4. In an equation with real coefficients, imaginary roots occur in conjugate pairs, i.e. if $\alpha+i\beta$ is a root of the equation $f(x)$, then $\alpha-i\beta$ must also be its root.

Similarly if $a+\sqrt{b}$ is an irrational root of an equation, then $a-\sqrt{b}$ must also be its root.

5.2 BISECTION METHOD

Bisection method is based on the repeated application of the intermediate value theorem. If we know that a root of $f(x)=0$ lies in the interval (a, b). Let $f(a)$ be negative and $f(b)$ be positive. We bisect (a,b) at the point $c=\dfrac{a+b}{2}$. Then the first approximation to the root is c.

If $f(c)=0$, then c is a root of $f(x)=0$. Otherwise. If $f(c)$ is positive, then root lies between a and c or if $f(c)$ is negative, then root lies between c and b. Then we bisect the interval as before and continue the process until the root is found to desired accuracy.

Example 1. Find the root of the equation $x^3-2x-5=0$, using bisection method upto three decimal places.

Solution. Let $f(x)=x^3-2x-5$

so that $f(2)=(2)^3-2(2)-5=-1$

and $f(3)=(3)^3-2(3)-5=16$

$\therefore$ Root lies between 2 and 3.

Let $a=2$ and $b=3$, $c=\dfrac{a+b}{2}=\dfrac{2+3}{2}=2.5$

We show the root of given equation by the following table:

$f(a)$	sign	$f(b)$	sign	$f(c)$	sign
$f(2)$	–	$f(3)$	+	$f(2.5)$	+
$f(2)$	–	$f(2.5)$	+	$f(2.25)$	+
$f(2)$	–	$f(2.25)$	+	$f(2.125)$	+
$f(2)$	–	$f(2.125)$	+	$f(2.0625)$	–
$f(2.0625)$	–	$f(2.125)$	+	$f(2.09375)$	–
$f(2.09375)$	–	$f(2.125)$	+	$f(2.109375)$	+
$f(2.09375)$	–	$f(2.109375)$	+	$f(2.1015625)$	+
$f(2.09375)$	–	$f(2.1015625)$	+	$f(2.09765)$	+
$f(2.09375)$	–	$f(2.09765)$	+	$f(2.0957)$	+
$f(2.09375)$	–	$f(2.0957)$	+	$f(2.094725)$	+
$f(2.09375)$	–	$f(2.094725)$	+	$f(2.0942375)$	-

Thus, we get 2.094 root correct upto three decimal place ($\because$ value of c is repeated).

Example 2. Find the root of the equation $x - \cos x = 0$, using bisection method correct to three decimal places.

Solution. Let $f(x) = x - \cos x$

so that $f(0) = 0 - \cos(0) = -1$

and $f(1) = 1 - \cos(1) = 1 - \cos(57.3°) = 1 - 0.54 = 0.46$

Therefore, root lies between 0 and 1.

Let $a = 0$ and $b = 1$, $c = \frac{a+b}{2} = \frac{0+1}{2} = 0.5$

We show the root of the given equation by the following table:

$f(a)$	sign	$f(b)$	sign	$f(c)$	sign
$f(0)$	–	$f(1)$	+	$f(0.5)$	–
$f(0.5)$	–	$f(1)$	+	$f(0.75)$	+
$f(0.5)$	–	$f(0.75)$	+	$f(0.625)$	–
$f(0.625)$	–	$f(0.75)$	+	$f(0.6875)$	–
$f(0.6875)$	–	$f(0.75)$	+	$f(0.71875)$	–
$f(0.71875)$	–	$f(0.75)$	+	$f(0.7344)$	–
$f(0.7344)$	–	$f(0.75)$	+	$f(0.7422)$	+
$f(0.7344)$	–	$f(0.7422)$	+	$f(0.7383)$	–
$f(0.7383)$	–	$f(0.7422)$	+	$f(0.74025)$	+
$f(0.7383)$	–	$f(0.74025)$	+	$f(0.739275)$	+
$f(0.7383)$	–	$f(0.739275)$	+	$f(0.73879)$	–
$f(0.73879)$	–	$f(0.739275)$	+	$f(0.7390)$	–
$f(0.7390)$	–	$f(0.739275)$	+	$f(0.7391)$	+

Thus, we get 0.739 root of given equation (because value of c is repeated).

Example 3. Find a root of the equation $x \log_{10} x = 1.2$, using bisection method.

Solution. Let $f(x) = x \log_{10} x - 1.2$

so that $f(2) = 2 \log_{10} 2 - 1.2 = 2\,(0.3010) - 1.2 = -0.598$

$f(3) = 3 \log_{10} 3 - 1.2 = 3\,(0.4771) - 1.2 = 0.2313$

Therefore, root lies between 2 and 3.

Let $a = 2$ and $b = 3$, $c = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$

We show the root of the given equation by the following table:

$f(a)$	sign	$f(b)$	sign	$f(c)$	sign
$f(2)$	–	$f(3)$	+	$f(2.5)$	–
$f(2.5)$	–	$f(3)$	+	$f(2.75)$	+
$f(2.5)$	–	$f(2.75)$	+	$f(2.625)$	–
$f(2.625)$	–	$f(2.75)$	+	$f(2.6875)$	–
$f(2.6875)$	–	$f(2.75)$	+	$f(2.71875)$	–
$f(2.71875)$	–	$f(2.75)$	+	$f(2.734375)$	–
$f(2.734375)$	–	$f(2.75)$	+	$f(2.7422)$	+
$f(2.734375)$	–	$f(2.7422)$	+	$f(2.7383)$	–
$f(2.7383)$	–	$f(2.7422)$	+	$f(2.74025)$	–
$f(2.74025)$	–	$f(2.7422)$	+	$f(2.7412)$	+
$f(2.74025)$	–	$f(2.7412)$	+	$f(2.741)$	+

Thus, we get 2.741 root of the given equation.

Example 4. Find root of the equation $\sin x = \frac{1}{x}$, using bisection method.

Solution. Given equation is $\sin x = \frac{1}{x}$ or $x \sin x - 1 = 0$

Let $f(x) = x \sin x - 1$

so that $f(1) = (1)\sin(1) - 1 = (1)\sin 57.3° - 1 = -0.158$

$f(2) = (2)\sin(2) - 1 = (2)\sin 114.6° - 1 = 0.818$

Thus, root lies between 1 and 2.

Let $a = 1, b = 2$ and $c = \frac{a+b}{2} = \frac{1+2}{2} = 1.5$

We show the root of given equation by the following table:

$f(a)$	sign	$f(b)$	sign	$f(c)$	sign
$f(1)$	–	$f(2)$	+	$f(1.5)$	+
$f(1)$	–	$f(1.5)$	+	$f(1.25)$	+
$f(1)$	–	$f(1.25)$	+	$f(1.125)$	+
$f(1)$	–	$f(1.125)$	+	$f(1.0625)$	–
$f(1.0625)$	–	$f(1.125)$	+	$f(1.09375)$	–
$f(1.09375)$	–	$f(1.125)$	+	$f(1.109375)$	–
$f(1.109375)$	–	$f(1.125)$	+	$f(1.117)$	+
$f(1.109375)$	–	$f(1.117)$	+	$f(1.113)$	–

We get 1.11 root of the given equation.

Example 5. Find the root of the equation $x - e^{-x} = 0$ by bisection method.

Solution. Let $f(x) = x - e^{-x}$

so that $f(0.5) = 0.5 - e^{-0.5} = 0.5 - 0.6065 = -0.1065$

$f(0.6) = 0.6 - e^{-0.6} = 0.6 - 0.5488 = 0.0512$

Therefore, roots lies between 0.5 and 0.6.

Let $a = 0.5$ and $b = 0.6$ $c = \dfrac{a+b}{2} = \dfrac{0.5+0.6}{2} = 0.55$

We show the root of the given equation by the following table:

$f(a)$	sign	$f(b)$	sign	$f(c)$	sign
$f(0.5)$	–	$f(0.6)$	+	$f(0.55)$	–
$f(0.55)$	–	$f(0.6)$	+	$f(0.575)$	+
$f(0.55)$	–	$f(0.575)$	+	$f(0.5625)$	–
$f(0.5625)$	–	$f(0.575)$	+	$f(0.56875)$	+
$f(0.5625)$	–	$f(0.56875)$	+	$f(0.565625)$	–
$f(0.565625)$	–	$f(0.56875)$	+	$f(0.5671875)$	+
$f(0.565625)$	–	$f(0.5671875)$	+	$f(0.56640625)$	–
$f(0.56640625)$	–	$f(0.5671875)$	+	$f(0.566796875)$	–

Thus, we get 0.566 root of the given equation

Example 6. Find the root of the equation $x\,e^{x} = 1$ b

Solution. Let $f(x) = x\,e^{x} - 1$

so that $f(0) = -1$ and $f(1) =$

Therefore, roots lies between 0 and 1..

Let $a = 0$ and $b = 1$, $c = \dfrac{a+b}{2} = \dfrac{0+1}{2} = 0.5$

We show the root of the given equation by the

$f(a)$	sign	$f(b)$
$f(0)$	–	$f(1)$
$f(0.5)$	–	$f(1)$
$f(0.5)$	–	$f(0.6$
$f(0.5625)$	–	$f(0.6$
$f(0.5625)$	–	$f(0.$
$f(0.5625)$	–	$f(0$

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$f(a)$	sign	$f(b)$	sign	$f(c)$	sign
$f(0.5625)$	–	$f(0.5703)$	+	$f(0.5664)$	–
$f(0.5664)$	–	$f(0.5703)$	+	$f(0.5683)$	+
$f(0.5664)$	–	$f(0.5683)$	+	$f(0.5673)$	+
$f(0.5664)$	–	$f(0.5673)$	+	$f(0.5668)$	–
$f(0.5668)$	–	$f(0.5673)$	+	$f(0.567)$	–
$f(0.567)$	–	$f(0.5673)$	+	$f(0.567)$	–

Thus, we get 0.567 root of the given equation.

EXERCISE 5.1

Find a root of the following equations, using the bisection method correct to three decimal places:

(i) $x^3 - 4x - 9 = 0$ (ii) $3x + \sin x = e^x$ (iii) $x^3 - x - 11 = 0$

(iv) $x^3 - x - 1 = 0$ (v) $x^3 - 26 = 0$ (vi) $2x^3 - x^3 - 20x + 12 = 0$

(vii) $2x - \log x = 6$ (viii) $x + \log_{10} x = 3.375$ (ix) $e^x - x = 2$

(x) $x^3 - x^2 + x + 7 = 0$ (xi) $e^x - 3x - \sin x = 0$ (xii) $x^3 + 2x^2 + 10x - 20 = 0$

(xiii) $x^3 - 4x - 10 = 0$ (xiv) $x^2 - 15 = 0$ (xv) $x \sin x + \cos x = 0$

(xvi) $x^3 - x - 4 = 0$ (xvii) $x^3 - 9x + 1 = 0$ (xviii) $9x^2 - \sin x - 1 = 0$

(xix) $x^4 - x^3 - 2x^2 - 6x - 4 = 0$

ANSWERS

(i) 2.7064 (ii) 0.34375 (iii) 2.373 (iv) 1.325 (v) 2.962 (vi) 2.522 (vii) 3.257

(viii) 2.910 (ix) 1.146 (x) –2.0625 (xi) 0.3604 (xii) 1.3688 (xiii) –1.7416

(xiv) 4.999 (xv) 2.7983 (xvi) 1.79625 (xvii) 2.9429 (xviii) 0.3918 (xix) 2.7065

5.3 REGULA–FALSI METHOD OR FALSE POSITION METHOD

In Regula–Falsi method, to find a real root of the equation $f(x) = 0$ and closely resembles the bisection method. We choose a sufficiently small ...terval (x_0, x_1) in which the root of the equation lies. ...raph of $y = f(x)$ crosses the X–axis between x_0 ...oints and hence $f(x_0)$ and $f(x_1)$ are of ...s i.e. $f(x_0)\, f(x_1) < 0$.

...h of the curve can be taken as a ...he x–coordiante of the point of ...ining points P and Q with ...value of the root.

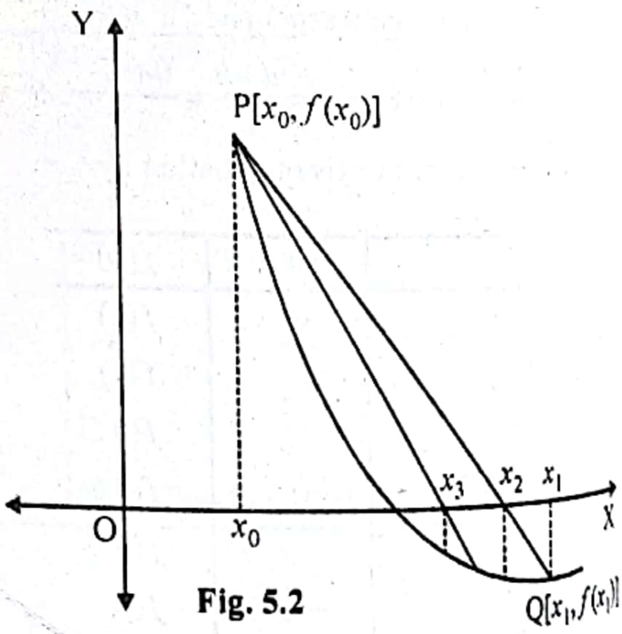

Fig. 5.2

Equation of the chord joining the points $P[(x_0, f(x_0)]$ and $Q[(x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \quad ....(1)$$

The x-coordinate of the point of intersection of this line with x–axis is given by putting $y = 0$ in (1), we get

$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)}.f(x_0) \quad ....(2)$$

Replacing x_2 by x_{n+1}, x_1 by x_n and x_0 by x_{n-1}, the above relation (2) can be written as

$$x_{n+1} = x_{n-1} - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}.f(x_{n-1})$$

or $$x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Example 1. Find a real root of the equation $x^3 + x - 1 = 0$ correct to three decimal palces, by Regula–Falsi method.

Solution. Let $f(x) = x^3 + x - 1$

so that $f(0.65) = (0.65)^3 + 0.65 - 1 = -0.075$

$f(0.71) = (0.71)^3 + 0.71 - 1 = 0.0679$

Thus, root lies between 0.65 and 0.71.

Taking $x_0 = 0.65$ and $x_1 = 0.71$

By Regula–Falsi method

$$x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad ....(1)$$

Putting $n = 1$, we get

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(0.65)(0.0679) - (0.71)(-0.075)}{0.0679 + 0.075} = 0.68149$$

Now $f(x_2) = f(0.68149) = (0.68149)^3 + 0.68149 - 1 = -0.002$

Thus, root lies between 0.68149 and 0.71.

Putting $n = 2$ in equation (1) and take $x_1 = 0.68149$, $f(x_1) = 0.002$

and $x_2 = 0.71$ and $f(x_2) = 0.0679$, we get

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(0.68149)(0.0679) + (0.71)(0.002)}{0.0679 + 0.002} = 0.68226$$

Now $f(x_3) = f(0.68226) = (0.68226)^3 + 0.68226 - 1 = -0.000162$

Thus, root lies between 0.68226 and 0.71.

Take $x_2 = 0.68226, f(x_2) = -0.000162, x_3 = 0.71, f(x_3) = 0.0679$

Putting $n = 3$ in (1), we get

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{(0.68226)(0.0679) + (0.71)(0.000162)}{0.0679 + 0.000162} = 0.6823$$

Hence root is 0.682 correct to three decimal places.

Example 2. Find a real root of the equation $x^3 - 2x - 5 = 0$ by method of False–Position, correct to four decimal places. [KUK 2007]

Solution. Let $f(x) = x^3 - 2x - 5$

so that $f(2.095) = (2.095)^3 - 2(2.095) - 5 = 0.005$

$f(2.005) = (2.005)^3 - 2(2.005) - 5 = -0.949$

Thus, root lies between 2.002 and 2.095

By method of False–position

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad ....(1)$$

Putting $n = 1$ in equation (1) and take $x_0 = 2.005$ and $x_1 = 2.095$, $f(x_0) = -0.949$ and $f(x_1) = 0.005$, we get

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(2.005)(0.005) + (2.095)(0.949)}{0.005 + 0.949} = 2.0945$$

$f(x_2) = f(2.0945) = (2.0945)^3 - 2(2.0945) - 5 = -0.0006$

Therefore, root lies between 2.0945 and 2.095.

Taking $x_1 = 2.0945$, $f(x_1) = 0.0006$, $x_2 = 2.095$. $f(x_2) = 0.005$

Putting $n = 2$ in equation (1), we get

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(2.0945)(0.005) + (2.095)(0.0006)}{0.005 + 0.0006} = 2.0945$$

Hence root is 2.0945 correct to four decimal places.

Example 3. Find the root of the equation $x e^x = \cos x$ using the Regula–Falsi method, correct to four decimal places [KUK 2009, 2008, 2006]

Solution. Let $f(x) = \cos x - xe^x$

so that
$$f(0.515) = \cos(0.515) - (0.515)(2.7)^{0.515}$$
$$= \cos 29.5095° - (0.515)(1.6736)$$
$$= 0.87027 - 0.861904$$
$$= 0.0083$$

$$\left(\because 1^r = 57.3° \quad e \propto 2.7\right)$$

$$f(0.525) = \cos(0.525) - (0.525)(2.7)^{0.525}$$
$$= \cos 30.0825° - (0.525)(1.69045)$$
$$= 0.8653 - 0.88748$$
$$= -0.02218$$

Thus, root lies between 0.525 and 0.515

Taking $x_0 = 0.525$ and $x_1 = 0.515$, $f(x_0) = -0.02218$, $f(x_1) = 0.0083$

By Regula–Falsi method

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \qquad ....(1)$$

Putting $n = 1$, we get

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(0.525)(0.0083) + (0.515)(0.02218)}{0.0083 + 0.02218} = 0.5177$$

$$f(x_2) = f(0.5177) = \cos(0.5177) - (0.5177)(2.7)^{0.5177}$$
$$= \cos 29.66° - (0.5177)(1.67816) = 0.000194$$

Thus, root lies between 0.525 and 0.5177

Taking $x_1 = 0.525$, $f(x_1) = -0.02218$, $x_2 = 0.5177$, $f(x_2) = 0.000194$

Putting $n = 2$ in (1), we get

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(0.525)(0.000194) + (0.5177)(0.02218)}{0.000194 + 0.02218} = 0.5177$$

Hence root is 0.5177 correct to four decimal places.

Example 4. Find the real root of the equation $x e^x = 2$, using Regula–Falsi method correct three decimal three places.

Solution. Let $f(x) = xe^x - 2$

so that
$$f(0.851) = (0.851)(2.7)^{0.851} - 2$$
$$= (0.851)(2.34198) - 2 = -0.00697$$

and $\quad f(0.855) = (0.855)(2.7)^{0.855} - 2$

$$= (0.855)(2.35137) - 2 = 0.01042$$

Thus, root lies between 0.851 and 0.855

By Regula–Falsi method

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad \text{....(1)}$$

Taking $x_0 = 0.851, f(x_0) = -0.00697, x_1 = 0.855, \; f(x_1) = 0.01042$

Putting $n = 1$ in equation (1), we get

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(0.851)(0.01042) + (0.855)(0.00697)}{0.01042 + 0.00697} = 0.8525$$

$$f(x_2) = (0.8525)(2.7)^{0.8525} - 2 = (0.8525)(2.3455) - 2 = 0.00046$$

Thus, root lies between 0.8525 and 0.855

Taking $x_1 = 0.8525, x_2 = 0.855, \; f(x_1) = -0.00046, \; f(x_2) = 0.01042$

Putting $n = 2$ in equation (1), we get

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(0.8525)(0.01042) + (0.855)(0.00046)}{0.01042 + 0.00046} = 0.852$$

Hence root is 0.852 correct to three decimal places.

Example 5. Find the fourth root of 12 correct to three decimal places using method of false position.

Solution. Let $\quad x = (12)^{\frac{1}{4}} \quad$ or $\quad x^4 = 12$

Let $\quad f(x) = x^4 - 12$

so that $\quad f(1.86) = (1.86)^4 - 12 = 11.9688 - 12 = -0.0312$

and $\quad f(1.863) = (1.863)^4 - 12 = 12.04623 - 12 = 0.004623$

Thus, root lies between 1.86 and 1.863.

Taking $x_0 = 1.86, \; f(x_0) = -0.0312, \; x_1 = 1.863, \; f(x_1) = 0.04623$

By method of false position

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad \text{....(1)}$$

Putting $n = 1$, we get

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.86)(0.04623) + (1.863)(0.0312)}{0.04623 + 0.0312} = 1.8612$$

Now $\quad f(x_2) = f(1.8612) = (1.8612)^4 - 12 = 11.9997 - 12 = -0.00025$

Thus, root lies between 1.8612 and 1.863

Taking $x_1 = 1.8612, f(x_1) = -0.00025, x_2 = 1.863,\ f(x_2) = 0.04623$

Putting $n = 2$ in (1), we get

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(1.8612)(0.04623) + (1.863)(0.00025)}{0.04623 + 0.00025} = 1.8611$$

Hence root is 1.8611 correct to three decimal places.

Example 6. Find the real root of the equation $x^4 - x = 10$ correct to three decimal places, using Regula–Falsi method.

Solution. Let $f(x) = x^4 - x - 10$

so that $f(1.852) = (1.852)^4 - 1.852 - 10 = 11.76424 - 11.852 = -0.0877$

and $f(1.857) = (1.857)^4 - 1.857 - 10 = 11.8918 - 11.857 = 0.0348$

Thus, root lies between 1.852 and 1.857

Taking $x_0 = 1.852,\ f(x_0) = -0.0877, x_1 = 1.857,\ f(x_1) = 0.0348$

By Regula–Falsi method

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \qquad ....(1)$$

Putting $n = 1$, we get

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.852)(0.0348) + (1.857)(0.0877)}{0.0348 + 0.0877} = 1.855$$

and $f(x_2) = f(1.855) = (1.855)^4 - 1.855 - 10 = 11.84065 - 11.855 = -0.0143$

Thus, root lies between 1.855 and 1.857.

Taking $x_1 = 1.855,\ f(x_1) = -0.0143, x_2 = 1.857, f(x_2) = 0.00348$

Putting $n = 2$ in (1), we get

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(1.855)(0.0348) + (1.857)(0.0143)}{0.0348 + 0.0143} = 1.855$$

Hence root 1.855 correct to three decimal places.

Example 7. Find the root of the equation $x \log_{10} x = 1.2$. Correct to three decimal places, using Regula Falsi method.

Solution. Let $f(x) = x \log_{10} x - 1.2$

so that $f(1) = 1 \log_{10} 1 - 1.2 = -1.2$

$f(2) = 2 \log_{10} 2 - 1.2 = 2\,(0.3010) - 1.2 = -0.598$

$f(3) = 3 \log_{10} 3 - 1.2 = 3\,(0.4771) - 1.2 = -0.2313$

Thus, roots lies between 2 and 3.

Taking $x_0 = 2,\ f(x_0) = -0.598,\ \ x_1 = 3,\ f(x_1) = 0.2313$

By Regula–Falsi method

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad \text{....(1)}$$

Putting $n = 1$, we get

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{2(0.2313) - 3(-0.598)}{0.2313 + 0.598} = 2.721014$$

and $\quad f(x_2) = f(2.721014) = 2.721014 \log_{10} 2.721014 - 1.2 = -0.017104$

Thus, root lies between 2.721014 and 3.

Taking $x_1 = 2.721014,\ f(x_1) = -0.017104,\ \ x_2 = 3, f(x_2) = 0.2313$

Putting $n = 2$ in (1), we get

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{2.721014(0.2313) - 3(-0.017104)}{0.2313 + 0.017104} = 2.740211$$

and $\quad f(x_3) = f(2.740211) = 2.740211 \log_{10} 2.740211 - 1.2 = -0.000389$

Thus, root lies between 2.740211 and 3.

Taking $x_2 = 2.740211,\ f(x_2) = -0.000389,\ \ x_3 = 3, f(x_3) = 0.2313$

Putting $n = 3$ in (1), we get

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{2.740211(0.2313) - 3(-0.000389)}{0.2313 + 0.000389} = 2.7406$$

and $\quad f(x_4) = f(2.7406) = 2.7406 \log_{10} 2.7406 - 1.2 = 0.00012$

Thus, root lies between 2.740211 and 2.7406.

Taking $x_3 = 2.740211,\ f(x_3) = -0.000389,\ \ x_4 = 2.7406, f(x_4) = 0.00012$

Putting $n = 4$ in (1), we get

$$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)} = \frac{2.740211(0.00012) - 2.7406(-0.000389)}{0.00012 + 0.000389} = 2.7405$$

Hence root 2.740 correct to three decimal places.

EXERCISE 5.2

1. Find a root of the following equations, using Regula-Falsi method correct to three decimal places:

(i) $x \log_{10} = 1.2$ (ii) $x.e^x = 3$ (iii) $x^3 - 4x - 9 = 0$
(iv) $\cos x = 3x - 1$ (v) $x^4 - x^3 - 2x^2 - 6x - 4 = 0$ (vi) $x^3 + x - 3 = 0$
(vii) $x^3 - 5x + 1 = 0$ (viii) $x^2 - 37 = 0$ (ix) $x^2 + 2x - 2 = 0$
(x) $\log x = \cos x$ (xi) $\tan x + x = 0$ (xii) $x = 2 \sin x$

2. Find the fourth root of 32 correct to three decimal places using Regula–Falsi method.
3. Find the value of $\sqrt{11}$ by Regula–Falsi method.

ANSWERS

1. (i) 2.741 (ii) 1.050 (iii) 2.706 (iv) 0.607 (v) 2.732 (vi) 1.213
(vii) 0.2016 (viii) 6.083 (ix) 0.7321 (x) 1.3 (xi) 2.03 (xii) 1.895
2. 2.378 3. 3.3166

5.4 NEWTON – RAPHSON METHOD [KUK 2005]

Let x_0 be an approximate root of the equation $f(x) = 0$

If $x_1 = x_0 + h$ be the exact root, then $f(x_1) = 0$ *i.e.* $f(x_0 + h) = 0$

Expanding by Taylor's series, we get

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots\dots\dots = 0$$

Since h is small, neglecting h^2 and higher powers of h, we get

$$f(x_0) + hf'(x_0) = 0 \qquad \text{or} \qquad h = -\frac{f(x_0)}{f'(x_0)}$$

Thus, A closer approxmition to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly, starting with x_1, a still better approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \dots(1)$$

5.4.1 Geometrical Interpretation

Let x_0 be a point near the root α of the equation $f(x) = 0$

Then the equation of the tangent at $P[x_0, f(x_0)]$ is

$$y - f(x_0) = f'(x_0)](x - x_0)$$

It cuts the x–axis i.e. $y = 0$, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

which is a first approximation to the root α. If P_1 is the point corresponding to x_1 on the curve, then the tangent at P_1 will cut the x–axis at x_2, which is nearer to α and is, therefore a second approximation to the root.

Repeating this process, we approach the root α quite rapidly.

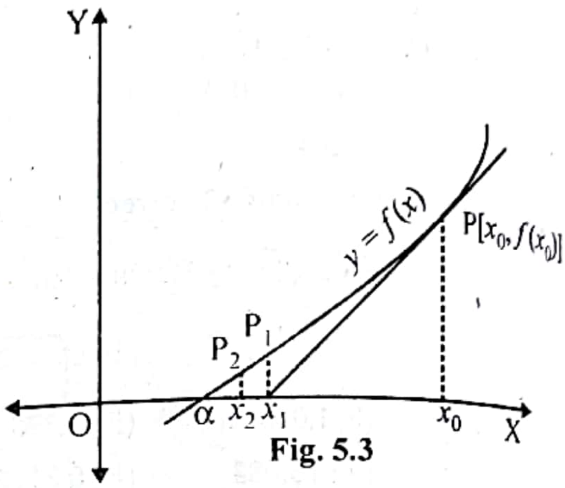

Fig. 5.3

5.4.2 Order of Convergence

[KUK 2005, 2006, 2009]

OR

Rate of Convergence of Newton–Raphson Method

Let a be the actual root of equation $f(x) = 0$ i.e. $f(a) = 0$. Let x_n and x_{n+1} be two successive approximations to the actual root a. If e_n and e_{n+1} are the corresponding errors, we have $x_n = a + e_n$ and $x_{n+1} = a + e_{n+1}$.

By Newton's –Raphson formula, we have

$$e_{n+1} = e_n - \frac{f(a+e_n)}{f'(a+e_n)}$$

$$e_{n+1} = e_n - \frac{f(a) + e_n f'(a) + \frac{e_n^2}{2!} f''(a) + \ldots}{f'(a) + e_n f''(a) + \frac{e_n^2}{2!} f'''(a) + \ldots}$$

$$e_{n+1} = e_n - \frac{e_n f'(a) + \frac{e_n^2}{2!} f''(a) + \ldots}{f'(a) + e_n f''(a) + \frac{e_n^2}{2!} f'''(a) + \ldots} \qquad [\because f(a) = 0]$$

$$e_{n+1} = \frac{e_n^2 f''(a)}{2\left[f'(a) + e_n f''(a)\right]} \qquad \text{(on neglecting high powers of } e_n\text{)}$$

$$= \frac{e_n^2}{2} - \frac{f''(a)}{f'(a)\left[1 + e_n \frac{f''(a)}{f'(a)}\right]}$$

$$= \frac{e_n^2}{2}\frac{f''(a)}{f'(a)}\left[1+e_n\frac{f''(a)}{f'(a)}\right]^{-1}$$

$$= \frac{e_n^2}{2}\frac{f''(a)}{f'(a)}\left[1-e_n\frac{f''(a)}{f'(a)}+\ldots\right]$$

$$= \frac{e_n^2}{2}\frac{f''(a)}{f'(a)}-\frac{e_n^3}{2}\left[\frac{f''(a)}{f'(a)}\right]^2+\ldots$$

$$\Rightarrow \quad \frac{e_{n+1}}{e_n^2}=\frac{1}{2}\frac{f''(a)}{f'(a)}-\frac{e_n}{2}\left[\frac{f''(a)}{f'(a)}\right]^2+\ldots$$

$$= \frac{f''(a)}{2f'(a)} \qquad \text{[Neglecting terms containing powers of } e_n]$$

By definition, the order of convergence of Newton Raphson method is 2.

Hence Newton–Raphson method has a quadratic convergence.

5.4.3 Condition of Convergence of Newton–Raphson Method [KUK. 2007, 08]

Comparing equation (1) with the relation $x_{n+1}=\phi(x_n)$ of the interation method.

We get $\quad \phi(x_n)=x_n-\dfrac{f(x_n)}{f'(x_n)}$

In general $\quad \phi(x)=x-\dfrac{f(x)}{f'(x)}$ which gives $\quad \phi'(x)=\dfrac{f(x)f''(x)}{[f'(x)]^2}$

Since the iteration method converges if $|\phi'(x)|<1$

Thus, Newton's formula will converge if $|f(x)f''(x)|<|f'(x)|^2$.

Theorem. If (i) α be a root of $f(x)=0$ which is equivalent to $x=\phi(x)$, (ii) I, be any interval containing the point $x=\alpha$, (iii) $|\phi'(x)|<1$ for all x in I, then the sequence of approximations $x_0, x_1, x_2, \ldots\ldots\ldots, x_n$ will converge to the root α provided the initial approximation x_0 is chosen in I.

Proof. Since α is the root of $x=\phi(x)$, we have $\alpha=\phi(\alpha)$

If x_{n-1} and x_n be 2 successive approximation to α, we have $x_n=\phi(x_{n-1})$. ....(1)

$$\therefore \quad x_n-\alpha=\phi(x_{n-1})-\phi(\alpha)$$

By mean value theorem, $\quad \dfrac{\phi(x_{n-1})-\phi(\alpha)}{x_{n-1}-\alpha}=\phi'(\xi) \quad$ where $\quad x_{n-1}<\xi<\alpha$

Hence equation (1) becomes $x_n - \alpha = \phi(x_{n-1} - \alpha)\phi'(\xi)$

If $|\phi'(x_i)| \le k < 1$ for all i, then

$$|x_n - \alpha| \le k|x_{n-1} - \alpha|$$

Similarly, $$|x_{n-1} - \alpha| \le k|x_{n-2} - \alpha|$$

i.e. $$|x_n - \alpha| \le k^2|x_{n-2} - \alpha|$$

Proceeding this way, $$|x_n - \alpha| \le k^n|x_0 - \alpha|$$

As $n \to \infty$, the R.H.S. tends to zero, therefore the sequence of approximations converses to root α.

Example 1. Find a real root of the equation $x = e^{-x}$ by Newton–Raphson method. [K.U.K. 2005]

Solution. Given equation is $x = e^{-x}$ or $x = \frac{1}{e^x}$ or $xe^x - 1 = 0$

Let $f(x) = xe^x - 1,\ f'(x) = (1+x)e^x$

so that $f(0.58) = (0.58)\,(2.7)^{0.58} - 1 = (0.58)(1.786) - 1$ $(\because e \propto 2.7)$

$= 1.0358 - 1 = 0.358$

and $f(0.53) = (0.53)(2.7)^{0.53} - 1 = (0.53)(1.6989) - 1 = 0.9004 - 1 = -0.099$

Thus, root lies between 0.53 and 0.58.

Take $x_0 = 0.53$

By Newton–Raphson formula, $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{....(1)}$$

Putting $n = 0$, we get, $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.53 - \frac{-0.099}{2.599} \quad \left[\because f'(x_0) = f'(0.53) = (1+0.53)e^{0.53} = (1.53)(1.6989) = 2.599\right]$$

$$= 0.53 + 0.038 = 0.568$$

Putting $n = 1$ in (1), we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.568 - \frac{f(0.568)}{f'(0.568)} \quad \left[\begin{array}{l}\because f(0.568) = (0.568)(2.7)^{0.568} - 1 = 0.0023 \\ f'(0.568) = (1+0.568)(2.7)^{0.568} = 2.767\end{array}\right]$$

$$= 0.568 - \frac{0.0023}{2.767} = 0.567$$

Putting $n=2$ in (1), we get

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.567 - \frac{f(0.567)}{f'(0.567)}$$

$$x_3 = 0.567 + \frac{0.0004}{2.7624} \qquad \left(\begin{array}{l} \because f(0.567) = (0.567)e^{0.567} - 1 = -0.0004 \\ f'(0.567) = (1+0.567)e^{0.567} = 2.7624 \end{array}\right)$$

$$= 0.567$$

Here $x_2 = x_3$

Hence the required root is 0.567 correct to three decimal places.

Example 2. Find a real root of the equation $x^3 - 3x + 1 = 0$ by Newton–Raphson method correct to three decimal places.

Solution. Let $f(x) = x^3 - 3x + 1, \quad f'(x) = 3x^2 - 3$

so that $f(1.53) = (1.53)^3 - 3(1.53) + 1 = -0.0085$

$f(1.54) = (1.54)^3 - 3(1.54) + 1 = 0.0323$

Thus, root lies the between 1.53 and 1.54

Take $x_0 = 1.53$, by Newton–Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{....(1)}$$

Putting $n = 0$, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1.53 + \frac{0.0085}{4.0227} \qquad \left(\because f'(x_0) = f'(1.53) = 3(1.53)^2 - 3 = 4.0227\right)$$

$$= 1.53 + 0.0021 = 1.532$$

Putting $n = 1$, in (1), we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.532 - \frac{f(1.532)}{f'(1.532)} = 1.532 + \frac{0.0004}{4.04107} = 1.532 + 0.000098 = 1.532$$

Here $x_1 = x_2$

Hence the required root is 1.532 correct to three decimal places.

Example 3. Find a real root of the equation $\log x = \cos x$ by Newton–Raphson method correct to two decimal places.

Solution. Let $f(x) = \log x - \cos x, \quad f'(x) = \frac{1}{x} + \sin x$

so that $f(1.5) = \log 1.5 - \cos 1.5 = 0.17609 - 0.0706 = 0.105$

and $f(1.2) = \log 1.2 - \cos 1.2 = 0.07918 - 0.36227 = -0.2831$

Take $x_0 = 1.5$, by Newton–Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \ldots(1)$$

Putting $n = 0$, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{0.105}{1.6641} \qquad \left(\because f'(1.5) = \frac{1}{1.5} + \sin 1.5 = 0.6666 + 0.997 = 1.6641\right.$$

$$= 1.5 - 0.0631 = 1.4369$$

$$f(x_1) = f(1.4369) = \log 1.4369 \mp \cos 1.4369 = 0.1574 - 0.1335 = 0.0239$$

$$f'(x_1) = f'(1.4369) = \frac{1}{1.4369} + \sin 1.4369 = 0.6959 + 0.9911 = 1.687$$

Putting $n = 1$ in (1), we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.4369 - \frac{f(1.4369)}{f'(1.4369)} = 1.4369 - \frac{0.0239}{1.687} = 1.4227$$

$$f(x_2) = f(1.4227) = \log(1.4227) - \cos(1.4227) = 0.1531 - 0.1475 = 0.0056$$

$$f'(x_2) = f'(1.4227) + \sin 14227 = 0.7028 + 0.9891 = 1.6919$$

Putting $n = 2$ in (1), we get

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.4227 - \frac{f(1.4227)}{f'(1.227)} = 1.4227 - \frac{0.0056}{1.6919} = 1.4227 - 0.0033 = 1.4119$$

$$f(1.4119) = \log(1.4119) - \cos(1.4119) = 0.1498 - 0.158015 = -0.0083$$

$$f'(1.4119) = \frac{1}{1.4119} + \sin(1.4119) = 0.7083 + 0.9874 = 1.6857$$

Putting $n = 3$ in (1), we get

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.4119 + \frac{0.0083}{1.6957} = 1.4119 + 0.0048 = 1.416$$

Here $x_3 = x_4$

Here the required root is 1.41 correct to two decimal places.

Example 4. Find a real root of the equation $3x = \cos x + 1$ by Newton–Raphson method correct to five decimal places.

Solution. Let $f(x) = 3x - \cos x - 1$, $f'(x) = 3 + \sin x$

so that
$$f(0.605) = 3(0.605) - \cos(0.605) - 1$$
$$= 1.815 - 0.8225 - 1 = -0.007$$

and $f(0.615) = 3(0.615) - \cos(0.615) - 1$

$= 1.845 - 0.8167 - 1 = 0.028$

Thus, root lies between 0.605 and 0.615

By Newton–Raphson method $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ ....(1)

Taking $x_0 = 0.605$, putting $n = 0$ in equation (1), we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.605 + \frac{0.007}{3.5687} \qquad [\because f'(0.605) = 3 + \sin(0.605) = 3 + 0.5687]$$

$$= 0.60696$$

$$f(x_1) = f(0.60696) = 3(0.60696) - \cos(0.60696) - 1$$

$$= 1.82088 - 0.82136 - 1 = -0.00048$$

and $f'(0.60696) = 3 + \sin(0.60696) = 3.5703$

Putting $n = 1$ in equation (1), we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.60696 + \frac{0.00048}{3.5703} = 0.60709$$

$$f(x_2) = f(0.60709) = 3(0.60709) - \cos(0.60709) - 1$$

$$= 1.82127 - 0.82128 - 1 = -0.00001$$

$$f'(x_2) = f'(0.60709) = 3 + \sin(0.60709) = 3.5705$$

Putting $n = 2$ in equation (1), we get

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.60709 + \frac{000001}{3.5705} = 0.60709 \qquad \therefore x_2 = x_3$$

Hence the required root is 0.60709 correct to five decimal places.

Example 5. Find a real root of the eqatuion $e^x \sin x = 1$ correct to four decimal places.

Solution. Let $f(x) = e^x \sin x - 1$, $f'(x) = e^x \cos x + \sin x e^x = e^x(\sin x + \cos x)$

so that $f(0.587) = e^{0.587} \sin(0.587) - 1 = (1.79858)(0.5539) - 1$

$= 0.9962 - 1 = -0.0038$

and $f(0.589) = e^{0.589} \sin(0.589) - 1 = (1.802185)(0.5555) - 1$

$= 1.001113 - 1 = 0.001113$

Thus, root lies between 0.587 and 0.587.

By Newton–Raphs on method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad ....(1)$$

Taking $x_0 = 589$ and $f(x_0) = 0.00113$

Putting $n = 0$ in equation (1), we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.589 - \frac{0.00113}{2.4996} = 0.5885$$

$$f(x_1) = e^{0.5885}\sin(0.5885) - 1 = (1.80128)(0.555) - 1 = -0.000289$$

$$f'(x_1) = e^{0.5885}\left[\sin(0.5885) + \cos(0.5885)\right]$$

$$= (1.80128)[0.555 + 0.83176] = 1.4979$$

Putting $n = 1$ in equation (1), we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.5885 + \frac{0.000289}{2.4979} = 0.5886$$

$$f(x_2) = e^{0.5886}\sin(0.5886) - 1 = (1.80146)(0.555) - 1 = -0.0001897$$

$$f'(x_2) = e^{0.5886}\left[\sin(0.5886) + \cos(0.5886)\right] = (1.80146)\left[0.555 + 0.8317\right] = 2.49808$$

Putting $n = 2$ in equation (1), we get

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 0.5886 + \frac{0.0001897}{2.49808} = 0.5886$$

Here $x_2 = x_3$. Hence the required root is 0.5886 of given equation.

Example 6. Find a real root of the equation $e^x = x^3 + \cos 25x$ by Newton–Raphson method correct to three decimal places.

Solution. Let $f(x) = e^x - x^3 - \cos 25x$, $f'(x) = e^x - 3x^2 + 25\sin 25x$

so that
$$f(4.55) = e^{4.55} - (4.55)^3 - \cos(113.75)$$
$$= 94.6324 - 94.196375 - 0.78935 = -0.353$$

and
$$f(4.57) = e^{4.57} - (4.57)^3 - \cos(114.25)$$
$$= 96.5441 - 95.4439 - 0.3987 = 0.7015$$

Thus, root lies between 4.55 and 4.57

By Newton–Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{....(1)}$$

Taking $x_0 = 4.55$ and putting $n = 0$ in equation (1), we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Now $f'(x_0) = f'(x_0) = f'(4.55) = e^{4.55} - 3(4.55)^2 + 25\sin(113.75)$

$$= 94.6324 - 62.1075 + 15.3785 = 47.8734$$

$$x_1 = 4.55 + \frac{0.353}{47.8734} = 4.557$$

$$f(x_1) = f(4.557) = e^{4.557} - (4.557)^3 - \cos(113.925)$$

$$= 95.2972 - 94.6318 - 0.6704 = -0.005$$

$$f'(x_1) = f'(4.557) = e^{4.557} - 3(4.557)^2 + 25\sin 113.925$$

$$= 95.2972 - 62.2987 + 18.5494 = 51.5479$$

Putting n=1 in, we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 4.557 + \frac{0.005}{51.5479} = 4.557$$

Here $x_2 = x_1$, hence the required root is 4.557 correct to the three decimal places.

Example 7. Find a real root of the equation $x\tan x + 1 = 0$ by Newton–Raphson method correct to three decimal places.

Solution. Let $f(x) = x\tan x + 1, f'(x) = x(1+\tan^2 x) + \tan x$

so that $f(2.75) = 2.75\tan(2.75) + 1 = -0.135$

and $f(2.81) = 2.81\tan(2.81) + 1 = 0.033$

Thus, root lies between 2.75 and 2.81.

By Newton–Raphson method $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad ....(1)$$

Taking $x_0 = 2.81$, putting $n = 0$ in (1), we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.81 - \frac{f(2.81)}{f'(2.81)} = 2.81 - \frac{0.033}{2.7985} = 2.798$$

$$f(x_1) = f(2.798) = 2.798\tan(2.798) + 1 = -0.00045$$

$$f'(x_1) = f'(2.798) = 2.798(1+\tan^2 2.798) + \tan 2.798 = 2.798$$

Putting $n = 1$ in (1), we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.798 + \frac{0.00045}{2.798} = 2.798$$

Here $x_1 = x_2$. Hence the required root is 2.798 correct to three decimal places.

5.5 SOME USEFUL RESULTS OF NEWTON–RAPHSON METHOD

(a) If we find the value of $\frac{1}{N}$, then the formula $x_{n+1} = x_n(2 - Nx_n)$

(b) If we find the value of $\sqrt{N}$, then use formula $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$

(c) If we find the value of $\frac{1}{\sqrt{N}}$, then use formula $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$

(d) If we find the value of $\sqrt[k]{N}$, then use formula $x_{n+1} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right]$

Example 1. Find the cube root of 41, using Newton–Raphson method correct to three decimal places.

Solution. Let $x = (41)^{1/3}$ or $x^3 - 41 = 0$

Let $f(x) = x^3 - 41$

so that $f(3.445) = (3.445)^3 - 41 = -0.1146$

and $f(3.449) = (3.449)^3 - 1 = 0.0279$

As we take $x_0 = 3.449$, use result $x_{n+1} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right]$ ....(1)

where $N = 41$, $k = 3$

Putting $n = 0$ in (1), we get

$$x_1 = \frac{1}{3}\left[2x_0 + \frac{41}{(x_0)^2}\right] = \frac{1}{3}\left[2(3.449) + \frac{41}{(3.449)^2}\right]$$

$$= \frac{1}{3}[6.898 + 3.44665] = 3.448$$

Putting $n = 1$ in (1), we get

$$x_2 = \frac{1}{3}\left[2x_1 + \frac{41}{x_1^2}\right] = \frac{1}{3}\left[2(3.448) + \frac{41}{(3.448)^2}\right] = \frac{1}{3}[6.896 + 3.44865] = 3.448$$

Here $x_1 = x_2$

Example 2. Find the value of $\frac{1}{\sqrt{15}}$, using Newton–Raphson method.

Solution. Let $x = \frac{1}{\sqrt{15}}$ or $15x^2 - 1 = 0$

Let $f(x) = 15x^2 - 1$

so that $f(0.256) = 15(0.256)^2 - 1 = -0.01696$

and $f(0.259) = 15(0.259)^2 - 1 = 0.0062$

Using result $x_{n+1} = \frac{1}{2}\left[x_n + \frac{1}{N\,x_n}\right]$ ....(1)

Taking $x_0 = 0.259$, N=15, Putting $n = 0$ in (1), we get

$$x_1 = \frac{1}{2}\left[x_0 + \frac{1}{15x_0}\right] = \frac{1}{2}\left[0.259 + \frac{1}{15(0.259)}\right] = \frac{1}{2}[0.259 + 0.2574] = 0.2582$$

Putting $n = 1$ in (1), we get

$$x_2 = \frac{1}{2}\left[x_1 + \frac{1}{15x_1}\right] = \frac{1}{2}\left[0.2582 + \frac{1}{15(0.2582)}\right] = \frac{1}{2}[0.2582 + 0.25819] = 0.2582$$

Here $x_1 = x_2$

$\therefore \quad \frac{1}{\sqrt{15}} = 0.2582$

EXERCISE 5.3

1. Find a real root of the following equation, using newton–Raphson method correct to three decimal places:

(i) $3x^3 - 9x^2 + 8 = 0$ (ii) $x^4 - x = 10$ (iii) $x = \cos x$

(iv) $x \sin x + \cos x = 0$ (v) $x^3 - 21x + 3500 = 0$ (vi) $x \,.\, x^x = 3$

(vii) $x^4 - 12x + 7 = 0$ (viii) $10^x - x - 4 = 0$ (ix) $e^{-x} \sin x = 0$

(x) $2x - \log_{10} x = 6$ (xi) $4x - e^x = 0$ (xii) $x^3 + 2x^2 + 50x + 7 = 0$

2. Develop an algorithm using Newton–Raphson method to find $\sqrt[4]{N}$ and hence find $\sqrt[4]{32}$.

3. Find the value of the following, using Newton–Raphson method.

(i) $\frac{1}{\sqrt{14}}$ (ii) $\sqrt[3]{24}$ (iii) $(30)^{\frac{-1}{5}}$ (iv) $(17)^{\frac{1}{3}}$

ANSWERS

1. (i) 1.226, (ii) 1.856, (iii) 0.739 (iv) 2.798, (v) −16.56, (vi) 1.050, (vii) 2.047, (viii) 0.539, (ix) 0.5885, (x) 3.256, (xi) 2.153, (xii) −0.1474

2. 2.3784

3. (i) 0.2673, (ii) 2.8845, (iii) 0.5065, (iv) 2.57128

5.6 SECANT METHOD

Secant method is an important over the method of Regula–Falsi method. In this method, it does not require the condition $f(x_0)$ and $f(x_1)$ are opposite sign i.e. $f(x_0)\ f(x_1) < 0$.

Taking x_0 and x_1 as the initial limits of interval, we write the equation of the chord joining these as

ANSWERS

4. 2.925, 0.225 5. 31, 129, 351 6. 60.05 7. 244 8. 571

3.7 INTERPOLATION

Interpolation is the method of estimating unknown values with the help of given set of observations. According to Theile Interpolation is, "The art of reading between the lines of the table." The study of interpolation is based on the calculus of finite differences.

3.7.1 Newton Gregory's Forward Interpolation Formula.

Let $y = f(x)$ be a function of x which takes the value $f(a), f(a+h), f(a+2h), \ldots\ldots, f(a+nh)$ for $(n+1)$ equally spaced value a, $a+h$, $a+2h, \ldots\ldots\ldots\ldots, a+nh$ of the independent variable x.

Assume that $f(x)$ be a polynomial of n^{th} degree,

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)(x-a-h) + \ldots\ldots + A_n(x-a)(x-a-h)\ldots[x-a-(n-1)h] \quad \ldots(1)$$

where $A_0, A_1, A_2, \ldots\ldots, A_n$ are to be determined.

Put $x = a$, $a+h$, $a+2h, \ldots\ldots\ldots, a+nh$ in equation (1), we get

$$f(a) = A_0$$

$$f(a+h) = A_0 + hA_1 \quad \text{or} \quad f(a+h) = f(a) + hA_1 \quad \Rightarrow \quad A_1 = \frac{f(a+h)-f(a)}{h}$$

$$\therefore \quad A_1 = \frac{\Delta f(a)}{h}$$

$$f(a+2h) = A_0 + A_1(2h) + A_2(2h)h = f(a) + 2h\left[\frac{\Delta f(a)}{h}\right] + 2h^2 A_2$$

$$\Rightarrow \quad 2h^2 A_2 = f(a+2h) - 2f(a+h) + f(a) = \Delta^2 f(a)$$

$$\therefore \quad A_2 = \frac{\Delta^2 f(a)}{2!h^2}$$

Similarly,

$$A_3 = \frac{\Delta^3 f(a)}{3!h^3}$$

$$\vdots$$

$$A_n = \frac{\Delta^n f(a)}{n!h^n}$$

Putting the values of $A_0, A_1, A_2, \ldots\ldots, A_n$ in equation (1), we get

$$f(x)=f(a)+(x-a)\frac{\Delta f(a)}{h}+(x-a)(x-a-h)\frac{\Delta^2 f(a)}{2!h^2}+\ldots+(x-a)\ldots[x-a-(n-1)h]\frac{\Delta^n f(a)}{n!h^n}$$

Put $x=a+hu \Rightarrow u=\frac{x-a}{h}$, we get

$$f(a+hu)=f(a)+u\Delta f(a)+\frac{u(u-1)}{2!}\Delta^2 f(a)+\ldots\ldots+\frac{u(u-1)(u-2)\ldots(u-n+1)}{n!}\Delta^n f(a)$$

Example 1. Find the value of y when x=1.85 by Newton's Forward interpolation formula from the table:

x	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$f(x)$	5.474	6.050	6.686	7.389	8.166	9.025	9.974

Solution. The difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
1.7	5.474						
		0.576					
1.8	6.050		0.06				
		0.636		0.007			
1.9	6.686		0.067		0		
		0.703		0.007		0.001	
2.0	7.389		0.074		0.001		–0.002
		0.777		0.008		–0.001	
2.1	8.166		0.082		0		
		0.859		0.008			
2.2	9.025		0.09				
		0.949					
2.3	9.974						

Take $a=1.8,\ x=1.85,\ h=0.1$

$$u=\frac{x-a}{h}=\frac{1.85-1.8}{0.1}=\frac{0.05}{0.1}=0.5$$

By Newton's forward interpolation formula

$$f(x)=f(a)+u\Delta f(a)+\frac{u(u-1)}{2!}\Delta^2 f(a)+\frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)+\ldots..$$

$$f(1.85) = 6.050 + (0.5)(0.636) + \frac{(0.5)(0.5-1)}{2!}(0.067) + \frac{(0.5)(0.5-1)(0.5-2)}{3!}(0.007)$$

$$= 6.050 + 0.318 - 0.0083 + 0.00043$$

$$= 6.36$$

Example 2. Calculate the approximate value of sin x for x=0.54, by Newton's interpolation formula, given that

x	0.5	0.7	0.9	1.1	1.3	1.5
$\sin x$	0.47943	0.64422	0.78333	0.89121	0.96356	0.99749

Solution. The difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0.5	0.47943					
		0.16479				
0.7	0.64422		–0.02568			
		0.13911		–0.00555		
0.9	0.78333		–0.03123		0.00125	
		0.10788		–0.00430		0.00016
1.1	0.89121		–0.03553		0.00141	
		0.07235		–0.00289		
1.3	0.96356		–0.03842			
		0.03393				
1.5	0.99749					

Take $a = 0.5$, $x = 0.54$, $u = \dfrac{x-a}{h} = \dfrac{0.54-0.5}{0.2} = 0.2$, $h = 0.2$

By Newton's forward interpolation formula

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \ldots$$

$$\Rightarrow \quad \sin(0.54) = 0.47943 + 0.2(0.16479) + \frac{(0.2)(0.2-1)}{2!}(-0.02568)$$

$$+ \frac{(0.2)(0.2-1)(0.2-2)}{3!}(-0.00555) + \frac{(0.2)(0.2-1)(0.2-2)(0.2-3)}{4!}(0.00125)$$

$$= 0.51386$$

Example 3. From the following table find the number of students who obtained less than 45 marks:

Marks	Number of students
30–40	31
40–50	42
50–60	51
60–70	35
70–80	31

[K.U.K. 2006]

Solution. The difference table is

x	$f(x)$ C.F.	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
40	31				
		42			
50	73		9		
		51		–25	
60	124		–16		37
		35		12	
70	159		–4		
		31			
80	190				

Taking $a = 40, x = 45, h = 10,\ u = \frac{x-a}{h} = \frac{45-40}{10} = \frac{5}{10} = 0.5$

Newton's forward interpolation formula

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \ldots\ldots$$

$$f(45) = 31 + 0.5(42) + \frac{(0.5)(0.5-1)}{2}(9) + \frac{(0.5)(0.5-1)(0.5-2)}{6}(-25)$$

$$+\frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24}(37)$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4453 = 47.87$$

The number of students with marks less than 45 is 47.87 i.e. 48. But the number of students with marks less than 40 is 31. Hence the number of students getting marks between 40 and 45 is 48–31=17

Example 4. From the following data, find y at $x = 43$

x	40	50	60	70	80	90
y	184	204	226	250	276	304

[K.U.K. 2009]

Also express y in terms of x.

Solution. The difference table is

x	y	Δy	$\Delta^2 y$
40	184		
		20	
50	204		2
		22	
60	226		2
		24	
70	250		2
		26	
80	276		2
		28	
90	304		

Taking $a = 40, x = 43,\ u = \frac{x-a}{h} = \frac{43-40}{10} = 0.3$

By Newton's forward interpolation formula

$$y_{43} = y_{40} + u\Delta y_{40} + \frac{u(u-1)}{2!}\Delta^2 y40$$

$$= 184 + (0.3)(20) + \frac{(0.3)(0.3-1)}{2}(2)$$

$$= 184 + 6 - 0.21 = 189.79$$

By Newton's forward interpolation formula

$$y = y_{40} + u\Delta y_{40} + \frac{u(u-1)}{2!}\Delta^2 y_{40}$$

$$y = 180 + \frac{x-40}{10}(20) + \frac{\left(\frac{x-40}{10}\right)\left(\frac{x-40}{10}-1\right)}{2!}(2)$$

$$y = 184 + 2x - 80 + \frac{x^2 - 90x + 2000}{100}$$

$$y = \frac{x^2 - 110x + 12400}{100}$$

Example 5. Find the value of cos (1.74) from the following data: **[K.U.K. 2009]**

x	1.70	1.74	1.78	1.82	1.86
$\sin x$	0.9916	0.9857	0.9781	0.9691	0.9584

Solution. The difference table is

x	$y=\sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.70	0.9916				
		–0.0059			
1.74	0.9857		–0.0017		
		–0.0076		0.0084	
1.78	0.9781		0.0067		–0.0249
		–0.0009		–0.0165	
1.82	0.9691		–0.0098		
		–0.0107			
1.86	0.9584				

First, we find expression for sin x by using Newton's forward interpolation formula.

Taking $a=1.70,\ h=0.04,\ u=\dfrac{x-a}{h}=\dfrac{x-1.70}{0.04}$

$$\sin x = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + ..$$

$$= 0.9916 + \left(\frac{x-1.70}{0.04}\right)(-0.0059) + \frac{\left(\frac{x-1.70}{0.04}\right)\left(\frac{x-1.70}{0.04}-1\right)}{2}(-0.0017)$$

$$+\frac{\left(\frac{x-1.70}{0.04}\right)\left(\frac{x-1.70}{0.04}-1\right)\left(\frac{x-1.70}{0.04}-2\right)}{6}(0.0084)$$

$$+\frac{\left(\frac{x-1.70}{0.04}\right)\left(\frac{x-1.70}{0.04}-1\right)\left(\frac{x-1.70}{0.04}-2\right)\left(\frac{x-1.70}{0.04}-3\right)}{24}(-0.0249)$$

$$\sin x = 0.9916 - 0.1475(x-1.70) - (1.0625)(x^2-3.44x+2.958)$$
$$+21.875(x^3-5.22x^2+9.0812x-5.26524)$$
$$-405.27(x^4-7.04x^3+18.5816x^2-21.79302x+9.58274)$$

$\therefore$ $$\cos x = -0.1475 - 1.0625(2x-3.44) + 21.875(3x^2-10.44x+9.0812)$$
$$-405.27(4x^3-21.12x^2+37.1632x-21.79302)$$

Hence $$\cos(1.74) = -0.1475 - 1.0625\left[2(1.74)-3.44\right] + 21.875\left[3(1.74)^2-10.44(1.74)+9.0812\right]$$
$$-405.27\left[4(1.74)^3-21.12(1.74)^2+37.1632(1.74)-21.79302\right]$$
$$=-0.27849.$$

3.7.2 Newton's Gregory Formula For Backward Interpolation

Let $y = f(x)$ be a funciton of x which takes the values $f(a), f(a+h), f(a+2h), \ldots\ldots, f(a+nh)$ for $n+1$ equally spaced values $a, a+h, a+2h, \ldots\ldots, a+nh$ of the independent variable x.

Assume that $f(x)$ be a polynomial of n^{th} degree

$$f(x) = A_0 + A_1(x-a-nh) + A_2[x-a-nh][x-a-(n-1)h] + \ldots\ldots$$
$$+ A_n(x-a-nh)[x-a-(n-1)h]\ldots\ldots(x-a-h) \quad \ldots(1)$$

where $A_0, A_1, A_2, \ldots\ldots, A_n$ are to be determined.

Put $x = a+nh,\ a+(n-1)h,\ a+(n-2)h, \ldots\ldots, a$ in equation (1) respectively, we get

$$f(a+nh) = A_0 \quad \ldots(2)$$

$$f(a+(n-1)h) = A_0 - hA_1 = f(a+nh) - hA_1 \quad [\text{from equation } (2)]$$

$$\Rightarrow \quad A_1 = \frac{\nabla f(a+nh)}{h}$$

$$f(a+(n-2)h) = A_0 - 2hA_1 + (-2h)(-h)A_2$$

$$\Rightarrow \quad 2!h^2 A_2 = f(a+(n-2)h) - f(a+nh) + 2\nabla f(a+nh) = \nabla^2 f(a+nh)$$

$$\therefore \quad A_2 = \frac{\nabla^2 f(a+nh)}{2!h^2}$$

Proceeding in similar way, we get

$$A_n = \frac{\nabla^n f(a+nh)}{n!h^n}$$

Substituting these values in equation (1), we get

$$f(x) = f(a+nh) + (x-a-nh)\frac{\nabla f(a+nh)}{h} + \ldots\ldots$$
$$+(x-a-nh)(x-a-(n-1)h)\ldots\ldots(x-a-h)\frac{\nabla^n f(a+nh)}{n!h^n}$$

Put $x = a+nh+uh$, then $x-a-nh = uh$ and $x-a-(n-1)h = (u+1)h$, $x-a-h = u+(n-1)h$

$$\therefore \quad f(x) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh) + \ldots\ldots$$
$$+\frac{u(u+1)(u+2)\ldots\ldots(u+(n-1)h)}{n!}\nabla^n f(a+nh)$$

Example 1. In the table below, find the value of tan 0.50, by Newton's interpolation formula.

x	0.10	0.15	0.20	0.25	0.30
$\tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

Solution. The difference table is

x	$y = \tan x$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0.10	0.1003				
		0.0508			
0.15	0.1511		0.008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

Taking $x = 0.50,\ h = 0.05,\ a = 0.30$

$$\therefore \qquad u = \frac{x-a}{h} = \frac{0.50-0.30}{0.05} = 4$$

By Newton's Backward interpolation formula

$$f(x) = f(a) + u\nabla f(a) + \frac{u(u+1)}{2!}\nabla^2 f(a) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a) + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(a)$$

$$\tan(0.50) = 0.3093 + 4(0.0540) + \frac{4(4+1)}{2}(0.0014) + \frac{4(4+1)(4+2)}{6}(0.0004) + \frac{4(4+1)(4+2)(4+3)}{24}(0.0002)$$

$$= 0.3093 + 0.216 + 0.014 + 0.008 + 0.007 = 0.5543$$

Example 2. Using Newton's backward difference formula, find the value of $e^{-1.9}$ from the following table.

x	1	1.25	1.50	1.75	2.00
e^{-x}	0.3679	0.2865	0.2231	0.1738	0.1353

Solution. The difference table is

x	$y = e^{-x}$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	0.3679				
		–0.0814			
1.25	0.2865		0.0180		
		–0.0634		–0.0039	
1.50	0.2231		0.0141		0.0006
		–0.0493		–0.0033	
1.75	0.1738		0.0108		
		–0.0385			
2.00	0.1353				

Taking $a = 2.00$, $x = 1.9$, $h = 0.25$

$$u = \frac{x-a}{h} = \frac{1.9-2.00}{0.25} = -0.4$$

By Newton's backward difference formula

$$y_x = y_a + u\nabla y_a + \frac{u(u+1)}{2!}\nabla^2 y_a + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_a + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_a$$

$$e^{-1.9} = 0.1353 + (-0.4)(-0.0385) + \frac{(-0.4)(-0.4+1)}{2}(0.0108) + \frac{(-0.4)(-0.4+1)(-0.4+2)}{6}(-0.0033)$$
$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)}{24}(0.0006)$$

$$= 0.1353 + 0.0154 - 0.001296 + 0.0002112 - 0.000024 = 0.14959$$

Example 3. Find the value of tan 48°15′ from the following table

x°	45°	46°	47°	48°	49°	50°
$\tan x^\circ$	1.00000	1.03553	1.07237	1.11061	1.15037	1.19175

Solution. The difference table is

x°	$y = \tan x$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
45°	1.00000					
		0.03553				
46°	1.03553		0.00131			
		0.03684		0.00009		
47°	1.07237		0.00140		0.00003	
		0.3824		0.00012		–0.00005
48°	1.11061		0.00152		–0.00002	
		0.03976		0.00010		
49°	1.15037		0.00162			
		0.04138				
50°	1.19175					

Taking $x = 48°15' = 48.25°$, $a = 50°$, $h = 1$ $\qquad \because 1' = \left(\frac{1}{60}\right)^{\circ}$

$$u = \frac{x-a}{h} = \frac{48.25-50}{1} = -1.75$$

By Newton's backward interpolation formula

$$y_x = y_a + u\nabla y_a + \frac{u(u+1)}{2!}\nabla^2 y_a + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_a + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_a + \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 y_a$$

$$\tan 48°15' = 1.19175 + (-1.75)(0.04138) + \frac{(-1.75)(-1.75+1)}{2}(0.00162) + \frac{(-1.75)(-1.75+1)(-1.75+2)}{6}(0.00010) + \frac{(-1.75)(-1.75+1)(-1.75+2)(-1.75+3)}{24}(-0.00002) + \frac{(-1.75)(-1.75+1)(-1.75+2)(-1.75+3)(-1.75+4)}{120}(-0.00005)$$

$$= 1.120402867$$

Example 4. (i) Find the value of u_0 and u_3, given $u_{-1} = 10$, $u_1 = 8$, $u_2 = 10$, $u_4 = 50$

(ii) Find $\nabla^5 y_0$, given $y_0 = 3, y_1 = 12, y_2 = 81, y_3 = 200, y_4 = 100, y_5 = 8$

without forming the difference table.

Solution (i). The fourth difference will be zero, because four observation are given, we get

$$\Delta^4 u_{-1} = 0 \quad \text{and} \quad \Delta^4 u_0 = 0$$

$$(E-1)^4 u_{-1} = 0 \qquad \text{and} \qquad (E-1)^4 u_0 = 0$$

$$(E^4 - 4E^3 + 6E^2 - 4E + 1)u_{-1} = 0 \quad \text{and} \quad (E^4 - 4E^3 + 6E^2 - 4E + 1)u_0 = 0$$

$$u_3 - 4u_2 + 6u_1 - 4u_0 + u_{-1} = 0 \quad \text{and} \quad u_4 - 4u_3 + 6u_2 - 4u_1 + u_0 = 0$$

$$u_3 - 4u_0 = -18 \quad ....(1) \qquad \text{and} \qquad -4u_3 + u_0 = -78 \quad ....(2)$$

Solving (1) and (2), we get $\qquad u_0 = 10, u_3 = 22$

(ii) Now $\quad \Delta^5 y_0 = (E-1)^5 y_0 = (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)y_0$

$$= y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0$$

$$= 8 - 5(100) + 10(200) - 10(81) + 5(12) - 3$$

$$= 8 - 500 + 2000 - 810 + 60 - 3 = 755$$

EXERCISE 3.3

1. Find the cubic polynomial which takes the following values:

x	0	1	2	3
y	1	2	1	10

2. Following are the marks obtained by 492 candidates in a certain examination:

Marks	0–40	40–50	45–50	50–55	55–60	60–65
No. of Candidates	210	43	54	74	32	79

Find out: (i) Number of candidates, if the secure more than 48 but less than 50 marks.
(ii) Less than 48 but not less than 45 marks.

3. Using Newton's backward difference formula to construct an interploting polynomial of degree 3 for the data:

x	–0.75	–0.5	–0.25	0
y	–0.0718	–0.0247	0.3349	1.1010

Find the value at $x = -0.33$.

4. Find the value of annuity of the age of 27.5, given value of annuities following table:

Age	25	26	27	28	29
Annuity	16.195	15.919	15.630	15.326	15.006

5. Construct the interpolating polynomial that fits the data:

x	0	0.1	0.2	0.3	0.4	0.5
y	–1.5	–1.27	–0.98	–0.63	–0.22	0.25

Using the Newton's forward or backward difference interpolation. Hence, or otherwise estimate the values of $f(x)$ at $x = 0.15, 0.25$ and 0.45.

6. Using the Newton's backward difference interpolation, construct the interpolating polynomial that fits the data:

x	0.1	0.3	0.5	0.7	0.9	1.1
y	–1.699	–1.073	–0.375	0.443	1.429	2.631

Estimate the values of $f(x)$ at $x = 0.6$ and $x = 1.0$.

7. The following data represents the function $f(x) = e^x$.

x	1	1.5	2.0	2.5
$f(x)$	2.7	4.4817	7.3891	12.1825

Estinate the value of $f(2.25)$ using Newton's backward difference interpolation formula.

8. Calculate the approximate value of $\sin x$ for $x = 1.36$, using the following table.

x	0.5	0.7	0.9	1.1	1.3	1.5
y	0.47943	0.64422	0.78333	0.89121	0.96356	0.99749

9. Find the cubic polynomial $f(x)$ which takes on the following table:

x	0	1	2	3	4	5
$f(x)$	–4	–1	2	11	32	71

Find the values of $f(2.5)$ and $f(6)$, using interpolation formula.

10. From the following data estimate to number of persons having income between 2000 and 2500, using interpolation formula:

Income	below 500	500–1000	1000–2000	2000–3000	3000–4000
Number of persons	6000	4250	3600	1500	650

ANSWERS

1. $2x^3-7x^2+6x+1$ 2. (i) 27 (ii) 27 3. 0.1745 4. 15.479

5. $3x^2+2x-1.5$; -1.1325; -0.8125; 0.0075 6. x^3+3x-2; 0.016; 2

7. 9.5037 8. 0.977849 9. x^3-3x^2+5x-4; 5.375; 134 10. 11106

3.8. CENTRAL DIFFERENCE FORMULAE

As earlier we study formulae for leading terms and differences. These formulae are fundamental and are applicable to nearly all cases of interpolation, but they do not coverage as rapidly as central difference formulae. The main advantage of central difference formulae is that they give more accurate result than other method of interpolation. Their disadvantages lies in complicated calculations and tedious expression, which are rather difficult to remember. These formulae are used for interpolation near the middle of a argument values. In this category we use the following formulae.

3.8.1 Gauss Forward Difference Formula

We know Newton's Gregory forward difference formula is given by

$$f(a+hu) = f(a)+u\Delta f(a)+\frac{u(u-1)}{2!}\Delta^2 f(a)+\frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)+\frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a)+.... \quad ....(1)$$

Substitute $a=0$, $h=1$ in (1), we get

$$f(u)= f(0)+u\Delta f(0)+\frac{u(u-1)}{2!}\Delta^2 f(0)+\frac{u(u-1)(u-2)}{3!}\Delta^3 f(0)+\frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(0)+.... \quad ....(2)$$

Now obtain the values of $\Delta^2 f(0)$, $\Delta^3 f(0)$, $\Delta^4 f(0)$ ....

To get these values,

$$\Delta^3 f(-1) = \Delta^2 f(0) - \Delta^2 f(-1)$$

$$\Rightarrow \quad \Delta^2 f(0) = \Delta^3 f(-1) + \Delta^2 f(-1)$$

Also, $\Delta^4 f(-1) = \Delta^3 f(0) - \Delta^3 f(-1)$

$$\Rightarrow \quad \Delta^3 f(0) = \Delta^4 f(-1) + \Delta^3 f(-1)$$

$$\Delta^5 f(-1) = \Delta^4 f(0) - \Delta^4 f(-1)$$

$$\Rightarrow \quad \Delta^4 f(0) = \Delta^5 f(-1) + \Delta^4 f(-1)$$

$$\Delta^6 f(-1) = \Delta^5 f(0) - \Delta^5 f(-1)$$

$$\Rightarrow \quad \Delta^5 f(0) = \Delta^6 f(-1) + \Delta^5 f(-1) \ldots\ldots$$ and so on.

Substituting these values in equation (2), we get

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\left[\Delta^3 f(-1) + \Delta^2 f(-1)\right] + \frac{u(u-1)(u-2)}{3!}\left[\Delta^4 f(-1) + \Delta^3 f(-1)\right]$$
$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\left[\Delta^5 f(-1) + \Delta^4 f(-1)\right]$$
$$+ \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\left[\Delta^6 f(-1) + \Delta^5 f(-1)\right] + \ldots\ldots\ldots\ldots$$

$$\Rightarrow \quad f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2}\Delta^2 f(-1) + \frac{u(u-1)}{2!}\left\{1 + \frac{(u-2)}{3}\right\}\Delta^3 f(-1)$$
$$+ \frac{u(u-1)(u-2)}{6}\left\{1 + \frac{(u-3)}{4}\right\}\Delta^4 f(-1)$$
$$+ \frac{u(u-1)(u-2)(u-3)}{24}\left\{1 + \frac{(u-4)}{5}\right\}\Delta^5 f(-1) + \ldots\ldots\ldots$$

$$\Rightarrow \quad f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1)$$
$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-1) + \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^5 f(-1) + \ldots \qquad \ldots(3)$$

But $\quad \Delta^5 f(-2) = \Delta^4 f(-1) - \Delta^4 f(-2)$

$\Rightarrow \quad \Delta^4 f(-1) = \Delta^4 f(-2) + \Delta^5 f(-2)$

and $\quad \Delta^6 f(-2) = \Delta^5 f(-1) + \Delta^5 f(-2)$

$\Rightarrow \quad \Delta^5 f(-1) = \Delta^5 f(-2) + \Delta^6 f(-2)$

The equation (3) becomes

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1) + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2)$$
$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^5 f(-2) + \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^5 f(-2)$$
$$+ \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^6 f(-2)$$

$$\therefore \quad f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1)$$
$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2) + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-2)$$
$$+ \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^6 f(-2) + \ldots$$

This formula is known as **Gauss forward difference formula.**

This formula is applicable when u lies between **0** and $\mathbf{\frac{1}{2}}$

$$\begin{tikzcd}[column sep=small]
f(0) \arrow[rr, dotted, no head] \arrow[dr] & & \Delta^2 f(-1) \arrow[rr, dotted, no head] \arrow[dr] & & \Delta^4 f(-2) \arrow[rr, dotted, no head] \arrow[dr] & & \Delta^6 f(-3) \arrow[r, dotted, no head] \arrow[dr] & \text{Central Line} \\
& \Delta f(0) \arrow[ur] & & \Delta^3 f(-1) \arrow[ur] & & \Delta^5 f(-2) \arrow[ur] & & \Delta^7 f(-3)
\end{tikzcd}$$

3.8.2 Gauss Backward Difference Formula

This formula is also solved by using Newton's forward difference formula.

Now, we know Newton's formula for forward interpolation is

$$f(a+hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$
$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a) + \ldots \quad \text{.....(1)}$$

Put $a = 0$, and $h = 1$, in equation (1), we get

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(0) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(0) + \ldots \quad \text{....(2)}$$

Now $\Delta f(0) = \Delta f(-1) + \Delta^2 f(-1)$

$\Delta^2 f(0) = \Delta^2 f(-1) + \Delta^3 f(-1)$

$\Delta^3 f(0) = \Delta^3 f(-1) + \Delta^4 f(-1)$

$\Delta^4 f(0) = \Delta^4 f(-1) + \Delta^5 f(-1)\ldots$ and so on.

On substituting these values in equation (2), we get

$$f(u) = f(0) + u\left[\Delta f(-1) + \Delta^2 f(-1)\right] + \frac{u(u-1)}{2!}\left[\Delta^2 f(-1) + \Delta^3 f(-1)\right] + \frac{u(u-1)(u-2)}{3!}\left[\Delta^3 f(-1) + \Delta^4 f(-1)\right] + \frac{u(u-1)(u-2)(u-3)}{4!}\left[\Delta^4 f(-1) + \Delta^5 f(-1)\right] + \ldots$$

$$\therefore \quad f(u) = f(0) + u\Delta(-1) + u\Delta^2 f(-1)\left[1 + \frac{(u-1)}{2}\right] + \frac{u(u-1)}{2}\Delta^3 f(-1)\left[1 + \frac{(u-2)}{3}\right] + \frac{u(u-1)(u-2)}{3}\Delta^4 f(-1)\left[1 + \frac{(u-3)}{4}\right] + \ldots$$

$$= f(0) + u\Delta(-1) + \frac{(u+1)}{2!}u\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1) + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-1) + \ldots.. \quad \text{....(3)}$$

Again $\Delta^3 f(-1) = \Delta^3 f(-2) + \Delta^4 f(-2)$

$\Delta^4 f(-1) = \Delta^4 f(-2) + \Delta^5 f(-2)$

$\Delta^5 f(-1) = \Delta^5 f(-2) + \Delta^6 f(-2)\ldots$ and so on

Therefore, equation (3) becomes

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\left[\Delta^3 f(-2) + \Delta^4 f(-2)\right] + \frac{(u+1)u(u-1)(u-2)}{4!}\left[\Delta^4 f(-2) + \Delta^5 f(-2)\right] + \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\left[\Delta^5 f(-2) + \Delta^6 f(-2)\right] + \ldots\ldots.$$

$$f(u)=f(0)+u\Delta f(-1)+\frac{(u+1)u}{2!}\Delta^2 f(-1)+\frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2)$$

$$+\frac{(u+1)u(u-1)}{3!}\left[1+\frac{(u-2)}{4}\right]\Delta^4 f(-2)$$

$$+\frac{(u+1)u(u-1)(u-2)}{4!}\left[1+\frac{(u-3)}{5}\right]\Delta^5 f(-2)+\ldots\ldots$$

$$f(u)=f(0)+u\Delta f(-1)+\frac{(u+1)u}{2!}\Delta^2(-1)+\frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2)$$

$$+\frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 f(-2)$$

$$+\frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-2)+\ldots\ldots$$

This is known as Gauss Backward difference formula and useful when u lies between $-\frac{1}{2}$ and 0.

Example 1. Using Gauss central difference formula, find the value of $e^{1.17}$ from the following data:

x	1	1.05	1.10	1.15	1.20	1.25	1.30
$e^x=f(x)$	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693

Solution. Taking $a=1.15$, $x=1.17$, $h=0.05$

$$u=\frac{x-a}{h}=\frac{1.17-1.15}{0.05}=0.4$$

The difference table is

x	u	$f(u)$	$\nabla f(u)$	$\nabla^2 f(u)$	$\nabla^3 f(u)$	$\nabla^4 f(u)$	$\nabla^5 f(u)$	$\nabla^6 f(u)$
1	–3	2.7183						
			0.1394					
1.05	–2	2.8577		0.0071				
			0.1465		0.0004			
1.10	–1	3.0042		0.0075		0		
			0.154		0.0004		0	
1.15	0	3.1582		0.0079		0		0.0001
			0.1619		0.0004		0.0001	
1.20	1	3.3201		0.0083		0.0001		
			0.1702		0.0005			
1.25	2	3.4903		0.0088				
			0.179					
1.30	3	3.6693						

By Gauss forward interpolation formula.

$$f(u)=f(0)+u\Delta f(0)+\frac{u(u-1)}{2!}\Delta^2 f(-1)+\frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1)$$

$$+\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2)$$

$$+\frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-2)+\ldots$$

$$f(0.4)=3.1582+(0.4)(0.1619)+\frac{(0.4)(-0.6)}{2}(0.0079)+\frac{(1.4)(0.4)(-0.6)}{6}(0.0004)$$
$$+\frac{(1.4)(0.4)(-0.6)(-1.6)(2.4)}{120}(0.0001)$$

$$=3.1582+0.06476-0.000948-0.0000224+0.000001075=3.22199$$

$\therefore \quad e^{1.17}=3.22199$

Example 2. Use Gauss's forward formula, evaluate at x =3.75 from the following table.

x	2.5	3.0	3.5	4.0	4.5	5.0
$f(x)$	24.145	22.043	20.225	18.644	17.262	16.047

Solution. The difference table is

x	u	$f(u)$	$\nabla f(u)$	$\nabla^2 f(u)$	$\nabla^3 f(u)$	$\nabla^4 f(u)$	$\nabla^5 f(u)$
2.5	–2	24.145					
			–2.102				
3.0	–1	22.043		0.284			
			–1.818		–0.047		
3.5	0	20.225		0.237		0.009	
			–1.581		–0.038		–0.003
4.0	1	18.664		0.199		0.006	
			–1.382		–0.032		
4.5	2	17.262		0.167			
			–1.215				
5.0	3	16.047					

Take $a=3.5,\ x=3.75,\ h=0.5,\ u=\frac{x-a}{h}=\frac{3.75-3.5}{0.5}=0.5$

By Gauss forward interpolation formula

$$f(u)=f(0)+u\Delta f(0)+\frac{u(u-1)}{2!}\Delta^2 f(-1)+\frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1)$$
$$+\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2)$$
$$+\frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-2)+\ldots$$

$$f(0.5)=20.225+(0.5)(-1.581)+\frac{(0.5)(-0.5)}{2}(0.237)+\frac{(1.5)(0.5)(-0.5)}{6}(-0.038)$$
$$+\frac{(1.5)(0.5)(-0.5)(-1.5)}{24}(0.009)$$
$$+\frac{(2.5)(1.5)(0.5)(-0.5)(-1.5)}{120}(-0.003)$$

$$=20.225-0.7905-0.029625+0.002375+0.00014-0.000035=19.40735$$

Example 3. Use Gauss's forward formula to find a polynomial of degree four takes the value of the following table.

x	1	2	3	4	5
$f(x)$	1	–1	1	–1	1

Solution. Taking $a = 3$ and $h = 1$, $u = \dfrac{x-a}{h} = x - 3$

The difference table is

x	u	$f(u)$	$\nabla f(u)$	$\nabla^2 f(u)$	$\nabla^3 f(u)$	$\nabla^4 f(u)$
1	–2	1				
			–2			
2	–1	–1		4		
			2			
3	0	1			–8	
				–4		16
			–2			
4	1	–1			8	
				4		
			2			
5	2	1				

By Gauss's forward interpolation formula

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1) + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2) + \ldots\ldots$$

$$= 1 + (x-3)(-2) + \frac{(x-3)(x-4)}{2}(-4) + \frac{(x-3)(x-4)(x-2)}{6}(8) + \frac{(x-2)(x-3)(x-4)(x-5)}{24}(16) + \ldots\ldots$$

$$= 1 - 2x + 6 - 2x^2 + 14x - 24 + \frac{4}{3}x^3 - 12x^2 - \frac{104}{3}x - 32 + \frac{2}{3}x^4 - \frac{28}{3}x^3 + \frac{142}{3}x^2 - \frac{308}{3}x + 80$$

$\therefore$

$$f(x) = \frac{2}{3}x^4 - 8x^3 + \frac{100}{3}x^2 - 56x + 31$$

Example 4. Apply Gauss's backward formula to find sin 45° from the following table:

$x°$	20	30	40	50	60	70	80
$f(x) = \sin x$	0.34202	0.502	0.64279	0.76604	0.86603	0.93969	0.98481

Solution. Take $a = 40, x = 45, h = 10$.

$$u = \frac{x-a}{h} = \frac{45-40}{10} = 0.5$$

The difference table is

x	u	$f(u)$	$\Delta f(u)$	$\Delta^2 f(u)$	$\Delta^3 f(u)$	$\Delta^4 f(u)$
20	–2	0.34202				
			0.15998			
30	–1	0.502		–0.01919		
			0.14079		0.00165	
40	0	0.64279		–0.01754		–0.00737
			0.12325		–0.00572	
50	1	0.76604		–0.02326		0.00265
			0.09999		–0.00307	
60	2	0.86603		–0.02633		0.00086
			0.07366		–0.00221	
70	3	0.93969		–0.02854		
			0.04512			
80	4	0.98481				

By Gauss's backward formula is

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2) + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 f(-2) + \ldots\ldots$$

$$f(0.5) = 0.64279 + (0.5)(0.14079) + \frac{(0.5+1)(0.5)}{2}(-0.01754) + \frac{(0.5+1)(0.5)(0.5-1)}{6}(0.00165) + \frac{(0.5+2)(0.5+1)(0.5)(0.5-1)}{24}(-0.00737)$$

$$= 0.64279 + 0.07039 - 0.00657 - 0.0001 + 0.00028 = 0.70679$$

$\therefore \quad \sin 45° = 0.70679$

Example 5. Use Gauss backward difference formula to find the value of $(1.06)^{19}$, given that $(1.06)^{10} = 1.79085$, $(1.06)^{15} = 2.39656$, $(1.06)^{20} = 3.20714$, $(1.06)^{25} = 4.29187$, $(1.06)^{30} = 5.74349$.

Solution. Take $a = 20, x = 19, h = 5, u = \dfrac{x-a}{h} = \dfrac{19-20}{5} = -0.2$

The difference table is

x	u	$f(u)$	$\Delta f(u)$	$\Delta^2 f(u)$	$\Delta^3 f(u)$	$\Delta^4 f(u)$
10	–2	1.79082				
			0.60571			
15	–1	2.39656		0.20487		
			0.81058		0.06928	
20	0	3.20714		0.27415		0.02346
			1.08473		0.09274	
25	1	4.29187		0.36689		
			1.45162			
30	2	5.74349				

From Gauss backward formula

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2) + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 f(-2) + \ldots\ldots$$

$$f(u) = 3.20714 - 0.2 \times 0.81058 - \frac{0.2(0.8)}{2} \times 0.27415 - \frac{0.2(0.8)(-1.2)}{6} \times 0.06928 + \frac{0.2(0.8)(-1.2)(1.8)}{24} \times 0.02346$$

$$f(u) = 3.20714 - 0.162116 - 0.021932 + 0.002216 + 0.00033782 = 3.02564$$

$\therefore \quad (1.06)^{19} = 3.02564$

Example 6. Using Gauss backward formula, Estimate the number of persons earning wages between Rs. 60 and Rs. 70 from the following data:

Wages (Rs.)	Below 40	40–60	60–80	80–100	100–120
No. of persons (in thousands)	250	120	100	70	50

Solution. Take $a = 80, x = 70, h = 20, u = \dfrac{x-a}{h} = \dfrac{70-80}{20} = -0.5$

The difference table is

Wages below (x)	u	$f(u)$	$\Delta f(u)$	$\Delta^2 f(u)$	$\Delta^3 f(u)$	$\Delta^4 f(u)$
40	–2	250				
			120			
60	–1	370		–20		
			100		–10	
80	0	470		–30		20
			70		10	
100	1	540		–20		
			50			
120	2	590				

By Gauss backward formula

$$f(u)=f(0)+u\Delta f(-1)+\frac{(u+1)u}{2!}\Delta^2 f(-1)+\frac{(u+1)(u-1)}{3!}\Delta^3 f(-2)$$
$$+\frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 f(-2)+\ldots\ldots$$

$$f(u)=470+(-0.05)\times 100+\frac{(-0.5)(0.5)}{2}\times(-30)+\frac{(-0.5)(0.5)(-1.5)}{6}\times(-10)$$
$$+\frac{(-0.5)(0.5)(-1.5)(1.5)}{24}\times(20)$$

$$=470-50+3.75-0.625+0.46875=423.59375$$

Hence number of person earning wages between Rs. 60 to Rs.70 is $423.59375-370=53.59375$

Example 7. If $f(x)$ is a polynomial of degree four find the value of $f(5.8)$ using Gauss's backward formula from the following data: $f(4)=270, f(5)=648,\ \Delta f(5)=682,\ \Delta^3 f(4)=132.$

Solution. Given $\Delta f(5)=682.$

$\Rightarrow\quad f(6)-f(5)=682$

$\Rightarrow\quad f(6)=682+648=1330$

Also $\quad \Delta^3 f(4)=132$

$\Rightarrow\quad (E-1)^3 f(4)=132$

$\Rightarrow\quad f(7)-3f(6)+3f(5)-f(4)=132$

$\Rightarrow\quad f(7)=3\times 1330-3\times 648+270+132=2448$

Now form difference table is

x	u	$f(u)$	$\Delta f(u)$	$\Delta^2 f(u)$	$\Delta^3 f(u)$
4	–2	270			
			378		
5	–1	648		304	
			682		132
6	0	1330		436	
			1118		
7	1	2448			

Take $a = 6$, $h = 1$, $x = 5.8$, $u = \frac{x-a}{h} = \frac{5.8-6}{1} = -0.2$ $\quad\therefore\quad u = -0.2$

From Gauss backward formula

$$f(-0.2) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u-1)u(u-1)}{3!}\Delta^3 f(-2)$$

$$= 1330 + (-0.2)\times 682 + \frac{(-0.2)(0.8)}{2}\times 436 + \frac{(-0.2)(0.8)(-1.2)}{6}\times 132$$

$$= 1330 - 136.4 - 34.88 + 4.224 = 1162.944$$

Hence $f(5.8) = 1162.944$

Example 8. Using Gauss backward interpolation formula that find the population for the year 1936. Given that

Year	1901	1911	1921	1931	1941	1951
Population (in thousands)	12	15	20	27	39	52

Solution. Here $h = 10$. Take origin at 1941 to evaluate population in1936 i.e. $x = 1936$, $a = 1941$.

$$\Rightarrow \quad u = \frac{x-a}{h} = \frac{1936-1941}{10} = \frac{-5}{10} - 0.5$$

Difference table for given data is as

u	$f(u)$	$\Delta f(u)$	$\Delta^2 f(u)$	$\Delta^3 f(u)$	$\Delta^4 f(u)$	$\Delta^5 f(u)$
–4	12					
		3				
–3	15		2			
		5		0		
–2	20		2		3	
		7		3		–10
–1	27		5		–7	
		12		–4		
0	39		1			
		13				
1	52					

By Gauss backward formula is

$$f(u)=f(0)+u\Delta f(-1)+\frac{(u+1)u}{2!}\Delta^2 f(-1)+\frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2)$$
$$+\frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 f(-2)+\frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-3)+\ldots\ldots$$

$$=39+(-0.5)\times 12+\frac{(-0.5)(0.5)}{2}\times 1+\frac{(-0.5)(0.5)(-1.5)}{6}\times(-4)$$

$$=39-6.0-0.125-0.25=32.625 \text{ thousands}$$

Hence, the population in 1936 is 32625.

EXERCISE 3.4

Using Gauss forward difference formula

1. Find the value of $e^{-1.7425}$, data are given by

x	1.72	1.73	1.74	1.75	1.76
e^{-x}	0.17907	0.17728	0.17552	0.17377	0.17204

2. Find $f(22)$ from the Gauss forward formula:

x	20	25	30	35	40	45
$f(x)$	354	332	291	260	261	204

3. Find the value of $f(32)$, given that $f(25)=0.2707$, $f(30)=0.3027$, $f(35)=0.3386$, $f(40)=0.3794$

4. Find the value of $f(30)$ for the following data:

x	21	25	29	33	37
$f(x)$	18.4708	17.8144	17.1070	16.3432	15.5154

5. Find the first term, in the following table the values of $f(x)$ are consecutive terms of a series

x	3	4	5	6	7	8	9
$f(x)$	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Using Gauss backward difference formula.

6. Given that $\sqrt{12500}=111.8033$, $\sqrt{12510}=111.8481$, $\sqrt{12520}=111.8928$, $\sqrt{12530}=111.9394$.

   Find the value of $\sqrt{12516}$.

7. Find the value of $f(3.75)$ for the following table.

x	2.5	3.0	3.5	4.0	4.5	5.0
$f(x)$	24.145	22.043	20.225	18.644	17.262	16.047

8. Find the population of a town for the year 1974, given that

Year	1939	1949	1959	1969	1979	1989
Population(in thousands)	12	15	20	27	39	52

9. Find the value of $f(1.0005)$, data are given by

x	1.00	1.01	1.02	1.03	1.04	1.05	1.06
$f(x)$	1.0000	1.0050	1.0100	1.0149	1.0198	1.0247	1.0296

10. Find the values of $f(1.01)$, $f(1.12)$ and $f(1.28)$ from thefollowing data:

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
$f(x)$	1.00000	1.02470	1.04881	1.07238	1.09544	1.11803	1.14017

ANSWERS

1. 0.175081
2. 347.98
3. 0.3165
4. 16.9216
5. 3.1
6. 111.8749
7. 19.704
8. 32.53
9. 1.0025
10. 1.00499, 1.05830, 1.13137

3.8.3 Stirling's Formula

This is another central difference formula and useful when $|u| < \frac{1}{2}$ or $-\frac{1}{2} < u < \frac{1}{2}$. It gives best estimation when $-\frac{1}{4} < u < \frac{1}{4}$. This formula is obtained by taken mean of Gauss forward and Gauss backward difference formula

Gauss forward formula for interpolating central difference is,

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1) + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2)$$
$$+ \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-2) + \frac{(u+1)u(u-1)(u-2)(u-3)}{6!}\Delta^6 f(-2) + \ldots\ldots \quad \ldots(1)$$

Gauss Backward difference is,

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2)$$
$$+ \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 f(-2) + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-2) + \ldots\ldots \quad \ldots(2)$$

Take mean of Equation (1) and (2), we get