5

Functions

DEFINITION

A function f from a set P into a set Q is a relation from P to Q such that each element of P is related to exactly one element of the set Q. It is denoted as $f : P \to Q$ and read as "f is a function from P to Q".

FUNCTIONS AS A SET

If P and Q are any two non-empty sets, then a function f from P to Q is a subset of $P \times Q$, with two important restrictions

(*i*) $\forall\, a \in P$, $(a, b) \in f$ for some $b \in Q$

(*ii*) If $(a, b) \in f$ and $(a, c) \in f$ then $b = c$.

Note 1. There may be some elements of the Q which are not related to any element of set P.

2. Every element of P must be related with at least one element of Q.

Example 1. *If a set A has n elements, how many functions are there from A to A.* $f : A \to B$ is fn. then no. of fn's $= (\text{Card } B)^{\text{Card } A}$

Sol. If a set A has n elements, then there are n^n functions from A to A.

Example 2. *If A has m elements and B has n elements, how many functions are there from A to B and from B to A.*

Sol. A has m elements and B has n elements. So, the total number of functions from A to B are n^m and the total number of functions from B to A are m^n.

Example 3. *If A = {2, 3, 4} and B = {5, 6}. Determine all functions from A to B.*

Sol. The total number of functions from $A \to B$ are $2^3 = 8$

(*i*) {(2, 5), (3, 5), (4, 5)}	(*ii*) {(2, 6), (3, 6), (4, 6)}
(*iii*) {(2, 5), (3, 5), (4, 6)}	(*iv*) {(2, 5), (3, 6), (4, 6)}
(*v*) {(2, 6), (3, 5), (4, 5)}	(*vi*) {(2, 6), (3, 6), (4, 5)}
(*vii*) {(2, 6), (3, 5), (4, 6)}	(*viii*) {(2, 5), (3, 6), (4, 5)}.

DOMAIN OF A FUNCTION

Let f be a function from P to Q. The set P is called domain of the function f.

CO-DOMAIN OF A FUNCTION

Let f be a function from P to Q. The set Q is called co-domain of the function f.

IMAGE OF AN ELEMENT

Set of all images of elements of A
Range(f) $\subseteq$ Co-domain

If the element x of P corresponds to y under function f, then y is the image of x u
and is written as

$$f(x) = y.$$

If $f(x) = y$, then we say that x is a per image of y.

If $f : X \to Y$, then each element of P has unique image in Q, whereas every elemen
need not be image of some x in P.

Example 4. *Let A = {1, 2, 3, 4} and the function $f : A \to A$ as shown in Fig. 1. Find image of each element of A (ii) the image f(A) of the function f.*

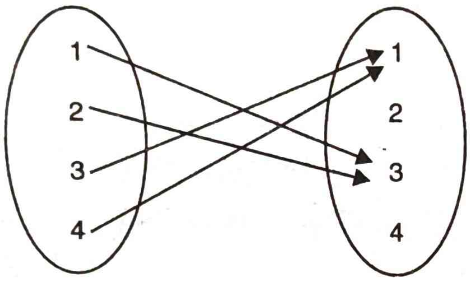


Fig. 1.

Sol. (*i*) The image of $f(1) = 3$, $f(2) = 3$, $f(3) = 1$ and $f(4) = 1$.

(*ii*) The image of $f(A)$ of f consists of all the image values.

Therefore, $f(A) = \{1, 3\}$.

RANGE OF A FUNCTION

The range of a function is the set of images of its domain. In other words, w
is a subset of its co-domain. It is denoted as f(domain).

If $f : P \to Q$, then $f(P) = \{f(x) : x \in P\} = \{y : y \in Q \mid \exists\, x \in P$, such that $f(x) = y\}$.

Example 5. *Let P = {x, y, z, u} and Q = {a, b, c, d} and $f : P \to Q$, such that*

$$f = \{(x, a), (y, b), (z, c), (u, c)\}.$$

Find the domain, co-domain and range of function.

Sol. Domain of function f is the set P

Co-domain of function f is the set Q and range of the function f is $\{a, b, c\}$.

Example 6. *Let A = {2, 3, 4} and B = {a, b, c} and f = {(2, a), (3, b), (4, b)}*
Find domain, co-domain and range of the function.

Sol. Domain of function is Domain $(f) = \{2, 3, 4\}$

Co-domain of function is co-domain $(f) = \{a, b, c\}$

Range of function is Range $(f) = \{a, b\}$.

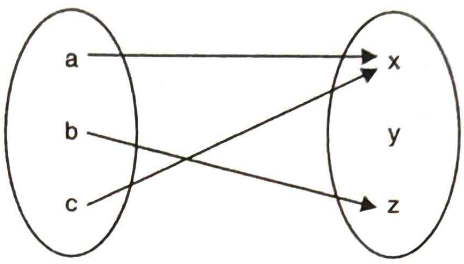
REPRESENTATION OF A FUNCTION

We can represent a function by a diagram also. The two sets P and Q are represented by two circles. The function $f : P \to Q$ is represented by a collection of arrows joining the points which represent the elements of P and corresponding elements of Q

e.g., Let $P = \{a, b, c\}$ and $Q = \{x, y, z\}$ and $f : P \to Q$ such that

$f = \{(a, x), (b, z), (c, x)\}$

Then f can be represented diagrammatically as follows (Fig. 2).



f : A→B is a fun if every element of A has unique image in B

Fig. 2.

Example 7. *Let $X = \{x, y, z, k\}$ and $Y = \{1, 2, 3, 4\}$. Determine which of the following are functions. Give reasons if it is not. Find range if it is a function*

(*i*) $f = \{(x, 1), (y, 2), (z, 3), (k, 4)\}$ (*ii*) $g = \{(x, 1), (y, 1), (k, 4)\}$

(*iii*) $h = \{(x, 1), (x, 2), (x, 3), (x, 4)\}$ (*iv*) $l = \{(x, 1), (y, 1), (z, 1), (k, 1)\}$

(*v*) $d = \{(x, 1), (y, 2), (y, 3), (z, 4), (z, 4)\}$.

Sol. (*i*) It is a function. Range $(f) = \{1, 2, 3, 4\}$.

(*ii*) It is not a function because every element of X does not relate with some element of Y *i.e.*, Z is not related with any element of Y.

(*iii*) h is not a function because $h(x) = \{1, 2, 3, 4\}$ *i.e.*, element x has more than one image in set Y.

(*iv*) It is a function. Range $(l) = \{1\}$.

(*v*) d is not a function because $d(y) = \{2, 3\}$ *i.e.*, element y has more than one image in set Y.

EVERYWHERE DEFINED FUNCTION

Consider a function f from A to B. Then the function f is everywhere defined if dom $(f) = A$.

Example 8. *Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$ and $C = \{\alpha, \beta, \gamma\}$. Consider the following two functions from B to C and A to C.*

(*i*) $f = \{(a, \alpha), (b, \beta), (c, \gamma)\}$ (*ii*) $g = \{(1, \alpha), (3, \beta), (2, \gamma)\}$

Determine whether or not each function is everywhere defined.

Sol. (*i*) f is everywhere defined since dom $(f) = B$.

(*ii*) g is not everywhere defined since dom $(f) = \{1, 2, 3\}$ and $A = \{1, 2, 3, 4\}$ which is not equal.

TYPES OF FUNCTIONS

(*a*) **Injective (One-to-One) Functions.** Let $f : X \to Y$. The function f is called o one or injective if different elements in X have different images in Y *i.e.*, if $f(a) = f(a') \Rightarrow$ $\forall\, a, a' \in X$.

i.e. $x \neq y$ - $f(x) \neq f(y)$

Another way of defining injective function is that every element of domain X unique image in the co-domain Y and there is no element of Y which is image of more tha element of domain X.

For example : Consider, X = {*x*, *y*, *z*, *k*} and Y = {1, 2, 3, 4} and *f* is function from such that

$$f = \{(x, 1), (y, 2), (z, 3), (k, 4)\}$$

The function *f* is injective function as every element of domain X has a unique im the co-domain Y (Fig. 3).

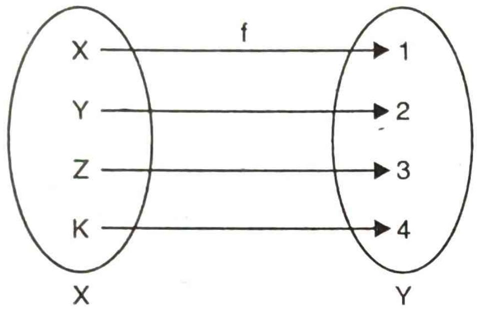


Fig. 3.

(*b*) **Surjective (Onto) Functions.** Let $f : X \to Y$. The function f is called sur function if each element in Y, is the image of at least one element in X. In other wor surjective functions, the range of *f* is equal to co-domain Y *i.e.*, $\forall\, b \in Y, b = f(a)$ for some

For example : Consider, X = {1, 2, 3, 4, 5}, Y = {*a*, *b*, *c*, *d*} and $f = \{(1, a), (2, a),$ $(4, c), (5, d)\}$.

It is a surjective function, as every element of Y is the image of some of X (Fig. 4

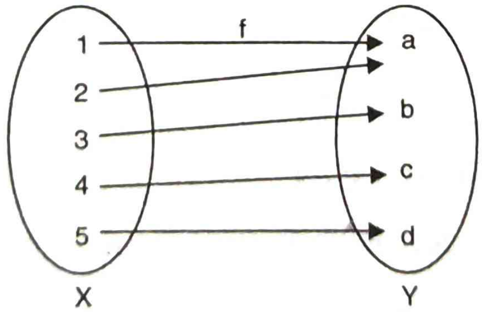


Fig. 4.

(*c*) **Bijective (One-to-One Onto) Functions.** A function which is both injectiv to-one) and surjective (onto) is called a bijective (one-to-one-onto) function.

For example : Consider, P = {*x*, *y*, *z*}, Q = {*a*, *b*, *c*} and $f : P \to Q$ such that

$$f = \{(x, a), (y, b), (z, c)\}$$

The f is one-to-one and also it is onto. So it is a bijective function (Fig. 5).

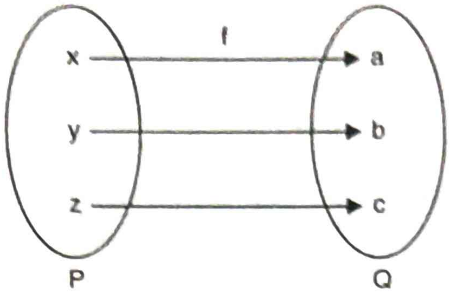


Fig. 5.

(*d*) **Into Functions.** Let $f : X \rightarrow Y$. The function f is called an into function if the range of f is not equal to the co-domain Y. Therefore, there must be an element of co-domain Y which is not the image of any element of domain X.

For example : Consider, $X = \{1, 2, 3\}$, $Y = \{k, l, m, n, p\}$ and $f : X \rightarrow Y$ such that

$$f = \{(1, k), (2, n), (3, p)\}.$$

In the function f, the range *i.e.*, $\{k, n, p\} \neq$ co-domain of Y *i.e.*, $\{k, l, m, n, p\}$

Therefore, it is an into function (Fig. 6).

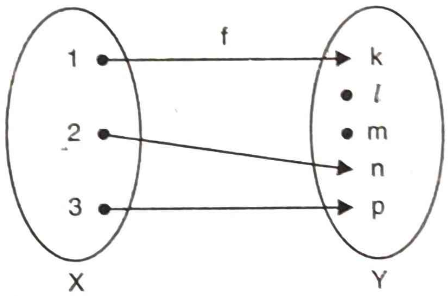


Fig. 6.

(*e*) **One-One Into Functions.** Let $f : X \rightarrow Y$. The function f is called one-one into function if different elements of X have different unique images of Y.

For example : Consider, $X = \{k, l, m\}$, $Y = \{1, 2, 3, 4\}$ and $f : X \rightarrow Y$ such that

$$f = \{(k, 1), (l, 3), (m, 4)\}$$

The function f is one-one into function (Fig. 7).

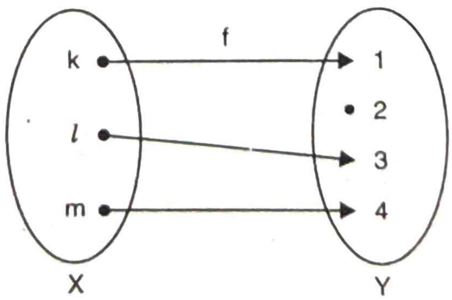


Fig. 7.

(*f*) **Many One Functions.** Let $f : X \rightarrow Y$. The function f is said to be many one function if there exists two or more than two different elements in X having the same image in Y.

For example : Consider $X = \{1, 2, 3, 4, 5\}$, $Y = \{x, y, z\}$ and $f : X \rightarrow Y$ such that

$$f = \{(1, x), (2, x), (3, x), (4, y), (5, z)\}$$

The function f is a many one function. (Fig. 8)

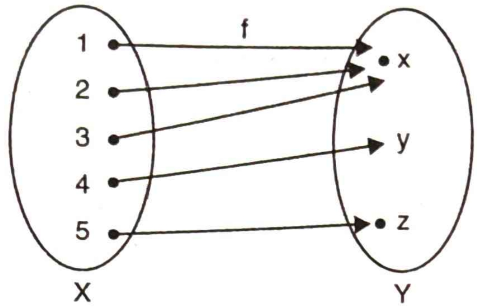


Fig. 8.

(*g*) **Many One Into Functions.** Let $f : X \to Y$. The function f is called many-one function if and only if it is both many one and into function.

For example : Consider $X = \{a, b, c\}$, $Y = \{1, 2\}$ and $f : X \to Y$ such that

$$f = \{(a, 1), (b, 1), (c, 1)\}$$

As the function f is many-one and into, so it is a many-one into function (Fig. 9).

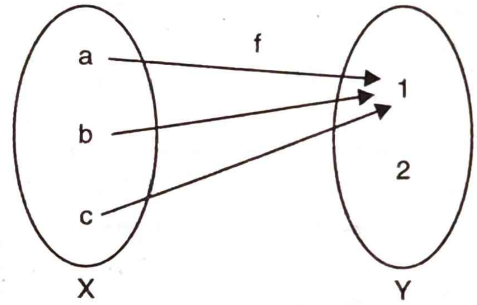


Fig. 9.

(*h*) **Many One Onto Functions.** Let $f : X \to Y$. The function f is called many-one function if and only if it is both many one and onto.

For example : Consider, $X = \{1, 2, 3, 4\}$, $Y = \{k, l\}$ and $f : X \to Y$ such that

$$f = \{(1, k), (2, k), (3, l), (4, l)\}$$

The function f is many-one (as two elements have the same image in Y) and it is ont every element of Y is the image of some element X). So, it is many-one onto function (Fig

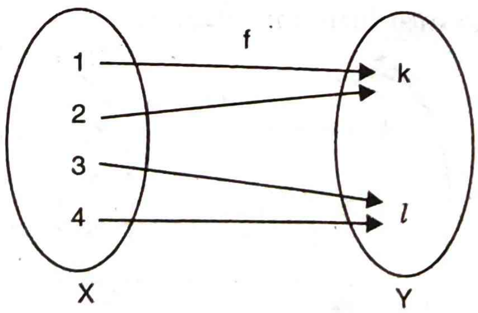


Fig. 10.

EQUAL FUNCTIONS

Consider two functions f and h from a set X to a set Y. The functions f and h are ca equal functions if and only if $f(a) = h(a)$, for every $a \in X$.

The functions f and h are called unequal functions if there exist at least one ele $a \in X$ such that $f(a) \neq h(a)$.

Example 9. *Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$. Consider the function $f : X \to Y$, $g : X \to Y$ and $h : X \to Y$ such that*

$$f = \{(1, a), (2, a), (3, c)\}$$
$$g = \{(1, b), (2, a), (3, c)\}$$
$$h = \{(1, a), (2, a), (3, c)\}$$

Determine which functions are equal and which are unequal.

Sol. The functions f and h are equal functions. The functions f and g and g and h are unequal functions.

IDENTITY FUNCTIONS

Consider any set A. Let the function $f : A \to A$. The function f is called the identity function if each element of set A has image on itself *i.e.,* $f(a) = a \ \forall \ a \in A$

It is denoted by I.

For example : Consider, $A = \{1, 2, 3, 4, 5\}$ and $f : A \to A$ such that

$$f = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}.$$

The function f is an identity function as each element of A is mapped onto itself. The function f is one-one and onto (Fig. 11).

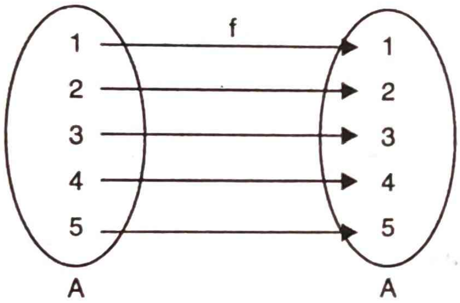


Fig. 11.

INVERTIBLE (INVERSE) FUNCTIONS

A function $f : X \to Y$ is invertible if and only if it is a bijective function.

Consider the bijective (one to one onto) function $f : X \to Y$. As f is one-to-one, therefore, each element of X corresponds to a distinct element of Y. As f is onto, there is no element of Y which is not the image of any element of X *i.e.,* range = co-domain Y.

The inverse function for f exists if f^{-1} is a function from Y to X.

For example : Consider, $X = \{1, 2, 3\}$, $Y = \{k, l, m\}$ and $f : X \to Y$, such that

$$f = \{(1, k), (2, m), (3, l)\} \text{ as in Fig. 12.}$$

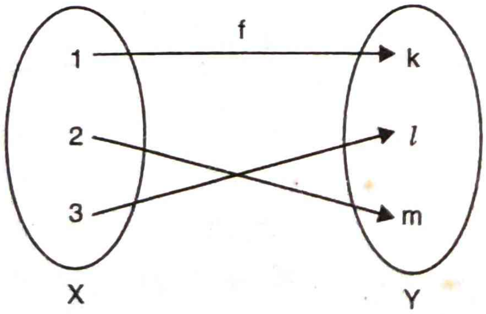


Fig. 12.

The inverse function of f is shown in Fig. 13.

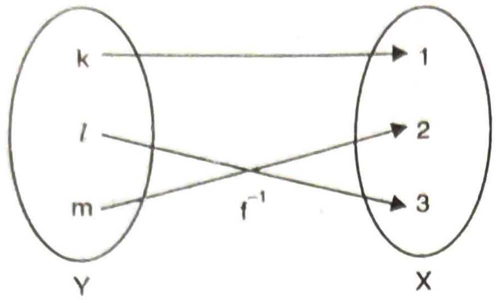


Fig. 13.

COMPOSITION OF FUNCTIONS

Consider functions, $f : A \to B$ and $g : B \to C$. The composition of f with g is a functio from A into C defined by $(gof)\ (x) = g[f(x)]$ and is denoted by gof.

To find the composition of f and g, first find the image of x under f and then find t image of $f(x)$ under g.

Example 10. *Let X = {1, 2, 3}, Y = {a, b} and Z = {5, 6, 7}. Consider the functio f = {(1, a), (2, a), (3, b)} and g = {(a, 5), (b, 7)} as in Figs. 14 and 15. Find the composition of*

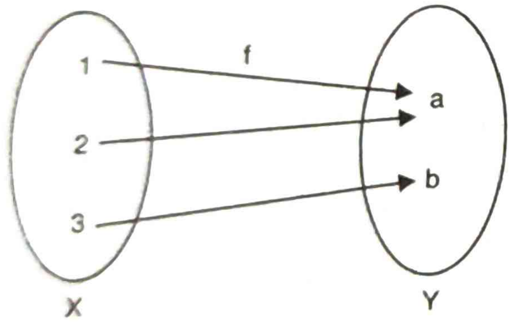


Fig. 14.

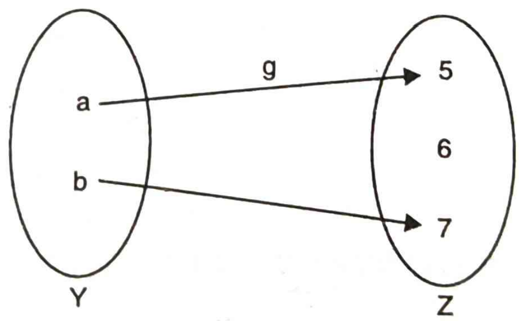


Fig. 15.

Sol. The composition function gof is shown in Fig. 16.

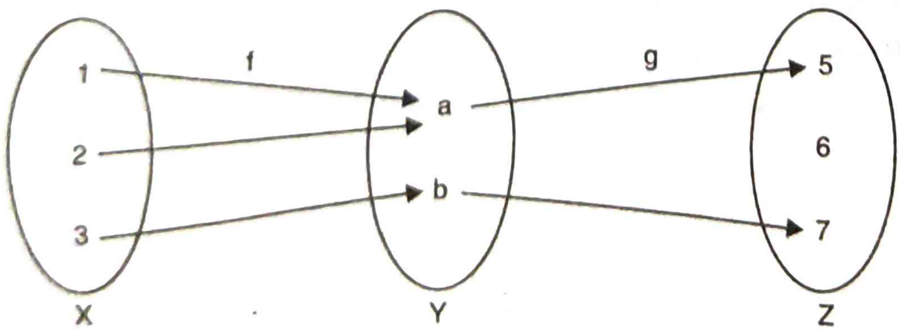


Fig. 16.

$(gof)\ (1) = g[f(1)] = g(a) = 5$, $(gof)\ (2) = g[f(2)] = g(a) = 5$

$(gof)\ (3) = g[f(3)] = g(b) = 7$.

Example 11. *Consider f, g and h, all functions on the integers, by $f(n) = n^2$, $g(n) = n + 1$ and $h(n) = n - 1$.*

Determine (i) hofog (ii) gofoh (iii) fogoh.

Sol. (*i*)

$$hofog(n) = n + 1,$$
$$hofog(n + 1) = (n + 1)^2$$
$$h[(n + 1)^2] = (n + 1)^2 - 1 = n^2 + 1 + 2n - 1 = n^2 + 2n.$$

(*ii*)

$$gofoh(n) = n - 1, \quad gof(n - 1) = (n - 1)^2$$
$$g[(n - 1)^2] = (n - 1)^2 + 1 = n^2 + 1 - 2n + 1 = n^2 - 2n + 2.$$

(*iii*)

$$fogoh(n) = n - 1$$
$$fog(n - 1) = (n - 1) + 1$$
$$f(n) = n^2.$$

Theorem I. *Prove that if $f: X \to Y$ and $g: Y \to Z$ are one to one functions, then gof is one-to-one.*

Proof. Consider two distinct elements a_1 and $a_2 \in X$.

Now $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$ ($\because$ f is one to one)

$\Rightarrow$ $g[f(a_1)] \neq g[f(a_2)]$ ($\because$ g is one to one)

$\Rightarrow$ $(gof)(a_1) \neq (gof)(a_2)$.

Therefore, the function *gof* has distinct elements $\in$ X that are mapped to distinct element of Z. Hence, *gof* is one-to-one function.

Theorem II. *Prove that if $f: X \to Y$ and $g: Y \to Z$ are into functions, then gof is onto.*

Proof. Consider an element $c \in Z$. As g is an onto function from $Y \to Z$, therefore, there exists $b \in Y$, such that $g(b) = c$.

Now, since f is onto function from X to Y, therefore, there exists $a \in X$ such that $f(a) = b$.

Thus $\forall\ c \in Z$, there exists $a \in X$ such that $C = g(b) = g[f(a)] = (gof)(a)$.

Thus, *gof* is a function of X onto Z. Hence, *gof* is an onto function.

Theorem III. *Prove that if $f: X \to Y$ and $g: Y \to Z$ be two one-to-one onto function, then gof is also one-to-one onto function.*

Proof. Assume two elements $a_1, a_2 \in \overset{X}{A}$ with $a_1 \neq a_2$.

As $f: X \to Y$ is a one-to-one onto function therefore, there exists distinct elements $b_1, b_2 \in Y$.

So, $f(a_1) = b_1, f(a_2) = b_2$ and $b_1 \neq b_2$ ...(1)

The function $g: Y \to Z$ is also one-to-one onto.

Therefore, there exists unique elements $c_1, c_2 \in Z$.

So $g(b_1) = c_1, g(b_2) = c_2$ and $c_1 \neq c_2$ ...(2)

The composition of (1) and (2) gives

$$(gof)(a_1) = g[f(a_1)] = g(b_1) = c_1$$

and

$$(gof)(a_2) = g[f(a_2)] = g(b_2) = c_2$$

We know $c_1 \neq c_2$ so $(gof)(a_1) \neq (gof)(a_2)$

As $a_1 \neq a_2 \Rightarrow (gof)(a_1) = (gof)(a_2)$.

Therefore, *gof* is one-to-one function.

Now, assume an element $c \in Z$.

As $g : Y \to Z$ is a one-to-one onto function.

Therefore, there exists a distinct element $b \in Y \Rightarrow g(b) = c$

As $f : X \to Y$ is one-to-one onto function.

Therefore, there exists a distinct element $a \in X \Rightarrow f(a) = b$

Therefore, $(gof)(a) = g[f(a)] = g(b) = c$

It implies every element of c has image under gof. Therefore, gof is an onto funct

So, we conclude that gof is one-to-one onto function.

FUNCTIONS APPLICABLE IN COMPUTER SCIENCE

The following are the functions which are widely used in computer science.

Characteristic Function of A. Consider a subset A of the universal set $U = \{u_1, u$ $u_4, \ldots\ldots, u_{n-1}, u_n\}$. The characteristic function of A is a function from U to {0, 1} defin follows :

$$f_A(u_j) = \begin{cases} 1, & \text{if } u_j \in A \\ 0, & \text{if } u_j \notin A \end{cases}$$

Note. The characteristic function of A *i.e.*, f_A is not one to one but is everywhere define onto.

Example 12. *Let A = {a, c, e} and U = {a, b, c, d, e, f, g}. Compute the following fun values :*

(*i*) $f_A(a)$ (*ii*) $f_A(g)$ (*iii*) $f_A(e)$ (*iv*) $f_A(t)$ (*v*) $f_A(d)$.

Sol. (*i*) $f_A(a) = 1$ (*ii*) $f_A(g) = 0$ (*iii*) $f_A(e) = 1$

(*iv*) $f_A(t)$ = undefined (*v*) $f_A(d) = 0$.

Floor Functions. The floor function for any real number x is defined as $f(x)$ is greatest integer 1 less than or equal to x. It is denoted by $\lfloor x \rfloor$.

Example 13. *Determine the value of*

(*i*) $\lfloor 3.5 \rfloor$ (*ii*) $\lfloor -2.4 \rfloor$ (*iii*) $\lfloor 3.143 \rfloor$.

Sol. (*i*) $\lfloor 3.5 \rfloor = 3$ (*ii*) $\lfloor -2.4 \rfloor = -3$ (*iii*) $\lfloor 3.143 \rfloor = 3$.

Example 14. *Determine the value of*

(*i*) $\lfloor \sqrt{27} \rfloor$ (*ii*) $\lfloor -13 \rfloor$ (*iii*) $\lfloor \sqrt{6} \rfloor$.

Sol. (*i*) $\lfloor \sqrt{27} \rfloor = 5$ (*ii*) $\lfloor -13 \rfloor = -13$ (*iii*) $\lfloor \sqrt{6} \rfloor = 2$.

Ceiling Function. The ceiling function for any real number x is defined as $h(x)$ is smallest integer greater than or equal to x. It is denoted by $\lceil x \rceil$.

Example 15. *Determine the value of*

(*i*) $\lceil 3.5 \rceil$ (*ii*) $\lceil -2.4 \rceil$ (*iii*) $\lceil 3.143 \rceil$.

Sol. (*i*) $\lceil 3.5 \rceil = 4$ (*ii*) $\lceil -2.4 \rceil = -2$ (*iii*) $\lceil 3.143 \rceil = 4$.

Example 16. *Determine the value of*

(*i*) $\lceil \sqrt{27} \rceil$ (*ii*) $\lceil -13 \rceil$ (*iii*) $\lceil \sqrt{6} \rceil$.

Now, the cycle (m, o, q) can be written as

$$(m, o, q) = (m, q) \circ (m, o).$$

Now, the permutation p in product of transpositions form can be written as

$$p = (k, l) \circ (m, q) \circ (m, o) \circ (n, p).$$

Even permutation. A permutation p of a finite set A is said to be EVEN if the permutation can be written as a product of an even no. of transpositions.

Odd Permutation. A permutation p of a finite set A is said to be odd if the permutation can be written as a product of an odd no. of transpositions.

Example 35. *Determine whether the following permutation is odd or even*

$$p = \begin{pmatrix} a & b & c & d & e & f & g & h \\ b & c & a & e & f & d & h & g \end{pmatrix}.$$

Sol. First of all, write p as a product of disjoint cycles, which is as follows :

$$p = (a, b, c) \circ (d, e, f) \circ (g, h)$$

Now, write all of the cycles as a product of transpositions, which is as follows :

$$(a, b, c) = (a, c) \circ (a, b)$$

$$(d, e, f) = (d, f) \circ (d, e) \quad \text{and} \quad (g, h) = (g, h)$$

Then, we have $p = (a, c) \circ (a, b) \circ (d, f) \circ (d, e) \circ (g, h)$. Since, p is a product of an odd no. of permutations, hence it is an odd permutation.

Note 1. The product of two odd permutations is even.

2. The product of two even permutations is even.

3. The product of an even and an odd permutation is odd.

SOLVED PROBLEMS

Problem 1. *Let $X = \{x, y, z, k\}$ and $Y = \{a, b, c\}$. Determine whether the relation S from X to Y is a function. If it is a function, give its domain and range.*

(*i*) $S = \{(x, a), (y, a), (x, b), (y, b)\}$ (*ii*) $S = \{(x, c), (y, a), (z, b)\}$

(*iii*) $S = \{(x, a), (y, b), (z, c), (k, b), (x, b)\}$ (*iv*) $S = \{(x, c), (y, a), (3, a), (k, c)\}$.

Sol. (*i*) Not a function because domain of $S \neq X$ *i.e.*, there is no image of Z, K in set Y.

(*ii*) Not a function because domain of $S \neq X$ *i.e.*, there is no image of K in set Y.

(*iii*) It is a function.

Domain $(S) = \{x, y, z, k\}$, Range $(S) = \{a, b, c\}$

(*iv*) It is a function.

Domain $(S) = \{x, y, z, k\}$, Range $(S) = \{a, c\}$.

Problem 2. *Let $A = B = \{1, 2, 3, 4\}$. Define functions $f : A \to B$ (if possible) such that*

(*i*) *f is one-to-one and onto*

(*ii*) *f is neither one-to-one nor onto*

(*iii*) *f is onto but not one-to-one*

(*iv*) *f is one-to-one but not onto.*

Sol. (*i*) The function $f = \{(1, 1), (2, 4), (3, 2), (4, 3)\}$ is one-to-one and onto (Fig. 18)

(*ii*) The function $f = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$ is neither one-to-one nor onto (Fig. 19)

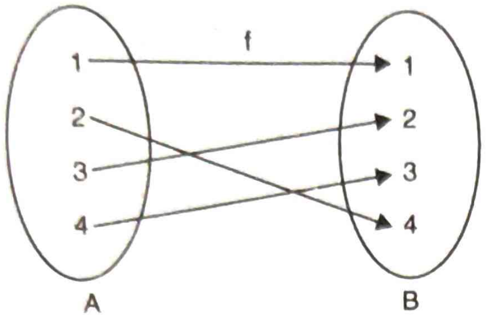


Fig. 18.

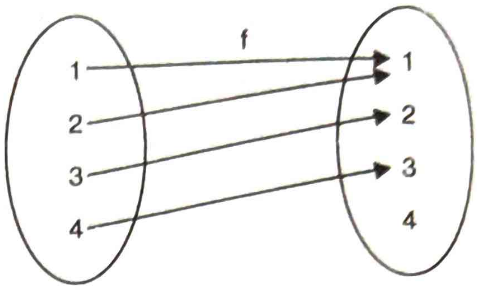


Fig. 19.

(*iii*) The function f which is onto but not one-to-one is not possible on the set A = B = {1, 2, 3, 4}.

(*iv*) The function f which is one-to-one but not onto is not possible on the set A = B = {1, 2, 3, 4}.

Problem 3. *Consider the sets X and Y and let $f : X \to Y$. Determine whether the following functions are :*

(*a*) *one-to-one function* (*b*) *onto function*

(*c*) *one-to-one onto* (*d*) *neither one-to-one nor onto.*

(*i*) $X = \{x, y, z\}$; $Y = \{1, 2, 3, 4\}$, $f = \{(x, 1), (y, 1), (z, 3)\}$

(*ii*) $X = \{a, b, c, d\} = Y$, $f = \{(a, a), (b, c), (c, d), (d, b)\}$

(*iii*) $X = \left\{\frac{1}{6}, \frac{1}{9}, \frac{1}{12}\right\}$; $Y = \{a, b, c, d\}$, $f = \left\{\left(\frac{1}{6}, b\right), \left(\frac{1}{9}, d\right), \left(\frac{1}{12}, a\right)\right\}$

(*iv*) $X = \{y_1, y_2, y_3\}$; $Y = \{k, l\}$, $f = \{(y_1, l), (y_2, l), (y_3, k)\}$.

Sol. (*i*) Neither one-to-one nor onto (*ii*) one-to-one onto

(*iii*) one-to-one (*iv*) onto.

Problem 4. *Consider the sets $X = \{k, l, m, n\}$ to $Y = \{7, 8, 9, 10\}$. Let $f : X \to Y$ such*

(*i*) $f = \{(k, 10), (l, 7), (m, 8), (n, 9)\}$ (*ii*) $f = \{(k, 7), (l, 8), (m, 7), (n, 8)\}$

Determine whether f^{-1} is a function.

Sol. (*i*) The f^{-1} is $f^{-1} = \{(10, k), (7, l), (8, m), (9, n)\}$

The f^{-1} is also a function.

(*ii*) The f^{-1} is $f^{-1} = \{(7, k), (8, l), (7, m), (8, n)\}$

The domain $(f) \neq$ set Y. So f^{-1} is not a function.

Problem 5. *Let X = Y = Z = R, be the set of reals.*

Consider the functions f and g such that

$$f(x) = x + 2,\ x \in R,\ g(y) = y^2,\ y \in \text{R}.$$

Find the composition function gof.

Sol. The composition of f with g is given by

$$gof(x) = g(x + 2) = (x + 2)^2,\ x \in \text{R}.$$

Problem 6. *Consider the functions* $f, g : R \to R$ *difined by*

$$f(x) = x^2 + 3x + 1,\ g(x) = 2x - 3.$$

Find the composition functions

(i) fof *(ii) fog* *(iii) gof.*

Sol. (*i*)
$$\begin{aligned}(fof)(x) &= f[f(x)] = f(x^2 + 3x + 1)\\ &= (x^2 + 3x + 1)^2 + 3(x^2 + 3x + 1) + 1\\ &= x^4 + 9x^2 + 1 + 6x^3 + 2x^2 + 6x + 3x^2 + 9x + 3 + 1\\ &= x^4 + 6x^3 + 14x^2 + 15x + 5.\end{aligned}$$

(*ii*)
$$\begin{aligned}(gof)(x) &= g[f(x)] = g(x^2 + 3x + 1)\\ &= g(x^2 + 3x + 1) - 3\\ &= 2x^2 + 6x + 2 - 3\\ &= 2x^2 + 6x - 1.\end{aligned}$$

(*iii*)
$$\begin{aligned}(fog)(x) &= f[g(x)] = f(2x - 3)\\ &= (2x - 3)^2 + 3(2x - 3) + 1\\ &= 4x^2 + 9 - 12x + 6x - 9 + 1\\ &= 4x^2 - 6x + 1.\end{aligned}$$

Problem 7. *Let f, g, h be functions from N to N, where N is the set of natural numbers so that*

$$f(n) = n + 1$$
$$g(n) = 2n$$
$$h(n) = \begin{cases} 0 & \text{when } n \text{ is even} \\ 1 & \text{when } n \text{ is odd}. \end{cases}$$

Determine fof, fog, gof, goh, hog, (fog)oh.

Sol. (*i*)
$$\begin{aligned}fof(n) &= n + 1\\ f[f(n)] &= n + 1\\ f(n + 1) &= (n + 1) + 1 = n + 2.\end{aligned}$$

(*ii*)
$$\begin{aligned}fog(n) &= 2n\\ f[g(n)] &= 2n\\ f[g(n)] &= 2(2n) = 4n.\end{aligned}$$

(*iii*)
$$\begin{aligned}gof(n) &= n + 1\\ g[f(n)] &= n + 1\\ g(n + 1) &= 2(n + 1) = 2n + 2.\end{aligned}$$

(*iv*)
$$goh(n) = 0 \text{ ; } n \text{ is even}$$
$$goh(n) = 1 \text{ ; } n \text{ is odd}$$
$$g(0) = 0 \text{ when } n \text{ is even}$$
$$g(1) = 2 \text{ when } n \text{ is odd.}$$

(*v*)
$$\begin{aligned}hog(n) &= 2n\\ h[g(n)] &= 2n\\ h[g(n)] &= 1, \text{ as } n \text{ is even every time.}\end{aligned}$$

(*vi*)
$$fogoh(n) = 0 \text{ ; } n \text{ is even}$$
$$fogoh(n) = 1 \text{ ; } n \text{ is odd}$$

$$fog[h(n)] = 0 \text{ and } fog[h(n)] = 1$$
$$fog(0) = 0 \text{ and } fog(1) = 2$$
$$f[g(0)] = 0 \text{ and } f[g(1)] = 2$$
$$f(0) = 1 \text{ and } f(2) = 2 + 1 = 3.$$

So, $(fog)oh = 1$ when n is even, $(fog)oh = 3$ when n is odd.

Problem 8. *Consider $A = B = C = R$ and let $f : A \to B$ and $g : B \to C$ be defin $f(x) = x + 9$ and $g(y) = y^2 + 3$.*

Find the following composition functions :

(i) $(fof)(a)$ *(ii) $(gog)(a)$*

(iii) $(fog)(b)$ *(iv) $(gof)(b)$*

(v) $(gof)(4)$ *(vi) $(fog)(-4)$.*

Sol. (i) $(fof)(a) = f[f(a)] = f(a + 9) = (a + 9) + 9 = a + 18$.

(ii) $(gog)(a) = g[g(a)] = g(a^2 + 3) = (a^2 + 3)^2 + 3 = a^4 + 6a^2 + 12$.

(iii) $(fog)(b) = f[g(b)] = f(b^2 + 3) = (b^2 + 3) + 9 = b^2 + 12$.

(iv) $(gof)(b) = g[f(b)] = g(b + 9) = (b + 9)^2 + 3 = b^2 + 18b + 84$.

(v) $(gof)(4) = g[f(4)] = g(13) = (13)^2 + 3 = 172$.

(vi) $(fog)(-4) = f[g(-4)] = f(19) = 19 + 9 = 28$.

Problem 9. *Let $X = \{a, b, c\}$. Define $f : X \to X$ such that*

$$f = \{(a, b), (b, a), (c, c)\}$$

Find (i) f^{-1} *(ii) f^2* *(iii) f^3* *(iv) f^4.*

Sol. (i) The inverse function $f^{-1} = \{(b, a), (a, b), (c, c)\}$

(ii) The f^2 is fof (Fig. 20).

$$(fof)(a) = f[f(a)] = f(b) = a, \qquad (fof)(b) = f[f(b)] = f(a) = b$$

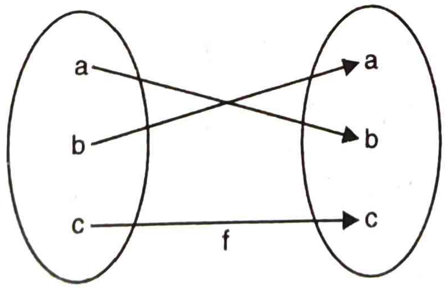


Fig. 20.

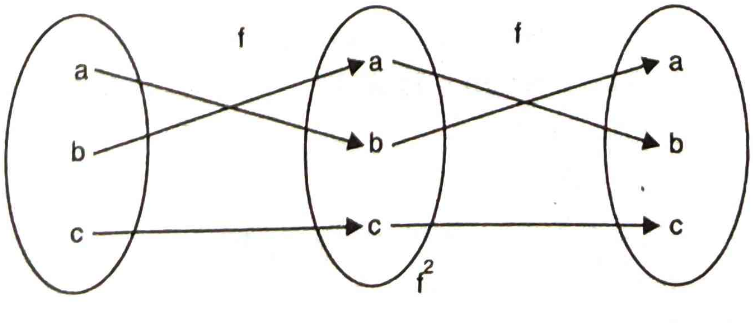


Fig. 21.

$$(fof)(c) = f[f(c)] = f(c) = c,$$

So, $\quad f^2 = \{(a, a), (b, b), (c, c)\}.$

(*iii*) The f^3 is $fofof$ i.e., fof^2 (Fig. 22).

$$(fof)^2\,(a) = f[f^2(a)] = f(a) = b, \qquad (fof^2)(b) = f[f^2(b)] = f(b) = a$$
$$(fof^2)(c) = f[f^2(c)] = f(c) = c$$

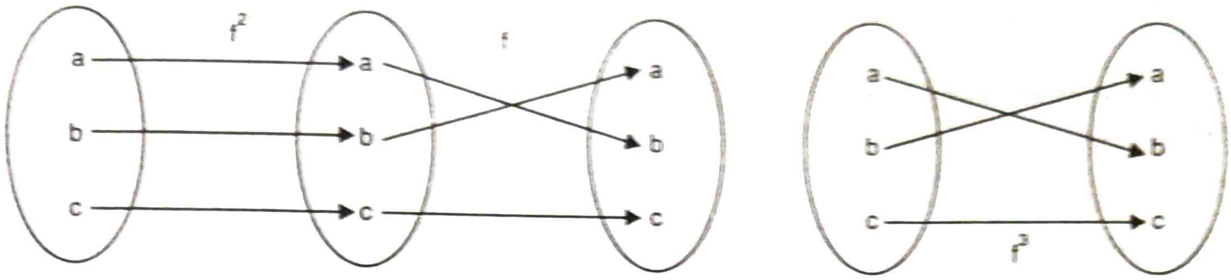


Fig. 22.

So, $\quad f^3 = \{(a, b), (b, a), (c, c)\}.$

(*iv*) The f^4 is $fofofof$ i.e., fof^3 (Fig. 23).

$$(fof^3)\,(a) = f[f^3\,(a)] = f(b) = a, \qquad (fof^3)\,(b) = f[f^3\,(b)] = f(a) = b$$
$$(fof^3)\,(c) = f[f^3\,(c)] = f(c) = c$$

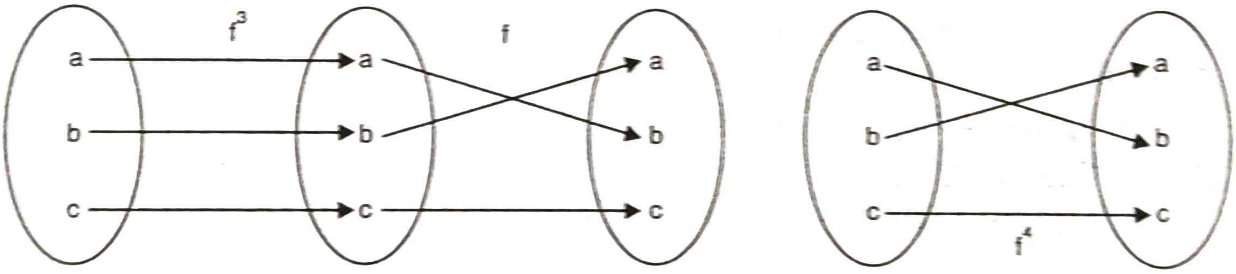


Fig. 23.

So, $\quad f^4 = \{(a, a), (b, b), (c, c)\}.$

Problem 10. *Consider the functions f, g and h as in Figs. 24, 25 and 26.*

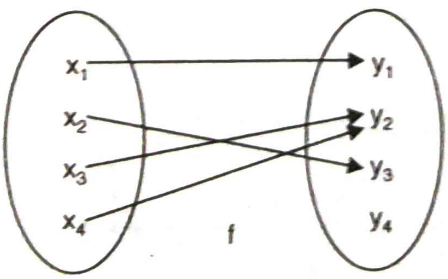


Fig. 24.

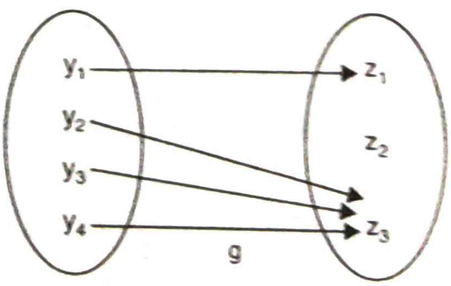


Fig. 25.

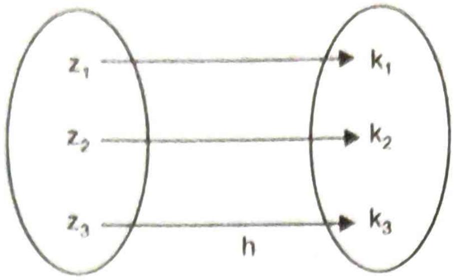


Fig. 26.

Determine (*i*) *gof* (*ii*) *ho*(*gof*) (*iii*) (*hog*)*of*.

Sol. (*i*) Consider Fig. 27.

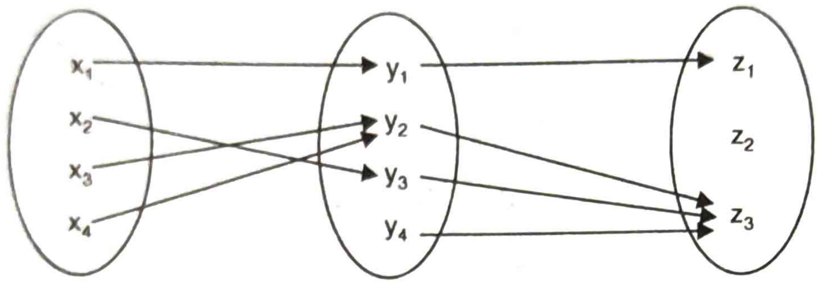


Fig. 27. *gof*.

$$(gof)(x_1) = g[f(x_1)] = g(y_1) = z_1, \qquad (gof)(x_2) = g[f(x_2)] = g(y_3) = z_3$$
$$(gof)(x_3) = g[f(x_3)] = g(y_2) = z_3, \qquad (gof)(x_4) = g[f(x_4)] = g(y_2) = z_3$$

So, $gof = \{(x_1, z_1), (x_2, z_3), (x_3, z_3), (x_4, z_3)\}$.

(*ii*) First find the composition of f with g and then with h (Fig. 28).

$$ho(gof)\,(x_1) = ho[g(f(x_1))] = ho[g(y_1)] = h(z_1) = k_1$$
$$ho(gof)\,(x_2) = ho[g(f(x_2))] = ho[g(y_3)] = h(z_3) = k_3$$
$$ho(gof)\,(x_3) = ho[g(f(x_3))] = ho[g(y_2)] = h(z_3) = k_3$$
$$ho(gof)\,(x_4) = ho[g(f(x_4))] = h[g(y_2)] = h(z_3) = k_3$$

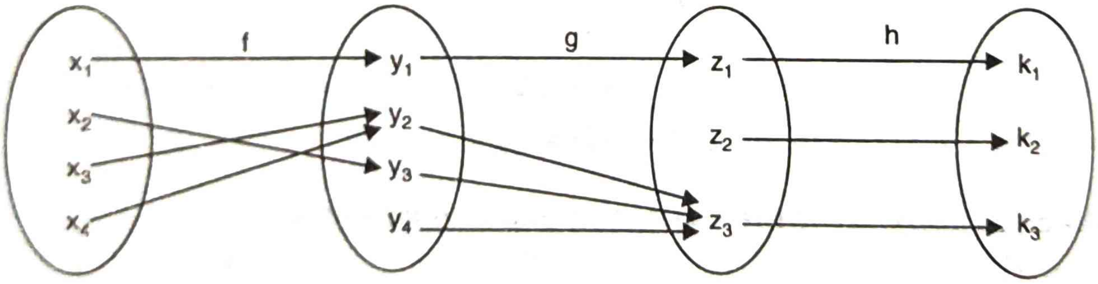


Fig. 28. *ho*(*gof*).

So, $ho(gof) = \{(x_1, k_1), (x_2, k_3), (x_3, k_3), (x_4, k_3)\}$.

(*iii*) First find the composition of g with h and then with f (Fig. 29).

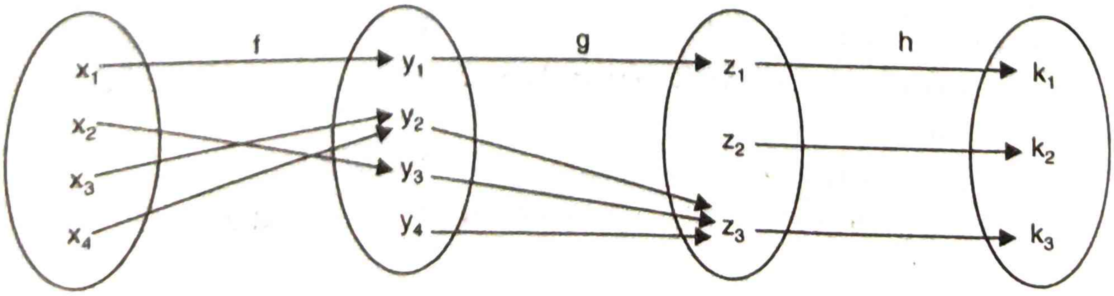


Fig. 29. (*hog*)*of*.

$(hog)(y_1) = h(g(y_1)) = h(z_1) = k_1,$ $\quad$ $(hog)(y_2) = h(g(y_2)) = h(z_3) = k_3$

$(hog)(y_3) = h(g(y_3)) = h(z_3) = k_3,$ $\quad$ $(hog)(y_4) = h(g(y_4)) = h(z_3) = k_3$

So, $\quad hog = \{(y_1, k_1), (y_2, k_2), (y_3, k_3), (y_4, k_4)\}$

Now,

$$((hog)of)\,(x_1) = (hog)(f(x_1)) = hog(y_1) = k_1$$
$$((hog)of)\,(x_2) = (hog)(f(x_2)) = hog(y_3) = k_3$$
$$((hog)of)\,(x_3) = (hog)(f(x_3)) = hog(y_2) = k_3$$
$$((hog)of)\,(x_4) = (hog)(f(x_4)) = hog(y_2) = k_3$$

So, $\quad (hog)of = \{(y_1, k_1), (y_2, k_3), (y_3, k_3), (y_4, k_3)\}.$

Problem 11. *Determine the value of each of the following floor functions :*

(*i*) $[8\,.\,2]$ $\quad$ (*ii*) $[19\,.\,1231]$ $\quad$ (*iii*) $[-9\,.\,6]$

(*iv*) $[8]$ $\quad$ (*v*) $[\sqrt{5}\,]$ $\quad$ (*vi*) $[-11\,.\,2]$.

Sol. (*i*) $[8\,.\,2] = 8$ $\quad$ (*ii*) $[19\,.\,1231] = 19$ $\quad$ (*iii*) $[-9\,.\,6] = -10$

(*iv*) $[8] = 8$ $\quad$ (*v*) $[\sqrt{15}\,] = 3$ $\quad$ (*vi*) $[-11\,.\,2] = -12$.

Problem 12. *Determine the value of each of the following ceiling functions :*

(*i*) $[8\,.\,3]$ $\quad$ (*ii*) $[19\,.\,1231]$ $\quad$ (*iii*) $[-9\,.\,7]$

(*iv*) $[9]$ $\quad$ (*v*) $[\sqrt{17}\,]$ $\quad$ (*vi*) $[-11\,.\,1]$.

Sol. (*i*) $[8\,.\,3] = 9$ $\quad$ (*ii*) $[19\,.\,1231] = 20$ $\quad$ (*iii*) $[-9\,.\,7] = -9$

(*iv*) $[9] = 9$ $\quad$ (*v*) $[\sqrt{17}\,] = 5$ $\quad$ (*vi*) $[-11\,.\,1] = -11$.

Problem 13. *Let $X = \{10, 20, 30\}$ and $U = \{10, 20, 30, \ldots\ldots, 100\}$. Compute the following function values :*

(*i*) $f \times (10)$ $\quad$ (*ii*) $f \times (80)$ $\quad$ (*iii*) $f \times (110)$

(*iv*) $f \times (30)$ $\quad$ (*v*) $f \times (40)$.

Sol. (*i*) $f \times (10) = 1$ $\quad$ (*ii*) $f \times (80) = 0$ $\quad$ (*iii*) $f \times (110) =$ Undefined

(*iv*) $f \times (30) = 1$ $\quad$ (*v*) $f \times (40) = 0$.

Problem 14. *Determine the value of the following functions :*

(*i*) *18(MOD 3)* $\quad$ (*ii*) *11(MOD 9)* $\quad$ (*iii*) *40(MOD 7)*

Sol. (*i*) 18(MOD 3) = 0 $\quad$ (*ii*) 11(MOD 9) = 2 $\quad$ (*iii*) 40(MOD 7) = 5.

Problem 15. *Let f be the function from $X = \{0, 1, 2, 3, 4\}$ to X defined by $f(x) = 4x$(MOD 5). Write f as a set of ordered pairs. Is f one-to-one ?*

Sol. We have $\quad f(x) = 4x$(MOD 5) $\quad$ (given)

Now put the value of X in the function $f(x)$ one by one and obtain the corresponding second value of the pair

e.g., $\quad f(0) = 4 \times 0$(MOD 5) = 0(MOD 5) = 0

$f(1) = 4 \times 1$(MOD 5) = 4(MOD 5) = 4 and so on.