



1. Introduction

Adv. Macro: Heterogenous Agent Models

Jeppe Druedahl & Patrick Moran

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 1. What explains the level and dynamics of heterogeneity/inequality?
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- Prerequisite:** *Intro. to Programming and Numerical Analysis*
- Complicated:** *Close to the research frontier*

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- **Plan for today:**
 1. More about the course
 2. Dynamic programming - theory
 3. Dynamic programming - practice

Macroeconomic Models with Heterogeneous Agents

- **Model components:**

1. Optimizing individual agents (households + firms)
2. Idiosyncratic and aggregate risk
3. Information flows (who knows what when \Rightarrow often everything)
4. Market clearing (Walras vs. search-and-match)

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Incomplete \rightarrow agents need to *self-insure*

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- **HANC:** Heterogeneous Agent *Neo-Classical* model

- **HANK:** Heterogeneous Agent *New Keynesian* model
(i.e. include price and wage setting frictions)

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2 hours of »normal« lecture
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Web: sites.google.com/view/numeconcph-advmacrohet/
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- **Code:**
 1. We provide code you will build upon
 2. Based on the **GEModelTools** package

- Individual **assignments** (hand-in on Absalon)

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Deadline for peer feedback: 16th of December (*exam requirement*)

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- **Exam**:
 1. Hand-in 3×**assignments**
 2. **48 hour take-home**: Programming of new extension
+ analysis of model + interpretation of results

1. **Assumed knowledge:** From **Introduction to Programming and Numerical Analysis** you are assumed to know the basics of
 - 1.1 Python
 - 1.2 JupyterLab
 - 1.3 VSCode
 - 1.4 git
2. **Updated Python:** Install (or re-install) newest Anaconda
3. **Packages:** `pip install quantecon, EconModel, consav`
4. **GEModel tools:**
 - 4.1 Clone the GEModelTools repository
 - 4.2 Locate repository in command prompt
 - 4.3 Run `pip install -e .`

See CoursePlan.pdf

1. Account for, formulate and interpret precautionary saving models
2. Account for stochastic and non-stochastic simulation methods
3. Account for, formulate and interpret general equilibrium models with ex ante and ex post heterogeneity, idiosyncratic and aggregate risk, and with and without pricing frictions
4. Discuss the difference between the stationary equilibrium, the transition path and the dynamic equilibrium
5. Discuss the relationship between various equilibrium concepts and their solution methods
6. Identify and account for methods for analyzing the dynamic distributional effects of long-run policy (e.g. taxation and social security) and short-run policy (e.g. monetary and fiscal policy)

1. Solve precautionary saving problems with dynamic programming and simulate behavior with stochastic and non-stochastic techniques
2. Solve general equilibrium models with ex ante and ex post heterogeneity, idiosyncratic and aggregate risk, and with and without pricing frictions (stationary equilibrium, transition path, dynamic equilibrium)
3. Analyze dynamics of income and wealth inequality
4. Analyze transitional and permanent structural changes (e.g. inequality trends and the long-run decline in the interest rate)
5. Analyze the dynamic distributional effects of long-run policy (e.g. taxation and social security) and short-run policy (e.g. monetary and fiscal policy)

1. Independently formulate, discuss and assess research on both the causes and effects of heterogeneity and risk for both long-run and short-run outcomes
2. Discuss and assess the importance of how heterogeneity and risk is modeled for questions about both long-run and short-run dynamics

Dynamic programming

From static to dynamic consumer optimization

- **Budget constraint** for $t \in \{0, 1, \dots, T - 1\}$

$$\text{assets}_t = (1 + \text{return rate}) \times \text{assets}_{t-1} + \text{wage} \times \text{productivity}_t - \text{consumption}_t$$

$$a_t = (1 + r)a_{t-1} + wz_t - c_t$$

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- **Static problem:**

1. **Information:** z_t is known for all t
2. **Target:** Discounted utility, $\sum_{t=0}^{T-1} \beta^t u(c_t)$, $\beta > 0$
3. **Behavior:** Choose c_0, c_1, \dots, c_{T-1} *simultaneously*
4. **Solution:** Sequence of consumption *choices* c_0, c_1, \dots, c_{T-1}

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4. **Solution:** Sequence of consumption *choices* c_0, c_1, \dots, c_{T-1}

- **Dynamic programming:**

1. **Information:** z_t is revealed period-by-period
2. **Target:** *Expected* discounted utility, $\sum_{t=0}^{T-1} \beta^t \mathbb{E}_t[u(c_t)]$, $\beta > 0$
3. **Behavior:** Choose c_t *sequentially* as information is revealed
4. **Solution:** Sequence of consumption *functions*, $c_t^*(z_t, a_{t-1})$

- **Substitution** implies *Intertemporal Budget Constraint* (IBC)

$$\begin{aligned}a_{T-1} &= (1+r)a_{T-2} + wz_{T-1} - c_{T-1} \\&= (1+r)^2 a_{T-3} + (1+r)wz_{T-2} - (1+r)c_{T-1} + wz_{T-1} - c_{T-1} \\&= (1+r)^T a_{-1} + \sum_{t=0}^{T-1} (1+r)^{T-1-t} (wz_t - c_t)\end{aligned}$$

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- Use **terminal condition** $a_{T-1} = 0$ (equality due utility max.)

$$(1+r)^{-(T-1)} a_{T-1} = 0 \Leftrightarrow b_0 + h_0 - \sum_{t=0}^{T-1} (1+r)^{-t} c_t = 0$$

where $b_0 = (1+r)a_{-1}$ and $h_0 \equiv \sum_{t=0}^{T-1} (1+r)^{-t} wz_t$

Static solution: FOC and consumption function

$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t \frac{c_t^{1-\rho}}{1-\rho} + \lambda \left[\sum_{t=0}^{T-1} (1+r)^{-t} c_t - b_0 - h_0 \right]$$

- **First order conditions:**

$$\forall t : 0 = \beta^t c_t^{-\rho} - \lambda(1+r)^{-t} \Leftrightarrow c_t^{-\rho} = \beta(1+r)c_{t+1}^{-\rho}$$

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- Insert **Euler** into **IBC** to get consumption choice

$$\begin{aligned} \sum_{t=0}^{T-1} (1+r)^{-t} (\beta(1+r))^{t/\rho} c_0 &= b_0 + h_0 \Leftrightarrow \\ c_0 &= \frac{1 - (\beta(1+r))^{1/\rho}/(1+r)}{1 - ((\beta(1+r))/(1+r))^T} (b_0 + h_0) \end{aligned}$$

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- **Question:** Is this the solution correct?

Dynamic solution: Bellman's Principle of Optimality

- **In words:** *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. (See Bellman, 1957, Chap. III.3.)*

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- **In math:**
 1. **Value function, v_t :** Defined recursively from

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$

$$\text{s.t. } a_t = (1 + r)a_{t-1} + wz_t - c_t \geq 0$$

with $v_T(\bullet) = 0$.

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2. **Policy function, c_t^* :** Is the same as

$$c_t^*(z_t, a_{t-1}) = \arg \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1 + r)a_{t-1} + wz_t - c_t \geq 0$$

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1 + r)a_{t-1} + wz_t - c_t \geq 0$$

1. **State variables:** z_t and a_{t-1}
2. **Control variable:** c_t
3. **Continuation value:** $\beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$
4. **Parameters:** r , w , and stuff in $u(\bullet)$

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- **End-of-period value function** (after realization):

$$\begin{aligned} v_t(z_t, a_{t-1}) &= \max_{c_t} u(c_t) + \beta \underline{v}_{t+1}(z_t, a_{t-1}) \\ \text{s.t. } a_t &= (1 + r)a_{t-1} + wz_t - c_t \geq 0 \end{aligned}$$

- **Discretization:** All state variable belong to discrete monotonically increasing sets \equiv *grids*,

$$z_t \in \mathcal{G}_z = \{z^0, z^1, \dots, z^{\#z-1}\}$$

$$a_t \in \mathcal{G}_a = \{a^0, a^1, \dots, a^{\#a-1}\}$$

Numerical value function iteration - basics

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- **Transition probabilities:** $\pi_{i_{z-}, i_z} = \Pr[z_t = z^{i_z} \mid z_t = z^{i_{z-}}]$

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- **Linear interpolation** (function approximation):

1. Assume \underline{v}_{t+1} is known on $\mathcal{G}_z \times \mathcal{G}_a$ (tensor product)
2. Evaluate $\underline{v}_{t+1}(z^{i_z}, a)$ for arbitrary a by

$$\check{\underline{v}}_{t+1}(z^{i_z}, a) = \underline{v}_{t+1}(z^{i_z}, a^{\iota}) + \omega_i \frac{a - a^{\iota}}{a^{\iota+1} - a^{\iota}}$$

$$\omega_i \equiv \frac{\underline{v}_{t+1}(z^{i_z}, a^{\iota+1}) - \underline{v}_{t+1}(z^{i_z}, a^{\iota})}{a^{\iota+1} - a^{\iota}}$$

$$\iota \equiv \text{largest } i_a \in \{0, 1, \dots, \#_a - 2\} \text{ such that } a^{i_a} \leq a$$

Deriving transition probabilities

- **Specification:** Assume $\log z_t$ follows the AR(1) process

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1}, \quad \psi_{t+1} \sim \mathcal{N}(\mu_\psi, \sigma_\psi)$$

where μ_ψ is used to ensure $\mathbb{E}[z_t] = 1$

- **Literature:** Tauchen (1986), Tauchen and Hussey (1991) and Rouwenhorst (1995) develops method for deriving \mathcal{G}_z and π_{i_z-, i_z} given ρ_z and σ_ψ , but we don't care about the details here

- Beginning-of-period value function:

$$\underline{v}_t(z^{i_z-}, a^{i_a-}) = \sum_{i_z=0}^{\#_z-1} \pi_{i_z-, i_z} v_t(z^{i_z}, a^{i_a-})$$

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- End-of-period value-of-choice:

$$v_t(z^{i_z}, a^{i_a-} | c_t) = u(c_t) + \beta \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z, i_{z+}} \check{v}_{t+1}(z^{i_{z+1}}, a_t)$$
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- **Nested loops:**

1. **Outer loop:** Backwards in time from $t = T - 1$ (note \underline{v}_T is known)
2. **Inner loop:** For each grid point in $\mathcal{G}_z \times \mathcal{G}_a$ find $c_t^*(z_t, a_{t-1})$ and therefore $v_t^*(z_t, a_{t-1})$ with a *numerical optimizer*

- **Example-notebook:** `Introduction.ipynb`
 1. Introduces `EconModel` package
 2. Show implementation of solution and simulation methods

Numerical Monte Carlo simulation

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- **Simulation:** Forwards in time from $t = 0$ and in each time period
 1. Draw z_{it} given transition probabilities
 2. Use linear interpolation to evaluate

$$c_{it} = \check{c}_t^*(z_{it}, a_{it-1})$$

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- **Review:**
 - **Pro:** Simple to implement
 - **Con:** Computationally costly and introduces randomness

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 1. **Distribute stochastic mass:** For each i_z and i_{a-} calculate

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- **Review:**
 1. **Pro:** Computationally efficient and no randomness
 2. **Con:** Introduces a non-continuous distribution

Side-note: Matrix formulation

- The histogram method can be written in **matrix form**:

$$\underline{D}_t = \Pi'_z \underline{D}_t$$

$$\underline{D}_{t+1} = \Lambda_t \underline{D}_t$$

where

\underline{D}_t is vector of length $\#_z \times \#_a$

D_t is vector of length $\#_z \times \#_a$

Π'_z is derived from the π_{i_z-, i_z} 's

Λ'_t is derived from the ι 's and ω 's

- **Note:** Example showed in notebook.

Infinite horizon: $T \rightarrow \infty$?

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq 0$$

- **Contraction mapping result:** *If β is low enough (strong enough impatience) then the value and policy function converge to $v(z_t, a_{t-1})$ and $c^*(z_t, a_{t-1})$ for large enough T*
- **Maximum upper limit for β :** $\frac{1}{1+r}$
- **In practice:** Solve backwards until value and policy functions does not change anymore (given some tolerance)

EGM



Euler-equation from variation argument

- **Case I:** If $c_t^{-\rho} > \beta(1+r)\mathbb{E}_t[c_{t+1}^{-\rho}]$:

Increase c_t by $\Delta > 0$, and lower c_{t+1} by $(1+r)$

1. **Feasible:** Yes, if $a_t > 0$
2. **Utility change:** $(c_t^{-\rho}) + \beta(-(1+r))\mathbb{E}_t[c_{t+1}^{-\rho}] > 0$

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- **Conclusion:** By contradiction
 1. **Constrained:** $a_t = 0$ and $c_t^{-\rho} \geq \beta(1+r)\mathbb{E}_t [c_{t+1}^{-\rho}]$, or
 2. **Unconstrained:** $a_t > 0$ and $c_t^{-\rho} = \beta(1+r)\mathbb{E}_t [c_{t+1}^{-\rho}]$

Endogenous grid-point method (EGM)

Alternative to value function iteration:

1. Calculate **post-decision marginal value of cash**:

$$q(z^{i_z}, a^{i_a}) = \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z, i_{z+}} c_+(z^{i_{z+}}, a^{i_a})^{-\sigma}$$

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$$c(z^{i_z}, a^{i_a}) = (\beta(1+r)q(z^{i_z}, a^{i_a}))^{-\frac{1}{\sigma}}$$

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$$m(z^{i_z}, a^{i_a}) = a^{i_a} + c_+(z^{i_z}, a^{i_a})$$

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4. **Consumption function**: $c^*(z^{i_z}, a^{i_a-})$ = interpolation of function from $m(z^{i_z}, :)$ to $c(z^{i_z}, :)$ at $m = (1+r)a^{i_a-} + wz^{i_z}$

Exercises

Exercises: Model extensions

- **Three exercises for you to do:**

1. Ensure the stationary distribution is found in the simulation
2. Make some borrowing allowed, b
3. Introduce transitory shock, ξ_t (hardest)

- **Full extended model:**

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$

s.t.

$$a_t = (1 + r)a_{t-1} + wz_t - c_t + \xi_t$$

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1}, \quad \mathbb{E}[z_t] = 1, \quad \text{Var}[\psi_t] = \sigma_\psi^2$$

$$\xi_t \sim \mathcal{N}(0, \sigma_\xi^2)$$

$$a_t \geq -b$$

- **General problem:** How can we calculate

$$\mathbb{E}(f(x)) = \int f(x)g(x)dx$$

- $f : \mathbb{R} \rightarrow \mathbb{R}$ some function
- $g(x)$ is the probability distribution function (PDF) for x and $G(x)$ is the CDF

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- **How to choose S and the *nodes* (x_i) and *weights* (ω_i)?**

Answer: Guassian quadrature

- **Gauss-Hermite** quadrature uses that

$$\int_{-\infty}^{\infty} f(x) e^{-x^2} dx = \sum_{i=1}^S \omega_i f(x_i) + \frac{S! \sqrt{\pi}}{S^S (2S)!} f^{(2S)}(\epsilon)$$

for some ϵ and where the (x_i, ω_i) 's can be easily found

Extra: Gauss-Hermite II

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- **Example: Random normal variable:** $Y \sim \mathcal{N}(\mu, \sigma^2)$ so that

$$\begin{aligned} \mathbb{E}[f(Y)] &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} f(y) e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy \\ &\approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^S \omega_i f(\sqrt{2}\sigma x_i + \mu) \end{aligned}$$

Summary

Summary and next week

- **Today:**

1. Introduction to course
2. Dynamic programming in theory
3. Dynamic programming in practice

- **Next week:** More on consumption-saving models and precautionary savings in partial equilibrium

- **Homework:**

1. **Work on:** Completing the model extension exercise
2. **Read:** Kaplan and Violante, 2014, »A Model of the Consumption Response to Fiscal Stimulus Payments«