CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

13. HANK-SAM

Adv. Macro: Heterogenous Agent Models

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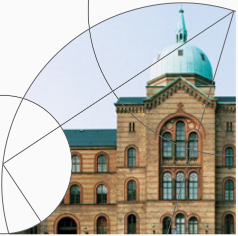

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Introduction

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- **From RANK to HANK:**
  1. Income effects more important relative to substitution effects
  2. Cash-flows more important relative to relative prices
- **Central:** High MPCs

  I. Idiosyncratic risk $+$ incomplete markets $\rightarrow$

  II. Precautionary saving and liquidity constraint $\rightarrow$

  III. Concave consumption function $\rightarrow$ high MPCs

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  New: *Endogenous fluctuations in idiosyncratic risk*

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  New: *Endogenous fluctuations in idiosyncratic risk*
- **Today:**

  GEModelTools: Model description of HANK-SAM

  Broer, Druedahl, Harmenberg and Öberg:

  2024: »Stimulus effects of common fiscal policies«
  2025: »The Unemployment-Risk Channel in Business-Cycle Fluctuations«

HANK-SAM

Overview

- Intermediate **producers:**
  1. Hire and fire in search-and-matching labor market
  2. Sell homogeneous good at price p_t^X.
- Wholesale **price-setters:**
  1. Set prices in monopolistic competition subject to adjustment costs
  2. Pay out dividends
- Final producers: Aggregate to final good
- **Government:**
  1. Pay transfers and unemployment insurance
  2. Collect taxes and issues debt
- **Central bank:** Sets nominal interest rate
- **Households:** Consume and save

Equilibrium dynamics

1. **Incomplete markets:** Unemployment risk $\rightarrow$ demand

   Complete markets / representative agent:
   Only total income matters

2. **Sticky prices:** Demand $\rightarrow$ profitability
3. **Frictional labor market:** Profitability $\rightarrow$ unemployment risk

Household problem

$$v_t(\beta_i, u_{it}, a_{it-1}) = \max_{c_{it}, a_{it}} \frac{c_{it}^{1-\sigma}}{1-\sigma} + \beta_i \mathbb{E}_t \left[v_{t+1}\left(\beta_i, u_{it+1}, a_{it}\right)\right]$$

$$\text{s.t. } a_{it} + c_{it} = (1+r_t)a_{it-1} + (1-\tau_t)y_t(u_{it}) + \text{div}_t + \text{transfer}_t$$

$$a_{it} \geq 0$$

1. **Dividends and government transfers:** div_t and transfer_t
2. **Real wage**: w_{ss}
3. **Income tax:** τ_t
4. **Separation rate** for employed: δ_{ss}
5. **Job-finding rate** for unemployed: $\lambda_t^{u,s} s(u_{it-1})$
   (where $s(u_{it-1})$ is exogenous search effectiveness)
6. US-style duration-dependent **UI system:**
   a) High replacement rate $\overline{\phi}$, first $\overline{u}$ months
   b) Low replacement rate $\underline{\phi}$, after $\overline{u}$ months

Income process

- **Income** is

$$y_{it}(u_{it}) = w_{ss} \cdot \begin{cases} 1 & \text{if } u_{it} = 0 \\ \overline{\phi}\mathrm{UI}_{it} + (1 - \mathrm{UI}_{it})\underline{\phi} & \text{else} \end{cases}$$

where the share of the month with UI is

$$\mathrm{UI}_{it} = \begin{cases} 0 & \text{if } u_{it} = 0 \\ 1 & \text{else if } u_{it} < \overline{u} \\ 0 & \text{else if } u_{it} > \overline{u} + 1 \\ \overline{u} - (u_{it} - 1) & \text{else} \end{cases}$$

- **Note:** *Hereby $\overline{u}$ becomes a continuous variable.*

Transition probabilities

- **Beginning-of-period value function:**

$$\underline{v}_t\left(\beta_i, u_{it-1}, a_{it-1}\right) = \mathbb{E}\left[v_t(\beta_i, u_{it}, a_{it-1}) \,\middle|\, u_{it-1}, a_{it-1}\right]$$

- **Grid:** $u_{it} \in \{0, 1, \ldots, \#_u - 1\}$
- **Employed** with $u_{it-1} = 0$: $u_{it} = \begin{cases} 0 & \text{with prob. } 1 - \delta_{ss} \\ 1 & \text{with prob. } \delta_{ss} \end{cases}$
- **Unemployed** with $u_{it-1} = 1$:

$$u_{it} = \begin{cases} 0 & \text{with prob. } \lambda_t^{u,s} s(u_{it-1}) \\ u_{it-1} + 1 & \text{with prob. } 1 - \lambda_t^{u,s} s(u_{it-1}) \end{cases}$$

  Trick: $u_{it} = \min\left\{u_{it-1} + 1, \#_u - 1\right\}$

- **All unemployed search:** $s(u_{it-1}) = \begin{cases} 0 & \text{if } u_{it-1} = 0 \\ 1 & \text{else} \end{cases}$

Aggregation

- **Distributions:**
  1. Beginning-of-period: $\underline{\boldsymbol{D}}_t$ over β_i, u_{it-1} and a_{it-1}
  2. At decision: $\boldsymbol{D}_t$ over β_i, u_{it} and a_{it-1}
- **Stochastic (time-varying) transition matrix:** $\Pi_{t,z} = \Pi_z(\lambda_t^u)$
- **Deterministic savings policy matrix:** Λ_t'
- **Transition steps:**

$$\begin{aligned} \boldsymbol{D}_t &= \Pi_{t,z}' \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} &= \Lambda_t' \boldsymbol{D}_t \end{aligned}$$

- **Searchers:** $S_t = \int s(\beta_i, u_{it-1}, a_{it-1}) d\underline{\boldsymbol{D}}_t$
- **Savings:** $A_t^{hh} = \int a_t^*(\beta_i, u_{it}, a_{it-1}) d\boldsymbol{D}_t$
- **Consumption:** $C_t^{hh} = \int c_t^*(\beta_i, u_{it}, a_{it-1}) d\boldsymbol{D}_t$

EGM

- **Beginning-of-period value function:**

$$\underline{v}_{a,t}(\beta_i, u_{it-1}, a_{it-1}) = \mathbb{E}_t\left[v_{a,t}(\beta_i, u_{it}, a_{it-1})\right] = \mathbb{E}_t\left[(1+r_t)c_{it}^{-\sigma}\right]$$

- **Endogenous grid method:** Vary u_{it} and a_{it} to find

$$c_{it} = \left(\beta \underline{v}_{a,t+1}(\beta_i, u_{it}, a_{it})\right)^{-\frac{1}{\sigma}}$$
$$m_{it} = c_{it} + a_{it}$$

- **Consumption:** Use linear interpolation to find

$$c_t^*(\beta_i, u_{it}, a_{it-1}) \text{ with } m_{it} = (1+r_t)a_{it-1}$$

- **Savings:** $a^*(u_{it}, a_{it-1}) = (1+r_t)a_{it-1} - c_t^*(\beta_i, u_{it}, a_{it-1})$

- **Job value:**

$$V_t^j = p_t^X Z_t - w_{ss} + \beta^{\text{firm}} \mathbb{E}_t \left[(1 - \delta_{ss}) V_{t+1}^j \right]$$

- **Vacancy value:**

$$V_t^v = -\kappa + \lambda_t^v V_t^j + (1 - \lambda_t^v)(1 - \delta_{ss}) \beta^{\text{firm}} \mathbb{E}_t \left[V_{t+1}^v \right]$$

- **Free entry** implies

$$V_t^v = 0$$

Labor market dynamics

- **Labor market tightness** is given by

$$\theta_t = \frac{\text{vacancies}_t}{\text{searchers}_t} = \frac{v_t}{S_t}$$

- **Cobb-Douglas matching function**

$$\text{matches}_t = AS_t^{\alpha} v_t^{1-\alpha}, \ \ \alpha \in (0,1)$$

implies the job-filling and job-finding rates:

$$\lambda_t^{v} = \frac{\text{matches}_t}{v_t} = A\theta_t^{-\alpha}$$

$$\lambda_t^{u,s} = \frac{\text{matches}_t}{S_t} = A\theta_t^{1-\alpha}$$

- **Law of motion for unemployment:**

$$u_t = u_{t-1} + \delta_t(1 - u_{t-1}) - \lambda_t^{u,s} S_t$$

Price setters

- **Intermediate goods price:** p_t^X
- Dixit-Stiglitz **demand curve** $\Rightarrow$ **Phillips curve** relating marginal cost, $MC_t = p_t^x$, and **final goods price inflation**, $\Pi_t = P_t/P_{t-1}$,

$$1 - \epsilon + \epsilon p_t^x = \phi\pi_t(1+\pi_t) - \phi\beta^{\text{firm}}\mathbb{E}_t\left[\pi_{t+1}(1+\pi_{t+1})\frac{Y_{t+1}}{Y_t}\right]$$

  with output $Y_t = Z_t(1-u_t)$

- **Flexible price limit:** $\phi \to 0$
- **Dividends:**

$$\text{div}_t = Y_t - w_t(1-u_t)$$

Central bank

- **Taylor rule:**

$$1 + i_t \quad = \quad (1 + i_{ss}) \left(\frac{1 + \pi_t}{1 + \pi_{ss}} \right)^{\delta_\pi}$$

Government

- **Unemployment insurance:** $\Phi_t = w_{ss}\left(\overline{\phi}\mathrm{UI}_t^{hh} + \underline{\phi}\left(u_t - \mathrm{UI}_t^{hh}\right)\right)$
- **Total expenses:** $X_t = \Phi_t + G_t + \text{transfer}_t$
- **Total taxes:** $\text{taxes}_t = \tau_t\left(\Phi_t + w_{ss}(1 - u_t)\right)$
- **Government budget** is

$$q_t B_t = (1 + q_t \delta_q) B_{t-1} + X_t - \text{taxes}_t$$

  Long-term debt: Real payment stream is $1, \delta, \delta^2, \ldots$.
  The real bond price is q_t.

- **Tax rule:**

$$\tilde{\tau}_t = \frac{(1 + q_t \delta_q) B_{t-1} + X_t - q_{ss} B_{ss}}{\Phi_t + w_{ss}(1 - u_t)}$$

$$\tau_t = \omega \tilde{\tau}_t + (1 - \omega)\tau_{ss}$$

- **Transfers:** $\text{transfer}_t = -\text{div}_{ss}$

Financial markets: No arbitrage

1. **Pricing of government debt:**

$$\frac{1+\delta_q q_{t+1}}{q_t} = \frac{1+i_t}{1+\pi_{t+1}} = 1 + r_{t+1}$$

2. **Ex post real return:**

$$1 + r_t = \begin{cases} \frac{(1+\delta_q q_0)B_{-1}}{A^{hh}_{-1}} & \text{if } t = 0 \\ \frac{1+i_{t-1}}{1+\pi_t} & \text{else} \end{cases}$$

Market clearing

1. Asset market: $A_t^{hh} = q_t B_t$
2. Goods market: $Y_t = C_t^{hh} + G_t$

Tip: *You should be able to verify Walras' law.*

Shocks, target, unknowns

1. **Shocks:** G_t
2. **Unknowns:** p_t^X, V_t^j, v_t, u_t, S_t, π_t, $\mathrm{UI}_t^{\text{guess}}$
3. **Targets:**
    3.1 Error in Job Value
    3.2 Error in Vacancy Value
    3.3 Error in Law-of-Motion for u_t
    3.4 Error in Philips Curve
    3.5 Error in Asset Market Clearing
    3.6 $u_t = U_t^{hh} = \int 1\{u_{it} > 0\} d\boldsymbol{D}_t$
    3.7 $\mathrm{UI}_t^{\text{guess}} = \mathrm{UI}_t^{hh} = \int \mathrm{UI}_{it} d\boldsymbol{D}_t$

Steady State

1. **Zero inflation:** $\pi_t = 0$
2. **SAM:** Choose A and κ to ensure $\delta_{ss} = 0.02$ and $\lambda^s_{u,ss} = 0.30$
3. **HANK:** Enforce *asset market clearing*
   3.1 Set r_{ss}
   3.2 Calculate implied A^{hh}_{ss}
   3.3 Adjust G_{ss} so $q_{ss} B_{ss} = A^{hh}_{ss}$

Calibration

1. **Real interest rate:** $1 + r_t = 1.02^{\frac{1}{12}}$
2. **Households:** $\sigma = 2.0$
   30%: $\beta_i = \beta^{\text{HtM}} = 0$
   60%: $\beta_i = \beta^{\text{BS}} = 0.94^{\frac{1}{12}}$
   10%: $\beta_i = \beta^{\text{PIH}} = 0.975^{\frac{1}{12}}$
3. **Matching and bargaining:** $\alpha = 0.60$**,** $\theta = 0.60$, $w_{ss} = 0.90$
4. **Producers:** $\beta^{\text{firm}} = 0.975^{\frac{1}{12}}$
5. **Price-setters:** $\epsilon = 6$ and $\phi = 600$
6. **Monetary policy:** $\phi = 1.5$
7. **Government:**
   Tax: $\tau = 0.30$
   Debt: $\delta_q = 1 - \frac{1}{36}$ and $\omega = 0.05$
   UI: $\overline{\phi} = 0.70$, $\underline{\phi} = 0.40$, and $\overline{u} = 6$

Steady state analysis

In **steady state:**

1. Look at the consumption functions
2. Look at the distribution of savings
3. Look at how consumption evolves in unemployment

Look at **household Jacobians**

$$d\boldsymbol{C} = \boldsymbol{M}^{r}\, d\boldsymbol{r} + \boldsymbol{M}^{\lambda}\, d\boldsymbol{\lambda}^{\boldsymbol{u},\boldsymbol{s}} + \boldsymbol{M}^{\tau}\, d\boldsymbol{\tau} + \boldsymbol{M}^{\Delta}\, d\boldsymbol{\Delta}$$

where $\Delta_t = \text{div}_t + \text{transfer}_t$

1. Liquidity effects
2. Precautionary saving effects

Policy analysis

Shock: Consider a 1% shock to government consumption

$$G_t - G_{ss} = 0.80^t \cdot 0.01 \cdot G_{ss}$$

Look at **impulse responses** for**:**

1. Output
2. Unemployment (risk)
3. Tax rate

What drives the consumption response?

1. Interest rate
2. Tax rate
3. Job-finding rate
4. Dividends

Is the effect from the job-finding rate larger than an equivalent change in income causes by wages? Why?

Stimulus Effects of Common Fiscal Policies

Motivation and question

- **Resurgence of countercyclical fiscal policy for stabilization**
- **Often:** What's the size of *the* fiscal multiplier to gov. spending?
  Christiano–Eichenbaum–Rebelo (2011); Ramey (2016);
  Auclert–Rognlie–Straub (2024)
- In practice, **policy design varies widely (often transfers)**
  - Households: Cash transfers, UI increases, UI duration extensions
  - Firms: Retention and hiring subsidies
- **Research question**
  Which fiscal transfer policies are most cost effective?

A HA-NK-SAM model

HA: Incomplete asset market

NK: Pricing frictions in the goods market

SAM: Search frictions in the labor market

- **Policy:**
  - Fiscal authority finances expenditures through taxes and debt
  - Monetary authority follows inflation-targeting Taylor-rule
- **Calibration:**
  1. Realistic degree of partial self-insurance
     $\Rightarrow$ Matches micro-level consumption profiles during unemployment
  2. Realistic hiring-and-firing dynamics
     $\Rightarrow$ Matches macro-level dynamics of job-finding and separation rates

Overview

- **Types of fiscal policy:**
  1. Government consumption, G_t
  2. Universal transfer, $T_t = \text{transfer}_t$
  3. Higher unemployment benefits, $\overline{\phi}_t$
  4. Longer unemployment benefit duration, $\overline{u}_t$
  5. Hiring subsidies, hs_t
  6. Retention subsidies, rs_t
- **Extended model:**
  1. Endogenous separations + sluggish entry
  2. Dividends distributed equally
  3. Decreasing search intensity/efficiency while unemployed
  4. Risk of no unemployment benefits
  5. More detailed calibration
- **Previous paper:** Broer et. al. (2025) in zero-liquidity

Model summary

- **Notation:** $\boldsymbol{x} = [X_0 - X_{ss}, X_1 - X_{ss}, \dots]$
- **Household policies:**

$$\boldsymbol{h} = \left[\boldsymbol{g}, \boldsymbol{t}, \overline{\boldsymbol{\phi}}, \overline{\boldsymbol{u}}\right]'$$

- **Firm policies:**

$$\boldsymbol{f} = [\boldsymbol{hs}, \boldsymbol{rs}]'$$

- **Income process:**

$$\boldsymbol{inc} = [\boldsymbol{\delta}, \boldsymbol{\lambda^u}, \boldsymbol{div}]'$$

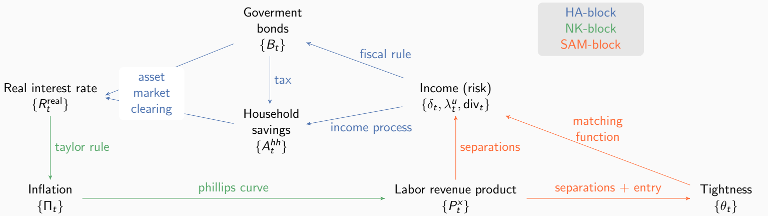
- **Model summary:**

$$\boldsymbol{r^{real}} = M_{HA}\boldsymbol{inc} + M_{h,r}\boldsymbol{h} + M_{f,r}\boldsymbol{f}, \tag{1}$$

$$\boldsymbol{p^x} = M_{NK}\boldsymbol{r^{real}}, \tag{2}$$

$$\boldsymbol{inc} = M_{SAM}\boldsymbol{p^x} + M_{s,inc}\boldsymbol{f}. \tag{3}$$

Directed Cycle Graph



Directed Cycle Process

If there is a unique solution to the system (1)-(3), it is given by

$$\boldsymbol{inc} = \underbrace{\mathcal{G}}_{\text{GE}} \times \left(\underbrace{M_{\text{SAM}} M_{\text{NK}} \underbrace{M_{h,r} \boldsymbol{h}}_{\text{direct}}}_{\text{first round, household transfer policy}} + \underbrace{M_{SAM} M_{NK} \underbrace{M_{f,r} \boldsymbol{f}}_{\text{direct}} + \underbrace{M_{f,\text{inc}} \boldsymbol{f}}_{\text{direct}}}_{\text{first round, firm transfer policy}} \right),$$

where $\mathcal{G}$ is defined by

$$\mathcal{G} = (I - M_{SAM} M_{NK} M_{HA})^{-1}.$$

Fiscal multipliers

- **Fiscal multiplier:**

$$\mathcal{M} = \text{cumulative fiscal multiplier} = \frac{\mathbf{1}'\boldsymbol{y}}{\mathbf{1}'\boldsymbol{taxes}}.$$

$$\boldsymbol{taxes} = M_{\text{inc,taxes}}\boldsymbol{inc} + M_{h,taxes}\boldsymbol{h}$$

- **Household policies 0 and 1:** If same direct PE real interest rate

$$M_{h,r}\boldsymbol{h}^0 = M_{h,r}\boldsymbol{h}^1$$

then output and income are the same $\boldsymbol{y}^0 = \boldsymbol{y}^1$ and $\boldsymbol{inc}^0 = \boldsymbol{inc}^1$. Differences in taxes are due to direct fiscal costs

$$\mathbf{1}'\boldsymbol{taxes}^0 - \mathbf{1}'\boldsymbol{taxes}^1 = \mathbf{1}'M_{h,\text{taxes}}\boldsymbol{h}^0 - \mathbf{1}'M_{h,\text{taxes}}\boldsymbol{h}^1,$$

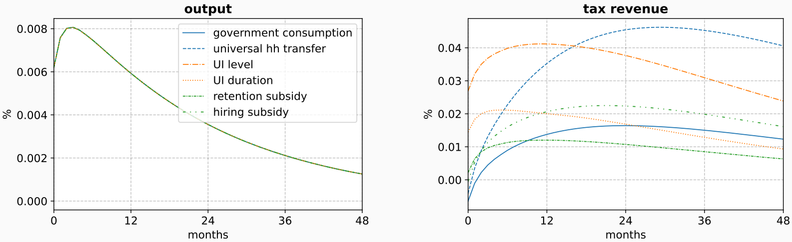
Fiscal multipliers are ordered by direct fiscal costs:

$$\mathcal{M}_{h^0} \gtreqless \mathcal{M}_{h^1} \iff \mathbf{1}'M_{h,\text{taxes}}\boldsymbol{h}^0 \lesseqgtr \mathbf{1}'M_{h,\text{taxes}}\boldsymbol{h}^1.$$

- **Firm policies:** Same result, but only with representative agent

Policy experiment

- **Experiment:** Same output path for different policies.



Different fiscal multipliers

	G [level]	Household transfers			Firm transfers	
		Transfer	Level	Duration	Retention	Hiring
1. Relative fiscal multiplier	1.0 [0.99]	0.28	0.44	1.03	1.64	0.72
2. Relative tax response	1.00	3.64	2.29	0.97	0.61	1.39
3. PE relative tax response	1.47	4.11	2.77	1.45	0.57	1.56
4. GE relative tax response	-0.47	-0.47	-0.47	-0.47	0.04	-0.17

- **Relative fiscal multiplier:** $\frac{\mathcal{M}_{hj}}{\mathcal{M}_{hG}}$
- **Relative tax respones:** $\frac{\mathbf{1}'\boldsymbol{taxes}^j}{\mathbf{1}'\boldsymbol{taxes}^G}$
  Decomposition for household transfers:

$$\boldsymbol{taxes}^j = M_{\text{inc},\text{taxes}}\boldsymbol{inc}^j + M_{h,taxes}\boldsymbol{h}^j$$

$$\boldsymbol{taxes}^{j,\text{PE}} = M_{h,\text{taxes}}\boldsymbol{h}^j$$

$$\boldsymbol{taxes}^{j,\text{GE}} = M_{\text{inc},\text{taxes}}\boldsymbol{inc}^j$$

Determinants of fiscal multipliers

	G [level]	Household transfers: Transfer	Household transfers: Level	Household transfers: Duration	Firm transfers: Retention	Firm transfers: Hiring
1. Baseline	1.0 [0.99]	0.28	0.44	1.03	1.64	0.72
2. Less sticky prices ($\phi = 178$)	1.0 [0.61]	0.30	0.47	1.03	3.43	1.15
3. More reactive mp ($\delta_\pi = 2$)	1.0 [0.64]	0.30	0.47	1.03	3.33	1.13
4. Representative agent	1.0 [0.54]	0.00	0.00	0.00	1.92	0.57
5. Fewer HtM (17.4%)	1.0 [0.80]	0.19	0.41	1.11	1.80	0.69
6. More tax financing ($\omega = 0.10$)	1.0 [0.84]	0.19	0.40	1.10	1.70	0.67
7. Exo. separations ($\psi = 0$)	1.0 [0.13]	0.35	0.52	1.02	1.39	3.38
8. Free entry ($\xi = \infty$)	1.0 [0.54]	0.31	0.47	1.03	1.50	1.21
9. Wage rule ($\eta_e = 0.50$)	1.0 [0.73]	0.29	0.46	1.03	1.55	0.74
10. 95% of div. to PIH	1.0 [0.82]	0.28	0.43	0.99	0.72	0.16

Endogenous search*

Endogenous search

- **Search decision:**
  1. Discrete search choice: $s_{it} \in \{0, 1\}$
  2. Search cost: λ if $s_{it} = 1$
  3. Taste shocks: $\varepsilon(s_{it}) \sim$ Extreme value (Iskhakov et. al., 2017)
- **Note:** Drop β_i for notational simplicity
- **Warning I:** *This is advanced! Only if time permits.*
- **Warning II:** *Not implemented in GEModelTools*

Discrete search decision

- **Search intensity matter for transition:**

$$\underline{v}_t\left(\beta_i, u_{it-1}, a_{it-1} \,|\, s_{it}\right) = \mathbb{E}\left[v_t(\beta_i, u_{it}, a_{it-1}) \,|\, u_{it-1}, a_{it-1}, s_{it}\right]$$

- **Standard logit formula:**

$$\begin{aligned}\underline{v}_t(u_{it-1}, a_{it-1}) &= \max_{s_{it}\in\{0,1\}} \left\{\underline{v}_t(u_{it-1}, a_{it-1} \,|\, s_{it}) - \lambda \mathbf{1}_{s_{it}=1} + \sigma_\varepsilon \varepsilon\left(s_{it}\right)\right\} \\ &= \sigma_\varepsilon \log\left(\exp\frac{\underline{v}_t(u_{it-1}, a_{it-1} \,|\, 0)}{\sigma_\varepsilon} + \exp\frac{\underline{v}_t(u_{it-1}, a_{it-1} \,|\, 1)}{\sigma_\varepsilon}\right)\end{aligned}$$

Envelope condition

- **Choice probabilities:**

$$P_t(s \mid u_{it-1}, a_{it-1}) = \frac{\exp \frac{\underline{v}_t(u_{it-1}, a_{it-1} \mid s)}{\sigma_\xi}}{\sum_{s' \in \{0,1\}} \exp \frac{\underline{v}_t(u_{it-1}, a_{it-1} \mid s')}{\sigma_\xi}}$$

- **Envelope condition:**

$$\begin{aligned}\underline{v}_{a,t}(u_{t-1}, a_{t-1}) &= \sum_{s \in \{0,1\}} P_t(s \mid u_{it-1}, a_{it-1}) \pi_t(u_{it} \mid u_{it-1}, s) v_{a,t}(u_{it}, a_{it-1}) \\ &= \sum_{s \in \{0,1\}} P_t(s \mid u_{it-1}, a_{it-1}) \pi_t(u_{it} \mid u_{it-1}, s) c_t^*(u_{it}, a_{it-1})^{-\sigma}\end{aligned}$$

- **Break of *monotonicity*** $\Rightarrow$ FOC still *necessary*, but not *sufficient*
  1. **Normally:** Savings $\uparrow$ $\Rightarrow$ future consumption $\uparrow$ $\Rightarrow$ marginal utility $\downarrow$
  2. **Now also:** Future search jump $\downarrow$ $\Rightarrow$ future income $\downarrow$
     $\Rightarrow$ future consumption $\downarrow$ $\Rightarrow$ marginal utility $\uparrow$

Upper envelope for given z^{i_z}

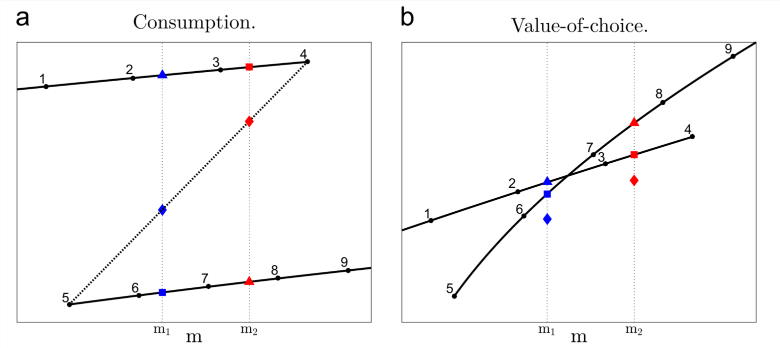
1. **Generate candidate points:** $\forall i_a \in \{0, 1, \dots, \#_a - 1\}$

$$
\begin{aligned}
w^{i_a} &= \beta \underline{v}_{t+1}(z^{i_z}, a^{i_a}) \\
c^{i_a} &= u'^{-1}\left(\beta \underline{v}_{a,t+1}(z^{i_z}, a^{i_a})\right) \\
m^{i_a} &= a^{i_a} + c^{i_a} \\
v^{i_a} &= u\left(c^{i_a}\right) + w^{i_a}
\end{aligned}
$$

2. **Apply upper-envelope:** $\forall i_{a-} \in \{0, 1, \dots, \#_a - 1\}$

$$
\begin{aligned}
c^*(a^{i_{a-}}) &= \max_{j \in \{0,1,\dots \#_a - 2\}} u\left(c^{i_{a-}}\right) + w^{i_{a-}} \text{ s.t.} \\
m^{i_{a-}} &= (1 + r_t) a^{i_{a-}} + w_t z^{i_z} \in \left[m^j, m^{j+1}\right] \\
c^{i_{a-}} &= \min\left\{\text{interp } \left\{m^{i_a}\right\} \to \left\{c^{i_a}\right\} \text{ at } m^{i_{a-}}, m^{i_{a-}}\right\} \\
a^{i_{a-}} &= m^{i_{a-}} - c^{i_{a-}} \\
w^{i_{a-}} &= \text{interp } \left\{a^{i_a}\right\} \to \left\{w^{i_a}\right\} \text{ at } a^{i_{a-}}
\end{aligned}
$$

Illustration



1. **Numbering:** Different levels of end-of-period assets, a^{i_a}
2. **Problem:** Find the consumption function at m_1 and m_2
3. **Largest value-of-choice:** Denoted by the *triangles*

Source: Druedahl and Jørgensen (2017), G^2EGM

Example

- **Beg.-of-period value function:**

$$\underline{v}_{t+1}(a_t) = \sqrt{m_{t+1}} + \eta \max\left\{m_{t+1} - \underline{m}, 0\right\}$$
$$\text{where } m_{t+1} = (1+r)a_t + 1$$

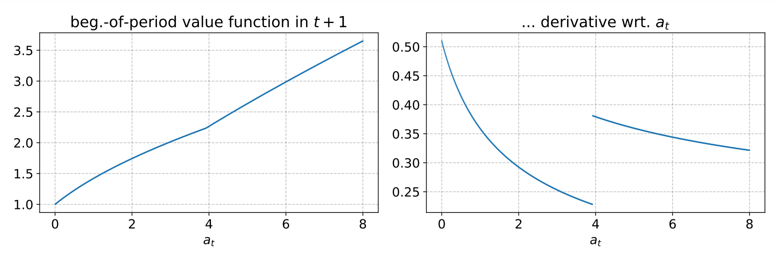
- **Derivative:**

$$\underline{v}_{a,t+1}(a_t) = \frac{1}{2}(1+r)m_{t+1}^{-\frac{1}{2}} + (1+r)\eta\mathbf{1}\left\{m_{t+1} > \underline{m}\right\}$$

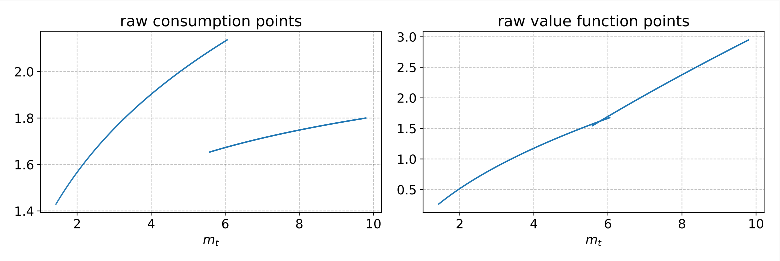
- **Budget constraint:**

$$a_t + c_t = (1+r)a_{t-1} + 1$$

Next-period values

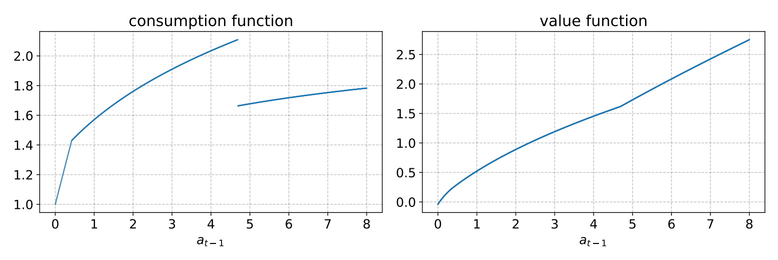


Raw values of c^{i_a} and v^{i_a}



Problem: Overlaps $\Rightarrow$ not a function m_t!

Result after upper envelope



General problem structure

- **General problem structure with *nesting*:**

$$\overline{v}_t\left(\overline{x}_t, d_t, e_t, m_t\right) = \max_{c_t \in [0,m_t]} u(c_t, d_t, e_t) + \beta \underline{v}_{t+1}\left(\underline{\Gamma}_t\left(\overline{x}_t, d_t, e_t, a_t\right)\right)$$

$$\text{with } a_t = m_t - c_t$$

$$v(x_t) = \max_{d_t \in \Omega^d(x_t)} \overline{v}_t\left(\overline{\Gamma}_t\left(x_t, d_t\right)\right)$$

$$\underline{v}_t\left(\underline{x}_t\right) = \max_{e_t \in \Omega^e(\underline{x}_t)} \mathbb{E}\left[v\left(\Gamma\left(\underline{x}_t, e_t\right)\right) \mid \underline{x}_t, e_t\right]$$

- **Finding c_t:** *EGM with upper envelope can (typically) still be used*
- **Finding d_t and e_t:**
  1. Combination of discrete and continuous choices
  2. Typically requires use of numerical optimizer or root-finder
- **Druedahl (2021),** »A Guide on Solving Non-Convex Consumption-Saving Models« (costly with extra states in $\overline{v}$)

Summary

Summary

- **HANK-SAM:**
  1. More realistic labor market and income process
  2. Allow for fluctuations in idiosyncratic risk
  3. Laboratory for studying e.g. fiscal policy
- **Solution methods:**
  1. Time-varying transition matrix is straightforward
  2. (Non-sufficient Euler-equation create serious problems)