CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

9. A Baseline HANK Model

Adv. Macro: Heterogenous Agent Models

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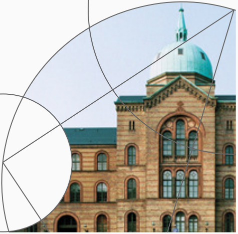

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Introduction

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- **Model:** Heterogeneous Agent New Keynesian (HANK) model
- **Code:**
  1. Based on the GEModelTools package
  2. Example from GEModelToolsNotebooks/HANK
- **Literature:**
  1. Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«
  2. Documentation for GEModelTools

HANK model

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- **Households:**
  1. Differ by stochastic idiosyncratic productivity and savings
  2. Supply labor and choose consumption
  3. Subject to a borrowing constraint

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- **Central bank**: Set nominal interest rate

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$$\max_{y_{jt}\ \forall j} P_t Y_t - \int_0^1 p_{jt} y_{jt} dj \text{ s.t. } Y_t = \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu}$$

for given output price, P_t, and input prices, p_{jt}

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- **Note:** Zero profits (can be used to derive price index)

Derivation of demand curve

- **FOC wrt.** y_{jt}

$$
\begin{aligned}
0 &= P_t \mu \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} \, dj \right)^{\mu - 1} \frac{1}{\mu} y_{jt}^{\frac{1}{\mu} - 1} - p_{jt} \Leftrightarrow \\
\frac{p_{jt}}{P_t} &= \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} \, dj \right)^{\mu - 1} y_{jt}^{\frac{1-\mu}{\mu}} \Leftrightarrow \\
\left(\frac{p_{jt}}{P_t} \right)^{\frac{\mu}{\mu - 1}} &= \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} \, dj \right)^{\mu} y_{jt}^{-1} \Leftrightarrow \\
y_{jt} &= \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu - 1}} Y_t
\end{aligned}
$$

Intermediary goods firms

- ***Dynamic*** **problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, n_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$

$$\text{s.t. } y_{jt} = Z_t n_{jt}, \quad y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t$$

$$\Omega(p_{jt}, p_{jt-1}) = \frac{\mu}{\mu - 1} \frac{1}{2\kappa} \left[\log \left(\frac{p_{jt}}{p_{jt-1}} \right) \right]^2$$

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- **NKPC** derived from FOC wrt. p_{jt} and envelope condition:

$$\log(1+\pi_t) = \kappa \left(\frac{w_t}{Z_t} - \frac{1}{\mu} \right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}}, \;\; \pi_t \equiv P_t / P_{t-1} - 1$$

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- **Implied production:** $Y_t = y_{jt}$, $N_t = n_{jt}$ (from symmetry)
- **Implied dividends:** $d_t = Y_t - w_t N_t - \frac{\mu}{\mu-1} \frac{1}{2\kappa} \left[\log(1 + \pi_t) \right]^2$

Derivation of NKPC

- **FOC wrt. p_{jt}:**

$$0 = \left(1 - \frac{\mu}{\mu - 1}\right) \left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{P_t} + \frac{\mu}{\mu - 1} \frac{w_t}{Z_t} \left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{p_{jt}}$$
$$- \frac{\mu}{\mu - 1} \frac{1}{\kappa} \frac{\log\left(\frac{p_{jt}}{p_{jt-1}}\right)}{p_{jt}} Y_t + \frac{J'_{t+1}(p_{jt})}{1 + r_{t+1}}$$

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Households

- **Household problem**: Distribution, $\boldsymbol{D}_t$, over z_t and a_{t-1}

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_t^{1+\nu}}{1+\nu} + \beta \mathbb{E}\left[v_{t+1}(z_{t+1}, a_t) \,|\, z_t, a_t\right]$$

$$\text{s.t. } a_t + c_t = (1+r_t)a_{t-1} + (w_t \ell_t - \tau_t + d_t) z_t \geq 0$$

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- **Effective labor-supply:** $n_t = z_t \ell_t$

EGM I

- **Beginning-of-period value function:**

$$\underline{v}_{a,t}(z_{t-1}, a_{t-1}) = \mathbb{E}_t\left[v_{a,t}(z_t, a_{t-1})\right] = \mathbb{E}\left[(1 + r_t)c_t^{-\rho}\right]$$

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- **Endogenous grid method:** Vary z_t and a_t to find

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- **Consumption and labor supply:** Use linear interpolation to find

$c^*(z_t, a_{t-1})$ and $\ell^*(z_t, a_{t-1})$ with $m_t = (1+r_t)a_{t-1} + (w_t \ell_t^* - \tau_t + d_t) z_t$

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- **Savings:** $a^*(z_t, a_{t-1}) = (1 + r_t)a_{t-1} - c_t^* + (w_t \ell_t^* - \tau_t + d_t) z_t$

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  1. Stop if $f(\ell^*) = \ell^* - \left(\frac{w_t z_t}{\varphi}\right)^{\frac{1}{\nu}} (c^*)^{-\frac{\sigma}{\nu}} <$ tol. where

$$
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  $$\begin{aligned} c^* &= (1 + r_t)a_{t-1} + (w_t \ell^* - \tau_t + d_t)z_t \\ n^* &= z_t \ell^* \end{aligned}$$

  2. Set

  $$\ell^* = \frac{f(\ell^*)}{f'(\ell^*)} = \frac{f(\ell^*)}{1 - \left(\frac{w_t z_t}{\varphi}\right)^{\frac{1}{\nu}} \left(-\frac{\sigma}{\nu}\right) (c^*)^{-\frac{\sigma}{\nu}} w_t z_t}$$

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  $$\begin{aligned} c^* &= (1 + r_t)a_{t-1} + (w_t \ell^* - \tau_t + d_t) z_t \\ n^* &= z_t \ell^* \end{aligned}$$

  2. Set

  $$\ell^* = \frac{f(\ell^*)}{f'(\ell^*)} = \frac{f(\ell^*)}{1 - \left(\frac{w_t z_t}{\varphi}\right)^{\frac{1}{\nu}} \left(-\frac{\sigma}{\nu}\right) (c^*)^{-\frac{\sigma}{\nu}} w_t z_t}$$

  3. Return to step 1

Government and central bank

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- **Government:** Choose τ_t to keep debt constant and finance exogenous public consumption

$$\tau_t = r_t B_{ss} + G_t$$

Market clearing

1. Labor: $N_t = \int n_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t$ (in effective units)
2. Assets: $B_{ss} = \int a_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t$
3. Goods: $Y_t = \int c_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t$

As an equation system

$$\boldsymbol{H}(\boldsymbol{\pi}, \boldsymbol{w}, \boldsymbol{Y}, \boldsymbol{i}^*, \boldsymbol{Z}, \underline{\boldsymbol{D}}_0) = \mathbf{0}$$

$$\begin{bmatrix} \log(1+\pi_t) - \left[\kappa\left(\frac{w_t}{Z_t} - \frac{1}{\mu}\right) + \frac{Y_{t+1}}{Y_t}\frac{\log(1+\pi_{t+1}}{1+r_{t+1}})\right] \\ N_t - \int n_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t \\ B_{ss} - \int a_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t \end{bmatrix} = \mathbf{0}$$

The rest of the model is given by

$$\boldsymbol{X} = \boldsymbol{M}(\boldsymbol{\pi}, \boldsymbol{w}, \boldsymbol{Y}, \boldsymbol{i}^*, \boldsymbol{Z})$$

Steady state

- **Analytically:**
  1. **Normalization:** $Z_{ss} = N_t = 1 \Rightarrow Y_{ss} = 1$
  2. **Zero-inflation:** $\pi_{ss} = 0 \Rightarrow i_{ss} = i_{ss}^* = r_{ss}$
  3. **Phillips-curve:** $w_{ss} = \frac{1}{\mu}$ and $d_t = 1 - w_{ss}$
  4. **Government:** $\tau_t = G_{ss}$
- **Numerically:** Choose β and φ to get market clearing

The HANK example from GEModelToolsNotebooks I

- **Presentation:** I go through the code for finding the transition path
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IRFs and simulation

Linearized IRFs

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- **Limitations:**

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  2. Imprecise in models with *aggregate non-linearities* (direct in aggregate equations or through micro-behavior)

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- **Equivalence:**
  1. Same result if we linearized all aggregated equations and write the model in MA(∞) form
  2. The state space form can also be recovered (not needed)

Advanced linearized simulation

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     where $\partial \boldsymbol{a}^*_{i_g} / \partial X^{hh}_k$ to a k-period ahead shock to input X^{hh} (calculated in fake news algorithm)

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  2. The policy function can there be simulated as

$$\boldsymbol{a}^*_{i_g} = \sum_{s=0}^{T} da^*_{i_g,s} \tilde{\boldsymbol{\epsilon}}_{t-s}$$

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$$\boldsymbol{a}^*_{i_g} = \sum_{s=0}^{T} da^*_{i_g,s} \tilde{\boldsymbol{\epsilon}}_{t-s}$$

  3. Distribution can then be simulated forwards

The HANK example from GEModelToolsNotebooks II

- **Presentation:** I go through the code for *finding the linearized IRFS and simulating the model*
- **In-class exercise:**
  1. Look at the code and talk about it with the person next to you for 10 minutes
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Exercises

Exercises: Model mechanism and extension

- **Understand the model dynamics**
    1. Illustrate through which channels tighter monetary policy affect aggregate consumption.
    2. Illustrate the effect of tighter monetary policy on inequality.

Exercises: Model mechanism and extension

- **Understand the model dynamics**
  1. Illustrate through which channels tighter monetary policy affect aggregate consumption.
  2. Illustrate the effect of tighter monetary policy on inequality.
- **Sticky prices vs. sticky wages:**
  1. Assume that a union chooses labor supply ($\ell_t = N_t$) and set wages
  2. Assume wage inflation, $\pi_t^w = \frac{w_t P_t}{w_{t-1} P_{t-1}}$, solves

     $$\log(1+\pi_t^w) = \kappa_w \left(\varphi N_t^{1+\nu} - \frac{w_t N_t}{\mu_w} \int z_{it} c_{it}^{-\sigma} d\boldsymbol{D}_t \right) + \beta \log(1+\pi_{t+1}^w)$$

     Micro-foundation: Based on unions maximizing average utility of household.
  3. Investigate how model dynamics change with changes in wage and price stickiness through κ and κ_w

Summary

Summary and next week

- **Today:**
  1. A baseline HANK model
  2. Linearized IRFs and simulation
- **Next week:** Adding unemployment to the HANK model
- **Homework:**
  1. Work on completing the model extension exercise
  2. Read: Auclert et al. (2018), »The Intertemporal Keynesian Cross«