Date :
Page No.

Infinite Series :-

Let $\{a_n\}$ is a sequence of real numbers, then $a_1+a_2+a_3+\cdots+a_n+\cdots$ is called infinite series.

The infinite series $a_1+a_2+a_3+\cdots+a_n+\cdots$ is denoted as

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

e.g. $1+2+3+\cdots+n+\cdots = \sum(n)$

→ If all the terms of the series are +ve then $\sum a_n$ is called series of positive terms.

→ If in a series $\sum a_n$ terms are alternatively positive and negative, the series

$$\sum (-1)^{n-1} a_n = a_1 - a_2 + a_3 - \cdots (-1)^{n-1} a_n + \cdots$$

when $a_n > 0 \ \forall n$ is known as Alternating Series

Partial Sum :-

If $\sum a_n = a_1 + a_2 + \cdots + a_n + \cdots$ be a infinite series where terms may be +ve or -ve, then

$S_n = a_1 + a_2 + \cdots a_n$ is called n^{th} partial sum of $\sum a_n$.

(Sum of first n terms)

Date :
Page No.

$S_1 \rightarrow$ first partial sum a_1

$S_2 \rightarrow$ second Partial sum $a_1 + a_2$

$S_3 \rightarrow$ Third partial sum $a_1 + a_2 + a_3$

$S_n \rightarrow n^{th}$ partial sum $a_1 + a_2 + \cdots + a_n$

$\{S_n\}$ is a sequence of partial sums of infinite series $\sum a_n$.

$\{S_1, S_2, S_3, S_4, \ldots\}$

→ An infinite series $\sum a_n$ converges, if sequence of partial sums of this series is Converge $\{S_n\}$

i.e. $\lim_{n \to \infty} S_n =$ finite

→ An infinite series $\sum a_n$ diverges if ~~its~~ sequence of partial sum of the series is divergent.

i.e. $\lim_{n \to \infty} S_n = +\infty$ or $-\infty$

Date :
Page No.

→ Σa_n oscillates finitely or infinitely as sequence of partial sums of this series oscillates.

i.e.

* Σa_n oscillates finitely if $\{S_n\}$ of its partial sums is bounded and neither converges nor diverges.

* Σa_n oscillates infinitely if $\{S_n\}$ of its partial sums is unbounded and neither converges or diverges.

e.g. Check for convergence or divergence

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots\frac{1}{n(n+1)}\cdots\infty$$

here $a_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$ (using partial fraction)

Now

$a_1 = 1-\frac{1}{2}$

$a_2 = \frac{1}{2}-\frac{1}{3}$

$a_3 = \frac{1}{3}-\frac{1}{4}$

— — —

$a_n = \frac{1}{n}-\frac{1}{n+1}$

$S_1 = a_1$

$S_2 = a_1+a_2$

$S_n = a_1+a_2+\cdots+a_n$

$S_n = 1-\frac{1}{n+1}$

Now sequence of n^{th} partial sums

$$\{S_1, S_2, \ldots S_n, \ldots\}$$

$$= \{S_n\} = \left\{1 - \frac{1}{n+1}\right\} \quad n = 1, 2, \ldots$$

here $S_n = 1 - \frac{1}{n+1}$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right) = 1 \text{ (finite)}$$

$\Rightarrow$ sequence $\{S_n\}$ converges

$\Rightarrow$ $\sum a_n = \sum \left(\frac{1}{n} - \frac{1}{n+1}\right)$ converges.

Practice Questions :-

① $1 - \frac{1}{5} + \frac{1}{5^2} - \frac{1}{5^3} + \ldots \infty$

$\cancel{\left\{(-1)^{n-1} \frac{1}{5^{n-1}}\right\}}$ $\sum a_n = \sum (-1)^{n-1} \cdot \frac{1}{5^{n-1}}$

where $a_n = (-1)^{n-1} \cdot \frac{1}{5^{n-1}}$

$$S_n = 1 - \frac{1}{5} + \frac{1}{5^2} - \frac{1}{5^3} \ldots (-1)^{n-1} \frac{1}{5^{n-1}}$$

S_n is n^{th} term of sequence of partial sums

$$S_n = 1 - \frac{1}{5} + \frac{1}{5^2} - \frac{1}{5^3} - \cdots + (-1)^n \cdot \frac{1}{5^{n-1}}$$

which is a G.P. series.

$a = 1, \; r = -\frac{1}{5}$

$$S_n = \frac{1\left(1 - \left(-\frac{1}{5}\right)^n\right)}{1 - \left(-\frac{1}{5}\right)}$$

$$= \frac{1 - \frac{(-1)^n}{5^n}}{\frac{6}{5}} = \frac{5}{6}\left(1 - \frac{(-1)^n}{5^n}\right)$$

$a + ar + ar^2 - \cdots ar^{n-1}$

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad \text{if } r < 1$$

$$= \frac{a(r^n - 1)}{r - 1} \quad \text{if } r > 1$$

Now $\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{5}{6}\left(1 - \frac{(-1)^n}{5^n}\right)$

$$= \frac{5}{6}(1 - 0) = \frac{5}{6} \text{ (finite)}$$

$\Rightarrow$ sequence $\{S_n\}$ of partial sums of given series is convergent

$\Rightarrow$ given series is convergent

(2) $\sum a_n = 7-4-3+7-4-3+7-4-3$ ----

here $S_n = 7-4-3+7-4-3+7-4-3+ \cdots$ upto n terms

$S_1 = 7$

$S_2 = 7-4 = 3$

$S_3 = 7-4-3 = 0$

$$= \underset{3m}{\underset{\downarrow}{0}},\ \underset{3m+1}{\underset{\downarrow}{7}} \text{ or } \underset{3m+2}{\underset{\downarrow}{3}}$$

as there are 3m, 3m+1, 3m+2 terms in the series

$m = 1, 2, \cdots$

clearly $\{S_n\}$ does not tends to a unique limits. It osillates b/w three values 0, 7 or 3.

hence $\sum a_n$ also oscillates finitely.

(3) Show that series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ converges to $\frac{3}{4}$

here $a_n = \frac{1}{n(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$ [using Partial fractions]

$a_1 = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) = \frac{1}{2}\left(\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)\right)$

$a_2 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}\left(\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)\right)$

$a_3 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{1}{2}\left(\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)\right)$

$a_n = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{1}{2}\left(\left(\frac{1}{n}-\frac{1}{n+1}\right)+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)\right)$

Now

$S_n = a_1 + a_2 + - - - + a_n$

nth term of sequence of partial sum.

$= \frac{1}{2}\left(1-\frac{1}{2}\right) + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + - - + \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

$+1/4$

$= \frac{1}{2}\left(-\frac{1}{4} + \frac{1}{2}\right) + \frac{1}{2}\cdot\frac{1}{n+1} - \frac{1}{2}\cdot\frac{1}{n+2}$

$= \frac{1}{2}\left(1+\frac{1}{n+1}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n+2}\right)$

Date :
Page No.

Now $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n+1}\right)+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{n+2}\right)$

$= \frac{1}{2}+\frac{1}{4} = \frac{3}{4}$

Ques (1) Show that $\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^{n-1}$ converges to 4

(2) Show that $1^2+2^2+3^2+ \text{———}$ diverges to ∞

(3) Test the convergence of series $\sum_{n=1}^{\infty}(-1)^{n-1}$

Geometric Series

The geometric series $1+x+x^2+x^3+ - - \cdot \infty$

(i) converges if $-1<x<1$ i.e. $|x|<1$

(ii) diverges if $x\geq 1$

(iii) oscillates finitely if $x=-1$

(iv) oscillates infinitely if $x<-1$

Proof (i) When $|x|<1$

$|x|<1$

$x^n \to 0$ as $n\to\infty$

Date :
Page No.

$$S_n = \underbrace{1 + x + x^2 + \cdots x^{n-1}}_{n\text{ terms}}$$

$$= \frac{1(1-x^n)}{1-x} = \frac{1}{1-x} - \frac{x^n}{1-x}$$

$$\lim_{n\to\infty} S_n = \frac{1}{1-x} \text{ (finite)}$$

$\Rightarrow \{S_n\}$ is convergent

hence series is convergent

(ii) when $x \geq 1$

when $x = 1$

$S_n = 1 + 1 + 1 + \cdots$ upto n terms

$S_n = n$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$$

$\Rightarrow \{S_n\}$ is divergent

hence series is divergent.

(ii) $x > 1$

$\left(\frac{a(r^n-1)}{r-1}\right)$ G.P series sum

$x^n \to \infty$ as $n \to \infty$

$S_n = 1 + x + x^2 \ldots n$ terms $= \frac{1(1+x^n)}{1+x}$

$= \frac{x^n}{x-1} - \frac{1}{x-1} = \frac{x^n - 1}{x-1}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{x^n-1}{x-1} = \infty$

$\Rightarrow \{S_n\}$ is divergent

hence series is divergent

(iii) When $x = -1$

$S_n = 1 - 1 + 1 - 1 + \ldots n$ terms

$= 1$ or 0 as n is odd or even

$\lim_{n\to\infty} S_n = 1$ or 0

hence $\{S_n\}$ oscillates finitely

$\Rightarrow$ series oscillates finitely

(iv) When $x < -1$

$\Rightarrow -x > 1$

Put $r = -x$

$r > 1$

$r^n \to \infty$ as $n \to \infty$

Date :
Page No.

$S_n = 1 + x + x^2$ - - n terms

$$= \frac{1-x^n}{1-x} = \frac{1-(-r)^n}{1-(-r)} = \frac{1-(-1)^n r}{1+r}$$

$$= \frac{1-r^n}{1+r}, \quad n \text{ is even}$$

$$\frac{1+r^n}{1+r}, \quad n \text{ is odd}$$

$\lim_{n\to\infty} S_n = -\infty$ or ∞

$\Rightarrow \{S_n\}$ osullates infinitely

hence geometric series also osullates infinitely.