1. Consider

$$\dot{\mathbf{x}} = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} \mathbf{x} + \begin{bmatrix} b_1 \\ \bar{b_1} \end{bmatrix} u \qquad y = \begin{bmatrix} c_1 & \bar{c_1} \end{bmatrix} \mathbf{x}$$

where the overbear denotes complex conjugate. Verify that the equation can be formed into

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{b}}u \qquad y = \bar{\mathbf{c}}\bar{\mathbf{x}}$$

with

$$\bar{\mathbf{A}} = \begin{bmatrix} 0 & 1 \\ -\lambda\bar{\lambda} & \lambda + \bar{\lambda} \end{bmatrix} \qquad \bar{\mathbf{b}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \bar{\mathbf{c}} = \begin{bmatrix} -2\mathrm{Re}\left(\bar{\lambda} b_1 c_1\right) & 2\mathrm{Re}\left(b_1 c_1\right) \end{bmatrix}$$

by using the transformation $\mathbf{x} = \mathbf{Q}\bar{\mathbf{x}}$ with

$$\mathbf{Q} = \begin{bmatrix} -\bar{\lambda} b_1 & b_1 \\ -\lambda \bar{b_1} & \bar{b_1} \end{bmatrix}$$

Solution
Solution on following page.

2. Are the two sets of state equations

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \mathbf{x}$$

and

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \mathbf{x}$$

equivalent? Are they zero-state equivalent?

Solution

The two systems are equivalent provided that their characteristic polynomials match. Thus:

$$\Delta_1(\lambda) = (2-\lambda)^2 (1-\lambda) = 0$$
$$\Delta_2(\lambda) = (2-\lambda)^2 (1+\lambda) = 0$$

Since the characteristic polynomials do not match, the systems are not equivalent. The two systems are zero-state equivalent if their transfer functions are the same, that is if $\mathbf{D}_1 + \mathbf{C}_1(s\mathbf{I} - \mathbf{A}_1)^{-1}\mathbf{B}_1$ is equal to $\mathbf{D}_2 + \mathbf{C}_2(s\mathbf{I} - \mathbf{A}_2)^{-1}\mathbf{B}_2$. Thus:

$$\mathbf{D}_1 + \mathbf{C}_1(s\mathbf{I} - \mathbf{A}_1)^{-1}\mathbf{B}_1 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -\frac{1}{(s-2)^2}$$

$$\mathbf{D}_2 + \mathbf{C}_2(s\mathbf{I} - \mathbf{A}_2)^{-1}\mathbf{B}_2 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -\frac{1}{(s-2)^2}$$

Since the transfer functions match, the systems are zero-state equivalent.

3. Find a realization for each column of $\hat{\mathbf{G}}(s)$ below and then connect them, as shown in Fig.4.4(a), to obtain a realization of $\hat{\mathbf{G}}(s)$. What is the dimension of this realization?

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{2s-3}{(s+1)(s+2)} \\ \frac{s-2}{s+1} & \frac{s}{s+2} \end{bmatrix}$$

Solution

A realization for $\hat{\mathbf{G}}(s)$ is:

$$\dot{\mathbf{x}} = \left[\begin{array}{cc|cc} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & -3 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right] \mathbf{x} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u \qquad y = \left[\begin{array}{cc|cc} 2 & 0 & 2 & -3 \\ -3 & 0 & -2 & -2 \end{array}\right] \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} u$$

This realization is a 4×4 realization.

4. Find a realization for each row of $\hat{\mathbf{G}}(s)$ below 4.11 and then connect them, as shown in Fig.4.4(b), to obtain a realization of $\hat{\mathbf{G}}(s)$. What is the dimension of this realization?

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{2s-3}{(s+1)(s+2)} \\ \frac{s-2}{s+1} & \frac{s}{s+2} \end{bmatrix}$$

Solution

A realization for $\hat{\mathbf{G}}(s)$ is:

$$\dot{\mathbf{x}} = \left[\begin{array}{cc|cc} -3 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -3 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right] \mathbf{x} + \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ \hline 0 & 1 \\ 0 & 0 \end{array}\right] u \qquad y = \left[\begin{array}{cc|cc} 2 & 2 & 2 & -3 \\ \hline -3 & -3 & -6 & -2 \end{array}\right] \mathbf{x} + \left[\begin{array}{cc} 0 & 0 \\ \hline 1 & 1 \end{array}\right] u$$

This realization is a 4×4 realization.

5. Find a realization for

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{-(12s+6)}{3s+34} & \frac{22s+23}{3s+34} \end{bmatrix}$$

Solution

A realization for $\hat{\mathbf{G}}(s)$ is:

$$\dot{\mathbf{x}} = \left[\begin{array}{cc|cc} -1 & 0 & -\frac{34}{3} & 0 \\ 0 & -1 & 0 & -\frac{34}{3} \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \mathbf{x} + \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right] u, \qquad y = \left[\begin{array}{cc|cc} 0 & 0 & -130 & \frac{22}{3} \end{array}\right] \mathbf{x} + \begin{bmatrix} -4 & \frac{22}{3} \end{bmatrix} u$$

6. Find the state transition matrix of

$$\dot{\mathbf{x}} = \begin{bmatrix} -\sin t & 0 \\ 0 & -\cos t \end{bmatrix} \mathbf{x}$$

Solution

The state transition matrix was found to be:

$$\phi(t, t_0) = \begin{bmatrix} e^{\cos t - \cos t_0} & 0 \\ 0 & e^{-\sin t + \sin t_0} \end{bmatrix}$$

7. Verify that $\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}e^{\mathbf{B}t}$ is the solution of

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{XB} \qquad \mathbf{X}(0) = \mathbf{C}$$

Solution
Solution on following page.

8. Find a time-varying realization and a time-invariant realization of the impulse response $g(t) = t^2 e^{\lambda t}$.

Solution

A time-varying realization of $g(t)$ is:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^{-\lambda t} \\ -2te^{-\lambda t} \\ t^2 e^{-\lambda t} \end{bmatrix} u(t), \qquad y(t) = \begin{bmatrix} t^2 e^{\lambda t} & te^{\lambda t} & e^{\lambda t} \end{bmatrix} \mathbf{x}(t)$$

A time-invariant realization $g(t)$ is:

$$\dot{\mathbf{x}} = \begin{bmatrix} 3\lambda & -3\lambda^2 & \lambda^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix} \mathbf{x}$$