CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

4. Stationary Equilibrium

Adv. Macro: Heterogenous Agent Models

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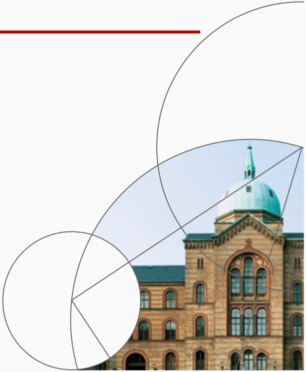

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    1. Single agent problems
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  2. You can help to improve interface
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- **Literature:** Aiyagari (1994)

HANC

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  3. The Standard Incomplete Market (SIM) model

Notation - central variables

- **Aggregate variables (quantities and prices):**
  1. Technology: Γ_t
  2. Capital: K_t
  3. Labor: L_t
  4. Consumption: C_t
  5. Investment: I_t
  6. Rental rate: r_t^k
  7. Real wage: w_t
- **Idiosyncratic variables:**
  1. Saving: a_t
  2. Consumption: c_t
  3. Productivity: z_t
- **Distributions:**
  1. $\underline{\boldsymbol{D}}_t$ over z_{t-1} and a_{t-1}
  2. $\boldsymbol{D}_t$ over z_t and a_{t-1}

Firms

- **Production function:** $Y_t = \Gamma_t K_{t-1}^{\alpha} L_t^{1-\alpha}$
- **Profits:** $\Pi_t = Y_t - w_t L_t - r_t^k K_{t-1}$
- **Profit maximization:** $\max_{K_{t-1},L_t} \Pi_t$
  1. Rental rate: $\frac{\partial \Pi_t}{\partial r_t^k} = 0 \Leftrightarrow r_t^k = \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha\text{-}1}$
  2. Real wage: $\frac{\partial \Pi_t}{\partial w} = 0 \Leftrightarrow w_t = (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha}$

Households - formulation

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}\left[v_{t+1}(z_{t+1}, a_t) \mid z_t, a_t\right]$$

$$\text{s.t. } a_t + c_t = (1 + r_t)a_{t-1} + w_t z_t \geq 0$$

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1} \quad , \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \, \mathbb{E}[z_t] = 1$$

with $r_t \equiv r_t^k - \delta$, where δ is the depreciation rate

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- **Aggregates:**

$$A_t^{hh} = \int a_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t = A^{hh}\left(\underline{\boldsymbol{D}}_t, \{r_\tau, w_\tau\}_{\tau \geq t}\right) = \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t$$

$$C_t^{hh} = \int c_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t = C^{hh}\left(\underline{\boldsymbol{D}}_t, \{r_\tau, w_\tau\}_{\tau \geq t}\right) = \boldsymbol{c}_t^{*\prime} \boldsymbol{D}_t$$

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- **Distributional dynamics** (with histogram method):

  1. Stochastic: $\boldsymbol{D}_t = \Pi_z' \underline{\boldsymbol{D}}_t$
  2. Choices: $\underline{\boldsymbol{D}}_{t+1} = \Lambda_t' \boldsymbol{D}_t, \quad \Lambda_t = \Lambda\left(\{r_\tau, w_\tau\}_{\tau \geq t}\right)$

Households - solution

- **Beginning-of-period value function:**

$$\underline{v}_t(z_{t-1}, a_{t-1}) = \mathbb{E}\left[v_t(z_t, a_{t-1}) \,|\, z_{t-1}, a_{t-1}\right]$$

Note: This re-formulation will be useful later in the course

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- **Envelope theorem:** Differentiate with fixed a_t choice

$$\underline{v}_{a,t} \equiv \frac{\partial \underline{v}_t}{\partial a_{t-1}} = \underline{\mathbb{E}}_t\left[(1 + r_t)c_t^{-\sigma}\right]$$

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- **EGM:** Find solution by

$$c_t = \left(\beta \underline{v}_{a,t+1}\right)^{\frac{1}{\sigma}} \Rightarrow m_t = a_t + c_t$$

Resource constraint and market clearing

- **Law-of-motion for capital**

$$K_t = (1-\delta)K_{t-1} + I_t$$

- **Market clearing:**
  1. Labor market: $L_t = \int z_t dD_t = 1$
  2. Goods market: $Y_t = C_t + I_t$
  3. Capital market: $K_t = \int a_t dD_t$

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  3. Aggregating across individual

$$
\begin{aligned}
C_t + Y_t &= \int c_t dD_t + (K_t - (1-\delta) K_{t-1}) \\
&= \int \left[(1 + r_t) a_{t-1} + w_t z_t - a_t \right] d\boldsymbol{D}_t + K_t - (1-\delta) K_{t-1} \\
&= (1 + r_t) K_{t-1} + w_t - K_t + K_t - (1-\delta) K_{t-1} \\
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\end{aligned}
$$

  4. Combined: Then *the goods market clears*

$$
Y_t = C_t + I_t
$$

Equation system

The model can be written as an **equation system**

$$\boldsymbol{H}\left(\{K_t, L_t; \Gamma_t\}_{t\geq 0}, \underline{\boldsymbol{D}}_0\right) = \begin{bmatrix} K_t - \boldsymbol{a}_t^{*\prime}\boldsymbol{D}_t \\ r_t - \alpha\Gamma_t(K_{t-1}/L_t)^{\alpha-1} \\ w_t - (1-\alpha)\Gamma_t(K_{t-1}/L_t)^{\alpha} \\ L_t - 1 \\ \boldsymbol{D}_t - \Pi_z^{\prime}\underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} - \Lambda_t^{\prime}\boldsymbol{D}_t \\ \forall t \in \{0, 1, \dots\} \end{bmatrix} = \boldsymbol{0}$$

where $\{\Gamma_t\}_{t\geq 0}$ is a given technology path and $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$

Remember: Policies and choice transitions depend on prices

1. Policy function: $a_t^* = a^*\left(\{r_\tau, w_\tau\}_{\tau\geq t}\right)$
2. Choice transition: $\Lambda_t = \Lambda\left(\{r_\tau, w_\tau\}_{\tau\geq t}\right)$

Stationary Equilibrium

Stationary equilibrium - equation system

The **stationary equilibrium** satisfies

$$H_{ss}\left(K_{ss}, L_{ss}; \Gamma_{ss}\right) = \begin{bmatrix} K_t - \boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss} \\ r_{ss} - \alpha\Gamma_{ss}(K_{ss}/L_{ss})^{\alpha-1} \\ w_{ss} - (1-\alpha)\Gamma_{ss}(K_{ss}/L_{ss})^{\alpha} \\ L_{ss} - 1 \\ \boldsymbol{D}_{ss} - \Pi_z^{\prime}\underline{\boldsymbol{D}}_{ss} \\ \underline{\boldsymbol{D}}_{ss} - \Lambda_{ss}^{\prime}\boldsymbol{D}_{ss} \end{bmatrix} = \boldsymbol{0}$$

Note I: Households still move around »inside« the distribution due to idiosyncratic shocks

Note II: Steady state for aggregates (quantities and prices) and the distribution as such

Stationary equilibrium - more verbal definition

For a given Γ_{ss}

1. Quantities K_{ss} and L_{ss},
2. prices r_{ss} and w_{ss},
3. the distribution $\boldsymbol{D}_{ss}$ over z_t and a_{t-1}
4. and the policy functions $a_{ss}^*(z_t, a_{t-1})$ and $c_{ss}^*(z_t, a_{t-1})$

are such that

1. Household maximize expected utility (policy functions)
2. Firms maximize profits (prices)
3. $\boldsymbol{D}_{ss}$ is the invariant distribution implied by the household problem
4. The labor market clears
5. The capital market clears
6. The goods market clears

Direct implementation

Root-finding problem in K_{ss} with the objective function:

1. Set $L_{ss} = 1$
2. Calculate $r_{ss} = \alpha\Gamma_{ss}(K_{ss})^{\alpha-1}$ and $w_{ss} = (1-\alpha)\Gamma_{ss}(K_{ss})^{\alpha}$
3. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
4. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
5. Return $K_{ss} - \boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss}$

Indirect implementation

1. Choose r_{ss} and w_{ss}
2. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
3. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
4. Set $K_{ss} = \boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss}$
5. Set $L_{ss} = 1$
6. Set $\Gamma_{ss} = \frac{w_{ss}}{(1-\alpha)\Gamma_{ss}(K_{ss})^{\alpha}}$
7. Set $r_{ss}^{k} = \alpha\Gamma_{ss}(K_{ss})^{\alpha-1}$
8. Set $\delta = r_{ss}^{k} - r_{ss}$

Steady interest rate

- **Complete markets / representative agent:** Derived from aggregate Euler-equation

$$C_t^{-\rho} = \beta(1+r)C_{t+1}^{-\rho} \Rightarrow C_{ss}^{-\rho} = \beta(1+r)C_{ss}^{-\rho} \Leftrightarrow \beta = \frac{1}{1+r}$$

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- **Heterogeneous agents:** *No such equation exists*
    1. Euler-equation replaced by asset market clearing condition
    2. Idiosyncratic income risk affects the steady state interest rate

Calibration

How to choose parameters?

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  1. **Informal:** Roughly match targets by hand
  2. **Formal:**
     2a. Solve root-finding problem
     2b. Minimize a squared loss function
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- **Complication:** *We must always solve for the steady state for each guess of the parameters to be calibrated*

Exercises

1. **Households:** Solve

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(z_{t+1}, a_t) \right]$$

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where r_t is the real-interest rate and τ_t is a tax rate

2. **Government:** Set taxes and government bonds follows

$$B_{t+1} = (1+r_t)B_t - \int \tau_t z_t d\boldsymbol{D}_t$$

3. **Bond market clearing:** $B_t = \int a_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t$
4. **Define and find the stationary equilibrium**
5. **What is the optimal level of τ_t?**

Summary

Summary and next week

- **Today:**
  1. The concept of a stationary equilibrium
  2. Introduction to the GEModelTools package
- **Next week:** More on models with interesting dynamics in the stationary equilibrium
- **Homework:**
  1. Work on completing the model extension exercise
  2. Read: TBA