4. Transition Path

Adv. Macro: Heterogenous Agent Models

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Introduction

Introduction

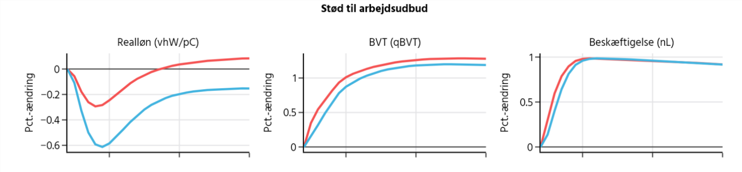
- **Last time:** *Stationary equilibrium (steady states)*
- **Today:** *Transition path (dynamic responses away from steady state)*
- **Model:** Heterogeneous Agent Neo-Classical (HANC) model
- **Code:**
  1. Based on the GEModelTools package
  2. Example from GEModelToolsNotebooks/HANC
- **Literature:**
  1. Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«
  2. Documentation for GEModelTools
  3. Kirkby (2017)

Outline

1. Introduction to transitions with the Ramsey model
2. Transition path in HA in partial equilibrium
3. Transition path in HA in general equilibrium: using sequence-space Jacobians
4. Fake news algorithm: computing SSJ fast
5. Exercises
6. First-order approximations of transition paths

Example I

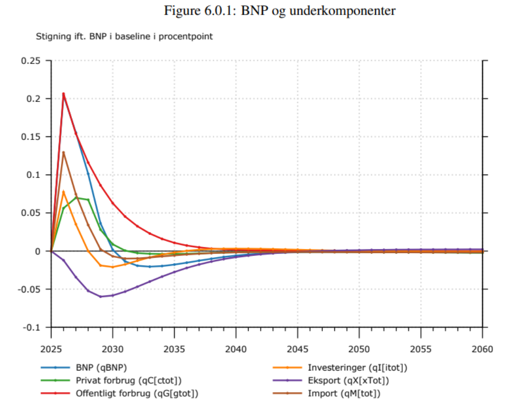
- What do we mean by transition path?
- Permanent shock to labor supply (think increase in retirement age) in the macroeconomic model of the Ministry of Finance:



- Note: Permanent shock, so transition path *between* two different steady states

Example II

- Temporary shock to public spending (i.e. fiscal stimulus during recessions)



- Note: Temporary shock, so model returns to the *same steady state*

Ramsey model

Ramsey: Summary

- **Simplified form:**

$$u'(C_t^{hh}) = \beta(1 + F_K(K_t, 1) - \delta)u'(C_{t+1}^{hh})$$
$$K_t = (1-\delta)K_{t-1} + F(K_{t-1}, 1) - C_t^{hh}$$

- **Production function:** $\Gamma_t K_t^{\alpha} L_t^{1-\alpha}$
- **Utility function:** $\frac{\left(C_t^{hh}\right)^{1-\sigma}}{1-\sigma}$
- **Steady state:**

$$K_{ss} = \left(\frac{\left(\frac{1}{\beta} - 1 + \delta\right)}{\Gamma_{ss}\alpha} \right)^{\frac{1}{\alpha-1}}$$
$$C_{ss}^{hh} = (1-\delta)K_{ss} + \Gamma_{ss}K_{ss}^{\alpha} - K_{ss}$$

Ramsey: As an equation system

$$\begin{bmatrix} r_t^K - \alpha \Gamma_t K_{t-1}^{\alpha-1} L_t^{1-\alpha} \\ w_t - (1-\alpha)\Gamma_t K_{t-1}^{\alpha} L_t^{-\alpha} \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ A_t^{hh} - \left((1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}\right) \\ C_t^{hh,-\sigma} - \beta(1+r_{t+1})C_{t+1}^{hh,-\sigma} \\ A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0,1,\dots\}, \text{given } K_{-1} \end{bmatrix} = \mathbf{0}$$

Remember: Perfect foresight w.r.t aggregate variables

Unknowns: $\{r_t^K, w_t, L_t, K_t, r_t, A_t, C_t^{hh}, A_t^{hh}\}$ for $\forall t \in \{0,1,\dots\}$

Recap: Newton's method I

- Before solving the dynamic Ramsey model, consider a simpler example
- Want to solve 1 eq. with 1 unknown (x is a scalar):

$$f(x) = 0$$

- **How to find** x? First-order Taylor approximation around current guess x^i:

$$f(x) \approx f\left(x^i\right) + f'\left(x^i\right)\left(x - x^i\right)$$

- Set $f(x) = 0$ and solve for x to get:

$$x = x^i - \frac{f\left(x^i\right)}{f'\left(x^i\right)}$$

Recap: Newton's method II

- Newton's method: Given initial guess x_0 update guess for x from i to $i+1$ as:

$$x^{i+1} = x^i - \frac{f\left(x^i\right)}{f'\left(x^i\right)}$$

  - until $\left|f\left(x^i\right)\right| < \epsilon$
- Can always get $f\left(x^i\right)$ by simply evaluating the function at current estimate. What about derivative $f'\left(x^i\right)$?
- Use numerical approximation:

$$f'\left(x^i\right) \approx \frac{f\left(x^i + h\right) - f\left(x^i\right)}{h}$$

  - For small h.
- How well does it work?
  - If $f(x)$ is linear this update solves $f(x) = 0$ in **1 iteration**
  - If $f(x)$ is non-linear we typically need more iterations, but works well if initial guess is within basis of attraction

Recap: Multivariate Newton's method

- Generalize to vector-valued, multivariate functions $[f_1(x_{1,}x_2), f_2(x_{1,}x_2)]' = \boldsymbol{f}(\boldsymbol{x})$ with $\boldsymbol{x} = (x_{1,}x_2)'$:

$$\boldsymbol{x}^{i+1} = \boldsymbol{x}^i - \boldsymbol{J}\left(\boldsymbol{x}^i\right)^{-1} \boldsymbol{f}\left(\boldsymbol{x}^i\right)$$

- Where $\boldsymbol{J}\left(\boldsymbol{x}^i\right)$ is the *Jacobian* of $f(\boldsymbol{x})$ w.r.t $\boldsymbol{x}^i$:

$$\boldsymbol{J}(\boldsymbol{x}_i) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1^i} & \frac{\partial f_1}{\partial x_2^i} \\ \frac{\partial f_2}{\partial x_1^i} & \frac{\partial f_2}{\partial x_2^i} \end{bmatrix}$$

- Can calculate this jacobian in the same way as $f'(x)$ in previous example, but need to so for every element in $\boldsymbol{x}$
- **Go through code**

Recap: Broyden's method I

- Newton's method updates Jacobian $\boldsymbol{J}$ in **every iteration**
- If $\boldsymbol{J}$ is expensive to calculate, this is a serious bottleneck
- Broyden's method solves this issue by only calculating J around some initial point.
- Then apply following (linear) update of $f'\left(x^{i+1}\right)$ at every iteration i:

$$f'\left(x^{i+1}\right) = f'\left(x^{i}\right) + \frac{\left[f\left(x^{i+1}\right) - f\left(x^{i}\right)\right] - f'\left(x^{i}\right)\left(x^{i+1} - x^{i}\right)}{x^{i+1} - x^{i}}$$

Recap: Broyden's method II

1. Guess $\boldsymbol{x}^0$ and set $i = 0$
2. Calculate the Jacobian around initial point $\boldsymbol{J_0}$
3. Calculate $\boldsymbol{f}^i = \boldsymbol{f}(\boldsymbol{x}^i)$.
4. Stop if $\left|\boldsymbol{f}^i\right|$ below tolerance ϵ
5. Calculate Jacobian by
$$\boldsymbol{J}^i = \begin{cases} \boldsymbol{J_0} & \text{if } i = 0 \\ \boldsymbol{J}^{i-1} + \frac{(\boldsymbol{f}^i - \boldsymbol{f}^{i-1}) - \boldsymbol{J}^{i-1}(\boldsymbol{x}^i - \boldsymbol{x}^{i-1})}{|\boldsymbol{x}^i - \boldsymbol{x}^{i-1}|_2} \left(\boldsymbol{x}^i - \boldsymbol{x}^{i-1}\right)' & \text{if } i > 0 \end{cases}$$
6. Update guess by $\boldsymbol{x}^{i+1} = \boldsymbol{x}^i - \left(\boldsymbol{J}^i\right)^{-1} \boldsymbol{f}^i$
7. Increment i and return to step 3

- **Go through code**

Back to Ramsey

$$\begin{bmatrix} r_t^K - \alpha \Gamma_t K_{t-1}^{\alpha-1} L_t^{1-\alpha} \\ w_t - (1-\alpha)\Gamma_t K_{t-1}^{\alpha} L_t^{-\alpha} \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ A_t^{hh} - \left((1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}\right) \\ C_t^{hh,-\sigma} - \beta(1+r_{t+1})C_{t+1}^{hh,-\sigma} \\ A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0,1,\dots\}, \text{given } K_{-1} \end{bmatrix} = \mathbf{0}$$

- 2 issues:
  - Many unknowns (8 eqs per period)
  - In fact, infinitely many since time is infinite, $T \to \infty$

Truncated Ramsey, reduced vector form

$$\boldsymbol{H}\left(\boldsymbol{K},\boldsymbol{L},\boldsymbol{\Gamma},K_{-1}\right)=\begin{bmatrix} A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0,1,\ldots,T-1\} \end{bmatrix}=\mathbf{0}$$

where $\boldsymbol{X}=(X_0,X_1,\ldots,X_{T-1})$, $A_{-1}^{hh}=K_{-1}$ and

$$\begin{aligned}
r_t^K &= \alpha\Gamma_t(K_{t-1}/L_t)^{\alpha-1} \\
w_t &= (1-\alpha)\Gamma_t(K_{t-1}/L_t)^{\alpha} \\
A_t &= K_t \\
r_t &= r_t^K - \delta \\
C_t^{hh} &= \left(\beta(1+r_{t+1})\right)^{-\sigma} C_{t+1}^{hh}\,\text{(backwards)} \\
L_t^{hh} &= 1 \\
A_t^{hh} &= (1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}\,\text{(forwards)}
\end{aligned}$$

Truncation: $T<\infty$ fine when $\Gamma_t=\Gamma_{ss}$ for all $t>\underline{t}$ with $\underline{t}\ll T$

Further reduced

$$\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{\Gamma}, K_{-1}) = \left[\boldsymbol{A} - \boldsymbol{A}^{hh}\right] = \boldsymbol{0}$$

where $\boldsymbol{X} = (X_0, X_1, \dots, X_{T-1})$, $A_{-1}^{hh} = K_{-1}$ and

$$\begin{aligned}
L_t = L_t^{hh} = 1 \\
r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\
w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\
A_t &= K_t \\
r_t &= r_t^K - \delta \\
C_t^{hh} &= (\beta(1+r_{t+1}))^{-\sigma} C_{t+1}^{hh} \text{ (backwards)} \\
A_t^{hh} &= (1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh} \text{ (forwards)}
\end{aligned}$$

for $\forall t \in \{0, 1, \dots, T-1\}$

Sequence space

- Note: We have now written the model in *sequence space*
  - Representing an entire timepath/*sequence* of variables as a function of timepath/*sequence* of other variables
- Example: Keynesian consumption function $C_t = a + bY_t$:

$$\begin{bmatrix} C_0 & C_1 & C_2 & \dots \end{bmatrix}' = a + b \begin{bmatrix} Y_0 & Y_1 & Y_2 & \dots \end{bmatrix}'$$
$$\Leftrightarrow \boldsymbol{C} = a + b\boldsymbol{Y}$$
$$\Leftrightarrow \boldsymbol{C} = f\left(\boldsymbol{Y}\right)$$

- Powerfull since it also applies *non-linear*, forward-looking and backwards-looking eqs:

$$C_t = a + b_0 Y_t + b_1 \log Y_{t-4} + b_2 Y_{t+4}^2$$
$$\Leftrightarrow \boldsymbol{C} = g\left(\boldsymbol{Y}\right)$$

- As long as we have the sequence $\boldsymbol{Y}$ we can calculate $\boldsymbol{C}$
  - Will leverage this later when working with the HA model

Solution in sequence space

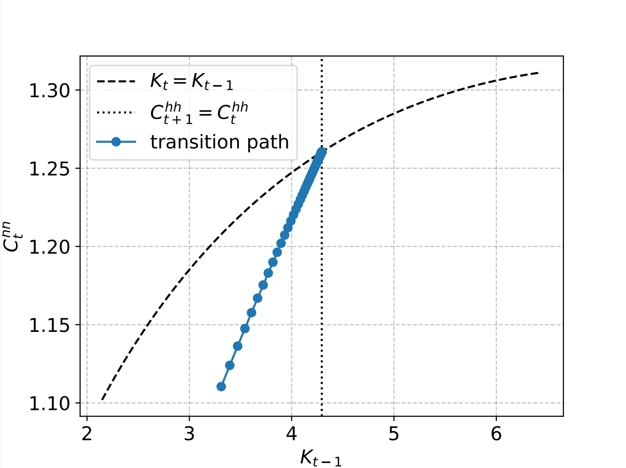
- **Truncation:** $T = 200$ (transition path should have converged to ss by then)
- **Jacobian:** Find $\boldsymbol{H_K}$ by *numerical differentiation*

$$\boldsymbol{H_K} = \begin{bmatrix} \frac{\partial(A_0 - A_0^{hh})}{\partial K_0} & \frac{\partial(A_0 - A_0^{hh})}{\partial K_1} & \cdots \\ \frac{\partial(A_1 - A_1^{hh})}{\partial K_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

- **Transition path:** Given $\boldsymbol{\Gamma}$ and K_{-1} solve $\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{\Gamma}, K_{-1})$ with non-linear equation system solver (e.g. broyden)
- **Two types of perfect foresight transitions:**
  1. *Transitory:* both the initial and terminal conditions are the steady-state values
  2. *Permanent:* the economy moves from one state to another state (the terminal state must be a stationary one)
- **Notebook:** *Ramsey.ipynb*

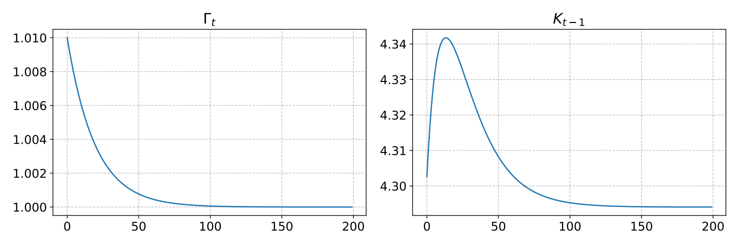
Example 1: permanent from low capital

Initially away from steady state: $K_{-1} = 0.75K_{ss}$



Example 2: transitory following technology shock

Technology shock: $\Gamma_t = 0.01 \times \Gamma_{ss} \times 0.95^t$ (i.e AR(1) with $\rho = 0.95$) (exogenous, deterministic)



Terminology: MIT-shock

Transition path in PE

Household model in a transition

Recall the household block in the HANC model

$$v_0(z_{it}, a_{it-1}) = \max_{\{c_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{it})$$

$$\text{s.t.}$$

$$\ell_{it} = z_{it}$$

$$a_{it} = (1 + r_t)a_{it-1} + w_t \ell_{it} - c_{it}$$

$$\log z_{it+1} = \rho_z \log z_{it} + \psi_{it+1}, \;\; \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \;\; \mathbb{E}[z_{it}] = 1$$

$$a_{it} \geq 0$$

Until now, we assume that $r_t = r_{ss}$ and $w_t = w_{ss}$ for all t.
What if they are time varying-instead?

Perfect foresight, initial and terminal conditions,

Important assumptions:

1. **Perfect foresight:** from $t = 0$, households know the future path of $\{r_t, w_t\}_{t=0}^{\infty}$
2. **Truncation:** the model converges to a stationary state after $t \geq T$, T large
3. **Initial conditions:** we compute the transition from a given distribution D_0 that we already know
4. **Terminal condition:** we compute a transition towards some stationary state where we know the value function (or its derivative)

Impulse reponses: backward and forward step

Our goal is to compute a sequence of impulse responses

$$A_t^{hh}(\{r_\tau, w_\tau\}_{\tau=0}^T = \int a_t(a,z) dD_t(a,z) \quad \forall t \in (0, T)$$

We thus need to obtain a sequence of policy functions $a_t(a,z)$ and distributions D_t. *(note the t subscript!)*

We will proceed in two steps:

1. **Backward step**: using the terminal condition on the value function, and going back in time, obtain the policy functions $a_t(a,z)$ and $c_t(a,z)$
2. **Forward step**: using the initial condition on the distribution, and going forward in time, simulate the distribution over time $D_t(a,z)$

Summing up the transition in PE

To solve the household problem, we need three objects:

1. An **exogenous path** of $\{r_t, w_t\}_{t=0}^{T}$
2. A **terminal condition** on the value function (or its derivative) $V_T^a(a, z)$
3. An **initial condition** on the distribution

We then do a:

1. **Backward step**, using $V_T^a(a, z)$ as a terminal condition, taking into account $\{r_t, w_t\}_{t=0}^{T}$ $\rightarrow$ this gives us $c_t(a, z)$ and $a_t(a, z)$
2. **Forward step**, using $D_0(a, z)$ as an initial condition, and $a_t(a, z)$ $\rightarrow$ this gives us $D_t(a, z)$

We can then obtain the aggregate values of the household as usual by computing $A_t = \int a_t(a, z) dD_t(a, z)$. This is the **impulse response**!

Let's code!

Transition path in GE

Equation system

The model can be written as an **equation system**

$$\begin{bmatrix} r_t^K - F_K(K_{t-1}, L_t) \\ w_t - F_L(K_{t-1}, L_t) \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ \boldsymbol{D}_t - \Pi_z \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} - \Lambda_t \boldsymbol{D}_t \\ A_t^{hh} - A_t \\ L_t^{hh} - L_t \\ \forall t \in \{0, 1, \dots\}, \text{given } \underline{\boldsymbol{D}}_0 \end{bmatrix} = \boldsymbol{0}$$

where $\{\Gamma_t\}_{t\geq 0}$ is a given technology path and $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$

Remember: Policies and choice transitions depend on prices

1. Policy function: $x_t^* = x^* \left(\{r_\tau, w_\tau\}_{\tau \geq t} \right)$ and $X_t^{hh} = \sum_i x_{it}^* D_{it} = \boldsymbol{x}_t^{*\prime} \boldsymbol{D}_t$
2. Choice transition: $\Lambda_t = \Lambda \left(\{r_\tau, w_\tau\}_{\tau \geq t} \right)$

Transition path - close to verbal definition

For a given $\underline{\boldsymbol{D}}_0$ and a path $\{\Gamma_t\}$

1. Quantities $\{K_t\}$ and $\{L_t\}$,
2. prices $\{r_t\}$ and $\{w_t\}$,
3. the distributions $\{\boldsymbol{D}_t\}$ over β_i, z_t and a_{t-1}
4. and the policy functions $\{a_t^*\}$, $\{\ell_t^*\}$ and $\{c_t^*\}$

are such that in all periods

1. Firms maximize profits (prices)
2. Household maximize expected utility (policy functions)
3. $\boldsymbol{D}_t$ is implied by simulating the household problem forwards from $\underline{\boldsymbol{D}}_0$
4. Mutual fund balance sheet is satisfied
5. The capital market clears
6. The labor market clears
7. The goods market clears

Reduce size of equation system

- In the equation system above we have many **unknowns** and many **equations**
- Makes finding the solution with Broyden's method since **Jacobian is large**
  - With truncation T and N equations/unknowns $\boldsymbol{J}$ has size $(T \times N, T \times N,)$

  $\Rightarrow$ Expensive to calculate
- We can typically **exploit model structure** to reduce size of system
  - Did this earlier for Ramsey
  - Now more formally

Truncated, reduced vector form

$$\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{\Gamma}, \underline{\boldsymbol{D}}_0) = \begin{bmatrix} A_t^{hh} - A_t \\ L_t^{hh} - L_t \\ \forall t \in \{0, 1, \dots, T-1\} \end{bmatrix} = \mathbf{0}$$

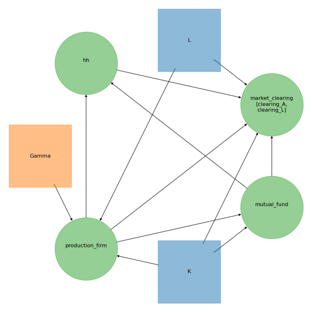
where $\boldsymbol{X} = (X_0, X_1, \dots, X_{T-1})$, $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$ and

$$\begin{aligned} r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\ w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\ r_t &= r_t^K - \delta \\ A_t &= K_t \\ \boldsymbol{D}_t &= \Pi_z' \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} &= \Lambda_t' \boldsymbol{D}_t \\ A_t^{hh} &= \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\ L_t^{hh} &= \boldsymbol{\ell}_t^{*\prime} \boldsymbol{D}_t \\ &\forall t \in \{0, 1, \dots, T-1\} \end{aligned}$$

Truncation: $T < \infty$ fine when $\Gamma_t = \Gamma_{ss}$ for all $t > \underline{t}$ with $\underline{t} \ll T$

DAG - Directed Acyclic Graph

- **Orange square:** Shocks (exogenous)
- **Blue square:** Unknowns (endogenous)
- **Green circles:** Blocks (with variables and targets inside)



- This DAG implies: Exo. input + guess ⇒ Firm block ⇒ Mutual fund ⇒HHs ⇒ Residuals

Further reduction

$$\boldsymbol{H}(\boldsymbol{K}, \Gamma, \underline{\boldsymbol{D}}_0) = \begin{bmatrix} A_t^{hh}(\boldsymbol{w}(\boldsymbol{K}), \boldsymbol{r}(\boldsymbol{K})) - K_t \\ \forall t \in \{0, 1, \dots, T-1\} \end{bmatrix} = \boldsymbol{0}$$

where $\boldsymbol{X} = (X_0, X_1, \dots, X_{T-1})$, $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$ and

$$\begin{aligned} L_t &= 1 \\ r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\ w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\ A_t &= K_t \\ r_t &= r_t^K - \delta \\ \boldsymbol{D}_t &= \Pi_z' \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} &= \Lambda_t' \boldsymbol{D}_t \\ A_t^{hh} &= \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\ & \forall t \in \{0, 1, \dots, T-1\} \end{aligned}$$

Truncation: $T < \infty$ fine when $\Gamma_t = \Gamma_{ss}$ for all $t > \underline{t}$ with $\underline{t} \ll T$

Solve with Broyden

- As with standard Ramsey model from before we have:
  - Equation system with T equations ($\boldsymbol{H}$)
  - And T unknowns ($\boldsymbol{K}$)
- If we can calculate the jacobian of $\boldsymbol{H}$ w.r.t $\boldsymbol{K}$ we can solve with Broyden's method as before

How to compute Jacobian?

- How do we compute the Jacobian of the residuals $\boldsymbol{H}$ w.r.t unknowns $\boldsymbol{K}$?
  - Before: Compute Jacobian of entire model using num. diff
  - **Now**: Use DAG structure + chain rule
- *Example*. Represent model in block form:

$$\boldsymbol{w}, \boldsymbol{r}^K = Firm\left(\boldsymbol{K}\right), \quad \boldsymbol{A}, \boldsymbol{r} = MutFund\left(\boldsymbol{K}, \boldsymbol{r}^K\right)$$

$$\boldsymbol{A}^{hh} = hh\left(\boldsymbol{r}, \boldsymbol{w}\right), \quad \boldsymbol{A} - \boldsymbol{A}^{hh} = \boldsymbol{H}\left(\boldsymbol{A}, \boldsymbol{A}^{hh}\right)$$

- Collapsing the previous equations, we write the asset-market clearing condition as

$$\boldsymbol{H} = \boldsymbol{A}^{hh}(\boldsymbol{w}(\boldsymbol{K}), \boldsymbol{r}(\boldsymbol{K})) - \boldsymbol{K}$$

What is a Jacobian

Let $\mathcal{J}^{y,x}$ be Jacobian of y w.r.t x. Then:

$$\boldsymbol{H_K} = \mathcal{J}^{A^{hh},r}\mathcal{J}^{r,K} + \mathcal{J}^{A^{hh},w}\mathcal{J}^{w,K} - \boldsymbol{I}$$

where

$$\mathcal{J}^{A^{hh},r} = \begin{bmatrix} \frac{\partial A_0^{hh}}{\partial dr_0} & \frac{\partial A_0^{hh}}{\partial dr_1} & \cdots & \frac{\partial A_0^{hh}}{\partial dr_T} \\ \frac{\partial A_1^{hh}}{\partial dr_0} & \frac{\partial A_1^{hh}}{\partial dr_1} & \cdots & \frac{\partial A_1^{hh}}{\partial dr_T} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial A_T^{hh}}{\partial dr_0} & \frac{\partial A_T^{hh}}{\partial dr_1} & \cdots & \frac{\partial A_T^{hh}}{\partial dr_T} \end{bmatrix}$$

Interpretation: row t of column s gives us the savings change at t in response to a shock on r at s. Not just a computational tool, also a lot of economic intuition behind it!

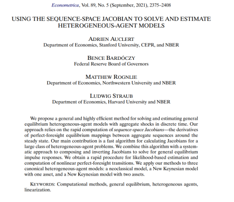
How to compute Jacobian?

- If we have individuals Jacobians, easy to compute $\boldsymbol{H_K}$
  - Also very efficient - just matrix mulitiplication
- How to get individual Jacobians?
  - Some are easy: For $\mathcal{J}^{w,K}, \mathcal{J}^{r,K}$ we just have to diff. $r_t^K = \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1}, w_t = (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha}$
    - Cheap, and can often be vectorized
  - What about HH Jacobians $\mathcal{J}^{A_{hh},r}, \mathcal{J}^{A_{hh},w}$?
    - Need to compute T impulse reponse!

Bottleneck: How do we find the Jacobian?

- **Naive approach:** For each input i into HH block $i \in \{r, w\}$
  - For each $s \in \{0, 1, ..., T-1\}$
    1. Shock input i in period s by small amount Δ
    2. Solve household problem backwards along transition path
    3. Simulate households forward along transition path
    4. Calculate column s, row t of jacobian as $\frac{\partial \mathcal{J}_t^{A_{hh},i}}{\partial i_s} = \frac{A_t^{hh} - A_{ss}^{hh}}{\Delta}$ for all t

  Bottleneck: We need T^2 solution steps and simulation steps for each input $\{r, w\}$!
- Solution: **Fake news algorithm** - only need T steps! (later today)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USING THE SEQUENCE-SPACE JACOBIAN TO SOLVE AND ESTIMATE HETEROGENEOUS-AGENT MODELS

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We propose a general and highly efficient method for solving and estimating general equilibrium heterogeneous-agent models with aggregate shocks in discrete time. Our approach relies on the rapid computation of *sequence-space Jacobians*—the derivatives of perfect-foresight equilibrium mappings between aggregate sequences around the steady state. Our main contribution is a fast algorithm for calculating Jacobians for a large class of heterogeneous-agent problems. We combine this algorithm with a systematic approach to composing and inverting Jacobians to solve for general equilibrium impulse responses. We obtain a rapid procedure for likelihood-based estimation and computation of nonlinear perfect-foresight transitions. We apply our methods to three canonical heterogeneous-agent models: a neoclassical model, a New Keynesian model with one asset, and a New Keynesian model with two assets.

KEYWORDS: Computational methods, general equilibrium, heterogeneous agents, linearization.

Summary

- Conditional on being able to compute HH jacobian efficiently we can compute **transition path** through following steps:
  1. Compute stationary state of model
  2. Formulate transition path as DAG
     - Reduce number of unknowns and residual equations
     - Not essential, but often good idea
  3. Compute Jacobian of residuals $\boldsymbol{H}$ w.r.t unknowns $\boldsymbol{K}$
  4. Formulate shock (i.e. TFP increases by 1% for 4 years)
  5. Use Broyden's method to solve for transition path

Let’s code!

Reminder

We are interested in the **dynamics** of our model What we have seen so far:

1. How to compute impulse response of the household block in partial equilibrium
   1.1 Backward step: EGM
   1.2 Forward step: histogram method
2. To move to general equilibrium: find the path of an endogenous variable (like $\boldsymbol{K}$) such that markets clear $\boldsymbol{H}(\boldsymbol{\Gamma}, \boldsymbol{K}) = 0$
3. To find $\boldsymbol{K}$, we use Newton's method, which require a **Sequence Space Jacobian**

Fake News Algorithm

Fake news algorithm

- **Household block:**

$$\boldsymbol{Y}^{hh} = hh(\boldsymbol{X}^{hh})$$

  - i.e. $\boldsymbol{Y}^{hh} = C^{hh}, A^{hh}$ and $\boldsymbol{X}^{hh} = w, r$
- **Goal:** Fast computation of

$$\mathcal{J}^{A^{hh},r} = \begin{bmatrix} \frac{\partial A_0^{hh}}{\partial dr_0} & \frac{\partial A_0^{hh}}{\partial dr_1} & \cdots & \frac{\partial A_0^{hh}}{\partial dr_T} \\ \frac{\partial A_1^{hh}}{\partial dr_0} & \frac{\partial A_1^{hh}}{\partial dr_1} & \cdots & \frac{\partial A_1^{hh}}{\partial dr_T} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial A_T^{hh}}{\partial dr_0} & \frac{\partial A_T^{hh}}{\partial dr_1} & \cdots & \frac{\partial A_T^{hh}}{\partial dr_T} \end{bmatrix}$$

- **Naive approach:**
  - Shock at time $s = 0$, solve + simulate HH block for T periods
  - Repeat until $s = T - 1$
  - Requires T^2 solution and simulation steps
- **Next slides:** *Sketch of much faster approach*

Initial step

- Note that aggregate is (matrix) product of individual policy function $\boldsymbol{y}_t$ and distribution $\boldsymbol{D}_t$.
- Linearize (first-order Taylor) around ss:

$$\boldsymbol{Y}^{hh} = (\boldsymbol{y}_t')\, \boldsymbol{D}_t$$
$$\Rightarrow \frac{d\boldsymbol{Y}^{hh}}{d\boldsymbol{X}^{hh}} = \left(\frac{d\boldsymbol{y}_t}{d\boldsymbol{X}^{hh}}'\right) \boldsymbol{D}_{ss} + (\boldsymbol{y}_{ss}')\, \frac{d\boldsymbol{D}_t}{d\boldsymbol{X}^{hh}}$$

- What can we say about policy function term $d\boldsymbol{y}_t$?

Pertubation of policy function

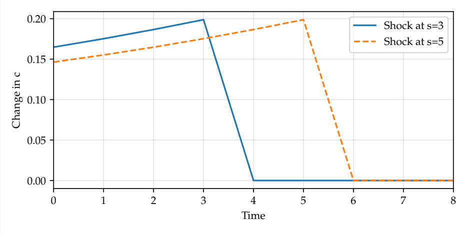
- The heart of the fake news algorithm is a central insight that allow us to compute $d\boldsymbol{y}_t/d\boldsymbol{X}^{hh}$ efficiently
- Let $\boldsymbol{y}_t^s$ be policy function at time t following a shock in period s. Then:

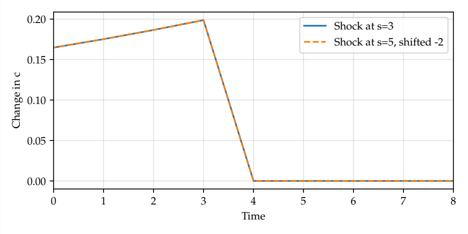
$$\boldsymbol{y}_t^s = \begin{cases} \boldsymbol{y}_{t+j}^{s+j} & t \leq s \\ \boldsymbol{y}_{ss} & t > s \end{cases}$$

- **Verbally**: The response of the policy function $\boldsymbol{y}$ at time t to a shock at s is the as the response at time $t+j$ to a shock at $s+j$
  - Policy function **does not depend on the absolute time of shock** only the relative distance between »today« and the shock, $s-t$.
- **Implication**: We need to only do a single backwards iteration to a shock at $s = T-1$.
  - Can then construct change in policy function $d\boldsymbol{y}_t^s/d\boldsymbol{X}^{hh}$ for different s by shifting policy function around

Numerical illustration

Graphically. Response of c_t to income shock at $s = 3, 5$





Implementation

Let's say you want to compute $\frac{d\boldsymbol{A}^{hh}}{d\boldsymbol{w}}$

Algorithm:

1. Compute the impulse response policy functions to a shock on w at $s = T \rightarrow$ obtain a vector of policy functions $\boldsymbol{a}'_t(a, z)$
2. Reconstruct the Jacobian: for all $s \in (0, T)$
   2.1 Let
   $$\boldsymbol{a}_t(a, z) = \begin{cases} \boldsymbol{a}_{t+j}(a, z) & t \leq s \\ \boldsymbol{a}_{ss}(a, z) & t > s \end{cases}$$
   2.2 Using this reconstructing policy vector, do the forward step
   2.3 Aggregate as usual using $\boldsymbol{A}_t = \int \boldsymbol{a}_t(a, z) dD_t(a, z)$
   2.4 Fill up the column s of the Jacobian as $(\boldsymbol{A}_t - \boldsymbol{A}_{ss})/h$

Let's code!

Fake news algorithm - summary

- Auclert et. al (2021) introduce an efficient algorithm to compute aggregate jacobians for models with heterogeneous agents
  - Can compute the linearized response of aggregate consumption, savings w.r.t aggregate variables such as wages, interest rates **fast**
- Central insight: Exploit structure of dynamic programming problems + histogram method
- Why did we need this?
  - Allows us to compute Jacobian of aggregate model by »chaining« together individual jacobians along DAG
  - Can then use Quasi-Newton methods to solve dynamic GE model!
- GEModeltools does all of this »under the hood« when you compute HH Jacobians
  - You just tell GEModeltools the inputs and outputs of the household block
  - Entire algorithm is automated

Linear transitions and aggregate uncertainty

Reminder of model class

- Unknowns: $\boldsymbol{U}$
- Shock: $\boldsymbol{Z}$
- Additional variables: $\boldsymbol{X}$
- Target equation system:

$$H(\boldsymbol{U}, \boldsymbol{Z}) = 0$$

- In deterministic, perfect foresigh model (MIT shocks), solve $H(\boldsymbol{U}, \boldsymbol{Z}) = 0$ by
  1. Calculating the Jacobian of H w.r.t $\boldsymbol{U}$ around s.s.
  2. Use Newton's method to find non-linear transition given $\boldsymbol{Z}$

  $\Rightarrow$ But we have abstracted from real aggregate uncertainty

Aggregate uncertainty

- In business cycle model common to have *aggregate uncertainty*
- I.e. underlying shocks (TFP, demand etc) x follow stochastic process with dist, f, $x_t \sim f$
- This implies that all variables which are functions of x are also random.
  - If TFP is random $\Rightarrow$ wages, interest rates, labor demand etc. are **random** until observed
- Implies that we need to compute expectation in Euler, NKPC and other forward looking equations:

$$u'(C_t) = \beta \mathbb{E}_t \left[R_{t+1}(x_{t+1}) \, u'(C_t(x_{t+1})) \right]$$

- Remember: So far in the course we have generally assumed **perfect foresight** w.r.t aggregate variables (w, r) so no expectation
  - Implies that aggregate shocks are not random process, but rather *MIT shocks*
  - Interpretation of MIT shocks generally hard to reconcile with business cycles

Stochastic vs deterministic models

- To see how the **stochastic** model and deterministic model are related consider the Euler with random x:

$$u'\left(C_t\right) = R\beta\mathbb{E}_t\left[u'\left(C_t\left(x_{t+1}\right)\right)\right]$$

- First-order Taylor approx. around deterministic ss (use $R\beta = 1$):

$$du'\left(C_t\right) \approx u''\left(C_{ss}\right)\cdot C'\left(x_{ss}\right)\cdot d\mathbb{E}_t x_{t+1}$$

- Assume $x_t = \rho^x x_{t-1} + \epsilon_t^x$ with $\mathbb{E}\epsilon_t^x = 0$. Period 0 solution in deterministic/perfect foresight model:

$$du'\left(C_0\right) \approx u''\left(C_{ss}\right)\cdot C'\left(x_{ss}\right)\cdot \rho^x d\epsilon_0^x$$

- Stochastic model we use:

$$\begin{aligned} d\mathbb{E}_0 x_1 &= d\mathbb{E}_0\left(\rho^x x_0 + \epsilon_1^x\right) \\ &= \rho^x d\mathbb{E}_0 x_0 = \rho^x d\epsilon_0^x = dx_1 \end{aligned}$$

- Same result! Aggregate uncertainty **does not matter to first-order** when linearizing w.r.t aggregate shock

Linearized IRFs

- Solve for IRfs for unknowns using first-order approximation

$$H(\boldsymbol{U}, \boldsymbol{Z}) = 0 \Rightarrow H_U d\boldsymbol{U} + H_{\boldsymbol{Z}} d\boldsymbol{Z} = 0 \Leftrightarrow d\boldsymbol{U} = \underbrace{-H_U^{-1} H_z}_{=G_U} d\boldsymbol{Z}$$

- We can find H_U and H_Z as before using fake-news
- Limitations:
  - Imprecise for large shocks
  - Imprecise in models with aggregate non-linearities
  - No real aggregate uncertainty (precautionary savings w.r.t. aggregate shocks, etc)

Linearized IRFs: example from HANC

You want to compute an IRF facing a TFP shock in HANC.
Recall that

$$H(\boldsymbol{K},\boldsymbol{\Gamma}) = \boldsymbol{A}(\boldsymbol{r}(\boldsymbol{K},\boldsymbol{\Gamma}),\boldsymbol{w}(\boldsymbol{K},\boldsymbol{\Gamma})) - \boldsymbol{K}$$

The derivative with respect to $\boldsymbol{K}$ is

$$H_{\boldsymbol{K}} = \mathcal{J}^{A^{hh},r}\mathcal{J}^{r,K} + \mathcal{J}^{A^{hh},w}\mathcal{J}^{w,K} - \boldsymbol{I}$$

The derivative with respect to $\boldsymbol{\Gamma}$ is

$$H_{\boldsymbol{\Gamma}} = \mathcal{J}^{A^{hh},r}\mathcal{J}^{r,\Gamma} + \mathcal{J}^{A^{hh},w}\mathcal{J}^{w,\Gamma}$$

Write a first-order approximation

$$H_{\boldsymbol{K}}d\boldsymbol{K} + H_{\boldsymbol{\Gamma}}d\boldsymbol{\Gamma} = 0 \Leftrightarrow d\boldsymbol{K} = -H_{\boldsymbol{K}}^{-1}H_{\boldsymbol{\Gamma}}d\boldsymbol{\Gamma}$$

This gives us the endogenous response $d\boldsymbol{K}$ for any shock $d\boldsymbol{\Gamma}$ by simply computing the product of two matrices!

Simulating a time-series using the linearized solution

We can also simulate the economy following a sequence of shocks:

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.
  - EX: If shock $\boldsymbol{Z}$ follows an AR(1) then $d\boldsymbol{Z}_s = \rho^{s-t}\boldsymbol{\epsilon}_{t-s}$
- **Linearized simulation:**
  1. Draw time series of innovations, $\tilde{\boldsymbol{\epsilon}}_t$
  2. Calculate the time series of shocks as $d\tilde{\boldsymbol{Z}}_t = \sum_{s=0}^{T-1} d\boldsymbol{Z}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$
     Note: $d\boldsymbol{Z}_s \tilde{\boldsymbol{\epsilon}}_{t-s} =$ effect of shock s periods ago today
  3. Calculate the time series of other aggregate variables as

$$d\tilde{\boldsymbol{X}}_t = \sum_{s=0}^{T-1} d\boldsymbol{X}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$$

     where $d\boldsymbol{X}_s$ is the IRF to a *unit-shock* after s periods (just needs jacobian of $\boldsymbol{X}$ w.r.t shocks $\boldsymbol{Z}$)

- **Intuition:** Sum of first order effects from all previous shocks

Let's code!

When does aggregate uncertainty matter?

- **Insight:** *The IRF from an MIT shock is equivalent to the IRF in a model with aggregate risk, which is linearized in the aggregate variables* (Boppart et. al., 2018)
- What about **high order**?
- Approximate Euler to **second** order:

$$du'(C_t) \approx u''(C_{ss}) \cdot C'(x_{ss}) \cdot d\mathbb{E}_t x_{t+1} + \frac{1}{2} u'''(C_{ss}) C''(x_{ss}) \cdot \mathbb{E}_t (x_{t+1} - x_{ss})^2$$

$$u''(C_{ss}) \cdot C'(x_{ss}) \cdot d\mathbb{E}_t x_{t+1} + \frac{1}{2} u'''(C_{ss}) C''(x_{ss}) \cdot \sigma_{x,t}^2$$

- In deterministic model $\sigma_{x,t}^2 = 0$ - not true in stochastic model!
  - Models deviate once we go beyond 1st order approximation (linearization)
- Still extremely usefull though - we may solve deterministic models to first-order and interpret as models with aggregate uncertainty
  - How do we linearize models numerically?

Calculating moments - variance

- Implications of prior slide:
  - Very easy to calculate business cycle moments
- Steps (variance of C) (1 shock):
  1. Formulate shock to e.g. public spending, $\{dG_t\}_{t=0}^{T} = d\boldsymbol{G}$ (could be an AR(1))
  2. Linearize and solve model to get IRF of $\{dC_t\}_{t=0}^{T} = d\boldsymbol{C}$ w.r.t $\{dG_t\}$
  3. Calculate variance $\text{var}(dC_t) = \sum_{s=0}^{T-1} (dC_s)^2$
- Same principle with more shocks

Calculating moments - covariance

- **Covariances:**

$$\text{cov}(dC_t, dY_{t+k}) = \sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1-k} dC_s^i dY_{s+k}^i$$

- **Covariance decomposition:**

$$\frac{\text{contribution from one shock}}{\text{contributions from all shocks}} = \frac{\sigma_j^2 \sum_{s=0}^{T-1-k} dC_s^j dY_{s+k}^j}{\sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1-k} dC_s^i dY_{s+k}^i}$$

Solving HA model with aggregate risk (advanced)

- To solve models with aggregate risk we need to write them in *state-space* form instead of *sequence-space*
  - Think of HA household problem - that is always in state-space form
  - Endogenous variables c_t, a_t as function of current states a_{t-1}, z_t
- **Aggregate stochastic variables:** $\boldsymbol{Z}$ follow some known process with innovations ϵ. *State space form*: RHS is what is known today

$$\begin{bmatrix} \boldsymbol{U}_t \\ \boldsymbol{Z}_t \end{bmatrix} = \mathcal{M}\left(\begin{bmatrix} \boldsymbol{U}_{t-1} \\ \boldsymbol{Z}_{t-1} \end{bmatrix}, \boldsymbol{\epsilon}_t \right)$$

  $\neq$ perfect foresight wrt. future agg. variables in *sequence-space*

- In standard NK model: no backward looking eqs. so number of state variables = Number of shocks

Example: Krussel-Smith

- What if we add heterogeneous agents? Canonical example: The Krussel-Smith model (1998)
  - HANC with aggregate uncertainty (TFP shocks)
- **Recursive formulation of household problem:**

$$v(\boldsymbol{D}_t, \Gamma_t, z_{it}, a_{it-1}) = \max_{a_{it}, c_{it}} u(c_{it}) + \beta \mathbb{E}_t \left[v(\boldsymbol{D}_{t+1}, \Gamma_{t+1}, z_{it+1}, a_{it}) \right]$$

s.t.

$$K_{t-1} = \int a_{it-1} d\boldsymbol{D}_t$$

$$r_t = \alpha \Gamma_t K_{t-1}^{\alpha-1} - \delta$$

$$w_t = (1-\alpha)\Gamma_t K_{t-1}^{\alpha}$$

$$a_{it} + c_{it} = (1+r_t)a_{it-1} + w_t z_{it}$$

$$\log z_{it+1} = \rho_z \log z_{it} + \psi_{it+1}, \;\; \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \;\; \mathbb{E}[z_{it}] = 1$$

$$a_{it} \geq 0,$$

- $\boldsymbol{D}_t$ is a state variable $\Rightarrow$ Massive state space

Comparisons

- **State-space approach with linearization**: Ahn et al. (2018); Bayer and Luetticke (2020); Bhandari et al. (2023); Bilal (2023)
  Con:
  1. Harder to implement
  2. Valuable to be able to interpret Jacobians

  Pro**:**
  1. Easier path to 2nd and higher order approximations
- **Global solution:** The distribution of households is a state variable for each household $\Rightarrow$ *explosion in complexity*
  1. Original: Krusell and Smith (1997, 1998); Algan et al. (2014);
  2. Deep learning: Fernández-Villaverde et al. (2021); Maliar et al. (2021); Han et al. (2021); Kase et al. (2022); Azinovic et al. (2022); Gu et al. (2023); Chen et al. (2023)
- **Discrete aggregate risk:** Lin and Peruffo (2023)

Summary

That was a lot! What you need to remember:

1. How to compute households impulse reponse in partial equilibrium: backward and forward steps
2. In general equilibrium: we use Newton's method and the Sequence Space Jacobian to find endogenous variables that clear markets (non-linear perfect foresight transitions)
3. Sequence Space Jacobian are costly to compute with 'brute-force', but fake-news algorithm is fast!
4. Linear approximations: first-order approximation to full model with aggregate risk, fast to compute once you have Jacobian

Exercises

Exercises: HANCGovModel

Same model. Your choice of τ_{ss}. New questions:

1. **Define the transition path.**
2. **Plot the DAG**
3. **What do the Jacobians look like?**
4. **Find the transition path for** $G_t = G_{ss} + 0.01G_{ss}0.95^t$
5. **What explains household savings behavior?**
6. **What happens to consumption inequality?**

Answers available at this link