14. Exam and Perspectives

Adv. Macro: Heterogenous Agent Models

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2025

Plan

1. Learning outcomes
2. Overview and perspectives
3. Exam
4. GEModelTools
5. Exam from 2023

Learning outcomes

Knowledge, Skills and Competencies

- **What do you need to know?**
  1. Understand and be able to use the *computational techniques* to solve and simulate heterogeneous agent models
  2. Understand and be able to discuss the *economic insights* heterogeneous agent models can provide
- **Next:** Learning outcomes in Knowledge, Skills and Competencies

Knowledge

1. Account for, formulate and interpret precautionary saving models
2. Account for stochastic and non-stochastic simulation methods
3. Account for, formulate and interpret general equilibrium models with ex ante and ex post heterogeneity, idiosyncratic and aggregate risk, and with and without pricing frictions
4. Discuss the difference between the stationary equilibrium, the transition path and the dynamic equilibrium
5. Discuss the relationship between various equilibrium concepts and their solution methods
6. Identify and account for methods for analyzing the dynamic distributional effects of long-run policy (e.g. taxation and social security) and short-run policy (e.g. monetary and fiscal policy)

Skills

1. Solve precautionary saving problems with dynamic programming and simulate behavior with stochastic and non-stochastic techniques
2. Solve general equilibrium models with ex ante and ex post heterogeneity, idiosyncratic and aggregate risk, and with and without pricing frictions (stationary equilibrium, transition path, dynamic equilibrium)
3. Analyze dynamics of income and wealth inequality
4. Analyze transitional and permanent structural changes (e.g. inequality trends and the long-run decline in the interest rate)
5. Analyze the dynamic distributional effects of long-run policy (e.g. taxation and social security) and short-run policy (e.g. monetary and fiscal policy)

Competencies

1. Independently formulate, discuss and assess research on both the causes and effects of heterogeneity and risk for both long-run and short-run outcomes
2. Discuss and assess the importance of how heterogeneity and risk is modeled for questions about both long-run and short-run dynamics

Overview and perspectives

Single-agent problems (»partial equilibrium«, PE)

- **Focus:** Consumption-saving (and labor supply) of households
- **Solution methods:**
  1. Value-Function Iteration (VFI)
  2. Endogenous Gridpoint method (EGM)
- **Simulation methods:**
  1. Monte Carlo
  2. Histogram
- **Perspectives:**
  1. Multiple assets (liquidity, riskiness, durables, housing)
  2. Similar analysis of *firm problems*
  3. Closer link to *empirical observations* (life-cycle dynamics, informational frictions, behavioral biases, social preferences, norms)
  4. More *complex models* (solved with e.g. »deep learning«)

Stationary equilibrium

- **Stationary:** *Time-constant aggregate variables and distribution*
    1. **Fixed point problem:** Choose all parameters (and normalizations) and search for prices or aggregate quantities
    2. **Combined with calibration:** Some parameters set residually and/or there are targets beyond market clearing (e.g. MPC)
- **Comparative statics:** How does equilibrium outcomes change when one or more parameters change?
    1. Keep all other parameters (and normalizations) fixed
    2. Re-solve fixed point problem for prices or aggregate quantities
- **Perspective:**
    1. Always a good starting point for analyzing dynamics
    2. Understanding of average outcomes in »stable periods«
    3. Deeper »welfare analysis« (maximization of social welfare function)

Dynamic equilibrium: Aggregate risk

- **Canonical state-space form with aggregate risk:**
  1. States summarize all relevant information
  2. Policy functions (idiosyncratic + aggregate) are functions of states

  $$\begin{bmatrix} \boldsymbol{X}_{t+1}, \boldsymbol{Z}_{t+1}, \underline{\boldsymbol{D}}_{t+1}, \boldsymbol{Y}_t \end{bmatrix}' = \mathcal{M}\left(\begin{bmatrix} \boldsymbol{X}_t, \boldsymbol{Z}_t, \underline{\boldsymbol{D}}_t \end{bmatrix}', \boldsymbol{\epsilon}_t\right)$$

  Expectations are hard:
  - Depend on current state (high-dimensional)
  - Requires knowing $\mathcal{M}$ (perceived law-of-motion, PLM)
- **Fluctuations:**
  1. Volatility of shocks $\rightarrow 0$: Deterministic steady state (DSS)
  2. Realization of shocks $\rightarrow 0$: Stochastic steady state (SSS)
  3. Otherwise: Convergence to *ergodic distribution*
- **Moving average form:** History of aggregate shocks (without any truncation) determine current states $\Rightarrow$

  $$\begin{bmatrix} \boldsymbol{X}_{t+1}, \boldsymbol{Z}_{t+1}, \underline{\boldsymbol{D}}_{t+1}, \boldsymbol{Y}_t \end{bmatrix}' = \tilde{\mathcal{M}}\left(\boldsymbol{\epsilon}_t, \boldsymbol{\epsilon}_{t-1}, \boldsymbol{\epsilon}_{t-2}, \ldots\right)$$

  Solve on MA form: Azinovic-Yand and Žemlička (2025, Sep 16)

Sequence space

- **Linearization in state space:**

$$\left[\begin{array}{c} \boldsymbol{X}_{t+1}, \boldsymbol{Z}_{t+1}, \underline{\boldsymbol{D}}_{t+1}, \boldsymbol{Y}_t \end{array} \right]' = \boldsymbol{A} \left[\begin{array}{c} \boldsymbol{X}_t, \boldsymbol{Z}_t, \underline{\boldsymbol{D}}_t \end{array} \right]' + \boldsymbol{B} \epsilon_t$$

- **Sequence space:** Perfect foresight wrt. aggregate variables
  Behavior depend on:
  1. Idiosyncratic state variables
  2. Current value and expected distribution of relevant »prices«
     $\Rightarrow$ collapses to sequence (time-path) of relevant »prices«
- **MIT shock:** One-time shock in a perfect foresight model
- **Insight:** *The first-order IRF from an MIT shock is <u>equivalent</u> to the IRF in a model with aggregate risk, which is linearized in the aggregate variables* (Boppart et. al., 2018)

GE and Household Jacobians

- **Target equation system** with implicit truncation

$$\boldsymbol{H}(\boldsymbol{U},\boldsymbol{Z})=\boldsymbol{0}$$

- **First-order IRF to $d\boldsymbol{Z}$:**

$$\boldsymbol{H}(\boldsymbol{U},\boldsymbol{Z})=0\Rightarrow\boldsymbol{H}_{\boldsymbol{U}}d\boldsymbol{U}+\boldsymbol{H}_{\boldsymbol{Z}}d\boldsymbol{Z}=0\Leftrightarrow d\boldsymbol{U}=-\boldsymbol{H}_{\boldsymbol{U}}^{-1}\boldsymbol{H}_{\boldsymbol{Z}}d\boldsymbol{Z}$$

  All other variables: Easy with $d\boldsymbol{U}$ (and $d\boldsymbol{Z}$)

- **Non-linear transition path:** Fully solve $\boldsymbol{H}(\boldsymbol{U},\boldsymbol{Z}_{ss}+d\boldsymbol{Z})=\boldsymbol{0}$ with quasi-Newton method initially using $\boldsymbol{H}_{\boldsymbol{U}}$ for updating guesses
- **Calculation of derivatives:**
  1. Derivatives of aggregate equations found with numerical derivative
  2. Derivatives of household block found with fake-news algorithm
     - 2.1 Backward: Only time to shock matters
     - 2.2 Forward: Apply chain-rule to reduce computation burden (parallelization is an alternative, especially in more complex models)

Aggregate consumption function

- **Household problem**

$$
\begin{aligned}
v_t(z_t, a_{t-1}) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(z_{t+1}, a_t) \right] \\
\text{s.t. } a_t + c_t &= (1+r_t)a_{t-1} + (1-\tau_t) w_t \ell_t z_t \\
\log z_{t+1} &= \rho_z \log z_t + \psi_{t+1} \quad , \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \, \mathbb{E}[z_t] = 1 \\
a_t &\geq 0
\end{aligned}
$$

- **Consumption function:**

$$
\begin{aligned}
C_t^{hh} &= C^{hh} \left(\{ r_s, \tau_s, w_s, \ell_s \}_{s \geq 0} \right) \Rightarrow \\
\boldsymbol{C}^{hh} &= C^{hh}(\boldsymbol{r}, \boldsymbol{\tau}, \boldsymbol{w}, \boldsymbol{\ell}) \Rightarrow \\
d\boldsymbol{C}^{hh} &= \boldsymbol{M}^r d\boldsymbol{r} + \boldsymbol{M}^\tau d\boldsymbol{\tau} + \boldsymbol{M}^w d\boldsymbol{w} + \boldsymbol{M}^\ell d\boldsymbol{\ell}
\end{aligned}
$$

where $\boldsymbol{M}^x = \left[\partial C_t / \partial x_s \right]_{ts}$ or with $Z_t \equiv (1-\tau_t) w_t \ell_t$

$$
d\boldsymbol{C}^{hh} = \boldsymbol{M}^r d\boldsymbol{r} + \boldsymbol{M} d\boldsymbol{Z}
$$

Intertemporal Keynesian Cross I

- **Linear production:** $Y_t = w_t L_t$ with $\int \ell_t d\boldsymbol{D}_t$
- **Taxes:** $T_t = \tau_t w_t L_t$
- **Unions:** $\ell_t = L_t$
- **Disposable income:** $Z_t = Y_t - T_t$
- **Market clearing:**

$$\begin{aligned} d\boldsymbol{Y} &= d\boldsymbol{G} + d\boldsymbol{C}^{hh} \\ &= d\boldsymbol{G} + \boldsymbol{M}^r d\boldsymbol{r} + \boldsymbol{M} d\boldsymbol{Z} \\ &= d\boldsymbol{G} + \boldsymbol{M}^r d\boldsymbol{r} + \boldsymbol{M}\left[d\boldsymbol{Y} - d\boldsymbol{T}\right] \end{aligned}$$

Requirement: $\sum_{t=0}^{\infty}(1+r_{ss})^{-t}(dG_t - dT_t) = 0$
Alternative: Specify explicit policy rule for $d\boldsymbol{T}$

Intertemporal Keynesian Cross II

- **Market clearing:**

$$d\boldsymbol{Y} = d\boldsymbol{G} + \boldsymbol{M}^r d\boldsymbol{r} + \boldsymbol{M}\left[d\boldsymbol{Y} - d\boldsymbol{T}\right]$$

- **Intertemporal Keynesian Cross:** $d\boldsymbol{r} = 0$ implies

$$\begin{aligned} d\boldsymbol{Y} &= d\boldsymbol{G} + \boldsymbol{M}\left[d\boldsymbol{Y} - d\boldsymbol{T}\right] \Leftrightarrow \\ \left[\boldsymbol{I} - \boldsymbol{M}\right] d\boldsymbol{Y} &= d\boldsymbol{G} - \boldsymbol{M} d\boldsymbol{T} \Leftrightarrow \\ d\boldsymbol{Y} &= \mathcal{M}\left[d\boldsymbol{G} - \boldsymbol{M} d\boldsymbol{T}\right] \end{aligned}$$

where $\mathcal{M}\left[\boldsymbol{I} - \boldsymbol{M}\right] = \boldsymbol{I}$ (left-inverse)

  - **Uniqueness:** Can be proven (single $\mathcal{M}$ imply bounded solution)
  - **Takeaway:** iMPCs in $\boldsymbol{M}$ (+ policy rule determining $d\boldsymbol{T}$) are sufficient statistics for fiscal multipliers.

From HANC to HANK

- **HANC:** *The real interest must ensure market clearing*
  1. Production: Endogenous labor supply and/or capital accumulation
  2. Exchange: Endowment or exogenous labor supply $\div$ capital acc.
- **Canonical HANK with demand-determined labor supply:** *CB controls real rate and output adjust to market clearing*
  1. Households and government: Exogenous labor supply without capital accumulation $\sim$ HANC exchange economy
  2. Central bank and production: Taylor-rule and New Keynesian Wage Phillips Curve link real rate to output
  3. Labor market and union/search-and-matching: Link output to idiosyncratic income process
- **Sticky prices, but fully flexible wages**: Empirically unrealistic large wealth effects on labor supply
- **More sticky wages than profits:** Pro-cyclical profits

Transition path

- **Transition path**
  1. Temporary shock: Non-linear IRF
  2. Permanent shock: Transition to new stationary equilibrium
- **Welfare analysis conditional on initial state**
  Example: Looser borrowing constraint
  1. Larger choice set $\Rightarrow$ higher utility
  2. Lower utility in stationary equilibrium possible due to dissaving...

  Details: Policy tools (e.g. restricted by functional forms) and degree of commitment (time-consistency)
- **More complex models:** New mechanisms? Inequality outcomes?
  1. Life-cycle and demographics
  2. Spatial dimension
  3. Structural transformation (agriculture $\rightarrow$ manufacturing $\rightarrow$ services)
  4. AI and automation
  5. Climate
- **Computationally:** *Imprecise solution become an issue*

Limitation of sequence space approach

- **First-order approximation:**
  1. Around deterministic steady state (DSS)
  2. No state and size-dependence of IRFs
     (the non-linear transition path to a MIT shock has size-dependence)
- **Second-order approximation:**
  1. Around stochastic steady state (SSS)
  2. Can be derived from MIT shock (Auclert, Rognlie, Straub, SED2025)
  3. Approximation need not be better
- **Full dynamic equilibrium:**
  1. Distribution affected by previous aggregate shocks
     (e.g. distribution of fixed rate mortgage contracts)
  2. Feedback effects from policy
     (e.g. lower self-insurance with counter-cyclical UI)
  3. Directly allow for fully non-linear dynamics
     (e.g. zero lower bound, capacity constraints etc.)

Business cycle questions

- **Importance of heterogeneity for business cycles:**
  1. Some aggregate neutrality results - still distributional concerns
  2. Size of mechanisms are different - cash-flow effects important
  3. High MPC and precautionary saving become of central importance
- **Investment:** Still not strongly demand-driven
  1. Positive correlation from investment shocks
     (due to cash-flow effects and less precautionary saving)
  2. Negative correlation from consumption (due higher real rate)
- **Other open questions:**
  1. Imperfect information and behavioral biases (e.g. wage negotiation in inflation surges, experimental/survey evidence)
  2. Asymmetric fluctuations (plucking models, efficient unemployment)
  3. Fundamental uncertainty (inflation due to demand or supply?)
  4. Coordination problems (Walras $\rightarrow$ search-and-matching $\rightarrow$? $\sim$ agent-based)

Exam

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- **Exam:** Similar to assignment I-II
    1. **Starting point:**
        1.1 Model definition in .py-files which can run
        1.2 Notebook with pre-defined structure and some plotting functions
    2. **Code questions:** Add a few lines of code in the right place
    3. **Math questions:** Derive analytical expressions
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  1. More comprehensive starting point
  2. More step-by-step questions

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- **Code not working:** Describe in words what you have tried to do

Hand-in

You should hand-in a single zip-file. The zip-file should have the following folder and file structure.

Assignment_I
Assignment_I.pdf – with text and all results
*files for producing the results

Assignment_II
Assignment_II.pdf – with text and all results
files for producing the results

Assignment_III
Assignment_III.pdf – with text and all results
files for producing the results

Exam
Exam.pdf – with text and all results
files for producing the results

Preparing for the exam

1. Read through slides and study the accompanying code and go through the exercises
2. Read the documentation for GEModelTools
3. Glance at the central papers
4. Optimize your Assignments I-III
5. Look at the Exam for 2022, 2023 and 2024 (in the Old folder)

GEModelTools

GEModelTools

- **Files:** MODELNAME.py importing blocks.py, household_problem.py, steady_state.py
- **Steady state:** .find_ss() ⇒ .ss
- **Tests:** .test_ss(), .test_path(),
- **Jacobians:** .compute_jacs(skip_hh=False,skip_shocks=False) ⇒ .jac and .jac_hh
- **First-order IRFs:** .find_IRFs(shocks=shocks) ⇒ .IRF
  1. shocks = ['x'] set x to AR(1) with jump_x and rho_x
  2. shocks = {dx:dxpath} set x to ss.x+dxpath
- **Non-linear response to MIT shock:** .find_transition_path(shocks=shocks) ⇒ .path
- **Decomposition of household response:** .decompose_hh_path
- **Troubleshooting in GEModelTools:** Check examples in notebook repository + look in documentation or source code

HANK-sticky-wages\simplified

Let us look at the code.

1. Compute the linear and non-linear IRFs to a AR(1) real rate shock
2. Verify clearing of asset or goods market does not matter
3. Plot household Jacobians
4. Plot linear and non-linear IRFs manually
5. Introduce multiple β types
6. Calibrate B_{ss} to an MPC target
7. Change to a balanced budget
8. Compute the linear and non-linear IRF to a custom real rate shock
9. Target a specific output response with government spending

Exam from 2023

Exam from 2023

Look at the exam from 2023