

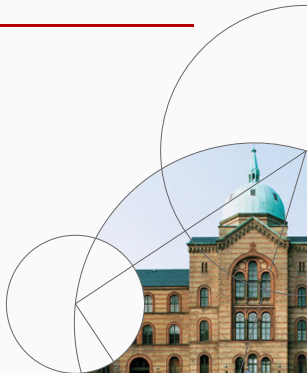


3. Stationary Equilibrium

Adv. Macro: Heterogenous Agent Models

Raphaël Huleux

2025



Introduction

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 1. Partial equilibrium
 2. No interactions

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- **Value function** (at decision)

$$v_t(\beta_i, z_{it}, a_{it-1}) = \max_{c_t} u(c_t) + \beta \underline{v}_{t+1}(\beta_i, z_{it}, a_{it})$$

s.t.

$$\ell_{it} = z_{it}$$

$$a_{it} = (1 + r_t)a_{it-1} + w_t \ell_{it} - c_{it} + \Pi_t$$

$$\log z_{it+1} = \rho_z \log z_{it} + \psi_{it+1}$$

$$a_{it} \geq 0$$

- **Beginning-of-period value function** (before shock realization):

$$\underline{v}_t(\beta_i, z_{it-1}, a_{it-1}) = \mathbb{E} [v_t(\beta_i, z_{it}, a_{it-1}) \mid \beta_i, z_{it-1}, a_{it-1}]$$

Distributions and aggregates

- Household policy function x^* where $x \in \{a, c, \ell\}$ function of:
 - Individuals states $(\beta_i, z_{it}, a_{it-1})$
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$$\chi_t^{hh} \left(\{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t} \right) = \int x_t^*(\beta_i, z_{it}, a_{it-1}, \{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t}) d\mathbf{D}_t$$

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- \Rightarrow If we know aggregates (w_t, Π_t, r_t) can calculate aggregate household behavior (consumption or savings)

Ramsey-recap

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- **Now:** Recap of the Ramsey model

- **Production function:** $Y_t = F(\Gamma_t, K_{t-1}, L_t)$ [note timing of capital]
where Γ_t is technology

Ramsey: Firms

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- **Profits:** $\Pi_t = Y_t - w_t L_t - r_t^K K_{t-1}$
- **Profit maximization:** $\max_{K_{t-1}, L_t} \Pi_t$
 1. Rental rate: $\frac{\partial \Pi_t}{\partial K_{t-1}} = 0 \Leftrightarrow r_t^K = F_K(\Gamma_t, K_{t-1}, L_t)$
 2. Real wage: $\frac{\partial \Pi_t}{\partial L_t} = 0 \Leftrightarrow w_t = F_L(\Gamma_t, K_{t-1}, L_t)$

With CRS we get zero profits: $\Pi_t = 0 \Rightarrow$

$$Y_t = w_t L_t + r_t^K K_{t-1} \text{ [functional income distribution]}$$

Ramsey: Zero-profit mutual fund

- Introduce **mutual fund**
 - Takes savings A_{t-1} from households and investment them in available assets
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- **Balance sheet:**

$$A_{t-1} = K_{t-1}$$

- **Utility maximization:**

$$v_0(A_{-1}^{hh}) = \max_{\{C_t^{hh}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t^{hh})$$

s.t.

$$C_t^{hh} + A_t^{hh} = (1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh}$$

Exogenous labor supply: $L_t^{hh} = 1$

- **Euler-equation** (implied by Lagrangian):

$$u'(C_t^{hh}) = \beta(1 + r_{t+1})u'(C_{t+1}^{hh})$$

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- **Walras:** Capital and labor market clears \Rightarrow goods market clears.

Start from

$$\begin{aligned}C_t^{hh} + A_t^{hh} &= (1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh} \\ \Leftrightarrow C_t^{hh} + I_t &= [(1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh} - A_t^{hh}] + (K_t - (1 - \delta)K_{t-1}) \\ &= [(1 + r_t)K_{t-1} + w_t L_t - K_t] + (K_t - (1 - \delta)K_{t-1}) \\ &= r_t^K K_{t-1} + w_t L_t \\ &= Y_t\end{aligned}$$

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- **Note:** Means that we can check if we have solved the numerical model correctly by:
 - Impose two of the market clearing conditions
 - Then check the third market clearing condition (should be zero)

- **Simplified form:**

$$\begin{aligned}u'(C_t^{hh}) &= \beta(1 + F_K(\Gamma_t, K_t, 1) - \delta)u'(C_{t+1}^{hh}) \\K_t &= (1 - \delta)K_{t-1} + F(\Gamma_t, K_{t-1}, 1) - C_t^{hh}\end{aligned}$$

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- **Extended form:**

$$r_t^K = F_K(\Gamma_t, K_{t-1}, L_t)$$

$$w_t = F_L(\Gamma_t, K_{t-1}, L_t)$$

$$r_t = r_t^K - \delta$$

$$A_t = K_t$$

$$A_t^{hh} = (1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}$$

$$u'(C_t^{hh}) = \beta(1 + r_{t+1})u'(C_{t+1}^{hh})$$

$$A_t = A_t^{hh}$$

$$L_t = L_t^{hh}$$

Ramsey: As an equation system

Eqs. system with unknowns $\{K_t, L_t, r_t^K, w_t, r_t, A_t, A_t^{hh}, C_t^{hh}\}_{t=0}^{\infty}$ and eqs:

$$\begin{bmatrix} r_t^K - F_K(\Gamma_t, K_{t-1}, L_t) \\ w_t - F_L(\Gamma_t, K_{t-1}, L_t) \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ A_t^{hh} - ((1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}) \\ u'(C_t^{hh}) - \beta(1 + r_{t+1})u'(C_{t+1}^{hh}) \\ A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0, 1, \dots\}, \text{ given } K_{-1} \end{bmatrix} = 0$$

Ramsey: Steady state

- **Euler-equation** can be solved for K_{ss} :

$$u'(C_{ss}) = \beta(1 + F_K(\Gamma_{ss}, K_{ss}, 1) - \delta)u'(C_{ss}) \Leftrightarrow$$
$$F_K(K_{ss}, 1) = \frac{1}{\beta} - 1 + \delta$$

- **Accumulation equation + goods mkt. clearing** then implies C_{ss} :

$$K_{ss} = (1 - \delta)K_{ss} + F(\Gamma_{ss}, K_{ss}, 1) - C_{ss} \Leftrightarrow$$
$$C_{ss} = (1 - \delta)K_{ss} + F(\Gamma_{ss}, K_{ss}, 1) - K_{ss}$$

HANC



- **Model blocks:**

1. **Firms:** Rent capital from mutual fund and hire labor from the households, produce with given technology, and sell output goods
2. **Zero-profit mutual funds:** Own capital and rent it to firms, take deposits and pay return to household
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3. The Standard Incomplete Market (SIM) model

Heterogeneous households

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s.t.

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- **Incomplete markets due to borrowing constraint**

(fancy words: partial self-insurance, lack of Arrow-Debreu securities)

- **Value function** (at decision)

$$v_t(\beta_i, z_{it}, a_{it-1}) = \max_{c_t} u(c_t) + \beta \underline{v}_{t+1}(\beta_i, z_{it}, a_{it})$$

s.t.

$$\ell_{it} = z_{it}$$

$$a_{it} = (1 + r_t)a_{it-1} + w_t \ell_{it} - c_{it} + \Pi_t$$

$$\log z_{it+1} = \rho_z \log z_{it} + \psi_{it+1}$$

$$a_{it} \geq 0$$

- **Beginning-of-period value function** (before shock realization):

$$\underline{v}_t(\beta_i, z_{it-1}, a_{it-1}) = \mathbb{E} [v_t(\beta_i, z_{it}, a_{it-1}) \mid \beta_i, z_{it-1}, a_{it-1}]$$

Distributions and aggregates

- Household policy function x^* where $x \in \{a, c, \ell\}$ function of:
 - Individuals states $(\beta_i, z_{it}, a_{it-1})$
 - Aggregates (w_t, Π_t, r_t)

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$$\chi_t^{hh} \left(\{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t} \right) = \int x_t^* (\beta_i, z_{it}, a_{it-1}, \{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t}) d\mathbf{D}_t$$

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 - Aggregate X_t^{hh} is only a function of $\{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t}$ in GE as long as exogenous states don't change
- \Rightarrow If we know aggregates (w_t, Π_t, r_t) can calculate aggregate household behavior (consumption or savings)

Equation system

$$\begin{bmatrix} r_t^K - F_K(\Gamma_t, K_{t-1}, L_t) \\ w_t - F_L(\Gamma_t, K_{t-1}, L_t) \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ A_t^{hh} - \int a_t d\mathbf{D}_t \\ L_t^{hh} - \int \ell_t d\mathbf{D}_t \\ \underline{\mathbf{D}}_{t+1} - \Lambda'_t \Pi'_z \underline{\mathbf{D}}_t \\ a_t - a_t^* \\ \forall t \in \{0, 1, \dots\}, \text{ given } \underline{\mathbf{D}}_0 \end{bmatrix} = \mathbf{0}$$

- **Note:** Much larger system compared to Ramsey due to last 2 eqs.

Equation system

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 - \mathbf{D}_t, a_t^* define mass and optimal savings policy at **the individual level**
 - Standard Ramsey model: 8 eqs. per period
 - HANC with $N_\beta = 3, N_z = 7, N_a = 300$:
 $8 + 2 \times 3 \times 7 \times 300 = 12.608$ per period

Market clearing

- **Capital market:** $K_t = A_t = \int a_t^*(\beta_i, z_{it}, a_{it-1}) d\mathbf{D}_t$
- **Labor market:** $L_t = \int \ell_t^*(\beta_i, z_{it}, a_{it-1}) d\mathbf{D}_t = \int z_{it} d\mathbf{D}_t = 1$
- **Goods market:** $Y_t = \int c_t^*(\beta_i, z_{it}, a_{it-1}) d\mathbf{D}_t + I_t$
- **Walras:** Capital and labor market clears \Rightarrow goods market clears

$$\begin{aligned} C_t^{hh} + I_t &= \int c_{it}^* d\mathbf{D}_t + [K_t - (1 - \delta)K_{t-1}] \\ &= \int [(1 + r_t)a_{it-1} + w_t z_{it} - a_{it}] d\mathbf{D}_t \\ &= [(1 + r_t)K_{t-1} + w_t L_t - K_t] + [K_t - (1 - \delta)K_{t-1}] \\ &= r_t^K K_{t-1} + w_t L_t \\ &= Y_t \end{aligned}$$

Stationary Equilibrium

Stationary equilibrium - equation system

The **stationary equilibrium** satisfies

$$\begin{bmatrix} r_{ss}^K - F_K(\Gamma_{ss}, K_{ss}, L_{ss}) \\ w_{ss} - F_L(\Gamma_{ss}, K_{ss}, L_{ss}) \\ r_{ss} - (r_{ss}^K - \delta) \\ A_{ss} - K_{ss} \\ A_{ss} - A_{ss}^{hh} \\ L_{ss} - L_{ss}^{hh} \\ A_{ss}^{hh} - \int a_{ss} d\mathbf{D}_{ss} \\ L_{ss}^{hh} - \int \ell_{ss} d\mathbf{D}_{ss} \\ \underline{\mathbf{D}}_{ss} - \Lambda'_{ss} \Pi'_Z \underline{\mathbf{D}}_{ss} \\ a_{ss} - a_{ss}^* \end{bmatrix} = \mathbf{0}$$

Note : Households still move around »inside« the distribution due to idiosyncratic shocks. Does not affect aggregates due to »law of large numbers«

Stationary equilibrium - more verbal definition

Given technology Γ_{ss}

1. Quantities K_{ss} and L_{ss} ,
2. prices r_{ss} and w_{ss} (always $\Pi_{ss} = 0$),
3. the distribution \mathbf{D}_{ss} over β_i , \mathbf{z}_{it} and \mathbf{a}_{it-1}
4. and the policy functions a_{ss}^* , ℓ_{ss}^* and c_{ss}^*

are such that

1. Households maximize expected utility (policy functions)
2. Firms maximize profits (prices)
3. \mathbf{D}_{ss} is the invariant distribution implied by the household problem
4. Mutual fund balance sheet is satisfied
5. The capital market clears
6. The labor market clears
7. The goods market clears

Direct implementation (K guess)

Technology: $F(K, L) = \Gamma K^\alpha L^{1-\alpha}$

Root-finding problem in K_{ss} with the objective function:

1. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
2. Calculate $r_{ss} = \alpha \Gamma_{ss} (K_{ss})^{\alpha-1} - \delta$ and $w_{ss} = (1 - \alpha) \Gamma_{ss} (K_{ss})^\alpha$
3. Solve infinite horizon household problem *backwards*, i.e. find \mathbf{a}_{ss}^*
4. Simulate households *forwards* until convergence, i.e. find \mathbf{D}_{ss}
5. Return $K_{ss} - \mathbf{a}_{ss}^{*'} \mathbf{D}_{ss}$

Note: $\mathbf{a}_{ss}^{*'} \mathbf{D}_{ss} = \sum_i a_{i,ss}^* D_i$

Direct implementation (r guess)

Technology: $F(K, L) = \Gamma K^\alpha L^{1-\alpha}$

Root-finding problem in r_{ss} with the objective function:

1. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
2. Calculate $K_{ss} = \left(\frac{r_{ss} + \delta}{\alpha \Gamma_{ss}} \right)^{\frac{1}{\alpha-1}}$ and $w_{ss} = (1 - \alpha) \Gamma_{ss} (K_{ss})^\alpha$
3. Solve infinite horizon household problem *backwards*, i.e. find \mathbf{a}_{ss}^*
4. Simulate households *forwards* until convergence, i.e. find \mathbf{D}_{ss}
5. Return $K_{ss} - \mathbf{a}_{ss}^{*'} \mathbf{D}_{ss}$

Indirect implementation

Technology: $F(K, L) = \Gamma K^\alpha L^{1-\alpha}$

Consider Γ_{ss} and δ as »free« parameters:

1. Choose r_{ss} and w_{ss}
2. Solve infinite horizon household problem *backwards*, i.e. find \mathbf{a}_{ss}^*
3. Simulate households *forwards* until convergence, i.e. find \mathbf{D}_{ss}
4. Set $K_{ss} = \mathbf{a}_{ss}^{*'} \mathbf{D}_{ss}$
5. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
6. Set $\Gamma_{ss} = \frac{w_{ss}}{(1-\alpha)(K_{ss})^\alpha}$
7. Set $r_{ss}^K = \alpha \Gamma_{ss} (K_{ss})^{\alpha-1}$
8. Set $\delta = r_{ss}^k - r_{ss}$

Code

- **Preferences:** $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$
 1. Discount factors: $\beta \in \{0.965, 0.975, 0.985\}$ in equal pop. shares
 2. Relative risk aversion: $\sigma = 2$
- **Income:**
 1. AR(1): $\rho_z = 0.95$
 2. Std.: $\sigma_\psi = 0.30\sqrt{(1 - \rho_z^2)}$
- **Technology:** $F(K, L) = \Gamma K^\alpha L^{1-\alpha}$
 1. Capital share: $\alpha = 0.36$
 2. TFP: $\Gamma_{ss} = 1.082$
 3. Depreciation: $\delta = 0.193$
- **Steady state:**
 1. Prices: $r_{ss} = 0.01$ and $w_{ss} = 1$
 2. Quantities: $K_{ss}/Y_{ss} = 1.776$

⇒ **Code example in repo**

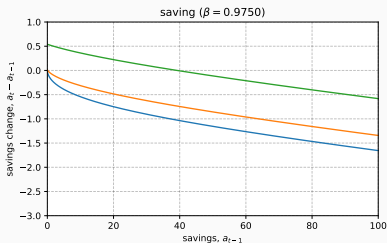
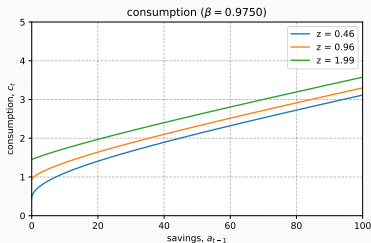
Consumption function

- Euler-equation still necessary for $a_{it} > 0$:

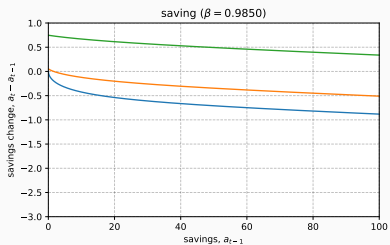
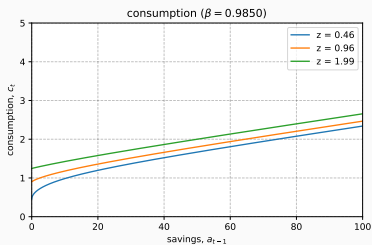
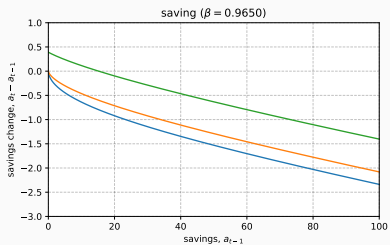
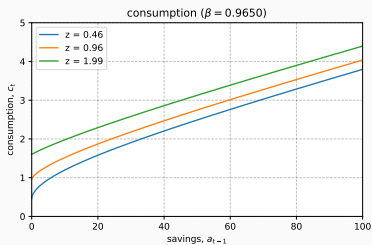
$$c_{it}^{-\sigma} = \beta_i(1 + r_{t+1})\mathbb{E}_t [c_{it+1}^{-\sigma}]$$

- Precautionary saving:

1. Low consumption for low cash-on-hand \rightarrow *buffer-stock target*
2. Steep slope for low cash-on-hand \rightarrow *high MPC*

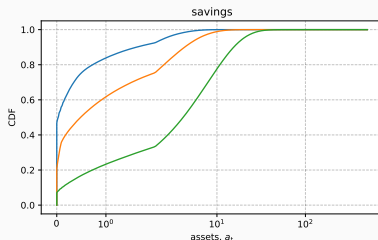
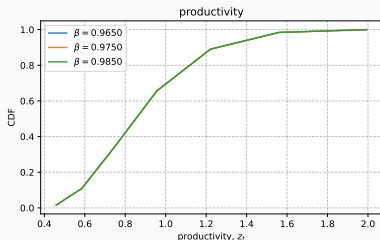


Low vs. high β_i



Distribution, D_t

- **Productivity:** Marginal distribution over only z_{it}
- **Savings:** Marginal distribution over a_{it} cond. on β_i



- **Drivers of wealth inequality:**
 1. Stochastic income
 2. Heterogeneous patience \rightarrow savings behavior

Steady state interest rate

- **Representative agent / complete markets:**

Derived from aggregate Euler-equation

$$C_t^{-\sigma} = \beta(1 + r_{t+1})C_{t+1}^{-\sigma} \Rightarrow C_{ss}^{-\sigma} = \beta(1 + r_{ss})C_{ss}^{-\sigma} \Leftrightarrow \beta = \frac{1}{1 + r_{ss}}$$

Steady state interest rate

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Derived from aggregate Euler-equation

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- **Heterogeneous agents:** *No such equation exists*

1. Euler-equation replaced by asset market clearing condition
2. Idiosyncratic income risk affects the steady state interest rate

| σ_ψ | PE ($r_{ss} = 1\%$), A^{hh} | GE, r_{ss} | GE, A^{hh} |
|---------------|---------------------------------|--------------|--------------|
| 0.09 | 2.78 | 1.00% | 2.78 |
| 0.14 | 7.39 | 0.12% | 2.97 |
| 0.19 | 13.68 | -1.11% | 3.30 |

Partial Equilibrium: Same interest rate.

General Equilibrium: Capital+labor market clearing.

Calibration

How to choose parameters?

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 1. **Informal:** Roughly match targets by hand
 2. **Formal:**
 - 2a. Solve root-finding problem
 - 2b. Minimize a squared loss function
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- **Complication:** *We must always solve for the steady state for each guess of the parameters to be calibrated*

Exercises

Exercise: HANCGovModel

- **No production.** No physical savings instrument
- **Households:** Get stochastic endowment z_{it} of consumption good
- **Government:**
 1. Choose government spending
 2. Collect taxes, τ_t , proportional to endowment
 3. Bonds: Pays 1 consumption good next period. Price is $p_t^B < 1$

$$p_t^B B_t + \int \tau_t z_{it} d\mathbf{D}_t = B_{t-1} + G_t$$
$$\tau_t = \tau_{ss} + \eta_t + \varphi (B_{t-1} - B_{ss})$$

where η_t is a tax-shifter

- **Market clearing:**

$$B_t = A_t^{hh}$$
$$C_t^{hh} + G_t = \int z_{it} d\mathbf{D}_t = 1$$

Exercise: Households

Households:

$$v_t(z_{it}, a_{it-1}) = \max_{c_{it}} \frac{c_{it}^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t [v_{it+1}(z_{it+1}, a_{it})]$$

$$\text{s.t. } p_t^B a_{it} + c_{it} = a_{it-1} + (1 - \tau_t) z_{it} \geq 0$$

$$\log z_{it+1} = \rho_z \log z_{it} + \psi_{it+1}, \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \mathbb{E}[z_{it}] = 1$$

Euler-equation:

$$c_{it}^{-\sigma} = \beta \frac{v_{a,t+1}(z_{it}, a_{it})}{p_t^B}$$

Envelope condition:

$$v_{a,t}(z_{it-1}, a_{it-1}) = c_{it}^{-\sigma}$$

Exercise: Questions

1. **Define the stationary equilibrium**
2. **Solve and simulate the household problem**
with $p_{ss}^B = 0.975$ and $\tau_{ss} = 0.12$.
3. **Find the stationary equilibrium**
with $G_{ss} = 0.10$ and $\tau_{ss} = 0.12$.
4. **What happens for $\tau_{ss} \in (0.11, 0.15)$?**
5. **When is average household utility maximized?**

Note: Full solution in repository folder
GEModelToolsNotebooks/HANCGovModel

Summary

Summary and next week

- **Today:**

1. The concept of a stationary equilibrium
2. Introduction to the **GEModelTools** package

- **Next week:** *Work on assignment*

- **Exercise/Homework:**

1. Work on completing the HANCGovModel exercise
2. Go through *Stationary-Equilibrium* notebook in repository
3. Assignment