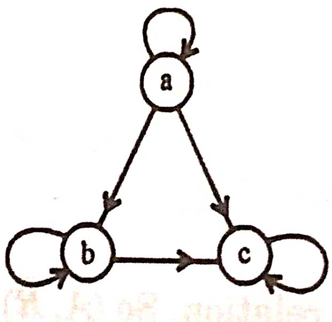
28. HASSE-DIAGRAMS

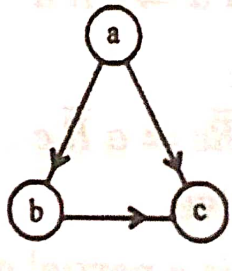
A diagram that is used to describe partial order relation associated with a set is called Hasse diagram. The steps given below are necessary to read Hasse diagrams for a relation.

Let A is a set and R is a relation.

(i) Since a partial order relation is reflexive therefore every vertex is related to itself, so the arrows from a vertex to itself in a diagraph are removed. Thus the diagraph shown in fig. (a) would be drawn as shown in fig. (b).

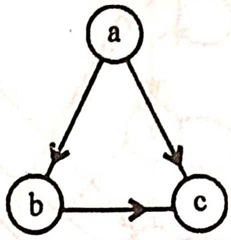


(a)



(b)

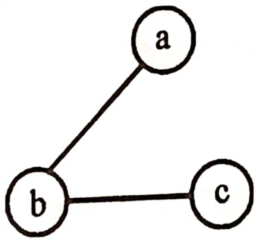
(ii) Also a partial order relation is antisymmetric so all arrows connecting two vertices are removed *i.e.*, if $a \leq b$ and $b \leq a$ is not necessarily unless $a = b$ in this case direction of arrows between a and b is not necessary as shown in figure.



(c)

(iii) Since a partial order relation is transitive so all edges that are implied by this property are removed thus if $a \leq b$ and $b \leq c$ then it implies that $a \leq c$ so the edge from a to c is removed as shown in the figure (d).

(iv) Circles in the diagraph representing vertices (or nodes) are replaced by dots (or points) thus the Hasse diagram shown in figure in the final form of figure (e).

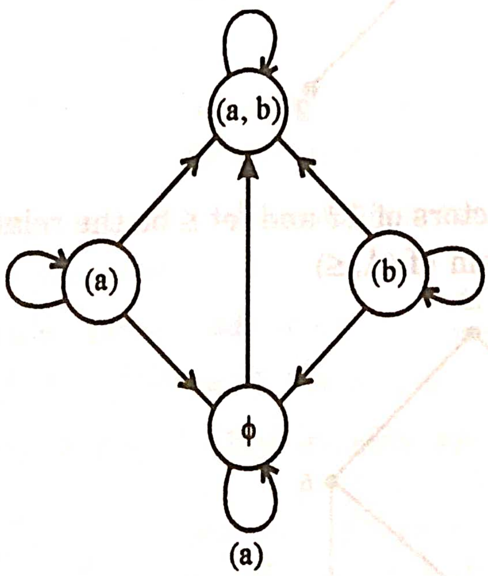


(d)

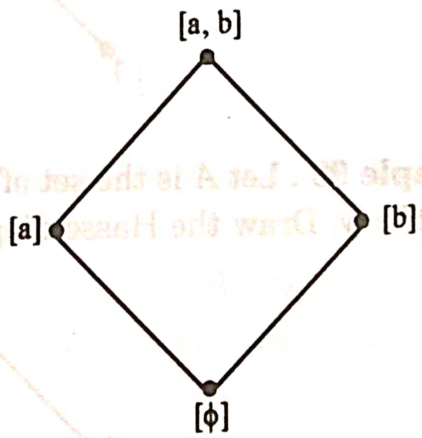
(e)

Example 90 : Let $A = [a, b]$ is a set and $P(A) = [(a), (b), \phi, (a, b)]$ be the power set of A. Consider the partial order relation of subsets ($\leq$) on A. Draw the diagraph and Hasse diagram of the poset $(A, \leq)$.

Solution : The diagraph of the given poset is shown in fig. (a) and the Hasse diagram is shown in figure as :



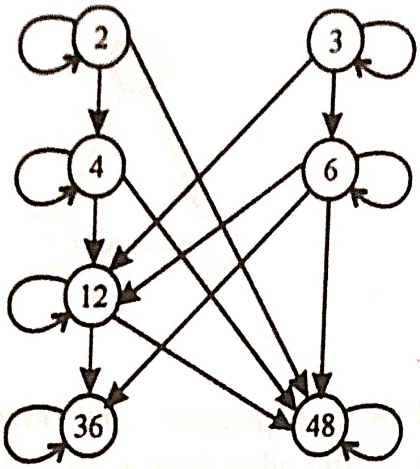
(a)



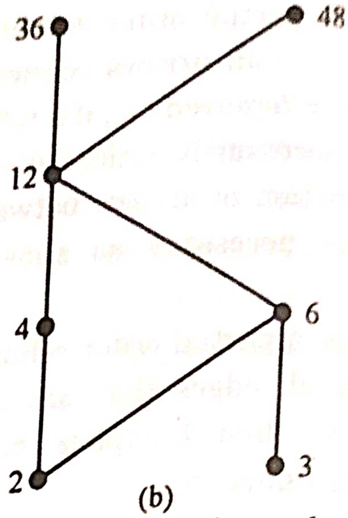
(b)

Example 91 : Let $A = (2, 3, 4, 6, 12, 36, 48)$ be a non empty set and R be a partial order relation of divisibility on A. That is if $a, b \in A$ then a divides b. Draw the diagraph and Hasse diagram of the relation.

Solution. The diagraph and the Hasse diagram of the given problem are :



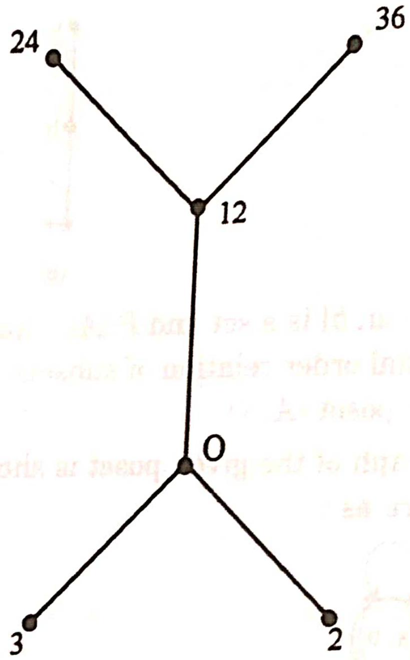
(a)



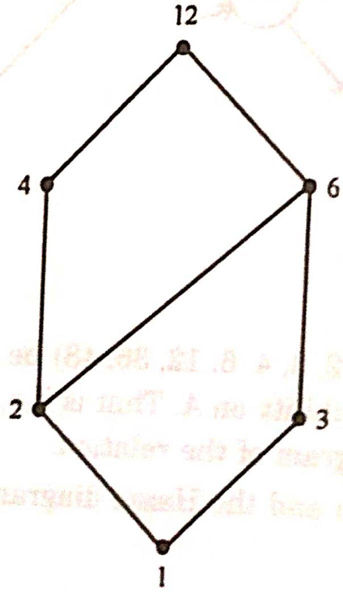
(b)

Example 92 : Let $A = (2, 3, 6, 12, 24, 36)$ and relation $\leq$ be such that $x \leq y$ if x divides y written as (x/y). Draw the Hasse-diagram of $(A, \leq)$.

Solution : The Hasse diagram of the given problem is



Example 93 : Let A is the set of factors of 12 and let $\leq$ be the relation divide i.e., $x \leq y$ iff x/y. Draw the Hasse diagram of $(A, \leq)$.



Example 93 : For set $A = [1, 2, 3, 4, 5, 6]$ and $\leq$ be the relation 'less than or equal to', Draw the Hasse diagram of $(A, \leq)$.

Solution : The Hasse diagram is shown in the given figure.

29. FUNCTIONS OR MAPPINGS

Let X and Y be two non empty sets. A rule or a correspondence which associates to each element $x \in X$ to a unique element $y \in Y$ is called a function or a mapping from X to Y and written as $f : X \to Y$.

i.e. $\quad f : X \to Y$ (reads "f maps X into Y")

The element y is called the **image** of x under f is denoted by $f(x)$ *i.e.* $y = f(x)$; and x is called the **pre-image** of y.

Domain : The set X is called domain of the function f.

Codomain : The set Y is called the co domain of the function f.

Range : The set $f[X] = [f(x) : x \in X]$ consisting all images of the elements of X under the mapping f is called the range of f. In general $f[X] \subseteq Y$.

→ If $f : X \to Y$, then a single element in X can not have more than one image in Y. However two or more than two elements in x may have the same image in Y.

→ Every element in X must have its image in Y but every element in Y may not have its pre image in X.

Example 95 : Let $X = [-1, 1, 2, 3]$, $Y = [0, 1, 4, 9]$. Consider the rule $f : X \to Y : f(x) = x^2 \; \forall \; x \in X$. Then it is clear that f associates to each $x \in X$ to a unique element $x^2 \in Y$. Thus we may write

$$f(1) = 1, \quad f(2) = 4; \quad f(3) = 9$$

Clearly	Domain	$(f) = (-1, 1, 2, 3)$
and	Range	$(f) = (1, 4, 9)$.

30. ALTERNATIVE DEFINITION OF FUNCTION (As a set of ordered pair)

Let X and Y be two non-empty set than the mapping or function f defined from X to Y is a subset $f(X)$ of $X \times Y$ satisfying the following conditions.

Posets, Lattices and Boolean Algebr

PARTIALLY ORDERED SET

Definition. Consider a relation R on a set S satisfying the following properties :

1. R is reflexive *i.e.*, $x R x$ for every $x \in S$.
2. R is antisymmetric *i.e.*, if $x R y$ and $y R x$, then $x = y$.
3. R is transitive *i.e.*, if $x R y$ and $y R z$, then $x R z$.

Then R is called a partially order relation and the set S together with partial or called a partially order set or POSET and is denoted by $(S, \leq)$.

For example :

1. The set N of natural numbers form a poset under the relation '$\leq$' because firstly secondly, if $x \leq y$ and $y \leq x$, then we have $x = y$ and lastly if $x \leq y$ and $y \leq z$, it implies $x \leq z$ $x, y, z \in N$.

2. The set N of natural numbers under divisibility *i.e.*, 'x divides y' forms a poset be x/x for every $x \in N$. Also if x/y and y/x, we have $x = y$. Again if x/y, y/z we have x/z, for ever $\in N$.

3. Consider a set S = {1, 2} and power set of S is P(S). The relation of set inclusion partial order. Since, for any sets A, B, C in P(S), firstly we have $A \subseteq A$, secondly, if $A \subseteq$ $B \subseteq A$, then we have A = B. Lastly, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. Hence, $(P(S), \subseteq)$ is a

Example 1. *Consider a set S = {a, b, c}. Is the relation of set inclusion '$\subseteq$' is a p order on P(S) where P(S) is a power set of S ?*

Sol. The power set of S is

$$P(S) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \phi\}$$

Now consider any sets A, B and C in P(S).

1. Since every $A \subseteq A$, hence it is reflexive.
2. If $A \subseteq B$ and $B \subseteq A$, we have A = B. Hence it is antisymmetric.
3. If $A \subseteq B$, $B \subseteq C$, we have $A \subseteq C$. Hence it is transitive.

$\therefore$ $(P(S), \subseteq)$ is a poset.

Example 2. *Consider a set A = {4, 9, 16, 36}. Is the relation 'divides' a partial ord*

Sol. The relation 'divides' is a partial order if it satisfies the property of reflex antisymmetry and transitivity.

1. Since for every $a \in A$, we have a/a. Hence, 'divides' is reflexive.
2. If a/b and b/a, we have $a = b$ for any $a, b \in A$. Hence, 'divides' is antisymmetric
3. If a/b and b/c, we have a/c for any $a, b, c \in A$. Hence, the relation 'divides' is a p order and (A, /) is a poset.

COMPARABLE ELEMENTS

Consider an ordered set A. Two elements a and b of set A are called comparable if

$$a \underset{R}{\leq} b \quad \text{or} \quad b \underset{R}{\leq} a.$$

NON-COMPARABLE ELEMENTS

Consider an ordered set A. Two elements a and b of set A are called non-comparable if neither $a \leq b$ nor $b \leq a$.

Example 3. *Consider A = {1, 2, 3, 5, 6, 10, 15, 30} is ordered by divisibility. Determine all the comparable and non-comparable pairs of elements of A.*

Sol. The comparable pairs of elements of A are :

{1, 2}, {1, 3}, {1, 5}, {1, 6}, {1, 10}, {1, 15}, {1, 30}
{2, 6}, {2, 10}, {2, 30}
{3, 6}, {3, 15}, {3, 30}
{5, 10}, {5, 15}, {5, 30}
{6, 30}, {10, 30}, {15, 30}.

The non-comparable pair of elements of A are :

{2, 3}, {2, 5}, {2, 15}
{3, 5}, {3, 10}, {5, 6}, {6, 10}, {6, 15}, {10, 15}.

LINEARLY ORDERED SET

Consider an ordered set A. The set A is called linearly ordered set or totally ordered set, if every pair of elements in A are comparable.

For Example : The set of positive integers I_+ with the usual order $\leq$ is a linearly ordered set.

Example 4. *Consider the set I = {1, 2, 3,......} is ordered by divisibility. Determine whether each of the following subsets of I are linearly ordered or not.*

(*i*) {2, 4, 8} 2 4 8 (*ii*) {3, 6, 9, 11}
(*iii*) {1} (*iv*) {2, 4, 6, 8, 10, ...}.

Sol. (*i*) The subset is linearly ordered, since every pair of elements is comparable *i.e.*, $2 | 4 | 8$.

(*ii*) The subset is not linearly ordered, since the pair (3, 11) is not comparable.

(*iii*) The subset is linearly ordered, since the set containing one element is always linearly ordered.

(*iv*) The subset is not linearly ordered since every pair of elements is not comparable *i.e.*, neither 4/6 nor 6/4.

HASSE DIAGRAMS

It is a useful tool, which completely describes the associated partial order. Therefore, it is also called an ordering diagram. It is very easy to convert a directed graph of a relation on a set A to an equivalent Hasse diagram. Therefore, while drawing a Hasse diagram following points must be remembered.

1. The vertices in Hasse diagram are denoted by points rather than by circles.

2. Since a partial order is reflexive, hence each vertex of A must be related to itself, so the edges from a vertex to itself are deleted in Hasse diagram.

3. Since a partial order is transitive, hence whenever $a\mathrm{R}b$, $b\mathrm{R}c$, we have $a\mathrm{R}c$. Eliminate all edges that are implied by the transitive property in Hasse diagram *i.e.*, Delete edge from a to c but retain the other two edges.

4. If a vertex 'a' is connected to vertex 'b' by an edge *i.e.*, $a\mathrm{R}b$, then vertex 'b' appears above vertex 'a'. Therefore, the arrows may be ommitted from the edges in Hasse diagram.

The Hasse diagram is much simpler than the directed graph of the partial order.

Example 5. *Consider the set A = {4, 5, 6, 7}. Let R be the relation ≤ on A. Draw the directed graph and the Hasse diagram of R.*

Sol. The relation ≤ on the set A is given by

R = {(4, 5), (4, 6), (4, 7), (5, 6), (5, 7), (6, 7), (4, 4), (5, 5), (6, 6), (7, 7)}

The directed graph of the relation R is as shown in Fig. 1.

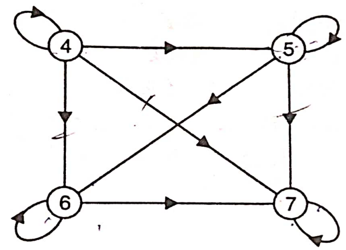


Fig. 1.

To draw the Hasse diagram of partial order, apply the following points :

1. Delete all edges implied by reflexive property *i.e.*,

(4, 4), (5, 5), (6, 6), (7, 7).

2. Delete all edges implied by transitive property *i.e.*,

(4, 7), (5, 7) and (4, 6).

3. Replace the circles representing the vertices by dots.

4. Omit the arrows.

The Hasse diagram is as shown in Fig. 2.

Fig. 2.

Example 6. *Draw the directed graph of relation determined by the Hasse diagram on the set A = {1, 4, 6, 8} as shown in Fig. 3.*

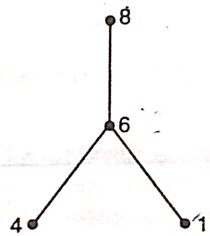


Fig. 3.

Sol. The directed graph is shown in Fig. 4.

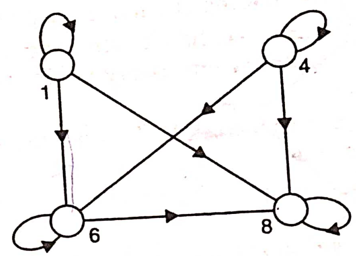


Fig. 4.

Example 7. *Determine the Hasse diagram of the partial order having the directed graph as shown in Fig. 5.*

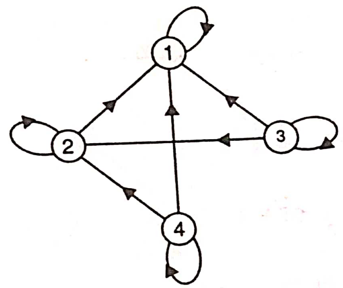


Fig. 5.

Sol. The Hasse diagram of the given partial order determine by the directed graph is as shown in Fig. 6.

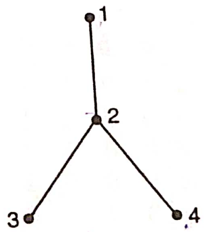


Fig. 6.

Example 8. *Consider the set A = {k, l, m, n, p} and the corresponding relation R = {(k, k), (l, l), (m, m), (n, n), (p, p), (k, m), (k, l), (k, n), (k, p), (m, n), (m, p), (n, p), (l, p)}.*

Construct the directed graph and the corresponding Hasse diagram of this partial orde

Sol. The directed graph of the partial order is as shown in Fig. 7.

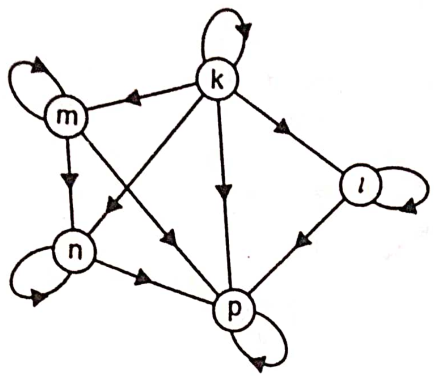


Fig. 7.

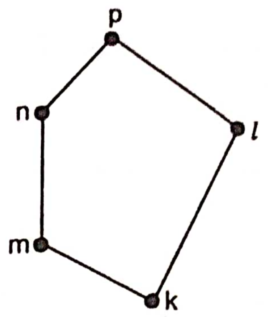


Fig. 8.

The Hasse diagram of the partial order is as shown in Fig. 8.

Example 9. *Consider the Hasse diagram as shown in Fig. 9. Determine the value of set A and also determine the set R.*

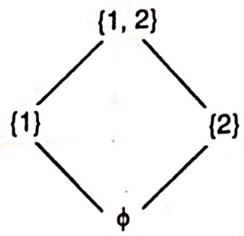


Fig. 9.

Sol. The set $A = \{\{1\}, \{2\}, \{1, 2\}, \phi\}$ and

$R = \{(\{1\}, \{1\}), (\{2\}, \{2\}), (\{1, 2\}, \{1, 2\}), \{\phi, \phi\}, (\{1\}, \{1, 2\}), (\{2\}, \{1, 2\}), (\phi, \{1\})$
$(\phi, \{2\}), (\phi, \{1, 2\})\}$

Here the relation R is (set inclusion) $\subseteq$.

Example 10. *Determine the Hasse diagram of the relation on A = {a, b, c, d, e} whose matrix is shown in Fig. 10.*

$$
\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}
\begin{array}{c}
\begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{bmatrix}
1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

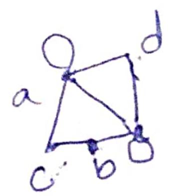


Fig. 10.

Sol. The Hasse diagram of the relation on A = {a, b, c, d, e} is shown in Fig. 11.

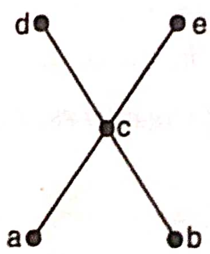


Fig. 11.

Example 11. *Let A = {1, 2, 3, 4, 5} be ordered by Hasse diagram.*

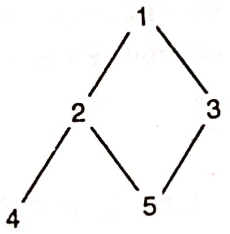


Fig. 12.

Insert correct symbol <, >, or || (not comparable) between :

(i) 1 – 5 *(ii) 2 – 3* *(iii) 4 – 1* *(iv) 3 – 4.*

Sol. *(i)* As there is a path from 1 to 5, hence 1 > 5.

(ii) As there is no path from 2 to 3 or vice-versa, hence 2 || 3.

(iii) As there is a path from 4 to 1, hence 4 < 1.

(iv) As there is no path from 3 to 4, hence 3 || 4.

ELEMENTS OF POSET

Maximal element. An element $a \in$ A is called a maximal element of A if there is no element c in A such that $a \leq c$.

Minimal element. An element $b \in$ A is called a minimal element of A if there is no element c in A such that $c \leq b$.

Note. There can be more than one maximal or more than one minimal element.

Example 12. *Determine all the maximal and minimal elements of the poset whose Hasse diagram is shown in Fig. 13.*

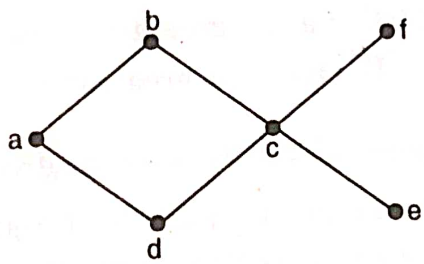


Fig. 13.

Sol. The maximal elements are b and f.

The minimal elements are d and e.

Example 13. *Let A = {2, 3, 4, 6, 8, 24, 48} with partial ordering of divisibility. Determine the maximal and minimal elements of A.*

Sol. The maximal element is 48.

The minimal elements are 2 and 3.

Theorem I. *Prove that the finite non empty poset (A, ≤) has at least one maximal and one minimal element in A.*

Proof. Consider an element $a \in$ A. Now if a is not maximal element, we will find ano element $b \in$ A such that $a < b$. Again if b is not a maximal element, we will find ano element $c \in$ A such that $b < c$. At last, we will stop finding the element, as we cannot conti this process indefinitely since A is a finite set. Hence, we obtain a finite sequence

$$a < b < c < d \text{ ...... } < n$$

which cannot be extended further.

Therefore, we cannot have $n < x$ for any $x \in$ A. So, we conclude that n is a maxi element of (A, ≤). By giving similar arguments we can prove that the finite non empty p (A, ≤) has atleast one minimal element in A.

Greatest element. An element $x \in$ A is called a greatest element of A if for all $a \in$ $\leq x$.

Least element. An element $y \in$ A is called a least element of A if, for all $a \in$ A, $y \leq$

* The greatest element of a poset is denoted by 1 and is called the unit element and least element of a poset is denoted by 0 and is called the zero element.

Example 14. *Determine the greatest and least elements of the poset whose Hasse diagr are shown in Fig. 14, if they exist.*

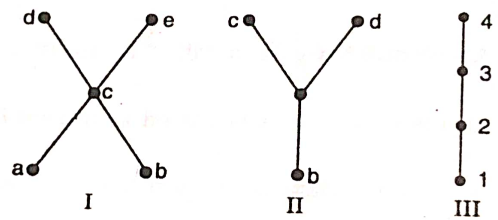


Fig. 14.

Sol. The poset shown in Fig. 14(I) has neither greatest nor least element.

The poset shown in Fig. 14(II) has no greatest element but b is the least element.

The poset shown in Fig. 14(III) has 4 as greatest element and 1 as least element.

Theorem II. *Prove that a poset has at the most one greatest element and one least elem*

Proof. Let x and y are two greatest elements of a poset A. Now, since y is a grea element, so we have $x \leq y$. Also, as x is a greatest element, so we have $y \leq x$ which implies t $x = y$ (*antisymmetric property*). Hence, we can say that if the poset has a greatest elem there exists only one such element.

By similar arguments, we can prove that if the poset has a least element, there ex only one such element.

UPPER BOUND

Consider B be a subset of a partially ordered set A. An element $x \in$ A is called an up bound of B if $y \leq x$ for every $y \in$ B.

LOWER BOUND

Again, consider B be a subset of a partially order set A. An element $z \in$ A is called a lower bound of B if $z \leq x$ for every $x \in$ B.

Example 15. *Consider the poset A = {a, b, c, d, e, f, g} be ordered as shown in Fig. 15. Also let B = {c, d, e}. Determine the upper and lower bounds of B.*

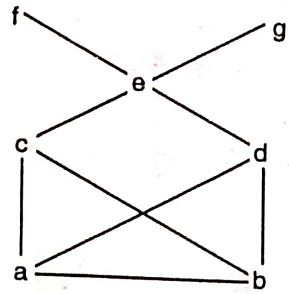


Fig. 15.

Sol. The upper bounds of B are e, f and g because every element of B is '≤' e, f and g. The lower bounds of B are a and b because a and b are '≤' every element of B.

Example 16. *Consider the poset A = {1, 2, 3, 4, 5, 6} be ordered as shown in Fig. 16. Also B = {3, 4}. Determine the upper and lower bounds of B.*

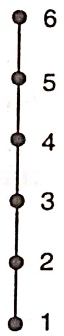


Fig. 16.

Sol. The upper bound of B is 4, 5 and 6 because every element of B is '≤' 4, 5 and 6. The lower bound of B is 3, 2 and 1 because 3, 2 and 1 are '≤' every element of B.

Example 17. *Consider the poset A = {1, 2, 3, 4, 5, 6, 7, 8} be ordered as shown in Fig. 17. let B = {3, 4, 5}. Determine the upper and lower bounds of B.*

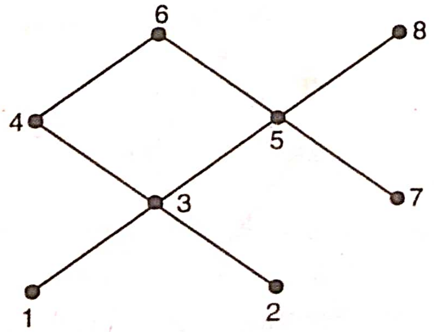


Fig. 17.

Sol. The upper bound of B is 6 because every element of B is '$\leq$' 6. The 8 is not an upper bound of B since $4 \nleq 8$.

The lower bound of B is 1, 2 and 3 because 1 and 2 are '$\leq$' every element of B. The 7 is not a lower bound of B since $7 \nleq 3$ and also $7 \nleq 4$.

Example 18. *Consider the poset A = {a, b, c, d} as shown in Fig. 18 and let B = {c*
Determine the upper and lower bounds of B.

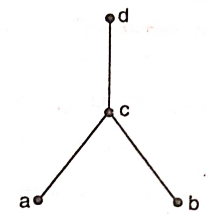


Fig. 18.

Sol. The upper bound of B is c, d.

The lower bound of B is a, b, c.

Note 1. A subset B of a poset A may or may not contain upper bounds or lower bounds in poset

2. An upper bound or lower bound of B may or may not belong to the subset B itself.

LEAST UPPER BOUND (SUPREMUM)

Consider B be a subset of a partially ordered set A. An element $x \in A$ is called a l upper bound of B or supremum written as L U B (B) or S U P(B) if x is an upper bound of B we have $x \leq x'$ for every x' which is an upper bound of B.

GREATEST LOWER BOUND (INFIMUM)

Consider B be a subset of a partially ordered set A. An element $y \in A$ is called a gre lower bound or infimum of B, written as GLB(B) or INF(B) if y is a lower bound of B an have $y' \leq y$, for every y' which is a lower bound of B.

Example 19. *Determine the least upper bound and greatest lower bound of B = {a if they exist, of the poset whose Hasse diagram is shown in Fig. 19.*

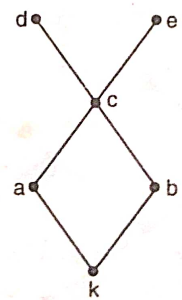


Fig. 19.

Sol. The least upper bound is c.
The greatest lower bound is k.

Example 20. *Let D_{100} = {1, 2, 4, 5, 10, 20, 25, 50, 100} and let the relation ≤ be the relation / (divides) be a partial ordering on D_{100}.*

(*a*) *Determine the GLB of B, where B = {10, 20}.*
(*b*) *Determine the LUB of B, where B = {10, 20}.*
(*c*) *Determine the GLB of B, where B = {5, 10, 20, 25}.*
(*d*) *Determine the LUB of B, where B = {5, 10, 20, 25}.*

Sol. The Hasse diagram of the poset D_{100} is as shown in Fig. 20.

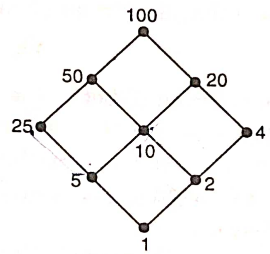


Fig. 20.

(*a*) The GLB (B) is 10. (*b*) The LUB (B) is 20.
(*c*) The GLB (B) is 5. (*d*) The LUB (B) is 100.

t2

Note. D_n denote the set of all positive integers which are divisors of n, where n is a + ve integer g.,

$$D_{10} = \{1, 2, 5, 10\} \text{ as all are divisors of 10.}$$

Example 21. *Determine the least upper bound and greatest lower bound of each B if ey exist, of the poset whose Hasse diagram is shown in Fig. 21.*

Fig. 21.

(*i*) *B = {2, 3}* (*ii*) *B = {2, 3, 4}*
(*iii*) *B = {1, 2, 3, 4}* (*iv*) *B = {1, 2, 3, 4, 5}.*

Sol. (*i*) The least upper bound of B is 3. The greatest lower bound of B is 2.
(*ii*) The least upper bound of B is 4. The greatest lower bound of B is 2.
(*iii*) The least upper bound of B is 4. The greatest lower bound of B is 1.
(*iv*) The least upper bound is 5. The greatest lower bound is 1.

Note. Let $B = \{b_1, b_2, ..., b_r\}$. If $a = LUB(B)$, then a is the first vertex that can be reached from b_1, $b_2, ..., b_r$ by upward paths. Similarly, if $a = GLB(B)$, then a is the first vertex that can be reached from $b_1, b_2, ..., b_r$ by downward paths.

LATTICES

Lattice

A lattice L is a poset in which every pair of elements has a least upper bound (LUB) or supremum and a greatest lower bound (GLB) or infimum.

Join

Consider a poset L under the ordering $\leq$. Let $a, b \in L$. Then LUB (a, b) or SUP (a, b) is denoted by $a \vee b$ or $a \cup b$ and is called the join of a and b *i.e.*, $a \vee b = SUP\,(a, b)$.

Meet

Consider a poset L under the ordering $\leq$. Let $a, b \in L$. Then GLB (a, b) or inf (a, b) is denoted by $a \wedge b$ or $a \cap b$ and is called the meet of a and b *i.e.*, $a \wedge b = \inf (a, b)$.

From the above, it follows that a lattice L is a mathematical structure with two binary operations $\vee$(Join) and $\wedge$ (meet). It is denoted by $\{L, \vee, \wedge\}$. The lattice L for any elements a, b and c satisfies the following properties :

(*a*) **Commutative Property**

(*i*) $a \wedge b = b \wedge a$

(*ii*) $a \vee b = b \vee a$.

(*b*) **Associative Property**

(*i*) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

(*ii*) $(a \vee b) \vee c = a \vee (b \vee c)$.

(*c*) **Absorption Property**

(*i*) $a \wedge (a \vee b) = a$

(*ii*) $a \vee (a \wedge b) = a$.

Theorem III. *Prove that if L be a lattice then $a \wedge b = a$ if and only if $a \vee b = b$.*

Proof. Let us first assume that $a \wedge b = a$.

Using absorption property, we have

$$b = b \vee (b \wedge a) = b \vee (a \wedge b) = b \vee a = a \vee b \quad ...(i)$$

Conversely, let us assume $a \vee b = b$.

Again using the absorption property, we have

$$a = a \wedge (a \vee b) = a \vee b \quad ...(ii)$$

From eqn. (*i*) and (*ii*), we have

$$a \wedge b = a \text{ if and only if } a \vee b = b.$$

Theorem IV. *Prove that for elements of lattice* — Idempotent property

(*i*) $a \wedge a = a$ (*ii*) $a \vee a = a$

Proof. (*i*) $a \wedge a = a \wedge (a \vee (a \wedge b))$ {Using absorption property c(ii)}

$= a$ {Using absorption property c(i)}

(*ii*) $a \vee a = a \vee (a \wedge (a \vee b))$ {Using absorption property c(i)}

$= a$ {Using absorption property c(ii)}

Theorem V. *Consider a lattice L. Prove that the relation $a \leq b$ defined by either $a \wedge b = a$ or $a \vee b = b$ is a partial ordering on lattice L.*

Proof. For any element $a \in L$, we have $a \wedge a = a$

i.e., $a \leq a$. Therefore, the relation $\leq$ is reflexive. Now assume $a \leq b$ and $b \leq a$. Then we have

$$a \wedge b = a \quad \text{and} \quad b \wedge a = b$$

Thus, $a = a \wedge b = b \wedge a = b$, therefore the relation $\leq$ is antisymmetric.

At last, we assume $a \leq b$ and $b \leq c$. So, we have

$$a \wedge b = a \text{ and } b \wedge c = b$$

$$a \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c)$$

i.e.,

$$= a \wedge b = a$$

Therefore, $a \leq c$. So, the relation $\leq$ is transitive. From above, we can say, $\leq$ is a partial order on L.

Example 22. *Let P(S) be the power set of the set S = {1, 2, 3}. Construct the Hasse diagram of the partial order induced on P(S) by the lattice (P(S), $\wedge$, $\vee$).*

Sol. The Hasse diagram obtained by a lattice is same as obtained under the partial ordering of set inclusion. In the lattice, $a \leq b$ whenever $a \wedge b = a$. Thus, in the above case $a \leq b$, whenever $a \wedge b = a$

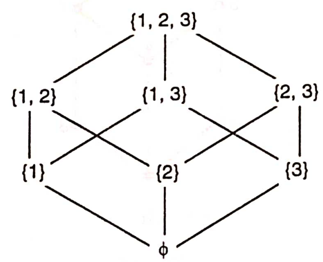


Fig. 22.

Example 23. *Determine which of the posets shown in Fig. 23 are lattices.*

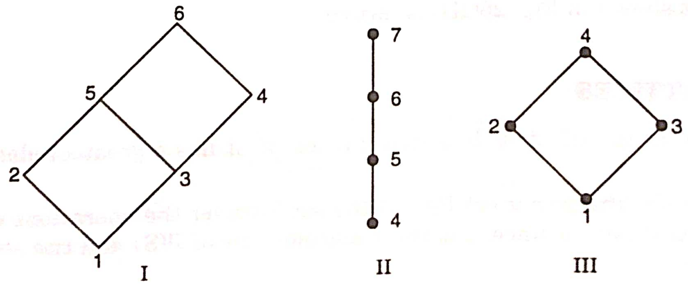


Fig. 23.

Sol. All the posets shown in Fig. 23 are lattices.

Example 24. *Determine whether the posets shown in Fig. 24 are lattices or not.*

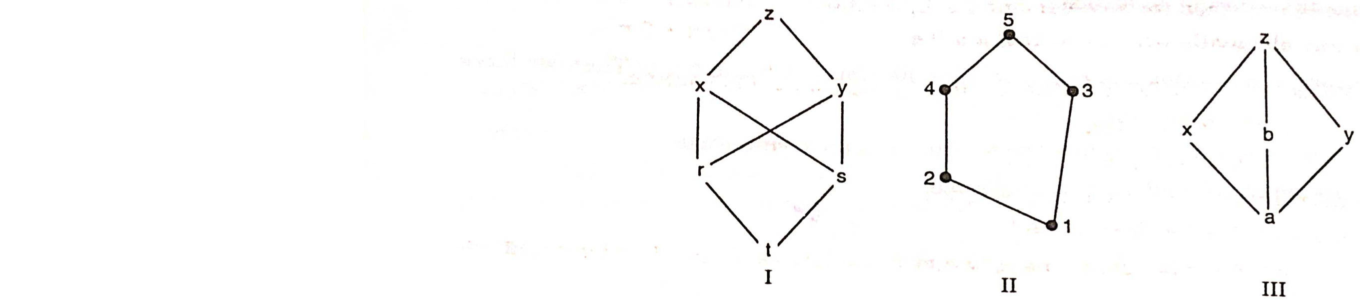


Fig. 24.

Sol. The posets shown in Figs. 24(II) and 24(III) are lattices. The posets shown in Fig. 24 is not lattice as (x, y) has three lower bounds *i.e.*, r, s and t but the inf (x, y) does not exist. Al sup (r, s) does not exist.

Example 25. *Determine whether the posets shown in Fig. 25 are lattices or not.*

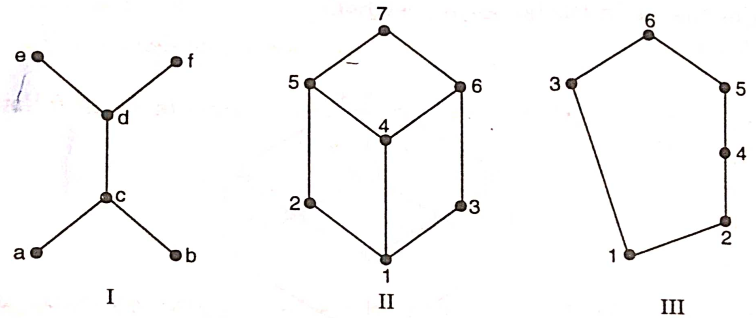


Fig. 25.

Sol. The poset shown in Fig. 25(II) is a lattice. The poset shown in Fig. 25(I) is not latt since the elements e and f have no upper bound, hence sup (e, f) does not exist. Similarly, t elements a and b have no lower bound, hence inf (a, b) does not exist.

The poset shown in Fig. 25(III) is lattice.

BOUNDED LATTICES

A lattice L is called a bounded lattice if it has a greatest element 1 and a le element 0.

Example (*i*). The power set P(S) of the set S under the operations of intersection a union is a bounded lattice since ϕ is the least element of P(S) and the set S is the great element of P(S).

(*ii*) The set of +ve integers I_+ under the usual order of $\leq$ is not a bounded lattice sinc has a least element 1 but the greatest element does not exist.

Properties of Bounded Lattices

If L is a bounded lattice, then for any element $a \in L$, we have the following identities :

(i) $a \vee 1 = 1$ (ii) $a \wedge 1 = a$ (iii) $a \vee 0 = a$ (iv) $a \wedge 0 = 0$.

Theorem VI. *Prove that every finite lattice $L = \{a_1, a_2, a_3, \ldots\ldots, a_n\}$ is bounded.*

Proof. We have given the finite lattice :

$$L = \{a_1, a_2, a_3, \ldots\ldots, a_n\}$$

Thus, the greatest element of lattice L is $a_1 \vee a_2 \vee a_3 \vee \ldots\ldots \vee a_n$.

Also, the least element of lattice L is $a_1 \wedge a_2 \wedge a_3 \wedge \ldots\ldots \wedge a_n$.

Since, the greatest and least elements exist for every finite lattice. Hence L is bounded.

SUBLATTICES

Consider a non-empty subset L_1 of a lattice L. Then L_1 is called a sublattice of L if L_1 itself is a lattice w.r.t. the operations of L *i.e.*, if $a \vee b \in L_1$ and $a \wedge b \in L_1$ whenever $a \in L_1$ and $b \in L_1$.

Example 26. *Consider the lattice of all +ve integers I_+ under the operation of divisibility. The lattice D_n of all divisors of $n > 1$ is a sublattice of I_+.*

Determine all the sublattices of D_{30} that contain at least four elements, $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.

Sol. The sublattices of D_{30} that contain at least four elements are as follows :

(i) {1, 2, 6, 30} (ii) {1, 2, 3, 30}
(iii) {1, 5, 15, 30} (iv) {1, 3, 6, 30}
(v) {1, 5, 10, 30} (vi) {1, 3, 15, 30}
(vii) {2, 6, 10, 30}.

Example 27. *Consider the lattice L = {1, 2, 3, 4, 5} as shown in Fig. 26. Determine all the sublattices with three or more elements.*

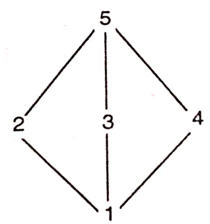


Fig. 26.

Sol. All the sublattices with three or more elements are those whose supremum and infimum exists for every pair of elements which are as follows :

(i) {1, 2, 5} (ii) {1, 3, 5}
(iii) {1, 4, 5} (iv) {1, 2, 3, 5}
(v) {1, 3, 4, 5} (vi) {1, 2, 3, 4, 5}
(vii) {1, 2, 4, 5}.

Example 28. *Consider the lattice L as shown in Fig. 27. Determine whether or no of the following is a sublattice of L.*

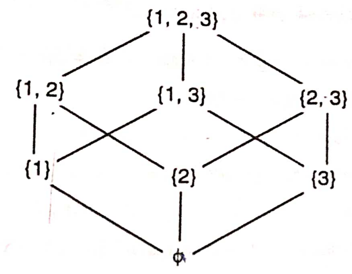


Fig. 27.

$A = \{\phi, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$
$B = \{\phi, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$
$C = \{\phi, \{3\}, \{1, 3\}, \{1, 2, 3\}\}$
$D = \{\{1\}, \{3\}, \{1, 3\}, \{1, 2, 3\}\}$
$E = \{\phi, \{3\}, \{1, 2\}, \{1, 2, 3\}\}$.

Sol. A is not a sublattice since $\{1, 2\} \wedge \{2, 3\} = \{2\}$ which does not exist in A.
B is a sublattice since sup and inf of every pair of elements exist.
C is a sublattice since sup and inf of every pair of elements exist.
D is not a sublattice since $\{1\} \wedge \{3\} = \phi$ which does not exist in D.
E is a sublattice since sup and inf of every pair of elements exist.

ISOMORPHIC LATTICES

Two lattices L_1 and L_2 are called isomorphic lattices if there is a bijection from I *i.e.*, $f : L_1 \to L_2$, such that

$$f(a \wedge b) = f(a) \wedge f(b) \text{ and } f(a \vee b) = f(a) \vee f(b)$$

for every element a, b belongs to L_1.

Example 29. *Determine whether the lattices shown in Fig. 28 are isomorphic.*

Sol. The lattices shown in Fig. 28 are isomorphic. Consider the mapping $f = \{(a, 1)$ $(c, 3), (d, 4)\}$. For example $f(b \wedge c) = f(a) = 1$. Also we have $f(b) \wedge f(c) = 2 \wedge 3 = 1$.

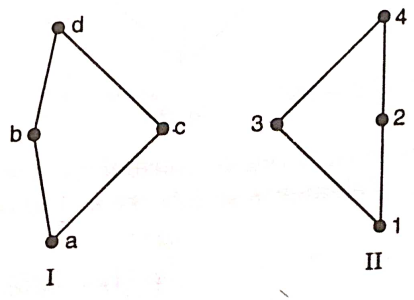


Fig. 28.

Example 30. *Determine whether the lattices shown in Fig. 29 are isomorphic.*

Sol. The lattices shown in Fig. 29 are not isomorphic since bijection is not possible as the element of the two lattices are not same.

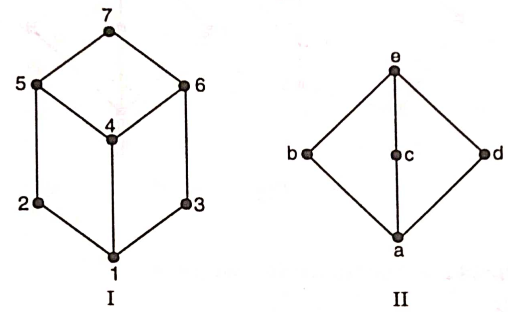


Fig. 29.

DISTRIBUTIVE LATTICE

A lattice L is called distributive lattice if for any elements a, b and c of L, it satisfies following distributive properties :

(*i*) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(*ii*) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

If the lattice L does not satisfies the above properties, it is called a non-distributive lattice.

For example :

1. The power set P(S) of the set S under the operations of intersection and union is a distributive function. Since,

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$$

and also $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$ for any sets a, b and c of P(S).

2. The lattice shown in Fig. 30 is distributive. Since it satisfies the distributive properties for all ordered triples which are taken from 1, 2, 3 and 4.

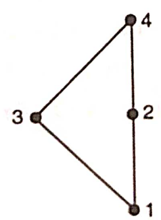
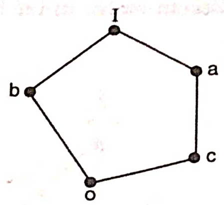


Fig. 30



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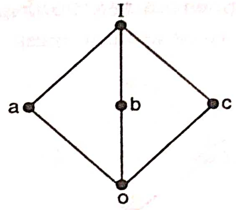


II

Fig. 31.

Sol.

I. From the (i) property of distributivity, we have

$$a \wedge (b \vee c) = a \wedge I = a$$

But, $(a \wedge b) \vee (a \wedge c) = 0 \vee c = c$

Since $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$

Hence, the lattice is not distributive.

II. Again, from the (i) property of distributivity, we have

$$a \wedge (b \vee c) = a \wedge I = a$$

But $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$

Since $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$

Hence, the lattice is not distributive.

Note. A lattice L is non-distributive if and only if it contains a sublattice that is isomorphic t one of the two lattices shown in Figs. 31(I) and (II).

Theorem VII. *Prove that in a distributive lattice* $(L, \wedge, \vee)$, $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$ $= (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$ *holds for all* $a, b, c \in L$.

Proof. We have given that L is distributive, so using distributive property, we have

$$\begin{aligned}
(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) &= \{[(a \vee b) \vee b] \wedge [(a \wedge b) \vee c)]\} \vee (c \wedge a)\\
&= [b \wedge \{(a \vee c) \wedge (b \vee c)\}] \vee (c \wedge a)\\
&= [(a \vee c) \wedge \{b \wedge (b \vee c)\}] \vee (c \wedge a)\\
&= [(a \vee c) \wedge b] \vee (c \wedge a)\\
&= [(a \vee c) \vee (c \wedge a)] \wedge [b \vee (c \wedge a)]\\
&= [\{(a \vee c) \vee c\} \wedge \{(a \vee c) \vee a\}] \wedge [(b \vee c) \wedge (b \vee a)]\\
&= (a \vee c) \wedge (a \vee c) \wedge (b \vee c) \wedge (a \vee b)\\
&= (a \vee c) \wedge (b \vee c) \wedge (a \vee b) = \text{R.H.S.}
\end{aligned}$$

JOIN-IRREDUCIBLE

Consider a lattice $(L, \wedge, \vee)$. An element $a \in L$ is called join-irreducible if it can be express as the join of two distinct elements of L *i.e.,* $a \in L$ is join-irreducible if

$$a = x \vee y \Rightarrow a = x \text{ or } a = y, \text{ where } x, y \in L.$$

Example 32. *Determine the join-irreducible elements of the lattices as shown in Fig. 32.*

Sol. 1. The join-irreducible elements of Fig. 32(I) are a, b and d.

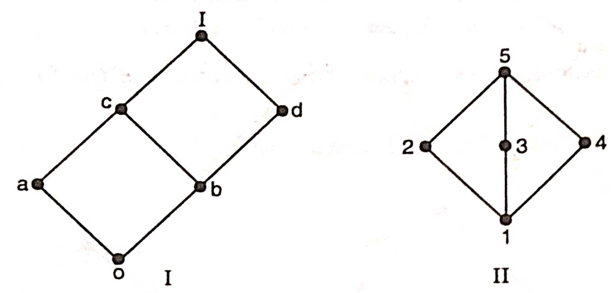


Fig. 32.

2. The join-irreducible elements of Fig. 32(II) are 2, 3 and 4.

EET-IRREDUCIBLE

Consider a lattice (L, ∧, ∨). An element $a \in L$ is called meet-irreducible if it can be pressed as the meet of two distinct elements of L *i.e.*, $a \in L$ is meet-irreducible if

$$a = x \wedge y \quad \Rightarrow \quad a = x \text{ or } a = y,$$

ere $x, y \in L$.

Example 33. *Determine the meet-irreducible elements of the lattices as shown in . 33.*

Sol. 1. The meet-irreducible elements of Fig. 33(I) are a, b and c.

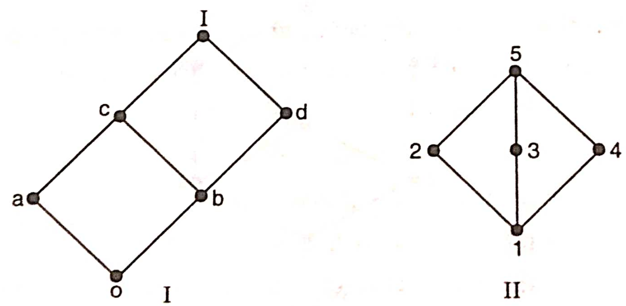


Fig. 33.

2. The meet-irreducible elements of Fig. 33(II) are 2, 3 and 4.

M

Consider a lattice (L, ∧, ∨) with a lower bound (1). An element a is called an atom if it is

ANTIATOM

Consider a lattice (L, $\wedge$, $\vee$) with an upper bound 1. An element a is called an antiatom if it is an immediate predecessor of 1 *i.e.*, $a \neq 1$ is an antiatom if

$$a \leq b \leq 1 \;\Rightarrow\; b = a \quad \text{or} \quad b = 1.$$

Example 34. *Determine the atoms and antiatoms of the following lattices shown Fig. 34.*

Sol. Atoms. 1. Atoms of Fig. 34(I) are a and b.

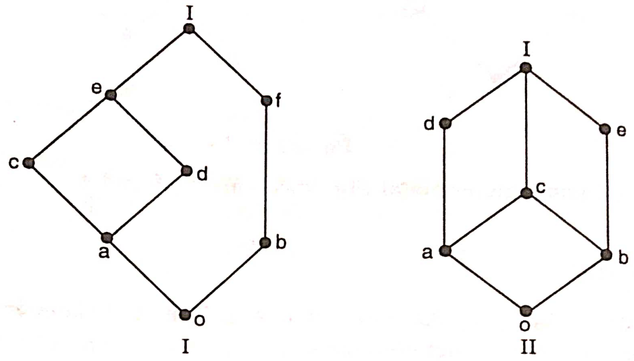


Fig. 34.

2. Atoms of Fig. 34(II) are a and b.

Antiatoms. 1. Antiatoms of Fig. 34(I) are e and f.

2. Antiatoms of Fig. 34(II) are d and e.

Example 35. *Construct the meet and join table of the lattice* (L, $\vee$, $\wedge$) *as shown in Fig*

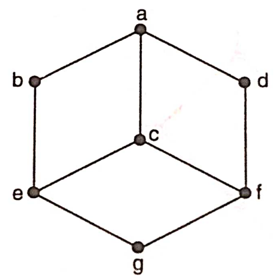


Fig. 35.

Sol. The following tables show the meet and join table of the lattice (L, $\wedge$, $\vee$).

$\vee$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	a	b	a	b
c	a	a	c	a	c	c	c
d	a	a	a	d	a	d	d
e	a	b	c	a	e	c	e
f	a	a	c	d	c	f	f
g	a	b	c	d	e	f	g

$\wedge$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	e	g	e	g	g
c	c	e	c	f	e	f	g
d	d	g	f	d	g	f	g
e	e	e	e	g	e	g	g
f	f	g	f	f	g	f	g
g	g	g	g	g	g	g	g

COMPLEMENTED LATTICES

Consider a bounded lattice L with greatest element I and the least element 0. An element $x \in$ L is called a complement of x if $x \vee x' = 1$ and $x \wedge x' = 0$.

From the definition of complement, if x' is a complement of x, then x is a complement of x'. It is not necessary that an element x has a complement. Also the complements need not be unique *i.e.*, an element have more than one complement.

Note. That $1' = 0$ and $0' = 1$.

Definition. A lattice L is called a complemented lattice if L is bounded and every element in L has a complement.

Example 36. *Determine the complement of a and c in Fig. 36.*

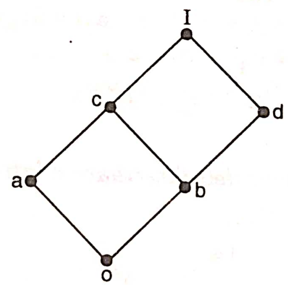


Fig. 36.

Sol. The complement of a is d. Since, $a \vee d = 1$ and $a \wedge d = 0$.

The complement of c does not exist. Since, there does not exist any element c such that $c \vee c' = 1$ and $c \wedge c' = 0$.

Theorem VIII. *Prove that 0 and 1 are complement of each other.*

Proof. To show that 1 is the only complement of 0, consider that $c \neq 1$ is a complement of 0 and $c \in$ L.

Then, $\quad 0 \wedge c = 0 \quad$ and $\quad 0 \vee c = 1$

But $\quad 0 \vee c = c \quad$ {Property of bounded lattice}

and $c \neq 1$ leads to a contradiction.

Similarly, we can show that 0 is the only complement of 1.

Example. The power set P(S) of the set S under the operations of intersection and union is a complemented lattice L since each element of L has a unique complement.

Example. The lattices shown below are complemented lattices Fig. 37.

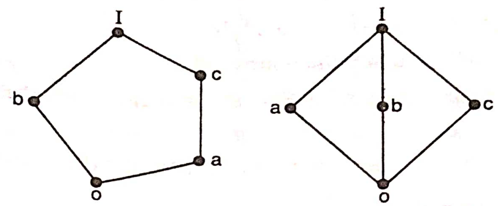


Fig. 37.

But the complements of some of the elements are not unique *e.g.*, b has two complements a and c in both the cases.

Theorem IX. *Prove that for a bounded distributive lattice L, the complements are unique if they exist.*

Proof. Consider a_1 and a_2 be complements of some elements $a \in$ L. Then, we have

$$a \vee a_1 = \text{I} \quad \text{and} \quad a \vee a_2 = \text{I}$$

Also
$$a \wedge a_1 = 0 \quad \text{and} \quad a \wedge a_2 = 0$$

Now using the distributive property, we have

$$a_1 = a_1 \vee 0 = a_1 \vee (a \wedge a_2)$$
$$= (a_1 \vee a) \wedge (a_1 \vee a_2) = (a \vee a_1) \wedge (a_1 \vee a_2) = \text{I} \wedge (a_1 \vee a_2) = a_1 \vee$$

Similarly,
$$a_2 = a_2 \vee 0 = a_2 \vee (a \wedge a_1) = (a_2 \vee a) \wedge (a_2 \vee a_1)$$
$$= (a \vee a_2) \wedge (a_1 \vee a_2)$$
$$= \text{I} \wedge (a_1 \vee a_2) = a_1 \vee a_2$$

Thus, $a_1 = a_2$. Hence proved.

Example 37. *Consider the bounded distributive lattice as shown in Fig. 38. Show t every complement is unique if it exists.*

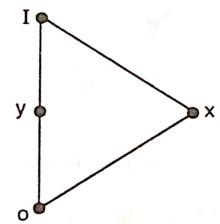


Fig. 38.

Sol. The complement of x is y and vice-versa. Similarly, the complement of 0 is I vice-versa. Hence, all the complements are unique.

MODULAR LATTICE

A lattice (L, $\wedge$, $\vee$) is called a modular lattice if $a \vee (b \wedge c) = (a \vee b) \wedge c$ whenever $a \leq$

Theorem X. *Show that* $a \vee (b \wedge c) = (a \vee b) \wedge c$ *whenever* $a \leq c$.

Proof. If $a \leq c$, then $a \vee c = c$

If L is distributive, then, we have

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge c \qquad | \because a \vee c = c$$

Note. (*i*) Every distributive lattice is modular.

(*ii*) Every modular lattice is not distributive.

Example. The lattice shown in Fig. 39 is a non-distributive modular lattice.

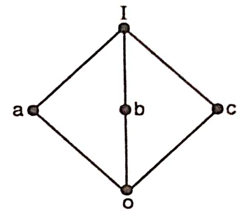


Fig. 39.

We can check that the lattice shown in Fig. 39 is not distributive.

$$a \wedge (b \vee c) = a \wedge 1 = a$$

but
$$(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$$

To, check the modularity holds for $a = c$.

DIRECT PRODUCT OF LATTICES

Let $(L_1, \vee_1, \wedge_1)$ and $(L_2, \vee_2, \wedge_2)$ be two lattices. Then $(L, \vee, \wedge)$ is the direct product of lattices, where $L = L_1 \times L_2$ in which the binary operations $\vee$(join) and $\wedge$(meet) on L are such that for any (a_1, b_1) and (a_2, b_2) in L.

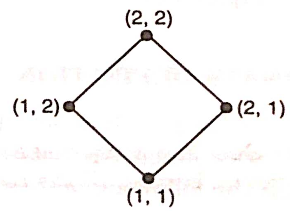
$$(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee_1 a_2, b_1 \vee_2 b_2)$$

and
$$(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge_1 a_2, b_1 \wedge_2 b_2).$$

Example 38. *Consider a lattice $(L, \leq)$ as shown in Fig. 40, where $L = \{1, 2\}$. Determine the lattice $(L^2, \leq)$, where $L^2 = L \times L$.*

Fig. 40.

Sol. The lattice $(L^2, \leq)$ is as shown in Fig. 41.



$(L^2, \leq)$

Fig. 41.

Example 39. *Let $(L_1, \leq)$ and $(L_2, \leq)$ be two lattices as shown in Figs. 42 and 43. Determine the lattice $(L, \leq)$, where $L = L_1 \times L_2$.*

Fig. 42.

Fig. 43.

Sol. The lattice $(L, \leq)$ is a direct product of the lattices $(L_1, \leq)$ and $(L_2, \leq)$ as shown in Fig. 44.

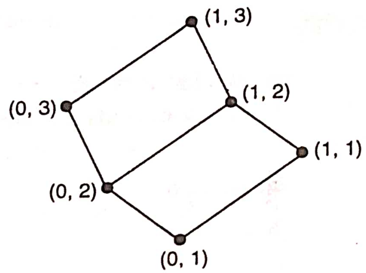


Fig. 44.

BOOLEAN ALGEBRA

Definition

A complemented distributive lattice is called a Boolean algebra. It is denoted by $(B, \wedge, \vee, ', 0, 1)$, where B is a set on which two binary operations $\wedge$ $(*)$ and $\vee$ $(+)$ and a unary operation $'$ (complement) are defined. Here 0 and 1 are two distinct elements of B.

Since $(B, \wedge, \vee)$ is a complemented distributive lattice, therefore each element of B has a unique complement.

ALTERNATE DEFINITION

Consider a set B on which two binary operations $*$ and $+$ and a uniary operation$'$ (complement) are defined. Also let 0 and 1 are two distinct elements of B. Then it is called a Boolean algebra, if the following properties are satisfied for any elements a, b and c of the set B by it

1. Commutative Properties

(*i*) $a + b = b + a$

(*ii*) $a * b = b * a$

2. Distributive Properties

(*i*) $a + (b * c) = (a + b) * (a + c)$

(*ii*) $a * (b + c) = (a * b) + (a * c)$