

Consumption-Saving

Adv. Macro: Heterogenous Agent Models

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Introduction

- **Generations of models:**
 1. **Permanent income hypothesis (PIH)** (Friedman, 1957)
or life-cycle model (Modigliani and Brumberg, 1954)
 2. **Buffer-stock consumption model**
Deaton (1991, 1992); Carroll (1992, 1997, 2019)
 3. **Multiple-asset buffer-stock consumption models**
e.g. Kaplan and Violante (2014); Harmenberg and Öberg (2021)
- **Consumption-and-saving over the life-cycle dynamic**
e.g. Gourinchas and Parker (2002); Druedahl and Martinello (2022)
- **Empirical MPCs and income risk**
e.g. Fagereng et. al. (2021); Guvenen et. al. (2021)

Book: **The Economics of Consumption**, Jappelli and Pistaferri (2017)

Plan

1. Introduction
2. PIH
3. Buffer-stock
4. 3-periods
5. EGM
6. Income process
7. Full household problem
8. Misc
9. Summary

PIH



$$v_0 = \max_{\{c_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t)$$

s.t.

$$a_t = (1 + r)a_{t-1} + wz_t - c_t$$

$$a_{T-1} \geq 0$$

- **Variables:**

Consumption: c_t

Productivity: z_t

End-of-period savings: a_t (*no debt at death*)

- **Parameters:**

Discount factor: β

Wage: w

Interest rate: r (define $R \equiv 1 + r$ as interest factor)

It is a *static* problem

$$v_0 = \max_{\{c_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t)$$

s.t.

$$a_t = (1 + r)a_{t-1} + wz_t - c_t$$

$$a_{T-1} \geq 0$$

- It is a *static* problem:

1. **Information:** z_t is known for all t at $t = 0$
2. **Target:** Discounted utility, $\sum_{t=0}^{T-1} \beta^t u(c_t)$
3. **Behavior:** Choose c_0, c_1, \dots, c_{T-1} *simultaneously*
4. **Solution:** Sequence of consumption *choices* $c_0^*, c_1^*, \dots, c_{T-1}^*$

- **Substitution** implies *Intertemporal Budget Constraint* (IBC)

$$\begin{aligned}
 a_{T-1} &= Ra_{T-2} + wz_{T-1} - c_{T-1} \\
 &= R(Ra_{T-3} + wz_{T-2} - c_{T-2}) + wz_{T-1} - c_{T-1} \\
 &= R^T a_{-1} + \sum_{t=0}^{T-1} R^{T-1-t} (wz_t - c_t)
 \end{aligned}$$

- Use **terminal condition** $a_{T-1} = 0$ (equality due utility max.)

$$R^{-(T-1)} a_{T-1} = 0 \Leftrightarrow \sum_{t=0}^{T-1} R^{-t} c_t = s_0 + h_0$$

where $s_0 \equiv Ra_{-1}$ (after-interest assets)
 and $h_0 \equiv \sum_{t=0}^{T-1} R^{-t} wz_t$ (human capital)

$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t u(c_t) + \lambda \left[\sum_{t=0}^{T-1} R^{-t} c_t - s_0 - h_0 \right]$$

- **First order conditions:**

$$\forall t : 0 = \beta^t u'(c_t) - \lambda(1+r)^{-t} \Leftrightarrow u'(c_t) = -\lambda(\beta R)^{-t}$$

- **Euler-equation** for $k \in \{1, 2, \dots\}$:

$$\frac{u'(c_t)}{u'(c_{t+k})} = \frac{-\lambda(\beta R)^{-t}}{-\lambda(\beta R)^{-(t+k)}} = (\beta R)^k$$

Consumption choice

- **CRRA:** $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$ imply Euler-equation

$$\frac{c_0^{-\sigma}}{c_t^{-\sigma}} = (\beta R)^t \Leftrightarrow c_t = (\beta R)^{\frac{t}{\sigma}} c_0$$

Constant consumption if:

1. $\beta R = 1$
 2. $\frac{1}{\sigma} \rightarrow 0$ (zero intertemporal elasticity of substitution)
- Insert **Euler** into **IBC** to get consumption choice

$$\sum_{t=0}^{T-1} \left((\beta R)^{1/\sigma} R^{-1} \right)^t c_0 = s_0 + h_0 \Leftrightarrow$$
$$c_0^* = \frac{1 - (\beta R)^{1/\sigma} R^{-1}}{1 - ((\beta R)^{1/\sigma} R^{-1})^T} (s_0 + h_0)$$

Infinite horizon: $T \rightarrow \infty$

- **Infinite horizon** for $(\beta R)^{1/\sigma} R^{-1} < 1$:

Let $T \rightarrow \infty$ to get

$$c_0^* = \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right) (s_0 + h_0)$$

- **Constant income**, $\forall z_t = 1$:

$$c_0^* = \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right) \left(Ra_{-1} + \frac{R}{R-1}w\right)$$

- **Consume annuity value** if also $\beta R = 1$

$$c_0^* = ra_{-1} + w$$

Propensities to consume ($\beta R \approx 1, z_t \approx 1$)

$$c_0^* \approx \frac{r}{1+r} \left((1+r)a_{-1} + \sum_{t=0}^{\infty} \frac{wz_t}{(1+r)^t} \right) \approx ra_{-1} + w$$

Different types of shocks:

1. MPC of *windfall* income: $\frac{\partial c_0}{\partial s_0} \approx \frac{r}{1+r}$
2. MPC of *future* income change: $\frac{\partial c_0}{\partial wz_t} \approx \frac{r}{1+r} (1+r)^{-t}$
3. MPC of *permanent* income change: $\frac{\partial c_0}{\partial w} \approx \frac{r}{1+r} \frac{1}{1-(1+r)^{-1}} = 1$

Dynamic affects: The same when $\beta R = 1$, for all $k > 0$

$$\begin{aligned} \frac{\partial c_k}{\partial s_0} &= \frac{\partial c_0}{\partial s_0} \\ \frac{\partial c_k}{\partial wz_t} &= \frac{\partial c_0}{\partial wz_t} \\ \frac{\partial c_k}{\partial w} &= \frac{\partial c_0}{\partial w} \end{aligned}$$

Savings ($\beta R = 1$)

- **Constant savings** $z_t = 1$:

$$c_t = ra_{t-1} + w \Rightarrow a_t = Ra_{t-1} + w - c_t = a_{t-1}$$

1. Decreasing savings with $\beta R < 1$: $c_t \uparrow \Rightarrow a_t < a_{t-1}$
2. Increasing savings with $\beta R > 1$: $c_t \downarrow \Rightarrow a_t > a_{t-1}$

- **Same consumption if NPV of wz_t is unchanged**
e.g. higher in one period, and lower in another

$$\frac{r}{1+r} \sum_{t=0}^{\infty} \frac{z_t}{(1+r)^t} = 1$$

\Rightarrow *savings change with income*

Empirical MPCs

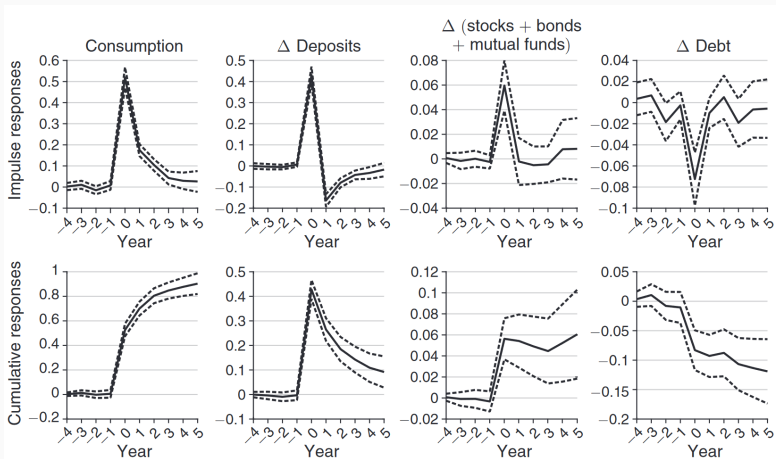


FIGURE 2. DYNAMIC HOUSEHOLD RESPONSES TO LOTTERY PRIZES

Source: Fagereng et. al. (2021)

Initial liquidity/borrowing constraint

- Implied period 0 **savings** are: $a_0 = s_0 + wz_0 - c_0$
- Hard **borrowing constraint**: $a_0 \geq -wb$
- **Maximum consumption**: $\bar{c}_0 = s_0 + wz_0 + wb$
- **Optimal consumption**: Constrained or unconstrained.

$$c_0^* = \min \left\{ \bar{c}_0, \left(1 - \frac{(\beta R)^{1/\sigma}}{R} \right) (s_0 + h_0) \right\}$$

- **Empirical realism.** MPC of constrained is one

$$c_0^* = \bar{c}_0 \Rightarrow \frac{\partial c_0^*}{\partial s_0} = \frac{\partial \bar{c}_0}{\partial s_0} = 1$$

- **Technical issue:** *Borrowing constraints further in the future complicates the analytical solution considerably.*

Buffer-stock

Uncertainty and always borrowing constraint

$$v_0(z_0, a_{-1}) = \max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

s.t.

$$a_t = (1 + r)a_{t-1} + wz_t - c_t$$

$$z_{t+1} \sim \mathcal{Z}(z_t)$$

$$a_t \geq -wb$$

$$\lim_{t \rightarrow \infty} (1 + r)^{-t} a_t \geq 0 \quad [\text{No-Ponzi game}]$$

- **Stochastic income** from 1st order Markov-process, \mathcal{Z}
- **A true dynamic problem:**
 1. **Information:** z_t is revealed period-by-period
 2. **Target:** Expected discounted utility, $\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$
 3. **Behavior:** Choose c_t *sequentially* as information is revealed
 4. **Solution:** Sequence of consumption *functions*, $c_t^*(z_t, a_{t-1})$

- **Substitution** still implies:

$$R^{-(T-1)}a_{T-1} = 0 \Leftrightarrow \sum_{t=0}^{\infty} R^{-t}c_t = s_0 + h_0$$

- **What if $T \rightarrow \infty$?** We must have $\lim_{T \rightarrow \infty} R^{-(T-1)}a_{T-1} = 0$
 1. $\lim_{T \rightarrow \infty} R^{-(T-1)}a_{T-1} > 0$: Consumption can be increased
 2. $\lim_{T \rightarrow \infty} R^{-(T-1)}a_{T-1} < 0$: Violates No-Ponzi game condition
- For $T \rightarrow \infty$ we have the **IBC**:

$$\sum_{t=0}^{\infty} R^{-t}c_t = Ra_{-1} + \sum_{t=0}^{\infty} R^{-t}wz_t$$

Natural borrowing limit

- Denote **minimum possible productivity** by \underline{z}
- **Consumption must be non-negative** \Rightarrow
interest payments must be less than minimum income

$$c_t \geq 0 \Rightarrow r(-a_t) \leq w\underline{z} \Leftrightarrow a_t \geq -\frac{w\underline{z}}{r}$$

If debt was larger it would in the worst case ($\forall z_t = \underline{z}$) grow without bound even with zero consumption ($\forall c_t = 0$)

$$a_0 = -\frac{w\underline{z}}{r} - \Delta$$

$$a_1 = (1+r)a_0 + w\underline{z} = a_0 - (1+r)\Delta$$

$$a_2 = (1+r)a_1 + w\underline{z} = a_0 - (1+r)^2\Delta$$

$$\vdots$$

- **Natural borrowing constraint:** $a_t \geq \underline{a} = -w \min \left\{ b, \frac{\underline{z}}{r} \right\}$

Euler-equation from variation argument

- **Case I:** If $u'(c_t) > \beta R \mathbb{E}_t [u'(c_{t+1})]$:
Increase c_t by marginal $\Delta > 0$, and lower c_{t+1} by $R\Delta$
 1. **Feasible:** Yes, if $a_t > \underline{a}$
 2. **Utility change:** For $\Delta \rightarrow 0$ $u'(c_t) + \beta(-R) \mathbb{E}_t [u'(c_{t+1})] > 0$
- **Case II:** If $u'(c_t) < \beta R \mathbb{E}_t [u'(c_{t+1})]$:
Lower c_t by marginal $\Delta > 0$, and increase c_{t+1} by $R\Delta$
 1. **Feasible:** Yes (always)
 2. **Utility change:** For $\Delta \rightarrow 0$ $u'(c_t) + \beta R \mathbb{E}_t [u'(c_{t+1})] > 0$
- **Conclusion:** By contradiction
 1. **Constrained:** $a_t = \underline{a}$ and $u'(c_t) \geq \beta R \mathbb{E}_t [u'(c_{t+1})]$, or
 2. **Unconstrained:** $a_t > \underline{a}$ and $u'(c_t) = \beta R \mathbb{E}_t [u'(c_{t+1})]$
- **Sufficiency:** Harder (\sim convexity of the choice set)

Special case I: Quadratic utility

- **Quadratic utility:** $u(c_t) = -\frac{1}{2}(\bar{c} - c)^2$ with $\beta R = 1$ and »large« \bar{c}
- **Euler-equation:** *Consumption = expected future consumption*

$$(\bar{c} - c_t) = \mathbb{E}_t [(\bar{c} - c_{t+k})] \Leftrightarrow c_t = \mathbb{E}_t [c_{t+k}]$$

- Use **IBC** in expectation to get **consumption function**:

$$\sum_{t=0}^{\infty} R^{-t} \mathbb{E}_0 [c_t] = Ra_{-1} + \sum_{t=0}^{\infty} R^{-t} w \mathbb{E}_0 [z_t] \Rightarrow$$
$$c^*(z_t, a_{t-1}) = c_0 = ra_{-1} + \frac{r}{R} \sum_{t=0}^T R^{-t} w \mathbb{E}_0 [z_t]$$

where we formally disregard the borrowing constraint

- **Certainty equivalence:** *Only expected income matter.*

Special case II: CARA utility

- **CARA utility:** $u(c_t) = -\frac{1}{\alpha} e^{-\alpha c}$
- **Productivity is absolute random walk:**

$$z_t = z_{t-1} + \psi_t$$

$$\psi_t \sim \mathcal{N}(0, \sigma_\psi^2)$$

- **Consumption function (see proof):**

$$c^*(a_{t-1}, z_t) = ra_{t-1} + wz_t - \frac{\log(\beta R)^{\frac{1}{\alpha}} + \alpha \frac{\sigma_\psi^2}{2}}{r^2}$$

where we formally disregard the borrowing constraint

- **Precautionary saving:** $\sigma_\psi^2 \uparrow$ implies $c_t^* \downarrow$ for given z_t and a_{t-1}
 \Rightarrow *accumulation of buffer-stock*

Dynamic solution: Bellman's Principle of Optimality

- Go back to a **finite horizon** problem with T periods
- Value function**, v_t : Defined *recursively* from $v_T(\bullet) = 0$

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1 + r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

- Policy function**, c_t^* : Is the same as

$$c_t^*(z_t, a_{t-1}) = \arg \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1 + r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

- Euler-equation:**

- FOC: $c_t^{-\sigma} = \beta \mathbb{E}_t[v_{t,a}(z_{t+1}, a_t)]$
- Envelope: $v_{t,a}(z_t, a_{t-1}) = (1 + r)c_t^{-\sigma}$ (fix a_t)

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1 + r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

1. **State variables:** z_t and a_{t-1}
2. **Control (choice) variable:** c_t
3. **Continuation value:** $\beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$
4. **Parameters:** r , w , and stuff in $u(\bullet)$

Note: Straightforward to extend to more goods, more assets or other states, more complex risk, bounded rationality etc.

Infinite horizon: $T \rightarrow \infty$

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1 + r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

- **Contraction mapping result:** *If β is low enough (strong enough impatience) then the value and policy functions converge to $v(z_t, a_{t-1})$ and $c^*(z_t, a_{t-1})$ for large enough T*
- **In practice:**
 1. Make arbitrary initial guess (e.g. $v_T = 0$)
 2. Solve backwards until the value (and/or policy function) does not change anymore (relative to chosen tolerance)

3-periods

3-period model

- **Expected discounted utility:** $v(z_0, a_{-1}) = \mathbb{E}_0 \sum_{t=0}^2 \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$
- **Income = wage \times productivity + transfer:**

$$y_t = wz_t + \chi_t$$

- **Cash-on-hand, savings and borrowing constraint:**

$$m_t = (1 + r)a_{t-1} + y_t$$

$$a_t = m_t - c_t$$

$$a_t \geq \underline{a}$$

- **Stochastic transition:** $\Pr[z_{t+1}|z_t] = \pi_t(z_t, z_{t+1})$ such that

$$\Pr[z_{t+1} = 1 | z_t = 1] = \pi$$

$$\Pr[z_{t+1} = 1 - \Delta | z_t = 1] = \Pr[z_{t+1} = 1 + \Delta | z_t = 1] = \frac{1 - \pi}{2}$$

$$\Pr[z_{t+1} = z_t | z_t \in \{1 - \Delta, 1 + \Delta\}] = 1$$

Bellman equation

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t [v_{t+1}(z_{t+1}, a_t)]$$

s.t.

$$y_t = wz_t + \chi_t$$

$$m_t = (1+r)a_{t-1} + y_t$$

$$a_t = m_t - c_t$$

$$\Pr[z_{t+1}|z_t] = \pi_t(z_t, z_{t+1})$$

$$a_t \geq \underline{a}$$

where

$$v_3(z_3, a_2) = 0$$

- **Discretization:** All state variables belong to discrete sets \equiv *grids*,

$$z_t \in \mathcal{G}_z = \{z^0, z^1, \dots, z^{\#z-1}\}$$

$$a_t \in \mathcal{G}_a = \{a^0, a^1, \dots, a^{\#a-1}\}$$

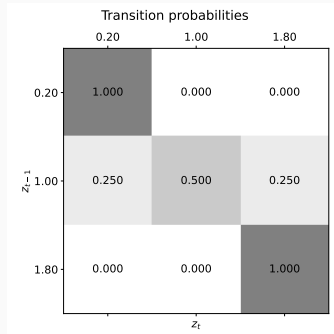
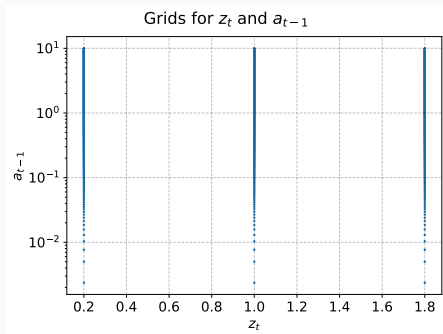
$$a^0 = \underline{a}$$

- **Expectation:** Numerical integration by

$$\mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)] = \sum_{z_{t+1} \in \{1-\Delta, 1, 1+\Delta\}} \pi_t(z_t, z_{t+1}) v_{t+1}(z_{t+1}, a_t)$$

- **ConSav:** `grids.nonlinspace`, `grids.equilogspace`
- **ConSavNotebooks:** 04. Tools/03. Grids.ipynb

Grids and transition probabilities



The size of risk is scaled by Δ

Baseline: $\Delta = 0.8$

Low risk: $\Delta = 0.4$

Linear interpolation

- **Linear interpolation** (function approximation):

1. Assume v_{t+1} is known on $\mathcal{G}_z \times \mathcal{G}_a$ (tensor product)
2. Evaluate $v_{t+1}(z^{i_z}, a)$ for arbitrary a by

$$\begin{aligned}\check{v}_{t+1}(z^{i_z}, a) &= \text{baseline} + \text{slope} \times \text{distance} \\ &= v_{t+1}(z^{i_z}, a^{\iota}) + \omega(a - a^{\iota})\end{aligned}$$

where

$$\omega \equiv \frac{v_{t+1}(z^{i_z}, a^{\iota+1}) - v_{t+1}(z^{i_z}, a^{\iota})}{a^{\iota+1} - a^{\iota}}$$

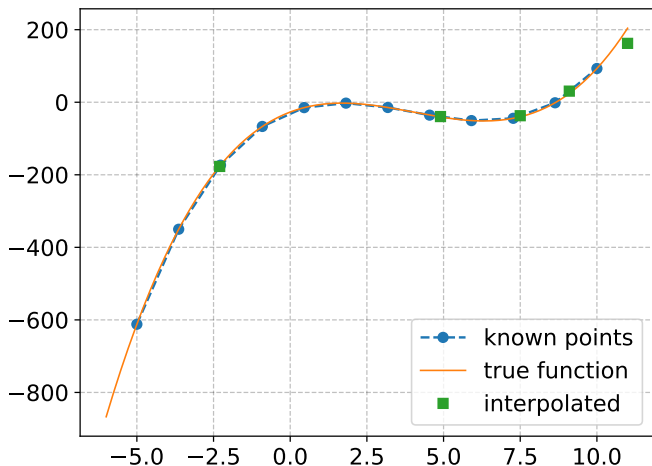
$$\iota \equiv \text{largest } i_a \in \{0, 1, \dots, \#_a - 2\} \text{ such that } a^{i_a} \leq a$$

- **ConSav:** `linear_interp.interp1d`

- **ConSavNotebooks:**

04. Tools/01. Linear interpolation.ipynb

Linear interpolation



Value function iteration (VFI)

- **Maximize value-of-choice:**

$$v_t(z^{i_z}, a^{i_a}) = \max_{c_t} v_t(z^{i_z}, a^{i_a} | c_t)$$

$$\text{with } c_t \in [0, (1+r)a^{i_a} + wz^{i_z} + \chi_t + \underline{a}]$$

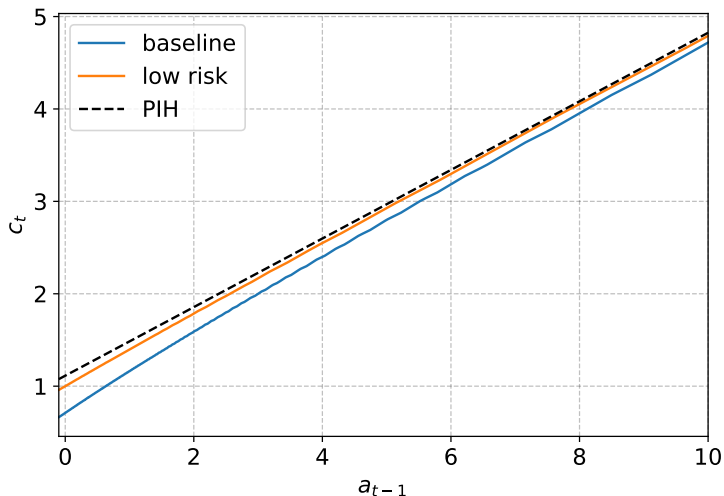
$$v_t(z^{i_z}, a^{i_a} | c_t) = u(c_t) + \beta \sum_{i_{z+1}=0}^{\#_z-1} \pi(i_z, i_{z+1}) \check{v}_{t+1}(z^{i_z}, a_t)$$

$$\text{with } a_t = (1+r)a^{i_a} + wz^{i_z} + \chi_t - c_t$$

- **Inner loop:** For each grid point in $\mathcal{G}_z \times \mathcal{G}_a$ find $c_t^*(z_t, a_{t-1})$ and therefore $v_t(z_t, a_{t-1})$ with a *numerical optimizer*
- **Outer loop:** Backwards from $t = T - 1$ (note $\underline{v}_T = 0$, or known)
- **ConSav+QuantEcon:** Various optimizers in numba
- **ConSavNotebooks:** 04. Tools/02. Optimization.ipynb

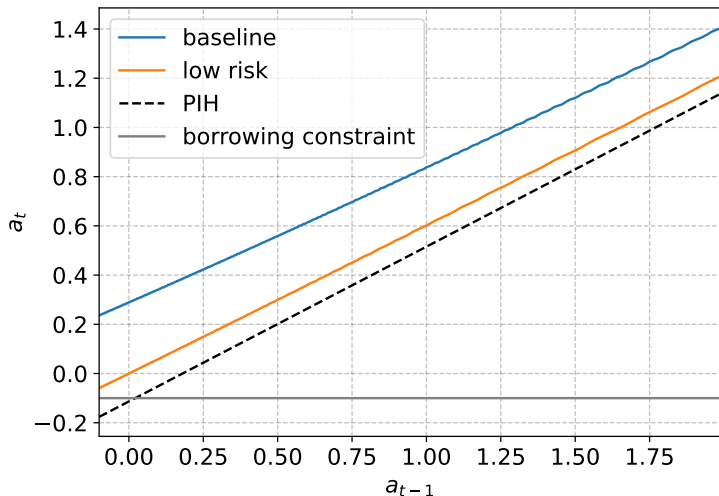
Consumption function

consumption function in $t = 0$



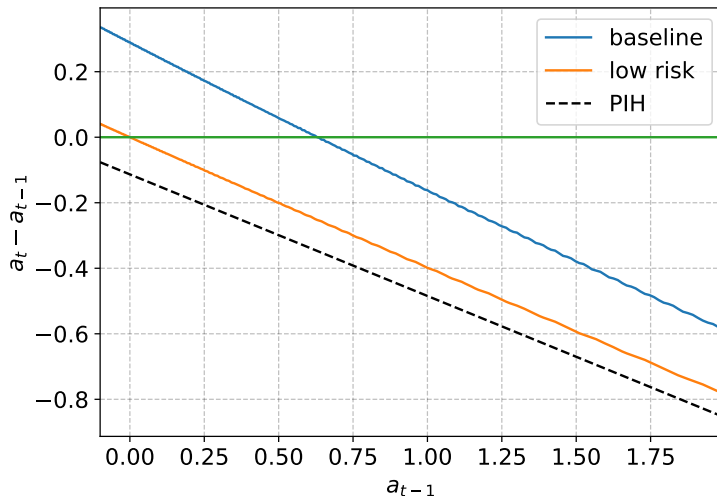
Savings function

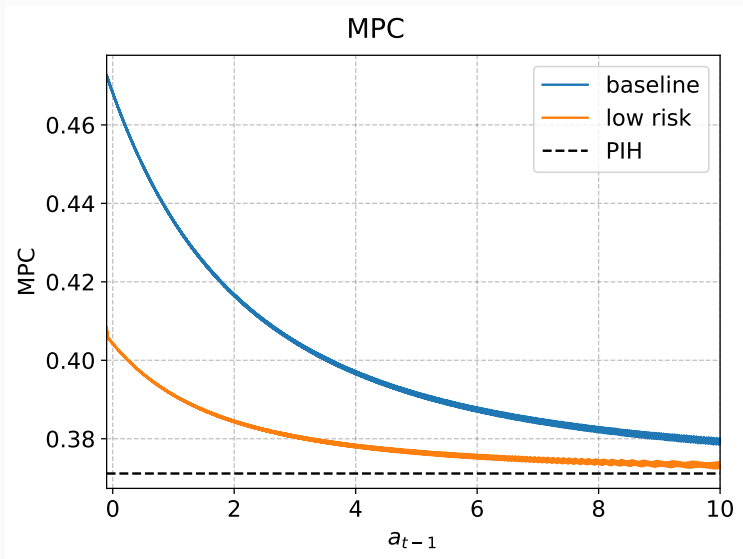
savings function in $t = 0$



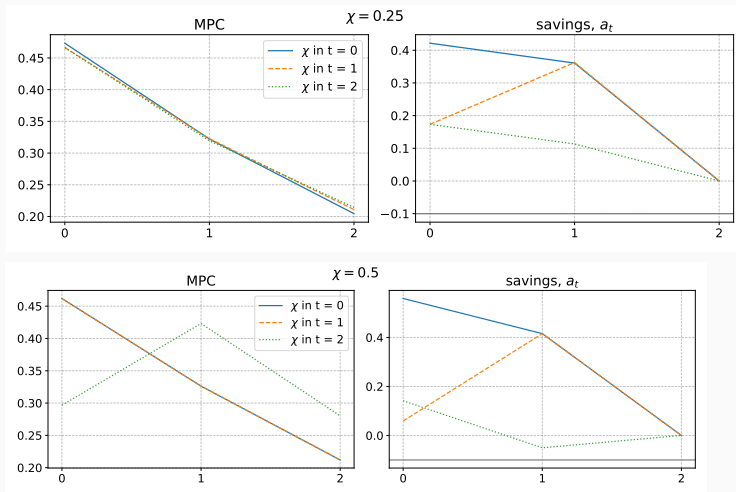
Change in savings function

savings diff. function in $t = 0$





Intertemporal MPC



Note: No wealth effect as $r = 0$

- **Notebook:** 01_ConSavModel_3periods/ConSavModel.ipynb

- **Consumption lower than under PIH and concave in assets**

Intuition: *Precautionary saving motive is relatively larger for asset poor households because income risk is the same for everybody*

Implications:

1. Windfall gives safety and increases average consumption
⇒ MPC decreasing in assets
2. Attraction towards a buffer-stock target $a_t = a_{t-1}$ despite $\beta R < 1$
3. Larger effective discounting of future income
(extreme: no effect of future income changes if constrained before)

Numerical Monte Carlo simulation

- **Initial distribution:** Draw $z_{i,-1}$ and $a_{i,-1}$ for $i \in \{0, 1, \dots, N-1\}$
- **Simulation:** Forwards in time from $t = 0$ and in each time period
 1. Draw z_{it} given transition probabilities
 2. Use linear interpolation to evaluate

$$c_{it} = \check{c}_t^*(z_{it}, a_{it-1})$$

$$a_{it} = (1 + r)a_{it-1} + wz_{it} - c_{it}$$

- **Review:**
 - **Pro:** Simple to implement
 - **Con:** Computationally costly and introduces randomness
- **Infinite horizon:**
 1. Assume z_{it} has an ergodic distribution
 2. Ergodic distribution of a_{it} around buffer-stock target

- **Value Function Iteration (VFI)**

1. Solve all consumption-saving models
2. Accurate with dense enough grids
3. Relatively simple code and easy to run in parallel
4. Finding optimal choices is the computational bottleneck
(especially with multi-starts in non-convex models)

EGM



Time iteration

- **Replace numerical optimization with root-finding**
- **Time iteration:** For each a_{t-1} and z_t find c_t to solve the Euler-equation

$$c_t^{-\sigma} = \beta(1+r)\mathbb{E}_t[c_{t+1}^{-\sigma}]$$

Note: *Necessary and sufficient* (for interior choices, else $a_t = \underline{a}$)

- **EGM:** No need for any numerical optimization or root-finding

Endogenous grid-point method (EGM)

1. Calculate **post-decision marginal value of cash**:

$$q(z^{i_z}, a^{i_a}) = \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z, i_{z+}} c_+^*(z^{i_{z+}}, a^{i_a})^{-\sigma}$$

2. **Invert Euler-equation**:

$$c(z^{i_z}, a^{i_a}) = (\beta(1+r)q(z^{i_z}, a^{i_a}))^{-\frac{1}{\sigma}}$$

3. **Endogenous cash-on-hand**:

$$m(z^{i_z}, a^{i_a}) = a^{i_a} + c(z^{i_z}, a^{i_a})$$

4. **Consumption function**: Calculate $m = (1+r)a^{i_{a-}} + wz^{i_z}$

- 4.1 Binding constraint: If $m \leq m(z^{i_z}, a^0)$ then

$$c^*(z^{i_z}, a^{i_{a-}}) = m + \underline{a}$$

- 4.2 Interior choice: Else

$$c^*(z^{i_z}, a^{i_{a-}}) = \text{interpolate } m(z^{i_z}, m) \rightarrow c(z^{i_z}, m)$$

Income process

- **Persistent-transitory income process:**

$$z_t = \tilde{z}_t \xi_t, \quad \log \xi_t \sim \mathcal{N}(\mu_\xi, \sigma_\xi)$$
$$\log \tilde{z}_{t+1} = \rho_z \log \tilde{z}_t + \psi_{t+1}, \quad \psi_{t+1} \sim \mathcal{N}(\mu_\psi, \sigma_\psi)$$

1. Transitory shock: ξ_t
2. Persistent shock: ψ_t
3. Normalization using μ_ψ and μ_ξ : $\mathbb{E}[z_t] = \mathbb{E}[\tilde{z}_t] = 1$

- **ConSav:** `qudarature.log_normal_gauss_hermite`
- **ConSavNotebook:** 04. Tools/04. Quadrature.ipynb

Transition probabilities

- **Discretization of ξ_t :** Derive \mathcal{G}_ξ and π_{i_ξ} given σ_ξ using Gauss-Hermite quadrature

$$x \sim \mathcal{N}(\mu, \sigma^2) : \mathbb{E}[h(x)] \approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^n \omega_i h(\sqrt{2}\sigma x_i + \mu)$$

where nodes, x_i , and weights, ω_i , have analytical expressions

- **Discretization of \tilde{z}_t :** Derive $\mathcal{G}_{\tilde{z}}$ and $\pi_{i_{\tilde{z}-}, i_{\tilde{z}}}$ given $\rho_z < 1$ and σ_ψ (using a method such as Tauchen (1986) or Rouwenhorst (1995))
If $\rho_z = 1$: Also use quadrature here.
- **Combined:** Derive $\mathcal{G}_z = \mathcal{G}_{\tilde{z}} \times \mathcal{G}_\xi$ (tensor product) and use independence of \tilde{z}_t and ξ_t to get transition probabilities π_{i_{z-}, i_z} (kronecker product)
- **ConSav:** `markov.log_rouwenhorst`, `markov.log_tauschen`
- **ConSavNotebook:** 04. Tools/05. Markov.ipynb

Full household problem

- **Utility maximization** for household i :

$$v_0(z_t, a_{t-1}) = \max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t.

$$a_t = (1 + r_t)a_{t-1} + wz_t - c_{it}$$

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1}, \quad \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \quad \mathbb{E}[z_t] = 1$$

$$a_t \geq -bw$$

- **Value function:**

$$v(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t [v(z_{t+1}, a_t)]$$

s.t.

$$a_t = (1 + r)a_{t-1} + wz_t - c_t$$

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1}$$

$$a_t \geq -bw$$

- **Same Euler-equation**
- **Borrowing constrained** has

$$c_{it} = m_{it} - bw$$

where $m_t = (1 + r)a_{t-1} + wz_t$

Recursive formulation II

- **Beginning-of-period value function**

$$\underline{v}(z_{t-1}, a_{t-1}) = \mathbb{E} [v(z_t, a_{t-1}) \mid z_{t-1}]$$

- **Value function:**

$$v(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \underline{v}(z_t, a_t)$$

s.t.

$$a_t = (1 + r)a_{t-1} + wz_t - c_t$$

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1}$$

$$a_t \geq -bw$$

- **Why do this?** *Only one interpolation per guess of c_t*

- **Problem with Monte Carlo simulation:** *Stochastic fluctuations in average wealth \Rightarrow stochastic error in asset market clearing condition.*
- **Alternative – Histogram:** Distribution as array of probabilities
 1. Beginning-of-period: \underline{D}_t over z_{it-1} and a_{it-1}
 2. Productivity transition: $D_t = \Pi'_z \underline{D}_t$ over z_{it} and a_{it-1}
 3. Savings transition: $\underline{D}_{t+1} = \Lambda' D_t$ where Λ derives from $a^*(z_t, a_{t-1})$

Numerical histogram simulation

- **Initial distribution:** Choose $\underline{D}_0(z_{-1}, a_{-1})$, which is defined on $\mathcal{G}_z \times \mathcal{G}_a$ and sum to 1 \equiv *histogram*

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- **Simulation:** Forwards in time from $t = 0$ and in each time period
 1. **Distribute stochastic mass:** For each i_z and i_{a-} calculate

$$D_t(z^{i_z}, a^{i_{a-}}) = \sum_{i_{z-}=0}^{\#_z-1} \pi_{i_{z-}, i_z} \underline{D}_t(z^{i_{z-}}, a^{i_{a-}})$$

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2. **Initial zero mass:** Set $\underline{D}_{t+1}(z^{i_z}, a^{i_a}) = 0$ for all i_z and i_a

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 - 3.1 Find $\iota \equiv$ largest $i_a \in \{0, 1, \dots, \#_a - 2\}$ such that $a^{i_a} \leq a^*(z^{i_z}, a^{i_{a-}})$

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Numerical histogram simulation

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 - 3.3 Increment $\underline{D}_{t+1}(z^{i_z}, a^\iota)$ with $\omega D_t(z^{i_z}, a^{i_{a-}})$

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 - 3.3 Increment $\underline{D}_{t+1}(z^{i_z}, a^{\iota})$ with $\omega D_t(z^{i_z}, a^{i_{a-}})$
 - 3.4 Increment $\underline{D}_{t+1}(z^{i_z}, a^{\iota+1})$ with $(1 - \omega) D_t(z^{i_z}, a^{i_{a-}})$

- **Toy example:** simple_histogram_simulation.xlsx
 - **Grids:** $\mathcal{G}_z = \{\underline{z}, \bar{z}\}$ and $\mathcal{G}_a = \{0, 1\}$
 - **Transition matrix:** $\pi_{0,0} = \pi_{1,1} = 0.5$
 - **Policy function:**
 - Low income: $a^*(\underline{z}, 0) = a^*(\underline{z}, 1) = 0$
 - High income: Let $a^*(\bar{z}, 0) = 0.5$ and $a^*(\bar{z}, 1) = 1$
 - **Initial distribution:** $\underline{D}_0(z_{it}, a_{it-1}) = \begin{cases} 1 & \text{if } z_{it} = \underline{z} \text{ and } a_{it} = 0 \\ 0 & \text{else} \end{cases}$
 - **Task:** Calculate by hand the transitions to

$$\underline{D}_0, \underline{D}_1, \underline{D}_1, \dots$$

Implementation

- **Toy example:** simple_histogram_simulation.xlsx
 - **Grids:** $\mathcal{G}_z = \{\underline{z}, \bar{z}\}$ and $\mathcal{G}_a = \{0, 1\}$
 - **Transition matrix:** $\pi_{0,0} = \pi_{1,1} = 0.5$
 - **Policy function:**
 - Low income: $a^*(\underline{z}, 0) = a^*(\underline{z}, 1) = 0$
 - High income: Let $a^*(\bar{z}, 0) = 0.5$ and $a^*(\bar{z}, 1) = 1$
 - **Initial distribution:** $\underline{D}_0(z_{it}, a_{it-1}) = \begin{cases} 1 & \text{if } z_{it} = \underline{z} \text{ and } a_{it} = 0 \\ 0 & \text{else} \end{cases}$
 - **Task:** Calculate by hand the transitions to

$$\underline{D}_0, \underline{D}_1, \underline{D}_1, \dots$$

- **Comparison with Monte Carlo:** See 02_ConSavModel_infConSavModel/
 1. **Pro:** Computationally efficient and no randomness
 2. **Con:** Introduces a non-continuous distribution

Misc



1. Life-cycle (I)

- **Basically:**

1. Born, working, retired, die
2. Age-varying parameters (esp. income)

- **Add-ons:**

1. Labor supply, human capital, occupation
 2. Portfolio choice and entrepreneurship
 3. Family formation
 4. Health, mortality
- etc.

- **Good starting example:** »Life-Cycle Consumption and Children: Evidence from a Structural Estimation«, Jørgensen (2017)

1. Life-cycle (II)

Paper: Gourinchas and Parker (2021)

Life-cycle consumption-saving model with retirement

- Young households:
Save for precautionary reasons (buffer)
- Older households:
Save for retirement (life-cycle)

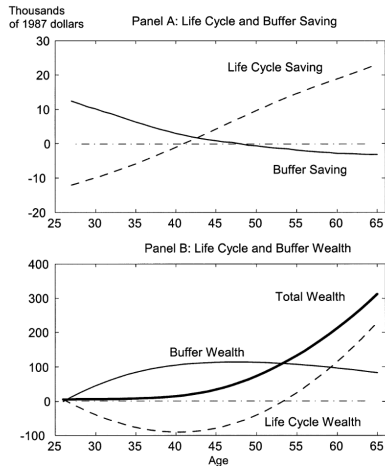
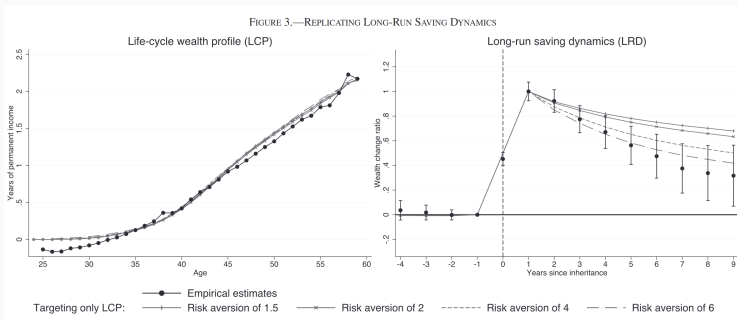


FIGURE 7.—The role of risk in saving and wealth accumulation.

1 Life-cycle (III)

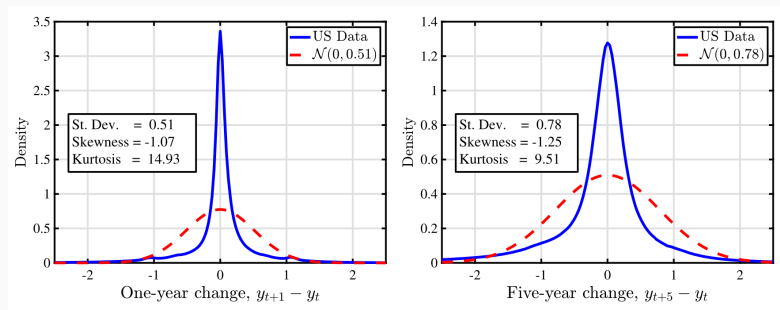
- **Natural experiment:** Wealth depletion after sudden inheritance
- **Results:**
 1. Life-cycle profile of wealth fitted for many levels of risk-aversion (by varying the discount factor)
 2. Fast wealth depletion requires high risk-aversion (or high perceived risk)



Source: Druedahl and Martinello (2022)

2. More realistic income risk (I)

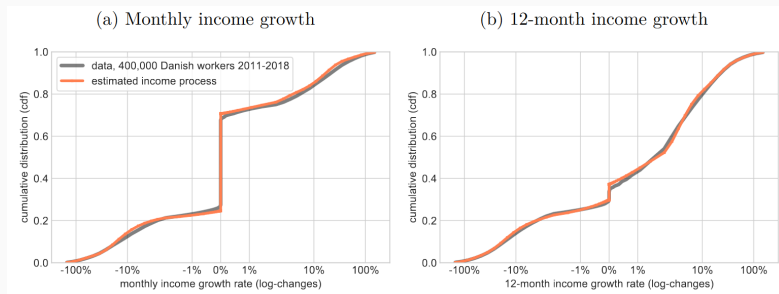
Annual earnings-changes are far from log-normal:



Source: Guvenen et. al. (2021)

2. More realistic income risk (II)

Many with zero-growth month-month:



Source: Druedahl et. al. (2021)

3. Epstein-Zin

$$\begin{aligned}v_t(z_t, m_t) &= \max_{c_t} \left[(1 - \beta) \cdot c_t^{1-\sigma} + \beta \cdot w_{t+1}^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \\ \text{s.t.} \quad w_{t+1} &\equiv \mathbb{E}_t \left[v_{t+1}(z_{t+1}, m_{t+1})^{1-\rho} \right]^{\frac{1}{1-\rho}} \\ m_{t+1} &= (1 + r)(m_t - c_t) + y_{t+1}\end{aligned}$$

- **Preferences:**

1. Patience: β
2. Intertemporal substitution: σ
3. Risk-aversion: ρ

- **Euler-equation:** $c_t^{-\sigma} = \beta R \cdot \mathbb{E}_t \left[c_{t+1}^{-\sigma} \cdot \left(\frac{w_{t+1}}{v_{t+1}} \right)^{\rho-\sigma} \right]$

1. FOC: $0 = v_t^\sigma \cdot \left[(1 - \beta) \cdot c_t^{-\sigma} - \beta R \cdot w_{t+1}^{\rho-\sigma} \cdot \mathbb{E}_t \left[v_{t+1}^{-\rho} \cdot \frac{\partial v_{t+1}}{\partial m_{t+1}} \right] \right]$
2. Envelope condition: $\frac{\partial v_t(z_t, m_t)}{\partial m_t} = v_t^\sigma \cdot (1 - \beta) \cdot c_t^{-\sigma}$

4. Deep learning

- **Curse of dimensionality:**
 1. Many states
 2. Many choices
 3. Many shocks
- **Deep (reinforcement) learning:**
 1. Approximate value and policy functions with *neural networks*
 2. Approximate on simulation sample rather than on grid
 3. Automatic differentiation (backpropagation) and GPUs for speed
- **Examples:** Maliar and Maliar (2021) and Azinovic and Scheidegger (2022)
- **Working paper:** Druedahl and Røpke (2025)
Python package: [EconDLSolvers](#)

Summary

Summary and what's next

- **This lecture:**

1. Consumption-saving models (precautionary-saving, buffer-stock target, intertemporal MPCs)
2. Basic numerical dynamic programming (discretization, numerical integration, interpolation, VFI)
3. EGM (time iteration, invert Euler-equation)
4. Simulation (monte carlo, histogram)

- **Next:** *Stationary equilibrium*

- **You should:**

1. Study the code from this lecture
2. Glance at Aiyagari (1994),
»Uninsured Idiosyncratic Risk and Aggregate Saving«