Consumption-Saving

Adv. Macro: Heterogenous Agent Models

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Introduction

Introduction

- **Generations of models:**
  1. **Permanent income hypothesis (PIH)** (Friedman, 1957) or life-cycle model (Modigliani and Brumburg, 1954)
  2. **Buffer-stock consumption model** Deaton (1991, 1992); Carroll (1992, 1997, 2019)
  3. **Multiple-asset buffer-stock consumption models** e.g. Kaplan and Violante (2014); Harmenberg and Öberg (2021)
- **Consumption-and-saving over the life-cycle dynamic** e.g. Gourinchas and Parker (2002); Druedahl and Martinello (2022)
- **Empirical MPCs and income risk** e.g. Fagereng et. al. (2021); Guvenen et. al. (2021)

Book: The Economics of Consumption, Jappelli and Pistaferri (2017)

Plan

PIH

Consumption-saving

$$v_0 = \max_{\{c_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t)$$

$$\text{s.t.}$$

$$a_t = (1+r)a_{t-1} + wz_t - c_t$$

$$a_{T-1} \geq 0$$

- **Variables:**
  Consumption: c_t
  Productivity: z_t
  End-of-period savings: a_t (*no debt at death*)
- **Parameters:**
  Discount factor: β
  Wage: w
  Interest rate: r (define $R \equiv 1 + r$ as interest factor)

It is a *static* problem

$$
\begin{aligned}
v_0 &= \max_{\{c_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t) \\
&\text{s.t.} \\
a_t &= (1+r)a_{t-1} + wz_t - c_t \\
a_{T-1} &\geq 0
\end{aligned}
$$

- **It is a *static* problem:**
  1. **Information:** z_t is known for all t at $t = 0$
  2. **Target:** Discounted utility, $\sum_{t=0}^{T-1} \beta^t u(c_t)$
  3. **Behavior:** Choose $c_0, c_1, \ldots, c_{T-1}$ *simultaneously*
  4. **Solution:** Sequence of consumption *choices* $c_0^*, c_1^*, \ldots, c_{T-1}^*$

IBC

- **Substitution** implies *Intertemporal Budget Constraint* (IBC)

$$
\begin{aligned}
a_{T-1} &= Ra_{T-2} + wz_{T-1} - c_{T-1} \\
&= R(Ra_{T-3} + wz_{T-2} - c_{T-2}) + wz_{T-1} - c_{T-1} \\
&= R^T a_{-1} + \sum_{t=0}^{T-1} R^{T-1-t}(wz_t - c_t)
\end{aligned}
$$

- Use **terminal condition** $a_{T-1} = 0$ (equality due utility max.)

$$
R^{-(T-1)} a_{T-1} = 0 \quad \Leftrightarrow \quad \sum_{t=0}^{T-1} R^{-t} c_t = s_0 + h_0
$$

where $s_0 \equiv Ra_{-1}$ (after-interest assets)
and $h_0 \equiv \sum_{t=0}^{T-1} R^{-t} wz_t$ (human capital)

FOC and Euler-equation

$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t u(c_t) + \lambda \left[\sum_{t=0}^{T-1} R^{-t} c_t - s_0 - h_0 \right]$$

- **First order conditions**:

$$\forall t : 0 = \beta^t u'(c_t) - \lambda(1+r)^{-t} \Leftrightarrow u'(c_t) = -\lambda \left(\beta R\right)^{-t}$$

- **Euler-equation** for $k \in \{1, 2, \dots\}$:

$$\frac{u'(c_t)}{u'(c_{t+k})} = \frac{-\lambda \left(\beta R\right)^{-t}}{-\lambda \left(\beta R\right)^{-(t+k)}} = \left(\beta R\right)^k$$

Consumption choice

- **CRRA:** $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$ imply Euler-equation

$$\frac{c_0^{-\sigma}}{c_t^{-\sigma}} = (\beta R)^t \Leftrightarrow c_t = (\beta R)^{\frac{t}{\sigma}} c_0$$

  Constant consumption if:

  1. $\beta R = 1$
  2. $\frac{1}{\sigma} \to 0$ (zero intertemporal elasticity of substitution)

- Insert **Euler** into **IBC** to get consumption choice

$$\begin{aligned} \sum_{t=0}^{T-1} \left((\beta R)^{1/\sigma} R^{-1} \right)^t c_0 &= s_0 + h_0 \Leftrightarrow \\ c_0^* &= \frac{1 - (\beta R)^{1/\sigma} R^{-1}}{1 - \left((\beta R)^{1/\sigma} R^{-1} \right)^T} (s_0 + h_0) \end{aligned}$$

Infinite horizon: $T \to \infty$

- **Infinite horizon** for $(\beta R)^{1/\sigma} R^{-1} < 1$:
  Let $T \to \infty$ to get

$$c_0^* = \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right)(s_0 + h_0)$$

- **Constant income**, $\forall z_t = 1$:

$$c_0^* = \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right)\left(Ra_{-1} + \frac{R}{R-1}w\right)$$

- **Consume annuity value** if also $\beta R = 1$

$$c_0^* = ra_{-1} + w$$

Propensities to consume ($\beta R \approx 1, z_t \approx 1$)

$$c_0^* \approx \frac{r}{1+r}\left((1+r)a_{-1} + \sum_{t=0}^{\infty}\frac{wz_t}{(1+r)^t}\right) \approx ra_{-1} + w$$

Different types of shocks:

1. MPC of *windfall* income: $\frac{\partial c_0}{\partial s_0} \approx \frac{r}{1+r}$
2. MPC of *future* income change: $\frac{\partial c_0}{\partial wz_t} \approx \frac{r}{1+r}(1+r)^{-t}$
3. MPC of *permanent* income change: $\frac{\partial c_0}{\partial w} \approx \frac{r}{1+r}\frac{1}{1-(1+r)^{-1}} = 1$

Dynamic affects: The same when $\beta R = 1$, for all $k > 0$

$$\frac{\partial c_k}{\partial s_0} = \frac{\partial c_0}{\partial s_0}$$
$$\frac{\partial c_k}{\partial wz_t} = \frac{\partial c_0}{\partial wz_t}$$
$$\frac{\partial c_k}{\partial w} = \frac{\partial c_0}{\partial w}$$

Savings ($\beta R = 1$)

- **Constant savings** $z_t = 1$:

$$c_t = ra_{t-1} + w \Rightarrow a_t = Ra_{t-1} + w - c_t = a_{t-1}$$

  1. Decreasing savings with $\beta R < 1$: $c_t \uparrow \Rightarrow a_t < a_{t-1}$
  2. Increasing savings with $\beta R > 1$: $c_t \downarrow \Rightarrow a_t > a_{t-1}$

- **Same consumption if NPV of wz_t is unchanged**
  e.g. higher in one period, and lower in another

$$\frac{r}{1+r} \sum_{t=0}^{\infty} \frac{z_t}{(1+r)^t} = 1$$

$\Rightarrow$ *savings change with income*

Empirical MPCs

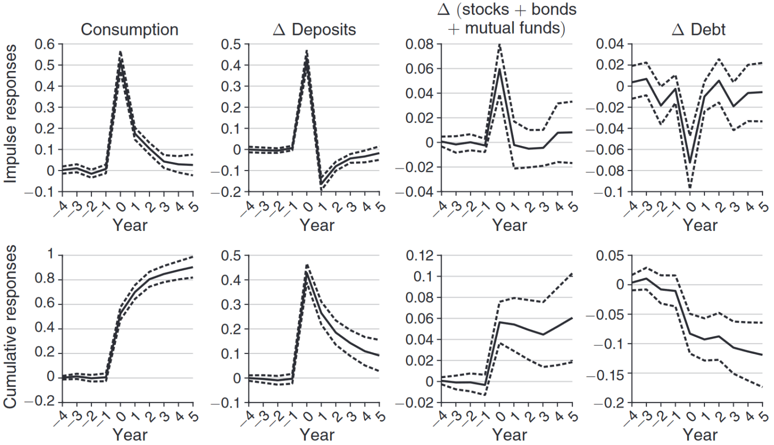


Figure 2. Dynamic Household Responses to Lottery Prizes

Source: Fagereng et. al. (2021)

Initial liquidity/borrowing constraint

- Implied period 0 **savings** are: $a_0 = s_0 + wz_0 - c_0$
- Hard **borrowing constraint:** $a_0 \geq -wb$
- **Maximum consumption:** $\overline{c}_0 = s_0 + wz_0 + wb$
- **Optimal consumption:** Constrained or unconstrained.

$$c_0^* = \min\left\{\overline{c}_0, \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right)(s_0 + h_0)\right\}$$

- **Empirical realism.** MPC of constrained is one

$$c_0^* = \overline{c}_0 \Rightarrow \frac{\partial c_0^*}{\partial s_0} = \frac{\partial \overline{c}_0}{\partial s_0} = 1$$

- **Technical issue:** *Borrowing constraints further in the future complicates the analytical solution considerably.*

Buffer-stock

Uncertainty and always borrowing constraint

$$v_0(z_0, a_{-1}) = \max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

s.t.

$$a_t = (1+r)a_{t-1} + wz_t - c_t$$

$$z_{t+1} \sim \mathcal{Z}(z_t)$$

$$a_t \geq -wb$$

$$\lim_{t\to\infty} (1+r)^{-t} a_t \geq 0 \quad \text{[No-Ponzi game]}$$

- **Stochastic income** from 1st order Markov-process, $\mathcal{Z}$
- ***A true dynamic problem:***
    1. **Information:** z_t is revealed period-by-period
    2. **Target:** *Expected* discounted utility, $\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$
    3. **Behavior:** Choose c_t *sequentially* as information is revealed
    4. **Solution:** Sequence of consumption *functions*, $c_t^*(z_t, a_{t-1})$

IBC

- **Substitution** still implies:

$$R^{-(T-1)}a_{T-1} = 0 \quad \Leftrightarrow \quad \sum_{t=0}^{\infty} R^{-t} c_t = s_0 + h_0$$

- **What if $T \to \infty$?** We must have $\lim_{T\to\infty} R^{-(T-1)}a_{T-1} = 0$
  1. $\lim_{T\to\infty} R^{-(T-1)}a_{T-1} > 0$: Consumption can be increased
  2. $\lim_{T\to\infty} R^{-(T-1)}a_{T-1} < 0$: Violates No-Ponzi game condition
- For $T \to \infty$ we have the **IBC**:

$$\sum_{t=0}^{\infty} R^{-t} c_t = Ra_{-1} + \sum_{t=0}^{\infty} R^{-t} w z_t$$

Natural borrowing limit

- Denote **minimum possible productivity** by $\underline{z}$
- **Consumption must be non-negative** $\Rightarrow$
  interest payments must be less than minimum income

$$c_t \geq 0 \Rightarrow r(-a_t) \leq w\underline{z} \Leftrightarrow a_t \geq -\frac{w\underline{z}}{r}$$

If debt was larger it would in the worst case ($\forall z_t = \underline{z}$) grow without bound even with zero consumption ($\forall c_t = 0$)

$$\begin{aligned}
a_0 &= -\frac{w\underline{z}}{r} - \Delta \\
a_1 &= (1+r)a_0 + w\underline{z} = a_0 - (1+r)\Delta \\
a_2 &= (1+r)a_1 + w\underline{z} = a_0 - (1+r)^2\Delta \\
&\vdots
\end{aligned}$$

- **Natural borrowing constraint:** $a_t \geq \underline{a} = -w \min\left\{b, \frac{\underline{z}}{r}\right\}$

Euler-equation from variation argument

- **Case I:** If $u'(c_t) > \beta R \mathbb{E}_t \left[u'(c_{t+1}) \right]$:
  Increase c_t by marginal $\Delta > 0$, and lower c_{t+1} by $R\Delta$
  1. **Feasible:** Yes, if $a_t > \underline{a}$
  2. **Utility change:** For $\Delta \rightarrow 0$ $u'(c_t) + \beta \left(-R \right) \mathbb{E}_t \left[u'(c_{t+1}) \right] > 0$
- **Case II:** If $u'(c_t) < \beta R \mathbb{E}_t \left[u'(c_{t+1}) \right]$:
  Lower c_t by marginal $\Delta > 0$, and increase c_{t+1} by $R\Delta$
  1. **Feasible:** Yes (always)
  2. **Utility change:** For $\Delta \rightarrow 0$ $u'(c_t) + \beta R \mathbb{E}_t \left[u'(c_{t+1}) \right] > 0$
- **Conclusion:** By contradiction
  1. **Constrained:** $a_t = \underline{a}$ and $u'(c_t) \geq \beta R \mathbb{E}_t \left[u'(c_{t+1}) \right]$, or
  2. **Unconstrained:** $a_t > \underline{a}$ and $u'(c_t) = \beta R \mathbb{E}_t \left[u'(c_{t+1}) \right]$
- **Sufficiency:** Harder ($\sim$ convexity of the choice set)

Special case I: Quadratic utility

- **Quadratic utility:** $u(c_t) = -\frac{1}{2}(\overline{c} - c)^2$ with $\beta R = 1$ and »large« $\overline{c}$
- **Euler-equation:** *Consumption = expected future consumption*

$$(\overline{c} - c_t) = \mathbb{E}_t\left[(\overline{c} - c_{t+k})\right] \Leftrightarrow c_t = \mathbb{E}_t\left[c_{t+k}\right]$$

- Use **IBC** in expectation to get **consumption function:**

$$\sum_{t=0}^{\infty} R^{-t}\mathbb{E}_0\left[c_t\right] = Ra_{-1} + \sum_{t=0}^{\infty} R^{-t} w\mathbb{E}_0\left[z_t\right] \Rightarrow$$

$$c^*(z_t, a_{t-1}) = c_0 = ra_{-1} + \frac{r}{R}\sum_{t=0}^{T} R^{-t} w\mathbb{E}_0\left[z_t\right]$$

where we formally disregard the borrowing constraint

- **Certainty equivalence:** *Only expected income matter.*

Special case II: CARA utility

- **CARA utility:** $u(c_t) = -\frac{1}{\alpha}e^{-\alpha c}$
- **Productivity is absolute random walk:**

$$z_t = z_{t-1} + \psi_t$$
$$\psi_t \sim \mathcal{N}(0, \sigma_\psi^2)$$

- **Consumption function** (see proof)**:**

$$c^*(a_{t-1}, z_t) = ra_{t-1} + wz_t - \frac{\log(\beta R)^{\frac{1}{\alpha}} + \alpha\frac{\sigma_\psi^2}{2}}{r^2}$$

  where we formally disregard the borrowing constraint

- **Precautionary saving:** $\sigma_\psi^2 \uparrow$ implies $c_t^* \downarrow$ for given z_t and a_{t-1}
  $\Rightarrow$ *accumulation of buffer-stock*

Dynamic solution: Bellman's Principle of Optimality

- Go back to a **finite horizon** problem with T periods
- **Value function, v_t:** Defined *recursively* from $v_T(\bullet) = 0$

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

- **Policy function, c_t^*:** Is the same as

$$c_t^*(z_t, a_{t-1}) = \arg\max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

- **Euler-equation:**
  1. FOC: $c_t^{-\sigma} = \beta \mathbb{E}_t \left[v_{t,a}(z_{t+1}, a_t) \right]$
  2. Envelope: $v_{t,a}(z_t, a_{t-1}) = (1+r)c_t^{-\sigma}$ (fix a_t)

Vocabulary

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

1. **State variables:** z_t and a_{t-1}
2. **Control** (choice) **variable:** c_t
3. **Continuation value:** $\beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$
4. **Parameters:** r, w, and stuff in $u(\bullet)$

Note: Straightforward to extend to more goods, more assets or other states, more complex risk, bounded rationality etc.

Infinite horizon: $T \to \infty$

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

- **Contraction mapping result:** *If β is low enough (strong enough impatience) then the value and policy functions converge to $v(z_t, a_{t-1})$ and $c^*(z_t, a_{t-1})$ for large enough T*
- **In practice:**
  1. Make arbitrary initial guess (e.g. $v_T = 0$)
  2. Solve backwards until the value (and/or policy function) does not change anymore (relative to chosen tolerance)

3-periods

3-period model

- **Expected discounted utility:** $v(z_0, a_{-1}) = \mathbb{E}_0 \sum_{t=0}^{2} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$
- **Income = wage $\times$ productivity + transfer:**

$$y_t = wz_t + \chi_t$$

- **Cash-on-hand, savings and borrowing constraint:**

$$\begin{aligned} m_t &= (1+r)a_{t-1} + y_t \\ a_t &= m_t - c_t \\ a_t &\geq \underline{a} \end{aligned}$$

- **Stochastic transition:** $\Pr[z_{t+1}|z_t] = \pi_t(z_t, z_{t+1})$ such that

$$\begin{aligned} \Pr[z_{t+1} = 1 \,|\, z_t = 1] &= \pi \\ \Pr[z_{t+1} = 1-\Delta \,|\, z_t = 1] = \Pr[z_{t+1} = 1+\Delta \,|\, z_t = 1] &= \frac{1-\pi}{2} \\ \Pr[z_{t+1} = z_t \,|\, z_t \in \{1-\Delta, 1+\Delta\}] &= 1 \end{aligned}$$

Bellman equation

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(z_{t+1}, a_t) \right]$$

$$\text{s.t.}$$

$$y_t = w z_t + \chi_t$$

$$m_t = (1+r) a_{t-1} + y_t$$

$$a_t = m_t - c_t$$

$$\Pr\left[z_{t+1} | z_t \right] = \pi_t(z_t, z_{t+1})$$

$$a_t \geq \underline{a}$$

where

$$v_3(z_3, a_2) = 0$$

Discretization

- **Discretization:** All state variables belong to discrete sets $\equiv$ *grids*,

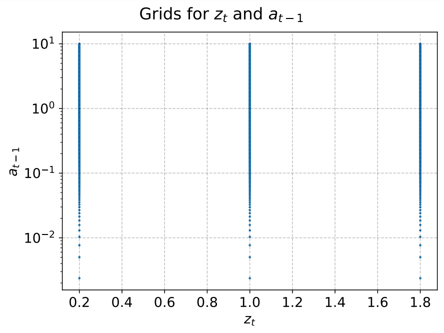
$$
\begin{aligned}
z_t \in \mathcal{G}_z &= \{z^0, z^1, \dots, z^{\#_z - 1}\} \\
a_t \in \mathcal{G}_a &= \{a^0, a^1, \dots, a^{\#_a - 1}\} \\
a^0 &= \underline{a}
\end{aligned}
$$

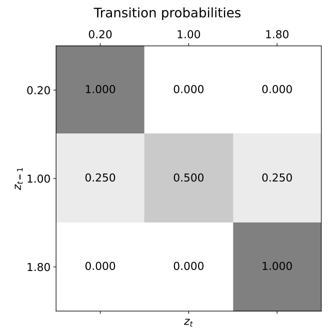
- **Expectation:** Numerical integration by

$$
\mathbb{E}_t \left[v_{t+1}(z_{t+1}, a_t) \right] = \sum_{z_{t+1} \in \{1-\Delta, 1, 1+\Delta\}} \pi_t(z_t, z_{t+1}) v_{t+1}(z_{t+1}, a_t)
$$

- **ConSav:** `grids.nonlinspace`, `grids.equilogspace`
- **ConSavNotebooks:** `04. Tools/03. Grids.ipynb`

Grids and transition probabilities





The size of risk is scaled by Δ

Baseline: $\Delta = 0.8$

Low risk: $\Delta = 0.4$

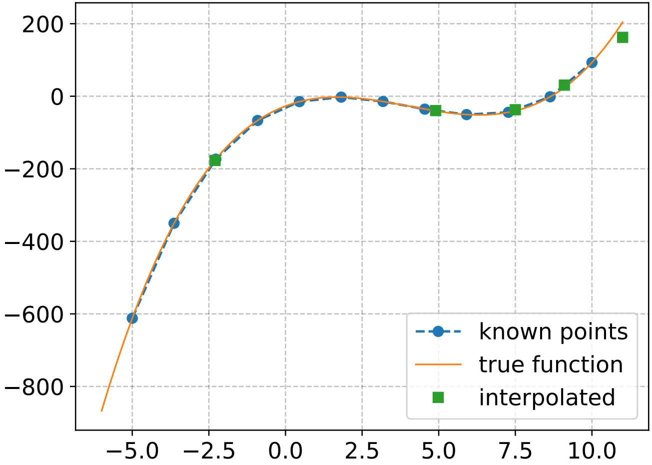
Linear interpolation

- **Linear interpolation** (function approximation):
  1. Assume v_{t+1} is known on $\mathcal{G}_z \times \mathcal{G}_a$ (tensor product)
  2. Evaluate $v_{t+1}(z^{i_z}, a)$ for arbitrary a by

$$
\begin{aligned}
\breve{v}_{t+1}(z^{i_z}, a) &= \text{baseline} + \text{slope} \times \text{distance} \\
&= v_{t+1}(z^{i_z}, a^{\iota}) + \omega(a - a^{\iota}) \\
&\text{where} \\
&\omega \equiv \frac{v_{t+1}(z^{i_z}, a^{\iota+1}) - v_{t+1}(z^{i_z}, a^{\iota})}{a^{\iota+1} - a^{\iota}} \\
&\iota \equiv \text{largest } i_a \in \{0, 1, \dots, \#_a - 2\} \text{ such that } a^{i_a} \leq a
\end{aligned}
$$

- **ConSav:** `linear_interp.interp1d`
- **ConSavNotebooks:**
  `04. Tools/01. Linear interpolation.ipynb`

Linear interpolation



Value function iteration (VFI)

- **Maximize value-of-choice:**

$$v_t(z^{i_z}, a^{i_{a-}}) = \max_{c_t} v_t(z^{i_z}, a^{i_{a-}}|c_t)$$

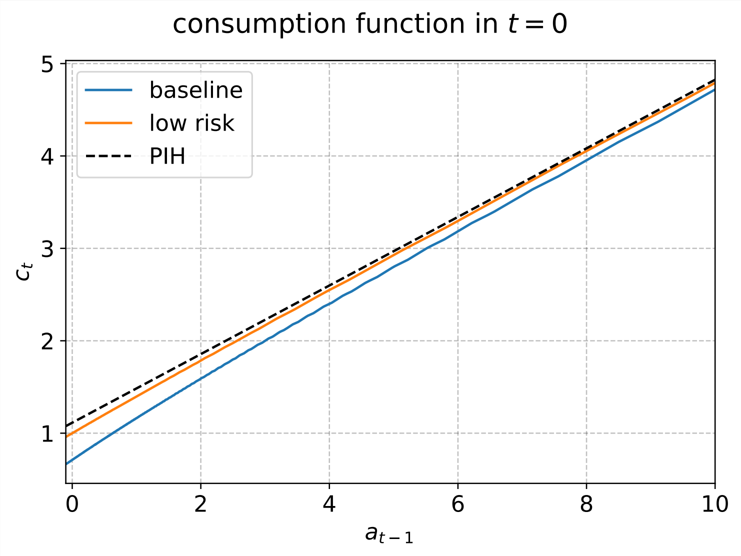
$$\text{with } c_t \in [0, (1+r)a^{i_{a-}} + wz^{i_z} + \chi_t + \underline{a}]$$

$$v_t(z^{i_z}, a^{i_{a-}}|c_t) = u(c_t) + \beta \sum_{i_{z+1}=0}^{\#_z-1} \pi\left(i_z, i_{z+1}\right) \breve{v}_{t+1}(z^{i_z}, a_t)$$

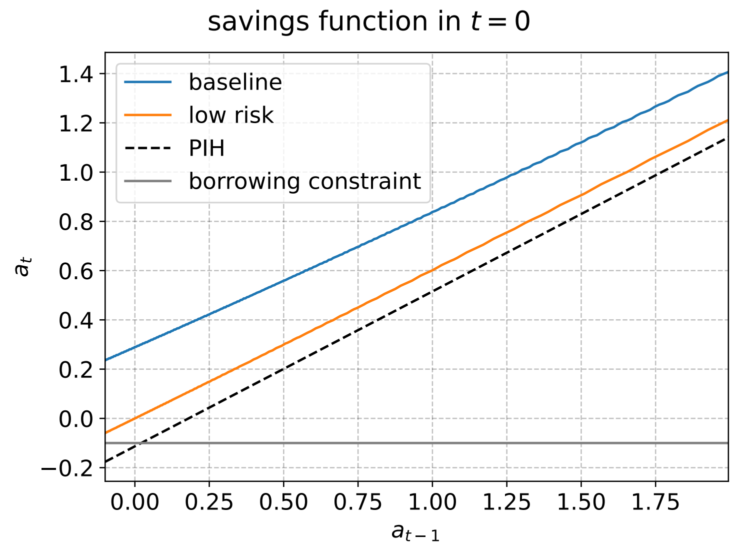
$$\text{with } a_t = (1+r)a^{i_{a-}} + wz^{i_z} + \chi_t - c_t$$

- **Inner loop:** For each grid point in $\mathcal{G}_z \times \mathcal{G}_a$ find $c_t^*(z_t, a_{t-1})$ and therefore $v_t(z_t, a_{t-1})$ with a *numerical optimizer*
- **Outer loop:** Backwards from $t = T-1$ (note $\underline{v}_T = 0$, or known)
- **ConSav+QuantEcon:** Various optimizers in numba
- **ConSavNotebooks:** `04. Tools/02. Optimization.ipynb`

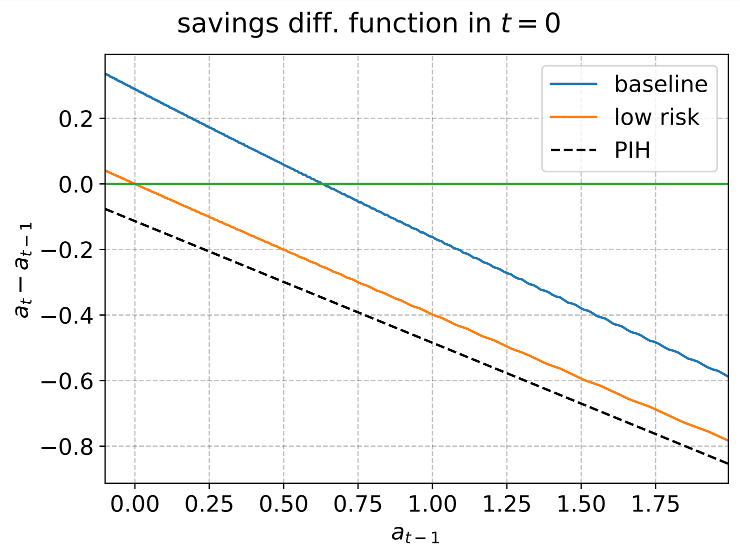
Consumption function



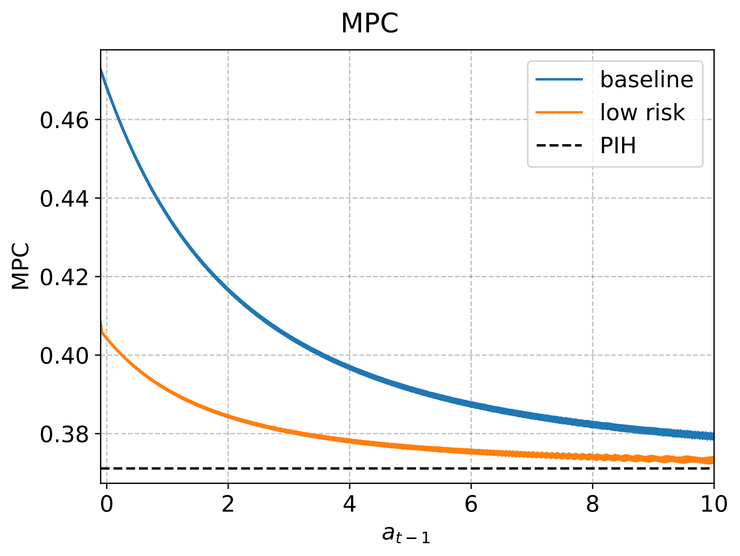
Savings function



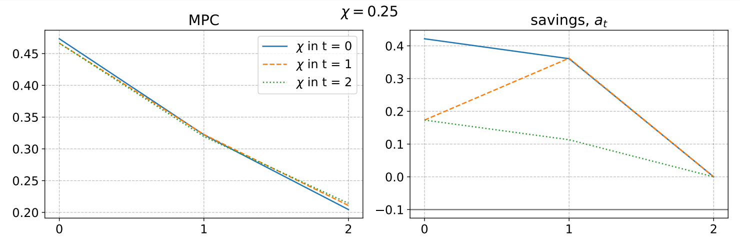
Change in savings function

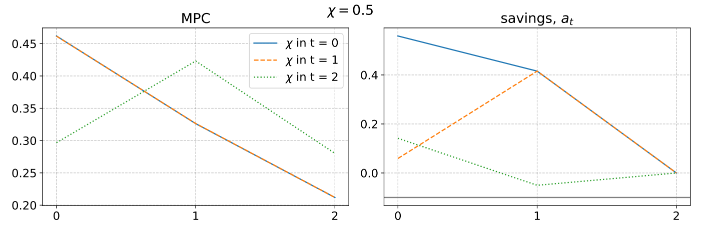


MPC



Intertemporal MPC





Note: No wealth effect as $r = 0$

Economic insights

- **Notebook:** 01_ConSavModel_3periods/ConSavModel.ipynb
- **Consumption lower than under PIH and concave in assets**
  Intuition: *Precautionary saving motive is relatively larger for asset poor households because income risk is the same for everybody*
  Implications:
  1. Windfall gives safety and increases average consumption
     $\Rightarrow$ MPC decreasing in assets
  2. Attraction towards a buffer-stock target $a_t = a_{t-1}$ despite $\beta R < 1$
  3. Larger effective discounting of future income
     (extreme: no effect of future income changes if constrained before)

Numerical Monte Carlo simulation

- **Initial distribution:** Draw $z_{i,-1}$ and $a_{i,-1}$ for $i \in \{0, 1, \ldots, N-1\}$
- **Simulation:** Forwards in time from $t = 0$ and in each time period
  1. Draw z_{it} given transition probabilities
  2. Use linear interpolation to evaluate

$$
\begin{aligned}
c_{it} &= \breve{c}_t^*(z_{it}, a_{it-1}) \\
a_{it} &= (1+r)a_{it-1} + wz_{it} - c_{it}
\end{aligned}
$$

- **Review:**
  - **Pro:** Simple to implement
  - **Con:** Computationally costly and introduces randomness
- **Infinite horizon:**
  1. Assume z_{it} has an ergodic distribution
  2. Ergodic distribution of a_{it} around buffer-stock target

Taking stock

- **Value Function Iteration (VFI)**
  1. Solve all consumption-saving models
  2. Accurate with dense enough grids
  3. Relatively simple code and easy to run in parallel
  4. Finding optimal choices is the computational bottleneck
     (especially with multi-starts in non-convex models)

EGM

Time iteration

- **Replace numerical optimization with root-finding**
- **Time iteration:** For each a_{t-1} and z_t find c_t to solve the Euler-equation

$$c_t^{-\sigma} = \beta(1+r)\mathbb{E}_t[c_{t+1}^{-\sigma}]$$

  Note: *Necessary and sufficient* (for interior choices, else $a_t = \underline{a}$)
- **EGM:** No need for any numerical optimization or root-finding

Endogenous grid-point method (EGM)

1. Calculate **post-decision marginal value of cash:**

$$q(z^{i_z}, a^{i_a}) = \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z, i_{z+}} c_{+}^{*}(z^{i_{z+}}, a^{i_a})^{-\sigma}$$

2. **Invert Euler-equation:**

$$c(z^{i_z}, a^{i_a}) = (\beta(1+r)q(z^{i_z}, a^{i_a}))^{-\frac{1}{\sigma}}$$

3. **Endogenous cash-on-hand:**

$$m(z^{i_z}, a^{i_a}) = a^{i_a} + c(z^{i_z}, a^{i_a})$$

4. **Consumption function:** Calculate $m = (1+r)a^{i_{a-}} + wz^{i_z}$

   4.1 Binding constraint: If $m \leq m(z^{i_z}, a^0)$ then

   $$c^*(z^{i_z}, a^{i_{a-}}) = m + \underline{a}$$

   4.2 Interior choice: Else

   $$c^*(z^{i_z}, a^{i_{a-}}) = \text{interpolate } m(z^{i_z}, m) \to c(z^{i_z}, m)$$

Income process

Permanent transitory income process

- **Persistent-transitory income process:**

$$z_t = \tilde{z}_t \xi_t, \;\; \log \xi_t \sim \mathcal{N}(\mu_\xi, \sigma_\xi)$$
$$\log \tilde{z}_{t+1} = \rho_z \log \tilde{z}_t + \psi_{t+1}, \;\; \psi_{t+1} \sim \mathcal{N}(\mu_\psi, \sigma_\psi)$$

  1. Transitory shock: ξ_t
  2. Persistent shock: ψ_t
  3. Normalization using μ_ψ and μ_ξ: $\mathbb{E}[z_t] = \mathbb{E}[\tilde{z}_t] = 1$

- **ConSav:** `qudarature.log_normal_gauss_hermite`
- **ConSavNotebook:** `04. Tools/04. Quadrature.ipynb`

Transition probabilities

- **Discretization of ξ_t:** Derive $\mathcal{G}_\xi$ and π_{i_ξ} given σ_ξ using Gauss-Hermite quadrature

$$x \sim \mathcal{N}(\mu, \sigma^2): \;\; \mathbb{E}[h(x)] \approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^{n} \omega_i h(\sqrt{2}\sigma x_i + \mu)$$

  where nodes, x_i, and weights, ω_i, have analytical expressions
- **Discretization of $\tilde{z}_t$:** Derive $\mathcal{G}_{\tilde{z}}$ and $\pi_{i_{\tilde{z}-}, i_{\tilde{z}}}$ given $\rho_z < 1$ and σ_ψ (using a method such as Tauchen (1986) or Rouwenhorst (1995))
  If $\rho_z = 1$: Also use quadrature here.
- **Combined:** Derive $\mathcal{G}_z = \mathcal{G}_{\tilde{z}} \times \mathcal{G}_\xi$ (tensor product) and use independence of $\tilde{z}_t$ and ξ_t to get transition probabilities π_{i_{z-}, i_z} (kronecker product)
- **ConSav:** `markov.log_rouwenhorst`, `markov.log_tauchen`
- **ConSavNotebook:** `04. Tools/05. Markov.ipynb`

Full household problem

Full household problem

- **Utility maximization** for household *i*:

$$v_0(z_t, a_{t-1}) = \max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t.

$$a_t = (1 + r_t)a_{t-1} + wz_t - c_{it}$$

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1}, \;\; \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \;\; \mathbb{E}[z_t] = 1$$

$$a_t \geq -bw$$

Recursive formulation I

- **Value function**:

$$v(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t \left[v(z_{t+1}, a_t) \right]$$

$$\text{s.t.}$$

$$a_t = (1+r)a_{t-1} + wz_t - c_t$$

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1}$$

$$a_t \geq -bw$$

- **Same Euler-equation**
- **Borrowing constrained** has

$$c_{it} = m_{it} - bw$$

where $m_t = (1+r)a_{t-1} + wz_t$

Recursive formulation II

- **Beginning-of-period value function**

$$\underline{v}\left(z_{t-1}, a_{t-1}\right) = \mathbb{E}\left[v(z_t, a_{t-1}) \,|\, z_{t-1}\right]$$

- **Value function**:

$$v(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \underline{v}\left(z_t, a_t\right)$$

$$\text{s.t.}$$

$$a_t = (1+r)a_{t-1} + wz_t - c_t$$

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1}$$

$$a_t \geq -bw$$

- **Why do this?** *Only one interpolation per guess of* c_t

Distribution

- **Problem with Monte Carlo simulation:** *Stochastic fluctuations in average wealth $\Rightarrow$ stochastic error in asset market clearing condition.*
- **Alternative – Histogram:** Distribution as array of probabilities
  1. Beginning-of-period: $\underline{\boldsymbol{D}}_t$ over z_{it-1} and a_{it-1}
  2. Productivity transition: $\boldsymbol{D}_t = \Pi_z' \underline{\boldsymbol{D}}_t$ over z_{it} and a_{it-1}
  3. Savings transition: $\underline{\boldsymbol{D}}_{t+1} = \Lambda' \boldsymbol{D}_t$ where Λ derives from $a^*(z_t, a_{t-1})$

Numerical histogram simulation

- **Initial distribution:** Choose $\underline{\boldsymbol{D}}_0(z_{-1}, a_{-1})$, which is defined on $\mathcal{G}_z \times \mathcal{G}_a$ and sum to $1 \equiv$ *histogram*

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  1. **Distribute stochastic mass:** For each i_z and i_{a-} calculate

$$\boldsymbol{D}_t(z^{i_z}, a^{i_{a-}}) = \sum_{i_{z-}=0}^{\#_z-1} \pi_{i_{z-},i_z} \underline{\boldsymbol{D}}_t(z^{i_{z-}}, a^{i_{a-}})$$

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     3.1 Find $\iota \equiv$ largest $i_a \in \{0, 1, \ldots, \#_a - 2\}$ such that $a^{i_a} \leq a^*(z^{i_z}, a^{i_{a-}})$

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     3.3 Increment $\underline{\boldsymbol{D}}_{t+1}(z^{i_z}, a^{\iota})$ with $\omega \boldsymbol{D}_t(z^{i_z}, a^{i_{a-}})$

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     3.4 Increment $\underline{\boldsymbol{D}}_{t+1}(z^{i_z}, a^{\iota+1})$ with $(1-\omega)\boldsymbol{D}_t(z^{i_z}, a^{i_{a-}})$

Implementation

- **Toy example:** simple_histogram_simulation.xlsx
  - **Grids:** $\mathcal{G}_z = \{\underline{z}, \overline{z}\}$ and $\mathcal{G}_a = \{0, 1\}$
  - **Transition matrix:** $\pi_{0,0} = \pi_{1,1} = 0.5$
  - **Policy function:**
    - Low income: $a^*(\underline{z}, 0) = a^*(\underline{z}, 1) = 0$
    - High income: Let $a^*(\overline{z}, 0) = 0.5$ and $a^*(\overline{z}, 1) = 1$
  - **Initial distribution**: $\underline{\boldsymbol{D}}_0(z_{it}, a_{it-1}) = \begin{cases} 1 & \text{if } z_{it} = \underline{z} \text{ and } a_{it} = 0 \\ 0 & \text{else} \end{cases}$
  - **Task:** Calculate by hand the transitions to

$$\boldsymbol{D}_0,\ \underline{\boldsymbol{D}}_1,\ \boldsymbol{D}_1, \ldots$$

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- **Comparison with Monte Carlo:** See 02_ConSavModel_infConSavModel/
  1. **Pro:** Computationally efficient and no randomness
  2. **Con:** Introduces a non-continuous distribution

Misc

1. Life-cycle (I)

- **Basically:**
  1. Born, working, retied, die
  2. Age-varying parameters (esp. income)
- **Add-ons:**
  1. Labor supply, human capital, occupation
  2. Portfolio choice and entrepreneurship
  3. Family formation
  4. Health, mortality

     etc.
- **Good starting example:** »Life-Cycle Consumption and Children: Evidence from a Structural Estimation«, Jørgensen (2017)

1. Life-cycle (II)

Paper: Gourinchas and Parker (2021)
Life-cycle consumption-saving model with retirement

- Young households:
  Save for precautionary reasons (buffer)
- Older households:
  Save for retirement (life-cycle)

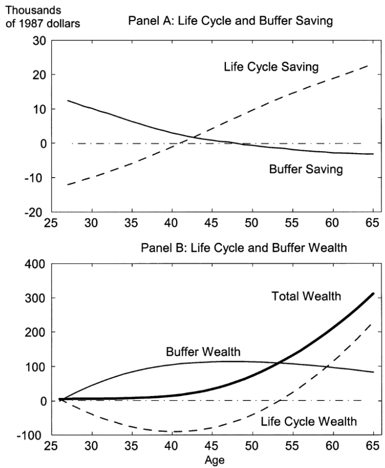
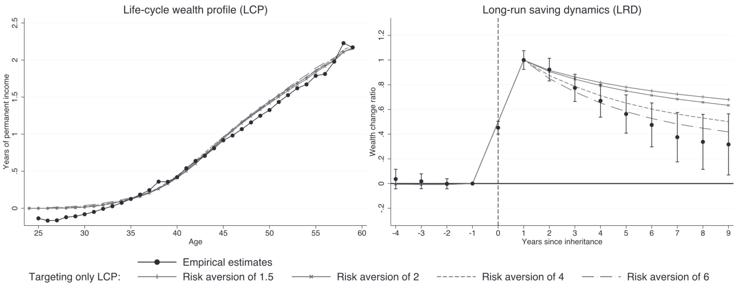


FIGURE 7.—The role of risk in saving and wealth accumulation.

- **Natural experiment:** Wealth depletion after sudden inheritance
- **Results:**
  1. Life-cycle profile of wealth fitted for many levels of risk-aversion (by varying the discount factor)
  2. Fast wealth depletation requires high risk-aversion (or high perceived risk)

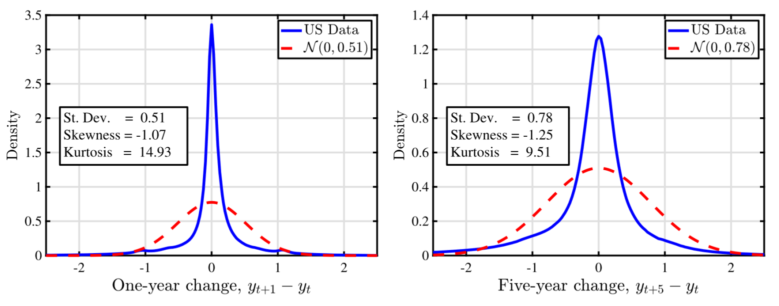
FIGURE 3.—REPLICATING LONG-RUN SAVING DYNAMICS



Source: Druedahl and Martinello (2022)

2. More realistic income risk (I)

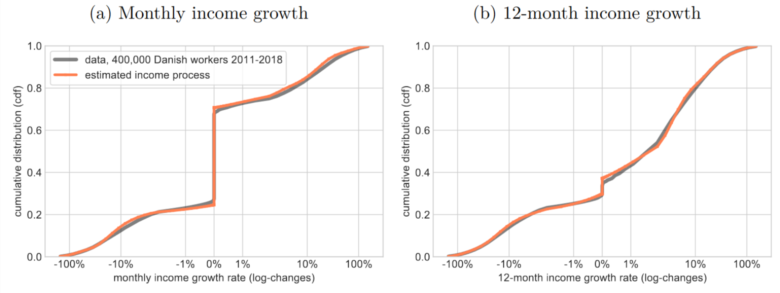
Annual earnings-changes are far from log-normal:



Source: Guvenen et. al. (2021)

2. More realistic income risk (II)

Many with zero-growth month-month:



Source: Druedahl et. al. (2021)

$$\begin{aligned} v_t(z_t, m_t) &= \max_{c_t} \left[(1-\beta) \cdot c_t^{1-\sigma} + \beta \cdot w_{t+1}^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \\ \text{s.t.} \quad & w_{t+1} \equiv \mathbb{E}_t \left[v_{t+1}(z_{t+1}, m_{t+1})^{1-\rho} \right]^{\frac{1}{1-\rho}} \\ & m_{t+1} = (1+r)(m_t - c_t) + y_{t+1} \end{aligned}$$

- **Preferences:**
    1. Patience: β
    2. Intertemporal substitution: σ
    3. Risk-aversion: ρ
- **Euler-equation:** $c_t^{-\sigma} = \beta R \cdot \mathbb{E}_t \left[c_{t+1}^{-\sigma} \cdot \left(\frac{w_{t+1}}{v_{t+1}} \right)^{\rho - \sigma} \right]$
    1. FOC: $0 = v_t^{\sigma} \cdot \left[(1-\beta) \cdot c_t^{-\sigma} - \beta R \cdot w_{t+1}^{\rho-\sigma} \cdot \mathbb{E}_t \left[v_{t+1}^{-\rho} \cdot \frac{\partial v_{t+1}}{\partial m_{t+1}} \right] \right]$
    2. Envelope condition: $\frac{\partial v_t(z_t, m_t)}{\partial m_t} = v_t^{\sigma} \cdot (1-\beta) \cdot c_t^{-\sigma}$

4. Deep learning

- **Curse of dimensionality:**
  1. Many states
  2. Many choices
  3. Many shocks
- **Deep (reinforcement) learning:**
  1. Approximate value and policy functions with *neural networks*
  2. Approximate on simulation sample rather than on grid
  3. Automatic differentiation (backpropagation) and GPUs for speed
- **Examples:** Maliar and Maliar (2021) and Azinovic and Scheidegger (2022)
- **Working paper:** Druedahl and Røpke (2025)
  Python package: EconDLSolvers

Summary

Summary and what's next

- **This lecture:**
  1. Consumption-saving models (precautionary-saving, buffer-stock target, intertemporal MPCs)
  2. Basic numerical dynamic programming (discretization, numerical integration, interpolation, VFI)
  3. EGM (time iteration, invert Euler-equation)
  4. Simulation (monte carlo, histogram)
- **Next:** *Stationary equilibrium*
- **You should:**
  1. Study the code from this lecture
  2. Glance at Aiyagari (1994),
     »Uninsured Idiosyncratic Risk and Aggregate Saving«