CENTER FOR ECONOMIC BEHAVIOR & INEQUALITY

3. Stationary Equilibrium

Adv. Macro: Heterogenous Agent Models

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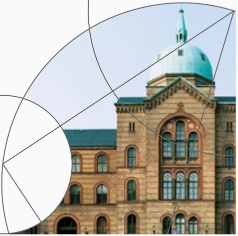

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Introduction

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- **Today:** Interaction through Walrassian markets
- **Model:** Heterogeneous Agent Neo-Classical (HANC) model
- **Equilibrium-concept:** Stationary equilibrium
- **Code:** Based on the GEModelTools package
  1. Is in active development
  2. You can help to improve interface
  3. You can help to find bugs
  4. You can help to add features

HANC

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  3. The Standard Incomplete Market (SIM) model

Notation - central variables

- **Aggregate variables (quantities and prices):**
  1. Technology: Γ_t
  2. Capital: K_t
  3. Labor: L_t
  4. Consumption: C_t
  5. Investment: I_t
  6. Rental rate: r_t^k
  7. Real wage: w_t
- **Idiosyncratic variables:**
  1. Saving: a_t
  2. Consumption: c_t
  3. Productivity: z_t
- **Distributions:**
  1. $\underline{\boldsymbol{D}}_t$ over z_{t-1} and a_{t-1}
  2. $\boldsymbol{D}_t$ over z_t and a_{t-1}

Firms

- **Production function:** $Y_t = \Gamma_t K_{t-1}^{\alpha} L_t^{1-\alpha}$
- **Profits:** $\Pi_t = Y_t - w_t L_t - r_t^k K_{t-1}$
- **Profit maximization:** $\max_{K_{t-1},L_t} \Pi_t$
  - Rental rate: $\frac{\partial \Pi_t}{\partial r_t^k} = 0 \Leftrightarrow r_t^k = \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha\text{-}1}$
  - Real wage: $\frac{\partial \Pi_t}{\partial w} = 0 \Leftrightarrow w_t = (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha}$

Households

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(z_{t+1}, a_t) \right]$$

$$\text{s.t. } a_t + c_t = (1 + r_t) a_{t-1} + w_t z_t \geq 0$$

$$\log z_t = \rho \log z_{t-1} + \varepsilon_t^z \ \ , \varepsilon_t^z \sim \mathcal{N}(\mu_z, \sigma_z), \ \mathbb{E}[z_t] = 1$$

with $r_t \equiv r_t^k - \delta$, where δ is the depreciation rate

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- **Aggregates:**

$$A_t^{hh} = \int a_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t = A^{hh}\left(\underline{\boldsymbol{D}}_t, \{r_\tau, w_\tau\}_{\tau \geq t}\right) = \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t$$

$$C_t^{hh} = \int c_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t = C^{hh}\left(\underline{\boldsymbol{D}}_t, \{r_\tau, w_\tau\}_{\tau \geq t}\right) = \boldsymbol{c}_t^{*\prime} \boldsymbol{D}_t$$

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- **Distributional dynamics** (with histogram method):
  1. Stochastic: $\boldsymbol{D}_t = \Pi_z' \underline{\boldsymbol{D}}_t$
  2. Choices: $\underline{\boldsymbol{D}}_{t+1} = \Lambda_t' \boldsymbol{D}_t, \quad \Lambda_t = \Lambda\left(\{r_\tau, w_\tau\}_{\tau \geq t}\right)$

Resource constraint and market clearing

- **Law-of-motion for capital**

$$K_t = (1-\delta)K_{t-1} + I_t$$

- **Market clearing:**
  1. Labor market: $L_t = \int z_t dD_t = 1$
  2. Goods market: $Y_t = C_t + I_t$
  3. Capital market: $K_t = \int a_t dD_t$

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  3. Aggregating across individual

$$
\begin{aligned}
C_t + Y_t &= \int c_t dD_t + (K_t - (1-\delta)K_{t-1}) \\
&= \int \left[(1+r_t)a_{t-1} + w_t z_t - a_t\right] d\boldsymbol{D}_t + K_t - (1-\delta)K_{t-1} \\
&= (1+r_t)K_{t-1} + w_t - K_t + K_t - (1-\delta)K_{t-1} \\
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  4. Combined: Then *the goods market clears*

$$Y_t = C_t + I_t$$

Equation system

The model can be written as an **equation system**

$$\boldsymbol{H}\left(\{K_t, L_t; \Gamma_t\}_{t\geq 0}, \underline{\boldsymbol{D}}_0\right) = \begin{bmatrix} K_t - \boldsymbol{a}_t^{*\prime}\boldsymbol{D}_t \\ r_t - \alpha\Gamma_t(K_{t-1}/L_t)^{\alpha-1} \\ w_t - (1-\alpha)\Gamma_t(K_{t-1}/L_t)^{\alpha} \\ L_t - 1 \\ \boldsymbol{D}_t - \Pi_z^{\prime}\underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} - \Lambda_t^{\prime}\boldsymbol{D}_t \\ \forall t \in \{0, 1, \ldots\} \end{bmatrix} = \boldsymbol{0}$$

where $\{\Gamma_t\}_{t\geq 0}$ is a given technology path and $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$

Remember: Policies and choice transitions depend on prices

1. Policy function: $a_t^* = a^*\left(\{r_\tau, w_\tau\}_{\tau\geq t}\right)$
2. Choice transition: $\Lambda_t = \Lambda\left(\{r_\tau, w_\tau\}_{\tau\geq t}\right)$

Stationary Equilibrium

Stationary equilibrium - equation system

The **stationary equilibrium** satisfies

$$H_{ss}\left(K_{ss}, L_{ss}; \Gamma_{ss}\right) = \begin{bmatrix} K_t - \boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss} \\ r_{ss} - \alpha\Gamma_{ss}(K_{ss}/L_{ss})^{\alpha-1} \\ w_{ss} - (1-\alpha)\Gamma_{ss}(K_{ss}/L_{ss})^{\alpha} \\ L_{ss} - 1 \\ \boldsymbol{D}_{ss} - \Pi_z^{\prime}\underline{\boldsymbol{D}}_{ss} \\ \underline{\boldsymbol{D}}_{ss} - \Lambda_{ss}^{\prime}\boldsymbol{D}_{ss} \end{bmatrix} = \mathbf{0}$$

Note I: Households still move around »inside« the distribution due to idiosyncratic shocks

Note II: Steady state for aggregates (quantities and prices) and the distribution as such

Stationary equilibrium - more verbal definition

For a given Γ_{ss}

1. Quantities K_{ss} and L_{ss},
2. prices r_{ss} and w_{ss},
3. the distribution $\boldsymbol{D}_{ss}$ over z_t and a_{t-1}
4. and the policy functions $a_{ss}^*(z_t, a_{t-1})$ and $c_{ss}^*(z_t, a_{t-1})$

are such that

1. Household maximize expected utility (policy functions)
2. Firms maximize profits (prices)
3. $\boldsymbol{D}_{ss}$ is the invariant distribution implied by the household problem
4. The labor market clears
5. The capital market clears
6. The goods market clears

Direct implementation

Root-finding problem in K_{ss} with the objective function:

1. Set $L_{ss} = 1$
2. Calculate $r_{ss} = \alpha\Gamma_{ss}(K_{ss})^{\alpha-1}$ and $w_{ss} = (1-\alpha)\Gamma_{ss}(K_{ss})^{\alpha}$
3. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
4. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
5. Return $K_{ss} - \boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss}$

Indirect implementation

1. Choose r_{ss} and w_{ss}
2. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
3. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
4. Set $K_{ss} = \boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss}$
5. Set $L_{ss} = 1$
6. Set $\Gamma_{ss} = \frac{w_{ss}}{(1-\alpha)\Gamma_{ss}(K_{ss})^{\alpha}}$
7. Set $r_{ss}^{k} = \alpha\Gamma_{ss}(K_{ss})^{\alpha-1}$
8. Set $\delta = r_{ss}^{k} - r_{ss}$

Steady interest rate

- **Complete markets / representative agent:** Derived from aggregate Euler-equation

$$C_t^{-\rho} = \beta(1+r)C_{t+1}^{-\rho} \Rightarrow C_{ss}^{-\rho} = \beta(1+r)C_{ss}^{-\rho} \Leftrightarrow \beta = \frac{1}{1+r}$$

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- **Heterogeneous agents:** *No such equation exists*
  1. Euler-equation replaced by asset market clearing condition
  2. Idiosyncratic income risk affects the steady state interest rate

Calibration

How to choose parameters?

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  1. **Informal:** Roughly match targets by hand
  2. **Formal:**
     - 2a. Solve root-finding problem
     - 2b. Minimize a squared loss function
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- **Complication:** *We must always solve for the steady state for each guess of the parameters to be calibrated*

Exercises

Exercises: Model extensions

1. **Households:** Solve

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(z_{t+1}, a_t) \right]$$
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   where r_t is the real-interest rate and τ_t is a tax rate

2. **Government:** Set taxes and government bonds follows the law-of-motion

$$B_{t+1} = (1+r_t)B_t - \int \tau_t z_t d\boldsymbol{D}_t$$

3. **Bond market clearing:** $B_t = \int a_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t$
4. **Define and find the stationary equilibrium**
5. **What is the optimal level of τ_t?**

Summary

Summary and next week

- **Today:**
  1. The concept of a stationary equilibrium
  2. Introduction to the GEModelTools package
- **Next week:** More on models with interesting dynamics in the stationary equilibrium
- **Homework:**
  1. Work on completing the model extension exercise
  2. Read TBA