log y - 1) - y + e + \log y - 1 = (y+1) \log y - 2y + e - 1$$

$$\therefore \quad I = \int_1^e [\log y \cdot (y+1) - 2y + e - 1] \, dy$$

$$= \left[\log y \cdot \left(\frac{y^2}{2} + y \right) \right]_1^e - \int_1^e \frac{1}{y} \left(\frac{y^2}{2} + y \right) dy - \left[y^2 \right]_1^e + (e-1) \left[y \right]_1^e$$

$$= \frac{e^2}{2} + e - \int_1^e \left(\frac{y}{2} + 1 \right) dy - (e^2 - 1) + (e-1)^2$$

$$= \frac{e^2}{2} + e - \left[\frac{y^2}{4} + y \right]_1^e - 2e + 2 = \frac{e^2}{2} + e - \left[\left(\frac{e^2}{4} + e \right) - \left(\frac{1}{4} + 1 \right) \right] - 2e + 2$$

$$= \frac{e^2}{4} - 2e + \frac{13}{4} = \frac{1}{4} (e^2 - 8e + 13).$$

EXERCISE 12.4

Evaluate the following integrals (1–11):

1. $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z} \, dx \, dy \, dz$. *(Kerala 2012)*

2. $\int_0^a \int_0^a \int_0^a (yz + zx + xy) \, dx \, dy \, dz$.

3. $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x + y + z) \, dy \, dx \, dz$.

4. $\int_1^3 \int_{1/x}^1 \int_0^{\sqrt{xy}} xyz \, dz \, dy \, dx$.

5. $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz \, dz \, dy \, dx.$

6. $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x \, dz \, dx \, dy.$

7. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \, dy \, dx.$ (M.D.U. Dec. 2010)

8. $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z} \, dz \, dy \, dx.$

9. $\int_0^{\pi/2} d\theta \int_0^{a \sin\theta} dr \int_0^{(a^2-r^2)/a} r dz.$

10. $\int_0^{\pi/2} \int_0^{a\cos\theta} \int_0^{\sqrt{a^2-r^2}} r dz \, dr \, d\theta.$

11. $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} \, dz \, dy \, dx.$ (K.U.K. 2009)

12. Evaluate $\iiint_V (x-2y+z) \, dz \, dy \, dx$, where V is the region determined by $0 \le x \le 1, 0 \le$ $0 \le z \le x+y$. (U.P.T.U

Answers

1. $(e-1)^3$ 2. $\frac{3}{4}a^5$ 3. 0 4. $\frac{13}{9} - \frac{1}{6}\log$

5. $\frac{1}{720}$ 6. $\frac{4}{35}$ 7. $\frac{1}{48}$ 8. $\frac{8}{3}\log 2 - \frac{19}{9}$

9. $\frac{5a^3\pi}{64}$ 10. $\frac{a^3}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right)$ 11. $\frac{1}{8}e^{4a} - \frac{3}{4}e^{2a} + e^a - \frac{3}{8}$ 12. $\frac{8}{35}$.

12.6. CHANGE OF VARIABLES

Quite often, the evaluation of a double or triple integral is greatly simplified by a able change of variables.

Let the variables x, y in the double integral $\iint_R f(x, y) \, dx \, dy$ be changed to u, v by mea the relations $x = \phi(u, v), y = \psi(u, v)$, then the double integral is transformed into $\iint_{R'} f\{\phi($ $\psi(u, v)\} \mid J \mid du \, dv$, where $J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

is the Jacobian of transformation from (x, y) to (u, v) co-ordinates and R′ is the region in uv-plane which corresponds to the region R in the xy-plane.

(i) *To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ).*

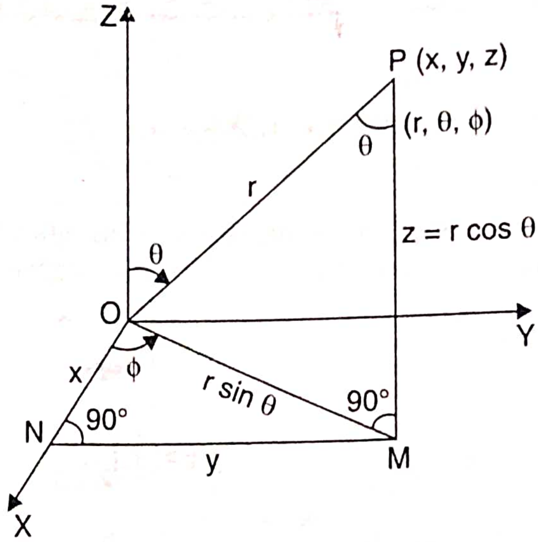
Here we have $x = r\cos\theta, y = r\sin\theta$ so that $x^2 + y^2 = r^2$.

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$\therefore \iint_R f(x, y) \, dx \, dy = \iint_{R'} f(r\cos\theta, r\sin\theta) \, r \, dr \, d\theta$$

i.e., replace x by $r\cos\theta$, y by $r\sin\theta$ and $dx \, dy$ by $r \, dr \, d\theta$.

(ii) *To change cartesian co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, φ).*



Here, we have $x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

so that $x^2 + y^2 + z^2 = r^2$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2 \sin\theta$$

$$\therefore \iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\, r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

Note. Equation of sphere $x^2 + y^2 + z^2 = a^2$ in spherical polar coordinates is $r = a$.

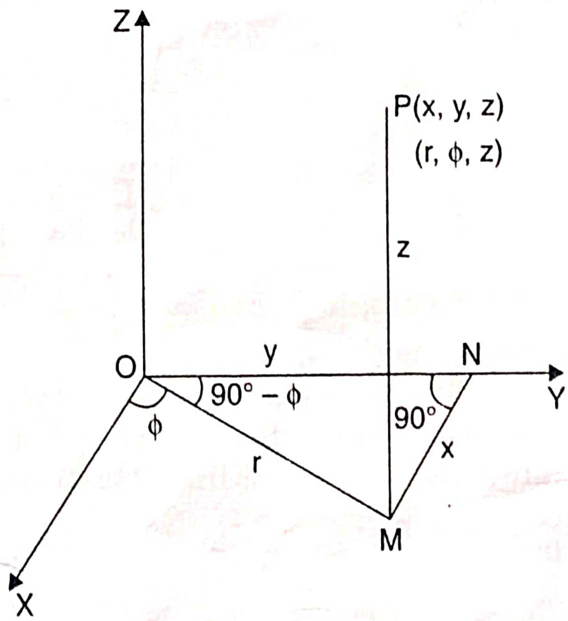
(i) If the region of integration is the whole sphere, then

$$0 \le r \le a, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi$$

(ii) If the region of integration is the positive octant, then

$$0 \le r \le a, \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \phi \le \frac{\pi}{2}.$$

(iii) *To change cartesian co-ordinates (x, y , z) to cylindrical polar co-ordinates (r, φ, z).*



Here we have $x = r\cos\phi$

$y = r\sin\phi$

$z = z$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r(\cos^2\phi + \sin^2\phi) = r$$

$$\therefore \iiint_V f(x, y, z)\,dx\,dy\,dz$$

$$= \iiint_{V'} f(r\cos\phi, r\sin\phi, z)\, r\, dr\, d\phi\, dz.$$

Note. For the cylinder $x^2 + y^2 = a^2$, $z = 0$, $z = h$, the limits of integration are

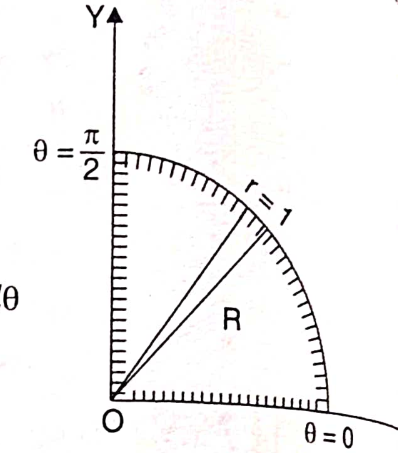
$$0 \le r \le a, \quad 0 \le \phi \le 2\pi, \; 0 \le z \le h$$

If the region of integration is a cylinder (or cone), change the problem to cylindrical polar coordinates.

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $\iint \sqrt{\dfrac{1-x^2-y^2}{1+x^2+y^2}}\, dx\, dy$ *over the positive quadrant of the circle* $x^2 + y^2 = 1$.

Sol. Changing to polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$; $x^2 + y^2 = 1$ transforms into $r = 1$. For the region of integration R, r varies from 0 to 1 and θ varies from 0 to $\dfrac{\pi}{2}$.



$$I = \iint_R \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\, dx\, dy$$

$$= \int_0^{\pi/2}\int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\; r\, dr\, d\theta$$

| $dx\, dy$ is replaced by $r\, dr\, d\theta$

$$= \int_0^{\pi/2}\int_0^1 \frac{r(1-r^2)}{\sqrt{1-r^4}}\, dr\, d\theta$$

Now
$$\int_0^1 \frac{r(1-r^2)}{\sqrt{1-r^4}}\, dr = \int_0^1 \left(\frac{r}{\sqrt{1-r^4}} - \frac{r^3}{\sqrt{1-r^4}}\right) dr$$

$$= \frac{1}{2}\int_0^1 \frac{2r}{\sqrt{1-r^4}}\, dr + \frac{1}{4}\int_0^1 -4r^3(1-r^4)^{-1/2}\, dr$$

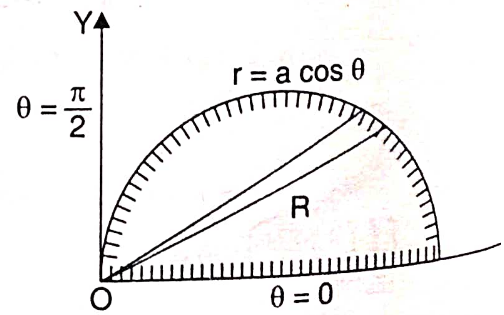
$$= \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{1-t^2}} + \frac{1}{4}\cdot\left[\frac{(1-r^4)^{1/2}}{1/2}\right]_0^1, \text{ where } t = r^2$$

$$= \frac{1}{2}\left[\sin^{-1} t\right]_0^1 + \frac{1}{2}(0-1) = \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}$$

$$\therefore \quad I = \int_0^{\pi/2}\left(\frac{\pi}{4} - \frac{1}{2}\right) d\theta = \left(\frac{\pi}{4} - \frac{1}{2}\right)\left[\theta\right]_0^{\pi/2} = \left(\frac{\pi}{4} - \frac{1}{2}\right)\frac{\pi}{2} = \frac{\pi^2}{8} - \frac{\pi}{4}.$$

Example 2. *Evaluate* $\iint \sqrt{a^2 - x^2 - y^2}\; dx\, dy$ *over the semi-circle* $x^2 + y^2 = ax$ *in the positive quadrant.*

Sol. Changing to polar co-ordinates, $x^2 + y^2 = ax$ transforms into $r = a\cos\theta$. For the region of integration R, r varies from 0 to $a\cos\theta$ and θ varies from 0 to $\dfrac{\pi}{2}$.



$$\therefore \quad \iint_R \sqrt{a^2 - x^2 - y^2}\; dx\, dy$$

$$= \int_0^{\pi/2}\int_0^{a\cos\theta} \sqrt{a^2 - r^2}\cdot r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{a\cos\theta} -\frac{1}{2}(a^2 - r^2)^{1/2}(-2r)\, dr\, d\theta$$

$$= \int_0^{\pi/2} -\frac{1}{2} \cdot \left[\frac{(a^2 - r^2)^{3/2}}{3/2} \right]_0^{a\cos\theta} d\theta$$

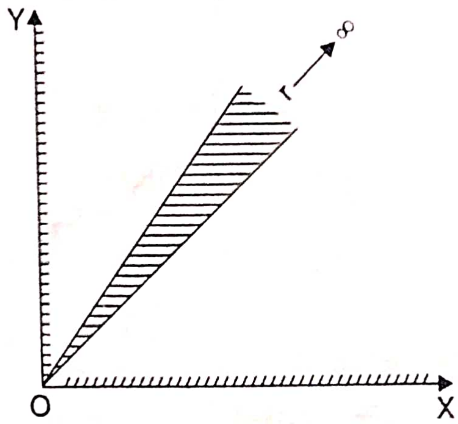
$$= -\frac{1}{3}\int_0^{\pi/2} (a^3 \sin^3 \theta - a^3)\, d\theta = -\frac{a^3}{3}\int_0^{\pi/2} (\sin^3 \theta - 1)\, d\theta$$

$$= -\frac{a^3}{3}\left[\frac{2}{3} - \frac{\pi}{2}\right] = \frac{a^3}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right).$$

Example 3. *Change into polar co-ordinates and evaluate* $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dy\, dx$.

Hence show that $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$. (K.U.K. Dec., 2013; P.T.U. May 2011, Dec. 2011)

Sol. For the region of integration in cartesian co-ordinates, y varies from 0 to ∞ and x also varies from 0 to ∞. Thus the region of integration is the plane XOY. Changing to polar co-ordinates by putting $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2 + y^2 = r^2$; for the region of integration r varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$.



$$\therefore \quad \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dy\, dx = \int_0^{\pi/2}\int_0^\infty e^{-r^2} \cdot r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^\infty \frac{1}{2} \cdot e^{-r^2} \cdot 2r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^\infty \frac{1}{2} e^{-t} dt\, d\theta, \text{where } t = r^2$$

$$= \int_0^{\pi/2} \left[-\frac{1}{2}e^{-t}\right]_0^\infty d\theta = -\frac{1}{2}\int_0^{\pi/2} (0-1)\, d\theta = \frac{1}{2}\Big[\theta\Big]_0^{\pi/2} = \frac{\pi}{4}.$$

Now the above result can be written as

$$\int_0^\infty e^{-x^2}\, dx \times \int_0^\infty e^{-y^2}\, dy = \frac{\pi}{4}$$

or

$$\int_0^\infty e^{-x^2}\, dx \times \int_0^\infty e^{-x^2}\, dx = \frac{\pi}{4} \qquad \text{or} \qquad \left[\int_0^\infty e^{-x^2}\, dx\right]^2 = \frac{\pi}{4}$$

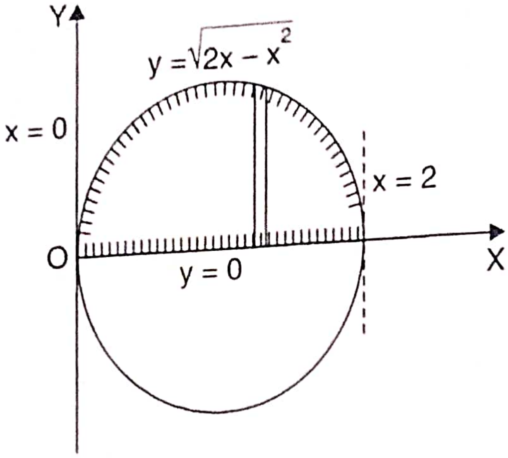
$$\therefore \qquad \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$

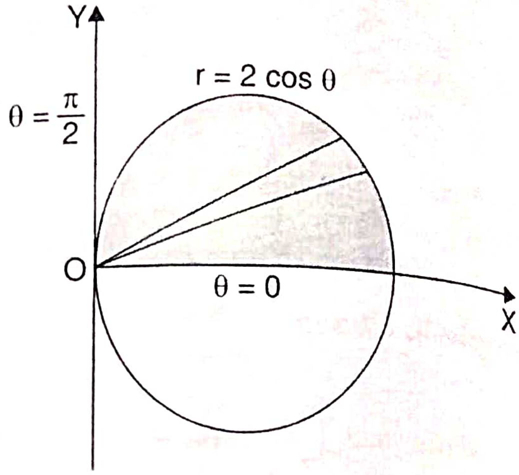
Example 4. *Evaluate* $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\, dy\, dx}{\sqrt{x^2+y^2}}$ *by changing to polar co-ordinates.*

Sol. In the given integral, y varies from 0 to $\sqrt{2x - x^2}$ and x varies from 0 to 2.

$$y = \sqrt{2x - x^2} \quad \Rightarrow \quad y^2 = 2x - x^2 \quad \Rightarrow \quad x^2 + y^2 = 2x.$$

In polar co-ordinates, we have $r^2 = 2r\cos\theta$ or $r = 2\cos\theta$.





$\therefore$ For the region of integration, r varies from 0 to $2\cos\theta$ and θ varies from 0 to $\frac{\pi}{2}$.

In the given integral, replacing x by $r\cos\theta$, y by $r\sin\theta$, $dy\,dx$ by $r\,dr\,d\theta$, we have

$$I = \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r\cos\theta \cdot r\,dr\,d\theta}{r} = \int_0^{\pi/2}\int_0^{2\cos\theta} r\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2} \cos\theta \left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta = \int_0^{\pi/2} 2\cos^3\theta\,d\theta = 2\cdot\frac{2}{3} = \frac{4}{3}.$$

Example 5. *Evaluate $\iiint z(x^2 + y^2 + z^2)\,dx\,dy\,dz$ through the volume of the cylinder $x^2 + y^2 = a^2$ intercepted by the planes $z = 0$ and $z = h$.*

Sol. Changing to cylindrical co-ordinates by changing x to $r\cos\phi$, y to $r\sin\theta$ and replacing $dx\,dy\,dz$ by $r\,dr\,d\phi\,dz$

$$I = \int_0^h\int_0^{2\pi}\int_0^a z(r^2 + z^2)\,r\,dr\,d\phi\,dz = \int_0^h\int_0^{2\pi}\int_0^a (zr^3 + z^3 r)\,dr\,d\phi\,dz$$

$$= \int_0^h\int_0^{2\pi}\left[z\cdot\frac{r^4}{4} + z^3\cdot\frac{r^2}{2}\right]_0^a d\phi\,dz = \int_0^h\int_0^{2\pi}\left(\frac{a^4}{4}z + \frac{a^2}{2}z^3\right)d\phi\,dz$$

$$= \int_0^h \left(\frac{a^4}{4}z + \frac{a^2}{2}z^3\right)\Big[\phi\Big]_0^{2\pi} dz = \int_0^h 2\pi\left(\frac{a^4}{4}z + \frac{a^2}{2}z^3\right)dz$$

$$= 2\pi\left[\frac{a^4 z^2}{8} + \frac{a^2 z^4}{8}\right]_0^h = \frac{\pi}{4}(a^4h^2 + a^2h^4) = \frac{\pi}{4}a^2h^2(a^2 + h^2).$$

Example 6. *Evaluate* $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1 - x^2 - y^2 - z^2}}$, *by changing to spherical polar co-ordinates.*

(Kerala 2008; M.D.U. Dec. 2008, Dec. 2010, Dec. 2012; P.T.U. May 20

Sol. Here the region of integration is bounded by

$z = 0,\quad z = \sqrt{1 - x^2 - y^2}\qquad$ (*i.e.*, $x^2 + y^2 + z^2 = 1$)

$y = 0,\quad y = \sqrt{1 - x^2}\qquad$ (*i.e.*, $x^2 + y^2 = 1$)

$x = 0,\quad x = 1$

which is the volume of the sphere $x^2 + y^2 + z^2 = 1$ in the positive octant.

Changing to spherical polar co-ordinates by putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ so that $x^2 + y^2 + z^2 = r^2$.

For the volume of sphere $x^2 + y^2 + z^2 = 1$ in the positive octant, r varies from 0 to 1, θ varies from 0 to $\frac{\pi}{2}$ and ϕ varies from 0 to $\frac{\pi}{2}$.

Replacing $dz\,dy\,dx$ by $r^2\sin\theta\,dr\,d\theta\,d\phi$, we have

$$\mathrm{I} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{\sqrt{1-r^2}}$$

Now $\displaystyle\int_0^1 \frac{r^2}{\sqrt{1-r^2}}\,dr$ Putting $r = \sin t$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^2 t}{\cos t}\cos t\,dt = \int_0^{\frac{\pi}{2}} \sin^2 t\,dt = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

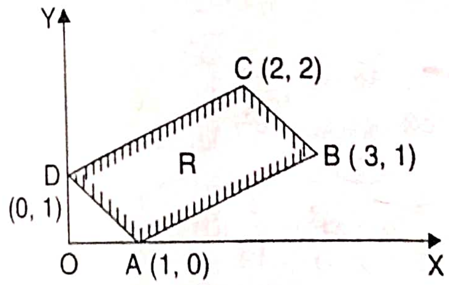
$$\therefore \quad \mathrm{I} = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\pi}{4}\sin\theta\,d\theta\,d\phi = \int_0^{\frac{\pi}{2}} \frac{\pi}{4}\Big[-\cos\theta\Big]_0^{\frac{\pi}{2}} d\phi = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} d\phi = \frac{\pi}{4}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}.$$

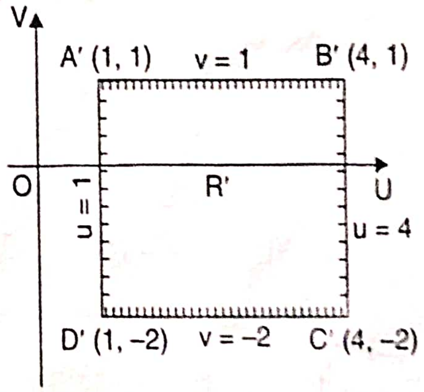
Note. For the whole volume of the sphere $x^2 + y^2 + z^2 = a^2$.

$$0 \le r \le a,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi.$$

Example 7. *Evaluate* $\iint_R (x+y)^2\,dx\,dy$, *where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1), using the transformation* $u = x + y$ *and* $v = x - 2y$.

Sol. The vertices A(1, 0), B(3, 1), C(2, 2), D(0, 1) of the parallelogram ABCD in the xy-plane become A′(1, 1), B′(4, 1), C′(4, –2), D′(1, –2) in the uv-plane under the given transformation.





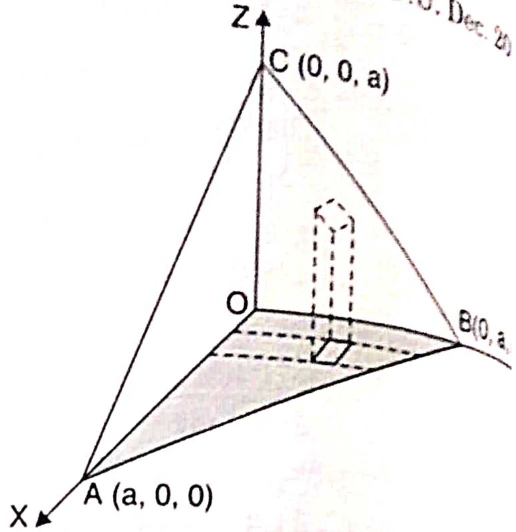
The region R in the xy-plane becomes the region R′ in the uv-plane which is a square bounded by the line $u = 1$, $u = 4$ and $v = -2$, $v = 1$.

Now $$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -3 \Rightarrow \mathrm{J} = \frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{3}$$

$$\therefore \quad \iint_R (x+y)^2\,dx\,dy = \iint_{R'} u^2\,|\mathrm{J}|\,du\,dv = \int_{-2}^{1}\int_1^4 u^2\cdot\frac{1}{3}\,du\,dv$$

$$= \int_{-2}^{1} \frac{1}{3}\left[\frac{u^3}{3}\right]_1^4 dv = \int_{-2}^{1} 7\,dv = 7\Big[v\Big]_{-2}^{1} = 7 \times 3 = 21.$$

Example 8. *Evaluate* $\iiint_R (x^2 + y^2 + z^2)\, dx\, dy\, dz$, *where R denotes the region bou by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a$, $(a > 0)$.* (M.D.U. Dec. 20



Sol. The plane $x + y + z = a$, $(a > 0)$ meets the coordinate axes in A $(a, 0, 0)$, B$(0, a, 0)$ and C$(0, 0, a)$. On the face ABC, $z = a - x - y$. The projection of plane ABC on the xy-plane is the triangle OAB bounded by the lines OB $(x = 0)$, OA $(y = 0)$ and AB $(x + y = a)$.

$$\therefore \ R = \{(x, y, z) : 0 \le x \le a,\ 0 \le y \le a - x,\ 0 \le z \le a - x - y\}$$

$$I = \iiint_R (x^2 + y^2 + z^2)\, dx\, dy\, dz$$

$$= \int_0^a \left[\int_0^{a-x} \left[\int_0^{a-x-y} (x^2 + y^2 + z^2)\, dz\right] dy\right] dx$$

$$= \int_0^a \left[\int_0^{a-x} \left[(x^2 + y^2)\, z + \frac{z^3}{3}\right]_0^{a-x-y} dy\right] dx$$

$$= \int_0^a \left[\int_0^{a-x} \left[(x^2 + y^2)(a - x - y) + \frac{1}{3}(a - x - y)^3\right] dy\right] dx$$

$$= \int_0^a \left[\int_0^{a-x} \left[(a - x)x^2 + (a - x)y^2 - x^2 y - y^3 + \frac{1}{3}(a - x - y)^3\right] dy\right] dx$$

$$= \int_0^a \left[(a - x)x^2 y + (a - x)\cdot\frac{y^3}{3} - x^2\cdot\frac{y^2}{2} - \frac{y^4}{4} + \frac{1}{3}\cdot\frac{(a - x - y)^4}{-4}\right]_0^{a-x} dx$$

$$= \int_0^a \left[x^2(a - x)^2 + \frac{1}{3}(a - x)^4 - \frac{1}{2}x^2(a - x)^2 - \frac{1}{4}(a - x)^4 - \frac{1}{12}(0) + \frac{1}{12}(a - x)^4\right] dx$$

$$= \int_0^a \left[\frac{1}{2}x^2(a - x)^2 + \frac{1}{6}(a - x)^4\right] dx = \int_0^a \left[\frac{1}{2}a^2x^2 - ax^3 + \frac{1}{2}x^4 + \frac{1}{6}(a - x)^4\right] dx$$

$$= \left[\frac{1}{2}a^2\cdot\frac{x^3}{3} - a\cdot\frac{x^4}{4} + \frac{1}{2}\cdot\frac{x^5}{5} + \frac{1}{6}\cdot\frac{(a - x)^5}{-5}\right]_0^a$$

$$= \frac{a^5}{6} - \frac{a^5}{4} + \frac{a^5}{10} - \frac{1}{30}(0) + \frac{1}{30}\cdot a^5$$

$$= \left(\frac{10 - 15 + 6 + 2}{60}\right)a^5 = \frac{3}{60}a^5 = \frac{a^5}{20}$$

EXERCISE 12.5

1. Evaluate $\iint \sin \pi (x^2 + y^2)\, dx\, dy$ over the region bounded by the circle $x^2 + y^2 = 1$ by changing to polar co-ordinates.

2. Evaluate $\iint (a^2 - x^2 - y^2)\, dx\, dy$ over the semi-circle $x^2 + y^2 = ax$ in the positive quadrant by changing to polar co-ordinates.

3. Evaluate $\iint (x^2 + y^2)^{7/2}\, dx\, dy$ over the circle $x^2 + y^2 = 1$.

4. Evaluate $\iint xy\,(x^2 + y^2)^{3/2}\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$. *(M.D.U. Dec. 2012)*

5. Evaluate the following by changing into polar co-ordinates:

(i) $\displaystyle\int_0^a \int_y^a \frac{x\, dx\, dy}{x^2 + y^2}$ (ii) $\displaystyle\int_{-a}^a \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} dy\, dx$

(iii) $\displaystyle\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2 - y^2}{x^2 + y^2}\, dx\, dy$ (iv) $\displaystyle\int_0^a \int_0^{\sqrt{a^2 - y^2}} (x^2 + y^2)\, dx\, dy$ *(M.D.U. May 2008)*

(v) $\displaystyle\int_0^a \int_0^{\sqrt{a^2 - y^2}} y^2 \sqrt{x^2 + y^2}\, dx\, dy$ (vi) $\displaystyle\int_0^a \int_y^a \frac{x^2\, dx\, dy}{\sqrt{x^2 + y^2}}$.

(vii) $\displaystyle\int_0^{\frac{a}{\sqrt{2}}} \int_y^{\sqrt{a^2 - y^2}} \log (x^2 + y^2)\, dx\, dy,\ (a > 0)$

6. Evaluate $\displaystyle\iint_D e^{-(x^2 + y^2)}\, dy\, dx$, where D is the region bounded by $x^2 + y^2 = a^2$.

7. Evaluate $\displaystyle\iint xy\,(x^2 + y^2)^{n/2}\, dx\, dy$ over the positive quadrant of $x^2 + y^2 = 4$, supposing $n + 3 > 0$.

8. Evaluate $\displaystyle\iint \sqrt{\frac{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}{1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}}}\, dx\, dy$ over the positive quadrant of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

9. Transform the following to cartesian form and hence evaluate $\displaystyle\int_0^\pi \int_0^a r^3 \sin\theta \cos\theta\, dr\, d\theta$.

10. Evaluate $\displaystyle\iiint (x + y + z)\, dx\, dy\, dz$ over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. *(M.D.U. May 2009)*

11. Evaluate $\displaystyle\iiint \frac{dx\, dy\, dz}{\sqrt{a^2 - x^2 - y^2 - z^2}}$,

(i) Over the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$.

(ii) Taken throughout the volume of the sphere $x^2 + y^2 + z^2 = a^2$. (M.D.U. Dec. 2009)

12. Evaluate $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ over the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$.

13. Evaluate $\iiint z(x^2 + y^2)\,dx\,dy\,dz$ over the volume of the cylinder $x^2 + y^2 = 1$ intercepted by the planes $z = 2$ and $z = 3$.

14. Evaluate the following integrals through the volume of the sphere $x^2 + y^2 + z^2 = 1$, by changing into spherical polar co-ordinates:

(i) $\iiint z^2\,dx\,dy\,dz$ (ii) $\iiint (x^2 + y^2 + z^2)^m\,dx\,dy\,dz$. $(m > 0)$

15. By using the transformation $x + y = u$, $y = uv$, show that $\int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}}\,dy\,dx = \frac{1}{2}(e-1)$.

[**Hint.** Here x varies from 0 to 1 and y varies from 0 to $1 - x$. The region D of integration is the triangle OAB in xy-plane. Under the given transformation

$$x = u - uv = u(1 - v),\ y = uv$$

Now,

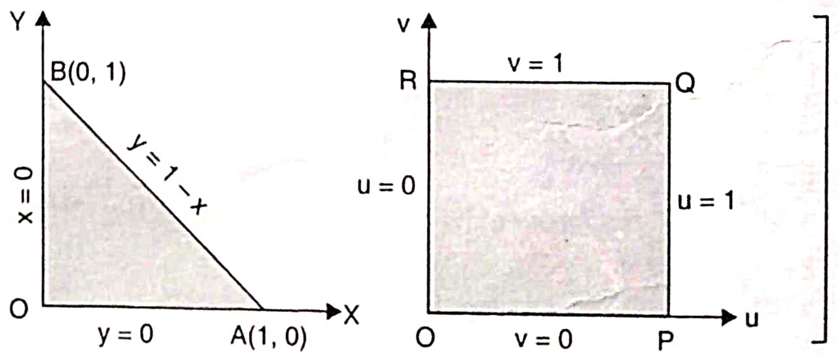
$$x = 0 \Rightarrow u = 0 \text{ or } v = 1$$
$$y = 0 \Rightarrow u = 0 \text{ or } v = 0$$
$$x + y = 1 \Rightarrow u = 1$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = u \text{ and } dxdy = |\,J\,|\,dudv = u\,dudv$$

The region D transforms into the region

$$D' = \{(u, v) : 0 \le u \le 1, 0 \le v \le 1\}$$

which is square OPQR is uv-plane.



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Answers

1. 2	2. $\frac{5\pi a^4}{64}$	3. $\frac{2\pi}{9}$	4. $\frac{1}{14}$
5. (i) $\frac{\pi a}{4}$	(ii) πa^2	(iii) $8\left(\frac{\pi}{2} - \frac{5}{3}\right)a^2$	(iv) $\frac{\pi a^4}{8}$

(v) $\frac{\pi a^5}{20}$ (vi) $\frac{a^3}{3}\log(\sqrt{2}+1)$ (vii) $\frac{\pi a^2}{4}\left(\log a - \frac{1}{2}\right)$ 6. $\pi(1-e^{-a^2})$

7. $\frac{2^{n+3}}{n+4}$ 8. $\frac{\pi ab}{8}(\pi - 2)$ 9. 0 10. $\frac{1}{8}$

11. (i) $\frac{\pi^2 a^2}{8}$ (ii) $\pi^2 a^2$ 12. $\frac{1}{2}\log 2 - \frac{5}{16}$ 13. $\frac{5\pi}{4}$

14. (i) $\frac{4\pi}{15}$ (ii) $\frac{4\pi}{2m+3}$.

12.7. AREA BY DOUBLE INTEGRATION

(a) *Cartesian co-ordinates*. The area A of the region bounded by the curves $y = f_1(x)$, $y = f_2(x)$ and the lines $x = a$, $x = b$ is given by $A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\,dx$

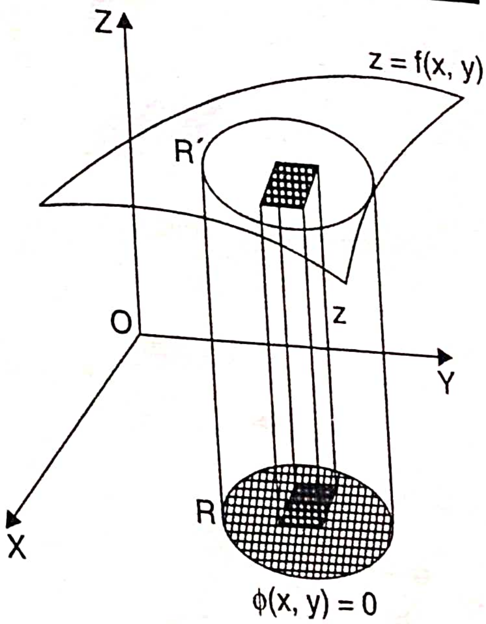
The area A of the region bounded by the curves $x = f_1(y)$, $x = f_2(y)$ and the lines $y = c$, $y = d$ is given by $A = \int_c^d \int_{f_1(y)}^{f_2(y)} dx\,dy$.

(b) *Polar co-ordinates*. The area A of the region bounded by the curves $f_1(\theta)$, $r = f_2(\theta)$ and the lines $\theta = \alpha$, $\theta = \beta$ is given by $A = \int_\alpha^\beta \int_{f_1(\theta)}^{f_2(\theta)} r\,dr\,d\theta$.

12.8. VOLUME AS A DOUBLE INTEGRAL

(a) *Cartesian co-ordinates*. Consider a surface $z = f(x, y)$ ...(1)

Let the orthogonal projection of its region R′ on the *xy*-plane be the region R given by $\phi(x, y) = 0$. ...(2)



Now (2) represents a cylinder with generators parallel to *z*-axis and the guiding curve given by (2). Let V be the volume of this cylinder between R and R′.

Divide R into elementary rectangles of area $\delta x\,\delta y$ by drawing lines parallel to the axis of *x* and *y*. On each of these rectangles, erect prisms of lengths parallel to *z*-axis. Volume of this prism between R and R′ is $z\,\delta x\,\delta y$. The volume V is composed of such prisms.

$$\therefore \quad V = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \sum z\,\delta x\,\delta y = \iint_R z\,dx\,dy.$$

(b) *Cylindrical co-ordinates*. Let the equation of the surface be $z = f(r, \phi)$. Replacing $dx\,dy$ by $r\,dr\,d\phi$, we get $V = \iint_R zr\,dr\,d\phi$.