CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

4. Transition Path

Adv. Macro: Heterogenous Agent Models

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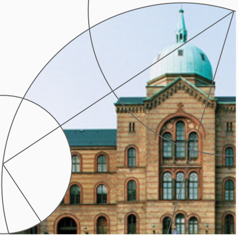
2022



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Introduction

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- **Code:**
    1. Based on the GEModelTools package
    2. Examples from GEModelToolsNotebooks/HANC
       (except stuff on *linearized solution* + *simulation*)
- **Literature:** Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«

Transition path

Transition path - close to verbal definition

For a given $\underline{\boldsymbol{D}}_0$ and a path $\{\Gamma_t\}$

1. Quantities $\{K_t\}$ and $\{L_t\}$,
2. prices $\{r_t\}$ and $\{w_t\}$,
3. the distributions $\{\boldsymbol{D}_t\}$ over z_t and a_{t-1}
4. and the policy functions $\{a_t^*(z_t, a_{t-1})\}$ and $\{c_t^*(z_t, a_{t-1})\}$

are such that

1. Household maximize expected utility (policy functions) in all periods
2. Firms maximize profits (prices) in all periods
3. $\boldsymbol{D}_t$ is implied by simulating the household problem forwards from $\underline{\boldsymbol{D}}_0$
4. The labor market clears in all periods
5. The capital market clears in all periods
6. The goods market clears in all periods

Equation system

The model can be written as an **equation system**

$$\boldsymbol{H}\left(\{K_t, L_t; \Gamma_t\}_{t\geq 0}, \underline{\boldsymbol{D}}_0\right) = \begin{bmatrix} K_t - \boldsymbol{a}_t^{*\prime}\boldsymbol{D}_t \\ r_t - \alpha\Gamma_t(K_{t-1}/L_t)^{\alpha-1} \\ w_t - (1-\alpha)\Gamma_t(K_{t-1}/L_t)^{\alpha} \\ L_t - 1 \\ \boldsymbol{D}_t - \Pi_z^{\prime}\underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} - \Lambda_t^{\prime}\boldsymbol{D}_t \\ \forall t \in \{0, 1, \dots\} \end{bmatrix} = \boldsymbol{0}$$

where $\{\Gamma_t\}_{t\geq 0}$ is a given technology path and $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$

Remember: Policies and choice transitions depend on prices

1. Policy function: $a_t^* = a^*\left(\{r_\tau, w_\tau\}_{\tau\geq t}\right)$
2. Choice transition: $\Lambda_t = \Lambda\left(\{r_\tau, w_\tau\}_{\tau\geq t}\right)$

Truncated, reduced vector form:

$$\boldsymbol{H}\left(\boldsymbol{K}, \boldsymbol{\Gamma}, \underline{\boldsymbol{D}}_0\right) = \left[\begin{array}{c} K_t - \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\ \forall t \in \{0, 1, \ldots, T-1\} \end{array} \right] = \mathbf{0}$$

where $\boldsymbol{K} = (K_0, K_1, \ldots, K_{T-1})$ and $\boldsymbol{\Gamma} = (\Gamma_0, \Gamma_1, \ldots, \Gamma_{T-1})$ and

$$\begin{aligned} L_t &= 1 \\ r_t &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\ w_t &= (1-\alpha) \Gamma_t (K_{t-1}/L_t)^{\alpha} \\ \boldsymbol{D}_t &= \Pi_z' \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} &= \Lambda_t' \boldsymbol{D}_t \\ & \forall t \in \{0, 1, \ldots, T-1\} \end{aligned}$$

Truncation: $T < \infty$ fine when $\Gamma_t = \Gamma_{ss}$ for all $t \ll T$

Could we solve it with a Newton method?

1. Guess $\boldsymbol{K}^0$ and set $i = 0$
2. Calculate $\boldsymbol{H}^i = \boldsymbol{H}_{\boldsymbol{K}}(\boldsymbol{K}^i, \boldsymbol{\Gamma})$.
3. Stop if $\left|\boldsymbol{H}^i\right|_\infty$ below chosen tolerance
4. Calculate the Jacobian $\boldsymbol{H}^i_{\boldsymbol{K}} = \boldsymbol{H}_{\boldsymbol{K}}(\boldsymbol{K}^0, \boldsymbol{\Gamma})$
5. Update guess by $\boldsymbol{K}^{i+1} = \left(\boldsymbol{H}^i_{\boldsymbol{K}}\right)^{-1} \boldsymbol{H}^i$
6. Increment i and return to step 2

Question: What is the problem?

Alternative: Use Broydens method?

1. Guess $\boldsymbol{K}^0$ and set $i = 0$
2. Calculate the steady state Jacobian $\boldsymbol{H}_{\boldsymbol{K},ss} = \boldsymbol{H}_{\boldsymbol{K}}(\boldsymbol{K}_{ss}, \boldsymbol{\Gamma}_{ss})$
3. Calculate $\boldsymbol{H}^i = \boldsymbol{H}_{\boldsymbol{K}}(\boldsymbol{K}^i, \boldsymbol{\Gamma})$.
4. Calculate Jacobian by
$$\boldsymbol{H}_{\boldsymbol{K}}^i = \begin{cases} \boldsymbol{H}_{\boldsymbol{K},ss} & \text{if } i = 0 \\ \boldsymbol{H}_{\boldsymbol{K}}^{i-1} + \frac{(\boldsymbol{H}^i - \boldsymbol{H}^{i-1}) - \boldsymbol{H}_{\boldsymbol{K}}^{i-1}(\boldsymbol{K}^i - \boldsymbol{K}^{i-1})}{\left|\boldsymbol{K}^i - \boldsymbol{K}^{i-1}\right|_2} \left(\boldsymbol{K}^i - \boldsymbol{K}^{i-1}\right)' & \text{if } i > 0 \end{cases}$$
5. Stop if $\left|\boldsymbol{H}^i\right|_\infty$ below tolerance
6. Update guess by $\boldsymbol{K}^{i+1} = \boldsymbol{K}^i + \left(\boldsymbol{H}_{\boldsymbol{K}}^i\right)^{-1} \boldsymbol{H}^i$
7. Increment i and return to step 3

Question: What are the benefits?

Bottleneck: How do we find the Jacobian?

1. **Naive approach:** For each $s \in \{0, 1, ..., T-1\}$ do
    1.1 Set $K_t = K_{ss} + \mathbf{1}\{t = s\} \cdot \Delta, \quad \Delta = 10^{-4}$
    1.2 Find $\boldsymbol{r}$ and $\boldsymbol{w}$
    1.3 Solve household problem backwards along transition path
    1.4 Simulate household forwards along transition path
    1.5 Calculate $\frac{\partial H_t}{\partial K_s} = \frac{(K_t - A_t^{hh}) - (K_{ss} - A_{ss}^{hh})}{\Delta}$ for all t

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   Bottleneck: We need T^2 solution steps and simulation steps!
2. **Fake news algorithm:** Only requires T solution steps and simulation steps $\Rightarrow$ *explained later today*

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What have we found?

- **Underlying assumption:** No aggregate uncertainty
- **»Shock«, Γ:** A fully unexpected non-recurrent event $\equiv$ *MIT shock*
- **Transition path, $\boldsymbol{K}$:** Non-linear perfect foresight response to
  1. Initial distribution, $\underline{\boldsymbol{D}}_0 \neq \boldsymbol{D}_{ss}$
  2. Shock, $\Gamma_t \neq \Gamma_{ss}$ for some t

  Also called: *Non-linear impulse-response*

Decomposition of GE response

- **GE transition path:** $\boldsymbol{r}^*$ and $\boldsymbol{w}^*$
- **PE response of each:**
  1. Set $(\boldsymbol{r}, \boldsymbol{w}) \in \{(\boldsymbol{r}^*, \boldsymbol{w}_{ss}), (\boldsymbol{r}_{ss}, \boldsymbol{w}^*)\}$
  2. Solve household problem backwards along transition path
  3. Simulate household forwards along transition path
  4. Calculate outcomes of interest
- **Additionally:** We can vary the initial distribution, $\underline{\boldsymbol{D}}_0$, to find the response of sub-groups

DAGs

General model class I

1. Time is discrete (index t).
2. There is a continuum of households (index i, when needed).
3. There is *perfect foresight* wrt. all aggregate variables, $\boldsymbol{X}$, indexed by $\mathcal{N}$, $\boldsymbol{X} = \{\boldsymbol{X}_t\}_{t=0}^{\infty} = \{\boldsymbol{X}^j\}_{j\in\mathcal{N}} = \{X_t^j\}_{t=0,j\in\mathcal{N}}^{\infty}$, where $\mathcal{N} = \mathcal{Z} \cup \mathcal{U} \cup \mathcal{O}$, and $\mathcal{Z}$ are *exogenous shocks,* $\mathcal{U}$ are *unknowns*, $\mathcal{O}$ are outputs, and $\mathcal{H} \in \mathcal{O}$ are *targets.*
4. The model structure is described in terms of a set of *blocks* indexed by $\mathcal{B}$, where each block has inputs, $\mathcal{I}_b \subset \mathcal{N}$, and outputs, $\mathcal{O}_b \subset \mathcal{O}$, and there exists functions $h^o(\{\boldsymbol{X}^i\}_{i\in\mathcal{I}_b})$ for all $o \in \mathcal{O}_b$.
5. The blocks are *ordered* such that (i) each output is *unique* to a block, (ii) the first block only have shocks and unknowns as inputs, and (iii) later blocks only additionally take outputs of previous blocks as inputs. This implies the blocks can be structured as a *directed acyclical graph* (DAG).

General model class II

6. The number of targets are equal to the number of unknowns, and an *equilibrium* implies $\boldsymbol{X}^{o} = 0$ for all $o \in \mathcal{H}$. Equivalently, the model can be summarized by an *target equation system* from the unknowns and shocks to the targets,

$$\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{Z}) = \boldsymbol{0},$$

and an *auxiliary model equation* to infer all variables

$$\boldsymbol{X} = \boldsymbol{M}(\boldsymbol{U}, \boldsymbol{Z}).$$

A *steady state* satisfy

$$\boldsymbol{H}(\boldsymbol{U}_{ss}, \boldsymbol{Z}_{ss}) = 0 \text{ and } \boldsymbol{X}_{ss} = \boldsymbol{M}(\boldsymbol{U}_{ss}, \boldsymbol{Z}_{ss}).$$

General model class III

7. The *discretized household block* can be written recursively as

$$
\begin{aligned}
\boldsymbol{v}_t &= v(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh}) \\
\underline{\boldsymbol{v}}_t &= \Pi(\boldsymbol{X}_t^{hh})\boldsymbol{v}_t \\
\boldsymbol{D}_t &= \Pi(\boldsymbol{X}_t^{hh})' \underline{\boldsymbol{D}}_t \\
\underline{\boldsymbol{D}}_{t+1} &= \Lambda(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh})' \boldsymbol{D}_t \\
\boldsymbol{a}_t^* &= \boldsymbol{a}^*(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh}) \\
\boldsymbol{Y}_t^{hh} &= \boldsymbol{y}(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh})' \boldsymbol{D}_t \\
&\underline{\boldsymbol{D}}_0 \text{ is given,} \\
&\boldsymbol{X}_t^{hh} = \{\boldsymbol{X}_t^i\}_{i \in \mathcal{I}_{hh}}, \boldsymbol{Y}_t^{hh} = \{\boldsymbol{X}_t^o\}_{o \in \mathcal{O}_{hh}},
\end{aligned}
$$

   where $\boldsymbol{Y}_t$ is aggregated outputs with $y(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh})$ as individual level measures.

8. Given the sequence of shocks, $\boldsymbol{Z}$,there exists a *truncation period*, T, such all variables return to steady state beforehand.

DAG: Directed Acyclical Growth

TBD

Fake News Algorithm

Fake news algorithm

TBD

Exercises

Exercises: Model extensions

1. **Firms:** Unchanged
2. **Households:** Solve

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(z_{t+1}, a_t) \right]$$

$$\text{s.t. } a_t + c_t = (1+r_t)a_{t-1} + w_t(1-\tau_t)z_t \geq 0$$

$$\log z_t = \rho \log z_{t-1} + \varepsilon_t^z \ , \varepsilon_t^z \sim \mathcal{N}(\mu_z, \sigma_z), \ \mathbb{E}[z_t] = 1$$

   where r_t is the real-interest rate and τ_t is a tax rate
3. **Government:** Set taxes and government consumption, and government bonds follows the law-of-motion

$$B_{t+1} = (1+r_t)B_t + G_t - \int \tau_t z_t d\boldsymbol{D}_t$$

4. **Asset market clearing:** $K_t + B_t = \int a_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t$
5. **Define and find the stationary equilibrium and transition path**
6. **How does the models result to a persistent shock to** G_t?

Summary

Summary and next week

- **Today:**
  1. The concept of a transition path
  2. Details to the GEModelTools package
- **Next week:** More on interesting heterogeneous agent models
- **Homework:**
  1. Work on completing the model extension exercise
  2. Read TBA