



9. A Baseline HANK Model

Adv. Macro: Heterogenous Agent Models

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Introduction

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- **Model:** Heterogeneous Agent New Keynesian (HANK) model
- **Code:**
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- **Literature:**
 1. Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«
 2. Documentation for GEModelTools

HANK model

- **Households:**

1. Differ by stochastic idiosyncratic productivity and savings
2. Supply labor and choose consumption
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- **Central bank:** Set nominal interest rate

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- **Static** problem for representative final good firm:

$$\max_{y_{jt} \forall j} P_t Y_t - \int_0^1 p_{jt} y_{jt} dj \text{ s.t. } Y_t = \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu}$$

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- **Note:** Zero profits (can be used to derive price index)

Derivation of demand curve

- FOC wrt. y_{jt}

$$0 = P_t \mu \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu-1} \frac{1}{\mu} y_{jt}^{\frac{1}{\mu}-1} - p_{jt} \Leftrightarrow$$

$$\frac{p_{jt}}{P_t} = \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu-1} y_{jt}^{\frac{1-\mu}{\mu}} \Leftrightarrow$$

$$\left(\frac{p_{jt}}{P_t} \right)^{\frac{\mu}{\mu-1}} = \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu} y_{jt}^{-1} \Leftrightarrow$$

$$y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t$$

- **Dynamic problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, n_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$
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$$\log(1 + \pi_t) = \kappa \left(\frac{w_t}{Z_t} - \frac{1}{\mu} \right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}}, \quad \pi_t \equiv P_t / P_{t-1} - 1$$

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- **Implied dividends:** $d_t = Y_t - w_t N_t - \frac{\mu}{\mu-1} \frac{1}{2\kappa} [\log(1 + \pi_t)]^2 Y_t$

Derivation of NKPC

- FOC wrt. p_{jt} :

$$0 = \left(1 - \frac{\mu}{\mu - 1}\right) \left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{P_t} + \frac{\mu}{\mu - 1} \frac{w_t}{Z_t} \left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{p_{jt}} \\ - \frac{\mu}{\mu - 1} \frac{1}{\kappa} \frac{\log\left(\frac{p_{jt}}{p_{jt-1}}\right)}{p_{jt}} Y_t + \frac{J'_{t+1}(p_{jt})}{1 + r_{t+1}}$$

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- FOC + Envelope + Symmetry + $\pi_t = P_t/P_{t-1} - 1$

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- **Household problem:** Distribution, \mathbf{D}_t , over z_t and a_{t-1}

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_t^{1+\nu}}{1+\nu} + \beta \mathbb{E}[v_{t+1}(z_{t+1}, a_t) | z_t, a_t]$$

$$\text{s.t. } a_t = (1 + r_t)a_{t-1} + (w_t \ell_t - \tau_t + d_t)z_t - c_t \geq 0$$

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- **Effective labor-supply:** $n_t = z_t \ell_t$

- **Beginning-of-period value function:**

$$\underline{v}_{a,t}(z_{t-1}, a_{t-1}) = \mathbb{E}_t [v_{a,t}(z_t, a_{t-1})] = \mathbb{E} [(1 + r_t)c_t^{-\sigma}]$$

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- **Endogenous grid method:** Vary z_t and a_t to find

$$c_t = (\beta \underline{v}_{a,t+1}(z_t, a_t))^{-\frac{1}{\sigma}}$$

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- **Consumption and labor supply:** Use linear interpolation to find

$$c^*(z_t, a_{t-1}) \text{ and } \ell^*(z_t, a_{t-1}) \text{ with } m_t = (1+r_t)a_{t-1} + (w_t \ell_t^* - \tau_t + d_t)z_t$$

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1. Stop if $f(\ell^*) = \ell^* - \left(\frac{w_t z_t}{\varphi}\right)^{\frac{1}{\nu}} (c^*)^{-\frac{\sigma}{\nu}} < \text{tol.}$ where

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3. Return to step 1

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- **Government:** Choose τ_t to keep debt constant and finance exogenous public consumption

$$\tau_t = r_t B_{ss} + G_t$$

Market clearing

1. Labor: $N_t = \int n_t^*(z_t, a_{t-1}) d\mathbf{D}_t$ (in effective units)
2. Assets: $B_{ss} = \int a_t^*(z_t, a_{t-1}) d\mathbf{D}_t$
3. Goods: $Y_t = \int c_t^*(z_t, a_{t-1}) d\mathbf{D}_t + G_t + \frac{\mu}{\mu-1} \frac{1}{2\kappa} [\log(1 + \pi_t)]^2 Y_t$

As an equation system

$$\begin{aligned} H(\pi, w, Y, i^*, Z, \underline{D}_0) &= 0 \\ \left[\begin{array}{c} \log(1 + \pi_t) - \left[\kappa \left(\frac{w_t}{Z_t} - \frac{1}{\mu} \right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}} \right] \\ N_t - \int n_t^*(z_t, a_{t-1}) d\mathbf{D}_t \\ B_{ss} - \int a_t^*(z_t, a_{t-1}) d\mathbf{D}_t \end{array} \right] &= 0 \end{aligned}$$

The rest of the model is given by

$$\mathbf{X} = M(\pi, w, Y, i^*, Z)$$

As a DAG (from Auclert et al., 2021)

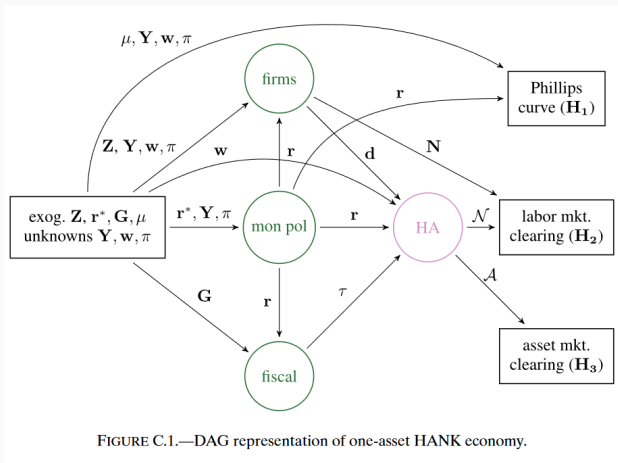


FIGURE C.1.—DAG representation of one-asset HANK economy.

Notation: $i^* = r^*$, μ is a shock, $A^{hh} = \mathcal{A}$, $N^{hh} = \mathcal{N}$

Steady state

- **Chosen:** B_{ss} , G_{ss} , r_{ss}
- **Analytically:**
 1. **Normalization:** $Z_{ss} = N_{ss} = 1$
 2. **Zero-inflation:** $\pi_{ss} = 0 \Rightarrow i_{ss} = i_{ss}^* = (1 + r_{ss})(1 + \pi_{ss}) - 1$
 3. **Firms:** $Y_{ss} = Z_{ss}N_{ss}$, $w_{ss} = \frac{Z_{ss}}{\mu}$ and $d_{ss} = Y_{ss} - w_{ss}N_{ss}$
 4. **Government:** $\tau_{ss} = r_{ss}B_{ss} + G_{ss}$
 5. **Assets:** $A_{ss} = B_{ss}$
- **Numerically:** Choose β and φ to get market clearing

The HANK example from GEModelToolsNotebooks I

- **Presentation:** I go through the code for finding the transition path
- **In-class exercise:**
 1. Look at the code and talk about it with the person next to you for 10 minutes
 2. Write at least one question on https://padlet.com/jeppe_drue Dahl/advmacrohet

Transmission mechanism to monetary policy shock

1. **Monetary policy shock:** $i_t^* \downarrow \Rightarrow i_t = i_t^* + \phi\pi_t \downarrow$
2. **Real interest rate:** $r_t = \frac{1+i_t-1}{1+\pi_t} \downarrow$
3. **Taxes:** $\tau_t = r_t B_{ss} \downarrow$
4. **Household consumption,** $C_t^{hh} \uparrow$, due to $r_t \downarrow$ and $\tau_t \downarrow$
5. **Firms production,** $Y_t \uparrow$, and **labor demand,** $N_t \uparrow$
6. **Inflation,** $\pi_t \uparrow$, and **wage,** $w_t \uparrow$ and **dividends,** $d_t \downarrow$
7. **Household labor supply,** $N_t^{hh} \uparrow$, due to $w_t \uparrow$ and $d_t \downarrow$,
but dampened $\tau_t \downarrow$
8. **Nominal rate,** $i_t \uparrow$ due to $\pi_t \uparrow$ implying $r_t \uparrow$
9. **Household consumption,** $C_t^{hh} \uparrow$, due to $w_t \uparrow$
but dampened by $d_t \downarrow$ and $r_t \uparrow$

IRFs and simulation

- **Previously:** Full non-linear transition path to an MIT-shock

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- **Today:** Just consider the first order solution

Linearized IRFs

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 1. Solve for IRFs for unknowns

$$H(\mathbf{U}, \mathbf{Z}) = 0 \Rightarrow \mathbf{H}_U d\mathbf{U} + \mathbf{H}_Z d\mathbf{Z} = 0 \Leftrightarrow d\mathbf{U} = \underbrace{-\mathbf{H}_U^{-1} \mathbf{H}_Z}_{\equiv \mathbf{G}_U} d\mathbf{Z}$$

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2. Derive all other IRFs for

$$\begin{aligned} \mathbf{X} = \mathbf{M}(\mathbf{U}, \mathbf{Z}) &\Rightarrow d\mathbf{X} = \mathbf{M}_U d\mathbf{U} + \mathbf{M}_Z d\mathbf{Z} \\ &= \underbrace{(-\mathbf{M}_U \mathbf{H}_U^{-1} \mathbf{H}_Z + \mathbf{M}_Z)}_{\equiv \mathbf{G}} d\mathbf{Z} \end{aligned}$$

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- **Limitations:**

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- **Limitations:**
 1. Imprecise for *large* shocks

Linearized IRFs

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- **Today:** Just consider the first order solution
 1. Solve for IRFs for unknowns

$$H(U, Z) = 0 \Rightarrow H_U dU + H_Z dZ = 0 \Leftrightarrow dU = \underbrace{-H_U^{-1} H_Z dZ}_{\equiv G_U}$$

2. Derive all other IRFs for

$$\begin{aligned} X = M(U, Z) \Rightarrow dX &= M_U dU + M_Z dZ \\ &= \underbrace{(-M_U H_U^{-1} H_Z + M_Z)}_{\equiv G} dZ \end{aligned}$$

- **Limitations:**

1. Imprecise for *large* shocks
2. Imprecise in models with *aggregate non-linearities*
(direct in aggregate equations or through micro-behavior)

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\mathbf{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation ϵ_t .

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where $d\mathbf{X}_s$ is the IRF to a unit-shock after s periods

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- **Intuition:** Sum of first order effects from all previous shocks

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- **Intuition:** Sum of first order effects from all previous shocks
- **Equivalence:**
 1. Same result if we linearize all aggregated equations and write the model in $MA(\infty)$ form
 2. The state space form can also be recovered (not needed)

- **Generality:** Extend the model with auxiliary variables (incl. distributional moments) to calculate additional IRFs and simulations

Advanced linearized simulation

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- **Full distribution** (advanced):

Advanced linearized simulation

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- **Full distribution** (advanced):
 1. The IRF for grid point i_g in a policy function can be calculated as

$$da_{i_g,s}^* = \sum_{s'=s}^{T-1} \sum_{X^{hh} \in \mathbf{X}^{hh}} \frac{\partial a_{i_g}^*}{\partial X_{s'-s}^{hh}} dX_{s'-s}^{hh}.$$

where $\partial a_{i_g}^* / \partial X_k^{hh}$ is the derivative to a k -period ahead shock to input X^{hh} (calculated in fake news algorithm)

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where $\partial a_{i_g}^* / \partial X_k^{hh}$ is the derivative to a k -period ahead shock to input X^{hh} (calculated in fake news algorithm)

2. The policy function can there be simulated as

$$a_{i_g}^* = \sum_{s=0}^T da_{i_g,s}^* \tilde{\epsilon}_{t-s}$$

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3. Distribution can then be simulated forwards

The HANK example from GEModelToolsNotebooks II

- **Presentation:** I go through the code for *finding the linearized IRFS and simulating the model*
- **In-class exercise:**
 1. Look at the code and talk about it with the person next to you for 10 minutes
 2. Write at least one question on https://padlet.com/jeppe_drue Dahl/advmacrohet

Exercise

Exercise = Assignment II

You can start working on Assignment II: The HANK model

Summary

Summary and next week

- **Today:**

1. A baseline HANK model
2. Linearized IRFs and simulation

- **Next week:** Analytical Properties of HANK models

- **Homework:**

1. Work on Assignment II
2. Read: Auclert et al. (2018), »The Intertemporal Keynesian Cross«