1 Define the stationary equilibrium for the model

The stationary equilibrium is defined in equation (1):

$$H_{ss}\left(K_{ss}, B_{ss}, L_{ss}; \tau^a_{ss}, \tau^\ell_{ss}\right) = \begin{bmatrix} B_{ss} + K_{ss} - \boldsymbol{a}^{*\prime}_{ss}\boldsymbol{D}_{ss} \\ L_{ss} - \int \ell^*_{ss}\zeta_i z_{ss} d\mathbf{D}_{ss} \\ r_{ss} - \frac{1}{B_{ss}}\left(\int\left(\tau^a_{ss} r_{ss} a^*_{ss} + \tau^\ell_{ss} w_{ss}\ell^*_{ss}\zeta_i z_{ss}\right) d\boldsymbol{D}_{ss} - G_t\right) \\ r^K_{ss} - \alpha\Gamma_{ss}(\frac{K_{ss}}{L_{ss}})^{\alpha-1} \\ w_{ss} - (1-\alpha)\Gamma_{ss}(K_{ss}/L_{ss})^{\alpha} \\ r_{ss} - \left(r^K_{ss} - \delta\right) \\ \boldsymbol{D}_{ss} - \Pi'_z \underline{\boldsymbol{D}}_{ss} \\ \underline{\boldsymbol{D}}_{ss} - \Lambda'_{ss}\boldsymbol{D}_{ss} \end{bmatrix} = \mathbf{0} \quad (1)$$

With policy functions, $a^*_t = a^*\left(\left\{w_\tau, r_\tau, \tau^a_\tau, \tau^\ell_\tau\right\}_{\tau<geqt}\right)$, $\ell^*_t = \ell^*\left(\left\{w_\tau, r_\tau, \tau^a_\tau, \tau^\ell_\tau\right\}_{\tau<geqt}\right)$, and the choice transition $\Lambda_t = \Lambda\left(\left\{w_\tau, r_\tau, \tau^a_\tau, \tau^\ell_\tau\right\}_{\tau<geqt}\right)$, and stochastic transition Π_z.

The first two conditions are the asset market and the labor market clearings, the goods market then clears by Walras' law. The next condition is a rewritten budget constraint for the government in steady state, using that $r^B_{ss} = r_{ss}$:

$$B_{ss} = (1 + r^B_{ss})B_{ss} + G_{ss} - \int \left[\tau^a_{ss} r_{ss} a^*_{ss} + \tau^\ell_{ss} w_{ss}\ell^*_{ss}\zeta_i z_t\right] d\boldsymbol{D}_t \quad (2)$$

$$\Leftrightarrow r_{ss} = -\frac{1}{B_{ss}}\left(G_{ss} - \int \left[\tau^a_{ss} r_{ss} a^*_{ss} + \tau^\ell_{ss} w_{ss}\ell^*_{ss}\zeta_i z_t\right] d\boldsymbol{D}_t\right) \quad (3)$$

Interestingly, this equation has a sort of reverse logic regarding deficits and government debt in steady state: If the government runs a permanent deficit they lend out money. Conversely, if they have a permanent surplus bonds are positive and the government is lending money from the public (assuming $r_{ss} > 0$). Intuitively, running a surplus should allow public savings and lending out money, but in the steady state, the surplus must be countered by paying interest on bonds to keep everything constant. When running a deficit they have to make money in order to finance their deficit and thus loan out money.

The 4th and 5th conditions come from the firm problem, and the 5th condition is the no-arbitrage condition between the bonds and the capital markets. The 6th condition is more of an assumption: the stochastic shocks do not change the overall distribution of productivity and assets. The 7th condition implies that the choice transition does not change the distribution.

When solving for the stationary equilibrium, the most straightforward thing is to use an optimizer to solve for both labor and assets clearings over K_{ss} and L_{ss}. This works perfectly well but is quite slow, and luckily there is a faster way.

The key to this trick is that the equilibrium variables that enter the household problem, w_{ss} and r_{ss} are both pinned down by the capital-labor ratio $\frac{K_{ss}}{L_{ss}}$, independent of the absolute sizes.

One can therefore find the equilibrium by only searching over $\frac{K_{ss}}{L_s}$. After solving the household problem the labor market is assumed to clear, setting L_{ss} equal to household labor supply. From the capital-labor ratio this gives an implied K_{ss}. With an implied B_{ss}, given tax income from household choices and r_{ss}, the assets market clearing can then be found, to check if that also clears.

In section 2 we will see that the model for many parameter values has 2 steady states. Luckily we will also see that they are relatively nicely ordered. The first thing to note is that the model is not well-defined when $r_{ss} = 0$, because of equation (2)[1]. This is because the government cannot clear its budget constraint if it is not possible to either make money or give them away in the bonds market. It is convenient for us to note which capital-labor ratio causes $r_{ss} = 0$ as the 2 steady states in the model are located on either side of the cut-off, from the firm optimization we have:

$$r_t^K = \alpha\Gamma \left(\frac{K_t}{L_t}\right)^{\alpha-1} \tag{4}$$

Which can then be inserted:

$$r_t = 0 \Leftrightarrow r_t^K - \delta = 0 \Leftrightarrow \alpha\Gamma \left(\frac{K_t}{L_t}\right)^{\alpha-1} - \delta = 0 \tag{5}$$

$$\Leftrightarrow \left(\frac{K_t}{L_t}\right)^{\alpha-1} = \frac{\delta}{\alpha\Gamma} \Leftrightarrow \frac{K_t}{L_t} = \left(\frac{\delta}{\alpha\Gamma}\right)^{\frac{1}{\alpha-1}} \tag{6}$$

2

2.1 Solving the model

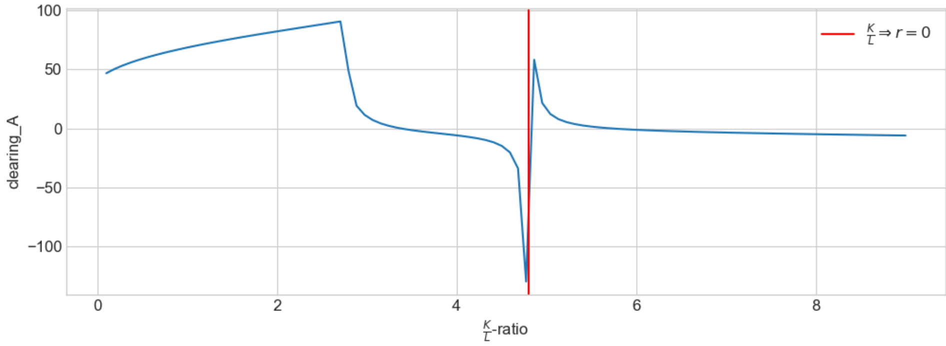
Figure 1 shows the problem with multiple steady states by plotting the assets clearing condition, $\boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss} - (B_{ss} + K_{ss})$, for different guesses of $\frac{K}{L}$. The asset condition is zero at two points. As $\frac{K}{L}$ pins down r, i have also noted the $\frac{K}{L}$ that makes $r = 0$. Testing around with some different calibration methods has led me to believe, that when there are two steady states they will be located on either side of the $r = 0$ cutoff, as shown in the figure.

This motivates my approach when searching for steady states, as I can bound the search based on the cutoff. When searching for the steady state with $r > 0$, I use Brent's method for finding the root between a number a little above 0 and a number a little below $\left(\frac{\delta}{\alpha\Gamma}\right)^{\frac{1}{\alpha-1}}$. When searching for the steady state with $r < 0$ I search between $1.05 \cdot \left(\frac{\delta}{\alpha\Gamma}\right)^{\frac{1}{\alpha-1}}$ and $6 \cdot \left(\frac{\delta}{\alpha\Gamma}\right)^{\frac{1}{\alpha-1}}$. The relative distance to the cutoff is a bit arbitrary but seems to work well.

Figure 2 shows a problem the model runs into when taxes are low. Namely that we go from having too many steady states to having none. I'm not entirely certain why this happens, but I suspect it has something to do with the clearing condition for the government's budget constraint.

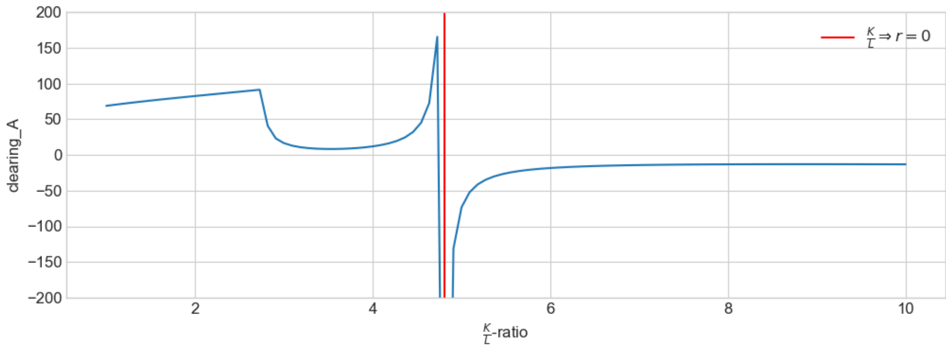
[1] Except in the edge case $G_{ss} = \int \left[\tau_{ss}^a r_{ss} a_{ss}^* + \tau_{ss}^\ell w_{ss} \ell_{ss}^* \zeta_i z_t\right] d\boldsymbol{D}_t$, where the government spending equals income.

Figure 1: Assets clearing across $\frac{K}{L}$, assignment calibration



In section 3 we will see that B_{ss} moves towards 0 as we approach the values of τ^a and τ^ℓ where a steady state is not found, I think this is because the lowered taxes pulls towards a balanced pre-interest payments government budget. In the case of a completely balanced budget, the clearing conditions puts $r_{ss} = 0 \vee B_{ss} = 0$, which are both ill-defined in the steady state conditions. $B_{ss} = 0$ is possible (with a small rewriting), but I suspect something fishy happens as we approach that state. Particularly because tax revenue is affected by household choices, so it is not certain whether the government budget is balanced for a given set of taxes that are not zero.

Figure 2: Assets clearing across $\frac{K}{L}$, $\tau^\ell = 0.1$



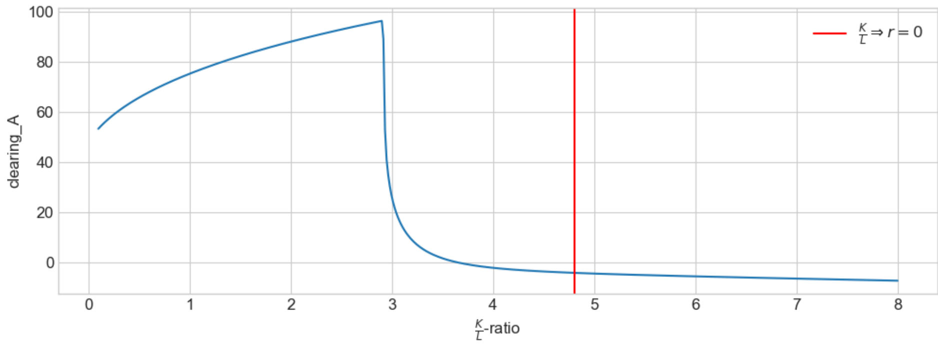
A possible solution is to make sure the government always run a surplus, by setting $G = 0$. Figure 3 shows that only 1 steady state with $r_{ss} > 0$ exists in the edge case with taxes set to zero[2].

2.2 Aggregate values

Table 1 shows aggregate values and prices for the steady states. With $r < 0$, the government still runs a pre-interest payments surplus, so they have to lend out money in order to balance. This significantly increases K, but also significantly decreases household assets as the capital now mainly comes from money borrowed from the government. Small net household assets also make sense as $r < 0$, which means that the only savings motive is precautionary. The no-government

[2]Steady states also exist with $r < 0$ for $\tau^\ell > 0$, but in order to limit the scope of this paper I will not discuss it

Figure 3: Assets clearing across $\frac{K}{L}$, $\tau^{\ell} = \tau^{a} = G = 0$



calibration has a vastly higher average utility, but this makes sense since wasteful government spending has been removed.

Table 1: The stationary steady states

	$r > 0$	$r < 0$	no gov
Y	1.458	1.747	1.364
C	0.816	0.855	1.025
I	0.342	0.593	0.339
K	3.419	5.928	3.391
L	1.012	1.035	0.923
KL	3.378	5.726	3.672
B	0.627	-5.775	-0.000
$\int \tau^{a}_{ss} r_{ss} a^{*}_{ss} d\mathbf{D}_{ss}$	0.011	-0.000	0.000
$\int \tau^{\ell}_{ss} w_{ss} \ell_{ss} \zeta_i z_{ss} d\mathbf{D}_{ss}$	0.306	0.367	0.000
w	1.009	1.182	1.034
r	0.028	-0.012	0.021
$\int a^{*}_{ss} d\mathbf{D}_{ss}$	4.045	0.153	3.391
$\int c^{*}_{ss} d\mathbf{D}_{ss}$	0.816	0.855	1.025
$\int \ell^{*}_{ss} d\mathbf{D}_{ss}$	1.081	1.138	0.994
$\int u\left(c^{*}_{ss}\right) d\mathbf{D}_{ss}$	-2.013	-2.000	-1.625
τ^{a}	0.100	0.100	0.000
τ^{ℓ}	0.300	0.300	0.000
G	0.300	0.300	0.000
Assets clearing	0.000	0.000	0.000
Goods clearing	-0.000	-0.000	-0.000
Labor clearing	0.000	0.000	0.000

Columns 1 and 2 show the two steady states of the model calibrated as suggested by the assignment. The third column shows a calibration with no government: $\tau^{\ell} = \tau^{a} = G = 0$.

2.3 Household behavior

This section displays household behavior and distribution, these plots are only done for the $r > 0$ steady state. Figure 4 shows optimal household behavior across a for different shocks. The most interesting point is probably the labor supply curves. For large values of a the most productive households choose to work more, which makes sense as their relative gain from working to leisure is larger. However, for low-asset households, a bad productivity shock forces them to work a lot more. I interpret this in an income/substitution framework.

Figure 4: Households behavior for different shocks, z_t

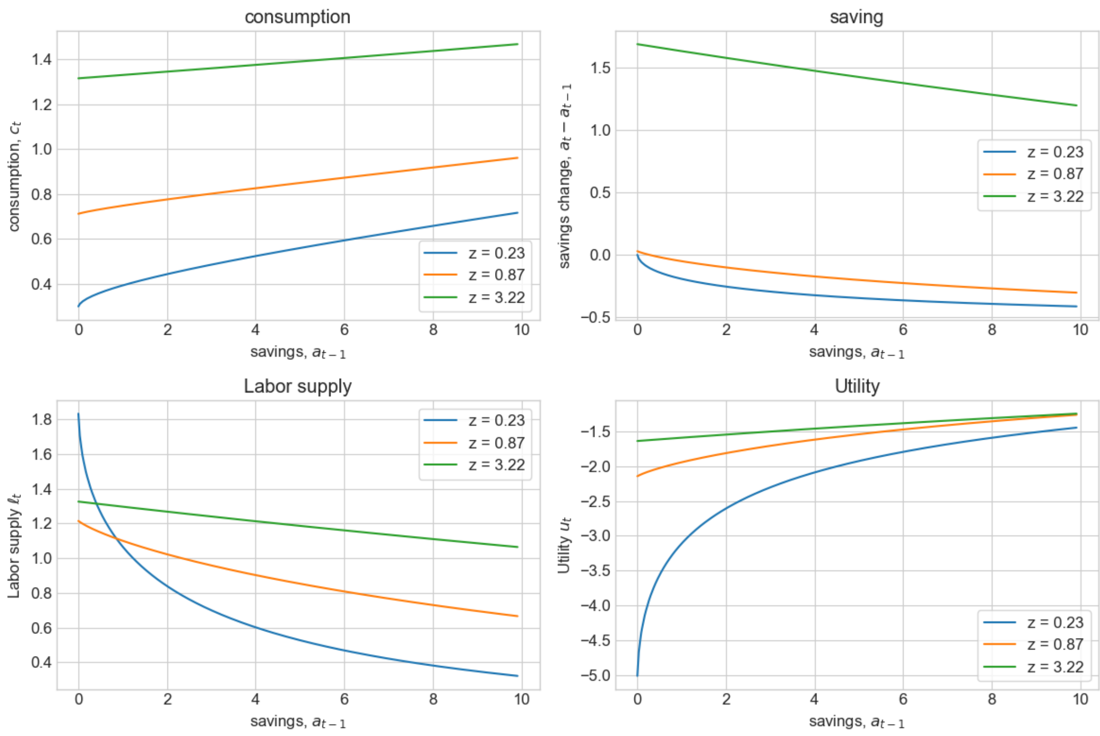


Figure 6 shows household behavior for the different fixed household types. Generally the households that has the lowest dis-utility from working, works more. The more productive households work less if they have the same φ.

Figure 6 shows the distribution of households. The households that derive the most dis-utility from working tend to hold the least savings, and the most productivity households holds the most savings.

3

Figures 7-9 extrapolates the most important values from table 1 to a grid over $\tau^a, \tau^\ell \in [0, 0.9]$. The missing dots indicate that the method described in section 2.1 was not able to find a solution for the parameter combination. As shown in section 2, for some parameter values the model does not have a steady state, however, some of the missing dots also might be due to failure of convergence if the bounds used in the root finder were not wide enough.

Average utility is clearly larger the lower the taxes are.

For the steady states with $r < 0$, τ^a seems to have little effect on the outcomes. This makes sense as net assets are really low, so household care less about their net return on their assets. It

Figure 5: The different household types, for a fixed shock

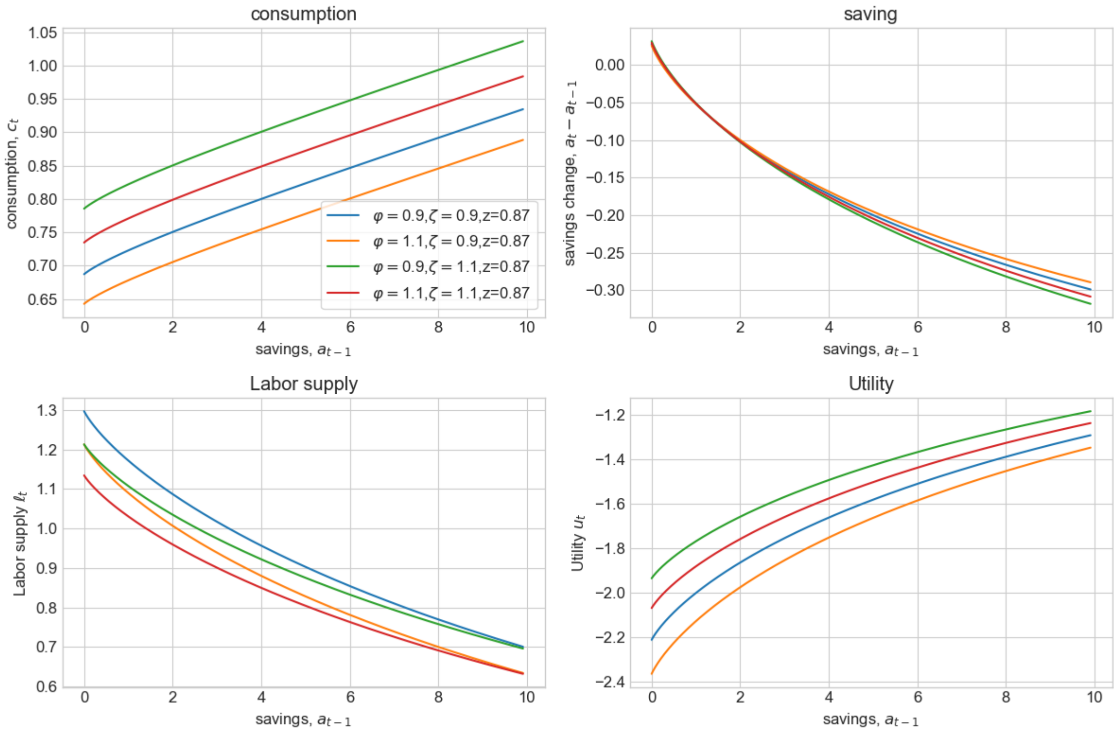
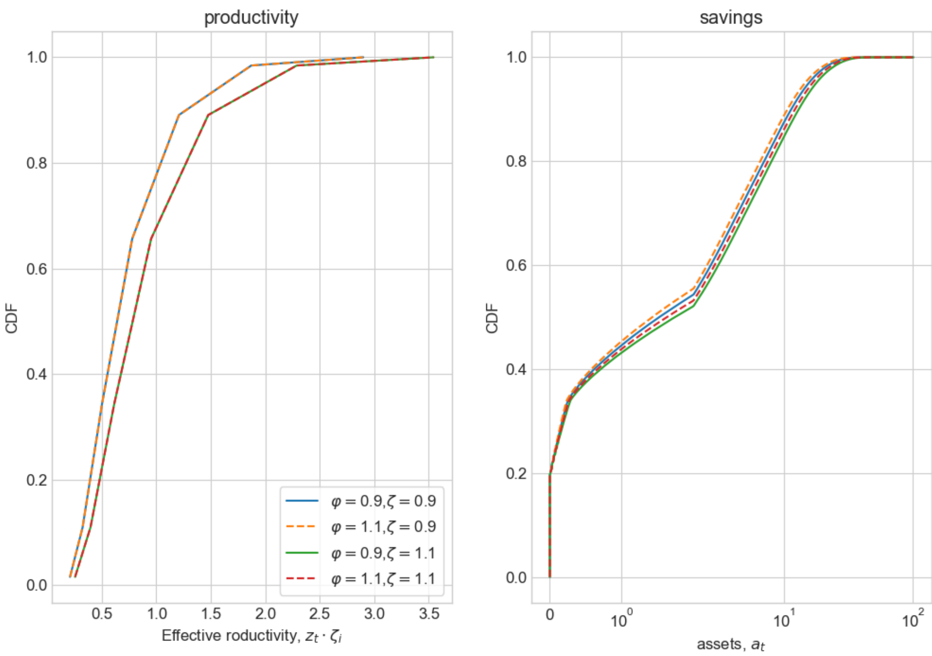


Figure 6: Distribution of households



should also be noted that when $r < 0$, τ^a is actually a subsidy that lowers the negative interest that households pay on their assets.

Figure 7: Equilibrium values across τ^a and τ^ℓ in steady states where $r > 0$

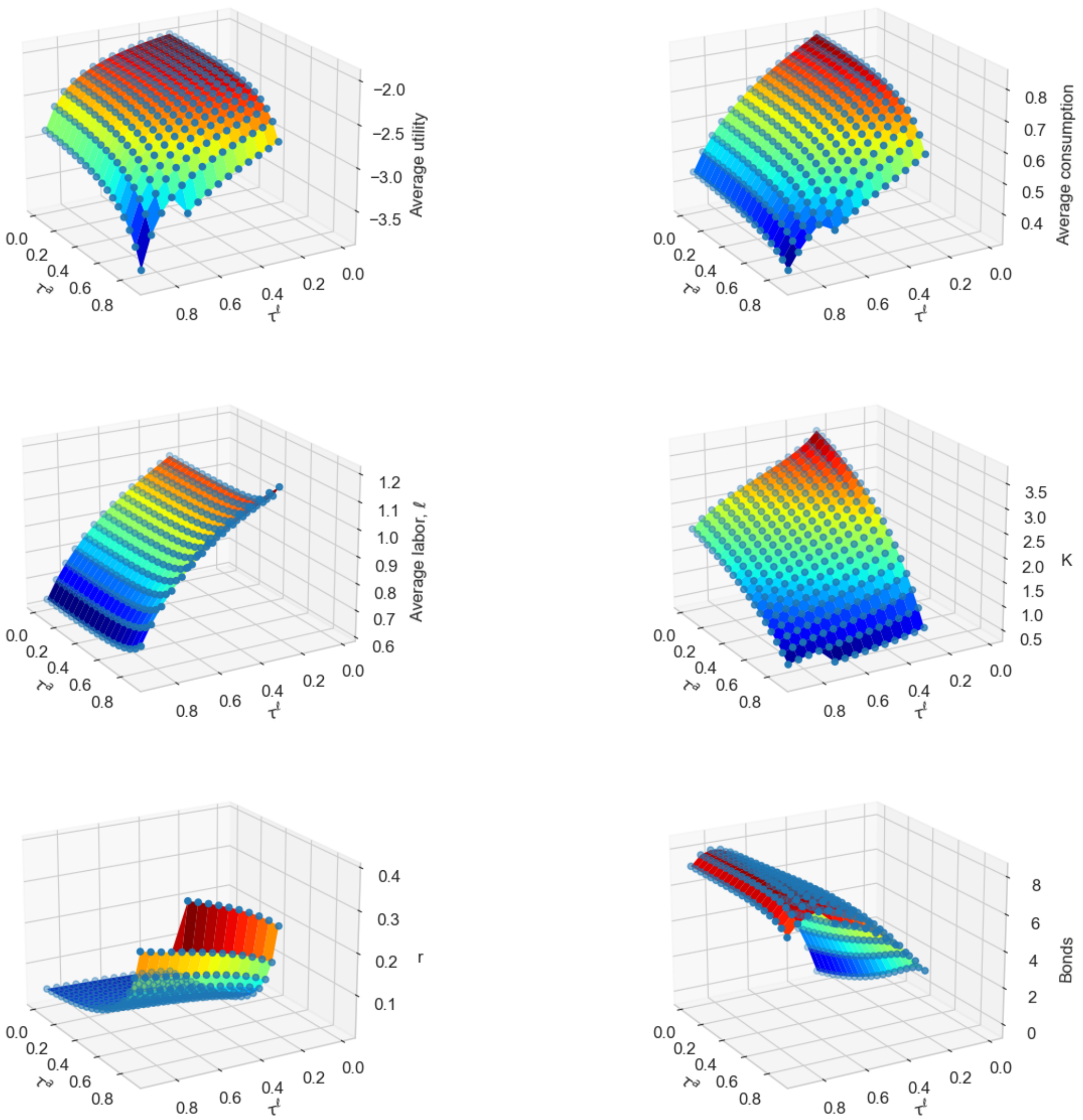
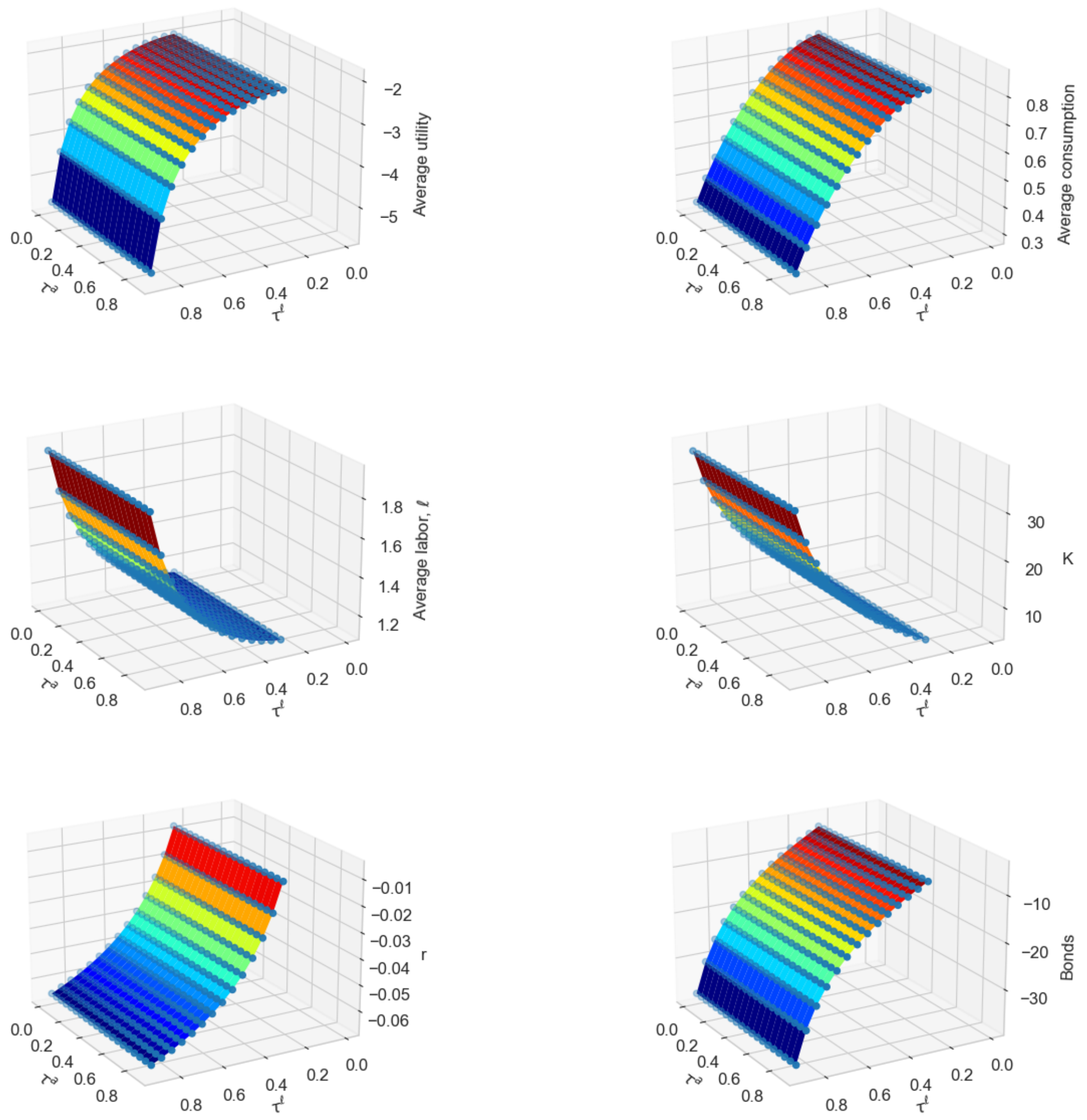


Figure 9 shows that model is more easily solved when $G = 0$. Average utility is clearly increasing for lower tax values across the spectrum.

4

Average utility is generally larger the lower the tax rates are. However, the model does not converge to a solution for small τ^ℓ when $G = 0.3$. So what the optimal tax level is a bit of a philosophical question. Figure 7 suggests that lower taxes are ambiguously better in all possible ranges, however since setting all taxes to zero is not possible as there is no steady state in. So the best policy advice within this model is to set $\tau^a = 0$, and τ^ℓ to the lowest number possible while not breaking

Figure 8: Equilibrium values across τ^a and τ^ℓ in steady states where $r < 0$



the model. This is a somewhat weird policy recommendation. I have founds this lower bound to be around $\tau^\ell \approx 25\%$.

For $r > 0$ the gains from taxes can only come from redistribution: Since the tax is proportional, a higher tax rate decreases the dispersion in after-tax earnings. However, that government surplus is paid back to the households through the bonds market through bonds payments. So in a sense, the redistribution is from high-earning households to high-wealth households, meaning its effect on average utility is likely to be low. The negative effects are the distortionary effects on optimal household behavior, and the fact that bonds are unproductive asset holdings (from the view of the total economy) that crowds out capital investments (when they are positive).

Figure 9: Equilibrium values across τ^a and τ^ℓ in steady states where $r > 0 \wedge G = 0$

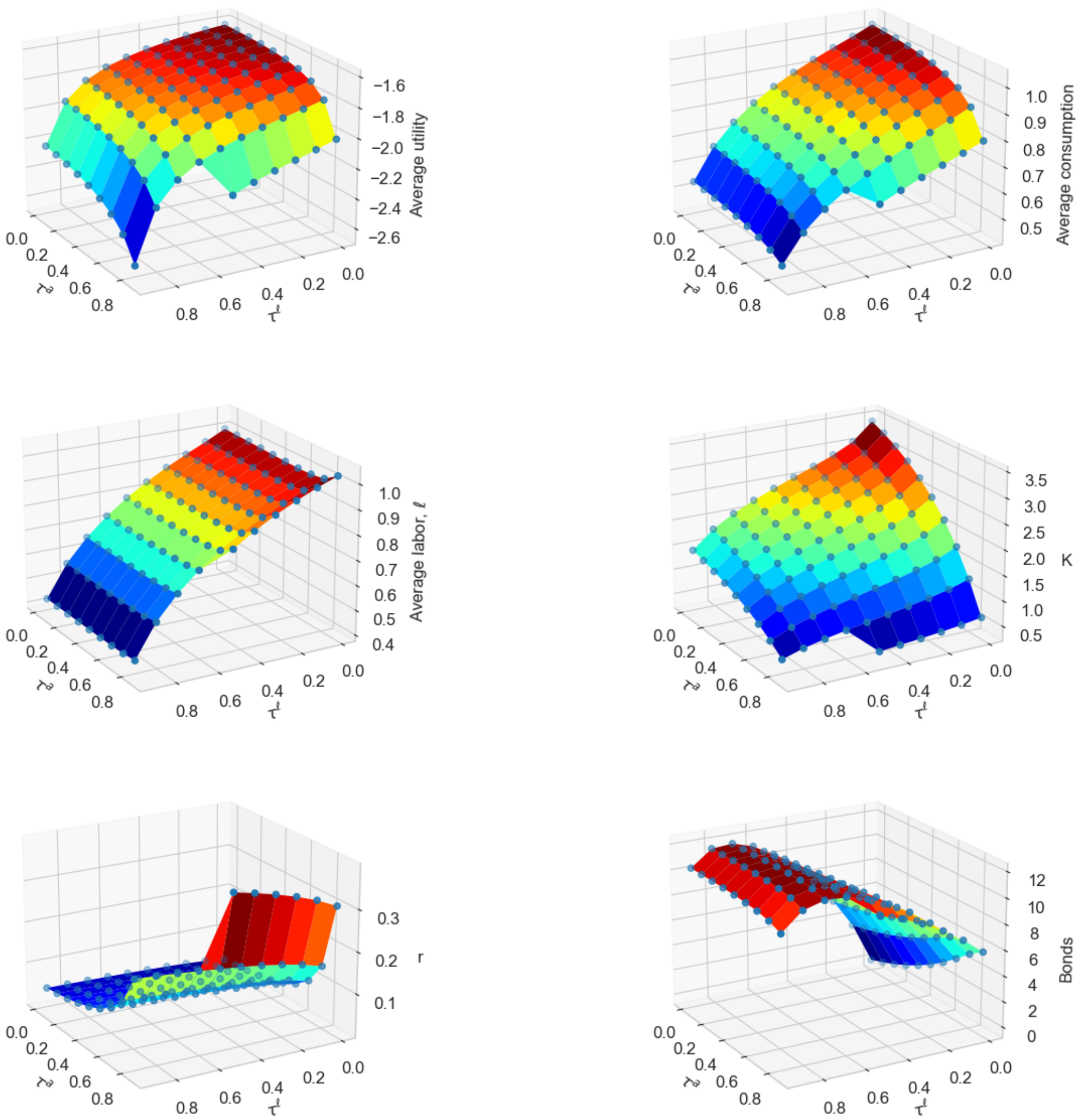


Figure 9 suggests that for $G = 0$ the optimal policy is to set $\tau^a = \tau^\ell = 0$ and using an optimizer also suggests that this is true.

For $r < 0$ the effect of the bonds market should be the reverse since the government pays back its surplus through negative bonds, which effectively allows households to borrow money from the government and use it as capital in production while having low net assets. I think the reason this does not happen is because the level of capital is inefficiently high. One of my reasons for thinking this is that consumption is not much higher than in the $r > 0$ steady state, despite a much larger stock of capital. The golden rule of savings would suggest that a capital level that pushes r_{ss}^K below δ in steady, is inefficient for maximizing average consumption. If we're only considering consumption, this means that capital is too high in the baseline steady-state $r > 0$. When negative, the bonds market increases the level of capital above what r_{ss}^K implies, and furthermore with dis-utility from working the optimal capital level when considering utility, is likely even lower.

5

My extension to the model is to make the marginal tax rate be increasing in income, such that the highest earner pay a larger proportion than the low earner, possibly even with a negative tax rate for the lowest earners. This might increase welfare as it lowers income uncertainty and smooths consumption across individuals more heavily than the proportional tax rate. It has the problem that it lowers the incentive for households to work, especially for those who are most productive. But this might not be a problem, as the optimal labor supply function plotted in figure 5, showed that higher productive households tended to work less.

Some cool continuously increasing marginal tax rates are discussed in Estévez Schwarz and Sommer (2018). However, I was however not able to implement any of them in the EGM framework when solving the household problem. The main challenge was equation (8) in the assignment, where the first-order condition also needed to include the derivative of the tax rate function, making isolating ℓ_t a problem.

A more salient approach is detailed below, using a differentiable post-tax income function. A further benefit from this function which I think gives a lot of the utility gains is that the tax rate becomes negative for very low incomes, meaning that the redistribution from rich to poor is direct and based entirely on income, as opposed to through the bonds market.

Consider the post tax income function, $\Theta(\cdot)$, with $\theta \in [0, 1]$:

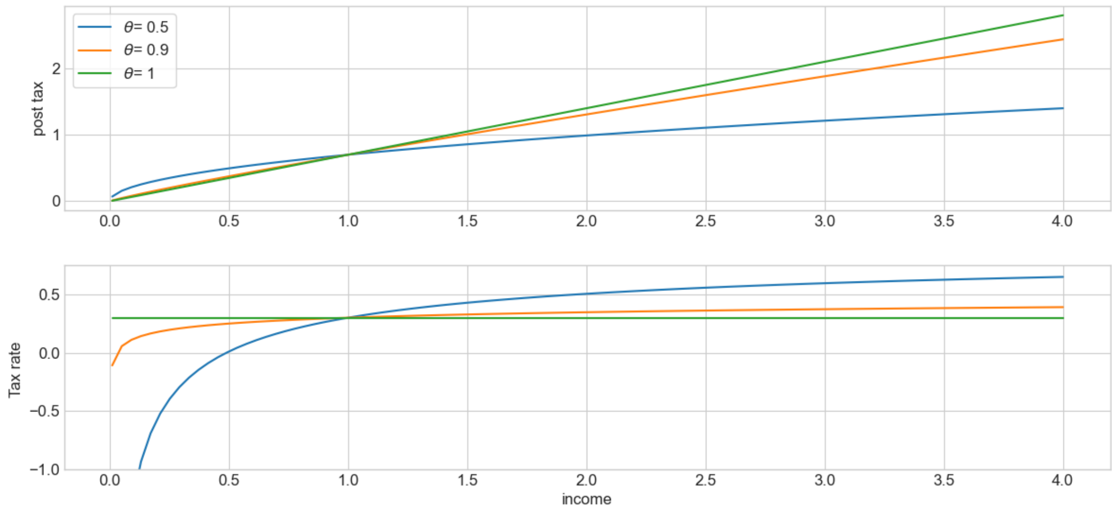
$$\Theta(x; \theta_1, \tau, x_h) = \frac{x^\theta}{x_h^{\theta-1}} \cdot \left(1 - \tau^\ell\right) \;\Rightarrow\; \frac{\partial \Theta}{\partial x} = \theta \frac{x^{\theta-1}}{x_h^{\theta-1}} \cdot \left(1 - \tau^\ell\right) \tag{7}$$

For $\theta = 1$ it simplifies to a constant marginal tax rate of τ^ℓ. For $\theta < 1$ the tax rate is only τ^ℓ if you earn the cutoff, x_h, otherwise the tax rate is either lower or higher depending on whether the income is above or below the cutoff. For $\theta = 0$ there is complete redistribution and everybody earns $x_h \left(1 - \tau^\ell\right)$ regardless of income. The function does not ensure a balanced budget, and tax revenue can therefore both be negative and positive depending on the tax rate and parameters. It should also be noted that there is still taxes on households that earn more than x_h when $\tau^\ell = 0$. A visualisation of the function is shown in Figure 10.

This post-tax function gives the budget constraint for the household:

$$a_t = (1 + \tilde{r}_t) a_{t-1} + \Theta\left(w_t \ell_t \zeta_i z_t\right) + \Pi_t - c_t \tag{8}$$

Figure 10: Post tax income and the effective tax rate for the post-tax income function defined in equation (7) with $x_h = 1$, $\tau^{\ell} = 0.3$



and

$$\frac{\partial a_t}{\partial \ell_t} = \theta \frac{(w_t \zeta_i z_t)^{\theta}}{x_h^{\theta-1}} \left(1-\tau^{\ell}\right) \ell_t^{\theta-1} \tag{9}$$

The first-order conditions of the households (equations (7) and (8) in the assignment) can be written as:

$$c_t = (\beta \underline{v}_{a,t+1}(z_t, a_t))^{-\frac{1}{\sigma}} \Leftrightarrow \beta \underline{v}_{a,t+1}(z_t, a_t) = c_t^{-\sigma} \tag{10}$$

$$\varphi_i \ell_t^{\nu} = (\beta \underline{v}_{a,t+1}(z_t, a_t)) \cdot \frac{\partial a_t}{\partial \ell_t} \tag{11}$$

Inserting equation (9) gives:

$$\varphi_i \ell_t^{\nu} = c_t^{-\sigma} \cdot \frac{\partial a_t}{\partial \ell_t} = c_t^{-\sigma} \theta \frac{(w_t \zeta_i z_t)^{\theta}}{x_h^{\theta-1}} \left(1-\tau^{\ell}\right) \ell_t^{\theta-1} \tag{12}$$

$$\Leftrightarrow \varphi_i \ell_t^{\nu-\theta+1} = c_t^{-\sigma} \theta \frac{(w_t \zeta_i z_t)^{\theta}}{x_h^{\theta-1}} \left(1-\tau^{\ell}\right) \Leftrightarrow \ell_t = c_t^{\frac{-\sigma}{\nu-\theta+1}} \left(\theta \frac{(w_t \zeta_i z_t)^{\theta}}{x_h^{\theta-1}} \frac{\left(1-\tau^{\ell}\right)}{\varphi_i}\right)^{\frac{1}{\nu-\theta+1}} \tag{13}$$

For the newton solver used for credit constrained households the $f(\ell_t)$-function needs to be replaced with:

$$f\left(\ell_t\right)=\ell_t-c_t^{\frac{-\sigma}{\nu-\theta+1}}\left(\theta\frac{\left(w_t\zeta_i z_t\right)^\theta}{x_h^{\theta-1}}\frac{\left(1-\tau^\ell\right)}{\varphi_i}\right)^{\frac{1}{\nu-\theta+1}} \tag{14}$$

While credit constrained we have:

$$c_t=\left(1+\tilde{r}\right)a_{t-1}+\Theta\left(w_t\ell_t\zeta_i z_t\right)\Rightarrow\frac{\partial c_t}{\partial\ell_t}=\theta\frac{\left(w_t\zeta_i z_t\right)^\theta}{x_h^{\theta-1}}\left(1-\tau^\ell\right)\ell_t^{\theta-1} \tag{15}$$

Giving the derivative of $f\left(\ell_t\right)$:

$$f'\left(\ell_t\right)=1-\frac{-\sigma}{\nu-\theta+1}c_t^{\frac{-\sigma}{\nu-\theta+1}-1}\theta\frac{\left(w_t\zeta_i z_t\right)^\theta}{x_h^{\theta-1}}\left(1-\tau^\ell\right)\ell_t^{\theta-1}\left(\theta\frac{\left(w_t\zeta_i z_t\right)^\theta}{x_h^{\theta-1}}\frac{\left(1-\tau^\ell\right)}{\varphi_i}\right)^{\frac{1}{\nu-\theta+1}} \tag{16}$$

Tax revenue from labor is the quite inelegant function below, which is simply total labor income, minus the post-tax income of the households:

$$\text{tax}^\ell=\int w_{ss}\ell_{ss}\zeta_i z_t-\Theta\left(w_{ss}\ell_{ss}\zeta_i z_t\right)d\boldsymbol{D}_{ss}=\int w_{ss}\ell_{ss}\zeta_i z_t-\frac{\left(w_{ss}\ell_{ss}\zeta_i z_t\right)^\theta}{x_h^{\theta-1}}\cdot\left(1-\tau^\ell\right)d\boldsymbol{D}_{ss} \tag{17}$$

5.1 Results from extension

I decided to only show the $r>0$-steady states for simplicity, even though the model also has steady states for $r<0$.

Figures 11 and 12 show that for $\theta=0.9$ or 0.5, average utility is still maximized by setting the tax rates as low as possible (for $\theta=0.5$ the model is not solvable for $\tau^{ell}<0.057$). However, even when $\tau^\ell=\tau^a=0$, the post-tax function ensures that there is still redistribution through taxes. In Figure 13 I set $\tau^a=0$ and show that utility is not maximized by setting $\theta=1$. For low τ^ℓ utility seems to be increasing for lower θ, but I'm not able to solve the model around the maximization point.

Table 2 shows how the new tax system can improve average welfare by comparison with the no-government calibration. Decreasing θ from 1 to 0.9 increases average utility while maintaining an approximately balanced government budget. This suggests that utility is increased through the redistribution and not through its effect on the capital level through the bonds market.

Column 3 with $\theta=0.5$ increases average utility further but it is not certain whether some of this effect is driven by the bonds market. Since tax revenue becomes negative, causing bonds to be negative and increase the capital level. As $r>0$ savings are below the golden savings rule for maximum consumption, it is unknown whether this has a positive effect on utility, meaning it could drive some of the increase in utility from column 2 to column 3.

Figure 11: Equilibrium values across τ^a and τ^ℓ in steady states where $r > 0 \wedge G = 0 \wedge \theta = 0.9$

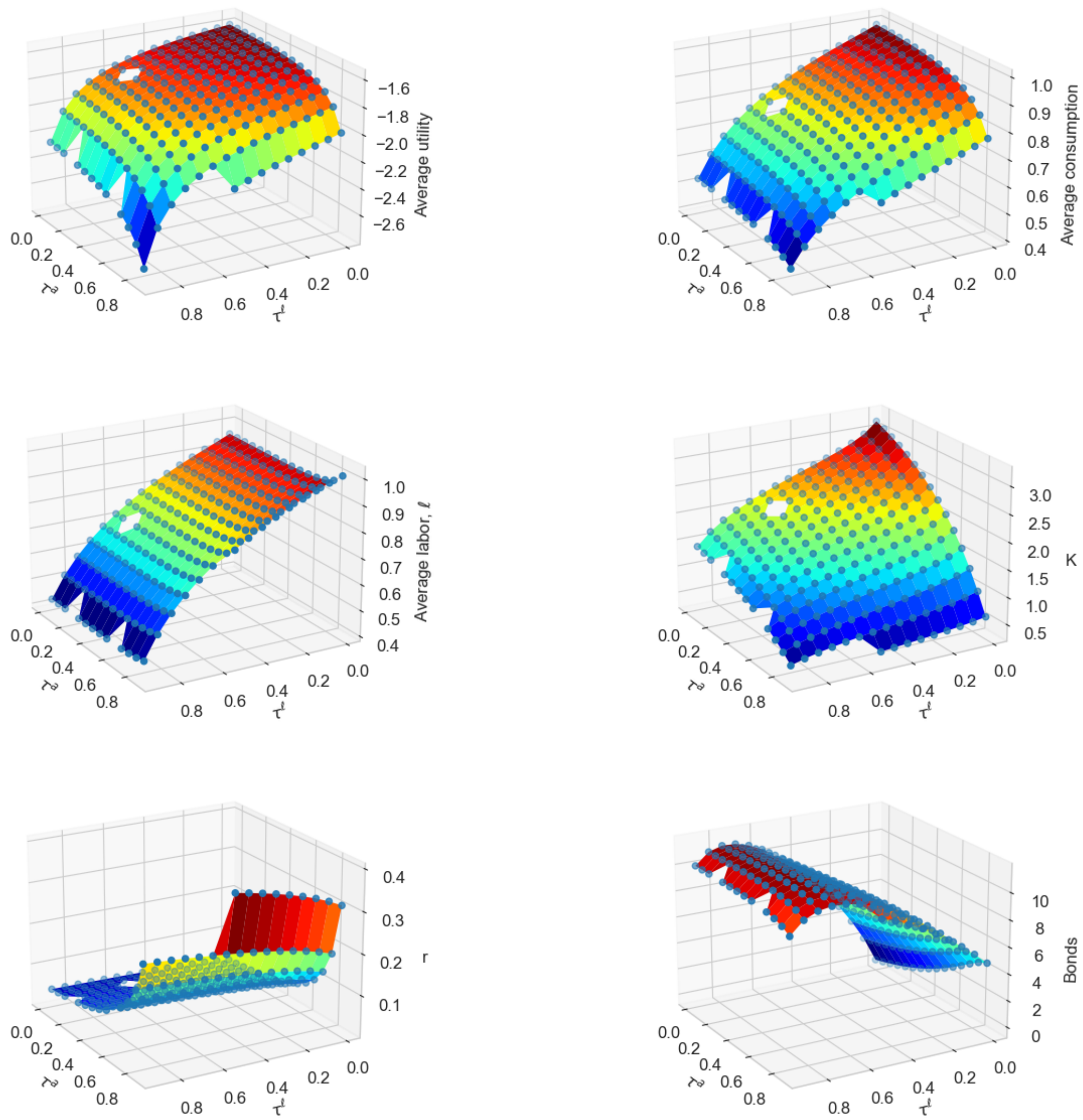


Figure 12: Equilibrium values across τ^a and τ^ℓ in steady states where $r > 0 \wedge G = 0 \wedge \theta = 0.5$

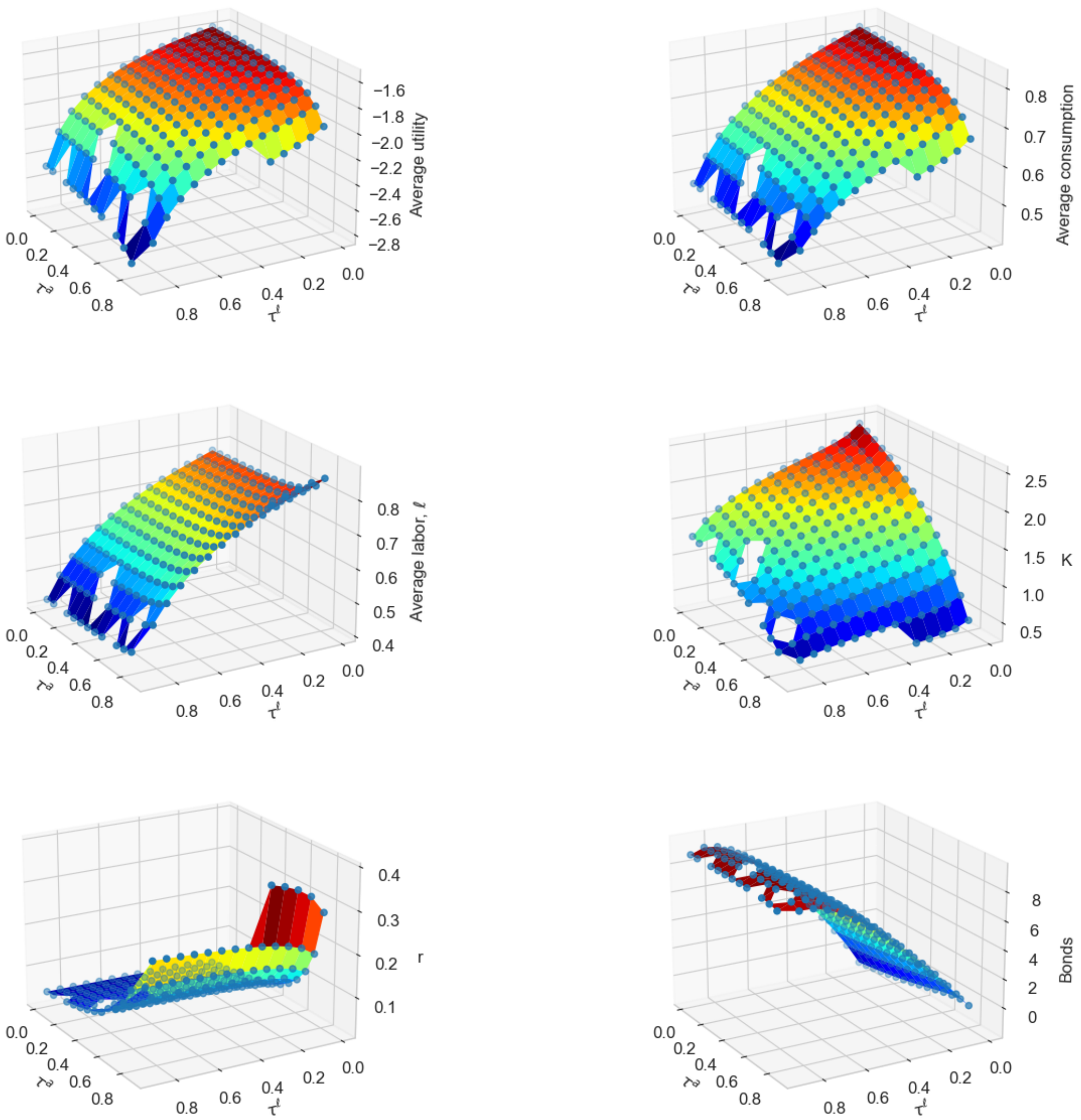


Figure 13: Equilibrium values across θ and τ^{ℓ} in steady states where $r > 0 \wedge G = 0 \wedge \tau^a = 0$

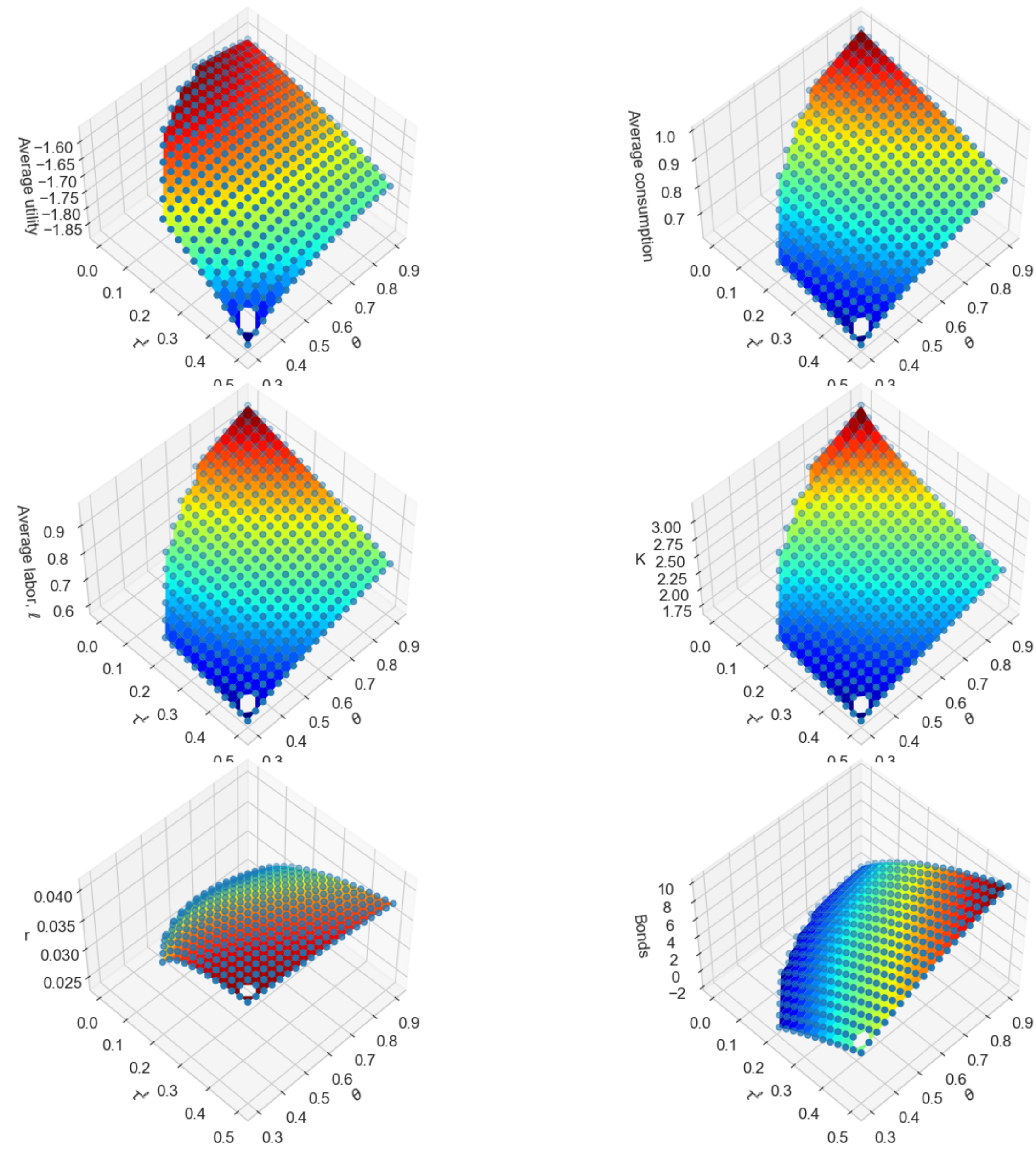


Table 2: Results from extended model

	No gov	$\theta = 0.9$	$\theta = 0.5$
Y	1.364	1.303	1.068
C	1.025	0.987	0.821
I	0.339	0.316	0.247
K	3.391	3.157	2.470
L	0.923	0.891	0.745
KL	3.672	3.541	3.313
B	0.000	0.006	-1.564
$\int \tau^a_{ss} r_{ss} a^*_{ss} d\mathbf{D}_{ss}$	0.000	0.000	0.000
$\int w_{ss}\ell_{ss}\zeta_i z_t - \frac{(w_{ss}\ell_{ss}\zeta_i z_t)^\theta}{x_h^{\theta-1}} \cdot \left(1-\tau^\ell\right) d\boldsymbol{D}_{ss}$	0.000	0.000	-0.046
w	1.034	1.023	1.003
r	0.021	0.024	0.030
$\int a^*_{ss} d\mathbf{D}_{ss}$	3.391	3.163	0.905
$\int c^*_{ss} d\mathbf{D}_{ss}$	1.025	0.987	0.821
$\int \ell^*_{ss} d\mathbf{D}_{ss}$	0.994	0.955	0.784
$\int u\left(c^*_{ss}\right) d\mathbf{D}_{ss}$	-1.625	-1.607	-1.589
τ^a	0.000	0.000	0.000
τ^ℓ	0.000	0.000	0.057
G	0.000	0.000	0.000
Assets clearing	0.000	-0.000	0.000
Goods clearing	-0.000	-0.000	-0.000
Labor clearing	0.000	0.000	0.000

$x_h = 1$ for both models with $\theta \neq 1$ (column 2 and 3)

References

Estévez Schwarz, D. and E. Sommer (2018). Smooth Income Tax Schedules: Derivation and Consequences. *SSRN Electronic Journal*.