CHAPTER

PERMUTATIONS AND COMBINATIONS

4

4.1. INTRODUCTION

We often come across the situations, where we have to select or arrange things out of a given number of things. In such situations, we can determine the number of possible ways of the desired selection with the help of *Permutations* and *Combinations* which we shall be studying in this chapter. To begin with, we shall introduce a notation $n\,!$ (n factorial) which is very helpful in the study and calculations of permutations and combinations.

4.2. FACTORIAL NOTATION

4.2.1. Factorial ($n\,!$)

The continued product of first n natural numbers (*i.e., product of* 1, 2, 3, ..., n) *is denoted by symbol* $n\,!$ *and read as factorial* n.

For example, $5\,! = 1.\,2.\,3.\,4.\,5 = 120$

In general, $n\,! = 1.\,2.\,3.\,4 \ldots\ldots (n-1)\,.\,n$

4.2.2. To prove that $n\,! = n\,(n-1)\,!$

$$n\,! = 1.2.3.4 \ldots (n-1).n$$
$$= n\,.\,[1.2.3 \ldots\ldots (n-1)] = n\,(n-1)\,!$$

Note :

1. *We define* $1\,! = 1$ *and* $0\,! = 1$.
2. $n\,!$ *is not defined when* n *is a negative integer or a fraction.*

SOLVED EXAMPLES

Example 1. ***Evaluate :***

(*i*) $\dfrac{30\,!}{28\,!}$ (*ii*) $\dfrac{9\,!}{5\,!\,3\,!}$ (*iii*) $\dfrac{12\,! - 10\,!}{9\,!}$ (*iv*) $\dfrac{20\,!}{18\,!\,(20-18)\,!}$

Solution. (*i*) $\dfrac{30!}{28!} = \dfrac{30\times 29\times 28!}{28!} = 30 \times 29 = \mathbf{870.}$

(*ii*) $\dfrac{9!}{5!\,3!} = \dfrac{9\times 8\times 7\times 6\times 5!}{5!\times 3\times 2\times 1} = \mathbf{504.}$

(*iii*) $\dfrac{12!-10!}{9!} = \dfrac{12\times 11\times 10!-10!}{9!} = \dfrac{10!}{9!}(132-1) = 10 \times 131 = \mathbf{1310.}$

(*iv*) $\dfrac{20!}{18!\,(20-18)!} = \dfrac{20!}{18!\,2!} = \dfrac{20\times 19\times 18!}{18!.\,(2\times 1)} = \dfrac{380}{2} = \mathbf{190.}$

Example 2. ***Evaluate :***

(i) $\dfrac{n!}{(n-r)!}$, ***when*** $n = 10, r = 4$ **(ii)** $\dfrac{n!}{r!\,(n-r)!}$, ***when*** $n = 5, r = 2$

(iii) $\dfrac{n!}{(n-r)!}$, ***when*** $r = 3$.

Solution. (*i*) Putting $n = 10, r = 4$, we have

$$\frac{n!}{(n-r)!} = \frac{10!}{(10-4)!} = \frac{10!}{6!} = \frac{10\times 9\times 8\times 7\times 6!}{6!}$$

$$= 10 \times 9 \times 8 \times 7 = \mathbf{5040.}$$

(*ii*) Putting $n = 5, r = 2$, we have

$$\frac{n!}{r!\,(n-r)!} = \frac{5!}{2!\,(5-2)!} = \frac{5!}{2!\,3!} = \frac{5\times 4\times 3!}{2\times 1\times 3!} = 5 \times 2 = \mathbf{10.}$$

(*iii*) Putting $r = 3$, $\dfrac{n!}{(n-r)!} = \dfrac{n!}{(n-3)!} = \dfrac{n(n-1)(n-2)(n-3)!}{(n-3)!}$

$$= n(n-1)(n-2) = n(n^2-3n+2)$$

$$= \mathbf{n^3 - 3n^2 + 2n.}$$

Example 3. ***If*** $\dfrac{1}{6!} + \dfrac{1}{7!} = \dfrac{x}{8!}$, ***find*** x.

Solution. We have $\dfrac{1}{6!} + \dfrac{1}{7!} = \dfrac{x}{8!}$

$$\Rightarrow \quad \frac{1}{6!} + \frac{1}{7\times 6!} = \frac{x}{8\times 7\times 6!}$$

$$\Rightarrow \quad 1 + \frac{1}{7} = \frac{x}{8\times 7} \quad \Rightarrow \quad \frac{7+1}{7} = \frac{x}{8\times 7}$$

$$\Rightarrow \quad 8 = \frac{x}{8} \quad \Rightarrow \quad x = 8 \times 8 = \mathbf{64.}$$

Example 4. ***Prove that $(2n)!\,.\,(n-1)! = 2(n)!\,(2n-1)!$***

Solution. L.H.S. $= (2n)!\,(n-1)!$

$= 2n(2n-1)!\,(n-1)!$

$= 2\,[n(n-1)!]\,(2n-1)!$

$= 2(n)!\,.\,(2n-1)! =$ R.H.S.

Example 5. ***Prove that :***

(i) $(2n)! = 2^n\,.\,n!\,.\,[1\,.\,3\,.\,5\ldots\ldots(2n-1)]$ [*M.D.U. 2009*]

(ii) $\dfrac{(2n+1)!}{n!} = 2^n[1\,.\,3\,.\,5\ldots\ldots(2n-1)(2n+1)].$

Solution. (*i*) $(2n)! = 1\,.\,2\,.\,3\,.\,4\ldots\ldots(2n-1)(2n)$

$= [2\,.\,4\,.\,6\ldots\ldots2n]\,[1\,.\,3\,.\,5\ldots\ldots(2n-1)]$

$= 2^n(1\,.\,2\,.\,3\ldots\ldots n).[1\,.\,3\,.\,5\ldots\ldots(2n-1)]$

$= \mathbf{2^n\,.\,n!\,[1\,.\,3\,.\,5\ldots\ldots(2n-1)].}$

(*ii*) L.H.S. $= \dfrac{(2n+1)!}{n!} = \dfrac{(2n+1)(2n)!}{n!}$

Now, $(2n)! = 2^n.\,n!\,[1\,.\,3\,.\,5\ldots\ldots(2n-1)]$ [*See Example* 5(*i*)]

$\therefore$ L.H.S. $= \dfrac{(2n+1)}{n!}\,.\,2^n\,.\,n!\,[1\,.\,3\,.\,5\ldots\ldots(2n-1)]$

$= 2^n[1\,.\,3\,.\,5\ldots(2n-1)(2n+1)] =$ R.H.S.

Example 6. ***Prove that :*** $\dfrac{n!}{r!(n-r)!} + \dfrac{n!}{(r-1)!(n-r+1)!} = \dfrac{(n+1)!}{r!(n-r+1)!}.$

Solution. L.H.S. $= \dfrac{n!}{r!(n-r)!} + \dfrac{n!}{(r-1)!(n-r+1)!}$

$$= n!\left[\frac{n-r+1}{r![(n-r)!\times(n-r+1)]} + \frac{r}{[(r-1)!\times r](n-r+1)!}\right]$$

$$= n!\left[\frac{n-r+1}{r!(n-r+1)!} + \frac{r}{r!(n-r+1)!}\right]$$

$$= \frac{n!\times(n+1)}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n-r+1)!} = \text{R.H.S.}$$

Example 7. ***Find the value of n if***

$$(n+3)! = 56\,((n+1)!).$$

Solution. We have $(n+3)\,! = 56\,((n+1)\,!)$

$$\Rightarrow \quad (n+3)(n+2)(n+1)\,! = 56\,((n+1)\,!)$$

$$\Rightarrow \quad (n+3)(n+2) = 56$$

$$\Rightarrow \quad n^2 + 5n - 50 = 0$$

$$\Rightarrow \quad (n+10)(n-5) = 0 \quad \Rightarrow \quad n = -10 \quad \text{or} \quad n = 5$$

[$\because$ *n cannot be negative*]

Hence $\quad n = 5$

Example 8. ***Find the value of n if*** $\dfrac{(2n)\,!}{3\,!\,(2n-3)\,!} : \dfrac{n\,!}{2\,!\,(n-2)\,!} = 44 : 3.$

Solution. We have $\dfrac{\dfrac{(2n)\,!}{3\,!\,(2n-3)\,!}}{\dfrac{n\,!}{2\,!\,(n-2)\,!}} = \dfrac{44}{3}$

$$\Rightarrow \quad \frac{2n(2n-1)(2n-2)(2n-3)\,!}{3\,!\,(2n-3)\,!} \times \frac{2\,!\,(n-2)\,!}{n(n-1)(n-2)\,!} = \frac{44}{3}$$

$$\Rightarrow \quad \frac{2n(2n-1)\,2(n-1)}{3n(n-1)} = \frac{44}{3}$$

$$\Rightarrow \quad \frac{4(2n-1)}{3} = \frac{44}{3}$$

$$\Rightarrow \quad 2n - 1 = 11 \quad \Rightarrow \quad n = \mathbf{6}.$$

EXERCISE 4.1

1. Evaluate $3\,!$, $4\,!$ and $7\,!$ and prove that $3\,! + 4\,! \neq 7\,!$

2. Compute $8\,!$; $4\,!$ and show that $\dfrac{8!}{4!} \neq 2\,!$.

3. Find the values of :

   (*i*) $\dfrac{11\,!}{8\,!}$ (*ii*) $\dfrac{6\,!}{4\,!\,2\,!}$ (*iii*) $\dfrac{(n+3)\,!}{n\,!}$

   (*iv*) $\dfrac{8\,!-7\,!}{6\,!}$ (*v*) $\dfrac{1}{8\,!} + \dfrac{1}{9\,!} + \dfrac{1}{10\,!}$

4. (*a*) Evaluate $(n-r)\,!$ when $n = 6, r = 2$.

   (*b*) Evaluate $\dfrac{n\,!}{r\,!\,(n-r)\,!}$ when (*i*) $n = 15, r = 12$ (*ii*) $n = 6,\; r = 2$.

5. Prove that $n(n-1)(n-2)......(n-r+1) = \frac{n!}{(n-r)!}$.

6. Prove that $(n-r+1)\frac{n!}{(n-r+1)!} = \frac{n!}{(n-r)!}$.

7. Prove that $\frac{(n-1)!}{(n-r-1)!} + r \cdot \frac{(n-1)!}{(n-r)!} = \frac{n!}{(n-r)!}$.

8. If $\frac{1}{9!} + \frac{1}{10!} = \frac{x}{11!}$; find x.

9. Find the L.C.M. of $4!, 5!, 6!$.

10. If $(n+1)! = 12(n-1)!$; find n.

11. If $(n+2)! = 2550\, n!$; find n.

12. If $\frac{n!}{2!(n-2)!}$ and $\frac{n!}{4!(n-4)!}$ are in the ratio 2 : 1, find the value of n.

ANSWERS

1. 6; 24; 5040
2. 40320; 24
3. (*i*) 990 (*ii*) 15 (*iii*) $(n+3)(n+2)(n+1)$
   (*iv*) 49 (*v*) $\frac{101}{10!}$
4. (*a*) 24 (*b*) (*i*) 455 (*ii*) 15
8. $x = 121$
9. $6! = 720$
10. 3
11. 49
12. 5

4.3. PRINCIPLE OF ASSOCIATION OF EVENTS OR FUNDAMENTAL PRINCIPLE OF COUNTING

If an event A can be performed in m ways, and another event B (independent of A) can be performed in n ways, then both the events A and B can be performed in $m \times n$ ways.

Note :

This principle can be generalised to any number of events.

SOLVED EXAMPLES

Example 1. ***If there are 20 buses plying between places A and B, in how many ways could the round trip from A be made if the return was made by***

(i) the same bus, (ii) a different bus.

Solution. (*i*) If the round trip from A is to be made by the same bus, then there are 20 buses plying between A and B and any one of these bus can be selected in 20 ways.

Hence, number of ways = 20.

(*ii*) In this case, the number of ways of making the journey from A to B are 20 and the number of ways of making the journey from B to A are 19.

[$\because$ *Journey is to be made by different bus from B to A*]

$\therefore$ By the fundamental principal of counting, the total number of ways for round trip from A = 20 × 19 = **380.**

Example 2. ***There are 6 candidates for 3 posts. In how many ways can the posts be filled.*** [*M.D.U. 2013*]

Solution. The 1st post can be filled up in 6 way.

The 2nd post can be filled up in 5 ways and then the 3rd post can be filled up in 4 ways.

$\therefore$ By the fundamental principle of counting, the three posts can be filled up in 6 × 5 × 4 = **120 ways.**

Example 3. ***From among the 36 teachers in a school, one principal and one vice-principal are to be appointed. In how many ways can this be done ?***

Solution. There are 36 teachers and every one has equal chance of being selected as a principal.

Hence, the principal can be appointed in 36 ways. When one person is appointed as principal, we are left with 35 teachers. Out of these 35 teachers, we can select one vice principal. So a vice-principal can be selected in 35 ways. Hence the number of ways in which a principal and vice-principal can be selected = 36 × 35 = **1260.**

Example 4. ***In a class there are 30 boys and 18 girls. The teacher wants to select one boy and one girl to represent the class for a quiz competition. In how many ways can the teacher make this selection ?***

Solution. Here the teacher has to make two choices, *viz.*

(*i*) selecting 1 boy from 30 boys

(*ii*) selecting 1 girl from 18 girls.

Now, one boy out of 30 can be selected in 30 ways as any one boy can be selected and similarly one girl out of 18 can be selected in 18 ways.

$\therefore$ By the fundamental principle of counting, the required number of ways is $30 \times 18 =$ **540**.

Example 5. ***Eight children are to be seated on a bench.***

(i) In how many ways can the children be seated ?

(ii) How many arrangements are possible if the youngest child sits at the left hand end of the bench ?

Solution. (*i*) Here the first child can take his seat on a bench in 8 ways.

Now there are 7 seats left and the second child can take his seat in $8 - 1 = 7$ ways

Similarly, the third child can take his seat in $7 - 1 = 6$ ways and so on.

$\therefore$ By the fundamental principle of counting, total number of ways

$$= 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \mathbf{40320}.$$

(*ii*) In this case, after the youngest child has taken his seat, which is fixed, the total number of seats left $= 8 - 1 = 7$.

$\therefore$ First child can be seated in 7 ways

Second child can be seated in $(7 - 1) = 6$ ways and so on.

$\therefore$ By the fundamental principle of counting, the total number of ways

$$= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \mathbf{5040}.$$

Example 6. ***A room has 5 doors. In how many ways can a man enter the room through one door and come out through a different door ?***

Solution. There are 5 doors in all. A person can enter the room through any one of the five doors. Thus, there are 5 ways of entering into the room.

After entering into the room, the man has to come out through a different door. Thus he can come out through any one of the remaining four doors which can be done in 4 ways.

$\therefore$ By the fundamental principle of counting, the total number of ways in which a man can enter a room through one door and come out through a different door $= 5 \times 4 =$ **20**.

Example 7. ***How many different numbers of two digits can be formed with the digits 1, 2, 3, 4, 5, 6; no digit being repeated.***

Solution. We have to fill up two places . [*Since numbers are of two digits*]

The first place can be filled up in 6 ways, as any one of the six digits can be placed in the first place. The 2nd place can be filled up in 5 ways as no digit is to be repeated.

Hence both places can be filled up in $6 \times 5 =$ **30 ways**.

Example 8. ***How many three digit odd numbers can be formed from the digits 1, 2, 3, 4, 5, 6 when***

(i) repetition of digits is not allowed

(ii) repetition of digits is allowed

Solution. (*i*) *When repetition of digits is not allowed* :

Since we have to form a three digit odd number, thus the digit at unit's place must be odd. Hence the unit's place can be filled up by 1, 3 or 5 *i.e.* in 3 ways.

Now the ten's digits can be filled up by any of the remaining 5 digits in 5 ways and then the hundred's place can be filled up by the remaining 4 digits in 4 ways.

Hence, the number of three digit odd numbers that can be formed $= 3 \times 5 \times 4 =$ **60.**

(*ii*) *When repetition of digits is allowed* :

Here again the unit's place can be filled up by 1, 3, 5 *i.e.*, in 3 ways. But the ten's and hundred's place can be filled up by any of the six given digits in 6 ways each

[*Since repetition is allowed*]

Hence, the number of three digit odd numbers that can be formed $= 3 \times 6 \times 6 =$ **108**.

Example 9. ***A class consists of 40 girls and 60 boys. In how many ways can a president, vice president, treasurer and secretary be chosen if the treasurer must be a girl, the secretary must be a boy and a student may not hold more than one office ?***

Solution. Here the treasurer must be a girl which can be selected out of 40 girls in 40 ways and the secretary must be a boy which can be selected out of 60 boys in 60 ways. Thus, treasurer and secretary can be selected in 40×60 ways. After the selection for these two posts has been made, 98 students are left and for the post of president and vice-president, either a boy or a girl may be selected. This selection can be made in 98×97 ways.

Hence the number of ways of forming the commitee $= 40 \times 60 \times 98 \times 97 =$ **22814400**.

Example 10. ***Given 5 flags of different colours. How many different signals can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff ?***

Solution. A signal can consist of either 2 flags, 3 flags, 4 flags or 5 flags. Now, we find the possible number of signals consisting of 2 flags, 3 flags, 4 flags and 5 flags.

Number of signals with 2 flags : There will be as many 2-flag signals as there are ways of filling up 2 vacant places in succession by the 5 available flags.

The upper place can be filled up in 5 ways and the lower place can be filled up in 4 ways.

$\therefore$ Number of signals that can be generated by 2 flags $= 5 \times 4 = 20$.

Number of signals with 3 flags : There will be as many 3-flag signals as there are ways of filling up 3 vacant places in succession by the 5 available flags.

Upper place can be filled up in 5 ways, the middle place can be filled up in 4 ways and the lower place can be filled up in 3 ways.

$\therefore$ Number of signals that can be generated by 3 flags $= 5 \times 4 \times 3 = 60$

Similarly, the number of signals by using 4 flags $= 5 \times 4 \times 3 \times 2 = 120$

and number of signals by using 5 flags $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

Hence, the total number of signals that can be generated by arranging atleast 2 flags $= 20 + 60 + 120 + 120 = \mathbf{320}$.

EXERCISE 4.2

1. In a hall, there are three entrance doors and two exit doors. In how many ways can a person enter the hall and then come out ? $3 \cdot 2 = 6$

2. In a railway compartment, 6 seats are vacant on a bench. In how many ways can 3 passengers sit on them ? $6 \times 5 \times 4$

3. In a monthly test, the teacher decides that there will be three questions, one from each of exercises 7, 8 and 9 of the text book. If there are 12 questions in exercise 7, 18 questions in exercise 8 and 9 questions in exercise 9, in how many ways can the three questions be selected ? $12 \times 18 \times 9$

4. There are 4 routes between Delhi and Mumbai. In how many different ways can a man go from Delhi to Mumbai and return, if for returning
   (*i*) any of the routes is taken 4×4
   (*ii*) the same route is not taken. 4×3

5. Each section of class in a college has exactly 40 students. If there are 4 sections, in how many ways can a set of 4 student representatives be selected, one from each section ? $40 \times 40 \times 40 \times 40$

6. Find the number of different signals that can be generated by arranging at least two flags in order (one below the other) on a vertical staff, if five different flags are available.

7. A team consists of 6 boys and 4 girls and the other has 5 boys and 3 girls. How many single matches can be arranged between the two teams when a boy plays against a boy and a girl against a girl ? $30 + 12 = 42$

8. How many 5 digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67, for example 67125 etc, and no digit appears more than once ? $8 \times 7 \times 6$

9. There are 6 multiple choice questions in an examination. How many sequences of answers are possible, if the first three questions have 4 choices each and the next three have 5 choices each ? $\cancel{12} + 15$

$4 \times 4 \times 4 \times 5 \times 5 \times 5$

5+2

10. How many 2 digit even numbers can be formed from the digits 1, 2, 3, 4 and 5 if the digits can be repeated ?
11. How many numbers are there between 100 and 1000 in which all the digits are distinct ?
12. How many numbers are there between 100 and 1000 such that every digit is either 2 or 9 ?
13. How many three-digit numbers can be formed without using the digits 0, 2, 3, 4, 5 and 6 ?
14. How many numbers are there between 100 and 1000 such that 7 is in the unit's place ?
15. How many odd numbers less than 1000 can be formed by using the digits 0, 2, 5, 7 when the repetition of digits is allowed ?
16. How many 3 digit numbers can be formed from the digits 1, 2, 3, 4 and 5 assuming
    (*i*) repetition of digits is allowed
    (*ii*) repetition of digits is not allowed ?

ANSWERS

1. 6	**2.** 120	**3.** 1944
4. (*i*) 16	(*ii*) 12	**5.** 2560000
6. 320	**7.** 42	**8.** 336
9. 8000	**10.** 10	**11.** 648
12. 8	**13.** 64	**14.** 90
15. 32	**16.** (*i*) 125	(*ii*) 60

4.4. DIFFERENCE BETWEEN PERMUTATIONS (ARRANGEMENTS) AND COMBINATIONS (GROUPS)

Each of the arrangements that can be made out of a given set of things, by taking some or all of them at a time, are called ***permutations.***

Each of the groups or selections that can be made out of a given set of things by taking some or all of them at a time are called ***combinations.***

eg., Two letters a and b together form one group (combination), but they can be arranged in two different ways as ab and ba and thus there are two arrangements (permutations).

Again, if we take three letters a, b, c, then the number of groups taking two letters at a time is three *i.e.*, ab, bc and ca.

But each group gives rise to two different arrangements, hence the total number of arrangements = 6 *i.e.*, ab, ba, bc, cb, ca and ac.

Further, if we take four letters a, b, c, d, then the *combinations* which can be made by taking two letters at a time are six in number *viz.*,

$$ab, \quad ac, \quad ad, \quad bc, \quad bd, \quad cd.$$

each of these presenting a different *selection* of two letters, and the *permutations* which can be made by taking two letters at a time are twelve in number, *viz.*,

$$ab, \quad ac, \quad ad, \quad bc, \quad bd, \quad cd$$
$$ba, \quad ca, \quad da, \quad cb, \quad db, \quad dc.$$

each of these presenting a different *arrangement* of two letters.

From above it is evident that in forming *combinations* we are only concerned with the number of things each selection contains; whereas in forming *permutations* we have also to consider the order of the things which make up each arrangement.

4.5. PERMUTATIONS (ARRANGEMENTS)

The different arrangements that can be made out of a given set of things, by taking some or all of them at a time are called permutations.

The number of permutations of n things taken r at a time is denoted by $^n\mathbf{P}_r$ or $\mathbf{P}(\boldsymbol{n}, \boldsymbol{r})$.

4.6. PERMUTATIONS WHEN ALL THE OBJECTS ARE DISTINCT

4.6.1. Theorem. ***The number of permutations of n different objects taken r at a time, where $0 < r \le n$ and the repetition of the objects in any permutation is not allowed, is given by***

$$^nP_r = n(n-1)(n-2) \,.....\, (n-r+1) = \frac{n\,!}{(n-r)\,!}.$$

Proof. There can be as many permutations as there are ways of filling up r vacant places $\square\ \square\ \square\ \ldots\ \square$ by the n objects.

$\leftarrow r$ vacant places $\rightarrow$

The 1st place can be filled up in n ways. Following which, the 2nd place can be filled up in $(n-1)$ ways, following which the 3rd place can be filled up in $(n-2)$ ways and so on. Proceeding like this, the rth place can be filled up in $(n-(r-1))$ ways. Thus, the number of ways of filling up r vacant places in succession $= n(n-1)(n-2) \,.......\, (n-(r-1))$

i.e.,
$$n(n-1)(n-2) \,.......\, (n-r+1).$$

$\therefore$ The number of permutations of n different things, taken r at a time

$$= n\,(n-1)\,(n-2) \,.......\, (n-r+1)$$
$$= \frac{[n\,(n-1)\,(n-2)\ldots(n-r+1)]\,(n-r)\,(n-r-1)\ldots3.2.1}{(n-r)\,(n-r-1)\ldots\ldots3.2.1.}$$
$$= \frac{n\,!}{(n-r)\,!}$$

Symbolically, $$\boxed{P(n, r) = {}^nP_r = n(n-1)(n-2) \ldots\ldots (n-r+1) = \frac{n!}{(n-r)!}, \text{ where } 0 < r \leq n.}$$

4.6.2. Theorem. ***The number of permutations of n different objects, taken all at a time, is*** $P(n, n) = {}^nP_n = n!$

Proof. The number of permutations of n different objects, taken all at a time, is the same as the number of arrangements of n objects in n places in a row.

The 1st place can be filled up in n ways as any one of the n objects can be placed in the 1st place.

The 2nd place can be filled up in $(n-1)$ ways, as only $(n-1)$ objects are left for the 2nd place.

Similarly, the third place can be filled up in $(n-2)$ ways. Proceeding like this, we are left with only one objects for the nth place.

$\therefore$ By the fundamental principle of counting, the number of permutations of n objects taken all at a time

$$= n \,.\, (n-1) \,.\, (n-2) \ldots\ldots 3 \,.\, 2 \,.\, 1$$
$$= n!$$

Symbolically, $$\boxed{P(n, n) = {}^nP_n = n!}$$

Cor. Value of 0 !

Since $${}^nP_r = \frac{n!}{(n-r)!}$$

$\therefore$ $${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} \quad \ldots(1)$$

But $${}^nP_n = n! \quad \ldots(2)$$

$\therefore$ From (1) and (2), $$n! = \frac{n!}{0!} \Rightarrow 0! = 1.$$

Remarks:

1. ${}^nP_r = n(n-1)(n-2) \ldots\ldots$ to r factors, which is known as product form. Thus, ${}^8P_3 = 8 \,.\, 7 \,.\, 6 = 336.$

   If r is small, we use the product form.

2. ${}^nP_0 = \dfrac{n!}{(n-0)!} = \dfrac{n!}{n!} = 1.$ Thus the number of permutations of n different things taken nothing at all is 1. Thus the formula for ${}^nP_r$ is applicable to $r = 0$ also.

   $\therefore$ $${}^nP_r = \frac{n!}{(n-r)!}, \; 0 \leq r \leq n.$$

4.6.3. To prove that

(*i*) $^nP_r = n\,.\,^{n-1}P_{r-1}$ (*ii*) $^nP_r = (n-r+1)\,.\,^nP_{r-1}$

Proof. (*i*) R.H.S. $= n\,.\,^{n-1}P_{r-1}$

$$= n\,.\,\frac{(n-1)!}{[(n-1)-(r-1)]!} \qquad \left[\because\ ^nP_r = \frac{n!}{(n-r)!}\right]$$

$$= n\,.\,\frac{(n-1)!}{(n-r)!}$$

$$= \frac{n(n-1)!}{(n-r)!} = \frac{n!}{(n-r)!} = {}^nP_r = \text{L.H.S.}$$

(*ii*) R.H.S. $= (n-r+1)\,.\,^nP_{r-1} = (n-r+1)\,.\,\dfrac{n!}{(n-r+1)!}$

$$= \frac{(n-r+1)\,.\,n!}{(n-r+1)\,.\,(n-r)!} = \frac{n!}{(n-r)!} \qquad [\because n! = n(n-1)!]$$

$$= {}^nP_r = \text{L.H.S.}$$

4.6.4. To prove that

(*i*) $^nP_r = {}^{n-1}P_r + r\,.\,^{n-1}P_{r-1}$ (*ii*) $(n+1)\,^nP_r = (n-r+1)\,.\,^{n+1}P_r$

Proof. (*i*) R.H.S. $= {}^{n-1}P_r + r\,.\,^{n-1}P_{r-1}$

$$= \frac{(n-1)!}{(n-1-r)!} + r\,.\,\frac{(n-1)!}{(n-1-r+1)!} = \frac{(n-1)!}{(n-r-1)!} + r\,.\,\frac{(n-1)!}{(n-r)!}$$

$$= (n-1)!\left[\frac{1}{(n-r-1)!} + \frac{r}{(n-r)(n-r-1)!}\right]$$

$$= \frac{(n-1)!}{(n-r-1)!}\left[1 + \frac{r}{n-r}\right] = \frac{(n-1)!}{(n-r-1)!}\left[\frac{n-r+r}{n-r}\right]$$

$$= \frac{n(n-1)!}{(n-r)(n-r-1)!} = \frac{n!}{(n-r)!} = {}^nP_r = \text{L.H.S.}$$

(*ii*) $$(n+1)\,^nP_r = (n+1)\frac{n!}{(n-r)!} = \frac{(n+1)!}{(n-r)!} \qquad \ldots(1)$$

and $$(n-r+1)\,^{n+1}P_r = (n-r+1)\,.\,\frac{(n+1)!}{(n-r+1)!}$$

$$= (n-r+1)\,.\,\frac{(n+1)!}{(n-r+1)(n-r)!} = \frac{(n+1)!}{(n-r)!} \qquad \ldots(2)$$

$\therefore$ From (1) and (2), $(n+1)\,^nP_r = (n-r+1)\,^{n+1}P_r$.

SOLVED EXAMPLES

Example 1. ***Evaluate***

(i) $^{12}P_3$ **(ii)** $^{20}P_4$ **(iii)** $^{8}P_8$

[M.D.U. 2014] [M.D.U. 2014]

Solution. (*i*) $$^{12}P_3 = \frac{12!}{(12-3)!} = \frac{12!}{9!}$$

$$= \frac{12 \times 11 \times 10 \times 9!}{9!} = \mathbf{1320}.$$

(*ii*) $$^{20}P_4 = \frac{20!}{(20-4)!} = \frac{20!}{16!}$$

$$= \frac{20 \times 19 \times 18 \times 17 \times 16!}{16!} = \mathbf{116280}.$$

(*iii*) $$^{8}P_8 = 8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \mathbf{40320}. \qquad [\because {}^nP_n = n!]$$

Example 2. ***Prove that :*** $^nP_n = {}^nP_{n-1}$

Solution. $$^nP_n = n! \qquad \text{...(1)}$$

$$^nP_{n-1} = \frac{n!}{[n-(n-1)]!}$$

$$= \frac{n!}{[n-n+1]!} = n! \qquad \text{...(2)}$$

$\therefore$ From (1) and (2), $^nP_n = {}^nP_{n-1}$.

Example 3. ***Find the value of n such that*** $^nP_5 = 42\, {}^nP_3$, $n > 4$.

Solution. We have $$^nP_5 = 42\, {}^nP_3$$

$$\Rightarrow \quad n(n-1)(n-2)(n-3)(n-4) = 42 \,.\, n(n-1)(n-2) \qquad \text{...(1)}$$

Now $n > 4$ *i.e.*, $n(n-1)(n-2) \neq 0$.

Dividing both sides of (1) by $n(n-1)(n-2)$, we get

$$(n-3)(n-4) = 42 \Rightarrow n^2 - 7n - 30 = 0$$

$$\Rightarrow \quad (n-10)(n+3) = 0 \Rightarrow n = 10, -3.$$

Hence, $n = \mathbf{10}$. [$\because$ *n cannot be* –ve]

Example 4. ***Find the value of n if $^nP_4 : {}^{n-1}P_3 = 9 : 1$.***

Solution.
$$\frac{^nP_4}{^{n-1}P_3} = \frac{\dfrac{n!}{(n-4)!}}{\dfrac{(n-1)!}{(n-4)!}} = \frac{9}{1}$$

$$\Rightarrow \quad \frac{n!}{(n-1)!} = 9 \quad \Rightarrow \quad \frac{n(n-1)!}{(n-1)!} = 9 \quad \Rightarrow \quad n = \mathbf{9}.$$

Example 5. ***If $^5P_r = 2 \,.\, {}^6P_{r-1}$, find r.*** *[M.D.U. 2017]*

Solution. We have $\quad {}^5P_r = 2.\,{}^6P_{r-1}$

$$\therefore \quad \frac{5!}{(5-r)!} = 2\,.\,\frac{6!}{[6-(r-1)]!}$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = 2\,.\,\frac{6.5!}{(7-r)!}$$

$$\Rightarrow \quad \frac{1}{(5-r)!} = \frac{12}{(7-r)(6-r)(5-r)!}$$

$$\Rightarrow \quad (7-r)(6-r) = 12 \quad \Rightarrow \quad r^2 - 13r + 30 = 0$$

$$\Rightarrow \quad (r-3)(r-10) = 10 \quad \Rightarrow \quad r = 3 \quad \text{or} \quad r = 10$$

Hence $r = \mathbf{3}$. *[Rejecting $r = 10$, as $r \le n$ and here $n = 5$ or 6]*

Example 6. ***If $^{22}P_{r+1} : {}^{20}P_{r+2} = 11 : 52$, find r.***

Solution. We have $\quad {}^{22}P_{r+1} : {}^{20}P_{r+2} = 11 : 52$

$$\Rightarrow \quad \frac{^{22}P_{r+1}}{^{20}P_{r+2}} = \frac{\dfrac{22!}{(21-r)!}}{\dfrac{20!}{(18-r)!}} = \frac{11}{52}$$

$$\Rightarrow \quad \frac{22!}{(21-r)!} \times \frac{(18-r)!}{20!} = \frac{11}{52}$$

$$\Rightarrow \quad \frac{22 \times 21 \times 20!}{(21-r)(20-r)(19-r)(18-r)!} \times \frac{(18-r)!}{20!} = \frac{11}{52}$$

$$\Rightarrow \quad \frac{22 \times 21}{(21-r)(20-r)(19-r)} = \frac{11}{52}$$

$$\Rightarrow \quad (21-r)(20-r)(19-r) = 2 \times 21 \times 52$$
$$\Rightarrow \quad (21-r)(20-r)(19-r) = 2 \times 7 \times 3 \times 13 \times 4$$
$$\Rightarrow \quad (21-r)(20-r)(19-r) = 14 \times 13 \times 12$$
$$\Rightarrow \quad (21-r)(20-r)(19-r) = (21-7)(20-7)(19-7)$$

Hence, $r = 7$

EXERCISE 4.3

1. Evaluate :
   (i) ${}^{7}P_{4}$ (ii) ${}^{18}P_{2}$ (iii) ${}^{6}P_{6}$
2. Prove that :
   (i) ${}^{n}P_{n} = 2\,{}^{n}P_{n-2}$ (ii) ${}^{10}P_{3} = {}^{9}P_{3} + 3\,{}^{9}P_{2}$
3. Find n if
   (i) ${}^{n}P_{6} = 3 \,.\, {}^{n}P_{5}$ (ii) ${}^{n}P_{2} = 20$
   (iii) $2 \,.\, {}^{n}P_{3} = {}^{n+1}P_{3}$ (iv) ${}^{2n}P_{3} = 100 \,.\, {}^{n}P_{2}$
   (v) $16 \,.\, {}^{n}P_{3} = 13 \,.\, {}^{n+1}P_{3}$ (vi) ${}^{n}P_{4} = 18 \,.\, {}^{n-1}P_{2}$
   (vii) ${}^{n}P_{5} : {}^{n-1}P_{4} = 6 : 1.$ (viii) ${}^{n}P_{4} : {}^{n-1}P_{4} = 5 : 3,\ n > 4$
   (ix) ${}^{10}P_{n-1} : {}^{11}P_{n-2} = 30 : 11$ *[M.D.U. 2010]*
4. Find r if
   (i) $4 \,.\, {}^{6}P_{r} = {}^{6}P_{r+1}$ (ii) ${}^{5}P_{r} = {}^{6}P_{r-1}$ (iii) ${}^{10}P_{r} = 2\,{}^{9}P_{r}$
   (iv) ${}^{11}P_{r} = {}^{12}P_{r-1}$ (v) $5 \,.\, {}^{4}P_{r} = 6 \,.\, {}^{5}P_{r-1}$ *[M.D.U. 2010*
5. Find r, if ${}^{10}P_{r+1} : {}^{11}P_{r} = 30 : 11.$ *[M.D.U. 2013*
6. Find n, if ${}^{2n-1}P_{n} : {}^{2n+1}P_{n-1} = 22 : 7.$ *[M.D.U. 2014*

ANSWERS

1. (i) 840 (ii) 306 (iii) 720
3. (i) 8 (ii) 5 (iii) 5
   (iv) 13 (v) 15 (vi) 6
   (vii) 6 (viii) 10 (ix) 7
4. (i) 2 (ii) 4 (iii) 5
   (iv) 9 (v) 3
5. 5
6. 10

4.7. RESTRICTED PERMUTATIONS

4.7.1. Permutations of n different things, taken all at a time, in which p things always occur together.

Proof. Since p given things always occur together, they may be regarded as one block. Thus now there are $(n-p+1)$ things, which can be arranged amongst themselves in $(n-p+1)!$ ways.

Also p things, which form one block, can be arranged amongst themselves in $p!$ ways.

$\therefore$ By the principle of association, the required number of arrangements $= p! \cdot (n-p+1)!$

4.7.2. Permutations of n different things, taken r at a time, when x particular things are always included.

First the x particular things can be arranged in r places in ${}^rP_x$ ways. The remaining $r-x$ places can be filled up by the remaining $n-x$ things in ${}^{n-x}P_{r-x}$ ways. Hence, the number of arrangements $= {}^rP_x \times {}^{n-x}P_{r-x}$.

4.7.3. Permutations of n different things, taken r at a time, when x particular things are always excluded.

In this case $(n-x)$ things will fill up r places.

Hence, the number of arrangements $= {}^{n-x}P_r$.

4.8. SOME PRACTICAL PROBLEMS ON PERMUTATIONS

SOLVED EXAMPLES

Example 1. *In how many ways can 10 people line up at a ticket window of a cinema hall ?*

Solution. The required number of ways

$= $ number of permutations of 10 people taking all 10 at a time

$= {}^{10}P_{10} = 10!$

$= 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= \mathbf{3628800}$.

Example 2. *How many words, with or without meaning can be formed using all letters of the word EQUATION, using each letter exactly one.*

Solution. The word EQUATION has 8 letters which are all different.

$\therefore$ Number of words that can be formed = number of permutations of 8 letters taken all at a time $= {}^8P_8 = 8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \mathbf{40320}$.

Example 3. ***There are 6 candidates contesting for a certain office in a municipal election. In how many ways can their names be listed on a ballot paper ?***

Solution. The required number of ways

= number of permutations of listing 6 candidates in 6 places on a ballot paper

= $^6P_6 = 6\,! = \mathbf{720}$.

Example 4. ***How many different signals can be given using any number of flags from 5 flags of different colours ?***

Solution. The signals can be made by using at a time, one or two or three or four or five flags.

The total number of signals of 5 flags taken r at a time is equal to the number of arrangements of 5, taking r at a time *i.e.*, ^5P_r.

Since r can take the values 1, 2, 3, 4, 5, therefore the total number of signals

$$= {}^5P_1 + {}^5P_2 + {}^5P_3 + {}^5P_4 + {}^5P_5$$

$$= 5 + 5 \times 4 + 5 \times 4 \times 3 + 5 \times 4 \times 3 \times 2 + 5 \times 4 \times 3 \times 2 \times 1$$

$$= 5 + 20 + 60 + 120 + 120 = \mathbf{325}.$$

Example 5. ***In how many ways can 5 men and 4 women be seated in a row so that the women occupy the even places.***

[*M.D.U. 2011*]

Solution. There are in all 9 people (5 men and 4 women) to be seated in a row consisting of 9 places. Out of 9 places, 4 are even (*i.e.*, 2nd, 4th, 6th and 8th)

$\therefore$ 4 women can occupy 4 even places in $^4P_4 = 4\,! = 24$ ways.

Now there are 5 odd places left. 5 men can be arranged in these 5 places in $^5P_5 = 5\,! = 120$ ways.

$\therefore$ Total number of ways = $24 \times 120 = \mathbf{2880}$.

Example 6. ***In how many ways can 5 children be arranged in a line such that (i) two particular children are always together (ii) two particular children are never together.***

Solution. (*i*) Here two children have to remain together. So we consider them as one block. Then, we are left with 4 children who can be arranged in $^4P_4 = 4\,! = 24$ ways.

Also, two particular children who are to remain together can be arranged amongst themselves in $2\,! = 2$ ways.

$\therefore$ Total number of ways = $24 \times 2 = \mathbf{48}$.

(*ii*) There are in all 5 children and the total number of ways in which they can be arranged in a line = 5 ! = 120.

Out of these arrangements we have seen above that the number of ways in which two particular children are always together = 48.

$\therefore$ Number of ways in which two particular children are never together = 120 – 48 = **72.**

Example 7. ***How many 4-digit numbers can be formed by using the digits 1 to 9, if repetition of digits is not allowed ?***

Solution. Here the digits cannot be repeated. Thus the number of four digit numbers that can be formed by using digits 1 to 9 is equal to number of permutations of 9 things taken 4 at a time

$$= {}^9P_4 = \frac{9\,!}{(9-4)\,!} = \frac{9\,!}{5\,!}$$

$$= 9.8.7.6. = \mathbf{3024.}$$

Example 8. ***(a) Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5, if no digit is repeated.***

(b) How many of these are even ?

(c) How many of these are divisible by 5 ?

Solution. (*a*) There can be as many numbers as there are permutations of five-digits 1, 2, 3, 4, 5 taken four at a time.

$\therefore$ The required 4-digit numbers = ${}^5P_4 = 5 \times 4 \times 3 \times 2 = \mathbf{120}$.

(*b*) For the numbers to be even, the digit in the unit's place can be any one of the two digits 2, 4. So, the unit's place can be filled up in 2 ways.

The remaining three places can be filled up by the remaining four digits in ${}^4P_3$ ways.

$\therefore$ The required number of 4-digit even numbers $= 2 \times {}^4P_3$

$$= 2 \times 4 \times 3 \times 2 = \mathbf{48}.$$

(*c*) For the numbers to be divisible by 5, the digit in the unit's place must have 5. So, the unit's place can be filled up in 1 way. The remaining three places can be filled up by the remaining four digits in ${}^4P_3$ ways.

$\therefore$ The required number of 4-digit numbers divisible by 5

$$= 1 \times {}^4P_3 = 1 \times 4 \times 3 \times 2 = \mathbf{24}.$$

Example 9. ***(i) How many different numbers of six digits can be formed with the numbers 3, 1, 7, 0, 9, 5 ?***

(ii) How many of them are divisible by 10 ?

Solution. (*i*) There can be as many such numbers as there are permutations of six digits 3, 1, 7, 0, 9, 5 taken all at a time = ${}^6P_6 = 6\,!$ ...(1)

But we have to neglect the numbers which begin with zero. Now the numbers in which zero comes in the 1st place = 5 ! ...(2)

Hence the required number = 6 ! – 5 ! = 720 – 120 = **600.**

(*ii*) Those numbers, which have zero in the units place, are divisible by 10.

If 0 is fixed at unit's place, we are left with 5 digits which have to be arranged in remaining 5 places.

Hence such numbers are 5 ! = **120.**

Example 10. ***In how many ways can the letters of the word 'UNIVERSAL' be arranged? In how many of these will E, R, S always occur together.***

Solution. (*i*) The word 'UNIVERSAL' has 9 letters which are all different. These 9 letters can be arranged amongst themselves in 9 ! ways.

(*ii*) Three letters E, R, S can be taken to form one block. Thus 7 letters U, N, I, V, ERS, A, L, can be arranged in 7 ! ways.

Also 3 letters E, R, S can be arranged amongst themselves in 3 ! ways.

∴ The total number of arrangements = 7 ! . 3 ! = **30240.**

Example 11. ***How many words, with or without meaning, can be made from the letters of the word MONDAY, assuming that no letter is repeated, if***

(i) 4 letters are used at a time ?

(ii) all letters are used at a time ?

(iii) all letters are used but first letter is a vowel ?

Solution. The given word 'MONDAY' has 6 different letters.

(*i*) The number of words, when 4 letters are used at a time

$$= {}^6P_4 = 6 \times 5 \times 4 \times 3 = \mathbf{360}.$$

(*ii*) The number of words when all letters are used at a time

$$= {}^6P_6 = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \mathbf{720}.$$

(*iii*) The first letter in each permutation to be a vowel *i.e.*, either O or A is possible in 2 ways.

The remaining five letters can be used without any restriction in ${}^5P_5$ ways

$$= 5 \times 4 \times 3 \times 2 \times 1 = 120$$

∴ Required number of words $= 2 \times 120 = \mathbf{240}.$

Example 12. ***How many words can be formed from the letters of the word DAUGHTER :***

(i) ***taking all the letters together***

(ii) ***beginning with D***

(iii) ***beginning with D and ending with R***

(iv) ***vowels being always together***

[*M.D.U. 2010*]

Solution. (*i*) There are 8 letters in the word 'DAUGHTER' out of which 3 are vowels namely, A, U and E.

$\therefore$ Number of words taking all the letters together = $^8P_8 = 8\,! =$ **40320.**

(*ii*) For the words to begin with D, the first place can be filled up by D only *i.e.*, in 1 way.

Now there are 7 letters left which have to be arranged in 7 remaining places. This can be done in $^7P_7 = 7\,! = 5040$ ways.

$\therefore$ Total number of words beginning with D = $1 \times 5040 =$ **5040.**

(*iii*) For the words to begin with D and end with R, the first and last places are fixed for D and R respectively, which can be done in 1 way.

Now there are 6 letters left which have to be arranged in 6 remaining places. This can be done in $^6P_6 = 6\,! = 720$ ways.

$\therefore$ Total number of words beginning with D and ending with R = $1 \times 720 =$ **720.**

(*iv*) Here the three vowels A, U, E are to remain together and can be considered as a block of one letter. Thus, we have 6 letters D, 'AUE', G, H, T, R which can be arranged in $^6P_6 = 6\,! = 720$ ways.

Also A, U, E can be arranged amongst themselves in $3\,! = 6$ ways.

$\therefore$ Total number of words = $720 \times 6 =$ **4320.**

EXERCISE 4.4

1. In how many ways can six children stand in a queue ?
2. There are 3 different rings to be worn in four fingers with at most one in each finger. In how many ways can this be done ?
3. A student wants to arrange 3 Economics, 2 Maths and 4 English books on a shelf. If the books on the same subject are different, find:
   (*i*) the number of possible arrangements.
   (*ii*) the number of possible arrangements, if all books on a particular subject are to be together.
4. How many 6 digited telephone numbers can be constructed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, if each number starts with 35 and no digit appears more than once.

5. If there are six periods in each working day of a school, in how many ways can one arrange 5 subjects such that each subject is allowed at least one period ?
6. In how many ways can 6 boys and 5 girls be arranged for a group photograph if the girls are to sit on chairs in a row and the boys are to stand in a row behind them ?
7. (i) In how many ways can 5 boys and 3 girls be arranged so that no two girls may sit together ?
   (ii) In how many ways can 7 boys and 5 girls arrange themselves in a row, so that no two girls are together ?
8. In how many ways can 9 examination papers be arranged so that the best and worst never come together. [M.D.U. 2012]
9. 6 candidates are to take examination, 3 in Mathematics and the remaining in different subjects. In how many ways can they be seated in a row so that all the 3 candidates in Mathematics may not sit together ?
10. In how many ways can the letters of the word DELHI be arranged so that the letters E and I occupy only even places.
11. The letters of the word 'TUESDAY' are arranged in a line, each arrangement ending with letter S. How many different arrangements are possible ? How many of them start with letter D ?
12. How many permutations can be made out of the letters of the word 'TRIANGLE'. How many of them begin with 'T' and end with 'E'.
13. How many different words can be formed of the letters of the word 'COMBINE' so that
    (i) vowels always remain together ?
    (ii) no two vowels are together ?
    (iii) vowels may occupy odd places ?
14. In how many ways can the letters of the word 'FRACTION' be arranged so that no two vowels are together ?
15. How many words can be formed from the letters of the word 'ORIENTAL' so that A and R always occupy odd places ?
16. In how many ways can the letters of the word 'PENCIL' be arranged so that (i) N is always next to E ? (ii) N and E are always together ?
17. How many four digit numbers can be formed having distinct digits using 0, 1, 2, 3, 4, 5.
18. (i) How many numbers greater than 20,000 can be formed by using the digits, 0, 1, 2, 3, 4, no digit being repeated in any number.
    (ii) How many odd numbers greater than 80,000 can be formed using the digits 2, 3, 4, 5 and 8 if each digit is used only once in a number.
19. How many different 6 digit numbers can be formed from the digits 4, 2, 5, 0, 6 and 7, no digit being repeated ? How many of these will have 0 in the ten's place.

20. How many numbers of four digits can be formed with the digits 3, 4, 5, 8, no digit being repeated ? How many of these are divisible by 2 and how many of these are divisible by 5 ?

ANSWERS

1. 720
2. 24
3. (*i*) 362880 (*ii*) 1728
4. 1680
5. 3600
6. 86400
7. (*i*) 14400 (*ii*) 33868800
8. 7 . 8 ! = 282240
9. 576
10. 12
11. 720; 120
12. 40320; 720
13. (*i*) 720 (*ii*) 1440 (*iii*) 576
14. 14400
15. 8640
16. (*i*) 120 (*ii*) 240
17. 300
18. (*i*) 72 (*ii*) 12
19. 600; 120
20. 24; 12; 6

4.9. PERMUTATIONS WITH REPITITIONS

4.9.1. Theorem. ***The number of arrangements of n dissimilar things taken r at a time, when each thing may be repeated any number of times in any arrangement is n^r.***

Proof. Suppose r places are to be filled up with n objects. The first place can be filled up in n ways because any one of the n objects can be put there. When the first place has been filled up, the second place can also be filled up in n ways because the object which has been placed in the first place can be repeated and placed in the second place also.

Hence by principle of association, the first two places can be filled up in $n \times n = n^2$ ways.

Similarly, the first three places can be filled up in n^3 ways.

Proceeding in this way the number of ways of filling up r places $= n^r$.

4.10. PERMUTATIONS OF OBJECTS NOT ALL DIFFERENT

4.10.1. Theorem. ***The number of permutations of n objects taken all at a time when p objects are alike and of one kind, q objects are alike and of another kind, r objects are alike of yet another kind, and the remaining objects are different is*** $\dfrac{n!}{p!\,q!\,r!}$.

Proof. Let the number of permutations be x and the p alike objects be made different from each other and from the rest, then these p objects can be arranged among themselves in p ! ways.

Since 1 arrangement gives rise to p ! ways.

$\therefore$ x arrangements will give rise to $x \cdot p$! ways.

Similarly, we replace q alike objects into different objects, which can be arranged in q ! ways.

$\therefore$ The number of ways $= x \cdot p! \cdot q!$.

Proceeding like this, replacing r similar objects into different objects, the number of permutations

$$= x \cdot p! \cdot q! \cdot r! \quad \text{...(1)}$$

All objects now being different, can be arranged in n ! ways. ...(2)

$$\therefore \quad x \cdot p! \cdot q! \cdot r! = n!$$

$$\therefore \quad x = \frac{n!}{p!\,q!\,r!}.$$

SOLVED EXAMPLES

Example 1. ***In how many ways can 6 different rings be worn on the four fingers of a hand.***

Solution. Each ring can be worn on any of the four fingers and this can be done in 4 ways.

$\therefore$ The required number of ways = $\mathbf{4^6}$.

Example 2. ***In how many ways can 6 apples be distributed among 3 boys, there being no restriction to the number of apples each boy may get.***

Solution. Each apple can be given to any one of the 3 boys and this can be done in 3 ways.

$\therefore$ The required number of ways = 3^6 = **729**.

Example 3. ***In how many ways can 3 prizes be distributed among 4 boys when***

(i) a boy may get any number of prizes.

(ii) no boy gets all the prizes.

Solution. (*i*) There are 3 prizes to be given to 4 boys. Now first prize can be given to any of the 4 boys in 4 ways.

Similarly, second and third prizes can again be given to any of the four boys in 4 ways as there is no restriction to the number of prizes a boy may get.

Hence the total number of ways $= 4^3 = \mathbf{64}$.

(*ii*) The number of ways in which one boy gets all the prizes is 4 as any one of the 4 boys may get all the prizes.

Hence the number of ways in which no boy gets all the prizes $= 64 - 4 = \mathbf{60}$.

Example 4. ***How many three digits numbers can be formed by using the digits***

(i) 1, 3, 6, 8 ***(ii) 0, 2, 3, 6, 8***

when the digits may be repeated any number of times.

Solution. (*i*) There are three places to be filled up to form a three digit number. Since any digit may be repeated any number of times, each one of these three places can be filled up by any of the given 4 digits in 4 ways.

Hence the number of three digit numbers that can be formed $= 4^3 = \mathbf{64}$.

(*ii*) There are in all 5 digits. Now 0 cannot be placed at the hundredth place as in that case the number will not be 3-digited. Thus hundredth place can be filled up by any of remaining 4 digits in 4 ways.

Since the digits may be repeated any number of times, each of the remaining two places can be filled up any of the 5 digits in 5 ways each. Thus the total number of such arrangements $= 5^2 = 25$.

Hence the total number of three digit numbers that can be formed $= 4 \times 25 = \mathbf{100}$.

Example 5. ***How many 5 digit even numbers can be formed using the digits 1, 2, 5, 5, 4.***

Solution. The 5 digit even numbers can be formed out of 1, 2, 5, 5, 4 by using either 2 or 4 in the unit's place. This can be done in 2 ways.

Corresponding to each such arrangement, the remaining 4 places can be filled up any of the remaining four digits in $\dfrac{4!}{2!} = 12$ ways. [$\because$ *5 occurs twice*]

Hence, the total number of words $= 2 \times 12 = 24$.

Example 6. ***How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3.***

Solution. A number greater than a million has 7 places, and thus all the 7 given digits are to be used.

But 2 is repeated twice and 3 is repeated thrice.

∴ Total number of ways of arranging these 7 digits amongst themselves

$$= \frac{7!}{2!.3!} = 420.$$

But numbers beginning with zero are not seven digited numbers, and thus we have reject those numbers which begin with zero and such numbers are $= \dfrac{6!}{2!3!} = 60.$

Hence the required number of arrangements = 420 – 60 = **360**.

Example 7. ***How many arrangements can be made of the letters of the word 'ARRANGEMENT'. In how many of these the vowels occur together.***

Solution. (*a*) The given word consists of 11 letters out of which A occurs 2 times, occurs 2 times, N occurs 2 times and E occurs 2 times and remaining three are different.

$$\therefore \text{ Number of arrangements} = \frac{11!}{2!2!2!2!} = 2491800$$

(*b*) Now there are 4 vowels in the given word (A, A, E, E). Let us treat these 4 vowels one letter (AAEE). Then there are 8 letters [(AAEE), R, R, N, G, M, N, T] out of which R ar N occur 2 times each.

$$\therefore \text{ Number of arrangements} = \frac{8!}{2!2!} = 10080$$

Corresponding to each such arrangement, the four vowels A, A, E, E out of which A an E occur 2 times each, can be arranged amongst themselves in $\dfrac{4!}{2!2!} = 6$ ways.

Hence the total number of arrangements = 10080 × 6 = **60480.**

Example 8. ***How many different words can be formed with the letters of the word 'BHARAT'***

(i) In how many of these B and H are never together.

(ii) How many of these begin with B and end with T ? [M.D.U. 2013

Solution. Out of letters in the word 'BHARAT', two letters *i.e.*, A's are alike.

$$\therefore \text{ Number of permutations} = \frac{6!}{2!} = 360.$$

(*i*) Number of words in which B and H are never together

= Total number of words – number of words in which B and H are together

$$= 360 - \frac{5!}{2!} . 2 = 360 - 120 = 240.$$

(*ii*) Those letters which begin with B and end at T, have only four places to be filled out of the remaining four letters, *i.e.*, H, A, R, A out of which 2 are alike

Number of such words $= \dfrac{4\,!}{2\,!} = \mathbf{12}$.

EXERCISE 4.5

1. In how many ways can 5 apples be distributed among 4 boys, there being no restriction to the number of apples each boy may get.

2. In how many ways can 4 prizes be given to 3 boys when a boy is eligible for all the prizes.

3. A boy has 3 pockets. In how many ways can he put 6 marbles in his pockets.

4. How many numbers less than 100 can be formed with the digits 1, 3, 5, 7, 9 when the digits may be repeated ? [*M.D.U. 2013*]

5. How many four digit numbers can be formed with the digits 1, 2, 3, 4, 5, 6 when a digit may be repeated any number of times in any arrangement ?

6. How many 3 digit numbers can be formed by using the digits 0, 1, 3, 5, 7 when the digits may be repeated any number of times.

7. Find the number of permutations that can be made out of the letters of the words taken all together :

   (*i*) INDIA (*ii*) EXAMINATION

   (*iii*) INDEPENDENCE (*iv*) ASSASSINATION

   (*v*) ACCOMODATION.

8. Find the number of arrangements that can be made out of the letters of the word MATHEMATICS. In how many of these vowels occur together.

9. Find the number of arrangements that can be made out of the letters of the word PERMUTATION. In how many of these 5 vowels occur together.

10. How many arrangements can be made with the letters of the word 'SERIES' ? How many of these begin and end with 'S' ?

11. In how many different ways, the letters of the word 'ALGEBRA' can be arranged in a row if

    (*i*) the two A's are together

    (*ii*) the two A's are not together.

12. How many 7 digit numbers can be formed using the digits 1, 2, 0, 2, 4, 2 and 4. [*M.D.U. 2012*]

13. How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 taken all together so that the odd digits always occupy the odd places.

14. How many different numbers of six digits can be formed out of the digits 1, 1, 1, 2, 2, 3 ? How many of them are greater than 300000 ?

15. How many numbers greater than 50,000 can be formed by the digits 3, 5, 6, 6, 7 ? [M.D.U. 2011, 01]

ANSWERS

1. 4^5	2. $3^4 = 81$	3. $3^6 = 729$
4. 25	5. 1296	6. 100
7. (i) 60	(ii) 4989600	(iii) 1663200
(iv) 10810800	(v) 19958400	
8. 4989600; 120960	9. 19958400; 302400	10. 180; 12
11. (i) 720	(ii) 1800	12. 360
13. 18	14. 60; 10	15. 48.

4.11. CIRCULAR PERMUTATIONS

To find the number of ways of arranging n persons at a round table (i.e., in a circle).

Let us consider the arrangement of five persons denoted by letters A, B, C, D, E sitting in a round table.

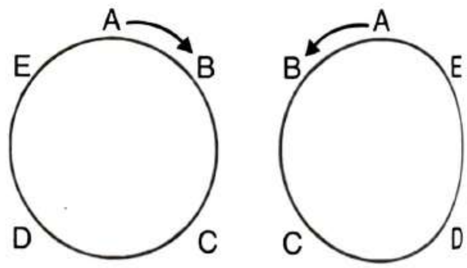

Fig. 4.1

Let us consider any one of the two arrangement as shown in the figures. Starting with different letters, this circular arrangement canbe read in the clock-wise direction in any one of the following five ways *viz.*, ABCDE, BCDEA, CDEAB, DEABC, EABCD.

These are five different linear arrangements, but are equivalent to a single circular arrangement as they are not distinct arrangements. Thus one circular arrangement of five persons gives rise to 5 different linear arrangements.

Now if the total number of circular arrangements of 5 persons be x, the total number of linear arrangements will be $5x$.

But the total number of linear arrangements of 5 persons = 5 !

$$\therefore \quad 5x = 5\,! \quad \Rightarrow \quad x = \frac{5!}{5} = 4\,!.$$

Hence the number of ways in which 5 persons can sit at a round table = 4 !.

Similarly, if there are n persons to be seated at a round table, the number of such arrangements = $(n-1)$!.

Alternative Method. When seats are not numbered at a round table, there is no distinction between the first and last seat, only changes in the relative positions are required. Therefore we are to imagine any one of the n persons to be fixed at one seat and then arrange the remaining $(n-1)$ persons which can be done in ${}^{n-1}P_{n-1}$ *i.e.*, $(n-1)$! ways. Hence the total number of ways = $(\boldsymbol{n-1})$ **!**.

4.12. CLOCKWISE AND ANTICLOCKWISE ARRANGEMENTS

In the circular permutations formed above, we have distinguished between clockwise and anticlockwise arrangements. In two corresponding arrangements of this type, every person has got the same neighbours. The only difference is that the right hand neighbour in one case changes into left hand neighbour in the other. If, however, no distinction be made in the clockwise and anticlockwise arrangements, then n persons can be seated round a table in $\frac{1}{2}(n-1)!$ ways.

In a similar way, if we have to form a necklace with n beads of different colours or a garland with n flowers of different colours, there is no distinction between clockwise and anticlockwise arrangements and thus the required number of arrangements is $\frac{1}{2}(n-1)!$.

SOLVED EXAMPLES

Example 1. ***In how many ways can 6 persons be seated***

(i) in a line ***(ii) at a round table.***

Solution. (*i*) The number of ways in which 6 persons can be seated in a line = 6 ! = 720.

(*ii*) The number of ways in which 6 persons can be seated at a round table

$$= (6-1)! = 5! = \mathbf{120}.$$

Example 2. ***In how many ways can 5 ladies and 5 gentlemen be seated at a round table, so that no two ladies are seated together ?***

Solution. 5 gentlemen can be seated round a table in $(5-1)! = 4!$ ways.

Since no two ladies are together, there are 5 places for the ladies (one between the two men).

$\therefore$ 5 ladies can be seated in 5 places in ${}^5P_5 = 5!$ ways

∴ By the principle of association, total number of arrangements

$$= 4\,! \times 5\,! = 24 \times 120 = \mathbf{2880}.$$

Example 3. ***3 boys and 3 girls are to be seated around a table in a circle. Among them, the boy X does not want any girl neighbour and the girl Y does not want any boy neighbour. How many such arrangements are possible ?***

Solution. Let the positions of boy X and girl Y be fixed as in adjoining figure, so that boy X does not have girl neighbour and girl Y does not have boy neighbour.

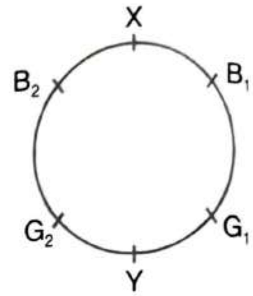

Fig. 4.2

Now the two boys B_1 and B_2 can be arranged amongst themselves in 2 ! ways.

Similarly, the two girls G_1 and G_2 can be arranged amongst themselves in 2 ! ways.

Hence the total number of arrangements = 2 ! × 2 ! = **4.**

Example 4. ***In how many ways can 5 beads of different colours form a necklace.***

Solution. Since the clockwise and anti-clockwise arrangements of the beads is the same, hence the total number of arrangements $= \frac{1}{2} . 4\,! = \mathbf{12.}$

EXERCISE 4.6

1. In how many ways can 8 students be arranged in
   (i) a line 8! (ii) a circle 7!
2. The 11 cabinet ministers of a state sit together to discuss a report. In how many ways can they seat themselves at a round table so that the Education and the Revenue Ministers sit together. [M.D.U. 2013]
3. The Prime Minister of 8 countries sit at a round table. In how many ways can they sit, if Indian and Pakistan Prime Minister are not to sit together.
4. There are 3 black, 4 blue and 5 red balls. In how many ways can they be arranged (i) in a row (ii) around a circle.
5. In how many ways can 5 persons A, B, C, D and E sit around a circular table if
   (i) B and D sit next to each other. 3!.2
   (ii) A and D do not sit next to each other.
6. Find the number of ways in which 8 different beads can be arranged to form a necklace.

ANSWERS

1. (i) 40320 (ii) 5040
2. 725760
3. 3600
4. (i) $\frac{12!}{3!4!5!}$ (ii) $\frac{11!}{3!4!5!}$
5. (i) 12 (ii) 12
6. 2520

4.13. COMBINATIONS

The different groups that can be formed by choosing r things from a given set of n different things, ignoring their order of arrangement, are called combinations of n things taken r at a time.

The number of all such combinations is denoted by $^n\mathbf{C}_r$ or $\mathbf{C}(n, r)$.

4.14. SOME THEOREMS ON COMBINATIONS

Theorem I. ***The number of combinations of n different things taken r at a time is given by*** $^n\mathbf{C}_r = \dfrac{n!}{r!(n-r)!}$, $0 < r \le n$

Proof. Let x be the required number of combinations of n things taken r at a time *i.e.*, $^n\mathrm{C}_r = x$.

Each combination contains r different things which can be arranged amongst themselves in $r!$ ways.

Thus each combination gives rise to $r!$ permutations.

$\therefore$ x combinations will give rise to $x \cdot r!$ permutations.

But the number of permutations of n things taken r at a time $= {}^n\mathrm{P}_r$

$$\therefore \quad x \cdot r! = {}^n\mathrm{P}_r$$

$$\Rightarrow \quad x = \frac{^n\mathrm{P}_r}{r!} = \frac{n!}{r!\,(n-r)!} \qquad \left[\because {}^nP_r = \frac{n!}{(n-r)!}\right]$$

$$\therefore \quad {}^n\mathbf{C}_r = \frac{n!}{r!(n-r)!}, \quad 0 < r \le n$$

Remarks:

1. We have $^n\mathrm{C}_r = \dfrac{n!}{r!(n-r)!}$

In particular, if $r = n$, then $^n\mathrm{C}_n = \dfrac{n!}{n!\ 0!} = \dfrac{1}{0!} = 1.$ $\qquad [\because 0! = 1]$

2. $^nC_0 = \dfrac{n!}{0!(n-0)!} = \dfrac{n!}{0!\,n!} = \dfrac{1}{0!} = 1$

Thus the formula $^nC_r = \dfrac{n!}{r!(n-r)!}$ is applicable for $r = 0$ also.

Hence, $\boldsymbol{^nC_r = \dfrac{n!}{r!(n-r)!}, 0 \le r \le n.}$

3. $^nC_r = \dfrac{n!}{r!(n-r)!} = \dfrac{n(n-1)(n-2)....(n-r+1)(n-r)(n-r-1)....2.1}{r!.[(n-r)(n-r-1)....3.2.1]}$

$$\therefore \qquad \boldsymbol{^nC_r = \frac{n(n-1)(n-2)......r \text{ factors}}{r!}}.$$

Theorem II. ***For*** $\boldsymbol{0 \le r \le n}$***, prove that*** $\boldsymbol{^nC_r = {}^nC_{n-r}}$

Proof. $^nC_{n-r} = \dfrac{n!}{(n-r)!\,[n-(n-r)]!}$

$$= \frac{n!}{(n-r)!\ r!} = {}^nC_r. \qquad \left[\because\ ^nC_r = \frac{n!}{r!(n-r)!}\right]$$

Hence, $\boldsymbol{^nC_r = {}^nC_{n-r}}$

i.e., selecting r objects out of n objects is same as rejecting $(n-r)$ objects.

Remark :

If $\quad ^nC_p = {}^nC_q \quad \Rightarrow \quad ^nC_p = {}^nC_q = {}^nC_{n-q} \qquad [\because\ ^nC_r = {}^nC_{n-r}]$

$\Rightarrow$ Either $\quad p = q \quad$ or $\quad p = n - q$

$\Rightarrow \quad p = q \quad$ or $\quad p + q = n$

Thus, $\quad \boldsymbol{^nC_p = {}^nC_q \quad \Rightarrow \quad p = q \quad \text{or} \quad p + q = n.}$

Theorem III. ***If n and r are natural numbers such that*** $\boldsymbol{1 \le r \le n}$***, then***

$$\boldsymbol{^nC_r + {}^nC_{r-1} = {}^{n+1}C_r.}$$

[M.D.U. 2010]

Proof. L.H.S. $= {}^nC_r + {}^nC_{r-1} = \dfrac{n!}{r!(n-r)!} + \dfrac{n!}{(r-1)!(n-r+1)!}$

$$= \frac{n!}{r(r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{1}{r}+\frac{1}{n-r+1}\right] = \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{n-r+1+r}{r(n-r+1)}\right]$$

$$= \frac{(n+1)\,n!}{[r(r-1)!][(n-r+1)(n-r)!]}$$

$$= \frac{(n+1)!}{r!\,(n-r+1)!} = {}^{n+1}C_r = \text{R.H.S.}$$

Hence, $\mathbf{{}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r}$.

Theorem IV. ***If n and r are natural numbers such that $1 \le r \le n$, then***

$$\boldsymbol{{}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}}.$$

Proof. L.H.S. $= {}^nC_r + {}^nC_{r+1} = \dfrac{n!}{r!\,(n-r)!} + \dfrac{n!}{(r+1)!\,(n-r-1)!}$

$$= \frac{n!}{r!\,(n-r)(n-r-1)!} + \frac{n!}{(r+1)r!\,(n-r-1)!}$$

$$= \frac{n!}{r!\,(n-r-1)!}\left[\frac{1}{n-r}+\frac{1}{r+1}\right]$$

$$= \frac{n!}{r!\,(n-r-1)!}\left[\frac{r+1+n-r}{(n-r)(r+1)}\right]$$

$$= \frac{n!\,(n+1)}{(r+1)\,r!\,(n-r)(n-r-1)!}$$

$$= \frac{(n+1)!}{(r+1)!\,(n-r)!} = {}^{n+1}C_{r+1} = \text{R.H.S.}$$

Hence, $\mathbf{{}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}}$.

Theorem V. ***If n and r are natural numbers, such that $1 \le r \le n$, then***

$$\boldsymbol{\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}}.$$

Proof. L.H.S. $= \dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n!}{r!\,(n-r)!} \times \dfrac{(r-1)!\,(n-r+1)!}{n!}$

$$= \frac{(r-1)!\,(n-r+1)(n-r)!}{r(r-1)!\,(n-r)!} = \frac{n-r+1}{r} = \text{R.H.S}$$

Hence, $\mathbf{\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}}$.

SOLVED EXAMPLES

Example 1. ***Evaluate :***

(i) $^{11}C_3$ **(ii)** $^{10}C_8$ **(iii)** $^{100}C_{98}$ **(iv)** $^{61}C_{57} - {}^{60}C_{56}$

[M.D.U. 2017]

Solution. (i) $^{11}C_3 = \dfrac{11!}{3!(11-3)!} = \dfrac{11!}{3!\,8!}$ $\left[\because {}^nC_r = \dfrac{n!}{r!(n-r)!}\right]$

$$= \frac{11\times10\times9\times8!}{3\times2\times8!} = \mathbf{165.}$$

(ii) $^{10}C_8 = {}^{10}C_{10-8} = {}^{10}C_2$ $[\because {}^nC_r = {}^nC_{n-r}]$

$$= \frac{10\times9}{2\times1} = \mathbf{45.}$$ $\left[\because {}^nC_r = \dfrac{n(n-1)\text{.......}r\text{ factors}}{r!}\right]$

(iii) $^{100}C_{98} = {}^{100}C_{100-98} = {}^{100}C_2$

$$= \frac{100\times99}{2\times1} = \mathbf{4950.}$$

(iv) We know that $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

$\therefore$ $^{60}C_{57} + {}^{60}C_{56} = {}^{61}C_{57}$ [Putting $n = 60, r = 57$]

$\therefore$ $^{61}C_{57} - {}^{60}C_{56} = {}^{60}C_{57} = {}^{60}C_3$ $[\because {}^nC_r = {}^nC_{n-r}]$

$\therefore$ $= \dfrac{60\times59\times58}{3\times2\times1} = \mathbf{34220.}$

Example 2. ***Prove that :*** $^2C_1 + {}^3C_1 + {}^4C_1 = {}^3C_2 + {}^4C_2$.

Solution. L.H.S. $= {}^2C_1 + {}^3C_1 + {}^4C_1 = 2 + 3 + 4 = 9$ $[\because {}^nC_1 = n]$

R.H.S. $= {}^3C_2 + {}^4C_2$

$$= \frac{3!}{2!(3-2)!} + \frac{4!}{2!(4-2)!}$$

$$= \frac{3.2.1}{2.1.1} + \frac{4.3.2.1}{2.1.2.1} = 3 + 6 = 9$$

$\therefore$ L.H.S. = R.H.S.

Example 3. ***If $^{18}C_r = {}^{18}C_{r+2}$, find $^rC_5$.***

Solution. $^{18}C_r = {}^{18}C_{18-r}$ $[\because {}^nC_r = {}^nC_{n-r}]$

But $^{18}C_r = {}^{18}C_{r+2}$ [*Given*]

$\therefore$ $^{18}C_{18-r} = {}^{18}C_{r+2}$

$\Rightarrow$ $18 - r = r + 2 \Rightarrow r = 8$

$\therefore$ $$^rC_5 = {}^8C_5 = \frac{8\,!}{5\,!\,3\,!} = \frac{8 \times 7 \times 6 \times 5\,!}{3 \times 2 \times 1 \times 5\,!} = \mathbf{56}.$$

Example 4. ***If $^nC_{14} = {}^nC_{16}$, find $^nC_{28}$.*** [*M.D.U. 2013*]

Solution. We have $^nC_{14} = {}^nC_{16}$

$\Rightarrow$ either $14 = 16$ or $14 + 16 = n$ $[\because {}^nC_p = {}^nC_q \Rightarrow p = q \text{ or } p + q = n]$

$\Rightarrow$ $n = 30$ $[\because 14 \neq 16]$

$\therefore$ $^nC_{28} = {}^{30}C_{28}$

$\Rightarrow$ $$^nC_{28} = \frac{30\,!}{28\,!\,(30-28)\,!} = \frac{30\,!}{2\,!\,28\,!} = \frac{30 \times 29}{2 \times 1} = \mathbf{435}.$$

Example 5. ***If $12 . {}^nC_2 = {}^{2n}C_3$, find n.***

Solution. We have $12 . {}^nC_2 = {}^{2n}C_3$

$\therefore$ $$12 . \frac{n\,!}{2\,!\,(n-2)\,!} = \frac{2n\,!}{3\,!\,(2n-3)\,!} \qquad \left[\because {}^nC_r = \frac{n\,!}{r\,!\,(n-r)\,!}\right]$$

or $$12 . \frac{n(n-1)(n-2)\,!}{2\,!\,(n-2)\,!} = \frac{(2n)(2n-1)(2n-2)(2n-3)\,!}{3\,!\,(2n-3)\,!}$$

or $$6 . n(n-1) = \frac{2n(2n-1)(2n-2)}{6}$$

or $$18 . (n-1) = (2n-1)(2n-2)$$

or $$9 . (n-1) = (2n-1)(n-1)$$

$\Rightarrow$ $$9 = 2n - 1 \Rightarrow n = \mathbf{5}.$$

Example 6. ***Determine n, if $^{2n}C_3 : {}^nC_3 = 12 : 1$.***

Solution. We have $$\frac{^{2n}C_3}{^nC_3} = \frac{12}{1}$$

$$\Rightarrow \quad \frac{2n(2n-1)(2n-2)}{3!} \times \frac{3!}{n(n-1)(n-2)} = 12$$

$$\Rightarrow \quad \frac{4n(2n-1)(n-1)}{n(n-1)(n-2)} = 12$$

$$\Rightarrow \quad 4(2n-1) = 12(n-2) \quad \Rightarrow \quad 2n-1 = 3(n-2) \quad \Rightarrow \quad n = \mathbf{5}.$$

Example 7. ***If*** $^{n+2}C_8 : {}^{n-2}P_4 = 57 : 16$***, find n.***

Solution. We have $^{n+2}C_8 : {}^{n-2}P_4 = 57 : 16$

$$\Rightarrow \quad \frac{^{n+2}C_8}{^{n-2}P_4} = \frac{57}{16}$$

$$\Rightarrow \quad \frac{\dfrac{(n+2)!}{(n-6)!\,8!}}{\dfrac{(n-2)!}{(n-6)!}} = \frac{57}{16}$$

$$\Rightarrow \quad \frac{(n+2)!\,(n-6)!}{(n-6)!\,.\,8!\,(n-2)!} = \frac{57}{16}$$

$$\Rightarrow \quad (n+2)(n+1)\,n\,(n-1) = \frac{57\,.\,8!}{16} = \frac{57\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{16}$$

$$\Rightarrow \quad (n+2)(n+1)\,n\,(n-1) = (19\times 3)\times 7\times 6\times 5\times 4\times 3$$

$$= (7\times 3)\times(5\times 4)\times 19\times(6\times 3)$$

$$= 21\times 20\times 19\times 18$$

Hence, $n = \mathbf{19}$.

Example 8. ***Prove that*** $^nC_r \times {}^rC_s = {}^nC_s \times {}^{n-s}C_{r-s}$.

Solution. L.H.S. $= {}^nC_r \times {}^rC_s$

$$= \frac{n!}{r!\,(n-r)!} \times \frac{r!}{s!\,(r-s)!} = \frac{n!}{s!\,(n-r)!\,(r-s)!} \qquad \text{... (1)}$$

R.H.S. $= {}^nC_s \times {}^{n-s}C_{r-s}$

$$= \frac{n!}{s!\,(n-s)!} \times \frac{(n-s)!}{(r-s)!\,[(n-s)-(r-s)]!}$$

$$= \frac{n!}{s!\,(n-s)!} \times \frac{(n-s)!}{(r-s)!\,(n-r)!} = \frac{n!}{s!\,(r-s)!\,(n-r)!} \qquad \text{...(2)}$$

From (1) and (2), L.H.S. = R.H.S.

Example 9. ***If $m = {}^nC_2$, show that ${}^mC_2 = 3\,.\,{}^{n+1}C_4$.***

Solution. $$m = {}^nC_2 = \frac{n(n-1)}{2.1} = \frac{n(n-1)}{2} \qquad \ldots(1)$$

Again, $${}^mC_2 = \frac{m(m-1)}{2} = \frac{\frac{n(n-1)}{2}\left[\frac{n(n-1)}{2} - 1\right]}{2} \qquad [\textit{Using (1)}]$$

$$= \frac{n(n-1)(n^2-n-2)}{2.2.2} = \frac{n(n-1)(n-2)(n+1)}{8}$$

$$= \frac{3(n+1)\,n(n-1)(n-2)}{24} = \frac{3(n+1)\,n(n-1)(n-2)}{4\,!}$$

$$= \mathbf{3\,.\,{}^{n+1}C_4}.$$

Example 10. ***If ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$, then find ${}^rC_2$.***

Solution. We know that $$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} \qquad \ldots(1)$$

$$\therefore \qquad \frac{n-r+1}{r} = \frac{84}{36} \quad \Rightarrow \quad \frac{n-r+1}{r} = \frac{7}{3}$$

$$\Rightarrow \qquad 3n - 3r + 3 = 7r$$

$$\Rightarrow \qquad 3n - 10r + 3 = 0 \qquad \ldots(2)$$

Replacing r by $r + 1$ in (1), we have

$$\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n-r}{r+1}$$

$$\therefore \qquad \frac{n-r}{r+1} = \frac{126}{84} \quad \Rightarrow \quad \frac{n-r}{r+1} = \frac{3}{2}$$

$$\Rightarrow \qquad 2n - 2r = 3r + 3$$

$$\Rightarrow \qquad 2n - 5r - 3 = 0 \qquad \ldots(3)$$

Solving (2) and (3), we get $r = 3$

Hence, $${}^rC_2 = {}^3C_2 = \frac{3\,!}{2\,!\,(3-2)\,!} = \frac{3\,.\,2\,!}{2\,!} = \mathbf{3}.$$

Example 11. ***Prove that : $\dfrac{n(n+1)\ldots\ldots(n+r-1)}{r\,!}$ is an integer.***

Solution. $\dfrac{n(n+1)(n+2)......(n+r-1)}{r!} = \dfrac{(n+r-1)(n+r-2)......(n+2)(n+1).\,n}{r!}$

$$= \frac{(n+r-1)(n+r-2)......(n+2)(n+1).\,n \times (n-1)!}{r!\,(n-1)!}$$

$$= \frac{(n+r-1)!}{r!\,(n-1)!} = {}^{n+r-1}C_r.$$

which must be an integer, as number of combinations of $(n + r - 1)$ things taken r at a time is denoted by ${}^{n+r-1}C_r$.

EXERCISE 4.7

1. Evaluate :

(i) ${}^{9}C_3$ (ii) ${}^{20}C_{18}$ (iii) ${}^{100}C_{97}$

(iv) ${}^{13}C_6 + {}^{13}C_5$ (v) ${}^{18}C_{14} - {}^{17}C_{13}$ (vi) ${}^{31}C_{26} - {}^{30}C_{26}$

2. Prove that :

(i) ${}^{8}C_4 + {}^{8}C_3 = {}^{9}C_4$ (ii) $1 + {}^{3}C_1 + {}^{4}C_2 = {}^{5}C_3$

3. (i) If ${}^{n}C_2 = {}^{n}C_6$, find n.

(ii) If ${}^{n}C_{10} = {}^{n}C_{12}$, find n and hence find ${}^{n}C_5$. [M.D.U. 2014]

4. (i) If ${}^{n+1}C_{n-1} = 36$, find n.

(ii) If ${}^{n-4}C_4 = 15$, find n.

5. Find n, if ${}^{2n}C_3 : {}^{n}C_3 = 11 : 1$.

6. (i) If ${}^{n+1}C_{r+1} : {}^{n}C_r : {}^{n-1}C_{r-1} = 11 : 6 : 3$, find n and r. [M.D.U. 2009]

(ii) If ${}^{n-1}C_r : {}^{n}C_r : {}^{n+1}C_r = 6 : 9 : 13$, find n and r. [M.D.U. 2014]

7. If $\dfrac{{}^{n}C_r}{{}^{n}C_{r+1}} = \dfrac{1}{2}$ and $\dfrac{{}^{n}C_{r+1}}{{}^{n}C_{r+2}} = \dfrac{2}{3}$; determine the values of n and r.

8. (i) If ${}^{4}P_2 = n\,.\,{}^{4}C_2$; find n.

(ii) If ${}^{n}P_r = 1320$, ${}^{n}C_r = 220$, find n and r. [M.D.U. 2015, 12]

9. If ${}^{n}P_r = {}^{n}P_{r+1}$ and ${}^{n}C_r = {}^{n}C_{r-1}$, find the values of n and r. [M.D.U. 2011]

10. Prove that :

(i) $r\,.\,{}^{n}C_r = n\,.\,{}^{n-1}C_{r-1}$ (ii) ${}^{n}C_r + 2\,.\,{}^{n}C_{r-1} + {}^{n}C_{r-2} = {}^{n+2}C_r$.

11. If $1 \le r \le n$; prove that $n \cdot {}^{n-1}C_{r-1} = (n - r + 1) \cdot {}^{n}C_{r-1}$.

12. Prove that $^{2n}C_n = \dfrac{2^n [1 . 3 . 5 ...... (2n-1)]}{n!}$.

ANSWERS

1. (i) 84 (ii) 190 (iii) 161700
(iv) 3003 (v) 680 (vi) 142506

3. (i) 8 (ii) 22; 26334

4. (i) 8 (ii) 10

5. 6

6. (i) $n = 10, r = 5$ (ii) $n = 12, r = 4$

7. $n = 14, r = 4$

8. (i) 2 (ii) $n = 12, r = 3$

9. $n = 3, r = 2$.

4.15. PRACTICAL PROBLEMS ON COMBINATIONS

In this section, we shall discuss problems related to real life where the formula for ^nC_r and its concept can be applied.

SOLVED EXAMPLES

Example 1. ***From a class of 30 students, 3 are to be chosen for a competition. In how many way can they be chosen.***

Solution. 3 students out of 30 can be chosen in

$$^{30}C_3 = \frac{30!}{3!\,(30-3)!} = \frac{30!}{3!\,27!}$$

$$= \frac{30 \times 29 \times 28 \times 27!}{3 \times 2 \times 1 \times 27!} = \frac{30 \times 29 \times 28}{3 \times 2 \times 1} = \textbf{4060 ways.}$$

Example 2. ***Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls, if each selection consists of 3 balls of each colour.***

Solution. The required number of ways of selecting 9 balls from 6 red, 5 white and 5 blue balls

$$= {}^6C_3 \times {}^5C_3 \times {}^5C_3 = \frac{6!}{3!\,(6-3)!} \times \frac{5!}{3!\,(5-3)!} \times \frac{5!}{3!\,(5-3)!}$$

$$= \frac{6 \times 5 \times 4 \times 3!}{3 \times 2 \times 1 \times 3!} \times \frac{5 \times 4 \times 3!}{2 \times 1 \times 3!} \times \frac{5 \times 4 \times 3!}{2 \times 1 \times 3!} = \textbf{2000.}$$

Example 3. ***In how many ways can 10 identical books on English and 14 identical books on Mathematics be placed in a row on a shelf so that no two books on English are together.***

Solution. We first arrange the 14 books on Mathematics in a row on a shelf and this can be done in 1 way.

Now, after arranging 14 books on Mathematics, we have to arrange the 10 books on English in such a manner that no two English books are together.

Since we have 14 books on Mathematics, therefore we have 15 places marked by X for arranging 10 books on English as shown below :

X M X M X M X M X M X M X M X M X M X M X M X M X M X M X

Now, 10 English books can be arranged in 15 places marked by X between the 14 Mathematics books in ${}^{15}C_{10}$ ways.

Hence, the required number of ways $= 1 \times {}^{15}C_{10} = {}^{15}C_5 = \dfrac{15 \times 14 \times 13 \times 12 \times 11}{5 \times 4 \times 3 \times 2 \times 1}$

$= 3 \times 7 \times 13 \times 11 = \mathbf{3003}$.

Example 4. ***In how many ways can a cricket team of 11 players be selected out of 16 players. Also find the number of ways***

(i) if two particular players are always to be included.

(ii) if one particular player is to be excluded.

(iii) if two particular players are to be included and one particular player is to be rejected.

Solution. 11 players can be selected out of 16 players in ${}^{16}C_{11}$ ways

Hence, the required number of ways

$$= {}^{16}C_{11} = {}^{16}C_5 = \frac{16 \times 15 \times 14 \times 13 \times 12}{5 \times 4 \times 3 \times 2 \times 1} = \mathbf{4368}$$

(*i*) When two particular players are always to be included, then 9 more players are to be selected out of remaining 14 players, which can be done in ${}^{14}C_9$ ways

Hence, the required number of ways $= {}^{14}C_5 = \dfrac{14 \times 13 \times 12 \times 11 \times 10}{5 \times 4 \times 3 \times 2 \times 1} = \mathbf{2002}$

(*ii*) If one particular player is to be excluded, then selection is to be made of 11 players out of 15 players and this can be done in ${}^{15}C_{11}$ ways

Hence, the required number of ways $= {}^{15}C_4 = \dfrac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1} = \mathbf{1365}$

(*iii*) If two particular players are to be included and one particular player is to be rejected, then we have to select 9 more out of 13 in ${}^{13}C_9$ ways

Hence, the required number of ways $= {}^{13}C_4 = \dfrac{13 \times 12 \times 11 \times 10}{4 \times 3 \times 2 \times 1} = \mathbf{715}$.

Example 5. ***There are 6 boys and 3 girls in a class. A committee of 5 is to be formed such that it contains 3 boys and 2 girls. In how many ways can this be done. Also find the number of ways if atleast one girl is always in the committee.*** [M.D.U. 2013]

Solution. (a) 3 boys can be selected out of 6 boys in ${}^6C_3$ ways and 2 girls can be selected out of 3 in ${}^3C_2$ ways.

$\therefore$ Total number of ways $= {}^6C_3 \times {}^3C_2 = 20 \times 3 = \mathbf{60}$.

(b) For including atleast one girl, the following cases arise :

(i) 1 girl + 4 boys which can be done in ${}^3C_1 \times {}^6C_4 = 3 \times 15 = 45$ ways

(ii) 2 girls + 3 boys which can be done in ${}^3C_2 \times {}^6C_3 = 3 \times 20 = 60$ ways

(iii) 3 girls + 2 boys which can be done in ${}^3C_3 \times {}^6C_2 = 1 \times 15 = 15$ ways

$\therefore$ Total number of ways, if atleast one girl is always in the committee $= 45 + 60 + 15 = \mathbf{120}$.

Example 6. ***A man has 7 friends. In how many ways can he invite one or more of them to a party.***

Solution. A man may invite (i) one of them (ii) two of them .... (iii) all of them and this can be done in ${}^7C_1, {}^7C_2, {}^7C_3, \ldots\ldots, {}^7C_7$ ways respectively.

$$\therefore \quad \text{Total number of ways} = {}^7C_1 + {}^7C_2 + {}^7C_3 + {}^7C_4 + {}^7C_5 + {}^7C_6 + {}^7C_7$$
$$= 7 + 21 + 35 + 35 + 21 + 7 + 1 = \mathbf{127}.$$

Example 7. ***From a class of 12 boys and 10 girls, 10 students are to be chosen for a competition, including atleast 4 boys and 4 girls. The two girls who won the prizes last year should be included. In how many ways can the selection be made ?***

Solution. There are 12 boys and 10 girls in the class. We have to select 10 students for a competition including atleast 4 boys and 4 girls. Two girls who were last year's winner are to be included. Since two girls are already selected, therefore we are left with 8 girls out of which atleast 2 girls are to be selected.

$\therefore$ We can make the selection in the following ways :

Choice	*Boys*	*Girls (+ 2 particular girls)*
I	4	4 + 2
II	5	3 + 2
III	6	2 + 2

First choice can be made in ${}^{12}C_4 \times {}^8C_4 = \dfrac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} \times \dfrac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 34650$ ways

Second choice can be made in ${}^{12}C_5 \times {}^8C_3 = \dfrac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2 \times 1} \times \dfrac{8 \times 7 \times 6}{3 \times 2 \times 1} = 44352$ ways

Third choice can be made in ${}^{12}C_6 \times {}^8C_2 = \dfrac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{6 \times 5 \times 4 \times 3 \times 2 \times 1} \times \dfrac{8 \times 7}{2 \times 1} = 25872$ ways

Hence, total number of possible selections $= 34650 + 44352 + 25872 = \mathbf{104874}$.

Example 8. ***A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of***

(*i*) ***no girl*** (*ii*) ***exactly 3 girls***

(*iii*) ***atleast 3 girls*** (*iv*) ***atmost 3 girls***

(*v*) ***atleast one boy and one girl.***

Solution. (*i*) Since the committee is not to consist of any girl, therefore, only boys are to be selected. 7 boys out of 9 boys can be selected in $^9C_7$ ways.

Hence, the required number of ways = $^9C_7 = {^9C_2}$

$$= \frac{9 \times 8}{2 \times 1} = \mathbf{36.}$$

(*ii*) A committee of 7, consisting of exactly 3 girls, can be formed by selecting 3 girls out of 4 and 4 boys out of 9. This can done in $^4C_3 \times {^9C_4}$ ways

$$= 4 \times \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 504 \text{ ways}$$

Hence, the required number of ways = **504**.

(*iii*) A committee of 7, consisting of atleast 3 girls, can be formed in the following ways:

(*a*) Selecting 3 girls out of 4 and 4 boys out of 9, which can be done in $^4C_3 \times {^9C_4}$ ways

$$= 4 \times \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 504 \text{ ways.}$$

(*b*) Selecting 4 girls out of 4 and 3 boys out of 9, which can be done in $^4C_4 \times {^9C_3}$ ways

$$= 1 \times \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 3 \times 4 \times 7 = 84 \text{ ways}$$

Hence, the required number of ways = 504 + 84 = **588**.

(*iv*) A committee of 7, consisting of at the most 3 girls, can be formed in the following ways :

(*a*) Selecting all 7 boys out of 9 and no girl, which can be done in $^9C_7$ ways

$$= {^9C_2} \text{ ways} = \frac{9 \times 8}{2 \times 1} = 36 \text{ ways.}$$

(*b*) Selecting 6 boys out of 9 and 1 girl out of 4, which can be done in $^9C_6 \times {^4C_1}$ ways

$$= {^9C_3} \times 4 \text{ ways} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} \times 4 = 336 \text{ ways.}$$

(c) Selecting 5 boys out of 9 and 2 girls out of 4 which can be done in ${}^9C_5 \times {}^4C_2$ ways

$$= {}^9C_4 \times {}^4C_2 \text{ ways} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \times \frac{4 \times 3}{2 \times 1} = 756 \text{ ways.}$$

(d) Selecting 4 boys out of 9 and 3 girls out of 4, which can be done in ${}^9C_4 \times {}^4C_3$ ways

$$= {}^9C_4 \times {}^4C_1 \text{ ways} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \times 4 = 504 \text{ ways}$$

Hence, the required number of ways = 36 + 336 + 756 + 504 = **1632**.

(v) Since the committee has to consist of atleast one boy and one girl, therefore, the possibilities are

(a) 3 boys and 4 girls

(b) 4 boys and 3 girls

(c) 5 boys and 2 girls

(d) 6 boys and 1 girl

3 boys and 4 girls can be selected in ${}^9C_3 \times {}^4C_4 = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} \times 1 = 84$ ways

4 boys and 3 girls can be selected in ${}^9C_4 \times {}^4C_3 = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \times 4 = 504$ ways

5 boys and 2 girls can be selected in ${}^9C_5 \times {}^4C_2 = \frac{9 \times 8 \times 7 \times 6 \times 5}{5 \times 4 \times 3 \times 2 \times 1} \times \frac{4 \times 3}{2 \times 1} = 756$ ways

6 boys and 1 girl can be selected in ${}^9C_6 \times {}^4C_1 = {}^9C_3 \times 4 = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} \times 4 = 336$ ways

Hence, the required number of ways = 84 + 504 + 756 + 336 = **1680.**

Example 9. ***An examination paper consists of 12 questions divided into parts A and B. Part A contains 7 questions and part B contains 5 questions. A candidate is required to attempt 8 questions, selecting at least 3 from each part. In how many ways can the student select the questions.***

Solution. The candidate may select the 8 questions in the following manner :

(a) 3 from part A and 5 from part B

(b) 4 from part A and 4 from part B

(c) 5 from part A and 3 from part B

(a) Numbers of ways of selecting 3 questions out of 7 from part A = ${}^7C_3$ and number of ways of selecting 5 questions out of 5 from part B = ${}^5C_5$

$\therefore$ Total number of ways $= {}^7C_3 \times {}^5C_5 = 35 \times 1 = 35.$

(*b*) Number of ways of selecting 4 questions out of 7 from part A = $^7C_4$ and number of ways of selecting 4 questions out of 5 from part B = $^5C_4$

$\therefore$ Total number of ways $= {}^7C_4 \times {}^5C_4 = {}^7C_3 \times {}^5C_1$

$= 35 \times 5 = 175$

(*c*) Number of ways of selecting 5 questions out of 7 from part A = $^7C_5$ and number of ways of selecting 3 questions out of 5 from part B = $^5C_3$

$\therefore$ Total number of ways $= {}^7C_5 \times {}^5C_3 = {}^7C_2 \times {}^5C_2 = 21 \times 10 = 210.$

Hence the required number of ways of selecting the questions

$= 35 + 175 + 210 =$ **420**.

Example 10. ***From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can they be chosen ?***

Solution. The possibilities are :

(*i*) the particular 3 students join

(*ii*) the particular 3 students do not join.

In the first case, we have to choose 7 more students out of the remaining 22 students and this can be done in $^{22}C_7$ ways

$$= \frac{22 \times 21 \times 20 \times 19 \times 18 \times 17 \times 16}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 170544$$

In the second case, we have to choose 10 students out of the remaining 22 students and this can be done in $^{22}C_{10}$ ways

$$= \frac{22 \times 21 \times 20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13}{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 646646$$

Hence the required number of ways = 170544 + 646646 = **817190**.

Example 11. ***Find the number of ways of choosing 4 cards from a pack of 52 playing cards ? In how many of these***

(a) four cards are of the same suit ?

(b) all are face cards ?

(c) two are red cards and two are black cards ?

(d) cards are of the same colour ?

Solution. There will be as many ways of choosing 4 cards out of 52 cards as there are combinations of 52 different things, taken 4 at a time. Therefore, the required number of ways

$$= {}^{52}C_4 = \frac{52 \times 51 \times 50 \times 49}{4 \times 3 \times 2 \times 1} = 270725.$$

(a) In a pack of cards, there are four suits (diamond, club, spade and heart) and there are 13 cards of each suit.

Therefore, there are $^{13}C_4$ ways of choosing all 4 diamonds, $^{13}C_4$ ways of choosing all 4 clubs, $^{13}C_4$ ways of choosing all 4 spades and $^{13}C_4$ ways of choosing all 4 hearts.

Hence, the required number of ways $= {^{13}C_4} + {^{13}C_4} + {^{13}C_4} + {^{13}C_4}$

$$= 4 \times {^{13}C_4} = 4 \times \frac{13 \times 12 \times 11 \times 10}{4 \times 3 \times 2 \times 1} = \mathbf{2860}.$$

(b) There are 12 face cards in a pack of cards (4 kings, 4 queens and 4 jacks). All four face cards can be chosen in $^{12}C_4$ ways.

Hence the required number of ways $= {^{12}C_4} = \dfrac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = \mathbf{495}$.

(c) There are 26 red cards and 26 black cards in a pack of cards.

∴ Two red cards and two black cards can be chosen in $= {^{26}C_2} \times {^{26}C_2}$ ways

Hence, the required number of ways $= \left(\dfrac{26 \times 25}{2 \times 1}\right)^2 = (325)^2 = \mathbf{105625}$.

(d) 4 red cards can be selected out of 26 red cards in $^{26}C_4$ ways.

4 black cards can be selected out of 26 black cards in $^{26}C_4$ ways.

Hence, the required number of ways $= {^{26}C_4} + {^{26}C_4}$

$$= 2 \times {^{26}C_4} = 2 \times \frac{26 \times 25 \times 24 \times 23}{4 \times 3 \times 2 \times 1} = \mathbf{26900}.$$

Example 12. ***How many words each of 3 vowels and 2 consonants can be formed from the letters of the word INVOLUTE.*** [M.D.U. 2015]

Solution. The given word INVOLUTE has 4 vowels (I, O, U, E) and 4 consonants (N, V, L, T). Out of 4 vowels, 3 can be selected in $^4C_3$ ways and out of 4 consonants, 2 can be selected in $^4C_2$ ways.

Thus the total number of combinations of 5 letter words $= {^4C_3} \times {^4C_2} = 4 \times 6 = 24$.

Take any one of these combinations. There are 5 letters in it and they can be arranged amongst themselves in 5 ! ways.

∴ For 24 combinations, number of words formed are $24 \times 5! = 24 \times 120 = \mathbf{2880}$.

Example 13. ***There are 15 points in a plane, no three of which are collinear except 6 of them which are all on a line. How many***

(i) straight lines [M.D.U. 2009]

(ii) triangles can be formed by joining them.

Solution. (*i*) When all the 15 points are non-collinear, the number of lines (by joining two points) = $^{15}C_2 = 105$.

Also 6 points, which are collinear would have given $^6C_2 = 15$ lines, but they give only one line.

$\therefore$ Loss of lines = 15 − 1 = 14

Hence, number of lines formed = 105 − 14 = **91**.

(*ii*) When all the 15 points are non-collinear, the number of triangles (by joining 3 points) = $^{15}C_3 = 455$.

Also 6 collinear points would have given = $^6C_3 = 20$ triangles but they give no triangle.

$\therefore$ Loss of triangles = 20

Hence, number of triangles formed = 455 − 20 = **435**.

Example 14. ***A polygon has 44 diagonals. Find the number of its sides.***

[*M.D.U. 2014, 12, 11*]

Solution. Let the number of sides of polygon = n.

$\therefore$ Number of angular points = n.

Number of straight lines joining any two of these n points (non-collinear) = $^nC_2$

Now the number of sides of polygon = n

$\therefore$ Number of diagonals = $^nC_2 - n$.

But it is given that number of diagonals = 44

$$\therefore \quad {}^nC_2 - n = 44 \quad \Rightarrow \quad \frac{n(n-1)}{2} - n = 44$$

$$\Rightarrow \quad n^2 - n - 2n = 88 \quad \Rightarrow \quad n^2 - 3n - 88 = 0$$

$$\Rightarrow \quad (n-11)(n+8) = 0 \quad \Rightarrow \quad n = 11 \text{ or } n = -8$$

Hence, the required number of sides = **11**. [*Rejecting negative quantity*]

EXERCISE 4.8

1. In how many ways can 5 sportsmen be selected from a group of 10. [*M.D.U. 2015*]
2. How many different kinds of return tickets must be prepared for a railway line with 16 stations.
   16c2
3. In how many ways can a party of 4 be selected out of 9, if one man is always to be
   (*i*) included 8c4 8c3
   (*ii*) excluded. 8c4

4. In how many ways can a hockey eleven be chosen out of 15 players so as to
   (i) include 3 particular players
   (ii) exclude 3 particular players.

5. Twelve persons meet in a room and each shake hands with all the others. Find the number of hand-shakes. $^{12}C_2$

6. A bookshelf contains 7 different Mathematics text-books and 5 different Physics text-books. How many groups of 3 Mathematics and 3 Physics text-books can be selected ? $^7C_3 \times {}^5C_3$

7. In how many ways can a student choose 5 courses out of 9 courses, if two courses are compulsory for every student. $^7C_3$

8. In how many ways can we select a cricket eleven from 17 players in which 5 players can bowl. Each team must include 2 bowlers. $^5C_2 \times {}^{12}C_9$

9. A father with 8 children takes 3 at a time to the zoo, as often as he can without taking the same 3 children together more than once. How often will he go and how often each child will go. $^8C_3$, $^7C_2$

10. A committee is to consist of 4 men and 3 women. How many different committees are possible if 7 men and 5 women are eligible ? $^7C_4 \times {}^5C_3$

11. A committee of 8 is to be selected out of 6 males and 8 females. In how many ways can it be formed so that the males may not be out-numbered ? 6M, 2F 5M, 3F 4M, 4F

12. From 6 gentlemen and 4 ladies, a committee of 5 is to be formed, so as to include atleast one lady. In how many ways can this be done.

13. A committee of 7 is to be formed from 9 boys and 5 girls. In how many ways can this be done if it contains (i) 2 girls (ii) at least 2 girls.

14. In an examination, a candidate is required to pass in 4 different subjects. In how many ways can he fail. $^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4$

15. In an examination, Ritika has to select 4 questions from each part. There are 6, 7 and 8 questions in Part I, Part II and Part III, respectively. What is the number of possible combinations in which she can choose the questions ?

16. A man has six friends. In how many ways may he invite one or more of them to dinner ?

17. A box contains a hundred rupee note, a ten rupee note, a five rupee note and four different coins. How many different amounts can be drawn from it.

18. In an election there are 5 candidates and 3 members are to be elected. A voter can vote for any number of candidates not greater than the number to be elected. In how many ways can a voter choose to vote.

19. Out of 7 consonants and 4 vowels, how many words each containing 3 consonants and 2 vowels can be formed.

20. How many (i) straight lines (ii) triangles can be formed by joining 12 points, 7 of which are in the same line. [M.D.U. 2017, 12, 10]

ANSWERS

1. 252	**2.** 120	**3.** (*i*) 56 (*ii*) 70	**4.** (*i*) 495 (*ii*) 12
5. 66	**6.** 350	**7.** 35	**8.** 2200
9. 56; 21	**10.** 350	**11.** 1414	**12.** 246
13. 1260; 2976	**14.** 15	**15.** 36750	**16.** 63
17. 127	**18.** 25	**19.** 25200	**20.** (*i*) 46 (*ii*) 185.

4.16. DIVISION INTO GROUPS

To find the number of ways in which (m + n) things can be divided into two groups containing m and n things respectively.

The number of ways in which m things can be selected out of $(m + n)$ things $= {}^{m+n}C_{m}$.

[*Each time n things are left out*]

$\therefore$ Number of ways of division of groups $= \dfrac{(m+n)!}{m!\,n!}$

Cor. 1. If $2m$ things are to be divided equally between two groups (say two persons) containing m things each, then the number of sub-divisions $= \dfrac{2m!}{(m!)^2}$.

If the two groups are equal and no distinction is possible between the groups, then the two groups can be interchanged without giving a new group.

$\therefore$ Number of different sub-divisions $= \dfrac{2m!}{2!\,(m!)^2}$.

Cor. 2. Similarly $(m + n + p)$ things can be divided into three groups containing m, n and p things respectively in $\dfrac{(m+n+p)!}{m!\,n!\,p!}$ sub-divisions.

4.17. THEOREM

To prove that the total number of ways in which a selection can be made out of (p + q + r + ...) things of which p are all alike and of one kind, q all alike and of 2nd kind, r all alike and of 3rd kind ... = [(p + 1)(q + 1)(r + 1)...] – 1.

Proof. There are $(p + 1)$ ways of dealing with p like things (we may take 0 or 1 or 2 or 4 ... or p).

Similarly q like things can be dealt with in $(q + 1)$ ways.

Proceeding like this, the total number of selections $= (p + 1)(q + 1)(r + 1)...$

But we have to reject the case in which none of the things is taken.

$\therefore$ Required number of ways = $[(p + 1)(q + 1)(r + 1)...] - 1$

Cor. 1. For n groups, each containing p similar things, number of selections

$$= (p + 1)^n - 1.$$

Cor. 2. For all n things different, number of selections

$$= [(1 + 1)(1 + 1) \ldots n \text{ times}] - 1 = 2^n - 1.$$

SOLVED EXAMPLES

Example 1. ***In how many ways can 11 distinct things be divided into two groups containing 5 and 6 things.***

Solution. Here 11 distinct things are to be divided into two groups of 5 and 6 things. Thus there will be 2 distinct groups.

Hence the number of ways $= \dfrac{11!}{5!\,6!} = \mathbf{462}$.

Example 2. ***In how many ways can 8 things be divided into two groups of 4 each.***

Solution. 8 things are to be divided into two groups of 4 each. Therefore no distinction is possible between the groups.

Hence the required number of ways $= \dfrac{2m!}{2!\,(m!)^2} = \dfrac{8!}{2!\,(4!)^2} = \mathbf{35}$.

Example 3. ***In how many ways can 52 playing cards be distributed equally among,***

(i) four heaps ***(ii) four players.***

Solution. (*i*) Here 52 cards are to be divided equally into 4 heaps of 13 cards each and thus the three heaps are identical.

Hence the required number of ways $= \dfrac{52!}{4!\,(13!)^4}$

(*ii*) Since 52 cards are to be divided equally among 4 players and thus the groups are distinct. Hence the required number of ways $= \dfrac{52!}{(13!)^4}$.

Example 4. ***A person invites 13 guests to dinner and places 8 of them at one round table and remaining 5 at the other. Find the number of ways in which he can arrange the guests.***

Solution. The 13 guests can be divided in two groups of 8 and 5 persons in

$$\frac{(8+5)!}{8!\,5!} = \frac{13!}{8!\,5!}$$

Now 8 persons can be arranged on one round table in $(8-1)! = 7!$ ways and 5 persons can be arranged on other round table in $(5-1)! = 4!$ ways.

Hence the total number of arrangements $= \dfrac{\mathbf{13!}}{\mathbf{8!\,5!}} \times \mathbf{7! \times 4!}$.

EXERCISE 4.9

1. In how many ways 9 distinct things can be divided into two groups containing 5 and 4 things. [*M.D.U. 2012*]
2. 7 distinct things are to be arranged in 3 groups of 2, 2 and 3 things respectively. Find the number of ways.
3. In how many ways can 6 things be divided into two groups of 3 each?
4. In how many ways can 12 things be equally divided among 4 persons.
5. In how many ways can 18 different coins be divided equally $\frac{18!}{(6!)^3}$
   (*i*) into 3 heaps $\frac{18!}{3!(6!)^3}$ (*ii*) among 3 persons.
6. In how many ways can a selection be made out of 4 red, 2 white and 2 blue identical balls ? $\frac{8!}{4!\,2!\,2!}$ [*M.D.U. 2016, 13*]
7. A box has 6 hundred rupees notes, 5 fifty rupee notes and 3 ten rupee notes. How many different amounts can be drawn from it.

ANSWERS

1. 126	**2.** 105	**3.** 10	**4.** 369600
5. (*i*) 2858856	(*ii*) 17153136	**6.** 44	**7.** 167

$\frac{28 \times 7 \times 6 \times 5}{2 \times 1}$