



# 1. Introduction

## Adv. Macro: Heterogenous Agent Models

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Jeppe Druedahl & Patrick Moran

2022



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  1. What explains the level and dynamics of heterogeneity/inequality?
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- Complicated:** *Close to the research frontier*

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- **Plan for today:**
  1. More about the course
  2. Dynamic programming - theory
  3. Dynamic programming - practice

# Macroeconomic Models with Heterogeneous Agents

- **Model components:**

1. Optimizing individual agents (households + firms)
2. Idiosyncratic and aggregate risk
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- **HANC:** Heterogeneous Agent *Neo-Classical* model

- **HANK:** Heterogeneous Agent *New Keynesian* model  
(i.e. include price and wage setting frictions)

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Web: [sites.google.com/view/numeconcph-advmacrohet/](https://sites.google.com/view/numeconcph-advmacrohet/)  
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Git: [github.com/numeconcopenhagen/adv-macro-het](https://github.com/numeconcopenhagen/adv-macro-het)
- **Code:**
  1. We provide code you will build upon
  2. Based on the **GEModelTools** package

- Individual **assignments** (hand-in on Absalon)



# Assignments and exam

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- **Exam**:
  1. Hand-in 3×**assignments**
  2. **48 hour take-home**: Programming of new extension  
+ analysis of model + interpretation of results

1. **Assumed knowledge:** From **Introduction to Programming and Numerical Analysis** you are assumed to know the basics of
  - 1.1 Python
  - 1.2 JupyterLab
  - 1.3 VSCode
  - 1.4 git
2. **Updated Python:** Install (or re-install) newest Anaconda
3. **Packages:** `pip install quantecon, EconModel, consav`
4. **GEModel tools:**
  - 4.1 Clone the GEModelTools repository
  - 4.2 Locate repository in command prompt
  - 4.3 Run `pip install -e .`

*See CoursePlan.pdf*

1. Account for, formulate and interpret precautionary saving models
2. Account for stochastic and non-stochastic simulation methods
3. Account for, formulate and interpret general equilibrium models with ex ante and ex post heterogeneity, idiosyncratic and aggregate risk, and with and without pricing frictions
4. Discuss the difference between the stationary equilibrium, the transition path and the dynamic equilibrium
5. Discuss the relationship between various equilibrium concepts and their solution methods
6. Identify and account for methods for analyzing the dynamic distributional effects of long-run policy (e.g. taxation and social security) and short-run policy (e.g. monetary and fiscal policy)



1. Solve precautionary saving problems with dynamic programming and simulate behavior with stochastic and non-stochastic techniques
2. Solve general equilibrium models with ex ante and ex post heterogeneity, idiosyncratic and aggregate risk, and with and without pricing frictions (stationary equilibrium, transition path, dynamic equilibrium)
3. Analyze dynamics of income and wealth inequality
4. Analyze transitional and permanent structural changes (e.g. inequality trends and the long-run decline in the interest rate)
5. Analyze the dynamic distributional effects of long-run policy (e.g. taxation and social security) and short-run policy (e.g. monetary and fiscal policy)

# Competencies

1. Independently formulate, discuss and assess research on both the causes and effects of heterogeneity and risk for both long-run and short-run outcomes
2. Discuss and assess the importance of how heterogeneity and risk is modeled for questions about both long-run and short-run dynamics

# Dynamic Programming

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# From static to dynamic consumer optimization

- **Budget constraint** for  $t \in \{0, 1, \dots, T - 1\}$

$$\text{assets}_t = (1 + \text{return rate}) \times \text{assets}_{t-1} + \text{wage} \times \text{productivity}_t - \text{consumption}_t$$

$$a_t = (1 + r)a_{t-1} + wz_t - c_t$$

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- **Static problem:**

1. **Information:**  $z_t$  is known for all  $t$
2. **Target:** Discounted utility,  $\sum_{t=0}^{T-1} \beta^t u(c_t)$ ,  $\beta > 0$
3. **Behavior:** Choose  $c_0, c_1, \dots, c_{T-1}$  *simultaneously*
4. **Solution:** Sequence of consumption *choices*  $c_0, c_1, \dots, c_{T-1}$

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- **Dynamic programming:**

1. **Information:**  $z_t$  is revealed period-by-period
2. **Target:** *Expected* discounted utility,  $\sum_{t=0}^{T-1} \beta^t \mathbb{E}_t[u(c_t)]$ ,  $\beta > 0$
3. **Behavior:** Choose  $c_t$  *sequentially* as information is revealed
4. **Solution:** Sequence of consumption *functions*,  $c_t^*(z_t, a_{t-1})$

- **Substitution** implies *Intertemporal Budget Constraint* (IBC)

$$\begin{aligned}a_{T-1} &= (1+r)a_{T-2} + wz_{T-1} - c_{T-1} \\&= (1+r)^2 a_{T-3} + (1+r)wz_{T-2} - (1+r)c_{T-1} + wz_{T-1} - c_{T-1} \\&= (1+r)^T a_{-1} + \sum_{t=0}^{T-1} (1+r)^{T-1-t} (wz_t - c_t)\end{aligned}$$



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- Use **terminal condition**  $a_{T-1} = 0$  (equality due utility max.)

$$(1+r)^{-(T-1)} a_{T-1} = 0 \Leftrightarrow b_0 + h_0 - \sum_{t=0}^{T-1} (1+r)^{-t} c_t = 0$$

where  $b_0 = (1+r)a_{-1}$  and  $h_0 \equiv \sum_{t=0}^{T-1} (1+r)^{-t} wz_t$

# Static solution: FOC and consumption function

$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t \frac{c_t^{1-\rho}}{1-\rho} + \lambda \left[ \sum_{t=0}^{T-1} (1+r)^{-t} c_t - b_0 - h_0 \right]$$

- **First order conditions:**

$$\forall t : 0 = \beta^t c_t^{-\rho} - \lambda(1+r)^{-t} \Leftrightarrow c_t^{-\rho} = \beta(1+r)c_{t+1}^{-\rho}$$

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- Insert **Euler** into **IBC** to get consumption choice

$$\begin{aligned} \sum_{t=0}^{T-1} (1+r)^{-t} (\beta(1+r))^{t/\rho} c_0 &= b_0 + h_0 \Leftrightarrow \\ c_0 &= \frac{1 - (\beta(1+r))^{1/\rho}/(1+r)}{1 - ((\beta(1+r))/(1+r))^T} (b_0 + h_0) \end{aligned}$$

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- **Question:** Is this the solution correct?

## Dynamic solution: Bellman's Principle of Optimality

- **In words:** *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. (See Bellman, 1957, Chap. III.3.)*

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- **In math:**
  1. **Value function,  $v_t$ :** Defined recursively from

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$

$$\text{s.t. } a_t = (1 + r)a_{t-1} + wz_t - c_t \geq 0$$

with  $v_T(\bullet) = 0$ .

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2. **Policy function,  $c_t^*$ :** Is the same as

$$c_t^*(z_t, a_{t-1}) = \arg \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
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1. **State variables:**  $z_t$  and  $a_{t-1}$
2. **Control variable:**  $c_t$
3. **Continuation value:**  $\beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$
4. **Parameters:**  $r$ ,  $w$ , and stuff in  $u(\bullet)$

- **Realization of shocks:** First in the period before choices are made

# Timing of shocks

- **Realization of shocks:** First in the period before choices are made
- **Beginning-of-period value function** (before realization):

$$\underline{v}_t(z_{t-1}, a_{t-1}) = \mathbb{E}_{t-1} [v_t(z_t, a_{t-1})]$$

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- **End-of-period value function** (after realization):

$$\begin{aligned} v_t(z_t, a_{t-1}) &= \max_{c_t} u(c_t) + \beta \underline{v}_{t+1}(z_t, a_{t-1}) \\ \text{s.t. } a_t &= (1 + r)a_{t-1} + wz_t - c_t \geq 0 \end{aligned}$$

- **Discretization:** All state variable belong to discrete monotonically increasing sets  $\equiv$  *grids*,

$$z_t \in \mathcal{G}_z = \{z^0, z^1, \dots, z^{\#z-1}\}$$

$$a_t \in \mathcal{G}_a = \{a^0, a^1, \dots, a^{\#a-1}\}$$

# Numerical value function iteration - basics

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- **Transition probabilities:**  $\pi_{i_{z-}, i_z} = \Pr[z_t = z^{i_z} \mid z_t = z^{i_{z-}}]$
- **Linear interpolation** (function approximation):

1. Assume  $\underline{v}_{t+1}$  is known on  $\mathcal{G}_z \times \mathcal{G}_a$  (tensor product)
2. Evaluate  $\underline{v}_{t+1}(z^{i_z}, a)$  for arbitrary  $a$  by

$$\check{\underline{v}}_{t+1}(z^{i_z}, a) = \underline{v}_{t+1}(z^{i_z}, a^{\iota}) + \omega_i \frac{a - a^{\iota}}{a^{\iota+1} - a^{\iota}}$$

$$\omega_i \equiv \frac{\underline{v}_{t+1}(z^{i_z}, a^{\iota+1}) - \underline{v}_{t+1}(z^{i_z}, a^{\iota})}{a^{\iota+1} - a^{\iota}}$$

$$\iota \equiv \text{largest } i_a \in \{0, 1, \dots, \#_a - 2\} \text{ such that } a^{i_a} \leq a$$

# Deriving transition probabilities

- **Specification:** Assume  $\log z_t$  follows the AR(1) process

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1}, \quad \psi_{t+1} \sim \mathcal{N}(\mu_\psi, \sigma_\psi)$$

where  $\mu_\psi$  is used to ensure  $\mathbb{E}[z_t] = 1$

- **Literature:** Tauchen (1986), Tauchen and Hussey (1991) and Rouwenhorst (1995) develops method for deriving  $\mathcal{G}_z$  and  $\pi_{i_z-, i_z}$  given  $\rho_z$  and  $\sigma_\psi$ , but we don't care about the details here



- Beginning-of-period value function:

$$\underline{v}_t(z^{i_z-}, a^{i_a-}) = \sum_{i_z=0}^{\#_z-1} \pi_{i_z-, i_z} v_t(z^{i_z}, a^{i_a-})$$

# Numerical value function iteration - loops

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- End-of-period value-of-choice:

$$v_t(z^{i_z}, a^{i_a-} | c_t) = u(c_t) + \beta \sum_{i_z+=0}^{\#_z-1} \pi_{i_z, i_z+} \check{v}_{t+1}(z^{i_z+1}, a_t)$$
$$a_t = (1 + r)a^{i_a-} + wz^{i_z} - c_t$$

# Numerical value function iteration - loops

- **Beginning-of-period value function:**

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- **End-of-period value-of-choice:**

$$v_t(z^{i_z}, a^{i_a-} | c_t) = u(c_t) + \beta \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z, i_{z+}} \check{v}_{t+1}(z^{i_{z+1}}, a_t)$$
$$a_t = (1 + r)a^{i_a-} + wz^{i_z} - c_t$$

- **Nested loops:**

1. **Outer loop:** Backwards in time from  $t = T - 1$  (note  $\underline{v}_T$  is known)
2. **Inner loop:** For each grid point in  $\mathcal{G}_z \times \mathcal{G}_a$  find  $c_t^*(z_t, a_{t-1})$  and therefore  $v_t^*(z_t, a_{t-1})$  with a *numerical optimizer*

- **Example-notebook:** `Introduction.ipynb`
  1. Introduces `EconModel` package
  2. Show implementation of solution and simulation methods

# Numerical Monte Carlo simulation

- **Initial distribution:** Draw  $z_{i,-1}$  and  $a_{i,-1}$  for  $i \in \{0, 1, \dots, N-1\}$

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- **Simulation:** Forwards in time from  $t = 0$  and in each time period
  1. Draw  $z_{it}$  given transition probabilities
  2. Use linear interpolation to evaluate

$$c_{it} = \check{c}_t^*(z_{it}, a_{it-1})$$

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- **Review:**
  - **Pro:** Simple to implement
  - **Con:** Computationally costly and introduces randomness

# Numerical histogram simulation

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- **Review:**
  1. **Pro:** Computationally efficient and no randomness
  2. **Con:** Introduces a non-continuous distribution

## Side-note: Matrix formulation

- The histogram method can be written in **matrix form**:

$$\begin{aligned}\underline{D}_t &= \Pi'_z \underline{D}_t \\ \underline{D}_{t+1} &= \Lambda_t \underline{D}_t\end{aligned}$$

where

$\underline{D}_t$  is vector of length  $\#_z \times \#_a$

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$\Pi'_z$  is derived from the  $\pi_{i_z-, i_z}$ 's

$\Lambda'_t$  is derived from the  $\iota$ 's and  $\omega$ 's

- **Note:** Example showed in notebook.

## Infinite horizon: $T \rightarrow \infty$ ?

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1 + r)a_{t-1} + wz_t - c_t \geq 0$$

- **Contraction mapping result:** *If  $\beta$  is low enough (strong enough impatience) then the value and policy function converge to  $v(z_t, a_{t-1})$  and  $c^*(z_t, a_{t-1})$  for large enough  $T$*
- **Maximum upper limit for  $\beta$ :**  $\frac{1}{1+r}$
- **In practice:** Solve backwards until value and policy functions does not change anymore (given some tolerance)

**EGM**



# Euler-equation from variation argument

- **Case I:** If  $c_t^{-\rho} > \beta(1+r)\mathbb{E}_t[c_{t+1}^{-\rho}]$ :

Increase  $c_t$  by  $\Delta > 0$ , and lower  $c_{t+1}$  by  $(1+r)$

1. **Feasible:** Yes, if  $a_t > 0$
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  1. **Feasible:** Yes (always)
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- **Conclusion:** By contradiction
  1. **Constrained:**  $a_t = 0$  and  $c_t^{-\rho} \geq \beta(1+r)\mathbb{E}_t [c_{t+1}^{-\rho}]$ , or
  2. **Unconstrained:**  $a_t > 0$  and  $c_t^{-\rho} = \beta(1+r)\mathbb{E}_t [c_{t+1}^{-\rho}]$

# Endogenous grid-point method (EGM)

Alternative to value function iteration:

1. Calculate **post-decision marginal value of cash**:

$$q(z^{i_z}, a^{i_a}) = \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z, i_{z+}} c_+(z^{i_{z+}}, a^{i_a})^{-\sigma}$$



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4. **Consumption function**:  $c^*(z^{i_z}, a^{i_a-})$  = interpolation of function from  $m(z^{i_z}, :)$  to  $c(z^{i_z}, :)$  at  $m = (1+r)a^{i_a-} + wz^{i_z}$

# Exercises

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# Exercises: Model extensions

- **Three exercises for you to do:**

1. Ensure the stationary distribution is found in the simulation
2. Make some borrowing allowed,  $b$
3. Introduce transitory shock,  $\xi_t$  (hardest)

- **Full extended model:**

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$

s.t.

$$a_t = (1 + r)a_{t-1} + wz_t - c_t + \xi_t$$

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1}, \quad \mathbb{E}[z_t] = 1, \quad \text{Var}[\psi_t] = \sigma_\psi^2$$

$$\xi_t \sim \mathcal{N}(0, \sigma_\xi^2)$$

$$a_t \geq -b$$

- **General problem:** How can we calculate

$$\mathbb{E}(f(x)) = \int f(x)g(x)dx$$

- $f : \mathbb{R} \rightarrow \mathbb{R}$  some function
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- **How to choose  $S$  and the *nodes* ( $x_i$ ) and *weights* ( $\omega_i$ )?**

**Answer:** Guassian quadrature



## Extra: Gauss-Hermite II

- **Gauss-Hermite** quadrature uses that

$$\int_{-\infty}^{\infty} f(x) e^{-x^2} dx = \sum_{i=1}^S \omega_i f(x_i) + \frac{S! \sqrt{\pi}}{S^S (2S)!} f^{(2S)}(\epsilon)$$

for some  $\epsilon$  and where the  $(x_i, \omega_i)$ 's can be easily found

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- **Example: Random normal variable:**  $Y \sim \mathcal{N}(\mu, \sigma^2)$  so that

$$\begin{aligned} \mathbb{E}[f(Y)] &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} f(y) e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy \\ &\approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^S \omega_i f(\sqrt{2}\sigma x_i + \mu) \end{aligned}$$

# Summary

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# Summary and next week

- **Today:**

1. Introduction to course
2. Dynamic programming in theory
3. Dynamic programming in practice

- **Next week:** More on consumption-saving models and precautionary savings in partial equilibrium

- **Homework:**

1. Work on completing the model extension exercise
2. Read: TBA