1.9 MEAN VALUE THEOREM

The roots of the given function as well as equality or inequality of any two or more than two functions can be determined using Mean Value Theorems. These theorems are Rolle's Theorem, Lagrange's Mean Value Theorem, and Cauchy's Mean Value Theorem. For better understanding of thes theorems, we shall first learn two type of functions.

1.9.1 Continuous and Differentiable Functions

A function $f(x)$ is said tobe continuous at $x = a$ if

(*i*) $f(a)$ is finite, i.e., $f(a)$ exists.

(*ii*) $\lim_{x \to a+} f(x) = \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = f(a)$

A function $f(x)$ is said to be differentiable at $x = a$, if Right Hand Derivative (RHD) and Left Hand Derivative (LHD) exists and RHD = LHD.

i.e.
$$\lim_{x\to a-}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a-}\frac{f(x)-f(a)}{x-a}$$

or
$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to a}\frac{f(a-h)-f(a)}{h}$$

Notes

(i) If function $f(x)$ is finitely differentiable (derivative is finite) in the interval (a, b), then it is continuous in the interval $[a, b]$, i.e., every differentiable function is continuous in the given interval. But converse is not necessarily true, i.e., a function may be continuous for a vlaue of x without being differentiable for that value. The interval (a, b) is also called as domain of the function.

(ii) Algebraic, trigonometric, inverse trigonometric, logarithmic and exponential function are ordinarily continuous and differentiable (with some exceptions).

(iii) Addition, subtraction, product and quotient of two or more continuous and differentiable functions are also continuous and differentiable.

(iv) $f(x)$ and $f'(x)$ are differentiable and continuos if $f''(x)$ exists.

(v) A function is said to be differentiable if its derivative is neither indetermines nor infinite.

1.10 ROLLE'S THEOREM

Statement: If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$

(ii) differentiable in the open interval (a, b)

(iii) $f(a) = f(b)$

then there exists at least one point c in the open interval (a, b) such that $f'(c) = 0$.

Proof : Since function $f(x)$ is continuous in the closed interval $[a, b]$, it attains its maxima and minima at some points in the interval. Let M and m be the maximum and minimum values of $f(x)$ respectively at some points c and d respectively in the interval $[a, b]$.

$$f(c) = \text{M} \quad \text{and} \quad f(d) = m$$

Now two cases arise :

Case I : If $\text{M} = m$

$$f(x) = M = m \text{ for all } x \text{ in } [a, b]$$
$$f(x) = \text{constant for all } x \text{ in } [a, b]$$
$$f(x) = 0 \text{ for all } x \text{ in } [a, b]$$

Hence, the theorem is true.

Case II : If $M \neq m$

Since $f(a) = f(b)$, either M or m must be different from $f(a)$ and $f(b)$.

Let M is different from $f(a)$ and $f(b)$.

$f(c)$ is different from $f(a)$ and $f(b)$.

$$f(c) \neq f(a) \quad \therefore \quad c \neq a$$

Also,
$$f(c) \neq f(b) \quad \therefore \quad c \neq b$$

Hence,
$$a < c < b$$

Now, since $f(x)$ is differentiable in the open interval (a, b), $f'(c)$ exists.
By definition,

$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

Since, $$f(c) = M, f(c+h) \le f(c)$$

$$\frac{f(c+h)-f(c)}{h} < 0 \quad \text{for} \quad h > 0 \quad \text{...(1)}$$

and $$\frac{f(c+h)-f(c)}{h} > 0 \quad \text{for} \quad h < 0 \quad \text{...(2)}$$

As $h \to 0$, Eq. (1) gives $f'(c) \le 0$ and Eq. (2) gives $f'(c) \ge 0$.
Since $f(x)$ is differentiable, $f'(c)$ must be unique.
Hence, $$f'(c) = 0 \quad \text{for} \quad a < c < b$$
Similarly, it can be proved that $f'(c) = 0$ for $a < c < b$ if m is different from $f(a)$ and $f(b)$.

Note :

(*i*) There may be more than one point c, such that, $f'(c) = 0$.

(*ii*) The converse of the theorem is not true, i.e., for some function $f(x)$, $f'(c) = 0$ but $f(x)$ may not satisfy the conditions of Rolle's theorem.

e.g., $$f(x) = 1 - 3\,(x-1)^{2/3} \text{ in } 0 \le x \le 10$$

$$f'(x) = 1 - \frac{2}{(x-1)^{1/3}}$$

$f'(c) = 0$ at $c = 9$. But $f'(x)$ does not exist at $x = 1$, i.e., not differentiable at $x = 1$. Hence, $f(x)$ does not satisfy the conditions of Rolle's theorem.

1.10.1 Another Form of Rolle's Theorem

If a function $f(x)$ is

(*i*) continuous in the closed interval $[a, a + h]$

(*ii*) differentiable in the open interval $(a, a + h)$

(*iii*) $f(a) = f(a + h)$, then there exists at least one real number θ between 0 and 1 such that $f'(a+\theta h) = 0$, for $0 < \theta < 1$.

1.10.2 Geometrical Interpretation of Rolle's Theorem

Let $y = f(x)$ represents a curve with $A\,[a, f(a)]$ and $B\,[b, f(b)]$ as end points and $C\,[c, f(c)]$ be any point between A and B.

$f'(c) =$ slope of the tangent at point C.

Thus, geometrically the theorem states that if

(*i*) curve is continuous at the points A, B and at every point between A and B, i.e., in the interval $[a, b]$

(*ii*) possesses unique tangent at every point between A and B.

(*iii*) ordinates of the points A and B are same, i.e., $f(a) = f(b)$, then there exists at least one point $C\,[c, f(c)]$ on the curve between A and B, tangent at which is parallel to x-axis.

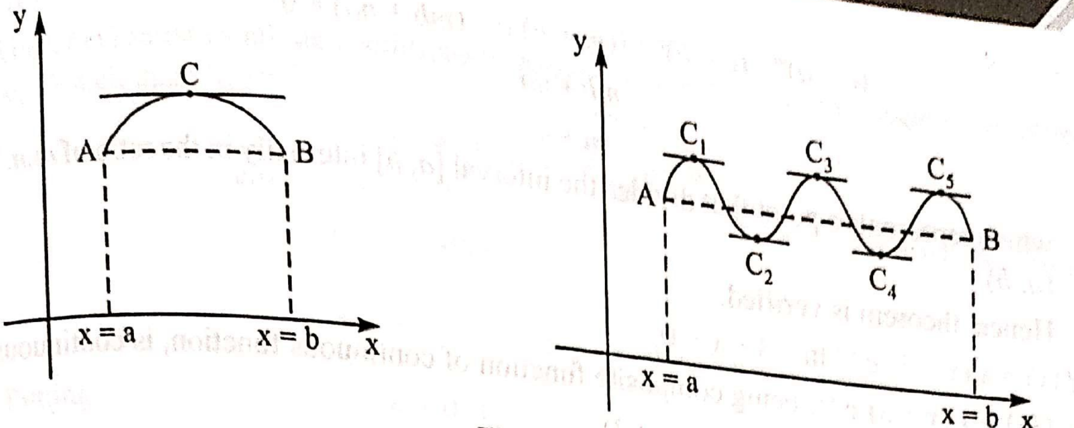

Fig. 1.25

1.10.3 Algebraic Interpretation of Rolle's Theorem

Let $f(x)$ be a polynomial in x. If $f(x) = 0$ satisfies all the conditions of Rolle's theorem and $x = a, x = b$ be the roots of the equation $f(x) = 0$, then at least one root of the equation $f'(x) = 0$ lies between a and b.

EXAMPLE 1.33 : Verify Rolle's theorem for the following functions :

(*i*) $f(x) = (x-a)^m (x-b)^n$ **in** $[a, b]$**, where** m, n **are positive integers.**

(*ii*) $f(x) = |x|$ **in** $[-1, 1]$

(*iii*) $f(x) = x(x+3)e^{-x/2}$ **in** $-3 \le x \le 0$.

(*iv*) $f(x) = \dfrac{\sin x}{e^x}$ **in** $[\theta, \pi]$

(*v*) $f(x) = ex(\sin x - \cos x)$ **in** $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$

(*vi*) $f(x) = \log\left[\dfrac{x^2+ab}{(a+b)x}\right]$ **in** $[a, b]$, $a > 0, b > 0$

(*vii*) $f(x) = x^2 + 1 \qquad 0 \le x \le 1$
$\quad = 3 - x \qquad 1 \le x \le 2$

(*viii*) $f(x) = x^2 - 2 \qquad -1 \le x \le 0$
$\quad = x - 2 \qquad 0 \le x \le 1$

Solution :

(*i*) $f(x) = (x-a)^m (x-b)^n$ in $[a, b]$, where m, n are positive integers.

(*a*) Since m and n are positive integers,

$f(x) = (x-a)^m (x-b)^n$, being a polynomial, is continuous in $[a, b]$.

(*b*) $f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m(x-b)^{n-1}$

$= (x-a)^{m-1}(x-b)^{n-1}[m(x-b) + n(x-a)]$

$= (x-a)^{m-1}(x-b)^{n-1}[(m+n)x - (mb+na)]$

exists for every value of x in (a, b). Therefore, $f(x)$ is differentiable in (a, b).

(*c*) $f(a) = f(b) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point c in (a, b) such that $f'(c) = 0$.

$$(c-a)^{m-1}(c-b)^{n-1}[(m+n)c-(mb+na)]=0$$

$$c=\frac{mb+na}{m+n}$$

which represents a point that divides the interval $[a, b]$ internally in the ratio of $m:n$. Thus, c lies in (a, b).

Hence, theorem is verified.

(ii) $f(x) = x(x+3)\,e^{-x/2}$ in $-3 \le x \le 0$.

(*a*) $f(x) = x(x+3)\,e^{-x/2}$, being composite function of continuous function, is continuous in $[-3, 0]$.

(*b*) $f'(x) = (x+3)\,e^{-x/2} + xe^{-x/2} - \dfrac{x(x+3)}{2}\,e^{-x/2}$

exists for every value of x in $(-3, 0)$. Therefore, $f(x)$ is differentiable in $(-3, 0)$.

(*c*) $f(-3) = f(0) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point c in $(-3, 0)$ such that $f'(c) = 0$.

$$(c+3)\,e^{-c/2} + ce^{-c/2} - \frac{c(c+3)}{2}\,e^{-c/2} = 0$$

$$2(c+3) + 2c - c(c+3) = 0 \qquad [\because e^{-c/2} \neq 0 \text{ for any finite value of } c]$$

$$-c^2 + c + 6 = 0,$$

$$c = -2, 3$$

$$c = -2 \text{ lies in } (-3, 0)$$

Hence, theorem is verified.

(iii) $f(x) = |x|$ in $[-1, 1]$

$$|x| = -x, \qquad -1 \le x \le 0$$
$$= x, \qquad 0 \le x \le 1$$

(*a*) $f(x)$ is continuous in $[-1, 1]$

(*b*) Left hand derivative at $x = 0$

$$f'(0^-) = \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{-x-0}{x} = -1$$

Right hand derivative at $x = 0$

$$f'(0^+) = \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{x-0}{x} = -1$$

$$f'(0^-) \neq f'(0^+)$$

Thus, function is not differentiable at $x = 0$ and hence, Rolle's theorem is not applicable.

(iv) $f(x) = \dfrac{\sin x}{e^x} = e^{-x}\sin x$

(*a*) $f(x) = e^{-x}\sin x$, being product of continuous functions, is continuous in $[0, \pi]$

(*b*) $f(x) = -e^{-x}\sin x + e^{-x}\cos x$

$$= e^{-x}(\cos x - \sin x)$$

exists for every value of x in $(0, \pi)$. Therefore, $f(x)$ is differentiable in $(0, \pi)$

(c) $f(0) = f(\pi) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one c in $(0, \pi)$ such that $f'(c) = 0$.

$$f'(c) = e^{-c}(\cos c - \sin c) = 0$$

$$\cos c - \sin c = 0 \qquad [\because e^{-c} \neq 0 \text{ for any finite value of } c]$$

$$\cos c = \sin c$$

$$\tan c = 1,\ c = n\pi + \frac{\pi}{4}, \text{ where } n \text{ is an integer.}$$

Putting $n = 0, 1, 2, \ldots$

$$c = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$$

$$c = \frac{\pi}{4} \text{ lies in the interval } (0, \pi)$$

Hence, theorem is verified.

(v) $f(x) = e^x(\sin x - \cos x)$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

(a) $f(x) = e^x(\sin x - \cos x)$, being composite function of continuous functions, is continuous in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

(b) $f'(x) = e^x(\sin x - \cos x) + ex(\cos x + \sin x)$

$= 2e^x \sin x$

exists for every value of x in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$. Therefore, $f(x)$ is differentiable in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

(c) $f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point c in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ such that $f'(c) = 0$

$$2e^c \sin c = 0$$

$$\sin c = 0 \qquad [\because e^c \neq 0 \text{ for any finite value of } c]$$

$$c = 0, \pi, 2\pi, \ldots\ldots\ldots$$

$$c = \pi \text{ lies in } \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$$

Hence, theorem is verified.

(vi) $f(x) = \log\left[\frac{x^2 + ab}{(a+b)x}\right]$ in $[a, b]$, $a > 0$, $b > 0$

(a) $f(x) = \log(x^2 + ab) - \log x - \log(a + b)$, being composite function of continuous functions, is continuous in $[a, b]$.

(b) $f(x) = \dfrac{2x}{x^2+ab} - \dfrac{1}{x}$

exists for every value of x in (a, b) [$\because a > 0, b > 0$]. Therefore, $f(x)$ is differentiable in (a, b)

(c)
$$f(a) = \log(a^2+ab) - \log a - \log(a+b)$$
$$= \log a + \log(a+b) - \log a - \log(a+b)$$
$$= 0$$
$$f(b) = \log(b^2+ab) - \log b - \log(a+b)$$
$$= \log b + \log(b+a) - \log b - \log(a+b)$$
$$= 0$$
$$f(a) = f(b)$$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point c in (a, b) such that $f'(c) = 0$.

$$\frac{2c}{c^2+ab} - \frac{1}{c} = 0$$
$$2c^2 - c^2 - ab = 0$$
$$c^2 - ab = 0,\ c = \pm\sqrt{ab}$$

Since, $c = \sqrt{ab}$ lies between a and b [being geometric mean of a and b].

Hence, theorem is verified.

(vii) $f(x) = x^2 + 1 \qquad 0 \le x \le 1$

$\qquad = 3 - x \qquad 1 \le x \le 2$

(a) $f(x) = x^2 + 1 \qquad 0 \le x \le 1$

$\qquad = 3 - x \qquad 1 \le x \le 2$

is defined every where in $[0, 2]$ and hence, continuous in $[0, 2]$.

(b) Left hand derivative at $x = 1$

$$f'(1^-) = \lim_{x\to1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to1^-} \frac{x^2+1-2}{x-1}$$
$$= \lim_{x\to1^-} \frac{x^2-1}{x-1} = \lim_{x\to1^-} (x+1) = 1+1 = 2$$

Right hand derivative at $x = 1$

$$f'(1^+) = \lim_{x\to1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to1^-} \frac{x^2+1-2}{x-1}$$
$$= \lim_{x\to1^-} \frac{x^2-1}{x-1} = \lim_{x\to1^-} (x+1) = 1+1 = 2$$

Thus, function is not differentiable at $x = 1$ and hence, Rolle's theorem is not applicable.

(viii) $f(x) = x^2 - 2 \qquad -1 \le x \le 0$

$\qquad = x - 2 \qquad 0 \le x \le 1$

(a) $f(x) = x^2 - 2 \qquad -1 \le x \le 0$

$\qquad = x - 2 \qquad 0 \le x \le 1$

is defined everywhere in $[-1, 1]$ and hence, is continuous in $[-1, 1]$.

(b) Left hand derivative at $x = 0$

$$f'(0^-) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{x^2+1-2}{x-1}$$

$$= \lim_{x\to 0^-} \frac{x^2-2+2}{x} = \lim_{x\to 0^-} \frac{x^2}{x} = 0$$

Right hand derivative at $x = 0$

$$f'(0^+) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{(x-2)-(-2)}{x}$$

$$= \lim_{x\to 0^+} \frac{x}{x} = 1$$

$$f'(0^-) = f'(0^+)$$

Thus, function is not differentiable at $x = 0$ and hence, Rolle's theorem is not applicable.

EXAMPLE 1.34 : Prove that between any two roots of $e^x \sin x = 1$ there exists at least one root of $e^x \cos x + 1 = 0$.

Solution : Let $f(x) = 1 - e^x \sin x$

(a) $f(x)$, being composite function of continuous functions, is continuous in a finite interval.

(b) $f'(x) = -(e^x \sin x + e^x \cos x) = -(1 + e^x \cos x)$ $[\because e^x \sin x = 1]$
exists for every finite value of x. Therefore, $f(x)$ is differentiable in a finite interval.

(c) Let α and β are two roots of the equation, $f(x) = 1 - e^x \sin x = 0$
Then, $f(\alpha) = f(\beta) = 0$
Thus, $f(x)$ satisfies all the conditions of Rolle's theorem in $[\alpha, \beta]$. Therefore, there exists at least one point c in (α, β) such that $f'(c) = 0$

$$1 + e^c \cos c = 0$$

This shows that c is the root of the equation $e^x \cos x + 1 = 0$ which lies between the root α and β of the equation $1 - e^x \sin x = 0$.

EXAMPLE 1.35 : Prove that the equation $2x^3 - 3x^2 - x + 1 = 0$ has atleast one root between 1 and 2.

Solution : Let us consider a function $f(x) = \frac{x^4}{2} - x^3 - \frac{x^2}{2} + x$ [obtained by integrating the given equation]

(a) $f(x)$, being an algebraic function, is continuous in $[1, 2]$

(b) $f'(x) = 2x^3 - 3x^2 - x + 1$ exist for every value of x in $(1, 2)$. Therefore, $f(x)$ is differentiable in $(1, 2)$.

(c) $f(1) = f(2) = 0$
Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point c in $(1, 2)$ such that $f'(c) = 0$.

$$2c^3 - 3c^2 - c + 1 = 0$$

This shows that c is the root of the equation $2x^3 - 3x^2 - x + 1 = 0$ which lies between 1 and 2.

EXAMPLE 1.36 : Prove that if $a_0, a_1, a_2, \ldots\ldots\ldots a_n$ are real numbers such the $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots\ldots + \frac{a_{n-1}}{2}$ $+ a_n = 0$, then there exists at least one real number x between 0 and 1 such that $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots a_n$ $= 0$.

Solution : Let us consider a function $f(x) = \dfrac{a_0 x^{n+1}}{n+1} + \dfrac{a_1 x^n}{n} + \dfrac{a_2 x^{n-1}}{n-1} + \ldots\ldots\ldots + a_n x$ defined in $[0, 1]$.

(*a*) $f(x)$, being an algebraic function, is continuous in $[0, 1]$.

(*b*) $f'(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots\ldots + a_n$ exists for every value of x in $(0, 1)$. Therefore, $f(x)$ is differentiable in $(0, 1)$.

(*c*) $f(0) = 0$

$$f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots\ldots\ldots + \frac{a_{n-1}}{2} + a_n = 0 \qquad \text{[given]}$$

$$f(0) = f(1)$$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point c in $(0, 1)$ such that $f'(c) = 0$.

$$a_0 c^n + a_1 c^{n-1} + a_2 c^{n-2} + \ldots\ldots\ldots\ldots\ldots + a_n = 0$$

Replacing c by x,

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots\ldots\ldots\ldots + a_n = 0$$

EXAMPLE 1.37 : If $f(x)$, $\phi(x)$, $\psi(x)$ are differentiable in (a, b), prove that there exists at least one point c in (a, b) such that $\begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(c) & \phi'(c) & \psi'(c) \end{vmatrix} = 0.$

Solution : Let us consider a function $F(x) = \begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f(x) & \phi(x) & \psi(x) \end{vmatrix}$

(*a*) Since $f(x)$, $\phi(x)$, $\psi(x)$ are differentiable in (a, b), therefore, will be continuous in $[a, b]$. $F(x)$, being composite function of continuous functions, is continuous in $[a, b]$.

(*b*) $F'(x) = \begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(x) & \phi'(x) & \psi'(x) \end{vmatrix}$ exists for every value of x in (a, b). Therefore, $f(x)$ is differentiable in (a, b).

(*c*) $f(a) = f(b) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point c in (a, b) such that $f'(c) = 0$

$$\begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(c) & \phi'(c) & \psi'(c) \end{vmatrix} = 0$$

EXAMPLE 1.38 : If $f(x) = x(x + 1)(x + 2)(x + 3)$, prove that $f'(x) = 0$ has three real roots.

Solution : $f(x) = x(x + 1)(x + 2)(x + 3)$

(*a*) $f(x)$, being polynomial is continuous, in the intervals $[-3, -2]$, $[-2, -1]$, $[-1, 0]$.

(*b*) $f'(x) = (x + 1)(x + 2)(x + 3) + x(x + 2)(x + 3) + x(x - 1)(x + 3) + x(x + 1)(x + 2)$ exists for every value of x in $[-3, -2]$, $[-2, -1]$ and $[-1, 0]$. Therefore, $f(x)$ is differentiable in $[-3, -2]$, $[-2, -1]$ and $[-1, 0]$.

(c) $f(-3) = f(-2) = f(-1) = f(0) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one c_1 in $(-3, -2)$, c_2 in $(-2, -1)$ and c_3 in $(-1, 0)$ such that $f'(c_1) = f(c_2) = f'(c_3) = 0$.

Thus, c_1, c_2 and c_3 are the roots of $f'(x) = 0$.

Hence, $f'(x) = 0$ has at least 3 real roots.

EXAMPLE 1.39 : If k is a real constant, prove that the equation $x^3 - 6x^2 + c = 0$ cannot have distinct roots in [0, 4].

Solution : Let $f(x) = x^3 - 6x^2 + c = 0$ has distinct roots a and b between 0 and 4 i.e.

$$0 \leq a \leq b \leq 4$$

Then,

$$f(a) = 0 = f(b)$$

Also, $f(x)$ being polynomial is, continuous in $[a, b]$ and $f(x) = 3x^2 - 12x$ exists for every value of x in $[a, b]$. Therefore, $f(x)$ is differentiable in (a, b).

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point c in (a, b) such that

$$f'(c) = 0$$

$$3c^2 - 12c = 0$$

$$3c\,(c - 4) = 0$$

$$c = 0, 4$$

But these values of c lies outside the interval (a, b). This is a contradiction to Rolle's theorem. Thus, our assumption is wrong.

Hence, $f(x) = 0$ cannot have distinct roots in [0, 4].

EXERICSE 1.3

1. Verify Rolle's Theorem for the following functions :

(i) $x^3 - 4x$ in $[-2, 2]$ $\left[\textbf{Ans. } c = \pm\frac{\sqrt{2}}{3}\right]$

(ii) $x^3 - 12x$ in $\left[0, 2\sqrt{3}\right]$ [**Ans.** $c = 2$]

(iii) x^2 in $[1, 2]$ [**Ans.** $f(1) \neq f(2)$, theorem is not applicable]

(iv) $2x^3 + x^2 - 4x - 2$ in $\left[-\sqrt{2}, \sqrt{2}\right]$ $\left[\textbf{Ans. } c = \frac{2}{3}, -1\right]$

(v) $2 + (x - 1)^{2/3}$ in $[0, 2]$ [**Ans.** Not differentiable at $x = 1$, theorem is not applicable]

(vi) $1-(x-3)^{2/3}$ in $[2, 4]$ [**Ans.** Not differentiable at $x = 3$, theorem is not applicable]

(vii) $\dfrac{x^2-4x}{x+2}$ in $[0, 4]$ [**Ans.** $c = 2\left(\sqrt{3}-1\right)$]

(viii) $(x+2)^3 (x-3)^4$ in $[-2, 3]$ [**Ans.** $c = \dfrac{1}{7}$]

(ix) $\log\left(\dfrac{x^2+6}{5x}\right)$ in $[2, 3]$ [**Ans.** $c = \sqrt{6}$]

(x) $\cos^2 x$ in $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$ [**Ans.** $c = 0$]

(xi) $\sin x$ in $[0, 2\pi]$ [**Ans.** $c = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$]

(xii) $|\cos x|$ in $[0, \pi]$ [**Ans.** Not differentiable at $x = \dfrac{\pi}{2}$, theorem is not applicable]

(xiii) $|\sin x|$ in $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$ [**Ans.** Not differentiable at $x = 0$, theorem is not applicable]

(xiv) $f(x) = 1 \quad x = 0$

$\quad = x \quad 0 < x \le 1$ [**Ans.** Discontinuous at $x = 0$, theorem is not applicable]

(xv) $1 - 3\sqrt{(x-1)^2}$ in $[0, 2]$ [**Ans.** Not differentiable at $x = 1$, theorem is not applicable]

(xvi) $f(x) = x^2 + 2 \quad -1 < x \le 0$

$\quad = x + 2 \quad 0 \le x \le 1$ [**Ans.** Not differentiable at $x = 1$, theorem is not applicable]

2. If c is a real constant, prove that the equation $x^3 - 12x + c = 0$ cannot have two distinct roots in the interval $[0, 4]$.

3. If $f(x) = a + b - 3bx^2 - 4ax^3$, $a \neq 0$, $b \neq 0$, prove that there exists at least one value c in $(0, 1)$ such that $f'(c) = 0$.

4. If c is a real constant, prove that the equation $x^3 + 3x + c = 0$ cannot have more than one real root.

5. Prove that one root of the equation $x \log x - 2 + x = 0$ lies in $(1, 2)$.

[**Hint :** Consider $f(x) = (x-2) \log x$]

6. Prove that the equation $\tan x = 1 - x$ has a real root in the interval $(0, 1)$.

1.12.1 L'Hospital's Rule

Statement : If $f(x)$ and $g(x)$ are two functions of x which can be expanded by Taylor's series in the neighbourhood of $x = a$ and

if $\lim_{x \to a} f(x) = f(a) = 0$, $\lim_{x \to a} g(x) = g(a) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Proof : Let $x = a + h$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{f(a+h)}{g(a+h)}$$

$$= \lim_{h \to 0} \frac{f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots}{g(a) + hg'(a) + \frac{h^2}{2!} g''(a) + \ldots\ldots} \quad \text{[By Taylor's theorem]}$$

$$= \lim_{h \to 0} \frac{hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots}{hg'(a) + \frac{h^2}{2!} g''(a) + \ldots\ldots} \quad [\because f(a) = 0, g(a) = 0]$$

$$= \lim_{h \to 0} \frac{f'(a) + \frac{h}{2!} f''(a) + \ldots\ldots}{g'(a) + \frac{h}{2!} g''(a) + \ldots\ldots}$$

$$= \frac{f'(a)}{g'(a)}$$

$$= \lim_{x \to a} \frac{f'(a)}{g'(a)}, \text{ provided } g'(a) \neq 0.$$

1.12.2 Standard Limits

Following standard limits can be used to solve the problems :

(1) $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (2) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

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(3) $\lim_{x\to 0} \cos x = 1$

(4) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log a$

(5) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(6) $\lim_{x\to 0} (1+x)^{1/x} = e$

(7) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$

(8) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1$

(9) $\lim_{x\to 0} \frac{\sinh x}{x} = 1$

(10) $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$

1.12.3 Type 1 : $\left(\frac{0}{0}\right)$

Problems under this type are solved by using L'Hospital's rule considering the fact that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \text{ if } \lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

EXAMPLE 1.55 : Evaluate $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$.

Solution : Let

$$l = \lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{e^x + xe^x - \frac{1}{1+x}}{2x} \quad \left[\frac{0}{0}\right] \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{e^x + e^x + xe^x + \frac{1}{(1+x)^2}}{2} \quad \text{[Applying L'Hospital's rule]}$$

$$= \frac{3}{2} \quad \textbf{Ans.}$$

EXAMPLE 1.56 : Evaluate $\lim_{y\to 0} \frac{\log(1+y^3)}{\sin^3 y}$.

Solution : Let

$$l = \lim_{y\to 0} \frac{\log(1+y^3)}{\sin^3 y} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{y\to 0} \frac{\frac{1}{1+y^3}.3y^2}{3\sin^2 y \cos y} \quad \text{[Applying L'Hospital's rule}$$

$$= \lim_{y\to 0} \left(\frac{y}{\sin y}\right)^2 \frac{1}{(1+y^3)\cos y} \quad \left[\lim_{y\to 0} \frac{y}{\sin y} = 1\right]$$

$$= 1 \quad \textbf{Ans.}$$

EXAMPLE 1.57 : Evaluate $\lim\limits_{x\to\frac{1}{2}} \dfrac{\cos^2 \pi x}{e^{2x} - 2xe}$.

Solution : Let $l = \lim\limits_{x\to\frac{1}{2}} \dfrac{\cos^2 \pi x}{e^{2x} - 2xe} \quad \left[\dfrac{0}{0}\right]$

$$= \lim_{x\to\frac{1}{2}} \frac{2\cos \pi x(-\pi \sin \pi x)}{2e^{2x} - 2e}$$ [Applying L'Hospital's rule]

$$= \lim_{x\to\frac{1}{2}} \frac{-\pi \sin 2\pi x}{2(e^{2x} - e)} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to\frac{1}{2}} \frac{-2\pi^2 \cos 2\pi x}{2.2e^{2x}}$$ [Applying L's Hospital rule]

$$= \frac{\pi^2}{2e}$$ **Ans.**

EXAMPLE 1.58 : Evaluate $\lim\limits_{x\to y} \dfrac{x^y - y^x}{x^x - y^y}$.

Solution : Let $l = \lim\limits_{x\to y} \dfrac{x^y - y^x}{x^x - y^y} \quad \left[\dfrac{0}{0}\right]$

$$= \lim_{x\to y} \frac{yx^{y-1} - y^x \log y}{x^x(1+\log x) - 0}$$ [Applying L's Hopital's rule]

$$= \frac{y^y - y^y \log y}{y^y(1+\log y)} = \frac{(1-\log y)}{(1+\log y)}$$ **Ans.**

EXAMPLE 1.59 : Evaluate $\lim\limits_{x\to 0} \dfrac{2^x - 1}{(1+x)^{1/2} - 1}$.

Solution : Let $l = \lim\limits_{x\to 0} \dfrac{2^x - 1}{(1+x)^{1/2} - 1} \quad \left[\dfrac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{2^x \log 2}{\frac{1}{2}(1+x)^{-\frac{1}{2}}} = 2 \log 2$$ **Ans.**

EXAMPLE 1.60 : Evaluate $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\sin(x \cos x)}{\cos(x \sin x)}$.

Solution : Let $l = \lim\limits_{x\to\frac{\pi}{2}} \dfrac{\sin(x\cos x)}{\cos(x \sin x)} \quad \left[\dfrac{0}{0}\right]$

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$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos(x\cos x)(\cos x - x\sin x)}{-\sin(x\sin x)(\sin x + x\cos x)}$$ [Applying L'sHospital rule]

$$= \frac{\pi}{2} \text{ Ans.}$$

EXAMPLE 1.61 : Prove that $\lim_{\theta\to\alpha} \frac{1-\cos(\theta-\alpha)}{(\sin\theta - \sin\alpha)^2} = \frac{1}{2}\sec^2\alpha.$

Solution : Let

$$l = \lim_{\theta\to\alpha} \frac{1-\cos(\theta-\alpha)}{(\sin\theta-\sin\alpha)^2} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{\theta\to\alpha} \frac{\sin(\theta-\alpha)}{2(\sin\theta-\sin\alpha)\cos\theta}$$ [Applying L's Hospital's rule]

$$= \lim_{\theta\to\alpha} \frac{\sin(\theta-\alpha)}{(\sin 2\theta - 2\sin\alpha\cos\theta)} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{\theta\to\alpha} \frac{\cos(\theta-\alpha)}{2\cos 2\theta + 2\sin\alpha\sin\theta}$$ [Applying L's Hospital's rule]

$$= \frac{\cos 0}{2\cos 2\alpha + 2\sin\alpha\sin\alpha}$$

$$= \frac{1}{2(1-2\sin^2\alpha)+2\sin^2\alpha} = \frac{1}{2-2\sin^2\alpha}$$

$$= \frac{1}{2\cos^2\alpha} = \frac{1}{2}\sec^2\alpha \quad \text{Ans.}$$

EXAMPLE 1.62 : Evaluate $\lim_{x\to 0} \frac{2x^2 - 2e^{x^2} + 2\cos(x^{3/2}) + \sin^3 x}{x^2}$.

Solution : Let

$$l = \lim_{x\to 0} \frac{2x^2 - 2e^{x^2} + 2\cos(x^{3/2}) + \sin^3 x}{x^2} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{4x - 2e^{x^2}(2x) - 2\sin x^{3/2}\left(\frac{3}{2}x^{1/2}\right) + 3\sin^2 x\cos x}{2x} \quad \left[\frac{0}{0}\right]$$

[Applying L's Hospital's rule]

$$= \lim_{x\to 0} \frac{4 - 4(e^{x^2} + xe^{x^2}.2x) - 3\left(\sqrt{x}\cos x^{3/2}.\frac{3}{2}x^{1/2} + \frac{1}{2\sqrt{x}}\sin x^{3/2}\right) + 6\sin x\cos^2 x - 3\sin^3 x}{2}$$

[Applying L'sHospital rule]

$$= \frac{4 - 4 - \lim_{x\to 0} \frac{\sin x^{3/2}}{2\sqrt{x}}.\frac{x}{x}}{2}$$

$$= \frac{-1}{2} \cdot \frac{1}{2} \lim_{x \to 0} \frac{\sin x^{3/2}}{x^{3/2}} \cdot x \qquad \left[\because \lim_{x \to 0} \frac{\sin x^{3/2}}{x^{3/2}} = 1\right]$$

$$= 0$$

EXAMPLE 1.63 : Evaluate $\lim_{x \to 0} \dfrac{\log_{\sec x} \cos \frac{x}{2}}{\log_{\sec \frac{x}{2}} \cos x}$.

Solution : Let

$$l = \lim_{x \to 0} \frac{\log_{\sec x} \cos \frac{x}{2}}{\log_{\sec \frac{x}{2}} \cos x}$$

$$= \lim_{x \to 0} \frac{\log \cos \frac{x}{2}}{\log \sec x} \cdot \frac{\log \cos \frac{x}{2}}{\log \cos x}$$

$$= \lim_{x \to 0} \frac{\log \cos \frac{x}{2}}{(-\log \cos x)} \cdot \frac{\left(-\log \cos \frac{x}{2}\right)}{\log \cos x}$$

$$= \lim_{x \to 0} \left(\frac{\log \cos \frac{x}{2}}{\log \cos x}\right)^2 \qquad \left[\frac{0}{0}\right]$$

Now,
$$\lim_{x \to 0} \frac{\log \cos \frac{x}{2}}{\log \cos x} = \lim_{x \to 0} \frac{\frac{1}{\cos \frac{x}{2}} \cdot \left(-\frac{1}{2} \sin \frac{x}{2}\right)}{\frac{1}{\cos x}(-\sin x)} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x \to 0} \frac{\tan \frac{x}{2}}{2 \tan x}$$

$$= \lim_{x \to 0} \frac{1}{4} \left(\frac{\tan \frac{x}{2}}{\frac{x}{2}}\right) \cdot \left(\frac{x}{\tan x}\right)$$

$$= \frac{1}{4} \qquad \left[\because \lim_{x \to 0} \frac{\tan x}{x} = 1\right]$$

$$\lim_{x \to 0} \left(\frac{\log \cos \frac{x}{2}}{\log \cos x}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}.$$

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EXAMPLE 1.64 : Evaluate $\lim\limits_{x\to 0} \dfrac{x^{1/2}\tan x}{(e^x-1)^{3/2}}$.

Solution : Let

$$l = \lim_{x\to 0} \frac{\sqrt{x}\tan x}{(e^x-1)^{3/2}} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{x\sqrt{x}}{(e^x-1)^{3/2}} \cdot \frac{\tan x}{x}$$

$$= \lim_{x\to 0} \frac{x\sqrt{x}}{(e^x-1)^{3/2}} \cdot \lim_{x\to 0}\left(\frac{\tan x}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{x}{e^x-1}\right)^{3/2} \qquad \left[\because \lim_{x\to 0}\frac{\tan x}{x} . = 1\right]$$

Now, $\lim\limits_{x\to 0}\dfrac{x}{e^x-1} = \lim\limits_{x\to 0}\dfrac{1}{e^x} = 1$ [Applying L's Hospital's rule]

Hence, $\lim\limits_{x\to 0}\left(\dfrac{x}{e^x-1}\right)^{3/2} = (1)^{3/2} = 1$

EXAMPLE 1.65 : Prove that $\lim\limits_{x\to 0}\dfrac{(1+x)^{1/x2}-e}{x} = -\dfrac{e}{2}$.

Solution : Let

$$l = \lim_{x\to 0}\frac{(1+x)^{1/x2}-e}{x} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{e^{\frac{1}{x}\log(1+x)}-e}{x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{e^{\frac{1}{x}\log(1+x)}\left[-\frac{1}{x^2}\log(1+x)+\frac{1}{x(1+x)}\right]}{1} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}(1+x)^{1/x}\lim_{x\to 0}\frac{[-\{\log(1+x)\}(1+x)+x]}{x^2(1+x)} \quad \left[\frac{0}{0}\right]$$

$$= e\lim_{x\to 0}\left[\frac{-\log(1+x)-1+1}{2x+3x^2}\right] \quad \left[\frac{0}{0}\right] \qquad \text{[Applying L'Hospital's rule]}$$

$$= e\lim_{x\to 0}\left(\frac{-\frac{1}{1+x}}{2+6x}\right) = -\frac{e}{2}.$$

EXAMPLE 1.66 : Prove that $\lim\limits_{x\to 0}\dfrac{(\sqrt{1-x}-1)^{2n}}{(1-\cos x)^n} = 2^{-n}$.

Solution : Let
$$l = \lim_{x\to 0} \frac{(\sqrt{1-x}-1)^{2n}}{(1-\cos x)^n} \cdot \frac{(\sqrt{1-x}+1)^{2n}}{(\sqrt{1-x}+1)^{2n}}$$

$$= \lim_{x\to 0} \frac{(1-x-1)^{2n}}{\left(2\sin^2\frac{x}{2}\right)^n \left(\sqrt{1-x}+1\right)^{2n}}$$

$$= \lim_{x\to 0} \frac{(-x)^{2n}}{2^n\left(\sin\frac{x}{2}\right)^{2n}\left(\sqrt{1-x}+1\right)^{2n}} \cdot \frac{2^n}{2^n}$$

$$= \lim_{x\to 0} \left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right)^{2n} \frac{2^n}{\left(\sqrt{1-x}+1\right)^{2n}} \qquad \left[\because (-x)^{2n} = \left\{(-x)^2\right\}^n = x^{2n}\right]$$

$$= \frac{1}{2^n}.$$

EXAMPLE 1.67 : If $\lim\limits_{x\to 0} \dfrac{\sin 2x + p\sin x}{x^3}$ is finite, find the value of p and hence, the limit.

Solution : Let
$$l = \lim_{x\to 0} \frac{\sin 2x + p\sin x}{x^3}, \text{ where } l \text{ is finite}$$

$$l = \lim_{x\to 0} \frac{\sin 2x + p\sin x}{x^3} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{2\cos 2x + p\cos x}{3x^2} = \frac{2+p}{0} \qquad \text{[Applying L'Hospital's rule]}$$

But limit is finite, therefore, numerator must be zero.

$$2+p=0, \quad p=-2$$

Thus,
$$l = \lim_{x\to 0} \frac{2\cos 2x - 2\cos x}{3x^2} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-4\sin 2x + 2\sin x}{6x} \quad \left[\frac{0}{0}\right] \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{-8\cos 2x + 2\cos x}{6}$$

$$= -1$$

Hence,
$$p = -2 \text{ and } l = -1$$

EXAMPLE 1.68 : Find the values of a and b such that $\lim\limits_{x\to 0} \dfrac{a\sin 2x + b\log\cos x}{x^4} = \dfrac{1}{2}$.

Solution :
$$\frac{1}{2} = \lim_{x\to 0} \frac{a\sin 2x + b\log\cos x}{x^4}$$

$$= \lim_{x\to 0} \frac{a\sin 2x + b\log\cos x}{x^4}$$

$$= \lim_{x\to 0} \frac{a.2\sin x\cos x + b.\frac{1}{\cos x}(-\sin x)}{4x^3} \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{a\sin 2x - b\tan x}{4x^3} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{2a\cos 2x - b\sec^2 x}{12x^2} \quad \text{[Applying L'Hospital's rule]}$$

$$= \frac{2a-b}{0}$$

But limit is finite, therefore, numerator must be zero.

$$2a - b = 0$$
$$b = 2a$$

Thus, $$\frac{1}{2} = \lim_{x\to 0} \frac{2a\cos 2x - 2a\sec^2 x}{12x^2} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-4a\sin 2x - 4a\sec^2 x\tan x}{24x} \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \left(\frac{-a}{3}\frac{\sin 2x}{2x} - \frac{a}{6}\sec^2 x.\frac{\tan x}{x}\right)$$

$$\frac{1}{2} = -\frac{a}{3} - \frac{a}{6} = -\frac{a}{2} \quad \left[\because \lim_{x\to 0}\frac{\sin 2x}{2x} = 1 \text{ and } \lim_{x\to 0}\frac{\tan x}{x} = 1\right]$$

Hence, $a = -1$, $b = -2$.

EXAMPLE 1.69 : Find a and b if $\lim\limits_{x\to 0} \dfrac{a\sin x - \sin 2x}{\tan^3 x} = b$.

Solution : $$b = \lim_{x\to 0} \frac{a\sin x - \sin 2x}{\tan^3 x} = \lim_{x\to 0} \frac{a\sin x - \sin 2x}{x^3\left(\frac{\tan x}{x}\right)^3}$$

$$= \lim_{x\to 0} \frac{a\sin x - \sin 2x}{x^3} \quad \left[\frac{0}{0}\right] \quad \left[\because \lim_{x\to 0}\frac{\tan x}{x} = 1\right]$$

$$= \lim_{x\to 0} \frac{a\cos x - 2\cos 2x}{3x^2} \quad \text{[Applying L'Hospital's rule]}$$

$$= \frac{a-2}{0}$$

But limit is finite, therefore, numerator must be zero.

$$a - 2 = 0, \quad a = 2$$

Thus, $$b = \lim_{x\to 0} \frac{2\cos x - 2\cos 2x}{3x^2} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-2\sin x + 4\sin 2x}{6x}$$ [Applying L'Hospital's rule]

$$= \lim_{x\to 0} \left[-\frac{2}{6}\left(\frac{\sin x}{x}\right) + \frac{4}{3}\left(\frac{\sin 2x}{2x}\right)\right]$$

$$= -\frac{2}{6} + \frac{4}{3} = 1$$ $\left[\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right]$

Hence, $a = 2$, $b = 1$.

EXAMPLE 1.70 : Find a, b, c if $\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$.

Solution :

$$2 = \lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x}$$

$$= \lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x.x\left(\frac{\sin x}{x}\right)}$$

$$= \lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x^2}$$ $\left[\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right]$

$$= \frac{a - b + c}{0}$$

But limit is finite, therefore, numerator must be zero.

$$a - b + c = 0 \qquad \text{...(1)}$$

Thus, $$2 = \lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x^2} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{ae^x + b\cos x - ce^{-x}}{2x}$$ [Applying L'Hospital's rule]

$$= \frac{a - c}{0}$$

But limit is finite, therefore, numerator must be zero.

$$a - c = 0 \quad \text{... (2)} \qquad a = 0$$

Thus, $$2 = \lim_{x\to 0} \frac{ae^x + b\sin x - ae^{-x}}{2x} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{ae^x + b\cos x + ae^{-x}}{2}$$ [Applying L'Hospital's rule]

$$= \frac{a + b + a}{2}$$

$$2a + b = 4 \qquad \text{...(3)}$$

From Eqs. (1) and (2), we have

$$2a - b = 0 \qquad \text{...(4)}$$

Solving Eqs. (3) and (4),

$$a = 1,\ b = 2 \text{ and } c = 1$$

EXERICSE 1.5

1. Prove that $\lim\limits_{x\to 0}\dfrac{\log(1-x^2)}{\log\cos x}=2$.

2. Prove that $\lim\limits_{x\to a}\dfrac{x^2\log a-a^2\log x}{x^2-a^2}=\log a-\dfrac{1}{2}$.

3. Prove that $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{1-\tan x}{1-\sqrt{2}\sin x}=2=2$.

4. Prove that $\lim\limits_{x\to 0}\dfrac{e^x+\log\left(\dfrac{1-x}{e}\right)}{\tan x-x}=-\dfrac{1}{2}$.

5. Prove that $\lim\limits_{x\to 0}\dfrac{e^x-\sqrt{1+2x}}{\log(1+x^2)}=1$.

6. Prove that $\lim\limits_{x\to 0}\dfrac{x^2+2\cos x-2}{x\sin x}=0$.

7. Prove that $\lim\limits_{x\to 0}\dfrac{6\sin x-6x+x^3}{2x^2\log(1+x)-2x^3+x^4}=\dfrac{3}{40}$.

8. Evaluate $\lim\limits_{x\to 0}\dfrac{\sqrt{a+x}\tan^{-1}\sqrt{a^2-x^2}}{\sqrt{a-x}}$.

$\left[\textbf{Hint: }\lim\limits_{x\to 0}(x+a)\dfrac{\tan^{-1}\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\text{ as } x\to a,\ a-x\to 0\right]$

9. Find the values of a and b, such that $\lim\limits_{x\to 0}\dfrac{a\sin 2x-b\tan x}{x^3}=1$.

10. Find a and b if $\dfrac{x(1+a\cos x)-b\sin x}{x^3}=1$.

11. Find the values of a, b and c so that $\lim\limits_{x\to 0}\dfrac{x(a+b\cos x)-c\sin x}{x^3}=1$.

12. Find the values of a and b so that $\lim\limits_{x\to 0}\dfrac{x(1+a\cos x)+b\sin x}{x^3}=\dfrac{1}{3}$.

13. Find the values of a, b and c such that $\lim\limits_{x\to 0}\dfrac{ae^x-be^{-x}-cx}{x-\sin x}=4$.

14. Evaluate $\lim\limits_{x\to 0}\dfrac{e^x+\log_e\dfrac{1-x}{e}}{\tan x-x}$.

15. Evaluate $\lim_{x\to 1}\dfrac{1-x+\log x}{1-\sqrt{2x-x^2}}$.

16. Prove that $\lim_{x\to 0}\dfrac{xe^x-\log(1+x)}{x^2}=\dfrac{3}{2}$.

17. Prove that $\lim_{x\to 0}\dfrac{e^x-\sqrt{1+2x}}{\log(1+x^2)}=1$.

18. Prove that $\lim_{x\to\frac{\pi}{4}}\dfrac{1-\tan x}{1-\sqrt{2}\sin x}=2$.

19. Prove that $\lim_{x\to 3}\dfrac{\sqrt{3x}-\sqrt{12-x}}{2x-3\sqrt{19-5x}}=\dfrac{8}{69}$.

20. Prove that $\lim_{x\to 1}\dfrac{a^{\log x}-x}{\log x}=\log\dfrac{a}{e}$.

21. Prove that $\lim_{x\to a}\dfrac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}=\dfrac{1}{\sqrt{2a}}$.

22. Prove that $\lim_{x\to 0}\dfrac{\tan x-x}{x^3}=\dfrac{1}{3}$.

ANSWERS

8. $2a$ **9.** $a=-\dfrac{1}{2}, b=-1$ **10.** $a=-\dfrac{5}{2}, b=-\dfrac{3}{2}$ **11.** $a=0, b=-3, c=-3$

12. $a=\dfrac{1}{2}, b=-\dfrac{1}{2}$ **13.** $a=2, b=2, c=4$ **14.** $-\dfrac{1}{2}$ **15.** -1

1.12.4 Type 2 : $\left(\dfrac{\infty}{\infty}\right)$

Problems under this type are also solved by using L'Hospital's rule considering the fact that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \text{ if } \lim_{x\to a}f(x)=\infty \text{ and } \lim_{x\to a}g(x)=\infty$$

EXAMPLE 1.71 : Prove that $\lim_{x\to\frac{\pi}{2}}\dfrac{\log\left(x-\frac{\pi}{2}\right)}{\tan x}=0.$

Solution : Let

$$l=\lim_{x\to\frac{\pi}{2}}\frac{\log\left(x-\frac{\pi}{2}\right)}{\tan x}\quad\left[\frac{\infty}{\infty}\right]$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\dfrac{1}{x-\frac{\pi}{2}}}{\sec^2 x}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\cos^2 x}{x-\frac{\pi}{2}}\quad\left[\frac{0}{0}\right]$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{2\cos x(-\sin x)}{1} = 0$$ [Applying L'Hospital's rule]

EXAMPLE 1.72 : Prove that $\lim\limits_{x\to\infty} \dfrac{\sinh^{-1} x}{\cosh^{-1} x} = 1.$

Solution : Let $$l = \lim_{x\to\infty} \frac{\sinh^{-1} x}{\cosh^{-1} x}$$

$$= \lim_{x\to\infty} \frac{\log\left(x+\sqrt{x^2+1}\right)}{\log\left(x+\sqrt{x^2-1}\right)} \quad \left[\frac{\infty}{\infty}\right]$$

$$l = \lim_{x\to\infty} \frac{\dfrac{1}{\left(x+\sqrt{x^2+1}\right)}\cdot\left(1+\dfrac{1}{2\sqrt{x^2+1}}.2x\right)}{\dfrac{1}{\left(x+\sqrt{x^2-1}\right)}\cdot\left(1+\dfrac{1}{2\sqrt{x^2+1}}.2x\right)}$$ [Applying L'Hospital's rule]

$$= \lim_{x\to\infty} \frac{\dfrac{\sqrt{x^2+1}+x}{\left(x+\sqrt{x^2+1}\right)\sqrt{x^2+1}}}{\dfrac{\sqrt{x^2-1}+x}{\left(x+\sqrt{x^2-1}\right)\sqrt{x^2-1}}} = \lim_{x\to\infty} \frac{\sqrt{x^2-1}}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{\sqrt{1-\dfrac{1}{x^2}}}{\sqrt{1+\dfrac{1}{x^2}}} = 1$$

EXAMPLE 1.73 : Prove that $\lim\limits_{x\to a} \dfrac{\log(x-a)}{\log(a^x-a^a)} = 1.$

Solution : Let $$l = \lim_{x\to a} \frac{\log(x-a)}{\log(a^x-a^a)} \quad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to a} \frac{\dfrac{1}{(x-a)}}{\dfrac{1}{a^x-a^a}.a^x\log a}$$ [Applying L'Hospital's rule]

$$= \lim_{x\to a}\left(\frac{a^x-a^a}{x-a}\right).\lim_{x\to a}\frac{1}{a^x\log a}$$

$$= \lim_{x\to a} \frac{a^x\log a}{1}\cdot\frac{1}{a^a\log a}$$ [Applying L'Hospital's rule for first term]

$$= a^a\log a.\frac{1}{a^a\log a} = 1$$

EXAMPLE 1.74 : Prove that $\lim_{x\to\infty} \log_x \sin x = 1$.

Solution : Let

$$l = \lim_{x\to\infty} \log_x \sin x$$

$$= \lim_{x\to\infty} \frac{\log \sin x}{\log x} \quad \left[\frac{\infty}{\infty}\right]$$ [Change of base property]

$$= \lim_{x\to 0} \frac{\frac{1}{\sin x}.\cos x}{\frac{1}{x}}$$ [Applying L'Hospital's rule]

$$= \lim_{x\to 0} \frac{x}{\sin x}.\lim_{x\to 0} \cos x$$

$$= 1$$ $\left[\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right]$

EXAMPLE 1.75 : Prove that $\lim_{x\to\infty} \dfrac{e^{\frac{1}{x}} + e^{\frac{2}{x}} + e^{\frac{3}{x}} + ... + e^{\frac{x}{x}}}{x} = e - 1$.

Solution : Let

$$l = \lim_{x\to\infty} \frac{e^{\frac{1}{x}} + e^{\frac{2}{x}} + e^{\frac{3}{x}} + ... + e^{\frac{x}{x}}}{x}$$

$$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}}\left[1 - \left(e^{\frac{1}{x}}\right)^x\right]}{1 - e^{\frac{1}{x}}} . \frac{1}{x}$$ [Sum of G.P.]

$$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}}(e-1)}{e^{\frac{1}{x}} - 1} . \frac{1}{x}$$

Putting $\frac{1}{x} = y$, when $x \to \infty$, $y \to 0$

$$l = \lim_{y\to 0} \frac{(e-1)e^y y}{e^y - 1} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{y\to 0} \frac{(e-1)(ye^y + e^y)}{e^y}$$ [Applying L'Hospital's rule]

$$= e - 1$$

EXAMPLE 1.76 : Prove that $\lim_{x\to\infty} \dfrac{x^n}{e^{kx}} = 0$.

Solution : Let

$$l = \lim_{x\to\infty} \frac{x^n}{e^{kx}} \quad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty} \frac{nx^{n-1}}{ke^{kx}} \quad \left[\frac{\infty}{\infty}\right]$$ [Applying L'Hospital's rule]

$$= \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{k^2 e^{kx}} \quad \left[\frac{\infty}{\infty}\right] \quad \text{[Applying L'Hospital's rule]}$$

Applying L'Hospital's rule n times,

$$l = \lim_{x\to\infty} \frac{n(n-1)(n-2)\ldots.2.1}{k^n e^{kx}} = \lim_{x\to\infty} \frac{n!}{k^n e^{kx}} = 0 \qquad \left[\because \lim_{x\to\infty} e^{kx} = \infty\right]$$

EXAMPLE 1.77 : Prove that $\lim\limits_{x\to\infty} \dfrac{1^2+2^2+3^2+\ldots..+x^2}{x^3} = \dfrac{1}{3}$.

Solution : Let

$$l = \lim_{x\to\infty} \frac{1^2+2^2+3^2+\ldots..+x^2}{x^3}$$

$$= \lim_{x\to\infty} \frac{x(x+1)(2x+1)}{6x^3} \qquad \left[\because \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{x\to\infty} \frac{2x^3+3x^2+x}{6x^3} = \lim_{x\to\infty} \frac{2+\frac{3}{x}+\frac{1}{x^2}}{6}$$

$$= \frac{2}{6} = \frac{1}{3}.$$

EXAMPLE 1.78 : Prove that $\lim\limits_{x\to\infty} \dfrac{e^x}{\left[\left(1+\frac{1}{x}\right)^x\right]^x} = e^{1/2}$.

Solution : Let

$$l = \lim_{x\to\infty} \frac{e^x}{\left[\left(1+\frac{1}{x}\right)^x\right]^x} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty} \frac{e^x}{\left(1+\frac{1}{x}\right)^{x^2}}$$

Taking logarithm on both the sides,

$$\log l = \lim_{x\to\infty} \left[\log e^x - \log\left(1+\frac{1}{x}\right)^{x^2}\right] = \lim_{x\to\infty} \left[x - x^2 \log\left(1+\frac{1}{x}\right)\right]$$

$$= \lim_{x\to\infty} x^2 \left[\frac{1}{x} - \log\left(1+\frac{1}{x}\right)\right] = \lim_{x\to\infty} \frac{\frac{1}{x} - \log\left(1+\frac{1}{x}\right)}{\frac{1}{x^2}} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to\infty} \frac{-\frac{1}{x^2} - \frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{2}{x^3}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to\infty} \frac{1-\dfrac{1}{1+\dfrac{1}{x}}}{\dfrac{2}{x}}$$

Hence, $l = e^{1/2}$

EXERICSE 1.6

1. Prove that $\lim_{x\to\infty} \frac{\log x}{\cot x} = 0.$

2. Prove that $\lim_{x\to\infty} \frac{\log x}{x^n} = 0\ (n > 0).$

3. Prove that $\lim_{x\to 0} \frac{\log_{\sin x} \cos x}{\log_{\sin\frac{x}{2}} \cos\frac{x}{2}} = 4.$

4. Prove that $\lim_{x\to 0} \log_{\tan x} \tan 2x = 1.$

5. Prove that $\lim_{x\to 1} \frac{\log(1-x)}{\cot \pi x} = 0.$

6. Prove that $\lim_{x\to\infty} \frac{\log(1+e^{3x})}{x} = 3.$

7. Prove that $\lim_{x\to 0} \log_{\sin x} \sin 2x = 1.$

8. Prove that $\lim_{x\to\infty} \frac{x^m}{e^x} = 0\ (m > 0).$

9. Prove that $\lim_{x\to 0} \frac{\log x^2}{\cot x^2} = 0.$ **[Hint : Put $x^2 = y$]**

10. Prove that $\lim_{n\to\infty} \frac{\left(\frac{1}{e}\right)+\left(\frac{1}{e}\right)^2+\left(\frac{1}{e}\right)^3+\left(\frac{1}{e}\right)^4+\left(\frac{1}{e}\right)^5+\dots\dots\dots+\left(\frac{1}{e}\right)^n}{n} = 0$

1.12.5 Type 3 : $(0 \times \infty)$

To solve the problems of the type

$$\lim_{x\to a} [f(x)\,.\,g(x)], \text{ when } \lim_{x\to a} f(x) = 0,\ \lim_{x\to a} g(x) = \infty \text{ (i.e., } 0\times\infty \text{ form)}$$

We write $\lim_{x\to a} f(x)\,.\,g(x) = \lim_{x\to a} \frac{f(x)}{\dfrac{1}{g(x)}}$ or $\lim_{x\to a} \frac{g(x)}{\dfrac{1}{f(x)}}$.

These new forms are of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ respectively, which can be solved using L'Hospital's rule.

EXAMPLE 1.79 : Prove that $\lim\limits_{x\to 0} \sin x \log x = 0$.

Solution : Let

$$l = \lim_{x\to 0} \sin x \log x \qquad [0\times\infty]$$

$$= \lim_{x\to 0} \frac{\log x}{\operatorname{cosec} x} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= -\lim_{x\to 0} \sin x . \frac{\tan x}{x}$$

$$= -\lim_{x\to 0} \sin x . \lim_{x\to 0} \frac{\tan x}{x}$$

$$= 0 \qquad \left[\because \lim_{x\to 0} \frac{\tan x}{x} = 1\right]$$

EXAMPLE 1.80 : $\lim\limits_{x\to\infty} 2^x . \sin\left(\frac{a}{2^x}\right) = a$.

Solution : Let

$$l = \lim_{x\to\infty} 2^x . \sin\left(\frac{a}{2^x}\right)$$

Taking $2x = \frac{1}{t}, \quad t = \frac{1}{2^x},$

When $x \to \infty, \; 2x \to \infty, \; t \to 0$

$$l = \lim_{t\to 0} \frac{\sin at}{t} = \lim_{t\to 0} \frac{a \sin at}{at} = a . 1 = a \qquad \left[\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right]$$

EXAMPLE 1.81 : Prove that $\lim\limits_{x\to\infty} \left(a^{1/x} - 1\right) x = \log a$.

Solution : Let

$$l = \lim_{x\to\infty} \left(a^{1/x} - 1\right) . x \qquad [0\times\infty]$$

$$= \lim_{x\to\infty} \frac{\left(a^{1/x} - 1\right)}{1/x} \qquad \left[\frac{0}{0}\right]$$

Taking $\frac{1}{x} = t$, when $x \to \infty, t \to 0$

$$l = \lim_{t\to 0} \frac{a^t - 1}{t} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{t\to 0} \frac{a^t \log a}{1} \qquad \text{[Applying L'Hospital's rule]}$$

$$= a^0 \log a = \log a$$

EXAMPLE 1.82 : $\lim\limits_{x\to 1} \tan^2\left(\frac{\pi x}{2}\right)(1+\sec \pi x) = -2.$

Solution : Let

$$l = \lim_{x\to 1} \tan^2\left(\frac{\pi x}{2}\right)(1+\sec \pi x) \qquad [\infty \times 0]$$

$$= \lim_{x\to 1} \frac{1+\sec \pi x}{\cot^2\left(\frac{\pi x}{2}\right)} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 1} \frac{\pi \sec \pi x \tan \pi x}{2\cot\left(\frac{\pi x}{2}\right)\left(-\operatorname{cosec}^2 \frac{\pi x}{2}\right)\frac{\pi}{2}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= -\left(\lim_{x\to 1} \frac{\sec \pi x}{\operatorname{cosec}^2 \frac{\pi x}{2}}\right)\left(\lim_{x\to 1} \frac{\tan \pi x}{\cot \frac{\pi x}{2}}\right)$$

$$= -\left(\frac{\sec \pi}{\operatorname{cosec}^2 \frac{\pi}{2}}\right) \lim_{x\to 1} \frac{\tan \pi x}{\cot \frac{\pi x}{2}} \qquad \left[\frac{0}{0}\right]$$

$$= -(-1) \lim_{x\to 1} \frac{\pi \sec^2 \pi x}{\left(-\operatorname{cosec}^2 \frac{\pi x}{2}\right)\frac{\pi}{2}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= -2\frac{\sec^2 \pi}{\operatorname{cosec}^2 \frac{\pi}{2}} = -2$$

EXAMPLE 1.83 : $\lim\limits_{x\to a} \sin^{-1}\sqrt{\frac{a-x}{a+x}} \operatorname{cosec}\sqrt{a^2-x^2} = \frac{1}{2a}.$

Solution : Let

$$l = \lim_{x\to a} \sin^{-1}\sqrt{\frac{a-x}{a+x}} \operatorname{cosec}\sqrt{a^2-x^2} \qquad [0 \times \infty]$$

$$= \lim_{x\to a} \frac{\sin^{-1}\sqrt{\frac{a-x}{a+x}}}{\sin\sqrt{a^2-x^2}}$$

Here applying L'Hospital's rule will make the expression complicated, so we rear-range the terms to apply the limits directly.

Let $\sqrt{\frac{a-x}{a+x}} = \alpha, \sqrt{a^2-x^2} = \beta$

When $x \to a$, $\alpha \to 0$ and $\beta \to 0$

Hence,

$$l = \lim_{\alpha\to 0} \sin^{-1}\alpha \lim_{\beta\to 0} \frac{1}{\sin\beta}$$

$$= \left[\lim_{\alpha\to 0}\left(\frac{\sin^{-1}\alpha}{\alpha}\right).\alpha\right]\lim_{\beta\to 0}\left(\frac{\beta}{\sin\beta}\right).\frac{1}{\beta}$$

$$= \lim_{\alpha\to 0}\alpha\,.\,\lim_{\beta\to 0}\frac{1}{\beta} \qquad \left[\because \lim_{x\to 0}\left(\frac{\sin^{-1}x}{x}\right)=1 \text{ and } \lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1\right]$$

$$= \lim_{x\to 0}\sqrt{\frac{a-x}{a+x}}\,.\,\frac{1}{\sqrt{a^2-x^2}} \qquad \text{[Resubstituting } \alpha \text{ and } \beta\text{]}$$

$$= \lim_{x\to a}\sqrt{\frac{a-x}{a+x}}\,.\,\frac{1}{\sqrt{a+x}\,\sqrt{a-x}}$$

$$= \lim_{x\to a}\frac{1}{a+x}=\frac{1}{2a}$$

EXAMPLE 1.84 : Evaluate $\lim_{x\to 0} x^m (\log x)^n$, where m and n are positive integers.

Solution : Let

$$l = \lim_{x\to 0} x^m (\log x)^n \qquad [0\times\infty]$$

$$= \lim_{x\to 0}\frac{(\log x)^n}{\frac{1}{x^m}} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0}\frac{n(\log x)^{n-1}\frac{1}{x}}{-m(x)^{-m-1}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{(-1)^1 n\,(\log x)^{n-1}}{m(x)^{-m}} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0}\frac{(-1)^1 n\,(n-1)(\log x)^{n-2}\,.\frac{1}{x}}{m(-m)^1(x)^{-m-1}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{(-1)^2 n\,(n-1)(\log x)^{n-2}}{m^2(x)^{-m}} \qquad \left[\frac{\infty}{\infty}\right]$$

Applying L'Hospital's rule $(n-2)$ times in the above expression,

$$l = \lim_{x\to 0}\frac{(-1)^2 n!\,(\log x)^0}{m^n(x)^{-m}}$$

$$= \lim_{x\to 0}\frac{(-1)^n n!}{m^n}\,.\,x^m = 0$$

EXERICSE 1.7

1. Prove that $\lim\limits_{x\to\infty} x^2 e^{-x} = 0.$

2. Prove that $\lim\limits_{x\to 0} x \log x = 0.$

3. Prove that $\lim\limits_{x\to 1}(x^2-1)\tan\left(\dfrac{\pi x}{2}\right) = -\dfrac{4}{\pi}.$

4. Prove that $\lim\limits_{x\to\infty} x^2\left(1-e^{\frac{-2gy}{x^2}}\right) = 2gy.$

5. Prove that $\lim\limits_{x\to 0} \tan x \log x = 0.$

6. Prove that $\log\left(2-\dfrac{x}{a}\right)\cot(x-a) = -\dfrac{1}{a}.$

7. Prove that $\lim\limits_{x\to 1}(1+\sec \pi x)\tan\dfrac{\pi x}{2} = 0.$

8. Prove that $\lim\limits_{x\to 0} \log\left(\dfrac{1+x}{1-x}\right)\cot x = 2.$

9. Prove that $\lim\limits_{x\to 1} \log(1-x)\cot\left(\dfrac{\pi x}{2}\right) = 0.$

10. Prove that $\lim\limits_{x\to a}\sqrt{\dfrac{a+x}{a-x}}\tan^{-1}\sqrt{a^2-x^2} = 2a$

1.12.6 Type 4 : $(\infty-\infty)$

To evaluate the limits of the type $\lim\limits_{x\to a}[f(x)-g(x)]$, when $\lim\limits_{x\to a} f(x)=\infty$ and $\lim\limits_{x\to a} g(x)=\infty$ [i.e., $(\infty-\infty)$ form], we reduce the expression in the form of $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ by taking LCM or by rearranging the terms and then applying L'Hospital's rule.

EXAMPLE 1.85 : Prove that $\lim\limits_{x\to\infty}(\cos h^{-1} x - \log x) = \log 2.$

Solution : Let

$$l = \lim_{x\to\infty}(\cos h^{-1} x - \log x) \quad [\infty-\infty]$$

$$= \lim_{x\to\infty}\left[\log\left(x+\sqrt{x^2-1}\right)-\log x\right]$$

$$= \lim_{x\to\infty} \log\left(\frac{x+\sqrt{x^2-1}}{x}\right)$$

$$= \lim_{x\to\infty} \log\left(1+\sqrt{1-\frac{1}{x^2}}\right)$$

$$= \log\left(1+\sqrt{1-0}\right) = \log 2$$

EXAMPLE 1.86 : Prove that $\lim\limits_{x\to 1}\left(\dfrac{1}{\log x}-\dfrac{x}{x-1}\right) = -\dfrac{1}{2}.$

Solution : Let

$$l = \lim_{x\to 1}\left(\frac{1}{\log x}-\frac{x}{x-1}\right) \qquad [\infty-\infty]$$

$$= \lim_{x\to 1}\left[\frac{x-1-x\log x}{(x-1)\log x}\right] \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 1}\frac{1-x.\frac{1}{x}-\log x}{(x-1).\frac{1}{x}+\log x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 1}\frac{-\log x}{1-\frac{1}{x}+\log x} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 1}\frac{-\frac{1}{x}}{\frac{1}{x^2}+\frac{1}{x}} = -\frac{1}{2} \qquad \text{[Applying L'Hospital's rule}$$

EXAMPLE 1.87 : Prove that $\lim\limits_{x\to 0}\left(\dfrac{a}{x}-\cot\dfrac{x}{a}\right) = 0.$

Solution : Let

$$l = \lim_{x\to 0}\left(\frac{a}{x}-\cot\frac{x}{a}\right) \qquad [\infty-\infty]$$

Taking $\dfrac{x}{a} = y$, when $x\to 0, y\to 0$

$$l = \lim_{y\to 0}\left(\frac{1}{y}-\cot y\right)$$

$$= \lim_{y\to 0}\left(\frac{1}{y}-\frac{1}{\tan y}\right) \qquad [\infty-\infty]$$

$$= \lim_{y\to 0}\left(\frac{\tan y - y}{y\tan y}\right) \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{y\to 0}\left(\frac{\tan y - y}{y^2}\right) \cdot \lim_{y\to 0}\frac{1}{\left(\frac{\tan y}{y}\right)}$$

$$= \lim_{y\to 0}\frac{\tan y - y}{y^2} \cdot 1 \qquad \left[\frac{0}{0}\right] \qquad \left[\because \lim_{y\to 0}\frac{\tan y}{y} = 1\right]$$

$$= \lim_{y\to 0}\frac{\sec^2 y - 1}{2y} \qquad \left[\frac{0}{0}\right] \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{y\to 0}\frac{2\sec y.\sec y\tan y}{2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= 0$$

EXAMPLE 1.88 : Prove that $\lim_{x\to 0}\left[\frac{1}{2x} - \frac{1}{x(e^{\pi x}+1)}\right] = \frac{\pi}{4}$.

Solution : Let

$$l = \lim_{x\to 0}\left[\frac{1}{2x} - \frac{1}{x(e^{\pi x}+1)}\right] \qquad [\infty - \infty]$$

$$= \lim_{x\to 0}\frac{e^{\pi x}+1-2}{2x(e^{\pi x}+1)} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{\pi e^{\pi x}}{2[(e^{\pi x}+1) + x(\pi e^{\pi x})]}$$

$$= \frac{\pi}{2}\frac{e^0}{(e^0+1)} = \frac{\pi}{4}$$

EXAMPLE 1.89 : Prove that $\lim_{x\to\infty}\left(x+\frac{1}{2}\right)\left[\log\left(x+\frac{1}{2}\right) - \log x\right] = \frac{1}{2}$.

Solution : Let

$$l = \lim_{x\to\infty}\left(x+\frac{1}{2}\right)\left[\log\left(x+\frac{1}{2}\right) - \log x\right] = \frac{1}{2}$$

$$= \lim_{x\to\infty}\left(x+\frac{1}{2}\right)\log\left(\frac{x+\frac{1}{2}}{x}\right)$$

$$= \lim_{x\to\infty}\left[x\log\left(1+\frac{1}{2x}\right) + \frac{1}{2}\log\left(1+\frac{1}{2x}\right)\right]$$

$$= \lim_{x\to\infty}\frac{1}{2}\log\left(1+\frac{1}{2x}\right)^{2x} + \frac{1}{2}\lim_{x\to\infty}\log\left(1+\frac{1}{2x}\right)$$

$$= \frac{1}{2}\log e + \frac{1}{2}\log 1 \qquad \left[\because \lim_{x\to\infty}\left(1+\frac{1}{ax}\right)^{ax} = e\right]$$

$$= \frac{1}{2}.$$

EXAMPLE 1.90 : If $\lim_{x\to 0}\left(\frac{a\cot x}{x}+\frac{b}{x^2}\right)=\frac{1}{3}$**, find a and b.**

Solution :

$$\frac{1}{3} = \lim_{x\to 0}\left(\frac{a\cot x}{x}+\frac{b}{x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{a}{x\tan x}+\frac{b}{x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{ax+b\tan x}{x^2\tan x}\right) \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{(ax+b\tan x)}{(x^2 . x)\left(\frac{\tan x}{x}\right)}$$

$$= \lim_{x\to 0}\frac{ax+b\tan x}{x^3} . \lim_{x\to 0}\left(\frac{x}{\tan x}\right)$$

$$= \lim_{x\to 0}\frac{ax+b\tan x}{x^3} .1 \qquad \left[\frac{0}{0}\right] \qquad \left[\because \lim_{x\to 0}\frac{x}{\tan x}=1\right]$$

$$= \lim_{x\to 0}\left(\frac{a+b\sec^2 x}{3x^3}\right) \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{a+b\sec 0}{0} = \frac{a+b}{0}$$

But limit is finite, therefore, numerator must be zero. ...(1)

$$a+b=0, \quad a=-b$$

Thus,

$$\frac{1}{3} = \lim_{x\to 0}\frac{-b+b\sec^2 x}{3x^2} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{b . 2\sec x \sec x \tan x}{6x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \left(\lim_{x\to 0}\frac{b}{3}\sec^2 x\right).\left(\lim_{x\to 0}\frac{\tan x}{x}\right) \qquad \left[\because \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

$$= \frac{b}{3}\sec 0.1$$

$$\frac{1}{3} = \frac{b}{3}, \qquad b=1$$

From Eq. (1),

$$a=-b=-1$$

Hence,

$$a=-1, b=1$$

EXERICSE 1.8

1. Prove that $\lim_{x \to a}\left[\frac{1}{x-a} - \cot(x-a)\right] = 0.$

2. Prove that $\lim_{x \to 0}\left(\cot x - \frac{1}{x}\right) = 0.$

3. Prove that $\lim_{x \to \frac{\pi}{2}}\left(\tan x - \frac{2x\sec x}{\pi}\right) = \frac{2}{\pi}.$

4. Prove that $\lim_{x \to \frac{\pi}{2}}(\sec x - \tan x) = 0.$

5. Prove that $\lim_{x \to 3}\left[\frac{1}{x-3} - \frac{1}{\log(x-2)}\right] = -\frac{1}{2}.$

6. Prove that $\lim_{x \to 0}\left[\frac{1}{x-a} - \frac{1}{\log(x+1-a)}\right] = \frac{1}{2}.$

7. Prove that $\lim_{x \to 0}\left(\frac{1}{x} - \frac{1}{e^x-1}\right) = \frac{1}{2}.$

8. Prove that $\lim_{x \to 0}\left[\frac{1}{x} - \frac{\log(1+x)}{x^2}\right] = \frac{1}{2}.$