CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

9. A Baseline HANK Model

Adv. Macro: Heterogenous Agent Models

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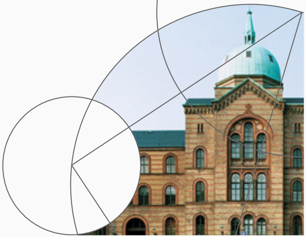

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Introduction

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- **Model:** Heterogeneous Agent New Keynesian (HANK) model
- **Code:**
  1. Based on the GEModelTools package
  2. Example from GEModelToolsNotebooks/HANK
- **Literature:**
  1. Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«
  2. Documentation for GEModelTools

HANK model

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- **Households:**
  1. Differ by stochastic idiosyncratic productivity and savings
  2. Supply labor and choose consumption
  3. Subject to a borrowing constraint

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- **Central bank**: Set nominal interest rate

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$$\max_{y_{jt}\ \forall j} P_t Y_t - \int_0^1 p_{jt} y_{jt} dj \text{ s.t. } Y_t = \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu}$$

for given output price, P_t, and input prices, p_{jt}

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- **Note:** Zero profits (can be used to derive price index)

Derivation of demand curve

- **FOC wrt.** y_{jt}

$$
\begin{aligned}
0 &= P_t \mu \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} \, dj \right)^{\mu - 1} \frac{1}{\mu} y_{jt}^{\frac{1}{\mu} - 1} - p_{jt} \Leftrightarrow \\
\frac{p_{jt}}{P_t} &= \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} \, dj \right)^{\mu - 1} y_{jt}^{\frac{1-\mu}{\mu}} \Leftrightarrow \\
\left(\frac{p_{jt}}{P_t} \right)^{\frac{\mu}{\mu - 1}} &= \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} \, dj \right)^{\mu} y_{jt}^{-1} \Leftrightarrow \\
y_{jt} &= \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu - 1}} Y_t
\end{aligned}
$$

Intermediary goods firms

- ***Dynamic* problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, n_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$

$$\text{s.t. } y_{jt} = Z_t n_{jt}, \;\; y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t$$

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$$\log(1+\pi_t) = \kappa \left(\frac{w_t}{Z_t} - \frac{1}{\mu} \right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}}, \;\; \pi_t \equiv P_t / P_{t-1} - 1$$

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- **Implied production:** $Y_t = y_{jt}$, $N_t = n_{jt}$ (from symmetry)
- **Implied dividends:** $d_t = Y_t - w_t N_t - \frac{\mu}{\mu-1}\frac{1}{2\kappa}\left[\log(1+\pi_t)\right]^2$

- **FOC wrt. p_{jt}:**

$$0 = \left(1 - \frac{\mu}{\mu - 1}\right)\left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{P_t} + \frac{\mu}{\mu - 1}\frac{w_t}{Z_t}\left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{p_{jt}}$$
$$- \frac{\mu}{\mu - 1}\frac{1}{\kappa}\frac{\log\left(\frac{p_{jt}}{p_{jt-1}}\right)}{p_{jt}} Y_t + \frac{J'_{t+1}(p_{jt})}{1 + r_{t+1}}$$

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- **FOC + Envelope + Symmetry +** $\pi_t = P_t/P_{t-1} - 1$

$$0 = \left(1 - \frac{\mu}{\mu - 1}\right)\frac{Y_t}{P_t} + \frac{\mu}{\mu - 1}\frac{w_t}{Z_t}\frac{Y_t}{P_t}$$

$$+ \frac{\mu}{\mu - 1}\frac{1}{\kappa}\log\left(1 + \pi_t\right)\frac{Y_t}{P_t} + \frac{\frac{\mu}{\mu-1}\frac{1}{\kappa}\log\left(1 + \pi_{t+1}\right)\frac{Y_{t+1}}{P_t}}{1 + r_{t+1}}$$

Households

- **Household problem**: Distribution, $\boldsymbol{D}_t$, over z_t and a_{t-1}

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_t^{1+\nu}}{1+\nu} + \beta \mathbb{E}\left[v_{t+1}(z_{t+1}, a_t) \,|\, z_t, a_t\right]$$

$$\text{s.t. } a_t + c_t = (1 + r_t)a_{t-1} + (w_t \ell_t - \tau_t + d_t)z_t \geq 0$$

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- **Effective labor-supply:** $n_t = z_t \ell_t$

EGM I

- **Beginning-of-period value function:**

$$\underline{v}_{a,t}(z_{t-1}, a_{t-1}) = \mathbb{E}_t\left[v_{a,t}(z_t, a_{t-1})\right] = \mathbb{E}\left[(1+r_t)c_t^{-\rho}\right]$$

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- **Endogenous grid method:** Vary z_t and a_t to find

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- **Consumption and labor supply:** Use linear interpolation to find

$c^*(z_t, a_{t-1})$ and $\ell^*(z_t, a_{t-1})$ with $m_t = (1+r_t)a_{t-1} + (w_t \ell_t^* - \tau_t + d_t) z_t$

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- **Savings:** $a^*(z_t, a_{t-1}) = (1 + r_t) a_{t-1} - c_t^* + (w_t \ell_t^* - \tau_t + d_t) z_t$

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  $$\begin{aligned} c^* &= (1 + r_t)a_{t-1} + (w_t \ell^* - \tau_t + d_t) z_t \\ n^* &= z_t \ell^* \end{aligned}$$

  2. Set

  $$\ell^* = \frac{f(\ell^*)}{f'(\ell^*)} = \frac{f(\ell^*)}{1 - \left(\frac{w_t z_t}{\varphi}\right)^{\frac{1}{\nu}} \left(-\frac{\sigma}{\nu}\right) (c^*)^{-\frac{\sigma}{\nu}} w_t z_t}$$

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  $$n^* = z_t \ell^*$$
  2. Set
  $$\ell^* = \frac{f(\ell^*)}{f'(\ell*)} = \frac{f(\ell^*)}{1 - \left(\frac{w_t z_t}{\varphi}\right)^{\frac{1}{\nu}} \left(-\frac{\sigma}{\nu}\right) (c^*)^{-\frac{\sigma}{\nu}} w_t z_t}$$
  3. Return to step 1

Government and central bank

- **Monetary policy:** Folow Taylor-rule:

$$i_t = i_t^* + \phi \pi_t$$

where i_t^* is a shock

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$$r_t = (1 + i_{t-1})/(1 + \pi_t) - 1$$

- **Government:** Choose τ_t to keep debt constant and finance exogenous public consumption

$$\tau_t = r_t B_{ss} + G_t$$

Market clearing

1. Labor: $N_t = \int n_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t$ (in effective units)
2. Assets: $B_{ss} = \int a_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t$
3. Goods: $Y_t = \int c_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t$

As an equation system

$$\boldsymbol{H}\left(\boldsymbol{\pi}, \boldsymbol{w}, \boldsymbol{Y}, \boldsymbol{i}^{*}, \boldsymbol{Z}, \underline{\boldsymbol{D}}_{0}\right) = \mathbf{0}$$

$$\begin{bmatrix} \log(1+\pi_t) - \left[\kappa\left(\frac{w_t}{Z_t} - \frac{1}{\mu}\right) + \frac{Y_{t+1}}{Y_t}\frac{\log(1+\pi_{t+1}}{1+r_{t+1}})\right] \\ N_t - \int n_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t \\ B_{ss} - \int a_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t \end{bmatrix} = \mathbf{0}$$

The rest of the model is given by

$$\boldsymbol{X} = \boldsymbol{M}(\boldsymbol{\pi}, \boldsymbol{w}, \boldsymbol{Y}, \boldsymbol{i}^{*}, \boldsymbol{Z})$$

As a DAG (from Auclert et al., 2021)

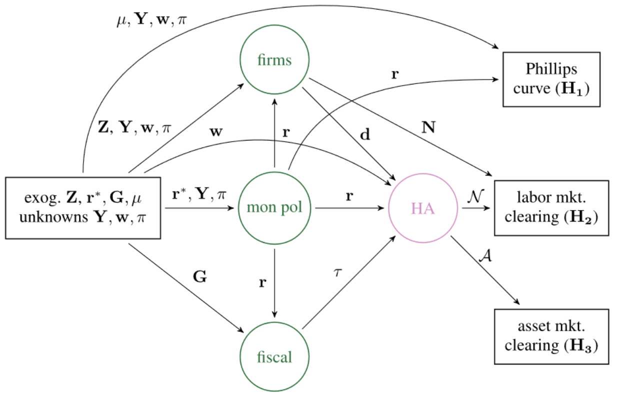

FIGURE C.1.—DAG representation of one-asset HANK economy.

Notation: $\boldsymbol{i}^* = \boldsymbol{r}^*$, $\boldsymbol{\mu}$ is a shock, $\boldsymbol{A}^{hh} = \mathcal{A}$, $\boldsymbol{N}^{hh} = \mathcal{N}$

Steady state

- **Chosen:** B_{ss}, G_{ss}, r_{ss}
- **Analytically:**
  1. **Normalization:** $Z_{ss} = N_{ss} = 1 \Rightarrow Y_{ss} = 1$
  2. **Zero-inflation:** $\pi_{ss} = 0 \Rightarrow i_{ss} = i^*_{ss} = r_{ss}$
  3. **Firms:** $Y_{ss} = Z_{ss} N_{ss}$, $w_{ss} = \frac{Z_{ss}}{\mu}$ and $d_{ss} = Y_{ss} - w_{ss} N_{ss}$
  4. **Government:** $\tau_{ss} = r_{ss} B_{ss} + G_{ss}$
  5. **Assets:** $A_{ss} = B_{ss}$
- **Numerically:** Choose β and φ to get market clearing

The HANK example from GEModelToolsNotebooks I

- **Presentation:** I go through the code for finding the transition path
- **In-class exercise:**
    1. Look at the code and talk about it with the person next to you for 10 minutes
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IRFs and simulation

Linearized IRFs

- **Previously:** Full non-linear transition path to an MIT-shock

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  1. Solve for IRFs for unknowns

$$\boldsymbol{H}(\boldsymbol{U},\boldsymbol{Z})=0 \Rightarrow \boldsymbol{H}_{\boldsymbol{U}}d\boldsymbol{U}+\boldsymbol{H}_{\boldsymbol{Z}}d\boldsymbol{Z}=0 \Leftrightarrow d\boldsymbol{U}=\underbrace{-\boldsymbol{H}_{\boldsymbol{U}}^{-1}\boldsymbol{H}_{\boldsymbol{Z}}}_{\equiv \boldsymbol{G}_{\boldsymbol{U}}}d\boldsymbol{Z}$$

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2. Derive all other IRFs for

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- **Limitations:**

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  1. Imprecise for *large* shocks

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- **Limitations:**

  1. Imprecise for *large* shocks
  2. Imprecise in models with *aggregate non-linearities* (direct in aggregate equations or through micro-behavior)

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.

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  where $d\boldsymbol{X}_s$ is the IRF to a unit-shock after s periods

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     where $d\boldsymbol{X}_s$ is the IRF to a unit-shock after s periods
- **Intuition:** Sum of first order effects from all previous shocks
- **Equivalence:**
  1. Same result if we linearize all aggregated equations and write the model in MA(∞) form
  2. The state space form can also be recovered (not needed)

Advanced linearized simulation

- **Generality:** Extend the model with auxiliary variables (incl. distributional moments) to calculate additional IRFs and simulations

Advanced linearized simulation

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- **Full distribution** (advanced)**:**

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- **Full distribution** (advanced)**:**
  1. The IRF for grid point i_g in a policy function can be calculated as

$$da^*_{i_g,s} = \sum_{s'=s}^{T-1} \sum_{X^{hh} \in \boldsymbol{X}^{hh}} \frac{\partial a^*_{i_g}}{\partial X^{hh}_{s'-s}} dX^{hh}_{s'}.$$

  where $\partial \boldsymbol{a}^*_{i_g} / \partial X^{hh}_k$ is the derivative to a k-period ahead shock to input X^{hh} (calculated in fake news algorithm)

Advanced linearized simulation

- **Generality:** Extend the model with auxiliary variables (incl. distributional moments) to calculate additional IRFs and simulations
- **Full distribution** (advanced)**:**
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     where $\partial \boldsymbol{a}_{i_g}^{*} / \partial X_k^{hh}$ is the derivative to a k-period ahead shock to input X^{hh} (calculated in fake news algorithm)

  2. The policy function can there be simulated as

$$\boldsymbol{a}_{i_g}^{*} = \sum_{s=0}^{T} da_{i_g,s}^{*} \tilde{\boldsymbol{\epsilon}}_{t-s}$$

Advanced linearized simulation

- **Generality:** Extend the model with auxiliary variables (incl. distributional moments) to calculate additional IRFs and simulations
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  2. The policy function can there be simulated as

$$\boldsymbol{a}^*_{i_g} = \sum_{s=0}^{T} da^*_{i_g,s} \tilde{\boldsymbol{\epsilon}}_{t-s}$$

  3. Distribution can then be simulated forwards

The HANK example from GEModelToolsNotebooks II

- **Presentation:** I go through the code for *finding the linearized IRFS and simulating the model*
- **In-class exercise:**
  1. Look at the code and talk about it with the person next to you for 10 minutes
  2. Write at least one question on https://padlet.com/jeppe_druedahl/advmacrohet

Exercise

Exercise = Assignment II

You can start working on Assignment II: The HANK model

Summary

Summary and next week

- **Today:**
    1. A baseline HANK model
    2. Linearized IRFs and simulation
- **Next week:** Analytical Properties of HANK models
- **Homework:**
    1. Work on Assignment II
    2. Read: Auclert et al. (2018), »The Intertemporal Keynesian Cross«