CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

1. Introduction

Adv. Macro: Heterogenous Agent Models

Jeppe Druedahl & Patrick Moran

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Grundforskningsfond
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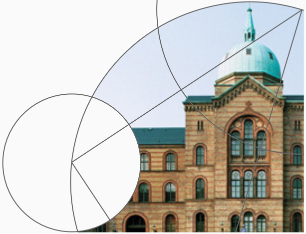

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  Prerequisite: *Intro. to Programming and Numerical Analysis*
  Complicated: *Close to the research frontier*

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- **Plan for today:**
  1. More about the course
  2. Dynamic programming - theory
  3. Dynamic programming - practice

- **Model components:**
  1. Optimizing individual agents (households + firms)
  2. Idiosyncratic and aggregate risk
  3. Information flows (who knows what when)
  4. Market clearing (Walras vs. search-and-match)

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- **HANC:** Heterogeneous Agent *Neo-Classical* model
- **HANK:** Heterogeneous Agent *New Keynesian* model
  (i.e. include price and wage setting frictions)

Teaching method

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  Web: sites.google.com/view/numeconcph-advmacrohet/
  Git: github.com/numeconcopenhagen/adv-macro-het

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  Git: github.com/numeconcopenhagen/adv-macro-het

- **Code:**

  1. We provide code you will build upon
  2. Based on the GEModelTools package

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- Individual **assignments** (hand-in on Absalon)

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     Deadline for proposal: 9th of December
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- **Exam:**
  1. Hand-in $3\times$**assignments**
  2. **48 hour take-home:** Programming of new extension
     $+$ analysis of model $+$ interpretation of results

Python

1. **Assumed knowledge:** From Introduction to Programming and Numerical Analysis you are assumed to know the basics of
   1.1 Python
   1.2 JupyterLab
   1.3 VSCode
   1.4 git
2. **Updated Python:** Install (or re-install) newest Anaconda
3. **Packages:**

```
pip install quantecon, EconModel, consav
pip install GEModelTools
```

Course plan

TBA

Knowledge

1. Account for, formulate and interpret precautionary saving models
2. Account for stochastic and non-stochastic simulation methods
3. Account for, formulate and interpret general equilibrium models with ex ante and ex post heterogeneity, idiosyncratic and aggregate risk, and with and without pricing frictions
4. Discuss the difference between the stationary equilibrium, the transition path and the dynamic equilibrium
5. Discuss the relationship between various equilibrium concepts and their solution methods
6. Identify and account for methods for analyzing the dynamic distributional effects of long-run policy (e.g. taxation and social security) and short-run policy (e.g. monetary and fiscal policy)

Skills

1. Solve precautionary saving problems with dynamic programming and simulate behavior with stochastic and non-stochastic techniques
2. Solve general equilibrium models with ex ante and ex post heterogeneity, idiosyncratic and aggregate risk, and with and without pricing frictions (stationary equilibrium, transition path, dynamic equilibrium)
3. Analyze dynamics of income and wealth inequality
4. Analyze transitional and permanent structural changes (e.g. inequality trends and the long-run decline in the interest rate)
5. Analyze the dynamic distributional effects of long-run policy (e.g. taxation and social security) and short-run policy (e.g. monetary and fiscal policy)

Competencies

1. Independently formulate, discuss and assess research on both the causes and effects of heterogeneity and risk for both long-run and short-run outcomes
2. Discuss and assess the importance of how heterogeneity and risk is modeled for questions about both long-run and short-run dynamics

Dynamic Programming

From static to dynamic consumer optimization

- **Budget constraint** for $t \in \{0, 1, \dots, T-1\}$

  $$\text{assets}_t = (1 + \text{return rate}) \times \text{assets}_{t-1} + \text{wage} \times \text{productivity}_t - \text{consumption}_t$$

  $$a_t = (1+r)a_{t-1} + wz_t - c_t$$

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- **Borrowing constraint:** $a_t \geq 0$
- **Static problem:**
  1. **Information:** z_t is known for all t
  2. **Target:** Discounted utility, $\sum_{t=0}^{T-1} \beta^t u(c_t)$, $\beta > 0$
  3. **Behavior:** Choose $c_0, c_1, \dots, c_{T-1}$ *simultaneously*
  4. **Solution:** Sequence of consumption *choices* $c_0, c_1, \dots, c_{T-1}$

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  3. **Behavior:** Choose $c_0, c_1, \ldots, c_{T-1}$ *simultaneously*
  4. **Solution:** Sequence of consumption *choices* $c_0, c_1, \ldots, c_{T-1}$
- **Dynamic programming:**
  1. **Information:** z_t is revealed period-by-period
  2. **Target:** *Expected* discounted utility, $\sum_{t=0}^{T-1} \beta^t \mathbb{E}_t[u(c_t)]$, $\beta > 0$
  3. **Behavior:** Choose c_t *sequentially* as information is revealed
  4. **Solution:** Sequence of consumption *functions*, $c_t^\star(z_t, a_{t-1})$

Static solution: IBC

- **Substitution** implies *Intertemporal Budget Constraint* (IBC)

$$
\begin{aligned}
a_{T-1} &= (1+r)a_{T-2} + wz_{T-1} - c_{T-1} \\
&= (1+r)^2 a_{T-3} + (1+r)wz_{T-2} - (1+r)c_{T-1} + wz_{T-1} - c_{T-1} \\
&= (1+r)^T a_{-1} + \sum_{t=0}^{T-1} (1+r)^{T-1-t}(wz_t - c_t)
\end{aligned}
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$$

- Use **terminal condition** $a_{T-1} = 0$ (equality due utility max.)

$$
(1+r)^{-(T-1)} a_{T-1} = 0 \quad \Leftrightarrow \quad b_0 + h_0 - \sum_{t=0}^{T-1} (1+r)^{-t} c_t = 0
$$

where $b_0 = (1+r)a_{-1}$ and $h_0 \equiv \sum_{t=0}^{T-1}(1+r)^{-t} wz_t$

Static solution: FOC and consumption function

$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t \frac{c_t^{1-\rho}}{1-\rho} + \lambda \left[\sum_{t=0}^{T-1} (1+r)^{-t} c_t - b_0 - h_0 \right]$$

- **First order conditions**:

$$\forall t : 0 = \beta^t c_t^{-\rho} - \lambda (1+r)^{-t} \Leftrightarrow c_t^{-\rho} = \beta(1+r) c_{t+1}^{-\rho}$$

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- Insert **Euler** into **IBC** to get consumption choice

$$\begin{aligned}
\sum_{t=0}^{T-1} (1+r)^{-t} (\beta(1+r))^{t/\rho} c_0 &= b_0 + h_0 \Leftrightarrow \\
c_0 &= \frac{1 - (\beta(1+r))^{1/\rho}/(1+r)}{1 - ((\beta(1+r))/(1+r))^T} (b_0 + h_0)
\end{aligned}$$

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- **Question:** Is this the solution correct?

Dynamic solution: Bellman's Principle of Optimality

- **In words:** *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision*. (See Bellman, 1957, Chap. III.3.)

Dynamic solution: Bellman's Principle of Optimality

- **In words:** *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision*. (See Bellman, 1957, Chap. III.3.)
- **In math:**
  1. **Value function, v_t:** Defined recursively from

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq 0$$

  with $v_T(\bullet) = 0$.

  2. **Policy function, $c_t^\star$:** Is the same as

$$c_t^\star(z_t, a_{t-1}) = \arg\max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq 0$$

Vocabulary

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq 0$$

1. **State variables:** z_t and a_{t-1}
2. **Control variable:** c_t
3. **Continuation value:** $\beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$
4. **Parameters:** r, w, and stuff in $u(\bullet)$

Timing of shocks

- **Realization of shocks:** Happens first in the period before choices are made

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- **Beginning-of-period value function** (before realization)**:**

$$\underline{v}_t(z_{t-1}, a_{t-1}) = \mathbb{E}_{t-1}\left[v_t(z_t, a_{t-1})\right]$$

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- **End-of-period value function** (after realization)**:**

$$\begin{aligned} v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \underline{v}_{t+1}(z_t, a_{t-1}) \\ \text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq 0 \end{aligned}$$

Numerical value function iteration - basics

- **Discretization:** All state variable belong to discrete monotonically increasing sets $\equiv$ *grids*,

$$z_t \in \mathcal{G}_z = \{z^0, z^1, \dots, z^{\#_z - 1}\}$$
$$a_t \in \mathcal{G}_a = \{a^0, a^1, \dots, a^{\#_a - 1}\}$$

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- **Transition probabilities:** $\pi_{i_{z-},i_z} = \Pr[z_t = z^{i_z} \,|\, z_t = z^{i_{z-}}]$

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- **Transition probabilities:** $\pi_{i_{z-},i_z} = \Pr[z_t = z^{i_z} \mid z_t = z^{i_z-}]$
- **Linear interpolation** (function approximation):
  1. Assume $\underline{v}_{t+1}$ is known on $\mathcal{G}_z \times \mathcal{G}_a$ (tensor product)
  2. Evaluate $\underline{v}_{t+1}(z^{i_z}, a)$ for arbitrary a by

$$\underline{\breve{v}}_{t+1}(z^{i_z}, a) = \underline{v}_{t+1}(z^{i_z}, a^{\iota}) + \omega_i \frac{a - a^{\iota}}{a^{\iota+1} - a^{\iota}}$$
$$\omega_i \equiv \frac{v_{t+1}(z^{i_z}, a^{\iota+1}) - v_{t+1}(z^{\iota_z}, a^{\iota})}{a^{\iota+1} - a^{\iota}}$$
$$\iota \equiv \text{largest } i_a \in \{0, 1, \dots, \#_a - 2\} \text{ such that } a^{i_a} \leq a$$

Deriving transition probabilities

- **Specification:** Assume $\log z_t$ follows the AR(1) process

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1}, \;\; \psi_{t+1} \sim \mathcal{N}(\mu_\psi, \sigma_\psi^2)$$

  where μ_ψ is used to ensure $\mathbb{E}[z_t] = 1$
- **Literature:** Tauchen (1986), Tauchen and Hussey (1991) and Rouwenhorst (1995) develops method for deriving $\mathcal{G}_z$ and π_{i_{z-},i_z} given ρ_z and σ_ψ, but we don't care about the details here

Numerical value function iteration - loops

- **Beginning-of-period value function:**

$$\underline{v}_t(z^{i_{z-}}, a^{i_{a-}}) = \sum_{i_z=0}^{\#_z-1} \pi_{i_{z-},i_z} v_t(z^{i_z}, a^{i_{a-}})$$

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- **End-of-period value-of-choice:**

$$v_t(z^{i_z}, a^{i_{a-}} | c_t) = u(c_t) + \beta \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z,i_{z+}} \check{v}_{t+1}(z^{i_{z+1}}, a_t)$$

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- **Nested loops:**
  1. **Outer loop:** Backwards in time from $t = T-1$ (note $\underline{v}_T$ is known)
  2. **Inner loop:** For each grid point in $\mathcal{G}_z \times \mathcal{G}_a$ find $c_t^\star(z_t, a_{t-1})$ and therefore $v_t^\star(z_t, a_{t-1})$ with a *numerical optimizer*

Numerical Monte Carlo simulation

- **Initial distribution:** Draw $z_{i,-1}$ and $a_{i,-1}$ for $i \in \{0, 1, \ldots, N-1\}$

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- **Simulation:** Forwards in time from $t = 0$ and in each time period
  1. Draw z_{it} given transition probabilities
  2. Use linear interpolation to evaluate

$$
\begin{aligned}
c_{it} &= \breve{c}^{\star}_{t}(z_{it}, a_{it-1}) \\
a_{it} &= (1+r)a_{it-1} + wz_{it} - c_{it}
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- **Review:**
  - **Pro:** Simple to implement
  - **Con:** Computationally costly and introduces randomness

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- **Simulation:** Forwards in time from $t = 0$ and in each time period
  1. **Distribute stochastic mass:** For each i_z and i_{a-} calculate
     $\boldsymbol{D}_t(z^{i_z}, a^{i_{a-}}) = \sum_{i_{z-}=0}^{\#_z-1} \pi_{i_{z-},i_z} \underline{\boldsymbol{D}}_t(z^{i_{z-}}, a^{i_{a-}})$

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- **Review:**
  1. **Pro:** Computationally efficient and no randomness
  2. **Con:** Introduces a non-continuous distribution

Side-note: Matrix formulation

- The histogram method can be written in **matrix form**:

$$\boldsymbol{D}_t = \Pi_z' \underline{\boldsymbol{D}}_t$$
$$\underline{\boldsymbol{D}}_{t+1} = \Lambda_t \boldsymbol{D}_t$$

where

$\underline{\boldsymbol{D}}_t$ is vector of length $\#_z \times \#_a$

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Π_z' is derived from the π_{i_{z-},i_z}'s

Λ_t' is derived from the ι's and ω's

- **Note:** Example showed in notebook.

Infinite horizon: $T \to \infty$?

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq 0$$

- **Contraction mapping result:** *If β is low enough (strong enough impatience) then the value and policy function converge to $v(z_t, a_{t-1})$ and $c^\star(z_t, a_{t-1})$ for large enough T*
- **Maximum upper limit for β:** $\frac{1}{1+r}$
- **In practice:** Solve backwards until value and policy functions does not change anymore (given some tolerance)

EGM

Euler-equation from variation argument

- **Case I:** If $c_t^{-\rho} > \beta(1+r)\mathbb{E}_t\left[c_{t+1}^{-\rho}\right]$:
  Increase c_t by $\Delta > 0$, and lower c_{t+1} by $(1+r)$
  1. **Feasible:** Yes, if $a_t > 0$
  2. **Utility change:** $\left(c_t^{-\rho}\right) + \beta\left(-(1+r)\right)\mathbb{E}_t\left[c_{t+1}^{-\rho}\right] > 0$

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  Lower c_t by $\Delta > 0$, and increase c_{t+1} by $(1+r)$
  1. **Feasible:** Yes (always)
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Euler-equation from variation argument

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- **Conclusion:** By contradiction
  1. **Constrained:** $a_t = 0$ and $c_t^{-\rho} \geq \beta(1+r)\mathbb{E}_t\left[c_{t+1}^{-\rho}\right]$, or
  2. **Unconstrained:** $a_t > 0$ and $c_t^{-\rho} = \beta(1+r)\mathbb{E}_t\left[c_{t+1}^{-\rho}\right]$

Endogenous grid-point method (EGM)

Alternative to value function iteration:

1. Calculate **post-decision marginal value of cash:**

$$q(z^{i_z}, a^{i_a}) = \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z, i_{z+}} c_+(z^{i_{z+}}, a^{i_a})^{-\sigma}$$

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$$m(z^{i_z}, a^{i_a}) = a^{i_a} + c_+(z^{i_z}, a^{i_a})$$

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4. **Consumption function:** $c^\star(z^{i_z}, a^{i_{a-1}})$ = interpolation of function from $m(z^{i_z}, :)$ to $c(z^{i_z}, :)$ at $m = (1+r)a^{i_{a-}} + wz^{i_z}$

Exercises

Exercises: Model extensions

- **Three exercises for you to do:**
  1. Ensure the stationary distribution is found in the simulation
  2. Make some borrowing allowed, b
  3. Introduce transitory shock, ξ_t (hardest)
- **Full extended model:**

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$

$$\begin{aligned}
&\text{s.t.} \\
&a_t = (1+r)a_{t-1} + wz_t - c_t + \xi_t \\
&\log z_t = \rho_z \log z_{t-1} + \psi_t,\ \mathbb{E}[z_t] = 1, \mathrm{Var}[\psi_t] = \sigma_\psi^2 \\
&\xi_t \sim \mathcal{N}(0, \sigma_\xi^2) \\
&a_t \geq -b
\end{aligned}$$

Extra: Gauss-Hermite I

- **General problem:** How can we calculate

$$\mathbb{E}(f(x)) = \int f(x)g(x)dx$$

  - $f : \mathbb{R} \to \mathbb{R}$ some function
  - $g(x)$ is the probability distribution function (PDF) for x and $G(x)$ is the CDF

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- **How to choose S and the *nodes* (x_i) and *weights* (ω_i)?**
  Answer: Guassian quadrature

Extra: Gauss-Hermite II

- **Gauss-Hermite** quadrature uses that

$$\int_{-\infty}^{\infty} f(x) e^{-x^2} dx = \sum_{i=1}^{S} \omega_i f(x_i) + \frac{S!\sqrt{\pi}}{s^S (2S)!} f^{(2S)}(\epsilon)$$

for some ϵ and where the (x_i, ω_i)'s can be easily found

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- **Example: Random normal variable:** $Y \sim \mathcal{N}(\mu, \sigma^2)$ so that

$$\mathbb{E}[f(Y)] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} f(y)e^{-\frac{(y-\mu)^2}{2\sigma^2}}dy$$

$$\approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^{S} \omega_i f(\sqrt{2}\sigma x_i + \mu)$$

Summary

Summary and next week

- **Today:**
  1. Introduction to course
  2. Dynamic programming in theory
  3. Dynamic programming in practice
- **Next week:** More on consumption-saving models and precautionary savings in partial equilibrium
- **Homework:**
  1. Work on completing the model extension exercise
  2. Read TBA