I've re-calibrated $\overline{\pi}_\chi = \underline{\pi}_\chi = 0.5$ to ensure $\mathbb{E}\left(r_{it}^K\right) = r_t^K$ and clearing the goods market.

1 a)

Results are shown in table 1. I've split income into labor income, $w_{ss}s_{i,ss}$, and capital income, $\left(r_{i,ss}^k - \delta\right) a_{i,ss}$.

σ_ψ directly determines dispersion of labor income. This can be seen by the higher standard deviation and Gini coefficient. The higher income uncertainty encourages precautionary savings raising the capital level. The standard deviation of $a_{i,ss}$ is also higher, but this is mainly through the scaling effect through a higher mean, as can be seen by the Gini coefficient only increasing slightly. Skewness is not much affected either. Average capital falls quite a lot as the falling interest rate due to a higher level of capital, outweighs the positive influence of having more assets to make money off. Through scaling this lowers the standard deviation, but we can see the Gini still increases, fitting with the change for $a_{i,ss}$. Although the Gini increases quite a bit more for capital income than for assets, which I'm a bit uncertain about. I think it's because the influence of differential returns becomes more important when r_{it}^K is low.

Table 1: Results for problem a)

	$w_{ss}s_{i,ss}$			$a_{i,ss}$				$\left(r_{i,ss}^k - \delta\right) a_{i,ss}$		
σ_ψ	$\mathbb{E}$	*Std*	Gini	$\mathbb{E}$	*Std*	Skewness	Gini	$\mathbb{E}$	*Std*	Gini
0.1	1.037	0.336	0.173	3.710	4.728	2.230	0.611	0.074	0.118	0.679
0.12	1.049	0.409	0.207	3.851	4.919	2.240	0.612	0.064	0.110	0.703
0.14	1.063	0.486	0.239	4.025	5.176	2.237	0.615	0.053	0.101	0.754

2 b)

The outcome variables for varying σ_β are shown in table 2. Labor income is barely affected, only a slight increase through higher capital, this makes sense as dispersion is pinned down by the distribution of s_{it}. The standard deviation and Gini increase dramatically for asset holding. This makes sense as the dispersion between the discounting rates of the household types becomes much larger, such that you have some households who care, relatively, much less about the future than others. Therefore there is also a much larger difference in the level of assets these households prefer to hold. Skewness only increases marginally as the increasing σ_β increases the mass in both ends of the distribution. Average capital earnings fall and the Gini increases substantially.

Figure 1 shows how the distribution spreads out as σ_β increases, substantially increasing the share of households with 0 assets.

Table 2: Results for problem b)

σ_β	$w_{ss}s_{i,ss}$ $\mathbb{E}$	Std	Gini	$a_{i,ss}$ $\mathbb{E}$	Std	Skewness	Gini	$\left(r^k_{i,ss}-\delta\right)a_{i,ss}$ $\mathbb{E}$	Std	Gini
0.01	1.037	0.336	0.173	3.710	4.728	2.230	0.611	0.074	0.118	0.679
0.02	1.065	0.345	0.173	4.052	6.364	2.525	0.700	0.051	0.117	0.814
0.03	1.100	0.356	0.173	4.515	7.733	2.471	0.745	0.020	0.100	2.051

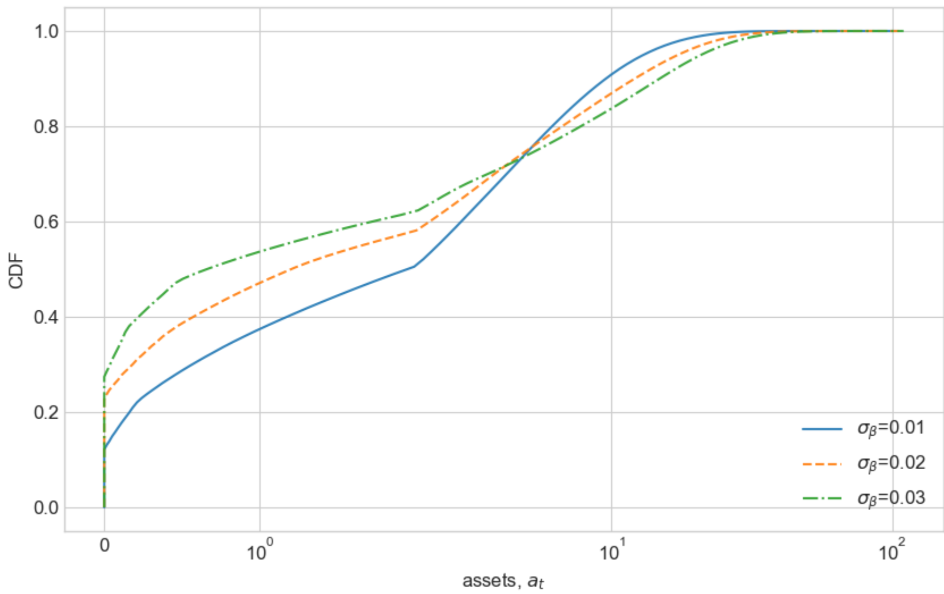


Figure 1: CDF of assets distribution across σ_β

3 c)

As σ_χ increases the spread of the differential return on assets increases. The capital stock is almost unchanged so there is no effect on labor income. The spread and Gini of assets increase slightly, while the skewness sees quite a large increase. I think this is because the 'lucky break' of getting $\chi_{i,ss} = 1$, matters more for high-asset households, while mattering relatively less for low-asset households, and not at all for 0-assets households. This means that σ_χ increases the dispersion of assets and capital income a lot in the high end of the distribution, while not changing that much in the low end of the distribution.

Table 3: Results for problem c)

σ_χ	$w_{ss}s_{i,ss}$ $\mathbb{E}$	Std	Gini	$a_{i,ss}$ $\mathbb{E}$	Std	Skewness	Gini	$\left(r^k_{i,ss}-\delta\right)a_{i,ss}$ $\mathbb{E}$	Std	Gini
0.1	1.037	0.336	0.173	3.710	4.728	2.230	0.611	0.074	0.118	0.679
0.2	1.037	0.336	0.173	3.708	4.802	2.406	0.614	0.074	0.174	0.973
0.3	1.037	0.336	0.173	3.704	4.895	2.639	0.617	0.074	0.242	1.458

4 d)

Wealth inequality being much higher than income inequality is definitely present in the model across all the considered parameter values. I think $\underline{a}$ contributes to making a_{it} more right-skewed, as it introduces holding assets directly to utility as a luxury good.

The parameter that most increased the skewness was increasing differential returns through σ_χ, while σ_β also increased it somewhat.

The finding in a) that increasing labor earnings dispersion can increase wealth inequality but cannot increase skewness fits well with De Nardi and Fella (2017). They discuss that it is theoretically possible to alter the earnings distribution to force the wealth distribution to fit the data better regarding skewness. However, they also find that an earnings process that fits the data better does not entail a higher skewness in the wealth distribution.

The finding in b) that preference heterogeneity in the discount factor can explain more dispersion and even skewness in assets fit with the survey of papers in De Nardi and Fella (2017). However, they also discuss how much preference heterogeneity is reasonable to put into this model, given that it is not a directly observable value, and that this approach still has difficulties replicating the very top end of the wealth distribution. Hubmer et al. (2021) finds that accounting for differences in returns and return risk across wealth levels is more important than heterogeneity in the discount factors for explaining the high end of the wealth distribution. They also argue for the positives of using observable factors like the distribution of return rate as opposed to preference heterogeneity, which they call "*residual explanations*".

In c) we are also able to increase the skewness using differential return rates on assets, however not much more than b) (in the given parameter space). One of the drawbacks of the reparametrization of $\overline{\pi}_\chi$ and $\underline{\pi}_\chi$ is that the difference in returns implied by $\chi_{i,ss}$ is no longer persistent for the household, which would likely have pushed the skewness further upward as the lucky households accumulate more assets anticipating that they are more likely to stay in the high-return group. Hubmer et al. (2021) allow return heterogeneity to vary directly with the assets of the household, this naturally implies persistence as assets holdings are persistent.

5 e)

The equation system for the model is:

$$\boldsymbol{H}(\boldsymbol{K};\boldsymbol{\alpha},\boldsymbol{\Gamma}) = \begin{bmatrix} w_t - (1-\alpha_t)\,\Gamma_t \left(\frac{K_{t-1}}{L_t}\right)^{\alpha_t} \\ r_t^K - \alpha\Gamma_t \left(\frac{K_{t-1}}{L_t}\right)^{\alpha_t - 1} \\ r_t - \left[r_t^K - \delta\right] \\ Y_t - \Gamma_t K_{t-1}^{\alpha_t} L_t^{1-\alpha_t} \\ K_t - \left[(1-\delta)\,K_{t-1} + I_t\right] \\ K_t - A_t^{hh} \\ Y_t - \left[C_t^{hh} + I_t\right] \end{bmatrix} = \mathbf{0} \tag{1}$$

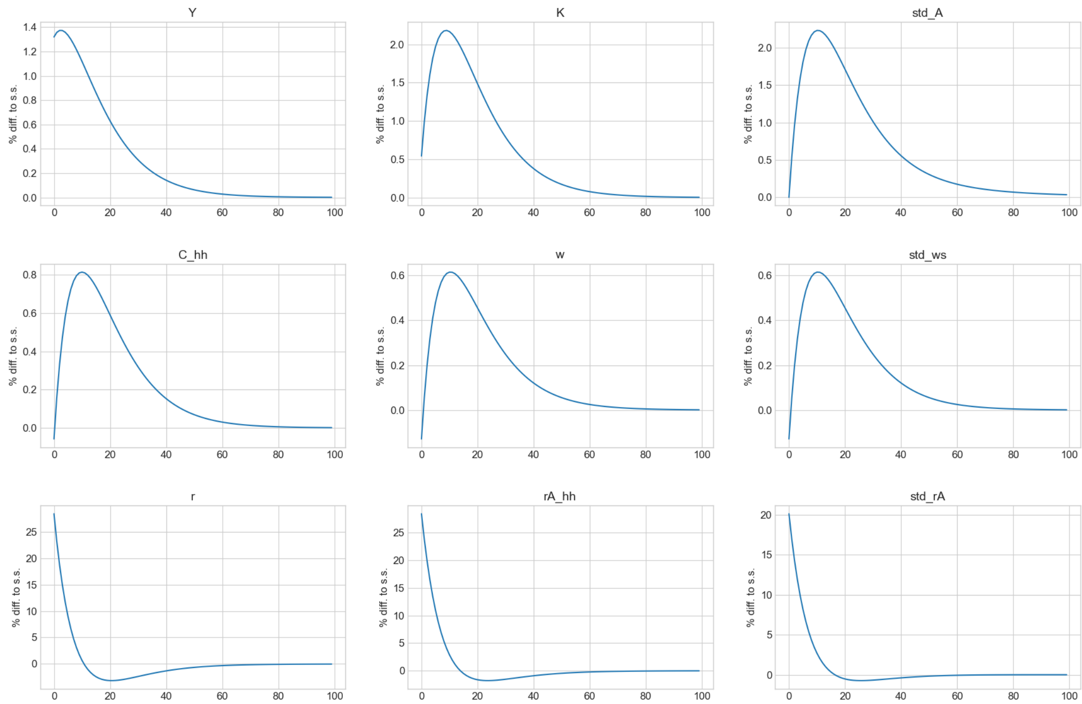
The IRFs are shown in figure 2. As the economy is relatively capital intensive, the positive shock to α_t immediately increases production. It also immediately increases the real interest rate, increasing the incentive for savings, so much that the consumption in the period of the shock is actually lower than in the steady state, despite total income being more than 1.3% above the steady state level. This increases the capital level, which is more persistent than the shock in α_t causing the interest rate to fall lower than the steady state level after about 15 periods. Capital income also falls below the steady state level a few periods later, but the dive is softened by the high capital level. The standard deviation of assets follows the level of capital. The standard deviation of capital income is actually smaller than the mean of capital income relative to their respective steady state values, indicating inequality in capital income is lower during the shock.

Ex ante I would have expected the shock to increase inequality, as it benefits asset holders by making capital temporarily more productive. This doesn't really seem to have happened, as the relative increases in standard deviations are either equal to or smaller than the relative increases in their corresponding means.

6 f)

I have proposed two different answers to this question. One where aggregate capital income is fixed by a tax on capital income, this allows the capital income of a household for a given level of assets to vary but fixes the aggregate amount. And one where the post-tax interest rate is kept constant, this means that for a household with a given level of assets the capital income is fixed, but as assets increase aggregate capital income is allowed to increase with it.

Figure 2: IRFs for a $\alpha_t = \alpha_{ss} + 0.01 \cdot 0.9^t$ shock in main calibration, e)



Note: std_A is the standard deviation of $a_{i,t-1}$(this is lagged to fit with capital income). std_ws is the standard deviation of $w_t s_{i,t}$. rA_hh is the average capital income, $\left(r^K_{i,t} - \delta\right) a_{i,t-1}$ or $r_t K_{t-1}$. std_rA is the standard deviation of capital income, $\left(r^K_{i,t} - \delta\right) a_{i,t-1}$.

6.1 Keeping aggregate capital income fixed

Introducing a tax on capital income, τ^t, the total post-tax capital income in steady state becomes:

$$(1 - \tau^a_{ss})\, r_{ss} K_{ss} \tag{2}$$

For a given period it will be:

$$(1 - \tau^a_t)\, r_t K_{t-1} \tag{3}$$

Now we define τ^a_t such aggregate capital income is always equal to the steady state level:

$$(1 - \tau^a_t)\, r_t K_{t-1} = (1 - \tau^a_{ss})\, r_{ss} K_{ss} \tag{4}$$

$$\Leftrightarrow \tau^a_t = 1 - (1 - \tau^a_{ss})\, \frac{r_{ss} K_{ss}}{r_t K_{t-1}} \tag{5}$$

$$\tag{6}$$

This tax rate keeps aggregate capital income fixed. I include an exponent, τ, just to be able to turn it off or on, by setting it to 0 or 1. But one could also set it somewhere between 0 and 1, which only curtails aggregate capital income to a certain degree.

$$\tau_t^a = 1 - (1 - \tau_{ss}^a) \left(\frac{r_{ss} K_{ss}}{r_t K_{t-1}} \right)^{\tau} \tag{7}$$

The transfers will be all the tax income

$$\xi = \tau_t^a r_t K_{t-1} \tag{8}$$

The household budget constraint becomes:

$$a_{it} + c_{it} = \left(1 + (1 - \tau_t^a) \left(r_{it}^K - \delta\right)\right) a_{it-1} + w_t s_{it} + \Pi_t \tag{9}$$

s I calibrate $\tau_{ss}^a = 0$, such that there are no capital income taxes in the steady state.

6.2 Keeping the post-tax return rate on capital constant

One can do something similar to the above, but instead of locking aggregate capital income, one could lock the post-tax interest rate. Thus allowing more capital to increase capital income, but only proportionally to the increase in capital.

I denote this form of tax θ_t^a, but is levied in the same way as τ_t^a. Keeping the post-tax interest rate equal to the steady state level implies:

$$(1 - \theta_t^a) r_t = (1 - \theta_{ss}^a) r_{ss} \tag{10}$$

$$\Leftrightarrow \theta_t^a = 1 - (1 - \theta_{ss}^a) \left(\frac{r_{ss}}{r_t} \right) \rightarrow \theta_t^a = 1 - (1 - \theta_{ss}^a) \left(\frac{r_{ss}}{r_t} \right)^{\vartheta} \tag{11}$$

With transfers $\xi = \theta_t^a r_t K_{t-1}$. Again, ϑ is just an on/off switch.

6.3 Results

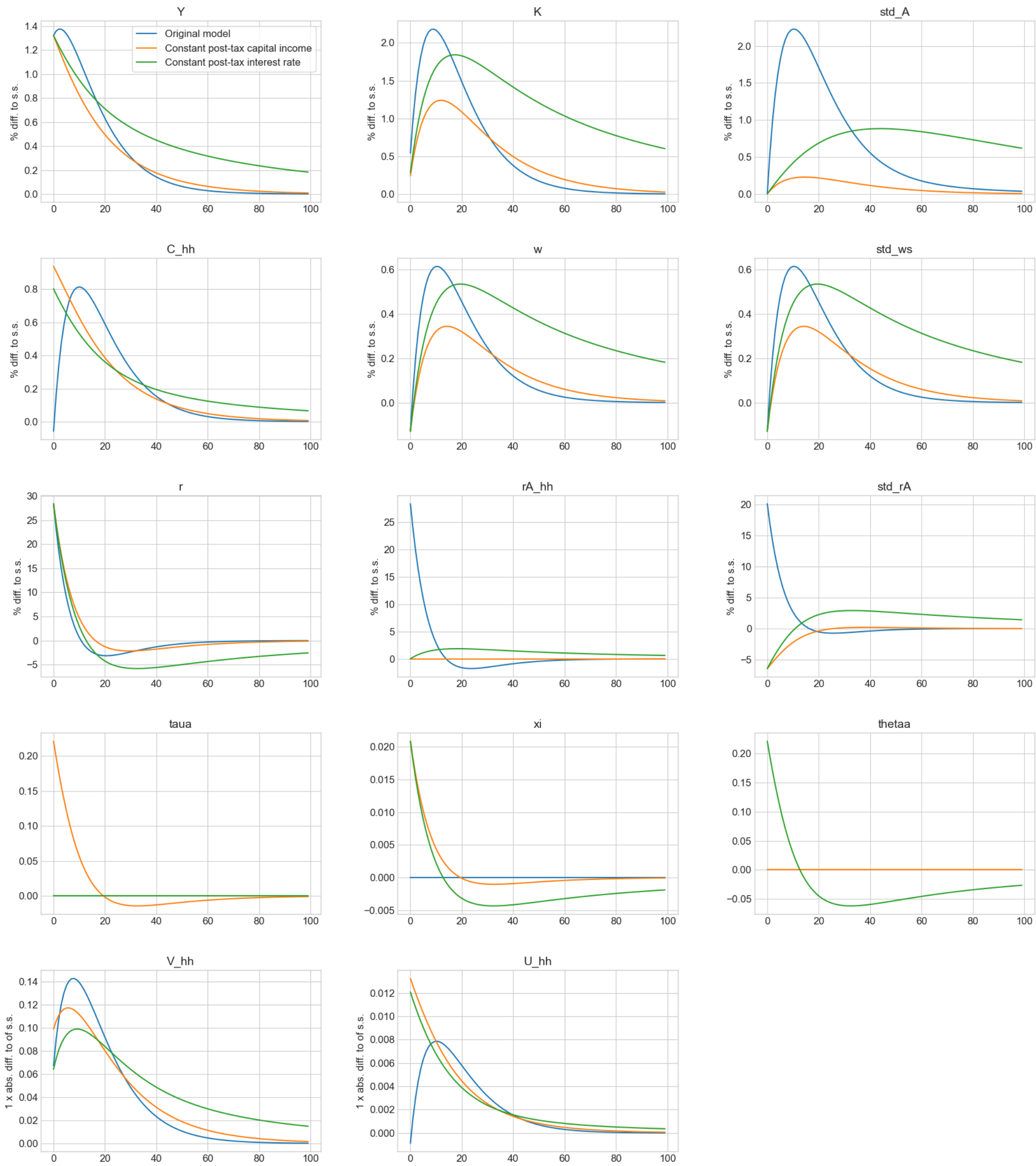
Figure 3 shows the IRFs of the original model and the ones with two proposed tax schemes. The fixed capital income tax scheme smooths out the shock's effect on Y over more periods. Because the higher productivity is offset by the tax scheme higher consumption begins immediately. This is bad from a 'maximize output' point of view, as the temporary increase in α_t is less utilized. The standard deviation of assets is almost completely neutralized, while the aggregate capital level still increases somewhat. Inequality in capital income is even lowered below the steady state level.

The fixed return rate tax scheme is in many ways similar. It smooths out the shock's effect over many more periods. This is because as the α_t disappears, capital is still above its steady state level, pushing r_t below the steady state level, which in turn pushes θ_t^a to become negative, i.e. a capital subsidy. The two factors of K above steady state, and $theta_t^a$ as a capital subsidy feeds back into each other, making the model very slow to converge (but it does so eventually). The scheme fares quite a bit worse on the standard deviation of assets, and slightly worse on the standard deviation of capital income than the other scheme.

Which policy is the best idea depends very much on which metric one is optimizing for. I the government literally dislikes increases in capital income the should implement one of the tax schemes, as they succeed in what they set out to do. If the government just dislikes inequality in general, there is also an argument for implementing a tax scheme, as both seem to lower inequality.

If one is a Benthamite utilitarian, then the metric to optimize is V_hh *in the period of the shock*. It is the average value function of the households taking into account all future discounted utility streams. Following this metric, the fixed capital income tax scheme is the winner by quite a big margin. Doing nothing is actually a tiny bit better than the fixed return rate tax scheme, as the gains from that scheme are pushed to the far future, which is too heavily discounted to matter.

Figure 3: IRFs for a $\alpha_t = \alpha_{ss} + 0.01 \cdot 0.9^t$ shock. Main and models with different tax systems, f)



Note: std_A is the standard deviation of $a_{i,t-1}$(this is lagged to fit with capital income). std_ws is the standard deviation of $w_t s_{i,t}$. rA_hh is the average post-tax capital income, $(1 - \tau_t^a - \theta_t^a)\left(r_{i,t}^K - \delta\right) a_{i,t-1}$. std_rA is the standard deviation of post-tax capital income, $(1 - \tau_t^a - \theta_t^a)\left(r_{i,t}^K - \delta\right) a_{i,t-1}$.

References

De Nardi, M. and G. Fella (2017, October). Saving and wealth inequality. *Review of Economic Dynamics 26*, 280–300.

Hubmer, J., P. Krusell, and A. A. Smith. (2021, May). Sources of US Wealth Inequality: Past, Present, and Future. *NBER Macroeconomics Annual 35*, 391–455. Publisher: The University of Chicago Press.