CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

1. Introduction

Adv. Macro: Heterogenous Agent Models

Jeppe Druedahl & Patrick Moran

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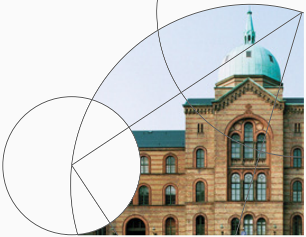
Danmarks
Grundforskningsfond
Danish National
Research Foundation



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  Prerequisite: *Intro. to Programming and Numerical Analysis*
  Complicated: *Close to the research frontier*

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- **Plan for today:**
  1. More about the course
  2. Dynamic programming - theory
  3. Dynamic programming - practice

Macroeconomic Models with Heterogeneous Agents

- **Model components:**
  1. Optimizing individual agents (households $+$ firms)
  2. Idiosyncratic and aggregate risk
  3. Information flows (who knows what when)
  4. Market clearing (Walras vs. search-and-match)

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- **HANK:** Heterogeneous Agent *New Keynesian* model (i.e. include price and wage setting frictions)

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- **Material:**

  Web: https://advmacrohet.netlify.app/

  Git: https://github.com/NumEconCopenhagen/Adv-Macro-Het-2022

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- **Code:**
  1. We provide code you will build upon
  2. Based on the GEModelTools package

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  3. **Free assignment** with essay on a relevant model extension of own choice and a simple implementation
     Deadline for proposal: 9th of December
     Deadline for peer feedback: 16th of December (*exam requirement*)

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- **Exam:**
  1. Hand-in 3×**assignments**
  2. **48 hour take-home:** Programming of new extension
     + analysis of model + interpretation of results

Python

1. **Assumed knowledge:** From Introduction to Programming and Numerical Analysis you are assumed to know the basics of
   1.1 Python
   1.2 JupyterLab
   1.3 VSCode
   1.4 git
2. **Updated Python:** Install (or re-install) newest Anaconda
3. **Packages:**

```
pip install quantecon, EconModel, consav
pip install GEModelTools
```

Course plan

TBA

Knowledge

1. Account for, formulate and interpret precautionary saving models
2. Account for stochastic and non-stochastic simulation methods
3. Account for, formulate and interpret general equilibrium models with ex ante and ex post heterogeneity, idiosyncratic and aggregate risk, and with and without pricing frictions
4. Discuss the difference between the stationary equilibrium, the transition path and the dynamic equilibrium
5. Discuss the relationship between various equilibrium concepts and their solution methods
6. Identify and account for methods for analyzing the dynamic distributional effects of long-run policy (e.g. taxation and social security) and short-run policy (e.g. monetary and fiscal policy)

Skills

1. Solve precautionary saving problems with dynamic programming and simulate behavior with stochastic and non-stochastic techniques
2. Solve general equilibrium models with ex ante and ex post heterogeneity, idiosyncratic and aggregate risk, and with and without pricing frictions (stationary equilibrium, transition path, dynamic equilibrium)
3. Analyze dynamics of income and wealth inequality
4. Analyze transitional and permanent structural changes (e.g. inequality trends and the long-run decline in the interest rate)
5. Analyze the dynamic distributional effects of long-run policy (e.g. taxation and social security) and short-run policy (e.g. monetary and fiscal policy)

Competencies

1. Independently formulate, discuss and assess research on both the causes and effects of heterogeneity and risk for both long-run and short-run outcomes
2. Discuss and assess the importance of how heterogeneity and risk is modeled for questions about both long-run and short-run dynamics

Dynamic Programming

From static to dynamic optimization

- **Budget constraint** for $t \in \{0, 1, \ldots, T-1\}$

$$\text{assets}_t = (1 + \text{return rate}) \times \text{assets}_{t-1} + \text{wage} \times \text{productivity}_t - \text{consumption}_t$$

$$a_t = (1 + r)a_{t-1} + wz_t - c_t$$

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- **Borrowing constraint:** $a_t \geq 0$
- **Static problem:**
  1. **Information:** z_t is known for all t
  2. **Target:** Discounted utility, $\sum_{t=0}^{T-1} \beta^t u(c_t), \;\; \beta > 0$
  3. **Behavior:** Choose $c_0, c_1, \ldots, c_{T-1}$ *simultaneously*
  4. **Solution:** Sequence of consumption *choices* $c_0, c_1, \ldots, c_{T-1}$

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  4. **Solution:** Sequence of consumption *choices* $c_0, c_1, \ldots, c_{T-1}$
- **Dynamic programming:**
  1. **Information:** z_t is revealed period-by-period
  2. **Target:** *Expected* discounted utility, $\sum_{t=0}^{T-1} \beta^t \mathbb{E}_t[u(c_t)], \;\; \beta > 0$
  3. **Behavior:** Choose c_t *sequentially* as information is revealed
  4. **Solution:** Sequence of consumption *functions*, $c_t^\star(z_t, a_{t-1})$

Static solution: IBC

- **Substitution** implies *Intertemporal Budget Constraint* (IBC)

$$
\begin{aligned}
a_{T-1} &= (1+r)a_{T-2} + wz_{T-1} - c_{T-1} \\
&= (1+r)^2 a_{T-3} + (1+r)wz_{T-2} - (1+r)c_{T-1} + wz_{T-1} - c_{T-1} \\
&= (1+r)^T a_{-1} + \sum_{t=0}^{T-1} (1+r)^{T-1-t}(wz_t - c_t)
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- Use **terminal condition** $a_{T-1} = 0$ with equality due utility maximization

$$
(1+r)^{-(T-1)} a_{T-1} = 0 \quad \Leftrightarrow \quad b_0 + h_0 = \sum_{t=0}^{T-1} (1+r)^{-t} c_t
$$

where $b_0 = (1+r)a_{-1}$ and $h_0 \equiv \sum_{t=0}^{T-1}(1+r)^{-t} wz_t$

Static solution: FOC and consumption function

$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t \frac{c_t^{1-\rho}}{1-\rho} + \lambda \left[\sum_{t=0}^{T-1} (1+r)^{-t} c_t - (1+r)a_{-1} - h_0 \right]$$

- **First order conditions**:

$$\forall t : 0 = \beta^t c_t^{-\rho} - \lambda (1+r)^{-t} \Leftrightarrow c_t^{-\rho} = \beta(1+r) c_{t+1}^{-\rho}$$

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- Insert **Euler** into **IBC** to get consumption choice

$$\begin{aligned} \sum_{t=0}^{T-1} (1+r)^{-t} (\beta(1+r))^{t/\rho} c_0 &= b_0 + h_0 \Leftrightarrow \\ c_0 &= \frac{1-(\beta(1+r))^{1/\rho}/(1+r)}{1-((\beta(1+r))/(1+r))^T}(b_0 + h_0) \end{aligned}$$

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- **Question:** Is the solution correct? For all b?

Dynamic solution: Bellman's Principle of Optimality

- **In words:** *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision*. (See Bellman, 1957, Chap. III.3.)

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- **In math:**
  1. **Value function, v_t:** Defined recursively from

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$

$$\text{s.t. } a_t = (1+r)a_{t-1} + wy - c_t \geq 0$$

     with $v_T(\bullet) = 0$.

  2. **Policy function, $c_t^\star$:** Is the same as

$$c_t^\star(z_t, a_{t-1}) = \arg\max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$

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Vocabulary

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq 0$$

1. **State variables:** z_t and a_{t-1}
2. **Control variable:** c_t
3. **Continuation value:** $\beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$
4. **Parameters:** r, w, and stuff in $u(\bullet)$

Numerical value function iteration - basics

- **Discretization:** All state variable belong to discrete monotonically increasing sets $\equiv$ grids,

$$z_t \in \mathcal{G}_z = \{z^0, z^1, \dots, z^{\#z-1}\}$$
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- **Transition probabilities:** $\pi_{i_z,i_{z+}} = \Pr[z_{t+1} = z^{i_{z+}} \mid z_t = z^{i_z}]$

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- **Transition probabilities:** $\pi_{i_z, i_{z+}} = \Pr[z_{t+1} = z^{i_{z+}} \mid z_t = z^{i_z}]$
- **Linear interpolation** (function approximation):
  1. Assume v_{t+1} is known on $\mathcal{G}_z \times \mathcal{G}_a$
  2. Evaluate $v_{t+1}(z^{i_{z+1}}, a)$ for arbitrary a by

$$\breve{v}_{t+1}(z^{i_{z+}}, a) = v_{t+1}(z^{i_{z+}}, a^i) + \omega_i \frac{a - a^i}{a^{i+1} - a^i}$$
$$\omega_i \equiv \frac{v_{t+1}(z^{i_{z+}}, a^{i+1}) - v_{t+1}(z^{i_{z+}}, a^i)}{a^{i+1} - a^i}$$
$$i \equiv \text{largest } \iota \in \{0, 1, \ldots, \#_a - 2\} \text{ such that } a^\iota \leq a$$

Deriving transition probabilities

- **Specification:** Assume $\log z_t$ follows the AR(1) process

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1}, \;\; \psi_{t+1} \sim \mathcal{N}(\mu_\psi, \sigma_\psi^2)$$

  where μ_ψ is used to ensure $\mathbb{E}[z_t] = 1$

- **Literature:** Tauchen (1986), Tauchen and Hussey (1991) and Rouwenhorst (1995) develops method for deriving $\mathcal{G}_z$ and $\pi_{i_z, i_{z+}}$, but we don't care about the details here

Numerical value function iteration - loops

- **Value-of-choice :**

$$v_t(z^{i_z}, a_{t-1}|c_t) = u(c_t) + \beta \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z,i_{z+}} \breve{v}_{t+1}(z^{i_{z+1}}, a_t)$$

$$a_t = (1+r)a_{t-1} + wz^{i_z} - c_t$$

- **Nested loops:**
  1. **Outer loop:** Backwards in time from $t = T-1$ (note v_T is known)
  2. **Inner loop:** For each grid point in $\mathcal{G}_z \times \mathcal{G}_a$ find $c_t^\star(z_t, a_{t-1})$ and therefore $v_t^\star(z_t, a_{t-1})$ with a numerical optimizer

Numerical Monte Carlo simulation

- **Initial distribution:** Draw $z_{i,0}$ and $a_{i,-1}$ for $i \in \{0, 1, \ldots, N-1\}$

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  1. If $t > 0$: Draw z_{it}
  2. Use linear interpolation to evaluate

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c_{it} &= c_t^\star(z_{it}, a_{it-1}) \\
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- **Review:**
  - **Pro:** Simple to implement
  - **Con:** Computationally costly and introduces randomness

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     2.2 Calculate $\omega = \frac{a^{i+1} - a^*(i_{z+}, i_{a-})}{a^{i+1} - a^i} \in [0, 1]$

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     - 2.2 Calculate $\omega = \frac{a^{i+1} - a^*(i_{z+}, i_{a-})}{a^{i+1} - a^i} \in [0, 1]$
     - 2.3 Increment $\underline{\boldsymbol{D}}_{t+1}(z^{i_z}, a^i)$ with $\omega \boldsymbol{D}_t(z^{i_z}, a^{i_a-})$

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  2. **Distribute endogenous mass:** For each i_z and i_{a-} do
     - 2.1 Find largest $i \in \{0, 1, \ldots, \#_a - 2\}$ such that $a_t^\star(i_z, i_{a-}) \geq a^i$
     - 2.2 Calculate $\omega = \frac{a^{i+1} - a^*(i_{z+}, i_{a-})}{a^{i+1} - a^i} \in [0, 1]$
     - 2.3 Increment $\underline{\boldsymbol{D}}_{t+1}(z^{i_z}, a^i)$ with $\omega \boldsymbol{D}_t(z^{i_z}, a^{i_{a-}})$
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Numerical histogram simulation

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- **Review:**
  1. **Pro:** Computationally efficient and no randomness
  2. **Con:** Introduces a non-continuous distribution

Infinite horizon: $T \to \infty$?

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq 0$$

- **Contraction mapping result:** *If β is low enough (strong enough impatience) then the value and policy function converge to $v(z_t, a_{t-1})$ and $c^\star(z_t, a_{t-1})$ so for large enough T*
- **Maximum upper limit for β:** $\frac{1}{1+r}$
- **In practice:** Solve backwards until value and policy functions does not change anymore (given some tolerance)

EGM

Euler-equation from variation argument

- **Case I:** If $c_t^{-\rho} > \beta(1+r)\mathbb{E}_t\left[c_{t+1}^{-\rho}\right]$:
  Increase c_t by $\Delta > 0$, and lower c_{t+1} by $(1+r)$
  1. **Feasible:** Yes, if $a_t > 0$
  2. **Utility change:** $\left(c_t^{-\rho}\right) + (1+r)\left(-\beta\mathbb{E}_t\left[c_{t+1}^{-\rho}\right]\right) > 0$

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- **Case II:** If $c_t^{-\rho} < \beta(1+r)\mathbb{E}_t\left[c_{t+1}^{-\rho}\right]$:
  Lower c_t by $\Delta > 0$, and increase c_{t+1} by $(1+r)$
  1. **Feasible:** Yes (always)
  2. **Utility change:** $\left(-c_t^{-\rho}\right) + (1+r)\left(\beta\mathbb{E}_t\left[c_{t+1}^{-\rho}\right]\right) > 0$

Euler-equation from variation argument

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- **Conclusion:**
  1. **Constrained:** $a_t = 0$ and $c_t^{-\rho} \geq \beta(1+r)\mathbb{E}_t\left[c_{t+1}^{-\rho}\right]$, or
  2. **Unconstrained:** $a_t > 0$ and $c_t^{-\rho} = \beta(1+r)\mathbb{E}_t\left[c_{t+1}^{-\rho}\right]$

Endogenous grid-point method (EGM)

Alternative to value function iteration:

1. Calculate **post-decision marginal value of cash:**

$$q(z^{i_z}, a^{i_a}) = \sum_{i_{z+}=0}^{\#_z - 1} \pi_{i_z, i_{z+}} c_{+}(z^{i_{z+}}, a^{i_a})^{-\sigma}$$

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4. **Consumption function:** $c^\star(z^{i_z}, a^{i_{a-1}}) =$ interpolation of function from $m(z^{i_z}, :)$ to $c(z^{i_z}, :)$ at $m = (1+r)a^{i_{a-}} + wz^{i_z}$

Exercises

- **Three exercises for you to do:**
  1. Ensure the stationary distribution is found in the simulation
  2. Make some borrowing allowed, b
  3. Introduce transitory shock, ξ_t
- **Extended model:**

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$

s.t.

$$a_t = (1+r)a_{t-1} + wz_t - c_t + \xi_t$$

$$\log z_t = \rho_z \log z_{t-1} + \psi_t,\ \mathbb{E}[z_t] = 1, \text{Var}[\psi_t] = \sigma_\psi^2$$

$$\xi_t \sim \mathcal{N}(0, \sigma_\xi^2)$$

$$a_t \geq -b$$

Details: Gauss-Hermite I

- **General problem:** How can we calculate

$$\mathbb{E}(f(x)) = \int f(x)g(x)dx$$

  - $f : \mathbb{R} \to \mathbb{R}$ some function
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- **How to choose S and the *nodes* (x_i) and *weights* (ω_i)?**
  Guassian quadrature

Details: Gauss-Hermite II

- **Gauss-Hermite** quadrature uses that

$$\int_{-\infty}^{\infty} f(x)e^{-x^2}dx = \sum_{i=1}^{S} \omega_i f(x_i) + \frac{S!\sqrt{\pi}}{s^S(2S)!} f^{(2S)}(\epsilon)$$

for some ϵ and where the (x_i, ω_i)'s can be easily found

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- **Example: Random normal variable:** $Y \sim \mathcal{N}(\mu, \sigma^2)$ so that

$$\begin{aligned}\mathbb{E}[f(Y)] &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} f(y)e^{-\frac{(y-\mu)^2}{2\sigma^2}}dy \\ &\approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^{S} \omega_i f(\sqrt{2}\sigma x_i + \mu)\end{aligned}$$

Summary

Summary and next week

- **Today:**
  1. Introduction to course
  2. Dynamic programming in theory
  3. Dynamic programming in practice
- **Next week:** More on consumption-saving models in partial equilibrium
- **Homework:**
  1. Work on completing the model extension exercise
  2. Read TBA