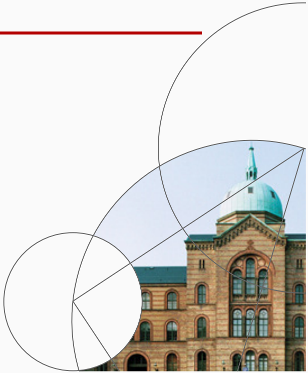
8. The New Keynesian Model

Adv. Macro: Heterogenous Agent Models



Nicolai Waldstrøm

2024

Introduction

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  1. Economic content: Long run trends and outcomes
  2. Methods: Stationary eq., Non-linear transition path and perfect foresight

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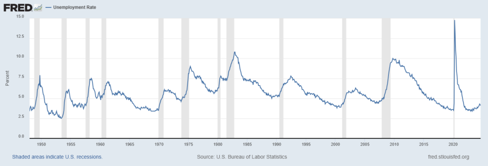
- **Today:**
  1. Business cycles in the New Keynesian model
  2. Linearized solution in models with aggregate risk
- **Literature:**
  - NK:
    1. Gali textbook ch. 3-4
    2. *Macroeconomics* textbook ch. 16
  - Solution methods:
    1. Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«
    2. Boppart et. al. (2018), »Exploiting MIT shocks in heterogeneous-agent economies: The impulse response as a numerical derivative«
    3. Documentation for GEModelTools

Business cycles

- Macro variables relatievly volatile around long-run trends



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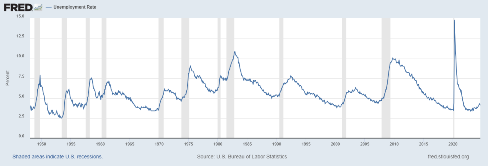

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  - Study how aggregate shocks cause business cycles
    - Does the transmission change with heterogeneous agents?
  - Implications for fiscal and monetary policy

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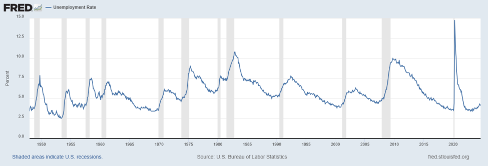

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  - Study how aggregate shocks cause business cycles
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  - Implications for fiscal and monetary policy
- First point on agenda: Need **role** for monetary policy
  - Models so far in the course have featured **monetary-non neutrality**
  - Monetary policy cannot affect real quantities (unemployment, GDP)

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  - Monopolistic competetion (price-setting)
  - Price rigidities

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  - Price rigidities
- The basic NK model is simple (can be reduced to 3 equations) but **extremely influential**

The New Keynesian model

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- The model consists of the following agents:
  - A representative household who consumes, saves and supplies labor
  - Firms with market power who produce output using labor and sets prices subject to nominal rigidities
  - A central bank which conduct monetary policy

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- **Central bank**: Sets nominal interest rate

Households

- Representative household solve the following problem:

$$\max_{C_t,A_t,L_t} E_0 \sum_{t=0}^{\infty} \beta^t \left[u\left(C_t\right) - \nu\left(L_t^{hh}\right) \right]$$

$$s.t.$$

$$C_t + A_t = \left(1 + r_t\right) A_{t-1} + \left(w_t L_t^{hh} + \Pi_t\right)$$

- Note: Expectation taken w.r.t **aggregate shocks** (TFP, monetary policy, markup etc.)
- Standard first-order conditions:

$$u'\left(C_t\right) = E_t \beta \left(1 + r_{t+1}\right) u'\left(C_{t+1}\right)$$
$$\nu'\left(L_t^{hh}\right) = w_t u'\left(C_t\right)$$

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- ***Static* problem for representative final good firm:**

$$\max_{y_{jt}\ \forall j} P_t Y_t - \int_0^1 p_{jt} y_{jt} dj \text{ s.t. } Y_t = \left(\int_0^1 y_{jt}^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}$$

for given output price, P_t, and input prices, p_{jt}

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- **Note:** Zero profits (can be used to derive price index)

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$$\text{s.t. } y_{jt} = \Gamma_t l_{jt}, \;\; y_{jt} = \left(\frac{p_{jt}}{P_t}\right)^{-\epsilon} Y_t, \;\; \Omega(p_{jt}, p_{jt-1}) = \frac{\theta}{2}\left[\frac{p_{jt}}{p_{jt-1}} - 1\right]^2$$

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- **Implied dividends:** $\Pi_t = Y_t - w_t L_t - \frac{\theta}{2}\left[\frac{p_{jt}}{p_{jt-1}} - 1\right]^2 Y_t$

Derivation of NKPC

- **FOC wrt. p_{jt}:**

$$0 = (1-\epsilon)\left(\frac{p_{jt}}{P_t}\right)^{-\epsilon}\frac{Y_t}{P_t} + \epsilon\frac{w_t}{\Gamma_t}\left(\frac{p_{jt}}{P_t}\right)^{-\epsilon-1}\frac{Y_t}{P_t}$$
$$-\theta\left[\frac{p_{jt}}{p_{jt-1}} - 1\right]\frac{Y_t}{p_{jt-1}} + E_t\frac{J'_{t+1}(p_{jt})}{1+r_{t+1}}$$

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- **FOC + Envelope + Symmetry +** $\pi_t = P_t/P_{t-1} - 1$

$$0 = \left[(1-\epsilon) + \epsilon\frac{w_t}{\Gamma_t}\right]\frac{Y_t}{P_t}$$
$$-\theta\left[\frac{P_t}{P_{t-1}} - 1\right]\frac{Y_t}{P_{t-1}} - E_t\frac{\theta\left[\frac{P_{t+1}}{P_t} - 1\right]\left(\frac{P_{t+1}}{P_t^2}\right)Y_{t+1}}{1+r_{t+1}}$$

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4. Note:
   - Sometimes a β^{firm} is used instead of $\frac{1}{1+r_{t+1}}$
   - $\pi_t(1+\pi_t) \approx \pi_t$ for small π_t

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- **Government:** In standard model Government simply supplies bonds that are in net-zero supply, $B = 0$
  - Note: HHs still make consumption-saving decisions (so cannot impose $A = 0$ in budget), but in equilibrium prices will adjust such that $A = B = 0$
  - Simplifying assumption, can easily incorporate more reaslitic government $\tau_t = r_t B_{ss} + G_t$ with $B_{ss} > 0$ (see HANK later)

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- Asset market:
$$A = 0$$
- As usual, in practice we will only impose market clearing in two of the markets when solving the model

Aggregate shocks

- In the standard NK model business cycles arise due to fluctuations in aggregate shocks:

1. TFP (supply)

$$\ln \Gamma_t = \overline{\Gamma} + \ln \Gamma_{t-1} + \epsilon_t^{\Gamma}, \quad \epsilon_t^{\Gamma} \sim \mathcal{N}\left(0, \sigma_{\Gamma}^2\right)$$

2. Discount factor (demand)

$$\ln \beta_t = \overline{\beta} + \ln \beta_{t-1} + \epsilon_t^{\beta}, \quad \epsilon_t^{\beta} \sim \mathcal{N}\left(0, \sigma_{\beta}^2\right)$$

3. Monetary policy

$$i_t^* = \overline{i^*} + \ln i_{t-1}^* + \epsilon_t^{i^*}, \quad \epsilon_t^{i^*} \sim \mathcal{N}\left(0, \sigma_{i^*}^2\right)$$

The 3 equation NK model

- Consider the deterministic, zero-inflation steady state of the model (with TFP and prices normalized to 1):

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- The model can be reduced to three equations:

$$\begin{aligned}
\hat{Y}_t &= -\sigma\left(i_t - \pi_{t+1}\right) + \hat{Y}_{t+1} + \epsilon_t^D && \text{(Euler/demand curve)} \\
\hat{\pi}_t &= \tilde{\kappa}\hat{Y}_t + \beta\hat{\pi}_{t+1} + \epsilon_t^S && \text{(NKPC/supply curve)} \\
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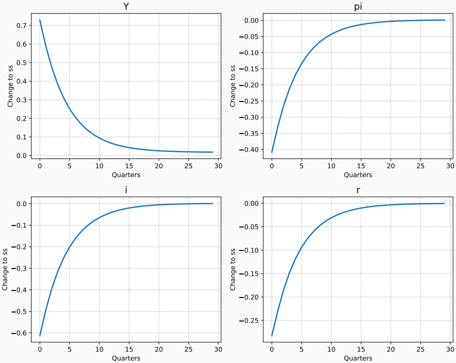
- With three unknowns (per period) $\hat{Y}_t, \hat{\pi}_t, \hat{i}_t$

Technology shocks in the NK model

- Effects of a positive TFP shock (increase Γ_t)



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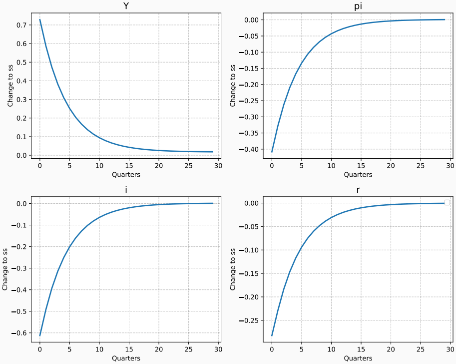

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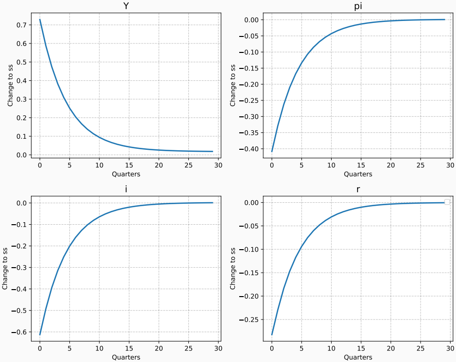

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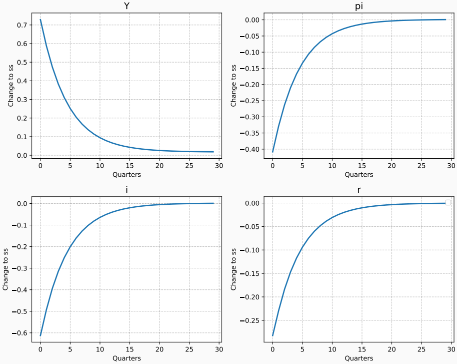

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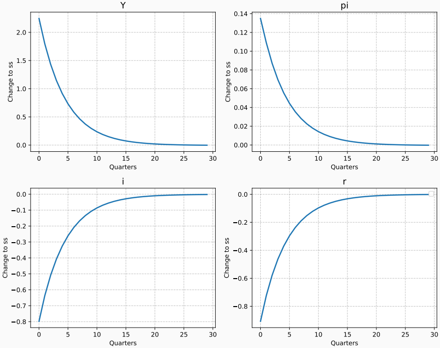
- Effects of accomodating monetary policy (easing) with persistent decline in i_t^*



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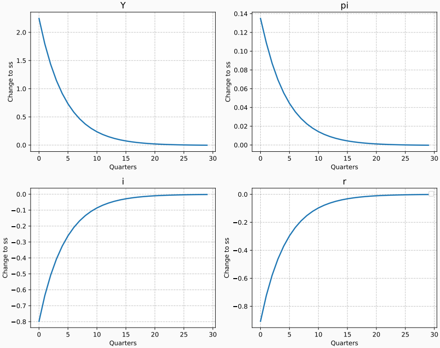

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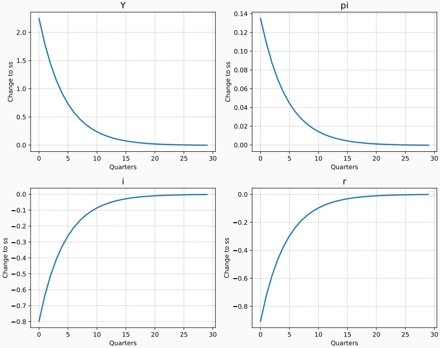

- Decrease real rate r which induce intertemporal substitution, so $C, Y \uparrow$
- Increase in employment pushes up wages (marginal costs), so inflation increases

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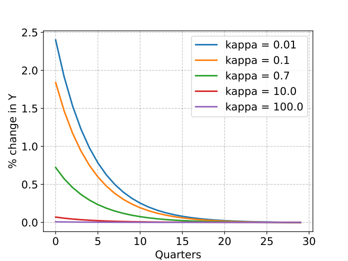

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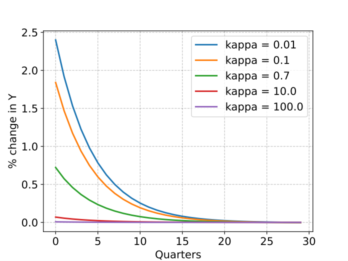

- Why? With completely flexible prices monetary policy just increases inflation 1-1 without affecting r

Review questions

- Consider the standard New Keynesian model

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  1. How does a positive demand shock ϵ_t^{β} (which decrease β) affect output Y, inflation π, and interest rates i, r?
  2. Are firm markups pro-cyclical or counter-cyclical (w.r.t Y) in response to the demand shock?
  3. Consider an extension with a government that spends G and raises lumpsum taxes τ
     - What is the effect of a shock to G? Is the fiscal multiplier $\frac{dY}{dG}$ above or below one?
     - Does the effects of the shock dependent on the method of financing (debt vs taxes)?

IRFs and simulation

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- Interpretation of MIT shocks generally hard to reconcile with business cycles

Stochastic vs deterministic models

- To see how the **stochastic** model and deterministic model are related consider the Euler with random x:

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- Same result! Aggregate uncertainty **does not matter to first-order** when linearizing w.r.t aggregate shock

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  - Models deviate once we go beyond 1st order approximation (linearization)
- Still extremely usefull though - we may solve deterministic models to first-order and interpret as models with aggregate uncertainty
  - How do we linearize models numerically?

Reminder of model class

- **Unknowns:** $\boldsymbol{U}$
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- In deterministic, perfect foresight model, solve $\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{Z}) = \mathbf{0}$ w.r.t $\boldsymbol{U}$ by:
  1. Calculating the jacobian of $\boldsymbol{H}$w.r.t $\boldsymbol{U}$around steady state
  2. Use Newton/Broyden's method to find non-linear transition path given shocks $\boldsymbol{Z}$

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- Next slide: **Can we solve model with aggregate risk globally** (i.e. to more than first-order)?

Aggregate risk (dynamic equilibrium)

- To solve models with aggregate risk we need to write them in *state-space* form instead of *sequence-space*
  - Think of HA household problem - that is always in state-space form
  - Endogenous variables c_t, a_t as function of current states a_{t-1}, z_t

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- In standard NK model: no backward looking eqs. so number of state variables = Number of shocks

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- $\boldsymbol{D}_t$ is a state variable $\Rightarrow$ Massive state space

Comparisons

- **State-space approach with linearization**: Ahn et al. (2018); Bayer and Luetticke (2020); Bhandari et al. (2023); Bilal (2023)
  Con:
  1. Harder to implement
  2. Valuable to be able to interpret Jacobians

  Pro**:**
  1. Easier path to 2nd and higher order approximations
- **Global solution:** The distribution of households is a state variable for each household $\Rightarrow$ *explosion in complexity*
  1. Original: Krusell and Smith (1997, 1998); Algan et al. (2014);
  2. Deep learning: Fernández-Villaverde et al. (2021); Maliar et al. (2021); Han et al. (2021); Kase et al. (2022); Azinovic et al. (2022); Gu et al. (2023); Chen et al. (2023)
- **Discrete aggregate risk:** Lin and Peruffo (2023)

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.
  - EX: If shock $\boldsymbol{Z}$ follows an AR(1) then $d\boldsymbol{Z}_s = \rho^{s-t}\boldsymbol{\epsilon}_{t-s}$

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- **Linearized simulation:**
  1. Draw time series of innovations, $\tilde{\boldsymbol{\epsilon}}_t$

Basic linearized simulation

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  - EX: If shock $\boldsymbol{Z}$ follows an AR(1) then $d\boldsymbol{Z}_s = \rho^{s-t}\boldsymbol{\epsilon}_{t-s}$
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  1. Draw time series of innovations, $\tilde{\boldsymbol{\epsilon}}_t$
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     Note: $d\boldsymbol{Z}_s \tilde{\boldsymbol{\epsilon}}_{t-s} =$ effect of shock s periods ago today

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     where $d\boldsymbol{X}_s$ is the IRF to a *unit-shock* after s periods (just needs jacobian of $\boldsymbol{X}$ w.r.t shocks $\boldsymbol{Z}$)

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- **Intuition:** Sum of first order effects from all previous shocks

Generalized linearized simulation [advanced]

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where $\partial \boldsymbol{a}^*_{i_g} / \partial X^{hh}_k$ is the derivative to a k-period ahead shock to input X^{hh} (calculated in fake news algorithm)

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  3. Distribution can then be simulated forwards using standard method

Calculating moments - variance

- **Identical and independent distributed innovations:**

$$\mathbb{E}\left[\epsilon_t^i \epsilon_{t'}^j\right] = \begin{cases} \sigma_i^2 & \text{if } t = t' \text{ and } i = j \\ 0 & \text{else} \end{cases}$$

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- **Calculating moments such as var**(dC_t) from the IRFs:

$$\begin{aligned} \text{var}(dC_t) &= \mathbb{E}\left[\left(\sum_{i \in \mathcal{Z}} \sum_{s=0}^{T-1} dC_s^i \epsilon_{t-s}^i\right)^2\right] \\ &= \sum_{i \in \mathcal{Z}} \sum_{s=0}^{T-1} \mathbb{E}\left[\epsilon_{t-s}^i \epsilon_{t-s}^i\right] \left(dC_s^i\right)^2 \\ &= \sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1} \left(dC_s^i\right)^2 \end{aligned}$$

where dC_s^i is the IRF to a unit-shock to i after s periods and σ_i is the standard deviation of shock i

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    2. Linearize and solve model to get IRF of $\{dC_t\}_{t=0}^{T} = d\boldsymbol{C}$ w.r.t $\{dG_t\}$
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- Same principle with more shocks

Calculating moments - covariance

- **Covariances:**

$$\text{cov}(dC_t, dY_{t+k}) = \sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1-k} dC_s^i dY_{s+k}^i$$

Calculating moments - covariance

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- **Covariance decomposition:**

$$\frac{\text{contribution from one shock}}{\text{contributions from all shocks}} = \frac{\sigma_j^2 \sum_{s=0}^{T-1-k} dC_s^j dY_{s+k}^j}{\sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1-k} dC_s^i dY_{s+k}^i}$$

Exercise

Exercise - NK model with government

1. Familiarize yourself with the model equations in *blocks.py*. Do you understand all the equations?
2. Compute the non-linear response to a temporary increase in government spending
   2.1 Use *model.find_transition_path()* for the non-linear response (results are in *model.path*)
   2.2 Use *model.find_IRFs()* for the linear response (results are in *model.IRF*)
3. Add a zero lower bound to the model:

   $$i_t = \max\{i_{ss} + \phi\pi_t, 0\}$$

   Compute linear and non-linear responses to a β-shock of size 0.05 and compare.
4. Assume that the government tries to stabilize the economy after the demand shock. Compute linear and non-linear responses to a simultaneous shock to β ($d\beta_0 = 0.05$) and G ($dG_0 = 0.03$).
5. Is stabilization policy more or less efficient once we take the ZLB into account? Hint: Compare the multipliers $\frac{dY^{\beta,G} - dY^{\beta}}{dG}$ for the linear and non-linear responses and compare.
   MORE ON NEXT SLIDE

6. Simulate a monetary policy shock of size 0.01. Calculate the variance of consumption using the analytical formula:

$$\mathrm{var}\left(dC\right) = \sum_{s=0}^{T-1} \left(dC_s\right)^2$$

Check that you get the same variance if you simulate a timeseries of consumption using model.simulate(skip_hh=True), and calculate the variance as:

$$\mathrm{var}\left(dC\right) = \frac{1}{N} \sum_{i=0}^{N} \left(dC_i^{sim}\right)^2$$

Hints: You can set the size of the shock for the IRFs using model.par.jump_eps_i, while the standard error of the shocks in the simulation is set using model.par.std_eps_i.

Make sure that the standard error of other shocks in the model are zero when you simulate. You can find the simulated series in model.sim.dC.

Summary

Summary and next week

- **Today:**
  1. The New Keynesian model
  2. Aggregate risk and linearized dynamics (IRF and simulation)
  3. Calculating aggregate moments (for calibration or estimation)
- **Next week:** HANK + Fiscal policy
- **Homework:**
  1. Work on exercise
  2. Skim-read Auclert et al. (2023),
     »The Intertemporal Keynesian Cross«