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Functions having point of discontinuity :-

let $f(x) = \begin{cases} f_1(x), & \alpha < x < a \\ f_2(x), & a < x < \alpha + 2\pi \end{cases}$

a is point of finite discontinuity

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\left[\int_{\alpha}^{a} f_1(x)\,dx + \int_{a}^{\alpha+2\pi} f_2(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{\alpha}^{a} f_1(x)\cos nx\,dx + \int_{a}^{\alpha+2\pi} f_2(x)\cos nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{\alpha}^{a} f_1(x)\sin nx\,dx + \int_{a}^{\alpha+2\pi} f_2(x)\sin nx\,dx\right]$$

Examples :→

① $f(x) = \begin{cases} x, & -\pi < x < 0 \\ -x, & 0 < x < \pi \end{cases}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx\,dx + \sum_{n=1}^{\infty} b_n \sin nx$$

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$$\rightarrow \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} x\,dx + \int_{0}^{\pi} -x\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_{-\pi}^{0} + \left(-\frac{x^2}{2}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left(0 - \frac{\pi^2}{2} + \left(-\frac{\pi^2}{2}\right) - 0\right)$$

$$= \frac{1}{\pi}\left(-\pi^2\right) = -\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} \underbrace{x}_{\mathrm{I}}\,\underbrace{\cos nx}_{\mathrm{II}}\,dx + \int_{0}^{\pi} \underbrace{-x}_{\mathrm{I}}\,\underbrace{\cos nx}_{\mathrm{II}}\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(x\frac{\sin nx}{n} - \int \frac{\sin nx}{n}\,dx\right)_{-\pi}^{0} - \left(x\frac{\sin nx}{n} - \int\frac{\sin nx}{n}\,dx\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right)_{-\pi}^{0} - \left(x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right)_{0}^{\pi}\right]$$

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$\frac{\cos n\pi}{n^2}$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{\cos n(-\pi)}{n^2} - \frac{\cos n\pi}{n^2} + \frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2}{n^2} - \frac{2(-1)^n}{n^2}\right] = \frac{2}{n^2\pi}\left(1-(-1)^n\right)$$

$$\rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} \underset{\mathrm{I}}{x}\,\underset{\mathrm{II}}{\sin nx}\,dx + \int_{0}^{\pi} \underset{\mathrm{I}}{-x}\,\underset{\mathrm{II}}{\sin nx}\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{x\cos nx}{n} - \int 1\cdot\left(-\frac{\cos nx}{n}\right)dx\right)_{-\pi}^{0} - \left(-\frac{x\cos nx}{n} - \int 1\cdot\left(-\frac{\cos nx}{n}\right)dx\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{x\cos nx}{n} + \underbrace{\frac{\sin nx}{n^2}}_{=0}\right)_{-\pi}^{0} + \left(\frac{x\cos nx}{n} - \underbrace{\frac{\sin nx}{n^2}}_{=0}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{\cancel{n}} + \frac{\pi\cos n\pi}{\cancel{n}}\right]$$

$$b_n = 0$$

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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = -\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left(1-(-1)^n\right)\cos nx$$

$$= -\frac{\pi}{2} + \frac{2}{\pi}\left(\frac{2\cos x}{1^2} + \frac{2\cos 3x}{3^2} + \frac{2\cos 5x}{5^2} \text{ --- } \infty\right)$$

$$f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} \text{ --- } \infty\right)$$

— A

From above result

hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \text{ --- } = \frac{\pi^2}{8}$

$$f(x) = \begin{cases} x, & -\pi < x < 0 \\ -x, & 0 < x < \pi \end{cases}$$

Put $x = 0$ in eqn A

$$0 = -\frac{\pi}{2} + \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \text{ --- } \infty\right)$$

$$\frac{\pi}{2} = +\frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \text{ --- } \infty\right)$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \text{ --- } \infty = \frac{\pi^2}{8}$$

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Odd and Even functions

Odd function :- A function $f(x)$ is said to be odd function if $f(-x) = -f(x)$

Even function :- A function $f(x)$ is said to be an even function if $f(-x) = f(x)$

In case of odd and even function some integral results :

$$\int_{-a}^{a} f(x)\,dx = 0 \quad , \text{ if } f(x) \text{ is an odd function.}$$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad , \text{ if } f(x) \text{ is an even function}$$

Use in fourier series →

1. If $f(x)$ is an odd function.

We know that $\cos nx$ is an even function
$\sin nx$ —— odd function.

odd f^n x even f^n = odd f^n
Even f^n x even f^n = even f^n
odd f^n x odd f^n = even function

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$f(x)\cos nx$ is odd function

$f(x)\sin nx$ is even function

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 \;, \int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 2\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$\Rightarrow$ If $f(x)$ is odd ~~in t~~ : $a_0 = 0$, $a_n = 0$ in the range $(-\pi, \pi)$

2) $f(x)$ is even f^n

$f(x)\cos nx \rightarrow$ Even f^n

$f(x)\sin nx \longrightarrow$ odd f^n.

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$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\sin nx}_{\text{odd function}}\,dx = 0$$

$\Rightarrow$ If $f(x)$ is even function then $b_n = 0$ in the range $(-\pi, \pi)$

Practice Question! -

(1) $f(x) = x\sin x$ in $-\pi \leq x \leq \pi$

$$f(-x) = -x\sin(-x) = -x(-\sin x) = x\sin x = f(x)$$

$$f(-x) = f(x)$$

$\Rightarrow$ $f(x)$ is an even function.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

Since $f(x)$ is even function

$\Rightarrow$ $b_n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \underbrace{f(x)}_{\text{even } f^n}\,dx = \frac{2}{\pi}\int_0^{\pi} \underbrace{x}_{I}\,\underbrace{\sin x}_{II}\,dx$$

$$= \frac{2}{\pi}\left(-x\cos x - \int 1\cdot(-\cos x)\,dx\right)_0^{\pi}$$

$$= \frac{2}{\pi}\left(-x\cos x + \sin x\right)_0^{\pi} = \frac{2}{\pi}\left(-\pi\cos\pi + 0 - (0+0)\right)$$

$$= \frac{2}{\pi}\left(-\pi(-1)\right) = 2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{x\sin x}_{\text{Even } f^n}\;\underbrace{\cos nx}_{\text{even } f^n}}_{\text{Even function}}\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\cdot(2\cos nx\sin x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left(\sin(n+1)x - \sin(n-1)x\right)dx$$

$$= \frac{1}{\pi}\left[\frac{-x\cos(n+1)x}{(n+1)} + \frac{\sin(n+1)x}{(n+1)^2} - \left(\frac{-x\cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{\pi}$$

$(n \neq 1)$

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$$a_n = \frac{1}{\pi}\left[\frac{-\pi\cos(n+1)\pi}{(n+1)} + \frac{\pi\cos(n-1)\pi}{n-1}\right]$$

$$= -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} \qquad (n \neq 1)$$

$$= -\frac{(-1)^{n-1}(-1)^2}{n+1} + \frac{(-1)^{n-1}}{n-1} = -\frac{(-1)^{n-1}}{n+1} + \frac{(-1)^{n-1}}{n-1}$$

$$= (-1)^{n-1}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) = (-1)^{n-1}\left(\frac{\cancel{n}+1-\cancel{n}+1}{(n-1)(n+1)}\right)$$

$$= (-1)^{n-1} \cdot \frac{2}{n^2-1} \qquad (n \neq 1)$$

for $n = 1$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[-\frac{x\cos 2x}{2} - \int 1 \cdot \left(\frac{-\cos 2x}{2}\right)dx\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-x\frac{\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{-\pi}{2}\right)$$

$$= -\frac{1}{2}$$

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Fourier Series is written as:

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$x \sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}}{n^2-1} \cos nx$$

Questions

① Obtain the Fourier Series for function f(x):

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \leq x \leq 0 \\ 1 - \frac{2x}{\pi}, & 0 \leq x \leq \pi \end{cases}$$

② $f(x) = x - x^2$ from $x = -\pi$ to $x = \pi$

hence show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots \infty = \frac{\pi^2}{12}$

③ $$f(x) = \begin{cases} x, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

hence show that $1 + \frac{1}{3^2} + \frac{1}{5^2} \cdots \infty = \frac{\pi^2}{8}$

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(4) $f(x) = \begin{cases} x, & 0 \leq x \leq \pi \\ 2\pi - x, & \pi \leq x \leq 2\pi \end{cases}$

hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} - - \infty = \frac{\pi^2}{8}$

(5) $f(x) = \begin{cases} 0, & -\pi \leq x \leq 0 \\ \sin x, & 0 \leq x \leq \pi \end{cases}$

Prove that $f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}$

hence show that $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - - \cdot = \frac{1}{2}$

$\frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - - - - \infty = \frac{\pi - 2}{4}$

(6) $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

(7) $f(x) = \begin{cases} x^2, & 0 \leq x \leq \pi \\ -x^2, & -\pi \leq x \leq 0 \end{cases}$

Solution $f(x) = \begin{cases} 1+\frac{2x}{\pi}, & -\pi \leq x \leq 0 \\ 1-\frac{2x}{\pi}, & 0 \leq x \leq \pi \end{cases}$

$$f(-x) = \begin{cases} 1-\frac{2x}{\pi}, & -\pi \leq -x \leq 0 \quad \pi \geq x \geq 0 \\ 1+\frac{2x}{\pi}, & 0 \leq -x \leq \pi \quad 0 \geq x \geq -\pi \end{cases}$$

$$= \begin{cases} 1-\frac{2x}{\pi}, & 0 \leq x \leq \pi \\ 1+\frac{2x}{\pi}, & -\pi \leq x \leq 0 \end{cases}$$

Now $f(-x) = f(x) \Rightarrow f(x)$ is an even function. Hence in fourier series expansion $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx = \frac{2}{\pi}\left(x - \frac{2\cdot x^2}{2\pi}\right)_0^{\pi}$$

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$$a_0 = \frac{2}{\pi}\left(\pi - \pi\right) = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \underbrace{\left(1 - \frac{2x}{\pi}\right)}_{I} \underbrace{\cos nx}_{II}\, dx$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \int\left(-\frac{2}{\pi}\right)\frac{\sin nx}{n}\,dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \frac{2}{n^2\pi}\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2}{n^2\pi}\cos n\pi + \frac{2}{n^2\pi}\right]$$

$$= \frac{4}{n^2\pi^2}\left(1 - (-1)^n\right) \checkmark$$

Now Fourier Series expansion is written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)\cos nx$$

$$= \frac{4}{\pi^2}\left[\frac{2}{1^2}\cos x + \frac{2}{3^2}\cos 3x + \frac{2}{5^2}\cos 5x - \infty\right]$$

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$$f(x) = \frac{8}{\pi^2}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} \text{——} \infty\right)$$