2*

Matrices and its Applications

2.1. ELEMENTARY TRANSFORMATIONS (Or Operations)

Any one of the following operations on a matrix is called an elementary transformation (or E-operation).

(*i*) **Interchange of two rows or two columns.**

The interchange of i^{th} and j^{th} rows is denoted by R_{ij} or $R_i \leftrightarrow R_j$

The interchange of i^{th} and j^{th} columns is denoted by C_{ij} or $C_i \leftrightarrow C_j$.

(*ii*) **Multiplication of (each element of) a row or column by a non-zero number k.**

The multiplication of i^{th} row by k is denoted by kR_i.

The multiplication of i^{th} column by k is denoted by kC_i.

(*iii*) **Addition of k times the elements of a row (or column) to the corresponding elements of another row (or column), $k \neq 0$.**

The addition of k times the j^{th} row to the ith row is denoted by $R_i + kR_j$.

The addition of k times the j^{th} column to the ith column is denoted by $C_i + kC_j$.

If a matrix B is obtained from a matrix A by one or more E-operations, then B is said to be equivalent to A.

Two equivalent matrices A and B are written as $A \sim B$.

2.2. ELEMENTARY MATRICES

The matrix obtained from a unit matrix I by subjecting it to one of the E-operations is called an elementary matrix.

For example, let $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(*i*) Operating R_{23} or C_{23} on I, we get the same elementary matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

It is denoted by E_{23}. Thus, the E-matrix obtained by either of the operations R_{ij} or C_{ij} on I is denoted by E_{ij}.

(*ii*) Operating $5R_2$ or $5C_2$ on I, we get the same elementary matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

*This chapter is not included in the syllabus of M.D.U., Rohtak.

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It is denoted by $5E_2$. Thus, the E-matrix obtained by either of the operations kR_i or kC_i is denoted by kE_i.

(*iii*) Operating $R_2 + 4R_3$ on I, we get the elementary matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

It is denoted by $E_{23}(4)$. Thus, the E-matrix obtained by the operation $R_i + kR_j$ is denoted by $E_{ij}(k)$.

(*iv*) Operating $C_2 + 4C_3$ on I, we get the elementary matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$, which is the transpose of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = E_{23}(4)$ and is, therefore, denoted by $E'_{23}(4)$. Thus, the E-matrix obtained by the operation $C_i + kC_j$ is denoted by $E'_{ij}(k)$.

2.3. THE FOLLOWING THEOREMS ON THE EFFECT OF E-OPERATIONS ON MATRICES HOLD GOOD

(*a*) *Any E-row operation on the product of two matrices is equivalent to the same E-row operation on the pre-factor.*

If the E-row operation is denoted by R, then R(AB) = R(A).B.

(*b*) *Any E-column operation on the product of two matrices is equivalent to the same E-column operation on the post-factor.*

If the E-column operation is denoted by C, then C(AB) = A . C(B).

(*c*) *Every E-row operation on a matrix is equivalent to pre-multiplication by the corresponding E-matrix.*

Thus, the effect of E-row operation R_{ij} on $A = E_{ij} . A$

The effect of E-row operation kR_i on $A = kE_i . A$

The effect of E-row operation $R_i + kR_j$ on $A = E_{ij}(k) . A$.

(*d*) *Every E-column operation on a matrix is equivalent to post-multiplication by the corresponding E-matrix.*

Thus, the effect of E-column operation C_{ij} on $A = A . E_{ij}$

The effect of *E*-column operation kC_i on $A = A . (kE_i)$

The effect of *E*-column operation $C_i + kC_j$ on $A = A . E'_{ij}(k)$.

2.4. INVERSE OF A MATRIX BY E-OPERATIONS (Gauss-Jordan Method)

The elementary row transformations which reduce a square matrix A to the unit matrix, when applied to the unit matrix, give the inverse matrix A^{-1}.

Let A be a non-singular square matrix. Then A = IA

Apply suitable E-row operations to A on the left hand side so that A is reduced to I.

Simultaneously, apply the same E-row operations to the pre-factor I on right hand side. Let I reduce to B, so that

$$I = BA$$

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Post-multiplying by A^{-1}, we get

$$IA^{-1} = BAA^{-1} \quad \Rightarrow \quad A^{-1} = B(AA^{-1}) = BI = B$$

$$\therefore \qquad B = A^{-1}.$$

Note. In practice, to find the inverse of A by E-row operations, we write A and I side by side in the form [A ⋮ I] and the same E-row operations are performed on both. As soon as A is reduced to I, I will reduce to A^{-1}.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the inverse of* $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ *by elementary row operations.*

Sol. Writing the given matrix side by side with unit matrix I_3, we get

$$[A \vdots I_3] = \left[\begin{array}{ccc:ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \qquad \text{operating } R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \qquad \text{operating } R_3 \to R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array}\right] \qquad \text{operating } R_1 \to R_1 - 2R_2,\ R_3 \to R_3 + 5R_2$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right] \qquad \text{operating } R_3 \to \tfrac{1}{2}R_3$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right] \qquad \text{operating } R_1 \to R_1 + R_3,\ R_2 \to R_2 - 2R_3$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right] = [I_3 : A^{-1}]$$

$$\therefore \qquad A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

Note. Check the answer.

$$A^{-1}A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

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$$= \begin{bmatrix} 0 - \frac{1}{2} + \frac{3}{2} & \frac{1}{2} - 1 + \frac{1}{2} & 1 - \frac{3}{2} + \frac{1}{2} \\ 0 + 3 - 3 & -4 + 6 - 1 & -8 + 9 - 1 \\ 0 - \frac{3}{2} + \frac{3}{2} & \frac{5}{2} - 3 + \frac{1}{2} & 5 - \frac{9}{2} + \frac{3}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Example 2. *Using Gauss-Jordan method, find the inverse of the matrix*

$$A = \begin{bmatrix} 2 & 4 & 3 & 2 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 14 \end{bmatrix}.$$

Sol. Writing the given matrix A and the unit matrix I_4 side by side, we get

$$[A : I_4] = \left[\begin{array}{cccc:cccc} 2 & 4 & 3 & 2 & 1 & 0 & 0 & 0 \\ 3 & 6 & 5 & 2 & 0 & 1 & 0 & 0 \\ 2 & 5 & 2 & -3 & 0 & 0 & 1 & 0 \\ 4 & 5 & 14 & 14 & 0 & 0 & 0 & 1 \end{array}\right]$$

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - 2R_1$

$$\sim \left[\begin{array}{cccc:cccc} 2 & 4 & 3 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -5 & -1 & 0 & 1 & 0 \\ 0 & -3 & 8 & 10 & -2 & 0 & 0 & 1 \end{array}\right]$$

Operating $R_1 \longleftrightarrow R_2$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 2 & 2 & 0 & -1 & 1 & 0 & 0 \\ 2 & 4 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -5 & -1 & 0 & 1 & 0 \\ 0 & -3 & 8 & 10 & -2 & 0 & 0 & 1 \end{array}\right]$$

Operating $R_2 \to R_2 - 2R_1$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 2 & 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 3 & -2 & 0 & 0 \\ 0 & 1 & -1 & -5 & -1 & 0 & 1 & 0 \\ 0 & -3 & 8 & 10 & -2 & 0 & 0 & 1 \end{array}\right]$$

Operating $R_1 \to R_1 - 2R_3$, $R_4 \to R_4 + 3R_3$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 4 & 10 & 1 & 1 & -2 & 0 \\ 0 & 0 & -1 & 2 & 3 & -2 & 0 & 0 \\ 0 & 1 & -1 & -5 & -1 & 0 & 1 & 0 \\ 0 & 0 & 5 & -5 & -5 & 0 & 3 & 1 \end{array}\right]$$

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Operating $R_2 \longleftrightarrow R_3$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 4 & 10 & 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & -5 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & 3 & -2 & 0 & 0 \\ 0 & 0 & 5 & -5 & -5 & 0 & 3 & 1 \end{array}\right]$$

Operating $R_1 \to R_1 + 4R_3$, $R_2 \to R_2 - R_3$, $R_4 \to R_4 + 5R_3$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 18 & 13 & -7 & -2 & 0 \\ 0 & 1 & 0 & -7 & -4 & 2 & 1 & 0 \\ 0 & 0 & -1 & 2 & 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 5 & 10 & -10 & 3 & 1 \end{array}\right]$$

Operating $R_3 \to (-1) R_3$, $R_4 \to 1/5 R_4$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 18 & 13 & -7 & -2 & 0 \\ 0 & 1 & 0 & -7 & -4 & 2 & 1 & 0 \\ 0 & 0 & 1 & -2 & -3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & 3/5 & 1/5 \end{array}\right]$$

Operating $R_1 \to R_1 - 18R_4$, $R_2 \to R_2 + 7R_4$, $R_3 \to R_3 + 2R_4$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & -23 & 29 & -64/5 & -18/5 \\ 0 & 1 & 0 & 0 & 10 & -12 & 26/5 & 7/5 \\ 0 & 0 & 1 & 0 & 1 & -2 & 6/5 & 2/5 \\ 0 & 0 & 0 & 1 & 2 & -2 & 3/5 & 1/5 \end{array}\right] = [I_4 \vdots A^{-1}]$$

$$\therefore \qquad A^{-1} = \begin{bmatrix} -23 & 29 & -64/5 & -18/5 \\ 10 & -12 & 26/5 & 7/5 \\ 1 & -2 & 6/5 & 2/5 \\ 2 & -2 & 3/5 & 1/5 \end{bmatrix}$$

Note. Check the answer.

2.5. RANK OF A MATRIX

(M.D.U. Dec. 2010 ; P.T.U. May 2008)

Let A be any $m \times n$ matrix. It has square sub-matrices of different orders. The determinants of these square sub-matrices are called minors of A. If all minors of order $(r + 1)$ are zero but there is at least one non-zero minor of order r, then r is called the rank of A. Symbolically, rank of A is written as $\rho(A)$.

From the definition of the rank of a matrix A, it follows that :

(*i*) If A is a null matrix, then $\rho(A) = 0$.

[$\because$ every minor of A has zero value.]

(*ii*) If A is not a null matrix, then $\rho(A) \geq 1$.

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(*iii*) If A is a non-singular $n \times n$ matrix, then $\rho(A) = n$

[$\because$ $|A| \neq 0$ is the largest minor of A.]

If I_n is the $n \times n$ unit matrix, then $|I_n| = 1 \neq 0 \Rightarrow \rho(I_n) = n$.

(*iv*) If A is an $m \times n$ matrix, then $\rho(A) \leq$ minimum of m and n.

(*v*) If all minors of order r are equal to zero, then $\rho(A) < r$.

To determine the rank of a matrix A, we adopt the following different methods :

(*i*) Start with the highest order minor (or minors) of A. Let their order be r. If any one of them is non-zero, then $\rho(A) = r$.

If all of them are zero, start with minors of next lower order $(r - 1)$ and so on till you get a non-zero minor. The order of that minor is the rank of A.

This method usually involves a lot of computational work since we have to evaluate several determinants.

(*ii*) Convert the given matrix A into lower triangular matrix by elementary column operations or upper triangular matrix by elementary row operations.

$$\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$$

Then $\rho(A)$ = number of non-zero columns in lower triangular matrix or number of non-zero rows in upper triangular matrix.

A non-zero row or column has at least one non-zero element.

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

(*iii*) If A is an $m \times n$ matrix and by a series of elementary (*row or column or both*) operations, it can be put into one of the following forms (called normal forms) :

$$\begin{bmatrix} I_r & \vdots & O \\ \cdots & \cdots & \cdots \\ O & \vdots & O \end{bmatrix}, \begin{bmatrix} I_r \\ O \end{bmatrix}, [I_r \,\vdots\, O], [I_r], \text{ where } I_r \text{ is the}$$

unit matrix of order r.

Since the rank of a matrix is not changed as a result of elementary transformations, it follows that

$$\rho(A) = r. \qquad [\because \ r\text{th order minor } |I_r| = 1 \neq 0]$$

Note. *For an $m \times n$ matrix A of rank r, to find square matrices P and Q of orders m and n respectively, such that PAQ is in the normal form* $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$.

Method. Write A = IAI.

Reduce the matrix on L.H.S. to normal form by affecting elementary row and/or column transformations.

Every elementary row transformation on A must be accompanied by the same transformation on the pre-factor on R.H.S.

Every elementary column transformation on A must be accompanied by the same transformation on the post-factor on R.H.S.

Example 3. *Find the rank of the matrix*

(*i*) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ **(M.D.U. Dec. 2010)**

(*iii*) $\begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$.

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Sol. (*i*) Here $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$ is a 2×4 matrix.

$\therefore \quad \rho(A) \le 2$, the smaller of 2 and 4.

The second order minor $\begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = 4 \neq 0 \qquad \therefore \quad \rho(A) = 2.$

(*ii*) Here $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ is a 3×3 matrix. $\qquad \therefore \quad \rho(A) \le 3$

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} \quad \text{operating } R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The only third order minor is zero, but the second order minor

$$\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2 \neq 0 \quad \therefore \quad \rho(A) = 2.$$

(*iii*) Here $A = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$ is a 3×4 matrix. $\quad \therefore \quad \rho(A) \le 3$

Operating C_{14}

$$A \sim \begin{bmatrix} -1 & 3 & 4 & 2 \\ -1 & 2 & 0 & 5 \\ -1 & 5 & 12 & -4 \end{bmatrix} \quad \text{operating } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} -1 & 3 & 4 & 2 \\ 0 & -1 & -4 & 3 \\ 0 & 2 & 8 & -6 \end{bmatrix} \quad \text{operating } R_1 \to R_1 + 3R_2,\ R_3 \to R_3 + 2R_2$$

$$\sim \begin{bmatrix} -1 & 0 & -8 & 11 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

All the third order minors are zero but the second order minor

$$\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1 \neq 0 \quad \therefore \quad \rho(A) = 2.$$

(*iv*) Here $\qquad A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ is a 4×4 matrix $\quad \therefore \quad \rho(A) \le 4$

Operating $R_4 \to R_4 - (R_3 + R_2 + R_1)$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{operating } R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$$

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$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{operating } R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The only fourth order minor is zero. Since the third order minor

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & 0 & -3 \end{vmatrix} = (1)(-4)(-3) = 12 \neq 0 \quad \therefore \quad \rho(A) = 3.$$

Example 4. *Reduce each of the following matrices to normal form and hence, find their ranks :*

$$(i) \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \qquad (ii) \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}.$$

(M.D.U. Dec. 2009)

Sol. (i) Let

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \quad \text{operating } R_4 \to R_4 - (R_1 + R_2 + R_3)$$

$$\sim \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{operating } R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{operating } C_2 \to C_2 + C_1,\ C_3 \to C_3 + 2C_1,\ C_4 \to C_4 + 4C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 3 & 7 \\ 3 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{operating } R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{operating } R_2 \to R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{operating } C_3 \to C_3 + 6C_2,\ C_4 \to C_4 + 3C_2$$

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$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{operating } R_3 \to R_3 - 4R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{operating } C_3 \to \frac{1}{33} C_3,\ C_4 \to \frac{1}{22} C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{operating } C_4 \to C_4 - C_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} I_3 & 0 \\ \hline 0 & 0 \end{array}\right] \quad \text{which is the required normal form}$$

$\therefore \quad \rho(A) = 3.$

(ii) Let
$$A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix} \quad \text{operating } C_1 \to \frac{1}{8} C_1$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -1 & -1 & -3 & 4 \end{bmatrix} \quad \text{operating } R_3 \to R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix} \quad \text{operating } C_2 \to C_2 - C_1,\ C_3 \to C_3 - 3C_1,\ C_4 \to C_4 - 6C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix} \quad \text{operating } C_2 \to \frac{1}{3} C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix} \quad \text{operating } C_3 \to C_3 - 2C_2,\ C_4 \to C_4 - 2C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \quad \text{operating } C_4 \to \frac{1}{10} C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{operating } C_3 \leftrightarrow C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \end{bmatrix} = [I_3 : O]$$

which is the required normal form

$$\rho(A) = 3.$$

Example 5. *For the matrix* $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$, *find non-singular matrices P and Q su that PAQ is in the normal form. Hence find the rank of A.* (M.D.U., Dec. 2010; K.U.K. 201

Sol. We write A = IAI

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_2 \to C_2 - C_1$, $C_3 \to C_3 - 2C_1$ (subjecting the *post-factor* on R.H.S. to sam operations)

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_2 \to R_2 - R_1$ (subjecting the *pre-factor* on R.H.S. to same operation)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_3 \to C_3 - C_2$ (subjecting the *post-factor* on R.H.S. to same operation)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 \to R_3 + R_2$ (subjecting the *pre-factor* on R.H.S. to same operation)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} = PAQ, \text{ where } P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Also, rank of $A = \rho(A) = 2$

Remarks 1. The matrices P and Q are not unique.

2. If the normal form of PAQ involves I_r, then $\rho(A) = r$.

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EXERCISE 2.1

1. Reduce to triangular form (i) $\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & -1 \\ 0 & -3 & 2 \\ -5 & 3 & 7 \end{bmatrix}$

2. (a) Use Gauss-Jordan method to find the inverse of each of the following matrices:

(i) $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$

(*K.U.K., 2008, 2011*)

(b) Compute the inverse of the following matrices by using Gauss-Jordan method:

(i) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

(*K.U.K., May 2012*)

3. By using elementary transformations, find the inverse of each of the following matrices:

(i) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

(*G.B.T.U., 2010*)

4. Find the ranks of the matrices

(i) $\begin{bmatrix} 2 & -1 & 0 & 5 \\ 0 & 3 & 1 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 10 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$ (vi) $\begin{bmatrix} 2 & -4 & 6 \\ -1 & 2 & -3 \\ 3 & -6 & 9 \end{bmatrix}$.

5. (i) Find rank of the matrix $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ by reducing it in its normal form.

(ii) Determine the rank of the matrix $A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$ by reducing it to the normal form.

6. Find the ranks of the following matrices:

(i) $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$ (vi) $\begin{bmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{bmatrix}$

(*K.U.K., 2008*)

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7. Find the ranks of the following matrices :

(i) $\begin{bmatrix} 3 & -4 & -1 & 2 \\ 1 & 7 & 3 & 1 \\ 5 & -2 & 5 & 4 \\ 9 & -3 & 7 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 4 & 3 & -2 \\ -3 & -2 & -1 & 4 \\ 6 & -1 & 7 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$

8. Reduce each of the following matrices to normal form and hence, find their ranks:

(i) $\begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 2 & -1 & 2 & 5 \\ 5 & 6 & 3 & 2 \\ 1 & 3 & -1 & -3 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$.

9. Find non-singular matrices P and Q such that PAQ is in the normal form for the matrices:

(i) $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

10. For the matrix

$$A = \begin{bmatrix} 2 & 1 & -3 & 6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

find non-singular matrices P and Q such that PAQ is in the normal form. Hence find the rank of A.

(M.D.U., May 2008)

11. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ find A^{-1}. Also find two non-singular matrices **P** and **Q** such that PAQ = I, where I is the unit matrix and verify that $A^{-1} = QP$.

Answers

2. (a) (i) $\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$

(b) (i) $\frac{1}{12}\begin{bmatrix} 3 & -3 & 3 \\ 5 & -1 & -3 \\ -1 & 5 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix}$

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3. (i) $\begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{2}{21} & \frac{1}{7} & -\frac{13}{21} \\ -\frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{5}{21} & -\frac{1}{7} & -\frac{1}{21} \end{bmatrix}$.

4. (i) 2 (ii) 2 (iii) 3 (iv) 2
(v) 3 (vi) 1

5. (i) 2 (ii) 2

6. (i) 2 (ii) 2 (iii) 4 (iv) 2
(v) 2 (vi) 3

7. (i) 3 (ii) 2 (iii) 3 (iv) 2

8. (i) $\begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$, 2 (ii) $[I_3 \quad O]$, 3 (iii) $\begin{bmatrix} I_3 & O \\ O & O \end{bmatrix}$, 3 (iv) $[I_3\ O]$, 3

10. $PAQ = [I_3 \quad O]$, $\rho(A) = 3$ 11. $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$.

2.6. SOLUTION OF A SYSTEM OF LINEAR EQUATIONS

Consider the system of equations $\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\}$ (3 equations in 3 unknowns)

In matrix notation, these equations can be written as

$$\begin{bmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

or

$$AX = B$$

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is called the co-efficient matrix,

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the column matrix of unknowns

$B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ is the column matrix of constants.

If $d_1 = d_2 = d_3 = 0$, then $B = O$ and the matrix equation $AX = B$ reduces to $AX = O$.
Such a system of equations is called a system of **homogeneous linear equations.**
If at least one of d_1, d_2, d_3 is non-zero, then $B \neq O$.
Such a system of equations is called a system of **non-homogeneous linear equations.**

Solving the matrix equation $AX = B$ means finding X, *i.e.*, finding a column matrix $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$

such that

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$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}. \text{ Then } x = \alpha,\ y = \beta,\ z = \gamma.$$

The matrix equation AX = B need not always have a solution. It may have no solution or a unique solution or an infinite number of solutions.

A system of equations having **no solution** is called an **inconsistent** system of equations.

A system of equations having **one or more solutions** is called a **consistent** system of equations.

For a system of non-homogeneous linear equations AX = B

(*i*) if $\rho[A : B] \neq \rho(A)$, the system is inconsistent.

(*ii*) if $\rho[A : B] = \rho(A)$ = number of unknowns, the system has a unique solution.

(*iii*) $\rho[A : B] = \rho(A)$ < number of unknowns, the system has an infinite number of solutions.

The matrix [A : B] in which the elements of A and B are written side by side is called the **augmented matrix.**

For a system of homogeneous linear equations AX = O

(*i*) X = O is always a solution. This solution in which each unknown has the value zero is called the **Null Solution** or the **Trivial Solution.** Thus a homogeneous system is always consistent.

A system of homogeneous linear equations has either the trivial solution or an infinite number of solutions.

(*ii*) if $\rho(A)$ = number of unknowns, the system has only the trivial solution.

(*iii*) if $\rho(A)$ < number of unknowns, the system has an infinite number of non-trivial solutions.

2.7. IF A IS A NON-SINGULAR MATRIX, THEN THE MATRIX EQUATION AX = B HAS A UNIQUE SOLUTION

The given equation is $\qquad AX = B \qquad$ ...(1)

$\because$ A is a non-singular matrix, $\therefore$ A^{-1} exists.

Pre-multiplying both sides of (1) by A^{-1}, we get

$$A^{-1}AX = A^{-1}B \quad \text{or} \quad (A^{-1}A)X = A^{-1}B$$

or

$$IX = A^{-1}B \quad \text{or} \quad X = A^{-1}B$$

which is the required unique solution (since A^{-1} is unique).

Another Method

Write the augmented matrix [A : B]. By **E-row operations on A and B, reduce A** to a diagonal matrix, thus getting

$$[A : B] \sim \begin{bmatrix} p_1 & 0 & 0 & : & q_1 \\ 0 & p_2 & 0 & : & q_2 \\ 0 & 0 & p_3 & : & q_3 \end{bmatrix}$$

Then $\qquad p_1x = q_1,\ p_2y = q_2,\ p_3z = q_3.$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve, with the help of matrices, the simultaneous equations :*

$$x + y + z = 3, \quad x + 2y + 3z = 4, \quad x + 4y + 9z = 6.$$

Sol. [In this question, there is no restriction that the solution must be obtained by finding A^{-1}].

$$\text{Augmented matrix } [A : B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{array}\right] \quad \text{operating } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{array}\right] \quad \text{operating } R_3 \to R_3 - 3R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right] \quad \text{operating } R_3 \to \tfrac{1}{2} R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \text{operating } R_1 \to R_1 - R_3,\ R_2 \to R_2 - 2R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \text{operating } R_1 \to R_1 - R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\therefore$ $x = 2,\ y = 1,\ z = 0.$

Example 2. *Solve the system of equations :*

$$2x_1 + x_2 + 2x_3 + x_4 = 6, \quad 6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$$
$$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1, \quad 2x_1 + 2x_2 - x_3 + x_4 = 10.$$

Sol. In matrix notation, the given system of equations can be written as AX = B

$$\text{where } A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},\ B = \begin{bmatrix} 6 \\ 36 \\ -1 \\ 10 \end{bmatrix}$$

Augmented matrix

$$[A : B] = \left[\begin{array}{cccc:c} 2 & 1 & 2 & 1 & 6 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{array}\right]$$

Operating $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$

$$\sim \left[\begin{array}{cccc:c} 2 & 1 & 2 & 1 & 6 \\ 0 & -9 & 0 & 9 & 18 \\ 0 & 1 & -1 & -5 & -13 \\ 0 & 1 & -3 & 0 & 4 \end{array}\right] \quad \text{operating } R_2 \to -\frac{1}{9} R_2$$

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$$\sim \begin{bmatrix} 2 & 1 & 2 & 1 & \vdots & 6 \\ 0 & 1 & 0 & -1 & \vdots & -2 \\ 0 & 1 & -1 & -5 & \vdots & -13 \\ 0 & 1 & -3 & 0 & \vdots & 4 \end{bmatrix}$$

Operating $R_1 \to R_1 - R_2$, $R_3 \to R_3 - R_2$, $R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 2 & 0 & 2 & 2 & \vdots & 8 \\ 0 & 1 & 0 & -1 & \vdots & -2 \\ 0 & 0 & -1 & -4 & \vdots & -11 \\ 0 & 0 & -3 & 1 & \vdots & 6 \end{bmatrix} \quad \text{operating } R_4 \to R_4 - 3R_3,\ R_1 \to \tfrac{1}{2}R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 & \vdots & 4 \\ 0 & 1 & 0 & -1 & \vdots & -2 \\ 0 & 0 & -1 & -4 & \vdots & -11 \\ 0 & 0 & 0 & 13 & \vdots & 39 \end{bmatrix} \quad \text{operating } R_1 \to R_1 + R_3,\ R_4 \to \frac{1}{13} R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -3 & \vdots & 7 \\ 0 & 1 & 0 & -1 & \vdots & -2 \\ 0 & 0 & -1 & -4 & \vdots & -11 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \quad \text{operating } R_1 \to R_1 + 3R_4,\ R_2 \to R_2 + R_4,\ R_3 \to R_3 + 4R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & 0 & \vdots & 1 \\ 0 & 0 & -1 & 0 & \vdots & 1 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \quad \text{operating } R_3 \to (-1)\,R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & 0 & \vdots & 1 \\ 0 & 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix}$$

Hence $x_1 = 2,\ x_2 = 1,\ x_3 = -1,\ x_4 = 3.$

Example 3. *Using matrix method, show that the equations $3x + 3y + 2z = 1$, $x + 2y = 4$, $10y + 3z = -2$, $2x - 3y - z = 5$ are consistent and hence obtain the solutions for x, y and z.*

(U.K.T.U. 2010; G.B.T.U. 2010)

Sol. In matrix notation, the given system of equations can be written as $AX = B$

where $A = \begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$

Augmented matrix

$$[A : B] = \begin{bmatrix} 3 & 3 & 2 & \vdots & 1 \\ 1 & 2 & 0 & \vdots & 4 \\ 0 & 10 & 3 & \vdots & -2 \\ 2 & -3 & -1 & \vdots & 5 \end{bmatrix} \quad \text{operating } R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & \vdots & 4 \\ 3 & 3 & 2 & \vdots & 1 \\ 0 & 10 & 3 & \vdots & -2 \\ 2 & -3 & -1 & \vdots & 5 \end{bmatrix} \quad \text{operating } R_2 \to R_2 - 3R_1,\ R_4 \to R_4 - 2R_1$$

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$$\sim \begin{bmatrix} 1 & 2 & 0 & \vdots & 4 \\ 0 & -3 & 2 & \vdots & -11 \\ 0 & 10 & 3 & \vdots & -2 \\ 0 & -7 & -1 & \vdots & -3 \end{bmatrix} \text{ operating } R_3 \to R_3 + 3R_2,\ R_4 \to R_4 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & \vdots & 4 \\ 0 & -3 & 2 & \vdots & -11 \\ 0 & 1 & 9 & \vdots & -35 \\ 0 & -1 & -5 & \vdots & 19 \end{bmatrix} \text{ operating } R_1 \to R_1 - 2R_3,\ R_2 \to R_2 + 3R_3,\ R_4 \to R_4 + R_3$$

$$\sim \begin{bmatrix} 1 & 0 & -18 & \vdots & 74 \\ 0 & 0 & 29 & \vdots & -116 \\ 0 & 1 & 9 & \vdots & -35 \\ 0 & 0 & 4 & \vdots & -16 \end{bmatrix} \text{ operating } R_2 \leftrightarrow R_3,\ R_4 \to \tfrac{1}{4} R_4$$

$$\sim \begin{bmatrix} 1 & 0 & -18 & \vdots & 74 \\ 0 & 1 & 9 & \vdots & -35 \\ 0 & 0 & 29 & \vdots & -116 \\ 0 & 0 & 1 & \vdots & -4 \end{bmatrix} \text{ operating } R_1 \to R_1 + 18R_4,\ R_2 \to R_2 - 9R_4,\ R_3 \to R_3 - 29R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & -4 \end{bmatrix} \text{ operating } R_3 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & -4 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

$\rho(A) = \rho(A : B) = 3$ = number of unknowns

$\Rightarrow$ The given system of equations is consistent and the unique solution is

$$x = 2,\ y = 1,\ z = -4.$$

Example 4. *For what values of parameters λ and μ do the system of equations*

$$x + y + z = 6,\ x + 2y + 3z = 10,\ x + 2y + \lambda z = \mu$$

have (i) no solution (ii) unique solution (iii) more than one solution ?

(M.T.U. 2013; K.U.K. 2008, 2012)

Sol. In matrix notation, the given system of equations can be written as AX = B

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$

Augmented matrix $[A : B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 1 & 2 & 3 & \vdots & 10 \\ 1 & 2 & \lambda & \vdots & \mu \end{bmatrix}$ operating $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 1 & \lambda - 1 & \vdots & \mu - 6 \end{bmatrix} \text{ operating } R_1 \to R_1 - R_2,\ R_3 \to R_3 - R_2$$

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$$\sim\begin{bmatrix}1 & 0 & -1 & \vdots & 2\\0 & 1 & 2 & \vdots & 4\\0 & 0 & \lambda-3 & \vdots & \mu-10\end{bmatrix}$$

Case I. *If $\lambda = 3, \mu \neq 10$*

$$\rho(A) = 2,\ \rho(A : B) = 3$$

$\because$ $$\rho(A) \neq \rho(A : B)$$

$\therefore$ The system has no solution.

Case II. *If $\lambda \neq 3$, μ may have any value*

$$\rho(A) = \rho(A : B) = 3 = \text{number of unknowns}$$

$\therefore$ The system has unique solution.

Case III. *If $\lambda = 3, \mu = 10$*

$$\rho(A) = \rho(A : B) = 2 < \text{number of unknowns}$$

$\therefore$ The system has an infinite number of solutions.

Example 5. *Solve the equations :*

$x_1 + 3x_2 + 2x_3 = 0,\ 2x_1 - x_2 + 3x_3 = 0,\ 3x_1 - 5x_2 + 4x_3 = 0,\ x_1 + 17x_2 + 4x_3 = 0.$

Sol. In matrix notation, the given system of equations can be written as AX = O

where $$A = \begin{bmatrix}1 & 3 & 2\\2 & -1 & 3\\3 & -5 & 4\\1 & 17 & 4\end{bmatrix},\ X = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Operating $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix}1 & 3 & 2\\0 & -7 & -1\\0 & -14 & -2\\0 & 14 & 2\end{bmatrix} \quad \text{operating } R_3 \to R_3 - 2R_2,\ R_4 \to R_4 + 2R_2$$

$$\sim \begin{bmatrix}1 & 3 & 2\\0 & -7 & -1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} \quad \text{operating } R_1 \to R_1 + 2R_2$$

$$\sim \begin{bmatrix}1 & -11 & 0\\0 & -7 & -1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

$\therefore$ $$\rho(A) = 2 < \text{number of unknowns}$$

$\Rightarrow$ The system has an infinite number of non-trivial solutions given by

$$x_1 - 11x_2 = 0, \qquad -7x_2 - x_3 = 0$$

i.e., $$x_1 = 11k, \qquad x_2 = k, \quad x_3 = -7k, \text{ where } k \text{ is arbitrary.}$$

Different values of k give different solutions.

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Example 6. *Show that the system of equations :*

$$2x_1 - 2x_2 + x_3 = \lambda x_1$$
$$2x_1 - 3x_2 + 2x_3 = \lambda x_2$$
$$-x_1 + 2x_2 = \lambda x_3$$

can possess a non-trivial solution only if $\lambda = 1$ or -3. Obtain the general solution in each case.

Sol. The given system of equation is

$$(2-\lambda)x_1 - 2x_2 + x_3 = 0$$
$$2x_1 - (3+\lambda)x_2 + 2x_3 = 0$$
$$-x_1 + 2x_2 - \lambda x_3 = 0$$

In matrix notation, it can be written as AX = O

$$\text{where} \quad A = \begin{bmatrix} 2-\lambda & -2 & 1 \\ 2 & -(3+\lambda) & 2 \\ -1 & 2 & -\lambda \end{bmatrix}, \; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

For non-trivial solution, $|A| = 0$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -(3+\lambda) & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)[\lambda(3+\lambda) - 4] + 2(-2\lambda + 2) + [4 - (3+\lambda)] = 0$$

$$\Rightarrow \quad \lambda^3 + \lambda^2 - 5\lambda + 3 = 0$$

$$\Rightarrow \quad (\lambda - 1)^2 (\lambda + 3) = 0$$

$$\therefore \quad \lambda = 1 \quad \text{or} \quad -3.$$

When $\lambda = 1$, the equations become

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_1 - 4x_2 + 2x_3 = 0$$
$$-x_1 + 2x_2 - x_3 = 0$$

which are identical.

The given system is equivalent to a single equation

$$x_1 - 2x_2 + x_3 = 0$$

Taking $x_2 = t$, $x_3 = s$, we get $x_1 = 2t - s$

$$\therefore \quad x_1 = 2t - s, \; x_2 = t, \; x_3 = s$$

which give an infinite number of non-trivial solutions, t and s being the parameters.

When $\lambda = -3$, the equations become

$$5x_1 - 2x_2 + x_3 = 0$$
$$2x_1 + 2x_3 = 0$$
$$-x_1 + 2x_2 + 3x_3 = 0$$

Solving the first two, we have

$$\frac{x_1}{-4} = \frac{x_2}{2-10} = \frac{x_3}{4} \quad \text{or} \quad x_1 = \frac{x_2}{2} = \frac{x_3}{-1}$$

$$\therefore \quad x_1 = t, \quad x_2 = 2t, \quad x_3 = -t$$

which give an infinite number of non-trivial solutions, t being the parameter.

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EXERCISE 2.2

1. Write the following equations in matrix form $AX = B$ and solve for X :

(i) $4x - 5y + z = 2$
$3x + y - 2z = 9$
$x + 4y + z = 5$

(ii) $x_1 + x_2 + x_3 = 1$
$2x_1 - x_2 + x_3 = 4$
$x_1 - 3x_2 - 2x_3 = 2$

2. Using the loop current method on a circuit, the following equations were obtained :

$$7i_1 - 4i_2 = 12, \; -4i_1 + 12i_2 - 6i_3 = 0, \; -6i_2 + 14i_3 = 0.$$

By matrix method, solve for i_1, i_2 and i_3

3. Solve the following systems of equations by matrix method :

(i) $x + y + z = 8, \quad x - y + 2z = 6, \quad 3x + 5y - 7z = 14$

(ii) $x + y + z = 6, \quad x - y + 2z = 5, \quad 3x + y + z = 8$

(iii) $x + 2y + 3z = 1, \quad 2x + 3y + 2z = 2, \; 3x + 3y + 4z = 1.$

4. Show that the equations $x + 2y - z = 3$, $3x - y + 2z = 1$, $2x - 2y + 3z = 2$, $x - y + z = -1$ are consistent and solve them. *(Calicut 2008)*

5. (a) Test for consistency the equations $2x - 3y + 7z = 5$, $3x + y - 3z = 13$, $2x + 19y - 47z = 32$.

(b) Examine if the following equations are consistent and solve them if they are consistent :

$2x + 6y + 11 = 0, \quad 6x + 20y - 6z + 3 = 0, \quad 6y - 18z + 1 = 0$ *(U.K.T.U. 2011)*

(c) Test for consistency the following equations and if consistent, obtain the solution :

$2x + 3y + 4z = 11, \quad x + 5y + 7z = 15, \quad 3x + 11y + 13z = 25$

(d) Test for consistency and solve

$5x + 3y + 7z = 4, \; 3x + 26y + 2z = 9, \; 7x + 2y + 10z = 5.$

6. (a) For what values of a and b do the equations $x + 2y + 3z = 6$, $x + 3y + 5z = 9$, $2x + 5y + az = b$ have (i) no solution (ii) a unique solution and (iii) more than one solution ?

(b) For what values of a and b, the equations : $x + y + 5z = 0$, $x + 2y + 3az = b$ and $x + 3y + az = 1$ have (i) no solution (ii) unique solution and (iii) infinitely many solutions. *(K.U.K. May 2013)*

7. Find the value of k so that the equations $x + ky + 3z = 0$, $4x + 3y + kz = 0$, $2x + y + 2z = 0$ have a non-trivial solution.

8. Show that the equations $3x + 4y + 5z = a$, $4x + 5y + 6z = b$, $5x + 6y + 7z = c$ do not have a solution unless $a + c = 2b$. *(U.P.T.U. 2008; M.T.U. 2009)*

9. Investigate the values of λ and μ so that the equations $2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$ have (i) no solution, (ii) a unique solution and (iii) an infinite number of solutions.

10. Determine the values of λ for which the following system of equations may possess non-trivial solution $3x_1 + x_2 - \lambda x_3 = 0$, $4x_1 - 2x_2 - 3x_3 = 0$, $2\lambda x_1 + 4x_2 + \lambda x_3 = 0$. For each permissible value of λ, determine the general solution.

11. Investigate for consistency the following equations and if possible, find the solutions :

$4x - 2y + 6z = 8, \; x + y - 3z = -1, \; 15x - 3y + 9z = 21.$

12. For what values of k the equations $x + y + z = 1$, $2x + y + 4z = k$, $4x + y + 10z = k^2$ have a solution and solve them completely in each case. *(M.D.U., Dec. 2010)*

13. Show that if $\lambda \neq -5$, the system of equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$, $6x + 5y + \lambda z = -3$ have a unique solution. If $\lambda = -5$, show that the equations are consistent. Determine the solutions in each case.

14. Find the values of a and b for which the equations $x + ay + z = 3$, $x + 2y + 2z = b$, $x + 5y + 3z = 9$ are consistent. When will these equations have a unique solution ? *(M.D.U., Dec. 2009; K.U.K., 2008)*

15. Find the value of λ for which the system of equations $x + y + 4z = 1$, $x + 2y - 2z = 1$ and $\lambda x + y + z = 1$ will have unique solution.

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16. Find the value of k such that the system of equations $4x + 9y + z = 0$, $kx + 3y + kz = 0$ and $x + 4y + 2z = 0$ has non-trivial solution. Hence find the solution of the system of equations.

17. Determine the values of a and b for which the system $\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 3 \\ -1 \end{bmatrix}$ has

(*i*) a unique solution (*ii*) no solution (*iii*) an infinite number of solutions.

18. Solve the equations
$$x + y + z + w = 0, x + 3y + 2z + w = 0, \quad 2x + z - w = 0.$$

19. Discuss the consistency of the system of equations :
$$2x - 3y + 6z - 5w = 3,\ y - 4z + w = 1,\ 4x - 5y + 8z - 9w = \lambda$$
for various values of λ. If consistent, find the solution. (*M.D.U., May 2008*)

20. Find the values of λ for which the equations
$$(2 - \lambda)x + 2y + 3 = 0, \quad 2x + (4 - \lambda)y + 7 = 0, \quad 2x + 5y + (6 - \lambda) = 0$$
are consistent and find the values of x and y corresponding to each of these values of λ.

21. Find the values of λ for which the equations
$$(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$$
$$(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$$
$$2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0$$
have non-trivial solution and find the ratios $x : y : z$, when λ has the smallest of these values. What happens when λ has the greatest of these values ?

Answers

1. (*i*) $x = 2, y = 1, z = -1$ (*ii*) $x_1 = 1, x_2 = -1, x_3 = 1$ 2. $i_1 = \dfrac{396}{175}, i_2 = \dfrac{24}{25}, i_3 = \dfrac{72}{175}$

3. (*i*) $x = 5, y = \dfrac{5}{3}, z = \dfrac{4}{3}$ (*ii*) $x = 1, y = 2, z = 3$ (*iii*) $x = -\dfrac{3}{7}, y = \dfrac{8}{7}, z = -\dfrac{2}{7}$

4. $x = -1, y = 4, z = 4$ 5. (*a*) Inconsistent (*b*) Inconsistent

(*c*) Consistent; $x = 2, y = -3, z = 4$ (*d*) Consistent ; $x = \dfrac{1}{11}(7 - 16k), y = \dfrac{1}{11}(3 + k), z = k$

6. (*a*) (*i*) $a = 8, b \neq 15$ (*ii*) $a \neq 8$, b may have any value (*iii*) $a = 8, b = 15$

(*b*) (*i*) $a = 1, b \neq \dfrac{1}{2}$ (*ii*) $a \neq 1$, b may have any value (*iii*) $a = 1, b = \dfrac{1}{2}$

7. $k = 0, 9/2$ 9. (*i*) $\lambda = 5, \mu \neq 9$ (*ii*) $\lambda \neq 5$, μ arbitrary (*iii*) $\lambda = 5, \mu = 9$

10. $\lambda = 1, -9$. For $\lambda = 1$, solution is $x = k, y = -k, z = 2k$. For $\lambda = -9$, solution is $x = 3k, y = 9k, z = -2k$

11. Consistent ; $x = 1, y = 3k - 2, z = k$, where k is arbitrary

12. $k = 1, x = -3\lambda, y = 1 + 2\lambda, z = \lambda$; $k = 2, x = 1 - 3\lambda, y = 2\lambda, z = \lambda$, where λ is arbitrary

13. $\lambda \neq -5, x = \dfrac{4}{7}, y = -\dfrac{9}{7}, z = 0$; $\lambda = -5, x = \dfrac{4 - 5k}{7}, y = \dfrac{13k - 9}{7}, z = k$, where k is arbitrary

14. If $a = -1, b = 6$, equations will be consistent and have infinitely many solutions
If $a = -1, b \neq 6$, equations will be inconsistent
If $a \neq -1$, b is arbitrary, equations will be consistent and have a unique solution

15. $\lambda \neq \dfrac{7}{10}$ 16. $k = 1; x = 2\lambda, y = -\lambda, z = \lambda$

17. (*i*) $a \neq -3$ (*ii*) $a = -3, b \neq \dfrac{1}{3}$ (*iii*) $a = -3, b = \dfrac{1}{3}$

18. $x = k, y = k, z = -2k, w = 0$ where k is arbitrary

19. Inconsistent for $\lambda \neq 7$. When $\lambda = 7$, the number of solutions is infinite $x = 3k_1 + k_2 + 3$, $y = 4k_1 - k_2 + 1$, $z = k_1$, $w = k_2$ where k_1, k_2 are arbitrary

20. $\lambda = -1, 1, 12$; $x = -\dfrac{1}{11}$, $y = -\dfrac{15}{11}$; $x = -5$, $y = 1$; $x = \dfrac{1}{2}$, $y = 1$

21. $\lambda = 0, 3$; $x : y : z = 1 : 1 : 1$; equations become identical.

2.8. VECTORS

Any ordered n-tuple of numbers is called an n-vector. By an ordered n-tuple, we mean a set consisting of n numbers in which the place of each number is fixed. If $x_1, x_2, \ldots\ldots, x_n$ be any n numbers then the ordered n-tuple $X = (x_1, x_2, \ldots, x_n)$ is called an n-vector. Thus the coordinates of a point in space can be represented by a 3-vector (x, y, z). Similarly $(1, 0, 2, -1)$ and $(2, 7, 5, -3)$ are 4-vectors. The n numbers $x_1, x_2, \ldots\ldots, x_n$ are called the components of the n-vector $X = (x_1, x_2, \ldots\ldots, x_n)$. A vector may be written either as a *row vector* or as a *column vector*. If A be a matrix of order $m \times n$, then each row of A will be an n-vector and each column of A will be an m-vector. A vector whose components are all zero is called a zero vector and is denoted by O. Thus $O = (0, 0, 0, \ldots\ldots, 0)$.

Let $\quad X = (x_1, x_2, \ldots\ldots, x_n)$ and $Y = (y_1, y_2, \ldots\ldots, y_n)$ be two vectors.

Then $X = Y$ if and only if their corresponding components are equal.

i.e., if
$$x_i = y_i, \text{ for } i = 1, 2, \ldots., n$$
$$X + Y = (x_1 + y_1, x_2 + y_2, \ldots\ldots, x_n + y_n)$$

If k be a scalar, then $\quad kX = (kx_1, kx_2, \ldots\ldots, kx_n)$.

2.9. LINEAR DEPENDENCE AND LINEAR INDEPENDENCE OF VECTORS

A set of r n-vectors $X_1, X_2, \ldots\ldots, X_r$ is said to be *linearly dependent* if there exist r scalars (numbers) $k_1, k_2, \ldots\ldots., k_r$ not all zero, such that
$$k_1X_1 + k_2X_2 + \ldots. + k_rX_r = O$$

If $\quad k_i \neq 0$, then $\quad k_iX_i = -k_1X_1 - k_2X_2 \ldots - k_rX_r$
$$\Rightarrow \qquad X_i = -\frac{k_1}{k_i}X_1 - \frac{k_2}{k_i}X_2 - \ldots\ldots - \frac{k_r}{k_i}X_r$$
or
$$X_i = \lambda_1X_1 + \lambda_2X_2 + \ldots + \lambda_rX_r$$

The vector X_i is called a linear combination of the remaining vectors $X_1, X_2, \ldots, X_r$.

If a set of vectors is linearly dependent, then at least one member of the set can be expressed as a linear combination of the remaining vectors.

A set of r n-vectors $X_1, X_2, \ldots\ldots, X_r$ is said to be *linearly independent* if every relation of the type
$$k_1X_1 + k_2X_2 + \ldots\ldots + k_rX_r = O \text{ implies } k_1 = k_2 = \ldots\ldots = k_r = 0$$

To test the linear dependence of r given vectors, write them as row vectors. Add suitable multiples of one vector to the others so that the resulting $(r - 1)$ vectors have their first component zero. Choose any one of these $(r - 1)$ vectors and add its multiples to the others so that the resulting $(r - 2)$ vectors have their second component zero. In this way continue reducing the successive components to zero. If the final reduction gives a vector all of whose components are zero, then the original vectors are linearly dependent. However, if the final reduction gives a vector all of whose components are not zero, then the original vectors are linearly independent.

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Example. *Show that the vectors* $x_1 = (1, 2, 4)$, $x_2 = (2, -1, 3)$, $x_3 = (0, 1, 2)$ *and* $x_4 = (-3, 7, 2)$ *are linearly dependent and find the relation between them.* (U.K.T.U. 2012)

Sol. Adding suitable multiples of x_1 to x_2 and x_4 so that the first component reduces to zero, we have

$$x_2 - 2x_1 = (2, -1, 3) - (2, 4, 8) = (0, -5, -5)$$
$$x_4 + 3x_1 = (-3, 7, 2) + (3, 6, 12) = (0, 13, 14)$$

Also $\quad x_3 = (0, 1, 2).$

Adding suitable multiples of x_3 to the above vectors so that the second component reduces to zero, we have

$$(x_2 - 2x_1) + 5x_3 = (0, -5, -5) + (0, 5, 10) = (0, 0, 5)$$
$$(x_4 + 3x_1) - 13x_3 = (0, 13, 14) - (0, 13, 26) = (0, 0, -12)$$

To reduce the third component to zero, multiplying the above vectors by 12 and 5 respectively and adding, we have

$$12(x_2 - 2x_1 + 5x_3) + 5(x_4 + 3x_1 - 13x_3) = (0, 0, 60) + (0, 0, -60)$$
$$\Rightarrow \quad -9x_1 + 12x_2 - 5x_3 + 5x_4 = (0, 0, 0)$$
$$\Rightarrow \quad 9x_1 - 12x_2 + 5x_3 - 5x_4 = 0 \qquad \ldots(1)$$

Thus, there exist numbers $k_1 = 9$, $k_2 = -12$, $k_3 = 5$, $k_4 = -5$ which are not all zero such that

$$k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 = 0$$

Hence the vectors x_1, x_2, x_3 and x_4 are linearly dependent. Also, (1) is the relation between them.

Second Method

Consider the relation $\quad k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 = 0$

i.e., $\quad k_1(1, 2, 4) + k_2(2, -1, 3) + k_3(0, 1, 2) + k_4(-3, 7, 2) = 0$

or $\quad (k_1, 2k_1, 4k_1) + (2k_2, -k_2, 3k_2) + (0, k_3, 2k_3) + (-3k_4, 7k_4, 2k_4) = 0$

or $\quad (k_1 + 2k_2 - 3k_4,\ 2k_1 - k_2 + k_3 + 7k_4,\ 4k_1 + 3k_2 + 2k_3 + 2k_4) = (0, 0, 0)$

Equating corresponding components on both sides, we have

$$k_1 + 2k_2 - 3k_4 = 0 \qquad \ldots(1)$$
$$2k_1 - k_2 + k_3 + 7k_4 = 0 \qquad \ldots(2)$$
$$4k_1 + 3k_2 + 2k_3 + 2k_4 = 0 \qquad \ldots(3)$$

We have a homogeneous system of 3 linear equations in 4 unknowns. The coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \text{ is } 3 \times 4.$$

$\rho(A) \le \min\{3, 4\} = 3$. Since rank of A is less than the number of unknowns, the homogeneous system has infinitely many non-zero solution. Thus there exist scalars k_1, k_2, k_3, k_4, not all zero, such that

$$k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 = 0 \qquad \ldots(4)$$

$\therefore$ The vectors x_1, x_2, x_3, x_4 are linearly dependent.

To find the relation between the given vectors

In matrix notation, the system of equations (1), (2) and (3) can be written as

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_2 \to R_2 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 0 & 12 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad k_1 + 2k_2 - 3k_4 = 0, \quad -5k_2 + 12\,k_4 = 0, \quad k_3 + k_4 = 0$$

$$\Rightarrow \quad k_1 + \frac{24}{5}\,k_4 - 3k_4 = 0, \quad k_2 = \frac{12}{5}\,k_4, \quad k_3 = -\,k_4$$

$$\Rightarrow \quad k_1 = -\frac{9}{5}k_4, \quad k_2 = \frac{12}{5}\,k_4, \quad k_3 = -\,k_4$$

For non-zero solutions, $k_4 \neq 0$

Putting these values in (4), we get

$$-\frac{9}{5}\,k_4\,x_1 + \frac{12}{5}\,k_4\,x_2 - k_4\,x_3 + k_4\,x_4 = 0$$

$$\Rightarrow \quad 9x_1 - 12x_2 + 5x_3 - 5x_4 = 0$$

Since k_1, k_2, k_3, k_4 are all non-zero, we can express each of the given vectors as a linear combination of the remaining three.

e.g.,
$$9x_1 = 12x_2 - 5x_3 + 5x_4$$

$$\therefore \quad x_1 = \frac{4}{3}x_2 - \frac{5}{9}x_3 + \frac{5}{9}x_4.$$

Note. The elements in the four columns of A are the components of the four given vectors.

2.10. LINEAR TRANSFORMATIONS

Let a point $P(x, y)$ in a plane transform to the point $P'(x', y')$ under reflection in the co-ordinate axes, or reflection in the line $y = x \tan \theta$ or rotation of OP through an angle θ about the origin or rotation of axes through an angle θ etc.

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Then the co-ordinates of P' can be expressed in terms of those of P by the linear relations of the form

$$\left.\begin{aligned} x' &= a_1x + b_1y \\ y' &= a_2x + b_2y \end{aligned}\right]$$

which in matrix notation is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ or $X' = AX$

Such transformations are called linear transformations in two dimensions.

Similarly, relations of the form $\left.\begin{aligned} x' &= a_1x + b_1y + c_1z \\ y' &= a_2x + b_2y + c_2z \\ z' &= a_3x + b_3y + c_3z \end{aligned}\right]$

which in matrix notation is $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ or $X' = AX$ gives a linear transformation $(x, y, z) \longrightarrow (x', y', z')$ in three dimensions.

In general, the relation $Y = AX$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

defines a linear transformation which carries any vector X into another vector Y over the matrix A which is called the linear operator of the transformation.

This transformation is called linear because $Y_1 = AX_1$ and $Y_2 = AX_2$ implies $aY_1 + bY_2 = A(aX_1 + bX_2)$ for all values of a and b.

Thus, if $X = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, then $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$

so that $(2, -3) \rightarrow (5, -5)$ under the transformation defined by A.

If the transformation matrix A is non-singular, *i.e.*, if $|A| \neq 0$, then the linear transformation is called *non-singular* or *regular*.

If the transformation matrix A is singular, *i.e.*, if $|A| = 0$, then the linear transformation is also called *singular*.

For a non-singular transformation $Y = AX$, since A is non-singular, A^{-1} exists and we can write the inverse transformation, which carries the vector Y back into the vector X, as $X = A^{-1}Y$.

Note. If a transformation from $(x_1, x_2, \ldots\ldots, x_n)$ to $(y_1, y_2, \ldots\ldots, y_n)$ is given by $Y = AX$ and another transformation from $(y_1, y_2, \ldots\ldots, y_n)$ to $(z_1, z_2, \ldots\ldots, z_n)$ is given by $Z = BY$, then the transformation from $(x_1, x_2, \ldots\ldots, x_n)$ to $(z_1, z_2, \ldots\ldots, z_n)$ is given by $Z = BY = B(AX) = (BA)X$.

2.11. ORTHOGONAL TRANSFORMATION

The linear transformation $Y = AX$, where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is said to be *orthogonal* if it transforms $y_1^2 + y_2^2 + \ldots\ldots + y_n^2$ into $x_1^2 + x_2^2 + \ldots\ldots + x_n^2$.

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The matrix A of this transformation is called an *orthogonal matrix.*

Now $$X'X = [x_1\ x_2\ \ldots\ldots\ x_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + \ldots\ldots + x_n^2$$

and similarly $$Y'Y = y_1^2 + y_2^2 + \ldots\ldots + y_n^2$$

$\therefore$ If $Y = AX$ is an orthogonal transformation, then

$$X'X = x_1^2 + x_2^2 + \ldots\ldots + x_n^2 = y_1^2 + y_2^2 + \ldots\ldots + y_n^2$$
$$= Y'Y = (AX)'(AX) = (X'A')(AX) \qquad [\because \ (AB)' = B'A']$$
$$= X'(A'A)X$$

which holds only when $A'A = I$ or when $A'A = A^{-1}A$ $\qquad [\because \ A^{-1}A = I]$

or when $A' = A^{-1}$.

Hence a square matrix A is said to be orthogonal if $AA' = A'A = I$

Also, *for an orthogonal matrix A, $A' = A^{-1}$.*

ILLUSTRATIVE EXAMPLES

Example 1. *Show that the transformation $y_1 = 2x_1 + x_2 + x_3$, $y_2 = x_1 + x_2 + 2x_3$, $y_3 = x_1 - 2x_3$ is regular. Write down the inverse transformation.*

Sol. In matrix notation, the given transformation is $Y = AX$,

where $$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Since $$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix} = -1 \neq 0,$$

the matrix A is non-singular and hence, the given transformation is **non-singular or regular.**

As usual, $$A^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

The inverse transformation is given by $X = A^{-1}Y$

$$\Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\Rightarrow \quad x_1 = 2y_1 - 2y_2 - y_3,\ x_2 = -4y_1 + 5y_2 + 3y_3,\ x_3 = y_1 - y_2 - y_3.$$

Example 2. *Prove that the matrix* $\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ ***is orthogonal.***

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Sol. Denoting the given matrix by A, we have

$$A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Now
$$AA' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Since AA' = I, A is an orthogonal matrix.

Example 3. *If A is an orthogonal matrix, prove that* $|A| = \pm 1$.

Sol. A is an orthogonal matrix

$\Rightarrow \quad AA' = I \quad \Rightarrow \quad |AA'| = |I|$

$\Rightarrow \quad |A||A'| = 1 \quad \Rightarrow \quad |A||A| = 1 \qquad [\because \ |A'| = |A|]$

$\Rightarrow \quad |A|^2 = 1 \quad \therefore \quad |A| = \pm 1.$

Example 4. *Prove that the inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal.*

Sol. (*i*) A is orthogonal $\Rightarrow$ AA' = I

$\Rightarrow \quad (AA')^{-1} = I^{-1} \quad \Rightarrow \quad (A')^{-1} A^{-1} = I \qquad [\because \ (AB)^{-1} = B^{-1}A^{-1}]$

$\Rightarrow \quad (A^{-1})' A^{-1} = I \qquad [\because \ (A')^{-1} = (A^{-1})']$

$\Rightarrow$ Product of A^{-1} and its transpose $(A^{-1})'$ is equal to I

$\Rightarrow$ A^{-1} is also orthogonal.

(*ii*) A is orthogonal $\Rightarrow$ AA' = I $\Rightarrow$ (AA')' = I' $\Rightarrow$ (A')' A' = I

$\Rightarrow$ Product of A' and its transpose (A')' is equal to I.

$\Rightarrow$ A' is also orthogonal.

EXERCISE 2.3

1. (*a*) Are the following vectors linearly dependent ? If so, find a relation between them.

(*i*) $x_1 = (1, 2, 1), x_2 = (2, 1, 4), x_3 = (4, 5, 6)$

(*ii*) $x_1 = (2, -1, 4), x_2 = (0, 1, 2), x_3 = (6, -1, 16)$

(*iii*) $x_1 = (3, 2, 7), x_2 = (2, 4, 1), x_3 = (1, -2, 6)$

(*iv*) $x_1 = (2, -1, 3, 2), x_2 = (1, 3, 4, 2), x_3 = (3, -5, 2, 2)$

(*v*) $x_1 = (2, 3, 1, -1), x_2 = (2, 3, 1, -2), x_3 = (4, 6, 2, 1)$

(*vi*) $x_1 = (1, 1, 1, 3), x_2 = (1, 2, 3, 4), x_3 = (2, 3, 4, 9)$

(*vii*) $X_1 = (1, 1, -1, 1), X_2 = (1, -1, 2, -1), X_3 = (3, 1, 0, 1)$.

(*b*) Verify that the following set of vectors in R^3 is linearly dependent : (1, 0, 1), (1, 1, 1), (1, 1, 2) and (1, 2, 1).

Also find the number of linearly independent vectors. (*M.D.U., May 2008*)

(*c*) Show that the row vectors of the matrix $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ are linearly independent.

(*G.B.T.U. 2010*)

2. Show that the transformation $y_1 = x_1 - x_2 + x_3$, $y_2 = 3x_1 - x_2 + 2x_3$, $y_3 = 2x_1 - 2x_2 + 3x_3$ is non-singular. Find the inverse transformation.

3. Represent each of the transformations $x_1 = 3y_1 + 2y_2$, $x_2 = -y_1 + 4y_2$, $y_1 = z_1 + 2z_2$ and $y_2 = 3z_1$ by the use of matrices and find the composite transformation which expresses x_1, x_2 in terms of z_1, z_2.

4. A transformation from the variables x_1, x_2, x_3 to y_1, y_2, y_3 is given by $Y = AX$, and another transformation from y_1, y_2, y_3 to z_1, z_2, z_3 is given by $Z = BY$, where $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$.

Obtain the transformation from x_1, x_2, x_3 to z_1, z_2, z_3.

5. Which of the following matrices are orthogonal ?

(i) $\dfrac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$.

6. (a) Prove that the matrix $\begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$ is orthogonal.

(b) Show that the transformation

$$y_1 = \frac{1}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3$$

$$y_2 = \frac{2}{3}x_1 + \frac{1}{3}x_2 - \frac{2}{3}x_3$$

$$y_3 = \frac{2}{3}x_1 - \frac{2}{3}x_2 + \frac{1}{3}x_3$$

is orthogonal.

7. (i) Verify that the matrix $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ is orthogonal

(ii) Prove that $\begin{bmatrix} l & m & n & 0 \\ 0 & 0 & 0 & -1 \\ n & l & -m & 0 \\ -m & n & -l & 0 \end{bmatrix}$ is orthogonal when $l = \dfrac{2}{7}$, $m = \dfrac{3}{7}$, $n = \dfrac{6}{7}$.

(iii) Show that $A = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ \sin\theta\sin\phi & \cos\theta & -\sin\theta\cos\phi \\ -\cos\theta\sin\phi & \sin\theta & \cos\theta\cos\phi \end{bmatrix}$ is an orthogonal matrix.

8. If A and B are orthogonal matrices, prove that AB is also orthogonal.

9. Find the values of a, b, c if $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal.

Answers

1. (a) (i) Yes ; $x_3 = 2x_1 + x_2$ (ii) Yes ; $x_3 = 3x_1 + 2x_2$
(iii) Yes ; $x_1 = x_2 + x_3$ (iv) Yes ; $2x_1 - x_2 - x_3 = 0$ (v) Yes ; $5x_1 - 3x_2 - x_3 = 0$ (vi) No
(vii) Yes ; $2X_1 + X_2 - X_3 = 0$
(b) 3

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2. $x_1 = \frac{1}{2}(y_1 + y_2 - y_3),\ x_2 = \frac{1}{2}(-5y_1 + y_2 + y_3),\ x_3 = -2y_1 + y_3$

3. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

4. $Z = (BA)X$, where $BA = \begin{bmatrix} 1 & 4 & -1 \\ -1 & 9 & -1 \\ -3 & 14 & -1 \end{bmatrix}$

5. (*i*) Orthogonal (*ii*) Not orthogonal

9. $a = \pm \frac{1}{\sqrt{2}},\ b = \pm \frac{1}{\sqrt{6}},\ c = \pm \frac{1}{\sqrt{3}}$.

2.12. CHARACTERISTIC EQUATION

If A is a square matrix of order n, we can form the matrix $A - \lambda I$, where λ is a scalar and I is the unit matrix of order n. The determinant of this matrix equated to zero, *i.e.*,

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots a_{nn} - \lambda \end{vmatrix} = 0$$ is called the *characteristic equation* of A.

On expanding the determinant, the characteristic equation can be written as a polynomial equation of degree n in λ of the form $(-1)^n \lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \ldots\ldots + k_n = 0$.

The roots of this equation are called the *characteristic roots* or *latent roots* or *eigen values* of A. The set of all the characteristic roots of A is called *spectrum* of A.

2.13. EIGEN VECTORS

Consider the linear transformation $Y = AX$ ...(1)

which transforms the column vector X into the column vector Y. In practice, we are often required to find those vectors X which transform into scalar multiples of themselves.

Let X be such a vector which transforms into λX(λ being a non-zero scalar) by the transformation (1).

Then $Y = \lambda X$ ...(2)

From (1) and (2), $AX = \lambda X \Rightarrow AX - \lambda IX = O \Rightarrow (A - \lambda I)X = O$ ...(3)

This matrix equation gives n homogeneous linear equations

$$\left.\begin{array}{l} (a_{11} - \lambda)x_1 + a_{12}x_2 \cdots\cdots + \cdots\cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdots\cdots + a_{2n}x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + (a_{nn} - \lambda)x_n = 0 \end{array}\right] \quad \ldots(4)$$

These equations will have a non-trivial solution only if the co-efficient matrix $A - \lambda I$ is singular

i.e., if $|A - \lambda I| = 0$ ...(5)

This is the characteristic equation of the matrix A and has n roots which are the eigen values of A. Corresponding to each root of (5), the homogeneous system (3) has a non-zero solution

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$ which is called an *eigen vector* or *latent vector*.

Note. If X is a solution of (3), then so is kX, where k is an arbitrary constant. Thus, the eigen vector corresponding to an eigen value is not unique.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the eigen values and eigen vectors of the matrix* $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$.

Sol. The characteristic equation of the given matrix is

$$|A - \lambda I| = 0 \quad \text{or} \quad \begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$$

or $$(1-\lambda)(4-\lambda) - 10 = 0 \quad \text{or} \quad \lambda^2 - 5\lambda - 6 = 0$$

or $$(\lambda - 6)(\lambda + 1) = 0 \quad \therefore \quad \lambda = 6, -1.$$

Thus, the eigen values of A are 6, – 1.

Corresponding to $\lambda = 6$, the eigen vectors are given by $(A - 6I)X = O$

or $$\begin{bmatrix} 1-6 & -2 \\ -5 & 4-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = O \quad \text{or} \quad \begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = O$$

Operating $R_2 \to R_2 - R_1 \sim \begin{bmatrix} -5 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

We get only one independent equation $-5x_1 - 2x_2 = 0$, since rank of co-efficient matrix is 1.

$\therefore \quad \dfrac{x_1}{2} = \dfrac{x_2}{-5}$ giving the eigen vector (2, – 5).

Corresponding to $\lambda = -1$, the eigen vectors are given by $\begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = O$

Operating $R_1 \to \dfrac{1}{2} R_1$ and $R_2 \to \dfrac{1}{5} R_2$.

$$\sim \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = O$$

Operating $R_2 \to R_2 + R_1$

$$\sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We get only one independent equation $2x_1 - 2x_2 = 0$, since rank of co-efficient matrix is 1.

$\therefore \quad x_1 = x_2$ giving the eigen vector (1, 1).

Example 2. *Find the eigen values and eigen vectors of the matrix* $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

(K.U.K. 2013; U.K.T.U. 2011)

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Sol. The characteristic equation of the given matrix is $|A - \lambda I| = 0$

or $$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

or $$(-2-\lambda)[-\lambda(1-\lambda)-12] - 2[-2\lambda-6] - 3[-4+1(1-\lambda)] = 0$$

or $$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

By trial, $\lambda = -3$ satisfies it.

$\therefore \quad (\lambda+3)(\lambda^2 - 2\lambda - 15) = 0 \Rightarrow (\lambda+3)(\lambda+3)(\lambda-5) = 0 \Rightarrow \lambda = -3, -3, 5$

Thus, the eigen values of A are $-3, -3, 5$.

Corresponding to $\lambda = -3$, the eigen vectors are given by

$$(A + 3I)\, X = O \quad \text{or} \quad \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$$

Operating $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get only one independent equation $x_1 + 2x_2 - 3x_3 = 0$, since rank of co-efficient matrix is 1.

Choosing $x_2 = 0$, we have $x_1 - 3x_3 = 0$

$\therefore \quad \dfrac{x_1}{3} = \dfrac{x_2}{0} = \dfrac{x_3}{1}$ giving the eigen vector (3, 0, 1)

Choosing $x_3 = 0$, we have $x_1 + 2x_2 = 0$

$\therefore \quad \dfrac{x_1}{2} = \dfrac{x_2}{-1} = \dfrac{x_3}{0}$ giving the eigen vector (2, – 1, 0)

Any other eigen vector corresponding to $\lambda = -3$ will be a linear combination of these two.

Corresponding to $\lambda = 5$, the eigen vectors are given by $\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$

Operating $R_1 \to R_1 - 7R_3$ and $R_2 \to R_2 + 2R_3$

$$\sim \begin{bmatrix} 0 & 16 & 32 \\ 0 & -8 & -16 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$$

Operating $R_1 \to \dfrac{1}{16} R_1$ and $R_2 \to -\dfrac{1}{8} R_2$

$$\sim \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$$

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Operating $R_1 \to R_1 - R_2$, $R_3 \to R_3 + 2R_2$

$$\sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$$

Operating $R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get only two independent equations $-x_1 - x_3 = 0$ and $x_2 + 2x_3 = 0$, since rank of co-efficient matrix is 2.

$\Rightarrow \qquad x_1 = -x_3$ and $x_2 = -2x_3$.

$\Rightarrow \qquad \dfrac{x_1}{1} = \dfrac{x_2}{2} = \dfrac{x_3}{-1}$ giving the eigen vector (1, 2, – 1).

2.14. PROPERTIES OF EIGEN VALUES

(a) *The eigen values of a square matrix A and its transpose A′ are the same.*

Let
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{bmatrix}$$

then
$$A' = \begin{bmatrix} a_{11} & a_{21} & \cdots\cdots & a_{n1} \\ a_{12} & a_{22} & \cdots\cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots\cdots & a_{nn} \end{bmatrix}$$

The characteristic equation of A is given by

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} - \lambda \end{vmatrix} = 0 \qquad \ldots(1)$$

Also, the characteristic equation of A′ is given by

$$|A' - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{21} & \cdots\cdots & a_{n1} \\ a_{12} & a_{22} - \lambda & \cdots\cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots\cdots & a_{nn} - \lambda \end{vmatrix} = 0 \qquad \ldots(2)$$

Since the value of a determinant remains unchanged when its rows and columns are interchanged, therefore, the characteristic equations (1) and (2) are same. Hence the eigen values of A and A′ are the same.

(b) *The sum of the eigen values of a matrix is the sum of the elements on the principal diagonal.* (M.D.U., May 2008)

We prove the property for a matrix of order 3. However, the result is true for a matrix of any order.

Let $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

so that $$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix}$$

$$= -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \ldots\ldots \quad \ldots(1)$$

If $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A, then

$$|A - \lambda I| = (-1)^3 (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

$$= -\lambda^3 + \lambda^2 (\lambda_1 + \lambda_2 + \lambda_3) - \ldots\ldots \quad \ldots(2)$$

From (1) and (2), we have

$$-\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) \ldots\ldots = -\lambda^3 + \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) \ldots\ldots$$

Comparing co-efficients of λ^2, we get

$$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$$

= The sum of the elements on the principal diagonal.

(c) *The product of the eigen values of a matrix A is equal to | A |.*

If $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ be the eigen values of a square matrix A of order n, then

$$|A - \lambda I| = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots\ldots (\lambda - \lambda_n)$$

Putting $\lambda = 0$, we get

$$|A| = (-1)^n . (-1)^n \lambda_1\lambda_2 \ldots\ldots \lambda_n$$

$\Rightarrow$ $\lambda_1\lambda_2 \ldots\ldots \lambda_n = |A|$ $\quad [\because \ (-1)^{2n} = 1]$

(d) *If λ is an eigen value of a non-singular matrix A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.*

λ is an eigen value of A

$\Rightarrow$ There exists a non-zero vector X such that $AX = \lambda X$

Pre-multiplying both sides by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}(\lambda X)$$

$\Rightarrow$ $(A^{-1}A)X = \lambda(A^{-1}X) \Rightarrow X = \lambda(A^{-1}X)$

$\Rightarrow$ $\frac{1}{\lambda} X = A^{-1}X \Rightarrow A^{-1}X = \frac{1}{\lambda} X$

$\Rightarrow$ $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

(e) *If λ is an eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also its eigen value.*

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Let λ be an eigen value of an orthogonal matrix A, then $\dfrac{1}{\lambda}$ is an eigen value of A^{-1}.

[Part (*d*) above]

But $\qquad A^{-1} = A' \qquad$ ($\because$ A is an orthogonal matrix)

$\therefore \quad \dfrac{1}{\lambda}$ is an eigen value of A'.

But the matrices A and A' have the same eigen values. [Part (*a*) above]

$\therefore \quad \dfrac{1}{\lambda}$ is also an eigen value of A.

(*f*) *If $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are the eigen values of a matrix A, then A^m has the eigen values $\lambda_1^m, \lambda_2^m, \ldots\ldots, \lambda_n^m$ (m being a positive integer).*

Let λ be an eigen value of a matrix A, then there exists a non-zero vector X such that

$$AX = \lambda X \qquad \ldots(1)$$

Now $\qquad A(AX) = A(\lambda X) \quad \Rightarrow \quad A^2X = \lambda(AX)$

$\Rightarrow \qquad A^2X = \lambda(\lambda X) \qquad$ [by (1)]

$\Rightarrow \qquad A^2X = \lambda^2 X$

Similarly, $A^3X = \lambda^3 X$. In general $A^mX = \lambda^m X$

$\Rightarrow \quad \lambda^m$ is an eigen value of A^m.

Since λ is an eigen value of A $\quad \Rightarrow \quad \lambda^m$ is an eigen value of A^m.

$\therefore$ If $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are the eigen values of A, then $\lambda_1^m, \lambda_2^m, \ldots\ldots, \lambda_n^m$ *are the eigen values of A^m.*

(*g*) *The eigen values of an idempotent matrix are either zero or unity.*

Let A be an idempotent matrix, then $A^2 = A$.

Let λ be an eigen value of A, then there exists a non-zero vector X such that

$$AX = \lambda X \qquad \ldots(1)$$

$\therefore \qquad A(AX) = A(\lambda X) \quad$ or $\quad A^2X = \lambda(AX)$

or $\qquad AX = \lambda(\lambda X) \qquad$ [$\because$ $A^2 = A$ and $AX = \lambda X$]

or $\qquad AX = \lambda^2 X \qquad \ldots(2)$

From (1) and (2), we have $\lambda^2 X = \lambda X$ or $(\lambda^2 - \lambda)X = O$

or $\qquad \lambda^2 - \lambda = 0 \qquad (\because \; X \neq O)$

or $\qquad \lambda(\lambda - 1) = 0 \qquad \therefore \; \lambda = 0$ or 1.

Example 3. *If λ be an eigen value of a non-singular matrix A, show that $\dfrac{|A|}{\lambda}$ is an eigen value of the matrix adj. A.*

Sol. λ is an eigen value of A.

$\Rightarrow$ There exists a non-zero matrix X such that $AX = \lambda X$

$\Rightarrow \qquad (\text{adj } A)(AX) = (\text{adj } A)(\lambda X)$

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$$\Rightarrow \qquad ((\text{adj } A)A)\,X = \lambda\,(\text{adj } A)\,X$$

$$\Rightarrow \qquad |A|\,IX = \lambda\,(\text{adj } A)\,X \qquad [\because \ (\text{adj } A)\,A = |A|\,I]$$

$$\Rightarrow \qquad |A|\,X = \lambda\,(\text{adj } A)\,X \ \Rightarrow \ \frac{|A|}{\lambda}X = (\text{adj } A)\,X$$

$$\Rightarrow \qquad (\text{adj } A)\,X = \frac{|A|}{\lambda}X \ \Rightarrow \ \frac{|A|}{\lambda} \text{ is an eigen value of adj A.}$$

2.15. CAYLEY HAMILTON THEOREM

Every square matrix satisfies its characteristic equation.

i.e., if the characteristic equation of the nth order square matrix A is

$$|A - \lambda I| = (-1)^n\lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \ldots\ldots + k_n = 0 \qquad \ldots(1)$$

then
$$(-1)^nA^n + k_1A^{n-1} + k_2A^{n-2} + \ldots\ldots + k_nI = O$$

Let
$$P = \text{adj}\,(A - \lambda I)$$

Since the elements of $A - \lambda I$ are at most of first degree in λ, the elements of $P = \text{adj}\,(A - \lambda I)$ are polynomials in λ of degree $(n-1)$ or less. We can, therefore, split up P into a number of matrices each containing the same power of λ and write

$$\text{adj}(A\lambda I) = P = P_1\lambda^{n-1} + P_2\lambda^{n-2} + P_{n-1}\lambda + P_n \qquad \ldots(2)$$

Also, we know that if M is a square matrix, then M (adj M) = | M | × I

$$\therefore \qquad (A - \lambda I)P = |A - \lambda I| \times I$$

By (1) and (2), we have

$$(A - \lambda I)(P_1\lambda^{n-1} + P_2\lambda^{n-2} + \ldots\ldots + P_{n-1}\lambda + P_n) = [(-1)^n\lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \ldots\ldots + k_{n-1}\lambda + k_n]I$$

Equating co-efficients of like powers of λ on both sides, we have

$$-P_1 = (-1)^n I \qquad [\because \ IP_1 = P_1]$$

$$AP_1 - P_2 = k_1 I$$

$$AP_2 - P_3 = k_2 I$$

....

$$AP_{n-1} - P_n = k_{n-1} I$$

$$AP_n = k_n I$$

Pre-multiplying these equations by $A^n, A^{n-1}, A^{n-2}, \ldots\ldots, A, I$ respectively and adding, we get

$$O = (-1)^nA^n + k_1A^{n-1} + k_2A^{n-2} + \ldots\ldots + k_{n-1}A + k_nI$$

[terms on the L.H.S. cancel in pairs]

or
$$(-1)^nA^n + k_1A^{n-1} + k_2A^{n-2} + \ldots\ldots + k_{n-1}A + k_nI = O \qquad \ldots(3)$$

which proves the theorem.

Note 1. Multiplying (3) by A^{-1}, we have $(-1)^n A^{n-1} + k_1A^{n-2} + \ldots\ldots + k_{n-1} I + k_nA^{-1} = O$

$$\Rightarrow \qquad A^{-1} = -\frac{1}{k_n}\,[(-1)^nA^{n-1} + k_1A^{n-2} + \ldots\ldots + k_{n-1}I]$$

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Thus Cayley-Hamilton theorem gives another method for computing the inverse of a matrix. Since this method expresses the inverse of a matrix of order n in terms of $(n-1)$ powers of A, it is most suitable for computing inverses of large matrices.

Note 2. If m be a positive integer such that $m > n$, then multiplying (3) by A^{m-n}, we get

$$(-1)^n A^m + k_1 A^{m-1} + k_2 A^{m-2} + \ldots\ldots + k_{n-1} A^{m-n+1} + k_n A^{m-n} = O$$

showing that any positive integral power A^m $(m > n)$ of A is linearly expressible in terms of those of lower degree.

Example 4. *Verify Cayley-Hamilton theorem for the matrix* $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

Hence compute A^{-1}. (K.U.K. 2011 ; G.B.T.U. 2012; U.K.T.U. 2011)

Sol. The characteristic equation of A is

$$|A - \lambda I| = 0 \quad i.e., \quad \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

or $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$ (on simplification)

To verify Cayley-Hamilton theorem, we have to show that

$$A^3 - 6A^2 + 9A - 4I = O \qquad \ldots(1)$$

Now
$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$\therefore \quad A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$+ 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

This verifies Cayley-Hamilton theorem.

Now, multiplying both sides of (1) by A^{-1}, we have

$$A^2 - 6A + 9I - 4A^{-1} = O \quad \Rightarrow \quad 4A^{-1} = A^2 - 6A + 9I$$

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$$\Rightarrow \qquad 4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore \qquad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$$

Example 5. *Using Cayley-Hamilton theorem, find A^8 if*

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Sol. The characteristic equation of A is

$$|A - \lambda I| = 0 \quad i.e., \quad \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

or $\qquad -(1-\lambda^2) - 4 = 0 \quad$ or $\qquad \lambda^2 = 5 \qquad$ ...(1)

By Cayley-Hamilton theorem. A satisfies its characteristic equation (1)

$$\therefore \qquad A^2 = 5I$$

$$\Rightarrow \qquad (A^2)^4 = (5I)^4 \quad \Rightarrow \quad A^8 = 625\ I \qquad (\because\ I^4 = I)$$

Example 6. *Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and, hence, find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.*

(M.T.U. 2013; U.K.T.U. 2010)

Sol. The characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley-Hamilton theorem, $A^3 - 5A^2 + 7A - 3I = O \qquad$ ...(1)

Now $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + (A^2 + A + I)$$

$$= A^2 + A + I \qquad \text{[Using (1)]}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}.$$

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EXERCISE 2.4

1. Find the sum and product of the eigen values of

(i) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 6 \\ 7 & 4 & 3 & 2 \\ 4 & 3 & 0 & 5 \end{bmatrix}$. (M.D.U., May 2008)

2. (a) Find the product of the eigen values of $\begin{bmatrix} 7 & 2 & 2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$.

(b) Find the sum of eigen values of the inverse of the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{bmatrix}$$ (Calicut 2008)

[**Hint.** Eigen values of A are 3, 4, 5 $\Rightarrow$ Eigen values of A^{-1} are $3^{-1}, 4^{-1}, 5^{-1}$]

(c) Find the eigen values of $2A^2$, if $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$. (Calicut 2010)

[**Hint.** Eigen values of A are 1, 5 $\Rightarrow$ Eigen values of A^2 are $1^2, 5^2$

$\Rightarrow$ Eigen values of $2A^2$ are $2 \times 1^2, 2 \times 5^2$]

3. Find the eigen values and the corresponding eigen vectors of the following matrices:

(i) $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

4. Find the eigen values and the corresponding eigen vectors of the following matrices:

(i) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (Calicut 2008; G.B.T.U. 2011) (ii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (M.D.U. Dec. 2010 ; K.U.K. 2008, 2011)

(iii) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(vi) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ (vii) $\begin{bmatrix} -9 & 2 & 6 \\ 5 & 0 & -3 \\ -16 & 4 & 11 \end{bmatrix}$ (viii) $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$

(ix) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ (x) $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ (xi) $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$.

5. Show that the eigen values of a triangular matrix A are equal to the elements of the principal diagonal of A.

6. Show that 0 is a characteristic root of a matrix if and only if the matrix is singular.

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7. Show that if λ is a characteristic root of a matrix A, then $\lambda + k$ is a characteristic root of the matrix $A + kI$.

8. If $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are the eigen values of a matrix A, then A^2 has the eigen values $\lambda_1^2, \lambda_2^2, \ldots\ldots, \lambda_n^2$.

9. Prove that the matrices A and $P^{-1}AP$ have the same eigen values, P being an invertible matrix of same order as A. *(M.T.U. 2011)*

10. (*a*) Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

Show that the equation is satisfied by A and hence obtain the inverse of the given matrix.

(*b*) Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$

Show that the equation is satisfied by A. Also find A^{-1}.

11. Verify Cayley-Hamilton theorem for the matrix A and find A^{-1} when A is

(*i*) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ *(K.U.K. May 2012)* (*ii*) $\begin{bmatrix} 2 & 6 & 1 \\ 0 & 1 & -6 \\ 3 & 4 & -2 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$. *(M.D.U. 2009; U.K.T.U. 2012)*

12. Using Cayley-Hamilton theorem, find the inverse of

(*i*) $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ *(M.D.U. Dec. 2010)* (*ii*) $\begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

13. Using Cayley-Hamilton theorem, find A^6 if $A = \begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix}$.

14. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and use it to find the matrix represented by

$$A^5 + 5A^4 - 6A^3 + 2A^2 - 4A + 7I.$$

Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A.

15. Show that the matrix $A = \begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix}$ satisfies Cayley-Hamilton theorem.

16. If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a linear polynomial in A.

Answers

1. (*i*) 5, 3 (*ii*) – 1, 45 (*iii*) 12, 32 (*iv*) 10, 262

2. (*a*) – 21 (*b*) 47/60 (*c*) 2, 50

3. (*i*) 1, 6 ; (1, – 1), (4, 1) (*ii*) 5, 0 ; (1, 2), (2, – 1) (*iii*) 5, – 2 ; (1, 1), (4, – 3)

4. (*i*) 0, 3, 15 ; (1, 2, 2), (2, 1, – 2), (2, – 2, 1) (*ii*) 2, 2, 8 ; (1, 0, – 2), (1, 2, 0), (2, – 1, 1)
(*iii*) 1, 2, 3 ; (1, 0, – 1), (0, 1, 0), (1, 0, 1) (*iv*) 1, 1, 3 ; (1, – 1, 0), (1, 0, – 1), (1, 1, 0)
(*v*) 2, 3, 5 ; (1, – 1, 0), (1, 0, 0), (2, 0, 1) (*vi*) – 2, 3, 6 ; (– 1, 0, 1), (1, – 1, 1), (1, 2, 1)
(*vii*) 1, – 1, 2 ; (1, – 1, 2), (2, – 1, 3), (2, – 1, 4) (*viii*) 0, 0, 5 ; (0, 1, 0), (2, 0, 1), (1, 0, – 2)
(*ix*) 2, 3, 6 ; (– 1, 0, 1), (1, 1, 1), (1, – 2, 1) (*x*) 1, 2, 3 ; (1, – 1, 0), (2, – 1, – 2), (1, – 1, – 2)
(*xi*) 1, $-\sqrt{5}$, $\sqrt{5}$; (– 1, 0, 1), $(1+\sqrt{5}, -1, 1)$, $(1-\sqrt{5}, -1, 1)$.

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9. [Hint. $|P^{-1}AP - \lambda I| = |P^{-1}AP - \lambda P^{-1}P| = |P^{-1}(A - \lambda I)P|$
$= |P^{-1}|\,|A - \lambda I|\,|P| = |A - \lambda I|\,|P^{-1}P| = |A - \lambda I|$]

10. (a) $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0,\ \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$ (b) $\lambda^3 - 3\lambda^2 - 14\lambda - 14 = 0,\ \frac{1}{14}\begin{bmatrix} -3 & 1 & 5 \\ 10 & -8 & 2 \\ -1 & 5 & -3 \end{bmatrix}$

11. (i) $\begin{bmatrix} -3 & 0 & 2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \end{bmatrix}$ (ii) $\frac{1}{67}\begin{bmatrix} -22 & -16 & 37 \\ 18 & 7 & -12 \\ 3 & -10 & -2 \end{bmatrix}$ (iii) $\frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$

12. (i) $\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$ (ii) $\frac{1}{50}\begin{bmatrix} -8 & 20 & -7 \\ -40 & 50 & -10 \\ 22 & -30 & 13 \end{bmatrix}$ (iii) $\frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$

13. $729I$ 14. $\begin{bmatrix} 1861 & 3676 \\ 1838 & 3699 \end{bmatrix}$; $A + 5I$ 16. $-4A + 5I$.

2.16. REDUCTION OF A MATRIX TO DIAGONAL FORM

If a square matrix A of order n has n linearly independent eigen vectors, then a matrix B can be found such that $B^{-1}AB$ is a diagonal matrix.

[We prove the result for a square matrix of order 3. The proof can be easily extended to matrices of higher order.]

Let A be a square matrix of order 3. Let $\lambda_1, \lambda_2, \lambda_3$ be its eigen values and

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix},\ X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix},\ X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \text{ be the corresponding eigen vectors.}$$

Since eigen vectors are non-trivial solutions of the matrix equation $AX = \lambda X$, we have

$$AX_1 = \lambda_1 X_1,\ AX_2 = \lambda_2 X_2,\ AX_3 = \lambda_3 X_3$$

Let
$$B = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = [X_1 X_2 X_3]$$

Consider $AB = A[X_1, X_2, X_3] = [AX_1, AX_2, AX_3] = [\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3]$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$= BD, \text{ where D is the diagonal matrix } \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$\Rightarrow$ $AB = BD \Rightarrow B^{-1}AB = B^{-1}BD = ID = D$ which proves the theorem.

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Note. The matrix B which diagonalises A is called the **modal matrix** of A and is obtained by grouping the eigen vectors of A into a square matrix. The diagonal matrix D is called the **spectral matrix** of A and has the eigen values of A as its diagonal elements.

2.17. CALCULATION OF POWERS OF A MATRIX

Given a square matrix A, it is quite tedious to find A^n (n being a positive integer) when n is large. On the other hand, it is quite easy to obtain any positive integral power of a diagonal matrix D, since

$$D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \quad \Rightarrow \quad D^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$$

Therefore, diagonalisation of a matrix is useful for finding positive integral powers of a square matrix.

Let A be a square matrix. Then we know that a non-singular matrix B can be found such that

$$B^{-1}AB = D \qquad ...(1)$$

where D is a diagonal matrix.

From (1), $\quad D^2 = (B^{-1}AB)(B^{-1}AB)$

$\quad = B^{-1}A^2B \qquad [\because BB^{-1} = I]$

Similarly, $\quad D^3 = B^{-1}A^3B$

In general, $\quad D^n = B^{-1}A^nB \qquad ...(2)$

To find A^n from (2), pre-multiply (2) by B and post-multiply by B^{-1}. Then

$$BD^nB^{-1} = BB^{-1}A^nBB^{-1}$$
$$= A^n$$

Hence $\quad A^n = BD^nB^{-1}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Show the the matrix* $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ *is diagonalizable. Hence find P such that* $P^{-1}AP$ *is a diagonal matrix.* (M.T.U. 2012; P.T.U. May 2008)

Sol. The characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

or $\quad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \qquad$ **(On simplification)**

Solving it, $\quad \lambda = 1, 2, 3$

Since the matrix A has three distinct eigen values, it has three linearly independent eigen vectors and hence A is diagonalizable.

When $\lambda = 1$, the corresponding eigen vector is given by

$$(A - I)X = O \quad \text{or} \quad \begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Operating $R_1 \to R_1 + R_2$

$$\sim \begin{bmatrix} 0 & 1 & 1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_2 \to \frac{1}{2}R_2$ and $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get only two independent equations $x_2 + x_3 = 0$ and $-x_1 + x_3 = 0$, since rank of co-efficient matrix is 2.

$\Rightarrow$ $x_2 = -x_3$ and $x_1 = x_3$

$\Rightarrow$ $\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$ giving the eigen vector (1, –1, 1).

When $\lambda = 2$, the corresponding eigen vector is given by

$$(A - 2I)\,X = O \quad \text{or} \quad \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_1 \to R_1 - R_3$ and $R_2 \to R_2 + R_3$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_2 \to R_2 + 2R_1$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get two independent equations $x_1 - x_3 = 0$ and $x_2 = 0$, since rank of co-efficient matrix is 2.

$\Rightarrow$ $x_1 = x_3$ and $x_2 = 0$ giving the eigen vector (1, 0, 1)

When $\lambda = 3$, the corresponding eigen vector is given by

$$(A - 3I)\,X = O \quad \text{or} \quad \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Operating $R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get two independent equations $x_2 - x_3 = 0$ and $-2x_1 = 0$, since rank of co-efficient matrix is 2.

$\Rightarrow$ $x_1 = 0$ and $x_2 = x_3$ giving the eigen vector (0, 1, 1)

$\therefore$ The modal matrix $P = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

and $$P^{-1} AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is the required diagonal matrix.

Example 2. *Diagonalise the matrix* $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ *and obtain the modal matrix.*

(M.D.U. Dec. 2009; U.K.T.U. 2011)

Sol. The characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0 \text{ or } \lambda^3 - \lambda^2 - 5\lambda + 5 = 0 \quad \text{(on simplification)}$$

Solving it, we have $\lambda = 1, \pm\sqrt{5}$.

When $\lambda = 1$, the corresponding eigen vector is given by

$$-2x_1 + 2y_1 - 2z_1 = 0$$
$$x_1 + y_1 + z_1 = 0$$
$$x_1 - y_1 + z_1 = 0$$

Solving the last two $\dfrac{x_1}{2} = \dfrac{y_1}{0} = \dfrac{z_1}{-2}$ giving the eigen vector (1, 0, – 1).

When $\lambda = \sqrt{5}$, the corresponding eigen vector is given by

$$(-1 - \sqrt{5})x_1 + 2y_1 - 2z_1 = 0$$
$$x_1 + (2 - \sqrt{5})y_1 + z_1 = 0$$
$$-x_1 - y_1 - \sqrt{5}z_1 = 0$$

Solving the last two $\dfrac{x_1}{6 - 2\sqrt{5}} = \dfrac{y_1}{-1 + \sqrt{5}} = \dfrac{z_1}{1 - \sqrt{5}}$

or $\dfrac{x_1}{(\sqrt{5} - 1)^2} = \dfrac{y_1}{\sqrt{5} - 1} = \dfrac{z_1}{1 - \sqrt{5}}$ or $\dfrac{x_1}{\sqrt{5} - 1} = \dfrac{y_1}{1} = \dfrac{z_1}{-1}$ giving the eigen vector $(\sqrt{5} - 1, 1, -1)$

Similarly, the eigen vector corresponding to $\lambda = -\sqrt{5}$ is $(\sqrt{5} + 1, -1, 1)$.

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$$\therefore \quad \text{The modal matrix } B = \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

and
$$B^{-1}AB = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$

is the required diagonal matrix.

Example 3. *Diagonalise the matrix* $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ *and hence find* A^4.

Sol. The characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

or $\quad \lambda^3 - 6\lambda^2 + 5\lambda + 12 = 0$ (On simplification)

or $\quad (\lambda + 1)(\lambda - 3)(\lambda - 4) = 0$

$\therefore \quad \lambda = -1, 3, 4$

When $\lambda = -1$, the corresponding eigen vector is given by

$$2x_1 + 6y_1 + z_1 = 0$$
$$x_1 + 3y_1 = 0$$
$$4z_1 = 0$$

Solving the last two, $\dfrac{x_1}{3} = \dfrac{y_1}{-1}$, $z_1 = 0$ giving the eigen vector (3, – 1, 0).

When $\lambda = 3$, the corresponding eigen vector is given by

$$-2x_1 + 6y_1 + z_1 = 0$$
$$x_1 - y_1 = 0$$

Solving $\dfrac{x_1}{1} = \dfrac{y_1}{1} = \dfrac{z_1}{-4}$ giving the eigen vector (1, 1, – 4).

When $\lambda = 4$, the corresponding eigen vector is given by

$$-3x_1 + 6y_1 + z_1 = 0$$
$$x_1 - 2y_1 = 0$$
$$z_1 = 0$$

Solving the last two, $\dfrac{x_1}{2} = \dfrac{y_1}{1}$, $z_1 = 0$ giving the eigen vector (2, 1, 0).

$$\therefore \quad \text{The modal matrix } B = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

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and $$B^{-1}AB = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

is the required diagonal matrix.

To find B^{-1}

$$|B| = \begin{vmatrix} 3 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & -4 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{(say)}$$

$$A_1 = 4,\ B_1 = 0,\ C_1 = 4,\ A_2 = -8,\ B_2 = 0,\ C_2 = 12,\ A_3 = -1,\ B_3 = -5,\ C_3 = 4$$

$\therefore$ $$|B| = a_1A_1 + b_1B_1 + c_1C_1 = (3)(4) + (1)(0) + 2(4) = 20$$

$$\text{adj. } B = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|}\text{adj } B = \frac{1}{20}\begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

Now $$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \Rightarrow D^4 = \begin{bmatrix} (-1)^4 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 4^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 256 \end{bmatrix}$$

$\therefore$ $$A^4 = BD^4B^{-1} = \frac{1}{20}\begin{bmatrix} 3 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 256 \end{bmatrix}\begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 3 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}\begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & -405 \\ 1024 & 3072 & 1024 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 2060 & 6120 & 1640 \\ 1020 & 3080 & 620 \\ 0 & 0 & 1620 \end{bmatrix} = \begin{bmatrix} 103 & 306 & 82 \\ 51 & 154 & 31 \\ 0 & 0 & 81 \end{bmatrix}$$

EXERCISE 2.5

1. Reduce the matrix $A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$ to the diagonal form.

2. By diagonalising the matrix $A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$, find A^4.

3. Diagonalise each of the following matrices :

(i) $\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

4. Diagonalise $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ and hence find A^3.

5. Diagonalise the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ and hence find A^4. *(M.T.U. 2013)*

6. Diagonalise $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ and hence find A^8.

7. Find a matrix P which transforms the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ to its diagonal form. Hence calculate A^3.

8. Show that the linear transformation $T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, where $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{b-a}\right)$ changes the matrix $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ to diagonal form $D = T^{-1}AT$.

Answers

1. $\begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$

2. $\begin{bmatrix} -29 & 45 \\ -30 & 46 \end{bmatrix}$.

3. (i) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -7 & 32 \\ 0 & 8 & -19 \\ 0 & 0 & 27 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$, $\begin{bmatrix} 251 & -405 & 235 \\ -405 & 891 & -405 \\ 235 & -405 & 251 \end{bmatrix}$

6. $\begin{bmatrix} -12099 & 12355 & 6305 \\ -12100 & 12356 & 6305 \\ -13120 & 13120 & 6561 \end{bmatrix}$

7. $P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 49 & 63 & 41 \\ 63 & 153 & 63 \\ 41 & 63 & 49 \end{bmatrix}$.

2.18. SIMILAR MATRICES

Let A and B be square matrices of the same order. The matrix A is said to be similar to the matrix B if there exists an invertible square matrix P such that $A = P^{-1}BP$.

The matrix P is called the **similarity matrix.**

Note. The matrices on the left and right of B are inverse of each other.

2.19. THEOREMS ON SIMILAR MATRICES

Theorem 1. *A square matrix A is similar to a square matrix B if and only if B is similar to A.*

Proof. A is similar to B if and only if there exists a non-singular matrix P such that

$$A = P^{-1}BP \quad \text{or} \quad PA = BP$$

Post-multiplying both sides by P^{-1}, we get

$$PAP^{-1} = B \quad \text{or} \quad B = PAP^{-1}$$

showing B is similar to A.

Therefore, A is similar to B if and only if B is similar to A.

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Theorem 2. *If A is similar to B and B is similar to C, then A is similar to C.* (Transitivity)

Proof. A is similar to B $\Rightarrow$ There exists an invertible matrix P such that $A = P^{-1}BP$ ...(1)

B is similar to C $\Rightarrow$ There exists an invertible matrix Q such that $B = Q^{-1}CQ$ ...(2)

Substituting for B from (2) in (1), we get

$$A = P^{-1}Q^{-1}CQP = (QP)^{-1}\,C\,(QP) = R^{-1}CR \text{ where } R = QP$$

$\Rightarrow$ A is similar to C.

Theorem 3. *Similar matrices have the same characteristic equation (and hence the same eigen values).*

Proof. Let A and B be two similar matrices. Then there exists a non-singular matrix P such that

$$B = P^{-1}AP$$

Now,
$$\begin{aligned}|B - \lambda I| &= |P^{-1}AP - \lambda I|\\ &= |P^{-1}AP - \lambda P^{-1}P| && (\because\ P^{-1}P = I)\\ &= |P^{-1}(A - \lambda I)P| = |P^{-1}|\ |A - \lambda I|\ |P|\\ &= |A - \lambda I|\ |P^{-1}|\ |P|\\ &= |A - \lambda I|\ |P^{-1}P| = |A - \lambda I|\ |I|\\ &= |A - \lambda I| && (\because\ |I| = 1)\end{aligned}$$

$\therefore$ Matrices A and B have the same characteristic polynomial and hence the same characteristic equation.

Example. *Show that the matrices*

$$A = \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \text{ are similar.}$$

Sol. A and B are similar if there exists a non-singular matrix P such that

$$A = P^{-1}BP \quad \text{or} \quad PA = BP$$

Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We shall find a, b, c, d such that PA = BP and then check whether P is non-singular.

Now, $$PA = BP$$

$$\Rightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 5a - 2b & 5a \\ 5c - 2d & 5c \end{bmatrix} = \begin{bmatrix} a + 2c & b + 2d \\ -3a + 4c & -3b + 4d \end{bmatrix}$$

Equating corresponding elements, we have

$$5a - 2b = a + 2c \quad \text{or} \quad 4a - 2b - 2c = 0$$

or $$2a - b - c = 0 \qquad ...(1)$$

$$5a = b + 2d \quad \text{or} \quad 5a - b - 2d = 0 \qquad ...(2)$$

$$5c - 2d = -3a + 4c \quad \text{or} \quad 3a + c - 2d = 0 \qquad ...(3)$$

$$5c = -3b + 4d \quad \text{or} \quad 3b + 5c - 4d = 0 \qquad ...(4)$$

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The co-efficient matrix $\begin{bmatrix} 2 & -1 & -1 & 0 \\ 5 & -1 & 0 & -2 \\ 3 & 0 & 1 & -2 \\ 0 & 3 & 5 & -4 \end{bmatrix}$ of this system of homogeneous equations has rank 2. A solution to this system of equations is $a = b = c = 1, d = 2$

$$\therefore \quad P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$ which is non-singular. Hence the matrices A and B are similar.

EXERCISE 2.6

1. Examine whether A is similar to B, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$ **(Ans. No)**

2. Show that the matrices

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -13 & -47 & -58 \\ 4 & 14 & 15 \\ 1 & 4 & 6 \end{bmatrix} \text{ are similar if}$$

$$P = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

[**Hint.** $P^{-1}AP = B$, $PBP^{-1} = A$]

3. Show that 'similarity of matrices' is an equivalence relation.
4. Let A be similar to B. Then show that

(i) A^{-1} is similar to B^{-1} (ii) $|A| = |B|$

2.20. QUADRATIC FORMS

Definition. A homogeneous polynomial of second degree in any number of variables is called a quadratic form. For example,

(i) $ax^2 + 2hxy + by^2$ (ii) $ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx$ and

(iii) $ax^2 + by^2 + cz^2 + du^2 + 2hxy + 2gyz + 2fzx + 2lxw + 2myw + 2nzw$

are quadratic forms in two, three and four variables.

In n-variables $x_1, x_2, \ldots\ldots, x_n$, the general quadratic form is $\sum_{j=1}^{n} \sum_{i=1}^{n} b_{ij} x_i x_j$

In the expansion, the co-efficient of $x_i x_j = (b_{ij} + b_{ji})$.

Suppose $2a_{ij} = b_{ij} + b_{ij}$ where $a_{ij} = a_{ji}$ and $a_{ii} = b_{ii}$

$$\therefore \quad \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ij} x_i x_j = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i x_j \text{ where } a_{ij} = \frac{1}{2}(b_{ij} + b_{ji}).$$

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Hence every quadratic form can be written as $\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j = X'AX$, so that the matrix A is always symmetric, where $A = [a_{ij}]$ and $X' = [x_1, x_2, \ldots\ldots, x_n]$.

Rule to write the matrix A of quadratic form

$$A = \begin{bmatrix} \text{co-eff. of } x_1^2 & \frac{1}{2}\text{co-eff. of } x_1x_2 & \frac{1}{2}\text{co-eff. of } x_1x_3 & \ldots \\ \frac{1}{2}\text{co-eff. of } x_2x_1 & \text{co. eff. of } x_2^2 & \frac{1}{2}\text{co-eff. of } x_2x_3 & \ldots \\ \frac{1}{2}\text{co-eff. of } x_3x_1 & \frac{1}{2}\text{co-eff. of } x_3x_2 & \text{co-eff. of } x_3^2 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$

Now writing the above said examples of quadratic forms in matrix form, we get

(*i*) $ax^2 + 2hxy + by^2 = [x\ y]\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(*ii*) $ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx = [x\ y\ z]\begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and (*iii*) $ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gyz + 2fzx + 2lxw + 2myw + 2nzw$

$$= [x\ y\ z\ w]\begin{bmatrix} a & h & f & l \\ h & b & g & m \\ f & g & c & n \\ l & m & n & d \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}.$$

2.21. LINEAR TRANSFORMATION OF A QUADRATIC FORM

Let $X'AX$ be a quadratic form in n-variables and let $X = PY$ ...(1)

where P is a non-singular matrix, be the non-singular transformation.

From (1), $X' = (PY)' = Y'P'$ and hence

$$X'AX = Y'P'APY = Y'(P'AP)Y = Y'BY \quad \text{...(2)}$$

where $B = P'AP$. Therefore $Y'BY$ is also a quadratic form in n-variables. Hence it is a linear transformation of the quadratic form $X'AX$ under the linear transformation $X = PY$ and $B = P'AP$.

2.22. CANONICAL FORM

If a real quadratic form be expressed as a sum or difference of the squares of new variables by means of any real non-singular linear transformation, then the latter quadratic expression is called a *canonical form* of the given quadratic form.

i.e., if the quadratic form $X'AX = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j$ can be reduced to the quadratic form

$$Y'BY = \sum_{i=1}^{n} \lambda_i y_i^2 \text{ by a non-singular linear transformation}$$

$X = PY$ then $Y'BY$ is called the canonical form of the given one.

$\therefore$ If $B = P'AP = \text{diag}(\lambda_1, \lambda_2, \ldots\ldots, \lambda_n)$ then $X'AX = Y'BY = \sum_{j=1}^{n} \lambda_i y_i^2$.

Note 1. Here some of λ_i (eigen values) may be positive or negative or zero.

2. A quadratic form is said to be real if the elements of the symmetric matrix are real.

3. If $\rho(A) = r$, then the quadratic form $X'AX$ will contain only r terms.

2.23. INDEX AND SIGNATURE OF THE QUADRATIC FORM

The number p of positive terms in the canonical form is called the *index* of the quadratic form.

(The number of positive terms)–(The number of negative terms) *i.e.*, $p - (r - p) = 2p - r$ is called *signature* of the quadratic form, where $\rho(A) = r$.

2.24. DEFINITE, SEMI-DEFINITE AND INDEFINITE REAL QUADRATIC FORMS

Let $X'AX$ be a real quadratic form in n-variables $x_1, x_2, \ldots\ldots, x_n$ with rank r and index p. Then we say that the quadratic form is

(*i*) *positive definite* if $r = n, p = r$

(*ii*) *negative definite* if $r = n, p = 0$

(*iii*) *positive semi-definite* if $r < n, p = r$

and (*iv*) *negative semi-definite* if $r < n, p = 0$

If the canonical form has both positive and negative terms, the quadratic form is said to be *indefinite*.

Note. If $X'AX$ is positive definite then $|A| > 0$.

2.25. LAW OF INERTIA OF QUADRATIC FORM

The index of a real quadratic form is invariant under real non-singular transformations.

ILLUSTRATIVE EXAMPLES

Example 1. *Reduce $3x^2 + 3z^2 + 4xy + 8xz + 8yz$ into canonical form.*

Sol. The given quadratic form can be written as $X'AX$ where $X' = [x, y, z]$ and the symmetric matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix}.$$

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Let us reduce A into diagonal matrix. We know that $A = I_3 A I_3$

i.e.,
$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_2 \to R_2 - \frac{2}{3} R_1$, $R_3 \to R_3 - \frac{4}{3} R_1$ (for A on L.H.S. and pre-factor on R.H.S.),

we get
$$\begin{bmatrix} 3 & 2 & 4 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & \frac{4}{3} & -\frac{7}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_2 \to C_2 - \frac{2}{3} C_1$, $C_3 \to C_3 - \frac{4}{3} C_1$ (for A on L.H.S. and post-factor on R.H.S.).

we get
$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & \frac{4}{3} & -\frac{7}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 \to R_3 + R_2$, we get
$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_3 \to C_3 + C_2$, we get
$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

or
$$\text{Diag}\left(3, -\frac{4}{3}, -1\right) = P'AP$$

∴ The canonical form of the given quadratic form is

$$Y'(P'AP)Y = [y_1, y_2, y_3] \begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 3y_1^2 - \frac{4}{3} y_2^2 - y_3^2.$$

Here $\rho(A) = 3$, index = 1, signature = 1 − 2 = − 1.

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Note. In this problem the non-singular transformation which reduces the given quadratic form into the canonical form is X = PY i.e., $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{3} & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, *i.e.*, $x = y_1 - \frac{2}{3} y_2 - 2y_3$, $y = y_2 + y_3$, $z =$

Note. The above example can also be questioned as 'Diagonalise' the quadratic form $3x^2 + 3z^2 + 4xy + 8xz + 8yz$ by linear transformation and write the linear transformation.

Or

Reduce the quadratic form $3x^2 + 3z^2 + 4xy + 8xz + 8yz$ into the sum of squares.

Example 2. *Reduce the quadratic form* $x^2 - 4y^2 + 6z^2 + 2xy - 4xz + 2w^2 - 6zw$ *into sum of squares.*

Sol. The matrix form of the given quadratic form is X'AX, where $X' = [x\ y\ z\ w]$

and
$$A = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

Let us reduce A to the diagonal matrix. We know that $A = I_4 A I_4$.

$$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 + 2R_1$, we get

$$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $C_2 \to C_2 - C_1$, $C_3 \to C_3 + 2C_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 \to R_3 + (2/5)R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 0 & \frac{14}{5} & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Operating $C_3 \to C_3 + (2/5)\, C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{8}{5} & 0 \\ 0 & 1 & \frac{2}{5} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $R_4 \to R_4 + \frac{15}{14} R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & -3 \\ 0 & 0 & 0 & -\frac{17}{14} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ \frac{12}{7} & \frac{3}{7} & \frac{15}{14} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{8}{5} & 0 \\ 0 & 1 & \frac{2}{5} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $C_4 \to C_4 + \frac{15}{14} C_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & 0 \\ 0 & 0 & 0 & -\frac{17}{14} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ \frac{12}{7} & \frac{3}{7} & \frac{15}{14} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{8}{5} & \frac{12}{7} \\ 0 & 1 & \frac{2}{5} & \frac{3}{7} \\ 0 & 0 & 1 & \frac{15}{14} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i.e., $\quad \text{diag}\left(1, -5, \frac{14}{5}, -\frac{17}{14}\right) = P'AP$

$\therefore$ the canonical form of the given quadratic form is

$$Y'(P'AP)Y = Y' \text{ diag}\left(1, -5, \frac{14}{5}, -\frac{17}{14}\right) Y$$

$$= [y_1 y_2 y_3 y_4] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & 0 \\ 0 & 0 & 0 & -\frac{17}{14} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$= y_1^2 - 5y_2^2 + \frac{14}{5} y_3^2 - \frac{17}{14} y_4^2,$$ which is the sum of the squares.

Note. Here $\rho(A) = 4$, index $= 2$, signature $= 2 - 2 = 0$.

2.26. REDUCTION TO CANONICAL FORM BY ORTHOGONAL TRANSFORMATION

Let $X'AX$ be a given quadratic form. The modal matrix B of A is that matrix whose columns are characteristic vectors of A. If B represents the orthogonal matrix of A (the

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normalised modal matrix of A whose column vectors are pairwise orthogonal) then $X = BY$ will reduce $X'AX$ to $Y'DY$ where $D = \text{diag}(\lambda_1, \lambda_2, \ldots\ldots, \lambda_n)$ and $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are characteristic roots of A.

Note. This method works successfully if the characteristic vectors of A are linearly independent which are pairwise orthogonal.

Example 3. *Reduce $8x^2 + 7y^2 + 3z^2 - 12xy + 4xz - 8yz$ into canonical form by orthogonal transformation.* (Calicut 2008)

Sol. The matrix of the quadratic form is

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

The characteristic roots of A are given by $|A - \lambda I| = 0$

i.e.,
$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda(\lambda - 3)(\lambda - 15) = 0$$

$\therefore$
$$\lambda = 0, 3, 15$$

Characteristic vector for $\lambda = 0$ is given by $[A - (0)I]\, X = O$

i.e.,
$$8x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 7x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 + 3x_3 = 0$$

Solving first two, we get $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$ giving the eigen vector $X_1 = k_1(1, 2, 2)'$

When $\lambda = 3$, the corresponding characteristic vector is given by $[A - 3I]X = O$

i.e.,
$$5x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 4x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 = 0$$

Solving any two equations, we get $X_2 = k_2(2, 1, -2)'$.

Similarly characteristic vector corresponding to

$$\lambda = 15 \text{ is } X_3 = k_3(2, -2, 1)'.$$

Now X_1, X_2, X_3 are pairwise orthogonal

i.e.,
$$X_1 \cdot X_2 = X_2 \cdot X_3 = X_3 \cdot X_1 = 0.$$

$\therefore$ The normalised modal matrix is

$$B = \left[\frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|}, \frac{X_3}{\|X_3\|}\right] = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

Now B is orthogonal matrix and $|B| = 1$

i.e.,
$$B^{-1} = B^T \text{ and } B^{-1}AB = D = \text{diag}(0, 3, 15)$$

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i.e.,
$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} A \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

$$X'AX = Y'(B^{-1}AB)\,Y = Y'DY$$

$$= [y_1\ y_2\ y_3] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0y_1^2 + 3\,y_2^2 + 15y_3^2$$

which is the required canonical form.

Note. Here the orthogonal transformation is X = BY, rank of the quadratic form = 2 ; index = 2, signature = 2. It is positive semi-definite.

Example 4. *Reduce $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$ into canonical form.*

Sol. The matrix of the quadratic form is $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic roots are given by $|A - \lambda I| = 0$

i.e.,
$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

or
$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

which on solving gives $\lambda = 8, 2, 2$.

The vector corresponding to $\lambda = 8$ is given by $[A - 8I]X = O$

i.e.,
$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or
$$-2x_1 - 2x_2 + 2x_3 = 0$$
$$-2x_1 - 5x_2 - x_3 = 0$$
$$2x_1 - x_2 - 5x_3 = 0$$

Solving any two of the equations, we get the vector as $k_1[2, -1, 1]'$.

The characteristic vector for $\lambda = 2$ is given by $[A - 2I]X = O$ which reduces to single equation

$$2x_1 - x_2 + x_3 = 0.$$

Putting $x_2 = 0$, we get $\frac{x_2}{1} = \frac{x_3}{1}$ or the vector is $k_2\,[0, 1, 1]'$. Again by putting $x_2 = 0$, we get $\frac{x_1}{1} = \frac{x_3}{-2}$ or the vector is $k_3\,[1, 0 - 2]'$.

Now $X_1 = k_1\,[2, -1, 1]'$; $X_2 = k_2\,[0, 1, 1]'$ and $X_3 = k_3\,[1, 0, -2]'$.

Here X_1, X_2, X_3 are not pairwise orthogonal

$\because$
$$X_1 \cdot X_2 = 0\ ; \quad X_2 \cdot X_3 \neq 0 \ \text{ and } \ X_3 \cdot X_1 = 0$$

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To get X_3 orthogonal to X_2 assume a vector $[u, v, w]'$ orthogonal to X_2 also satisfying

$2x_1 - x_2 + x_3 = 0$ *i.e.*, $2u - v + w = 0$ and $0.u + 1.v + 1.w = 0$

Solving $[u, v, w]' = [1, 1, -1]' = X_3$ so that $X_1 . X_2 = X_2 . X_3 = X_3 . X_1 = 0$

$\therefore$ The normalised modal matrix is

$$B = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

Now B is orthogonal matrix and $|B| = -1$

i.e., $$B' = B^{-1} \text{ and } B^{-1}AB = D \text{ where } D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\therefore$ $$X'AX = Y'(B^{-1}AB)Y = Y'DY$$

$$= [y_1\ y_2\ y_3] \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 8y_1^2 + 2y_2^2 + 2y_3^2$$

which is the required canonical form.

Note. Here rank of the quadratic form is 3, index = 3. signature = 3. It is positive definite.

EXERCISE 2.7

1. Write down the matrices of the following quadratic forms

   (*i*) $2x^2 + 3y^2 + 6xy$ (*ii*) $2x^2 + 5y^2 - 6z^2 - 2xy - yz + 8zx$

   (*iii*) $x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 + 2x_1x_2 + 4x_1x_3 - 6x_1x_4 - 4x_2x_3 - 8x_2x_4 + 12x_3x_4$.

2. Write down the quadratic forms corresponding to the following matrices:

   (*i*) $\begin{bmatrix} 2 & 4 & 5 \\ 4 & 3 & 1 \\ 5 & 1 & 1 \end{bmatrix}$ (*M.D.U. Dec. 2009*) (*ii*) $\begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$

3. Reduce the following quadratic forms to canonical forms or to sum of squares by linear transformation. Write also the rank, index and signature.

   (*i*) $2x^2 + 2y^2 + 3z^2 + 2xy - 4yz - 4xz$ (*M.D.U. Dec. 2010*) (*ii*) $12x_1^2 + 4x_2^2 + 5x_3^2 - 4x_2x_3 + 6x_1x_3 - 6x_1x_2$

   (*iii*) $2x^2 + 9y^2 + 6z^2 + 8xy + 8yz + 6zx$ (*iv*) $x^2 + 4y^2 + z^2 + 4xy + 6yz + 2zx$.

4. Reduce the following quadratic forms to canonical forms or to sum of squares by orthogonal transformation. Write also rank, index, signature.

   (*i*) $3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2zx$ (*K.U.K. 2008*) (*ii*) $2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 + 2x_1x_3 - 2x_2x_3$

   (*iii*) $3x^2 - 2y^2 - z^2 - 4xy + 8xz + 12yz$ (*iv*) $x^2 + 3y^2 + 3z^2 - 2yz$. (*Calicut 2009*)

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Answers

$\begin{bmatrix} \cdots & \\ & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 4 \\ -1 & 5 & -\frac{1}{2} \\ 4 & -\frac{1}{2} & -6 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & 2 & -3 \\ 1 & 2 & -2 & -4 \\ 2 & -2 & 3 & 6 \\ -3 & -4 & 6 & 4 \end{bmatrix}$

$2x^2 + 3y^2 + z^2 + 8xy + 2yz + 10xz$ (ii) $x_1^2 - 4x_2^2 + 6x_3^2 + 2x_4^2 + 2x_1x_2 - 4x_1x_3 - 6x_3x_4$

3. (i) $2y_1^2 + \frac{3}{2}y_2^2 + \frac{1}{3}y_3^2$; Rank = 3, index = 3, Sig. = 3

(ii) $12y_1^2 + \frac{13}{4}y_2^2 + \frac{49}{13}y_3^2$; Rank = 3, index = 3, Sig. = 3

(iii) $2y_1^2 + y_2^2 - \frac{5}{2}y_3^2$; Rank = 3, index = 2, Sig. = 1

(iv) $y_1^2 + 2y_2^2 - \frac{1}{2}y_3^2$; Rank = 3, index = 2, Sig. = 1.

4. (i) $2y_1^2 + 3y_2^2 + 6y_3^2$; Rank = 3 , index = 3, Sig. = 3

(ii) $4y_1^2 + y_2^2 + y_3^2$; Rank = 3, index = 3, Sig. = 3

(iii) $3y_1^2 + 6y_2^2 - 9y_3^2$; Rank = 3, index = 2, Sig. = 1

(iv) $y_1^2 + 2y_2^2 + 4y_3^2$; Rank = 3, index = 3, Sig. = 3.

2.27. COMPLEX MATRICES

If *all* the elements of a matrix are real numbers, then it is called *a real matrix* or a matrix over R. On the other hand, if *at least one* element of a matrix is a complex number $a + ib$, where a, b are real and $i = \sqrt{-1}$, then the matrix is called a *complex matrix*.

The matrix obtained by replacing the elements of a complex matrix A by the corresponding conjugate complex numbers is called the *conjugate of the matrix* A and is denoted by $\overline{A}$.

Thus, if $A = \begin{bmatrix} 2+3i & -7i \\ 5 & 1-i \end{bmatrix}$, then $\overline{A} = \begin{bmatrix} 2-3i & 7i \\ 5 & 1+i \end{bmatrix}$

It is easy to see that the *conjugate of the transpose of* A *i.e.*, $(\overline{A'})$ and the *transposed conjugate of* A *i.e.*, $(\overline{A})'$ are equal. Each of them is denoted by A^*.

Thus, $(\overline{A'}) = (\overline{A})' = A^*$.

A square matrix A is said to be *Hermitian* if $A^* = A$.

A square matrix A is said to be *skew-Hermitian* if $A^* = -A$.

In a Hermitian matrix, the diagonal elements are all real, while every other element is the conjugate complex of the element in the transposed position. For example,

$$A = \begin{bmatrix} 5 & 2+i & -3i \\ 2-i & -3 & 1-i \\ 3i & 1+i & 0 \end{bmatrix} \text{ is a Hermitian matrix.}$$

In a skew-Hermitian matrix, the diagonal elements are zero or purely imaginary numbers of the form $i\beta$, where β is real. Every other element is the negative of the cojugate complex of the element in the transposed position.

For example, $$B = \begin{bmatrix} 3i & 1+i & 7 \\ -1+i & 0 & -2-i \\ -7 & 2-i & -i \end{bmatrix} \text{ is a skew-Hermitian matrix.}$$

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2.28. UNITARY MATRIX

A square matrix A is said to be *unitary* if $A^*A = I = AA^*$. *The determinant of a unitary matrix is of unit modulus. For a matrix to be unitary, it must be non-singular.*

Note. The following results hold:

(*i*) $\overline{(\overline{A})} = A$ (*ii*) $\overline{A + B} = \overline{A} + \overline{B}$ (*iii*) $\overline{\lambda A} = \overline{\lambda}\,\overline{A}$

(*iv*) $\overline{AB} = \overline{A}\,\overline{B}$ (*v*) $(A^*)^* = A$ (*vi*) $(A + B)^* = A^* + B^*$

(*vii*) $(\lambda A)^* = \overline{\lambda} A^*$ (*viii*) $(AB)^* = B^*A^*$.

ILLUSTRATIVE EXAMPLES

Example 1. *If* $A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$, *verify that A^*A is a Hermitian matrix.*

Sol.
$$A' = \begin{bmatrix} 2+i & -5 \\ 3 & i \\ -1+3i & 4-2i \end{bmatrix}$$

$$A^* = \overline{(A')} = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$$

$$\therefore \qquad A^*A = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}\begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 6-8i & -19-17i \\ 6+8i & 10 & -5+5i \\ -19+17i & -5+5i & 30 \end{bmatrix} = B \text{ (say)}$$

Now
$$B' = \begin{bmatrix} 30 & 6+8i & -19-17i \\ 6-8i & 10 & -5-5i \\ -19+17i & -5+5i & 30 \end{bmatrix}$$

$$B^* = \overline{(B')} = \begin{bmatrix} 30 & 6-8i & -19+17i \\ 6+8i & 10 & -5+5i \\ -19-17i & -5-5i & 30 \end{bmatrix} = B$$

$\Rightarrow$ $B = A^*A$ is a Hermitian matrix.

Example 2. *If A and B are Hermitian, show that AB – BA is skew-Hermitian.*

Sol. A and B are Hermitian $\Rightarrow$ $A^* = A$ and $B^* = B$

Now $(AB - BA)^* = (AB)^* - (BA)^*$

$= B^*A^* - A^*B^* = BA - AB = -(AB - BA)$

$\Rightarrow$ AB – BA is skew-Hermitian.

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Example 3. *If A is a skew-Hermitian matrix, then show that iA is Hermitian.*

Sol. A is a skew-Hermitian matrix $\Rightarrow$ $A^* = -A$

Now $(iA)^* = \bar{i}\ A^* = (-i)(-A) = iA$

$\Rightarrow$ iA is a Hermitian matrix.

Example 4. *Prove that a unitary matrix over R is an orthogonal matrix.*

Sol. Let A be a unitary matrix over R, then $\bar{A} = A$

$$\therefore \quad A^* = (\bar{A})' = A' \qquad \ldots(1)$$

Since A is unitary, $AA^* = I = A^*A$

$$\Rightarrow \quad AA' = I = A'A \qquad [\text{Using } (1)]$$

$\Rightarrow$ A is orthogonal.

Example 5. *Prove that the transpose of a unitary matrix is unitary.*

Sol. Let A be a unitary matrix, then

$$AA^* = I = A^*A$$

$$\Rightarrow \quad (AA^*)' = I' = (A^*A)'$$

$$\Rightarrow \quad (A^*)'A' = I = A'(A^*)' \qquad [\because \ (AB)' = B'A']$$

$$\Rightarrow \quad ((\overline{A'}))'\ A' = I = A'((\overline{A'}))'$$

$$\Rightarrow \quad (A')^*\ A' = I = A'(A')^* \qquad [\because \ (\bar{P})' = P^*]$$

$\Rightarrow$ A' is unitary.

Example 6. *Prove that the conjugate of a unitary matrix is unitary.*

Sol. Let A be a unitary matrix, then

$$AA^* = I = A^*A$$

$$\Rightarrow \quad \overline{AA^*} = \bar{I} = \overline{A^*A}$$

$$\Rightarrow \quad \bar{A}(\overline{A^*}) = I = (\overline{A^*})\bar{A} \qquad [\because \ \overline{AB} = \bar{A}\ \bar{B}]$$

$$\Rightarrow \quad \bar{A}(\overline{(\bar{A})'}) = I = (\overline{(\bar{A})'})\bar{A}$$

$$\Rightarrow \quad \bar{A}(\bar{A})^* = I = (\bar{A})^*\bar{A} \qquad [\because \ (\bar{P}') = P^*]$$

$\Rightarrow$ $\bar{A}$ is unitary.

Example 7. *Prove that the product of two unitary matrices is a unitary matrix.* (K.U.K. 2013)

Sol. Let A and B be two unitary matrices, then

$$AA^* = I = A^*A \quad \text{and} \quad BB^* = I = B^*B \qquad \ldots(1)$$

Now
$$\begin{aligned}(AB)(AB)^* &= (AB)(B^*A^*)\\ &= A(BB^*)A^* \qquad [\text{Using } (1)]\\ &= AIA^*\\ &= (AI)A^*\\ &= AA^* \qquad [\text{Using } (1)]\\ &= I\end{aligned}$$

Similarly $(AB)^*(AB) = I$

Since $(AB)(AB)^* = I = (AB)^*(AB)$

The matrix AB is unitary.

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Example 8. *Show that every square matrix is expressible as the sum of a Hermitian matrix and a skew-Hermitian matrix.*

Sol. Let A be any square matrix.

Since $(A + A^*)^* = A^* + (A^*)^* = A^* + A = A + A^*$

and $(A - A^*)^* = A^* - (A^*)^* = A^* - A = -(A - A^*)$

$\therefore$ $A + A^*$ is Hermitian and $A - A^*$ is skew-Hermitian.

Now $$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) = P + Q \text{ (say)}$$

where P is Hermitian and Q is skew-Hermitian. Thus, every square matrix can be expressed as the sum of a Hermitian matrix and a skew-Hermitian matrix.

Example 9. *Show that every square matrix A can be expressed as $P + iQ$, where P and Q are Hermitian matrices.*

Sol. Let A be any square matrix.

Consider $$A = \frac{1}{2}(A + A^*) + i \cdot \frac{1}{2i}(A - A^*) = P + iQ$$

where $$P = \frac{1}{2}(A + A^*) \quad \text{and} \quad Q = \frac{1}{2i}(A - A^*)$$

Now $$P^* = \left[\frac{1}{2}(A + A^*)\right]^* = \frac{1}{2}(A + A^*)^*$$

$$= \frac{1}{2}(A^* + (A^*)^*) = \frac{1}{2}(A^* + A) = \frac{1}{2}(A + A^*) = P$$

and $$Q^* = \left[\frac{1}{2i}(A - A^*)\right]^* = \overline{\left(\frac{1}{2i}\right)}(A - A^*)^*$$

$$= -\frac{1}{2i}(A^* - (A^*)^*) = -\frac{1}{2i}(A^* - A) = \frac{1}{2i}(A - A^*) = Q$$

Thus P and Q both are Hermitian matrices.

Example 10. *Show that every Hermitian matrix A can be expressed as $P + iQ$ where P is real and symmetric while Q is real and skew symmetric.*

Sol. Let A be a Hermitian matrix so that $A^* = A$

$\Rightarrow$ $(\bar{A})' = A \Rightarrow \bar{A} = A'$

Consider $$A = \frac{1}{2}(A + \bar{A}) + i \cdot \frac{1}{2i}(A - \bar{A}) = P + iQ$$

where $$P = \frac{1}{2}(A + \bar{A}) \quad \text{and} \quad Q = \frac{1}{2i}(A - \bar{A})$$

If $a_{ij} = x + iy$ then $\bar{a}_{ij} = x - iy$ so that $a_{ij} + \bar{a}_{ij} = 2x$

and $a_{ij} - \bar{a}_{ij} = 2iy$

Thus P and Q are real matrices.

Now $$P' = \left[\frac{1}{2}(A + \bar{A})\right]' = \frac{1}{2}(A + \bar{A})' = \frac{1}{2}(A' + (\bar{A})')$$

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$$= \frac{1}{2}(A' + A^*) = \frac{1}{2}(\bar{A} + A) = \frac{1}{2}(A + \bar{A}) = P$$

$\Rightarrow$ P is symmetric.

Also
$$Q' = \left[\frac{1}{2i}(A - \bar{A})\right]' = \frac{1}{2i}(A - \bar{A})' = \frac{1}{2i}(A' - (\bar{A})')$$

$$= \frac{1}{2i}(A' - A^*) = \frac{1}{2i}(\bar{A} - A) = -\frac{1}{2i}(A - \bar{A}) = -Q$$

$\Rightarrow$ Q is skew symmetric.

Example 11. *If* $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, *obtain the matrix* $(I - N)(I + N)^{-1}$ *and show that it is unitary.* (M.T.U. 2013; G.B.T.U. 2011)

Sol.
$$I - N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$I + N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$|I + N| = \begin{vmatrix} 1 & 1+2i \\ -1+2i & 1 \end{vmatrix} = 1 - (4i^2 - 1) = 6$$

$$\text{adj}(I + N) = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$(I + N)^{-1} = \frac{1}{|I + N|}\,\text{adj}(I + N) = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$\therefore \quad (I - N)(I + N)^{-1} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} = A \text{ (say)}$$

$$A' = \frac{1}{6}\begin{bmatrix} -4 & 2-4i \\ -2-4i & -4 \end{bmatrix}$$

$$\overline{(A')} = A^* = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$$

$$A^*A = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix} \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$$

$$= \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\Rightarrow$ $A = (I - N)(I + N)^{-1}$ is unitary.

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EXERCISE 2.8

1. Show that the matrix $A = \begin{bmatrix} 2 & 3-4i \\ 3+4i & 2 \end{bmatrix}$ is Hermitian and iA is skew-Hermitian. *(U.K.T.U. 2012)*

2. If $H = \begin{bmatrix} 3 & 5+2i & -3 \\ 5-2i & 7 & 4i \\ -3 & -4i & 5 \end{bmatrix}$, prove that H is Hermitian.

   Also verify that iH is skew-Hermitian.

3. If $A = \begin{bmatrix} -1 & 2+i & 5-3i \\ 2-i & 7 & 5i \\ 5+3i & -5i & 2 \end{bmatrix}$, show that A is a Hermitian matrix and iA is a skew-Hermitian matrix.

4. If A is any square matrix, prove that $A + A^*$, AA^*, A^*A are all Hermitian and $A - A^*$ is skew-Hermitian.

5. If A, B are Hermitian or skew-Hermitian, then so is $A + B$.

6. Show that the matrix $B^* AB$ is Hermitian or skew-Hermitian according as A is Hermitian or skew-Hermitian.

7. Show that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary. *(G.B.T.U. 2013)*

8. Prove that $\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitary matrix and find A^{-1}. *(K.U.K., May 2011, May 2012)*

9. If A is a Hermitian matrix, then show that iA is a skew-Hermitian matrix. *(M.T.U. 2013)*

10. Show that the matrix $\begin{bmatrix} \alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma \end{bmatrix}$ is unitary if $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$. *(G.B.T.U. 2011)*

11. Show that $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ is skew-Hermitian as well as unitary. *(M.T.U. 2012)*

12. Express the matrix $A = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$ as the sum of a Hermitian and a skew-Hermitian matrix. *(G.B.T.U. 2010)*

13. Express the Hermitian matrix $A = \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}$ as $P + iQ$ where P is a real symmetric and Q is a real skew symmetric matrix. *(G.B.T.U. 2012)*

14. Prove that the conjugate transpose of a unitary matrix is unitary.

15. Prove that the inverse of a unitary matrix is unitary.

16. Prove that the determinant of a unitary matrix has absolute value 1.

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