1*

Infinite Series

1.1. SEQUENCE

A sequence is a function whose domain is the set N of all natural numbers whereas the range may be any set S. In other words, a sequence in a set S is a rule which assigns to each natural number a unique element of S.

1.2. REAL SEQUENCE

A real sequence is a function whose domain is the set N of all natural numbers and range a subset of the set R of real numbers.

Symbolically $\quad f : N \to R \quad$ (or $x : N \to R$ or $a : N \to R$)
is a real sequence.

Note. If $x : N \to R$ be a sequence, the image of $n \in N$ instead of denoting it by $x(n)$, we shall generally denote it by x_n. Thus x_1, x_2, x_3 etc. are the real numbers associated to 1, 2, 3, etc. by this mapping. Also, the sequence $x : N \to R$ is denoted by $\{x_n\}$ or $< x_n >$.

$x_1, x_2, \ldots$ are called the first, second ... terms of the sequence. The mth and nth terms x_m and x_n for $m \neq n$ are treated as distinct event if $x_m = x_n$ *i.e.*, the terms occurring at different positions are treated as distinct terms even if they have the same value.

1.3. RANGE OF A SEQUENCE

The set of all **distinct** terms of a sequence is called its range.

Note. In a sequence $\{x_n\}$, since $n \in N$ and N is an infinite set, **the number of terms of a sequence is always infinite.** The range of a sequence may be a finite set. *e.g.* if $x_n = (-1)^n$ then $\{x_n\} = \{-1, 1, -1, 1, \ldots\}$

The range of sequence $\{x_n\} = \{-1, 1\}$ which is a finite set.

1.4. CONSTANT SEQUENCE

A sequence $\{x_n\}$ defined by $x_n = c$, where c is a fixed real number, $\forall\ n \in N$ is called a constant sequence.

e.g. $\quad \{x_n\} = \{c, c, c, \ldots\}$ is a constant sequence with range $= \{c\}$.

1.5. BOUNDED AND UNBOUNDED SEQUENCES

Bounded above sequence. A sequence $\{a_n\}$ is said to be bounded above if there exists a real number K such that $\quad a_n \leq K \quad \forall\ n \in N$.

*This chapter is not included in the syllabus of KU, Kurukshetra.

Bounded below sequence. A sequence $\{a_n\}$ is said to be bounded below if there exists a real number k such that $a_n \geq k \quad \forall\ n \in N$.

Bounded sequence. A sequence $\{a_n\}$ is said to be bounded when it is bounded both above and below.

$\Rightarrow$ A sequence $\{a_n\}$ is bounded if there exist two real numbers k and K ($k \leq$ K) such that

$$k \leq a_n \leq K \quad \forall\ n \in N.$$

Choosing M = maximum {| k |, | K |}, we can also define a sequence $\{a_n\}$ to be bounded if

$$|\ a_n\ | \leq M \quad \forall \quad n \in N.$$

Unbounded sequence. If there exists no real number M such that $|\ a_n\ | \leq M\ \forall\ n \in N$, then the sequence $\{a_n\}$ is said to be unbounded.

Examples (1). The sequence $\{a_n\}$ defined by $a_n = \frac{1}{n}$.

Here $\quad \{a_n\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$

$\because \quad 0 < a_n \leq 1 \quad \forall \quad n \in N$

$\therefore \quad \{a_n\}$ is bounded.

(2) The sequence $\{a_n\}$ defined by $a_n = 2^{n-1}$

Here $\quad \{a_n\} = \{1, 2, 2^2, 2^3, \ldots\}$.

Although $a_n \geq 1$, $\forall \quad n \in N$, there exists no real number K such that $a_n \leq K$.

$\therefore$ The sequence is unbounded above.

1.6. CONVERGENT, DIVERGENT, OSCILLATORY SEQUENCES

Convergent sequence. A sequence $\{a_n\}$ is said to be convergent if $\lim\limits_{n\to\infty} a_n$ is finite.

For example, consider the sequence $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n} \ldots$

Here $a_n = \frac{1}{2^n}$, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{2^n} = 0$, which is finite.

$\Rightarrow$ the sequence $\{a_n\}$ is convergent.

Divergent sequence. A sequence $\{a_n\}$ is said to be divergent if $\lim\limits_{n\to\infty} a_n$ is not finite,

i.e., if $\quad \lim\limits_{n\to\infty} a_n = +\infty$ or $-\infty$.

For example

(*i*) Consider the sequence $\{n^2\}$

Here $a_n = n^2$, $\lim\limits_{n\to\infty} a_n = +\infty \Rightarrow$ The sequence $\{n^2\}$ is divergent.

(*ii*) Consider the sequence $\{-2^n\}$.

Here $\quad a_n = -2^n$, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (-2^n) = -\infty$

$\Rightarrow$ the sequence $\{-2^n\}$ is divergent.

Oscillatory sequence. If a sequence $\{a_n\}$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$, it is called an oscillatory sequence. Oscillatory sequences are of two types:

(*i*) A bounded sequence which does not converge is said to **oscillate finitely.**

For example, consider the sequence $\{(-1)^n\}$.

Here $\quad a_n = (-1)^n$

It is a bounded sequence. $\lim\limits_{n\to\infty} a_{2n} = \lim\limits_{n\to\infty} (-1)^{2n} = 1$

$$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} (-1)^{2n+1} = -1.$$

Thus $\lim\limits_{n\to\infty} a_n$ does not exist $\Rightarrow$ the sequence does not converge.

Hence this sequence oscillates finitely.

(*ii*) An unbounded sequence which does not diverge is said to **oscillate infinitely.**

For example, consider the sequence $\{(-1)^n\ n\}$.

Here $\quad a_n = (-1)^n\ n$.

It is an unbounded sequence.

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} (-1)^{2n} \cdot 2n = \lim_{n\to\infty} 2n = +\infty$$

$$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} (-1)^{2n+1} (2n+1) = \lim_{n\to\infty} -(2n+1) = -\infty.$$

Thus the sequence does not diverge.

Hence this sequence oscillates infinitely.

Note. When we say $\lim\limits_{n\to\infty} a_n = l$, it means $\lim\limits_{n\to\infty} a_{2n} = \lim\limits_{n\to\infty} a_{2n+1} = l$

Similarly $\quad \lim\limits_{n\to\infty} a_n = +\infty$ means $\lim\limits_{n\to\infty} a_{2n} = \lim\limits_{n\to\infty} a_{2n+1} = +\infty$.

1.7. MONOTONIC SEQUENCES

(*i*) A sequence $\{a_n\}$ is said to be **monotonically increasing** if $a_{n+1} \geq a_n \quad \forall \quad n \in N$.

i.e., if $\quad a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n \leq a_{n+1} \leq \ldots$

(*ii*) A sequence $\{a_n\}$ is said to be **monotonically decreasing** if $a_{n+1} \leq a_n \quad \forall \quad n \in N$.

i.e., if $\quad a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n \geq a_{n+1} \geq \ldots$

(*iii*) A sequence $\{a_n\}$ is said to be **monotonic** if it is either monotonically increasing or monotonically decreasing.

(*iv*) A sequence $\{a_n\}$ is said to be **strictly monotonically increasing if**

$$a_{n+1} > a_n \quad \forall \quad n \in N.$$

(*v*) A sequence $\{a_n\}$ is said to be **strictly monotonically decreasing** if

$$a_{n+1} < a_n \quad \forall \quad n \in N.$$

(*vi*) A sequence $\{a_n\}$ is said to be **strictly monotonic** if it is either strictly monotonically increasing or strictly monotonically decreasing.

1.8. LIMIT OF A SEQUENCE

A sequence $\{a_n\}$ is said to approach the limit l (say) when $n \to \infty$, if for each $\varepsilon > 0$, there exists a +ve integer m (depending upon ε) such that $|\ a_n - l\ | < \varepsilon \quad \forall \quad n \geq m$.

In symbols, we write $\lim\limits_{n\to\infty} a_n = l$.

Note. $|\ a_n - l\ | < \varepsilon\ \forall\ n \geq m \Rightarrow l - \varepsilon < a_n < l + \varepsilon$ for $n = m, m+1, m+2, \ldots$

1.9. EVERY CONVERGENT SEQUENCE IS BOUNDED

Let the sequence $\{a_n\}$ be convergent. Let it tend to the limit l.

Then given $\varepsilon > 0$, there exists a +ve integer m, such that

$$| a_n - l | < \varepsilon \quad \forall \quad n \geq m$$

$$\Rightarrow \qquad l - \varepsilon < a_n < l + \varepsilon \quad \forall \quad n \geq m.$$

Let k and K be the least and the greatest of $a_1, a_2, a_3, \ldots, a_{m-1}, l - \varepsilon, l + \varepsilon$

Then $\qquad k \leq a_n \leq K \quad \forall \quad n \in N,$

$\Rightarrow$ the sequence $\{a_n\}$ is bounded.

The converse is not always true *i.e.*, a sequence may be bounded, yet it may not be convergent. *e.g.*, Consider $a_n = (-1)^n$, then the sequence $\{a_n\}$ is bounded but not convergent since it does not have a unique limit.

1.10. CONVERGENCE OF MONOTONIC SEQUENCES

Theorem I. *The necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.*

A monotonic increasing sequence which is bounded above converges.

A monotonic decreasing sequence which is bounded below converges.

Theorem II. *If a monotonic increasing sequence is not bounded above, it diverges to* $+\infty$.

Theorem III. *If a monotonic decreasing sequence is not bounded below, it diverges to* $-\infty$.

Theorem IV. *If* $\{a_n\}$ *and* $\{b_n\}$ *are two convergent sequences, then sequence* $\{a_n + b_n\}$ *is also convergent.*

Or

If $\quad \lim a_n = A$ *and* $\lim b_n = B$, *then* $\lim (a_n + b_n) = A + B$.

$\langle a_n \rangle = \langle -1, 1, -1, \ldots \rangle$

$\langle b_n \rangle = \langle 1, -1, 1, \ldots \rangle$

$a_n + b_n = \langle 0, \ldots \rangle$

Theorem V. *If* $\{a_n\}$ *and* $\{b_n\}$ *are two convergent sequences such that* $\lim a_n = A$ *and* $\lim b_n = B$, *then*

$(-1)^n = a_n$

$(-1)^n = b_n$

(i) sequence $\{a_n b_n\}$ *is also convergent and converges to AB.*

(ii) sequence $\left\{\frac{a_n}{b_n}\right\}$ *is also convergent and converges to* $\frac{A}{B}$, $(B \neq 0)$.

$\langle a_n \rangle = \langle n \rangle$

$\langle b_n \rangle = \langle \frac{1}{n} \rangle$

$a_n b_n = 1$

$a_n = \langle n \rangle$

$b_n = \langle n \rangle$

Theorem VI. *The sequence* $\{| a_n |\}$ *converges to zero if and only if the sequence* $\{a_n\}$ *converges to zero.*

Theorem VII. *If a sequence* $\{a_n\}$ *converges to a and* $a_n \geq 0 \quad \forall\, n$, *then* $a \geq 0$.

Theorem VIII. *If* $a_n \to a$, $b_n \to b$ *and* $a_n \leq b_n \quad \forall\, n$, *then* $a \leq b$.

Theorem IX. *If* $a_n \to l$, $b_n \to l$, *and* $a_n \leq c_n \leq b_n$, $\forall\, n$, *then* $c_n \to l$. *(Squeeze Principle)*

Squeeze Principle

ILLUSTRATIVE EXAMPLES

Example 1. *Give an example of a monotonic increasing sequence which is (i) convergent, (ii) divergent.*

Sol. (*i*) Consider the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$

Since $\frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \ldots$ the sequence is monotonic increasing.

$$a_n = \frac{n}{n+1}, \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

which is finite. $\therefore$ The sequence is convergent.

(*ii*) Consider the sequence $1, 2, 3, \cdots, n, \cdots$

Since $1 < 2 < 3 < \cdots$, the sequence is monotonic increasing.

$$a_n = n, \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty$$

$\therefore$ The sequence diverges to $+\infty$.

Example 2. *Give an example of a monotonic decreasing sequence which is (i) convergent, (ii) divergent.*

Sol. (*i*) Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$

Since $1 > \frac{1}{2} > \frac{1}{3} > \ldots$, the sequence is monotonic decreasing.

$$a_n = \frac{1}{n}, \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

$\therefore$ The sequence converges to 0.

(*ii*) Consider the sequence $-1, -2, -3, \cdots, -n, \cdots$

Since $-1 > -2 > -3 > \ldots$, the sequence is monotonic decreasing.

$$a_n = -n, \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} (-n) = -\infty$$

$\therefore$ The sequence diverges to $-\infty$.

Example 3. *Discuss the convergence of the sequence* $\{a_n\}$ *where*

(*i*) $a_n = \frac{n+1}{n}$ $\qquad$ (*ii*) $a_n = \frac{n}{n^2+1}$ $\qquad$ (*iii*) $a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n}$.

Sol. (*i*) Here $\qquad a_n = \frac{n+1}{n} = 1 + \frac{1}{n}$

$$\therefore \qquad a_{n+1} - a_n = \left(1 + \frac{1}{n+1}\right) - \left(1 + \frac{1}{n}\right) = \frac{-1}{n(n+1)} < 0 \quad \forall\, n$$

$$\Rightarrow \qquad a_{n+1} < a_n \quad \forall\, n$$

$\Rightarrow$ $\{a_n\}$ is a decreasing sequence.

Also $\qquad a_n = \frac{n+1}{n} = 1 + \frac{1}{n} > 1 \quad \forall\, n$

$\Rightarrow$ $\{a_n\}$ is bounded below by 1,

$\because$ $\{a_n\}$ is decreasing and bounded below, it is convergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1.$$

$\therefore$ The sequence $\{a_n\}$ converges to 1.

(*ii*) Here $\qquad a_n = \dfrac{n}{n^2+1}$

$$\therefore \quad a_{n+1} - a_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} = \frac{(n+1)(n^2+1) - n(n^2+2n+2)}{(n^2+2n+2)(n^2+1)}$$

$$= \frac{-n^2-n+1}{(n^2+2n+2)(n^2+1)} < 0 \quad \forall \ n \ \Rightarrow \ a_{n+1} < a_n \ \forall \ n$$

$\Rightarrow$ $\{a_n\}$ is a decreasing sequence.

Also $\qquad a_n = \dfrac{n}{n^2+1} > 0 \ \ \forall \ n \ \Rightarrow$ $\{a_n\}$ is bounded below by 0.

$\because$ $\{a_n\}$ is decreasing and bounded below, it is convergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{n^2+1} = \lim_{n\to\infty}\frac{\frac{1}{n}}{1+\frac{1}{n^2}} = 0.$$

$\therefore$ The sequence $\{a_n\}$ converges to 0.

(*iii*) Here $a_n = 1 + \dfrac{1}{3} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{3^n}$

= sum of $(n+1)$ terms of a G.P. whose first term is 1 and common ratio is $\frac{1}{3}$

$$= \frac{1\left(1-\frac{1}{3^{n+1}}\right)}{1-\frac{1}{3}} \qquad \left| \ S_n = \frac{a(1-r^n)}{1-r} \right.$$

$$= \frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right)$$

Now $\quad a_{n+1} = 1 + \dfrac{1}{3} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{3^n} + \dfrac{1}{3^{n+1}}$

$\therefore \ a_{n+1} - a_n = \dfrac{1}{3^{n+1}} > 0 \ \ \forall n \ \Rightarrow \ a_{n+1} > a_n \ \ \forall n$

$\Rightarrow$ $\{a_n\}$ is an increasing sequence.

Also $\quad a_n = \dfrac{3}{2}\left(1-\dfrac{1}{3^{n+1}}\right) < \dfrac{3}{2} \ \ \forall n \ \Rightarrow$ $\{a_n\}$ is bounded above by $\dfrac{3}{2}$.

$\because$ $\{a_n\}$ is increasing and bounded above, it is convergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right) = \frac{3}{2}(1-0) = \frac{3}{2}.$$

$\therefore$ The sequence $\{a_n\}$ converges to $\dfrac{3}{2}$.

1.11. INFINITE SERIES

If $\{u_n\}$ is a sequence of real numbers, then the expression $u_1 + u_2 + u_3 + \cdots + u_n \cdots$ [*i.e.*, the sum of the terms of the sequence, which are infinite in number] is called an infinite series.

The infinite series $u_1 + u_2 + \cdots + u_n + \cdots$ is denoted by $\sum\limits_{n=1}^{\infty} u_n$ or more briefly, by Σu_n.

1.12. SERIES OF POSITIVE TERMS

If all the terms of the series $\Sigma u_n = u_1 + u_2 + \cdots + u_n + \cdots$ are positive *i.e.* if $u_n > 0 \ \forall \ n$, then the series Σu_n is called a series of positive terms.

1.13. ALTERNATING SERIES

A series in which the terms are alternately positive and negative is called an alternating series. Thus, the series $\Sigma(-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1} u_n + \cdots$ where $u_n > 0 \ \forall \ n$ is an alternating series.

1.14. PARTIAL SUMS

If $\Sigma u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is an infinite series where the terms may be +ve or –ve, then

$S_n = u_1 + u_2 + \cdots + u_n$ is called the nth partial sum of Σu_n. Thus, the nth partial sum of an infinite series is the sum of its first n terms.

$S_1, S_2, S_3, \ldots$ are the first, second, third, ... partial sums of the series.

Since $n \in N$, $\{S_n\}$ is a sequence called the **sequence of partial sums** of the infinite series Σu_n.

$\therefore$ To every infinite series Σu_n, there corresponds a sequence $\{S_n\}$ of its partial sums.

1.15. BEHAVIOUR OF AN INFINITE SERIES (Convergence, Divergence and Oscillation)

An infinite series Σu_n converges, diverges or oscillates (finitely or infinitely) according as the sequence $\{S_n\}$ of its partial sums converges, diverges or oscillates (finitely or infinitely).

(*i*) The series Σu_n converges (or is said to be convergent) if the sequence $\{S_n\}$ of its partial sums converges.

Thus, Σu_n is convergent if $\lim\limits_{n\to\infty} S_n$ = a finite quantity.

(*ii*) The series Σu_n diverges (or is said to be divergent) if the sequence $\{S_n\}$ of its partial sums diverges.

Thus, Σu_n is divergent if $\lim\limits_{n\to\infty} S_n = +\infty$ or $-\infty$

(*iii*) The series Σu_n oscillates finitely if the sequence $\{S_n\}$ of its partial sums oscillates finitely.

Thus, Σu_n oscillates finitely if $\{S_n\}$ is bounded and neither converges nor diverges.

(*iv*) The series Σu_n oscillates infinitely if the sequence $\{S_n\}$ of its partial sums oscillates infinitely.

Thus, Σu_n oscillates infinitely if $\{S_n\}$ is unbounded and neither converges nor diverges.

ILLUSTRATIVE EXAMPLES

Example 1. *Discuss the convergence or otherwise of the series*

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots+\frac{1}{n(n+1)}+\dots\infty.$$

Sol. Here $u_n = \dfrac{1}{n(n+1)} = \dfrac{(n+1)-n}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$

Putting $n = 1, 2, 3, \dots, n$, we get

$$u_1 = \frac{1}{1} - \frac{1}{2}$$
$$u_2 = \frac{1}{2} - \frac{1}{3}$$
$$u_3 = \frac{1}{3} - \frac{1}{4}$$
$$\dots\dots\dots$$
$$\dots\dots\dots$$
$$u_n = \frac{1}{n} - \frac{1}{n+1}$$

$S_n = u_1 + u_2 + \text{—} + u_n$

Adding, $S_n = 1 - \dfrac{1}{n+1}$

$$\lim_{n\to\infty} S_n = 1 - 0 = 1.$$

$\Rightarrow$ $\{S_n\}$ converges to 1 $\Rightarrow$ Σu_n converges to 1.

Note. For another method, see Comparison Test.

Example 2. *Show that the series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ *converges to* $\dfrac{3}{4}$.

Sol. Let $u_n = \dfrac{1}{n(n+2)} = \dfrac{1}{2}\cdot\dfrac{(n+2)-n}{n(n+2)} = \dfrac{1}{2}\left(\dfrac{1}{n} - \dfrac{1}{n+2}\right)$

$$= \frac{1}{2}\left[\left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\right]$$

Putting $n = 1, 2, 3, \dots, n$, we get

$$u_1 = \frac{1}{2}\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)\right]$$
$$u_2 = \frac{1}{2}\left[\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)\right]$$
$$u_3 = \frac{1}{2}\left[\left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right)\right]$$
$$\dots\dots\dots$$

$u_{n-1} = \frac{1}{2}\left[\left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)\right]$

$$u_n = \frac{1}{2}\left[\left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\right]$$

Adding, $S_n = \dfrac{1}{2}\left[\left(1 - \dfrac{1}{n+1}\right) + \left(\dfrac{1}{2} - \dfrac{1}{n+2}\right)\right]$

$$\lim_{n\to\infty} S_n = \frac{1}{2}\left(1 - 0 + \frac{1}{2} - 0\right) = \frac{3}{4}, \text{ a finite quantity.}$$

$\Rightarrow$ the sequence $< S_n >$ converges to $\dfrac{3}{4}$.

$\Rightarrow$ the infinite series $\sum\limits_{n=1}^{\infty} u_n$ converges to $\dfrac{3}{4}$.

[**Note.** For another method, see Comparison Test.]

Example 3. *Show that the series* $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^{n-1}$ *converges to 4.*

Sol. Let $u_n = \left(\dfrac{3}{4}\right)^{n-1}$

then $S_n = u_1 + u_2 + u_3 + \dots + u_n$

$$= 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \dots + \left(\frac{3}{4}\right)^{n-1}$$

$$= \frac{1\times\left[1 - \left(\frac{3}{4}\right)^n\right]}{1 - \frac{3}{4}} \qquad \left[\because \text{ In a G.P. } S_n = \frac{a(1-r^n)}{1-r}\right]$$

$$= 4\left[1 - \left(\frac{3}{4}\right)^n\right]$$

$$\lim_{n\to\infty} S_n = 4\,[1 - 0] \qquad \left[\because \text{ if } |x| < 1, \text{ then } x^n \to 0 \text{ as } n \to \infty\right]$$

$= 4$, a finite quantity.

$\Rightarrow$ the sequence $< S_n >$ converges to 4.

$\Rightarrow$ the infinite series $\sum\limits_{n=1}^{\infty} u_n$ converges to 4.

Example 4. *Show that the series* $1^2 + 2^2 + 3^2 + \dots + n^2 + \dots$ *diverges to* $+\infty$.

Sol. Here $S_n = 1^2 + 2^2 + 3^2 + \dots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

$$\lim_{n\to\infty} S_n = +\infty$$

$\Rightarrow$ $< S_n >$ diverges to $+\infty$.

$\Rightarrow$ the given series diverges to $+\infty$.

Example 5. *Show that the series* $-1-2-3-\dots-n-\dots$ *diverges to* $-\infty$.

Sol. Here $S_n = -1-2-3-\dots-n$

$$= -(1+2+3+\dots+n) = -\frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} S_n = -\infty \Rightarrow < S_n > \text{ diverges to } -\infty$$

$\Rightarrow$ the given series diverges to $-\infty$.

Example 6. *Test the convergence or otherwise of* $\sum_{n=1}^{\infty} (-1)^{n-1}$.

$S_{2n} \to 0$
$S_{2n-1} \to 1$

Sol. Here $S_n = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$ to n terms
$= 1$ or 0 according as n is odd or even.

The subsequence $< S_{2n-1} >$ converges to 1, while the subsequence $< S_{2n} >$ converges to 0.

$\Rightarrow$ $< S_n >$ is not convergent.

Since $< S_n >$ is bounded.

$\therefore$ $< S_n >$ oscillates finitely. $\Rightarrow$ $\sum_{n=1}^{\infty} (-1)^{n-1}$ oscillates finitely.

Example 7. *Test the convergence of the series*

$$5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \cdots \text{ to } \infty.$$

Sol. Here $S_n = 5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \cdots$ to n terms $= 0$, 5 or 1 according as the number of terms is $3m$, $3m + 1$ or $3m + 2$ respectively.

Clearly S_n does not tend to a unique limit. Since $< S_n >$ is bounded, it oscillates finitely.

$\Rightarrow$ The given series oscillates finitely.

Example 8. *Show that the series* $\sum_{n=1}^{\infty} n(-1)^n$ *oscillates infinitely.*

Sol. Here $S_n = -1 + 2 - 3 + 4 - 5 + 6 + \cdots$ to n terms

$$= \begin{cases} -\left(\dfrac{n+1}{2}\right), & \text{if } n \text{ is odd} \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

The subsequence $< S_{2n-1} >$ diverges to $-\infty$, while the subsequence $< S_{2n} >$ diverges to $+\infty$.

$\therefore$ $< S_n >$ oscillates infinitely.

$\Rightarrow$ $\sum_{n=1}^{\infty} n(-1)^n$ oscillates infinitely.

EXERCISE 1.1

Test the nature of the following series:

1. $1 + 2 + 3 + 4 + \cdots \infty$
2. $1 + 3 + 5 + 7 + \cdots \infty$
3. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \infty$
4. $1 - \frac{1}{5} + \frac{1}{5^2} - \frac{1}{5^3} + \cdots \infty$
5. $1^2 + 3^2 + 5^2 + 7^2 + \cdots \infty$
6. $7 - 4 - 3 + 7 - 4 - 3 + 7 - 4 - 3 + \cdots \infty$
7. $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots \infty$
8. $-1 - 8 - 27 - 64 - \cdots \infty$.

Answers

1. Divergent
2. Divergent
3. Convergent
4. Convergent
5. Divergent
6. Oscillatory
7. Convergent
8. Divergent.

1.16. THE GEOMETRIC SERIES $1 + x + x^2 + x^3 + \cdots \infty$ *(M.D.U. Dec. 2010)*

(*i*) converges if $-1 < x < 1$, *i.e.*, $|x| < 1$ (*ii*) diverges if $x \geq 1$
(*iii*) oscillates finitely if $x = -1$ (*iv*) oscillates infinitely if $x < -1$.

Proof. (*i*) When $|x| < 1$

Since $|x| < 1$, $x^n \to 0$ as $n \to \infty$

$$S_n = 1 + x + x^2 + \cdots \text{ to } n \text{ terms} = \frac{1(1-x^n)}{1-x} = \frac{1}{1-x} - \frac{x^n}{1-x}$$

$$\lim_{n\to\infty} S_n = \frac{1}{1-x} \Rightarrow \text{the sequence } \{S_n\} \text{ is convergent}$$

$\Rightarrow$ the given series is convergent.

(*ii*) When $x \geq 1$

Sub-case I. When $x = 1$

$$S_n = 1 + 1 + 1 + \cdots \text{ to } n \text{ terms} = n$$

$$\lim_{n\to\infty} S_n = \infty \Rightarrow \text{the sequence } \{S_n\} \text{ diverges to } \infty.$$

$\Rightarrow$ the given series diverges to ∞.

Sub-case II. When $x > 1$, $x^n \to \infty$ as $n \to \infty$

$$S_n = 1 + x + x^2 + \cdots \text{ to } n \text{ terms} = \frac{1(x^n - 1)}{x - 1}$$

$$\lim_{n\to\infty} S_n = \infty \Rightarrow \text{the sequence } \{S_n\} \text{ diverges to } \infty$$

$\Rightarrow$ the given series diverges to ∞.

(*iii*) When $x = -1$

$$S_n = 1 - 1 + 1 - 1 + \cdots \text{ to } n \text{ terms}$$
$= 1$ or 0 according as n is odd or even.

$\Rightarrow$ $\lim_{n\to\infty} S_n = 1$ or 0 $\Rightarrow$ the sequence $\{S_n\}$ oscillates finitely.

$\Rightarrow$ the given series oscillates finitely.

(*iv*) When $x < -1$

$x < -1 \Rightarrow -x > 1$

Let $r = -x$, then $r > 1$

$\therefore$ $r^n \to \infty$ as $n \to \infty$

$$S_n = 1 + x + x^2 + x^3 + \cdots \text{ to } n \text{ terms} = \frac{1-x^n}{1-x} = \frac{1-(-r)^n}{1+r} \qquad [\because x = -r]$$

$$= \frac{1-r^n}{1+r} \text{ or } \frac{1+r^n}{1+r} \text{ according as } n \text{ is even or odd}$$

$$\lim_{n\to\infty} S_n = \frac{1-\infty}{1+r} \text{ or } \frac{1+\infty}{1+r} = -\infty \text{ or } +\infty$$

$\Rightarrow$ the sequence $\{S_n\}$ oscillates infinitely

$\Rightarrow$ the given series oscillates infinitely.

Examples. 1. Consider $1 - \frac{3}{4} + \frac{9}{16} - \frac{27}{64} + \cdots$

It is a geometric series with common ratio $r = -\frac{3}{4}$.

Since $|r| = \left|-\frac{3}{4}\right| = \frac{3}{4} < 1$, the given series is convergent.

2. Consider $2 + 3 + \frac{9}{2} + \frac{27}{4} + \cdots$

It is a geometric series with $r = \frac{3}{2} > 1$

$\Rightarrow$ the given series is divergent.

3. Consider $\frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \cdots$

It is a geometric series with $r = 1$

$\Rightarrow$ the given series is divergent.

4. Consider $\frac{3}{4} - \frac{3}{4} + \frac{3}{4} - \frac{3}{4} + \cdots$

It is a geometric series with $r = -1$

$\Rightarrow$ the given series oscillates finitely.

5. Consider $1 - \frac{5}{2} + \frac{25}{4} - \frac{125}{8} + \cdots$

It is a geometric series with $r = -\frac{5}{2} < -1$

$\Rightarrow$ the given series oscillates infinitely.

1.17. NECESSARY CONDITION FOR CONVERGENCE

(M.D.U. Dec. 2009)

If a series Σu_n is convergent, then $\lim\limits_{n\to\infty} u_n = 0$.

Proof. Let S_n denote the nth partial sum of the series Σu_n.

Then Σu_n is convergent $\Rightarrow$ $\{S_n\}$ is convergent

$\Rightarrow$ $\lim\limits_{n\to\infty} S_n$ is finite and unique $= s$ (say). $\Rightarrow$ $\lim\limits_{n\to\infty} S_{n-1} = s$

Now $S_n - S_{n-1} = u_n$

$\therefore$ $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} (S_n - S_{n-1}) = \lim\limits_{n\to\infty} S_n - \lim\limits_{n\to\infty} S_{n-1} = s - s = 0.$

Hence Σu_n is convergent $\Rightarrow$ $\lim\limits_{n\to\infty} u_n = 0$.

The converse of the above theorem is not always true, *i.e.*, the nth term may tend to zero as $n \to \infty$ even if the series is not convergent.

For example, consider the series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \cdots$

Here $S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$

$$> \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}} \qquad \left(\because m < n \Rightarrow \frac{1}{\sqrt{m}} > \frac{1}{\sqrt{n}}\right)$$

$$= \frac{n}{\sqrt{n}} = \sqrt{n}$$

$\therefore$ $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \sqrt{n} \to \infty$ 4341 $\lim\limits_{n\to\infty} S_n$ 7 $\sqrt{n} = \infty$

$\Rightarrow$ the series is divergent, though $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Thus $\lim\limits_{n\to\infty} u_n = 0$ is a necessary condition but not a sufficient condition for convergence of Σu_n.

Note 1. Σu_n is convergent $\Rightarrow$ $\lim\limits_{n\to\infty} u_n = 0$

Note 2. $\lim\limits_{n\to\infty} u_n = 0$ $\Rightarrow$ Σu_n may or may not be convergent

Note 3. $\lim\limits_{n\to\infty} u_n \neq 0$ $\Rightarrow$ Σu_n is not convergent.

1.18. A POSITIVE TERM SERIES EITHER CONVERGES OR DIVERGES TO $+\infty$

Proof. Let Σu_n be a positive term series and S_n be its nth partial sum.

Then $S_{n+1} = u_1 + u_2 + \cdots + u_n + u_{n+1} = S_n + u_{n+1}$

$\Rightarrow$ $S_{n+1} - S_n = u_{n+1} > 0 \quad \forall\ n$ $\qquad [\because\ u_n > 0\ \forall\ n]$

$\Rightarrow$ $S_{n+1} > S_n \quad \forall\ n$

$\Rightarrow$ $\{S_n\}$ is a monotonic increasing sequence.

Two cases arise. The sequence $\{S_n\}$ may be bounded or unbounded above.

Case I. *When $\{S_n\}$ is bounded above.*

Since $\{S_n\}$ is monotonic increasing and bounded above, it is convergent $\Rightarrow$ Σu_n is convergent.

Case II. *When $\{S_n\}$ is not bounded above.*

Since $\{S_n\}$ is monotonic increasing and not bounded above, it diverges to $+\infty$ $\Rightarrow$ Σu_n diverges to $+\infty$.

Hence a positive term series either converges or diverges to $+\infty$.

Cor. If $u_n > 0\ \forall\ n$ and $\lim\limits_{n\to\infty} u_n \neq 0$, then the series Σu_n diverges to $+\infty$.

Proof. $u_n > 0\ \forall\ n$ $\Rightarrow$ Σu_n is a series of +ve terms.

$\Rightarrow$ Σu_n either converges or diverges to $+\infty$

Since $\lim\limits_{n\to\infty} u_n \neq 0$ (given)

$\therefore$ Σu_n does not converge.

Hence Σu_n must diverge to $+\infty$.

1.19. (a) THE NECESSARY AND SUFFICIENT CONDITION FOR THE CONVERGENCE OF A POSITIVE TERM SERIES Σu_n IS THAT THE SEQUENCE $\{S_n\}$ OF ITS PARTIAL SUMS IS BOUNDED ABOVE

Proof. (*i*) Suppose the sequence $\{S_n\}$ is bounded above. Since the series Σu_n is of positive terms, the sequence $\{S_n\}$ is monotonically increasing. Since every monotonically increasing sequence which is bounded above, converges, therefore $\{S_n\}$ and hence Σu_n converges.

(*ii*) Conversely, suppose Σu_n converges. Then the sequence $\{S_n\}$ of its partial sums also converges. Since every convergent sequence is bounded, $\{S_n\}$ is bounded. In particular, $\{S_n\}$ is bounded above.

1.19. (b) CAUCHY'S GENERAL PRINCIPLE OF CONVERGENCE OF SERIES

The necessary and sufficient condition for the infinite series $\sum_{n=1}^{\infty} u_n$ to converge is that given $\varepsilon > 0$, however small, there exists a positive integer m such that

$$| u_{m+1} + u_{m+2} + \dots + u_n | < \varepsilon \quad \forall \quad n > m.$$

Example. *Prove with the help of Cauchy's general principle of convergence that the series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots \textit{ does not converge.}$$

Sol. If possible, suppose the given series is convergent.

Take $\quad \varepsilon = \dfrac{1}{2}$

By Cauchy's general principle of convergence, there exists a positive integer m such that

$$\left| \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} \right| < \frac{1}{2} \quad \forall \quad n > m$$

or

$$\frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} < \frac{1}{2} \quad \forall \quad n > m \qquad \dots(1)$$

By taking $n = 2m$, we observe that $\dfrac{1}{m+1} + \dfrac{1}{m+2} + \dots + \dfrac{1}{n}$

$$= \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{2m} > \frac{1}{2m} + \frac{1}{2m} + \dots + \frac{1}{2m} = \frac{m}{2m} = \frac{1}{2}$$

i.e.,
$$\frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} > \frac{1}{2}$$

where $n = 2m > m$. This contradicts (1).

$\Rightarrow$ our supposition is wrong.

$\Rightarrow$ the given series does not converge.

1.20. IF m IS A GIVEN POSITIVE INTEGER, THEN THE TWO SERIES $u_1 + u_2 + \dots + u_{m+1} + u_{m+2} + \dots$ AND $u_{m+1} + u_{m+2} + \dots$ CONVERGE OR DIVERGE TOGETHER

Proof. Let S_n and s_n denote the nth partial sums of the two series.

Then $\quad S_n = u_1 + u_2 + \dots + u_n$

$$s_n = u_{m+1} + u_{m+2} + \dots + u_{m+n}$$
$$= (u_1 + u_2 + \dots + u_{m+n}) - (u_1 + u_2 + \dots + u_m)$$
$$= S_{m+n} - S_m \quad \Rightarrow \quad s_n = S_{m+n} - S_m \qquad \dots(1)$$

S_m being the sum of a finite number of terms of Σu_n is a fixed finite quantity.

(*i*) If $S_{m+n} \to$ a finite limit as $n \to \infty$, then from (1), so does s_n.

(*ii*) If $S_{m+n} \to +\infty$ as $n \to \infty$, so does s_n.

(*iii*) If $S_{m+n} \to -\infty$ as $n \to \infty$, so does s_n.

(*iv*) If S_{m+n} does not tend to any limit (finite or infinite), so does s_n.

$\Rightarrow$ The sequences $\{S_n\}$ and $\{s_n\}$ converge or diverge together.

$\Rightarrow$ The two given series converge or diverge together. Hence the result.

Note. The above theorem shows that the convergence, divergence or oscillation of a series is not affected by addition or omission of a finite number of its terms.

1.21. IF Σu_n AND Σv_n CONVERGE TO u AND v RESPECTIVELY, THEN $\Sigma(u_n + v_n)$ CONVERGES TO (u + v)

Proof. Let $\quad U_n = u_1 + u_2 + \dots + u_n$

$\quad V_n = v_1 + v_2 + \dots + v_n$

and $\quad S_n = (u_1 + v_1) + (u_2 + v_2) + \dots + (u_n + v_n)$

Then $\quad S_n = (u_1 + u_2 + \dots + u_n) + (v_1 + v_2 + \dots + v_n) = U_n + V_n.$

Since Σu_n converges to u, $\lim_{n \to \infty} U_n = u$

Σv_n converges to v, $\lim_{n \to \infty} V_n = v$

$$\therefore \quad \lim_{n \to \infty} S_n = \lim_{n \to \infty} (U_n + V_n) = \lim_{n \to \infty} U_n + \lim_{n \to \infty} V_n = u + v.$$

$\Rightarrow \quad \Sigma(u_n + v_n)$ converges to $(u + v)$.

1.22. COMPARISON TESTS

Test I. *If Σu_n and Σv_n are series of positive terms and Σv_n is convergent and there is a positive constant k such that $\boxed{u_n \le k v_n}$ $\forall$ $n > m$, then Σu_n is also convergent.*

Proof. Let $\quad U_n = u_1 + u_2 + \dots + u_n \quad$ and $\quad V_n = v_1 + v_2 + \dots + v_n$

Now $\quad u_n \le k v_n \; \forall \; n > m$

$\Rightarrow \quad u_{m+1} \le k v_{m+1}$

$\quad u_{m+2} \le k v_{m+2}$

$\quad \dots\dots\dots\dots\dots\dots$

Adding $u_{m+1} + u_{m+2} + \dots + u_n \le k(v_{m+1} + v_{m+2} + \dots + v_n)$

$\Rightarrow \quad U_n - U_m \le k(V_n - V_m) \quad \forall \quad n > m$

$\Rightarrow \quad U_n \le kV_n + (U_m - kV_m) \quad \forall \quad n > m$

$\Rightarrow \quad U_n \le kV_n + k_0 \quad \forall \quad n > m \qquad \dots(1)$

where $k_0 = U_m - kV_m$ is a fixed number. Since Σv_n is convergent, the sequence $\{V_n\}$ is convergent and hence bounded above.

$\therefore$ From (1), the sequence $\{U_n\}$ is bounded above.

$\because$ Σu_n is a series of +ve terms, $\{U_n\}$ is monotonic increasing.

$\because$ $\{U_n\}$ is monotonic increasing sequence and is bounded above.

$\therefore$ It is convergent.

$\Rightarrow$ Σu_n is convergent.

Test II. *If Σu_n and Σv_n are two series of positive terms and Σv_n is divergent and there is a positive constant k such that $u_n > kv_n$, $\forall\ n > m$, then Σu_n is also divergent.*

Proof. Let $U_n = u_1 + u_2 + \cdots + u_n$

and $V_n = v_1 + v_2 + \cdots + v_n$

Now $u_n > kv_n \quad \forall \quad n > m$

$\Rightarrow$ $u_{m+1} > kv_{m+1}$

$u_{m+2} > kv_{m+2}$

....

....

$u_n > kv_n$

Adding $u_{m+1} + u_{m+2} + \cdots + u_n > k(v_{m+1} + v_{m+2} + \cdots + v_n)$

$\Rightarrow$ $U_n - U_m > k(V_n - V_m) \quad \forall \quad n > m$

$\Rightarrow$ $U_n > kV_n + (U_m - kV_m) \quad \forall \quad n > m$

$\Rightarrow$ $U_n > kV_n + k_0 \quad \forall \quad n > m \quad \ldots(1)$

where $k_0 = U_m - kV_m$ is a fixed number.

Since Σv_n is divergent, the sequence $\{V_n\}$ is divergent.

$\Rightarrow$ for each positive real number k_1, however large, there exists a +ve integer m' such that

$V_n > k_1 \quad \forall \quad n > m'$

Let $m^* = \max.\{m, m'\}$, then $V_n > k_1 \quad \forall \quad n > m^*$

From (1), $U_n > kk_1 + k_0 = K \quad \forall \quad n > m^*$

$\Rightarrow$ $\{U_n\}$ is divergent

$\Rightarrow$ Σu_n is divergent.

Test III. *If Σu_n and Σv_n are two positive term series and there exist two positive constants H and K (independent of n) and a positive integer m such that $H < \frac{u_n}{v_n} < K\ \forall\ n > m$, then the two series Σu_n and Σv_n converge or diverge together.*

Proof. Since Σv_n is a series of + ve terms, $v_n > 0, \quad \forall\ n$

$\therefore$ $H < \frac{u_n}{v_n} < K \quad \forall\ n > m$

$\Rightarrow$ $Hv_n < u_n < Kv_n \quad \forall\ n > m \quad \ldots(1)$

Case I. *When Σv_n is convergent*

From (1), $u_n < Kv_n \quad \forall\ n > m$ and Σv_n is convergent

$\Rightarrow$ Σu_n is convergent. [See Test I]

Case II. *When Σv_n is divergent*

From (1), $u_n > Hv_n \quad \forall\ n > m$ and Σv_n is divergent.

$\Rightarrow$ Σu_n is divergent. [See Test II]

Case III. *When Σu_n is convergent*

From (1), $Hv_n < u_n \quad \forall\ n > m$

$\Rightarrow$ $v_n < \frac{1}{H} u_n \quad \forall\ n > m \quad (\because\ H > 0)$

Since Σu_n is convergent $\therefore$ Σv_n is convergent. [See Test I]

Case IV. *When Σu_n is divergent*

From (1), $Kv_n > u_n \quad \forall\ n > m$

$\Rightarrow$ $v_n > \frac{1}{K} u_n. \quad \forall\ n > m \quad (\because\ K > 0)$

Since Σu_n is divergent $\therefore$ Σv_n is divergent. [See Test II]

Particular Case of Test III (When m = 0)

If Σu_n and Σv_n are two positive term series and there exist two positive constants H and K (independent of n) such that $H < \frac{u_n}{v_n} < K \quad \forall\ n$,

then the two series Σu_n and Σv_n converge or diverge together.

Test IV. *Let Σu_n and Σv_n be two positive term series.*

(i) If $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = l$ (finite and non-zero), then Σu_n and Σv_n both converge or diverge together.

(ii) If $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = 0$ and Σv_n converges, then Σu_n also converges.

(iii) If $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \infty$ and Σv_n diverges, then Σu_n also diverges.

Proof. (*i*) Since $u_n > 0$, $v_n > 0$ $\therefore$ $\frac{u_n}{v_n} > 0$

$\therefore$ $\lim\limits_{n\to\infty} \frac{u_n}{v_n} \geq 0$

But $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = l \neq 0 \Rightarrow l > 0$

Now $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = l$

$\Rightarrow$ given $\varepsilon > 0$, there exists a +ve integer m such that $\left|\frac{u_n}{v_n} - l\right| < \varepsilon \quad \forall \quad n > m$

$\Rightarrow$ $l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \quad \forall\ n > m$

$\Rightarrow$ $(l - \varepsilon)\, v_n < u_n < (l + \varepsilon)\, v_n \quad \forall\ n > m \quad (\because\ v_n > 0)$

Choose $\varepsilon > 0$ such that $l - \varepsilon > 0$.

Let $l - \varepsilon = H$, $l + \varepsilon = K$, where H, K are > 0

$\therefore$ $Hv_n < u_n < Kv_n \quad \forall\ n > m \quad \ldots(1)$

Case I. *When Σu_n is convergent*

From (1), $Hv_n < u_n \quad \forall\ n > m$

$\Rightarrow$ $v_n < \frac{1}{H} u_n \quad \forall\ n > m \quad (\because\ H > 0)$

Since Σu_n is convergent, Σv_n is also convergent.

Case II. *When Σu_n is divergent*

From (1), $Kv_n > u_n \quad \forall\ n > m$

$\Rightarrow$ $v_n > \frac{1}{K} u_n \quad \forall\ n > m \quad (\because\ K > 0)$

Since Σu_n is divergent, Σv_n is also divergent.

Case III. *When Σv_n is convergent* $\quad \forall\, n > m$

From (1), $\quad u_n < K v_n$

Since Σv_n is convergent, Σu_n is also convergent.

Case IV. *When Σv_n is divergent* $\quad \forall\, n > m$

From (1), $\quad u_n > H v_n$

Since Σv_n is divergent, Σu_n is also divergent.

Hence Σu_n and Σv_n converge or diverge together.

(*ii*) Here $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = 0$

$\therefore$ Given $\varepsilon > 0$, there exists a +ve integer m such that $\left|\dfrac{u_n}{v_n} - 0\right| < \varepsilon \quad \forall\, n > m$

$$\Rightarrow \quad -\varepsilon < \frac{u_n}{v_n} < \varepsilon \qquad \forall\, n > m \qquad (\because\ v_n > 0)$$

$$\Rightarrow \quad u_n < \varepsilon\, v_n \qquad \forall\, n > m$$

Since Σv_n is convergent, Σu_n is also convergent.

(*iii*) Here $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \infty$

$\therefore$ Given M > 0, however large, there exists a +ve integer m such that $\dfrac{u_n}{v_n} > M \ \forall\, n > m$

$$\Rightarrow \quad u_n > M v_n \qquad \forall\, n > m$$

Since Σv_n is divergent, Σu_n is also divergent.

Test V. *Let Σu_n and Σv_n be two positive term series.*

(*i*) *If* $\dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}}$ $\ \forall\, n > m$ *and Σv_n is convergent, then Σu_n is also convergent.*

(*ii*) *If* $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$ $\ \forall\, n > m$ *and Σv_n is divergent, then Σu_n is also divergent.*

Proof. (*i*) $\quad \dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}} \qquad \forall\, n > m$

$$\Rightarrow \quad \frac{u_{m+1}}{u_{m+2}} > \frac{v_{m+1}}{v_{m+2}}$$

$$\frac{u_{m+2}}{u_{m+3}} > \frac{v_{m+2}}{v_{m+3}}$$

$$\frac{u_{m+3}}{u_{m+4}} > \frac{v_{m+3}}{v_{m+4}}$$

....

....

$$\frac{u_{n-1}}{u_n} > \frac{v_{n-1}}{v_n}$$

Multiplying the corresponding sides of the above inequalities, we have

$$\frac{u_{m+1}}{u_n} > \frac{v_{m+1}}{v_n} \qquad \forall\, n > m$$

$$\Rightarrow \quad u_n < \left(\frac{u_{m+1}}{v_{m+1}}\right) v_n \qquad \forall\, n > m$$

$$\Rightarrow \quad u_n < k v_n \qquad \forall\, n > m,$$

where $k = \dfrac{u_{m+1}}{v_{m+1}}$ is a fixed +ve quantity.

Since Σv_n is convergent, so is Σu_n.

(*ii*) Using $\quad \dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}} \qquad \forall\, n > m$

and proceeding as in part (*i*), we have $\dfrac{u_{m+1}}{u_n} < \dfrac{v_{m+1}}{v_n} \qquad \forall\, n > m$

$$\Rightarrow \quad u_n > \left(\frac{u_{m+1}}{v_{m+1}}\right) v_n \qquad \forall\, n > m$$

$$\Rightarrow \quad u_n > k v_n \qquad \forall\, n > m,$$

where $k = \dfrac{u_{m+1}}{v_{m+1}}$ is a fixed +ve quantity.

Since Σv_n is divergent, so is Σu_n.

1.23. AN IMPORTANT TEST FOR COMPARISON $\sum \frac{1}{n^p}$ [HYPER HARMONIC SERIES OR P-SERIES]

Statement: The series $\sum \dfrac{1}{n^p} = \dfrac{1}{1^p} + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \dots + \dfrac{1}{n^p} + \dots \infty$ **converges if p > 1 and diverges if p ≤ 1.** (M.D.U. Dec. 2013)

Proof. Case I. *When p > 1*

1 $\quad \dfrac{1}{1^p} = 1$

2 $\quad \dfrac{1}{2^p} + \dfrac{1}{3^p} < \dfrac{1}{2^p} + \dfrac{1}{2^p} = \dfrac{2}{2^p} = \dfrac{1}{2^{p-1}} \qquad \left[\because\ \dfrac{1}{3^p} < \dfrac{1}{2^p}\right]$

4 $\quad \dfrac{1}{4^p} + \dfrac{1}{5^p} + \dfrac{1}{6^p} + \dfrac{1}{7^p} < \dfrac{1}{4^p} + \dfrac{1}{4^p} + \dfrac{1}{4^p} + \dfrac{1}{4^p}$

$$= \frac{4}{4^p} = \frac{1}{4^{p-1}} = \frac{1}{(2^{p-1})^2} \qquad \left[\because\ \frac{1}{5^p} < \frac{1}{4^p}, \frac{1}{6^p} < \frac{1}{4^p} \text{ etc.}\right]$$

Similarly, the sum of next eight terms

$$= \frac{1}{8^p} + \frac{1}{9^p} + \dots + \frac{1}{15^p} < \frac{1}{8^p} + \frac{1}{8^p} + \dots + \frac{1}{8^p} = \frac{8}{8^p}$$

$$= \frac{1}{8^{p-1}} = \frac{1}{(2^{p-1})^3} \text{ and so on.}$$

Now $\displaystyle\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots$

$$= \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \frac{1}{9^p} + \dots + \frac{1}{15^p}\right) + \dots \qquad \dots(1)$$

$\therefore$ Each term of (1) after the first is less than the corresponding term in

$$1+\frac{1}{2^{p-1}}+\frac{1}{(2^{p-1})^2}+\frac{1}{(2^{p-1})^3} \quad \ldots(2)$$

But (2) is a G.P. whose common ratio $= \frac{1}{2^{p-1}} < 1$ $\quad (\because \; p > 1)$

$\therefore$ (2) is convergent $\Rightarrow$ (1) is convergent.

Hence the given series is convergent.

Case II. *When* $p = 1$

$$\sum \frac{1}{n^p} = \sum \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

$$1+\frac{1}{2} = 1+\frac{1}{2}$$

$$\frac{1}{3}+\frac{1}{4} > \frac{1}{4}+\frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{4}{8} = \frac{1}{2} \quad \text{and so on.}$$

Now $\quad \sum \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots$

$$= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots \quad \ldots(1)$$

Each term of (1) after the second is greater than the corresponding term in

$$1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots \quad \ldots(2)$$

But after the second term (2) is a G.P. whose common ratio = 1.

$\therefore$ (2) is divergent. $\Rightarrow$ (1) is divergent.

Hence the given series is divergent.

Case III. *When* $p < 1$

$$p < 1 \;\Rightarrow\; n^p < n \;\Rightarrow\; \frac{1}{n^p} > \frac{1}{n} \quad \forall \; n$$

But the series $\sum \frac{1}{n}$ is divergent (Case II).

Hence $\sum \frac{1}{n^p}$ is also divergent.

Note. The above test can also be proved by using Cauchy's Integral test.

ILLUSTRATIVE EXAMPLES

Example 1. *Examine the convergence of the series:*

(i) $\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+\cdots\infty$ $\qquad$ (ii) $1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+\cdots$

Sol. (i) $\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+\cdots\infty$

$$= \left(\frac{3}{5}+\frac{3}{5^3}+\cdots\infty\right)+\left(\frac{4}{5^2}+\frac{4}{5^4}+\cdots\infty\right) = \Sigma u_n + \Sigma v_n \text{ (say)}$$

Now Σu_n is a G.P. with common ratio $= \frac{1}{5^2}$ which is numerically less than 1.

$\therefore$ Σu_n is convergent.

Σv_n is also a G.P. with common ratio $= \frac{1}{5^2}$ which is numerically less than 1.

$\therefore$ Σv_n is convergent.

$\therefore$ The given series *viz.* $\Sigma(u_n + v_n)$ is also convergent.

(ii) $1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+\cdots\infty = 1+\frac{1}{(2^2)^{2/3}}+\frac{1}{(3^2)^{2/3}}+\frac{1}{(4^2)^{2/3}}+\cdots\infty$

$$= \frac{1}{1^{4/3}}+\frac{1}{2^{4/3}}+\frac{1}{3^{4/3}}+\frac{1}{4^{4/3}}+\cdots\infty = \sum \frac{1}{n^{4/3}} = \sum \frac{1}{n^p} \text{ with } p = \frac{4}{3} > 1$$

$\therefore$ By p-series test, the given series is convergent.

Example 2. *Test the convergence of the series:* $\frac{1}{1\cdot2\cdot3}+\frac{3}{2\cdot3\cdot4}+\frac{5}{3\cdot4\cdot5}+\cdots$

Sol. Here
$$u_n = \frac{T_n \text{ of } 1, 3, 5, \ldots}{n(n+1)(n+2)} = \frac{2n-1}{n(n+1)(n+2)} = \frac{n\left(2-\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$= \frac{2-\frac{1}{n}}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

Let us compare Σu_n with Σv_n, where $v_n = \frac{1}{n^2}$

$$\frac{u_n}{v_n} = \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = \frac{2}{(1)(1)} = 2 \text{ which is finite and non-zero.}$$

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \frac{1}{n^2}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = 2 > 1$.

$\therefore$ Σv_n is convergent $\Rightarrow$ Σu_n is convergent.

Example 3. *Test the convergence of the following series:*

(i) $\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots$ $\qquad$ (ii) $\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{6}}+\sqrt{\frac{3}{8}}+\cdots+\sqrt{\frac{n}{2(n+1)}}+\cdots$

Sol. (*i*) Here $\quad u_n = \dfrac{1}{\sqrt{n}+\sqrt{n+1}} = \dfrac{1}{\sqrt{n}\left[1+\sqrt{1+\dfrac{1}{n}}\right]}$

Let us compare Σu_n with Σv_n, where $v_n = \dfrac{1}{\sqrt{n}}$

$$\frac{u_n}{v_n} = \frac{1}{1+\sqrt{1+\dfrac{1}{n}}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{1+\sqrt{1+\dfrac{1}{n}}} = \frac{1}{1+1} = \frac{1}{2} \quad \text{which is finite and non-zero.}$$

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma\dfrac{1}{n^{1/2}}$ is of the form $\Sigma\dfrac{1}{n^p}$ with $p = \dfrac{1}{2} < 1$

$\therefore$ Σv_n is divergent $\Rightarrow$ Σu_n is divergent.

(*ii*) Here $\quad u_n = \sqrt{\dfrac{n}{2(n+1)}} = \sqrt{\dfrac{1}{2\left(1+\dfrac{1}{n}\right)}}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\sqrt{2\left(1+\dfrac{1}{n}\right)}} = \frac{1}{\sqrt{2}} \neq 0$$

$\Rightarrow$ Σu_n does not converge.

Since the given series is a series of +ve terms, it either converges or diverges. Since it does not converge, it must diverge.

Hence the given series is divergent.

Example 4. *Test the convergence of the series:* $1+\dfrac{1}{2^2}+\dfrac{2^2}{3^3}+\dfrac{3^3}{4^4}+\dfrac{4^4}{5^5}+\cdots$

(M.D.U. Dec. 2008)

Sol. *Leaving aside the first term* ($\because$ *addition or deletion of a finite number of terms does not alter the nature of the series*), we have $u_n = \dfrac{n^n}{(n+1)^{n+1}} = \dfrac{n^n}{n^{n+1}\left(1+\dfrac{1}{n}\right)^{n+1}} = \dfrac{1}{n\left(1+\dfrac{1}{n}\right)^{n+1}}$

Take $\quad v_n = \dfrac{1}{n}$.

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\left(1+\dfrac{1}{n}\right)^{n+1}} = \lim_{n\to\infty}\frac{1}{\left(1+\dfrac{1}{n}\right)^{n}}\times\frac{1}{\left(1+\dfrac{1}{n}\right)}$$

$$= \frac{1}{e}\cdot\frac{1}{1} \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

$$= \frac{1}{e} \quad \text{which is finite and non-zero.}$$

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma\dfrac{1}{n}$ is of the form $\Sigma\dfrac{1}{n^p}$ with $p = 1$

$\therefore$ Σv_n is divergent. $\Rightarrow$ Σu_n is divergent.

Example 5. *Examine the convergence of the series:* $\dfrac{\sqrt{2}-1}{3^3-1}+\dfrac{\sqrt{3}-1}{4^3-1}+\dfrac{\sqrt{4}-1}{5^3-1}+\cdots$

(M.D.U. Dec. 2013)

Sol. Here $\quad u_n = \dfrac{\sqrt{n+1}-1}{(n+2)^3-1} = \dfrac{\sqrt{n}\left(\sqrt{1+\dfrac{1}{n}}-\dfrac{1}{\sqrt{n}}\right)}{n^3\left[\left(1+\dfrac{2}{n}\right)^3-\dfrac{1}{n^3}\right]} = \dfrac{\sqrt{1+\dfrac{1}{n}}-\dfrac{1}{\sqrt{n}}}{n^{5/2}\left[\left(1+\dfrac{2}{n}\right)^3-\dfrac{1}{n^3}\right]}$

Take $\quad v_n = \dfrac{1}{n^{5/2}}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\sqrt{1+\dfrac{1}{n}}-\dfrac{1}{\sqrt{n}}}{\left(1+\dfrac{2}{n}\right)^3-\dfrac{1}{n^3}} = \frac{\sqrt{1+0}-0}{(1-0)^3-0} = 1 \quad \text{which is finite and non-zero.}$$

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma\dfrac{1}{n^{5/2}}$ is of the form $\Sigma\dfrac{1}{n^p}$ with $p = \dfrac{5}{2} > 1$.

Σv_n is convergent. $\Rightarrow$ Σu_n is convergent.

Example 6. *Examine the convergence of the series:*

(*i*) $\sum\dfrac{\sqrt{n+1}-\sqrt{n}}{n^p}$ $\qquad$ (*ii*) $\sum\limits_{n=1}^{\infty}\left(\sqrt[3]{n^3+1}-n\right)$. $\qquad$ (M.D.U. Dec. 2014)

Sol. (*i*) Here $\quad u_n = \dfrac{\sqrt{n+1}-\sqrt{n}}{n^p} = \dfrac{\sqrt{n+1}-\sqrt{n}}{n^p}\times\dfrac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$ $\qquad$ (Rationalising)

$$= \frac{(n+1)-n}{n^p\cdot\sqrt{n}\left(\sqrt{1+\dfrac{1}{n}}+1\right)} = \frac{1}{n^{p+1/2}\left(\sqrt{1+\dfrac{1}{n}}+1\right)}$$

Take $\quad v_n = \dfrac{1}{n^{p+1/2}}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\dfrac{1}{n}}+1} = \frac{1}{1+1} = \frac{1}{2} \quad \text{which is finite and non-zero.}$$

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\sum v_n = \sum\dfrac{1}{n^{p+1/2}}$ is convergent if $p+\dfrac{1}{2} > 1$ and divergent if $p+\dfrac{1}{2} \le 1$.

i.e., convergent if $p > \frac{1}{2}$ and divergent if $p \le \frac{1}{2}$.

$\therefore\ \Sigma u_n$ is convergent if $p > \frac{1}{2}$ and divergent if $p \le \frac{1}{2}$.

(*ii*) Here
$$u_n = (n^3+1)^{1/3} - n = \left[n^3\left(1+\frac{1}{n^3}\right)\right]^{1/3} - n$$
$$= n\left(1+\frac{1}{n^3}\right)^{1/3} - n = n\left[\left(1+\frac{1}{n^3}\right)^{1/3} - 1\right]$$
$$= n\left[1+\frac{1}{3}\cdot\frac{1}{n^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot\left(\frac{1}{n^3}\right)^2+\dots-1\right]$$
$$= \frac{n}{n^3}\left[\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{n^3}\dots\right] = \frac{1}{n^2}\left[\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{n^3}\dots\right]$$

Take
$$v_n = \frac{1}{n^2}.$$
$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{n^3}\dots\right) = \frac{1}{3}$$

which is finite and non-zero.

$\therefore\ \Sigma u_n$ and Σv_n converge or diverge together.

Since $\sum v_n = \sum \frac{1}{n^2}$ is of the form $\sum \frac{1}{n^p}$ with $p = 2 > 1$

$\therefore\ \Sigma v_n$ is convergent $\Rightarrow$ Σu_n is convergent.

Note. Rationalisation is effective only when square roots are involved whereas this method of Binomial Expansion is general.

Example 7. *Test the convergence of the series* $\sum_{n=1}^{\infty}\frac{1}{x^n+x^{-n}}$, $x > 0$.

Sol. Here
$$u_n = \frac{1}{x^n+x^{-n}} = \frac{x^n}{x^{2n}+1}$$

Case I. *When* $0 < x < 1$, *take* $v_n = x^n$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{x^{2n}+1} = \frac{1}{0+1} = 1 \qquad [\because\ x^{2n}\to 0 \text{ as } n\to\infty]$$

which is non-zero, finite.

$\therefore\ \Sigma u_n$ and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma x^n = x + x^2 + x^3 + \cdots$ is an infinite geometric series with common ratio $x < 1$.

$\therefore\ \Sigma v_n$ is convergent. $\Rightarrow$ Σu_n is convergent.

Case II. *When* $x > 1$ *so that* $0 < \frac{1}{x} < 1$

$$u_n = \frac{x^n}{x^{2n}+1} = \frac{\frac{1}{x^n}}{1+\frac{1}{x^{2n}}}$$

Take
$$v_n = \frac{1}{x^n}$$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{x^{2n}}} = \frac{1}{1+0} = 1 \qquad \left[\because\ \frac{1}{x^{2n}}\to 0 \text{ as } n\to\infty\right]$$

which is non-zero, finite.

$\therefore\ \Sigma u_n$ and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma\frac{1}{x^n}$ is an infinite geometric series with common ratio $\frac{1}{x} < 1$.

$\therefore\ \Sigma v_n$ is convergent $\Rightarrow$ Σu_n is convergent.

Case III. *When* $x = 1$, $u_n = \frac{1}{2}$

$\lim_{n\to\infty} u_n = \frac{1}{2} \ne 0$. Therefore, Σu_n is divergent.

Hence Σu_n converges for $x < 1$ and $x > 1$ but diverges for $x = 1$.

Example 8. *Show that* $\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{1}{n}$ *converges.* (M.D.U. Dec. 2012)

Sol. Here
$$u_n = \frac{1}{n}\sin\frac{1}{n} = \frac{1}{n}\left[\frac{1}{n}-\frac{1}{3!n^3}+\frac{1}{5!n^5}\dots\right]$$

Take $v_n = \frac{1}{n^2}$ $\qquad \left(\because\ \sin x = x - \frac{x^3}{3!}+\frac{x^5}{5!}\dots\right)$

Then
$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left[1-\frac{1}{3!n^2}+\frac{1}{5!n^4}-\dots\right] = 1$$

which is finite and non-zero.

$\therefore\ \Sigma u_n$ and Σv_n converge or diverge together.

Since $\Sigma v_n = \sum\frac{1}{n^2}$ is of the form $\sum\frac{1}{n^p}$ with $p = 2 > 1$

$\therefore\ \Sigma v_n$ is convergent $\Rightarrow$ Σu_n is convergent.

EXERCISE 1.2

Test the convergence or divergence of the following series:

1. $1-\frac{1}{3}+\frac{1}{2}-\frac{1}{3^3}+\frac{1}{2^2}-\frac{1}{3^5}+\cdots$ (*M.D.U. Dec. 2011*)
2. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots\infty$
3. $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\cdots\infty$
4. $\frac{1}{1.2}+\frac{2}{3.4}+\frac{3}{5.6}+\cdots\infty$
5. $\frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+\cdots\infty$
6. $\frac{1}{\sqrt{1.2}}+\frac{1}{\sqrt{2.3}}+\frac{1}{\sqrt{3.4}}+\cdots\infty$

7. $\dfrac{1}{1^p}+\dfrac{1}{3^p}+\dfrac{1}{5^p}+\cdots\infty$

8. $\dfrac{2}{1^p}+\dfrac{3}{2^p}+\dfrac{4}{3^p}+\cdots\infty$ (M.D.U. May 2008)

9. $\sum_{n=1}^{\infty}\dfrac{n+1}{n(2n-1)}$

10. $\sum_{n=1}^{\infty}\dfrac{1}{n^p(n+1)^p}$

11. $\dfrac{2^p}{1^q}+\dfrac{3^p}{2^q}+\dfrac{4^p}{3^q}+\cdots\infty$

(p and q are positive numbers).

12. $\sum_{n=1}^{\infty}\dfrac{\sqrt{n}}{n^2+1}$

13. $\sum\dfrac{2n^3+5}{4n^5+1}$

14. $\sum\dfrac{\sqrt{n^2-1}}{n^3+1}$

15. (i) $\dfrac{1}{5}+\dfrac{\sqrt{2}}{7}+\dfrac{\sqrt{3}}{9}+\dfrac{\sqrt{4}}{11}+\cdots\infty$ (ii) $\dfrac{1}{1\cdot2\cdot3}+\dfrac{1}{2\cdot3\cdot4}+\dfrac{1}{3\cdot4\cdot5}+\cdots\infty$

16. $\sum\left(\sqrt{n^2+1}-n\right)$

17. $\sum\left(\sqrt{n^3+1}-\sqrt{n^3}\right)$

18. $\sum\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$ (M.D.U. May 2009, Dec. 2012; Kerala 2010)

19. $\dfrac{\sqrt{2}-\sqrt{1}}{1}+\dfrac{\sqrt{3}-\sqrt{2}}{2}+\dfrac{\sqrt{4}-\sqrt{3}}{3}+\cdots$

20. $\sum\left\{(\sqrt[3]{n+1}-\sqrt[3]{n})\right\}$

21. $\sum\dfrac{1}{\sqrt{n}}\sin\dfrac{1}{n}$

22. $\sum\cot^{-1}n^2$ (M.D.U. Dec. 2009)

23. $\sum\left(\dfrac{1}{n}-\log\dfrac{n+1}{n}\right)$

24. $\sum\dfrac{\sqrt[3]{3n^2+1}}{\sqrt[4]{4n^3+2n+7}}$

25. $\sum\dfrac{x^{n-1}}{1+x^n}, x>0$

26. $\dfrac{1}{1^2}+\dfrac{1+2}{1^2+2^2}+\dfrac{1+2+3}{1^2+2^2+3^2}+\cdots$

27. $\dfrac{1}{1+\sqrt{2}}+\dfrac{2}{1+2\sqrt{3}}+\dfrac{3}{1+3\sqrt{4}}+\cdots$

28. $\sum_{n=1}^{\infty}\left(1-\cos\dfrac{\pi}{n}\right)$

29. $\dfrac{1}{a(a+b)}+\dfrac{1}{(a+2b)(a+3b)}+\dfrac{1}{(a+4b)(a+5b)}+\cdots\infty, a>0, b>0.$

Answers

1. Convergent 2. Convergent 3. Convergent 4. Divergent
5. Divergent 6. Divergent 7. Convergent for $p > 1$, divergent for $p \le 1$
8. Convergent for $p > 2$, divergent for $p \le 2$ 9. Divergent
10. Convergent for $p > \frac{1}{2}$, divergent for $p \le \frac{1}{2}$ 11. Convergent for $q > p + 1$, divergent for $q \le p+1$
12. Convergent 13. Convergent 14. Convergent
15. (i) Divergent (ii) Convergent 16. Divergent 17. Convergent
18. Convergent 19. Convergent 20. Divergent 21. Convergent
22. Convergent 23. Convergent 24. Divergent
25. Convergent for $x < 1$, divergent for $x \ge 1$ 26. Divergent
28. Convergent 29. Convergent. 27. Divergent

1.24. D'ALEMBERT'S RATIO TEST

Statement. *If Σu_n is a positive term series, and $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}=l$, then*

(i) Σu_n is convergent if $l < 1$. *(ii) Σu_n is divergent if $l > 1$.* (M.D.U. Dec. 2012)

Note. If $l = 1$, the test fails, *i.e.*, no conclusion can be drawn about the convergence or divergence of the series. The series may converge, it may diverge.

Proof. Since $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}=l,$

$\therefore$ Given $\varepsilon > 0$, however small, there exists a positive integer m such that

$$\left|\frac{u_{n+1}}{u_n}-l\right|<\varepsilon \quad \forall \quad n\ge m$$

$\Rightarrow$ $$l-\varepsilon<\frac{u_{n+1}}{u_n}<1+\varepsilon \quad \forall \quad n\ge m \qquad ...(i)$$

(i) **When $l < 1$**

Choose $\varepsilon > 0$ such that $l < l + \varepsilon < 1$

Put $l + \varepsilon = r$, then $0 < r < 1$.

From (i), $\dfrac{u_{n+1}}{u_n} < r \;\forall\; n \ge m \;\Rightarrow\; u_{n+1} < ru_n \;\forall\; n\ge m$ $(\because\; u_n > 0)$

Putting $n = m, m + 1, m + 2, \cdots$, we get

$$u_{m+1} < ru_m$$
$$u_{m+2} < ru_{m+1} < r^2u_m$$
$$u_{m+3} < ru_{m+2} < r^3u_m \quad \text{and so on.}$$

Adding $u_{m+1} + u_{m+2} + u_{m+3} + \cdots < u_m(r + r^2 + r^3 + \cdots)$

$\Rightarrow$ each term of the given series Σu_n after leaving the first m terms (*i.e.* a finite number of terms) is less than the corresponding term of a geometric series which is convergent ($\because$ its common ratio $r < 1$). Hence the given series is also convergent.

(ii) **When $l > 1$**

Choose $\varepsilon > 0$ such that $l - \varepsilon > 1$

Put $l - \varepsilon = R$, then $R > 1$.

From (i), $\dfrac{u_{n+1}}{u_n} > R \;\forall\; n\ge m \;\Rightarrow\; u_{n+1} > Ru_n \;\forall\; n\ge m$ $(\because\; u_n > 0)$

Putting $n = m, m + 1, m + 2, \cdots$, we get

$$u_{m+1} > Ru_m$$
$$u_{m+2} > Ru_{m+1} > R^2u_m$$
$$u_{m+3} > Ru_{m+2} > R^3u_m \quad \text{and so on.}$$

Adding $u_{m+1} + u_{m+2} + u_{m+3} + \cdots > u_m(R + R^2 + R^3 + \cdots)$

$\Rightarrow$ each term of the given series Σu_n after leaving the first m terms (*i.e.* a finite number of terms) is greater than the corresponding term of a geometric series which is divergent ($\because$ its common ratio R > 1). Hence the given series is also divergent.

Practical Form of D'Alembert's Ratio Test

In practice, Ratio Test is used in the following form:

If Σu_n is a positive term series, and $\lim\limits_{n \to \infty} \dfrac{u_n}{u_{n+1}} = l$, then

(*i*) Σu_n is convergent if $l > 1$ (*ii*) Σu_n is divergent if $l < 1$.

ILLUSTRATIVE EXAMPLES

Example 1. *Discuss the convergence of the following series:*

(*i*) $1 + \dfrac{2^p}{2!} + \dfrac{3^p}{3!} + \dfrac{4^p}{4!} + \cdots, (p > 0)$ (*ii*) $\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{5} + \cdots + \dfrac{1}{2^{n-1}+1} + \cdots$

(*iii*) $\dfrac{1^2 \cdot 2^2}{1!} + \dfrac{2^2 \cdot 3^2}{2!} + \dfrac{3^2 \cdot 4^2}{3!} + \dfrac{4^2 \cdot 5^2}{4!} + \cdots$

Sol. (*i*) Here $u_n = \dfrac{n^p}{n!}$ $\left[\because 1 = \dfrac{1^p}{1!}\right]$

$\therefore$ $u_{n+1} = \dfrac{(n+1)^p}{(n+1)!}$

$$\frac{u_n}{u_{n+1}} = \frac{n^p}{n!} \cdot \frac{(n+1)!}{(n+1)^p} = \frac{n^p \cdot (n+1)\, n!}{n!\,(n+1)^p} = \frac{n^p}{(n+1)^{p-1}}$$

$$= \frac{n^p}{n^{p-1}\left(1+\frac{1}{n}\right)^{p-1}} = \frac{n}{\left(1+\frac{1}{n}\right)^{p-1}}$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{n}{\left(1+\frac{1}{n}\right)^{p-1}} = \infty > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

(*ii*) Here $u_n = \dfrac{1}{2^{n-1}+1}$ $\therefore$ $u_{n+1} = \dfrac{1}{2^n + 1}$

$$\frac{u_n}{u_{n+1}} = \frac{2^n+1}{2^{n-1}+1} = \frac{2^n\left(1+\frac{1}{2^n}\right)}{2^{n-1}\left(1+\frac{1}{2^{n-1}}\right)} = 2 \cdot \frac{1+\frac{1}{2^n}}{1+\frac{1}{2^{n-1}}}$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} 2 \cdot \frac{1+\frac{1}{2^n}}{1+\frac{1}{2^{n-1}}} = 2 > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

(*iii*) Here $u_n = \dfrac{n^2(n+1)^2}{n!}$ $\therefore$ $u_{n+1} = \dfrac{(n+1)^2(n+2)^2}{(n+1)!}$

$$\frac{u_n}{u_{n+1}} = \frac{n^2(n+1)^2}{n!} \cdot \frac{(n+1)!}{(n+1)^2(n+2)^2} = \frac{n^2 \cdot (n+1)\, n!}{n!\,(n+2)^2}$$

$$= \frac{n^3\left(1+\frac{1}{n}\right)}{n^2\left(1+\frac{2}{n}\right)^2} = n \cdot \frac{1+\frac{1}{n}}{\left(1+\frac{2}{n}\right)^2}$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} n \cdot \frac{1+\frac{1}{n}}{\left(1+\frac{2}{n}\right)^2} = \infty > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

Example 2. *Test the convergence of the following series:*

(*i*) $\sum \dfrac{n^3 + a}{2^n + a}$ (*ii*) $\sum \dfrac{n!\,3^n}{n^n}$.

Sol. (*i*) Here $u_n = \dfrac{n^3+a}{2^n+a}$ $\therefore$ $u_{n+1} = \dfrac{(n+1)^3+a}{2^{n+1}+a}$

$$\frac{u_n}{u_{n+1}} = \frac{n^3+a}{(n+1)^3+a} \cdot \frac{2^{n+1}+a}{2^n+a}$$

$$= \frac{n^3\left(1+\frac{a}{n^3}\right)}{n^3\left[\left(1+\frac{1}{n}\right)^3+\frac{a}{n^3}\right]} \cdot \frac{2^{n+1}\left(1+\frac{a}{2^{n+1}}\right)}{2^n\left(1+\frac{a}{2^n}\right)} = \frac{1+\frac{a}{n^3}}{\left(1+\frac{1}{n}\right)^3+\frac{a}{n^3}} \cdot \frac{2\left(1+\frac{a}{2^{n+1}}\right)}{1+\frac{a}{2^n}}$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1+0}{1+0} \cdot 2 \cdot \frac{1+0}{1+0} = 2 > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

(*ii*) Here $u_n = \dfrac{n!\,3^n}{n^n}$ $\therefore$ $u_{n+1} = \dfrac{(n+1)!\,3^{n+1}}{(n+1)^{n+1}}$

$$\frac{u_n}{u_{n+1}} = \frac{n!\,3^n}{(n+1)!\,3^{n+1}} \cdot \frac{(n+1)^{n+1}}{n^n} = \frac{1}{3(n+1)} \cdot \frac{(n+1)^{n+1}}{n^n}$$

$$= \frac{1}{3} \cdot \frac{(n+1)^n}{n^n} = \frac{1}{3}\left(\frac{n+1}{n}\right)^n = \frac{1}{3}\left(1+\frac{1}{n}\right)^n$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^n = \frac{e}{3} < 1$$ ($\because$ $e =$

$\therefore$ Σu_n is divergent.

Example 3. *Discuss the convergence of the series:* $\sum \frac{n!}{n^n}$.

Sol. Here
$$u_n = \frac{n!}{n^n}$$
$$u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}} = \frac{(n+1)\,n!}{(n+1)^{n+1}} = \frac{n!}{(n+1)^n}$$
$$\frac{u_n}{u_{n+1}} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1 \qquad (\because\ e = 2.7)$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

Example 4. *Test the convergence of the following series:*

(i) $1 + \frac{1^2\cdot 2^2}{1\cdot 3\cdot 5} + \frac{1^2\cdot 2^2\cdot 3^2}{1\cdot 3\cdot 5\cdot 7\cdot 9} + \cdots \infty$ (ii) $\sum_{n=1}^{\infty} \frac{3\cdot 6\cdot 9 \cdots 3n}{4\cdot 7\cdot 10 \cdots (3n+1)} \cdot \frac{5^n}{3n+2}$

Sol. (i) Here
$$u_n = \frac{1^2\cdot 2^2\cdot 3^2 \ldots n^2}{1\cdot 3\cdot 5 \ldots (4n-5)(4n-3)}$$

[Note that two new factors are added with every term in the denominator and u_n has $(2n-1)$ factors in the denominator.]
$$u_{n+1} = \frac{1^2\cdot 2^2\cdot 3^2 \ldots n^2 (n+1)^2}{1\cdot 3\cdot 5 \ldots (4n-5)(4n-3)(4n-1)(4n+1)}$$
$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(4n-1)(4n+1)}{(n+1)^2} = \frac{16n^2-1}{(n+1)^2} = \frac{16-\frac{1}{n^2}}{\left(1+\frac{1}{n}\right)^2}$$
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{16-\frac{1}{n^2}}{\left(1+\frac{1}{n}\right)^2} = \frac{16-0}{(1+0)^2} = 16 > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

(ii) Here
$$u_n = \frac{3\cdot 6\cdot 9 \ldots 3n}{4\cdot 7\cdot 10 \ldots (3n+1)} \cdot \frac{5^n}{3n+2}$$
$$u_{n+1} = \frac{3\cdot 6\cdot 9 \ldots (3n)(3n+3)}{4\cdot 7\cdot 10 \ldots (3n+1)(3n+4)} \cdot \frac{5^{n+1}}{3n+5}$$
$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{3n+4}{3n+3}\cdot\frac{3n+5}{3n+2}\cdot\frac{1}{5} = \frac{\left(1+\frac{4}{3n}\right)\left(1+\frac{5}{3n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{3n}\right)}\cdot\frac{1}{5}$$
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{4}{3n}\right)\left(1+\frac{5}{3n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{3n}\right)}\cdot\frac{1}{5} = \frac{1\times 1}{1\times 1}\cdot\frac{1}{5} = \frac{1}{5} < 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is divergent.

Example 5. *Discuss the convergence of the series:* $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}}\, x^n \cdot (x > 0)$.

Sol. Here
$$u_n = \sqrt{\frac{n}{n^2+1}}\; x^n$$
$$\therefore \quad u_{n+1} = \sqrt{\frac{n+1}{(n+1)^2+1}} \cdot x^{n+1}$$
$$\frac{u_n}{u_{n+1}} = \sqrt{\frac{n}{n+1}\cdot\frac{n^2+2n+2}{n^2+1}}\cdot\frac{1}{x} = \sqrt{\frac{1}{1+\frac{1}{n}}\cdot\frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}}\cdot\frac{1}{x}$$
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}}\cdot\frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}}\cdot\frac{1}{x} = \frac{1}{x}$$

$\therefore$ By D'Alembert's Ratio Test, $\sum u_n$ converges if $\frac{1}{x} > 1$ *i.e.,* $x < 1$ and diverges if $\frac{1}{x} < 1$ *i.e.,* $x > 1$

When $x = 1$, the Ratio Test fails.

When $x = 1$, $u_n = \sqrt{\frac{n}{n^2+1}} = \sqrt{\frac{n}{n^2\left(1+\frac{1}{n^2}\right)}} = \frac{1}{\sqrt{n}}\cdot\frac{1}{\sqrt{1+\frac{1}{n^2}}}$

Take $v_n = \frac{1}{\sqrt{n}}$, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$

which is finite and non-zero.

$\therefore$ By Comparison Test, Σu_n and Σv_n converge or diverge together.

Since $\sum v_n = \sum \frac{1}{\sqrt{n}}$ is of the form $\sum \frac{1}{n^p}$ with $p = \frac{1}{2} < 1$

Σv_n diverges $\Rightarrow$ Σu_n diverges.

Hence the given series Σu_n converges if $x < 1$ and diverges if $x \geq 1$.

Example 6. *Examine the convergence or divergence of the following series for* $x > 0$:
$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots$$
(M.D.U. Dec. 2008)

Sol. Here $u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$ $\therefore$ $u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$
$$\frac{u_n}{u_{n+1}} = \frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}}\cdot\frac{1}{x^2} = \frac{1+\frac{2}{n}}{1+\frac{1}{n}}\cdot\sqrt{1+\frac{1}{n}}\cdot\frac{1}{x^2}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{1}{x^2}$$

By D'Alembert's Ratio Test, $\sum u_n$ converges if $\frac{1}{x^2}>1$, *i.e.*, $x^2<1$

and diverges if $\frac{1}{x^2}<1$ *i.e.* $x^2>1$.

When $x^2=1$, $u_n=\frac{1}{(n+1)\sqrt{n}}=\frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$

Take $v_n=\frac{1}{n^{3/2}}$, $\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}=1$ which is finite and non-zero. By comparison test Σu_n is convergent. $(\because p>1)$

Hence Σu_n is convergent if $x^2\le 1$ and divergent if $x^2>1$.

Example 7. *Examine the convergence or divergence of the following series:*

$$1+\frac{2}{5}x+\frac{6}{9}x^2+\frac{14}{17}x^3+\dots+\frac{2^{n+1}-2}{2^{n+1}+1}x^n+\cdots\ (x>0).$$

Sol. Here, leaving the first term, $u_n=\frac{2^{n+1}-2}{2^{n+1}+1}x^n$

$$\therefore\quad u_{n+1}=\frac{2^{n+2}-2}{2^{n+2}+1}x^{n+1}$$

$$\frac{u_n}{u_{n+1}}=\frac{2^{n+1}-2}{2^{n+1}+1}\cdot\frac{2^{n+2}+1}{2^{n+2}-2}\cdot\frac{1}{x}=\frac{2^{n+1}\left(1-\frac{2}{2^{n+1}}\right)}{2^{n+1}\left(1+\frac{1}{2^{n+1}}\right)}\cdot\frac{2^{n+2}\left(1+\frac{1}{2^{n+2}}\right)}{2^{n+2}\left(1-\frac{2}{2^{n+2}}\right)}\cdot\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{1-\frac{1}{2^n}}{1+\frac{1}{2^{n+1}}}\cdot\frac{1+\frac{1}{2^{n+2}}}{1-\frac{1}{2^{n+1}}}\cdot\frac{1}{x}=\frac{1}{x}$$

$\therefore$ By D'Alembert's Ratio Test, $\sum u_n$ converges if $\frac{1}{x}>1$, *i.e.*, $x<1$

and diverges if $\frac{1}{x}<1$, *i.e.*, $x>1$.

When $x=1$, $u_n=\frac{2^{n+1}-2}{2^{n+1}+1}=\frac{2^{n+1}\left(1-\frac{2}{2^{n+1}}\right)}{2^{n+1}\left(1+\frac{1}{2^{n+2}}\right)}=\frac{1-\frac{1}{2^n}}{1+\frac{1}{2^{n+1}}}$

$\lim_{n\to\infty}u_n=1\ne 0\ \Rightarrow\ \Sigma u_n$ does not converge. Being a series of + ve terms, it must diverge.

Hence Σu_n is convergent if $x<1$ and divergent if $x\ge 1$.

Example 8. *Test for convergence the positive term series:*

$$1+\frac{\alpha+1}{\beta+1}+\frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)}+\frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)}+\cdots\quad\text{(M.D.U. Dec. 2013)}$$

Sol. Leaving the first term $u_n=\frac{(\alpha+1)(2\alpha+1)\dots(n\alpha+1)}{(\beta+1)(2\beta+1)\dots(n\beta+1)}$

$$\therefore\quad u_{n+1}=\frac{(\alpha+1)(2\alpha+1)\dots(n\alpha+1)[(n+1)\alpha+1]}{(\beta+1)(2\beta+1)\dots(n\beta+1)[(n+1)\beta+1]}$$

$$\frac{u_n}{u_{n+1}}=\frac{(n+1)\beta+1}{(n+1)\alpha+1}=\frac{\left(1+\frac{1}{n}\right)\beta+\frac{1}{n}}{\left(1+\frac{1}{n}\right)\alpha+\frac{1}{n}}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\beta+\frac{1}{n}}{\left(1+\frac{1}{n}\right)\alpha+\frac{1}{n}}=\frac{\beta}{\alpha}$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n converges if $\frac{\beta}{\alpha}>1$ *i.e.*, $\beta>\alpha>0$

and diverges if $\frac{\beta}{\alpha}<1$ *i.e.*, $\beta<\alpha$ or $\alpha>\beta>0$

When $\alpha=\beta$, the Ratio Test fails.

When $\alpha=\beta$, $u_n=1$ $\lim_{n\to\infty}u_n=1\ne 0$

$\Rightarrow$ Σu_n does not converge. Being a series of +ve terms, it must diverge.

Hence the given series is convergent if $\beta>\alpha>0$ and divergent if $\alpha\ge\beta>0$.

EXERCISE 1.3

Discuss the convergence of the following series:

1. $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\cdots\infty$
2. $1+\frac{2!}{2^2}+\frac{3!}{3^3}+\frac{4!}{4^4}+\cdots\infty$
3. $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\cdots\infty$
4. $\frac{2}{1}+\frac{2\cdot5\cdot8}{1\cdot5\cdot9}+\frac{2\cdot5\cdot8\cdot11}{1\cdot5\cdot9\cdot13}+\cdots\infty$
5. $\frac{2!}{3}+\frac{3!}{3^2}+\frac{4!}{3^3}+\cdots\infty$
6. $\left(\frac{1}{3}\right)^2+\left(\frac{1\cdot2}{3\cdot5}\right)^2+\left(\frac{1\cdot2\cdot3}{3\cdot5\cdot7}\right)^2+\left(\frac{1\cdot2\cdot3\cdot4}{3\cdot5\cdot7\cdot9}\right)^2+\cdots\infty$
7. $\sum\frac{2^{n-1}}{3^n+1}$
8. $\sum\frac{1}{n!}$
9. $\frac{1}{1\cdot2\cdot3}+\frac{x}{4\cdot5\cdot6}+\frac{x^2}{7\cdot8\cdot9}+\cdots\infty\ (x>0)$.
10. $\sum\frac{x^n}{3^n\cdot n^2}$, $x>0$

11. $\sum \frac{x^n}{n}, x > 0$

12. $\sum \frac{n}{n^2+1} x^n, x > 0$

13. $\sum \left(\frac{n^2}{2^n} + \frac{1}{n^2}\right)$

14. $\sum \frac{n^3 - n + 1}{n!}$

15. $\sum \sqrt{\frac{n+1}{n^3+1}} \cdot x^n, x > 0$

16. $x + 2x^2 + 3x^3 + 4x^4 + \cdots \infty$ (M.D.U. Dec. 2009, Dec. 2014)

17. $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \cdots + \frac{x^n}{n^2+1} + \cdots \infty$

18. $\frac{x}{1.3} + \frac{x^2}{3.5} + \frac{x^3}{5.7} + \cdots \infty$

19. $x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \frac{15}{17}x^4 + \cdots + \frac{n^2-1}{n^2+1}x^n + \cdots \infty$

20. $\frac{x}{2\sqrt{3}} + \frac{x^2}{3\sqrt{4}} + \frac{x^3}{4\sqrt{5}} + \cdots \infty$

21. $\sum \frac{3^n - 2}{3^n + 1} \cdot x^{n-1}, x > 0$

22. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ (M.D.U. Dec. 2011)

23. $\sum_{n=1}^{\infty} \frac{x^n}{(2n)!}$

24. $\frac{x}{1+\sqrt{1}} + \frac{x^2}{2+\sqrt{2}} + \frac{x^3}{3+\sqrt{3}} + \cdots \infty$

25. $\sum_{n=1}^{\infty} \frac{n\,x^n}{(n+1)(n+2)}$

26. $\sum_{n=1}^{\infty} \frac{x^n}{(2n-1)^2 \cdot 2^n}$

27. $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \cdots \infty$ (M.D.U. Dec. 2011)

28. $1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \cdots \infty$

29. $\frac{4}{18} + \frac{4 \cdot 12}{18 \cdot 27} + \frac{4 \cdot 12 \cdot 20}{18 \cdot 27 \cdot 36} + \cdots \infty$

30. $\sum_{0}^{\infty} \frac{(3x+5)^n}{(n+1)!}$

31. $\sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1)\sqrt{n}}$ (M.D.U. Dec. 2012)

Answers

1. Convergent
2. Convergent
3. Convergent
4. Convergent
5. Divergent
6. Convergent
7. Convergent
8. Convergent
9. Convergent if $x \le 1$, Divergent if $x > 1$
10. Convergent for $x \le 3$, divergent for $x > 3$
11. Convergent for $x < 1$, divergent for $x \ge 1$
12. Convergent for $x < 1$, divergent for $x \ge 1$
13. Convergent
14. Convergent
15. Convergent for $x < 1$, divergent for $x \ge 1$
16. Convergent for $x < 1$, divergent for $x \ge 1$
17. Convergent for $x \le 1$, divergent for $x > 1$
18. Convergent for $x \le 1$, divergent for $x > 1$
19. Convergent for $x < 1$, divergent for $x \ge 1$
20. Convergent for $x \le 1$, divergent for $x > 1$
21. Convergent for $x < 1$, divergent for $x \ge 1$
22. Convergent
23. Convergent
24. Convergent for $x < 1$, divergent for $x \ge 1$
25. Convergent for $x < 1$, divergent for $x \ge 1$
26. Convergent for $x \le 2$, divergent for $x > 2$
27. Convergent
28. Convergent for $x < 1$, divergent for $x \ge 1$
29. Convergent
30. Convergent for all x
31. Convergent for $x \le 1$, divergent for $x > 1$.

1.25. CAUCHY'S ROOT TEST

Statement. *If Σu_n is a positive term series and $\lim_{n \to \infty} (u_n)^{1/n} = l$, then*

(i) Σu_n is convergent if $l < 1$ *(ii) Σu_n is divergent if $l > 1$.*

Note. If $l = 1$, the test fails *i.e.* no conclusion can be drawn about the convergence or divergence of the series. The series may converge, it may diverge.

Proof. Since $\lim_{n \to \infty} (u_n)^{1/n} = l$,

$\therefore$ Given $\varepsilon > 0$, however small, there exists a +ve integer m such that

$$| (u_n)^{1/n} - l \,| < \varepsilon \qquad \forall \quad n \ge m$$

$$\Rightarrow \qquad l - \varepsilon < (u_n)^{1/n} < l + \varepsilon \qquad \forall \quad n \ge m$$

$$\Rightarrow \qquad (l - \varepsilon)^n < u_n < (l + \varepsilon)^n \qquad \forall \quad n \ge m \qquad \ldots(1)$$

(i) When $l < 1$

Choose $\varepsilon > 0$ such that $l < l + \varepsilon < 1$

Put $l + \varepsilon = r$, then $0 < r < 1$

From (1), $\quad u_n < r^n \quad \forall \quad n \ge m$

Putting $n = m, m + 1, m + 2, \ldots$, we get $u_m < r^m$, $u_{m+1} < r^{m+1}$, $u_{m+2} < r^{m+2}$, ... and so on.

Adding $\quad u_m + u_{m+1} + u_{m+2} + \cdots + < r^m + r^{m+1} + r^{m+2} + \cdots$

$\Rightarrow$ each term of the given series Σu_n after leaving the first $(m - 1)$ terms, (*i.e.* a finite number of terms) is less than the corresponding term of a geometric series which is convergent ($\because$ its common ratio $r < 1$). Hence the given series is also convergent.

(ii) When $l > 1$

Choose $\varepsilon > 0$ such that $l - \varepsilon > 1$

Put $l - \varepsilon = R$, then $R > 1$

From (1), $\quad u_n > R^n \quad \forall \quad n \ge m$

Putting $n = m, m + 1, m + 2, \ldots$, we get $u_m > R^m$, $u_{m+1} > R^{m+1}$, $u_{m+2} > R^{m+2}$, ... and so on.

Adding $\quad u_m + u_{m+1} + u_{m+2} + \cdots > R^m + R^{m+1} + R^{m+2} + \cdots$

$\Rightarrow$ each term of the series Σu_n after leaving the first $(m - 1)$ terms, (*i.e.* a finite number of terms) is greater than the corresponding term of a geometric series which is divergent. ($\because$ its common ratio $R > 1$). Hence the given series is also divergent.

ILLUSTRATIVE EXAMPLES

Example 1. *Test the convergence of the following series:*

$$\sum \left(\frac{n}{n+1}\right)^{n^2} \quad or \quad \sum \left(1 + \frac{1}{n}\right)^{-n^2}. \qquad \text{(M.D.U. Dec. 2010)}$$

Sol. Here $\quad u_n = \left(\frac{n}{n+1}\right)^{n^2}$

$$\therefore \qquad (u_n)^{1/n} = \left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n} = \left(\frac{n}{n+1}\right)^{n} = \left(\frac{n+1}{n}\right)^{-n} = \left[\left(1 + \frac{1}{n}\right)^{n}\right]^{-1}$$

$$\lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{-1} = e^{-1} = \frac{1}{e} < 1 \qquad (\because\ e = 2.7)$$

$\therefore$ By Cauchy's Root Test, the given series Σu_n is convergent.

Example 2. *Examine the convergence of the series:*

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3} + \cdots$$ (Kerala 2010)

Sol. Here
$$u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$$

$$(u_n)^{1/n} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-1} = \left[\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$= \left[\left(1+\frac{1}{n}\right)^n \cdot \left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$\lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$= (e.1-1)^{-1} = \frac{1}{e-1} < 1 \qquad (\because\ e = 2.7)$$

$\therefore$ By Cauchy's Root Test, Σu_n is convergent.

Example 3. *Test the following series for convergence:*

(i) $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2 x^2+\left(\frac{4}{5}\right)^3 x^3+\cdots\ (x>0)$ (M.D.U. Dec. 2013) (ii) $\sum_{n=1}^{\infty}\frac{(n+1)^n\, x^n}{n^{n+1}}$. (M.D.U. Dec. 2012)

Sol. (i) Neglecting the first term, we have

$$u_n = \left(\frac{n+1}{n+2}\right)^n x^n$$

$$\Rightarrow \qquad (u_n)^{1/n} = \left(\frac{n+1}{n+2}\right)x = \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)x$$

$$\Rightarrow \qquad \lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)x = x$$

$\therefore$ By Cauchy's root test, the series is convergent if $x < 1$ and divergent if $x > 1$. The test fails when $x = 1$.

When $x = 1$,
$$u_n = \left(\frac{n+1}{n+2}\right)^n = \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)^n = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}$$

$$\Rightarrow \qquad \lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n}{\left[\left(1+\frac{2}{n}\right)^{n/2}\right]^2} = \frac{e}{e^2} = \frac{1}{e} \neq 0$$

Since $\lim_{n\to\infty} u_n \neq 0$, Σu_n cannot converge. Being a series of positive terms, it must diverge.

Hence the given series is convergent if $x < 1$ and divergent if $x \geq 1$.

(ii) Here
$$u_n = \frac{(n+1)^n \cdot x^n}{n^{n+1}} = \left[\frac{(n+1)\,x}{n}\right]^n \cdot \frac{1}{n}$$

$$\Rightarrow \qquad (u_n)^{1/n} = \frac{(n+1)\,x}{n}\cdot\frac{1}{n^{1/n}} = \left(1+\frac{1}{n}\right)x\cdot\frac{1}{n^{1/n}}$$

$$\Rightarrow \qquad \lim_{n\to\infty}(u_n)^{1/n} = \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)x\right]\left[\lim_{n\to\infty}\frac{1}{n^{1/n}}\right]$$

$$= (1+0)\,x\cdot\frac{1}{1} = x \qquad \left[\because\ \lim_{n\to\infty} n^{1/n} = 1\right]$$

$\therefore$ By Cauchy's root test, Σu_n is convergent if $x < 1$ and divergent if $x > 1$. The test fails when $x = 1$.

When $x = 1$,
$$u_n = \frac{(n+1)^n}{n^{n+1}} = \frac{1}{n}\cdot\frac{(n+1)^n}{n^n} = \frac{1}{n}\left(1+\frac{1}{n}\right)^n$$

Taking
$$v_n = \frac{1}{n}, \quad \frac{u_n}{v_n} = \left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \quad \text{which is finite and non-zero.}$$

$\therefore$ By comparison test, Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \sum \frac{1}{n}$ is of the form $\sum \frac{1}{n^p}$ with $p = 1$, Σv_n is divergent.

$\Rightarrow$ Σu_n is divergent.

Hence Σu_n is convergent if $x < 1$ and divergent if $x \geq 1$.

EXERCISE 1.4

Discuss the convergence of the following series:

1. $\sum\left(1+\frac{1}{n}\right)^{n^2}$ *(M.D.U. Dec. 2011)*
2. $\sum_{n=2}^{\infty}\frac{1}{(\log n)^n}$
3. $\sum\left(\frac{n+1}{3n}\right)^n$
4. $\sum\frac{(n-\log n)^n}{2^n \cdot n^n}$
5. $\sum\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$
6. $\sum\left(\frac{nx}{n+1}\right)^n$
7. $\sum 5^{-n-(-1)^n}$
8. $\sum\frac{(1+nx)^n}{n^n}$
9. $\sum\frac{(x+2)^n}{3^n \cdot n}$
10. $1+\frac{x}{2}+\frac{x^2}{3^2}+\frac{x^3}{4^3}+\cdots \ (x>0)$
11. $\frac{2}{1^2}x+\frac{3^2}{2^3}x^2+\frac{4^3}{3^4}x^3+\cdots+\frac{(n+1)^n}{n^{n+1}}x^n+\cdots$

Answers

1. Divergent
2. Convergent
3. Convergent
4. Convergent
5. Convergent
6. Convergent for $x<1$, divergent for $x\geq 1$
7. Convergent
8. Convergent for $x<1$, divergent for $x\geq 1$
9. Convergent for $x<1$, divergent for $x\geq 1$
10. Convergent
11. Convergent for $x<1$, divergent for $x\geq 1$.

1.26. RAABE'S TEST

Statement. *If Σu_n is a series of positive terms and $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=l$, then the series is convergent if $l>1$ and divergent if $l<1$.*

Proof. Let us compare the given series Σu_n with an auxiliary series $\Sigma v_n=\Sigma\frac{1}{n^p}$ which we know converges if $p>1$ and diverges if $p\leq 1$.

Now
$$\frac{v_n}{v_{n+1}}=\frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}}=\left(\frac{n+1}{n}\right)^p=\left(1+\frac{1}{n}\right)^p=1+\frac{p}{n}+\frac{p(p-1)}{2!}\cdot\frac{1}{n^2}+\cdots$$

Case I. *Let $\Sigma v_n=\Sigma\frac{1}{n^p}$ be convergent, so that $p>1$.*

Then Σu_n will also converge if $\frac{u_n}{u_{n+1}}>\frac{v_n}{v_{n+1}}$

or if
$$\frac{u_n}{u_{n+1}}>1+\frac{p}{n}+\frac{p(p-1)}{2!}\cdot\frac{1}{n^2}+\cdots$$

or if
$$n\left(\frac{u_n}{u_{n+1}}-1\right)>p+\frac{p(p-1)}{2!}\cdot\frac{1}{n}+\cdots$$

or if
$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)>p \quad\text{or}\quad \text{if } l>p$$

But p is itself greater than 1, $\therefore$ Σu_n is convergent if $l>1$.

Case II. *Let Σv_n be divergent, so that $p\leq 1$*

Then Σu_n will also diverge if $\frac{u_n}{u_{n+1}}<\frac{v_n}{v_{n+1}}$

or if
$$\frac{u_n}{u_{n+1}}<1+\frac{p}{n}+\frac{p(p-1)}{2!}\cdot\frac{1}{n^2}+\cdots$$

or if
$$n\left(\frac{u_n}{u_{n+1}}-1\right)<p+\frac{p(p-1)}{2!}\cdot\frac{1}{n}+\cdots$$

or if
$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)<p \quad\text{or}\quad \text{if } l<p.$$

But p itself ≤ 1. Thus the given series Σu_n diverges if $l<1$. This proves the result.

Note 1. If $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=1$, then Raabe's test fails.

Note 2. Raabe's test is used when D'Alembert's Ratio test fails and when in the ratio test, $\frac{u_n}{u_{n+1}}$ does not involve the number e. When $\frac{u_n}{u_{n+1}}$ involves e, we apply logarithmic test after the ratio test and not Raabe's test.

1.27. LOGARITHMIC TEST

Statement. *A positive term series Σu_n converges or diverges according as*
$$\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}}>1 \text{ or } <1.$$

Proof. Let us compare the given series Σu_n with an auxiliary series $\Sigma v_n=\Sigma\frac{1}{n^p}$ which we know converges if $p>1$ and diverges if $p\leq 1$.

Now
$$\frac{v_n}{v_{n+1}}=\frac{(n+1)^p}{n^p}=\left(1+\frac{1}{n}\right)^p$$

Case I. *Let Σv_n be convergent, so that $p>1$*

Then Σu_n will also be convergent if $\frac{u_n}{u_{n+1}}>\frac{v_n}{v_{n+1}}$

or if
$$\frac{u_n}{u_{n+1}}>\left(1+\frac{1}{n}\right)^p$$

or if
$$\log\frac{u_n}{u_{n+1}}>\log\left(1+\frac{1}{n}\right)^p=p\log\left(1+\frac{1}{n}\right)$$

or if $\log \dfrac{u_n}{u_{n+1}} > p\left[\dfrac{1}{n} - \dfrac{1}{2n^2} + \dfrac{1}{3n^3} - \ldots\right]$ $\left[\because \log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots\right]$

or if $n \log \dfrac{u_n}{u_{n+1}} > p\left[1 - \dfrac{1}{2n} + \dfrac{1}{3n^2} - \ldots\right]$ or if $\lim_{n\to\infty} n \log \dfrac{u_n}{u_{n+1}} > p$

or if $\lim_{n\to\infty} n \log \dfrac{u_n}{u_{n+1}} > 1$ $\quad | \because p > 1$ (given)

Case II. *Let Σv_n be divergent, so that $p \le 1$*

Then Σu_n also diverges if $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$ or if $\log \dfrac{u_n}{u_{n+1}} < \log \dfrac{v_n}{v_{n+1}}$

or if $\log \dfrac{u_n}{u_{n+1}} < \log\left(1+\dfrac{1}{n}\right)^p = p \log\left(1+\dfrac{1}{n}\right)$

or if $\log \dfrac{u_n}{u_{n+1}} < p\left(\dfrac{1}{n} - \dfrac{1}{2n^2} + \dfrac{1}{3n^3} - \ldots\right)$

or if $n \log \dfrac{u_n}{u_{n+1}} < p\left[1 - \dfrac{1}{2n} + \dfrac{1}{3n^2} - \ldots\right]$ or if $\lim_{n\to\infty} n \log \dfrac{u_n}{u_{n+1}} < p$

or if $\lim_{n\to\infty} n \log \dfrac{u_n}{u_{n+1}} < 1$ $\quad | \because p \le 1$

Thus the series Σu_n converges or diverges according as $\lim_{n\to\infty} n \log \dfrac{u_n}{u_{n+1}} > 1$ or < 1.

Note 1. The test fails if $\lim_{n\to\infty} n \log \dfrac{u_n}{u_{n+1}} = 1$.

Note 2. The test is applied after the failure of Ratio test and generally when in Ratio test, $\dfrac{u_n}{u_{n+1}}$ involves 'e'.

1.28. GAUSS TEST

Statement. *If for the series Σu_n of positive terms, $\dfrac{u_n}{u_{n+1}}$ can be expanded in the form*

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)$$

then Σu_n converges if $\lambda > 1$ and diverges if $\lambda \le 1$. (M.D.U. Dec. 2010)

Note. The test never fails as we know that the series diverges for $\lambda = 1$. Moreover the test is applied after the failure of Ratio test and when it is possible to expand $\dfrac{u_n}{u_{n+1}}$ in powers of $\dfrac{1}{n}$ by Binomial Theorem or by any other method.

ILLUSTRATIVE EXAMPLES

Example 1. *Discuss the convergence of the series:* $\dfrac{1}{2} + \dfrac{1\cdot 3}{2\cdot 4} + \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} + \ldots$

Sol. Here $\quad u_n = \dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}$.

$\therefore \quad u_{n+1} = \dfrac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{2\cdot 4\cdot 6\cdots(2n)(2n+2)}$

$\therefore \quad \dfrac{u_n}{u_{n+1}} = \dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n} \times \dfrac{2\cdot 4\cdot 6\cdots 2n(2n+2)}{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}$

$$= \frac{2n+2}{2n+1} = \frac{1+\frac{1}{n}}{1+\frac{1}{2n}} \to 1 \text{ as } n \to \infty.$$

$\therefore$ D'Alembert's Ratio test fails.

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] = n\left[\frac{2n+2}{2n+1} - 1\right] = \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$

$\therefore \quad \lim_{n\to\infty} n\left[\dfrac{u_n}{u_{n+1}} - 1\right] = \dfrac{1}{2} < 1.$

$\therefore$ By Raabe's test, Σu_n diverges.

Example 2. *Discuss the convergence of the series:* $\dfrac{1^2}{2^2} + \dfrac{1^2\cdot 3^2}{2^2\cdot 4^2} + \dfrac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2} + \ldots\ldots$

(M.D.U. Dec. 2009, Dec. 2010)

Sol. Here $\quad u_n = \dfrac{1^2\cdot 3^2\cdot 5^2\ldots(2n-1)^2}{2^2\cdot 4^2\cdot 6^2\ldots(2n)^2}$

and $\quad u_{n+1} = \dfrac{1^2\cdot 3^2\cdot 5^2\ldots(2n-1)^2(2n+1)^2}{2^2\cdot 4^2\cdot 6^2\ldots(2n)^2(2n+2)^2}$

$\therefore \quad \dfrac{u_n}{u_{n+1}} = \dfrac{(2n+2)^2}{(2n+1)^2} = \dfrac{4n^2\left(1+\frac{1}{n}\right)^2}{4n^2\left(1+\frac{1}{2n}\right)^2} = \dfrac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2}$

$\therefore \quad \lim_{n\to\infty} \dfrac{u_n}{u_{n+1}} = 1.$ Hence the ratio test fails

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{(2n+2)^2}{(2n+1)^2} - 1\right]$$

$$= n\left[\frac{4n^2 + 8n + 4 - (4n^2 + 4n + 1)}{(2n+1)^2}\right] = n\,\frac{(4n+3)}{(2n+1)^2} = \frac{4n^2+3n}{(2n+1)^2}$$

$$= \frac{1+\frac{3}{4n}}{\left(1+\frac{1}{2n}\right)^2} \to 1 \text{ as } n \to \infty$$

$\therefore$ Raabe's test also fails.

When D'Alembert ratio test fails, we can directly apply Gauss test.

Now, $$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2} = \left(1+\frac{1}{n}\right)^2\left(1+\frac{1}{2n}\right)^{-2}$$

$$= \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1-\frac{2}{2n}+\frac{3}{4n^2}-\dots\right) = 1+\frac{1}{n}+\frac{1}{n^2}\left(1-2+\frac{3}{4}\right)+\dots$$

$$= 1+\frac{1}{n}-\frac{1}{4n^2}+\dots = 1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)$$

Comparing it with $\frac{u_n}{u_{n+1}} = 1+\frac{\lambda}{n}+O\left(\frac{1}{n^2}\right)$

we have $\lambda = 1$. Thus by Gauss test, the series Σu_n diverges.

Example 3. *Discuss the convergence of the series:* $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}+\dots$ $(x > 0)$.

(M.D.U. May 2008, Dec. 2012)

Sol. Neglecting the first term, we have $u_n = \frac{1\cdot 3\cdot 5\dots(2n-1)}{2\cdot 4\cdot 6\dots(2n)}\cdot\frac{x^{2n+1}}{2n+1}$

and $$u_{n+1} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\frac{x^{2n+1}}{2n+1}$$

$\therefore$ $$\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}\cdot\frac{2n+3}{2n+1}\cdot\frac{1}{x^2} = \frac{2n\left(1+\frac{1}{n}\right)\cdot 2n\left(1+\frac{3}{2n}\right)}{2n\left(1+\frac{1}{2n}\right)\cdot 2n\left(1+\frac{1}{2n}\right)}\cdot\frac{1}{x^2}$$

$$= \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{2n}\right)^2}\cdot\frac{1}{x^2}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{2n}\right)^2}\cdot\frac{1}{x^2} = \frac{1}{x^2}$$

$\therefore$ By Ratio Test, Σu_n is convergent if $\frac{1}{x^2} > 1$ *i.e.,* $x^2 < 1$ and divergent if $\frac{1}{x^2} < 1$ *i.e.,* $x^2 > 1$.

If $x^2 = 1$, then Ratio Test fails.

When $x^2 = 1$, we have $\frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2} = \frac{4n^2+10n+6}{4n^2+4n+1}$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{4n^2+10n+6}{4n^2+4n+1}-1\right)$$

$$= \lim_{n\to\infty}\frac{6n^2+5n}{4n^2+4n+1} = \lim_{n\to\infty}\frac{6+\frac{5}{n}}{4+\frac{4}{n}+\frac{1}{n^2}} = \frac{6}{4} = \frac{3}{2} > 1.$$

$\therefore$ By Raabe's Test, the series converges.

Hence Σu_n is convergent if $x^2 \le 1$ *i.e.,* $0 < x \le 1$ and divergent if $x^2 > 1$ *i.e.,* $x > 1$.

Example 4. *Discuss the convergence of the series:* $1+\frac{x}{2}+\frac{2!}{3^2}x^2+\frac{3!}{4^3}x^3+\frac{4!}{5^4}x^4+\dots$

Sol. Neglecting the first term, we have

$$u_n = \frac{n!}{(n+1)^n}x^n \quad\text{and}\quad u_{n+1} = \frac{(n+1)!}{(n+2)^{n+1}}\cdot x^{n+1}.$$

$\therefore$ $$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{n!}{(n+1)^n}\cdot\frac{(n+2)^{n+1}}{(n+1)!}\cdot\frac{1}{x}$$

$$= \lim_{n\to\infty}\frac{1}{n^n\left(1+\frac{1}{n}\right)^n}\cdot\frac{n^{n+1}\left(1+\frac{2}{n}\right)^{n+1}}{(n+1)}\cdot\frac{1}{x}$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)^n\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}\cdot\frac{1}{x} = \frac{e^2}{e}\cdot\frac{1}{x} = \frac{e}{x}$$

$$\left[\because \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = \lim_{n\to\infty}\left[\left(1+\frac{a}{n}\right)^{n/a}\right]^a = e^a\right]$$

$\therefore$ By D'Alembert's ratio test, the series converges if $\frac{e}{x} > 1$ or if $x < e$ and diverges if $\frac{e}{x} < 1$ or if $x > e$.

If $x = e$, the ratio test fails, $\because$ $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = 1$.

Now when $x = e$.

$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{2}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}}\cdot\frac{1}{e}.$$

Since the expression $\frac{u_n}{u_{n+1}}$ involves the number e, so we do not apply Raabe's test but apply logarithmic test.

$\therefore$ $$\log\frac{u_n}{u_{n+1}} = (n+1)\log\left(1+\frac{2}{n}\right)-(n+1)\log\left(1+\frac{1}{n}\right)-\log e$$

$$= (n+1)\left[\log\left(1+\frac{2}{n}\right) - \log\left(1+\frac{1}{n}\right)\right] - 1$$

$$= (n+1)\left[\left(\frac{2}{n} - \frac{1}{2}\cdot\frac{4}{n^2} + \frac{1}{3}\cdot\frac{8}{n^3} - \cdots\right) - \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \cdots\right)\right] - 1$$

$$= (n+1)\left[\frac{1}{n} - \frac{3}{2n^2} + \cdots\right] - 1$$

$$= 1 - \frac{3}{2n} + \frac{1}{n} - \frac{3}{2n^2} + \cdots - 1 = -\frac{1}{2n} - \frac{3}{2n^2} + \cdots$$

$$\therefore \quad \lim_{n\to\infty} n \log\frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n\left[-\frac{1}{2n} - \frac{3}{2n^2} + \cdots\right] = \lim_{n\to\infty}\left(-\frac{1}{2} - \frac{3}{2n} + \cdots\right) = -\frac{1}{2} < 1$$

$\therefore$ By log test, the series *diverges*.

Hence the given series Σu_n converges if $x < e$ and diverges if $x \geq e$.

Example 5. *Test the following series for convergence:*

$$\frac{a+x}{1!} + \frac{(a+2x)^2}{2!} + \frac{(a+3x)^3}{3!} + \cdots \infty.$$

Sol. Here $\quad u_n = \dfrac{(a+nx)^n}{n!} \quad \therefore \quad u_{n+1} = \dfrac{[a+(n+1)x]^{n+1}}{(n+1)!}$

$$\frac{u_n}{u_{n+1}} = \frac{(a+nx)^n}{[a+(n+1)x]^{n+1}}\cdot\frac{(n+1)!}{n!} = \frac{n^n x^n\left(1+\frac{a}{nx}\right)^n}{(n+1)^{n+1}x^{n+1}\left[1+\frac{a}{(n+1)x}\right]^{n+1}}\cdot(n+1)$$

$$= \frac{n^n\cdot\left(1+\frac{a}{nx}\right)^n}{(n+1)^n\left[1+\frac{a}{(n+1)x}\right]^{n+1}}\cdot\frac{1}{x} = \frac{\left(1+\frac{a}{nx}\right)^n}{\left(1+\frac{1}{n}\right)^n\left[1+\frac{a}{(n+1)x}\right]^{n+1}}\cdot\frac{1}{x} \quad \ldots(1)$$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left[\left(1+\frac{a}{nx}\right)^{\frac{nx}{a}}\right]^{a/x}}{\left(1+\frac{1}{n}\right)^n\left[\left\{1+\frac{a}{(n+1)x}\right\}^{\frac{(n+1)x}{a}}\right]^{a/x}}\cdot\frac{1}{x} = \frac{e^{a/x}}{e\cdot e^{a/x}}\cdot\frac{1}{x} = \frac{1}{ex}$$

$\therefore$ By D'Alembert's ratio test, the series converges if $\dfrac{1}{ex} > 1$, *i.e.*, if $x < \dfrac{1}{e}$ and diverges if $\dfrac{1}{ex} < 1$, *i.e.*, if $x > \dfrac{1}{e}$.

The test fails when $\quad x = \dfrac{1}{e}$.

When $\quad x = \dfrac{1}{e}$, we have from (1)

$$\frac{u_n}{u_{n+1}} = \frac{e\left(1+\frac{ae}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{ae}{n+1}\right)^{n+1}}$$

Since $\dfrac{u_n}{u_{n+1}}$ involves e, we use logarithmic test after the failure of ratio test.

$$\log\frac{u_n}{u_{n+1}} = \log\left[\frac{e\left(1+\frac{ae}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{ae}{n+1}\right)^{n+1}}\right]$$

$$= \log e + n\log\left(1+\frac{ae}{n}\right) - n\log\left(1+\frac{1}{n}\right) - (n+1)\log\left(1+\frac{ae}{n+1}\right)$$

$$= 1 + n\left(\frac{ae}{n} - \frac{a^2e^2}{2n^2} + \frac{a^3e^3}{3n^3} - \cdots\right)$$

$$- n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right) - (n+1)\left[\frac{ae}{n+1} - \frac{a^2e^2}{2(n+1)^2} + \frac{a^3e^3}{3(n+1)^3} - \cdots\right]$$

$$= 1 + \left(ae - \frac{a^2e^2}{2n} + \frac{a^3e^3}{3n^2} - \cdots\right) - \left(1 - \frac{1}{2n} + \frac{1}{3n^2} - \cdots\right)$$

$$- \left[ae - \frac{a^2e^2}{2(n+1)} + \frac{a^3e^3}{3(n+1)^2} - \cdots\right]$$

$$= \left(-\frac{a^2e^2}{2n} + \frac{a^3e^3}{3n^2} - \cdots\right) + \left(\frac{1}{2n} - \frac{1}{3n^2} + \cdots\right)$$

$$+ \left[\frac{a^2e^2}{2(n+1)} - \frac{a^3e^3}{3(n+1)^2} + \cdots\right]$$

$$\Rightarrow n\log\frac{u_n}{u_{n+1}} = \left(-\frac{a^2e^2}{2} + \frac{a^3e^3}{3n} - \cdots\right) + \left(\frac{1}{2} - \frac{1}{3n} + \cdots\right)$$

$$+ \left[\frac{a^2e^2}{2}\cdot\frac{n}{n+1} - \frac{a^3e^3}{3(n+1)}\cdot\frac{n}{n+1} + \cdots\right]$$

$$= \left(-\frac{a^2e^2}{2} + \frac{a^3e^3}{3n} - \cdots\right) + \left(\frac{1}{2} - \frac{1}{3n} + \cdots\right)$$

$$+ \left[\frac{a^2e^2}{2}\cdot\frac{1}{1+\frac{1}{n}} - \frac{a^3e^3}{3(n+1)}\cdot\frac{1}{1+\frac{1}{n}} + \cdots\right]$$

$$\Rightarrow \quad \lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = \left(-\frac{a^2e^2}{2}\right) + \left(\frac{1}{2}\right) + \left(\frac{a^2e^2}{2}\right) = \frac{1}{2} < 1$$

$\therefore$ By Logarithmic Test, the series is divergent.

Hence the given series converges for $x < \dfrac{1}{e}$ and diverges for $x \geq \dfrac{1}{e}$.

Example 6. *Test the following series for convergence:*

$$x^2(\log 2)^q + x^3(\log 3)^q + x^4(\log 4)^q + \cdots \infty.$$

Sol. Here $u_n = x^{n+1}[\log(n+1)]^q \quad \therefore \quad u_{n+1} = x^{n+2}[\log(n+2)]^q$

$$\frac{u_n}{u_{n+1}} = \frac{1}{x}\cdot\left[\frac{\log(n+1)}{\log(n+2)}\right]^q = \frac{1}{x}\left[\frac{\log n\left(1+\frac{1}{n}\right)}{\log n\left(1+\frac{2}{n}\right)}\right]^q$$

$$= \frac{1}{x}\left[\frac{\log n + \log\left(1+\frac{1}{n}\right)}{\log n + \log\left(1+\frac{2}{n}\right)}\right]^q = \frac{1}{x}\left[\frac{\log n + \frac{1}{n} - \frac{1}{2n^2} + \cdots}{\log n + \frac{2}{n} - \frac{1}{2}\left(\frac{2}{n}\right)^2 + \cdots}\right]^q$$

$$= \frac{1}{x}\left[\frac{1 + \frac{1}{n\log n} - \frac{1}{2n^2\log n} + \cdots}{1 + \frac{2}{n\log n} - \frac{2}{n^2\log n} + \cdots}\right]^q$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{1}{x}\left[\frac{1 + \frac{1}{n\log n} - \frac{1}{2n^2\log n} + \cdots}{1 + \frac{2}{n\log n} - \frac{2}{n^2\log n} + \cdots}\right]^q = \frac{1}{x}$$

$\therefore$ By Ratio Test, the series Σu_n converges if $\frac{1}{x} > 1$, *i.e.*, if $x < 1$ and diverges if $\frac{1}{x} <$ 1 *i.e.*, if $x > 1$.

Ratio test fails when $x = 1$.

When $x = 1$, from (1) we have

$$\frac{u_n}{u_{n+1}} = \left[\frac{1 + \frac{1}{n\log n} - \frac{1}{2n^2\log n} + \cdots}{1 + \frac{2}{n\log n} - \frac{2}{n^2\log n} + \cdots}\right]^q$$

$$= \left[1 + \left(\frac{1}{n\log n} - \frac{1}{2n^2\log n} + \cdots\right)\right]^q\left[1 + \left(\frac{2}{n\log n} - \frac{2}{n^2\log n} + \cdots\right)\right]^{-q}$$

Expanding by Binomial Theorem

$$= \left[1 + q\left(\frac{1}{n\log n} - \frac{1}{2n^2\log n} + \cdots\right) + \cdots\right]\left[1 - q\left(\frac{2}{n\log n} - \frac{2}{n^2\log n} + \cdots\right) + \cdots\right]$$

$$= 1 + q\left(\frac{1}{n\log n} - \frac{2}{n\log n}\right) + \cdots$$

$\Rightarrow \quad \frac{u_n}{u_{n+1}} - 1 = -\frac{q}{n\log n}$ + (terms containing squares and higher power of n and $\log n$ in the denominator)

$\Rightarrow \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = -\frac{q}{\log n}$ + (terms containing n, n^2, ..., $(\log n)^2$, $(\log n)^3$ ... in the denominator)

$$\Rightarrow \quad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = 0 < 1$$

$\therefore$ By Raabe's test, the series Σu_n is divergent.

Hence the given series Σu_n is convergent for $x < 1$ and divergent for $x \geq 1$.

Example 7. *Discuss the convergence of the hyper-geometric series:*

$$1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + \cdots$$

Sol. Neglecting the first term,

$$u_n = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{1\cdot 2\cdot 3\cdots n\cdot\gamma(\gamma+1)\cdots(\gamma+n-1)}\cdot x^n$$

$$u_{n+1} = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)(\alpha+n)\beta(\beta+1)\cdots(\beta+n-1)(\beta+n)}{1\cdot 2\cdot 3\cdots n(n+1)\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1)(\gamma+n)}\cdot x^{n+1}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}\cdot\frac{1}{x} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}\cdot\frac{1}{x}$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

$\therefore$ By D'Alembert's Ratio test the series Σu_n converges if $\frac{1}{x} > 1$ *i.e.*, if $x < 1$ and diverges if $\frac{1}{x} < 1$ or if $x > 1$.

If $\quad x = 1$, $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = 1 \quad \therefore$ Ratio test fails.

Putting $x = 1$ in $\frac{u_n}{u_{n+1}}$, we have $\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left(1+\frac{\alpha}{n}\right)^{-1}\left(1+\frac{\beta}{n}\right)^{-1} \qquad \textbf{[Expand by Binomial Theorem]}$$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left(1-\frac{\alpha}{n}+\frac{\alpha^2}{n^2}+\cdots\right)\left(1-\frac{\beta}{n}+\frac{\beta^2}{n^2}\cdots\right)$$

$$= \left(1+\frac{1}{n}+\frac{\gamma}{n}+\frac{\gamma}{n^2}\right)\left(1-\frac{\alpha}{n}-\frac{\beta}{n}+\frac{\alpha\beta}{n^2}+\frac{\alpha^2}{n^2}+\frac{\beta^2}{n^2}\cdots\right)$$

$$= 1 + \frac{1}{n}(1+\gamma-\alpha-\beta) + O\left(\frac{1}{n^2}\right).$$

$\therefore$ By Gauss test, the series Σu_n converges if $1+\gamma-\alpha-\beta > 1$ *i.e.*, if $\gamma > \alpha+\beta$ and diverges if $1+\gamma-\alpha-\beta \le 1$ *i.e.*, if $\gamma \le \alpha+\beta$.

Thus the given series *converges if $x < 1$ and diverges if $x > 1$. If $x = 1$, then the series converges if $\gamma > \alpha+\beta$ and diverges if $\gamma \le \alpha+\beta$.*

EXERCISE 1.5

Discuss the convergence of the following series:

1. $1 + \frac{2^2}{3^2} + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2} + \cdots \infty$

2. $\frac{1^2}{4^2} + \frac{1^2 \cdot 5^2}{4^2 \cdot 8^2} + \frac{1^2 \cdot 5^2 \cdot 9^2}{4^2 \cdot 8^2 \cdot 12^2} + \cdots \infty$

3. $1 + \frac{3}{7}x + \frac{3 \cdot 6}{7 \cdot 10}x^2 + \frac{3 \cdot 6 \cdot 9}{7 \cdot 10 \cdot 13}x^3 + \cdots \infty$

4. $1 + \frac{2}{1} \cdot \frac{1}{2} + \frac{2 \cdot 4}{1 \cdot 3} \cdot \frac{1}{3} + \frac{2 \cdot 4 \cdot 6}{1 \cdot 3 \cdot 5} \cdot \frac{1}{4} + \cdots \infty$

5. $\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}x + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}x^2 + \ldots \infty$

6. $x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \frac{5^5 x^5}{5!} + \cdots \infty$

(*M.D.U. May 2009, Dec. 2010, Dec. 2013*)

7. $1 + \frac{2x}{2!} + \frac{3^2 x^2}{3!} + \frac{4^3 x^3}{4!} + \cdots \infty$

8. $1 + \frac{1}{2} \cdot \frac{x^2}{4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^4}{8} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \cdot \frac{x^6}{12} + \cdots \infty$

9. $1^p + \left(\frac{1}{2}\right)^p + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^p + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^p + \cdots \infty$

10. $\frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \cdots \infty$

11. $\frac{a}{b} + \frac{a(a+d)}{b(b+d)}x + \frac{a(a+d)(a+2d)}{b(b+d)(b+2d)}x^2 + \cdots \infty \quad (a > 0, b > 0, x > 0)$

12. $\sum \frac{n!}{x(x+1)(x+2)\ldots(x+n-1)}$

13. $1 + \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \cdots \infty \quad (x > 0)$

14. $\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} + \cdots \infty \quad (x > 0)$

15. $\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \frac{x^4}{4 \cdot 5} + \cdots \infty$

16. $1 + \frac{a}{1!} + \frac{a(a+1)}{2!} + \frac{a(a+1)(a+2)}{3!} + \cdots \infty \quad (a > 0)$

17. $\sum \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots (2n)} \cdot \frac{x^{2n}}{2n}, \ x > 0$

18. $\sum \frac{(n!)^2}{(2n)!} x^{2n}, \ x > 0.$ (*M.D.U. Dec. 2010*)

19. $\sum \frac{4 \cdot 7 \cdot 10 \ldots (3n+1)}{1 \cdot 2 \cdot 3 \ldots n} x^n$

20. $1 + \frac{(1!)^2}{2!}x^2 + \frac{(2!)^2}{4!}x^4 + \frac{(3!)^2}{6!}x^6 + \cdots \infty \quad (x > 0)$

21. $\frac{1}{2}x + x^2 + \frac{9}{8}x^3 + x^4 + \frac{25}{32}x^5 + \cdots \infty$

22. $\frac{x^2}{2 \log 2} + \frac{x^3}{3 \log 3} + \frac{x^4}{4 \log 4} + \cdots \infty$

(*M.D.U. Dec. 2011, Dec. 2014*)

23. $1 + \frac{2}{3}x + \frac{2 \cdot 3}{3 \cdot 5}x^2 + \frac{2 \cdot 3 \cdot 4}{3 \cdot 5 \cdot 7}x^3 + \cdots \infty$

24. $1 + \frac{2^2}{3^2}x + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2}x^2 + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2}x^3 + \cdots \infty$

25. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$

26. $1 + \frac{a}{b}x + \frac{a(a+1)}{b(b+1)}x^2 + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}x^3 + \cdots \infty.$

27. $\sum \frac{n^2(n+1)^2}{n!}$

28. $x^2 + \frac{2^2}{3 \cdot 4}x^4 + \frac{2^2 \cdot 4^2}{3 \cdot 4 \cdot 5 \cdot 6}x^6 + \frac{2^2 \cdot 4^2 \cdot 6^2}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}x^8 + \cdots \infty \quad (x > 0)$

29. $\frac{1}{(\log 2)^k} + \frac{1}{(\log 3)^k} + \frac{1}{(\log 4)^k} + \cdots \infty.$

Answers

1. Divergent
2. Convergent
3. Convergent for $x \le 1$, divergent for $x > 1$
4. Divergent
5. Convergent for $x < 1$, divergent for $x \ge 1$
6. Convergent for $x < \frac{1}{e}$, divergent for $x \ge \frac{1}{e}$
7. Convergent for $x \le \frac{1}{e}$, divergent for $x > \frac{1}{e}$
8. Convergent for $x^2 \le 1$, divergent for $x^2 > 1$

   $\left[\textbf{Hint.} \text{ Here } u_n = \frac{1 \cdot 3 \cdot 5 \cdots (4n-3)}{2 \cdot 4 \cdot 6 \cdots (4n-2)} \cdot \frac{x^{2n}}{4n} \text{ and } u_{n+1} = \frac{1 \cdot 3 \cdot 5 \cdots (4n-3)(4n-1)(4n+1)}{2 \cdot 4 \cdot 6 \cdots (4n-2)(4n)(4n+2)} \cdot \frac{x^{2n+2}}{4n+4}\right]$
9. Convergent for $p > 2$, divergent for $p \le 2$
10. Convergent for $b > a + 1$, divergent for $b \le a + 1$
11. Convergent for $x < 1$ or $x = 1$ and $b > a + d$, divergent for $x > 1$ or $x = 1$ and $b \le a + d$
12. Convergent for $x > 2$, divergent for $x \le 2$
13. Convergent for $x < 1$, divergent for $x \ge 1$
14. Convergent for $x \le 1$, divergent for $x > 1$
15. Convergent for $x \le 1$, divergent for $x > 1$
16. Divergent
17. Convergent for $x^2 \le 1$, divergent for $x^2 > 1$
18. Convergent for $x^2 < 4$, divergent for $x^2 \ge 4$
19. Convergent for $x < \frac{1}{3}$, divergent for $x \ge \frac{1}{3}$
20. Convergent for $x^2 < 4$, divergent for $x^2 \ge 4$
21. Convergent for $x < 2$, divergent for $x \ge 2$
22. Convergent for $x < 1$, divergent for $x \ge 1$
23. Convergent for $x < 2$, divergent for $x \ge 2$
24. Convergent for $x < 1$, divergent for $x \ge 1$
25. Convergent for $x < \frac{1}{4}$, divergent for $x \ge \frac{1}{4}$
26. Convergent for $x < 1$, divergent for $x > 1$, when $x = 1$, convergent for $b - a > 1$, divergent for $b - a \le 1$
27. Convergent
28. Convergent if $x^2 \le 1$, divergent if $x^2 > 1$
29. Divergent.

1.29. CAUCHY'S INTEGRAL TEST

Statement. *If for $x \ge 1$, $f(x)$ is a non-negative, monotonic decreasing function of x such that $f(n) = u_n$ for all positive integral values of n, then the series Σu_n and the integral $\int_1^\infty f(x)\,dx$ converge or diverge together.*

Proof. Let r be a +ve integer. Choose x such that $r + 1 \ge x \ge r \ge 1$

Since $f(x)$ is a monotonic decreasing function of x.

$$\therefore \qquad f(r+1) \le f(x) \le f(r) \ \Rightarrow\ u_{r+1} \le f(x) \le u_r \qquad [\because\ f(n) = u_n,\ n \in N]$$