Hey, so before we begin: I got a bit caught up with my model extension and didn't start writing until way too late. So a lot of this just results with no comments, I apologize in advance.

1 Define the stationary equilibrium for the model

The stationary equilibrium is defined in equation (1):

$$H_{ss}\left(B_{ss};\tau_{ss}^{a},\tau_{ss}^{\ell}\right)=\begin{bmatrix} B_{ss}+K_{ss}-\boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss} \\ L_{ss}-\int \ell_t\zeta_i z_t d\mathbf{D}_{ss} \\ r_{ss}-\frac{1}{B_{ss}}\left(\int \tau_{ss}^{a}r_{ss}a_{ss}^{*}+\tau_{ss}^{\ell}w_{ss}\ell_{ss}\zeta_i z_t d\boldsymbol{D}_{ss}-G_t\right) \\ r_{ss}^{K}-\alpha\Gamma_{ss}(\frac{K_{ss}}{L_{ss}})^{\alpha-1} \\ w_{ss}-(1-\alpha)\Gamma_{ss}(K_{ss}/L_{ss})^{\alpha} \\ r_{ss}-\left(r_{ss}^{K}-\delta\right) \\ \boldsymbol{D}_{ss}-\Pi_z^{\prime}\underline{\boldsymbol{D}}_{ss} \\ \underline{\boldsymbol{D}}_{ss}-\Lambda_{ss}^{\prime}\boldsymbol{D}_{ss}\end{bmatrix}=\mathbf{0} \tag{1}$$

With Policy function, a_t^*, and Choice transition Λ_t.

The first two condition are the asset market and the labor market clearings, the goods market then clears by Walras' law. The next condition is a rewriting of the budget constraint for the government:

$$\begin{aligned} B_{ss}&=(1+r_t^B)B_{ss}+G_{ss}-\int\left[\tau_{ss}^{a}r_{ss}a_{ss}^{*}+\tau_{ss}^{\ell}w_{ss}\ell_{ss}^{*}\zeta_i z_t\right]d\boldsymbol{D}_t \\ \Leftrightarrow r_{ss}&=-\frac{1}{B_{ss}}\left(G_{ss}-\int\left[\tau_{ss}^{a}r_{ss}a_{ss}^{*}+\tau_{ss}^{\ell}w_{ss}\ell_{ss}^{*}\zeta_i z_t\right]d\boldsymbol{D}_t\right)\end{aligned}$$

The 4th and 5th conditions comes from the firm problem, and the 5th condition is the no arbitrage condition between the bonds and the capital markets.

When solving for the stationary equilibrium, the most straightforward thing is to use an optimizer to solve for both labor and assets clearings over K_{ss} and L_{ss}. This works perfectly well but is quit slow and luckily there is a faster way. The key to this trick is that the equilibrium variables that enter the household problem, w_{ss} and r_{ss} are both pinned down by the capital-labor ratio $\frac{K_{ss}}{L_s}$, independent of the absolute sizes. One can therefore find the equilibrium just searching over $\frac{K_{ss}}{L_s}$. After solving the household problem the labor market is assumed to clear, setting L_{ss} equal to household labor supply. From the capital-labor ratio this gives an implied K_{ss}. With an implied B_{ss} given tax income from household choices and r_{ss} the assets market clearing can the found to check if that also clears.

In section 2 we will see that the model for many parameter values have 2 equilibria. Luckily we will also see that they are relatively nicely ordered. The first thing to note is that the model is not well-defined when $r_{ss} = 0$[1], this is because the government cannot clear its budget constraint when it is not possible to either make money or give them away in the bonds market. It is convenient for us to note which capital-labor ratio causes $r_{ss} = 0$ as the 2 equilibria in the model are located on either side of the cut-off:

$$r_t^K = \alpha\Gamma \left(\frac{K_t}{L_t}\right)^{\alpha-1} \tag{2}$$

$$r_t = 0 \Leftrightarrow r_t^K - \delta = 0 \Leftrightarrow \alpha\Gamma \left(\frac{K_t}{L_t}\right)^{\alpha-1} - \delta = 0 \tag{3}$$

$$\Leftrightarrow \left(\frac{K_t}{L_t}\right)^{\alpha-1} = \frac{\delta}{\alpha\Gamma} \Leftrightarrow \frac{K_t}{L_t} = \left(\frac{\delta}{\alpha\Gamma}\right)^{\frac{1}{\alpha-1}} \tag{4}$$

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Table 1: The stationary equilibria

	$r > 0$	$r < 0$	no gov
Y	1.458	1.747	1.364
C	0.816	0.855	1.025
I	0.342	0.593	0.339
K	3.419	5.928	3.391
L	1.012	1.035	0.923
KL	3.378	5.726	3.672
B	0.627	-5.775	-0.000
$\int \tau^a_{ss} r_{ss} a^*_{ss} d\mathbf{D}_{ss}$	0.011	-0.000	0.000
$\int \tau^\ell_{ss} w_{ss} \ell_{ss} \zeta_i z_{ss} d\mathbf{D}_{ss}$	0.306	0.367	0.000
w	1.009	1.182	1.034
r	0.028	-0.012	0.021
$\int a^*_{ss} d\mathbf{D}_{ss}$	4.045	0.153	3.391
$\int c^*_{ss} d\mathbf{D}_{ss}$	0.816	0.855	1.025
ℓ_hh	1.081	1.138	0.994
$\int u\left(c^*_{ss}\right) d\mathbf{D}_{ss}$	-2.013	-2.000	-1.625
τ^a	0.100	0.100	0.000
τ^ℓ	0.300	0.300	0.000
G	0.300	0.300	0.000
Assets clearing	0.000	0.000	0.000
Goods clearing	-0.000	-0.000	-0.000
Labor clearing	0.000	0.000	0.000

Column 1 and 2 shows the two equilibria of the model calibrated as suggested by the assignment. The third column show a calibration with no government $tau^\ell = \tau^a = G = 0$.

Figure 1: Assets clearing across $\frac{K}{L}$, assignment calibration

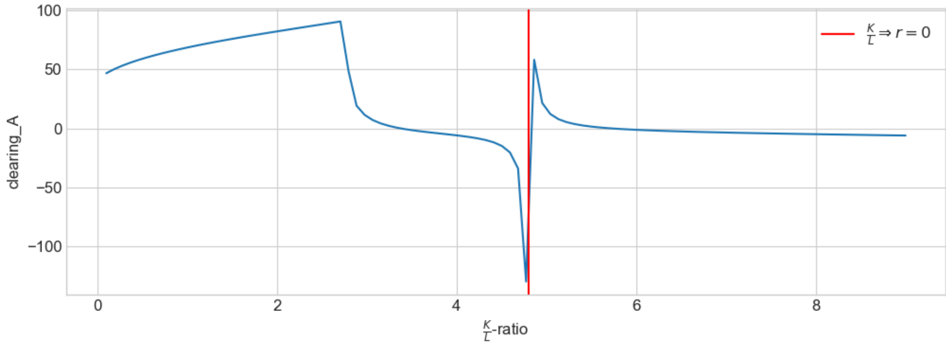


Figure 2: Assets clearing across $\frac{K}{L}$, $\tau^{\ell} = 0.1$

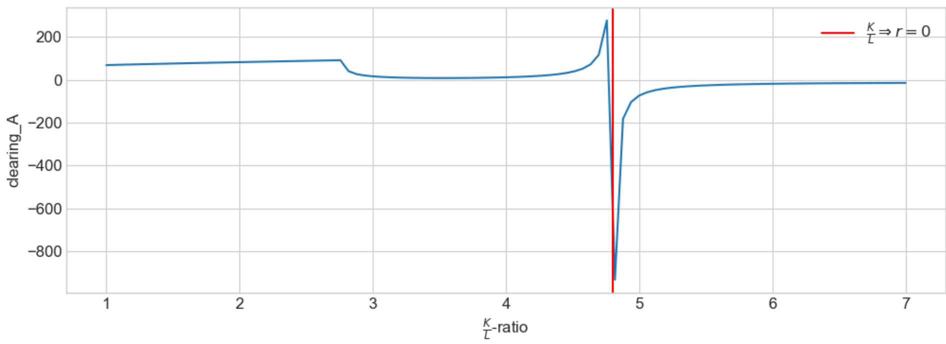
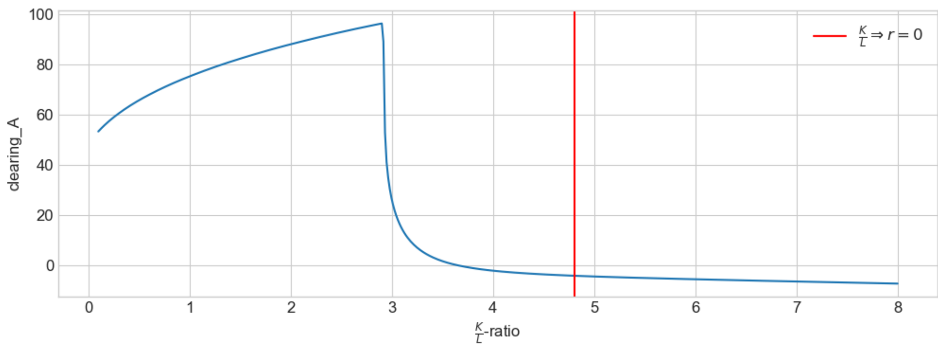


Figure 3: Assets clearing across $\frac{K}{L}$, $tau^{\ell} = \tau^{a} = G = 0$



2.1 Household behavior

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Average utility is generally larger the lower the tax rates are. However the model does not converge

[1] Except in the edge case $G_{ss} = \int \left[\tau^{a}_{ss} r_{ss} a^{*}_{ss} + \tau^{\ell}_{ss} w_{ss} \ell^{*}_{ss} \zeta_i z_t\right] d\boldsymbol{D}_t$, where the government spending equals income.

Figure 4: Households for different shocks, z_t

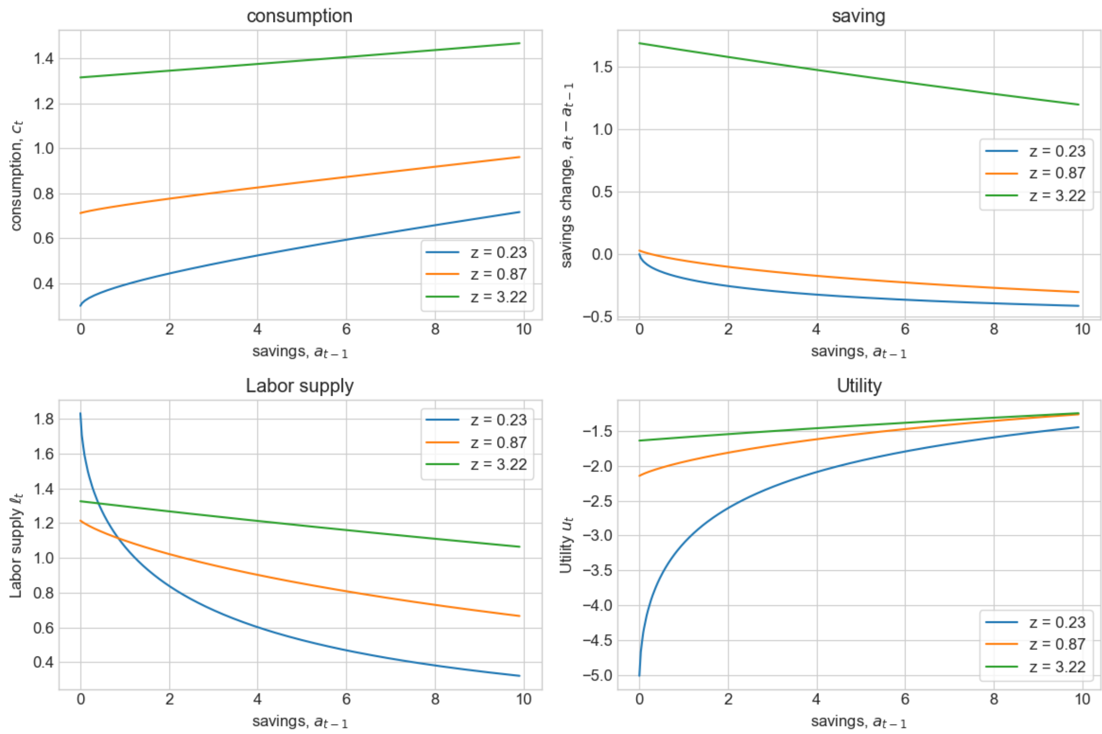
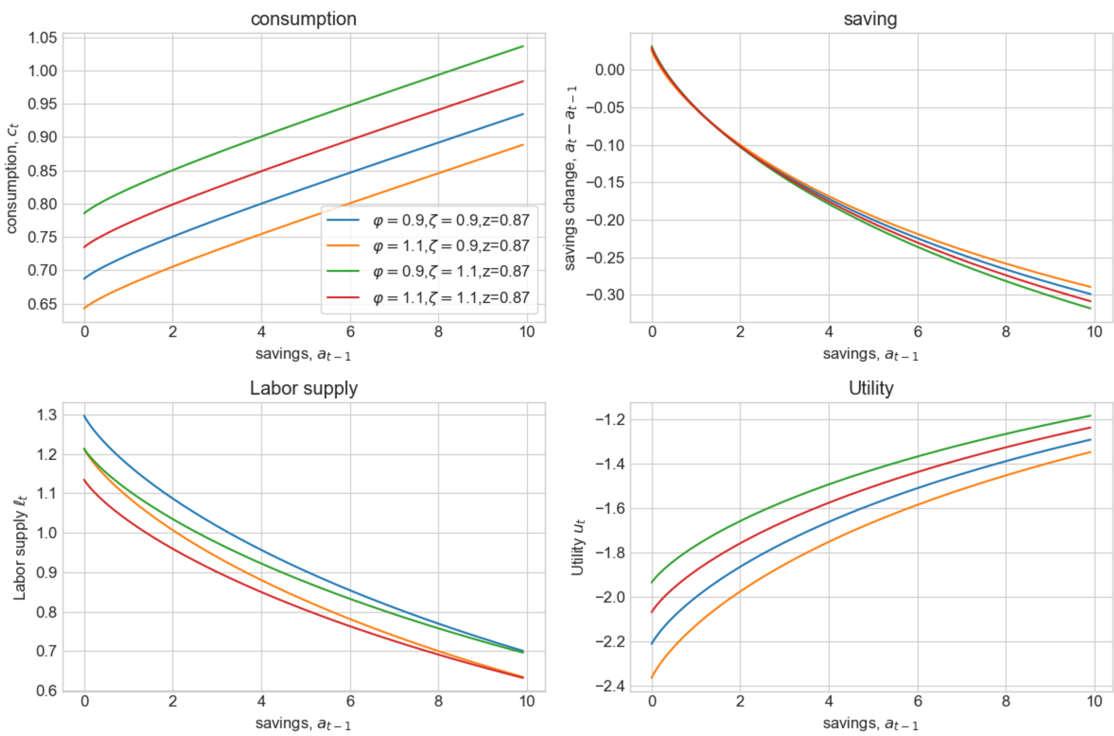


Figure 5: The different household types, for a fixed shock



to a solution for small τ^{ℓ} when $G = 0.3$.

Figure 6: Distribution of households

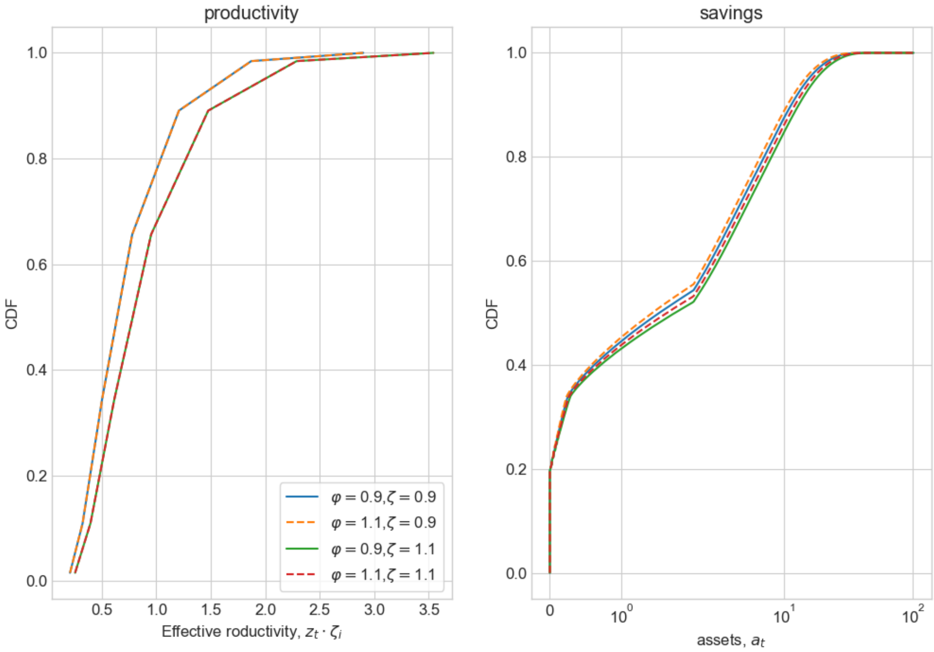


Figure 7: Equilibrium values across τ^a and τ^ℓ in equilibria where $r > 0$

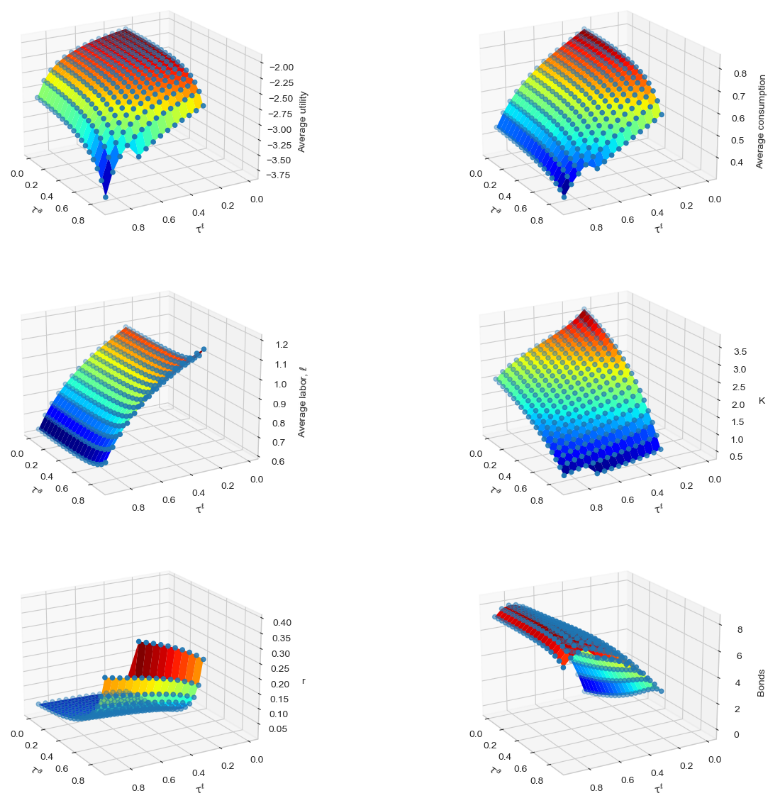


Figure 8: Equilibrium values across τ^a and τ^ℓ in equilibria where $r < 0$

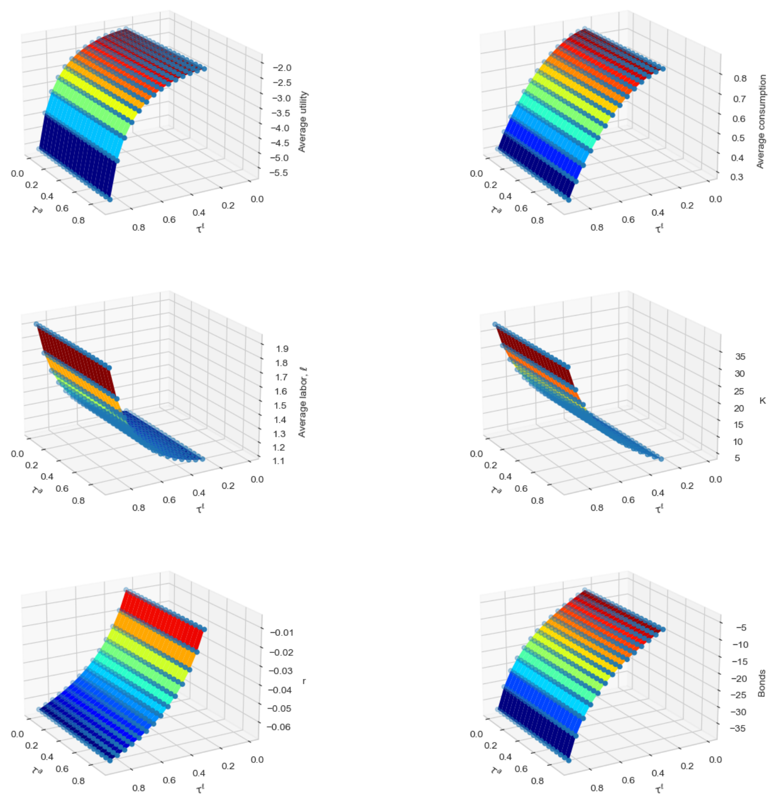
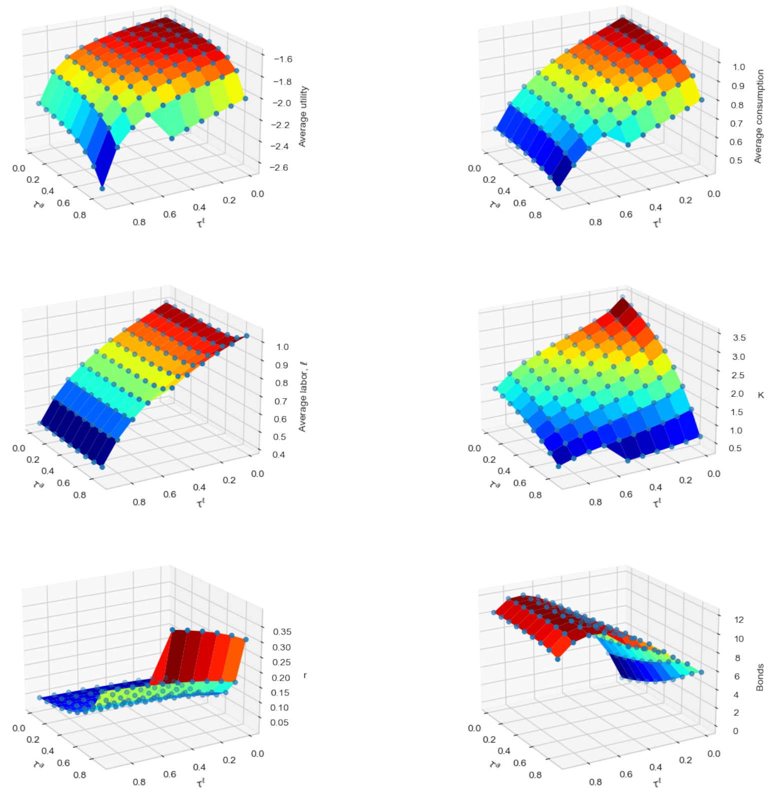


Figure 9: Equilibrium values across τ^a and τ^ℓ in equilibria where $r > 0 \wedge G = 0$



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My idea was to implement an increasing marginal tax rate, such that the highest earner pay are larger proportion than the low earner, possible even with a negative tax rate for the lower earners. And that this might increase welfare as it lowers income uncertainty and smooths consumption across individuals. It has the problem that it lowers the incentive for households to work, especially for those who are most productive. Some cool continuous increasing marginal tax rates are discussed in Estévez Schwarz and Sommer (2018), I was however not able to implement any of them in the EGM framework when solving the household problem.

A more salient approach is detailed below, using a differentiable post-tax income function. When finding my solution the goods market does not clear however.

Post tax income function with $\theta \in [0,1]$

$$\Theta_t^{\ell}(x;\theta_1,\tau,x_h) = \frac{x^{\theta}}{x_h^{\theta-1}} \cdot \left(1-\tau^{\ell}\right) \; \Rightarrow \; \frac{\partial \Theta}{\partial x} = \theta \frac{x^{\theta-1}}{x_h^{\theta-1}} \cdot \left(1-\tau^{\ell}\right) \tag{5}$$

For $\theta = 1$ it simplifies to a constant marginal tax rate of τ^{ℓ}. For $\theta < 1$ the tax rate is only τ^{ℓ} if you earn x_h, otherwise you have a lower tax rate

Which gives the budget constraint

$$a_t = (1+\tilde{r}_t)a_{t-1} + \Theta\left(w_t \ell_t \zeta_i z_t\right) + \Pi_t - c_t \tag{6}$$

and

$$\frac{\partial a_t}{\partial \ell_t} = \frac{\left(w_t \zeta_i z_t\right)^{\theta}}{x_h^{\theta-1}} \left(1-\tau^{\ell}\right) \ell_t^{\theta-1} \tag{7}$$

$$c_t = \left(\beta \underline{v}_{a,t+1}(z_t,a_t)\right)^{-\frac{1}{\sigma}} \Leftrightarrow \beta \underline{v}_{a,t+1}(z_t,a_t) = c_t^{-\sigma} \tag{8}$$

$$\varphi_i \ell_t^{\nu} = \left(\beta \underline{v}_{a,t+1}(z_t,a_t)\right) \cdot \frac{\partial a_t}{\partial \ell_t} \tag{9}$$

$$\varphi_i \ell_t^{\nu} = c_t^{-\sigma} \cdot \frac{\partial a_t}{\partial \ell_t} = c_t^{-\sigma} \frac{\left(w_t \zeta_i z_t\right)^{\theta}}{x_h^{\theta-1}} \left(1-\tau^{\ell}\right) \ell_t^{\theta-1} \tag{10}$$

$$\Leftrightarrow \varphi_i \ell_t^{\nu-\theta+1} = c_t^{-\sigma} \frac{\left(w_t \zeta_i z_t\right)^{\theta}}{x_h^{\theta-1}} \left(1-\tau^{\ell}\right) \Leftrightarrow \ell_t = c_t^{\frac{-\sigma}{\nu-\theta+1}} \left(\frac{\left(w_t \zeta_i z_t\right)^{\theta}}{x_h^{\theta-1}} \frac{\left(1-\tau^{\ell}\right)}{\varphi_i} \right)^{\frac{1}{\nu-\theta+1}} \tag{11}$$

For the newton solver for credit constrained:

$$f\left(\ell_t\right)=\ell_t-c_t^{\frac{-\sigma}{\nu-\theta+1}}\left(\frac{\left(w_t\zeta_i z_t\right)^{\theta}}{x_h^{\theta-1}}\frac{\left(1-\tau^{\ell}\right)}{\varphi_i}\right)^{\frac{1}{\nu-\theta+1}} \tag{12}$$

$$f'\left(\ell_t\right)=1-\frac{-\sigma}{\nu-\theta+1}c_t^{\frac{-\sigma}{\nu-\theta+1}-1}\frac{\left(w_t\zeta_i z_t\right)^{\theta}}{x_h^{\theta-1}}\left(1-\tau^{\ell}\right)\ell_t^{\theta-1}\left(\frac{\left(w_t\zeta_i z_t\right)^{\theta}}{x_h^{\theta-1}}\frac{\left(1-\tau^{\ell}\right)}{\varphi_i}\right)^{\frac{1}{\nu-\theta+1}} \tag{13}$$

References

Estévez Schwarz, D. and E. Sommer (2018). Smooth Income Tax Schedules: Derivation and Consequences. *SSRN Electronic Journal*.