3-4. Stationary Equilibrium

Adv. Macro: Heterogenous Agent Models

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Recap of the two last classes

What did we learn so far?

- The buffer-stock model captures some facts about consumption and MPCs
- We can solve it with dynamic programming:
  - The Value Function Iteration algorithm (slow)
  - The Endogenous Grid Method (fast)
  - The goal is to obtain the policy functions $c(a, z)$, $a'(a, z)$
- How to simulate a distribution of households using
  - The Monte-Carlo method (slow and imprecise)
  - The histogram method (fast and precise)

Today and next week: use those methods to solve the steady-state of a simple heterogeneous-agent model.

Introduction

Introduction

- **Last time:**
  1. Partial equilibrium
  2. No interactions
- **Today:** Interaction through markets
- **Model:** Heterogeneous Agent Neo-Classical (HANC) model
- **Equilibrium-concept:** *Stationary equilibrium*
  1. What determines income and wealth inequality *in the long run*?
  2. What determines the real interest rate *in the long run*?
- **Code:** Based on the GEModelTools package
  1. Is in active development
  2. You can help to improve interface, find bugs and features

  Documentation: See GEModelToolsNotebooks
  - Many examples in repo, so look if you have issues
- **Literature:** Aiyagari (1994)

Outline of this lecture

1. Recap of the Ramsey (Neo-Classical) model
2. Overview of the Heterogeneous-Agent Neo-Classical model (HANC)
3. How to compute the stationary equilibrium
4. Some economic properties of the HANC stationary equilibrium

Ramsey-recap

The Ramsey model

- We will study the *stationary equilibrium* in the Heterogeneous Agent Neo-Classical (HANC) model
- Merges two well known models in the literature:
  - Standard Ramsey–Cass–Koopman model (*NC*)
    - What do we mean by Neo-classical?
  - One-asset Buffer-stock model (*HA*)
- Went through the Buffer-stock model over the last two lectures
- **Now**: Recap of the Ramsey model

Ramsey: Firms

- **Production function:** $Y_t = F(\Gamma_t, K_{t-1})$ [capital chosen in $t-1$ is used for production at t]
  where Γ_t is technology
- **Profits:** $\Pi_t = Y_t - w_t L_t - r_t^K K_{t-1}$
- **Profit maximization:** $\max_{K_{t-1}, L_t} \Pi_t$
  1. Rental rate: $\frac{\partial \Pi_t}{\partial K_{t-1}} = 0 \Leftrightarrow r_t^K = F_K(\Gamma_t, K_{t-1}, L_t)$
  2. Real wage: $\frac{\partial \Pi_t}{\partial L_t} = 0 \Leftrightarrow w_t = F_L(\Gamma_t, K_{t-1}, L_t)$

  With CRS we get zero profits: $\Pi_t = 0 \Rightarrow$

  $Y_t = w_t L_t + r_t^K K_{t-1}$ [functional income distribution]

Ramsey: Zero-profit mutual fund

- Introduce **mutual fund**
  - Takes savings A_{t-1} from households and invest them in available assets
  - In the Ramsey model: Only capital K_{t-1} but could also include gov. bonds, firm equity etc.
  - Receive income from firms and redistribute it to households
- **Capital depreciate** with rate $\delta \in (0, 1)$,

$$K_t = (1 - \delta)K_{t-1} + I_t$$

- **Deposits** (from households), A_{t-1}: The rate of return is

$$r_t = r_t^K - \delta$$

- **Balance sheet:**

$$A_{t-1} = K_{t-1}$$

Ramsey: Households

- **Utility maximization:**

$$v_0(A_{-1}^{hh}) = \max_{\{C_t^{hh}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t^{hh})$$

s.t.

$$C_t^{hh} + A_t^{hh} = (1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh}$$

  Exogenous labor supply: $L_t^{hh} = 1$

- **Euler-equation** (implied by Lagrangian):

$$u'(C_t^{hh}) = \beta(1 + r_{t+1})u'(C_{t+1}^{hh})$$

Ramsey: Market Clearing

- **Capital market:** $K_t = A_t = A_t^{hh}$
- **Labor market:** $L_t = L_t^{hh} = 1$
- **Goods market:** $Y_t = C_t^{hh} + I_t$

- **Walras:** Capital and labor market clears $\Rightarrow$ goods market clears. Start from

$$
\begin{aligned}
C_t^{hh} + A_t^{hh} &= (1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh}) \\
\Leftrightarrow C_t^{hh} + I_t &= \left[(1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh} - A_t^{hh})\right] + (K_t - (1 - \delta)K_{t-1}) \\
&= [(1 + r_t)K_{t-1} + w_t L_t - K_t)] + (K_t - (1 - \delta)K_{t-1}) \\
&= r_t^K K_{t-1} + w_t L_t \\
&= Y_t
\end{aligned}
$$

- Note: Means that we can check if we have solved the numerical model correctly by:
  - Impose two of the market clearing conditions
  - Then check the third market clearing condition (should be zero)

Ramsey: Summary

- **Simplified form:**

$$u'(C_t^{hh}) = \beta(1 + F_K(\Gamma_t, K_t, 1) - \delta)u'(C_{t+1}^{hh})$$
$$K_t = (1-\delta)K_{t-1} + F(\Gamma_t, K_{t-1}, 1) - C_t^{hh}$$

- **Extended form:**

$$\begin{aligned}
r_t^K &= F_K(\Gamma_t, K_{t-1}, L_t) \\
w_t &= F_L(\Gamma_t, K_{t-1}, L_t) \\
r_t &= r_t^K - \delta \\
A_t &= K_t \\
A_t^{hh} &= (1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh} \\
u'(C_t^{hh}) &= \beta(1+r_{t+1})u'(C_{t+1}^{hh}) \\
A_t &= A_t^{hh} \\
L_t &= L_t^{hh}
\end{aligned}$$

Ramsey: As an equation system

Eqs. system with unknowns $\left\{K_t, L_t, r_t^K, w_t, r_t, A_t, A_t^{hh}, C_t^{hh}\right\}_{t=0}^{\infty}$ and eqs:

$$\begin{bmatrix} r_t^K - F_K(\Gamma_t, K_{t-1}, L_t) \\ w_t - F_L(\Gamma_t, K_{t-1}, L_t) \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ A_t^{hh} - \left((1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}\right) \\ u'(C_t^{hh}) - \beta(1+r_{t+1})u'(C_{t+1}^{hh}) \\ A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0,1,\dots\}, \text{given } K_{-1} \end{bmatrix} = \mathbf{0}$$

- **Euler-equation** can be solved for r_{ss} and hence K_{ss}:

$$u'(C_{ss}) = \beta(1 + F_K(\Gamma_{ss}, K_{ss}, 1) - \delta)u'(C_{ss}) \Leftrightarrow$$
$$F_K(K_{ss}, 1) = \frac{1}{\beta} - 1 + \delta$$

- **Accumulation equation + goods mkt. clearing** then implies C_{ss}:

$$K_{ss} = (1 - \delta)K_{ss} + F(\Gamma_{ss}, K_{ss}, 1) - C_{ss} \Leftrightarrow$$
$$C_{ss} = (1 - \delta)K_{ss} + F(\Gamma_{ss}, K_{ss}, 1) - K_{ss}$$

- Important thing to note: the steady-state asset supply is completely inelastic!

HANC

HANC model overview

- **Model blocks:**
  1. **Firms:** Rent capital from mutual fund and hire labor from the households, produce with given technology, and sell output goods
  2. **Zero-profit mutual funds:** Own capital and rent it to firms, take deposits and pay return to household
  3. **Households:** Face idiosyncratic productivity shocks, supplies labor exogenously and makes consumption-saving decisions
  4. **Markets:** Perfect competition in labor, goods and capital markets
- **Add-on to Ramsey-Cass-Koopman:** *Heterogeneous households subject to idiosyncratic shocks* $\rightarrow$ *generate precautionary savings!*
- **Other names:**
  1. The Aiyagari-model
  2. The Aiyagari-Bewley-Hugget-Imrohoroglu-model
  3. The Standard Incomplete Market (SIM) model

Heterogeneous households

- **Utility maximization** for household i:

$$v_0(z_{it}, a_{it-1}) = \max_{\{c_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{it})$$

s.t.

$$\ell_{it} = z_{it}$$

$$a_{it} = (1 + r_t)a_{it-1} + w_t \ell_{it} - c_{it}$$

$$\log z_{it+1} = \rho_z \log z_{it} + \psi_{it+1}, \;\; \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \;\; \mathbb{E}[z_{it}] = 1$$

$$a_{it} \geq 0$$

- **Where does heterogeneity enter?**

- **Incomplete markets due to borrowing constraint**
(fancy words: partial self-insurance, lack of Arrow-Debreu securities)

Recursive formulation

- **Value function** (at decision)

$$
\begin{aligned}
v(z_{it}, a_{it-1}) &= \max_{c_t} u(c_t) + \beta \mathbb{E}\left[v(z_{it+1}, a_{it})\right] \\
&\text{s.t.} \\
\ell_{it} &= z_{it} \\
a_{it} &= (1 + r_t) a_{it-1} + w_t \ell_{it} - c_{it} \\
\log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1} \\
a_{it} &\geq 0
\end{aligned}
$$

Distributions and aggregates

- Household policy function x^* where $x \in \{a, c, \ell\}$ function of:
  - Individuals states (z_{it}, a_{it-1})
  - Aggregates (w_t, r_t)
- Aggregate policy:

$$X_t^{hh}\left(\{r_\tau, w_\tau\}_{\tau \geq t}\right) = \int x_t^*(z_{it}, a_{it-1}, \{r_\tau, w_\tau\}_{\tau \geq t}) d\boldsymbol{D}_t$$

- When aggregating we **integrate** out individual states
  - Aggregate X_t^{hh} is only a function of $\{r_\tau, w_\tau\}_{\tau \geq t}$ in GE as long as exogenous states don't change
- $\Rightarrow$ If we know aggregates (w_t, r_t) can calculate aggregate household behavior (consumption or savings)

Equation system

$$\begin{bmatrix} r_t^K - F_K(\Gamma_t, K_{t-1}, L_t) \\ w_t - F_L(\Gamma_t, K_{t-1}, L_t) \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ A_t^{hh} - \int a_t d\boldsymbol{D}_t \\ L_t^{hh} - \int \ell_t d\boldsymbol{D}_t \\ \underline{\boldsymbol{D}}_{t+1} - \Lambda_t' \Pi_z' \underline{\boldsymbol{D}}_t \\ a_t - a_t^* \\ \forall t \in \{0, 1, \dots\}, \text{given } \underline{\boldsymbol{D}}_0 \end{bmatrix} = \mathbf{0}$$

- **Note**: Much larger system compared to Ramsey due to last 2 eqs.
  - $\boldsymbol{D}_t, a_t^*$ define mass and optimal savings policy at **the individual level**
  - Standard Ramsey model: 8 eqs. per period
  - HANC with $N_z = 7, N_a = 300$: $8 + 7 \times 300 = 2108$ per period

Market clearing

- **Capital market:** $K_t = A_t = \int a_t^*(z_{it}, a_{it-1}) d\boldsymbol{D}_t$
- **Labor market:** $L_t = \int \ell_t^*(z_{it}, a_{it-1}) d\boldsymbol{D}_t = \int z_{it} d\boldsymbol{D}_t = 1$
- **Goods market:** $Y_t = \int c_t^*(z_{it}, a_{it-1}) d\boldsymbol{D}_t + I_t$

- **Walras:** Capital and labor market clears $\Rightarrow$ goods market clears (using Euler's theorem)

$$
\begin{aligned}
C_t^{hh} + I_t &= \int c_{it}^* d\boldsymbol{D}_t + [K_t - (1-\delta)K_{t-1}] \\
&= \int \left[(1+r_t)a_{it-1} + w_t z_{it} - a_{it}\right] d\boldsymbol{D}_t \\
&= [(1+r_t)K_{t-1} + w_t L_t - K_t] + [K_t - (1-\delta)K_{t-1}] \\
&= r_t^K K_{t-1} + w_t L_t \\
&= Y_t
\end{aligned}
$$

Computing the Stationary Equilibrium

Stationary equilibrium - equation system

The **stationary equilibrium** satisfies

$$\begin{bmatrix} r^K_{ss} - F_K(\Gamma_{ss}, K_{ss}, L_{ss}) \\ w_{ss} - F_L(\Gamma_{sst}, K_{ss}, L_{ss}) \\ r_{ss} - (r^K_{ss} - \delta) \\ A_{ss} - K_{ss} \\ A_{ss} - A^{hh}_{ss} \\ L_{ss} - L^{hh}_{ss} \\ A^{hh}_{ss} - \int a_{ss} d\boldsymbol{D}_{ss} \\ L^{hh}_{ss} - \int \ell_{ss} d\boldsymbol{D}_{ss} \\ \underline{\boldsymbol{D}}_{ss} - \Lambda'_{ss}\Pi'_z\underline{\boldsymbol{D}}_{ss} \\ a_{ss} - a^*_{ss} \end{bmatrix} = \mathbf{0}$$

Note : Households still move around »inside« the distribution due to idiosyncratic shocks. Does not affect aggregates due to »law of large numbers«

Stationary equilibrium - more verbal definition

Given technology Γ_{ss}

1. Quantities K_{ss} and L_{ss},
2. prices r_{ss} and w_{ss} (always $\Pi_{ss} = 0$),
3. the distribution $\boldsymbol{D}_{ss}$ over β_i, z_{it} and a_{it-1}
4. and the policy functions a^*_{ss}, ℓ^*_{ss} and c^*_{ss}

are such that

1. Households maximize expected utility (policy functions)
2. Firms maximize profits (prices)
3. $\boldsymbol{D}_{ss}$ is the invariant distribution implied by the household problem
4. Mutual fund balance sheet is satisfied
5. The capital market clears
6. The labor market clears
7. The goods market clears

How do we solve the household block in practice?

The hard part is to solve the household block! How do we do it?

- Use EGM to obtain the policy functions a_{ss}, c_{ss} for a given r, w
  - The »backward« step
- Use the histogram method to obtain the stationary distribution $\boldsymbol{D}_{ss}$
  - The »forward« step
- Aggregate policy:

$$A_{ss}^{hh}\left(\{r_{ss}, w_{ss}\}\right) = \int a_{ss}^{*}(z_{it}, a_{it-1}, \{r_{ss}, w_{ss}\}) d\boldsymbol{D}_{ss}$$

Time to code!

Direct implementation (K guess)

Technology: $F(K,L) = \Gamma K^{\alpha} L^{1-\alpha}$

Root-finding problem in K_{ss} with the objective function:

1. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
2. Calculate $r_{ss} = \alpha \Gamma_{ss} (K_{ss})^{\alpha-1} - \delta$ and $w_{ss} = (1-\alpha)\Gamma_{ss}(K_{ss})^{\alpha}$
3. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
4. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
5. Return $K_{ss} - \boldsymbol{a}_{ss}^{*\prime} \boldsymbol{D}_{ss}$

Note: $\boldsymbol{a}_{ss}^{*\prime} \boldsymbol{D}_{ss} = \sum_i a_{i,ss}^{*} D_i$

Direct implementation (r guess)

Technology: $F(K,L) = \Gamma K^{\alpha} L^{1-\alpha}$

Root-finding problem in r_{ss} with the objective function:

1. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
2. Calculate $K_{ss} = \left(\frac{r_{ss}+\delta}{\alpha\Gamma_{ss}}\right)^{\frac{1}{\alpha-1}}$ and $w_{ss} = (1-\alpha)\Gamma_{ss}(K_{ss})^{\alpha}$
3. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}^{*}_{ss}$
4. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
5. Return $K_{ss} - \boldsymbol{a}^{*\prime}_{ss}\boldsymbol{D}_{ss}$

Indirect implementation

Technology: $F(K,L) = \Gamma K^{\alpha} L^{1-\alpha}$

Consider Γ_{ss} and δ as »free« parameters:

1. Choose r_{ss} and w_{ss}
2. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
3. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
4. Set $K_{ss} = \boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss}$
5. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
6. Set $\Gamma_{ss} = \frac{w_{ss}}{(1-\alpha)(K_{ss})^{\alpha}}$
7. Set $r_{ss}^{K} = \alpha\Gamma_{ss}(K_{ss})^{\alpha-1}$
8. Set $\delta = r_{ss}^{k} - r_{ss}$

Direct implementation (calibration)

Set $r_{ss} = r^{target}$, $K_{ss} = K^{target}$, $Y_{ss} = Y^{target}$, and back out

1. $\Gamma_{ss} = Y^{target}/(L_{ss}^{1-\alpha} K_{ss}^{\alpha})$
2. $\delta = \alpha Y^{target}/K^{target} - r^{target}$

We know that $w_{ss} = (1-\alpha)Y^{target}$. Then find the β that clears the market

Root-finding problem in β with the objective function:

1. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$),
2. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$for a given β
3. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
4. Return $K_{ss} - \boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss}$
5. Update β

How to choose parameters?

- **External calibration:** Set subset of parameters to the *standard values in the literature* or *directly from data estimates* (e.g. income process)
- **Internal calibration:** Set remaining parameters so the model fit to a number of chosen *macro-level and/or micro-level targets* based on empirical estimates
    1. **Informal:** Roughly match targets by hand
    2. **Formal:**
        2a. Solve root-finding problem
        2b. Minimize a squared loss function
    3. **Estimation:** Formal with squared loss function (think GMM) or likelihood function + standard errors
- **Complication:** *We must always solve for the steady state for each guess of the parameters to be calibrated*

Calibration at the quarterly level

- $r_{ss} = 0.05/4$ to match 5% annual interest rate
- $Y_{ss} = 1$ as a normalization
- $K_{ss} = 16$ to match annual wealth-to-output ratio of 4
- $\alpha = 1/3$ to match labor share of roughly 2/3
- $\sigma_\psi = 0.5$, $\rho = 0.9$: data on income inequality and risk

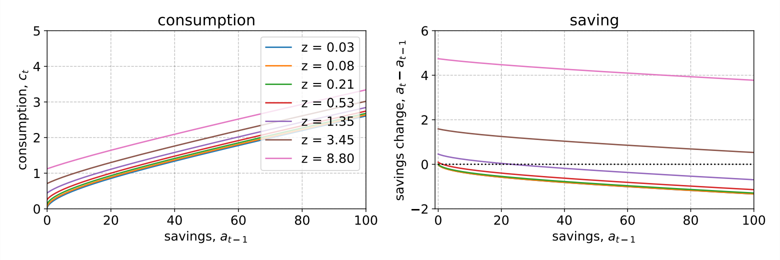
Some Properties of the HANC steady-state

Consumption function

- **Euler-equation still necessary for $a_{it} > 0$:**

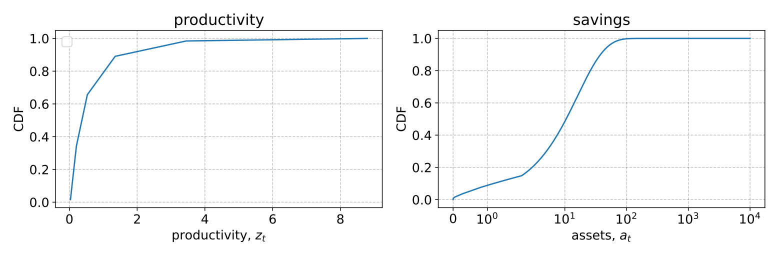
$$c_{it}^{-\sigma} = \beta_i(1+r_{t+1})\mathbb{E}_t\left[c_{it+1}^{-\sigma}\right]$$

- **Precautionary saving:**
  1. Low consumption for low cash-on-hand $\rightarrow$ *buffer-stock target*
  2. Steep slope for low cash-on-hand $\rightarrow$ *high MPC*



Some amount of inequality

- **Productivity:** Marginal distribution over only z_{it}
- **Savings:** Marginal distribution over a_{it} cond. on β_i



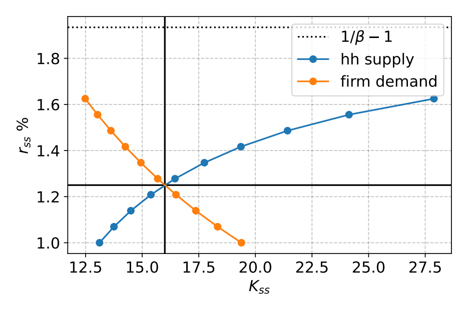
- **Drivers of wealth inequality here: income shocks**

Steady state interest rate

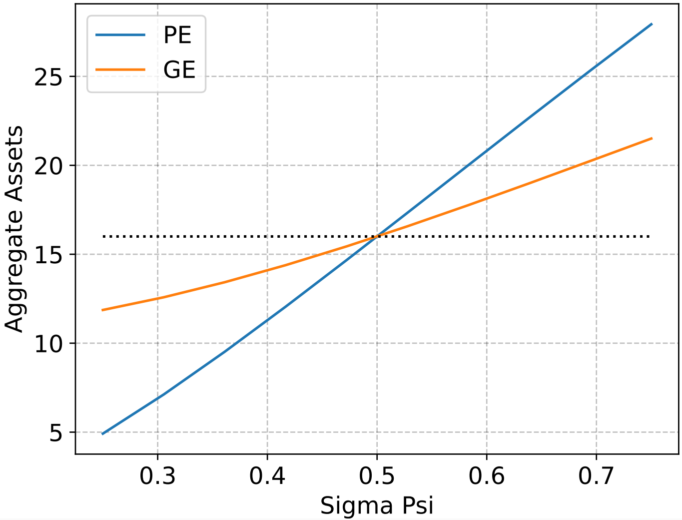
- **Representative agent / complete markets:**
  Derived from aggregate Euler-equation

$$C_t^{-\sigma} = \beta(1 + r_{t+1})C_{t+1}^{-\sigma} \Rightarrow C_{ss}^{-\sigma} = \beta(1 + r_{ss})C_{ss}^{-\sigma} \Leftrightarrow \beta = \frac{1}{1 + r_{ss}}$$

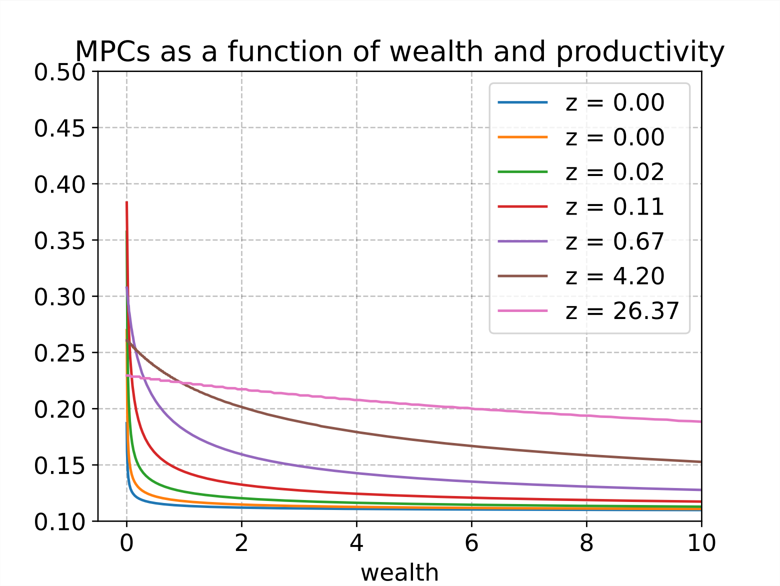
- **Heterogeneous agents:** *No such equation exists*



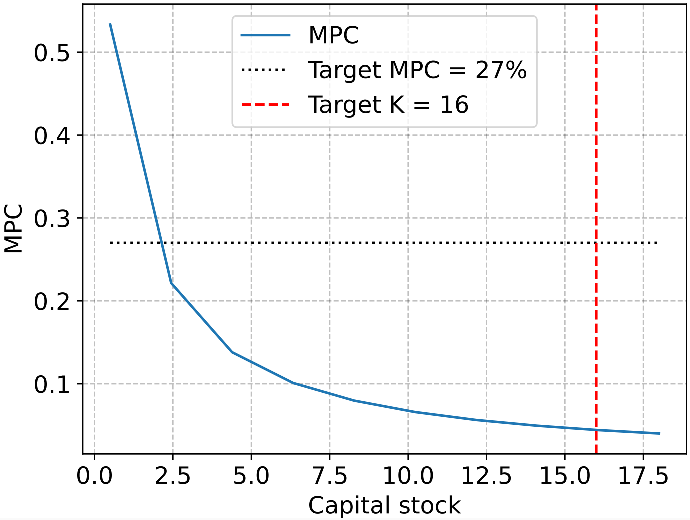
Risk drives wealth accumulation



Marginal Propensity to Consume



Tradeoff between matching aggregate wealth and MPCs



Exercises

Exercise 1: HANC with ex-ante heterogeneity

Add permanent β heterogeneity to the HANC model:

$$
\begin{aligned}
v_t(\beta_i, z_{it}, a_{it-1}) &= \max_{c_{it}} \frac{c_{it}^{1-\sigma}}{1-\sigma} + \beta_i \mathbb{E}_t \left[v_{it+1}(z_{it+1}, a_{it}) \right] \\
\text{s.t. } & a_{it} + c_{it} = (1+r_t)a_{it-1} + w_t z_{it} \geq 0 \\
& \log z_{it+1} = \rho_z \log z_{it} + \psi_{it+1} \;\; , \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \, \mathbb{E}[z_{it}] = 1
\end{aligned}
$$

Assume that we have three types of households:
$\beta_i \in (\beta - \delta, \beta, \beta + \delta)$. Find δ such that $MPC = 0.27$ and $K/Y = 16$.

Exercise 2: HANCGovModel

- **No production.** No physical savings instrument
- **Households:** Get stochastic endowment z_{it} of consumption good
- **Government:**
  1. Choose government spending
  2. Collect taxes, τ_t, proportional to endowment
  3. Bonds: Pays 1 unit of the consumption good next period. Price is $p_t^B < 1$

$$p_t^B B_t + \int \tau_t z_{it} d\boldsymbol{D}_t = B_{t-1} + G_t$$

$$\tau_t = \tau_{ss} + \eta_t + \varphi \left(B_{t-1} - B_{ss} \right)$$

  where η_t is a tax-shifter

- **Market clearing:**

$$B_t = A_t^{hh}$$

$$C_t^{hh} + G_t = \int z_{it} d\boldsymbol{D}_t = 1$$

Exercise 2: Households

Households:

$$v_t(z_{it}, a_{it-1}) = \max_{c_{it}} \frac{c_{it}^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{it+1}(z_{it+1}, a_{it}) \right]$$

$$\text{s.t. } p_t^B a_{it} + c_{it} = a_{it-1} + (1-\tau_t) z_{it} \geq 0$$

$$\log z_{it+1} = \rho_z \log z_{it} + \psi_{it+1} \; , \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \; \mathbb{E}[z_{it}] = 1$$

Euler-equation:

$$c_{it}^{-\sigma} = \beta \frac{\underline{v}_{a,t+1}(z_{it}, a_{it})}{p_t^B}$$

Envelope condition:

$$\underline{v}_{a,t}(z_{it-1}, a_{it-1}) = c_{it}^{-\sigma}$$

Exercise 2: Questions

1. **Define the stationary equilibrium**
2. **Solve and simulate the household problem**
   with $p^B_{ss} = 0.975$ and $\tau_{ss} = 0.12$.
3. **Find the stationary equilibrium**
   with $G_{ss} = 0.10$ and $\tau_{ss} = 0.12$.
4. **What happens for** $\tau_{ss} \in (0.11, 0.15)$?
5. **When is average household utility maximized?**

Note: Full solution in repository folder
GEModelToolsNotebooks/HANCGovModel