



4. Stationary Equilibrium

Adv. Macro: Heterogenous Agent Models

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Introduction

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- **Code:** Based on the **GEModelTools** package
 1. Is in active development
 2. You can help to improve interface
 3. You can help to find bugs
 4. You can help to add features

Documentation: See **GEModelToolsNotebooks**

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- **Literature:** Aiyagari (1994)

HANC



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3. The Standard Incomplete Market (SIM) model

Notation - aggregate variables

- **Aggregate variables (quantities and prices):**

1. Output: Y_t
2. Consumption: C_t
3. Investment: I_t
4. Technology: Γ_t
5. Capital: K_t
6. Labor: L_t
7. Rental rate: r_t^k
8. Real wage: w_t
9. Real interest rate: r_t
10. Profit: Π_t

Notation - idiosyncratic variables

- **Idiosyncratic variables:**

1. Saving: a_t
2. Consumption: c_t
3. Productivity: z_t

- **Distributions:**

1. \underline{D}_t over z_{t-1} and a_{t-1}
2. D_t over z_t and a_{t-1}

- **Production function:** $Y_t = \Gamma_t K_{t-1}^\alpha L_t^{1-\alpha}$
- **Profits:** $\Pi_t = Y_t - w_t L_t - r_t^k K_{t-1}$
- **Profit maximization:** $\max_{K_{t-1}, L_t} \Pi_t$
 1. Rental rate: $\frac{\partial \Pi_t}{\partial r_t^k} = 0 \Leftrightarrow r_t^k = \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1}$
 2. Real wage: $\frac{\partial \Pi_t}{\partial w} = 0 \Leftrightarrow w_t = (1 - \alpha) \Gamma_t (K_{t-1}/L_t)^\alpha$

Households - formulation

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E} [v_{t+1}(z_{t+1}, a_t) \mid z_t, a_t]$$

$$\text{s.t. } a_t + c_t = (1 + r_t)a_{t-1} + w_t z_t + \Pi_t \geq 0$$

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1}, \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \mathbb{E}[z_t] = 1$$

with $r_t \equiv r_t^k - \delta$, where δ is the depreciation rate

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- **Aggregates:**

$$A_t^{hh} = \int a_t^*(z_t, a_{t-1}) d\mathbf{D}_t = A^{hh}(\mathbf{D}_t, \{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t}) = \mathbf{a}_t^{*'} \mathbf{D}_t$$

$$C_t^{hh} = \int c_t^*(z_t, a_{t-1}) d\mathbf{D}_t = C^{hh}(\mathbf{D}_t, \{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t}) = \mathbf{c}_t^{*'} \mathbf{D}_t$$

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- **Distributional dynamics** (with histogram method):

1. Stochastic: $\mathbf{D}_t = \Pi'_z \mathbf{D}_t$
2. Choices: $\mathbf{D}_{t+1} = \Lambda'_t \mathbf{D}_t, \quad \Lambda_t = \Lambda(\{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t})$

- **Beginning-of-period value function:**

$$\underline{v}_t(z_{t-1}, a_{t-1}) = \mathbb{E} [v_t(z_t, a_{t-1}) \mid z_{t-1}, a_{t-1}]$$

Note: This re-formulation will be useful later in the course

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- **Envelope theorem:** Differentiate with fixed a_t choice

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- **EGM:** Find solution from FOC

$$c_t^{-\sigma} = \beta \underline{v}_{a,t+1} \Leftrightarrow c_t = (\beta \underline{v}_{a,t+1})^{-\frac{1}{\sigma}} \text{ and } m_t = a_t + c_t$$

- Law-of-motion for capital

$$K_t = (1 - \delta)K_{t-1} + I_t$$

- Market clearing:

1. Labor market: $L_t = \int z_t d\mathbf{D}_t = 1$
2. Goods market: $Y_t = C_t + I_t$
3. Capital market: $K_t = \int a_t d\mathbf{D}_t$

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3. Aggregating across individuals

$$\begin{aligned} C_t &= \int c_t dD_t \\ &= \int [(1 + r_t) a_{t-1} + w_t z_t - a_t] dD_t \\ &= (1 + r_t) K_{t-1} + w_t - K_t \end{aligned}$$

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4. Combined: Then *the goods market clears*

$$\begin{aligned} C_t + I_t &= [(1 + r_t) K_{t-1} + w_t - K_t] + [K_t - (1 - \delta) K_{t-1}] \\ &= w_t + (r_t + \delta) K_{t-1} \\ &= Y_t \end{aligned}$$

Equation system

The model can be written as an **equation system**

$$H(\{K_t, L_t; \Gamma_t\}_{t \geq 0}, \underline{D}_0) = \begin{bmatrix} K_t - a_t^{*'} \underline{D}_t \\ r_t - \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\ w_t - (1 - \alpha) \Gamma_t (K_{t-1}/L_t)^{\alpha} \\ L_t - 1 \\ \underline{D}_t - \Pi_z' \underline{D}_t \\ \underline{D}_{t+1} - \Lambda_t' \underline{D}_t \\ \forall t \in \{0, 1, \dots\} \end{bmatrix} = \mathbf{0}$$

where $\{\Gamma_t\}_{t \geq 0}$ is a given technology path and $K_{-1} = \int a_{t-1} d\underline{D}_0$

Remember: Policies and choice transitions depend on prices

1. Policy function: $a_t^* = a^* \left(\{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t} \right)$
2. Choice transition: $\Lambda_t = \Lambda \left(\{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t} \right)$

Stationary Equilibrium

Stationary equilibrium - equation system

The **stationary equilibrium** satisfies

$$H_{ss}(K_{ss}, L_{ss}; \Gamma_{ss}) = \begin{bmatrix} K_t - \mathbf{a}_{ss}' \mathbf{D}_{ss} \\ r_{ss} - \alpha \Gamma_{ss} (K_{ss}/L_{ss})^{\alpha-1} \\ w_{ss} - (1-\alpha) \Gamma_{ss} (K_{ss}/L_{ss})^{\alpha} \\ L_{ss} - 1 \\ \mathbf{D}_{ss} - \Pi'_z \underline{\mathbf{D}}_{ss} \\ \underline{\mathbf{D}}_{ss} - \Lambda'_{ss} \mathbf{D}_{ss} \end{bmatrix} = \mathbf{0}$$

Note I: Households still move around »inside« the distribution due to idiosyncratic shocks

Note II: Steady state for aggregates (quantities and prices) and the distribution as such

Stationary equilibrium - more verbal definition

For a given Γ_{ss}

1. Quantities K_{ss} and L_{ss} ,
2. prices r_{ss} and w_{ss} (always $\Pi_{ss} = 0$),
3. the distribution \mathbf{D}_{ss} over z_t and a_{t-1}
4. and the policy functions $a_{ss}^*(z_t, a_{t-1})$ and $c_{ss}^*(z_t, a_{t-1})$

are such that

1. Household maximize expected utility (policy functions)
2. Firms maximize profits (prices)
3. \mathbf{D}_{ss} is the invariant distribution implied by the household problem
4. The labor market clears
5. The capital market clears
6. The goods market clears

Root-finding problem in K_{ss} with the objective function:

1. Set $L_{ss} = 1$
2. Calculate $r_{ss} = \alpha \Gamma_{ss}(K_{ss})^{\alpha-1}$ and $w_{ss} = (1 - \alpha) \Gamma_{ss}(K_{ss})^{\alpha}$
3. Solve infinite horizon household problem *backwards*, i.e. find \mathbf{a}_{ss}^*
4. Simulate households *forwards* until convergence, i.e. find \mathbf{D}_{ss}
5. Return $K_{ss} - \mathbf{a}_{ss}^* \mathbf{D}_{ss}$

Indirect implementation

1. Choose r_{ss} and w_{ss}
2. Solve infinite horizon household problem *backwards*, i.e. find \mathbf{a}_{ss}^*
3. Simulate households *forwards* until convergence, i.e. find \mathbf{D}_{ss}
4. Set $K_{ss} = \mathbf{a}_{ss}^{*'} \mathbf{D}_{ss}$
5. Set $L_{ss} = 1$
6. Set $\Gamma_{ss} = \frac{w_{ss}}{(1-\alpha)\Gamma_{ss}(K_{ss})^\alpha}$
7. Set $r_{ss}^k = \alpha\Gamma_{ss}(K_{ss})^{\alpha-1}$
8. Set $\delta = r_{ss}^k - r_{ss}$

- **Complete markets / representative agent:**

Derived from aggregate Euler-equation

$$C_t^{-\sigma} = \beta(1+r)C_{t+1}^{-\sigma} \Rightarrow C_{ss}^{-\sigma} = \beta(1+r)C_{ss}^{-\sigma} \Leftrightarrow \beta = \frac{1}{1+r}$$

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- **Heterogeneous agents:** *No such equation exists*

1. Euler-equation replaced by asset market clearing condition
2. Idiosyncratic income risk affects the steady state interest rate

Calibration

How to choose parameters?

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 1. **Informal:** Roughly match targets by hand
 2. **Formal:**
 - 2a. Solve root-finding problem
 - 2b. Minimize a squared loss function
 3. **Estimation:** Formal with squared loss function + standard errors

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- **Complication:** *We must always solve for the steady state for each guess of the parameters to be calibrated*

Exercises

Exercises: Model extensions

1. Households: Solve

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t [v_{t+1}(z_{t+1}, a_t)]$$

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where r_t is the real-interest rate and τ_t is a tax rate

2. Government: Set taxes and government bonds follows

$$B_{t+1} = (1 + r_t)B_t - \int \tau_t z_t d\mathbf{D}_t$$

$$3. \text{ Bond market clearing: } B_t = \int a_t^*(z_t, a_{t-1}) d\mathbf{D}_t$$

4. Define and find the stationary equilibrium

5. What is the optimal level of τ_t ?

Summary

Summary and next week

- **Today:**
 1. The concept of a stationary equilibrium
 2. Introduction to the **GEModelTools** package
- **Next week:** More on models with interesting dynamics in the stationary equilibrium
- **Homework:**
 1. Work on completing the model extension exercise
 2. Read: Hubmer et al. (2021)