Solutions Manual

PRINCIPLES OF DYNAMICS

second edition

Donald T. Greenwood

Professor, Department of Aerospace Engineering
The University of Michigan

PRENTICE HALL, *Englewood Cliffs, New Jersey* 07632

SOLUTIONS MANUAL

PRINCIPLES OF DYNAMICS

Second Edition

Donald T. Greenwood

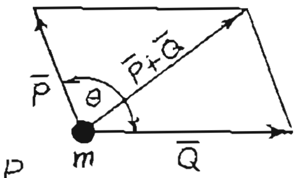
A Division of Simon & Schuster
Englewood Cliffs, N.J. 07632

ISBN: 0-13-710005-1

Printed in the United States of America

CHAPTER 1

1-1. Let $Q > P$.



$a_{max} = \frac{1}{m}(Q+P)$, $a_{min} = \frac{1}{m}(Q-P)$

$\frac{a_{max}}{a_{min}} = \frac{Q+P}{Q-P} = 3$, giving $Q = 2P$.

$F = ma$, $a = \frac{F}{m}$

Mean acceleration $= \frac{1}{2}(a_{max}+a_{min}) = \frac{Q}{m} = \frac{1}{m}|\bar{P}+\bar{Q}|$

Hence $Q = |\bar{P}+\bar{Q}|$. By cosine law, $|\bar{P}+\bar{Q}|^2 = Q^2+P^2+2QP\cos\theta = Q^2$

Then $P^2 = -2QP\cos\theta = -4P^2\cos\theta$ or $\cos\theta = -1/4$, $\underline{\theta = 104.48°}$

1-2. (a) Auto position $\bar{r}_a = 30t\,\bar{i}$ meters

Plane position $\bar{r}_p = 1000\,\bar{i} + 90t\,\bar{j} + 1000\,\bar{k}$ meters

Relative position $\bar{r}_{p/a} = \bar{r}_p - \bar{r}_a = (1000-30t)\,\bar{i} + 90t\,\bar{j} + 1000\,\bar{k}$

Relative velocity $\bar{v}_{p/a} = \frac{d}{dt}(\bar{r}_{p/a}) = \underline{-30\,\bar{i} + 90\,\bar{j}\ \ m/sec}$

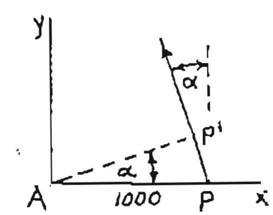
(b) $AP' = 1000\cos\alpha$, $\alpha = \tan^{-1}\frac{1}{3}$

or $AP' = 1000\,\frac{3}{\sqrt{10}}$

Separation $d_{min} = \sqrt{(AP')^2 + 1000^2}$

$= 1000\sqrt{\frac{9}{10}+1}$

$= \underline{1378.4 \text{ meters}}$



1-3. Given $\bar{e}_1 = l_1\bar{i} + l_2\bar{j} + l_3\bar{k}$

$\bar{e}_2 = m_1\bar{i} + m_2\bar{j} + m_3\bar{k}$

$\bar{e}_3 = n_1\bar{i} + n_2\bar{j} + n_3\bar{k}$

(a) $l_1^2 + l_2^2 + l_3^2 = 1$

$m_1^2 + m_2^2 + m_3^2 = 1$

$n_1^2 + n_2^2 + n_3^2 = 1$

(b)

$$\bar{e}_1\cdot(\bar{e}_2\times\bar{e}_3) = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 0$$

(c) Equations of part (a) and

$l_1m_1 + l_2m_2 + l_3m_3 = 0$

$l_1n_1 + l_2n_2 + l_3n_3 = 0$

$m_1n_1 + m_2n_2 + m_3n_3 = 0$

4. $\bar{A} = A_x \bar{i} + A_y \bar{j} + A_z \bar{k}$
$= A_1 \bar{e}_1 + A_2 \bar{e}_2 + A_3 \bar{e}_3$ where $\begin{cases} \bar{e}_1 = \bar{i} \\ \bar{e}_2 = \frac{1}{\sqrt{2}}(\bar{i}+\bar{j}) \\ \bar{e}_3 = \frac{1}{\sqrt{2}}(\bar{i}+\bar{k}) \end{cases}$

quating $\bar{i}$ components,

$$\left.\begin{aligned} A_1 + \frac{A_2}{\sqrt{2}} + \frac{A_3}{\sqrt{2}} &= A_x \\ \text{rom } \bar{j} \text{ components, } \frac{A_2}{\sqrt{2}} &= A_y \\ \text{rom } \bar{k} \text{ components, } \frac{A_3}{\sqrt{2}} &= A_z \end{aligned}\right\} \text{solving,} \quad \begin{aligned} A_1 &= A_x - A_y - A_z \\ A_2 &= \sqrt{2}\, A_y \\ A_3 &= \sqrt{2}\, A_z \end{aligned}$$

5. Given $[L]$, $[M]$, $[F]$ are fundamental.
$= ma$, so the dimensions of acceleration $[A] = [M^{-1} F]$
$A] = [L T^{-2}]$ or $[T] = [L^{\frac{1}{2}} A^{-\frac{1}{2}}] = [L^{\frac{1}{2}} M^{\frac{1}{2}} F^{-\frac{1}{2}}]$

-6. Fundamental units: mass — kg, length — ell, time — tee

.et 1 ell = a meters
1 tee = b sec

Density of water $\rho = 1\ g/cm^3 = 1000\ kg/m^3$

Specific wt. $w = \rho g = 9810 \frac{kg}{m^2 sec^2} = 1 \frac{kg}{ell^2\ tee^2}$

Hence $1 = 9810 \left(\frac{ell}{m}\right)^2 \left(\frac{tee}{sec}\right)^2 = 9810\, a^2 b^2$

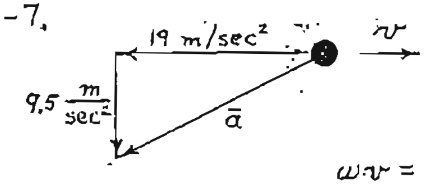
Accel. of gravity $= 9.81 \frac{m}{sec^2} = 1 \frac{ell}{tee^2}$ or $9.81 = \frac{a}{b^2}$

Multiply equations. $9.81 = 9810\, a^3$ giving $a = 0.1$

Then $b = \sqrt{\frac{a}{9.81}} = \frac{1}{\sqrt{98.1}} = 0.1010$

1 ell = 0.1000 meters
1 tee = 0.1010 sec

1-7.

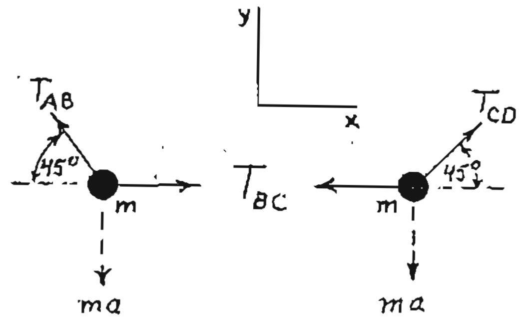


$a = \sqrt{19^2 + 9.5^2} = 21.243 \frac{m}{sec^2}$

For $v = 3000$ m/sec,

$\omega v = 9.5 \frac{m}{sec^2}$, $\omega = \frac{9.5}{3000} = 3.167 \times 10^{-3} \frac{rad}{sec}$

1-8. Use d'Alembert's principle.



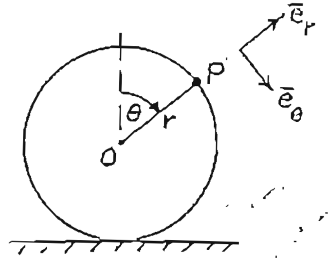
Sum y forces on system. $\frac{1}{\sqrt{2}}(T_{AB} + T_{CD}) = 2ma$

Sum x forces on each particle. $T_{BC} = \frac{T_{AB}}{\sqrt{2}} = \frac{T_{CD}}{\sqrt{2}}$

Then $T_{AB} = T_{CD} = \sqrt{2}\, ma$, $T_{BC} = ma$

2-1. $\omega = \alpha t$ where $\alpha = \text{const.}$

Integrate to obtain $\theta = \frac{1}{2}\alpha t^2$



Acceleration of P is

$$\bar{a} = \bar{a}_o + \bar{a}_{P/O}$$

where the horizontal acceleration $\bar{a}_o = r\alpha(\sin\theta\,\bar{e}_r + \cos\theta\,\bar{e}_\theta)$

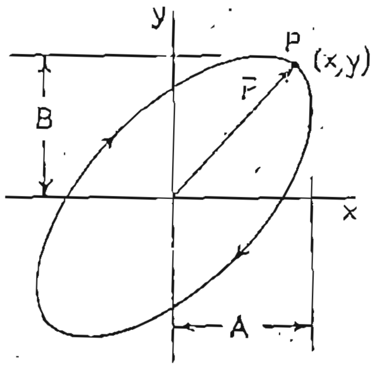
P moves in a circular path relative to O.

$$\bar{a}_{P/O} = -r(\alpha t)^2\,\bar{e}_r + r\alpha\,\bar{e}_\theta$$

Hence. $\underline{\bar{a} = r\alpha\left[(\sin\theta - \alpha t^2)\,\bar{e}_r + (1+\cos\theta)\,\bar{e}_\theta\right]}$

2-2. Given $x = A\cos\omega t$

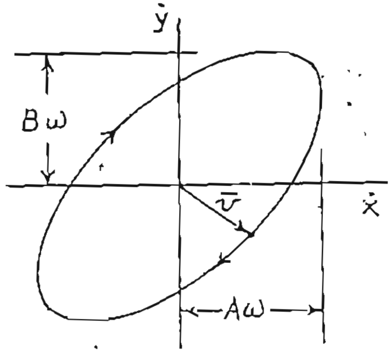
$y = B\cos(\omega t + \beta)$



Then $\dot{x} = -A\omega\sin\omega t$

$\dot{y} = -B\omega\sin(\omega t + \beta)$

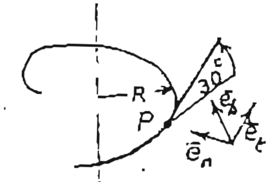
The hodograph is a plot of $\bar{v} = \dot{x}\,\bar{i} + \dot{y}\,\bar{j}$



Note that the two ellipses have the same shape.

2-3. Speed $s = v_o$ and $\dot{s} = a_o$

$$\bar{a} = \ddot{s}\,\bar{e}_t + \frac{\dot{s}^2}{\rho}\,\bar{e}_n$$



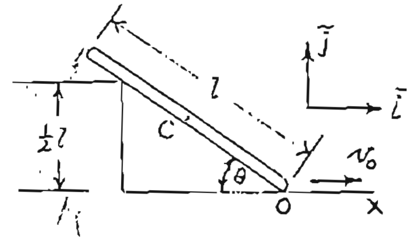
From Eq. (2-61), $\rho = R(1+k^2)$

$= R(1+\tan^2 30^\circ) = \frac{4}{3}R$

Then we obtain $\underline{\bar{a} = a_o\,\bar{e}_t + \frac{3v_o^2}{4R}\,\bar{e}_n}$

2nd method: Use the horizontal component of velocity and a circular path of radius R to obtain $a_n = \frac{(v_o\cos 30^\circ)^2}{R} = \frac{3v_o^2}{4R}$

2-4. Position x of lower end O is



$$x = \frac{1}{2}l\cot\theta$$

giving $\dot{x} = -\frac{1}{2}l\dot{\theta}\csc^2\theta = v_o$

or

$$\underline{\dot{\theta} = -\frac{2v_o}{l}\sin^2\theta}$$

(b) Differentiate again. $\ddot{\theta} = \frac{-4v_o\dot{\theta}}{l}\sin\theta\cos\theta$

Using $\dot{\theta}$ expression, $\underline{\ddot{\theta} = \frac{8v_o^2}{l^2}\sin^3\theta\cos\theta}$

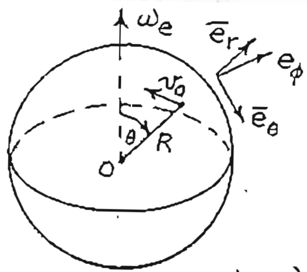
(c) $\bar{v}_c = \bar{v}_o + \bar{v}_{c/o} = v_o\bar{i} + \frac{1}{2}l\dot{\theta}(\sin\theta\,\bar{i} + \cos\theta\,\bar{j})$

$= v_o\bar{i} + \frac{1}{2}l\left(\frac{-2v_o}{l}\sin^2\theta\right)(\sin\theta\,\bar{i} + \cos\theta\,\bar{j})$

or

$$\underline{\bar{v}_c = v_o\left[(1-\sin^3\theta)\,\bar{i} - \sin^2\theta\cos\theta\,\bar{j}\right]}$$

2-5. Let moving frame rotate with the earth. $\bar{\omega}=\bar{\omega}_e$, $\theta = 45°$



$\bar{a}=\ddot{\bar{R}}+\dot{\bar{\omega}}\times\bar{\rho}+\bar{\omega}\times(\bar{\omega}\times\bar{\rho})+(\ddot{\bar{\rho}})_r+2\bar{\omega}\times(\dot{\bar{\rho}})_r$

Acceleration of O is $\ddot{\bar{R}}=0$

$\dot{\bar{\omega}}=0$, $\bar{\rho}=R\bar{e}_r$, so $\dot{\bar{\omega}}\times\bar{\rho}=0$

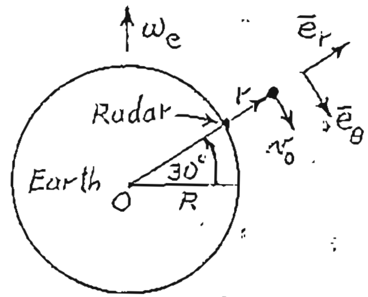
$\bar{\omega}\times(\bar{\omega}\times\bar{\rho})=\frac{R}{\sqrt{2}}\omega_e^2\left(-\frac{1}{\sqrt{2}}\bar{e}_r-\frac{1}{\sqrt{2}}\bar{e}_\theta\right)$ $\qquad \bar{\omega}=\omega_e\left(\frac{1}{\sqrt{2}}\bar{e}_r-\frac{1}{\sqrt{2}}\bar{e}_\theta\right)$

$(\ddot{\bar{\rho}})_r=-\frac{v_0^2}{R}\bar{e}_r$ $\qquad (\dot{\bar{\rho}})_r=v_0\left(\frac{-1}{\sqrt{2}}\bar{e}_\theta-\frac{1}{\sqrt{2}}\bar{e}_\phi\right)$

$2\bar{\omega}\times(\dot{\bar{\rho}})_r=\omega_e v_0\left(-\bar{e}_r\times\bar{e}_\theta-\bar{e}_r\times\bar{e}_\phi+\bar{e}_\theta\times\bar{e}_\phi\right)=\omega_e v_0(\bar{e}_r+\bar{e}_\theta-\bar{e}_\phi)$

Adding, $\underline{\bar{a}=\left(-\frac{1}{2}R\omega_e^2-\frac{v_0^2}{R}+\omega_e v_0\right)\bar{e}_r+\left(-\frac{1}{2}R\omega_e^2+\omega_e v_0\right)\bar{e}_\theta-\omega_e v_0\bar{e}_\phi}$

2-6. Rotating frame is fixed in the earth, with origin at O.



$\bar{\rho}=r\bar{e}_r$, $\quad\bar{\omega}=\omega_e\left(\frac{1}{2}\bar{e}_r-\frac{\sqrt{3}}{2}\bar{e}_\theta\right)$

From (2-102), $\bar{v}=\dot{\bar{R}}+(\dot{\bar{\rho}})_r+\bar{\omega}\times\bar{\rho}$.

We know $\bar{v}=v_0\bar{e}_\theta$. Solve for $(\dot{\bar{\rho}})_r$.

$\dot{\bar{R}}=0$, $\bar{\omega}\times\bar{\rho}=\frac{\sqrt{3}}{2}r\omega_e\bar{e}_\phi$, so $\qquad \underline{(\dot{\bar{\rho}})_r=v_0\bar{e}_\theta-\frac{\sqrt{3}}{2}r\omega_e\bar{e}_\phi}$

From (2-106),

$\bar{a}=\ddot{\bar{R}}+\dot{\bar{\omega}}\times\bar{\rho}+\bar{\omega}\times(\bar{\omega}\times\bar{\rho})+(\ddot{\bar{\rho}})_r+2\bar{\omega}\times(\dot{\bar{\rho}})_r$, $\qquad \ddot{\bar{R}}=0$, $\dot{\bar{\omega}}\times\bar{\rho}=0$

We know $\bar{a}=-\frac{v_0^2}{r}\bar{e}_r$. Solve for $(\ddot{\bar{\rho}})_r$.

Now $\quad\bar{\omega}\times(\bar{\omega}\times\bar{\rho})=\frac{\sqrt{3}}{2}r\omega_e^2\left(-\frac{\sqrt{3}}{2}\bar{e}_r-\frac{1}{2}\bar{e}_\theta\right)=-\frac{3}{4}r\omega_e^2\bar{e}_r-\frac{\sqrt{3}}{4}r\omega_e^2\bar{e}_\theta$

$2\bar{\omega}\times(\dot{\bar{\rho}})_r=2\omega_e\left(\frac{1}{2}\bar{e}_r-\frac{\sqrt{3}}{2}\bar{e}_\theta\right)\times\left(v_0\bar{e}_\theta-\frac{\sqrt{3}}{2}r\omega_e\bar{e}_\phi\right)$

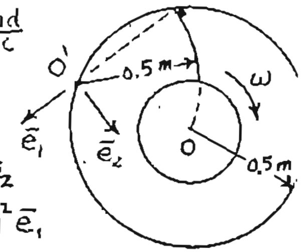
$=\omega_e v_0\bar{e}_\phi+\frac{\sqrt{3}}{2}r\omega_e^2\bar{e}_\theta+\frac{3}{2}r\omega_e^2\bar{e}_r$

Hence we obtain

$\underline{(\ddot{\bar{\rho}})_r=\left(-\frac{v_0^2}{r}-\frac{3}{4}r\omega_e^2\right)\bar{e}_r-\frac{\sqrt{3}}{4}r\omega_e^2\bar{e}_\theta-\omega_e v_0\bar{e}_\phi}$

Note that the result is independent of earth's radius R.

*2-7. Moving system has origin at O' and rotates with impeller at $\omega=40\pi\ \frac{\text{rad}}{\text{sec}}$



From (2-106),

$\bar{a}=\ddot{\bar{R}}+\dot{\bar{\omega}}\times\bar{\rho}+\bar{\omega}\times(\bar{\omega}\times\bar{\rho})+(\ddot{\bar{\rho}})_r+2\bar{\omega}\times(\dot{\bar{\rho}})_r$

$\ddot{\bar{R}}=0.5(40\pi)^2\left(-\frac{1}{2}\bar{e}_1+\frac{\sqrt{3}}{2}\bar{e}_2\right)=-3948\bar{e}_1+6838\bar{e}_2$

$\dot{\bar{\omega}}=0$, $\quad\bar{\rho}=-0.5\,\bar{e}_1$, $\quad\bar{\omega}\times(\bar{\omega}\times\bar{\rho})=0.5(40\pi)^2\bar{e}_1$
$=7896\,\bar{e}_1$

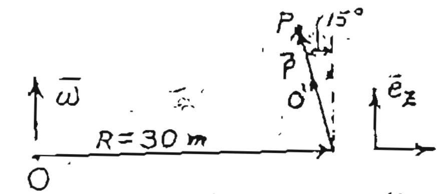
$(\ddot{\bar{\rho}})_r=\frac{(20)^2}{0.5}\bar{e}_1=800\bar{e}_1$, $\quad(\dot{\bar{\rho}})_r=-20\,\bar{e}_2$

$2\bar{\omega}\times(\dot{\bar{\rho}})_r=-(80\pi)(20)\bar{e}_1=-5027\,\bar{e}$

Adding terms, $\quad\underline{\bar{a}=-278.7\,\bar{e}_1+6838\,\bar{e}_2\ \ m/sec^2}$

2-8. $\rho=0.4$ m

$\bar{\omega}=\frac{10}{30}\bar{e}_z$



Take O' at hub of wheel.

From (2-106), $\bar{a}=\ddot{\bar{R}}+\dot{\bar{\omega}}\times\bar{\rho}+\bar{\omega}\times(\bar{\omega}\times\bar{\rho})+(\ddot{\bar{\rho}})_r+2\bar{\omega}\times(\dot{\bar{\rho}})_r$

$\bar{R}=(30-0.4\sin 15°)\bar{e}_r+0.4\cos 15°\bar{e}_z=29.896\,\bar{e}_r+0.3864$

$\bar{\rho}=0.4\sin 15°\,\bar{e}_r+0.4\cos 15°\,\bar{e}_z=-0.1035\,\bar{e}_r+0.3864$

$\ddot{\bar{R}}=-29.896\left(\frac{1}{3}\right)^2\bar{e}_r=-3.322\,\bar{e}_r$, $\qquad\dot{\bar{\omega}}\times\bar{\rho}=0$

$\bar{\omega}\times(\bar{\omega}\times\bar{\rho})=0.1035\left(\frac{1}{3}\right)^2\bar{e}_r=0.0115\,\bar{e}_r$

$(\ddot{\bar{\rho}})_r=\frac{10^2}{0.4}\left(\sin 15°\,\bar{e}_r-\cos 15°\bar{e}_z\right)=64.705\,\bar{e}_r-241.481$

since the relative velocity $(\dot{\bar{\rho}})_r=10\,\bar{e}_\phi$

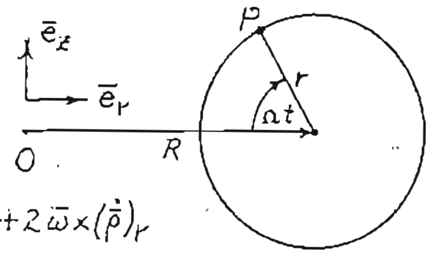
Then $\quad 2\bar{\omega}\times(\dot{\bar{\rho}})_r=-\frac{20}{3}\bar{e}_r$

Adding terms, we obtain

$\underline{\bar{a}=54.73\,\bar{e}_r-241.5\,\bar{e}_z\ \ m/sec^2}$

2-9. $\bar R = R\bar e_r$, $\bar\omega = \frac{v}{R}\bar e_z = \text{const.}$ $\quad \bar e_\phi$ is into page

$\bar\rho = r(-\cos\Omega t\,\bar e_r + \sin\Omega t\,\bar e_z)$



$(\dot{\bar\rho})_r = r\Omega(\sin\Omega t\,\bar e_r + \cos\Omega t\,\bar e_z)$

From (2-10b),

$\bar a = \ddot{\bar R} + \dot{\bar\omega}\times\bar\rho + \bar\omega\times(\bar\omega\times\bar\rho) + (\ddot{\bar\rho})_r + 2\bar\omega\times(\dot{\bar\rho})_r$

$\ddot{\bar R} = -R\omega^2\bar e_r = -\frac{v^2}{R}\bar e_r, \quad \dot{\bar\omega}\times\bar\rho = 0$

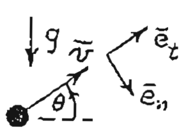
$\bar\omega\times(\bar\omega\times\bar\rho) = r\left(\frac{v}{R}\right)^2\cos\Omega t\,\bar e_r, \quad (\ddot{\bar\rho})_r = r\Omega^2(\cos\Omega t\,\bar e_r - \sin\Omega t\,\bar e_z)$

$2\bar\omega\times(\dot{\bar\rho})_r = \frac{2r}{R}v\Omega\sin\Omega t\,\bar e_\phi$

Adding, $\bar a = \left[-\frac{v^2}{R} + r\left(\frac{v^2}{R^2}+\Omega^2\right)\cos\Omega t\right]\bar e_r + 2\frac{r}{R}v\Omega\sin\Omega t\,\bar e_\phi - r\Omega^2\sin\Omega t\,\bar e_z$

2-10. (a) $a_n = g\cos\theta$

(b) $a_t = -g\sin\theta$



(c) Also $a_n = \frac{v^2}{\rho}$ or $\rho = \frac{v^2}{a_n} = \frac{v^2}{g\cos\theta}$

(d) Also $a_n = -v\dot\theta$ or $\dot\theta = -\frac{g}{v}\cos\theta$

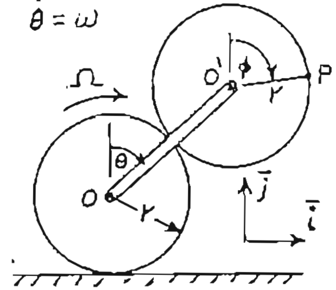
(e) $\dot\rho = \frac{2v\dot v}{g\cos\theta} + \frac{v^2\dot\theta\sin\theta}{g\cos^2\theta}$ where $\dot v = a_t = -g\sin\theta$

or $\dot\rho = -2v\tan\theta - v\tan\theta = -3v\tan\theta$

(f) $\ddot\theta = \frac{g}{v^2}\dot v\cos\theta + \frac{g\dot\theta}{v}\sin\theta = -\frac{2g^2}{v^2}\sin\theta\cos\theta$

or $\ddot\theta = -\frac{g^2}{v^2}\sin 2\theta$

2-11. (a) $v_{0/0'} = 2r\dot\theta = 2r\omega$



Also $v_{0/0'} = r\Omega + r\dot\phi$

i.e., the velocity of the contact point of disks relative to O plus the velocity of O' relative to the contact point. Hence

$\dot\phi = 2\omega - \Omega$

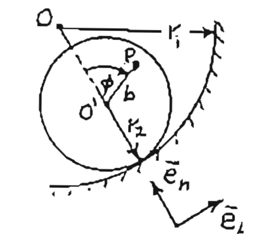
(b) $\bar a = \bar a_{0'} + \bar a_{P/0'}$ $\quad$ Note that $\bar a_0 = 0$.

Then $\bar a_{0'} = 2r\omega^2(-\sin\theta\,\bar i - \cos\theta\,\bar j)$

$\bar a_{P/0'} = r\dot\phi^2(-\sin\phi\,\bar i - \cos\phi\,\bar j)$

$\bar a = [-2r\omega^2\sin\theta - r(2\omega-\Omega)^2\sin\phi]\bar i + [-2r\omega^2\cos\theta - r(2\omega-\Omega)^2\cos\phi]\bar j$

2-12. Moving system rotates with OO'. Its angular velocity ω is found from



$v_{0'} = (r_1-r_2)\omega = r_2(\dot\phi-\omega), \quad \bar\omega = \frac{r_2}{r_1}\dot\phi\,\bar e_b$

Use $\bar a = \ddot{\bar R} + \dot{\bar\omega}\times\bar\rho + \bar\omega\times(\bar\omega\times\bar\rho) + (\ddot{\bar\rho})_r + 2\bar\omega\times(\dot{\bar\rho})_r$

$\bar\rho = b(\sin\phi\,\bar e_t + \cos\phi\,\bar e_n)$

$\bar e_b$ out of page

$(\dot{\bar\rho})_r = b\dot\phi(\cos\phi\,\bar e_t - \sin\phi\,\bar e_n)$

$\ddot{\bar R} = (r_1-r_2)\left(\frac{r_2}{r_1}\dot\phi\right)^2\bar e_n + (r_1-r_2)\left(\frac{r_2}{r_1}\ddot\phi\right)\bar e_t, \quad \dot{\bar\omega}\times\bar\rho = \frac{br_2}{r_1}\ddot\phi(-\cos\phi\,\bar e_t + \sin\phi\,\bar e_n)$

$\bar\omega\times(\bar\omega\times\bar\rho) = b\left(\frac{r_2}{r_1}\dot\phi\right)^2(-\sin\phi\,\bar e_t - \cos\phi\,\bar e_n)$

$(\ddot{\bar\rho})_r = b\ddot\phi(\cos\phi\,\bar e_t - \sin\phi\,\bar e_n) + b\dot\phi^2(-\sin\phi\,\bar e_t - \cos\phi\,\bar e_n)$

$2\bar\omega\times(\dot{\bar\rho})_r = 2\frac{br_2}{r_1}\dot\phi^2(\sin\phi\,\bar e_t + \cos\phi\,\bar e_n)$

Adding, and collecting terms,

$\bar a = \left[\left(1-\frac{r_2}{r_1}\right)(r_2 + b\cos\phi)\ddot\phi - \left(1-\frac{r_2}{r_1}\right)^2 b\dot\phi^2\sin\phi\right]\bar e_t$

$+ \left\{-\left(1-\frac{r_2}{r_1}\right)b\ddot\phi\sin\phi + \left[\left(1-\frac{r_2}{r_1}\right)\frac{r_2^2}{r_1} - \left(1-\frac{r_2}{r_1}\right)^2 b\cos\phi\right]\dot\phi^2\right\}\bar e_n$

*13. Origin of moving frame is at O'.

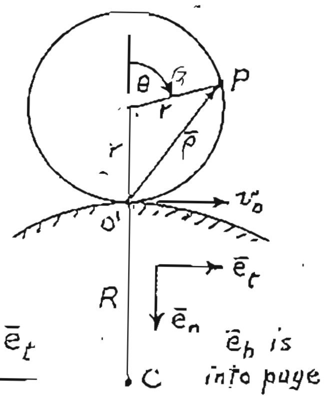
$\bar{\omega} = \frac{v_0}{R}\bar{e}_b$, $\bar{\rho} = -r(1+\cos\theta)\bar{e}_n + r\sin\theta\,\bar{e}_t$

$\bar{v} = \dot{\bar{R}} + (\dot{\bar{\rho}})_r + \bar{\omega}\times\bar{\rho}$ where $\dot{\bar{R}} = v_0\bar{e}_t$

$(\dot{\bar{\rho}})_r = r\dot{\theta}(\sin\theta\,\bar{e}_n + \cos\theta\,\bar{e}_t)$

$= v_0(\sin\theta\,\bar{e}_n + \cos\theta\,\bar{e}_t)$ since $r\dot{\theta} = v_0$

$\bar{\omega}\times\bar{\rho} = \frac{r}{R}v_0[\sin\theta\,\bar{e}_n + (1+\cos\theta)\bar{e}_t]$



Then $\bar{v} = v_0(1+\frac{r}{R})\sin\theta\,\bar{e}_n + v_0(1+\frac{r}{R})(1+\cos\theta)\bar{e}_t$

$\bar{a} = \ddot{\bar{R}} + \dot{\bar{\omega}}\times\bar{\rho} + \bar{\omega}\times(\bar{\omega}\times\bar{\rho}) + (\ddot{\bar{\rho}})_r + 2\bar{\omega}\times(\dot{\bar{\rho}})_r$, $\dot{\bar{\omega}} = -\frac{v_0}{R^2}\dot{R}\,\bar{e}_b$

$\ddot{\bar{R}} = \frac{v_0^2}{R}\bar{e}_n$; $\dot{\bar{\omega}}\times\bar{\rho} = -\frac{r\dot{R}}{R^2}v_0[\sin\theta\,\bar{e}_n + (1+\cos\theta)\bar{e}_t]$

$\bar{\omega}\times(\bar{\omega}\times\bar{\rho}) = \frac{r}{R^2}v_0^2[(1+\cos\theta)\bar{e}_n - \sin\theta\,\bar{e}_t]$

$(\ddot{\bar{\rho}})_r = v_0\dot{\theta}(\cos\theta\,\bar{e}_n - \sin\theta\,\bar{e}_t) = \frac{v_0^2}{r}(\cos\theta\,\bar{e}_n - \sin\theta\,\bar{e}_t)$

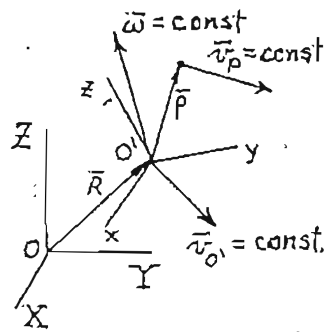
$\bar{\omega}\times(\dot{\bar{\rho}})_r = 2\frac{v_0^2}{R}(\cos\theta\,\bar{e}_n - \sin\theta\,\bar{e}_t)$

Adding, $\bar{a} = \left[\frac{v_0^2}{r}\left(1+\frac{r}{R}\right)^2\cos\theta + \frac{v_0^2}{R}\left(1+\frac{r}{R}\right) - \frac{r v_0}{R^2}\dot{R}\sin\theta\right]\bar{e}_n$
$- \left[\frac{v_0^2}{r}\left(1+\frac{r}{R}\right)^2\sin\theta + \frac{r v_0}{R^2}\dot{R}(1+\cos\theta)\right]\bar{e}_t$

2-14. (a) $\dot{\bar{\rho}} = \bar{v}_P - \bar{v}_{O'}$ = const

(b) $\dot{\bar{\rho}} = (\dot{\bar{\rho}})_r + \bar{\omega}\times\bar{\rho}$ or

$(\dot{\bar{\rho}})_r = \bar{v}_P - \bar{v}_{O'} - \bar{\omega}\times\bar{\rho}$



(c) $\dot{\bar{v}}_P = \dot{\bar{v}}_0 = 0$ so $\ddot{\bar{\rho}} = \dot{\bar{v}}_P - \dot{\bar{v}}_0 = 0$

(d) $\bar{a}_P = 0$. Using (2-106), we obtain

$(\ddot{\bar{\rho}})_r = -\bar{\omega}\times(\bar{\omega}\times\bar{\rho}) - 2\bar{\omega}\times(\dot{\bar{\rho}})_r = -\bar{\omega}\times(\bar{\omega}\times\bar{\rho}) - 2\bar{\omega}\times(\bar{v}_P - \bar{v}_{O'} - \bar{\omega}\times\bar{\rho})$

$= \bar{\omega}\times(\bar{\omega}\times\bar{\rho}) - 2\bar{\omega}\times(\bar{v}_P - \bar{v}_{O'})$

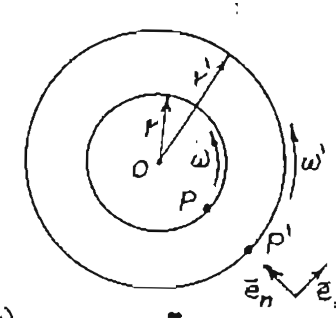
(e) Using (b), then (a),

$\frac{d}{dt}[(\dot{\bar{\rho}})_r] = \dot{\bar{v}}_P - \dot{\bar{v}}_{O'} - \dot{\bar{\omega}}\times\bar{\rho} - \bar{\omega}\times\dot{\bar{\rho}} = -\bar{\omega}\times(\bar{v}_P - \bar{v}_{O'})$

2-15. (a) Nonrotating observer.

$\bar{v}_{P'/P} = \bar{v}_{P'} - \bar{v}_P = (r'\omega' - r\omega)\bar{e}_t$

$\bar{a}_{P'/P} = \bar{a}_{P'} - \bar{a}_P = r'\omega'^2\bar{e}_n - r\omega^2\bar{e}_n$
$= (r'\omega'^2 - r\omega^2)\bar{e}_n$

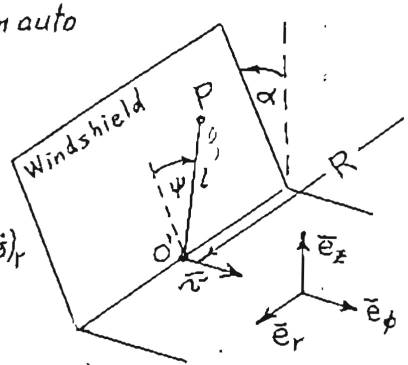


(b) Observer, rotating with unprimed system, sees the primed system rotating counterclockwise at $(\omega' - \omega)$.

$\bar{v}_{P'/P} = r'(\omega'-\omega)\bar{e}_t$, $\bar{a}_{P'/P} = r'(\omega'-\omega)^2\bar{e}_n$

2-16. Moving system is fixed in auto with origin at O'. $\bar{\omega} = \frac{v}{R}\bar{e}_z$

$\bar{\rho} = l(-\sin\psi\,\bar{e}_r - \cos\psi\sin\alpha\,\bar{e}_\phi + \cos\psi\cos\alpha\,\bar{e}_z)$



where $\psi = \psi_0\sin\beta t$. Use

$\bar{a} = \ddot{\bar{R}} + \dot{\bar{\omega}}\times\bar{\rho} + \bar{\omega}\times(\bar{\omega}\times\bar{\rho}) + (\ddot{\bar{\rho}})_r + 2\bar{\omega}\times(\dot{\bar{\rho}})_r$

$\ddot{\bar{R}} = -\frac{v^2}{R}\bar{e}_r$, $\dot{\bar{\omega}}\times\bar{\rho} = 0$

$\bar{\omega}\times(\bar{\omega}\times\bar{\rho}) = \frac{l v^2}{R^2}(\sin\psi\,\bar{e}_r + \cos\psi\sin\alpha\,\bar{e}_\phi)$

Now $(\dot{\bar{\rho}})_r = l\dot{\psi}(-\cos\psi\,\bar{e}_r + \sin\psi\sin\alpha\,\bar{e}_\phi - \sin\psi\cos\alpha\,\bar{e}_z)$

$2\bar{\omega}\times(\dot{\bar{\rho}})_r = \frac{2vl\dot{\psi}}{R}(-\sin\psi\sin\alpha\,\bar{e}_r - \cos\psi\,\bar{e}_\phi)$

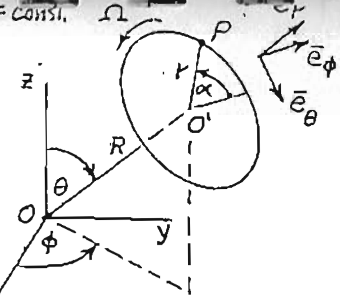
$(\ddot{\bar{\rho}})_r = l\ddot{\psi}(-\cos\psi\,\bar{e}_r + \sin\psi\sin\alpha\,\bar{e}_\phi - \sin\psi\cos\alpha\,\bar{e}_z)$
$+ l\dot{\psi}^2(\sin\psi\,\bar{e}_r + \cos\psi\sin\alpha\,\bar{e}_\phi - \cos\psi\cos\alpha\,\bar{e}_z)$

Adding terms, we obtain

$\bar{a} = \left[-\frac{v^2}{R} + \frac{v^2}{R^2}l\sin\psi + l\dot{\psi}^2\sin\psi - l\ddot{\psi}\cos\psi - \frac{2vl\dot{\psi}}{R}\sin\psi\sin\alpha\right]\bar{e}_r$
$+ \left[\frac{v^2}{R^2}l\cos\psi\sin\alpha + l\dot{\psi}^2\cos\psi\sin\alpha + l\ddot{\psi}\sin\psi\sin\alpha - \frac{2vl\dot{\psi}}{R}\cos\psi\right]\bar{e}_\phi$
$+ \left[-l\ddot{\psi}\sin\psi\cos\alpha - l\dot{\psi}^2\cos\psi\cos\alpha\right]\bar{e}_z$

Largest force occurs for the largest acceleration normal to windshield (outward). $\bar{\omega}\times(\bar{\omega}\times\bar{\rho}) + 2\bar{\omega}\times(\dot{\bar{\rho}})_r$ max. normal when $\dot{\psi} = -\beta$

2-[illegible] Given [illegible] $\dot\alpha =$ [illegible] const., r = const., θ = const.



Moving system with origin O' rotates at $\bar\omega = \dot\phi\,\bar e_z$ or $\bar\omega = \dot\phi(\cos\theta\,\bar e_r - \sin\theta\,\bar e_\theta)$

Also $O'P$ or $\bar\rho = r(-\sin\alpha\,\bar e_\theta + \cos\alpha\,\bar e_\phi)$

Use

$$\bar a = \ddot{\bar R} + \dot{\bar\omega}\times\bar\rho + \bar\omega\times(\bar\omega\times\rho) + (\ddot{\bar\rho})_r + 2\bar\omega\times(\dot{\bar\rho})_r$$

$$\ddot{\bar R} = R\dot\phi^2\sin\theta\,(-\sin\theta\,\bar e_r - \cos\theta\,\bar e_\theta)$$

$$\dot{\bar\omega}\times\bar\rho = 0$$

Now $$\bar\omega\times\bar\rho = r\dot\phi\begin{vmatrix}\bar e_r & \bar e_\theta & \bar e_\phi\\ \cos\theta & -\sin\theta & 0\\ 0 & -\sin\alpha & \cos\alpha\end{vmatrix} = r\dot\phi(-\cos\alpha\sin\theta\,\bar e_r - \cos\alpha\cos\theta\,\bar e_\theta - \sin\alpha\cos\theta\,\bar e_\phi)$$

and we obtain

$$\bar\omega\times(\bar\omega\times\bar\rho) = r\dot\phi^2\begin{vmatrix}\bar e_r & \bar e_\theta & \bar e_\phi\\ \cos\theta & -\sin\theta & 0\\ -\cos\alpha\sin\theta & -\cos\alpha\cos\theta & -\sin\alpha\cos\theta\end{vmatrix}$$

$$= r\dot\phi^2(\sin\alpha\sin\theta\cos\theta\,\bar e_r + \sin\alpha\cos^2\theta\,\bar e_\theta - \cos\alpha\,\bar e_\phi)$$

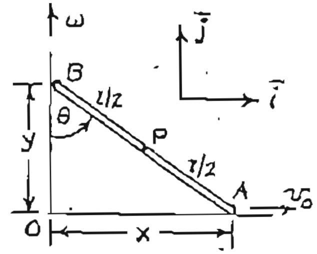
$$(\dot{\bar\rho})_r = r\Omega(-\cos\alpha\,\bar e_\theta - \sin\alpha\,\bar e_\phi),\quad (\ddot{\bar\rho}) = r\Omega^2(\sin\alpha\,\bar e_\theta - \cos\alpha\,\bar e_\phi)$$

$$2\bar\omega\times(\dot{\bar\rho})_r = 2r\Omega\dot\phi(\sin\alpha\sin\theta\,\bar e_r + \sin\alpha\cos\theta\,\bar e_\theta - \cos\alpha\cos\theta\,\bar e_\phi)$$

Adding terms,

$$\bar a = [-R\dot\phi^2\sin^2\theta + r\dot\phi^2\sin\alpha\sin\theta\cos\theta + 2r\Omega\dot\phi\sin\alpha\sin\theta]\,\bar e_r$$
$$+ [-R\dot\phi^2\sin\theta\cos\theta + r\dot\phi^2\sin\alpha\cos^2\theta + r\Omega^2\sin\alpha + 2r\Omega\dot\phi\sin\alpha\cos\theta]\,\bar e_\theta$$
$$+ [-r\dot\phi^2\cos\alpha - r\Omega^2\cos\alpha - 2r\Omega\dot\phi\cos\alpha\cos\theta]\,\bar e_\phi$$

2-18. Take origin of rotating system at O.



$x = l\sin\theta$, $\dot x = l\dot\theta\cos\theta$ or $\dot\theta = \dfrac{v_0}{l\cos\theta}$

$\bar\omega = \omega\bar j$ = const, $\bar\rho = \frac{l}{2}(\sin\theta\,\bar i + \cos\theta\,\bar j)$

$$\bar v = \dot{\bar R} + (\dot{\bar\rho})_r + \bar\omega\times\bar\rho$$

$$\dot{\bar R} = 0,\quad (\dot{\bar\rho})_r = \frac{l\dot\theta}{2}(\cos\theta\,\bar i - \sin\theta\,\bar j) = \frac{v_0}{2\cos\theta}(\cos\theta\,\bar i - \sin\theta\,\bar j)$$

$$\bar\omega\times\bar\rho = -\frac{\omega l}{2}\sin\theta\,\bar k$$

Hence $$\bar v = \tfrac{1}{2}v_0\,\bar i - \tfrac{1}{2}v_0\tan\theta\,\bar j - \tfrac{1}{2}\omega l\sin\theta\,\bar k$$

2-18. (cont'd.) From (2-106),

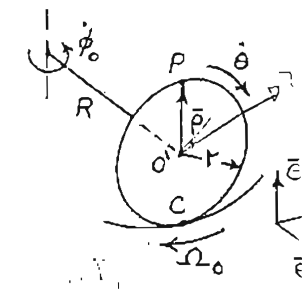
$$\bar a = \ddot{\bar R} + \dot{\bar\omega}\times\bar\rho + \bar\omega\times(\bar\omega\times\bar\rho) + (\ddot{\bar\rho})_r + 2\bar\omega\times(\dot{\bar\rho})_r$$

$$\ddot{\bar R} = 0,\quad \dot{\bar\omega}\times\bar\rho = 0,\quad \bar\omega\times(\bar\omega\times\bar\rho) = -\tfrac{1}{2}l\omega^2\sin\theta\,\bar i$$

$$(\ddot{\bar\rho})_r = \frac{-v_0\dot\theta}{2\cos^2\theta}\,\bar j = -\frac{v_0^2}{2l\cos^3\theta}\,\bar j,\quad 2\bar\omega\times(\dot{\bar\rho})_r = -\omega\cdot v_0$$

Adding, $$\bar a = -\tfrac{1}{2}l\omega^2\sin\theta\,\bar i - \frac{v_0^2}{2l\cos^3\theta}\,\bar j - \omega\cdot v_0\,\bar k$$

2-19. Take the moving frame with the origin at O' and with $\bar\omega = \dot\phi_0\,\bar e_z$



$$v_{0'} - v_c = R(\dot\phi_0 + \Omega_0) = r\dot\theta$$

giving: $$\dot\theta = \frac{R}{r}(\dot\phi_0 + \Omega_0)$$

Use $$\bar a = \ddot{\bar R} + \dot{\bar\omega}\times\bar\rho + \bar\omega\times(\bar\omega\times\bar\rho) + (\ddot{\bar\rho})_r + 2\bar\omega\times(\dot{\bar\rho})_r$$

$$\ddot{\bar R} = -R\dot\phi_0^2\,\bar e_r,\quad \dot{\bar\omega}\times\bar\rho = 0,\quad \bar\omega\times(\bar\omega\times\bar\rho) = 0$$

$$(\ddot{\bar\rho})_r = -r\dot\theta^2\,\bar e_z = -\frac{R^2}{r}(\dot\phi_0 + \Omega_0)^2\,\bar e_z$$

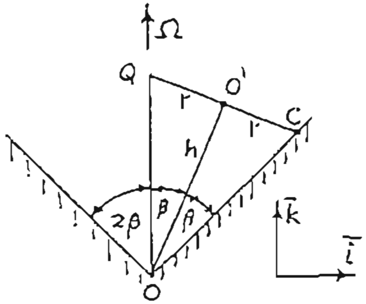
Now $(\dot{\bar\rho})_r = r\dot\theta\,\bar e_\phi$ so we obtain

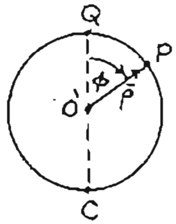
$$2\bar\omega\times(\dot{\bar\rho})_r = -2R\dot\phi_0(\dot\phi_0 + \Omega_0)\,\bar e_r$$

Adding terms,

$$\bar a = (-3R\dot\phi_0^2 - 2R\dot\phi_0\Omega_0)\,\bar e_r - \frac{R^2}{r}(\dot\phi_0 + \Omega_0)^2\,\bar e_z$$

2-20.





The moving system has origin at O' and angular velocity $\bar{\omega} = \Omega \bar{k}$. Relative to this frame, point Q on the cone has the velocity $v_Q = r\dot{\phi}$. But we also know that $v_Q = 2v_{O'}$ since point C is instantaneously at rest and QO'C is a diameter. Hence we can write

$$2v_{O'} = 2(r\Omega\cos\beta) = r\dot{\phi} \quad \text{or} \quad \dot{\phi} = 2\Omega\cos\beta$$

Also we see that $\bar{\rho} = r(-\cos\phi\cos\beta\,\bar{i} + \sin\phi\,\bar{j} + \cos\phi\sin\beta\,\bar{k})$

Then we use $\quad \bar{a} = \ddot{\bar{R}} + \dot{\bar{\omega}}\times\bar{\rho} + \bar{\omega}\times(\bar{\omega}\times\bar{\rho}) + (\ddot{\bar{\rho}})_r + 2\bar{\omega}\times(\dot{\bar{\rho}})_r$

$\ddot{\bar{R}} = -r\Omega^2\cos\beta\,\bar{i}$, $\quad \dot{\bar{\omega}}\times\bar{\rho} = 0$, $\bar{\omega}\times(\bar{\omega}\times\bar{\rho}) = r\Omega^2(\cos\phi\cos\beta\,\bar{i} - \sin\phi\,\bar{j})$

Now $(\dot{\bar{\rho}})_r = r\dot{\phi}(\sin\phi\cos\beta\,\bar{i} + \cos\phi\,\bar{j} - \sin\phi\sin\beta\,\bar{k})$

$\qquad = 2r\Omega(\sin\phi\cos^2\beta\,\bar{i} + \cos\phi\cos\beta\,\bar{j} - \sin\phi\sin\beta\cos\beta\,\bar{k})$

and $(\ddot{\bar{\rho}})_r = -\dot{\phi}^2\bar{\rho} = r\dot{\phi}^2(\cos\phi\cos\beta\,\bar{i} - \sin\phi\,\bar{j} - \cos\phi\sin\beta\,\bar{k})$

$\qquad = 4r\Omega^2(\cos\phi\cos^3\beta\,\bar{i} - \sin\phi\cos^2\beta\,\bar{j} - \cos\phi\sin\beta\cos^2\beta\,\bar{k})$

$2\bar{\omega}\times(\dot{\bar{\rho}})_r = 4r\Omega^2(-\cos\phi\cos\beta\,\bar{i} + \sin\phi\cos^2\beta\,\bar{j})$

Adding, $\quad \bar{a} = r\Omega^2\cos\beta(-1 + \cos\phi - 4\cos\phi\sin^2\beta)\,\bar{i} - r\Omega^2\sin\phi\,\bar{j}$
$\qquad\qquad - 4r\Omega^2\cos\phi\sin\beta\cos^2\beta\,\bar{k}$

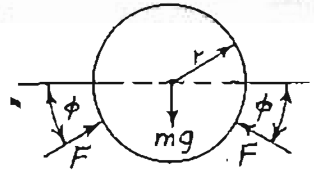
(b) For $\phi = 0$, $\quad \bar{a} = -4r\Omega^2\sin\beta\cos\beta(\sin\beta\,\bar{i} + \cos\beta\,\bar{k})$

Acceleration magnitude $a = \frac{v^2}{\rho} = 4r\Omega^2\sin\beta\cos\beta$ where ρ = radius of curvature and $v = v_Q = 2r\Omega\cos\beta$.

Hence $\rho = \frac{v^2}{a} = r\cot\beta = h$

CHAPTER 3

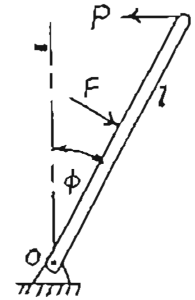
3-1. The upward acceleration a is found from



$$ma = 2F\sin\phi - mg$$

or

$$a = \frac{2F}{m}\sin\phi - g$$



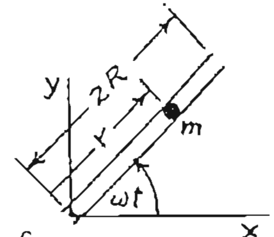
Consider the massless lever. Taking moments about O,

$$Fr\cot\phi - Pl\cos\phi = 0$$

giving $\quad F = \frac{Pl}{r}\sin\phi$

Then $\quad a = \frac{2Pl}{mr}\sin^2\phi - g$

For $\phi = 30°$, $\quad a = \frac{Pl}{2mr} - g$



3-2. The radial acceleration is

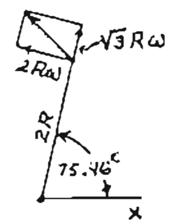
$$a_r = \ddot{r} - \omega^2 r = 0$$

since the radial force is zero.

This equation has a solution of the form

$$r = A\cosh\omega t + B\sinh\omega t \quad \text{with } r(0) = R,\ \dot{r}(0) = 0$$

Hence $A = R$ and $B = 0$.



$r = R\cosh\omega t$, $\quad \dot{r} = R\omega\sinh\omega t$

When $r = 2R$, $\quad \cosh\omega t = 2$, $\sinh\omega t = \sqrt{3}$

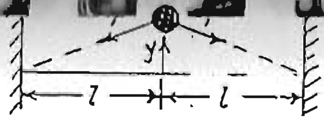
$\omega t = \cosh^{-1}2 = 1.3170 \text{ rad} = 75.456°$

Radial velocity $v_r = \dot{r} = \sqrt{3}R\omega$
Transverse velocity $v_t = 2R\omega$
$\quad v = \sqrt{v_r^2 + v_t^2} = \sqrt{7}R\omega$

Direction (counterclockwise from x axis) $\theta = 75.46 + \tan^{-1}\frac{2}{\sqrt{3}}$
$\qquad = 124.56°$

Cartesian components: $\bar{v} = (-1.5010\,\bar{i} + 2.1788)R\omega$

Equation of motion: $m\ddot{y} = -2T\frac{y}{l}$

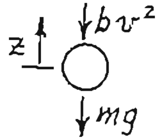


or $\ddot{y} + \frac{2T}{ml}y = 0$

Natural frequency $\omega_n = \sqrt{\frac{2T}{ml}}$

3-4. (a) Eq^{n}. of motion: $m\dot{v} = -bv^2 - mg$

Integrating,



$$-\int_{v_0}^{v}\frac{dv}{g+\frac{b}{m}v^2} = t = \sqrt{\frac{m}{bg}}\left(\tan^{-1}\sqrt{\frac{b}{mg}}\,v_0 - \tan^{-1}\sqrt{\frac{b}{mg}}\,v\right)$$

Then, solving for v, $v = \sqrt{\frac{mg}{b}}\tan\sqrt{\frac{bg}{m}}\,(t^*-t)$

where the time of z_{max} is $t^* = \sqrt{\frac{m}{bg}}\tan^{-1}\sqrt{\frac{b}{mg}}\,v_0$

Expanding $\tan\sqrt{\frac{bg}{m}}(t^*-t)$ yields

$$v = \left(\frac{v_0 - \sqrt{\frac{mg}{b}}\tan\sqrt{\frac{bg}{m}}\,t}{1+\sqrt{\frac{b}{mg}}\,v_0\tan\sqrt{\frac{bg}{m}}\,t}\right)$$

(b) Let $\dot{v} = v\frac{dv}{dz}$ Then $mv\frac{dv}{dz} = -bv^2 - mg$

Integrating, $z = -\int_{v_0}^{v}\frac{v\,dv}{g+\frac{b}{m}v^2} = \frac{-m}{2b}\left[\ln\left(\frac{mg}{b}+v^2\right) - \ln\left(\frac{mg}{b}+v_0^2\right)\right]$

or $e^{\frac{-2bz}{m}} = \frac{mg+bv^2}{mg+bv_0^2}$ giving $v^2 = \frac{mg}{b}\left[\left(1+\frac{bv_0^2}{mg}\right)e^{-\frac{2b}{m}z} - 1\right]$

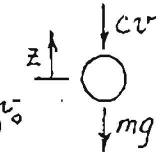
For $v=0$, we obtain $z_{max} = \frac{m}{2b}\ln\left(1+\frac{bv_0^2}{mg}\right)$

(c) For square-law damping with $bv_0^2 = mg$, we obtain

$$z_{max} = \frac{m}{2b}\ln 2 = 0.3466\,\frac{v_0^2}{g}$$

of motion is

$m\dot{v} = -cv - mg$ or $\dot{v} = v\frac{dv}{dz} = -\frac{c}{m}v - g$



Integrating,

$$z_{max} = -\int_{v_0}^{0}\frac{v\,dv}{g+\frac{c}{m}v} = \frac{m^2}{c^2}\left[g+\frac{c}{m}v - g\ln\left(g+\frac{c}{m}v\right)\right]_0^{v_0}$$

$$= \frac{mv_0}{c} - \frac{m^2g}{c^2}\ln\left(1+\frac{cv_0}{mg}\right)$$

For $c = bv_0 = \frac{mg}{v_0}$, $z_{max} = \frac{v_0^2}{g} - \frac{v_0^2}{g}\ln 2 = 0.3069\frac{v_0^2}{g}$

Also see Eqs. (3-63) and (3-64).

3-5. $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 0$ where $\omega_n = \sqrt{\frac{k}{m}}$, $\zeta = \frac{c}{2\sqrt{km}}$

From (3-171), the solution has the form

$x = Ce^{-\zeta\omega_n t}\cos(\omega_d t+\epsilon)$ where $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Hence $\dot{x} = Ce^{-\zeta\omega_n t}\left[-\zeta\omega_n\cos(\omega_d t+\theta) - \omega_d\sin(\omega_d t+\theta)\right]$

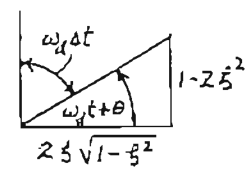
$|\dot{x}|_{max}$ occurs when $\ddot{x}=0$ or, from differential equation,

when $\frac{\dot{x}}{x} = -\frac{\omega_n}{2\zeta} = -\zeta\omega_n - \omega_d\tan(\omega_d t+\theta)$

Hence $\tan(\omega_d t+\theta) = \frac{1-2\zeta^2}{2\zeta\sqrt{1-\zeta^2}}$ for $|\dot{x}|_{max}$.

But $x=0$ at $\cos(\omega_d t+\theta)=0$ or $\omega_d t+\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

Thus $\tan\omega_d\Delta t = \frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2}$



or $\Delta t = \frac{1}{\omega_n\sqrt{1-\zeta^2}}\tan^{-1}\frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2}$

-6. From (3-142), the weighting function is

$$h(t) = \frac{\omega_n e^{-\zeta\omega_n t}}{k\sqrt{1-\zeta^2}} \sin \omega_d t \quad \text{where } \omega_d = \omega_n\sqrt{1-\zeta^2}$$

Using (3-195),

$$x(t) = \int_0^\infty h(\tau)F(t-\tau)\,d\tau = \frac{F_0\omega_n}{k\sqrt{1-\zeta^2}} \int_0^\infty e^{-\zeta\omega_n\tau} \sin \omega_d\tau \cos\omega(t-\tau)\,d\tau$$

where the upper limit is ∞ since the steady-state solution is desired. Next expand $\cos \omega(t-\tau)$ and obtain

$$x(t) = \frac{F_0\omega_n}{k\sqrt{1-\zeta^2}} \left\{ \cos\omega t \int_0^\infty e^{-\zeta\omega_n\tau} \sin\omega_d\tau \cos\omega\tau\,d\tau + \sin\omega t \int_0^\infty e^{-\zeta\omega_n\tau} \sin\omega_d\tau \sin\omega\tau\,d\tau \right\}$$

Evaluating the integrals,

$$\int_0^\infty e^{-\zeta\omega_n\tau} \sin\omega_d\tau \cos\omega\tau\,d\tau = \frac{1}{2}\int_0^\infty e^{-\zeta\omega_n\tau}\left[\sin(\omega_d+\omega)\tau + \sin(\omega_d-\omega)\tau\right]d\tau$$

$$= \frac{1}{2}\left[\frac{e^{-\zeta\omega_n\tau}\left[-\zeta\omega_n \sin(\omega_d+\omega)\tau - (\omega_d+\omega)\cos(\omega_d+\omega)\tau\right]}{\zeta^2\omega_n^2 + (\omega_d+\omega)^2}\right]_0^\infty$$

$$+ \frac{1}{2}\left[\frac{e^{-\zeta\omega_n\tau}\left[-\zeta\omega_n \sin(\omega_d-\omega)\tau - (\omega_d-\omega)\cos(\omega_d-\omega)\tau\right]}{\zeta^2\omega_n^2 + (\omega_d-\omega)^2}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{\omega_d+\omega}{\zeta^2\omega_n^2+(\omega_d+\omega)^2} + \frac{\omega_d-\omega}{\zeta^2\omega_n^2+(\omega_d-\omega)^2}\right]$$

Similarly,

$$\int_0^\infty e^{-\zeta\omega_n\tau} \sin\omega_d\tau \sin\omega\tau\,d\tau = \frac{1}{2}\int_0^\infty e^{-\zeta\omega_n\tau}\left[\cos(\omega_d-\omega)\tau - \cos(\omega_d+\omega)\tau\right]d\tau$$

$$= \frac{1}{2}\left[\frac{\zeta\omega_n}{\zeta^2\omega_n^2+(\omega_d-\omega)^2} - \frac{\zeta\omega_n}{\zeta^2\omega_n^2+(\omega_d+\omega)^2}\right]$$

Now $\omega_d = \omega_n\sqrt{1-\zeta^2}$, so we find that

$$\zeta^2\omega_n^2 + (\omega_d+\omega)^2 = \zeta^2\omega_n^2 + \omega_n^2(1-\zeta^2) + 2\omega\omega_n\sqrt{1-\zeta^2} + \omega^2 = \omega_n^2 + 2\omega\omega_n\sqrt{1-\zeta^2} + \omega^2$$

$$\zeta^2\omega_n^2 + (\omega_d-\omega)^2 = \omega_n^2 - 2\omega\omega_n\sqrt{1-\zeta^2} + \omega^2$$

Then the above integrals can be expressed in the form

3-6. (cont'd.)

$$\int_0^\infty e^{-\zeta\omega_n\tau} \sin\omega_d\tau \cos\omega\tau\,d\tau = \frac{1}{2}\left[\frac{\omega_n\sqrt{1-\zeta^2}+\omega}{\omega_n^2+\omega^2+2\omega\omega_n\sqrt{1-\zeta^2}} + \frac{\omega_n\sqrt{1-\zeta^2}-\omega}{\omega_n^2+\omega^2-2\omega\omega_n\sqrt{1-\zeta^2}}\right]$$

$$= \frac{\omega_n(\omega_n^2-\omega^2)\sqrt{1-\zeta^2}}{(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega^2\omega_n^2}$$

$$\int_0^\infty e^{-\zeta\omega_n\tau} \sin\omega_d\tau \sin\omega\tau\,d\tau = \frac{1}{2}\left[\frac{\zeta\omega_n}{\omega_n^2+\omega^2-2\omega\omega_n\sqrt{1-\zeta^2}} - \frac{\zeta\omega_n}{\omega_n^2+\omega^2+2\omega\omega_n\sqrt{1-\zeta^2}}\right]$$

$$= \frac{2\zeta\omega\omega_n^2\sqrt{1-\zeta^2}}{(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega^2\omega_n^2}$$

Finally,

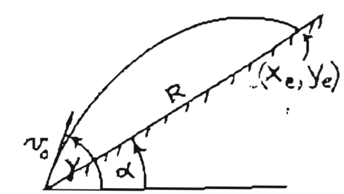
$$x(t) = \frac{F_0\omega_n}{k}\left[\frac{\omega_n(\omega_n^2-\omega^2)\cos\omega t + 2\zeta\omega\omega_n^2 \sin\omega t}{(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega^2\omega_n^2}\right]$$

$$= \frac{F_0}{k}\left[\frac{\left(1-\frac{\omega^2}{\omega_n^2}\right)\cos\omega t + 2\zeta\frac{\omega}{\omega_n}\sin\omega t}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}\right]$$

$$= \frac{(F_0/k)\cos(\omega t+\phi)}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \quad \text{where } \tan\phi = \frac{-2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}$$

3-7. From (3-27), the envelope of possible trajectories is

$$x_e^2 = -\frac{2v_0^2}{g}\left(y_e - \frac{v_0^2}{2g}\right)$$



and any point is a point of maximum slant range.

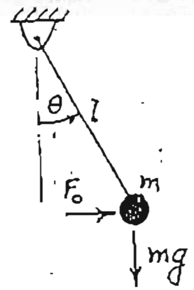
$y_e = x_e \tan\alpha$ and, from (3-26), $\frac{v_0^2}{g} = x_e \tan\gamma$

Hence $x_e^2(1 + 2\tan\alpha\tan\gamma - \tan^2\gamma) = 0$, $\tan\gamma = \tan\alpha + \sqrt{\tan^2\alpha+1} = \tan\alpha + \sec\alpha$

where $\gamma > \alpha$ and $\frac{\pi}{4} \le \gamma < \frac{\pi}{2}$.

Then $R_{max} = \frac{x_e}{\cos\alpha} = \frac{v_0^2}{g\cos\alpha(\tan\alpha+\sec\alpha)} = \frac{v_0^2}{g(1+\sin\alpha)}$

-8. Conservative system. Represent $F_0 = \frac{1}{2}mg$ by an additional potential energy term.



$$V = -mgl\cos\theta - \tfrac{1}{2}mgl\sin\theta$$

$$T = \tfrac{1}{2}ml^2\dot\theta^2$$

From conservation of energy,

$$\therefore \tfrac{1}{2}ml^2\dot\theta^2 - mgl(\cos\theta + \tfrac{1}{2}\sin\theta) = -mgl$$

$\dot\theta = 0$ at $\theta = \theta_{max}$, so $\cos\theta + \frac{1}{2}\sin\theta = 1$

But $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$ and $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

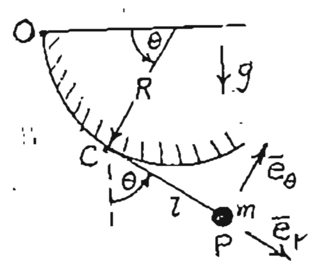
$\cos\theta = 1 - \frac{1}{2}\sin\theta$

Hence $\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\sin^2\frac{\theta}{2}$

The root $\theta = 0$ represents θ_{min}, so $\tan\frac{\theta_{max}}{2} = \frac{1}{2}$, $\underline{\theta_{max} = 53.13°}$

2nd Method: The two constant forces are equivalent to the force of an effective gravity of magnitude $\sqrt{5}g/2$ at $\theta = \tan^{-1}\frac{1}{2}$. The pendulum will swing equally on either side of equilibrium, giving $\theta_{max} = 2\tan^{-1}\frac{1}{2} = 53.13°$.

3-9. $l = 2R - R\theta = (2-\theta)R$



Using tangential and normal components, from (2-41), $\bar{a}_p = \ddot{s}\bar{e}_\theta - \frac{\dot{s}^2}{\rho}\bar{e}_r$

where $\dot{s} = l\dot\theta = (2-\theta)R\dot\theta$ and $\rho = l$

$\ddot{s} = (2-\theta)R\ddot\theta - R\dot\theta^2$

Now $ma_\theta = F_\theta$ or $m[(2-\theta)R\ddot\theta - R\dot\theta^2] = -mg\sin\theta$

giving the differential equation $\underline{(2-\theta)R\ddot\theta - R\dot\theta^2 + g\sin\theta = 0}$

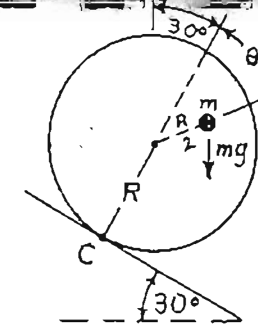
(b) Use conservation of energy. $\theta(0) = 0$, $\dot\theta(0) = \sqrt{\frac{g}{2R}}$

$$T + V = \tfrac{1}{2}m(2-\theta)^2R^2\dot\theta^2 - mgR[\sin\theta + (2-\theta)\cos\theta] = mgR - 2mgR = -mgR$$

θ_{max} occurs when $\dot\theta = 0$.

$\sin\theta + (2-\theta)\cos\theta = 1$ or $\underline{\theta_{max} = \frac{\pi}{2}}$

3-10. Conservative system. The contact point C is the instantaneous center. Thus



$$v^2 = \left[R^2 + \left(\tfrac{R}{2}\right)^2 + 2R\left(\tfrac{R}{2}\right)\cos\theta\right]\dot\theta^2$$

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}mR^2\dot\theta^2\left(\tfrac{5}{4} + \cos\theta\right)$$

$$V = -\tfrac{1}{2}mgR\theta + \tfrac{1}{2}mgR\cos(\theta + 30°)$$

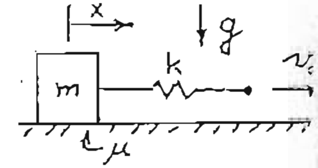
Using conservation of energy,

$$T + V = \tfrac{1}{8}mR^2\dot\theta^2(5 + 4\cos\theta) + \tfrac{1}{2}mgR[\cos(\theta+30°) - \theta] = \tfrac{\sqrt{3}}{4}mg$$

Solving for $\dot\theta$, we obtain $\underline{\dot\theta = \sqrt{\frac{g}{R}\left[\frac{2\sqrt{3} + 4\theta - 4\cos(\theta+30°)}{5 + 4\cos\theta}\right]}}$

(b) The speed of the particle is momentarily constant when a i.e., when the tangent to the path is horizontal, and gravity d no work. $\frac{dV}{d\theta} = -\frac{1}{2}mgR - \frac{1}{2}mgR\sin(\theta+30°) = 0$, $\underline{\theta = 240°}$

3-11. Sliding begins at time $t = t_1$ when the spring force equals μmg.



$kv_0t_1 = \mu mg$ or $t_1 = \frac{\mu mg}{kv_0}$

Thus $\underline{x = 0, \quad 0 \le t \le \frac{\mu mg}{kv_0}}$

For $t \ge t_1$ and $\dot{x} \ge 0$, we have the differential equatio of motion $m\ddot{x} = k(v_0t - x) - \mu mg$ or $\ddot{x} + \frac{k}{m}x = \frac{kv_0}{m}t - \mu g$

Steady-state solution $x_s = C_1t + C_2$

From differential eq^n, $\frac{k}{m}(C_1t + C_2) = \frac{kv_0}{m}t - \mu g$, $C_1 =$ $C_2 = -$

Transient sol^n has the form $x_t = A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t$

Thus we obtain

$$x = x_s + x_t = v_0t - \frac{\mu mg}{k} + A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t$$

3-11. (cont'd.) The initial conditions at $t=t_1$ are

$x(t_1)=0$, $\dot{x}(t_1)=0$. Hence

$$x(t_1)=\frac{\mu m g}{k}-\frac{\mu m g}{k}+A\cos\frac{\mu g}{v_0}\sqrt{\frac{m}{k}}+B\sin\frac{\mu g}{v_0}\sqrt{\frac{m}{k}}=0$$

$$\dot{x}(t_1)=v_0-A\sqrt{\frac{k}{m}}\sin\frac{\mu g}{v_0}\sqrt{\frac{m}{k}}+B\sqrt{\frac{k}{m}}\cos\frac{\mu g}{v_0}\sqrt{\frac{m}{k}}=0$$

Solving these equations for A and B, we obtain

$$A=v_0\sqrt{\frac{m}{k}}\sin\frac{\mu g}{v_0}\sqrt{\frac{m}{k}},\quad B=-v_0\sqrt{\frac{m}{k}}\cos\frac{\mu g}{v_0}\sqrt{\frac{m}{k}}$$

Then

$$x=v_0\sqrt{\frac{m}{k}}\left[\sin\frac{\mu g}{v_0}\sqrt{\frac{m}{k}}\cos\sqrt{\frac{k}{m}}\,t-\cos\frac{\mu g}{v_0}\sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}\,t\right]+v_0t-\frac{\mu m g}{k}$$

or

$$\underline{x=v_0\left[t-\sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}\left(t-\frac{\mu m g}{k v_0}\right)\right]-\frac{\mu m g}{k},\quad t\ge\frac{\mu m g}{k v_0}}$$

3-12. Given that $\frac{\partial F_x}{\partial y}=\frac{\partial F_y}{\partial x}$, $\frac{\partial F_x}{\partial z}=\frac{\partial F_z}{\partial x}$, $\frac{\partial F_y}{\partial z}=\frac{\partial F_z}{\partial y}$ and

$\bar{F}=F_x\bar{i}+F_y\bar{j}+F_z\bar{k}$ which are exactness conditions,

implying that $F_x dx+F_y dy+F_z dz=-dV$, $V=V(x,y,z)$

where the minus sign was chosen arbitrarily.

Now
$$dV=\frac{\partial V}{\partial x}dx+\frac{\partial V}{\partial y}dy+\frac{\partial V}{\partial z}dz$$

so
$$F_x=-\frac{\partial V}{\partial x},\quad F_y=-\frac{\partial V}{\partial y},\quad F_z=-\frac{\partial V}{\partial z}\quad\text{and}\quad \bar{F}=-\nabla V$$

Thus $\bar{F}\cdot d\bar{r}=-dV$ and we see that

(1) $V(x,y,z)$ is a function of position only.

(2) $\int_A^B \bar{F}\cdot d\bar{r}=-\int_A^B dV=V_A-V_B$ which is a function of the end-points only, independent of the path.

Therefore, from Sec. 3 3, $V(x,y,z)$ represents a conservative force.

3-13. Given $\bar{F}=F_x\bar{i}+F_y\bar{j}+F_z\bar{k}$ where $F_x=-x+y$, $F_y=x-y+y^2$, $F_z=0$. First check exactness of $F_x dx+F_y dy$.

$$\frac{\partial F_x}{\partial y}=\frac{\partial F_y}{\partial x}=1\quad\text{so}\quad(-x+y)dx+(x-y+y^2)dy=-dV$$

since the differential form is exact. Thus we have

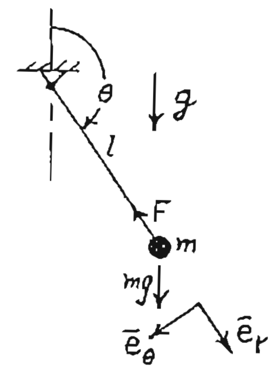
$$-\frac{\partial V}{\partial x}=-x+y\quad\text{which integrates to}\quad -V=-\tfrac{1}{2}x^2+xy+f_1(y)$$

$$-\frac{\partial V}{\partial y}=x-y+y^2\quad\text{which yields}\quad -V=xy-\tfrac{1}{2}y^2+\tfrac{1}{3}y^3+f_2(x)$$

These results are consistent with

$$\underline{V=\tfrac{1}{2}x^2-xy+\tfrac{1}{2}y^2-\tfrac{1}{3}y^3+C},\quad C=\text{arbitrary const.}$$

3-14. Using (2-33),



$$a_\phi=r\ddot{\phi}\sin\theta+2\dot{r}\dot{\phi}\sin\theta+2r\dot{\theta}\dot{\phi}\cos\theta=0$$

since there is no force in the ϕ direction. Now $r=l$, $\dot{r}=0$, so at $t=0$,

$$\underline{\ddot{\phi}(0)=-2\dot{\theta}_0\dot{\phi}_0\cot\theta}$$

Taking θ components of force and acceleration, $mg\sin\theta=ma_\theta$ or

$$a_\theta=r\ddot{\theta}+2\dot{r}\dot{\theta}-r\dot{\phi}^2\sin\theta\cos\theta=g\sin\theta.$$

Then, at $t=0$,
$$\underline{\ddot{\theta}(0)=\dot{\phi}_0^2\sin\theta\cos\theta+\frac{g}{l}\sin\theta}$$

(b) Taking radial components of force and acceleration,

$$ma_r=-F-mg\cos\theta$$

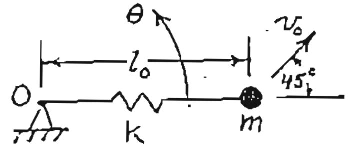
where
$$a_r=\ddot{r}-r\dot{\theta}^2-r\dot{\phi}^2\sin^2\theta$$

At $t=0$,
$$\underline{F=ml(\dot{\theta}_0^2+\dot{\phi}_0^2\sin^2\theta)-mg\cos\theta}$$

3-15. Use polar coordinates.

$T=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)$

$V=\frac{1}{2}k(r-l_0)^2$



Conservation of energy requires

$$\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+\frac{1}{2}k(r-l_0)^2=\frac{1}{2}mv_0^2$$

At max stretch, $r=4l_0/3$, $\dot{r}=0$, so we obtain

$$\frac{1}{2}m\left(\frac{4l_0}{3}\right)^2\dot{\theta}^2+\frac{1}{2}k\left(\frac{l_0}{3}\right)^2=\frac{1}{2}mv_0^2$$

Using conservation of angular momentum,

$$mr^2\dot{\theta}=\frac{ml_0}{\sqrt{2}}v_0 \quad \text{or} \quad \dot{\theta}=\frac{l_0v_0}{\sqrt{2}r^2}=\frac{9v_0}{16\sqrt{2}l_0} \quad \text{at } r=r_{max}$$

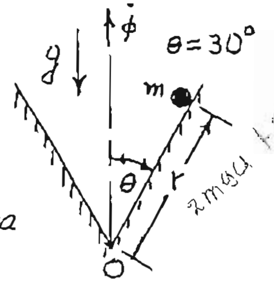
Then $$\frac{8}{9}ml_0^2\left(\frac{9v_0}{16\sqrt{2}l_0}\right)^2+\frac{kl_0^2}{18}=\frac{1}{2}mv_0^2$$

or $$k=18\frac{mv_0^2}{l_0^2}\left(\frac{1}{2}-\frac{9}{64}\right)=\frac{207mv_0^2}{32l_0^2}=\underline{6.4688\frac{mv_0^2}{l_0^2}}$$

3-16. $r(0)=a$, $\dot{r}(0)=0$, $\dot{\phi}(0)=4\sqrt{\frac{g}{a}}$

$T=\frac{1}{2}m(\dot{r}^2+r^2\sin^2\theta\,\dot{\phi}^2)=\frac{1}{2}m\dot{r}^2+\frac{1}{8}mr^2\dot{\phi}^2$

$V=mgr\cos\theta=\frac{\sqrt{3}}{2}mgr$



Using conservation of energy,

$$\frac{1}{2}m\dot{r}^2+\frac{1}{8}mr^2\dot{\phi}^2+\frac{\sqrt{3}}{2}mgr=\left(2+\frac{\sqrt{3}}{2}\right)mga$$

Angular momentum is conserved about the vertical axis.

$$H_v=\frac{1}{4}mr^2\dot{\phi}=ma^{3/2}g^{1/2} \quad \text{giving} \quad \dot{\phi}=\frac{4a}{r^2}\sqrt{ag}$$

Then, from the energy eqn, noting that $\dot{r}=0$ at $r=r_{max}$,

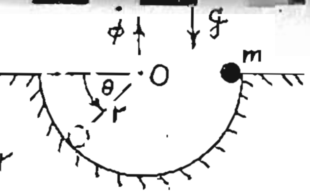
$$\frac{2mga^3}{r^2}+\frac{\sqrt{3}}{2}mgr=\left(2+\frac{\sqrt{3}}{2}\right)mga \quad \text{or} \quad \sqrt{3}r^3-(4+\sqrt{3})ar^2+4a^3=0$$

or $(r-a)(\sqrt{3}r^2-4ar-4a^2)=0$ The root $r=a$ is r_{min} at $t=0$.

$$\underline{r_{max}=\frac{2a}{\sqrt{3}}\left(1+\sqrt{1+\sqrt{3}}\right)=3.0633a}$$

3-17. $T=\frac{1}{2}mr^2(\dot{\theta}^2+\dot{\phi}^2\cos^2\theta)$

$V=-mgr\sin\theta$, $v(0)=\sqrt{2gr}$



Using conservation of energy,

$$\frac{1}{2}mr^2(\dot{\theta}^2+\dot{\phi}^2\cos^2\theta)-mgr\sin\theta=mgr$$

Using conservation of angular momentum about a vertical axis through the center O,

$$H_v=mr^2\cos^2\theta\,\dot{\phi}=mr\sqrt{2gr} \quad \text{or} \quad \dot{\phi}=\frac{1}{\cos^2\theta}\sqrt{\frac{2g}{r}}$$

$\dot{\theta}=0$ at θ_{max}, so the energy equation becomes

$$\frac{1}{2}mr^2\left(\frac{2g}{r\cos^2\theta}\right)=mgr(1+\sin\theta)$$

Noting that $\cos^2\theta=1-\sin^2\theta$, we obtain

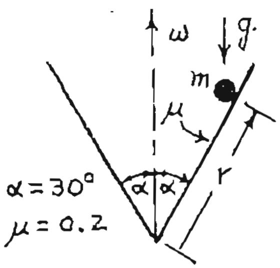
$$(1+\sin\theta)(1-\sin^2\theta)=1 \quad \text{or} \quad \sin\theta(\sin^2\theta+\sin\theta-1)=0$$

$\theta_{min}=0$, $\sin\theta_{max}=\frac{1}{2}(-1+\sqrt{5})=0.6180$, $\underline{\theta_{max}=38.17^\circ}$

3-18. Centrifugal force $=mr\omega^2\sin\alpha$

Equating normal components,

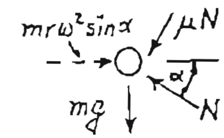
$$N=mg\sin\alpha+mr\omega^2\sin\alpha\cos\alpha$$



Case 1 – Sliding upward. Consider force components in the r direction. Sliding occurs if

$$mr\omega^2\sin^2\alpha-mg\cos\alpha>\mu N$$

or $$\omega^2>\frac{g}{r}\left(\frac{\mu+\cot\alpha}{\sin\alpha-\mu\cos\alpha}\right)$$



sliding up

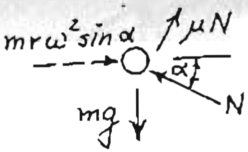
Hence ω_{max} for no sliding is

$$\underline{\omega_{max}=\sqrt{\frac{g(\mu+\cot\alpha)}{r(\sin\alpha-\mu\cos\alpha)}}=2.4315\sqrt{\frac{g}{r}}}$$

Case 2 – Sliding downward. The forces are the same as in Case 1, except that the friction force μN is reversed.

3-18. (cont'd.) Thus we obtain

$$mr\omega^2\sin^2\alpha + \mu N < mg\cos\alpha$$



sliding down

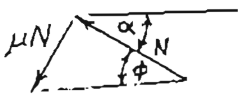
or
$$\omega^2 < \frac{g}{r}\left(\frac{\cot\alpha-\mu}{\sin\alpha+\mu\cos\alpha}\right)$$

if sliding downward occurs.

$$\omega_{min} = \sqrt{\frac{g(\cot\alpha-\mu)}{r(\sin\alpha+\mu\cos\alpha)}} = 1.5086\sqrt{\frac{g}{r}}$$

These results are also obtained by noting that incipient sliding occurs when the resultant of the centrifugal and gravitational forces lies on the cone of friction.

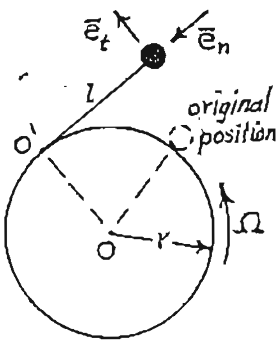
(b) The cone of friction angle ϕ is given by
$$\tan\phi = \mu$$



If $\phi > \alpha$, or $\mu > \tan\alpha$, then the resultant of N and μN has a downward component. Gravity also is downward. Thus, no amount of centrifugal force will cause the particle to slide upward. Also, the expression for ω_{max}, Case 1, becomes imaginary.

$$\mu_{min} = \tan\alpha = 1/\sqrt{3}$$

3-19. $l = r\theta$, $\dot{l} = r\dot{\theta}$ or $\dot{\theta} = \frac{\dot{l}}{r}$



Choose the moving system with origin at O' and rotating at $\omega = \Omega + \dot{\theta} = \Omega + \frac{\dot{l}}{r}$.

Using (2-106),

$$\bar{a} = \ddot{\bar{R}} + \dot{\bar{\omega}}\times\bar{\rho} + \bar{\omega}\times(\bar{\omega}\times\bar{\rho}) + (\ddot{\bar{\rho}})_r + 2\bar{\omega}\times(\dot{\bar{\rho}})_r$$

$R = r\bar{e}_t$, $\bar{\rho} = -l\bar{e}_n$, $(\dot{\bar{\rho}})_r = -\dot{l}\,\bar{e}_n$

$\bar{\omega} = \left(\Omega + \frac{\dot{l}}{r}\right)\bar{e}_b$ (out of page), $\dot{\bar{\omega}} = \frac{\ddot{l}}{r}\bar{e}_b$

3-19. (cont'd.) $\ddot{\bar{R}} = r\dot{\omega}\,\bar{e}_n - r\omega^2\bar{e}_t = \ddot{l}\,\bar{e}_n - r\left(\Omega + \frac{\dot{l}}{r}\right)^2\bar{e}_t$

$\dot{\bar{\omega}}\times\bar{\rho} = \frac{l\ddot{l}}{r}\bar{e}_t$; $\bar{\omega}\times(\bar{\omega}\times\bar{\rho}) = l\left(\Omega+\frac{\dot{l}}{r}\right)^2\bar{e}_n$

$(\ddot{\bar{\rho}})_r = -\ddot{l}\,\bar{e}_n$, $2\bar{\omega}\times(\dot{\bar{\rho}})_r = 2\dot{l}\left(\Omega+\frac{\dot{l}}{r}\right)\bar{e}_t$

Adding, $\bar{a} = \left[-r\left(\Omega+\frac{\dot{l}}{r}\right)^2 + \frac{l\ddot{l}}{r} + 2\dot{l}\left(\Omega+\frac{\dot{l}}{r}\right)\right]\bar{e}_t + l\left(\Omega+\frac{\dot{l}}{r}\right)^2\bar{e}_n$

There is no force on the particle in the $\bar{e}_t$ direction, so

$a_t = 0$ or $-\frac{1}{r}(\dot{l}^2 + 2r\Omega\dot{l} + r^2\Omega^2) + \frac{l\ddot{l}}{r} + \frac{1}{r}(2\dot{l}^2 + 2r\Omega\dot{l}) = 0$

giving the differential equation

$$l\ddot{l} + \dot{l}^2 - r^2\Omega^2 = 0 \quad \text{or} \quad \frac{d}{dt}(l\dot{l}) = r^2\Omega^2$$

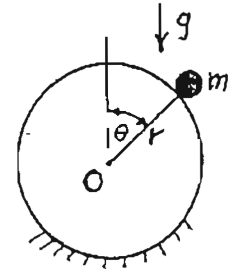
Initial conditions: $l(0) = 0$, $\dot{l}(0) = r\Omega$. Integration of the differential equation yields $l\dot{l} = r^2\Omega^2 t$. A second integration results in

$$\int l\,dl = \int r^2\Omega^2 t\,dt \quad \text{or} \quad \frac{1}{2}l^2 = \frac{1}{2}r^2\Omega^2 t^2, \quad \text{so } l = r\Omega t$$

Tensile force $F = ma_n = ml\left(\Omega + \frac{\dot{l}}{r}\right)^2 = 4ml\Omega^2$

But $l = r\Omega t$ so we obtain $F = 4mr\Omega^3 t$

3-20. $T = \frac{1}{2}mr^2\dot{\theta}^2$, $V = mgr\cos\theta$

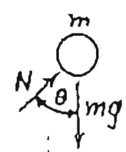


Using conservation of energy,

$\frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta = mgr$, $\dot{\theta}^2 = \frac{2g}{r}(1-\cos\theta)$

Now $ma_r = F_r$ or $-mr\dot{\theta}^2 = N - mg\cos\theta$

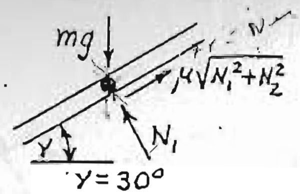
resulting in $N = mg(3\cos - 2)$



The particle will leave the sphere when the normal force N decreases to zero.

$$\cos\theta = \frac{2}{3} \quad \text{or} \quad \theta = 48.19°$$

3-21. Normal force $N=\sqrt{N_1^2+N_2^2}$

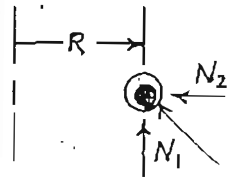


For constant speed, the acceleration is centripetal (horizontal).

$$N_2=ma=\frac{m}{R}\left(\frac{\sqrt{3}}{2}v\right)^2$$

Balancing forces in the N_1 direction,

$$N_1=\frac{\sqrt{3}}{2}mg$$



Friction force $F_f=\mu N=\mu\sqrt{N_1^2+N_2^2}$

View looking upward into tube.

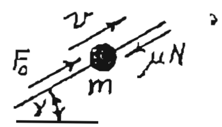
Balancing forces along tube,

$$\tfrac{1}{2}mg=\mu\sqrt{N_1^2+N_2^2} \quad\text{or}\quad N_2^2=\left(\frac{mg}{2\mu}\right)^2-\frac{3}{4}m^2g^2$$

Equating expressions for N_2,

$$N_2=\frac{3mv^2}{4R}=\tfrac{1}{2}mg\sqrt{\frac{1}{\mu^2}-3} \quad\text{giving}\quad \underline{v=\sqrt{\frac{2Rg}{3}}\left(\frac{1}{\mu^2}-3\right)^{1/4}}$$

3-22. The normal force N is directed radially inward, perpendicular to the axis.



$$N=\frac{m(v\cos\gamma)^2}{R}$$

Taking tangential components of force and acceleration,

$$m\dot{v}=F_0-\frac{\mu m}{R}v^2\cos^2\gamma \quad\text{where}\quad \dot{v}=v\frac{dv}{ds}$$

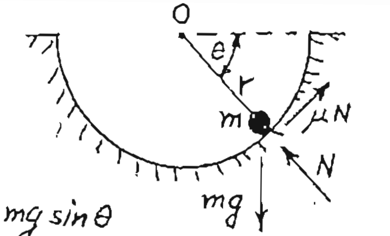
Then $v\dfrac{dv}{ds}+\dfrac{\mu\cos^2\gamma}{R}v^2=\dfrac{F_0}{m}$ or $\dfrac{d(v^2)}{ds}+\dfrac{2\mu\cos^2\gamma}{R}v^2=\dfrac{2F_0}{m}$

which is linear in v^2. The form of the solution is

$$v^2=Ce^{-\frac{2\mu\cos^2\gamma}{R}s}+\frac{F_0R}{\mu m\cos^2\gamma} \quad\text{where } v=0 \text{ at } s=0$$

$$C=\frac{-F_0R}{\mu m\cos^2\gamma} \qquad \underline{v^2=\frac{F_0R}{\mu m\cos^2\gamma}\left(1-e^{-\frac{2\mu\cos^2\gamma}{R}s}\right)}$$

3-23. The radial equation is



$$mr\dot{\theta}^2=N-mg\sin\theta$$

giving $\quad N=mr\dot{\theta}^2+mg\sin\theta$

θ eq^n: $\quad mr\ddot{\theta}=mg\cos\theta-\mu N$

$$=mg\cos\theta-\mu mr\dot{\theta}^2-\mu mg\sin\theta$$

Thus, the differential equation is

$$\underline{\ddot{\theta}+\mu\dot{\theta}^2+\frac{g}{r}(\mu\sin\theta-\cos\theta)=0}$$

(b) Change independent variable to θ. $\ddot{\theta}=\dot{\theta}\dfrac{d\dot{\theta}}{d\theta}=\dfrac{1}{2}\dfrac{d(\dot{\theta}^2)}{d\theta}$

Then $\dfrac{d\dot{\theta}^2}{d\theta}+2\mu\dot{\theta}^2=\dfrac{2g}{r}(\cos\theta-\mu\sin\theta)$

which is linear in $\dot{\theta}^2$. The steady-state solution has the form $(\dot{\theta}^2)_s=C_1\cos\theta+C_2\sin\theta$

Substituting into the differential equation,

$$(C_2+2\mu C_1)\cos\theta+(-C_1+2\mu C_2)\sin\theta=\frac{2g}{r}\cos\theta-\frac{2\mu g}{r}\sin\theta$$

Comparing coefficients of $\cos\theta$ and of $\sin\theta$,

$$\left.\begin{aligned}2\mu C_1+C_2&=\frac{2g}{r}\\ -C_1+2\mu C_2&=-\frac{2\mu g}{r}\end{aligned}\right\}\text{solving,}\quad C_1=\frac{6\mu g}{r(1+4\mu^2)},\quad C_2=\frac{2g}{r}\left(\frac{1-2\mu^2}{1+4\mu^2}\right)$$

The transient solution has the form

$$(\dot{\theta}^2)_t=Ae^{-2\mu\theta}$$

Initial condition $\dot{\theta}(0)=0$ so $A+C_1=0$, $\quad A=\dfrac{-6\mu g}{r(1+4\mu^2)}$

Finally,

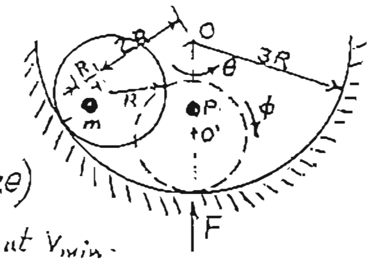
$$\underline{\dot{\theta}^2=\frac{-6\mu g\,e^{-2\mu\theta}}{r(1+4\mu^2)}+\frac{2g}{r(1+4\mu^2)}\left[3\mu\cos\theta+(1-2\mu^2)\sin\theta\right]}$$

3-24. $v_{0'} = 2R\dot\theta = R\dot\phi$

so $\dot\phi = 2\dot\theta$. Let $\phi = 2\theta$

with $\phi=0$, $\theta=0$ at the lowest

position. $T = \frac{1}{2}mv^2$



$$V = mgR\left(-2\cos\theta + \tfrac{1}{4}\cos 2\theta\right)$$

Conservative system. T_{max} occurs at V_{min}.

$$\frac{dV}{d\theta} = mgR\left(2\sin\theta - \tfrac{1}{2}\sin 2\theta\right) = 0 \quad \text{giving} \quad V_{min} = -\tfrac{7}{4}mgR$$

at $\theta = 0$.

Now $V(0) = -\frac{9}{8}mgR$ at $\theta = -60°$

Using conservation of energy, $\frac{1}{2}mv^2 - \frac{7}{4}mgR = -\frac{9}{8}mgR$

$$v^2 = \tfrac{5}{4}gR, \quad v_{max} = \tfrac{1}{2}\sqrt{5gR}$$

(b) When $\theta = 0$, $\phi = 0$ we see that $\dot\phi = \dfrac{v_{max}}{\frac{5}{4}R} = 2\sqrt{\dfrac{g}{5R}}$

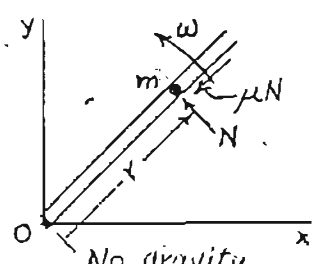
$$\dot\theta = \tfrac{1}{2}\dot\phi = \sqrt{\frac{g}{5R}}$$

$a_{0'} = 2R\dot\theta^2 = \frac{2}{5}g$ upward. $a_{P/0'} = \frac{1}{4}R\dot\phi^2 = \frac{1}{5}g$ downward

$a_P = a_{0'} + a_{P/0'} = \frac{1}{5}g$ upward. $F - mg = m(\frac{1}{5}g)$ or $F = \frac{6}{5}mg$

3-25. Let $\dot\phi = \omega = \text{const.}$

Transverse acceleration $a_\phi = r\ddot\phi + 2\dot r\dot\phi$
$= 2\dot r\omega$



Hence $N = ma_\phi = 2m\omega\dot r$

Radial acceleration $a_r = \ddot r - r\omega^2$

$F_r = ma_r$ or $-\mu N = m(\ddot r - r\omega^2)$

giving the differential equation

$$\ddot r + 2\mu\omega\dot r - \omega^2 r = 0$$

Assume solutions of form $e^{\lambda t}$. The characteristic equation is $\lambda^2 + 2\mu\omega\lambda - \omega^2 = 0$ with roots $\lambda_{1,2} = (-\mu \pm \sqrt{1+\mu^2})\,\omega$

3-25. (cont'd.) The solution is of the form

$$r = e^{-\mu\omega t}\left(A\cosh\sqrt{1+\mu^2}\,\omega t + B\sinh\sqrt{1+\mu^2}\,\omega t\right)$$

Given $r(0) = R$, so $A = R$. Also, $\dot r(0) = 0 = -\mu A + \sqrt{1+\mu^2}\,B$

yielding $B = \dfrac{\mu R}{\sqrt{1+\mu^2}}$. Hence we obtain

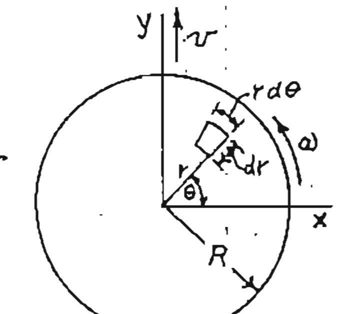
$$r(t) = Re^{-\mu\omega t}\left(\cosh\sqrt{1+\mu^2}\,\omega t + \frac{\mu}{\sqrt{1+\mu^2}}\sinh\sqrt{1+\mu^2}\,\omega t\right)$$

Alternate form is

$$r(t) = \tfrac{1}{2}Re^{-\mu\omega t}\left[\left(1+\frac{\mu}{\sqrt{1+\mu^2}}\right)e^{\sqrt{1+\mu^2}\,\omega t} + \left(1-\frac{\mu}{\sqrt{1+\mu^2}}\right)e^{-\sqrt{1+\mu^2}\,\omega t}\right]$$

3-26. $\bar v = v\bar j$ where $v << R\omega$.

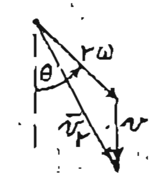
At an area element $dA = r\,dr\,d\theta$ of the shaft, the relative velocity of the plate is $\bar v_r = r\omega\sin\theta\,\bar i - (r\omega\cos\theta + v)\bar j$



The friction stress $\tau_f = \mu N/\pi R^2$ acts on the shaft in the direction of $\bar v_r$.

Hence the total frictional force is

$$\bar F_f = \frac{\mu N}{\pi R^2}\int_0^{2\pi}\!\!\int_0^R \left[\frac{\sin\theta\,\bar i - (\cos\theta + \frac{v}{r\omega})\bar j}{\sqrt{1+\frac{2v}{r\omega}\cos\theta + \frac{v^2}{r^2\omega^2}}}\right] r\,dr\,d\theta$$



By symmetry, the x component of $\bar F_f$ vanishes.

For $\frac{v}{r\omega} << 1$, we can use the approximation

$$\left[1 + \frac{2v}{r\omega}\cos\theta + \left(\frac{v}{r\omega}\right)^2\right]^{-1/2} \cong 1 - \frac{v}{r\omega}\cos\theta$$

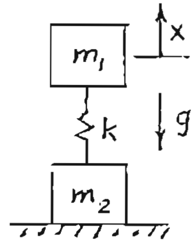
Then, to first order in $v/r\omega$, we obtain

$$\bar F_f = \frac{-\mu N}{\pi R^2}\int_0^R\!\!\int_0^{2\pi} r\left(\cos\theta + \frac{v}{r\omega}\sin^2\theta\right)\bar j\,d\theta\,dr = \frac{-\mu N}{\pi R^2}\int_0^R \frac{v\pi}{r\omega}\bar j\,r\,dr = \frac{-\mu N}{R\omega}\bar v$$

Note: We were given that $\frac{v}{R\omega} = \epsilon << 1$. But we assumed that $\frac{v}{r\omega} << 1$. This last assumption is clearly not valid in a circle of radius r where $r/R = \epsilon$. But the frictional force on this small circle is, at most, $\epsilon^2\mu N$ compared to the total frictional force $\epsilon\mu N$, giving a negligible error.

CHAPTER 4

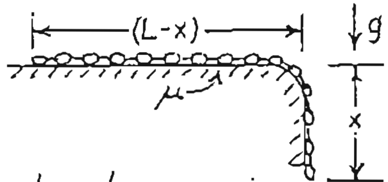
4-1. Let x be measured from the unstressed position. m_2 leaves the floor if $k x_{max} > m_2 g$. The static deflection $x_{st} = -\frac{1}{k}(m_1 g)$ is the average value of x in the sinusoidal oscillations of m_1. In the borderline case,



$$x_{st} = \frac{1}{2}(x_{max} + x_{min}) \quad \text{or} \quad \frac{-1}{k} m_1 g = \frac{1}{2}\left(\frac{m_2 g}{k} - \delta\right)$$

since $\delta = -x_{min}$. Thus we obtain $\underline{\delta = \frac{1}{k}(2m_1 + m_2)g}$

4-2. Linear density $\rho = \frac{m}{L}$.



$x(0) = a$. The initial potential energy $V_i = -\frac{1}{2}\rho a^2 = \frac{-1}{2L} m g a^2$

The final potential energy $V_f = -\frac{1}{2} m g L$.

Initial kinetic energy $T_i = 0$. Final kinetic energy $T_f = \frac{1}{2} m v^2$.

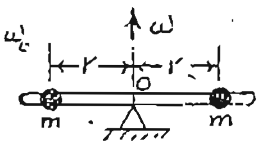
Energy lost in friction $W_f = \int_a^L \frac{\mu m g}{L}(L-x)\,dx = \frac{\mu m g}{2L}(L-a)^2$

From the work and energy principle of (4-26),

$$T_i + V_i = T_f + V_f + W_f \quad \text{or} \quad -\frac{m g a^2}{2L} = \frac{1}{2} m v^2 - \frac{1}{2} m g L + \frac{\mu m g}{2L}(L-a)^2$$

$$v = \sqrt{\frac{g}{L}\left[L^2 - a^2 - \mu(L-a)^2\right]}$$

4-3. Given $r = r_0 + A\sin\beta t$, $A < r_0$, $\omega(0) = \omega_0$



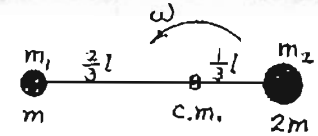
Conservation of angular momentum about O yields

$$H = 2 m r^2 \omega = 2 m r_0^2 \omega_0 \quad \text{or} \quad \omega = \frac{r_0^2}{r^2}\omega_0$$

Radial acceleration $a_r = \ddot{r} - r\omega^2 = -A\beta^2 \sin\beta t - \frac{r_0^4 \omega_0^2}{r^3}$

Radial force $F_r = m a_r = -m\left[A\beta^2 \sin\beta t + \frac{r_0^4 \omega_0^2}{(r_0 + A\sin\beta t)^3}\right]$

4-4. Using conservation of angular momentum about the c.m.,



$$\left[m\left(\tfrac{2}{3}l\right)^2 + 2m\left(\tfrac{1}{3}l\right)^2\right]\omega = \frac{2}{3} m l_0^2 \omega_0$$

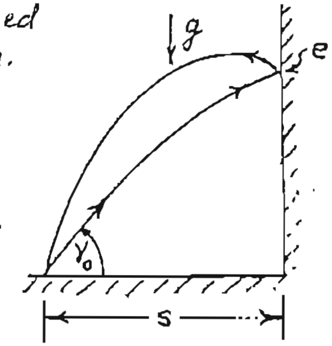
$l(0) = l_0$, $\omega(0) = \omega_0$

When $l = \frac{1}{2} l_0$, we have $\frac{1}{6} m l_0^2 \omega = \frac{2}{3} m l_0^2 \omega_0$ or $\underline{\omega = 4\omega_0}$

(b) The c.m. is fixed in an inertial frame. The string tension is

$$F = m_1 \frac{l_0}{3}\omega^2 = \frac{m l_0}{3}(4\omega_0)^2 = \underline{\frac{16}{3} m l_0 \omega_0^2}$$

4-5. The vertical velocity is unchanged upon impact because there no friction.



Total time $t = \frac{2 v_0 \sin\gamma_0}{g}$

Time to reach wall $t_1 = \frac{s}{v_0 \cos\gamma_0}$

Time to return $t_2 = \frac{1}{e} t_1$

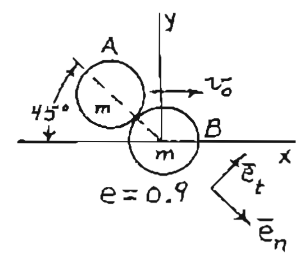
since its horizontal velocity is $e v_0 \cos\gamma_0$.

Then $t = t_1 + t_2 = \left(1 + \frac{1}{e}\right)\frac{s}{v_0 \cos\gamma_0} = \frac{2 v_0 \sin\gamma_0}{g}$

Solving for γ_0, we obtain $\underline{\gamma_0 = \frac{1}{2}\sin^{-1}\left[\left(1 + \frac{1}{e}\right)\frac{g s}{v_0^2}\right]}$

for $v_0^2 \geq \left(1 + \frac{1}{e}\right) g s$. If the inequality applies, there are two values of γ_0, one in the range $0 < \gamma_0 < 45°$; the other $45° < \gamma_0 \leq 90°$.

4-6. Before impact,



$$v_{nA} = v_{tA} = \frac{v_0}{\sqrt{2}}, \quad v_{nB} = v_{tB} = 0$$

The tangential velocities are unchanged by the impact.

$$v'_{tA} = \frac{v_0}{\sqrt{2}}, \quad v'_{tB} = 0$$

-6. (cont'd.) Using (4-83), the normal velocities after npact are

$$v'_{nA} = \left(\frac{1-e}{2}\right)v_{nA} + \left(\frac{1+e}{2}\right)v_{nB} = \frac{v_0}{20\sqrt{2}}$$

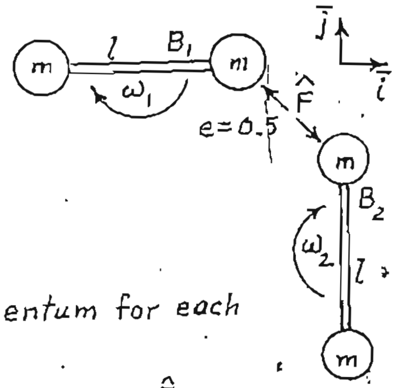
$$v'_{nB} = \left(\frac{1+e}{2}\right)v_{nA} + \left(\frac{1-e}{2}\right)v_{nB} = \frac{19 v_0}{20\sqrt{2}}$$

aking $\bar{i}$ and $\bar{j}$ components,

$$\bar{v}'_A = v_0\left[\left(\frac{1}{2}+\frac{1}{40}\right)\bar{i} + \left(\frac{1}{2}-\frac{1}{40}\right)\bar{j}\right] = \underline{v_0(0.525\,\bar{i} + 0.475\,\bar{j})}$$

$$\bar{v}'_B = v_0\left[\frac{19}{40}\bar{i} - \frac{19}{40}\bar{j}\right] = \underline{v_0(0.475\,\bar{i} - 0.475\,\bar{j})}$$

-7. Using linear impulse and momentum, the c.m. velocities are



$$\bar{v}'_1 = \left(v_0 - \frac{\hat{F}}{2\sqrt{2}\,m}\right)\bar{i} + \frac{\hat{F}}{2\sqrt{2}\,m}\bar{j}$$

$$\bar{v}'_2 = \frac{\hat{F}}{2\sqrt{2}\,m}\bar{i} - \frac{\hat{F}}{2\sqrt{2}\,m}\bar{j}$$

sing angular impulse and momentum for each lumbbell about its center,

$$\hat{M}_1 = -\frac{\hat{F}l}{2\sqrt{2}} = 2m\left(\frac{l}{2}\right)^2\omega'_1 \quad \text{or} \quad \omega'_1 = \frac{-\hat{F}}{\sqrt{2}\,ml}$$

$$\hat{M}_2 = \frac{\hat{F}l}{2\sqrt{2}} = 2m\left(\frac{l}{2}\right)^2\omega'_2 \quad \text{or} \quad \omega'_2 = \frac{\hat{F}}{\sqrt{2}\,ml}$$

rom the definition of e,

$$e\frac{v_0}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\frac{\hat{F}}{\sqrt{2}\,m} - v_0\right) + \frac{1}{\sqrt{2}}\left(\frac{\hat{F}}{\sqrt{2}\,m}\right) + \frac{1}{\sqrt{2}}\left(\frac{l}{2}\right)\left(\frac{2\hat{F}}{\sqrt{2}\,ml}\right)$$

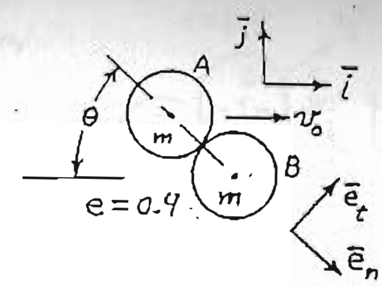
or

$$\frac{v_0}{2\sqrt{2}} = -\frac{v_0}{\sqrt{2}} + \frac{3\hat{F}}{2m} \quad \text{giving} \quad \hat{F} = \frac{mv_0}{\sqrt{2}}$$

Then $\underline{\bar{v}'_1 = \frac{v_0}{4}(3\bar{i} + \bar{j})}$, $\underline{\bar{v}'_2 = \frac{v_0}{4}(\bar{i} - \bar{j})}$

$\underline{\omega'_1 = -\frac{v_0}{2l}}$, $\underline{\omega'_2 = \frac{v_0}{2l}}$

4-8. $v_{nA} = v_0\cos\theta$



$v_{tA} = v_0\sin\theta$

$$v'_{nA} = \frac{1-e}{2}v_{nA} = \frac{v_0}{20}\cos\theta$$

$$v'_{tA} = v_{tA} = v_0\sin\theta$$

Shifting to $\bar{i}, \bar{j}$ unit vectors,

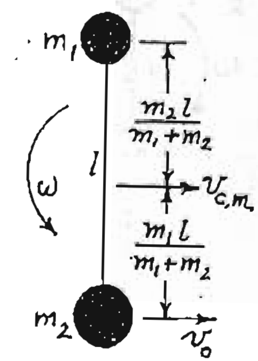
$$\bar{v}'_A = v_0\left[\left(\frac{1}{20}\cos^2\theta + \sin^2\theta\right)\bar{i} + \left(-\frac{1}{20}\sin\theta\cos\theta + \sin\theta\cos\theta\right)\bar{j}\right]$$

Final velocity $\bar{v}'_A$ has $\bar{i}$ and $\bar{j}$ components equal.

$$\frac{1}{20}\cos^2\theta + \sin^2\theta = \frac{19}{20}\sin\theta\cos\theta \quad \text{or} \quad \tan^2\theta - \frac{19}{20}\tan\theta + \frac{1}{20} = 0$$

$$\tan\theta = \frac{19}{40} \pm \sqrt{\left(\frac{19}{40}\right)^2 - \frac{1}{20}} = 0.05592,\ 0.8941 \qquad \underline{\theta = 3.20^\circ,\ 41.80^\circ}$$

4-9. The initial velocity of m_2 is v_0, due to the impulse $\hat{F} = m_2 v_0$. The initial velocity of m_1 is zero.



After the initial impulse, the velocity of the c.m. is constant and is found from

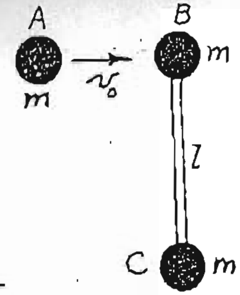
$$\hat{F} = (m_1+m_2)v_{c.m.} \quad \text{or} \quad \underline{v_{c.m.} = \frac{m_2 v_0}{m_1+m_2}}$$

After the impulse, the angular momentum is constant, so the angular velocity is constant at its initial value. $\underline{\omega = \frac{v_0}{l}}$

Consider the rotational motion about the c.m. The tensile force is

$$F = m_2\left(\frac{m_1 l}{m_1+m_2}\right)\omega^2 = \underline{\frac{m_1 m_2 v_0^2}{(m_1+m_2)l}}$$

4-10. Using (4-83), the velocities immediately after impact are



$$v_A' = \left(\frac{1-e}{2}\right)v_0\,, \quad v_B' = \left(\frac{1+e}{2}\right)v_0$$

and $v_C' = 0$, positive to the right.

For the dumbbell, $v_{c.m.}' = \frac{1}{2}v_B' = \left(\frac{1+e}{4}\right)v_0$

$$\omega' = \frac{v_B'}{l} = \left(\frac{1+e}{2l}\right)v_0 \quad \text{clockwise}$$

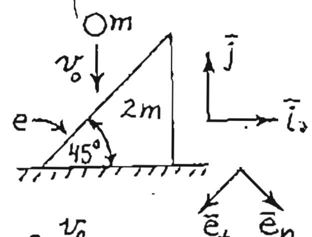
(b) In order that A and C collide, $v_A' = v_{c.m.}'$

or $\frac{1-e}{2} = \frac{1+e}{4}$ or $e = \frac{1}{3}$

Just before the collision, $v_A = \frac{1}{3}v_0$, $v_C = v_{c.m.}' + \frac{l}{2}\omega' = \frac{2}{3}v_0$.

4-11. Let $\bar{v}_p$ = particle velocity

$\bar{v}_b$ = velocity of block



Then $v_{pt}' = v_{pt} = \frac{v_0}{\sqrt{2}}$

From the definition of e, $\frac{v_b'}{\sqrt{2}} - v_{pn}' = e\frac{v_0}{\sqrt{2}}$

From momentum conservation in the $\bar{i}$ direction,

$$m\left(\frac{v_{pn}'}{\sqrt{2}} - \frac{v_{pt}'}{\sqrt{2}}\right) + 2m v_b' = 0 \quad \text{or} \quad 2\sqrt{2}\,v_b' + v_{pn}' = \frac{v_0}{\sqrt{2}}$$

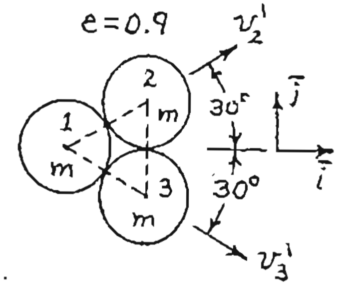
Adding equations, $\left(\frac{1}{\sqrt{2}} + 2\sqrt{2}\right)v_b' = (1+e)\frac{v_0}{\sqrt{2}}$

$$\bar{v}_b' = \left(\frac{1+e}{5}\right)v_0\,\bar{i}$$

$$v_{pn}' = \frac{v_0 - 4v_b'}{\sqrt{2}} = \left(\frac{1-4e}{5\sqrt{2}}\right)v_0$$

$$\bar{v}_p' = v_{pn}'\left(\frac{\bar{i}-\bar{j}}{\sqrt{2}}\right) + v_{pt}'\left(\frac{-\bar{i}-\bar{j}}{\sqrt{2}}\right) = v_0\left[-\frac{2}{5}(1+e)\,\bar{i} - \left(\frac{3-2e}{5}\right)\bar{j}\right]$$

4-12. From conservation of momentum in the $\bar{i}$ direction,



$$m\left(v_1' + \frac{\sqrt{3}}{2}v_2' + \frac{\sqrt{3}}{2}v_3'\right) = m v_0$$

By symmetry, $v_2' = v_3'$, so

$$v_1' + \sqrt{3}\,v_2' = v_0$$

From the definition of e,

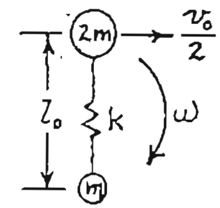
$$v_2' - \frac{\sqrt{3}}{2}v_1' = \frac{\sqrt{3}}{2}e\,v_0 \quad \text{or} \quad -\frac{3}{2}v_1' + \sqrt{3}\,v_2' = 1.35\,v_0$$

Subtracting equations, $\frac{5}{2}v_1' = -0.35\,v_0$ or $\bar{v}_1' = -0.1400\,v_0\,\bar{i}$

Then $v_2' = v_3' = \frac{1.14}{\sqrt{3}}v_0 = 0.6582\,v_0$, $\bar{v}_2' = (0.5700\,\bar{i} + 0.3291\,\bar{j})v_0$

$\bar{v}_3' = (0.5700\,\bar{i} - 0.3291\,\bar{j})v_0$

4-13. The situation just after impact is as shown. $\omega_i = \frac{v_0}{2l_0}$, $v_{c.m.} = \frac{v_0}{3}$



The total energy is $E = \frac{1}{2}(2m)\left(\frac{v_0}{2}\right)^2 = \frac{mv_0^2}{4}$

The total angular momentum about the c.m. is

$$H_c = 2m\left(\frac{l_0}{3}\right)\left(\frac{v_0}{2} - \frac{v_0}{3}\right) + m\left(\frac{2l_0}{3}\right)\left(\frac{v_0}{3}\right) = \frac{1}{3}m l_0 v_0$$

At maximum stretch, using conservation of H_c,

$$\left[2m l_0^2 + m(2l_0)^2\right]\omega = \frac{1}{3}m l_0 v_0 \quad \text{or} \quad \omega = \frac{v_0}{18\,l_0}$$

Then, using conservation of energy,

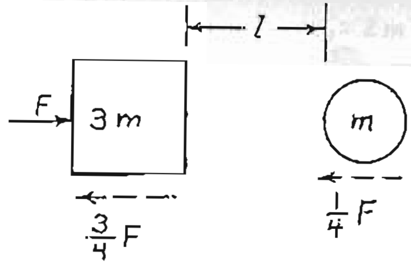
$$\frac{1}{2}k(2l_0)^2 + \frac{3m}{2}\left(\frac{v_0}{3}\right)^2 + \frac{6m l_0^2}{2}\left(\frac{v_0}{18\,l_0}\right)^2 = \frac{mv_0^2}{4}$$

$$2k l_0^2 = m v_0^2\left(\frac{1}{4} - \frac{1}{6} - \frac{1}{108}\right) = \frac{2}{27}m v_0^2$$

$$v_0 = 3l_0\sqrt{\frac{3k}{m}}$$

4-14. The acceleration of the c.m. is

$$a_{c.m.} = \frac{F}{4m}$$



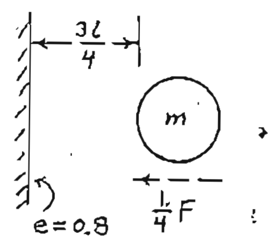
Take a reference frame accelerating with the c.m. and include inertia forces.

The block and sphere each have a net force $\frac{1}{4}F$ acting to move them toward each other, but the sphere moves three times as far as the block since the sphere has a smaller mass by a factor of three. Hence the sphere moves $\frac{3}{4}l$ to first impact and rebounds with $e = 0.8$.

The motion of the sphere relative to the c.m. frame is equivalent to its motion in an artificial gravitational field with

$$g = a_{c.m.} = \frac{F}{4m}$$

and $e = 0.8$ at a fixed surface.

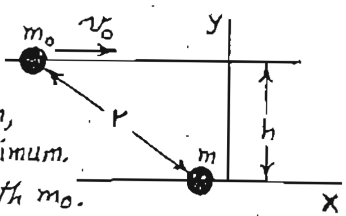


Using the results of Example 4-5, the time to stop bouncing is

$$t = \sqrt{\frac{2h}{g}}\left(\frac{1+e}{1-e}\right) = 9\sqrt{\frac{6ml}{F}}$$

At this time, $v = v_{c.m.} = a_{c.m.}t = \frac{9}{4}\sqrt{\frac{6Fl}{m}}$ in fixed frame.

4-15. Particle m has a negative acceleration $\ddot{x}$ due to the mutual attraction until r reaches a minimum, at which time the speed of m is maximum. Choose an inertial frame moving with m_0. In this frame, energy is conserved as particle m starts at $+\infty$ and has an initial velocity $-v_0$. The potential



4-15. (cont'd.) energy is $V = -K/r$. Then, at minimum separation,

$$-\frac{K}{h} + \frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 \quad \text{or} \quad v = -\sqrt{v_0^2 + \frac{2K}{mh}}$$

But, relative to the fixed xy frame,

$$\dot{x} = v_0 + v = v_0\left(1 - \sqrt{1 + \frac{2K}{mv_0^2 h}}\right)$$

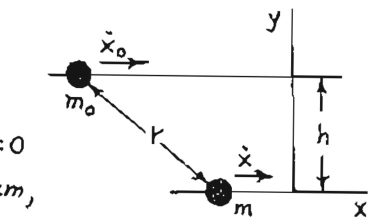
giving a maximum speed

$$v_{max} = |\dot{x}| = v_0\left(\sqrt{1 + \frac{2K}{mv_0^2 h}} - 1\right) \text{ to the left.}$$

4-16. Let $\dot{x}$ = velocity of m

$\dot{x}_0$ = velocity of m_0

$x_0(0) = -\infty$, $\dot{x}_0(0) = v_0$, $x(0) = 0$, $\dot{x}(0) = 0$



Using conservation of momentum,

$$m\dot{x} + m_0\dot{x}_0 = m_0 v_0 \quad \text{or} \quad \dot{x}_0 = v_0 - \frac{m}{m_0}\dot{x}$$

Using conservation of energy,

$$\frac{1}{2}m_0\dot{x}_0^2 + \frac{1}{2}m\dot{x}^2 - \frac{K}{r} = \frac{1}{2}m_0 v_0^2$$

or, substituting for $\dot{x}_0$,

$$\frac{1}{2}m_0\left(v_0 - \frac{m}{m_0}\dot{x}\right)^2 + \frac{1}{2}m\dot{x}^2 - \frac{K}{r} = \frac{1}{2}m_0 v_0^2$$

giving

$$\frac{1}{2}\left(m + \frac{m^2}{m_0}\right)\dot{x}^2 - mv_0\dot{x} - \frac{K}{r} = 0$$

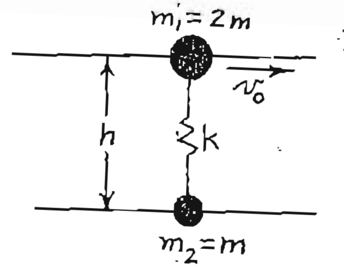
Noting that $\dot{x}$ is actually negative,

$$\dot{x} = \left[mv_0 - \sqrt{m^2v_0^2 + 2\frac{m(m_0+m)K}{m_0 r}}\right]\frac{m_0}{m(m_0+m)}$$

At $r_{min} = h$,

$$v_{max} = |\dot{x}|_{max} = \frac{m_0 v_0}{m_0+m}\left(\sqrt{1 + \frac{2K(m_0+m)}{mm_0v_0^2 h}} - 1\right)$$

4-17. Using conservation of linear momentum,



$$2mv_1 + mv_2 = 2mv_0$$

or $v_1 = v_0 - \frac{1}{2}v_2$

Using conservation of energy,

$mv_1^2 + \frac{1}{2}mv_2^2 + \frac{1}{2}k\delta^2 = mv_0^2$ or $m\left(v_0 - \frac{v_2}{2}\right)^2 + \frac{1}{2}mv_2^2 + \frac{1}{2}k\delta^2 = mv_0^2$

This results in

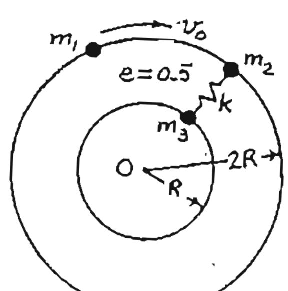
$\frac{3}{4}v_2^2 - v_0 v_2 + \frac{k}{2m}\delta^2 = 0$ or $v_2 = \dfrac{v_0 \pm \sqrt{v_0^2 - 3k\delta^2/2m}}{3/2}$

$(v_2)_{max}$ occurs for $\delta = 0$ with + sign. $\underline{(v_2)_{max} = \frac{4}{3}v_0}$

(b) δ_{max} occurs when $\dot{\delta} = 0$ and $v_1 = v_2 = \frac{2}{3}v_0$ from the momentum equation. Then, using the energy equation,

$\frac{3}{4}\left(\frac{2}{3}v_0\right)^2 - \frac{2}{3}v_0^2 + \frac{k}{2m}\delta^2$ giving $\underline{\delta_{max} = \sqrt{\frac{2m}{3k}}\,v_0}$

4-18. (a) From (4-83), $v_2' = \left(\frac{1+e}{2}\right)v_0 = \frac{3}{4}v_0$



(b) Consider m_2 and m_3 after impact. Using conservation of angular momentum about O, $2Rmv_2 + Rmv_3 = \frac{3}{2}Rmv_0$

giving $v_2 = \frac{3}{4}v_0 - \frac{1}{2}v_3$

Using conservation of energy,

$\frac{1}{2}mv_2^2 + \frac{1}{2}mv_3^2 + \frac{1}{2}k\delta^2 = \frac{1}{2}m\left(\frac{3v_0}{4}\right)^2$ or $v_2^2 + v_3^2 + \frac{k}{m}\delta^2 = \frac{9}{16}v_0^2$

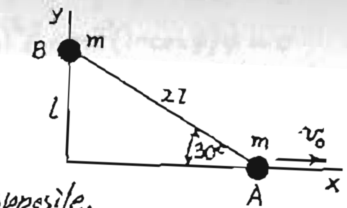
For $(v_3)_{max}$, spring force is zero. Hence $\delta = 0$ and we obtain

$\left(\frac{3}{4}v_0 - \frac{1}{2}v_3\right)^2 + v_3^2 = \frac{9}{16}v_0^2$ or $\frac{v_3}{4}(5v_3 - 3v_0) = 0$, $\underline{(v_3)_{max} = \frac{3}{5}v_0}$

(c) At $\delta = \delta_{max}$, $\omega_2 = \omega_3$ or $v_2 = 2v_3$. Then the angular momentum equation gives $v_3 = \frac{3}{10}v_0$. Then, using the energy equation,

$\frac{k}{m}\delta^2 = \left(\frac{9}{16} - \frac{9}{25} - \frac{9}{100}\right)v_0^2$ Hence $\delta_{max} = \frac{3}{4}\sqrt{\frac{m}{5k}}\,v_0$

4-19. There are no external forces on the system. The cord applies equal and opposite impulses to particles A and B. Hence, the velocity changes are equal and opposite, and are in the direction of the cord. Furthermore, the final velocity components along the cord are equal.

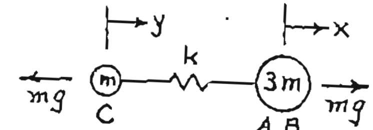


$v_B = \frac{\sqrt{3}}{2}v_0 - v_B$ so $v_B = \frac{\sqrt{3}}{4}v_0$, $\bar{v}_B = \frac{\sqrt{3}}{4}v_0\left(\frac{\sqrt{3}}{2}\bar{i} - \frac{1}{2}\bar{j}\right)$

$= \frac{v_0}{8}(3\bar{i} - \sqrt{3}\bar{j})$

Then $\bar{v}_A = v_0\bar{i} - \bar{v}_B = \underline{\frac{v_0}{8}(5\bar{i} + \sqrt{3}\bar{j})}$

4-20. Particles A and B move together under a net gravitational force mg, opposed by a force mg acting on C. The equivalent system is shown. The c.m. of this system remains fixed since the net force is zero.



$$x = -\frac{1}{3}y \quad \text{and} \quad \dot{x} = -\frac{1}{3}\dot{y}$$

When the stretch $\delta = x - y$ is maximum, the work done by gravity is equal to the energy stored in the spring.

$mg\,\delta_{max} = \frac{1}{2}k\delta_{max}^2$ or $\underline{\delta_{max} = 2mg/k}$

(b) In general, using work and energy,

$mg\delta = \frac{1}{2}k\delta^2 + \frac{3m}{2}\dot{x}^2 + \frac{m}{2}\dot{y}^2$ or $-\frac{4}{3}mgy = \frac{1}{2}k\left(\frac{4}{3}y\right)^2 + \left(\frac{1}{6} + \frac{1}{2}\right)m\dot{y}^2$

Thus $\dot{y}^2 = -2gy - \frac{4k}{3m}y^2$ For $|\dot{y}|_{max}$, we have

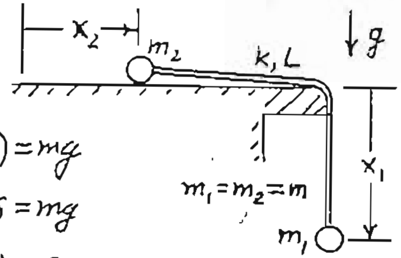
$\frac{d(\dot{y}^2)}{dy} = 0$ or $-2g - \frac{8k}{3m}y = 0$ giving $y = \frac{-3mg}{4k}$

Then $\dot{y}^2 = -2g\left(\frac{-3mg}{4k}\right) - \frac{4k}{3m}\left(\frac{-3mg}{4k}\right)^2 = \frac{3mg^2}{4k}$

$\underline{\dot{y}_{max} = \frac{g}{2}\sqrt{\frac{3m}{k}}}$

4-21. Equations of motion:

$$m\ddot{x}_1 + kx_1 - kx_2 = mg$$
$$m\ddot{x}_2 + kx_2 - kx_1 = 0$$



Subtracting, $m(\ddot{x}_1 - \ddot{x}_2) + 2k(x_1 - x_2) = mg$

stretch $\delta = x_1 - x_2$, so $m\ddot{\delta} + 2k\delta = mg$

Initial conditions: $\delta(0) = 0$, $\dot{\delta}(0) = 0$

Solution has form $\delta = \frac{mg}{2k} + A\cos\sqrt{\frac{2k}{m}}\,t + B\sin\sqrt{\frac{2k}{m}}\,t$

$\delta(0) = \frac{mg}{2k} + A = 0$, $\dot{\delta}(0) = \sqrt{\frac{2k}{m}}\,B = 0$, $A = \frac{-mg}{2k}$, $B = 0$.

Then $\delta = \frac{mg}{2k}\left(1 - \cos\sqrt{\frac{2k}{m}}\,t\right)$ and $\delta_{max} = \frac{mg}{k}$

$$F_{max} = k\,\delta_{max} = \underline{mg}$$

(b) Consider the equivalent system. The midpoint of the cord is also the c.m.

$a_{c.m.} = \frac{1}{2}g$ and $v_c^2 = 2a_{c.m.}x_c$ where $x_c = \frac{1}{2}(x_1 + x_2)$

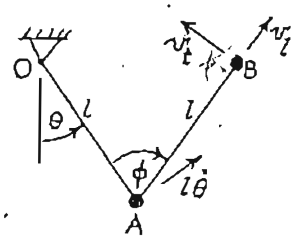
For $x_c = \frac{1}{2}L$, $v_c^2 = 2(\frac{1}{2}g)(\frac{1}{2}L)$ or $\underline{v_c = \sqrt{\frac{gL}{2}}}$

(c) Cord returns to length L when $\delta = 0$ and $t = 2\pi\sqrt{\frac{m}{2k}}$.

At this time, $x_2 = L = x_c$ where $x_c = \frac{1}{2}a_{cm}t^2$ or $L = \left(\frac{1}{4}g\right)\left(4\pi^2\frac{m}{2k}\right)$

$$= \underline{\frac{\pi^2 mg}{2k}}$$

4-22. Let v_l and v_t be the longitudinal and transverse components of the absolute velocity of B.



$v_l = l\dot{\theta}\sin\phi$, $v_t = -l\dot{\theta}\cos\phi - l(\dot{\phi} - \dot{\theta})$

Angular momentum about O is conserved.

4-22. (cont'd.) $H_O = ml^2\dot{\theta} + mv_l\,l\sin\phi + mv_t\,l(1-\cos\phi) = 0$

or $ml^2\dot{\theta} + ml^2\dot{\theta}\sin^2\phi + ml^2(1-\cos\phi)^2\dot{\theta} - ml^2(1-\cos\phi)\dot{\phi} = 0$

or $ml^2(1 + \sin^2\phi + 1 - 2\cos\phi + \cos^2\phi)\dot{\theta} = ml^2(1-\cos\phi)\dot{\phi}$

$$\dot{\theta} = \left(\frac{1-\cos\phi}{3-2\cos\phi}\right)\dot{\phi}$$

Energy is conserved.

$\frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}m(v_l^2 + v_t^2) = \frac{1}{2}ml^2\omega_0^2$, $\theta(0) = \phi(0) = 0$, $\dot{\theta}(0) = 0$, $\dot{\phi}(0) = \omega_0$

or

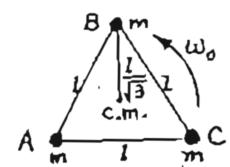
$$\frac{1}{2}ml^2\left\{\dot{\theta}^2\left[1 + \sin^2\phi + (1-\cos\phi)^2\right] + \dot{\phi}^2 - 2\dot{\theta}\dot{\phi}(1-\cos\phi)\right\} = \frac{1}{2}ml^2\omega_0^2$$

Substituting for $\dot{\theta}$,

$$\left[\frac{(1-\cos\phi)^2}{3-2\cos\phi} + 1 - \frac{2(1-\cos\phi)^2}{3-2\cos\phi}\right]\dot{\phi}^2 = \omega_0^2 \text{ or } \left(\frac{1+\sin^2\phi}{3-2\cos\phi}\right)\dot{\phi}^2 = \omega_0^2$$

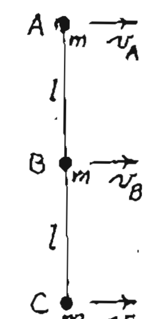
$$\underline{\dot{\phi} = \sqrt{\frac{3-2\cos\phi}{1+\sin^2\phi}}\,\omega_0}, \quad \underline{\dot{\theta} = \frac{(1-\cos\phi)\,\omega_0}{\sqrt{(1+\sin^2\phi)(3-2\sin\phi)}}}$$

4-23. The angular momentum about the c.m. is $H_c = 3m\left(\frac{l}{\sqrt{3}}\right)^2\omega_0 = ml^2\omega_0$



In the straight configuration,

$H_c = ml(v_c - v_A) = ml^2\omega_0$ or $-v_A + v_c = l\omega_0$



From conservation of linear momentum,

$$v_A + v_B + v_c = 0$$

Using conservation of kinetic energy,

$$\frac{1}{2}m(v_A^2 + v_B^2 + v_c^2) = \frac{1}{2}ml^2\omega_0^2$$

Substituting $v_c = l\omega_0 + v_A$ and $v_B = -2v_A - l\omega_0$ in the energy equation,

$$v_A^2 + (4v_A^2 + 4l\omega_0 v_A + l^2\omega_0^2) + (v_A^2 + 2l\omega_0 v_A + l^2\omega_0^2) = l^2\omega_0^2$$

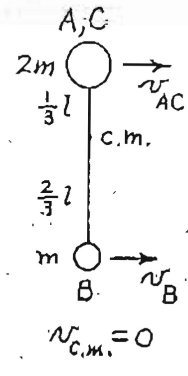
or $6v_A^2 + 6l\omega_0 v_A + l^2\omega_0^2 = 0$, $v_A = \left(-\frac{1}{2} \pm \frac{1}{\sqrt{12}}\right)l\omega_0$

4-23. (cont'd.) On the first pass through the straight-line configuration, the + sign applies since the original exterior angle at B is decreasing.

$$\underline{v_A = \left(-\tfrac{1}{2} + \tfrac{1}{\sqrt{12}}\right) l\omega_0 = -0.2113\, l\omega_0}, \quad \underline{v_B = \frac{-l\omega_0}{\sqrt{3}} = -0.5774\, l\omega_0}$$

$$\underline{v_C = \left(\tfrac{1}{2} + \tfrac{1}{\sqrt{12}}\right) l\omega_0 = 0.7887\, l\omega_0}.$$

(b) Suppose A and C should just barely collide, i.e., have the same position and equal velocities.



From angular momentum conservation,

$$m\left(\tfrac{2}{3}l\right)v_B - 2m\left(\tfrac{1}{3}l\right)v_{AC} = ml^2\omega_0, \quad v_B - v_{AC} = \tfrac{3}{2} l\omega_0$$

From linear momentum conservation,

$$2m\, v_{AC} + m v_B = 0, \qquad v_B + 2v_{AC} = 0$$

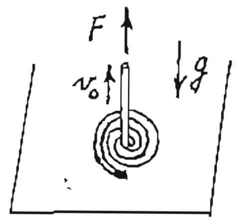
Hence $v_{AC} = -\frac{1}{2} l\omega_0$, $v_B = l\omega_0$.

The kinetic energy is

$$T = \tfrac{1}{2} m (l\omega_0)^2 + m\left(-\tfrac{1}{2} l\omega_0\right)^2 = \tfrac{3}{4} m l^2 \omega_0^2$$

This is larger than the actual energy $\frac{1}{2} m l^2 \omega_0^2$. Hence, A and C cannot collide because there is insufficient energy. A nonzero collision velocity would require even more energy.

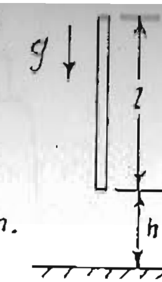
* 4-24. The net force on the vertical portion of the rope is the applied force F minus its weight $\rho g v_0 t$.



Its rate of increase of momentum is the mass per second ρv_0 times its velocity change v_0. Equating the net force to the rate of change of momentum,

$$F - \rho g v_0 t = \rho v_0^2 \quad \text{or} \quad \underline{F = \rho v_0^2 + \rho g v_0 t}$$

4-25. Time t when rope first hits floor is found from $\frac{1}{2} g t^2 = h$ or $t = \sqrt{\frac{2h}{g}}$



$$\underline{F = 0, \qquad 0 \le t \le \sqrt{\frac{2h}{g}}}$$

The length of rope on the floor is $\frac{1}{2} g t^2 - h$. Its weight is $\rho g(\frac{1}{2} g t^2 - h)$. The impact force is ρv^2 since a mass ρv having a velocity v hits each second. $v = gt$. The total force is equal to the weight of the rope on the floor plus the impact force.

$$\underline{F = \rho g\left(\tfrac{1}{2} g t^2 - h\right) + \rho g^2 t^2 = \tfrac{3}{2} \rho g^2 t^2 - \rho g h}$$

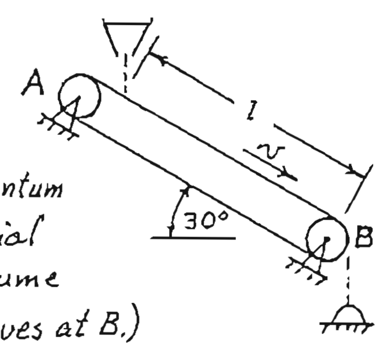
$$\sqrt{\frac{2h}{g}} \le t \le \sqrt{\frac{2(h+l)}{g}}$$

When all the rope is on the floor,

$$\underline{F = \rho g l \quad \text{for } t > \sqrt{\frac{2(h+l)}{g}}}$$

4-26. The gravitational force component along the belt is

$$F_t = \frac{bgl}{2v}$$

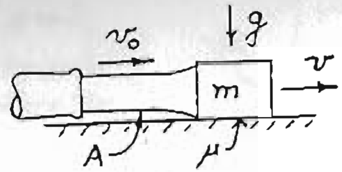


since the mass per unit length is b/v. The force F_t results in a rate of momentum increase bv of the material reaching the belt. (We assume no reaction as material leaves at B.)

Hence

$$\frac{bgl}{2v} = bv \quad \text{or} \quad \underline{v = \sqrt{\frac{gl}{2}}}$$

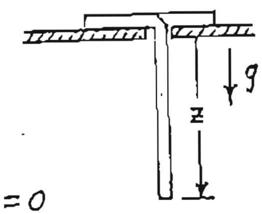
-27. The mass of water
hitting the block per unit time
$\rho A(v_0 - v)$. It



undergoes a velocity change
$v_0 - v$. Hence the force on the block is

$$\rho A(v_0 - v)^2 = \mu m g \quad \text{giving.} \quad v = v_0 - \sqrt{\frac{\mu m g}{\rho A}}$$

-28. Consider the entire rope as
the system. The net force acting
on the system is the gravitational
force $\rho g z$ acting downward. The
momentum is $\rho z \dot{z}$ downward.



Hence $\frac{d}{dt}(\rho z \dot{z}) = \rho g z$ or $z\ddot{z} + \dot{z}^2 - gz = 0$

) Let $\dot{z} = v$ and $\ddot{z} = v\frac{dv}{dz}$. Divide the differential
equation by z and obtain

$$v\frac{dv}{dz} + \frac{v^2}{z} = g \quad \text{or} \quad \frac{1}{2}\frac{d(v^2)}{dz} + \frac{v^2}{z} = g$$

This is linear in v^2. The homogeneous equation is

$$\frac{1}{2}\frac{d(v^2)}{dz} + \frac{v^2}{z} = 0 \quad \text{or} \quad \frac{d(v^2)}{v^2} = -2\frac{dz}{z}$$

$\ln(v^2) = -2 \ln z + \text{const}$ giving $(v^2)_t = \frac{A}{z^2}$.

To obtain the particular integral (steady-state solⁿ), use variation of parameters. $v^2 = \frac{V(z)}{z^2}$, $\frac{d(v^2)}{dz} = \frac{-2V}{z^3} + \frac{V'}{z^2}$

Then the differential equation yields

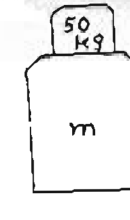
$$\frac{1}{2z^2}\frac{dV}{dz} = g, \text{ so } V(z) = \frac{2}{3}g z^3 \text{ and } (v^2)_s = \frac{2}{3}g z$$

where $v = 0$ when $z = z_0$

Then $v^2 = (v^2)_t + (v^2)_s = \frac{A}{z^2} + \frac{2}{3}gz$

$A = -\frac{2}{3}g z_0^3$ giving $v^2 = \frac{-2g z_0^3}{3z^2} + \frac{2}{3}gz$ or $v = \sqrt{\frac{2g}{3}\left(z - \frac{z_0^3}{z^2}\right)}$

4-29. (a) The required burnout velocity
is found using conservation of energy.

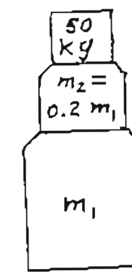


$$\frac{1}{2}mv^2 - mgR = \frac{-mgR^2}{1.25R}, \quad v = \sqrt{\frac{2}{5}gR} = 5000.77 \frac{m}{sec}$$

But $v = I_{sp} g \ln\left(\frac{m+50}{.1m+50}\right)$ or $\ln\left(\frac{m+50}{.1m+50}\right) = 2.0390$

$\frac{m+50}{0.1m+50} = 7.6833$; $m = 1442.41$ kg, $m + 50 = 1492.4$ kg

(b) Two stages, $m_1 = 5m_2$. Same burnout $v = 5000.77 \frac{m}{sec}$
Assume the first rocket is discarded after its burnout.



$$\ln\left(\frac{1.2m_1 + 50}{0.3m_1 + 50}\right) + \ln\left(\frac{0.2m_1 + 50}{0.02m_1 + 50}\right) = 2.0390$$

or $\left(\frac{1.2m_1 + 50}{0.3m_1 + 50}\right)\left(\frac{0.2m_1 + 50}{0.02m_1 + 50}\right) = 7.6833$

or $0.1939\, m_1^2 - 52.933\, m_1 - 16{,}708 = 0$

$m_1 = 136.494 + 323.728 = 460.22$

Total mass $1.2m_1 + 50 = 602.27$ kg

(c) Single stage. Let F = rocket thrust.
Total impulse $10F = I_{sp}(0.9mg) = 3.1838 \times 10^6$ N·sec

At burnout, $a_{max} = \frac{F}{0.1m + 50} - g = 166.08g = 1629.3 \frac{m}{sec^2}$

Two stages. The first-stage thrust is

$F_1 = \frac{1}{10} I_{sp}(0.9m_1 g) = 1.0158 \times 10^5$ N

$(a_1)_{max} = \frac{F_1}{0.3m_1 + 50} - g = 54.060g = 530.33 \frac{m}{sec^2}$

second-stage thrust is

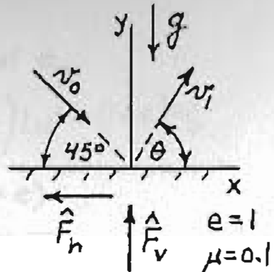
$F_2 = \frac{1}{10} I_{sp}(0.9m_2 g) = 0.2F_1 = 20{,}317$ N

$(a_2)_{max} = \frac{F_2}{0.02m_1 + 50} - g = 33.981g = 333.35 \frac{m}{sec^2}$

4-30. On any bounce,

$\hat{F}_h = \mu \hat{F}_v$ if impulse $\mu \hat{F}_v \le m\dot{x}$, where $\dot{x}$ is the horizontal velocity just before the bounce.

$0 \le \hat{F}_h < \mu \hat{F}_v$ if $\mu \hat{F}_v > m\dot{x}$



First bounce. $\hat{F}_v = 2m\frac{v_0}{\sqrt{2}}$, $\hat{F}_h = \frac{m v_0}{5\sqrt{2}}$

$$\bar{v}_1 = \frac{v_0}{\sqrt{2}}\left(\frac{4}{5}\bar{i} + \bar{j}\right), \quad \tan\theta = \frac{5}{4}, \quad \underline{\theta = 51.34°}$$

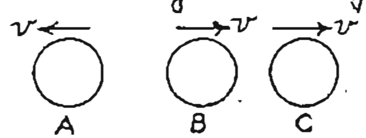
$$v_1 = \frac{v_0}{\sqrt{2}}\sqrt{\left(\frac{4}{5}\right)^2 + 1} = \underline{0.9055\, v_0}$$

(b) Assume the first bounce occurs at $x = y = 0$. On each bounce, the horizontal velocity decreases by $v_0/5\sqrt{2}$ but the vertical velocity is unchanged. After five bounces the horizontal velocity is zero. The time between bounces is constant and equal to $\sqrt{2}\,v_0/g$.

$$\text{Final } x = \frac{v_0}{\sqrt{2}}(0.8 + 0.6 + 0.4 + 0.2)\frac{\sqrt{2}\,v_0}{g} = \underline{\frac{2v_0^2}{g}}$$

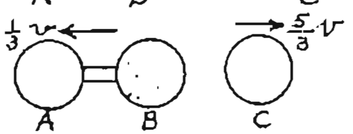
Since $e=1$, the motion after the fifth bounce is entirely vertical with bounce height $h = \frac{1}{2g}\left(\frac{v_0}{\sqrt{2}}\right)^2 = \frac{v_0^2}{4g}$.

4-31. (a) A and B not connected. The final motion is, for $e = 1$,



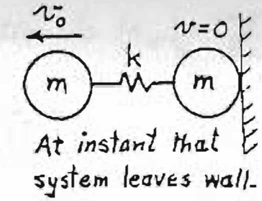
(b) A and B connected by a short inelastic string. The final motion is

(c) A and B connected by a rigid massless rod.



Mass 2m and mass m collide.

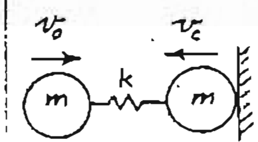
4-32.

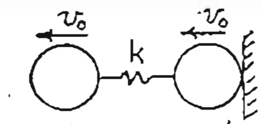


(a) $e' = 0$

$v_{c.m.} = \frac{1}{2}v_0$ to left, so $\underline{e = \frac{1}{2}}$.

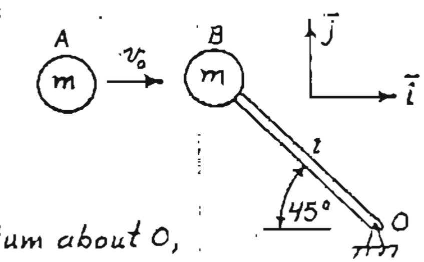
(b) $e' = 1$





Final $v_{c.m.} = v_0$ to left so $\underline{e = 1}$

4-33. Let ω' be the clockwise angular velocity of B about O after impact. From the definition of e,



$$\frac{l\omega'}{\sqrt{2}} - v_A' = e\,v_0$$

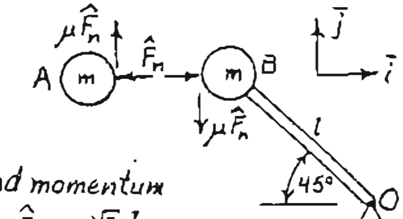
Conserving angular momentum about O,

$$ml^2\omega' + \frac{ml}{\sqrt{2}}v_A' = \frac{ml}{\sqrt{2}}v_0$$

Hence $\frac{3l}{\sqrt{2}}\omega' = (1+e)v_0$ or $\underline{\omega' = \frac{\sqrt{2}\,v_0}{3l}(1+e)}$

Then $v_A' = \frac{l}{\sqrt{2}}\omega' - e v_0 = \left(\frac{1-2e}{3}\right)v_0$ $\quad \underline{\bar{v}_A' = \left(\frac{1-2e}{3}\right)v_0\,\bar{i}}$

(b) Assume slipping at impact. For A, using linear impulse and momentum, $\bar{v}_A' = \left(v_0 - \frac{\hat{F}_n}{m}\right)\bar{i} + \frac{\mu\hat{F}_n}{m}\bar{j}$



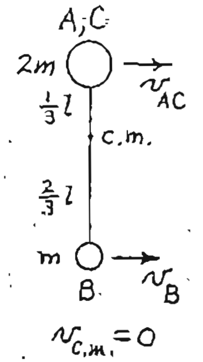
For B, using angular impulse and momentum about O, $(1-\mu)\hat{F}_n\frac{l}{\sqrt{2}} = ml^2\omega'$ or $\frac{\hat{F}_n}{m} = \frac{\sqrt{2}\,l}{1-\mu}\omega'$

4-23. (cont'd.) On the first pass through the straight-line configuration, the + sign applies since the original exterior angle at B is decreasing.

$$v_A = \left(-\frac{1}{2}+\frac{1}{\sqrt{12}}\right) l\omega_0 = -0.2113\, l\omega_0\,, \quad v_B = \frac{-l\omega_0}{\sqrt{3}} = -0.5774\, l\omega_0$$

$$v_C = \left(\frac{1}{2}+\frac{1}{\sqrt{12}}\right) l\omega_0 = 0.7887\, l\omega_0\,.$$

(b) Suppose A and C should just barely collide, i.e., have the same position and equal velocities.



From angular momentum conservation,

$$m\left(\tfrac{2}{3}l\right)v_B - 2m\left(\tfrac{1}{3}l\right)v_{AC} = ml^2\omega_0\,, \quad v_B - v_{AC} = \tfrac{3}{2}l\omega_0$$

From linear momentum conservation,

$$2m\,v_{AC} + m v_B = 0\,, \qquad v_B + 2v_{AC} = 0$$

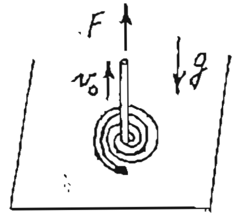
Hence $v_{AC} = -\frac{1}{2}l\omega_0\,, \quad v_B = l\omega_0$

The kinetic energy is

$$T = \tfrac{1}{2}m\left(l\omega_0\right)^2 + m\left(-\tfrac{1}{2}l\omega_0\right)^2 = \tfrac{3}{4}ml^2\omega_0^2$$

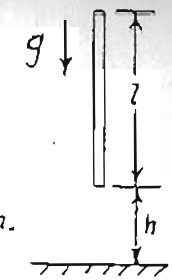
This is larger than the actual energy $\frac{1}{2}ml^2\omega_0^2$. Hence, A and C cannot collide because there is insufficient energy. A nonzero collision velocity would require even more energy.

4-24. The net force on the vertical portion of the rope is the applied force F minus its weight $\rho g v_0 t$. Its rate of increase of momentum is the mass per second ρv_0 times its velocity change v_0. Equating the net force to the rate of change of momentum,



$$F - \rho g v_0 t = \rho v_0^2 \quad \text{or} \quad F = \rho v_0^2 + \rho g v_0 t$$

4-25. Time t when rope first hits floor is found from $\frac{1}{2}gt^2 = h$ or $t = \sqrt{\frac{2h}{g}}$



$$F = 0\,, \qquad 0 \le t \le \sqrt{\frac{2h}{g}}$$

The length of rope on the floor is $\frac{1}{2}gt^2 - h$. Its weight is $\rho g(\frac{1}{2}gt^2 - h)$. The impact force is ρv^2 since a mass ρv having a velocity v hits each second. $v = gt$. The total force is equal to the weight of the rope on the floor plus the impact force.

$$F = \rho g\left(\tfrac{1}{2}gt^2 - h\right) + \rho g^2 t^2 = \tfrac{3}{2}\rho g^2 t^2 - \rho g h$$

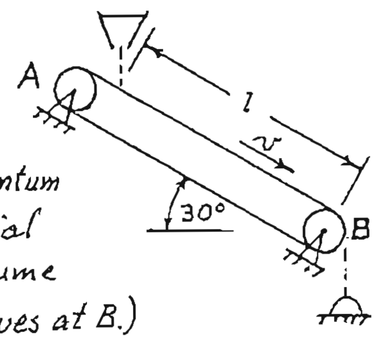
$$\sqrt{\frac{2h}{g}} \le t \le \sqrt{\frac{2(h+l)}{g}}$$

When all the rope is on the floor,

$$F = \rho g l \quad \text{for } t > \sqrt{\frac{2(h+l)}{g}}$$

4-26. The gravitational force component along the belt is

$$F_t = \frac{bgl}{2v}$$

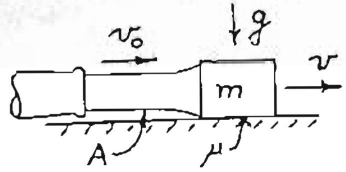


since the mass per unit length is b/v. The force F_t results in a rate of momentum increase bv of the material reaching the belt. (We assume no reaction as material leaves at B.)

Hence

$$\frac{bgl}{2v} = bv \quad \text{or} \quad v = \sqrt{\frac{gl}{2}}$$

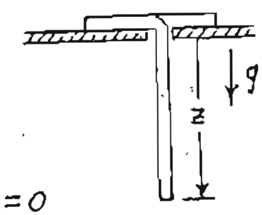
-27. The mass of water
'ting the block per unit time
$\rho A(v_0 - v)$. It
dergoes a velocity change



$-v$. Hence the force on the block is

$$\rho A(v_0-v)^2 = \mu m g \quad \text{giving} \quad v = v_0 - \sqrt{\frac{\mu m g}{\rho A}}$$

-28. Consider the entire rope as
the system. The net force acting
on the system is the gravitational
force $\rho g z$ acting downward. The
momentum is $\rho z \dot{z}$ downward.



Hence $\frac{d}{dt}(\rho z\dot{z}) = \rho g z$ or $z\ddot{z} + \dot{z}^2 - gz = 0$

) Let $\dot{z} = v$ and $\ddot{z} = v\frac{dv}{dz}$. Divide the differential equation by z and obtain

$$v\frac{dv}{dz} + \frac{v^2}{z} = g, \text{ or } \frac{1}{2}\frac{d(v^2)}{dz} + \frac{v^2}{z} = g$$

This is linear in v^2. The homogeneous equation is

$$\frac{1}{2}\frac{d(v^2)}{dz} + \frac{v^2}{z} = 0 \quad \text{or} \quad \frac{d(v^2)}{v^2} = -2\frac{dz}{z}$$

$\ln(v^2) = -2\ln z + \text{const}$ giving $(v^2)_t = \frac{A}{z^2}$.

To obtain the particular integral (steady-state sol^n), use variation of parameters. $v^2 = \frac{V(z)}{z^2}$, $\frac{d(v^2)}{dz} = \frac{-2V}{z^3} + \frac{V'}{z^2}$

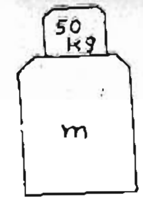
Then the differential equation yields

$$\frac{1}{2z^2}\frac{dV}{dz} = g, \text{ so } V(z) = \frac{2}{3}g z^3 \text{ and } (v^2)_s = \frac{2}{3}g z$$

Then $v^2 = (v^2)_t + (v^2)_s = \frac{A}{z^2} + \frac{2}{3}g z$ where $v=0$ when $z=z_0$

$A = -\frac{2}{3}g z_0^3$ giving $v^2 = \frac{-2gz_0^3}{3z^2} + \frac{2}{3}g z$ or $v = \sqrt{\frac{2g}{3}\left(z - \frac{z_0^3}{z^2}\right)}$

4-29. (a) The required burnout velocity is found using conservation of energy.



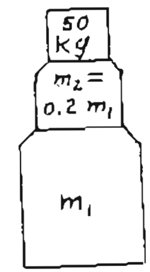
$$\frac{1}{2}m v^2 - mgR = \frac{-mgR^2}{1.25R}, \quad v = \sqrt{\frac{2}{5}gR} = 5000.77\ \frac{m}{sec}$$

But $v = I_{sp}\, g \ln\left(\frac{m+50}{.1m+50}\right)$ or $\ln\left(\frac{m+50}{.1m+50}\right) = 2.0390$

$\frac{m+50}{0.1m+50} = 7.6833$, $m = 1442.41$ kg, $m+50 = 1492.4$ kg

(b) Two stages, $m_1 = 5m_2$. Same burnout $v = 5000.77\ \frac{m}{sec}$

Assume the first rocket is discarded after its burnout.



$$\ln\left(\frac{1.2m_1+50}{0.3m_1+50}\right) + \ln\left(\frac{0.2m_1+50}{0.02m_1+50}\right) = 2.0390$$

or $\left(\frac{1.2m_1+50}{0.3m_1+50}\right)\left(\frac{0.2m_1+50}{0.02m_1+50}\right) = 7.6833$

or $0.1934\, m_1^2 - 52.933\, m_1 - 16{,}708 = 0$

$m_1 = 136.494 + 323.728 = 460.22$

Total mass $1.2m_1 + 50 = 602.27$ kg

(c) Single stage. Let F = rocket thrust.

Total impulse $10F = I_{sp}(0.9\, mg) = 3.1838\times10^6$ N-sec

At burnout, $a_{max} = \frac{F}{0.1m+50} - g = 166.08g = 1629.3\ \frac{m}{sec^2}$

Two stages. The first-stage thrust is

$$F_1 = \frac{1}{10} I_{sp}(0.9\, m_1 g) = 1.0158\times10^5 \text{ N}$$

$$(a_1)_{max} = \frac{F_1}{0.3m_1+50} - g = 54.060g = 530.33\ \frac{m}{sec^2}$$

second-stage thrust is

$$F_2 = \frac{1}{10} I_{sp}(0.9\, m_2 g) = 0.2F_1 = 20{,}317 \text{ N}$$

$$(a_2)_{max} = \frac{F_2}{0.02m_1+50} - g = 33.981g = 333.35\ \frac{m}{sec^2}$$

4-33. (cont'd.) From the definition of e,

$$\frac{l\omega'}{\sqrt2} - \left(v_0 - \frac{\hat F_n}{m}\right) = e v_0 \quad\text{or}\quad \left(\frac{1}{\sqrt2}+\frac{\sqrt2}{1-\mu}\right) l\omega' = (1+e)v_0$$

which results in $\omega' = \frac{\sqrt2\, v_0}{l}\left(\frac{1-\mu}{3-\mu}\right)(1+e)$

Then $\frac{\hat F_n}{m} = \frac{2v_0}{3-\mu}(1+e)$ and we obtain

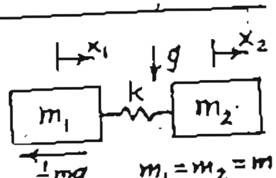
$$\bar v_A' = \left[1-\left(\frac{2}{3-\mu}\right)(1+e)\right]v_0\bar i + \left(\frac{2\mu}{3-\mu}\right)(1+e)v_0\bar j = \frac{v_0}{3-\mu}\left[(1-\mu-2e)\bar i + 2\mu(1+e)\bar j\right]$$

slipping actually occurs in the sense shown if the tangential velocity of B is greater than the tangential velocity of A, i.e, if $\frac{l\omega'}{\sqrt2} > v'_{Ay}$ or $\left(\frac{1-\mu}{3-\mu}\right)(1+e) > \frac{2\mu(1+e)}{3-\mu}$ or $1-\mu > 2\mu$ or $\mu<\frac13$.

(c) If $\mu \ge \frac13$, there is no slip in the tangential direction, and the motion is independent of μ. Setting $\mu=\frac13$,

$$\omega' = \frac{v_0}{2\sqrt2\, l}(1+e), \qquad \bar v_A' = \frac{v_0}{4}\left[(1-3e)\bar i + (1+e)\bar j\right]$$

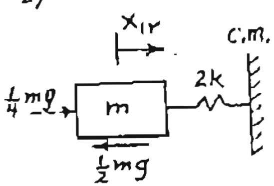
4-34. The initial spring force is mg, so both m_1 and m_2 start sliding toward each other. For $\dot x_1>0$, the acceleration of the c.m. is



$$a_{c.m.} = \frac{-\frac12 mg}{2m} = -\frac14 g \quad\text{so}\quad v_{c.m.} = -\frac{gt}{4},\quad x_{c.m.} = \frac{mg}{2k} - \frac{gt^2}{8}$$

since $x_1(0)=0$, $x_2(0)=\frac{mg}{k}$ and $x_{c.m.}=\frac12(x_1+x_2)$

Now consider the motion of m_1 relative to the accelerating c.m. frame. Including inertia forces and defining



$$x_{1r} = x_1 - x_{c.m.}$$

we obtain the differential equation

$$m\ddot x_{1r} + 2kx_{1r} = -\frac14 mg \quad\text{with initial conditions}\quad x_{1r}(0) = \frac{-mg}{2k},\quad \dot x_{1r}(0)=0$$

4-34. (cont'd.) The steady-state solution is $(x_{1r})_s = \frac{-mg}{8k}$.

The complete solution has the form

$$x_{1r} = -\frac{mg}{8k} + C_1\cos\sqrt{\frac{2k}{m}}\,t + C_2\sin\sqrt{\frac{2k}{m}}\,t$$

From the initial conditions, $C_1=-\frac{3mg}{8k}$ and $C_2=0$.

$$x_{1r} = -\frac{mg}{8k}\left(1+3\cos\sqrt{\frac{2k}{m}}\,t\right)$$

Now transform to the absolute frame.

$$x_1 = x_{1r}+x_{c.m.} = \frac{3mg}{8k}\left(1-\cos\sqrt{\frac{2k}{m}}\,t\right) - \frac18 gt^2$$

$$\dot x_1 = -\frac14 gt + \frac{3g}{8}\sqrt{\frac{2m}{k}}\sin\sqrt{\frac{2k}{m}}\,t, \qquad \ddot x_1 = -\frac14 g + \frac34 g\cos\sqrt{\frac{2k}{m}}\,t$$

$(\dot x_1)_{max}$ occurs when $\ddot x_1$ first reaches zero.

$$\cos\sqrt{\frac{2k}{m}}\,t = \frac13 \quad\text{or}\quad \sqrt{\frac{2k}{m}}\,t = 1.2310 \text{ rad } (70.53°)$$

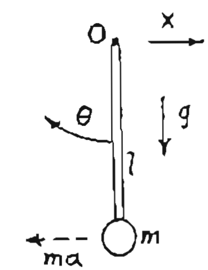
Then $(\dot x_1)_{max} = -\frac g4\left(1.2310\sqrt{\frac{m}{2k}}\right) + \frac{3g}{4}\sqrt{\frac{m}{2k}}\sqrt{\frac89} = 0.2824\sqrt{\frac mk}\,g$

(b) Sliding stops when $\dot x_1 = 0$ or $\sin\sqrt{\frac{2k}{m}}\,t = \frac13\sqrt{\frac{2k}{m}}\,t$

Let $\theta=\sqrt{\frac{2k}{m}}\,t$. Then $\sin\theta - \frac13\theta = 0$, $\theta = 2.2789$ rad (130.57°)

$$t = \sqrt{\frac{m}{2k}}\,\theta = 1.6114\sqrt{\frac mk}$$

4-35. Consider an accelerating frame in which the support point O is fixed. In this frame, there is an artificial gravity a as well as the natural gravity g, with the resultant at an angle $\beta=\tan^{-1}\frac ag$. The pendulum will swing an equal angle beyond the equilibrium position at $\theta=\beta$, giving $\theta_{min}=0$, $\theta_{max}=2\tan^{-1}\frac ag$

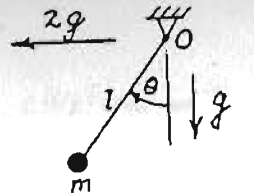


(b) $a=\sqrt3\, g$ so $\beta=\tan^{-1}\sqrt3 = 60°$, $\sqrt{a^2+g^2}=2g$

From (3-230), period $T=4\sqrt{\frac{l}{2g}}\,K(k)$

$k=\sin\frac\beta2=\frac12$. From tables, $K(\frac12)=1.6858$. $T=4.7682\sqrt{l/g}$

4-36. Consider the support point O to be fixed and add an artificial horizontal gravity of strength $2g$. In this frame, the potential energy is



$$V=-mgl\,(\cos\theta+2\sin\theta)$$

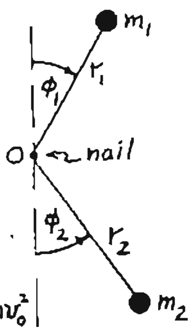
$\theta(0)=0,\ \dot\theta(0)=0$

Using conservation of energy,

$$\tfrac12 ml^2\dot\theta^2-mgl(\cos\theta+2\sin\theta)=-mgl$$

At $\theta=90^\circ$, $\tfrac12 ml^2\dot\theta^2=mgl$ giving $\dot\theta=\sqrt{\dfrac{2g}{l}}$

4-37. $r_1(0)=l$, $r_2(0)=3l$.



Using conservation of angular momentum about O, $mr_1^2\dot\phi_1=mlv_0$, $\dot\phi_1=\dfrac{lv_0}{r_1^2}$

$$mr_2^2\dot\phi_2=3mlv_0,\quad \dot\phi_2=\frac{3lv_0}{r_2^2}=\frac{3lv_0}{(4l-r_1)^2}$$

From conservation of kinetic energy, noting that $\dot r_2=-\dot r_1$,

$$T=\tfrac12 m(\dot r_1^2+r_1^2\dot\phi_1^2+\dot r_2^2+r_2^2\dot\phi_2^2)=\frac m2\left(2\dot r_1^2+\frac{l^2v_0^2}{r_1^2}+\frac{9l^2v_0^2}{(4l-r_1)^2}\right)=mv_0^2$$

$m_1=m_2=m$

$r_1+r_2=4l$

Now $(r_1)_{max}$ occurs for $\dot r_1=0$. Then

$$l^2v_0^2\left(\frac1{r_1^2}+\frac9{(4l-r_1)^2}\right)=2v_0^2 \quad\text{or}\quad r_1^4-8lr_1^3+11l^2r_1^2+4l^3r_1-8l^4=0$$

The root $r=l$ is known from initial conditions.

$(r_1-l)(r_1^3-7lr_1^2+4l^2r_1+8l^3)=0$. Roots $1.6532l$, $6.1355l$, and $-0.7887l$

Only first applies. $(r_1)_{max}=1.6532l$

(b) Consider m_1. Tensile force $P=m(-\ddot r_1+r_1\dot\phi_1^2)=m\left(-\ddot r_1+\dfrac{l^2v_0^2}{r_1^3}\right)$

Consider m_2, noting $\ddot r_2=-\ddot r_1$. $P=m(-\ddot r_2+r_2\dot\phi_2^2)=m\left(\ddot r_1+\dfrac{9l^2v_0^2}{(4l-r_1)^3}\right)$

Hence $\ddot r_1=\dfrac{l^2v_0^2}{2}\left(\dfrac1{r_1^3}-\dfrac9{(4l-r_1)^3}\right)$ giving $P=\dfrac{ml^2v_0^2}{2}\left(\dfrac1{r_1^3}+\dfrac9{(4l-r_1)^3}\right)$

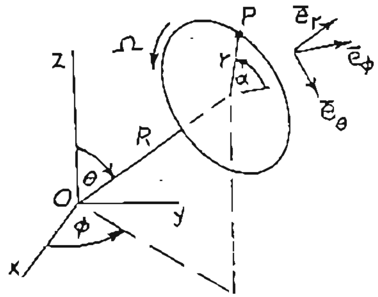
4-37. (cont'd.)

To find P_{min}, $\dfrac{dP}{dr_1}=\dfrac{ml^2v_0^2}{2}\left(-\dfrac3{r_1^4}+\dfrac{27}{(4l-r_1)^4}\right)=0$

Hence $(4l-r_1)^4=9r_1^4$ or $4l-r_1=\sqrt3\,r_1$, $r_1=\dfrac{4l}{1+\sqrt3}$

Then, $P_{min}=\dfrac{ml^2v_0^2}{2}\left[\dfrac{(1+\sqrt3)^3}{64l^3}\left(1+\dfrac9{3\sqrt3}\right)\right]=\dfrac{mv_0^2}{128l}(1+\sqrt3)^4=0.4353\dfrac{mv_0^2}{l}$

4-38. The gravitational moment is $\bar M_g=mgR\sin\theta\,\bar e_\phi$



Consider the inertia forces acting on an element $d\alpha$ of the rim. The corresponding inertia moment about O is

$$\bar M_i=-\frac{m}{2\pi}\int_0^{2\pi}(a_r r\sin\alpha\,\bar e_\phi+a_\theta R\,\bar e_\phi-a_\phi R\,\bar e_\theta)\,d\alpha$$

where, assuming that $\dot\theta=0$, we obtain from problem 2-17 that

$$a_r=-R\dot\phi^2\sin^2\theta+r\dot\phi^2\sin\alpha\sin\theta\cos\theta+2r\Omega\dot\phi\sin\alpha\sin\theta$$

$$a_\theta=-R\dot\phi^2\sin\theta\cos\theta+r\dot\phi^2\sin\alpha\cos^2\theta+r\Omega^2\sin\alpha+2r\Omega\dot\phi\sin\alpha\cos\theta$$

$$a_\phi=-r\dot\phi^2\cos\alpha-r\Omega^2\cos\alpha-2r\Omega\dot\phi\cos\alpha\cos\theta$$

Evaluating the integral, we obtain

$$\bar M_i=-\frac{m}{2\pi}\left(\pi r^2\dot\phi^2\sin\theta\cos\theta+2\pi r^2\Omega\dot\phi\sin\theta-2\pi R^2\dot\phi^2\sin\theta\cos\theta\right)\bar e_\phi$$

$$=-\tfrac12 m\left[(r^2-2R^2)\dot\phi^2\sin\theta\cos\theta+2r^2\Omega\dot\phi\sin\theta\right]\bar e_\phi$$

But $\bar M_g+\bar M_i=0$, or

$$mgR\sin\theta+\tfrac12 m(2R^2-r^2)\dot\phi^2\sin\theta\cos\theta-mr^2\Omega\dot\phi\sin\theta=0$$

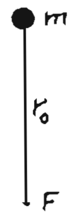
This equation is quadratic in $\dot\phi$, and we obtain

$$\dot\phi=\frac{r^2\Omega}{(2R^2-r^2)\cos\theta}\left[1\pm\sqrt{1-\frac{2Rg}{r^4\Omega^2}(2R^2-r^2)\cos\theta}\right]$$

CHAPTER 5

5-1. The orbit can be considered to be elliptical with $2a = r_0$, $\epsilon \cong 1$, and $r_p = 0$. The period is found from (5-74).

$$P = 2\pi\sqrt{\frac{a^3}{\mu}} = \pi\sqrt{\frac{r_0^3}{2\mu}}$$



The time to reach the focus is

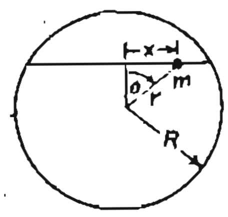
$$t = \frac{1}{2}P = \frac{\pi}{2}\sqrt{\frac{r_0^3}{2\mu}}$$

5-2. From (5-16) we see that the gravitational field within a homogeneous sphere varies linearly with the radial distance r from the center. Furthermore, the acceleration of gravity is $g_0 = 9.814\ \text{m/sec}^2$ at the earth's surface where $r = R = 6373$ km. Hence

$$\ddot{r} + \frac{g_0}{R} r = 0$$

giving a period $$P = 2\pi\sqrt{\frac{R}{g_0}} = 2\pi\sqrt{\frac{6.373\times10^6}{9.814}} = \underline{5063\text{ sec}} = 84.39\text{ min}$$

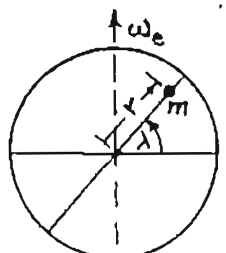
(b) The acceleration along the chord is equal to the component of the local acceleration in that direction.



$$\ddot{x} = -\frac{g_0}{R} r\sin\theta \quad\text{or}\quad \ddot{x} + \frac{g_0}{R}x = 0$$

$$P = 2\pi\sqrt{\frac{R}{g_0}} = 5063\text{ sec, as before.}$$

(c) $\omega_e = \dfrac{2\pi}{86,164} = 7.2921\times10^{-5}$ rad/sec



r component of acceleration due to ω_e is $-r\omega_e^2\cos^2\lambda$. Hence $\ddot{r} + \left(\frac{g_0}{R} - \omega_e^2\cos^2\lambda\right)r = 0$

$$P + \Delta P = 2\pi\sqrt{\frac{R}{g_0}}\left(1 - \frac{R\omega_e^2}{g_0}\cos^2\lambda\right)^{-\frac{1}{2}}$$

$$\cong 2\pi\sqrt{\frac{R}{g_0}}\left(1 + \frac{R\omega_e^2}{2g_0}\cos^2\lambda\right) \quad\text{or}\quad \frac{\Delta P}{P} = \frac{R\omega_e^2}{2g_0}\cos^2\lambda = \underline{1.7265\times10^{-3}\cos^2\lambda}$$

5-3. Earth's gravitational constant $\mu_e = 3.986\times10^{14}\ \text{m}^3/\text{sec}^2$

radius $R = 6.373\times10^6$ m

orbital radius $r_e = 1.496\times10^{11}$ m

orbital velocity $v_0 = \dfrac{2\pi r_e}{(365.25)(86400)} = 29,786$ m/sec

Sun's gravitational constant $\mu_s = r_e v_0^2 = 1.3272\times10^{20}\ \text{m}^3/\text{sec}^2$

1st Method. Consider a unit particle. After escape from the earth, but at the orbit of the earth, the particle's escape velocity from the solar system is

$$v_1 = \sqrt{2}\,v_0 = 42,123\text{ m/sec}$$

Relative to the earth, this velocity is

$$v_2 = v_1 - v_0 = 12,338\text{ m/sec}$$

Now consider escape from the earth with a residual velocity v_2. Using conservation of energy relative to the earth,

$$\frac{1}{2}v_r^2 - \frac{\mu_e}{R} = \frac{1}{2}v_2^2 \quad\text{or}\quad v_r^2 = \frac{2\mu_e}{R} + v_2^2 = 2.7731\times10^8$$

$$\underline{v_r = 16,653\text{ m/sec}}$$

2nd Method. Use absolute energies.

$$\frac{1}{2}(v_0 + v_r)^2 - \frac{\mu_e}{R} - \frac{\mu_s}{r_e} = W$$

where W is the work done by the unit particle on the earth during escape, due to the gravitational force and the earth's orbital velocity. To find W, first obtain the mutual impulse during escape from the earth.

$$\hat{F} = v_0 + v_r - v_1 \qquad \text{Then } W = \hat{F}v_0 = (1-\sqrt{2})v_0^2 + v_r v_0$$

Thus we obtain $$\frac{1}{2}v_0^2 + v_0 v_r + \frac{1}{2}v_r^2 - \frac{\mu_e}{R} - \frac{\mu_s}{r_e} = (1-\sqrt{2})v_0^2 + v_r v_0$$

or $$v_r^2 = (1-2\sqrt{2})v_0^2 + \frac{2\mu_e}{R} + \frac{2\mu_s}{r_e} = 2.7731\times10^8\ \text{m}^2/\text{sec}^2$$

$$\underline{v_r = 16,653\text{ m/sec}}$$

5-4. (a) From (5-53), $r = \dfrac{p}{1+\epsilon\cos\theta}$ so $\dot r = \dfrac{p\epsilon\sin\theta\,\dot\theta}{(1+\epsilon\cos\theta)^2}$

But $h = r^2\dot\theta$. Then $\dot r = \dfrac{\epsilon h}{p}\sin\theta$, $\underline{|\dot r|_{max}\text{ at } \theta = \pm\frac{\pi}{2}}$

(b) From (5-74), $\left.\begin{aligned} v^2 &= \mu\left(\tfrac{2}{r}-\tfrac{1}{a}\right) \text{ for ellipse} \\ v^2 &= \tfrac{\mu}{r} \text{ for circle}\end{aligned}\right\}$ $\underline{r = a}$ (At ends of minor axis).

5-5. $m = 1000$ kg , $r = R + 150{,}000 = 6.523\times10^6$ meters

$\mu = 3.986\times10^{14}\ \frac{m^3}{sec^2}$, $\dot r = -1$ km/day $= -1.1574\times10^{-2}$ m/sec

The orbit is approximately circular. $e \cong -\frac{\mu}{r}$, $\dot e = \frac{\mu}{2r^2}\dot r$

Hence $\dot e = -5.4212\times10^{-2}\ \frac{N\cdot m}{kg\cdot sec}$ $\quad v_c = 7909$ m/sec

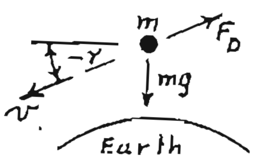
Orbital velocity $v = v_c\sqrt{\frac{R}{r}} = 7817.5$ m/sec

Using work and energy, $F_D v = -m\dot e$

drag force $F_D = \frac{-m\dot e}{v} = \underline{6.9347\times10^{-3} N}$

Flight path angle $\gamma = \sin^{-1}\frac{\dot r}{v} \cong \frac{\dot r}{v}$

$= -1.4805\times10^{-6}$ rad



$mg = \frac{\mu m}{r^2} = 9.3679\times10^3 N$

Component of mg along path is

$mg\sin(-\gamma) = \frac{-\mu m \dot r}{r^2 v} = 2F_D = 1.3869\times10^{-2} N$

Hence there is a net force F_D accelerating particle forward.

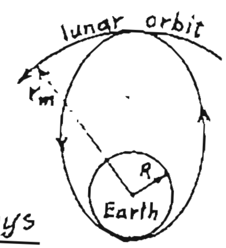
5-6. Hohmann transfer orbit

$r_a = r_m = 3.844\times10^8$ m ; $\mu = 3.986\times10^{14}\ \frac{m^3}{sec^2}$

$r_p = R = 6.373\times10^6$ m

$a = \frac12(r_a + r_p) = 1.9534\times10^8$ m

$t = \frac12 P = \pi\sqrt{\frac{a^3}{\mu}} = 4.2976\times10^5$ sec $= \underline{4.974 \text{ days}}$



5-7. Hohmann transfer orbit.

$r_a = 3.844\times10^8$ m $\qquad \mu = 3.986\times10^{14}\ \frac{m^3}{sec^2}$

$r_p = 6.373\times10^6$ m

Using (5-170),

$\Delta v_a = \sqrt{\frac{\mu}{r_a}}\left[1 - \sqrt{\frac{2}{1 + r_a/r_p}}\right] = \underline{834.39 \text{ m/sec}}$ opposite to orbital veloci

5-8. For the original circular orbit,

$r = 2R$, $v = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{\mu}{2R}}$, $h = rv = \sqrt{2\mu R}$, $e = -\frac{\mu}{2r} = -\frac{\mu}{4R}$

After $F = \frac18 mg_0$ is applied, h remains constant. At r_{min} or r_{max}, $\dot r = 0$, so $\quad h = rv = \sqrt{2\mu R}$ and $v = \frac{\sqrt{2\mu R}}{r}$

Using work and energy, per unit mass,

$$-\frac{\mu}{4R} + \frac18 g_0(2R - r) = -\frac{\mu}{r} + \frac12 v^2 = -\frac{\mu}{r} + \frac{\mu R}{r^2}$$

Noting that $g_0R^2 = \mu$, we obtain

$$\frac18\frac{\mu}{R^2}r - \frac{\mu}{r} + \frac{\mu R}{r^2} = 0 \text{ or } r^3 - 8R^2r + 8R^3 = 0$$

A known root is $r_{max} = 2R$, $\quad (r-2R)(r^2 + 2Rr - 4R^2) = 0$

$r_{min} = (-1+\sqrt5)R = \underline{1.2361\ R}$

v_{max} occurs at r_{min}. Using conservation of h, noting that $\mu = 3.986\times10^{14}$ m³/sec², $R = 6.373\times10^6$ m.,

$v_{max} = \sqrt{2\mu R}/r_{min} = \underline{9048 \text{ m/sec}}$

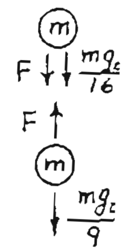
5-9. Inner particle: $\frac{mg_0}{9} - F = m(3R)\omega^2$

Outer particle: $\frac{mg_0}{16} + F = m(4R)\omega^2$

Adding, $mg_0\left(\frac19 + \frac1{16}\right) = 7mR\omega^2$ or $\omega = \frac{5}{12}\sqrt{\frac{g_0}{7R}}$

Period $P = \frac{2\pi}{\omega} = \frac{24\pi}{5}\sqrt{\frac{7R}{g_0}} = 32{,}151$ sec $= 535.84$ min

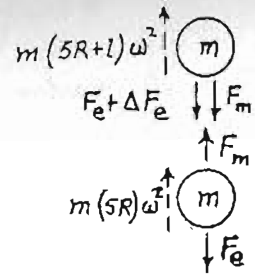
Then $F = \left(\frac19 - \frac{(25)(3)}{(144)(7)}\right)mg_0 = \frac{37}{1008}mg_0 = \underline{0.0367\ mg_0}$ N



5-10. $G = 6.673\times 10^{-11}\ \frac{m^3}{kg\ sec^2}$

$\mu = 3.986\times 10^{14}\ \frac{m^3}{sec^2}$

$m = 20$ kg , $R = 6.373\times 10^{6}$ meters



Balance the forces, including the centrifugal forces, the gravitational attraction between the spheres, and the earth's gravity.

Assume a sphere separation $l << R$. The earth's gravitational force on the inner sphere is

$$F_e = \frac{\mu m}{r^2} = \frac{\mu m}{25R^2} \quad \text{and} \quad \Delta F_e = \frac{\partial F_e}{\partial r}\Delta r = \frac{-2\mu m}{r^3}l = \frac{-2\mu m l}{125R^3}$$

The mutual gravitational force between the spheres is

$$F_m = \frac{Gm^2}{l^2}$$

Now $\omega^2 = \frac{\mu}{r^3} \cong \frac{\mu}{125R^3}$ and a balance of forces on the inner sphere yields

$$m(5R)\omega^2 + F_m = F_e \quad \text{or} \quad 5mR\omega^2 + \frac{Gm^2}{l^2} = \frac{\mu m}{25R^2}$$

For the outer sphere,

$$5mR\omega^2 + ml\omega^2 - \frac{Gm^2}{l^2} = \frac{\mu m}{25R^2} - \frac{2\mu m l}{125R^3}$$

Subtracting equations, we obtain

$$-ml\omega^2 + \frac{2Gm^2}{l^2} = \frac{2\mu m l}{125R^3} \quad \text{or} \quad \frac{2Gm^2}{l^2} = \frac{3\mu m l}{125R^3}$$

giving

$$l^3 = \frac{250\,GmR^3}{3\mu}, \quad l = \left(\frac{250\,Gm}{3\mu}\right)^{1/3} R = \underline{0.4164 \text{ meters}}$$

5-11. From (5-69), $\epsilon = \sqrt{1+\frac{2eh^2}{\mu^2}}$

$$\frac{\partial\epsilon}{\partial v} = \frac{1}{\sqrt{1+\frac{2eh^2}{\mu^2}}}\left[\frac{h^2}{\mu^2}\left(\frac{\partial e}{\partial v}\right) + \frac{2eh}{\mu^2}\left(\frac{\partial h}{\partial v}\right)\right]$$

where $e = \frac{1}{2}v^2 - \frac{\mu}{r}$, $\frac{\partial e}{\partial v} = v$

$h = rv\cos\gamma$, $\frac{\partial h}{\partial v} = r\cos\gamma = \frac{h}{v}$

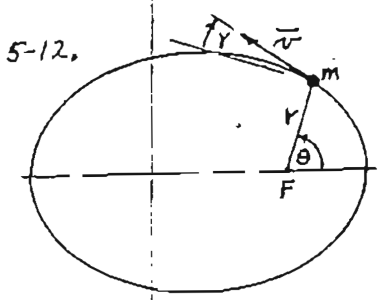
In terms of the burnout conditions r, v, γ as well as the geometrical constants a, ϵ, we note that

$$e = \frac{-\mu}{2a}, \quad h = \sqrt{a\mu(1-\epsilon^2)}; \quad v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

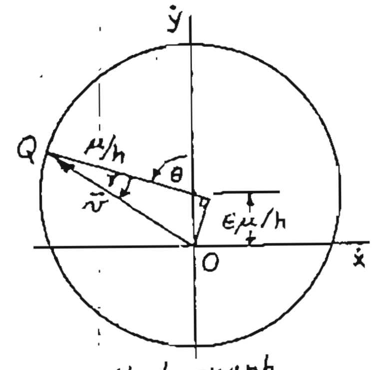
Then we obtain

$$\frac{\partial\epsilon}{\partial v} = \frac{1}{\epsilon}\left[\frac{a(1-\epsilon^2)}{\mu}v + \frac{2}{\mu^2}\left(\frac{-\mu}{2a}\right)\left(\frac{a\mu(1-\epsilon^2)}{v}\right)\right] = \underline{\frac{2(1-\epsilon^2)}{\epsilon v}\left(\frac{a}{r}-1\right)}$$

5-12.



Orbit



Hodograph

$$r = \frac{p}{1+\epsilon\cos\theta}, \quad \dot{r} = \frac{p\dot{\theta}\epsilon\sin\theta}{(1+\epsilon\cos\theta)^2}$$

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = p^2\dot{\theta}^2\left[\frac{\epsilon^2\sin^2\theta}{(1+\epsilon\cos\theta)^4} + \frac{1}{(1+\epsilon\cos\theta)^2}\right] = p^2\dot{\theta}^2\left[\frac{1+2\epsilon\cos\theta+\epsilon^2}{(1+\epsilon\cos\theta)^4}\right]$$

From (5-10), $\mu p = h^2 = r^4\dot{\theta}^2 = \frac{p^4\dot{\theta}^2}{(1+\epsilon\cos\theta)^4}$ or $\frac{p^2\dot{\theta}^2}{(1+\epsilon\cos\theta)^4} = \frac{\mu}{p}$

Hence $v^2 = \frac{\mu}{p}(1+2\epsilon\cos\theta+\epsilon^2)$

5-12. (cont'd.) From the figure, using the cosine law,

$$\overline{OQ}^2 = \left(\frac{\epsilon\mu}{h}\right)^2 + \left(\frac{\mu}{h}\right)^2 + 2\left(\frac{\epsilon\mu}{h}\right)\left(\frac{\mu}{h}\right)\cos\theta$$

$$= \left(\frac{\mu}{h}\right)^2\left(\epsilon^2 + 1 + 2\epsilon\cos\theta\right) \qquad \text{where } \frac{\mu^2}{h^2} = \frac{\mu}{p}$$

Hence $\overline{OQ}^2 = \frac{\mu}{p}\left(1 + 2\epsilon\cos\theta + \epsilon^2\right) = v^2$

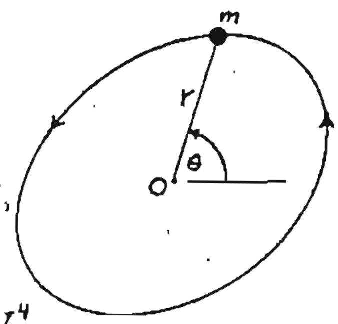
To show that $\overline{OQ}$ has the direction of $\bar{v}$,

$$\tan\gamma = \frac{\frac{\epsilon\mu}{h}\sin\theta}{\frac{\mu}{h} + \frac{\epsilon\mu}{h}\cos\theta} = \frac{\epsilon\sin\theta}{1+\epsilon\cos\theta} = \frac{\dot r}{r\dot\theta}$$

Since $\overline{OQ}$ has the magnitude and direction of $\bar{v}$, the path of Q represents the hodograph with origin at O.

5-13. $r(0) = l \qquad \theta(0) = 0$

$\dot r(0) = l\sqrt{\frac{k}{m}} \qquad \dot\theta(0) = \sqrt{\frac{k}{m}}$



$F_r = -kr = m(\ddot r - r\dot\theta^2)$

Conservation of angular momentum.

$h = r^2\dot\theta = l^2\sqrt{\frac{k}{m}}$ or $\dot\theta = \frac{l^2}{r^2}\sqrt{\frac{k}{m}}$

$\ddot r - r\dot\theta^2 + \frac{k}{m}r = 0$ or $\underline{\ddot r + \frac{k}{m}r - \frac{kl^4}{mr^3} = 0}$

(b) Use conservation of energy.

$$T + V = \tfrac{1}{2}m\left(\dot r^2 + r^2\dot\theta^2\right) + \tfrac{1}{2}kr^2 = \tfrac{3}{2}kl^2$$

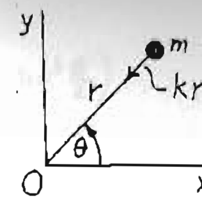
At r_{max} or r_{min}, $\dot r = 0$ giving $\frac{1}{2}mr^2\left(\frac{l^2}{r^2}\sqrt{\frac{k}{m}}\right)^2 + \frac{1}{2}kr^2 = \frac{3}{2}kl^2$

or $\frac{1}{2}k\frac{l^4}{r^2} + \frac{1}{2}kr^2 - \frac{3}{2}kl^2 = 0$ or $r^4 - 3l^2r^2 + l^4 = 0$

yielding $r^2 = \frac{l^2}{2}(3 \pm \sqrt{5})$, $r_{max}^2 = 2.6180\,l^2$, $\underline{r_{max} = 1.6180\,l}$

$r_{min}^2 = 0.3820\,l^2$, $\underline{r_{min} = 0.6180\,l}$

5-13. (cont'd) The orbit is an ellipse centered at O. To see this, use Cartesian coordinates.



$m\ddot x = F_x = -kr\cos\theta = -kx$ or $m\ddot x + kx = 0$

$m\ddot y = F_y = -kr\sin\theta = -ky$ or $m\ddot y + ky = 0$

This is two-dimensional harmonic motion with $\omega = \sqrt{k/m}$. In general, the path is an ellipse with period

$$P = \frac{2\pi}{\omega} = \underline{2\pi\sqrt{\frac{m}{k}}}$$

5-14. Use polar coordinates (r, θ).

Given that $F_r = -\frac{K}{r^n}$ where K is a positive constant. The equations of motion are

$$m(\ddot r - r\dot\theta^2) + \frac{K}{r^n} = 0$$

$$r^2\dot\theta = h = \text{const}$$

Eliminating $\dot\theta$, we obtain $m\left(\ddot r - \frac{h^2}{r^3}\right) + \frac{K}{r^n} = 0$

The corresponding perturbation equation is

$$m\,\delta\ddot r + \left(\frac{3mh^2}{r^4} - \frac{nK}{r^{n+1}}\right)_0 \delta r = \left(\frac{2mh}{r^3}\right)_0 \delta h$$

For the circular reference orbit,

$$\frac{K}{r^n} = mr\dot\theta^2 = \frac{mh^2}{r^3} \quad \text{so} \quad K = mh^2 r^{n-3}$$

The orbit is stable if the coefficient of δr is positive, that is, if $\frac{3mh^2}{r^4} > \frac{mh^2 n}{r^4}$ or $\underline{n < 3}$

5-15. $r(0)=r_0$, $v(0)=v_0$, $\gamma(0)=\gamma_0$

Transverse velocity $v_t=\frac{h}{r}=\frac{r_0 v_0\cos\gamma_0}{r}$

Using conservation of energy,

$$\frac{1}{2}(v_r^2+v_t^2)-\frac{\mu}{r}=\frac{1}{2}v_0^2-\frac{\mu}{r_0}$$

Solving for v_r, we obtain

$$v_r=v_0\sqrt{1+\frac{2\mu}{v_0^2}\left(\frac{1}{r}-\frac{1}{r_0}\right)-\frac{r_0^2\cos^2\gamma_0}{r^2}}$$

(b) $(v_t)_{max}$ occurs at $r_{min}=r_p=\frac{p}{1+\epsilon}=\frac{h^2}{\mu(1+\epsilon)}=\frac{r_0^2v_0^2\cos^2\gamma_0}{\mu(1+\epsilon)}$.

From (5-114), $\epsilon=\sqrt{1+\frac{r_0v_0^2}{\mu}\left(\frac{r_0v_0^2}{\mu}-2\right)\cos^2\gamma_0}$

noting $\mu=Rv_c^2$,

Then $(v_t)_{max}=\frac{r_0v_0\cos\gamma_0}{r_p}=\frac{\mu+\sqrt{\mu^2+r_0v_0^2(r_0v_0^2-2\mu)\cos^2\gamma_0}}{r_0v_0\cos\gamma_0}$ at $\theta=0$

To find $(v_r)_{max}$, set $\frac{d(v_r^2)}{dr}=v_0^2\left(\frac{-2\mu}{v_0^2r^2}+\frac{2r_0^2\cos^2\gamma_0}{r^3}\right)=0$

$r=\frac{r_0^2v_0^2\cos^2\gamma_0}{\mu}=\frac{h^2}{\mu}=p$, that is, at $\theta=\frac{\pi}{2}$.

Then $(v_r)_{max}=\sqrt{v_0^2-\frac{2\mu}{r_0}+\frac{\mu^2}{r_0^2v_0^2\cos^2\gamma_0}}$ at $\theta=\frac{\pi}{2}$

5-16. From (5-81), $M=2\pi\frac{t}{P}=E-\epsilon\sin E$

At the ends of the minor axis, $E=\pm\frac{\pi}{2}$.

Hence $\frac{t}{P}=\frac{2(\frac{\pi}{2}-\epsilon)}{2\pi}=\frac{1}{2}-\frac{\epsilon}{\pi}$

5-17. From (5-92),

$$t=\sqrt{\frac{2r_p^3}{\mu}}\left(\tan\frac{\theta}{2}+\frac{1}{3}\tan^3\frac{\theta}{2}\right)$$

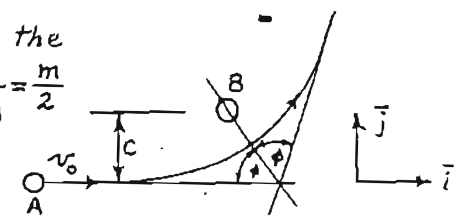
The time from $\theta=-\frac{\pi}{2}$ to $\frac{\pi}{2}$ is twice the time from 0 to $\frac{\pi}{2}$.

$$t=\frac{8}{3}\sqrt{\frac{2r_p^3}{\mu_s}}=1.0353\times10^7\text{ sec}=119.82\text{ days}$$

5-18. $m_A=m_B=m$, $v_0=\sqrt{2Gm/c}$

Consider B fixed and use the reduced mass $m'=\frac{m_Am_B}{m_A+m_B}=\frac{m}{2}$

$\mu=G(m_A+m_B)=2Gm$



Then $e=\frac{1}{2}v_0^2=\frac{Gm}{c}$, $h=cv_0=\sqrt{2Gmc}$

$$\epsilon=\sqrt{1+\frac{2eh^2}{\mu^2}}=\sqrt{2}$$

In B's frame, the deflection angle $\pi-2\phi=\pi-2\cos^{-1}\frac{1}{\epsilon}=\frac{\pi}{2}$

and A's final velocity is $\bar{v}_{A/B}=v_0\bar{j}$.

But $\bar{v}_{cm}=\frac{1}{2}v_0\bar{i}=\frac{1}{2}(\bar{v}_A+\bar{v}_B)$ in the fixed frame,

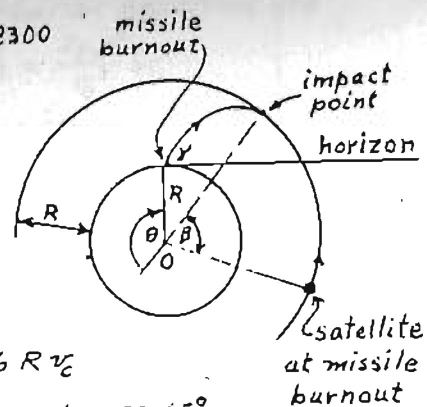
Also $\bar{v}_A-\bar{v}_B=\bar{v}_{A/B}=v_0\bar{j}$

Solving, $\bar{v}_A=\frac{v_0}{2}(\bar{i}+\bar{j})$ Deflection angle $=\pi/4$

$\bar{v}_B=\frac{v_0}{2}(\bar{i}-\bar{j})$

5-19. $v_c = 7909$ m/sec, $v = 8300$

$e = \frac{1}{2}v_c^2\left[\left(\frac{v}{v_c}\right)^2 - 2\frac{R}{r}\right] = -0.4493\, v_c^2$



At impact, $r = 2R$, and

$\left(\frac{v}{v_c}\right)^2 = 2(-0.4493) + 1 = 0.10132$

or $v = 0.3183\, v_c$

$h = 2R(0.3183\, v_c) = 0.6366\, R v_c$

$\cos\gamma' = \frac{0.6366\, v_c}{8300} = 0.6066$, $\underline{\gamma = 52.65^\circ}$

(b) Use (5-117) to find θ and the impact point. At burnout,

$\tan\theta = \frac{\sin\gamma\cos\gamma}{\cos^2\gamma - \frac{R}{r}\left(\frac{v_c}{v}\right)^2} = -0.8931$, $\theta = 138.23^\circ$

The impact point is at $180^\circ - \theta = 41.77^\circ$ from burnout.

$\epsilon = \sqrt{1 + \frac{2eh^2}{\mu^2}} = 0.7974$, $\tan\frac{E}{2} = \sqrt{\frac{1-\epsilon}{1+\epsilon}}\tan\frac{\theta}{2}$, $E = 82.70^\circ$

$M = E - \epsilon\sin E = 0.6525$, $a = \frac{r_a}{1+\epsilon} = 1.1127\, R$

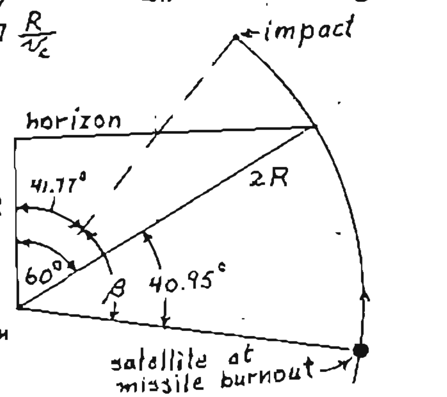
$\mu = R v_c^2$, Period $P = 2\pi\sqrt{\frac{a^3}{\mu}} = 7.3752\,\frac{R}{v_c}$

At burnout, the time since perigee is $t = \frac{M}{2\pi}P = 0.7659\,\frac{R}{v_c}$

Time of flight $t_f = \frac{1}{2}P - t = 2.9217\,\frac{R}{v_c}$

Satellite period

$P_s = 2\pi\sqrt{\frac{8R^3}{R v_c^2}} = 17.772\,\frac{R}{v_c}$



During time t_f, satellite travels $\beta = \frac{2.9217}{17.772}360 = 59.19^\circ$

At time of burnout, satellite is $\beta + 41.77^\circ - 60^\circ = 40.95^\circ$ below horizon

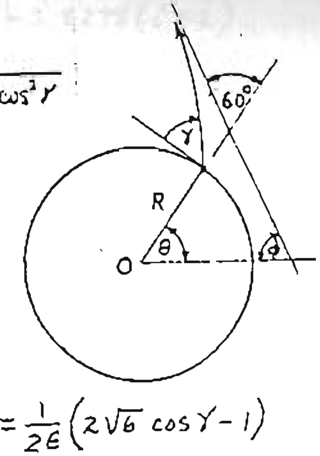
Corresponding time is

$\frac{40.95}{360}17.772\,\frac{R}{v_c} = 2.0216\,\frac{R}{v_c} = 1629$ sec $= \underline{27.15 \text{ min}}$ before horizon.

5-20. $v(0) = 2v_c$, $r(0) = R$

Using (5-114),

$\epsilon = \sqrt{\sin^2\gamma + \left(1 - \frac{r v^2}{R v_c^2}\right)^2\cos^2\gamma} = \sqrt{1 + 8\cos^2\gamma}$



For the final path 30° above the horizon, $\theta + 60^\circ + \phi = 180^\circ$, $\theta = 120^\circ - \phi$

where $\cos\phi = \frac{1}{\epsilon}$, $\sin\phi = \frac{\sqrt{\epsilon^2 - 1}}{\epsilon}$

$\cos\theta = -\frac{1}{2}\cos\phi + \frac{\sqrt{3}}{2}\sin\phi = \frac{-1}{2\epsilon} + \frac{\sqrt{3}}{2}\frac{\sqrt{\epsilon^2-1}}{\epsilon}$

where $\epsilon^2 - 1 = 8\cos^2\gamma$, so $\cos\theta = \frac{1}{2\epsilon}\left(2\sqrt{6}\cos\gamma - 1\right)$

But from (5-115), $\cos\theta = \frac{1}{\epsilon}\left[\frac{a}{r}(\epsilon^2 - 1) - 1\right]$

where, from (5-112), $a = \frac{-R}{2\frac{R}{r} - \left(\frac{v}{v_c}\right)^2} = \frac{R}{2}$ so $\cos\theta = \frac{8\cos^2\gamma - 2}{2\epsilon}$

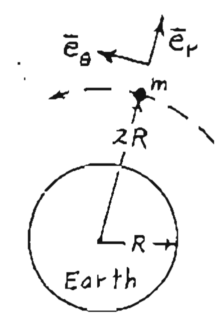
Equating the two expressions for $\cos\theta$,

$8\cos^2\gamma - 2 = 2\sqrt{6}\cos\gamma - 1$ or $\cos^2\gamma - \frac{\sqrt{6}}{4}\cos\gamma - \frac{1}{8} = 0$

$\cos\gamma = \frac{\sqrt{6}}{8} + \sqrt{\frac{3}{32} + \frac{1}{8}} = 0.7739$, $\underline{\gamma = 39.30^\circ}$

$\epsilon^2 = 1 + 8\cos^2\gamma = 5.7913$, $\underline{\epsilon = 2.4065}$

5-21. $F_\theta = 10^{-4} m g_0$ for one revolution



Equations of motion:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{g_0 R^2}{r^2}$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = F_\theta / m$$

In the reference orbit with $F_\theta = 0$,

$r_0 = 2R$, $2R\dot{\theta}_0^2 = g_0/4$ or $\dot{\theta}_0 = \sqrt{\frac{g_0}{8R}}$

$\dot{r}_0 = \ddot{r}_0 = \ddot{\theta}_0 = 0$, $v_0 = 2R\dot{\theta}_0 = \sqrt{\frac{g_0 R}{2}}$

5-21. (cont'd.) The perturbation equations are

$$\delta\ddot r - \dot\theta_0^2\,\delta r - 2r_0\dot\theta_0\,\delta\dot\theta = \frac{2g_0R^2}{r_0^3}\,\delta r$$

or $$\delta\ddot r - \frac{3g_0}{8R}\,\delta r - 2r_0\dot\theta_0\,\delta\dot\theta = 0$$

and $$r_0\,\delta\ddot\theta + 2\dot\theta_0\,\delta\dot r = \frac{\delta F_\theta}{m} = 10^{-4}g_0$$

Noting that $\delta\dot\theta(0)=0$ and $\delta r(0)=0$, and integrating the $\delta\theta$ equation,

$$r_0\,\delta\dot\theta + 2\dot\theta_0\,\delta r = 10^{-4}g_0 t$$

Substitute for $\delta\dot\theta$ in the δr equation and obtain

$$\delta\ddot r + \left[\frac{-3g_0}{8R} + 4\left(\frac{g_0}{8R}\right)\right]\delta r = 2\times10^{-4}\,\dot\theta_0 g_0 t$$

or $\delta\ddot r + \dot\theta_0^2\,\delta r = \frac{2g_0}{10^4}\dot\theta_0 t$ which has the solution

$$\delta r = \frac{-2g_0}{10^4\dot\theta_0^2}\sin\dot\theta_0 t + \frac{2g_0}{10^4\dot\theta_0}t \quad \text{satisfying } \delta r(0)=0,\ \delta\dot r(0)=0$$

After one cycle, $\delta r = \left(\frac{2g_0}{10^4\dot\theta_0}\right)\left(\frac{2\pi}{\dot\theta_0}\right) = \frac{32\pi R}{10^4}$, $\underline{\frac{\delta r}{r_0} = 16\pi\times10^{-4}}$

$v = r\dot\theta$ so $\delta v = \dot\theta_0\,\delta r + r_0\,\delta\dot\theta$ giving $\frac{\delta v}{v_0} = \frac{\delta r}{r_0} + \frac{\delta\dot\theta}{\dot\theta_0}$

For $t = \frac{2\pi}{\dot\theta_0}$ we have $\frac{\delta\dot\theta}{\dot\theta_0} = -2\frac{\delta r}{r_0} + \frac{g_0}{10^4 r_0}\frac{2\pi}{\dot\theta_0^2} = \frac{-24\pi}{10^4}$, $\underline{\frac{\delta v}{v_0} = -8\pi\times10^{-4}}$

5-22. $\omega_e = \frac{2\pi}{86164}\bar k = 7.2921\times10^{-5}\ \frac{\text{rad}}{\text{sec}}$

Period $P = 3$ hr $= 10{,}800$ sec

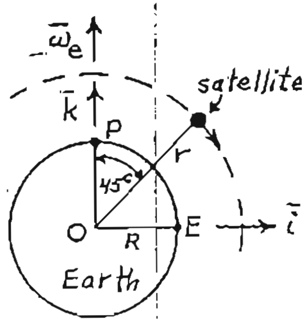
From Kepler's 3rd law, noting $P_c = 5063$ sec,

$\left(\frac{r}{R}\right)^3 = \left(\frac{10{,}800}{5063}\right)^2$, $\frac{r}{R} = 1.6571$

where $R = 6.373\times10^6$ m.

$v_{abs} = \frac{2\pi(1.6571R)}{10{,}800} = 6143.9\ \frac{\text{m}}{\text{sec}}$

$\bar v_{abs} = 4344.4(\bar i - \bar k)\ \frac{\text{m}}{\text{sec}}$



5-22. (cont'd.) $r = 1.6571\,R = 1.0561\times10^7$ m.

$a_{abs} = \frac{v^2}{r} = 3.5744\ \frac{\text{m}}{\text{sec}^2}$, $\bar a_{abs} = -2.5275(\bar i + \bar k)$

(a) For a frame fixed in the earth, both observers see the same velocity and acceleration.

$$\bar v_{S/P} = \bar v_{S/E} = \bar v_{abs} - \bar\omega_e\times\bar r = \underline{4344\,\bar i - 544.5\,\bar j - 4344\,\bar k\ \text{m/sec}}$$

$$\bar a_{S/P} = \bar a_{S/E} = \bar a_{abs} - \bar\omega_e\times(\bar\omega_e\times\bar r) - 2\,\bar\omega_e\times\bar v_{S/P}$$

where $\bar\omega_e\times(\bar\omega_e\times\bar r) = -0.03471\,\bar i$

$2\,\bar\omega_e\times\bar v_{S/P} = 0.0794\,\bar i + 0.6336\,\bar j$

$$\bar a_{S/P} = \bar a_{S/E} = -2.5672\,\bar i - 0.6336\,\bar j - 2.5275\,\bar k\ \text{m/sec}^2$$

(b) For nonrotating frames, $\bar v_{S/P} = \bar v_{abs} = \underline{4344.4(\bar i - \bar k)\ \frac{\text{m}}{\text{sec}}}$

$\bar a_{S/P} = \bar a_{abs} = \underline{-2.5275(\bar i + \bar k)\ \frac{\text{m}}{\text{sec}^2}}$

Observer E has an absolute velocity $v_E = \omega_e R\,\bar j = 464.7\,\bar j\ \frac{\text{m}}{\text{sec}}$

$\bar v_{S/E} = \bar v_{abs} - \bar v_E = \underline{4344.4\,\bar i - 464.7\,\bar j - 4344.4\,\bar k\ \text{m/sec}}$

E has an absolute acceleration $\bar a_E = -\omega_e^2 R\,\bar i = -0.03389\,\bar i\ \frac{\text{m}}{\text{sec}^2}$

$\bar a_{S/E} = \bar a_{abs} - \bar a_E = \underline{-2.4936\,\bar i - 2.5275\,\bar k\ \text{m/sec}^2}$

5-23. From (5-168) and (5-170) for a Hohmann transfer,

$$\Delta v = \Delta v_p + \Delta v_a = \sqrt{\frac{\mu}{r_p}}\left(\sqrt{\frac{2r_a/r_p}{1+r_a/r_p}} - 1\right) + \sqrt{\frac{\mu}{r_p}}\left[\sqrt{\frac{r_p}{r_a}} - \sqrt{\frac{2}{\frac{r_a}{r_p}+\left(\frac{r_a}{r_p}\right)^2}}\right]$$

Let $x = \frac{r_a}{r_p}$ and obtain $$\Delta v = \sqrt{\frac{\mu}{r_p}}\left(\sqrt{\frac{2x}{1+x}} - 1 + \frac{1}{\sqrt x} - \sqrt{\frac{2}{x+x^2}}\right)$$

For escape from a circular orbit of radius r_p $(r_a=\infty)$,

$$\Delta v = \sqrt{\frac{\mu}{r_p}}\left(\sqrt2 - 1\right)$$

5-23. (cont'd.) Equating the two expressions for Δv,

$$\sqrt{\frac{2x}{1+x}} + \frac{1}{\sqrt{x}} - \sqrt{\frac{2}{x+x^2}} = \sqrt{2} \quad \text{or} \quad \frac{\sqrt{2}\,x + \sqrt{1+x} - \sqrt{2}}{\sqrt{x(1+x)}} = \sqrt{2}$$

Squaring, $\left(\sqrt{2}\,(x-1) + \sqrt{1+x}\right)^2 = 2x(1+x)$

or $\quad 5x - 3 = 2\sqrt{2}\,(x-1)\sqrt{1+x}$

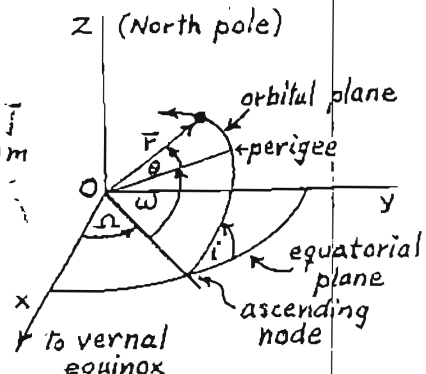
Squaring again, $25x^2 - 30x + 9 = 8x^3 - 8x^2 - 8x + 8$

or $\quad 8x^3 - 33x^2 + 22x - 1 = 0 \quad$ Roots are 3.3042, 0.7718, 0.04902

Only root > 1 is $x = 3.3042$ so $\quad \frac{r_a}{r_p} = 3.3042$

For larger values of $\frac{r_a}{r_p}$, $\quad \Delta v > \sqrt{\mu / r_p}\,(\sqrt{2} - 1)$.

5-24. Given $\bar{r}$ and $\dot{\bar{r}}$ at $t=0$, where



$\bar{r} = 1000\bar{i} + 3000\bar{j} + 6000\bar{k}$ km
$= 6.7823 \times 10^6 (0.14744\,\bar{i} + 0.44233\,\bar{j} + 0.88465\,\bar{k})$ m

$\dot{\bar{r}} = -6000\,\bar{i} - 4000\,\bar{j} + 3500\,\bar{k}$ m/sec
$= 8015.6\,(-0.74854\,\bar{i} - 0.44903\,\bar{j} + 0.43665\,\bar{k})$ m/sec

$\bar{h} = \bar{r} \times \dot{\bar{r}}$
$= 10^9 (34.5\,\bar{i} - 34.5\,\bar{j} + 14\,\bar{k})\ \frac{m^2}{sec}$
$= 5.4282 \times 10^{10} (0.63557\,\bar{i} - 0.72769\,\bar{j} + 0.25791\,\bar{k})$ ← perpendicular to orbital plane

$i = \cos^{-1} \frac{h_z}{h} = \cos^{-1} 0.25791 = \underline{75.05^\circ}$

$\Omega = \tan^{-1} \frac{h_x}{-h_y} = \tan^{-1} 0.87342 = \underline{41.13^\circ}$

$R = 6.373 \times 10^6$ m $\qquad a = \dfrac{R}{2\left(\frac{R}{r}\right) - \left(\frac{v}{v_c}\right)^2} = 1.17350\,R = \underline{7478.7 \text{ km}}$

$v_c = 7909$ m/sec

$\mu = R v_c^2 = 3.9865 \times 10^{14}\ \frac{m^3}{sec^2}$

5-24. (cont'd.)

Now $\quad p = \frac{h^2}{\mu} = \frac{h^2}{R v_c^2} = a(1 - \epsilon^2) \quad$ or $\quad \epsilon = \sqrt{1 - \frac{h^2}{a R v_c^2}}$

$\underline{\epsilon = 0.10811}$

$\gamma = \sin^{-1} \frac{\bar{r} \cdot \dot{\bar{r}}}{r v} = \sin^{-1} 0.05518 = 3.16^\circ$

$\tan\theta = \dfrac{\sin\gamma \cos\gamma}{\cos^2\gamma - \frac{R v_c^2}{r v^2}} = 0.67082, \quad \theta = 33.85^\circ$

$\tan\frac{E}{2} = \sqrt{\frac{1-\epsilon}{1+\epsilon}}\,\tan\frac{\theta}{2} = 0.27304, \quad E = 30.54^\circ = 0.5331$ rad

The time since perigee, using (5-86), is

$$t = \sqrt{\frac{a^3}{R v_c^2}}\,(E - \epsilon \sin E) = 489.78 \text{ sec}$$

$\underline{T = -489.78 \text{ sec}}$

A unit vector along the line of nodes is

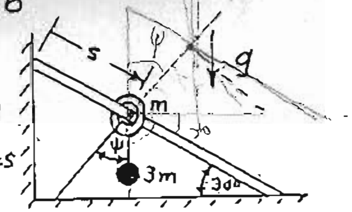
$$\bar{e}_n = \cos\Omega\,\bar{i} + \sin\Omega\,\bar{j} = 0.75317\,\bar{i} + 0.65793\,\bar{j}$$

$\cos(\omega + \theta) = \bar{e}_r \cdot \bar{e}_n = 0.40202, \quad \omega + \theta = 66.30^\circ$

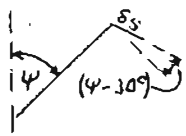
$\omega = 66.30 - \theta = \underline{32.44^\circ}$

CHAPTER 6

6-1. Consider a virtual displacement δs. The slanted length of string increases by $\delta s \sin(\psi - 30^\circ)$ and the vertical section becomes shorter by the same amount.



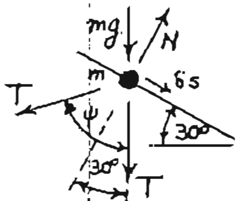
Thus, m goes down $\delta s \sin 30^\circ$ and $3m$ rises $[\sin(\psi-30^\circ) - \sin 30^\circ]\delta s$, resulting in a change of potential energy



$$\delta V = \{3mg[\sin(\psi-30^\circ) - \sin 30^\circ] - mg \sin 30^\circ\}\delta s = 0$$

Hence $\quad \sin(\psi - 30^\circ) = \frac{2}{3}$ or $\underline{\psi = 71.81^\circ}$

2nd Method: Consider the forces acting on the ring of mass m.

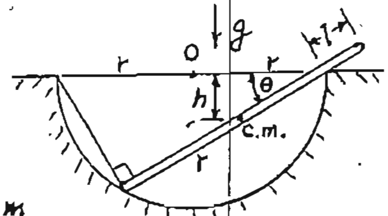


$$\delta W = [(T+mg)\sin 30^\circ - T\sin(\psi - 30^\circ)]\delta s = 0$$

where $T = 3mg$. $\quad \frac{4mg}{2} = 3mg \sin(\psi - 30^\circ)$

$$\sin(\psi - 30^\circ) = 2/3 \quad \text{or} \quad \underline{\psi = 71.81^\circ}$$

6-2. $\quad l = 2r(1-\cos\theta)$



$h = (r - l)\sin\theta$

$\quad = -r\sin\theta + 2r\cos\theta\sin\theta$

$V = -mgh$, $\frac{dV}{d\theta} = 0$ at static equilibrium

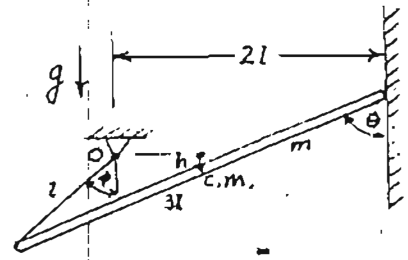
$$\frac{dh}{d\theta} = r(-\cos\theta + 2\cos^2\theta - 2\sin^2\theta) = r(4\cos^2\theta - \cos\theta - 2) = 0$$

$$\cos\theta = \frac{1+\sqrt{33}}{8} = 0.8431, \quad \theta = 32.53^\circ$$

$$l = 2r(1-\cos\theta) = \underline{0.3139\,r}$$

6-3. Let the vertical distance of c.m. below point O be

$$h = l\cos\phi - \tfrac{3}{2} l\cos\theta$$



From the geometry,

$$3l\sin\theta - l\sin\phi = 2l$$

or $\quad \sin\phi = 3\sin\theta - 2$

Then

$$\cos\phi = \sqrt{-3 + 12\sin\theta - 9\sin^2\theta} \quad \text{and we obtain}$$

$$h = l\left[\sqrt{-3+12\sin\theta - 9\sin^2\theta} - \tfrac{3}{2}\cos\theta\right]$$

Now $V = -mgh$ so $\frac{dV}{d\theta} = 0$ or $\frac{dh}{d\theta} = 0$ at static equilibrium.

$$\frac{dh}{d\theta} = l\left[\frac{6\cos\theta - 9\sin\theta\cos\theta}{\sqrt{-3+12\sin\theta - 9\sin^2\theta}} + \tfrac{3}{2}\sin\theta\right] = 0$$

Upon squaring, we obtain

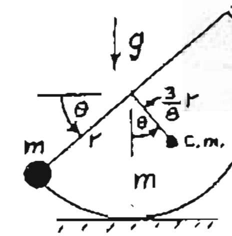
$$16\cos^2\theta\left(1 - 3\sin\theta + \tfrac{9}{4}\sin^2\theta\right) = \sin^2\theta(-3 + 12\sin\theta - 9\sin^2\theta)$$

or $\quad 27\sin^4\theta - 36\sin^3\theta - 23\sin^2\theta + 48\sin\theta - 16 = 0$

The root for stable equilibrium is $\sin\theta = 0.9216$, $\underline{\theta = 67.17^\circ}$

6-4.

$$V = -\tfrac{3}{8} mgr\cos\theta - mgr\sin\theta$$

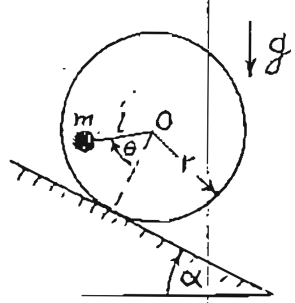


At equilibrium,

$$\frac{\partial V}{\partial\theta} = \tfrac{3}{8} mgr\sin\theta - mgr\cos\theta = 0$$

$$\tan\theta = \frac{8}{3} \quad \text{or} \quad \underline{\theta = 69.44^\circ}$$

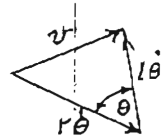
6-5. Use the cosine law to obtain v^2



$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \left(r^2 + l^2 - 2rl\cos\theta\right)\dot{\theta}^2$$

$$V = -mg\left[r\theta\sin\alpha + l\cos(\theta+\alpha)\right]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad \text{where } L = T - V$$



$$\frac{\partial L}{\partial \dot{\theta}} = m(r^2 + l^2 - 2rl\cos\theta)\dot{\theta}$$

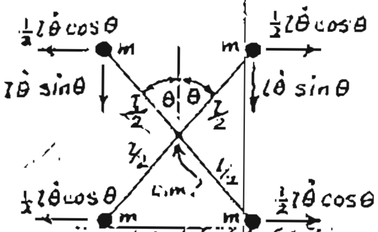
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m(r^2 + l^2 - 2rl\cos\theta)\ddot{\theta} + 2mrl\sin\theta\,\dot{\theta}^2$$

$$\frac{\partial L}{\partial \theta} = mrl\sin\theta\,\dot{\theta}^2 + mgr\sin\alpha - mgl\sin(\theta+\alpha)$$

The θ equation is

$$m(r^2 + l^2 - 2rl\cos\theta)\ddot{\theta} + mrl\dot{\theta}^2\sin\theta - mgr\sin\alpha + mgl\sin(\theta+\alpha) = 0$$

6-6. Choose an inertial frame in which the c.m. moves vertically. The velocity components of the particles are as shown.



$$T = \frac{m}{2}\left[4\left(\frac{l\dot{\theta}}{2}\right)^2\cos^2\theta + 2(l\dot{\theta})^2\sin^2\theta\right]$$

$$= \frac{1}{2} m l^2 (1 + \sin^2\theta)\dot{\theta}^2, \qquad V = 2mgl\cos\theta, \quad L = T - V$$

$$\frac{\partial T}{\partial \dot{\theta}} = ml^2\dot{\theta}(1+\sin^2\theta), \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}(1+\sin^2\theta) + 2ml^2\dot{\theta}^2\sin\theta\cos\theta$$

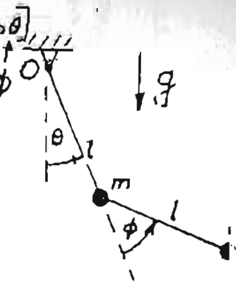
$$\frac{\partial T}{\partial \theta} = ml^2\dot{\theta}^2\sin\theta\cos\theta, \quad \frac{\partial V}{\partial \theta} = -2mgl\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \text{or} \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0$$

θ equation:

$$ml^2\ddot{\theta}(1+\sin^2\theta) + ml^2\dot{\theta}^2\sin\theta\cos\theta - 2mgl\sin\theta = 0$$

6-7. $v_1 = l\dot{\theta}, \quad v_2^2 = l^2\left[\dot{\theta}^2 + (\dot{\theta}+\dot{\phi})^2 + 2\dot{\theta}(\dot{\theta}+\dot{\phi})\cos\phi\right]$

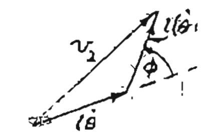


$$T = \frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2$$

$$= \frac{1}{2} m l^2\left[3\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}(1+\cos\phi) + 2\dot{\theta}^2\cos\phi\right]$$

$$V = -2mgl\cos\theta - mgl\cos(\theta+\phi)$$

$$L = T - V, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$



$$\frac{\partial T}{\partial \dot{\theta}} = ml^2\left[(3 + 2\cos\phi)\dot{\theta} + (1+\cos\phi)\dot{\phi}\right]$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = ml^2\left[(3+2\cos\phi)\ddot{\theta} + (1+\cos\phi)\ddot{\phi} - 2\dot{\theta}\dot{\phi}\sin\phi - \dot{\phi}^2\sin\phi\right]$$

$$\frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial V}{\partial \theta} = mgl\left[2\sin\theta + \sin(\theta+\phi)\right]$$

θ equation:

$$ml^2\left[(3+2\cos\phi)\ddot{\theta} + (1+\cos\phi)\ddot{\phi} - (\dot{\phi}^2 + 2\dot{\theta}\dot{\phi})\sin\phi\right] + mgl\left[2\sin\theta + \sin(\theta+\phi)\right] = 0$$

$$\frac{\partial T}{\partial \dot{\phi}} = ml^2\left[(1+\cos\phi)\dot{\theta} + \dot{\phi}\right], \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = ml^2\left[(1+\cos\phi)\ddot{\theta} + \ddot{\phi} - \dot{\theta}\dot{\phi}\sin\phi\right]$$

$$\frac{\partial T}{\partial \phi} = -ml^2(\dot{\theta}^2 + \dot{\theta}\dot{\phi})\sin\phi, \quad \frac{\partial V}{\partial \phi} = mgl\sin(\theta+\phi)$$

ϕ equation:

$$ml^2\left[(1+\cos\phi)\ddot{\theta} + \ddot{\phi} + \dot{\theta}^2\sin\phi\right] + mgl\sin(\theta+\phi) = 0$$

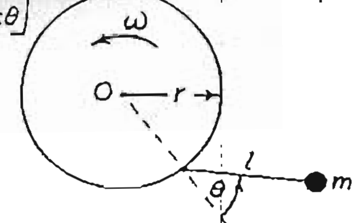
(b) For small θ and ϕ, let $\cos\theta \simeq 1$, $\cos\phi \simeq 1$, $\sin\theta \simeq \theta$, $\sin\phi \simeq \phi$, $\sin(\theta+\phi) \simeq \theta+\phi$, and neglect products of small angles or their derivatives. Then the linearized equations of motion are:

$$ml^2(5\ddot{\theta} + 2\ddot{\phi}) + mgl(3\theta + \phi) = 0$$

$$ml^2(2\ddot{\theta} + \ddot{\phi}) + mgl(\theta + \phi) = 0$$

6-8. $T=\frac{1}{2}mv^2$. Using cosine law,

$$T=\frac{1}{2}m\left[r^2\omega^2+l^2(\omega+\dot\theta)^2+2rl\omega(\omega+\dot\theta)\cos\theta\right]$$

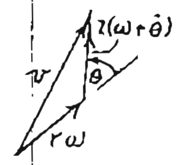


$$\frac{\partial T}{\partial\dot\theta}=ml^2(\omega+\dot\theta)+mrl\omega\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right)=ml^2\ddot\theta-mrl\omega\dot\theta\sin\theta$$

$$\frac{\partial T}{\partial\theta}=-mrl\omega(\omega+\dot\theta)\sin\theta$$

$V=0$



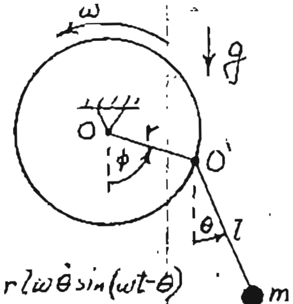
$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right)-\frac{\partial T}{\partial\theta}=0$$

or

$$ml^2\ddot\theta+mrl\omega^2\sin\theta=0$$

6-9. $\phi=\omega t$

$$T=\frac{1}{2}mv^2=\frac{m}{2}\left[l^2\dot\theta^2+r^2\omega^2+2rl\omega\dot\theta\cos(\omega t-\theta)\right]$$

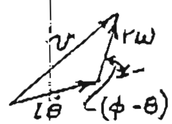


$$V=-mg(r\cos\omega t+l\cos\theta)$$

$$L=T-V,\quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right)-\frac{\partial L}{\partial\theta}=0$$

$$\frac{\partial T}{\partial\dot\theta}=ml^2\dot\theta+mrl\omega\cos(\omega t-\theta),\quad \frac{\partial T}{\partial\theta}=mrl\omega\dot\theta\sin(\omega t-\theta)$$

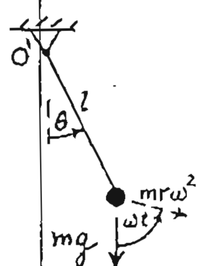
$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right)=ml^2\ddot\theta-mrl\omega^2\sin(\omega t-\theta)+mrl\omega\dot\theta\sin(\omega t-\theta)$$



$\frac{\partial V}{\partial\theta}=mgl\sin\theta$. The θ equation is

$$ml^2\ddot\theta-mrl\omega^2\sin(\omega t-\theta)+mgl\sin\theta=0$$

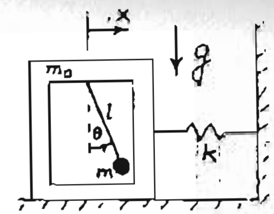
Second Method. Use a noninertial reference frame translating with O' but not rotating. Include an inertia force due to the acceleration of O' and applied at the particle. Taking force and acceleration components perpendicular to the pendulum,



$$ml\ddot\theta=mr\omega^2\sin(\omega t-\theta)-mg\sin\theta$$

6-10.

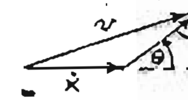
$$T=\frac{1}{2}m_0\dot x^2+\frac{1}{2}m(\dot x^2+l^2\dot\theta^2+2l\dot x\dot\theta\cos\theta)$$



$$V=\frac{1}{2}kx^2-mgl\cos\theta$$

$$\frac{\partial T}{\partial\dot x}=(m_0+m)\dot x+ml\dot\theta\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot x}\right)=(m_0+m)\ddot x+ml\ddot\theta\cos\theta-ml\dot\theta^2\sin\theta$$



$$\frac{\partial T}{\partial x}=0,\quad \frac{\partial V}{\partial x}=kx,\quad \frac{d}{dt}\left(\frac{\partial T}{\partial\dot q_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}=0$$

x equation: $(m_0+m)\ddot x+ml\ddot\theta\cos\theta-ml\dot\theta^2\sin\theta+kx=0$

$$\frac{\partial T}{\partial\dot\theta}=ml^2\dot\theta+ml\dot x\cos\theta,\quad \frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right)=ml^2\ddot\theta+ml\ddot x\cos\theta-ml\dot x\dot\theta\sin\theta$$

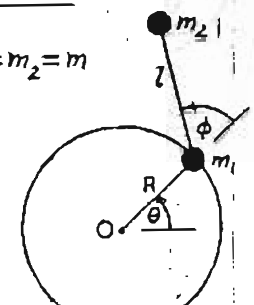
$$\frac{\partial T}{\partial\theta}=-ml\dot x\dot\theta\sin\theta,\quad \frac{\partial V}{\partial\theta}=mgl\sin\theta$$

θ equation: $ml^2\ddot\theta+ml\ddot x\cos\theta+mgl\sin\theta=0$

(b) Let $\cos\theta\cong 1$, $\sin\theta\cong\theta$, and omit higher-order terms.

$$(m_0+m)\ddot x+ml\ddot\theta+kx=0$$
$$ml\ddot x+ml^2\ddot\theta+mgl\theta=0$$

6-11. $m_1=m_2=m$

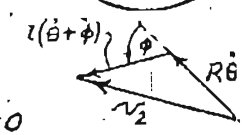


$$T=\frac{1}{2}mR^2\dot\theta^2+\frac{1}{2}mv_2^2$$
$$=\frac{1}{2}m\left[2R^2\dot\theta^2+l^2(\dot\theta+\dot\phi)^2+2Rl\dot\theta(\dot\theta+\dot\phi)\cos\phi\right]$$

$$\frac{\partial T}{\partial\dot\theta}=2mR^2\dot\theta+ml^2(\dot\theta+\dot\phi)+mRl(2\dot\theta+\dot\phi)\cos\phi$$

$$\frac{\partial T}{\partial\theta}=0,\quad V=0$$

Hence we obtain $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right)=0$ or



$$m(2R^2+l^2+2Rl\cos\phi)\ddot\theta+m(l^2+Rl\cos\phi)\ddot\phi-mRl\dot\phi(2\dot\theta+\dot\phi)\sin\phi=0$$

5-11. (cont'd.)

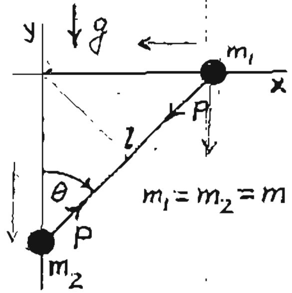
$$\frac{\partial T}{\partial \dot\phi} = ml^2(\dot\theta+\dot\phi) + mRl\dot\theta\cos\phi, \quad \frac{\partial T}{\partial \phi} = -mRl\dot\theta(\dot\theta+\dot\phi)\sin\phi$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\phi}\right) = ml^2(\ddot\theta+\ddot\phi) + mRl\ddot\theta\cos\phi - mRl\dot\theta\dot\phi\sin\phi$$

ϕ equation: $m(l^2+Rl\cos\phi)\ddot\theta + ml^2\ddot\phi + mRl\dot\theta^2\sin\phi = 0$

6-12. $x = l\sin\theta$, $y = -l\cos\theta$

$\dot x = l\dot\theta\cos\theta$, $\dot y = l\dot\theta\sin\theta$



$$T = \tfrac{1}{2}m(\dot x^2 + \dot y^2) = \tfrac{1}{2}ml^2\dot\theta^2$$

$$V = mgy = -mgl\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) = ml^2\ddot\theta, \quad \frac{\partial T}{\partial\theta} = 0, \quad \frac{\partial V}{\partial\theta} = mgl\sin\theta$$

θ equation: $ml^2\ddot\theta + mgl\sin\theta = 0$ or $\ddot\theta + \frac{g}{l}\sin\theta = 0$

This is identical with Eq. (3-220) for a simple pendulum.

$x(0) = l$, $\dot x(0) = 0$, $y(0) = 0$, $\dot y(0) = 0$.

Use conservation of energy. $\frac{1}{2}ml^2\dot\theta^2 - mgl\cos\theta = 0$

or $\dot\theta^2 = \frac{2g}{l}\cos\theta$

Consider m_1. $P\sin\theta = -m\ddot x = -m(l\ddot\theta\cos\theta - l\dot\theta^2\sin\theta)$

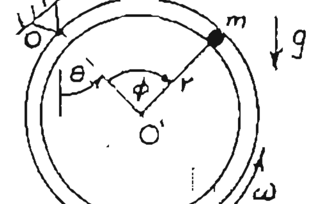
$= -m(-g\sin\theta\cos\theta) + ml\left(\frac{2g}{l}\sin\theta\cos\theta\right)$

or $P = 3mg\cos\theta$

$k = \sin\theta_0/2 = \sin\frac{\pi}{4} = \frac{1}{\sqrt2}$

From (3-230), period $T = 4\sqrt{\frac{l}{g}}\,K(k) = 7.4164\sqrt{\frac{l}{g}}$

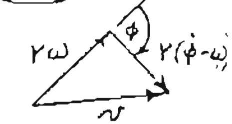
6-13. $\dot\theta = \omega = \text{const}$



$$T = \tfrac{1}{2}mv^2$$
$$= \tfrac{1}{2}mr^2\left[\omega^2 + (\dot\phi-\omega)^2 + 2\omega(\dot\phi-\omega)\cos\phi\right]$$

$$V = mgr\left[-\cos\theta + \cos(\phi-\theta)\right]$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\phi}\right) - \frac{\partial T}{\partial\phi} + \frac{\partial V}{\partial\phi} = 0$$



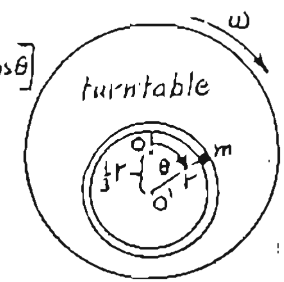
$$\frac{\partial T}{\partial\dot\phi} = mr^2(\dot\phi - \omega + \omega\cos\phi)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\phi}\right) = mr^2\ddot\phi - mr^2\omega\dot\phi\sin\phi$$

$$\frac{\partial T}{\partial\phi} = -mr^2\omega(\dot\phi-\omega)\sin\phi, \quad \frac{\partial V}{\partial\phi} = -mgr\sin(\phi-\theta)$$

ϕ equation: $mr^2\ddot\phi - mr^2\omega^2\sin\phi - mgr\sin(\phi-\theta) = 0$

6-14. $\theta(0) = 0$, $\dot\theta(0) = 0$

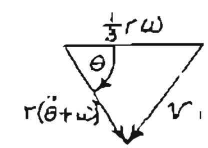


$$T = \tfrac{1}{2}mv^2 = \frac{m}{2}\left[\frac{r^2\omega^2}{9} + r^2(\dot\theta+\omega)^2 - \frac{2}{3}r^2\omega(\dot\theta+\omega)\cos\theta\right]$$

$$V = 0, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} = 0$$

$$\frac{\partial T}{\partial\dot\theta} = mr^2(\dot\theta+\omega) - \frac{mr^2}{3}\omega\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) = mr^2\ddot\theta + \frac{mr^2}{3}\omega\dot\theta\sin\theta$$



$$\frac{\partial T}{\partial\theta} = \frac{mr^2}{3}\omega(\dot\theta+\omega)\sin\theta$$

θ equation: $mr^2\ddot\theta - \frac{mr^2}{3}\omega^2\sin\theta = 0$

(b) $\ddot\theta = \dot\theta\frac{d\dot\theta}{d\theta} = \frac{\omega^2}{3}\sin\theta$. Integrate, $\frac{1}{2}\dot\theta^2 = \frac{\omega^2}{3}(1-\cos\theta)$

or $\dot\theta = \omega\sqrt{\frac{2}{3}(1-\cos\theta)}$

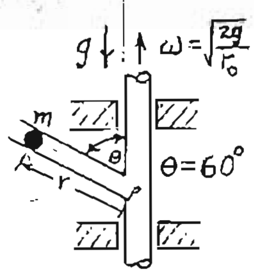
(c) Radial acceleration of m in direction $O'm$ is $a_r = -r(\dot\theta+\omega)^2 + \frac{r\omega^2}{3}\cos\theta$

$\dot\theta_{max}$ and $|a_r|_{max}$ occur at $\theta = \pi$. $|a_r|_{max} = r\omega^2\left[\left(1+\frac{2}{\sqrt3}\right)^2 + \frac{1}{3}\right]$

Max constraint force $F_{max} = m|a_r|_{max} = \frac{4}{3}(2+\sqrt3)mr\omega^2 = 4.9761\,mr\omega^2$

6-15.

$T = \frac{1}{2}m\left(\dot r^2 + r^2\omega^2\sin^2\theta\right) = \frac{1}{2}m\left(\dot r^2 + \frac{3r^2 g}{2r_0}\right)$



$V = mgr\cos\theta = \frac{1}{2}mgr$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot r}\right) - \frac{\partial T}{\partial r} + \frac{\partial V}{\partial r} = 0$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot r}\right) = m\ddot r$, $\frac{\partial T}{\partial r} = \frac{3mgr}{2r_0}$, $\frac{\partial V}{\partial r} = \frac{1}{2}mg$

r equation: $m\left(\ddot r - \frac{3g}{2r_0}r + \frac{1}{2}g\right) = 0$

(b) $r(0) = r_0$, $\dot r(0) = -\sqrt{gr_0/2}$

Let $\ddot r = \dot r\frac{d\dot r}{dr} = \frac{3g}{2r_0}r - \frac{1}{2}g$

Integrate. $\frac{1}{2}\dot r^2 = \frac{3g}{4r_0}r^2 - \frac{1}{2}gr + C$ where $C = 0$ from initial conditions.

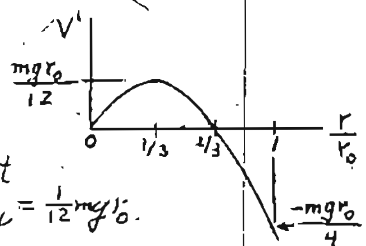
At $r = r_{min}$, we have $\dot r = 0$ so $3r^2 - 2r_0 r = 0$, $r = 0, \frac{2}{3}r_0$

At $r = 0$, $\ddot r < 0$ but at $r = \frac{2}{3}r_0$, $\ddot r > 0$ so $r_{min} = \frac{2}{3}r_0$

(c) Conservative system.

$E' = T' + V'$ where $T' = T_2 = \frac{1}{2}m\dot r^2$

$V' = V - T_0 = \frac{1}{2}mgr - \frac{3mg}{4r_0}r^2$



To reach the origin, the particle must go over the V' "hill". $E' = V'_{max} = \frac{1}{12}mgr_0$.

At $r = r_0$, $T' + V' = \frac{1}{2}m\dot r^2 - \frac{1}{4}mgr_0 = \frac{1}{12}mgr_0$, $\dot r^2 = \frac{2}{3}gr_0$.

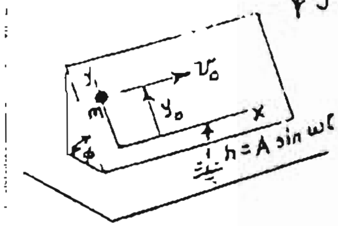
To reach the shaft, $\dot r < 0$. $\dot r(0) = -\sqrt{\frac{2}{3}gr_0}$

When $r = 0$, $E' = \frac{1}{2}m\dot r^2 = \frac{1}{12}mgr_0$ or $\dot r = -\sqrt{\frac{gr_0}{6}}$.

6-16.

$T = \frac{1}{2}m\left[\dot x^2 + \dot y^2 + A^2\omega^2\cos^2\omega t + 2\dot y A\omega\cos\omega t\sin\phi\right]$

$V = mg(y\sin\phi + A\sin\omega t)$



$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot x}\right) = m\ddot x$, $\frac{\partial T}{\partial x} = 0$, $\frac{\partial V}{\partial x} = 0$

x equation: $m\ddot x = 0$ where $x(0) = 0$, $\dot x(0) = v_0$ so $x = v_0 t$

$\frac{\partial T}{\partial \dot y} = m\dot y + mA\omega\sin\phi\cos\omega t$; $\frac{\partial T}{\partial y} = 0$, $\frac{\partial V}{\partial y} = mg\sin\phi$

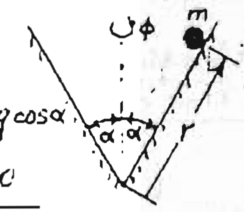
$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot y}\right) = m\ddot y - mA\omega^2\sin\phi\sin\omega t$

y equation: $m\ddot y - mA\omega^2\sin\phi\sin\omega t + mg\sin\phi = 0$ with $y(0) = y_0$, $\dot y(0) = 0$

After an integration, $\dot y = -A\omega\sin\phi\cos\omega t - gt\sin\phi + A\omega\sin\phi$

After second integration, $y = y_0 - A\sin\phi\sin\omega t - \frac{1}{2}gt^2\sin\phi + A\omega t\sin\phi$

6-17. $T = \frac{1}{2}m\left(\dot r^2 + r^2\sin^2\alpha\,\dot\phi^2\right)$, $V = mgr\cos\alpha$



$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot r}\right) = m\ddot r$, $\frac{\partial T}{\partial r} = mr\sin^2\alpha\,\dot\phi^2$, $\frac{\partial V}{\partial r} = mg\cos\alpha$

r equation: $m\ddot r - mr\dot\phi^2\sin^2\alpha + mg\cos\alpha = 0$

$\frac{\partial T}{\partial \dot\phi} = mr^2\dot\phi\sin^2\alpha$, $\frac{\partial T}{\partial \phi} = \frac{\partial V}{\partial \phi} = 0$ so $\frac{\partial T}{\partial \dot\phi} = $ const

φ equation: $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\phi}\right) = mr^2\ddot\phi\sin^2\alpha + 2mr\dot r\dot\phi\sin^2\alpha = 0$

or $r\ddot\phi + 2\dot r\dot\phi = 0$

(b) $r(0) = r_0$, $\dot r(0) = 0$, $\dot\phi(0) = 4\sqrt{g/r_0}$, $\alpha = 30°$

6-17. (cont'd.) $r^2\dot{\phi} = 4r_0^2\sqrt{\frac{g}{r_0}}$ or $\dot{\phi} = 4\frac{r_0^2}{r^2}\sqrt{\frac{g}{r_0}}$

This means that the angular momentum about the vertical axis is conserved. From the r equation,

$$\ddot{r} = \dot{r}\frac{d\dot{r}}{dr} = \frac{r}{4}\left(\frac{16r_0^3 g}{r^4}\right) - g\cos\alpha$$

Integrating, $\frac{1}{2}\dot{r}^2 = \frac{-2r_0^3 g}{r^2} - \frac{\sqrt{3}}{2}gr + \left(2+\frac{\sqrt{3}}{2}\right)gr_0$

$\dot{r}_{max}$ occurs when $\ddot{r}=0$ or $r^3 = \frac{8}{\sqrt{3}}r_0^3$, $r = \frac{2r_0}{3^{1/6}}$

$$\dot{r}^2 = gr_0\left(-3^{1/3} - 2\cdot 3^{1/3} + 4 + \sqrt{3}\right) = 1.4053\, gr_0$$

$$\dot{r}_{max} = 1.1855\sqrt{gr_0}$$

6-18. From problem 6-17 solution, the equations of motion are

$$\ddot{r} - r\dot{\phi}^2\sin^2\alpha + g\cos\alpha = 0$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \quad \text{or} \quad r^2\dot{\phi} = \text{const}$$

For circular motion at $r=r_0$ with $\dot{r}=0, \ddot{r}=0$, we obtain $\dot{\phi}_0^2 = \frac{g\cos\alpha}{r_0\sin^2\alpha}$ Then $\dot{\phi} = \frac{r_0^2}{r^2}\dot{\phi}_0 = \frac{r_0^2}{r^2}\sqrt{\frac{g\cos\alpha}{r_0\sin^2\alpha}}$

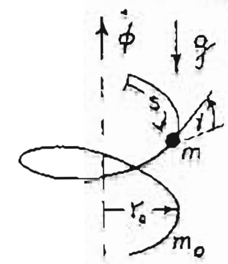
Substituting for $\dot{\phi}$ in the r equation, we obtain

$$\ddot{r} - \frac{r_0^3}{r^3}g\cos\alpha + g\cos\alpha = 0, \qquad \delta\ddot{r} + \frac{3g}{r_0}\cos\alpha\,\delta r = 0$$

The circular frequency is $\omega = \sqrt{\frac{3g}{r_0}\cos\alpha}$

6-19. Let s = relative displacement of particle downward along helix.
ϕ = absolute rotation angle of helix about its fixed vertical axis.

6-19. (cont'd.) The absolute velocity of the particle is given by

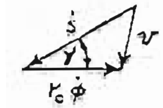


$$v^2 = \dot{s}^2 + r_0^2\dot{\phi}^2 - 2r_0\dot{s}\dot{\phi}\cos\gamma$$

All points on the helix have speed $r_0\dot{\phi}$.

$$T = \frac{1}{2}m_0 r_0^2\dot{\phi}^2 + \frac{1}{2}m\left(\dot{s}^2 + r_0^2\dot{\phi}^2 - 2r_0\dot{s}\dot{\phi}\cos\gamma\right)$$

$V = -mgs\sin\gamma$, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$



$\frac{\partial T}{\partial \dot{s}} = m\dot{s} - mr_0\dot{\phi}\cos\gamma$, $\frac{\partial V}{\partial s} = -mg\sin\gamma$, $\frac{\partial T}{\partial s} = 0$

s equation: $m\ddot{s} - mr_0\ddot{\phi}\cos\gamma - mg\sin\gamma = 0$

$\frac{\partial T}{\partial \dot{\phi}} = (m_0+m)r_0^2\dot{\phi} - mr_0\dot{s}\cos\gamma$, $\frac{\partial T}{\partial \phi} = 0$, $\frac{\partial V}{\partial \phi} = 0$

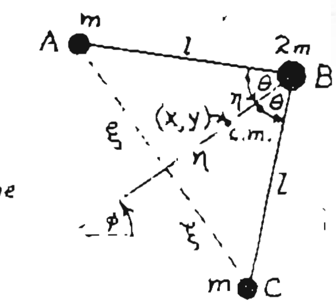
ϕ equation: $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = (m_0+m)r_0^2\ddot{\phi} - mr_0\ddot{s}\cos\gamma = 0$

$$\ddot{s} = \frac{(m_0+m)r_0}{m\cos\gamma}\ddot{\phi} = r_0\ddot{\phi}\cos\gamma + g\sin\gamma$$

$$\ddot{\phi} = \frac{mg\sin\gamma\cos\gamma}{r_0(m_0+m\sin^2\gamma)} \quad \text{and} \quad \ddot{s} = \frac{(m_0+m)g\sin\gamma}{m_0+m\sin^2\gamma}$$

Vertical acceleration $\ddot{z} = -\ddot{s}\sin\gamma = \frac{-(m_0+m)g\sin^2\gamma}{m_0+m\sin^2\gamma}$

6-20. Introduce the notation



$\eta = \frac{1}{2}l\cos\theta$, $\xi = l\sin\theta$

$\dot{\eta} = -\frac{1}{2}l\dot{\theta}\sin\theta$, $\dot{\xi} = l\dot{\theta}\cos\theta$

The particle velocities relative to the c.m. are

$$v_A^2 = (\xi\dot{\phi} + \dot{\eta})^2 + (\eta\dot{\phi} - \dot{\xi})^2$$

$$v_B^2 = \dot{\eta}^2 + \eta^2\dot{\phi}^2$$

$$v_C^2 = (\xi\dot{\phi} - \dot{\eta})^2 + (\eta\dot{\phi} + \dot{\xi})^2$$

6-20. (cont'd.) $T = \frac{1}{2}m(v_A^2 + v_B^2) + m v_B^2$, $V = 0$

$T = m(2\dot\eta^2 + \dot\xi^2 + 2\eta^2\dot\phi^2 + \xi^2\dot\phi^2) = \frac{1}{2}ml^2[(1+\cos^2\theta)\dot\theta^2 + (1+\sin^2\theta)\dot\phi^2]$

Use $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} = 0$

$\frac{\partial T}{\partial\dot\theta} = ml^2(1+\cos^2\theta)\dot\theta$, $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) = ml^2[(1+\cos^2\theta)\ddot\theta - 2\dot\theta^2\sin\theta\cos\theta]$

$\frac{\partial T}{\partial\theta} = \frac{ml^2}{2}(-2\sin\theta\cos\theta\,\dot\theta^2 + 2\sin\theta\cos\theta\,\dot\phi^2)$

θ equation: $\underline{ml^2[(1+\cos^2\theta)\ddot\theta - \sin\theta\cos\theta(\dot\theta^2+\dot\phi^2)] = 0}$

$\frac{\partial T}{\partial\dot\phi} = ml^2(1+\sin^2\theta)\dot\phi$, $\frac{\partial T}{\partial\phi} = 0$ so $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\phi}\right) = 0$

ϕ equation: $\underline{ml^2[(1+\sin^2\theta)\ddot\phi + 2\dot\theta\dot\phi\sin\theta\cos\theta] = 0}$

(b) $\theta(0) = \frac{\pi}{2}$, $\dot\theta(0) = -1$, $\dot\phi(0) = 1$

Conservation of $H_{c.m.}$, or $\frac{\partial T}{\partial\dot\phi}$ = const, gives $\dot\phi = \frac{2}{1+\sin^2\theta}$

Conservation of energy. $(1+\cos^2\theta)\dot\theta^2 + (1+\sin^2\theta)\dot\phi^2 = 3$

At $\theta = \theta_{min}$, $\dot\theta = 0$, so $\frac{4}{1+\sin^2\theta} = 3$, $\sin\theta = \frac{1}{\sqrt{3}}$, $\underline{\theta_{min} = 35.26°}$

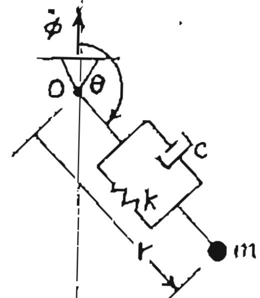
If $\dot x(0)$ and $\dot y(0)$ are nonzero, a uniform translation is added.

6-21. $L = T - V$ and $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} = Q_i'$

where $Q_r' = -c\dot r$, $Q_\theta' = Q_\phi' = 0$

$T = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2 + r^2\dot\phi^2\sin^2\theta)$

$V = mgr\cos\theta + \frac{1}{2}k(r-l)^2$



6-21. (cont'd.) $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot r}\right) = m\ddot r$, $\frac{\partial T}{\partial r} = mr\dot\theta^2 + mr\dot\phi^2\sin^2\theta$

$\frac{\partial V}{\partial r} = mg\cos\theta + k(r-l)$

r equation: $\underline{m\ddot r - mr\dot\theta^2 - mr\dot\phi^2\sin\theta + mg\cos\theta + k(r-l) + c\dot r = 0}$

$\frac{\partial T}{\partial\dot\theta} = mr^2\dot\theta$, $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) = mr^2\ddot\theta + 2mr\dot r\dot\theta$, $\frac{\partial T}{\partial\theta} = mr^2\dot\phi^2\sin\theta\cos\theta$

$\frac{\partial V}{\partial\theta} = -mgr\sin\theta$

θ equation: $\underline{mr^2\ddot\theta + 2mr\dot r\dot\theta - mr^2\dot\phi^2\sin\theta\cos\theta - mgr\sin\theta = 0}$

$\frac{\partial T}{\partial\dot\phi} = mr^2\dot\phi\sin^2\theta$, $\frac{\partial T}{\partial\phi} = \frac{\partial V}{\partial\phi} = 0$ so $\frac{d}{dt}(mr^2\dot\phi\sin^2\theta) = 0$

ϕ equation: $\underline{mr^2\ddot\phi\sin^2\theta + 2mr\dot r\dot\phi\sin^2\theta + 2mr^2\dot\theta\dot\phi\sin\theta\cos\theta = 0}$

(b) $r(0) = l$, $\dot r(0) = 0$, $\theta(0) = \frac{3\pi}{4}$, $\dot\theta(0) = 0$, $\dot\phi(0) = 2\sqrt{g/l}$

In general, $H_{vert} = mr^2\dot\phi\sin^2\theta = ml^2\sqrt{\frac{g}{l}}$ = const.

As $t\to\infty$, we assume $r \cong l$, and there can be no energy dissipation in this steady-state motion. Hence $\dot r = 0$, $\ddot r = 0$.

$\dot\phi = \frac{1}{\sin^2\theta}\sqrt{\frac{g}{l}}$. From the r equation for constant r,

$-r(\dot\theta^2 + \dot\phi^2\sin^2\theta) + g\cos\theta$ = const

But $T+V$ = const as $t\to\infty$. Multiplying by $\frac{2}{mr}$,

$r(\dot\theta^2 + \dot\phi^2\sin^2\theta) + 2g\cos\theta$ = const

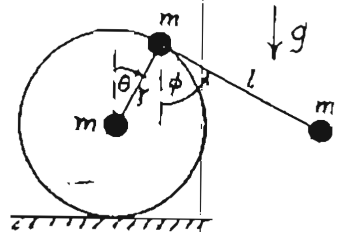
Adding equations, $3g\cos\theta$ = const, so θ = const.

Then, from the θ equation, $-ml^2\left(\frac{g}{l\sin^4\theta}\right)\sin\theta\cos\theta - mgl\sin\theta =$

or $\sin^4\theta = -\cos\theta$ or $\cos^4\theta - 2\cos^2\theta + \cos\theta + 1 = 0$

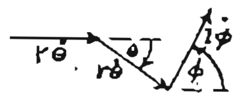
$\cos\theta = -0.52489$, $\underline{\theta = 121.66°}$, $\underline{\dot\phi = 1.3803\sqrt{\frac{g}{l}}}$

6-22. Adding individual kinetic energies,



$$T = \tfrac{1}{2}mr^2\dot\theta^2 + \tfrac{1}{2}m\left[(r\dot\theta)^2 + (r\dot\theta)^2 + 2(r\dot\theta)^2\cos\theta\right]$$
$$+\tfrac{1}{2}m\left[(r\dot\theta + r\dot\theta\cos\theta + l\dot\phi\cos\phi)^2 + (-r\dot\theta\sin\theta + l\dot\phi\sin\phi)^2\right]$$

or



$$T = \tfrac{1}{2}m\left\{r^2\dot\theta^2(5+4\cos\theta) + l^2\dot\phi^2 + 2rl\dot\theta\dot\phi\left[\cos(\theta+\phi)+\cos\phi\right]\right\}$$

$$V = mg(2r\cos\theta - l\cos\phi), \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$$

$$\frac{\partial T}{\partial\dot\theta} = mr^2\dot\theta(5+4\cos\theta) + mrl\dot\phi\left[\cos(\theta+\phi)+\cos\phi\right], \quad \frac{\partial V}{\partial\theta} = -2mgr\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) = mr^2\ddot\theta(5+4\cos\theta) + mrl\ddot\phi\left[\cos(\theta+\phi)+\cos\phi\right] - 4mr^2\dot\theta^2\sin\theta$$
$$- mrl\dot\phi\left[(\dot\theta+\dot\phi)\sin(\theta+\phi) + \dot\phi\sin\phi\right]$$

$$\frac{\partial T}{\partial\theta} = -2mr^2\dot\theta^2\sin\theta - mrl\dot\theta\dot\phi\sin(\theta+\phi)$$

θ equation: $mr^2\ddot\theta(5+4\cos\theta) + mrl\ddot\phi\left[\cos(\theta+\phi)+\cos\phi\right] - 2mr^2\dot\theta^2\sin\theta$
$-mrl\dot\phi^2\left[\sin(\theta+\phi)+\sin\phi\right] - 2mgr\sin\theta = 0$

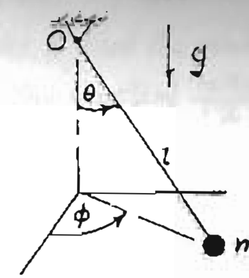
$$\frac{\partial T}{\partial\dot\phi} = ml^2\dot\phi + mrl\dot\theta\left[\cos(\theta+\phi)+\cos\phi\right], \quad \frac{\partial V}{\partial\phi} = mgl\sin\phi$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\phi}\right) = ml^2\ddot\phi + mrl\ddot\theta\left[\cos(\theta+\phi)+\cos\phi\right] - mrl\dot\theta\left[(\dot\theta+\dot\phi)\sin(\theta+\phi)+\dot\phi\sin\phi\right]$$

$$\frac{\partial T}{\partial\phi} = -mrl\dot\theta\dot\phi\left[\sin(\theta+\phi)+\sin\phi\right]$$

ϕ equation: $ml^2\ddot\phi + mrl\ddot\theta\left[\cos(\theta+\phi)+\cos\phi\right] - mrl\dot\theta^2\sin(\theta+\phi)$
$+ mgl\sin\phi = 0$

6-23. $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot q_i}\right) - \dfrac{\partial L}{\partial q_i} = 0, \quad L = T - U$



$$T = \tfrac{1}{2}ml^2(\dot\theta^2 + \dot\phi^2\sin^2\theta)$$
$$U = -mgl\cos\theta - \tfrac{1}{2}Cl^2\dot\phi\sin^2\theta$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = ml^2\ddot\theta$$
$$\frac{\partial L}{\partial\theta} = ml^2\dot\phi^2\sin\theta\cos\theta - mgl\sin\theta + Cl^2\dot\phi\sin\theta\cos\theta$$

θ equation: $ml^2\ddot\theta - ml^2\dot\phi^2\sin\theta\cos\theta - Cl^2\dot\phi\sin\theta\cos\theta + mgl\sin\theta = 0$

$\dfrac{\partial L}{\partial\dot\phi} = ml^2\dot\phi\sin^2\theta + \tfrac{1}{2}Cl^2\sin^2\theta = \text{const}$ since $\dfrac{\partial L}{\partial\phi} = 0$

ϕ equation: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot\phi}\right) = ml^2\ddot\phi\sin^2\theta + (2ml^2\dot\phi + Cl^2)\dot\theta\sin\theta\cos\theta = 0$

(b) $\theta(0) = \frac{\pi}{4}, \quad \dot\theta(0) = 0, \quad \dot\phi(0) = \sqrt{\frac{g}{l}}, \quad C = m\sqrt{\frac{2g}{l}}$

Use conservation of energy. $T + V = \text{const}$ where $V = -mgl\cos\theta$

$$\tfrac{1}{2}ml^2(\dot\theta^2 + \dot\phi^2\sin^2\theta) - mgl\cos\theta = \left(\tfrac{1}{4} - \tfrac{1}{\sqrt2}\right)mgl$$

$p_\phi = \dfrac{\partial L}{\partial\dot\phi} = \text{const}$ or $\left(ml^2\dot\phi + \tfrac{1}{2}Cl^2\right)\sin^2\theta = \left(\tfrac{1}{2} + \tfrac{\sqrt2}{4}\right)ml^2\sqrt{\tfrac{g}{l}}$

Hence $\dot\phi = \frac{1}{2}\sqrt{\frac{g}{l}}\left(\dfrac{1+\frac{\sqrt2}{2}}{\sin^2\theta} - \sqrt2\right)$. At $\theta = \theta_{max}$, we have $\dot\theta = 0$

From energy equation,
$$mgl\left\{\frac{1}{4}\left[\frac{\left(1+\frac{\sqrt2}{2}\right) - \sqrt2\sin^2\theta}{\sin\theta}\right]^2 - 2\cos\theta - \left(\tfrac{1}{2} - \sqrt2\right)\right\} = 0$$

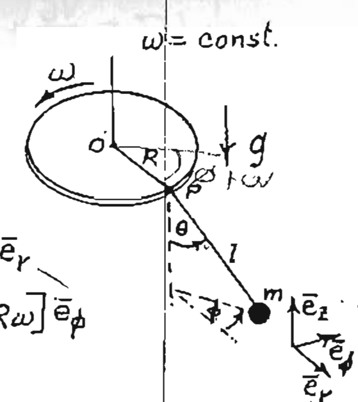
or $2\sin^4\theta - 2(2-\sqrt2)\sin^2\theta - 8\sin^2\theta\cos\theta + \left(\frac{3}{2} + \sqrt2\right) = 0$

Let $\sin^2\theta = 1 - \cos^2\theta$ and obtain $\cos^4\theta + 4\cos^3\theta - \sqrt2\cos^2\theta - 4\cos\theta + \left(\frac{3\sqrt2}{2} - \frac{1}{4}\right) -$

A known root, from initial conditions, is $\cos\theta = \frac{1}{\sqrt2}$. The only other root with $|\cos\theta| < 1$ is $\cos\theta = 0.5504$ $\quad \theta_{max} = 56.60^\circ$

Note: Without the velocity-dependent U, $\theta(0)$ would be θ_{max} rather than its actual θ_{min}.

6-24. Take cylindrical unit vectors $\bar{e}_r, \bar{e}_\phi, \bar{e}_z$ at the attachment point P. The absolute velocity of the particle is



$$\bar{v} = \left[-l(\dot\phi+\omega)\sin\theta\sin\phi + l\dot\theta\cos\theta\cos\phi\right]\bar{e}_r$$
$$+\left[l(\dot\phi+\omega)\sin\theta\cos\phi + l\dot\theta\cos\theta\sin\phi + R\omega\right]\bar{e}_\phi$$
$$+\ l\dot\theta\sin\theta\ \bar{e}_z$$

$$T = \tfrac{1}{2}mv^2 = \tfrac{m}{2}\left[l^2\dot\theta^2 + l^2(\dot\phi+\omega)^2\sin^2\theta + R^2\omega^2 + 2Rl\omega\dot\theta\cos\theta\sin\phi + 2Rl\omega(\dot\phi+\omega)\sin\theta\cos\phi\right]$$

$$V = -mgl\cos\theta$$

(a) From the T expression, $T_2 = \frac{1}{2}ml^2(\dot\theta^2+\dot\phi^2\sin^2\theta)$

$$T_1 = ml^2\omega\left[\dot\phi\sin^2\theta + \frac{R}{l}(\dot\theta\cos\theta\sin\phi + \dot\phi\sin\theta\cos\phi)\right]$$
$$T_0 = \tfrac{1}{2}m\omega^2(l^2\sin^2\theta + R^2 + 2Rl\sin\theta\cos\phi)$$

(b) $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0, \quad \frac{\partial V}{\partial \theta} = mgl\sin\theta$

$$\frac{\partial T}{\partial\dot\theta} = ml^2\dot\theta + mRl\omega\cos\theta\sin\phi,\quad \frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) = ml^2\ddot\theta + mRl\omega(-\dot\theta\sin\theta\sin\phi + \dot\phi\cos\theta\cos\phi)$$

$$\frac{\partial T}{\partial\theta} = ml^2(\dot\phi^2+2\omega\dot\phi+\omega^2)\sin\theta\cos\theta + mRl\omega[-\dot\theta\sin\theta\sin\phi + (\dot\phi+\omega)\cos\theta\cos\phi]$$

θ equation: $ml^2\ddot\theta - ml^2(\dot\phi+\omega)^2\sin\theta\cos\theta - mRl\omega^2\cos\theta\cos\phi + mgl\sin\theta = 0$

$$\frac{\partial T}{\partial\dot\phi} = ml^2\dot\phi\sin^2\theta + ml^2\omega\sin^2\theta + mRl\omega\sin\theta\cos\phi,\quad \frac{\partial V}{\partial\phi} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\phi}\right) = ml^2\ddot\phi\sin^2\theta + 2ml^2\dot\theta(\dot\phi+\omega)\sin\theta\cos\theta + mRl\omega(\dot\theta\cos\theta\cos\phi - \dot\phi\sin\theta\sin\phi)$$

$$\frac{\partial T}{\partial\phi} = mRl\omega(\dot\theta\cos\theta\cos\phi - \dot\phi\sin\theta\sin\phi - \omega\sin\theta\sin\phi)$$

φ equation:

$$ml^2\ddot\phi\sin^2\theta + 2ml^2\dot\theta(\dot\phi+\omega)\sin\theta\cos\theta + mRl\omega^2\sin\theta\sin\phi = 0$$

6-24. (cont'd.) (c) $R = l,\quad \omega^2 = g/2l,\quad \theta(0)=0;\ \dot\theta(0)=0$

Conservative system. Use the energy integral to find the maximum possible θ. $T_2 - T_0 + V = \text{const}$. Omit the constant $\frac{mR\omega^2}{2}$ and obtain

$$\tfrac{1}{2}ml^2(\dot\theta^2+\dot\phi^2\sin^2\theta) - mgl\left(\frac{\sin^2\theta}{4} + \frac{1}{2}\sin\theta\cos\phi\right) - mgl\cos\theta = -mgl$$

$\dot\theta = 0$ at $\theta = \theta_{max}$. θ_{max} implies V_{max} or $(T_2 - T_0)_{min}$.

$(T_2)_{min} = 0$ for $\dot\phi = \dot\theta = 0$. Look for $(-T_0)_{min}$ or $(T_0)_{max}$:

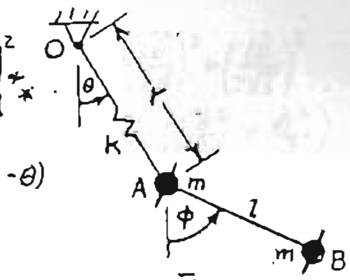
Assume $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$. Then, for any value of θ, $(T_0)_{max}$ occurs for $\phi = 0$. Then the energy equation becomes $\frac{1}{4}\sin^2\theta + \frac{1}{2}\sin\theta + \cos\theta - 1 = 0$

or $\left(\frac{1}{4}\sin^2\theta + \frac{1}{2}\sin\theta - 1\right)^2 = 1 - \sin^2\theta$ or $\frac{\sin^4\theta}{16} + \frac{\sin^3\theta}{4} - \frac{\sin^2\theta}{4} - \sin\theta = -\sin^2\theta$

Eliminate the zero root and obtain $\sin^3\theta + 4\sin^2\theta + 12\sin\theta - 16 = 0$

$$\sin\theta_{max} = 0.9559,\quad \theta_{max} = 72.93°$$

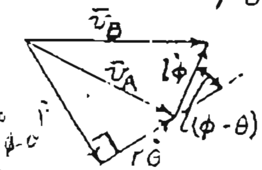
6-25. Nonholonomic system.



$$T = \tfrac{1}{2}m(\dot r^2 + r^2\dot\theta^2) + \tfrac{1}{2}m\left\{[r\dot\theta + l\dot\phi\cos(\phi-\theta)]^2 + [\dot r - l\dot\phi\sin(\phi-\theta)]^2\right\}$$

or
$$T = \tfrac{1}{2}m\left[2\dot r^2 + 2r^2\dot\theta^2 + l^2\dot\phi^2 + 2lr\dot\theta\dot\phi\cos(\phi-\theta) - 2l\dot r\dot\phi\sin(\phi-\theta)\right]$$

$$V = \tfrac{1}{2}k(r-l)^2$$



Nonholonomic constraint:

$$\cos(\phi-\theta)\,\dot r + r\sin(\phi-\theta)\,\dot\theta = 0$$

Using order (r,θ,ϕ), $a_{11} = \cos(\phi-\theta)$
$a_{12} = r\sin(\phi-\theta)$
$a_{13} = a_{1t} = 0$

$\bar{R} = r\bar{e}_r$
$\dot{\bar{R}} = \dot r\bar{e}_r + r\dot{\bar{e}}_r$
$\dot{\bar{R}} = \dot r\bar{e}_r + r\dot\theta\bar{e}_\theta$
$|\dot{\bar{R}}|^2 = \dot r^2 + (r\dot\theta)^2$

$\dot{\bar{e}}_r = \bar{\Theta}\times\bar{e}_r = \dot\theta\bar{e}_\theta$

$\bar{R}_{B/A} = l\cos(\phi-\theta)\bar{e}_r + l\sin(\phi-\theta)\bar{e}_\theta$
$\dot{\bar{R}}_{B/A} = -l\sin(\phi-\theta)(\dot\phi-\dot\theta)\bar{e}_r + l\cos(\phi-\theta)\dot{\bar{e}}_r + l\cos(\phi-\theta)(\dot\phi - \dot\theta)\bar{e}_\theta + l\sin(\phi-\theta)\dot{\bar{e}}_\theta$

$\dot{\bar{e}}_r = \bar{\Theta}\times\bar{e}_r$, $\dot{\bar{e}}_\theta = \bar{\Theta}\times\bar{e}_\theta$

$\bar{v}_B = \bar{v}_A + \bar{v}_{B/A}$

6-25. (cont'd.) $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = \sum_j \lambda_j a_{ji}$

$\frac{\partial T}{\partial \dot r} = 2m\dot r - ml\dot\phi\sin(\phi-\theta)$, $\frac{\partial T}{\partial r} = 2mr\dot\theta^2 + ml\dot\theta\dot\phi\cos(\phi-\theta)$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot r}\right) = 2m\ddot r - ml\ddot\phi\sin(\phi-\theta) - ml\dot\phi(\dot\phi-\dot\theta)\cos(\phi-\theta)$, $\frac{\partial V}{\partial r} = k(r-l)$

r equation: $2m\ddot r - ml\ddot\phi\sin(\phi-\theta) - ml\dot\phi^2\cos(\phi-\theta) - 2mr\dot\theta^2 + k(r-l) = \lambda\cos(\phi-\theta)$

$\frac{\partial T}{\partial \dot\theta} = 2mr^2\dot\theta + mlr\dot\phi\cos(\phi-\theta)$, $\frac{\partial V}{\partial\theta} = 0$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) = 2mr^2\ddot\theta + mlr\ddot\phi\cos(\phi-\theta) + 4mr\dot r\dot\theta + ml\dot r\dot\phi\cos(\phi-\theta) - mlr\dot\phi(\dot\phi-\dot\theta)\sin(\phi-\theta)$$

$\frac{\partial T}{\partial\theta} = mlr\dot\theta\dot\phi\sin(\phi-\theta) + ml\dot r\dot\phi\cos(\phi-\theta)$

θ equation: $2mr^2\ddot\theta + mlr\ddot\phi\cos(\phi-\theta) + 4mr\dot r\dot\theta - mlr\dot\phi^2\sin(\phi-\theta) = \lambda r\sin(\phi-\theta)$

$\frac{\partial T}{\partial\dot\phi} = ml^2\dot\phi + mlr\dot\theta\cos(\phi-\theta) - ml\dot r\sin(\phi-\theta)$, $\frac{\partial V}{\partial\phi} = 0$

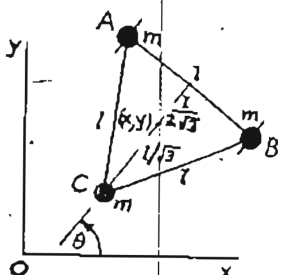
$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\phi}\right) = ml^2\ddot\phi + mlr\ddot\theta\cos(\phi-\theta) - ml\ddot r\sin(\phi-\theta) + ml\dot r\dot\theta\cos(\phi-\theta) - ml\dot r(\dot\phi-\dot\theta)\cos(\phi-\theta) - mlr\dot\theta(\dot\phi-\dot\theta)\sin(\phi-\theta) = 0$$

$\frac{\partial T}{\partial\phi} = -mlr\dot\theta\dot\phi\sin(\phi-\theta) - ml\dot r\dot\phi\cos(\phi-\theta)$

ϕ equation: $ml^2\ddot\phi + mlr\ddot\theta\cos(\phi-\theta) - ml\ddot r\sin(\phi-\theta) + 2ml\dot r\dot\theta\cos(\phi-\theta) + mlr\dot\theta^2\sin(\phi-\theta) = 0$

6-26. $T = \frac{3m}{2}(\dot x^2+\dot y^2) + \frac{3m}{2}\left(\frac{l}{\sqrt3}\dot\theta\right)^2 = \frac32 m(\dot x^2+\dot y^2) + \frac12 ml^2\dot\theta^2$

$V=0$, $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot q_i}\right) - \frac{\partial T}{\partial q_i} = \sum_j\lambda_j a_{ji}$



Constraint: velocity of A or B is zero in AB direction.

$\sin\theta\,\dot x - \cos\theta\,\dot y - \frac{l}{2\sqrt3}\dot\theta = 0$ Take order (x, y, θ). $a_{1t} = 0$

$a_{11} = \sin\theta$, $a_{12} = -\cos\theta$, $a_{13} = -l/2\sqrt3$

6-26. (cont'd.) The equations of motion are

$$3m\ddot x = \lambda\sin\theta$$
$$3m\ddot y = -\lambda\cos\theta$$
$$ml^2\ddot\theta = -\lambda l/2\sqrt3$$

With the constraint equation, there are 4 equations and the 4 unknowns (x, y, θ, λ).

(b) Initial conditions: $x(0)=0$, $y(0)=0$, $\theta(0)=0$; $\dot x(0)=0$, $\dot y(0)=v_0$, $\dot\theta(0) = \frac{-2\sqrt3 v_0}{l}$ From const.

Try to eliminate all variables except θ and its derivatives.

From θ equation, $\lambda = -2\sqrt3\, ml\ddot\theta$

Differentiate constraint equation,

$$(\dot x\cos\theta + \dot y\sin\theta)\dot\theta + \ddot x\sin\theta - \ddot y\cos\theta - \frac{l}{2\sqrt3}\ddot\theta = 0$$

Using the x and y equations and the λ expression,

$$(\dot x\cos\theta + \dot y\sin\theta)\dot\theta = \frac{l}{2\sqrt3}\ddot\theta - \frac{\lambda}{3m}(\cos^2\theta + \sin^2\theta) = \frac{5}{2\sqrt3}l\ddot\theta$$

The velocity of the c.m. is given by

$$v^2 = (\dot x\cos\theta + \dot y\sin\theta)^2 + (\dot x\sin\theta - \dot y\cos\theta)^2 = \left(\frac{5l\ddot\theta}{2\sqrt3\,\dot\theta}\right)^2 + \left(\frac{l\dot\theta}{2\sqrt3}\right)^2 = l^2\left(\frac{25\ddot\theta^2}{12\dot\theta^2} + \frac{\dot\theta^2}{12}\right)$$

Using conservation of energy,

$$T = \frac32 mv^2 + \frac12 ml^2\dot\theta^2 = \frac{ml^2}{8}\left(25\frac{\ddot\theta^2}{\dot\theta^2} + 5\dot\theta^2\right) = \frac{15}{2}mv_0^2$$

Then $\ddot\theta = \dot\theta\frac{d\dot\theta}{d\theta} = -\sqrt{\frac{12v_0^2}{5l^2} - \frac{\dot\theta^2}{5}}\;\dot\theta$ or $\theta = -\int_{\dot\theta(0)}^{\dot\theta}\frac{\sqrt5\,d\dot\theta}{\sqrt{\frac{12v_0^2}{l^2} - \dot\theta^2}}$

The negative sign is chosen for the square root because, for small t (small negative θ), $\dot\theta$ is negative and $\ddot\theta$ is positive, as seen from equations of motion and differentiate constraint.

6-26. (cont'd.) Integrating, we obtain

$$\theta = \sqrt{5}\left[\cos^{-1}\frac{l\dot\theta}{\sqrt{12}\,v_0}\right]_{-\frac{\sqrt{12}\,v_0}{l}}^{\dot\theta} = \sqrt{5}\left(-\pi + \cos^{-1}\frac{l\dot\theta}{\sqrt{12}\,v_0}\right)$$

or $\quad \dot\theta = -\frac{2\sqrt{3}}{l}\,v_0 \cos\frac{\theta}{\sqrt{5}}$

The constraint force λ is proportional to $\ddot\theta$ where

$$\ddot\theta = 2\sqrt{\frac{3}{5}}\,\frac{v_0}{l}\,\dot\theta\,\sin\frac{\theta}{\sqrt{5}} = \frac{-12\,v_0^2}{\sqrt{5}\,l^2}\sin\frac{\theta}{\sqrt{5}}\cos\frac{\theta}{\sqrt{5}}$$

$$\lambda = -2\sqrt{3}\,ml\ddot\theta = 12\sqrt{\frac{3}{5}}\,\frac{mv_0^2}{l}\sin\frac{2\theta}{\sqrt{5}} = 9.295\,\frac{mv_0^2}{l}\sin\frac{2\theta}{\sqrt{5}}$$

$\dot\theta$ remains negative until $\cos\frac{\theta}{\sqrt{5}}$ reaches zero, or $\theta = -\sqrt{5}\,\frac{\pi}{2}$.
At this time, $\lambda = 0$ and $\ddot\theta = 0$, so θ remains constant at
$\theta = -\frac{\sqrt{5}}{2}\pi = -201.25°$. The final velocity v_f is found using
conservation of energy. $\frac{3}{2}mv_f^2 = \frac{15}{2}mv_0^2$ or $v_f = \sqrt{5}\,v_0$.
Final motion is constant translation, $v_f = \sqrt{5}\,v_0$ at $\theta = -201.25°$
with particle C leading.

Note: The $\dot\theta$ equation can be integrated. $\theta = 2\sqrt{5}\left[\tan^{-1}\left(e^{-2\sqrt{\frac{3}{5}}\frac{v_0}{l}t}\right) - \frac{\pi}{4}\right]$

6-27. For a system described by n independent q's,
Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \qquad (i = 1,2,\cdots,n)$$

where, from (6-62), $\quad \frac{\partial T}{\partial \dot q_i} = p_i$

and, from (6-49), the generalized force due to the k
Cartesian force components is

$$Q_i = \sum_{j=1}^{k} F_j \frac{\partial x_j}{\partial q_i}$$

6-27. (cont'd.) Hence we obtain

$$\dot p_i - \frac{\partial T}{\partial q_i} = \sum_{j=1}^{k} F_j \frac{\partial x_j}{\partial q_i} \qquad (i = 1,2,\cdots,n)$$

For a set of impulsive F's applied over a small time interval Δt, integrate the equation over Δt, obtaining

$$\Delta p_i - \frac{\partial T}{\partial q_i}\Delta t = \sum_{j=1}^{k} F_j \frac{\partial x_j}{\partial q_i}\Delta t$$

As $\Delta t \to 0$, $\lim_{\Delta t\to 0} \frac{\partial T}{\partial q_i}\Delta t = 0$ since $\frac{\partial T}{\partial q_i}$ remains finite,
but $\lim_{\Delta t\to 0} F_j\Delta t = \hat F_j$ which is nonzero. Now $\frac{\partial x_j}{\partial q_i}$
is finite and essentially constant during Δt, so

$$\Delta p_i = \sum_{j=1}^{k} \hat F_j \frac{\partial x_j}{\partial q_i} \qquad (i = 1,2,\cdots,n)$$

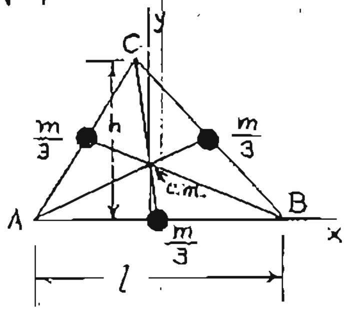
Also, we can write

$$\Delta p_i = \hat Q_i$$

where the generalized impulse is

$$\hat Q_i = \sum_{j=1}^{k} \hat F_j \frac{\partial x_j}{\partial q_i}$$

CHAPTER 7



7-1. First note that the total mass m of the particles equals that of the triangle. The c.m. of the triangle lies at a point which is 2/3 of the distance from any vertex to the midpoint of the opposite side. The c.m. of the particles lies at a point 2/3 of the distance from any particle to the mid-point of a line connecting the other two. Hence, the two c.m.'s coincide.

The moment of inertia of the triangle about side AB is

$$I_{AB} = \int_0^h \rho l\left(1-\frac{y}{h}\right)y^2 dy = \rho l \frac{h^3}{12} = \frac{mh^2}{6}$$

where ρ is the density and $m = \frac{1}{2}\rho l h$. For the particles, also,

$$I_{AB} = \frac{2}{3}m\left(\frac{h}{2}\right)^2 = \frac{mh^2}{6}.$$

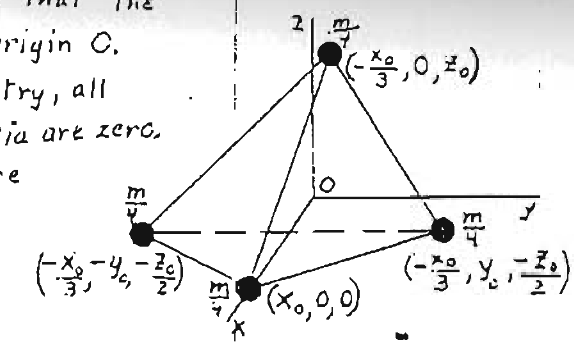
Now I_{xx} about the c.m. $= I_{AB} - m\left(\frac{h}{3}\right)^2 = \frac{mh^2}{18}$ in either case.

Using similar reasoning, the moments of inertia about axes through the c.m. and parallel to BC and CA are the same for the two systems. Hence, the moments of inertia for the two systems must be equal for any given axis through the c.m. and in the xy plane.

Recall that for a planar body in the xy plane,

$$I_{zz} = I_{xx} + I_{yy}$$

so the moments of inertia are also equal for the two systems about the z axis. Hence, the inertia ellipsoids are equal. Thus, the two systems are dynamically equivalent.



7-2. First note that the c.m. is at the origin O. From the symmetry, all products of inertia are zero. The xyz axes are principal axes.

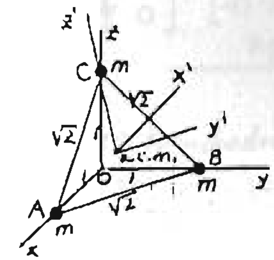
$$\frac{m}{4}\left(2y_0^2 + \frac{3}{2}z_0^2\right) = I_{xx}$$

$$\frac{m}{4}\left(\frac{4}{3}x_0^2 + \frac{3}{2}z_0^2\right) = I_{yy}$$

$$\frac{m}{4}\left(\frac{4}{3}x_0^2 + 2y_0^2\right) = I_{zz}$$

Solving these equations for x_0, y_0, z_0, we obtain

$$x_0 = \sqrt{\frac{3}{2m}\left(I_{yy}+I_{zz}-I_{xx}\right)},\quad y_0 = \sqrt{\frac{1}{m}\left(I_{zz}+I_{xx}-I_{yy}\right)},\quad z_0 = \sqrt{\frac{4}{3m}\left(I_{xx}+I_{yy}-I_{zz}\right)}$$



7-3. In the xyz system,

$$I_{xx} = \sum_i (y_i^2 + z_i^2), \text{ etc.}$$

$$I_{xy} = -\sum_i m_i x_i y_i, \text{ etc.}$$

giving the inertia matrix

$$[I] = m\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(b) This is a planar system. Two principal axes at the c.m. will lie in that plane with the third axis perpendicular. Since $l_{x'x}, l_{x'y}, l_{x'z}$ are all positive, the x' axis lies in the first octant of the xyz frame. Since x' is a principal axis, it is perpendicular to the plane ABC. $l_{x'x} = l_{x'y} = l_{x'z} = \frac{1}{\sqrt{3}}$

7-3. (cont'd.) The y' axis is parallel to the xy plane because $l_{y'z} = 0$. $l_{y'x} = \frac{-1}{\sqrt{2}}$, $l_{y'y} = \frac{1}{\sqrt{2}}$.

Recall that the sum of the squares of the elements in any row or column equals one. Knowing this, the elements of the third row of $[l]$ are determined in magnitude. The sign is obtained from the figure, $l_{z'x} = l_{z'y} = \frac{-1}{\sqrt{6}}$, $l_{z'z} = \sqrt{\frac{2}{3}}$.

$$[l] = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{bmatrix}$$

(c) The c.m. is at $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. The distance from particle C to the c.m. is $\sqrt{\frac{2}{3}}$. Hence $I_{x'x'} = 3m\left(\frac{2}{3}\right) = 2m$.

$I_{x'x'} = I_{y'y'} + I_{z'z'}$ where $I_{y'y'} = I_{z'z'}$ because of the threefold symmetry. Hence $I_{y'y'} = I_{z'z'} = m$.

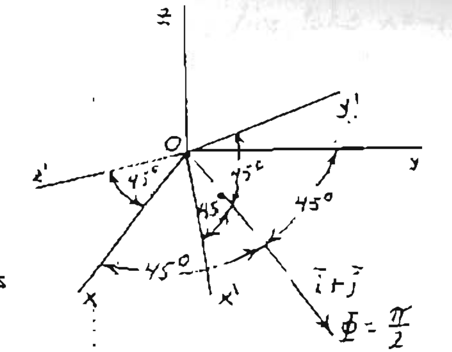
$$[I'] = m \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2nd Method: If the xyz frame is translated to the c.m., we obtain

$$[I_{cm}] = \frac{m}{3} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

Then use $[I'] = [l][I_{cm}][l]^T$ to obtain the result given above.

7-4. Assume the xy plane is horizontal and the z axis is vertical. After a 90° rotation about $\mathbf{i} + \mathbf{j}$, the $x'y'$ plane is vertical and the z' axis is horizontal.



By inspection, one obtains

$$[l] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

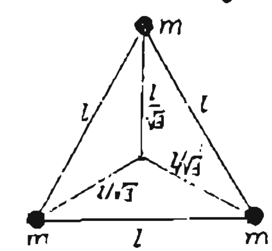
An alternate method to obtain $[l]$ is to multiply the rotation matrices for the following sequence of simpler rotations:

(1) 45° about z axis.
(2) 90° about new x axis.
(3) -45° about new z axis.

Then

$$[I'] = [l][I][l]^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 7 & -1 & -\sqrt{2} \\ -1 & 7 & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & 6 \end{bmatrix} \text{kg}\cdot\text{m}^2$$

7-5. Because of the symmetry of a regular tetrahedron, the inertia ellipsoid is actually spherical. So consider an axis perpendicular to a face (equilateral triangle) and passing through the c.m.



This axis passes through one particle and is at a distance $l/\sqrt{3}$ from the other three.

The principal moments of inertia are

$$I_1 = I_2 = I_3 = 3\left(\frac{ml^2}{3}\right) = ml^2$$

7-6. Given

$$[I] = \begin{bmatrix} 450 & -60 & 100 \\ -60 & 500 & 7 \\ 100 & 7 & 550 \end{bmatrix} \text{ kg}\cdot\text{m}^2$$

Characteristic equation: $\begin{vmatrix} (450-I) & -60 & 100 \\ -60 & (500-I) & 7 \\ 100 & 7 & (550-I) \end{vmatrix} = 0$

or $I^3 - 1500 I^2 + 733{,}851 I - 116{,}663{,}950 = 0$

Roots are $\underline{I_1 = 365.49,\ I_2 = 516.47,\ I_3 = 618.04 \text{ kg}\cdot\text{m}^2}$

(b) Let (x, y, z) be a point on a principal axis. Then

$$-60\frac{y}{x} + 100\frac{z}{x} = -(450-I)$$

$$(500-I)\frac{y}{x} + 7\frac{z}{x} = 60$$

$$7\frac{y}{x} + (550-I)\frac{z}{x} = -100$$

Solving the first two equations, and then first and third,

$$\frac{y}{x} = \frac{9150 - 7I}{50420 - 106I}, \quad \frac{z}{x} = \frac{9150 - 7I}{60I - 33{,}700}$$

For $\underline{I_1 = 365.49}$, $\frac{y}{x} = 0.4752$, $\frac{z}{x} = -0.5600$

Normalizing, $\sqrt{1 + \left(\frac{y}{x}\right)^2 + \left(\frac{z}{x}\right)^2} = 1.2407$. To minimize Φ, we maximize $\text{tr}[l] = 1 + 2\cos\Phi$, that is, we maximize $l_{x'x} + l_{y'y} + l_{z'z}$. This determines the sign of x in each case. For this case, take $x = 1$. Then, for the x' axis,

$$l_{x'x} = \frac{1}{1.2407} = 0.8060, \quad l_{x'y} = \frac{0.4752}{1.2407} = 0.3830$$

$$l_{x'z} = \frac{-0.5600}{1.2407} = -0.4513$$

7-6. (cont'd.) For $\underline{I_2 = 516.47}$, $\frac{y}{x} = -4.5097$, $\frac{z}{x} = -2.0411$

$\sqrt{1 + \left(\frac{y}{x}\right)^2 + \left(\frac{z}{x}\right)^2} = 5.0501$. Since $\frac{y}{x}$ is negative, take $x = -1$ so that $l_{y'y}$ will be positive. $l_{y'x} = \frac{-1}{5.0501} = -0.1980$

$$l_{y'y} = \frac{4.5097}{5.0501} = 0.8930, \quad l_{y'z} = \frac{2.0411}{5.0501} = 0.4042$$

For $\underline{I_3 = 618.04}$, $\frac{y}{x} = -0.42373$, $\frac{z}{x} = 1.4262$. Take $x = 1$

$\sqrt{1 + \left(\frac{y}{x}\right)^2 + \left(\frac{z}{x}\right)^2} = 1.7926$, $l_{z'x} = \frac{1}{1.7926} = 0.5578$

$$l_{z'y} = \frac{-0.42373}{1.7926} = -0.2364, \quad l_{z'z} = \frac{1.4262}{1.7926} = 0.7956$$

Summarizing, the rotation matrix is

$$\underline{[l] = \begin{bmatrix} 0.8060 & 0.3830 & -0.4513 \\ -0.1980 & 0.8930 & 0.4042 \\ 0.5578 & -0.2364 & 0.7956 \end{bmatrix}}$$

(c) $\text{tr}[l] = 2.4946 = 1 + 2\cos\Phi$, $\cos\Phi = 0.7473$

$$\underline{\Phi = 41.64^\circ}$$

The direction cosines of the rotation axis are

$$a_x = \frac{l_{y'z} - l_{z'y}}{2\sin\Phi} = \underline{0.4826}$$

$$a_y = \frac{l_{z'x} - l_{x'z}}{2\sin\Phi} = \underline{0.7593}$$

$$a_z = \frac{l_{x'y} - l_{y'x}}{2\sin\Phi} = \underline{0.4372}$$

7-6. Given

$$[I] = \begin{bmatrix} 450 & -60 & 100 \\ -60 & 500 & 7 \\ 100 & 7 & 550 \end{bmatrix} \text{kg}\cdot\text{m}^2$$

Characteristic equation: $\begin{vmatrix} (450-I) & -60 & 100 \\ -60 & (500-I) & 7 \\ 100 & 7 & (550-I) \end{vmatrix} = 0$

or $I^3 - 1500 I^2 + 733{,}851 I - 116{,}663{,}950 = 0$

Roots are $I_1 = 365.49,\ I_2 = 516.47,\ I_3 = 618.04$ kg·m²

(b) Let (x,y,z) be a point on a principal axis. Then

$$-60\frac{y}{x} + 100\frac{z}{x} = -(450-I)$$

$$(500-I)\frac{y}{x} + 7\frac{z}{x} = 60$$

$$7\frac{y}{x} + (550-I)\frac{z}{x} = -100$$

Solving the first two equations, and then first and third,

$$\frac{y}{x} = \frac{9150 - 7I}{50420 - 100 I}, \quad \frac{z}{x} = \frac{9150 - 7I}{60I - 33{,}700}$$

For $I_1 = 365.49$, $\frac{y}{x} = 0.4752$, $\frac{z}{x} = -0.5600$

Normalizing, $\sqrt{1+\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2} = 1.2407$. To minimize Φ, we maximize $tr[l] = 1 + 2\cos\Phi$, that is, we maximize $l_{x'x} + l_{y'y} + l_{z'z}$. This determines the sign of x in each case. For this case, take $x=1$. Then, for the x' axis,

$$l_{x'x} = \frac{1}{1.2407} = 0.8060, \quad l_{x'y} = \frac{0.4752}{1.2407} = 0.3830$$

$$l_{x'z} = \frac{-0.5600}{1.2407} = -0.4513$$

7-6. (cont'd.) For $I_2 = 516.47$, $\frac{y}{x} = -4.5091$, $\frac{z}{x} = -2.0411$

$\sqrt{1+\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2} = 5.0501$. Since $\frac{y}{x}$ is negative, take $x=-1$ so that $l_{y'y}$ will be positive. $l_{y'x} = \frac{-1}{5.0501} = -0.1980$

$$l_{y'y} = \frac{4.5091}{5.0501} = 0.8930, \quad l_{y'z} = \frac{2.0411}{5.0501} = 0.4042$$

For $I_3 = 618.04$, $\frac{y}{x} = -0.42373$, $\frac{z}{x} = 1.4262$. Take $x=1$

$\sqrt{1+\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2} = 1.7926$, $\quad l_{z'x} = \frac{1}{1.7926} = 0.5578$

$$l_{z'y} = \frac{-0.42373}{1.7926} = -0.2364, \quad l_{z'z} = \frac{1.4262}{1.7926} = 0.7956$$

Summarizing, the rotation matrix is

$$[l] = \begin{bmatrix} 0.8060 & 0.3830 & -0.4513 \\ -0.1980 & 0.8930 & 0.4042 \\ 0.5578 & -0.2364 & 0.7956 \end{bmatrix}$$

(c) $tr[l] = 2.4946 = 1 + 2\cos\Phi$, $\cos\Phi = 0.7473$

$$\Phi = 41.64°$$

The direction cosines of the rotation axis are

$$a_x = \frac{l_{y'z} - l_{z'y}}{2\sin\Phi} = 0.4820$$

$$a_y = \frac{l_{z'x} - l_{x'z}}{2\sin\Phi} = 0.7593$$

$$a_z = \frac{l_{x'y} - l_{y'x}}{2\sin\Phi} = 0.4372$$

7-7.

Given $[I] = \begin{bmatrix} 25 & -1 & -2 \\ -1 & 20 & 2 \\ -2 & 2 & 10 \end{bmatrix} kg\cdot m^2$

Characteristic equation: $\begin{vmatrix} (25-I) & -1 & -2 \\ -1 & (20-I) & 2 \\ -2 & 2 & (10-I) \end{vmatrix} = 0$

or $I^3 - 55I^2 + 941I - 4818 = 0$ giving $I_1 = 9.4106$

$I_2 = 20.034$

$\underline{I_3 = 25.555}$

(b) Taking (x, y, z) as a point on a principal axis, we have

$$\left.\begin{aligned} (25-I) - \frac{y}{x} - 2\frac{z}{x} &= 0 \\ -1 + (20-I)\frac{y}{x} + 2\frac{z}{x} &= 0 \\ -2 + 2\frac{y}{x} + (10-I)\frac{z}{x} &= 0 \end{aligned}\right\} \text{Solving,} \quad \frac{y}{x} = \frac{24-I}{I-19}, \quad \frac{z}{x} = \frac{48-2I}{I-6}$$

For $\underline{I_1 = 9.4106}$, $\frac{y}{x} = -1.5214$, $\frac{z}{x} = 8.5554$, since $\frac{z}{x}$ has the largest magnitude, call this principal axis the z' axis. Let $x=1$.

$\sqrt{1+\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2} = 8.7470$, $l_{z'x} = \frac{1}{8.7470} = 0.1143$,

$l_{z'y} = \frac{-1.5214}{8.7470} = -0.1739$, $l_{z'z} = \frac{8.5554}{8.7470} = 0.9781$

In each row of $[l]$, the largest element should lie on the main diagonal, in order to maximize $tr[l]$, or minimize Φ.

For $\underline{I_2 = 20.034}$, $\frac{y}{x} = 3.8356$, $\frac{z}{x} = 0.5652$. Let $x=1$. This is the y' axis.

$\sqrt{1+\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2} = 4.0039$, $l_{y'x} = \frac{1}{4.0039} = 0.2498$

$l_{y'y} = \frac{3.8356}{4.0039} = 0.9580$, $l_{y'z} = \frac{0.5652}{4.0039} = 0.1412$

7-7. (cont'd.) $\underline{I_3 = 25.555}$, $\frac{y}{x} = -0.2373$, $\frac{z}{x} = -0.1541$

Let $x=1$. This principal axis is the x' axis.

$\sqrt{1+\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2} = 1.0400$, $l_{x'x} = \frac{1}{1.0400} = 0.9615$

$l_{x'y} = \frac{-0.2373}{1.0400} = -0.2281$, $l_{x'z} = \frac{-0.1541}{1.0400} = -0.1530$

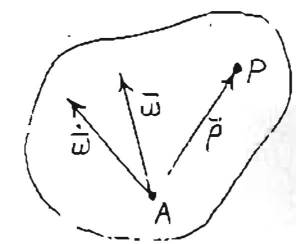
Summarizing, $\underline{[l] = \begin{bmatrix} 0.9615 & -0.2281 & -0.1530 \\ 0.2498 & 0.9580 & 0.1412 \\ 0.1143 & -0.1739 & 0.9781 \end{bmatrix}}$

(c) $tr[l] = 2.8976 = 1 + 2\cos\Phi$, $\cos\Phi = 0.9488$, $\underline{\Phi = 18.41^\circ}$

The direction cosines of the axis of rotation are

$a_x = \frac{l_{y'z} - l_{z'y}}{2\sin\Phi} = \underline{0.4983}$, $a_y = \frac{l_{z'x} - l_{x'z}}{2\sin\Phi} = \underline{0.4231}$, $a_z = \frac{l_{x'y} - l_{y'x}}{2\sin\Phi} = \underline{-0.7565}$

7-8. (a) Velocity. If $\bar{v}_A = 0$, then $\bar{v}_P = \bar{\omega} \times \bar{\rho}$. Hence, any point on the axis of rotation through A will have $\bar{v}_P = 0$ since $\bar{\omega}$ and $\bar{\rho}$ are parallel in that case.



(b) Acceleration. If $\bar{a}_A = 0$, then $\bar{a}_P = \dot{\bar{\omega}} \times \bar{\rho} + \bar{\omega} \times (\bar{\omega} \times \bar{\rho})$. Assume $\bar{\rho} \neq 0$. $\bar{a}_P$ will be zero only if (1) both terms are zero, or if (2) the terms are equal in magnitude and opposite in direction.

Case 1. Suppose $\bar{\omega} \times (\bar{\omega} \times \bar{\rho}) = 0$. This will occur only if $\bar{\omega}$ and $\bar{\rho}$ are parallel. But then $\dot{\bar{\omega}} \times \bar{\rho} \neq 0$ since $\bar{\omega}$ and $\dot{\bar{\omega}}$ are not parallel. Hence, both terms cannot be zero.

Case 2. Suppose $\bar{\omega} \times (\bar{\omega} \times \bar{\rho}) \neq 0$. In general, $\bar{\omega} \times (\bar{\omega} \times \bar{\rho})$ lies in the plane of $\bar{\omega}$ and $\bar{\rho}$, and is perpendicular to $\bar{\omega}$. $\dot{\bar{\omega}} \times \bar{\rho}$ is perpendicular to $\bar{\rho}$, so it cannot cancel $\bar{\omega} \times (\bar{\omega} \times \bar{\rho})$.

7-8. (cont'd.) Since both cases do not allow $\bar{a}_P$ to equal zero, we conclude that A is the only point having zero acceleration.

7-9. The moment of inertia of a solid sphere of radius r and density ρ about a diameter is

$$I_{sp} = \frac{2}{5} m_s r^2 = \frac{2}{5}\left(\frac{4}{3}\pi r^3 \rho\right) r^2 = \frac{8}{15}\pi\rho r^5$$

$$dI_{sp} = \frac{8}{3}\pi\rho r^4\,dr \quad \text{and} \quad m = 4\pi r^2 \rho\,dr$$

for a shell of thickness $dr << r$. Replace dI_{sp} by the I of the shell. $I = \frac{2}{3} m r^2$.

7-10. For a solid ellipsoid of density ρ, the mass is

$m_e = \frac{4}{3}\pi\rho abc$ where a, b, c are the semiaxes.

$I_e = \frac{1}{5} m_e (b^2+c^2) = \frac{4}{15}\pi\rho abc(b^2+c^2)$ about the x axis.

In order to vary the size of the ellipsoid, let $a \to ka$, $b \to kb$, $c \to kc$, where a, b, c are now constants, but k is variable with a nominal value of one. Then

$$I_e = \frac{4}{15}\pi\rho abc(b^2+c^2)k^5, \quad dI_e = \frac{4}{3}\pi\rho abc(b^2+c^2)k^4\,dk$$

For an ellipsoid, $V = \frac{4}{3}\pi abc k^3$, $dV = 4\pi abc\,k^2\,dk$

Thus, the shell mass is $m = 4\pi\rho abc\,dk$, $k=1$.

Replace dI_e by I_{xx} of the shell and set $k=1$.

$$I_{xx} = \frac{1}{3} m (b^2 + c^2)$$

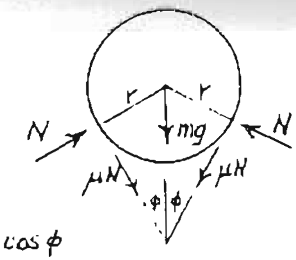
Similarly,

$$I_{yy} = \frac{1}{3} m (a^2 + c^2), \quad I_{zz} = \frac{1}{3} m (a^2 + b^2)$$

7-11. Vertical motion of sphere—

$$m\ddot{y} = 2N(\sin\phi - \mu\cos\phi) - mg$$

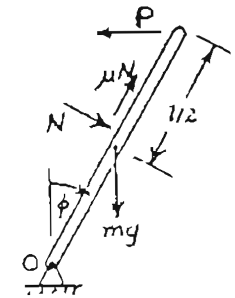
or $$N = \frac{m(\ddot{y}+g)}{2(\sin\phi - \mu\cos\phi)}$$



Rotation equation of rod. $l = 3r$.

$$3mr^2\ddot{\phi} = \frac{3}{2} mgr\sin\phi + Nr\cot\phi - 3Pr\cos\phi$$

$$P = \frac{1}{\cos\phi}\left[\frac{1}{2} mg\sin\phi + \frac{N}{3}\cot\phi - mr\ddot{\phi}\right]$$



Now

$$y = \frac{r}{\sin\phi}, \quad \dot{y} = \frac{-r\cos\phi}{\sin^2\phi}\dot{\phi}$$

$$\ddot{y} = \frac{-\cos\phi}{\sin^2\phi} r\ddot{\phi} + \frac{(1+\cos^2\phi)}{\sin^3\phi} r\dot{\phi}^2$$

$|\dot{\phi}| << 1$ so neglect $\dot{\phi}^2$ term.

Substitute for N and $\ddot{\phi}$ to obtain

$$P = \frac{mg\sin\phi}{2\cos\phi} + \frac{m(\ddot{y}+g)}{6\sin\phi(\sin\phi - \mu\cos\phi)} + \frac{m\sin^2\phi}{\cos^2\phi}\ddot{y}$$

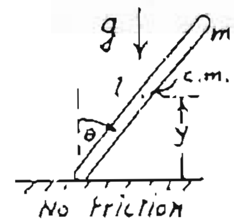
Now set $\phi = 30°$, $\ddot{y} = g$, and $\mu = 0.20$.

$$P = \left[\frac{1}{2\sqrt{3}} + \frac{2}{3\left(\frac{1}{2} - \frac{\sqrt{3}}{10}\right)} + \frac{1}{3}\right]mg = 2.6620\,mg$$

7-12. $y = \frac{1}{2} l\cos\theta$, $\dot{y} = -\frac{1}{2} l\sin\theta\,\dot{\theta}$

$$T = \frac{1}{2} m\dot{y}^2 + \frac{1}{2}\left(\frac{ml^2}{12}\right)\dot{\theta}^2 = \frac{ml^2}{24}(1 + 3\sin^2\theta)\dot{\theta}^2$$

$$V = mgy = \frac{1}{2} mgl\cos\theta$$



Use Lagrange's equation.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} + \frac{\partial V}{\partial\theta} = 0$$

7-12. (cont'd.) $\frac{\partial T}{\partial \dot\theta} = \frac{ml^2}{12}(1+3\sin^2\theta)\dot\theta$, $\frac{\partial V}{\partial\theta} = -\frac{1}{2}mgl\sin\theta$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) = \frac{ml^2}{12}\left[(1+3\sin^2\theta)\ddot\theta + 6\sin\theta\cos\theta\,\dot\theta^2\right]$$

$$\frac{\partial T}{\partial\theta} = \frac{ml^2}{4}\sin\theta\cos\theta\,\dot\theta^2$$

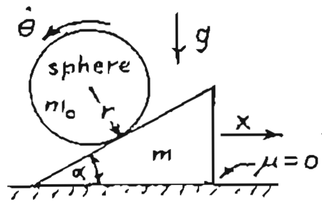
θ equation: $\underline{\frac{ml^2}{12}(1+3\sin^2\theta)\ddot\theta + \frac{ml^2}{4}\dot\theta^2\sin\theta\cos\theta - \frac{1}{2}mgl\sin\theta = 0}$

(b) Use conservation of energy.

$$T+V = \frac{ml^2}{24}(1+3\sin^2\theta)\dot\theta^2 + \frac{1}{2}mgl\cos\theta = \frac{1}{2}mgl$$

$$\dot\theta^2 = \frac{12g(1-\cos\theta)}{l(1+3\sin^2\theta)} \quad \text{or} \quad \underline{\dot\theta = \pm\sqrt{\frac{12g(1-\cos\theta)}{l(1+3\sin^2\theta)}}}$$

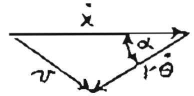
7-13. The sphere velocity v is given by $v^2 = \dot x^2 + r^2\dot\theta^2 - 2r\dot\theta\dot x\cos\alpha$



$$T = \frac{1}{2}m\dot x^2 + \frac{1}{2}m_0(\dot x^2 + r^2\dot\theta^2 - 2r\dot\theta\dot x\cos\alpha) + \frac{1}{5}m_0r^2\dot\theta^2$$

$$V = -m_0gr\theta\sin\alpha$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$$



$\frac{\partial T}{\partial\dot x} = m\dot x + m_0\dot x - m_0r\dot\theta\cos\alpha$, $\frac{\partial T}{\partial x} = 0$, $\frac{\partial V}{\partial x} = 0$

x equation: $(m+m_0)\ddot x - m_0r\ddot\theta\cos\alpha = 0$

$\frac{\partial T}{\partial\dot\theta} = \frac{7}{5}m_0r^2\dot\theta - m_0r\dot x\cos\alpha$, $\frac{\partial T}{\partial\theta} = 0$, $\frac{\partial V}{\partial\theta} = -m_0gr\sin\alpha$

θ equation: $\frac{7}{5}m_0r^2\ddot\theta - m_0r\ddot x\cos\alpha - m_0gr\sin\alpha = 0$

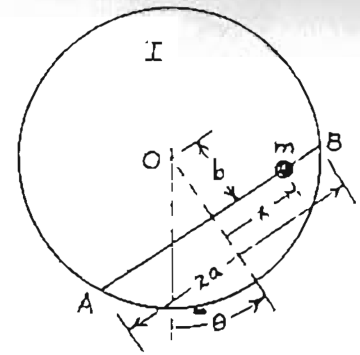
Solving the two equations for $\ddot x$, $\ddot x = \frac{5m_0g\sin\alpha\cos\alpha}{7(m+m_0)-5m_0\cos\alpha}$

Integrate with $\dot x(0) = 0$,

$$\underline{\dot x = \left(\frac{5m_0g\sin\alpha\cos\alpha}{7(m+m_0)-5m_0\cos\alpha}\right)t}$$

7-14. $x(0) = -a$ $\quad\theta(0) = 0$

$\dot x(0) = 0$ $\quad\dot\theta(0) = 0$



Angular momentum is conserved about O.

$$[I + m(b^2+x^2)]\dot\theta + mb\dot x = 0$$

or

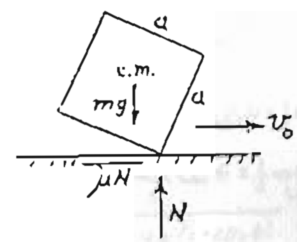
$$\dot\theta = \frac{-mb\dot x}{I+m(b^2+x^2)}$$

Integrating,

$$\theta = -\int_{-a}^{a}\frac{b\,dx}{\frac{I}{m}+b^2+x^2} = \underline{\frac{-2b}{\sqrt{\frac{I}{m}+b^2}}\tan^{-1}\frac{a}{\sqrt{\frac{I}{m}+b^2}}}$$

7-15. Take moments about the c.m.

For a positive (clockwise) tipping moment,



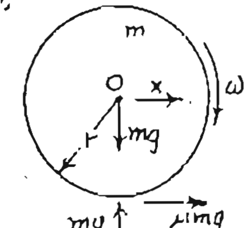
$$\mu N\frac{a}{2} > N\frac{a}{2}$$

giving $\underline{\mu_{min} = 1}$

Once tipping begins, the lever arm for μN increases, while that for N decreases, causing an increasing rotation rate. For a slope α, where $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, the results are unchanged because the tangential to normal force ratio μ is unchanged.

7-16. Using linear impulse and momentum,

$$\mu mg\,\Delta t = mv \quad\text{or}\quad v = \dot x = \underline{\mu g\,\Delta t}$$



Using angular impulse and momentum about O,

$$-\mu mgr\,\Delta t = \frac{mr^2}{2}\omega \quad\text{or}\quad \underline{\omega = \frac{-2\mu g\,\Delta t}{r}}$$

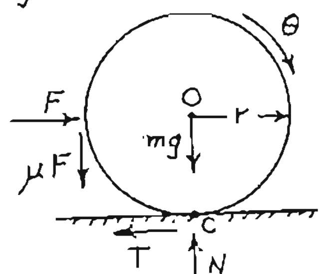
7-16. (cont'd.)

(b) After the paper is removed, the friction force is reversed, as are the accelerations. Hence, the cylinder comes to rest after another inteval Δt. The total displacement is

$$x_f = 2\left(\frac{v}{2}\right)\Delta t = \mu g (\Delta t)^2, \qquad v_f = 0$$

Note: The angular momentum about a point fixed on the floor remains zero throughout the motion.

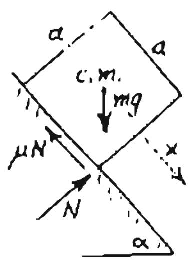
7-17. In order to avoid calculating the tangential road force T, take the fixed point C as the reference point for the rotational equation.



$$\frac{3}{2} m r^2 \ddot{\theta} = F(1-\mu) r$$

Acceleration $\quad a = r\ddot{\theta} = \frac{2F}{3m}(1-\mu)$

7-18. Assuming that sliding occurs, the cube will tip if there is a net clockwise moment about the c.m.



$$\mu N\left(\frac{a}{2}\right) > N\left(\frac{a}{2}\right) \quad \text{or} \quad \mu > 1.$$

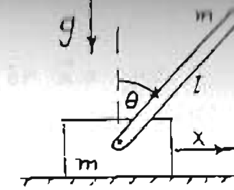
$N = mg\cos\alpha, \quad m\ddot{x} = mg\sin\alpha - \mu N$

$\qquad = mg(\sin\alpha - \mu\cos\alpha)$

Sliding occurs if $\ddot{x} > 0$ or $\mu < \tan\alpha$

Hence, both sliding and tipping occur if $\quad 1 < \mu < \tan\alpha$

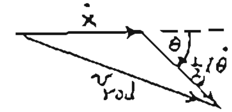
7-19.



$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\left(\dot{x}^2 + \frac{l^2\dot{\theta}^2}{4} + l\dot{x}\dot{\theta}\cos\theta\right) + \frac{ml^2}{24}\dot{\theta}^2$$

$$= m\dot{x}^2 + \frac{ml^2}{6}\dot{\theta}^2 + \frac{1}{2} ml\dot{x}\dot{\theta}\cos\theta$$

$V = \frac{1}{2} mgl\cos\theta, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$



$\frac{\partial T}{\partial \dot{x}} = 2m\dot{x} + \frac{1}{2} ml\dot{\theta}\cos\theta = p_x = \text{const.}$

since $\frac{\partial T}{\partial x} = 0, \quad \frac{\partial V}{\partial x} = 0,$ x equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = 2m\ddot{x} + \frac{1}{2} ml\ddot{\theta}\cos\theta - \frac{1}{2} ml\dot{\theta}^2\sin\theta = 0$$

$\frac{\partial T}{\partial \dot{\theta}} = \frac{ml^2}{3}\dot{\theta} + \frac{1}{2} ml\dot{x}\cos\theta, \quad \frac{\partial T}{\partial \theta} = -\frac{1}{2} ml\dot{x}\dot{\theta}\sin\theta, \quad \frac{\partial V}{\partial \theta} = -\frac{1}{2} mgl\sin\theta$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{ml^2}{3}\ddot{\theta} + \frac{1}{2} ml\ddot{x}\cos\theta - \frac{1}{2} ml\dot{x}\dot{\theta}\sin\theta$

θ equation: $\quad \frac{1}{3} ml^2\ddot{\theta} + \frac{1}{2} ml\ddot{x}\cos\theta - \frac{1}{2} mgl\sin\theta = 0$

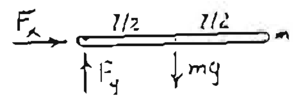
(b) $\dot{\theta}(0) = 0, \quad \theta(0) = 0, \quad \dot{x}(0) = 0$

Conservation of energy. $m\dot{x}^2 + \frac{1}{6} ml^2\dot{\theta}^2 + \frac{1}{2} ml\dot{x}\dot{\theta}\cos\theta + \frac{1}{2} mgl\cos\theta = \frac{1}{2} mgl$

But $p_x = 0$ or $\dot{x} = -\frac{1}{4} l\dot{\theta}\cos\theta$ so $ml^2\dot{\theta}^2\left(\frac{1}{6} - \frac{\cos^2\theta}{16}\right) + \frac{1}{2} mgl\cos\theta = \frac{1}{2} mgl$

or $\frac{ml^2\dot{\theta}^2}{48}(8 - 3\cos^2\theta) = \frac{1}{2} mgl(1-\cos\theta), \qquad \dot{\theta} = \sqrt{\frac{24g(1-\cos\theta)}{l(8-3\cos^2\theta)}}$

(c) $\theta = 90°. \qquad \dot{\theta} = \sqrt{\frac{3g}{l}}$



From the equations of motion,

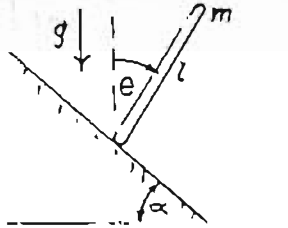
$\ddot{x} = \frac{3}{4} g, \quad \ddot{\theta} = \frac{3g}{2l}$

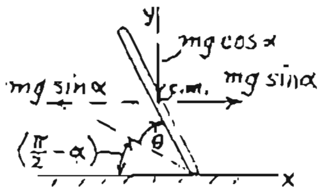
Looking at the rod,

$F_x = m\left(\ddot{x} - \frac{l}{2}\dot{\theta}^2\right) = \frac{3}{4} mg - \frac{3}{2} mg = -\frac{3}{4} mg$

$F_y \frac{l}{2} = \frac{ml^2}{12}\ddot{\theta} = \frac{mgl}{8}, \quad F_y = \frac{1}{4} mg$

7-20. Take a noninertial frame translating parallel to the inclined plane and moving down the plane with the c.m. Take the x axis along the plane and the y axis perpendicular to it, as shown. The c.m. moves along the y axis due to an effective gravity $g\cos\alpha$. The inertia force $mg\sin\alpha$, due to the acceleration of the frame, balances the actual gravitational force component.





Use conservation of energy in the xy frame. The c.m. is at $y=\frac{1}{2}l\cos(\theta-\alpha)$. Then $\dot{y}=-\frac{1}{2}l\dot{\theta}\sin(\theta-\alpha)$.

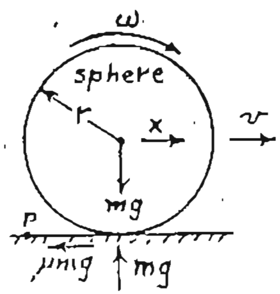
$$T+V=\frac{1}{2}m\left[-\frac{l\dot{\theta}}{2}\sin(\theta-\alpha)\right]^2+\frac{ml^2}{24}\dot{\theta}^2+\frac{1}{2}mgl\cos\alpha\cos(\theta-\alpha)$$
$$=\frac{1}{2}mgl\cos^2\alpha$$

or $$ml^2\dot{\theta}^2\left[\frac{1+3\sin^2(\theta-\alpha)}{24}\right]=\frac{1}{2}mgl\cos\alpha\left[\cos\alpha-\cos(\theta-\alpha)\right]$$

Solving for $\dot{\theta}$, we obtain $$\dot{\theta}=-\sqrt{\frac{12g\cos\alpha\left[\cos\alpha-\cos(\theta-\alpha)\right]}{l\left[1+3\sin^2(\theta-\alpha)\right]}}$$

7-21. $x(0)=0$ $\quad\omega(0)=\omega_0$

$\dot{x}(0)=v_0$ $\quad v_0>r\omega_0$



Conservation of angular momentum about the fixed point P results in

$$\frac{7}{5}mr^2\left(\frac{v_f}{r}\right)=mrv_0+\frac{2}{5}mr^2\omega_0$$

so the final velocity is $$v_f=\frac{5v_0+2r\omega_0}{7}$$

7-21, (cont'd.) (b) $m\ddot{x}=-\mu mg$ or $\ddot{x}=-\mu g$

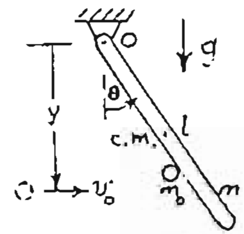
Time of sliding $t=\frac{v_0-v_f}{\mu g}$ and $v_{av}=\frac{v_0+v_f}{2}$

sliding displacement $x=v_{av}t=\frac{v_0^2-v_f^2}{2\mu g}$

$$x=\frac{1}{2\mu g}\left[v_0^2-\frac{(5v_0+2r\omega_0)^2}{49}\right]$$

$$x=\frac{2}{49\mu g}\left(6v_0^2-5r\omega_0v_0-r^2\omega_0^2\right)$$

7-22. The particle must strike the rod at the center of percussion relative to the pivot O.



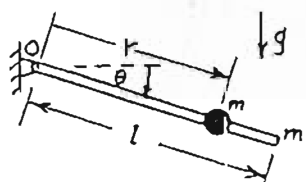
$$\frac{1}{2}l\left(y-\frac{1}{2}l\right)=\frac{l^2}{12} \text{ or } y=\frac{2}{3}l$$

(b) Conserve angular momentum about O during impact.

$$m_0v_0y=\left(\frac{ml^2}{3}+m_0y^2\right)\dot{\theta}$$

$$\dot{\theta}=\frac{\frac{2}{3}m_0v_0l}{\frac{1}{3}ml^2+\frac{4}{9}m_0l^2}=\frac{6m_0v_0}{(3m+4m_0)l}$$

7-23. Use Lagrange's equation.



$$T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+\frac{ml^2}{6}\dot{\theta}^2$$

$V=-mg\left(r+\frac{l}{2}\right)\sin\theta$, $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}=0$

$\frac{\partial T}{\partial\dot{r}}=m\dot{r}$, $\frac{\partial T}{\partial r}=mr\dot{\theta}^2$, $\frac{\partial V}{\partial r}=-mg\sin\theta$

r equation: $m\ddot{r}-mr\dot{\theta}^2-mg\sin\theta=0$

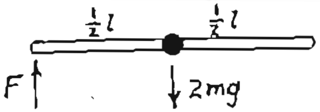
$\frac{\partial T}{\partial\dot{\theta}}=mr^2\dot{\theta}+\frac{ml^2}{3}\dot{\theta}$, $\frac{\partial T}{\partial\theta}=0$, $\frac{\partial V}{\partial\theta}=-mg\left(r+\frac{l}{2}\right)\cos\theta$

θ equation: $m\left(r^2+\frac{1}{3}l^2\right)\ddot{\theta}+2mr\dot{r}\dot{\theta}-mg\left(r+\frac{1}{2}l\right)\cos\theta=0$

-23. (cont'd.) (b) $\theta(0)=0$, $\dot\theta(0)=0$, $r(0)=\frac{1}{2}l$, $\dot r(0)=0$

itial accelerations are

$\ddot r(0)=0$, $\quad \ddot\theta(0) = \dfrac{g}{(\frac{1}{4}+\frac{1}{3})l} = \dfrac{12g}{7l}$

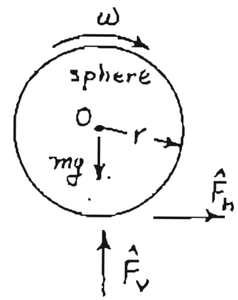


itial c.m. acceleration is

$$\tfrac{1}{2}l\ddot\theta = \tfrac{6}{7}g \text{ downward}. \qquad 2mg - F = 2m\left(\tfrac{6}{7}g\right)$$

$$\underline{F = \tfrac{2}{7}mg \text{ upward}}$$

-24. $\mu = 0.1$, $\quad e = 0.5$



he final motion is rolling with no ipping, so $\quad v = r\omega$

ing linear impulse and momentum, e total horizontal impulse is

$$\hat F_h = mv = mr\omega$$

sing angular impulse and momentum,

$$\hat F_h r = \tfrac{2}{5}mr^2(\omega_0-\omega) \quad \text{or} \quad \tfrac{7}{5}mr^2\omega = \tfrac{2}{5}mr^2\omega_0, \quad \omega = \tfrac{2}{7}\omega_0$$

Final velocity $\underline{v = \frac{2}{7}v_0}$

ouncing time after the first impact is

$$t_b = 2\frac{ev_0}{g} + 2\frac{e^2v_0}{g} + \cdots = \frac{2ev_0}{g}(1+e+e^2+\cdots) = \frac{2v_0}{g}\left(\frac{e}{1-e}\right)$$

or $e = 0.5$, $\quad t_b = \dfrac{2v_0}{g}$

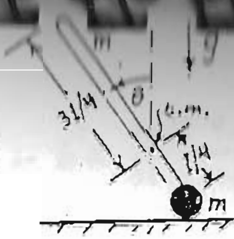
otal vertical impulse up to time t_b exerted by the floor is $\hat F_v$, where $\hat F_v - mgt_b = mv_0$ giving $\hat F_v = 3mv_0$.

f sliding continues until bouncing stops, then we btain $\quad \hat F_h = \mu\hat F_v = 3\mu m v_0 = 0.3\,mv_0$

nd a final velocity $v = 0.3v_0$. This is larger than he actual $v = \frac{2}{7}v_0$, so $\underline{\text{slipping stops first.}}$

7- H_{cm} is conserved during impact,

$$I_{cm} = \frac{ml^2}{12} + m\left(\frac{l}{4}\right)^2 + m\left(\frac{l}{4}\right)^2 = \frac{5}{24}ml^2$$



$$\frac{5}{24}ml^2\dot\theta = m\left(\frac{v_0}{2}\right)\left(\frac{l}{4}\right) + m\left(\frac{v_0}{2}\right)\left(\frac{l}{4}\right) = \frac{1}{4}mv_0l$$

where we note that $v_{cm} = \frac{1}{2}v_0$. Hence, just after impact, $\dot\theta(0^+) = \dfrac{6v_0}{5l}$.

Take an inertial frame translating horizontally with the c.m. at $\frac{1}{2}v_0$. There is conservation of energy after impact. Relative to this frame, the c.m. is at

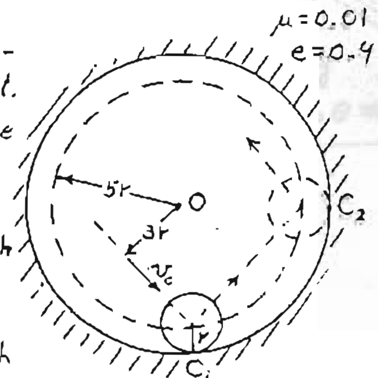
$$y = \tfrac{1}{4}l\cos\theta \quad \text{and then} \quad \dot y = -\tfrac{1}{4}l\dot\theta\sin\theta$$

$$T+V = \tfrac{1}{2}(2m)\left(-\tfrac{1}{4}l\dot\theta\sin\theta\right)^2 + \tfrac{1}{2}\left(\tfrac{5}{24}ml^2\right)\dot\theta^2 + \tfrac{1}{2}mgl\cos\theta = \tfrac{1}{2}\left(\tfrac{5}{24}ml^2\right)\left(\tfrac{6v_0}{5l}\right)^2 + \tfrac{1}{2}mgl$$

or $\left(\frac{5}{48}+\frac{1}{16}\sin^2\theta\right)ml^2\dot\theta^2 = \frac{1}{2}mgl(1-\cos\theta) + \frac{3}{20}mv_0^2$

resulting in $\quad \underline{\dot\theta = \sqrt{\dfrac{12[10gl(1-\cos\theta)+3v_0^2]}{5l^2(5+3\sin^2\theta)}}}$

7-26. At each impact, there is conservation of H about the contact point. Furthermore, this H value is the same for succeeding contact points since the angle of departure from one bounce equals the angle of approach to the next, measured relative to the local radial direction. The radial velocity decreases at each bounce and finally goes to zero. Also, sliding at the wall will stop, resulting in pure rolling along the wall.

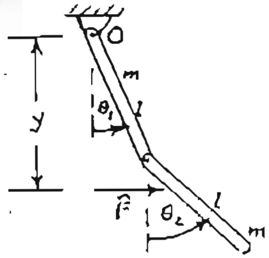


$$H_c = mv_0\left(\tfrac{3}{5}r\right) = \tfrac{3}{2}mr^2\omega, \quad \underline{\omega = \frac{2v_0}{5r} \text{ and } v = r\omega = \frac{2}{5}v_0}$$

7-27. Initially $\theta_1 = \theta_2 = 0$.

Just after the impulse, $\dot\theta_1 = \dot\theta_2 = \dot\theta$

Using angular impulse and momentum about O for the system,



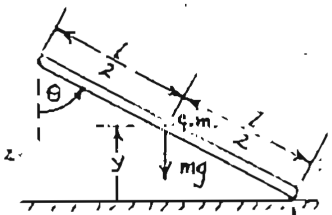
$$\frac{1}{3}(2m)(2l)^2\dot\theta = \hat F y$$

Use angular impulse and momentum for the lower rod with a fixed reference point at the initial position of the joint. This avoids having to calculate the interaction impulse between the rods.

$$\frac{ml^2}{12}\dot\theta + m\frac{l}{2}\left(\frac{3}{2}l\dot\theta\right) = \hat F(y-l) \quad \text{or} \quad \frac{5}{6}ml^2\dot\theta = \hat F(y-l)$$

Divide equations $\frac{16}{5} = \frac{y}{y-l}$ or $\underline{y = \frac{16}{11}l}$

7-28. Choose an inertial frame which translates horizontally with the c.m. Relative to this frame,



$$T = \frac{1}{2}m\dot y^2 + \frac{ml^2}{24}\dot\theta^2 = \frac{ml^2}{24}(1+3\sin^2\theta)\dot\theta^2$$

$$V = mgy = \frac{1}{2}mgl\cos\theta \qquad y = \frac{1}{2}l\cos\theta, \quad \dot y = -\frac{1}{2}l\dot\theta\sin\theta$$

$$\frac{\partial T}{\partial\dot\theta} = \frac{ml^2}{12}\dot\theta(1+3\sin^2\theta), \quad \frac{\partial T}{\partial\theta} = \frac{ml^2\dot\theta^2}{4}\sin\theta\cos\theta, \quad \frac{\partial V}{\partial\theta} = -\frac{1}{2}mgl\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) = \frac{ml^2}{12}\ddot\theta(1+3\sin^2\theta) + \frac{ml^2}{2}\dot\theta^2\sin\theta\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} + \frac{\partial V}{\partial\theta} = 0$$

θ equation: $\frac{ml^2}{12}\ddot\theta(1+3\sin^2\theta) + \frac{ml^2}{4}\dot\theta^2\sin\theta\cos\theta - \frac{1}{2}mgl\sin\theta = 0$

We need to obtain $\dot\theta^2$ as a function of θ. Use conservation of energy. From Example 7-8, the conditions when the bar leaves the wall are $\theta = \cos^{-1}\left(\frac{2}{3}\right)$, $\dot\theta = \sqrt{\frac{g}{l}}$.

7-28. (cont'd.) Thus we obtain

$$\frac{ml^2}{24}\dot\theta^2(1+3\sin^2\theta) + \frac{1}{2}mgl\cos\theta = \frac{4}{9}mgl \quad \text{or} \quad \dot\theta^2 = \frac{4g(8-9\cos\theta)}{3l(1+3\sin^2\theta)}$$

Then the θ equation becomes

$$\frac{ml^2}{12}\ddot\theta(1+3\sin^2\theta) = mgl\sin\theta\left[\frac{1}{2} - \frac{(8-9\cos\theta)\cos\theta}{3(1+3\sin^2\theta)}\right]$$

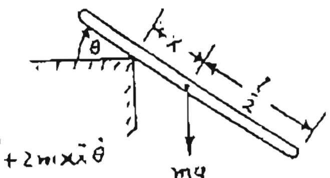
$$= mgl\sin\theta\left[\frac{12-16\cos\theta+9\cos^2\theta}{6(1+3\sin^2\theta)}\right]$$

Solving for $\ddot\theta$, $\underline{\ddot\theta = \frac{2g\sin\theta(12-16\cos\theta+9\cos^2\theta)}{l(1+3\sin^2\theta)^2}}$

(b) The rotational equation is

$$\frac{ml^2}{12}\ddot\theta = \frac{1}{2}F_y l\sin\theta, \quad F_y = \frac{ml\ddot\theta}{6\sin\theta} \quad \text{or} \quad \underline{F_y = \frac{mg(12-16\cos\theta+9\cos^2\theta)}{3(1+3\sin^2\theta)^2}}$$

7-29. $T = \frac{1}{2}m(\dot x^2 + x^2\dot\theta^2) + \frac{ml^2}{24}\dot\theta^2$



$V = -mgx\sin\theta$, $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$

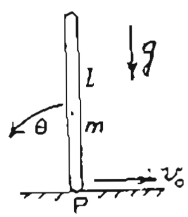
$\frac{\partial T}{\partial\dot\theta} = mx^2\dot\theta + \frac{ml^2}{12}\dot\theta$, $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) = m\left(x^2+\frac{l^2}{12}\right)\ddot\theta + 2mx\dot x\dot\theta$

$\frac{\partial T}{\partial\theta} = 0$, $\frac{\partial V}{\partial\theta} = -mgx\cos\theta$, $\underline{m\left(x^2+\frac{l^2}{12}\right)\ddot\theta + 2mx\dot x\dot\theta - mgx\cos\theta = 0}$

$\frac{\partial T}{\partial\dot x} = m\dot x$, $\frac{\partial T}{\partial x} = mx\dot\theta^2$, $\frac{\partial V}{\partial x} = -mg\sin\theta$

$$\underline{m\ddot x - mx\dot\theta^2 - mg\sin\theta = 0}$$

7-30. Choose an inertial frame translating to the right with velocity v_0. Relative to this frame, the rod translates to the left at v_0 when, suddenly at $t=0$, the lower end P is fixed. Angular momentum about P is conserved during impact.



$$\frac{ml^2}{3}\dot\theta(0^+) = \frac{mlv_0}{2} \quad \text{or} \quad \dot\theta(0^+) = \frac{3v_0}{2l}$$

Using conservation of energy,

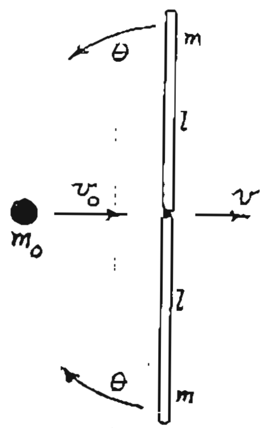
$$\frac{ml^2}{6}\dot\theta^2+\frac{1}{2}mgl\cos\theta=\frac{ml^2}{6}\left(\frac{3v_0}{2l}\right)^2+\frac{1}{2}mgl$$

Solving, $$\dot\theta^2=\frac{9v_0^2}{4l^2}+\frac{3g}{l}(1-\cos\theta)$$

At $\theta=90^\circ$, $$\dot\theta=\frac{3v_0}{2l}\sqrt{1+\frac{4gl}{3v_0^2}}$$

7-31. By symmetry, the angular velocities of the two rods are equal and opposite. Each rod receives an impulse at its pivot end and begins to rotate about the center of percussion at $\frac{2}{3}l$ from the pivot. Hence

$$v=\frac{2}{3}l\dot\theta \quad\text{or}\quad \dot\theta=\frac{3v}{2l}$$



Using conservation of linear momentum,

$$m_0v_0=m_0v+2m\left(v-\frac{1}{2}l\dot\theta\right)$$
$$=m_0v+2m\left(\frac{1}{4}v\right) \quad\text{or}\quad v=\frac{2m_0v_0}{2m_0+m}$$

$$\dot\theta=\frac{3v}{2l}=\left(\frac{3m_0}{2m_0+m}\right)\frac{v_0}{l}$$

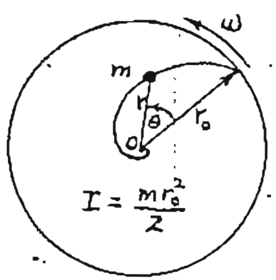
7-32. $\omega(0)=0 \qquad \theta(0)=0 \qquad r(0)=r_0$

$r=r_0e^{-\frac{1}{2}\theta} \qquad \dot\theta(0)=\dot\theta_0$

$\dot r=-\frac{1}{2}r_0\dot\theta e^{-\frac{1}{2}\theta}=-\frac{1}{2}r\dot\theta, \qquad \dot r(0)=-\frac{1}{2}r_0\dot\theta_0$

Using conservation of H about O,

$$(I+mr^2)\omega+mr^2\dot\theta=mr_0^2\dot\theta_0$$



$$\frac{1}{2}I\omega^2+\frac{1}{2}m\left[\dot r^2+r^2(\omega+\dot\theta)^2\right]=\frac{5}{8}mr_0^2\dot\theta_0^2$$

When $r=r_{min}$, $\dot r=0$ and $\dot\theta=0$, so

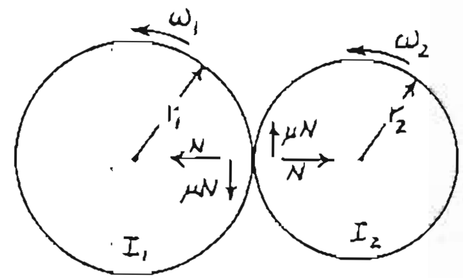
$$(I+mr^2)\omega=mr_0^2\dot\theta_0$$
$$\frac{1}{2}(I+mr^2)\omega^2=\frac{5}{8}mr_0^2\dot\theta_0^2$$

Dividing these equations, $\omega=\frac{5}{4}\dot\theta_0$

Then $$r^2=\frac{1}{m}\left(mr_0^2\frac{\dot\theta_0}{\omega}-I\right)=\frac{3}{10}r_0^2, \qquad r_{min}=\sqrt{0.3}\,r_0=0.5477r_0$$

7-33. ω_1 and ω_2 are initial angular velocities. Let ω_{1f} and ω_{2f} be the corresponding final values when sliding stops at time t.

Using angular impulse and momentum,



$$I_1(\omega_1-\omega_{1f})=\mu Nr_1t \quad\text{and}\quad I_2(\omega_2-\omega_{2f})=\mu Nr_2t$$

Dividing, $$\frac{I_1(\omega_1-\omega_{1f})}{I_2(\omega_2-\omega_{2f})}=\frac{r_1}{r_2}$$

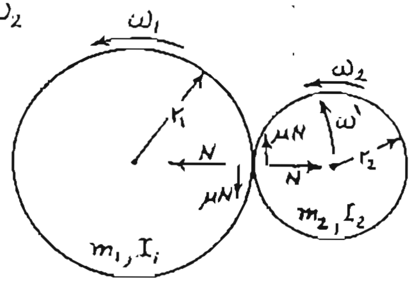
When slipping stops, $r_1\omega_{1f}=-r_2\omega_{2f}$ or $\omega_{2f}=-\frac{r_1}{r_2}\omega_{1f}$

$$I_1r_2(\omega_1-\omega_{1f})=I_2r_1\left(\omega_2+\frac{r_1}{r_2}\omega_{1f}\right) \text{ or } \left(I_1r_2+I_2\frac{r_1^2}{r_2}\right)\omega_{1f}=I_1r_2\omega_1-I_2r_1\omega_2$$

Solving,
$$\omega_{1f}=\frac{r_2(I_1r_2\omega_1-I_2r_1\omega_2)}{I_1r_2^2+I_2r_1^2} \quad\text{and similarly}\quad \omega_{2f}=\frac{r_1(I_2r_1\omega_2-I_1r_2\omega_1)}{I_1r_2^2+I_2r_1^2}$$

(b) $$t=\frac{I_1(\omega_1-\omega_{1f})}{\mu Nr_1}=\frac{I_1I_2(r_1\omega_1+r_2\omega_2)}{\mu N(I_1r_2^2+I_2r_1^2)}$$

7-34. Given that ω_1 and ω_2 are initial values. Let ω_{1f}, ω_{2f}, and ω' be final values, where ω' is the absolute angular velocity of the line of centers,



$$\frac{m_2}{m_1} = \frac{r_1}{r_2} = 2, \quad I_1 = I_2 = \frac{1}{2} m_1 r_1^2$$

Since $m_1 r_1 = m_2 r_2$, the c.m. is at the contact point. Let $v_{cm} = 0$.
Let t be the time to stop slipping. Using angular impulse and momentum on each wheel separately.

$$\left.\begin{aligned} I_1(\omega_1 - \omega_{1f}) &= \mu N r_1 t \\ I_2(\omega_2 - \omega_{2f}) &= \mu N r_2 t \end{aligned}\right\} I_1 r_2(\omega_1 - \omega_{1f}) = I_2 r_1(\omega_2 - \omega_{2f})$$

Angular momentum is conserved about the c.m.

$$I_1\omega_{1f} + I_2\omega_{2f} + (m_1 r_1^2 + m_2 r_2^2)\omega' = I_1\omega_1 + I_2\omega_2$$

Slipping stops when the contact points on the two wheels have identical velocities.

$$r_1(\omega_{1f} - \omega') = -r_2(\omega_{2f} - \omega')$$

Using the given parameter ratios, the three above equations become

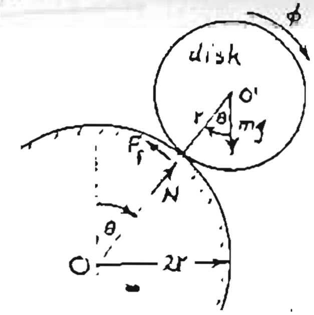
$$\left.\begin{aligned} \omega_1 - \omega_{1f} &= 2(\omega_2 - \omega_{2f}) \\ \omega_{1f} + \omega_{2f} + 3\omega' &= \omega_1 + \omega_2 \\ 2\omega_{1f} + \omega_{2f} - 3\omega' &= 0 \end{aligned}\right\} \text{Solving,} \quad \begin{aligned} \omega_{1f} &= \tfrac{1}{2}\omega_1 - \tfrac{1}{4}\omega_2 \\ \omega_{2f} &= -\tfrac{1}{4}\omega_1 + \tfrac{7}{8}\omega_2 \end{aligned}$$

Also $\omega' = \frac{1}{4}\omega_1 + \frac{1}{8}\omega_2$

(b) Time to stop slipping is

$$t = \frac{I_1(\omega_1 - \omega_{1f})}{\mu N r_1} = \frac{m_1 r_1}{2\mu N}\left(\tfrac{1}{2}\omega_1 + \tfrac{1}{4}\omega_2\right) = \frac{m r_1}{8\mu N}(2\omega_1 + \omega_2)$$

7-35. $v_{O'} = 3r\dot\theta = r\dot\phi$ so $\dot\phi = 3\dot\theta$



$$T = \frac{1}{2}\left(\frac{3}{2}mr^2\right)\dot\phi^2 = \frac{27}{4}mr^2\dot\theta^2$$

$$V = 3mgr\cos\theta$$

Use conservation of energy.

$$\frac{27}{4}mr^2\dot\theta^2 + 3mgr\cos\theta = 3mgr$$

giving $\dot\theta^2 = \frac{4g}{9r}(1 - \cos\theta)$

For radial components, $N - mg\cos\theta = -3mr\dot\theta^2$

$$N = mg\left[\cos\theta - \frac{4}{3}(1 - \cos\theta)\right] = \frac{mg}{3}(7\cos\theta - 4)$$

Disk leaves sphere when $N = 0$, going negative.

$$\theta = \cos^{-1}\frac{4}{7} = 55.15°$$

(b) $\frac{\partial T}{\partial\dot\theta} = \frac{27}{2}mr^2\dot\theta$, $\frac{\partial T}{\partial\theta} = 0$, $\frac{\partial V}{\partial\theta} = -3mgr\sin\theta$

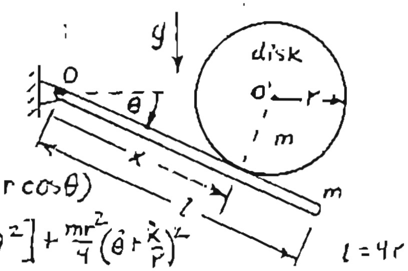
θ equation: $\frac{27}{2}mr^2\ddot\theta - 3mgr\sin\theta = 0$

Disk rotational equation: $\frac{mr^2}{2}\ddot\phi = F_f r$ where $\ddot\phi = 3\ddot\theta$

Hence $F_f = \frac{3}{2}mr\ddot\theta = \frac{1}{3}mg\sin\theta$

7-36. The absolute angular velocity of the disk is

$$\omega_d = \dot\theta + \frac{\dot x}{r}$$



$$V = -2mgr\sin\theta + mg(-x\sin\theta + r\cos\theta)$$

$$T = \frac{1}{2}m\frac{(4r)^2}{3}\dot\theta^2 + \frac{1}{2}m\left[(\dot x + r\dot\theta)^2 + x^2\dot\theta^2\right] + \frac{mr^2}{4}\left(\dot\theta + \frac{\dot x}{r}\right)^2$$

$$= \frac{3}{4}m\dot x^2 + \frac{41}{12}mr^2\dot\theta^2 + \frac{3}{2}mr\dot x\dot\theta + \frac{1}{2}mx^2\dot\theta^2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$$

7-36. (cont'd.)

$\frac{\partial T}{\partial \dot x} = \frac{3}{2}m\dot x + \frac{3}{2}mr\dot\theta$, $\frac{\partial T}{\partial x} = mx\dot\theta^2$, $\frac{\partial V}{\partial x} = -mg\sin\theta$

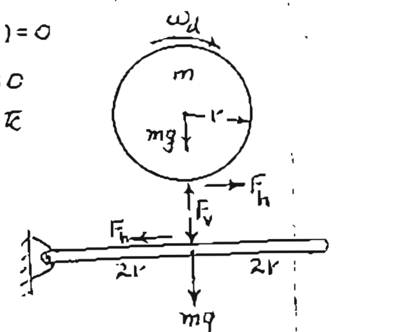
x equation: $\frac32 m\ddot x + \frac32 mr\ddot\theta - mx\dot\theta^2 - mg\sin\theta = 0$

$\frac{\partial T}{\partial\dot\theta} = \frac{41}{6}mr^2\dot\theta + \frac32 mr\dot x + mx^2\dot\theta$, $\frac{\partial T}{\partial\theta}=0$, $\frac{\partial V}{\partial\theta} = mg[-2r\cos\theta - x\cos\theta - r\sin\theta]$

$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) = m\left(\frac{41}{6}r^2+x^2\right)\ddot\theta + \frac32 mr\ddot x + 2mx\dot x\dot\theta$

θ equation: $m\left(\frac{41}{6}r^2+x^2\right)\ddot\theta + \frac32 mr\ddot x + 2mx\dot x\dot\theta - mg[(2r+x)\cos\theta + r\sin\theta] = 0$

(b) $\theta(0)=0$, $\dot\theta(0)=0$, $x(0)=2r$, $\dot x(0)=0$



From x equation, $\ddot x + r\ddot\theta = 0$ or $\dot\omega_d = 0$ at $t=0$. Hence, the moment applied to the disk is zero, or $F_h = 0$.

From the θ equation, at $t=0$,

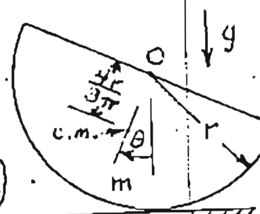
$\left(\frac{41}{6}+4\right)r\ddot\theta + \frac32\ddot x = 4g$

or $\left(\frac{65}{6}-\frac32\right)r\ddot\theta = 4g$, $\ddot\theta = \frac{3g}{7r}$

Rotational equation of rod: $\frac{16}{3}mr^2\left(\frac{3g}{7r}\right) = (F_v+mg)2r$, $F_v = \frac17 mg$ downward on rod

7-37. $I_o = \frac12 mr^2$, $I_{cm} = mr^2\left[\frac12 - \left(\frac{4}{3\pi}\right)^2\right]$

(a) No slipping. $\theta \ll 1$. $v_{cm} = r\left(1-\frac{4}{3\pi}\right)\dot\theta$



$T = \frac12 mv_{cm}^2 + \frac12 I_{cm}\dot\theta^2$

$= \frac12 mr^2\dot\theta^2\left[\left(1-\frac{4}{3\pi}\right)^2 + \frac12 - \left(\frac{4}{3\pi}\right)^2\right] = mr^2\dot\theta^2\left(\frac34 - \frac{4}{3\pi}\right)$

$V = -\frac{4}{3\pi}mgr\cos\theta$, $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} + \frac{\partial V}{\partial\theta} = 0$

$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) = \left(\frac32 - \frac{8}{3\pi}\right)mr^2\ddot\theta$, $\frac{\partial T}{\partial\theta}=0$, $\frac{\partial V}{\partial\theta} = \frac{4}{3\pi}mgr\sin\theta \cong \frac{4}{3\pi}mgr\theta$

7-37. (cont'd.) The θ equation is

$\left(\frac32 - \frac{8}{3\pi}\right)mr^2\ddot\theta + \frac{4}{3\pi}mgr\theta = 0$ or $\ddot\theta + \left(\frac{8}{9\pi-16}\right)\frac{g}{r}\theta = 0$

or $\ddot\theta + 0.6518\frac{g}{r}\theta = 0$

(b) Smooth plane. The c.m. moves vertically since the horizontal force is zero. The height of the c.m. is

$y = r\left(1-\frac{4}{3\pi}\cos\theta\right)$ so $\dot y = \frac{4r}{3\pi}\dot\theta\sin\theta \cong \frac{4r}{3\pi}\theta\dot\theta$

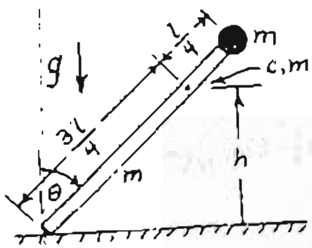
$\dot y^2$ is fourth order in small quantities and can be neglected in the kinetic energy expression. V is unchanged.

$T = \frac12 mr^2\dot\theta^2\left[\frac12 - \left(\frac{4}{3\pi}\right)^2\right]$, $\frac{\partial T}{\partial\dot\theta} = mr^2\dot\theta\left[\frac12-\left(\frac{4}{3\pi}\right)^2\right]$

θ equation: $\left[\frac12 - \left(\frac{4}{3\pi}\right)^2\right]mr^2\ddot\theta + \frac{4}{3\pi}mgr\theta = 0$

or $\ddot\theta + \left(\frac{24\pi}{9\pi^2-32}\right)\frac{g}{r}\theta = 0$ or $\ddot\theta + 1.3268\frac{g}{r}\theta = 0$

7-38. $h = \frac34 l\cos\theta$, $\dot h = -\frac34 l\dot\theta\sin\theta$



$I_{cm} = \frac{ml^2}{12} + m\left(\frac{l}{4}\right)^2 + m\left(\frac{l}{4}\right)^2 = \frac{5}{24}ml^2$

$T = m\left(\frac34 l\dot\theta\sin\theta\right)^2 + \frac{5}{48}ml^2\dot\theta^2$

$= \frac{ml^2\dot\theta^2}{48}(5+27\sin^2\theta)$

$V = \frac32 mgl\cos\theta$ Use conservation of energy. $\theta(0)=0$, $\dot\theta(0)=0$

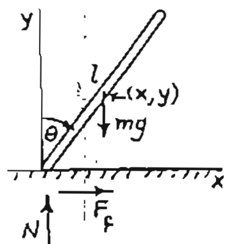
$\frac{ml^2\dot\theta^2}{48}(5+27\sin^2\theta) + \frac32 mgl\cos\theta = \frac32 mgl$

$\dot\theta = \sqrt{\frac{72g}{l}\left(\frac{1-\cos\theta}{5+27\sin^2\theta}\right)}$

Note: As θ increases from 0 to π/2, $\dot\theta$ increases continuously, so the applied moment is positive (clockwise) throughout. Hence, the rod does not leave the floor.

7-39. Previous to slipping, $M = \dot{H}$ about the contact point.

$$\frac{ml^2}{3}\ddot{\theta} = \frac{1}{2}mgl\sin\theta, \quad \ddot{\theta} = \frac{3g}{2l}\sin\theta$$



Use conservation of energy. $\theta(0)$, $\dot{\theta}(0) = 0$.

$$\frac{ml^2}{6}\dot{\theta}^2 + \frac{1}{2}mgl\cos\theta = \frac{1}{2}mgl, \quad \dot{\theta}^2 = \frac{3g}{l}(1-\cos\theta)$$

$$y = \frac{1}{2}l\cos\theta, \quad \dot{y} = -\frac{1}{2}l\dot{\theta}\sin\theta, \quad \ddot{y} = -\frac{1}{2}l\ddot{\theta}\sin\theta - \frac{1}{2}l\dot{\theta}^2\cos\theta$$

Vertical motion: $N - mg = m\ddot{y} = -\frac{1}{2}ml(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$

$$= -\frac{3mg}{2}\left[\frac{\sin^2\theta}{2} + \cos\theta(1-\cos\theta)\right]$$

or $N = mg\left[1 - \frac{3}{4}(1-\cos^2\theta) + \frac{3}{2}\cos^2\theta - \frac{3}{2}\cos\theta\right] = \frac{mg}{4}(1-3\cos\theta)^2$

The condition for incipient slipping to the left is that

$$F_f = \mu N = m\ddot{x} = \frac{ml}{2}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$$

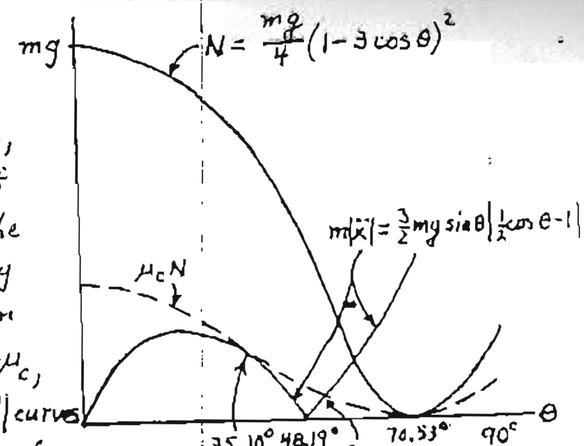
$$= \frac{mg}{2}\left[\frac{3}{2}\sin\theta\cos\theta - 3\sin\theta(1-\cos\theta)\right] = \frac{3mg}{2}\sin\theta\left(\frac{3}{2}\cos\theta - 1\right)$$

Solving for μ, $\mu = \dfrac{6\sin\theta\left(\frac{3}{2}\cos\theta - 1\right)}{(1-3\cos\theta)^2}$ For $\theta = 30°$: $\mu = \dfrac{9\sqrt{3}-12}{31-12\sqrt{3}} = \underline{0.3513}$

(b) Assuming $0 < \theta < \frac{\pi}{2}$, slipping to the left implies F_f and $\ddot{x}$ are positive, or $\frac{3}{2}\cos\theta - 1 > 0$, or $\cos\theta > \frac{2}{3}$ or $\theta < 48.19°$. On the other hand, slipping to the right implies that F_f and $\ddot{x}$ are negative, or $\cos\theta < \frac{2}{3}$, or $\theta > 48.19°$. Now slipping begins when $m|\ddot{x}|$ first equals μN, and N goes to zero at $\cos\theta = \frac{1}{3}$ or $\theta = 70.53°$. Hence slipping must begin in the range $0 < \theta \le 70.53°$.

The possibilities are best visualized by plotting N and $m|\ddot{x}|$ versus θ. If $\mu < \mu_c$, then the μN curve

7-39. (cont'd.)



intersects the $m|\ddot{x}|$ curve in three places, and the smallest of these values of θ is the angle at which sliding to the left begins. On the other hand, if $\mu > \mu_c$, then the μN and $m|\ddot{x}|$ curves intersect only once at the value of θ where sliding to the right begins. The critical value $\mu = \mu_c$ that divides the two regimes is found by equating slopes of the μN and $m|\ddot{x}|$ curves at some value of θ corresponding to the tangency point.

$$\mu\frac{dN}{d\theta} = m\frac{d\ddot{x}}{d\theta} \quad\text{or}\quad \frac{3}{2}\mu mg\sin\theta(1-3\cos\theta) = \frac{3}{2}mg\left(\frac{1}{2}\cos^2\theta - \frac{3}{2}\sin^2\theta - \cos\theta\right)$$

where $\mu = \dfrac{6\sin\theta\left(\frac{3}{2}\cos\theta - 1\right)}{(1-3\cos\theta)^2}$

This results in $9\cos\theta - 6 = \frac{1}{2}\cos\theta - \frac{3}{2}$ or $\cos\theta = \frac{9}{11}$, $\theta = 35.10$

Then $\mu_c = \dfrac{15\sqrt{10}}{128} = 0.3706$.

Now consider the intersection of the $\mu_c N$ curve and the right-hand branch of the $m|\ddot{x}|$ curve. We have

$$\mu_c = \frac{6\sin\theta\left(1 - \frac{3}{2}\cos\theta\right)}{(1-3\cos\theta)^2} = \frac{15\sqrt{10}}{128}. \quad \text{Solving,} \quad \theta = 51.25°$$

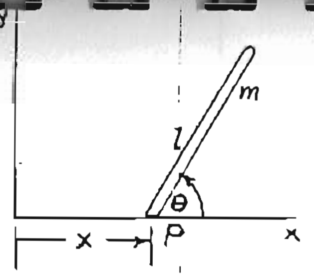
Summary: For $\mu \le 0.3706$, sliding occurs first to the left at an angle $0 < \theta \le 35.10°$. For $\mu > 0.3706$, sliding occurs first to the right for $51.25° < \theta < 70.53°$. Sliding cannot begin for $\underline{35.10° < \theta < 51.25°}$ or $\underline{\theta > 70.53°}$ regardless of the value of μ.

7-40. From (4-11),

$$T=\frac{1}{2}m\dot{\bar r}_p^2+\frac{1}{2}\sum_{i=1}^{n}m_i\dot{\bar\rho}_i^2+\dot{\bar r}_p\cdot m\dot{\bar\rho}_c$$

or, for this case,

$$T=\frac{1}{2}m\dot x^2+\frac{ml^2}{6}\dot\theta^2-m\frac{l}{2}\dot x\dot\theta\sin\theta$$



$$V=0,\quad \frac{d}{dt}\left(\frac{\partial T}{\partial\dot q_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}=0$$

$$\frac{\partial T}{\partial\dot x}=m\dot x-\frac{1}{2}ml\dot\theta\sin\theta,\quad \frac{\partial T}{\partial x}=0,\quad \frac{\partial V}{\partial x}=0$$

x equation: $\frac{d}{dt}\left[m(\dot x-\frac{1}{2}l\dot\theta\sin\theta)\right]=0$ or $\ddot x-\frac{1}{2}l(\ddot\theta\sin\theta+\dot\theta^2\cos\theta)=0$

$$\frac{\partial T}{\partial\dot\theta}=\frac{ml^2}{3}\dot\theta-\frac{1}{2}ml\dot x\sin\theta,\quad \frac{\partial T}{\partial\theta}=-\frac{1}{2}ml\dot x\dot\theta\cos\theta,\quad \frac{\partial V}{\partial\theta}=0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right)=\frac{ml^2}{3}\ddot\theta-\frac{1}{2}ml\ddot x\sin\theta-\frac{1}{2}ml\dot x\dot\theta\cos\theta$$

θ equation: $\frac{ml^2}{3}\ddot\theta-\frac{1}{2}ml\ddot x\sin\theta=0$ or $\ddot\theta-\frac{3}{2l}\ddot x\sin\theta=0$

(b) $\theta(0)=0$, $\dot\theta(0)=\dot\theta_0$. From the x equation,

$p_x=m(\dot x-\frac{1}{2}l\dot\theta\sin\theta)=\text{const.}=m\dot x_0$ where $\dot x(0)=\dot x_0$.

Then $\dot x=\dot x_0+\frac{1}{2}l\dot\theta\sin\theta$ Using conservation of energy,

$$T=\frac{1}{2}m\left(\dot x_0^2+l\dot x_0\dot\theta\sin\theta+\frac{1}{4}l^2\dot\theta^2\sin^2\theta\right)+\frac{ml^2}{6}\dot\theta^2-\frac{1}{2}ml\dot\theta\sin\theta\left(\dot x_0+\frac{1}{2}l\dot\theta\sin\theta\right)=\frac{ml^2}{6}\dot\theta_0^2+\frac{1}{2}m\dot x_0^2$$

Note that the $\dot x_0$ terms cancel, so the θ motion is independent of $\dot x_0$.

There remains $ml^2\dot\theta^2\left(\frac{1}{8}\sin^2\theta-\frac{1}{4}\sin^2\theta+\frac{1}{6}\right)=\frac{ml^2}{6}\dot\theta_0^2$, or

$$\dot\theta=\frac{\dot\theta_0}{\sqrt{1-\frac{3}{4}\sin^2\theta}}$$

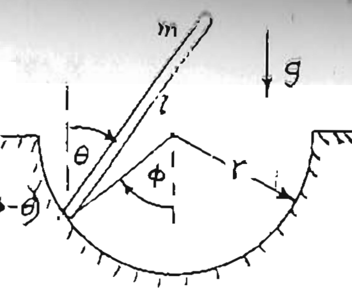
7-41. The velocity of the c.m. is given by

$$v^2=r^2\dot\phi^2+\frac{1}{4}l^2\dot\theta^2-rl\dot\theta\dot\phi\cos(\phi-\theta)$$

so

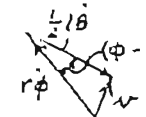
$$T=\frac{m}{2}\left[r^2\dot\phi^2+\frac{1}{4}l^2\dot\theta^2-rl\dot\theta\dot\phi\cos(\phi-\theta)\right]+\frac{ml^2}{24}\dot\theta^2$$

or $T=\frac{ml^2}{6}\dot\theta^2+\frac{1}{2}mr^2\dot\phi^2-\frac{1}{2}mrl\dot\theta\dot\phi\cos(\phi-\theta)$



$$V=\frac{1}{2}mgl\cos\theta-mgr\cos\phi$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot q_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}=0$$



$$\frac{\partial T}{\partial\dot\theta}=\frac{ml^2}{3}\dot\theta-\frac{1}{2}mrl\dot\phi\cos(\phi-\theta),\quad \frac{\partial T}{\partial\theta}=-\frac{1}{2}mrl\dot\theta\dot\phi\sin(\phi-\theta)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right)=\frac{ml^2}{3}\ddot\theta-\frac{1}{2}mrl\ddot\phi\cos(\phi-\theta)+\frac{1}{2}mrl\dot\phi(\dot\phi-\dot\theta)\sin(\phi-\theta)$$

$$\frac{\partial V}{\partial\theta}=-\frac{1}{2}mgl\sin\theta$$

θ equation is

$$\frac{ml^2}{3}\ddot\theta-\frac{1}{2}mrl\ddot\phi\cos(\phi-\theta)+\frac{1}{2}mrl\dot\phi^2\sin(\phi-\theta)-\frac{1}{2}mgl\sin\theta=0$$

$$\frac{\partial T}{\partial\dot\phi}=mr^2\dot\phi-\frac{1}{2}mrl\dot\theta\cos(\phi-\theta),\quad \frac{\partial T}{\partial\phi}=\frac{1}{2}mrl\dot\theta\dot\phi\sin(\phi-\theta)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\phi}\right)=mr^2\ddot\phi-\frac{1}{2}mrl\ddot\theta\cos(\phi-\theta)+\frac{1}{2}mrl\dot\theta(\dot\phi-\dot\theta)\sin(\phi-\theta)$$

$$\frac{\partial V}{\partial\phi}=mgr\sin\phi$$

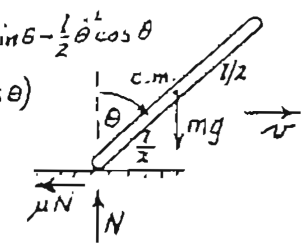
φ equation is

$$mr^2\ddot\phi-\frac{1}{2}mrl\ddot\theta\cos(\phi-\theta)-\frac{1}{2}mrl\dot\theta^2\sin(\phi-\theta)+mgr\sin\phi=0$$

7-42. Vertical position of c.m. is

$y=\frac{1}{2}l\cos\theta$. $\dot y=-\frac{1}{2}l\dot\theta\sin\theta$, $\ddot y=-\frac{l}{2}\ddot\theta\sin\theta-\frac{l}{2}\dot\theta^2\cos\theta$

$N-mg=m\ddot y$ or $N=mg-\frac{ml}{2}(\ddot\theta\sin\theta+\dot\theta^2\cos\theta)$



Rotational equation is

$$\frac{ml^2}{12}\ddot\theta=\frac{1}{2}Nl(\sin\theta+\mu\cos\theta)$$

7-42. (cont'd.) Substituting the expression for N,

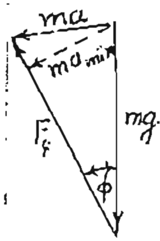
$$\frac{ml^2}{12}\ddot{\theta} = \frac{l}{2}(\sin\theta+\mu\cos\theta)\left(mg-\frac{ml}{2}\ddot{\theta}\sin\theta-\frac{ml}{2}\dot{\theta}^2\cos\theta\right)$$

or
$$\frac{ml^2}{12}\ddot{\theta}\left(1+3\sin^2\theta+3\mu\sin\theta\cos\theta\right)+\frac{ml^2}{4}\dot{\theta}^2\cos\theta(\sin\theta+\mu\cos\theta)$$
$$-\frac{1}{2}mgl(\sin\theta+\mu\cos\theta)=0$$

(b) When $\theta=0$, $N=mg$ since $\dot{\theta}(0)=0$ and $\theta(0)=0$.

When $\theta=90°$, $N=\frac{1}{4}mg$ since $\ddot{\theta}=\frac{3g}{2l}$

The total floor reaction force F_f is the sum of N and μN and lies on the cone of friction with a constant angle ϕ with the vertical and a decreasing magnitude as N decreases.



$\phi=\tan^{-1}\mu$

Initially, $\theta=0$, $N=mg$, $F_f=\sqrt{1+\mu^2}\,mg$

$a=\mu g$ and is horizontal to the left.

At $\theta=90°$, $N=\frac{1}{4}mg$, $F_f=\sqrt{1+\mu^2}\,\frac{mg}{4}$, $a=\sqrt{\left(\frac{3}{4}g\right)^2+\left(\frac{\mu g}{4}\right)^2}$
$=\sqrt{9+\mu^2}\;g/4$

$a_{min}=g\sin\phi=\frac{\mu}{\sqrt{1+\mu^2}}g$ for $0<\mu<\sqrt{3}$

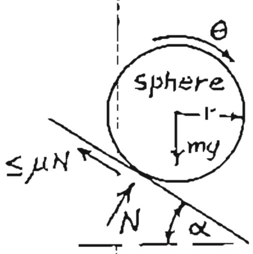
For $\mu=\frac{1}{2}$, $a_{min}=\frac{0.5}{\sqrt{1.25}}g=\underline{0.4472\,g}$ when $N=\frac{4}{5}mg$

7-43. The rotational equation about the contact point is

$$\frac{7}{5}mr^2\ddot{\theta}=mgr\sin\alpha$$

or
$$\ddot{\theta}=\frac{5g}{7r}\sin\alpha$$

assuming no slipping.



7-43. (cont'd.) $N=mg\cos\alpha$

For incipient slipping, the rotational equation about the c.m. is

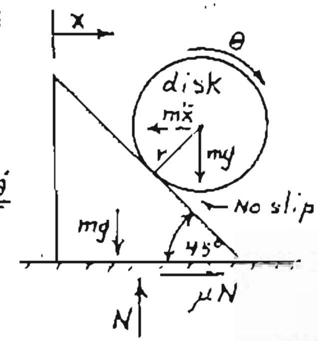
$$\frac{2}{5}mr^2\ddot{\theta}=\mu Nr=\mu mgr\cos\alpha$$
$$\ddot{\theta}=\frac{5\mu g}{2r}\cos\alpha$$

Equating the two expressions for $\ddot{\theta}$,

$$\frac{5g}{7r}\sin\alpha=\frac{5\mu g}{2r}\cos\alpha \quad\text{or}\quad \underline{\mu_{min}=\frac{2}{7}\tan\alpha}$$

7-44. The disk rolls downward and to the right. The block slides to the left. The vertical acceleration equation is

$$N-2mg=\frac{-mr\ddot{\theta}}{\sqrt{2}},\quad N=2mg-\frac{mr\ddot{\theta}}{\sqrt{2}}$$



The horizontal acceleration equation is

$$m\ddot{x}+m\left(\ddot{x}+\frac{r\ddot{\theta}}{\sqrt{2}}\right)=\mu N$$

or, substituting for N, $2m\ddot{x}+mr\ddot{\theta}\left(\frac{1+\mu}{\sqrt{2}}\right)=2\mu mg$

To avoid solving for the interaction forces between the disk and the block, take a reference frame fixed in the block, and include an inertia force at the c.m. of the disk. Then the rotation equation about the contact point is

$$\frac{3}{2}mr^2\ddot{\theta}=m(g-\ddot{x})\frac{r}{\sqrt{2}},\quad\text{or}\quad \ddot{x}=g-\frac{3}{\sqrt{2}}r\ddot{\theta}$$

Substitute for $\ddot{x}$ in the horizontal acceleration equation, obtaining $2g-3\sqrt{2}\,r\ddot{\theta}+\left(\frac{1+\mu}{\sqrt{2}}\right)r\ddot{\theta}=2\mu g$ or $\underline{\ddot{\theta}=2\sqrt{2}\left(\frac{1-\mu}{5-\mu}\right)\frac{g}{r}}$

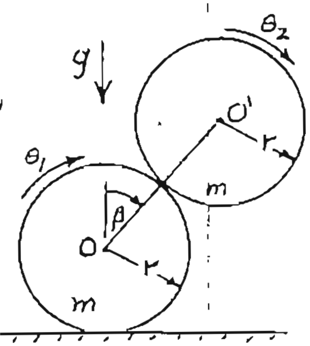
Then
$$\ddot{x}=\left[1-6\left(\frac{1-\mu}{5-\mu}\right)\right]g=\underline{-\left(\frac{1-5\mu}{5-\mu}\right)g}$$

-44. (cont'd.)

b) From the expression for $\ddot{x}$, the upper limit on μ is $\frac{1}{5}$ if there is no contradiction with the original assumption that the block slides to the left.

The range of μ for slipping of block is $\underline{0 \le \mu < 0.20}$

-45. Let $\dot\theta_2$ = clockwise angular velocity of the upper disk



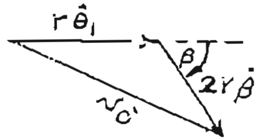
$v_{O'/O} = 2r\dot\beta = r\dot\theta_1 + r\dot\theta_2, \quad \dot\theta_2 = 2\dot\beta - \dot\theta_1$

$T = \frac{1}{2}\left(\frac{3}{2}mr^2\right)\dot\theta_1^2 + \frac{1}{2}\left(\frac{mr^2}{2}\right)(2\dot\beta - \dot\theta_1)^2$
$\quad + \frac{1}{2}m\left[r^2\dot\theta_1^2 + 4r^2\dot\beta^2 + 4r^2\dot\theta_1\dot\beta\cos\beta\right]$

$V = 2mgr\cos\beta$

Simplifying the T expression,

$T = mr^2\left[\frac{3}{2}\dot\theta_1^2 + 3\dot\beta^2 + \dot\theta_1\dot\beta(2\cos\beta - 1)\right]$



$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0, \quad \frac{\partial T}{\partial \theta_1} = \frac{\partial V}{\partial \theta_1} = 0$

$\frac{\partial T}{\partial \dot\theta_1} = 3mr^2\dot\theta_1 + mr^2\dot\beta(2\cos\beta - 1)$

θ_1 equation: $\underline{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta_1}\right) = 3mr^2\ddot\theta_1 + mr^2\ddot\beta(2\cos\beta - 1) - 2mr^2\dot\beta^2\sin\beta = 0}$

$\frac{\partial T}{\partial \dot\beta} = 6mr^2\dot\beta + mr^2\dot\theta_1(2\cos\beta - 1), \quad \frac{\partial T}{\partial \beta} = -2mr^2\dot\theta_1\dot\beta\sin\beta$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\beta}\right) = 6mr^2\ddot\beta + mr^2\ddot\theta_1(2\cos\beta - 1) - 2mr^2\dot\theta_1\dot\beta\sin\beta$

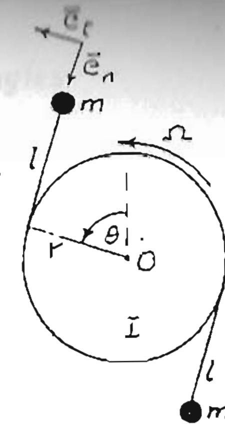
$\frac{\partial V}{\partial \beta} = -2mgr\sin\beta$

β equation: $\underline{6mr^2\ddot\beta + mr^2\ddot\theta_1(2\cos\beta - 1) - 2mgr\sin\beta = 0}$

b) For $|\beta| << 1$, the linearized θ_1 equation is

$3mr^2\ddot\theta_1 + mr^2\ddot\beta = 0$ resulting in $\underline{\frac{\ddot\beta}{\ddot\theta_1} = -3}$

7-46. Let θ be the unwinding angle of the rope relative to the circular cross-section. $l = r\theta$.



The absolute angular velocity of the straight portion of the rope is

$$\omega = \dot\theta + \Omega = \frac{\dot l}{r} + \Omega$$

The velocity of a particle is

$\bar v = r\Omega\bar e_n + l\omega\bar e_t = r\Omega\bar e_n + \left(\frac{l\dot l}{r} + l\Omega\right)\bar e_t$

$T = mv^2 + \frac{1}{2}I\Omega^2, \quad V = 0, \quad \Omega(0) = \Omega_0, \quad l(0) = 0$

Use conservation of energy.

$$\frac{1}{2}I\Omega^2 + m\left[r^2\Omega^2 + \left(\frac{l\dot l}{r} + l\Omega\right)^2\right] = \frac{1}{2}(I + 2mr^2)\Omega_0^2$$

Using conservation of angular momentum about O,

$$(I + 2mr^2)\Omega + 2ml^2\left(\frac{\dot l}{r} + \Omega\right) = (I + 2mr^2)\Omega_0^2$$

When $\Omega = 0$, we have

$$\frac{ml^2\dot l^2}{r^2} = \frac{1}{2}(I + 2mr^2)\Omega_0^2$$

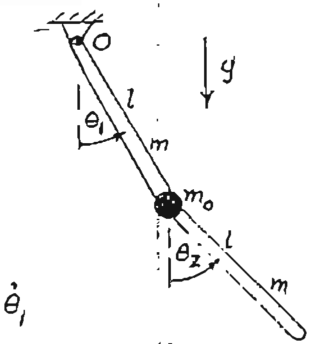
$$\frac{2ml^2\dot l}{r} = (I + 2mr^2)\Omega_0$$

Dividing these two equations, $\frac{\dot l}{2r} = \frac{\Omega_0}{2}$ or $\dot l = r\Omega_0$

Then $2ml^2\Omega_0 = (I + 2mr^2)\Omega_0$ or $\underline{l = \sqrt{\frac{I + 2mr^2}{2m}}}$

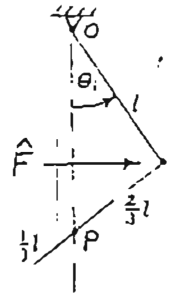
Thus, the unwinding length l when $\Omega = 0$ is independent of the initial angular velocity Ω_0.

7-47. Consider the lower bar. A horizontal impulse is applied at its upper end. Its center of percussion, which is located at $\frac{2}{3}l$ from its upper end, has zero velocity just after the impulse.



Hence $\quad v_p = l\dot\theta_1 + \frac{2}{3}l\dot\theta_2 = 0, \quad \dot\theta_2 = -\frac{3}{2}\dot\theta_1$

During impact, H is conserved about O.



$$m_0 l^2\dot\theta_1 + \frac{ml^2}{3}\dot\theta_1 + ml\left(\dot\theta_1 + \frac{1}{2}\dot\theta_2\right)\frac{3}{2}l + \frac{ml^2}{12}\dot\theta_2 = m_0 v_0 l$$

or $\quad \left(m_0 + \frac{11}{6}m\right)l^2\dot\theta_1 + \frac{5}{6}ml^2\dot\theta_2 = m v_0 l$

Substitute $\dot\theta_2 = -\frac{3}{2}\dot\theta_1$ and obtain

$$\left(m_0 + \frac{7}{12}m\right)l^2\dot\theta = m v_0 l \quad \text{or} \quad \dot\theta_1 = \frac{12 m_0 v_0}{(12m_0 + 7m)l}$$

Then we obtain $\quad \dot\theta_2 = \dfrac{-18 m_0 v_0}{(12 m_0 + 7m)l}$

7-48. $\qquad [l] = [\psi][\theta][\phi]$

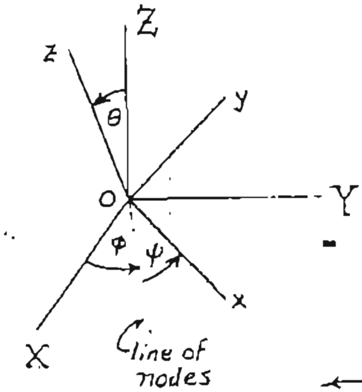
In detail,

$$[l] = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

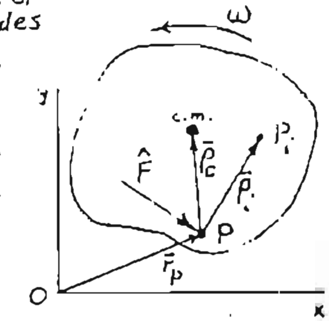
or

$$[l] = \begin{bmatrix} (\cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi) & (\sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi) & \sin\theta\sin\psi \\ (-\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi) & (-\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi) & \sin\theta\cos\psi \\ \sin\phi\sin\theta & -\cos\phi\sin\theta & \cos\theta \end{bmatrix}$$

7-48. (cont'd.) Type II Euler Angles



7-49. Use the noninertial reference point P. From (4-70),



$$\bar M_p - \bar\rho_c \times m\ddot{\bar r}_p = \dot{\bar H}_p = I_p\dot{\bar\omega}$$

$\bar M_p = 0$. Integrate over the time interval of the impulse and obtain

$$I_p\Delta\bar\omega = -m\bar\rho_c \times \Delta\bar v_p \quad \text{or} \quad \Delta\bar\omega = \frac{-m}{I_p}\bar\rho_c \times \Delta\bar v_p$$

Now

$$\bar v_i = \bar v_p + \bar\omega \times \bar\rho_i \quad \text{or} \quad \Delta\bar v_i = \Delta\bar v_p + \Delta\bar\omega \times \bar\rho_i$$

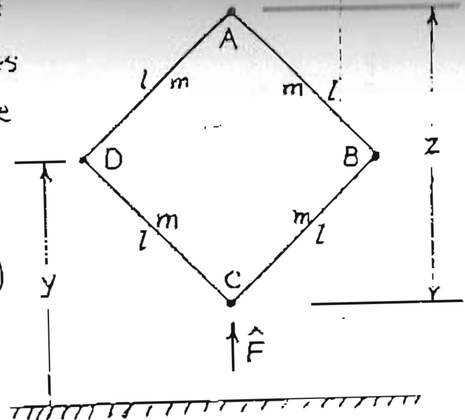
Then we obtain

$$\Delta\bar v_i = \Delta\bar v_p + \frac{m}{I_p}\bar\rho_i \times (\bar\rho_c \times \Delta\bar v_p)$$

$$= \Delta\bar v_p + \frac{m}{I_p}\left[(\bar\rho_i \cdot \Delta\bar v_p)\bar\rho_c - (\bar\rho_i \cdot \bar\rho_c)\Delta\bar v_p\right]$$

7-50. To avoid having to find the interaction impulses between the bars, let us use generalized impulse and momentum. Note that

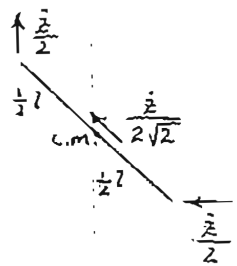
$v_C = \dot y - \frac{1}{2}\dot z$ (positive upward)



Then

$\hat Q_y = \hat F$, $\hat Q_z = -\frac{1}{2}\hat F$

For each bar, $\omega = \frac{\dot z}{\sqrt{2}\, l}$



The total kinetic energy is

$$T = \frac{1}{2}(4m)\dot y^2 + 4\left[\frac{m}{2}\left(\frac{\dot z}{2\sqrt{2}}\right)^2 + \frac{ml^2}{24}\left(\frac{\dot z}{\sqrt{2}\, l}\right)^2\right] = 2m\dot y^2 + \frac{1}{3}m\dot z^2$$

$p_y = \frac{\partial T}{\partial \dot y} = 4m\dot y$, $p_z = \frac{\partial T}{\partial \dot z} = \frac{2}{3}m\dot z$

Then $\Delta p_y = \hat Q_y$ or $4m\,\Delta\dot y = \hat F$

$\Delta p_z = \hat Q_z$ or $\frac{2}{3}m\,\Delta\dot z = -\frac{1}{2}\hat F$

Dividing these equations,

$6\frac{\Delta\dot y}{\Delta\dot z} = -2$ or $\Delta\dot z = -3\Delta\dot y$

When point C hits the floor, $\Delta v_C = \sqrt{2gh} = \Delta\dot y - \frac{1}{2}\Delta\dot z = \frac{5}{2}\Delta\dot y$

Hence $\Delta\dot y = \frac{2}{5}\sqrt{2gh}$ and $\Delta\dot z = \frac{-6}{5}\sqrt{2gh}$

Immediately after impact, $\underline{v_C = \Delta\dot z = \frac{-6}{5}\sqrt{2gh}}$

8-1. Using angular impulse and momentum,

$$\{\hat M\} = \{\Delta H\} = [I]\{\Delta\omega\} \text{ where } [I] = \begin{bmatrix} 10 & 5 & 0 \\ 5 & 15 & -4 \\ 0 & -4 & 20 \end{bmatrix}$$

Solving for $\{\Delta\omega\}$,

$$\{\Delta\omega\} = [I]^{-1}\{\hat M\} \text{ where } [I]^{-1} = \frac{\begin{bmatrix} 284 & -100 & -20 \\ -100 & 200 & 40 \\ -20 & 40 & 125 \end{bmatrix}}{2340}$$

Hence we obtain

$$\{\Delta\omega\} = \frac{1}{2340}\begin{bmatrix} 284 & -100 & -20 \\ -100 & 200 & 40 \\ -20 & 40 & 125 \end{bmatrix}\begin{Bmatrix} 4 \\ 5 \\ 6 \end{Bmatrix} = \begin{Bmatrix} 43/195 \\ 14/39 \\ 29/78 \end{Bmatrix} = \begin{Bmatrix} 0.2205 \\ 0.3590 \\ 0.3718 \end{Bmatrix}$$

Adding the initial components, $\omega_x = 1.2205$ rad/sec

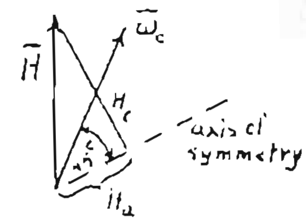
$\omega_y = 1.3590$

$\omega_z = 1.3718$

8-2. $I_t = 2I_a$

$\alpha = 45^0$ so $\omega_a = \omega_t = \frac{\omega_c}{\sqrt{2}}$

$H_a = I_a\omega_a = \frac{I_a\omega_c}{\sqrt{2}}$



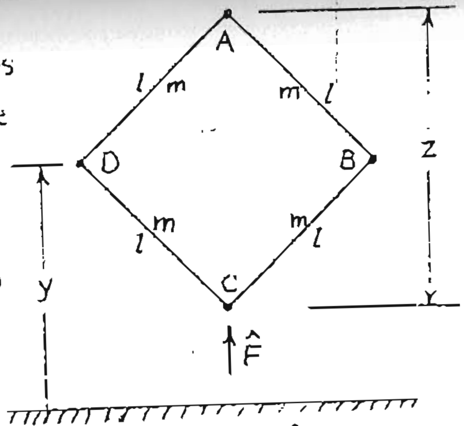
$H_t = I_t\omega_t = \sqrt{2}\, I_a\omega_c$.

If $\alpha = 0$, then $\omega_t = 0$ and $H_t = 0$.

Now $\Delta\bar H = \hat M$ and $\hat M_{min} = \Delta H_t = \underline{\sqrt{2}\, I_a\omega_c}$

in a direction opposite to the initial H_t.

7-50. To avoid having to find the interaction impulses between the bars, let us use generalized impulse and momentum. Note that

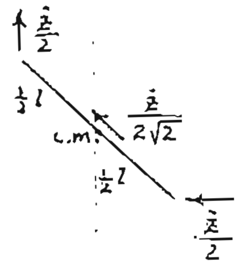


$$v_C = \dot y - \tfrac{1}{2}\dot z \quad \text{(positive upward)}$$

Then

$$\hat Q_y = \hat F \,, \quad \hat Q_z = -\tfrac{1}{2}\hat F$$

For each bar, $\omega = \dfrac{\dot z}{\sqrt{2}\, l}$



The total kinetic energy is

$$T = \tfrac{1}{2}(4m)\dot y^2 + 4\left[\frac{m}{2}\left(\frac{\dot z}{2\sqrt2}\right)^2 + \frac{ml^2}{24}\left(\frac{\dot z}{\sqrt2\, l}\right)^2\right] = 2m\dot y^2 + \tfrac{1}{3}m\dot z^2$$

$$p_y = \frac{\partial T}{\partial \dot y} = 4m\dot y \,, \qquad p_z = \frac{\partial T}{\partial \dot z} = \tfrac{2}{3}m\dot z$$

Then $\Delta p_y = \hat Q_y$ or $4m\,\Delta\dot y = \hat F$

$\Delta p_z = \hat Q_z$ or $\tfrac{2}{3}m\,\Delta\dot z = -\tfrac{1}{2}\hat F$

Dividing these equations,

$$6\,\frac{\Delta\dot y}{\Delta\dot z} = -2 \quad \text{or} \quad \Delta\dot z = -3\Delta\dot y$$

When point C hits the floor, $\Delta v_C = \sqrt{2gh} = \Delta\dot y - \tfrac{1}{2}\Delta\dot z = \tfrac{5}{2}\Delta\dot y$

Hence $\Delta\dot y = \tfrac{2}{5}\sqrt{2gh}$ and $\Delta\dot z = \tfrac{-6}{5}\sqrt{2gh}$

Immediately after impact, $v_C = \Delta\dot z = \dfrac{-6}{5}\sqrt{2gh}$

CHAPTER 8

8-1. Using angular impulse and momentum,

$$\{\hat M\} = \{\Delta H\} = [I]\{\Delta\omega\} \quad \text{where} \quad [I] = \begin{bmatrix} 10 & 5 & 0 \\ 5 & 15 & -4 \\ 0 & -4 & 20 \end{bmatrix}$$

Solving for $\{\Delta\omega\}$,

$$\{\Delta\omega\} = [I]^{-1}\{\hat M\} \quad \text{where} \quad [I]^{-1} = \frac{\begin{bmatrix} 284 & -100 & -20 \\ -100 & 200 & 40 \\ -20 & 40 & 125 \end{bmatrix}}{2340}$$

Hence we obtain

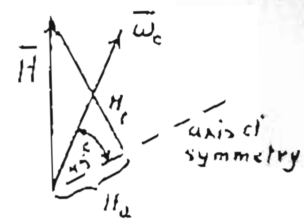
$$\{\Delta\omega\} = \frac{1}{2340}\begin{bmatrix} 284 & -100 & -20 \\ -100 & 200 & 40 \\ -20 & 40 & 125 \end{bmatrix}\begin{Bmatrix} 4 \\ 5 \\ 6 \end{Bmatrix} = \begin{Bmatrix} 43/195 \\ 14/39 \\ 29/78 \end{Bmatrix} = \begin{Bmatrix} 0.2205 \\ 0.3590 \\ 0.3718 \end{Bmatrix}$$

Adding the initial components, $\omega_x = 1.2205$ rad/sec

$\omega_y = 1.3590$

$\omega_z = 1.3718$

8-2. $I_t = 2I_a$



$\alpha = 45^\circ$ so $\omega_a = \omega_t = \dfrac{\omega_0}{\sqrt2}$

$H_a = I_a\omega_a = \dfrac{I_a\omega_0}{\sqrt2}$

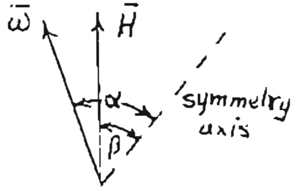
$H_t = I_t\omega_t = \sqrt2\, I_a\omega_0$.

If $\alpha = 0$, then $\omega_t = 0$ and $H_t = 0$.

Now $\Delta\bar H = \hat M$ and $\hat M_{min} = \Delta H_t = \sqrt2\, I_a\omega_0$

in a direction opposite to the initial H_t.

8-3. $I_a = 2 I_t$



$H_t = I_t \omega_t$ and $H_a = I_a \omega_a$

Hence $\frac{H_t}{H_a} = \frac{\omega_t}{2\omega_a}$ or $\tan\alpha = 2\tan\beta$

Now $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{\tan\beta}{1+2\tan^2\beta}$

Let $u = \tan\beta$. Then $\tan(\alpha-\beta) = \frac{u}{1+2u^2}$

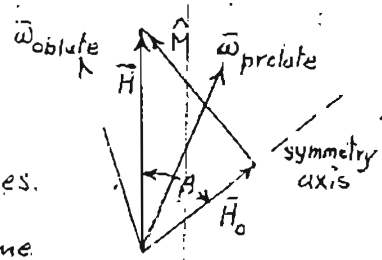
To maximize $\alpha-\beta$, maximize $\tan(\alpha-\beta)$.

$$\frac{d}{du}\left(\frac{u}{1+2u^2}\right) = \frac{1+2u^2-4u^2}{(1+2u^2)^2} = \frac{1-2u^2}{(1+2u^2)^2} = 0$$

$u = \tan\beta = \frac{1}{\sqrt2}$ or $\beta = 35.26°$, $\tan\alpha = \sqrt2$ or $\alpha = 54.74°$

$$\underline{(\alpha-\beta)_{max} = 19.47°}$$

8-4. $H_0 = I_a \Omega$



H_0, Ω, and I_a are the same for oblate and prolate cases.

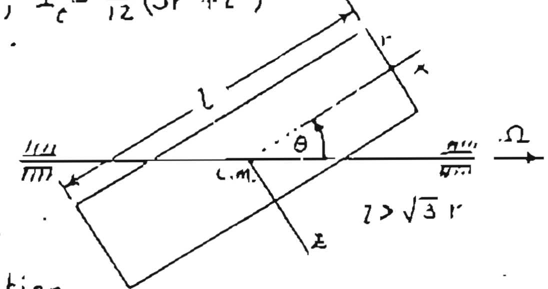
$\Delta \bar H = \hat M$ is the same in both cases.

Therefore $\beta = \tan^{-1}\frac{\hat M}{H_0}$ is the same in both cases. The maximum angular deviation of symmetry axis is 2β in each case, i.e., the maximum angular deviations are equal.

The precession rate $\dot\psi = -\frac{I_a\Omega}{I_t\cos\beta}$ where

$\left(\frac{I_a}{I_t}\right)_{oblate} > \left(\frac{I_a}{I_t}\right)_{prolate}$. Hence $\underline{|\dot\psi|_{oblate} > |\dot\psi|_{prolate}}$

8-5. $I_a = \frac{1}{2}mr^2$, $I_t = \frac{m}{12}(3r^2+l^2)$



$\omega_x = \Omega\cos\theta$

$\omega_y = \dot\theta$

$\omega_z = \Omega\sin\theta$

Use Euler's equation.

$$I_{yy}\dot\omega_y + (I_{xx}-I_{zz})\omega_x\omega_z = M_y = 0$$

This leads to

$$\frac{m}{12}(3r^2+l^2)\ddot\theta + \frac{m}{12}(3r^2-l^2)\Omega^2\sin\theta\cos\theta = 0$$

The perturbation equation about $\theta = \frac{\pi}{2}$ is

$$\frac{m}{12}(3r^2+l^2)\delta\ddot\theta - \frac{m}{12}(3r^2-l^2)\Omega^2\delta\theta = 0$$

which results in the circular frequency $\underline{\omega = \sqrt{\frac{l^2-3r^2}{l^2+3r^2}}\,\Omega}$

(b) Let $\ddot\theta = \dot\theta\frac{d\dot\theta}{d\theta}$ and integrate the differential equation.

$$\int_0^{\dot\theta}\dot\theta\, d\dot\theta = \frac{l^2-3r^2}{l^2+3r^2}\int_0^{\theta}\Omega^2\sin\theta\cos\theta\, d\theta$$

or

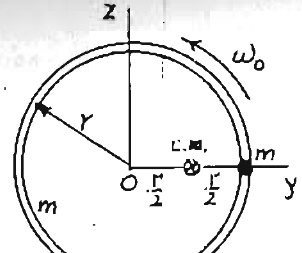
$$\frac{1}{2}\dot\theta^2 = \frac{1}{2}\left(\frac{l^2-3r^2}{l^2+3r^2}\right)\Omega^2\sin^2\theta$$

At $\theta = \frac{\pi}{2}$, $\dot\theta^2 = \left(\frac{l^2-3r^2}{l^2+3r^2}\right)\Omega^2$

or

$$\underline{\dot\theta = \sqrt{\frac{l^2-3r^2}{l^2+3r^2}}\,\Omega}$$

8-6. Conserve angular momentum about the c.m. The c.m. velocity is $\bar{v}_{cm} = -\frac{1}{2} v_c \bar{i}$ for all t.



Taking velocities relative to the c.m.,

$$\bar{H} = mr^2\omega_0 \bar{i} + \left[m\left(\frac{r}{2}\right)\left(\frac{v_c}{2}\right) + m\left(\frac{r}{2}\right)\left(\frac{v_c}{2}\right)\right]\bar{k}$$

$$= mr^2\omega_0 \bar{i} + \frac{1}{2} m r v_c \bar{k} \quad \text{before impact.}$$

After impact, $I_{xx} = mr^2 + m\left(\frac{r}{2}\right)^2 + m\left(\frac{r}{2}\right)^2 = \frac{3}{2}mr^2$

$$I_{yy} = \frac{1}{2}mr^2, \quad I_{zz} = \frac{mr^2}{2} + m\left(\frac{r}{2}\right)^2 + m\left(\frac{r}{2}\right)^2 = mr^2$$

Hence

$$H_x = \frac{3}{2}mr^2\omega_x = mr^2\omega_0, \qquad \omega_x = \frac{2}{3}\omega_0$$

$$H_y = \frac{1}{2}mr^2\omega_y = 0, \qquad \omega_y = 0$$

$$H_z = mr^2\omega_z = \frac{1}{2}m r v_0, \qquad \omega_z = \frac{v_0}{2r}$$

Note: The same result is obtained more directly by choosing a fixed reference point at the c.m. position at the time of impact.

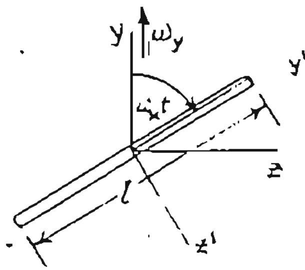
(b) The translational velocity of the ring just after impact is

$$\bar{v} = \left(-\frac{v_c}{2} + \frac{r}{2}\omega_z\right)\bar{i} - \frac{1}{2}r\omega_x \bar{k} = -\frac{1}{4}v_0\bar{i} - \frac{1}{3}r\omega_0\bar{k}$$

Using linear impulse and momentum for the ring,

$$\hat{F} = \Delta\bar{p} = -\frac{1}{4}mv_0\bar{i} - \frac{1}{3}mr\omega_0\bar{k}$$

8-7. $x'y'z'$ is a principal axis system. Use the Euler equations



$$I_{x'x'}\dot{\omega}_{x'} + (I_{z'z'} - I_{y'y'})\omega_{y'}\omega_{z'} = M_{x'}$$

where $\omega_{x'} = \omega_x = \text{const}$ so $\dot{\omega}_{x'} = 0$

$$\omega_{y'} = \omega_y \cos\omega_x t, \quad \omega_{z'} = -\omega_y \sin\omega_x t$$

8-7. (cont'd.) $I_{x'x'} = I_{z'z'} = \frac{ml^2}{12}$, $I_{y'y'} = 0$

Hence we obtain

$$M_x = M_{x'} = -\frac{ml^2}{12}\omega_y^2 \sin\omega_x t\cos\omega_x t = -\frac{ml^2}{24}\omega_y^2 \sin 2\omega_x t$$

Also,

$$M_{y'} = I_{y'y'}\dot{\omega}_{y'} + (I_{x'x'} - I_{z'z'})\omega_{z'}\omega_{x'} = 0$$

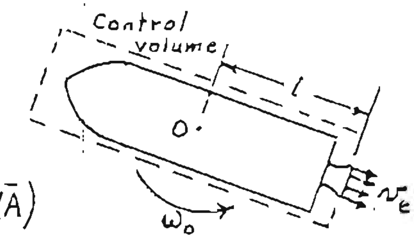
$$M_{z'} = I_{z'z'}\dot{\omega}_{z'} + (I_{y'y'} - I_{x'x'})\omega_{x'}\omega_{y'}$$

$$= -\frac{ml^2}{12}\omega_x\omega_y\cos\omega_x t - \frac{ml^2}{12}\omega_x\omega_y\cos\omega_x t = -\frac{ml^2}{6}\omega_x\omega_y\cos\omega_x t$$

Now $M_y = M_{y'}\cos\omega_x t - M_{z'}\sin\omega_x t = \frac{ml^2}{6}\omega_x\omega_y\sin\omega_x t\cos\omega_x t$

or $M_y = \frac{ml^2}{12}\omega_x\omega_y \sin 2\omega_x t$

8-8. Choose the fixed point O as the reference and use (4-106).



$$\bar{M} = \frac{d}{dt}\int_V \rho\bar{\rho}\times\bar{v}\,dV + \int_A \rho\bar{\rho}\times\bar{v}(\bar{v}_r\cdot d\bar{A})$$

where $\bar{v}$ is the absolute velocity of the material at a position $\bar{\rho}$ relative to O. The volume integral represents the angular momentum of the material within the control volume.

$$\frac{d}{dt}\int_V \rho\bar{\rho}\times\bar{v}\,dV = \frac{d}{dt}(I_t\bar{\omega}_0) = \frac{\dot{m}l^2}{3}\bar{\omega}_0 = \frac{-bl^2}{3}\bar{\omega}_0$$

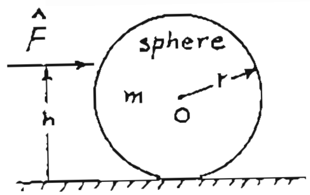
Here we assume that the internal flow relative to the rocket is axially symmetric and does not influence the volume integral.

$$\int_A \rho\bar{\rho}\times\bar{v}(\bar{v}_r\cdot d\bar{A}) = bl^2\bar{\omega}_0$$

This is the angular momentum outflow rate. Finally, adding terms,

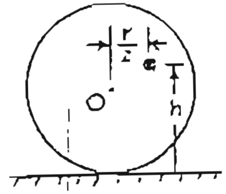
$$\bar{M} = \frac{2}{3}bl^2\bar{\omega}_0$$

8-9. For no slipping, $\hat{F}$ must pass through the center of percussion relative to the contact point.



$$(h-r)r = \frac{2}{5}r^2, \qquad \underline{h = \frac{7}{5}r}$$

(b) Use angular impulse and momentum about the c.m. The horizontal component of $\bar{\omega}$ is



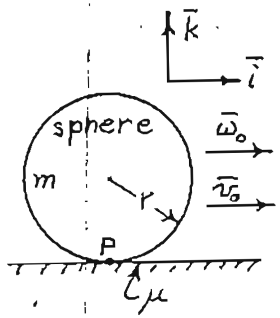
$$\frac{2}{5}r\hat{F} = \frac{2}{5}mr^2\omega_h \quad \text{or} \quad \omega_h = \frac{\hat{F}}{mr}$$

The vertical component of $\bar{\omega}$ is

$$\frac{r}{2}\hat{F} = \frac{2}{5}mr^2\omega_v \quad \text{or} \quad \omega_v = \frac{5\hat{F}}{4mr}$$

Then $\omega = \sqrt{\omega_h^2+\omega_v^2} = \sqrt{1+\frac{25}{16}}\,\frac{\hat{F}}{mr} = \underline{1.6008\,\frac{\hat{F}}{mr}}$

8-10. In general, the velocity of the contact point P on the ball is



$$\bar{v}_p = \bar{v} + \bar{\omega}\times(-r\bar{k})$$

At $t=0$, we have

$$\bar{v}_p = v_0\bar{i} + r\omega_0\bar{j}$$

Friction force $\bar{F}_f = -\mu mg\dfrac{\bar{v}_p}{v_p}$

Acceleration $\dot{\bar{v}} = \dfrac{\bar{F}_f}{m} = \underline{-\mu g\dfrac{v_0\bar{i}+r\omega_0\bar{j}}{\sqrt{v_0^2+r^2\omega_0^2}}}$

Friction moment $\bar{M}_f = -r\bar{k}\times\bar{F}_f = \mu mgr\dfrac{v_0\bar{j}-r\omega_0\bar{i}}{\sqrt{v_0^2+r^2\omega_0^2}}$

Then $\dot{\bar{\omega}} = \dfrac{\bar{M}_f}{\frac{2}{5}mr^2} = \underline{\dfrac{5\mu g(-r\omega_0\bar{i}+v_0\bar{j})}{2r\sqrt{v_0^2+r^2\omega_0^2}}}$

8-10. (cont'd.) (b) In general, we see that

$$\dot{\bar{v}}_p = \dot{\bar{v}} + \dot{\bar{\omega}}\times(-r\bar{k}) = -\mu g\frac{\bar{v}_p}{v_p} + \frac{(-r\bar{k}\times\bar{F}_f)\times(-r\bar{k})}{I}$$

$$= -\mu g\frac{\bar{v}_p}{v_p} + \frac{5}{2mr^2}(r^2\bar{F}_f) = -\frac{7}{2}\mu g\frac{\bar{v}_p}{v_p}$$

Since $\dot{\bar{v}}_p$ is opposite in direction to $\bar{v}_p$, the direction of $\bar{v}_p$ will not change, and therefore the direction of $\bar{F}_f$ will be unchanged as long as sliding continues.

(c) $|\dot{\bar{v}}_p| = \frac{7}{2}\mu g = \text{const}$

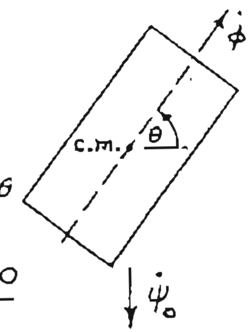
Time to stop sliding is $t_s = \dfrac{v_p(0)}{|\dot{\bar{v}}_p|} = \dfrac{\sqrt{v_0^2+r^2\omega_0^2}}{\frac{7}{2}\mu g}$

The velocity when sliding stops is

$$\bar{v} = v_0\bar{i} + \dot{\bar{v}}t_s = v_0\bar{i} - \mu g\frac{(v_0\bar{i}+r\omega_0\bar{j})}{\sqrt{v_0^2+r^2\omega_0^2}}\,\frac{2\sqrt{v_0^2+r^2\omega_0^2}}{7\mu g}$$

$$\bar{v} = \frac{5}{7}v_0\bar{i} - \frac{2}{7}r\omega_0\bar{j}$$

8-11. $T = \frac{1}{2}I_a(\dot{\phi}-\dot{\psi}_0\sin\theta)^2 + \frac{1}{2}I_t(\dot{\theta}^2+\dot{\psi}_0^2\cos^2\theta)$



$V = 0$, $\quad \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot{q}_i}\right) - \dfrac{\partial T}{\partial q_i} = 0$

$\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot{\theta}}\right) = I_t\ddot{\theta}$, $\quad \dfrac{\partial T}{\partial\theta} = -I_a\dot{\psi}_0\cos\theta(\dot{\phi}-\dot{\psi}_0\sin\theta) - I_t\dot{\psi}_0^2\sin\theta\cos\theta$

θ equation is

$$\underline{I_t\ddot{\theta} + I_a\dot{\psi}_0\cos\theta(\dot{\phi}-\dot{\psi}_0\sin\theta) + I_t\dot{\psi}_0^2\sin\theta\cos\theta = 0}$$

$\dfrac{\partial T}{\partial\dot{\phi}} = I_a(\dot{\phi}-\dot{\psi}_0\sin\theta)$, $\quad \dfrac{\partial T}{\partial\phi} = 0$

ϕ equation: $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot{\phi}}\right) = \underline{I_a\ddot{\phi} - I_a\dot{\psi}_0\dot{\theta}\cos\theta = 0}$

These equations of motion could also have been obtained fro (8-198).

8-11. (b) $\theta(0)=0$, $\dot\theta(0)=0$, $\dot\phi(0)=\Omega_0>0$, $\dot\psi_0>0$.

From the ϕ equation, $\frac{\partial T}{\partial\dot\phi}=\text{const}$ or $\dot\phi-\dot\psi_0\sin\theta=\Omega_0$

Then the θ equation becomes

$$I_t\ddot\theta+I_a\dot\psi_0\Omega_0\cos\theta+I_t\dot\psi_0^2\sin\theta\cos\theta=0$$

Let $\ddot\theta=\dot\theta\frac{d\dot\theta}{d\theta}$ and integrate.

$$\int_0^{\dot\theta}I_t\dot\theta\,d\dot\theta=-\int_0^{\theta}\left[I_a\dot\psi_0\Omega_0\cos\theta+I_t\dot\psi_0^2\sin\theta\cos\theta\right]d\theta$$

or $\frac{1}{2}I_t\dot\theta^2=-I_a\dot\psi_0\Omega_0\sin\theta-\frac{1}{2}I_t\dot\psi_0^2\sin^2\theta$

resulting in $\dot\theta^2=-\frac{2I_a}{I_t}\dot\psi_0\Omega_0\sin\theta-\dot\psi_0^2\sin^2\theta$

(c) To find limits of θ, set $\dot\theta=0$.

Case 1, $\Omega_0=\frac{I_t\dot\psi_0}{5I_a}$, $\dot\theta^2=-\frac{2}{5}\dot\psi_0^2\sin\theta-\dot\psi_0^2\sin^2\theta=0$

or $\sin\theta\left(\frac{2}{5}+\sin\theta\right)=0$

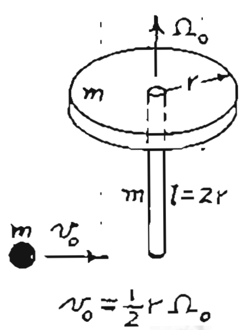
$\theta=0,\ -\sin^{-1}0.4$ $\qquad 0\le\theta\le-23.58^\circ$

Case 2, $\Omega_0=\frac{I_t\dot\psi_0}{I_a}$, $\dot\theta^2=-2\dot\psi_0^2\sin\theta-\dot\psi_0^2\sin^2\theta=0$

or $\sin\theta(2+\sin\theta)=0$

$\theta=0,\ -\pi$ $\qquad 0\le\theta\le-180^\circ$

8-12. There is conservation of $\bar H$ about the system c.m. which is at the center of the rod after impact. $v_{cm}=\frac{1}{3}v_0$



$H_a=I_a\omega_a=\frac{1}{2}mr^2\Omega_0$

$H_t=m\left(\frac{2}{3}v_0\right)r+m\left(\frac{v_0}{3}\right)r=mv_0r$

$=\frac{1}{2}mr^2\Omega_0$

8-12. (cont'd.) $\tan\beta=\frac{H_t}{H_a}=1$ so $\beta=45^\circ$

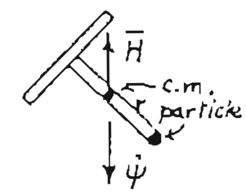
Max angular deviation $2\beta=90^\circ$

(b) $\dot\psi=\frac{-H}{I_t}$ where $I_t=mr^2+m\frac{(2r)^2}{12}+mr^2+\frac{mr^2}{4}=\frac{31}{12}mr^2$

Period of precession $=\frac{2\pi}{|\dot\psi|}=\frac{2\pi I_t}{H}=\frac{2\pi\left(\frac{31}{12}mr^2\right)}{\frac{\sqrt2}{2}mr^2\Omega_0}=\frac{31\sqrt2\pi}{6\Omega_0}=\frac{22.955}{\Omega_0}$

(c) Use linear impulse and momentum of particle.

$$\dot\psi=\frac{-H}{I_t}=\frac{-\frac{1}{\sqrt2}mr^2\Omega_0}{\frac{31}{12}mr^2}=\frac{-12\Omega_0}{31\sqrt2}$$



The velocity of the particle after impact is

$$v_p=v_{cm}-\frac{r}{\sqrt2}\dot\psi=\frac{v_0}{3}+\frac{6}{31}r\Omega_0=\frac{67}{186}r\Omega_0$$

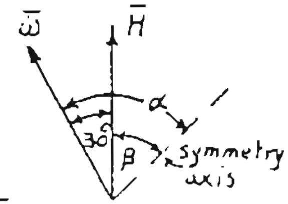
The impulse acting on the particle is

$$\hat F=m(v_0-v_p)=m\left(\frac{1}{2}-\frac{67}{186}\right)r\Omega_0=\frac{13}{93}mr\Omega_0=0.1398\,mr\Omega_0$$

This is also the impulse of the particle acting on the rod.

$\hat F=0.2796\,mv_0$

8-13. Since $|\dot\psi|>\omega$, $I_a>I_t$ and $\alpha>\beta$.



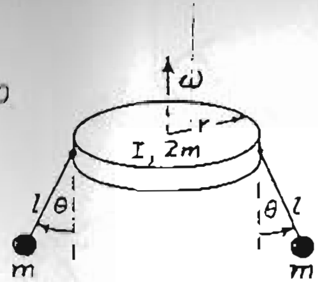
Symmetry axis changes direction by $2\beta=90^\circ$, so $\beta=45^\circ$. Then $\alpha=\beta+30^\circ=75^\circ$.

From (8-91), $\frac{I_a}{I_t}=\frac{\tan\alpha}{\tan\beta}=2+\sqrt3=3.7321$

Using (8-93), $\dot\psi=\frac{-\sin\alpha}{\sin\beta}\omega=-\left(\frac{1+\sqrt3}{2}\right)\omega=-1.3660\,\omega$

Using (8-95), $\dot\phi=\frac{\sin(\alpha-\beta)}{\sin\alpha}\dot\psi=-\frac{\sin(\alpha-\beta)}{\sin\beta}\omega=\frac{-\omega}{\sqrt2}=-0.7071\,\omega$

8-14. $\omega(0)=\omega_0$, $\theta(0)=0$, $\dot\theta(0)=0$



Conservation of the axial H gives

$$H_a=\left[I+2m(r+l\sin\theta)^2\right]\omega=(I+2mr^2)\omega_0$$

or
$$\omega=\frac{(I+2mr^2)\omega_0}{I+2m(r+l\sin\theta)^2}$$

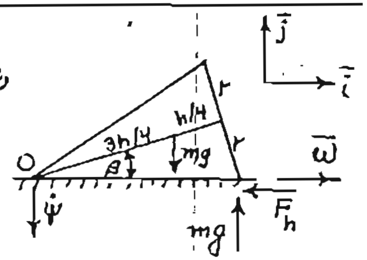
Use conservation of energy and note that the system c.m. is fixed. Hence, the downward velocity v_d of the disk is equal to the upward velocity of the particles.
$$v_d=\tfrac{1}{2}l\dot\theta\sin\theta$$

Then
$$T=\tfrac{1}{2}\left[I+2m(r+l\sin\theta)^2\right]\omega^2+m\left(\tfrac{1}{2}l\dot\theta\sin\theta\right)^2$$
$$+m\left[\left(\tfrac{1}{2}l\dot\theta\sin\theta\right)^2+\left(l\dot\theta\cos\theta\right)^2\right]=\frac{I+2mr^2}{2}\omega_0^2$$

Substitute for ω and obtain
$$T=\frac{(I+2mr^2)^2\omega_0^2}{2\left[I+2m(r+l\sin\theta)^2\right]}+\frac{ml^2\dot\theta^2}{2}(1+\cos^2\theta)=\tfrac{1}{2}(I+2mr^2)\omega_0^2$$

Solving for $\dot\theta^2$,
$$\dot\theta^2=\frac{(I+2mr^2)\omega_0^2}{ml^2(1+\cos^2\theta)}\left[1-\frac{I+2mr^2}{I+2m(r+l\sin\theta)^2}\right]$$

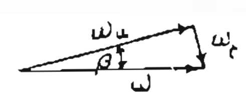
8-15. When the vertex begins to rise, the forces acting on the cone are as shown. Take the fixed point at O as a reference point and use
$$\bar M=\dot{\bar H}_0$$



The velocity of the center of the base is
$$h\dot\psi\cos\beta=h\omega\sin\beta \quad\text{or}\quad \omega=\dot\psi\cot\beta$$

Then
$$\bar H_0=I_a\omega_a(\cos\beta\,\bar i+\sin\beta\,\bar j)+I_t\omega_t(\sin\beta\,\bar i-\cos\beta\,\bar j)$$

8-15. (cont'd.) where ω_a and ω_t are defined as shown. $\bar H_0$ has constant magnitude and changing direction, so



$$\dot{\bar H}_0=\omega\dot\psi\left(I_a\cos^2\beta+I_t\sin^2\beta\right)\bar k$$

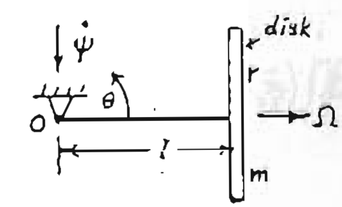
The applied moment is due to the couple
$$\bar M=mg\left(\tfrac{h}{4}\cos\beta+r\sin\beta\right)\bar k=\frac{mgh}{4\cos\beta}\left(\cos^2\beta+4\sin^2\beta\right)\bar k$$

Now $\quad I_a=\frac{3}{10}mr^2=\frac{3}{10}mh^2\tan^2\beta,\quad I_t=\frac{3m}{20}(r^2+4h^2)=\frac{3}{20}mh^2(4+\tan^2\beta)$

Then, from $\bar M=\dot{\bar H}_0$,
$$\dot\psi^2\cot\beta\left[\tfrac{3}{10}mh^2\sin^2\beta+\tfrac{3}{20}mh^2\sin^2\beta(4+\tan^2\beta)\right]$$
$$=\frac{mgh}{4\cos\beta}(1+3\sin^2\beta)$$

resulting in
$$\dot\psi^2=\frac{5g(1+3\sin^2\beta)}{3h\sin\beta(1+5\cos^2\beta)}$$

8-16. $\theta(0)=0$, $\dot\theta(0)=0$, $\dot\phi(0)=\Omega$



Because there is axial symmetry and no axial moment, the total spin Ω is constant.
Conserve the vertical component of $\bar H$ about O.

At $\theta=0$, $H_{vert}=\left(\frac{mr^2}{4}+ml^2\right)\dot\psi(0)$. At $\theta=-\frac{\pi}{2}$, $H_{vert}=\frac{mr^2}{2}\Omega$

where we take positive downward. Hence we obtain
$$\frac{m}{4}(4l^2+r^2)\dot\psi(0)=\frac{mr^2}{2}\Omega \quad\text{or}\quad \dot\psi(0)=\frac{2r^2\Omega}{4l^2+r^2}$$

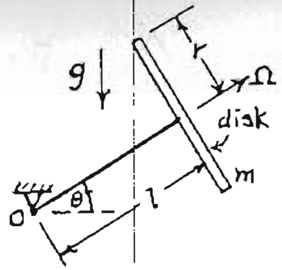
From conservation of energy,
$$E'=\frac{m}{2}\left(l^2+\frac{r^2}{4}\right)\dot\theta^2-mgl=\frac{m}{2}\left(l^2+\frac{r^2}{4}\right)\dot\psi^2(0)=\frac{mr^4\Omega^2}{2(4l^2+r^2)}$$

Then
$$\dot\theta^2=\frac{8}{4l^2+r^2}\left[gl+\frac{r^4\Omega^2}{2(4l^2+r^2)}\right] \quad\text{or,}$$

at $\theta=-\frac{\pi}{2}$, $\quad |\dot\theta|=\frac{2r^2\Omega}{4l^2+r^2}\sqrt{1+\frac{2gl(4l^2+r^2)}{r^4\Omega^2}}$

8-17. $l = 2r$, $\Omega = 10\sqrt{\frac{g}{r}}$

$I_a = \frac{1}{2}mr^2$, $I_t = \frac{1}{4}mr^2 + ml^2 = \frac{17}{4}mr^2$



For slow precession,

$$\dot\psi = \frac{-b}{2\sin\theta}\left[1 - \sqrt{1 - \frac{2c\sin\theta}{b^2}}\right]$$

where $b = \frac{I_a\Omega}{I_t} = \frac{20}{17}\sqrt{\frac{g}{r}}$, $c = \frac{2mgl}{I_t} = \frac{16g}{17r}$, $\theta = 30°$

Hence $\dot\psi = -\frac{20}{17}\sqrt{\frac{g}{r}}\left(1 - \sqrt{1 - \frac{17}{25}}\right) = \underline{-0.5110\sqrt{\frac{g}{r}}}$

(b) When support breaks, $\bar\omega$ is unchanged. For the free motion that follows, $I_a = \frac{1}{2}mr^2$, $I_t = \frac{1}{4}mr^2$ about the center of the disk. At the time of the break,

$\omega_a = \Omega$ and $\omega_t = |\dot\psi_{old}|\cos\theta = 0.4425\sqrt{\frac{g}{r}}$

Then the angular momentum about the c.m. of the disk is

$$H = \sqrt{(I_a\omega_a)^2 + (I_t\omega_t)^2} = I_a\Omega\sqrt{1 + \left(\frac{0.04425}{2}\right)^2} = 1.00024\, I_a\Omega$$

Measuring $\dot\psi$ about the new $\bar H$ vector,

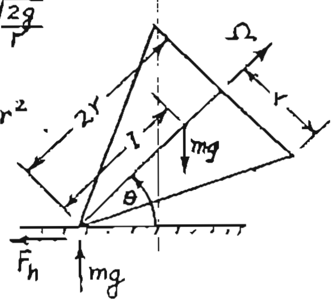
$\dot\psi = \frac{-H}{I_t} = -2.00048\,\Omega = \underline{-20.005\sqrt{\frac{g}{r}}}$

$\tan\beta = \frac{H_t}{H_a} = 0.02213$. Max deviation $\underline{2\beta = 2.535°}$

8-18. $l = \frac{3}{4}h = \frac{3}{2}r$; $\theta = 45°$, $\Omega = 10\sqrt{\frac{2g}{r}}$

(a) Fixed vertex. $I_a = \frac{3}{10}mr^2$

$I_t = \frac{3m}{20}(r^2 + 4h^2) = \frac{51}{20}mr^2$



$b = \frac{I_a}{I_t}\Omega = \frac{20}{17}\sqrt{\frac{2g}{r}}$, $c = \frac{2mgl}{I_t} = \frac{20g}{17r}$

$$\dot\psi = \frac{-b}{2\sin\theta}\left(1 - \sqrt{1 - \frac{2c\sin\theta}{b^2}}\right)$$

or

$$\dot\psi = \frac{-20}{17}\sqrt{\frac{g}{r}}\left(1 - \sqrt{1 - \frac{17}{20\sqrt2}}\right) = \underline{-0.4334\sqrt{\frac{g}{r}}}$$

8-18. (cont'd.) (b) No friction. $\theta = 45°$. $F_h = 0$.

The c.m. is fixed in an inertial frame, so take c.m. as the reference point. The applied moment is the same as in (a), so take $l = \frac{3}{2}r$. $I_a = \frac{3}{10}r^2$, $I_t = \frac{3m}{80}(4r^2 + h^2) = \frac{3}{10}mr^2$

$b = \Omega = 10\sqrt{\frac{2g}{r}}$, $c = 10\frac{g}{r}$

$$\dot\psi = \frac{-b}{2\sin\theta}\left(1 - \sqrt{1 - \frac{2c\sin\theta}{b^2}}\right) = -10\sqrt{\frac{g}{r}}\left(1 - \sqrt{1 - \frac{1}{10\sqrt2}}\right) = \underline{-0.3600\sqrt{\frac{g}{r}}}$$

8-19. Complex notation method. Starting with (8-201),

$$\dot w - i\frac{I_a\Omega}{I_t}\dot w = -i\frac{M_t}{I_t}$$

with $M_t = i\,mgl\,w$ results in (8-209).

$$\ddot w - i\frac{I_a\Omega}{I_t}\dot w - \frac{mgl}{I_t}w = 0$$

Assuming solutions of the form $Ce^{\lambda t}$, the characteristic equation is

$$\lambda^2 - i\frac{I_a\Omega}{I_t}\lambda - \frac{mgl}{I_t} = 0$$ with roots $\lambda_1 = i\left(2 - \frac{4}{\sqrt5}\right)\sqrt{\frac{g}{l}}$

$\lambda_2 = i\left(2 + \frac{4}{\sqrt5}\right)\sqrt{\frac{g}{l}}$

The nutation period is (see p. 444)

$$T = \frac{2\pi i}{\lambda_2 - \lambda_1} = \frac{\sqrt5}{4}\pi\sqrt{\frac{l}{g}} = \underline{1.7562\sqrt{\frac{l}{g}}}$$

Elliptic integral method. This is Case 2 of cuspidal motion.

$b = \frac{I_a\Omega}{I_t} = 4\sqrt{\frac{g}{l}}$, $c = \frac{2mgl}{I_t} = \frac{8g}{5l}$, $e = \frac{2E'}{I_t} = \frac{56g}{25l}$

$u^2 - \left(\frac{b^2 + e}{c} - 1\right)u + \frac{b^2 - e}{c} = 0$ or $u^2 - 10.4u + 8.6 = 0$ (see p. 433)

Roots: $u_1 = 0.4058$ and $u_{max} = u_2 = 1$. $k = \sqrt{\frac{u_2 - u_1}{u_3 - u_1}} = 0.1047$

$u_3 = 9.4442$

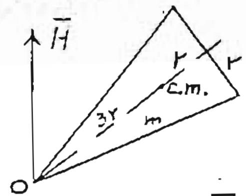
Then, using (8-121),

$$T = \frac{4K(k)}{\sqrt{c(u_3 - u_1)}} = 1.0791\sqrt{\frac{l}{g}}\,K(0.1047) = \underline{1.6946\sqrt{\frac{l}{g}}}$$

8-20. $h = 4r$, $\omega_a = \Omega_0$, $\omega_t = \frac{1}{2}\Omega_0$

$v_{cm} = 0$, $I_a = \frac{3}{10} mr^2$, $I_t = \frac{3}{4} mr^2$

$H_a = \frac{3}{10} mr^2 \Omega_0$, $H_t = \frac{3}{8} mr^2 \Omega_0$



For free motion, $\dot\psi = -\frac{H}{I_t}$ where $H = \sqrt{H_a^2 + H_t^2} = \frac{3\sqrt{41}}{40} mr^2 \Omega_0$

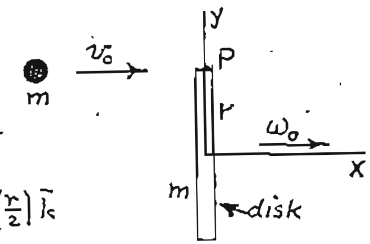
$$\dot\psi = -\frac{\sqrt{41}}{10}\Omega_0 = \underline{-0.6403\,\Omega_0}$$

(b) When vertex O is fixed, $\bar H_o$ is conserved since the reaction impulse goes through O. Since $v_{cm} = 0$ before impact, we see that $\bar H_{cm}$ before impact equals $\bar H_o$ after impact.

After impact, $I_t = \frac{3}{20} m (r^2 + 4h^2) = \frac{39}{4} mr^2$

$$\dot\psi = \frac{-H}{I_t} = \left(-\frac{3\sqrt{41}}{40}\right)\left(\frac{4}{39}\right)\Omega_0 = \underline{-0.04925\,\Omega_0}$$

8-21. Angular momentum is conserved about the system c.m. Particle and disk each translate at $v_0/2$ before impact.



Before $\bar H_{c.m.} = \frac{mr^2}{2}\omega_0 \bar i - 2m\left(\frac{v_0}{2}\right)\left(\frac{r}{2}\right)\bar k$

After $I_{xx} = \frac{mr^2}{2} + m\left(\frac{r}{2}\right)^2 + m\left(\frac{r}{2}\right)^2 = mr^2$ (about c.m.)

$I_{yy} = \frac{mr^2}{4}$, $I_{zz} = \frac{mr^2}{4} + m\left(\frac{r}{2}\right)^2 + m\left(\frac{r}{2}\right)^2 = \frac{3}{4} mr^2$

$$\bar H_{cm} = I_{xx}\omega_x \bar i + I_{yy}\omega_y \bar j + I_{zz}\omega_z \bar k = \frac{mr^2}{2}\omega_0 \bar i - \frac{mrv_0}{2}\bar k$$

giving $\underline{\omega_x = \frac{1}{2}\omega_0, \quad \omega_y = 0, \quad \omega_z = -\frac{2v_0}{3r}}$

(b) $v_0 = r\omega_0$. Relative to the c.m.,

$$H^2 = \left(\frac{mr^2}{2}\omega_0\right)^2 + \left(\frac{mrv_0}{2}\right)^2 = \frac{1}{2} m^2 r^4 \omega_0^2$$

$$2T_{rot} = I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 = mr^2\omega_0^2\left(\frac{1}{4} + \frac{1}{3}\right) = \frac{7}{12} mr^2 \omega_0^2$$

8-21. (cont'd.) $D = \frac{H^2}{2T_{rot}} = \frac{6}{7} mr^2$

Since $D > I_{zz}$, where I_{zz} is the intermediate moment of inertia, we know that the polhode encircles the x axis, corresponding to the maximum moment of inertia. The point on the polhode corresponding to ω_{max} is either in the xy plane or the xz plane. Recall that $2T_{rot}$ and H^2 are both conserved along a polhode.

For <u>xy plane</u>, $\omega_z = 0$. $2T_{rot} = mr^2\left(\omega_x^2 + \frac{1}{4}\omega_y^2\right) = \frac{7}{12} mr^2 \omega_0^2$

$$H^2 = (mr^2)^2\left(\omega_x^2 + \frac{1}{16}\omega_y^2\right) = (mr^2)^2 \frac{\omega_0^2}{2}$$

Solving, $\omega_x^2 = \frac{17}{36}\omega_0^2$, $\omega_y^2 = \frac{4}{9}\omega_0^2$, $\omega^2 = \frac{33}{36}\omega_0^2$

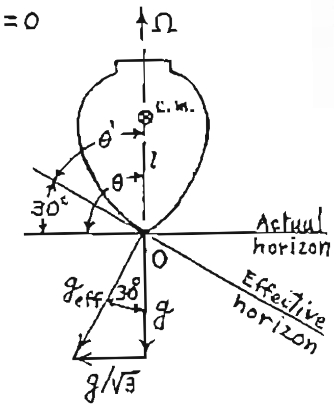
For <u>xz plane</u>, $\omega_y = 0$, i.e., just after impact.

$$\omega_x^2 = \frac{1}{4}\omega_0^2, \quad \omega_z^2 = \frac{4}{9}\omega_0^2 \text{ giving } \omega^2 = \frac{25}{36}\omega_0^2$$

Hence ω_{max} occurs in the xy plane. $\underline{\omega_{max} = \frac{\sqrt{33}}{6}\omega_0 = 0.9574\,\omega_0}$

8-22. $\Omega^2 = 80 g/l$, $\theta(0) = \frac{\pi}{2}$, $\dot\theta(0) = 0$

$I_a = \frac{1}{4} ml^2$, $I_t = \frac{5}{4} ml^2$



Consider a noninertial frame accelerating with point O at $g/\sqrt{3}$ to the right. In this frame, point O is fixed and there is an artificial gravity $g/\sqrt{3}$ to the left, as well as the actual gravity g downward, giving an effective gravity

$$g_{eff} = \sqrt{g^2 + (g/\sqrt{3})^2} = \frac{2}{\sqrt{3}} g \text{ at } 30° \text{ to the left of vertical.}$$

8-22. (cont'd.) In this frame, the effective horizontal plane is tipped 30° from the actual horizontal plane. Let θ' be the Euler angle relative to the effective horizontal plane. Then we can use the equations for Case 1 of cuspidal motion.

$$u_0 = \sin\theta'(0) = \sin 60° = \frac{\sqrt{3}}{2}$$

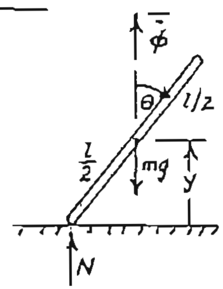
$$\lambda = \frac{I_a^2\Omega^2}{4I_t m g_{eff} l} = \frac{\left(\frac{ml^2}{4}\right)^2\left(\frac{80g}{l}\right)}{(5ml^2)\, m\left(\frac{2g}{\sqrt{3}}\right)l} = \frac{\sqrt{3}}{2}$$

$$u_1 = \lambda - \sqrt{\lambda^2 - 2\lambda u_0 + 1} = \frac{\sqrt{3}}{2} - \sqrt{\frac{3}{4} - \frac{3}{2} + 1} = \frac{\sqrt{3}-1}{2} = 0.3660$$

The corresponding $\theta_1' = \sin^{-1} 0.3660 = 21.47°$. This value of θ' will be reached periodically as the top precesses about the effective vertical axis. The actual θ_{min} will occur for a precession angle of 180°, i.e., in the forward direction.

$$\theta_{min} = \theta_1' - 30° = \underline{-8.53°}$$

8-23. $y = \frac{1}{2} l\cos\theta$, $\quad \dot y = -\frac{1}{2} l\dot\theta\sin\theta$



$$T = \frac{1}{2}m\left(\frac{l}{2}\dot\theta\sin\theta\right)^2 + \frac{ml^2}{24}\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right)$$

$$= \frac{ml^2}{24}\left[(1+3\sin^2\theta)\dot\theta^2 + \dot\phi^2\sin^2\theta\right]$$

$V = \frac{1}{2}mgl\cos\theta$, $\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) = \frac{d}{dt}\left[\frac{ml^2}{12}(1+3\sin^2\theta)\dot\theta\right] = \frac{ml^2}{12}\left[(1+3\sin^2\theta)\ddot\theta + 6\dot\theta^2\sin\theta\cos\theta\right]$$

$\frac{\partial T}{\partial\theta} = \frac{ml^2}{12}\left(3\dot\theta^2\sin\theta\cos\theta + \dot\phi^2\sin\theta\cos\theta\right)$, $\quad \frac{\partial V}{\partial\theta} = -\frac{1}{2}mgl\sin\theta$

θ equation: $\underline{\frac{ml^2}{12}\ddot\theta(1+3\sin^2\theta) + \frac{ml^2}{12}(3\dot\theta^2 - \dot\phi^2)\sin\theta\cos\theta - \frac{mgl}{2}\sin\theta = 0}$

Since $\frac{\partial T}{\partial\phi} = 0$ and $\frac{\partial V}{\partial\phi} = 0$, we have the

φ equation: $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\phi}\right) = \underline{\frac{ml^2}{12}\ddot\phi\sin^2\theta + \frac{ml^2}{6}\dot\theta\dot\phi\sin\theta\cos\theta = 0}$

8-23. (cont'd.) $\theta(0) = \frac{\pi}{4}$, $\dot\theta(0) = 0$, $\dot\phi(0) = \omega_0$.

(b) $N - mg = m\ddot y = -\frac{ml}{2}\ddot\theta\sin\theta - \frac{ml}{2}\dot\theta^2\cos\theta$

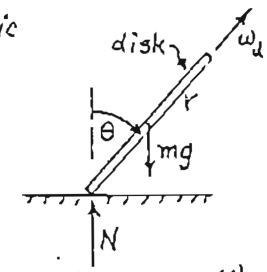
From the θ equation at t=0,

$$\frac{ml^2}{12}\left[\left(1+\frac{3}{2}\right)\ddot\theta - \frac{1}{2}\omega_0^2\right] = \frac{mgl}{2\sqrt{2}} \quad \text{or} \quad \ddot\theta = \frac{6\sqrt{2}}{5}\frac{g}{l} + \frac{\omega_0^2}{5}$$

Then, since $\dot\theta(0) = 0$,

$$N = mg - \frac{3}{5}mg - \frac{ml\omega_0^2}{10\sqrt{2}} = \underline{\frac{2}{5}mg - \frac{ml\omega_0^2}{10\sqrt{2}}} \quad \text{where } \omega_0^2 \le 4\sqrt{2}\frac{g}{l}$$

8-24. The disk is an axially symmetric body with no applied moment about the symmetry axis. Hence $\omega_a = 0 = \text{const.}$ Let ω_d be the angular velocity about a diameter through the contact point.



$$H_{vert} = \frac{mr^2}{4}\omega_d\cos\theta = \frac{mr^2}{4}\omega_0 = \text{const.}, \quad \omega_d = \frac{\omega_0}{\cos\theta}$$

Let height of c.m. be $y = r\cos\theta$. Then $\dot y = -r\dot\theta\sin\theta$.

From conservation of energy, noting $\theta(0) = 0$, $\dot\theta(0) = 0$,

$$\frac{1}{2}\left(\frac{mr^2}{4}\right)\left(\dot\theta^2 + \frac{\omega_0^2}{\cos^2\theta}\right) + \frac{1}{2}m\left(-r\dot\theta\sin\theta\right)^2 + mgr\cos\theta = \frac{mr^2}{8}\omega_0^2 + mgr$$

At $\theta = \theta_{max}$, we have $\dot\theta = 0$.

$$\left(\frac{mr^2}{8}\right)\left(\frac{\omega_0^2}{\cos^2\theta}\right) + mgr\cos\theta = \frac{mr^2\omega_0^2}{8} + mgr$$

or $mgr\cos^2\theta(1-\cos\theta) - \frac{mr^2\omega_0^2}{8}(1-\cos^2\theta) = 0$

Divide by the known factor $(1-\cos\theta)$ and note that $\omega_0^2 = \frac{g}{r}$.

Then $\cos^2\theta - \frac{1}{8}\cos\theta - \frac{1}{8} = 0$, $\quad \cos\theta = \frac{1}{16} + \sqrt{\left(\frac{1}{16}\right)^2 + \frac{1}{8}} = 0.4215$

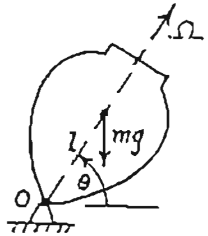
$$\underline{\theta_{max} = 65.07°}$$

(b) To stabilize, $\cos\theta = 1$ is second root.

$mgr\cos^2\theta - \frac{mr^2}{8}\omega_0^2(1+\cos\theta) = 0$ giving $\omega_0^2 = \frac{4g}{r}$, $\quad \underline{(\omega_0)_{min} = 2\sqrt{\frac{g}{r}}}$

8-25. $I_t = 4I_a$, $\theta(0)=30°$, $\dot\theta(0)=0$

$\Omega = 8\sqrt{\frac{mgl}{I_t}}$



H_{vert} = const or $I_a\Omega\sin\theta - I_t\dot\psi\cos^2\theta = \frac{I_a\Omega}{2} - \frac{3}{4}I_t\dot\psi(0)$

For $\theta=90°$, $I_a\Omega = \frac{I_a\Omega}{2} - \frac34 I_t\dot\psi(0)$, $\underline{\dot\psi(0) = \frac{-\Omega}{6}}$

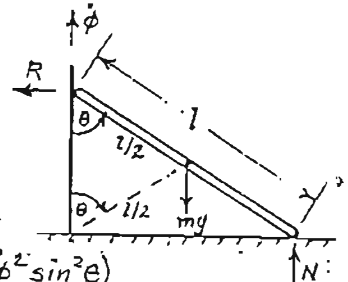
(b) E' = const or $\frac12 I_t(\dot\theta^2+\dot\psi^2\cos^2\theta)+mgl\sin\theta = \frac{I_t}{2}\left(\frac{\Omega}{6}\right)^2\frac34+\frac{mgl}{2}$

For $\theta=90°$, $\frac12 I_t\dot\theta^2 + mgl = \frac{I_t\Omega^2}{96}+\frac{mgl}{2} = \frac76 mgl$

Hence $\dot\theta^2 = \frac{mgl}{3I_t}$ or $|\dot\theta| = \sqrt{\frac{mgl}{3I_t}} = \underline{\frac{\Omega}{8\sqrt3} = 0.07217\Omega}$

8-26. H is conserved about the vertical axis. Since $\theta(0)=45°$, $\dot\theta(0)=0$, $\dot\phi(0)=\omega_0$, we obtain

$\frac{ml^2}{3}\dot\phi\sin^2\theta = \frac{ml^2}{6}\omega_0$ or $\underline{\dot\phi = \frac{\omega_0}{2\sin^2\theta}}$



$$T = \frac m2\left[\left(\frac{l\dot\theta}{2}\right)^2+\left(\frac{l\dot\phi}{2}\sin\theta\right)^2\right]+\frac{ml^2}{24}(\dot\theta^2+\dot\phi^2\sin^2\theta)$$

$$= \frac{ml^2}{6}(\dot\theta^2+\dot\phi^2\sin^2\theta), \quad V=\frac12 mgl\cos\theta$$

Use conservation of energy and the expression for $\dot\phi$ to obtain

$$\frac{ml^2}{6}\left(\dot\theta^2+\frac{\omega_0^2}{4\sin^2\theta}\right)+\frac{mgl}{2}\cos\theta = \frac{ml^2}{12}\omega_0^2+\frac{mgl}{2\sqrt2}$$

which leads to $\underline{\dot\theta^2 = \frac{\omega_0^2}{2}\left(1-\frac{1}{2\sin^2\theta}\right)+\frac{3g}{\sqrt2 l}(1-\sqrt2\cos\theta)}$

The rod will leave the floor when N drops to zero.

$N = mg + m\ddot y$ where y = vertical position of c.m.

$y=\frac l2\cos\theta$, $\dot y = -\frac{l\dot\theta}{2}\sin\theta$, $\ddot y = \frac{-l}{2}(\ddot\theta\sin\theta+\dot\theta^2\cos\theta)$

We need $\ddot\theta$ as a function of θ. Differentiate the $\dot\theta^2$ expression.

8-26. (cont'd.) $2\dot\theta\ddot\theta = \frac{\omega_0^2\dot\theta\cos\theta}{2\sin^3\theta}+\frac{3g}{l}\dot\theta\sin\theta$

$$\ddot\theta = \frac{3g}{2l}\sin\theta+\frac{\omega_0^2\cos\theta}{4\sin^3\theta}$$

Then we obtain

$$\ddot y = -\frac{3g}{4}\sin^2\theta - \frac{l\omega_0^2\cos\theta}{8\sin^2\theta} - \frac{l\cos\theta}{2}\left[\frac{\omega_0^2}{2}\left(1-\frac{1}{2\sin^2\theta}\right)+\frac{3g}{\sqrt2 l}(1-\sqrt2\cos\theta)\right]$$

$$= g\left(-\frac34\sin^2\theta - \frac{3}{2\sqrt2}\cos\theta+\frac32\cos^2\theta\right)+l\omega_0^2\left(\frac{-\cos\theta}{8\sin^2\theta}-\frac{\cos\theta}{4}+\frac{\cos\theta}{8\sin^2\theta}\right)$$

$$= g\left(\frac94\cos^2\theta-\frac{3}{2\sqrt2}\cos\theta-\frac34\right)-\frac{l\omega_0^2}{4}\cos\theta$$

and
$$N = \frac94 mg\cos^2\theta - \left(\frac{3mg}{2\sqrt2}+\frac{ml\omega_0^2}{4}\right)\cos\theta+\frac14 mg$$

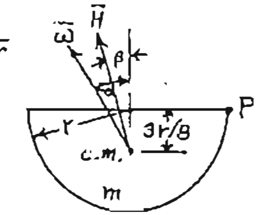
Setting $N=0$, we obtain
$$\cos^2\theta - \left(\frac{\sqrt2}{3}+\frac{l\omega_0^2}{9g}\right)\cos\theta+\frac19=0$$

The value of ω_0 for which N just reaches zero occurs if this quadratic equation has a double root. Now

$$\left(\cos\theta-\frac13\right)^2 = \cos^2\theta-\frac23\cos\theta+\frac19=0, \quad \theta=70.53°$$

so $\frac{\sqrt2}{3}+\frac{l\omega_0^2}{9g}=\frac23$ or $\omega_0^2 = \frac{3(2-\sqrt2)g}{l}$, $\underline{(\omega_0)_{min} = 1.3257\sqrt{\frac gl}}$

8-27. $I_a = \frac25 mr^2$, $I_t = \frac{83}{320}mr^2$, $\hat F = m\sqrt{gr}$



The impulse $\hat F$ is applied into the paper at P.

$\hat M_a = mr\sqrt{gr}$

$\hat M_t = \frac38 mr\sqrt{gr}$

$\omega_a = \frac{\hat M_a}{I_a} = \frac52\sqrt{\frac gr}$, $\omega_t = \frac{\hat M_t}{I_t} = \frac{120}{83}\sqrt{\frac gr}$, $\omega=\sqrt{\omega_a^2+\omega_t^2}$

$\underline{\omega = 2.8880\sqrt{\frac gr}}$

$\underline{\alpha = \tan^{-1}\frac{\omega_t}{\omega_a} = \tan^{-1}\frac{48}{83} = 30.04°}$

(b) $\beta = \tan^{-1}\frac{H_t}{H_a} = \tan^{-1}\frac{\hat M_t}{\hat M_a} = \tan^{-1}\frac38 = 20.556°$

The maximum deviation of the symmetry axis is $2\beta = \underline{41.11°}$

8-28. $I_a = \frac{2}{5}mr^2$, $I_t = \frac{83}{320}mr^2$, $\hat{F} = m\sqrt{gr}$

$\hat{M}_a = \hat{F}r = mr\sqrt{gr}$, $\omega_a = \frac{\hat{M}_a}{I_a} = \frac{5}{2}\sqrt{\frac{g}{r}}$

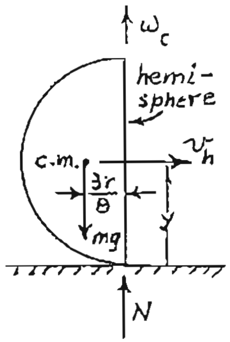
$\hat{M}_t = \frac{3}{8}\hat{F}r = \frac{3}{8}mr\sqrt{gr}$, $\omega_t = \frac{\hat{M}_t}{I_t} = \frac{120}{83}\sqrt{\frac{g}{r}}$

$v_{cm} = \frac{\hat{F}}{m} = \sqrt{gr}$. Now $v_p = v_{cm} + r\omega_a + \frac{3}{8}r\omega_t$

$$= \left(1 + \frac{5}{2} + \frac{45}{83}\right)\sqrt{gr} = \underline{4.0422\sqrt{gr}}$$

(b) Total energy is conserved after the impulse.

$T = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_a\omega_a^2 + \frac{1}{2}I_t\omega_t^2$, $V = mgy$



Immediately after the impulse,

$T(0^+) = \left(\frac{1}{2} + \frac{5}{4} + \frac{45}{166}\right)mgr = 2.0211\, mgr$

which, we note, equals $\frac{1}{2}\hat{F}v_p$.

$V(0^+) = \frac{5}{8}mgr$. Hence $E(0^+) = 2.6461\, mgr$.

Now suppose that the rim just touches the floor and the flat face is vertical. Then, since $H_{vert} = \text{const}$, the value of ω_t at this time is found from $I_t\omega_t = \hat{F}r = mr\sqrt{gr}$ or $\omega_t = \frac{320}{83}\sqrt{\frac{g}{r}}$.

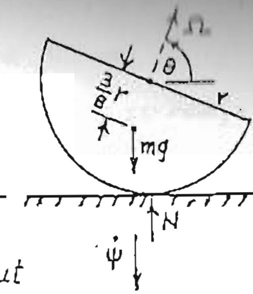
Noting that the horizontal component of v_{cm} and also the axial angular velocity component ω_a are constant throughout the motion for $t>0$, we obtain a final kinetic energy

$$T_f = \left(\frac{1}{2} + \frac{5}{4} + \frac{160}{83}\right)mgr = 3.6777\, mgr$$

Also $V_f = mgr$ so $E_f = 4.6777\, mgr$.

Since $E_f > E(0^+)$, the required final energy is greater than the available energy $E(0^+)$, and so it is impossible for the face to become vertical.

8-29. $I_a = \frac{2}{5}mr^2$, $I_t = \frac{83}{320}mr^2$



(a) Use angular impulse and momentum.

$I_t\dot\theta = -\hat{F}r$ or $\dot\theta = \frac{-320\hat{F}}{83mr} = \underline{-3.8554\frac{\hat{F}}{mr}}$

(b) $\Omega = \sqrt{\frac{mgr}{I_a}}$. Conserve H_{vert} about the c.m. as θ goes from 90° to 0°:

$I_a\Omega = -I_t\dot\psi$, or $\dot\psi = -\frac{I_a}{I_t}\Omega = -\frac{128}{83}\Omega = \underline{-2.4384\sqrt{\frac{g}{r}}}$

(c) Use conservation of E' after the impulse.

$$\frac{1}{2}I_t\dot\theta^2 + \frac{5}{8}mgr = \frac{1}{2}I_t\dot\psi^2 + mgr$$

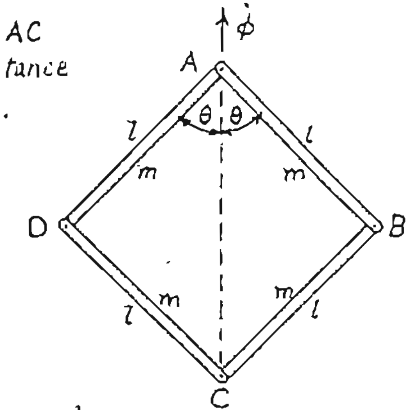
$\dot\theta^2 = \dot\psi^2 + \frac{2}{I_t}\left(\frac{3}{8}mgr\right)$ or $\left(\frac{320\hat{F}}{83\,mr}\right)^2 = \left(\frac{128}{83}\right)^2\frac{5g}{2r} + \frac{3}{4}\left(\frac{320}{83}\right)\frac{g}{r}$

giving $\hat{F}^2 = \left[\left(\frac{128}{320}\right)^2\frac{5}{2} + \frac{3}{4}\left(\frac{83}{320}\right)\right]m^2gr$, $\underline{\hat{F} = 0.7711\, m\sqrt{gr}}$

8-30. $\dot\phi(0) = \omega_0$, $\theta(0) = 45°$, $\dot\theta(0) = -\omega_0/\sqrt{2}$

The moment of inertia about AC varies as the square of the distance of any mass element from AC.

Thus, $I_{AC} = \frac{4}{3}ml^2\sin^2\theta$



Use conservation of H_{AC}.

$\frac{4}{3}ml^2\dot\phi\sin^2\theta = \frac{2}{3}ml^2\omega_0$

or $\dot\phi = \frac{\omega_0}{2\sin^2\theta}$

Use conservation of energy.

$T = \frac{2}{3}ml^2\sin^2\theta\,\dot\phi^2 + 4\left[\frac{m}{2}\left(\frac{l\dot\theta}{2}\right)^2 + \frac{ml^2}{24}\dot\theta^2\right]$

giving $\frac{2}{3}ml^2\dot\phi^2\sin^2\theta + \frac{2}{3}ml^2\dot\theta^2 = \frac{2}{3}ml^2\omega_0^2$

8-30. (cont'd.) Now substitute the expression for $\dot\phi$ and obtain

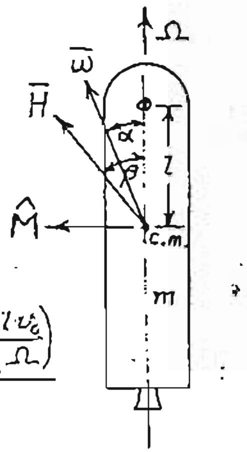
$$\frac{ml^2\omega_0^2}{6\sin^2\theta} + \frac{2}{3}ml^2\dot\theta^2 = \frac{2}{3}ml^2\omega_0^2$$

$$\dot\theta^2 = \left(1-\frac{1}{4\sin^2\theta}\right)\omega_0^2 \quad \text{or} \quad \dot\theta = \pm\,\omega_0\sqrt{1-\frac{1}{4\sin^2\theta}}$$

(b) To find θ_{min}, set $\dot\theta = 0$ and obtain $\sin^2\theta = \frac{1}{4}$

$$\theta_{min} = 30°$$

8-31. The linear impulse acting on the rocket is $\hat F = m_0 v_0$ into page.



$\hat M = m_0 l v_0$ so $\omega_t = \dfrac{\hat M}{I_t} = \dfrac{m_0 l v_0}{I_t}$

(b) $\tan\beta = \dfrac{H_t}{H_a} = \dfrac{\hat M}{I_a\Omega} = \dfrac{m_0 l v_0}{I_a\Omega}$

Maximum angular deviation $2\beta = 2\tan^{-1}\left(\dfrac{m_0 l v_0}{I_a\Omega}\right)$

(c) $\dot\psi = -\dfrac{H}{I_t}$ where $H = \sqrt{(I_a\Omega)^2 + (m_0 l v_0)^2}$

$$\dot\psi = -\frac{m_0 l v_0}{I_t}\sqrt{1+\left(\frac{I_a\Omega}{m_0 l v_0}\right)^2} \qquad \text{precession period } T = \frac{2\pi}{|\dot\psi|}$$

$v_{cm} = \dfrac{\hat F}{m} = \dfrac{m_0}{m}v_0$. Distance $d = v_{cm}T = \dfrac{2\pi I_t}{ml\sqrt{1+\left(\dfrac{I_a\Omega}{m_0 l v_0}\right)^2}}$

8-32. Given $\beta = 45°$ where $\tan\beta = \dfrac{H_t}{H_a}$, so $H_a = H_t$.

Now $I_t = 10 I_a$ so $\omega_a = \dfrac{I_t}{I_a}\omega_t = 10\omega_t$.

Initially, $H^2 = (I_a\omega_a)^2 + (I_t\omega_t)^2 = 2I_t^2\omega_t^2$ which is constant

$$T(0) = \tfrac{1}{2}I_a\omega_a^2 + \tfrac{1}{2}I_t\omega_t^2 = \tfrac{1}{2}\left[\tfrac{I_t}{10}(10\omega_t)^2 + I_t\omega_t^2\right] = \tfrac{11}{2}I_t\omega_t^2$$

8-32. (cont'd.) At the final time, we have

$I_a' = \frac{1}{2}I_a$, $I_t' = \frac{4}{5}I_t$. Because H^2 is conserved,

$$I_a'^2\omega_a'^2 + I_t'^2\omega_t'^2 = \frac{I_t^2}{400}\omega_a'^2 + \frac{16}{25}I_t^2\omega_t'^2 = 2I_t^2\omega_t^2$$

or $\omega_a'^2 + 256\,\omega_t'^2 = 800\,\omega_t^2$

Also, T is tripled, so

$$\frac{I_t}{40}\omega_a'^2 + \frac{2}{5}I_t\omega_t'^2 = 3\left(\frac{11}{2}I_t\omega_t^2\right)$$

or $\omega_a'^2 + 16\,\omega_t'^2 = 660\,\omega_t^2$

Solving, $\omega_t'^2 = \dfrac{140}{240}\omega_t^2$ or $\omega_t' = \sqrt{\dfrac{7}{12}}\,\omega_t$

$\omega_a'^2 = \dfrac{1952}{3}\omega_t^2$ or $\omega_a' = \sqrt{\dfrac{1952}{3}}\,\omega_t$

Final $\tan\beta = \dfrac{I_t'\omega_t'}{I_a'\omega_a'} = 16\sqrt{\dfrac{(7)(3)}{(12)(1952)}} = 0.4791$, $\beta = 25.60°$

8-33. $\theta(0) = 0$, $\dot\theta(0) = 0$, $\psi(0) = 0$, $\dot\phi(0) = \Omega = 100\sqrt{\dfrac{k}{I_t}}$, $\dfrac{I_t}{I_a} = 10$

$T = \frac{1}{2}I_a(\dot\phi - \dot\psi\sin\theta)^2 + \frac{1}{2}I_t(\dot\theta^2 + \dot\psi^2\cos^2\theta)$, $V = \frac{1}{2}k\left(\frac{\pi}{2}-\theta\right)^2$

$p_\psi = \dfrac{\partial T}{\partial\dot\psi} = -I_a(\dot\phi - \dot\psi\sin\theta)\sin\theta + I_t\dot\psi\cos^2\theta = \text{const} = 0$ since $\dfrac{\partial T}{\partial\psi} = \dfrac{\partial V}{\partial\psi} = 0$

$\Omega = \dot\phi - \dot\psi\sin\theta = \text{const}$, so $\dot\psi = \dfrac{I_a\Omega\sin\theta}{I_t\cos^2\theta}$

Using conservation of E,

$$\tfrac{1}{2}I_t\dot\theta^2 + \frac{I_a^2\Omega^2\sin^2\theta}{2I_t\cos^2\theta} + \frac{k\pi^2}{8} - \frac{k\pi}{2}\theta + \tfrac{1}{2}k\theta^2 = \frac{k\pi^2}{8}$$

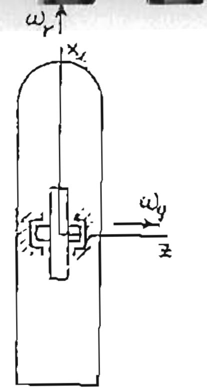
Assuming $\theta \ll 1$, let $\sin\theta \cong \theta$, $\cos\theta \cong 1$, and obtain

$$\dot\theta^2 = -\left[\left(\frac{I_a\Omega}{I_t}\right)^2 + \frac{k}{I_t}\right]\theta^2 + \frac{k\pi}{I_t}\theta = \left(-101\,\theta^2 + \pi\theta\right)\frac{k}{I_t}$$

To find θ_{max}, set $\dot\theta = 0$ and obtain $\theta_{max} = \dfrac{\pi}{101}\text{ rad} = 1.782°$

Then $\dot\psi_{max} = \dfrac{I_a\Omega}{I_t}\theta_{max} = 10\sqrt{\dfrac{k}{I_t}}\,\theta_{max} = 0.3110\sqrt{\dfrac{k}{I_t}}$ rad/sec.

34. Rocket: $I_a = 10^3\ kg\cdot m^2$, $I_t = 10^4\ kg\cdot m^2$, $\omega_r = 10$ rad/sec Gyro: $I_a = 0.1\ kg\cdot m^2$, $I_t = 0.05\ kg\cdot m^2$, $\omega_y = 10^4$ rad/sec



(a) Use conservation of angular momentum.

$$\bar H = 10^4\bar i + 10^3\bar k = const = 10^3\omega_x\bar i + 10^4\omega_z\bar k$$

Hence, $\omega_x = 10$ rad/sec and $\omega_z = 0.1$ rad/sec after clamping gyro. $\omega_y = 0$. $I_a \cong 10^3$, $I_t \cong 10^4$.

The angular impulse applied to the rocket is the negative of that applied to the gyro.

$$\hat M = -\left[(0.05)(10)\bar i - (0.1)(10^4)\bar k\right] = \underline{-0.5\bar i + 10^3\bar k}\ \text{N-m}\cdot\text{sec}$$

(b) $\beta = \tan^{-1}\frac{H_t}{H_a} = \tan^{-1}\frac{H_z}{H_x} = \tan^{-1}0.1 = 5.71°$

Max deviation $2\beta = \underline{11.42°}$

(c) $H = \sqrt{(10^4)^2 + (10^3)^2} = 1.0050\times10^4$, $\dot\phi = -\frac{H}{I_t} = -1.0050\ \frac{rad}{sec}$

Period $T = \frac{2\pi}{|\dot\phi|} = \underline{6.2520\ sec}$

8-35. Same figure as in 8-34, and also the same rocket and gyro parameters. Rocket $\begin{cases} I_a = 10^3\ kg\cdot m^2 \\ I_t = 10^4\ kg\cdot m^2\end{cases}$ Gyro $\begin{cases} I_a = 0.1\ kg\cdot m^2 \\ I_t = 0.05\ kg\cdot m^2\end{cases}$

$\omega_r = 10$ rad/sec and $\omega_y = 10^4$ rad/sec before clamping. $\bar H$ is constant in space throughout the motion.

$$\bar H = 10^4\bar i + 10^3\bar k = (10^3 + 0.05)\omega_x\bar i + 10^3\bar k \text{ during clamping.}$$

$\omega_x = 9.9995 \cong 10$ rad/sec just after clamping. $\omega_y = 10^4$ = const since there is no axial moment applied to the gyro. The angular impulse applied to the rocket is the negative of that applied to the gyro. Due to the sudden change of ω_t of the gyro, the rocket angular impulse is

$$\hat M = -(0.05)(10)\bar i = \underline{-0.5000\bar i\ \text{N-m}\cdot\text{sec}}$$

8-35. (cont'd.)(b) Assuming small motion of the symmetry axis (x axis) of the rocket, let us use the complex notation method with $\Omega = \omega_x = 10$ rad/sec. The moment applied to the rocket is $-\dot{\bar H}_{gyro} = (0.1)(10^4)(10)\bar j$ and rotates with the rocket. Hence, in the fixed y'z' complex plane,

$$M_t = 10^4 e^{i10t}$$

Using (8-207), and recalling that $w = y' + iz'$, we have

$$\ddot w - i\frac{I_a\Omega}{I_t}\dot w = -i\frac{M_t}{I_t} \quad \text{or} \quad \ddot w - i\dot w = -i e^{i10t}$$

with the initial conditions $w(0) = 0$, $\dot w(0) = 0$.

The solution for $\dot w$ has the form

$$\dot w = Ae^{it} + Be^{i10t}$$

where $B = -\frac19$ in steady-state solution (particular integral) and $A = -B = \frac19$ from $\dot w(0) = 0$. Then

$$\dot w = \frac19\left(e^{it} - e^{i10t}\right)$$

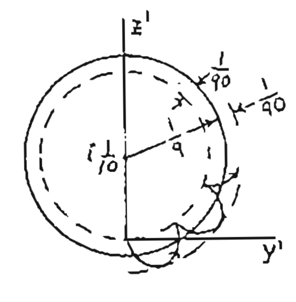
and, upon integration,

$$w = i\left(\frac{1}{10} - \frac19 e^{it} + \frac{1}{90}e^{i10t}\right)$$

The maximum deviation after half a precession cycle is

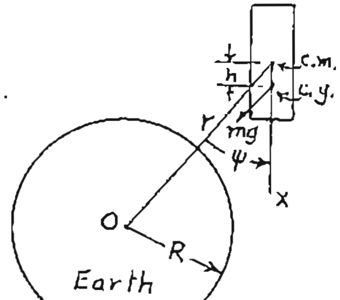
$$|w|_{max} = \frac{1}{10} + \frac19 + \frac{1}{90} = \frac29$$

Maximum angular deviation $= \frac29$ rad $= \underline{12.73°}$



(c) The precession period $\underline{T = 2\pi\ sec}$

8-36. $I_t > I_a$. Take the z axis out of the page at the c.m. of the satellite. From (8-312),



$$M_z = -3\omega_0^2(I_t - I_a)\psi = -\frac{3 g_c R^2}{r^3}(I_t - I_a)\psi$$

where we assume that $\psi \ll 1$.

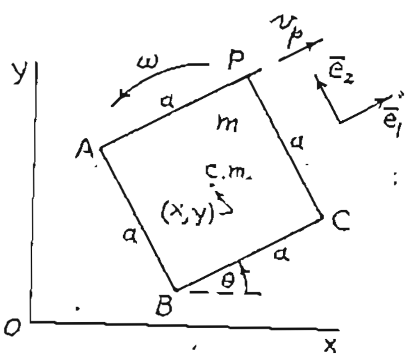
The center of gravity, as discussed in Example 5-6, is defined such that

$$M_z = -mgh\psi \quad \text{where} \quad g = \frac{g_c R^2}{r^2}$$

Hence $M_z = -\dfrac{m g_0 R^2}{r^2} h\psi$ and, comparing expressions for M_z,

$$\underline{h = \frac{3(I_t - I_a)}{mr}}$$

8-37. Choose P as the reference point and use (8-251).



$$\sum_{i=1}^{N}\left[m_i(\dot{\bar v}_i + \ddot{\bar\rho}_{ci})\cdot\bar\gamma_{ij} + \left(\bar{\bar I}_i\cdot\dot{\bar\omega}_i + \bar\omega_i\times\bar{\bar I}_i\cdot\bar\omega_i + m_i\bar\rho_{ci}\times\dot{\bar v}_i\right)\cdot\bar\beta_{ij}\right] = Q_j$$

First note that

$$\dot{\bar e}_1 = \omega\bar e_2, \quad \ddot{\bar e}_1 = \dot\omega\bar e_2 - \omega^2\bar e_1$$
$$\dot{\bar e}_2 = -\omega\bar e_1, \quad \ddot{\bar e}_2 = -\dot\omega\bar e_1 - \omega^2\bar e_2$$

Then $\bar v_p = v_p\bar e_1$, $\dot{\bar v}_p = \dot v_p\bar e_1 + v_p\omega\bar e_2$, $\bar\rho_c = \frac{a}{2}(-\bar e_1 - \bar e_2)$

$$\dot{\bar\rho}_c = \frac{a\omega}{2}(\bar e_1 - \bar e_2), \quad \ddot{\bar\rho}_c = \frac{a}{2}(\dot\omega + \omega^2)\bar e_1 + \frac{a}{2}(-\dot\omega + \omega^2)\bar e_2$$

$\bar\omega = \omega\bar k$ where $\bar k$ is out of paper. Choose (v_p, ω) as generalized speeds in that order. Then

$$\bar\gamma_{11} = \frac{\partial\bar v_p}{\partial v_p} = \bar e_1, \quad \bar\gamma_{12} = \frac{\partial\bar v_p}{\partial\omega} = 0, \quad \bar\beta_{11} = \frac{\partial\bar\omega}{\partial v_p} = 0, \quad \bar\beta_{12} = \frac{\partial\bar\omega}{\partial\omega} = \bar k$$

$$I_p = \frac{ma^2}{6} + m\left(\frac{a}{\sqrt2}\right)^2 = \frac{2}{3}ma^2, \quad \bar{\bar I}\cdot\dot{\bar\omega} = \frac{2}{3}ma^2\dot\omega\bar k$$

8-37. (cont'd.) $\bar\omega\times\bar{\bar I}\cdot\bar\omega = 0$, $\bar\rho_c\times\dot{\bar v}_p = \frac{a}{2}(\dot v_p - v_p\omega)\bar k$

$Q_1 = Q_2 = 0$. The dynamical equations are as follows:

The v_p equation is $\quad \underline{m\left(\dot v_p + \frac{a}{2}\dot\omega + \frac{a}{2}\omega^2\right) = 0}$

The ω equation is $\quad \underline{\frac{2}{3}ma^2\dot\omega + \frac{1}{2}ma\dot v_p - \frac{1}{2}ma v_p\omega = 0}$

The nonholonomic constraint equation is

$$\underline{\dot x\sin\theta - \dot y\cos\theta - \frac{1}{2}a\omega = 0}$$

Also, we have the kinematic relations

$$\underline{\dot x\cos\theta + \dot y\sin\theta - \frac{1}{2}a\omega = v_p}$$

$$\underline{\dot\theta = \omega}$$

These are the required 5 first-order differential equations. An equivalent set, obtained by solving for the first derivatives, is

$$\dot v_p = -\frac{3}{5}v_p\omega - \frac{4}{5}a\omega^2$$
$$\dot\omega = \frac{6}{5a}v_p\omega + \frac{3}{5}\omega^2$$
$$\dot x = v_p\cos\theta + \frac{1}{2}a\omega(\sin\theta + \cos\theta)$$
$$\dot y = v_p\sin\theta + \frac{1}{2}a\omega(\sin\theta - \cos\theta)$$
$$\dot\theta = \omega$$

(b) $v_p(0) = v_0$, $\theta(0) = 0$, $\omega(0) = \dfrac{v_0}{a}$

To find ω_{max}, set $\dot\omega = 0$ giving $v_p = -\frac{1}{2}a\omega$

Now $\bar v_{cm} = \left(v_p + \frac{a}{2}\omega\right)\bar e_1 - \frac{a}{2}\omega\bar e_2$

Use conservation of energy to obtain

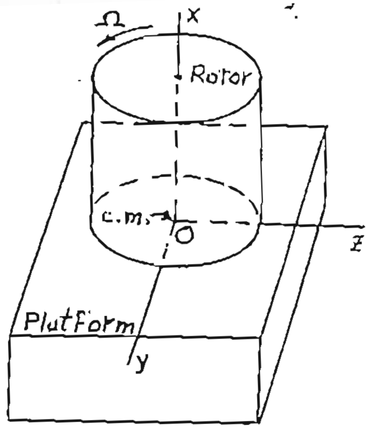
$$\frac{1}{2}m\left[\left(v_p + \frac{a}{2}\omega\right)^2 + \left(\frac{a\omega}{2}\right)^2\right] + \frac{ma^2}{12}\omega^2 = \frac{4}{3}mv_0^2$$

Substituting for v_p, $\left(\frac{1}{8} + \frac{1}{12}\right)a^2\omega^2 = \frac{4}{3}v_0^2$ or $\omega^2 = \dfrac{32\,v_0^2}{5\,a^2}$

$$\underline{\omega_{max} = \sqrt{\frac{32}{5}}\,\frac{v_0}{a} = 2.5298\,\frac{v_0}{a}}$$

8-38. The total angular momentum about O is

$$\bar{H} = [I_{xx}\omega_x + I_a(\omega_x + \Omega)]\bar{i} + (I_{yy}+I_t)\omega_y\bar{j} + (I_{zz}+I_t)\omega_z\bar{k}$$



Since $\bar{H}$ is constant, we obtain

$$\dot{\bar{H}} = (\dot{\bar{H}})_r + \bar{\omega}\times H = 0$$

where $\bar{\omega}$ is the angular velocity of the xyz frame (platform).

$$(\dot{\bar{H}})_r = [(I_{xx}+I_a)\dot{\omega}_x + I_a\dot{\Omega}]\bar{i} + (I_{yy}+I_t)\dot{\omega}_y\bar{j} + (I_{zz}+I_t)\dot{\omega}_z\bar{k}$$

$$\bar{\omega}\times\bar{H} = (\omega_y H_z - \omega_z H_y)\bar{i} + (\omega_z H_x - \omega_x H_z)\bar{j} + (\omega_x H_y - \omega_y H_x)\bar{k}$$

Setting each component equal to zero, we obtain the three first-order differential equations:

$$(I_{xx}+I_a)\dot{\omega}_x + I_a\dot{\Omega} + (I_{zz}-I_{yy})\omega_y\omega_z = 0$$

$$(I_{yy}+I_t)\dot{\omega}_y + (I_{xx}+I_a-I_{zz}-I_t)\omega_z\omega_x + I_a\Omega\omega_z = 0$$

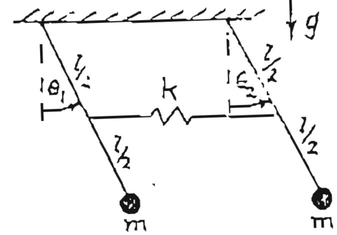
$$(I_{zz}+I_t)\dot{\omega}_z + (I_{yy}+I_t-I_{xx}-I_a)\omega_x\omega_y - I_a\Omega\omega_y = 0$$

In addition, there is the rotor equation

$$I_a(\dot{\omega}_x + \dot{\Omega}) = M_x(t)$$

9-1. $T = \frac{1}{2}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2)$

$V = \frac{1}{2}k\left(\frac{l}{2}\right)^2(\theta_1-\theta_2)^2 + \frac{1}{2}mgl(\theta_1^2+\theta_2^2)$



Symmetric Mode, $\theta_2 = -\theta_1$

Let the amplitude of θ_1 be A_1 and let the frequency be λ.

$T_{max} = ml^2\lambda^2A_1^2$ Rayleigh's method gives $T_{max} = V_{max}$

$V_{max} = \left(\frac{1}{2}kl^2 + mgl\right)A_1^2$ or $\lambda^2 = \dfrac{\frac{1}{2}kl^2+mgl}{ml^2}$, $\lambda_2 = \sqrt{\dfrac{k}{2m} + \dfrac{g}{l}}$

$A_2/A_1 = -1$

Antisymmetric Mode, $\theta_2 = \theta_1$

$T_{max} = ml^2\lambda^2A_1^2$ $\lambda^2 = \dfrac{g}{l}$ $\lambda_1 = \sqrt{\dfrac{g}{l}}$

$V_{max} = mgl A_1^2$ $A_2/A_1 = 1$

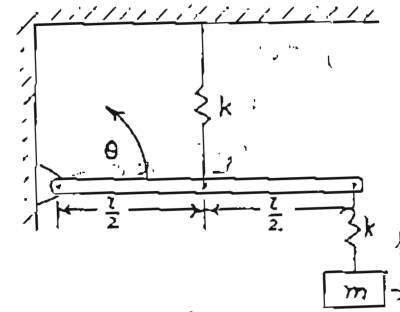
spring is unstressed.

9-2. $x=0$, $\theta=0$ at equilibrium.

$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{6}ml^2\dot{\theta}^2$

$V = \frac{1}{2}k\left(\frac{l\theta}{2}\right)^2 + \frac{1}{2}k(x - l\theta)^2$

$= k\left(\frac{1}{2}x^2 - lx\theta + \frac{5}{8}l^2\theta^2\right)$



Use the order (x, θ). Then

$$[m] = \left[\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right] = \begin{bmatrix} m & 0 \\ 0 & \frac{ml^2}{3} \end{bmatrix}, \quad [k] = \left[\frac{\partial^2 V}{\partial q_i \partial q_j}\right] = \begin{bmatrix} k & -kl \\ -kl & \frac{5}{4}kl^2 \end{bmatrix}$$

9-2. (cont'd.) Equations of motion are

$$m\ddot{x} + kx - kl\theta = 0$$
$$\frac{1}{3}ml^2\ddot{\theta} - klx + \frac{5}{4}kl^2\theta = 0$$

Assume solutions of the form $\cos(\lambda t + \phi)$. The characteristic equation is

$$\begin{vmatrix} (k - m\lambda^2) & -kl \\ -kl & (\frac{5}{4}kl^2 - \frac{1}{3}ml^2\lambda^2) \end{vmatrix} = \frac{m^2l^2}{3}\lambda^4 - \frac{19}{12}kml^2\lambda^2 + \frac{k^2l^2}{4} = 0$$

Roots: $\lambda^2 = \left(\frac{19 \pm \sqrt{313}}{8}\right)\frac{k}{m}$ or $\lambda_1^2 = 0.1635\,\frac{k}{m}$, $\lambda_2^2 = 4.5865\,\frac{k}{m}$

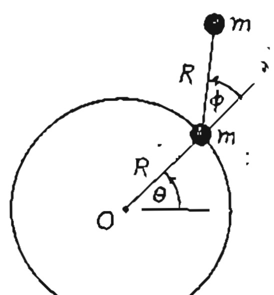
$$\frac{A_\theta}{A_x} = \frac{k - m\lambda^2}{kl} = \frac{1}{l}\left(1 - \frac{m\lambda^2}{k}\right)$$

1st Mode: $\lambda_1 = 0.4044\sqrt{\frac{k}{m}}$

$$\frac{A_{\theta 1}}{A_{x1}} = \frac{0.8365}{l}$$

2nd Mode: $\lambda_2 = 2.1416\sqrt{\frac{k}{m}}$

$$\frac{A_{\theta 2}}{A_{x2}} = \frac{-3.5865}{l}$$



9-3. $T = \frac{1}{2}m(v_1^2 + v_2^2)$

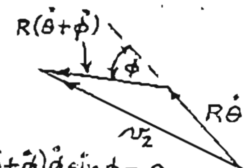
where $v_1^2 = R^2\dot{\theta}^2$

$v_2^2 = R^2[\dot{\theta}^2 + (\dot{\theta}+\dot{\phi})^2 + 2\dot{\theta}(\dot{\theta}+\dot{\phi})\cos\phi]$

So we obtain

$T = \frac{1}{2}mR^2[3\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi} + 2(\dot{\theta}^2 + \dot{\theta}\dot{\phi})\cos\phi]$, $V = 0$.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = 0$$



$\frac{\partial T}{\partial \dot{\theta}} = mR^2[3\dot{\theta} + \dot{\phi} + (2\dot{\theta}+\dot{\phi})\cos\phi]$, $\frac{\partial T}{\partial \theta} = 0$

θ equation: $(3 + 2\cos\phi)\ddot{\theta} + (1+\cos\phi)\ddot{\phi} - (2\dot{\theta}+\dot{\phi})\dot{\phi}\sin\phi = 0$

$\frac{\partial T}{\partial \dot{\phi}} = mR^2(\dot{\phi} + \dot{\theta} + \dot{\theta}\cos\phi)$, $\frac{\partial T}{\partial \phi} = -mR^2(\dot{\theta}^2 + \dot{\theta}\dot{\phi})\sin\phi$

ϕ equation: $mR^2[(1+\cos\phi)\ddot{\theta} + \ddot{\phi} - \dot{\theta}\dot{\phi}\sin\phi + (\dot{\theta}^2 + \dot{\theta}\dot{\phi})\sin\phi] = 0$

or $(1+\cos\phi)\ddot{\theta} + \ddot{\phi} + \dot{\theta}^2\sin\phi = 0$

9-3. (cont'd.) Reference condition: $\dot{\theta} = \dot{\theta}_0$, $\ddot{\theta} = 0$, $\phi = 0$, $\dot{\phi} = 0$, $\ddot{\phi} = 0$

The perturbation equations are

$$5\delta\ddot{\theta} + 2\delta\ddot{\phi} = 0$$
$$2\delta\ddot{\theta} + \delta\ddot{\phi} + \dot{\theta}_0^2\delta\phi = 0$$

Assume solutions of form $\cos(\lambda t + \phi)$. The characteristic equation is

$$\begin{vmatrix} -5\lambda^2 & -2\lambda^2 \\ -2\lambda^2 & (\dot{\theta}_0^2 - \lambda^2) \end{vmatrix} = \lambda^2(\lambda^2 - 5\dot{\theta}_0^2) = 0 \quad \text{with roots} \quad \lambda_1^2 = 0, \; \lambda_2^2 = 5\dot{\theta}_0^2$$

Amplitude ratios are obtained from

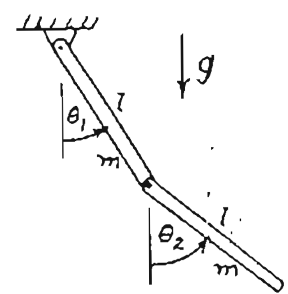
$$-5\lambda^2 A_\theta - 2\lambda^2 A_\phi = 0$$
$$-2\lambda^2 A_\theta + (\dot{\theta}_0^2 - \lambda^2)A_\phi = 0$$

Non-zero frequency mode is

$$\lambda_2 = \sqrt{5}\,\dot{\theta}_0, \quad \frac{A_\phi}{A_\theta} = -\frac{5}{2}$$

9-4. Assuming small motions, the velocity of the c.m. of the lower rod is

$$v_2 = l(\dot{\theta}_1 + \tfrac{1}{2}\dot{\theta}_2)$$



Then

$$T = \frac{ml^2}{6}\dot{\theta}_1^2 + \frac{m}{2}l^2(\dot{\theta}_1 + \tfrac{1}{2}\dot{\theta}_2)^2 + \frac{ml^2}{24}\dot{\theta}_2^2$$
$$= ml^2\left(\tfrac{2}{3}\dot{\theta}_1^2 + \tfrac{1}{6}\dot{\theta}_2^2 + \tfrac{1}{2}\dot{\theta}_1\dot{\theta}_2\right)$$

Also

$$V = -\frac{mgl}{2}\cos\theta_1 - mgl(\cos\theta_1 + \tfrac{1}{2}\cos\theta_2)$$
$$= -mgl\left(\tfrac{3}{2}\cos\theta_1 + \tfrac{1}{2}\cos\theta_2\right)$$

The reference equilibrium condition is $\theta_1 = \theta_2 = 0$.

$$[m] = \left[\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right] = ml^2\begin{bmatrix} \frac{4}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}, \quad [k] = \left[\frac{\partial^2 V}{\partial q_i \partial q_j}\right] = mgl\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Characteristic equation:

$$\begin{vmatrix} (\frac{3}{2}mgl - \frac{4}{3}ml^2\lambda^2) & -\frac{1}{2}ml^2\lambda^2 \\ -\frac{1}{2}ml^2\lambda^2 & (\frac{1}{2}mgl - \frac{1}{3}ml^2\lambda^2) \end{vmatrix} = 0$$

or $\frac{7}{36}m^2l^4\lambda^4 - \frac{7}{6}m^2gl^3\lambda^2 + \frac{3}{4}m^2g^2l^2 = 0$

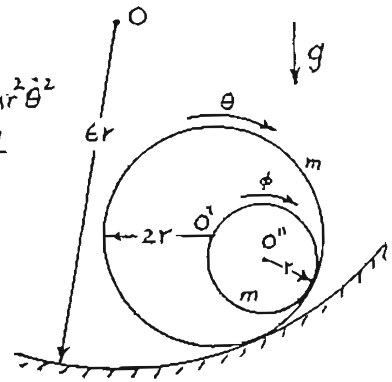
or $7\lambda^2 - 42\frac{g}{l}\lambda^2 + 27\frac{g^2}{l^2} = 0$, $\lambda_1^2 = 0.7322\frac{g}{l}$, $\lambda_2^2 = 5.2678\frac{g}{l}$

9-4. (cont'd.) $\frac{A_2}{A_1} = \frac{\frac{3}{2}mgl - \frac{4}{3}ml^2\lambda^2}{\frac{1}{2}ml^2\lambda^2} = \frac{\frac{3}{2}\frac{g}{l} - \frac{4}{3}\lambda^2}{\frac{1}{2}\lambda^2} = \frac{9\frac{g}{l}-8\lambda^2}{3\lambda^2}$

1st Mode: $\lambda_1 = 0.8557\sqrt{g/l}$ 2nd Mode: $\lambda_2 = 2.2952\sqrt{g/l}$

$\frac{A_{21}}{A_{11}} = 1.4305$ $\frac{A_{22}}{A_{12}} = -2.0972$

9-5.



Shell $I_s = 4mr^2$, $v_s = v_{o'} = 2r\dot{\theta}$

Kinetic energy $T_s = \frac{m}{2}v_s^2 + \frac{I_s}{2}\dot{\theta}^2 = 4mr^2\dot{\theta}^2$

Potential energy $V_s = -4mgr\cos\frac{\theta}{2}$

since $v_{o'} = 4r\dot{\alpha} = 2r\dot{\theta}$ so $\dot{\alpha} = \dot{\theta}/2$

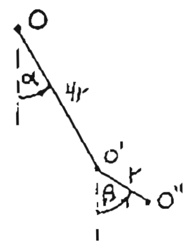
$\alpha = \theta/2$

Assuming small motion,

$v_{o''} = v_{o'} + r\dot{\beta} = r\dot{\phi}$ or $\dot{\beta} = \dot{\phi} - 2\dot{\theta}$

$\beta = \phi - 2\theta$

Cylinder $I_c = \frac{mr^2}{2}$, $v_c = r\dot{\phi}$



$T_c = \frac{1}{2}m(r\dot{\phi})^2 + \frac{mr^2}{4}\dot{\phi}^2 = \frac{3}{4}mr^2\dot{\phi}^2$

$V_c = -mg(4r\cos\alpha + r\cos\beta)$

$= -mgr\left[4\cos\frac{\theta}{2} + \cos(\phi - 2\theta)\right]$

$\alpha = \frac{1}{2}\theta$

$\beta = \phi - 2\theta$

Equilibrium is at $\theta = 0$, $\phi = 0$

For the system,

$T = T_s + T_c = mr^2\left(4\dot{\theta}^2 + \frac{3}{4}\dot{\phi}^2\right)$

$V = V_s + V_c = -mgr\left[8\cos\frac{\theta}{2} + \cos(\phi - 2\theta)\right]$

Using the order (θ, ϕ),

$$[m] = \left[\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right] = mr^2\begin{bmatrix} 8 & 0 \\ 0 & 3/2 \end{bmatrix}, \quad [k] = \left[\frac{\partial^2 V}{\partial q_i \partial q_j}\right] = mgr\begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix}$$

9-5. (cont'd.) The characteristic equation is

$$\begin{vmatrix} (6mgr - 8mr^2\lambda^2) & -2mgr \\ -2mgr & (mgr - \frac{3}{2}mr^2\lambda^2) \end{vmatrix} = 0 \text{ or } 12\lambda^4 - 17\frac{g}{r}\lambda^2 + 2\frac{g^2}{r^2} = 0$$

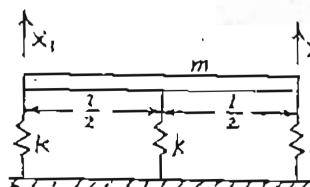
$$\lambda_{1,2}^2 = \left(\frac{17 \mp \sqrt{193}}{24}\right)\frac{g}{r}$$

First Mode: $\lambda_1^2 = 0.1295\frac{g}{r}$ Second Mode: $\lambda_2^2 = 1.2872$

$\frac{A_\phi}{A_\theta} = 3 - 4\frac{r}{g}\lambda^2$ $\lambda_1 = 0.3598\sqrt{\frac{g}{r}}$ $\lambda_2 = 1.1345\sqrt{\frac{g}{r}}$

$\frac{A_{\phi 1}}{A_{\theta 1}} = 2.4821$ $\frac{A_{\phi 2}}{A_{\theta 2}} = -2.1487$

9-6.



$T = \frac{1}{2}m\left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)^2 + \frac{ml^2}{24}\left(\frac{\dot{x}_2 - \dot{x}_1}{l}\right)^2$

$= \frac{m}{6}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_1\dot{x}_2)$

$V = \frac{k}{2}\left[x_1^2 + x_2^2 + \left(\frac{x_1 + x_2}{2}\right)^2\right] = \frac{k}{8}(5x_1^2 + 5x_2^2 + 2x_1x_2)$

Symmetric Mode $x_2 = x_1, \dot{x}_2 = \dot{x}_1$, $T = \frac{1}{2}m\dot{x}_1^2$, $V = \frac{3}{2}kx_1^2$

Equation of motion: $m\ddot{x}_1 + 3kx_1 = 0$, $\lambda_1 = \sqrt{\frac{3k}{m}}$

$A_{21}/A_{11} = 1$

Antisymmetric Mode $x_2 = -x_1$ $T = \frac{1}{6}m\dot{x}_1^2$

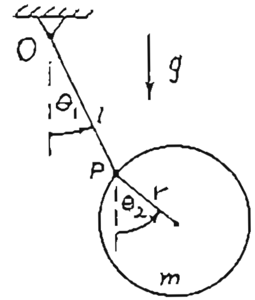
$\dot{x}_2 = -\dot{x}_1$ $V = kx_1^2$

Equation of motion: $\frac{1}{3}m\ddot{x}_1 + 2kx_1 = 0$

$\lambda_1 = \sqrt{\frac{6k}{m}}$

$A_{22}/A_{12} = -1$

9-7. (a) Assume small motion about $\theta_1 = \theta_2 = 0$.



$$T = \tfrac{1}{2}m(l\dot{\theta}_1 + r\dot{\theta}_2)^2 + \tfrac{1}{4}mr^2\dot{\theta}_2^2$$
$$= \tfrac{1}{2}ml^2\dot{\theta}_1^2 + \tfrac{3}{4}mr^2\dot{\theta}_2^2 + mlr\dot{\theta}_1\dot{\theta}_2$$

$$V = -mg(l\cos\theta_1 + r\cos\theta_2)$$

$$m_{ij} = \left(\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right)_0, \quad k_{ij} = \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0$$

$$[m] = \begin{bmatrix} ml^2 & mlr \\ mlr & \frac{3}{2}mr^2 \end{bmatrix}, \quad [k] = \begin{bmatrix} mgl & 0 \\ 0 & mgr \end{bmatrix}$$

Equations of motion:

$$[m]\{\ddot{q}\} + [k]\{q\} = \{0\} \quad \text{or} \quad \begin{aligned} &ml^2\ddot{\theta}_1 + mlr\ddot{\theta}_2 + mgl\theta_1 = 0 \\ &mlr\ddot{\theta}_1 + \tfrac{3}{2}mr^2\ddot{\theta}_2 + mgr\theta_2 = 0 \end{aligned}$$

(b) Let $l = 2r$. The characteristic equation is

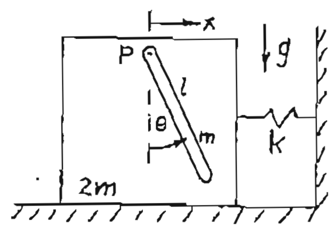
$$\begin{vmatrix} (2mgr - 4mr^2\lambda^2) & -2mr^2\lambda^2 \\ -2mr^2\lambda^2 & (mgr - \frac{3}{2}mr^2\lambda^2) \end{vmatrix} = 2m^2r^4\lambda^4 - 7m^2gr^3\lambda^2 + 2m^2g^2r^2 = 0$$

$$\lambda_{1,2}^2 = \left[\frac{7}{4} \pm \sqrt{\frac{49}{16} - 1}\right]\frac{g}{r} = 0.3139\frac{g}{r}, \; 3.1861\frac{g}{r}, \quad \frac{A_2}{A_1} = \frac{\frac{g}{r} - 2\lambda^2}{\lambda^2}$$

1st Mode: $\lambda_1 = 0.5602\sqrt{\frac{g}{r}}$ 2nd Mode: $\lambda_2 = 1.7850\sqrt{\frac{g}{r}}$

$$\frac{A_{21}}{A_{11}} = 1.1861 \qquad \frac{A_{22}}{A_{12}} = -1.6861$$

9-8. $k/m = g/l$



$$T = m\dot{x}^2 + \tfrac{1}{2}m(\dot{x} + \tfrac{1}{2}l\dot{\theta})^2 + \frac{ml^2}{24}\dot{\theta}^2$$
$$= \tfrac{3}{2}m\dot{x}^2 + \tfrac{1}{6}ml^2\dot{\theta}^2 + \tfrac{1}{2}ml\dot{x}\dot{\theta}$$

$$V = \tfrac{1}{2}kx^2 - \tfrac{1}{2}mgl\cos\theta$$

9-8. (cont'd.) $\therefore m_{ij} = \left(\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right)_0, \quad k_{ij} = \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0$

Use the order (x, θ).

$$[m] = \begin{bmatrix} 3m & \frac{1}{2}ml \\ \frac{1}{2}ml & \frac{1}{3}ml^2 \end{bmatrix}, \quad [k] = \begin{bmatrix} k & 0 \\ 0 & \frac{1}{2}mgl \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & \frac{1}{2}kl^2 \end{bmatrix}$$

The characteristic equation is

$$\begin{vmatrix} (k - 3m\lambda^2) & -\frac{1}{2}ml\lambda^2 \\ -\frac{1}{2}ml\lambda^2 & (\frac{1}{2}kl^2 - \frac{1}{3}ml^2\lambda^2) \end{vmatrix} = \tfrac{3}{4}m^2l^2\lambda^4 - \tfrac{11}{6}mkl^2\lambda^2 + \tfrac{1}{2}k^2l^2 = 0$$

$$\text{or} \quad \lambda^4 - \frac{22}{9}\frac{k}{m}\lambda^2 + \frac{2}{3}\frac{k^2}{m^2} = 0$$

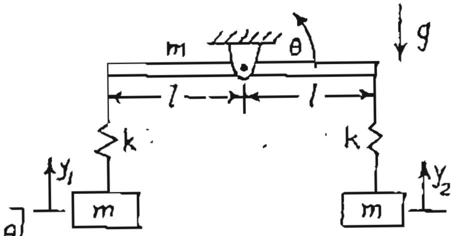
$$\frac{A_\theta}{A_x} = \frac{1 - 3\frac{m}{k}\lambda^2}{\frac{1}{2}l\frac{m}{k}\lambda^2}$$

$$\lambda_{1,2}^2 = \left[\frac{11}{9} \pm \sqrt{\left(\frac{11}{9}\right)^2 - \frac{2}{3}}\right]\frac{k}{m} = 0.3127\frac{k}{m}, \; 2.1317\frac{k}{m}$$

1st Mode: $\lambda_1 = 0.5592\sqrt{\frac{k}{m}}$ 2nd Mode: $\lambda_2 = 1.4600\sqrt{\frac{k}{m}}$

$$\frac{A_{\theta 1}}{A_{x1}} = \frac{0.3951}{l} \qquad \frac{A_{\theta 2}}{A_{x2}} = \frac{-5.0618}{l}$$

9-9. $I = \dfrac{m(2l)^2}{12} = \dfrac{ml^2}{3}$



$$T = \tfrac{1}{2}m\dot{y}_1^2 + \tfrac{1}{2}m\dot{y}_2^2 + \tfrac{1}{6}ml^2\dot{\theta}^2$$

$$V = \tfrac{1}{2}k(y_1 + l\theta)^2 + \tfrac{1}{2}k(y_2 - l\theta)^2$$
$$= \tfrac{1}{2}k\left[y_1^2 + y_2^2 + 2l^2\theta^2 + 2l(y_1 - y_2)\theta\right]$$

Symmetric Mode: $y_2 = y_1$, $\theta = 0$. $T = m\dot{y}_1^2$, $V = ky^2$

$$\lambda^2 = \frac{k}{m} \qquad \lambda_2 = \sqrt{\frac{k}{m}}, \quad \frac{A_{22}}{A_{12}} = 1, \quad \frac{A_{\theta 2}}{A_{12}} = 0$$

Antisymmetric Modes: $y_2 = -y_1$

$$T = m\dot{y}_1^2 + \tfrac{1}{6}ml^2\dot{\theta}^2, \quad V = k(y_1^2 + l^2\theta^2 + 2ly_1\theta)$$

9-9. (cont'd.) Take the order (y_1, θ).

$$[m] = \left[\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right] = \begin{bmatrix} 2m & 0 \\ 0 & \frac{1}{3}ml^2 \end{bmatrix}, \quad [k] = \left[\frac{\partial^2 V}{\partial q_i \partial q_j}\right] = \begin{bmatrix} 2k & 2kl \\ 2kl & 2kl^2 \end{bmatrix}$$

The characteristic equation is

$$\begin{vmatrix} (2k-2m\lambda^2) & 2kl \\ 2kl & (2kl^2 - \frac{1}{3}ml^2\lambda^2) \end{vmatrix} = \frac{2}{3}m^2l^2\lambda^4 - \frac{14}{3}mkl^2\lambda^2 = 0$$

$$\lambda_1^2 = 0, \quad \lambda_3^2 = \frac{7k}{m}$$

$$\frac{A_\theta}{A_1} = \frac{1}{l}\left(\frac{m}{k}\lambda^2 - 1\right)$$

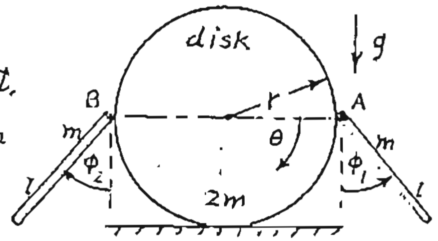
First Mode: $\lambda_1 = 0$

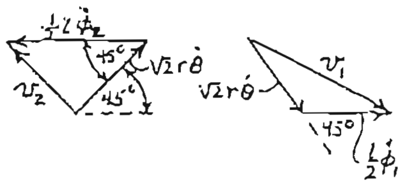
$$\frac{A_{21}}{A_{11}} = -1, \quad \frac{A_{\theta 1}}{A_{11}} = \frac{-1}{l}$$

3rd Mode: $\lambda_3 = 2.6458\sqrt{\frac{k}{m}}$

$$\frac{A_{23}}{A_{13}} = -1, \quad \frac{A_{\theta 3}}{A_{13}} = \frac{6}{l}$$

9-10. The velocity of each pivot is $\sqrt{2}r\dot{\theta}$ and is due to rotation about the contact point. The velocity of the c.m. of each rod is the vector sum of its pivot velocity and the velocity of its c.m. relative to the pivot. Assuming small ϕ_1, ϕ_2, we have





$$v_1^2 = 2r^2\dot{\theta}^2 + \frac{1}{4}l^2\dot{\phi}_1^2 + rl\dot{\theta}\dot{\phi}_1$$

$$v_2^2 = 2r^2\dot{\theta}^2 + \frac{1}{4}l^2\dot{\phi}_2^2 - rl\dot{\theta}\dot{\phi}_2$$

The kinetic energy of the disk is $T_d = \frac{3}{2}mr^2\dot{\theta}^2$.
Including the rotational kinetic energy of the rods, we obtain

$$T = \frac{7}{2}mr^2\dot{\theta}^2 + \frac{ml^2}{6}(\dot{\phi}_1^2 + \dot{\phi}_2^2) + \frac{1}{2}mrl\dot{\theta}(\dot{\phi}_1 - \dot{\phi}_2)$$

as the total kinetic energy of the system.

9-10. (cont'd.) $V = -\frac{1}{2}mgl(\cos\phi_1 + \cos\phi_2)$

Symmetric Mode: $\theta = 0, \quad \phi_2 = \phi_1$

$$T = \frac{ml^2}{3}\dot{\phi}_1^2, \quad V = -mgl\cos\phi_1$$

$$m_{11} = \left(\frac{\partial^2 T}{\partial \dot{\phi}_1^2}\right)_0 = \frac{2ml^2}{3}, \quad k_{11} = \left(\frac{\partial^2 V}{\partial \phi_1^2}\right)_0 = mgl, \quad \lambda_2^2 = \frac{k_{11}}{m_{11}} = \frac{3g}{2l}$$

$$\lambda_2 = \sqrt{\frac{3g}{2l}}, \quad \frac{A_{22}}{A_{12}} = 1, \quad \frac{A_{\theta 2}}{A_{12}} = 0$$

Antisymmetric Modes: $\phi_2 = -\phi_1$

$$T = \frac{7}{2}mr^2\dot{\theta}^2 + \frac{ml^2}{3}\dot{\phi}_1^2 + mrl\dot{\theta}\dot{\phi}_1, \quad V = -mgl\cos\phi_1$$

Use the coordinate order (ϕ_1, θ).

$$[m] = \left[\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right] = \begin{bmatrix} \frac{2}{3}ml^2 & mrl \\ mrl & 7mr^2 \end{bmatrix}, \quad [k] = \left[\frac{\partial^2 V}{\partial q_i \partial q_j}\right] = \begin{bmatrix} mgl & 0 \\ 0 & 0 \end{bmatrix}$$

The characteristic equation (antisymmetric modes) is

$$\begin{vmatrix} (mgl - \frac{2}{3}ml^2\lambda^2) & -mrl\lambda^2 \\ -mrl\lambda^2 & -7mr^2\lambda^2 \end{vmatrix} = \frac{11}{3}m^2r^2l^2\lambda^4 - 7m^2glr^2\lambda^2 = 0$$

$$\text{or} \quad \lambda^2\left(\lambda^2 - \frac{21g}{11l}\right) = 0$$

1st Mode: $\lambda_1^2 = 0$

$$(mgl - \tfrac{2}{3}ml^2\lambda^2)A_1 - mrl\lambda^2 A_\theta = 0$$

$$A_{11} = A_{21} = 0, \quad A_{\theta 1} = 1$$

$$\lambda_1 = 0$$

3rd Mode: $\lambda_3^2 = \frac{21g}{11l}$

$$\frac{A_\theta}{A_1} = \frac{mgl - \frac{2}{3}ml^2\lambda^2}{mrl\lambda^2}$$

$$\frac{A_{\theta 3}}{A_{13}} = \frac{-l}{7r}, \quad \frac{A_{23}}{A_{13}} = -1$$

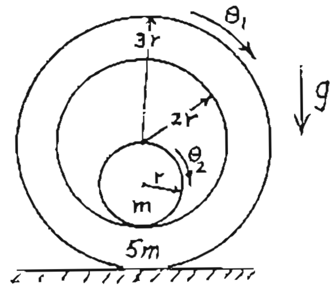
$$\lambda_3 = \sqrt{\frac{21g}{11l}}$$

9-11. Inner cylinder. $I_i = \frac{1}{2}mr^2$

The velocity of its c.m. is

$$v_i = r(\dot\theta_1+\dot\theta_2)$$

which is the velocity of its contact point plus the velocity of its center relative to its contact point. Its kinetic energy is



$$T_i = \frac{1}{2}mr^2(\dot\theta_1+\dot\theta_2)^2 + \frac{1}{4}mr^2\dot\theta_2^2$$
$$= \frac{1}{2}mr^2\dot\theta_1^2 + \frac{3}{4}mr^2\dot\theta_2^2 + mr^2\dot\theta_1\dot\theta_2$$

Outer cylinder. The square of the radius of gyration, assuming a constant density ρ (mass per unit area), is

$$\frac{\int_{2r}^{3r} 2\pi\rho r^3 dr}{\pi\rho[(3r)^2-(2r)^2]} = \frac{1}{5r^2}\left[\frac{r^4}{2}\right]_{2r}^{3r} = \frac{65r^4}{10r^2} = 6.5r^2$$

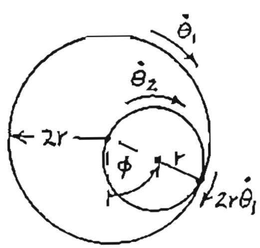
Hence, the moment of inertia about its center is $I_o = 32.5mr^2$

Then its kinetic energy is $T_o = \frac{5m}{2}(3r\dot\theta_1)^2 + \frac{32.5mr^2}{2}\dot\theta_1^2 = \frac{155}{4}mr^2\dot\theta_1^2$

The total kinetic energy of the system is

$$T = T_i + T_o = \frac{mr^2}{4}\left(157\dot\theta_1^2 + 3\dot\theta_2^2 + 4\dot\theta_1\dot\theta_2\right)$$

Potential Energy. The velocity of the center of the solid cylinder relative to the center of the outer cylinder is $r\dot\phi = r\dot\theta_2 - 2r\dot\theta_1$ giving $\phi = \theta_2 - 2\theta_1$.



$$V = -mgr\cos\phi = -mgr\cos(\theta_2-2\theta_1)$$

$$[m] = \left[\frac{\partial^2 T}{\partial\dot q_i\,\partial\dot q_j}\right] = mr^2\begin{bmatrix}\frac{157}{2} & 1\\ 1 & \frac{3}{2}\end{bmatrix},\quad [k] = \left[\frac{\partial^2 V}{\partial q_i\,\partial q_j}\right] = mgr\begin{bmatrix}4 & -2\\ -2 & 1\end{bmatrix}$$

We assume small motion about $\theta_1=\theta_2=0$, or $\phi=0$.

9-11. (cont'd.) The characteristic equation is

$$\begin{vmatrix}(4mgr - \frac{157}{2}mr^2\lambda^2) & (-2mgr - mr^2\lambda^2)\\ (-2mgr - mr^2\lambda^2) & (mgr - \frac{3}{2}mr^2\lambda^2)\end{vmatrix} = \frac{467}{4}m^2r^4\lambda^4 - \frac{177}{2}m^2gr^3\lambda^2$$

$$\lambda_{1,2}^2 = 0,\ \frac{354\,g}{467\,r}$$

$$\frac{A_2}{A_1} = \frac{8 - 157\frac{r}{g}\lambda^2}{4 + 2\frac{r}{g}\lambda^2}$$

1st Mode: $\lambda_1 = 0$

$$\frac{A_{21}}{A_{11}} = 2$$

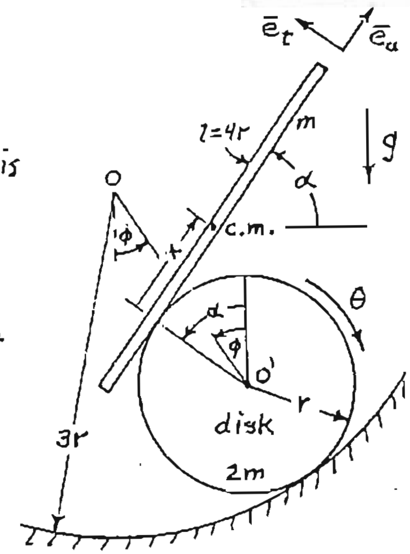
2nd Mode: $\lambda_2 = \sqrt{\frac{354\,g}{467\,r}} = 0.8706\sqrt{\frac{g}{r}}$

$$\frac{A_{22}}{A_{12}} = \frac{-161}{8} = -20.125$$

9-12. $x = r(\theta+\alpha)$

$v_{o'} = r\dot\theta = 2r\dot\phi,\quad \dot\phi = \frac{1}{2}\dot\theta$

The velocity of the rod c.m. is



$$\bar v_c = \bar v_{o'} + v_{c/o'}$$
$$= r\dot\theta[1+\cos(\alpha-\theta)]\,\bar e_\alpha + [x\dot\alpha - r\dot\theta\sin(\alpha-\theta)]\,\bar e_t$$
$$\cong 2r\dot\theta\,\bar e_\alpha \text{ to first order}$$

$$I_{rod} = \frac{m(4r)^2}{12} = \frac{4}{3}mr^2$$

Hence

$$T_{rod} = 2mr^2\dot\theta^2 + \frac{2}{3}mr^2\dot\alpha^2$$

$$V_{rod} = mg(-2r\cos\phi + r\cos\alpha + x\sin\alpha) = mgr\left[-2\cos\frac{\theta}{2} + \cos\alpha + (\theta+\alpha)\sin\alpha\right]$$

$I_{disk} = mr^2,\quad v_{o'} = r\dot\theta,\quad T_{disk} = \frac{3}{2}mr^2\dot\theta^2$

$$V_{disk} = -4mgr\cos\phi = -4mgr\cos\frac{\theta}{2}$$

Total $T = \frac{7}{2}mr^2\dot\theta^2 + \frac{2}{3}mr^2\dot\alpha^2$

Total $V = mgr\left[-6\cos\frac{\theta}{2} + \cos\alpha + (\theta+\alpha)\sin\alpha\right]$

9-12. (cont'd.) Assume the order (θ, α) and obtain

$$[m]=\left[\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right]=mr^2\begin{bmatrix}7 & 0\\ 0 & \frac{4}{3}\end{bmatrix}, \quad [k]=\left[\frac{\partial^2 V}{\partial q_i \partial q_j}\right]=mgr\begin{bmatrix}\frac{3}{2} & 1\\ 1 & 1\end{bmatrix}$$

Characteristic equation:

$$\begin{vmatrix}(\frac{3}{2}mgr-7mr^2\lambda^2) & mgr\\ mgr & (mgr-\frac{4}{3}mr^2\lambda^2)\end{vmatrix}=\frac{28}{3}m^2r^4\lambda^4-9m^2gr^3\lambda^2+\frac{1}{2}m^2g^2r^2=0$$

or $\lambda^4-\frac{27}{28}\frac{g}{r}\lambda^2+\frac{3g^2}{56r^2}=0$, $\quad \lambda^2_{1,2}=0.0592\frac{g}{r}, \ 0.9051\frac{g}{r}$

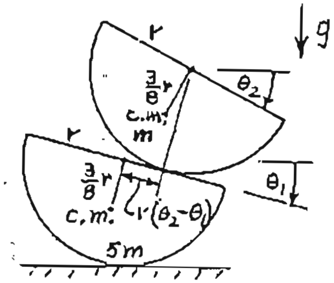
$\frac{A_\alpha}{A_\theta}=7\frac{r}{g}\lambda^2-\frac{3}{2}$,

First Mode: $\lambda_1=0.2433\sqrt{\frac{g}{r}}$, $\quad \frac{A_{\alpha 1}}{A_{\theta 1}}=-1.0857$

Second Mode: $\lambda_2=0.9514\sqrt{\frac{g}{r}}$, $\quad \frac{A_{\alpha 2}}{A_{\theta 2}}=4.8357$

9-13. About the individual c.m.,

$I_1=\frac{83}{64}mr^2, \quad I_2=\frac{83}{320}mr^2$



The c.m. velocities are

$v_1=\frac{5}{8}r\dot{\theta}_1, \quad v_2=r\dot{\theta}_1+\frac{5}{8}r\dot{\theta}_2$

$$T=\frac{5m}{2}\left(\frac{5}{8}r\dot{\theta}_1\right)^2+\frac{83}{128}mr^2\dot{\theta}_1^2+\frac{1}{2}m\left(r\dot{\theta}_1+\frac{5}{8}r\dot{\theta}_2\right)^2+\frac{83}{640}mr^2\dot{\theta}_2^2=mr^2\left(\frac{17}{8}\dot{\theta}_1^2+\frac{13}{40}\dot{\theta}_2^2+\frac{5}{8}\dot{\theta}_1\dot{\theta}_2\right)$$

$$V=5mg\left(-\frac{3}{8}r\cos\theta_1\right)+mg\left[-r(\theta_2-\theta_1)\sin\theta_1+r\cos\theta_1-\frac{3}{8}r\cos\theta_2\right]$$
$$=-\frac{7}{8}mgr\cos\theta_1-\frac{3}{8}mgr\cos\theta_2-mgr(\theta_2-\theta_1)\sin\theta_1$$

$$[m]=\left[\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right]=mr^2\begin{bmatrix}\frac{17}{4} & \frac{5}{8}\\ \frac{5}{8} & \frac{13}{20}\end{bmatrix}, \quad [k]=\left[\frac{\partial^2 V}{\partial q_i \partial q_j}\right]=mgr\begin{bmatrix}\frac{23}{8} & -1\\ -1 & \frac{3}{8}\end{bmatrix}$$

9-13. (cont'd.) The characteristic equation is

$$\begin{vmatrix}(\frac{23}{8}mgr-\frac{17}{4}mr^2\lambda^2) & (-mgr-\frac{5}{8}mr^2\lambda^2)\\ (-mgr-\frac{5}{8}mr^2\lambda^2) & (\frac{3}{8}mgr-\frac{13}{20}mr^2\lambda^2)\end{vmatrix}=\frac{759}{320}m^2r^4\lambda^4-\frac{377}{80}m^2gr^3\lambda^2+\frac{5}{64}m^2g^2r^2=0$$

or

$$\lambda^4-\frac{1508}{759}\frac{g}{r}\lambda^2+\frac{25}{759}\frac{g^2}{r^2}=0$$

$\lambda_1^2=0.01672\frac{g}{r}, \quad \lambda_2^2=1.9701\frac{g}{r}$

$$\frac{A_2}{A_1}=\frac{1+\frac{5}{8}\frac{r}{g}\lambda^2}{\frac{3}{8}-\frac{13}{20}\frac{r}{g}\lambda^2}$$

1st Mode:

$\lambda_1=0.1293\sqrt{\frac{g}{r}}$

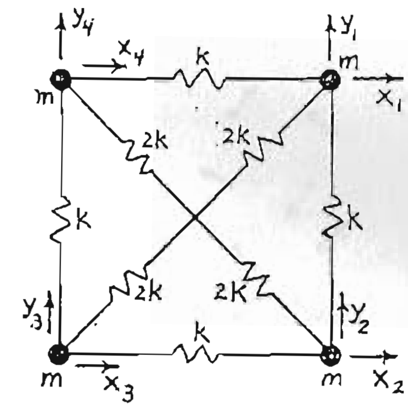
$\frac{A_{21}}{A_{11}}=2.7749$

2nd Mode:

$\lambda_2=1.4036\sqrt{\frac{g}{r}}$

$\frac{A_{22}}{A_{12}}=-2.4640$

9-14. Use symmetry. First, consider a vertical plane of symmetry through the center.



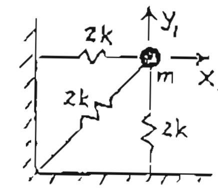
Symmetric Modes:

$x_4=-x_1, \quad y_4=y_1$

$x_3=-x_2, \quad y_3=y_2$

Now use a horizontal plane of symmetry through the center to obtain two cases.

Case 1: Symmetric-Symmetric.



$x_2=x_1 \qquad y_2=-y_1$

$x_3=-x_1 \qquad y_3=-y_1$

$x_4=-x_1 \qquad y_4=y_1$

$$T=\frac{1}{2}m(\dot{x}_1^2+\dot{y}_1^2), \quad V=k\left[x_1^2+y_1^2+\left(\frac{x_1+y_1}{\sqrt{2}}\right)^2\right]=k\left(\frac{3}{2}x_1^2+\frac{3}{2}y_1^2+x_1y_1\right)$$

$$[m]=\left[\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right]=\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}, \quad [k]=\left[\frac{\partial^2 V}{\partial q_i \partial q_j}\right]=\begin{bmatrix}3k & k\\ k & 3k\end{bmatrix}$$

9-14. (cont'd.) The characteristic equation is

$$\begin{vmatrix} (3k-m\lambda^2) & k \\ k & (3k-m\lambda^2) \end{vmatrix} = m^2\lambda^4 - 6mk\lambda^2 + 8k^2 = 0\,, \quad \lambda^2 = \frac{2k}{m}\,,\ \frac{4k}{m}$$

$$\frac{A_y}{A_x} = -3 + \frac{m}{k}\lambda^2$$

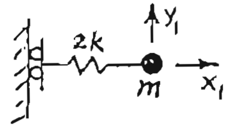
For $\lambda = \sqrt{\frac{2k}{m}}$, $\frac{A_{y1}}{A_{x1}} = -1$ giving

$A_{x1} = A_{x2} = 1$, $A_{x3} = A_{x4} = -1$
$A_{y1} = A_{y4} = -1$, $A_{y2} = A_{y3} = 1$

For $\lambda = 2\sqrt{\frac{k}{m}}$, $\frac{A_{y1}}{A_{x1}} = 1$ giving

$A_{x1} = A_{x2} = 1$, $A_{x3} = A_{x4} = -1$
$A_{y1} = A_{y4} = 1$, $A_{y2} = A_{y3} = -1$

Case 2: Symmetric - Antisymmetric.



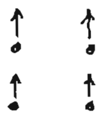
$x_3 = x_1$ $\qquad y_2 = y_1$
$x_2 = -x_1$ $\qquad y_3 = y_1$
$x_4 = -x_1$ $\qquad y_4 = y_1$

$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2)$, $V = kx_1^2$, $[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$, $[k] = \begin{bmatrix} 2k & 0 \\ 0 & 0 \end{bmatrix}$

Characteristic equation:

$$\begin{vmatrix} (2k-m\lambda^2) & 0 \\ 0 & -m\lambda^2 \end{vmatrix} = m^2\lambda^4 - 2mk\lambda^2 = 0\,, \quad \lambda^2 = 0,\ \frac{2k}{m}$$

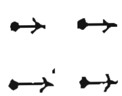
For $\lambda = 0$, $A_{x1} = 0$, $A_{y1} = 1$
$A_{x2} = A_{x3} = A_4 = 0$, $A_{y2} = A_{y3} = A_{y4} = 1$



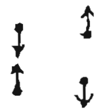
For $\lambda = \sqrt{\frac{2k}{m}}$, $A_{x1} = A_{x3} = 1$, $A_{x2} = A_{x4} = -1$
$A_{y1} = A_{y2} = A_{y3} = A_{y4} = 0$

Case 3: Antisymmetric - Symmetric. This is similar to Case 2, but with x and y interchanged.

For $\lambda = 0$, $A_{x1} = A_{x2} = A_{x3} = A_{x4} = 1$
$A_{y1} = A_{y2} = A_{y3} = A_{y4} = 0$

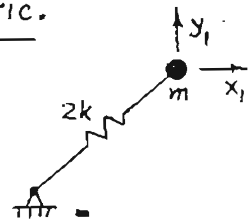


For $\lambda = \sqrt{\frac{2k}{m}}$, $A_{x1} = A_{x2} = A_{x3} = A_{x4} = 0$
$A_{y1} = A_{y3} = 1$, $A_{y2} = A_{y4} = -1$



9-14. (cont'd.) For antisymmetry about the vertical plane, antisymmetry about the horizontal plane, we have

Case 4: Antisymmetric - Antisymmetric.



$x_2 = -x_1$ $\qquad y_2 = y_1$
$x_3 = -x_1$ $\qquad y_3 = -y_1$
$x_4 = x_1$ $\qquad y_4 = -y_1$

$$T = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2)\,, \quad V = k\left(\frac{x_1 + y_1}{\sqrt{2}}\right)^2 = \frac{k}{2}(x_1^2 + y_1^2 + 2x_1y_1)$$

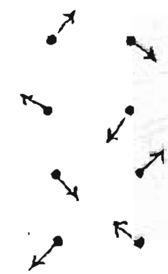
$$[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad [k] = \begin{bmatrix} k & k \\ k & k \end{bmatrix}, \quad \begin{vmatrix} (k-m\lambda^2) & k \\ k & (k-m\lambda^2) \end{vmatrix} = m^2\lambda^4 - 2mk\lambda^2 = 0$$

$$\lambda^2 = 0,\ \frac{2k}{m}$$

$$\frac{A_{y1}}{A_{x1}} = -1 + \frac{m}{k}\lambda^2$$

For $\lambda = 0$, $A_{x1} = A_{x4} = 1$, $A_{x2} = A_{x3} = -1$
$A_{y1} = A_{y2} = -1$, $A_{y3} = A_{y4} = 1$

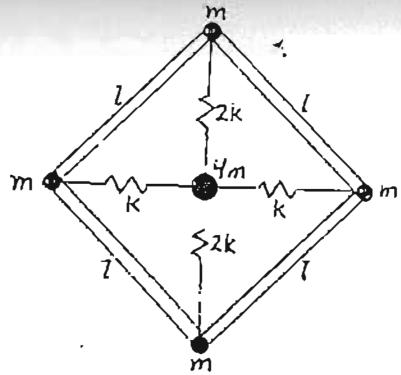
For $\lambda = \sqrt{\frac{2k}{m}}$, $A_{x1} = A_{x4} = 1$, $A_{x2} = A_{x3} = -1$
$A_{y1} = A_{y2} = 1$, $A_{y3} = A_{y4} = -1$



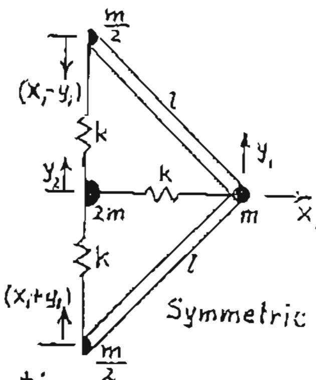
Altogether, there are three roots $\lambda = 0$, four roots $\lambda = \sqrt{\frac{2k}{m}}$, and one root $\lambda = 2\sqrt{\frac{k}{m}}$. For the case of repeated roots, any linear combination of the given A's is also a possible result.

Note that we could have used diagonal symmetry from the outset.

9-15. There are 6 degrees of freedom. The particle $4m$ has 2 degrees of freedom; the frame has 4, namely, 2 translational, one rotational, and one distortional.



Note the vertical plane of symmetry and consider the right-hand half of the system. There is also a horizontal plane of symmetry.



Symmetric - Symmetric Case.

$y_1 = y_2 = 0$. $2m$ is fixed.

$T = m\dot{x}_1^2$, $V = \frac{3}{2}kx_1^2$

Using Rayleigh's method.

$\lambda^2 = \frac{3k}{2m}$ or $\lambda = \sqrt{\frac{3k}{2m}}$ (frame distortion mode)

Symmetric-Antisymmetric Case. $x_1 = 0$. Frame is rigid.

$T = m(\dot{y}_1^2+\dot{y}_2^2)$, $V = k(y_1-y_2)^2 = k(y_1^2+y_2^2-2y_1y_2)$

$[m] = \begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix}$, $[k] = \begin{bmatrix} 2k & -2k \\ -2k & 2k \end{bmatrix}$ since $m_{ij} = \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}$, $k_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}$

Characteristic equation is

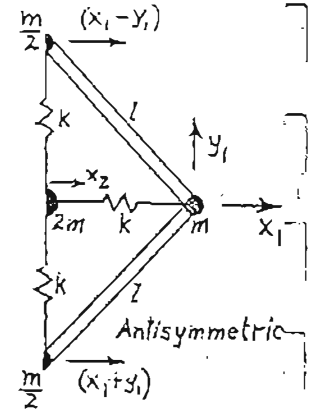
$$\begin{vmatrix} (2k-2m\lambda^2) & -2k \\ -2k & (2k-2m\lambda^2) \end{vmatrix} = 4m^2\lambda^4 - 8mk\lambda^2 = 0, \quad \lambda^2 = 0, \ \frac{2k}{m}$$

For $\lambda = 0$, $y_1 = y_2$ (y translation mode)   For $\lambda = \sqrt{\frac{2k}{m}}$, $y_2 = -y_1$

Antisymmetric-Antisymmetric Case. $x_1 = x_2 = 0$

$\lambda = 0$ Rigid body rotation. (See antisymmetric figure.)

9-15. (cont'd.)



Antisymmetric - Symmetric Case.

$y_1 = 0$. $T = m(\dot{x}_1^2+\dot{x}_2^2)$

$V = \frac{1}{2}k(x_1-x_2)^2 = \frac{1}{2}k(x_1^2+x_2^2-2x_1x_2)$

$[m] = \left[\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right] = \begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix}$

$[k] = \left[\frac{\partial^2 V}{\partial q_i \partial q_j}\right] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

Characteristic equation is

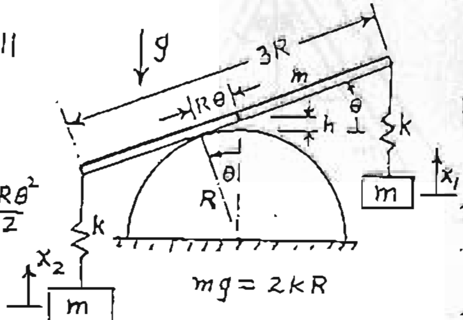
$$\begin{vmatrix} (k-2m\lambda^2) & -k \\ -k & (k-2m\lambda^2) \end{vmatrix} = 4m^2\lambda^4 - 4mk\lambda^2 = 0, \quad \lambda^2 = 0, \ \frac{k}{m}$$

For $\lambda = 0$, $x_1 = x_2$ (x translation mode), For $\lambda = \sqrt{\frac{k}{m}}$, $x_2 = -x_1$

9-16. x_1, x_2, and θ are small and are measured from equilibrium. To second order,

$$h = R\theta\sin\theta - R(1-\cos\theta) \cong \frac{R\theta^2}{2}$$

$$\dot{h} = R\theta\dot{\theta}$$



$mg = 2kR$

Noting that the $\dot{h}^2$ term can be neglected, we obtain

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{3}{8}mR^2\dot{\theta}^2$$

The stretch of the spring from the unstressed condition:

$$\delta_1 = \frac{mg}{k} + \frac{3}{2}R\theta - x_1 + h$$

$$\delta_2 = \frac{mg}{k} - \frac{3}{2}R\theta - x_2 + h$$

The total potential energy is

$$V = \frac{1}{2}k(\delta_1^2+\delta_2^2) + mg(x_1+x_2) + mgh$$

Omitting constant terms and terms of higher than second ord

$$V = \frac{1}{2}k(x_1^2+x_2^2) + \frac{9}{4}kR^2\theta^2 + \frac{3}{2}mgR\theta^2 + \frac{3}{2}kR\theta(x_2-x_1)$$

9-16. (cont'd.) $\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$

The equations of motion are

x_1 equation: $\quad m\ddot x_1 + kx_1 - \frac32 kR\theta = 0$

x_2 equation: $\quad m\ddot x_2 + kx_2 + \frac32 kR\theta = 0$

θ equation: $\quad \frac34 mR^2\ddot\theta + \left(\frac92 kR^2 + 3mgR\right)\theta + \frac32 kR(x_2 - x_1) = 0$

Assuming solutions of the form $\cos(\lambda t+\phi)$, the characteristic equation is

$$\begin{vmatrix} (k-m\lambda^2) & 0 & -\frac32 kR \\ 0 & (k-m\lambda^2) & \frac32 kR \\ -\frac32 kR & \frac32 kR & \left(\frac92 kR^2 + 3mgR - \frac34 mR^2\lambda^2\right) \end{vmatrix} = 0$$

or $\quad (k-m\lambda^2)\left[\frac34 m^2R^2\lambda^4 - \left(\frac{21}{4}mkR^2 + 3m^2gR\right)\lambda^2 + 3mkgR\right] = 0$

But $mg = 2kR$, so the quartic becomes

$$\lambda^4 - 15\frac{k}{m}\lambda^2 + 8\frac{k^2}{m^2} = 0 \quad \text{with} \quad \lambda^2 = \left(\frac{15}{2} \pm \frac{\sqrt{193}}{2}\right)\frac{k}{m} = \begin{cases} 0.5538\,\frac{k}{m} \\ 14.446\,\frac{k}{m} \end{cases}$$

$\frac{A_\theta}{A_1} = \frac{2}{3R}\left(1 - \frac{m}{k}\lambda^2\right)$, $\quad \frac{A_\theta}{A_2} = \frac{-2}{3R}\left(1 - \frac{m}{k}\lambda^2\right)$, $\quad A_2 = A_1$ if $A_\theta = 0$.

First Mode:	Second Mode:	Third Mode:
$\lambda_1 = 0.7442\sqrt{\frac{k}{m}}$	$\lambda_2 = \sqrt{\frac{k}{m}}$	$\lambda_3 = 3.8008\sqrt{\frac{k}{m}}$
$\frac{A_{\theta 1}}{A_{11}} = \frac{0.2975}{R}$	$\frac{A_{\theta 2}}{A_{12}} = 0$	$\frac{A_{\theta 3}}{A_{13}} = \frac{-8.9641}{R}$
$\frac{A_{21}}{A_{11}} = -1$	$\frac{A_{22}}{A_{12}} = 1$	$\frac{A_{23}}{A_{13}} = -1$

$$[m] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac34 mR^2 \end{bmatrix}, \qquad [k] = \begin{bmatrix} k & 0 & -\frac32 kR \\ 0 & k & \frac32 kR \\ -\frac32 kR & \frac32 kR & \frac{21}{2}kR^2 \end{bmatrix}$$

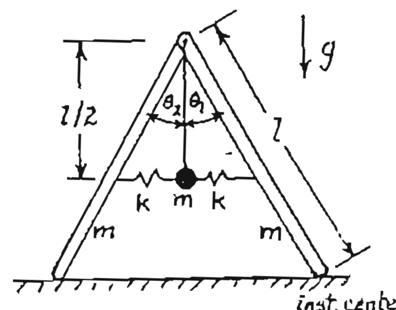
9-16. (cont'd.)

$$[A] = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ \frac{0.2975}{R} & 0 & \frac{-8.9641}{R} \end{bmatrix}$$

$$[M] = [A]^T[m][A] = \begin{bmatrix} 2.0664m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 62.267m \end{bmatrix}$$

$$[K] = [A]^T[k][A] = \begin{bmatrix} 1.1443k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 899.52k \end{bmatrix}$$

9-17. $\theta_1 = \theta_2 = 30^\circ$ at equilibrium.

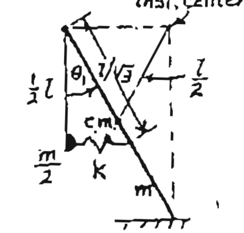


Symmetric Mode. $\theta_1 = \theta_2$.

Particle moves vertically only.

Consider half the system.

$$T = \frac{m}{4}\left(l\dot\theta_1\sin\theta_1\right)^2 + \frac12\left[\frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2\right]\dot\theta_1^2$$
$$= \frac{11}{48}ml^2\dot\theta_1^2$$



Let l_0 = unstressed length of spring.

$$V = \frac12 mgl\cos\theta_1 + mg\frac{l}{2}\cos\theta_1 + \frac12 k\left(\frac{l}{\sqrt3}\sin\theta_1 - l_0\right)^2$$

At equilibrium, $\theta_1 = 30^\circ$, and

$$\frac{\partial V}{\partial \theta_1} = -mgl\sin\theta_1 + k\left(\frac{l}{\sqrt3}\sin\theta_1 - l_0\right)\frac{l}{\sqrt3}\cos\theta_1 = 0$$

with the result $l_0 = \frac{l}{2\sqrt3} - \frac{mg}{k}$.

The stiffness coefficient is

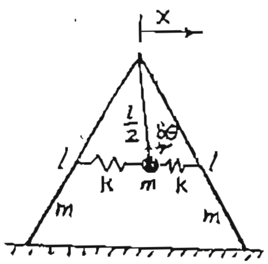
$$k_{11} = \left(\frac{\partial^2 V}{\partial \theta_1^2}\right)_0 = \left[-mgl\cos\theta_1 + \frac{kl^2}{3}\left(\cos^2\theta_1 - \sin^2\theta_1\right) + \frac{kl}{\sqrt3}l_0\sin\theta_1\right]_{\theta_1 = 30^\circ}$$
$$= \frac{kl^2}{4} - \frac{2}{\sqrt3}mgl$$

9-17. (cont'd.) $m_{11} = \frac{\partial^2 T}{\partial \dot\theta_1^2} = \frac{11}{24} m l^2$

$$\lambda^2 = \frac{k_{11}}{m_{11}} = \frac{\frac{kl^2}{4} - \frac{2}{\sqrt3} mgl}{\frac{11}{24} m l^2} = \frac{6k}{11m} - \frac{48 g}{11\sqrt3\, l}$$

The symmetric mode is $\lambda_1 = \sqrt{\frac{6}{11}\left(\frac{k}{m} - \frac{8g}{\sqrt3\, l}\right)}$, $\frac{A_{21}}{A_{11}} = 1$

Antisymmetric Mode. $\delta\theta_2 = -\delta\theta_1$. The two rods form a rigid frame. Let x = horizontal displacement of frame and let $\delta\theta_2$ = angular displacement of pendulum. For a non-zero frequency mode, the translational momentum must be zero.



$$2m\dot x + m\left(\dot x + \frac{l}{2}\delta\dot\theta_2\right) = 0 \quad \text{or} \quad \dot x = -\frac{l}{6}\delta\dot\theta_2$$

Then $T = m\dot x^2 + \frac{m}{2}\left(\dot x + \frac{l}{2}\delta\dot\theta_2\right)^2 = \frac{ml^2}{12}(\delta\dot\theta_2)^2$, $m_{22} = \frac{\partial^2 T}{\partial(\delta\dot\theta_2)^2} = \frac{ml^2}{6}$

$$V = k\left(\frac{l}{2}\delta\theta_2\right)^2 - mg\frac{l}{2}\cos(\delta\theta_2), \quad k_{22} = \left[\frac{\partial^2 V}{\partial(\delta\theta_2)^2}\right]_0 = \frac{kl^2}{2} + \frac{mgl}{2}$$

Antisymmetric mode is $\lambda_2^2 = \frac{k_{22}}{m_{22}} = 3\left(\frac{k}{m} + \frac{g}{l}\right)$

giving $\lambda_2 = \sqrt{3\left(\frac{k}{m} + \frac{g}{l}\right)}$, $\frac{A_{22}}{A_{12}} = -1$

9-18. Use symmetry and consider the right-hand half of the system. At equilibrium, the tensile force in the slanted spring is

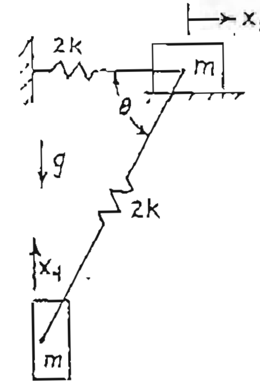
$$P_0 = \frac{2}{\sqrt3} mg$$

and therefore its unstressed length is

$$l_0 = l - \frac{P_0}{2k} = l - \frac{mg}{\sqrt3\, k}$$

9-18. (cont'd.)

Symmetric Modes. $x_2 = -x_1$, $x_3 = 0$.
$\theta = 60°$ at equilibrium.



In general, the length of the slanted spring is

$$L = \sqrt{\left(\frac{l}{2} + x_1\right)^2 + \left(\frac{\sqrt3}{2} l - x_4\right)^2}$$

and its elongation is $\delta = L - l_0$.

Similarly, at equilibrium, the horizontal spring has a compressive force $\frac12 P_0$ and a corresponding initial compression $P_0/4k = mg/2\sqrt3\, k$.

The potential energy for the half-system is

$$V = mgx_4 + k\left(\frac{mg}{2\sqrt3\, k} - x_1\right)^2 + k\delta^2$$

where

$$\delta = L - l_0 = \sqrt{l^2 + x_1^2 + x_4^2 + lx_1 - \sqrt3\, l x_4} - l_0$$

Then

$$\frac{\partial V}{\partial x_1} = -2k\left(\frac{mg}{2\sqrt3\, k} - x_1\right) + 2k(\sqrt{\cdots} - l_0)\left(\frac{2x_1 + l}{2\sqrt{\cdots}}\right)$$

$$= -\frac{mg}{\sqrt3} + 4kx_1 + kl - kl_0\left(\frac{2x_1 + l}{\sqrt{\cdots}}\right)$$

$$k_{11} = \left(\frac{\partial^2 V}{\partial x_1^2}\right)_0 = 4k - kl_0\left[\frac{2}{l} - \frac{(\frac12)(l)(l)}{l^3}\right] = 4k - \frac{3kl_0}{2l}$$

$$= \frac52 k + \frac{\sqrt3\, mg}{2l}$$

$$k_{14} = k_{41} = \left(\frac{\partial^2 V}{\partial x_4 \partial x_1}\right)_0 = -kl_0 l\left[\frac{-\frac12\sqrt3\, l}{l^3}\right] = -\frac{\sqrt3\, k l_0}{2l} = \frac{mg}{2l} - \frac{\sqrt3}{2}k$$

Also,

$$\frac{\partial V}{\partial x_4} = mg + 2k(\sqrt{\cdots} - l_0)\left(\frac{2x_4 - \sqrt3\, l}{2\sqrt{\cdots}}\right)$$

$$= mg + k(2x_4 - \sqrt3\, l) - \frac{kl_0}{\sqrt{\cdots}}(2x_4 - \sqrt3\, l)$$

$$k_{44} = \left(\frac{\partial^2 V}{\partial x_4^2}\right)_0 = 2k - \frac{2kl_0}{l} + \sqrt3\, k l_0 l \frac{(-\frac12)(-\sqrt3\, l)}{l^3} = 2k - \frac{kl_0}{2l} = \frac32 k + \frac{mg}{2\sqrt3\, l}$$

9-18. (cont'd.) Noting that $\frac{mg}{l} = \frac{\sqrt{3}}{8}k$,

the stiffness matrix is

$$[k] = \begin{bmatrix} \left(\frac{5}{2}k + \frac{\sqrt{3}mg}{2l}\right) & \left(\frac{mg}{2l} - \frac{\sqrt{3}}{2}k\right) \\ \left(\frac{mg}{2l} - \frac{\sqrt{3}}{2}k\right) & \left(\frac{3}{2}k + \frac{mg}{2\sqrt{3}l}\right) \end{bmatrix} = \begin{bmatrix} \frac{43}{16}k & \frac{-7\sqrt{3}}{16}k \\ -\frac{7\sqrt{3}}{16}k & \frac{25}{16}k \end{bmatrix}$$

The kinetic energy is $T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_4^2)$

which leads to the mass matrix

$$[m] = \left[\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

The characteristic equation is

$$\begin{vmatrix} \left(\frac{43}{16}k - m\lambda^2\right) & \frac{-7\sqrt{3}}{16}k \\ \frac{-7\sqrt{3}}{16}k & \left(\frac{25}{16}k - m\lambda^2\right) \end{vmatrix} = m^2\lambda^4 - \frac{17}{4}mk\lambda^2 + \frac{29}{8}k^2 = 0$$

$$\lambda^2 = \frac{17}{8} \pm \sqrt{\left(\frac{17}{8}\right)^2 - \frac{29}{8}} = \begin{cases} 1.1813\frac{k}{m} \\ 3.0687\frac{k}{m} \end{cases}$$

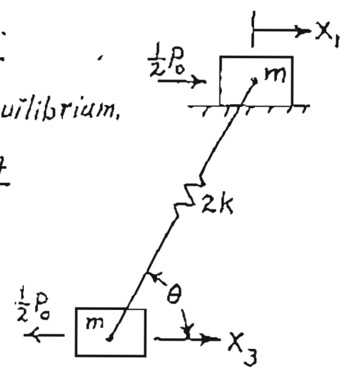
$$\frac{A_4}{A_1} = \frac{43}{7\sqrt{3}} - \frac{16}{7\sqrt{3}}\frac{m}{k}\lambda^2$$

Second Mode:

$\lambda_2 = 1.0869\sqrt{\frac{k}{m}}$, $\frac{A_{42}}{A_{12}} = 1.9877$

Fourth Mode: $\lambda_4 = 1.7518\sqrt{\frac{k}{m}}$, $\frac{A_{44}}{A_{14}} = -0.5031$

Antisymmetric Modes. $x_2 = x_1$, $x_4 = 0$.



As before, $P_o = \frac{2}{\sqrt{3}}mg$, $\theta = 60°$ at equilibrium.

$$L = \sqrt{\left(\frac{\sqrt{3}}{2}l\right)^2 + \left(\frac{l}{2} + x_1 - x_3\right)^2}, \quad l_o = l - \frac{mg}{\sqrt{3}k}$$

Let $\delta = L - l_o$, $\delta_o = \frac{mg}{\sqrt{3}k}$

$$V = \frac{1}{2}P_o(x_3 - x_1) + k\delta^2$$

$$\frac{\partial V}{\partial x_1} = -\frac{1}{2}P_o + 2k\delta\frac{\partial \delta}{\partial x_1} = -\frac{1}{2}P_o + 2k\delta\frac{\left(\frac{1}{2}l + x_1 - x_3\right)}{L}$$

9-18. (cont'd.)

$$k_{11} = \left(\frac{\partial^2 V}{\partial x_1^2}\right)_o = k\left(\frac{\partial \delta}{\partial x_1}\right)_o + \frac{2}{\sqrt{3}}mg\left[\frac{1}{l} - \left(\frac{1}{2}l\right)\frac{\left(\frac{\partial L}{\partial x_1}\right)_o}{l^2}\right]$$

where $\left(\frac{\partial \delta}{\partial x_1}\right)_o = \left(\frac{\partial L}{\partial x_1}\right)_o = \frac{1}{2}$. Hence $k_{11} = \frac{k}{2} + \frac{\sqrt{3}mg}{2l}$

$$k_{13} = k_{31} = \left(\frac{\partial^2 V}{\partial x_3 \partial x_1}\right)_o = k\left(\frac{\partial \delta}{\partial x_3}\right)_o + \frac{2}{\sqrt{3}}mg\left[\frac{-1}{l} - \frac{1}{2l}\left(\frac{\partial L}{\partial x_3}\right)_o\right]$$

where $\left(\frac{\partial \delta}{\partial x_3}\right)_o = \left(\frac{\partial L}{\partial x_3}\right)_o = -\frac{1}{2}$. $k_{13} = k_{31} = -\frac{k}{2} - \frac{\sqrt{3}mg}{2l}$

$$\frac{\partial V}{\partial x_3} = \frac{1}{2}P_o + 2k\delta\frac{\partial \delta}{\partial x_3} = \frac{1}{2}P_o - 2k\delta\frac{\left(\frac{1}{2}l + x_1 - x_3\right)}{L}$$

$$k_{33} = \left(\frac{\partial^2 V}{\partial x_3^2}\right)_o = -k\left(\frac{\partial \delta}{\partial x_3}\right)_o - \frac{2}{\sqrt{3}}mg\left[\frac{-1}{l} - \frac{1}{2l}\left(\frac{\partial L}{\partial x_3}\right)_o\right] = \frac{k}{2} + \frac{\sqrt{3}mg}{2l}$$

The stiffness matrix is

$$[k] = \begin{bmatrix} \left(\frac{k}{2} + \frac{\sqrt{3}mg}{2l}\right) & \left(-\frac{k}{2} - \frac{\sqrt{3}mg}{2l}\right) \\ \left(-\frac{k}{2} - \frac{\sqrt{3}mg}{2l}\right) & \left(\frac{k}{2} + \frac{\sqrt{3}mg}{2l}\right) \end{bmatrix} = \begin{bmatrix} \frac{11}{16}k & -\frac{11}{16}k \\ -\frac{11}{16}k & \frac{11}{16}k \end{bmatrix}$$

$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_3^2)$ so $[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$

The characteristic equation is

$$\begin{vmatrix} \left(\frac{11}{16}k - m\lambda^2\right) & \frac{-11}{16}k \\ -\frac{11}{16}k & \left(\frac{11}{16}k - m\lambda^2\right) \end{vmatrix} = m^2\lambda^4 - \frac{11}{8}mk\lambda^2 = 0, \quad \lambda^2 = 0, \ \frac{11k}{8m}$$

$$\frac{A_3}{A_1} = 1 - \frac{16}{11}\frac{m}{k}\lambda^2$$

First Mode:

$\lambda_1 = 0$

$\frac{A_{31}}{A_{11}} = 1$

Third Mode:

$\lambda_3 = 1.1726\sqrt{\frac{k}{m}}$

$\frac{A_{33}}{A_{13}} = -1$

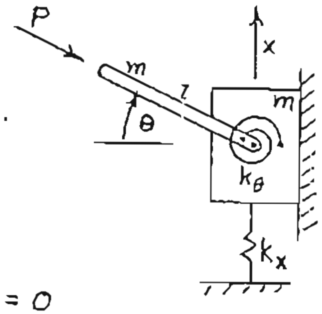
Complete Modal Matrix:

$$[A] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1.9877 & 0 & -0.5031 \end{bmatrix}$$

Note: The perturbation method of Example 9-5 can be used.

9-19. $x=0,\ \theta=0$ at equilibrium.

$$T=\tfrac{1}{2}m\dot{x}^2+\tfrac{1}{2}m\left(\dot{x}+\tfrac{1}{2}l\dot{\theta}\right)^2+\frac{ml^2}{24}\dot{\theta}^2$$
$$=m\dot{x}^2+\tfrac{1}{6}ml^2\dot{\theta}^2+\tfrac{1}{2}ml\dot{x}\dot{\theta}$$
$$V=\tfrac{1}{2}k_x x^2+\tfrac{1}{2}k_\theta\theta^2$$



Due to P, there are the generalized forces

$$Q'_x=-P\sin\theta\cong-P\theta\ ,\quad Q'_\theta=0$$

Use $$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}=Q'_i$$

The equations of motion are

$$2m\ddot{x}+\tfrac{1}{2}ml\ddot{\theta}+k_x x=-P\theta$$
$$\tfrac{1}{2}ml\ddot{x}+\tfrac{1}{3}ml^2\ddot{\theta}+k_\theta\theta=0$$

(b) Assume solutions of the form $\cos(\lambda t+\phi)$. The characteristic equation is

$$\begin{vmatrix}(k_x-2m\lambda^2) & (P-\tfrac{1}{2}ml\lambda^2)\\ -\tfrac{1}{2}ml\lambda^2 & (k_\theta-\tfrac{1}{3}ml^2\lambda^2)\end{vmatrix}=\frac{5}{12}m^2l^2\lambda^4+\left(\tfrac{1}{2}mlP-\tfrac{1}{3}ml^2k_x-2mk_\theta\right)\lambda^2+k_xk_\theta=0$$

or $$\lambda^4+\left(\frac{6P}{5ml}-\frac{28k_x}{5m}\right)\lambda^2+\frac{12k_x^2}{5m^2}=0$$ where $k_\theta=k_xl^2$

This has the form $\lambda^4+b\lambda^2+c=0$ where $c>0$.

$$\lambda^2=\frac{-b\pm\sqrt{b^2-4c}}{2}$$ For stability, λ^2 must be real and positive. Hence $b=\frac{6P}{5ml}-\frac{28k_x}{5m}<0$ and $b^2>4c$ or $-b>2\sqrt{c}$ or $\frac{28k_x}{5m}-\frac{6P}{5ml}>2\sqrt{\frac{12}{5}}\frac{k_x}{m}$

System is stable for $P<\frac{k_xl}{3}(14-2\sqrt{15})$ or $P<2.0847\,k_xl$

9-19. (cont'd.) (c) For unit coefficients, we have

$$\lambda^4-\frac{22}{5}\lambda^2+\frac{12}{5}=0\ ,\quad \lambda^2=\frac{11}{5}\pm\sqrt{\left(\frac{11}{5}\right)^2-\frac{12}{5}}$$

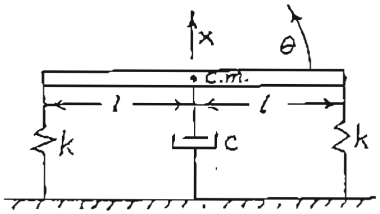
$$\lambda_1^2=0.6380\ ,\quad \lambda_2^2=3.7620$$

$$\frac{A_\theta}{A_x}=\frac{\frac{1}{2}\lambda^2}{1-\frac{1}{3}\lambda^2}$$

First Mode: $\lambda_1=0.7987$, $\frac{A_{\theta1}}{A_{x1}}=0.4051$

Second Mode: $\lambda_2=1.9396$, $\frac{A_{\theta2}}{A_{x2}}=-7.4051$

9-20. $x(0)=x_0$, $\theta(0)=\theta_0$
$\dot{x}(0)=0$, $\dot{\theta}(0)=0$



$$T=\tfrac{1}{2}m\dot{x}^2+\tfrac{1}{6}ml^2\dot{\theta}^2$$
$$V=\tfrac{1}{2}k\left[(x+l\theta)^2+(x-l\theta)^2\right]=k(x^2+l^2\theta^2)$$

The generalized forces due to the damper are

$$Q'_x=-c\dot{x}\ ,\quad Q'_\theta=0$$

$$[m]=\left[\frac{\partial^2T}{\partial\dot{q}_i\partial\dot{q}_j}\right]=\begin{bmatrix}m&0\\0&\frac{1}{3}ml^2\end{bmatrix},\quad [k]=\left[\frac{\partial^2V}{\partial q_i\partial q_j}\right]=\begin{bmatrix}2k&0\\0&2kl^2\end{bmatrix}$$

Since the [m] and [k] matrices are both diagonal, the x and θ motions are completely uncoupled.

x equation: $m\ddot{x}+2kx=-c\dot{x}$

θ equation: $\tfrac{1}{3}ml^2\ddot{\theta}+2kl^2\theta=0$

Assuming exponential solutions of the form $e^{\lambda t}$, the characteristic equation for x is

$$m\lambda^2+c\lambda+2k=0 \text{ giving } \lambda_{1,2}=-\frac{c}{2m}\pm i\sqrt{\frac{2k}{m}-\frac{c^2}{4m^2}}$$

The undamped natural frequency $\omega_n=\sqrt{\frac{2k}{m}}$ and the damping ratio $\zeta=\frac{c}{\sqrt{8km}}<1$.

9-20. (cont'd.) Then, using (9-11), we obtain

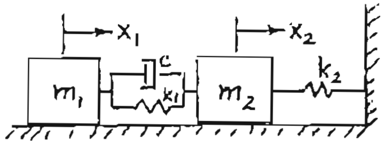
$$x = x_0 e^{-\frac{c}{2m}t}\left(\cos\sqrt{\frac{2k}{m}-\frac{c^2}{4m^2}}\,t + \frac{c}{\sqrt{8km-c^2}}\sin\sqrt{\frac{2k}{m}-\frac{c^2}{4m^2}}\,t\right)$$

From the differential equation, we see that θ has simple harmonic motion with frequency $\omega_n = \sqrt{\frac{2kl^2}{\frac{1}{3}ml^2}} = \sqrt{\frac{6k}{m}}$.

Noting that $\theta(0)=\theta_0$ and $\dot{\theta}(0)=0$, we obtain

$$\theta = \theta_0 \cos\sqrt{\frac{6k}{m}}\,t$$

9-21. Considering the forces acting on each of the blocks, the differential equations of motion are



$m_1 = 4$ kg, $k_1 = 1000$ N/m

$m_2 = 1$ kg, $k_2 = 1000$ N/m

$c = 50$ N-sec/m

$$m_1\ddot{x}_1 + c(\dot{x}_1-\dot{x}_2)+k_1(x_1-x_2)=0$$
$$m_2\ddot{x}_2 + c(\dot{x}_2-\dot{x}_1)+k_1(x_2-x_1)+k_2x_2=0$$

Assume solutions of the form $x_j = A_j C e^{\lambda t}$ and obtain

$$(4\lambda^2+50\lambda+1000)A_1-(50\lambda+1000)A_2=0$$
$$-(50\lambda+1000)A_1+(\lambda^2+50\lambda+2000)A_2=0$$

The characteristic equation is

$$4\lambda^4+250\lambda^3+9000\lambda^2+50{,}000\lambda+10^6=0$$

which has the roots

$$\lambda_{1,2} = -1.2410 \pm i\,11.184, \quad \lambda_{3,4} = -30.01 \pm i\,32.77$$

$$\frac{A_2}{A_1} = \frac{4\lambda^2+50\lambda+1000}{50\lambda+1000} = 0.5592 \pm i\,0.1444 \quad \text{for } \lambda_{1,2}.$$

$$\frac{A_2}{A_1} = -3.274 \pm i\,1.7284 \quad \text{for } \lambda_{3,4}.$$

9-22. Given the system described by

$$[m]\{\ddot{q}\}+[c]\{\dot{q}\}+[k]\{q\}=\{0\}$$

where $[m]$, $[c]$, and $[k]$ are real and symmetric. Assume solutions of the form $q_j = A_j C e^{\lambda t}$. For the k^{th} mode,

$$\lambda_k^2[m]\{A^{(k)}\}+\lambda_k[c]\{A^{(k)}\}+[k]\{A^{(k)}\}=\{0\}$$

Similarly, for the l^{th} mode, using symmetry,

$$\lambda_l^2\{A^{(l)}\}^T[m]+\lambda_l\{A^{(l)}\}^T[c]+\{A^{(l)}\}^T[k]=\{0\}$$

Premultiply the k^{th} mode equation by $\{A^{(l)}\}^T$, postmultiply the l^{th} mode equation by $\{A^{(k)}\}$, and subtract.

$$(\lambda_k^2-\lambda_l^2)\{A^{(l)}\}^T[m]\{A^{(k)}\}+(\lambda_k-\lambda_l)\{A^{(l)}\}^T[c]\{A^{(k)}\}=0$$

or

$$(\lambda_k+\lambda_l)\{A^{(l)}\}^T[m]\{A^{(k)}\}+\{A^{(l)}\}^T[c]\{A^{(k)}\}=0, \quad \lambda_k\neq\lambda_l$$

9-23. From (8-320),

$$I_{zz}\ddot{\psi}+3\omega_0^2(I_{yy}-I_{xx})\psi=0$$
$$I_{yy}\ddot{\theta}+4\omega_0^2(I_{zz}-I_{xx})\theta-(I_{xx}+I_{yy}-I_{zz})\omega_0\dot{\phi}=0$$
$$I_{xx}\ddot{\phi}+\omega_0^2(I_{zz}-I_{yy})\phi+(I_{xx}+I_{yy}-I_{zz})\omega_0\dot{\theta}=0$$

Since $I_{xx}:I_{yy}:I_{zz}=3:4:5$, we can write

pitch equation: $5\ddot{\psi}+3\omega_0^2\psi=0$

roll equation: $4\ddot{\theta}+8\omega_0^2\theta-2\omega_0\dot{\phi}=0$

yaw equation: $3\ddot{\phi}+\omega_0^2\phi+2\omega_0\dot{\theta}=0$

where ω_0 is the orbital angular velocity. Note that the pitch ψ is uncoupled from the other motions.

9-23. (cont'd.) Its frequency is

$$\omega_2 = \sqrt{\tfrac{3}{5}}\,\omega_0 = 0.7746\,\omega_0\,, \quad \frac{A_{\theta 2}}{A_{\psi 2}} = \frac{A_{\phi 2}}{A_{\psi 2}} = 0$$

For the θ and ϕ equations, assume solutions of the form $A_j C e^{\lambda t}$. Then we obtain

$$(4\lambda^2 + 8\omega_0^2)A_\theta - 2\omega_0\lambda A_\phi = 0$$
$$2\omega_0\lambda A_\theta + (3\lambda^2 + \omega_0^2)A_\phi = 0$$

The characteristic equation is

$$\begin{vmatrix} (4\lambda^2 + 8\omega_0^2) & -2\omega_0\lambda \\ 2\omega_0\lambda & (3\lambda^2+\omega_0^2) \end{vmatrix} = 12\lambda^4 + 32\omega_0^2\lambda^2 + 8\omega_0^4 = 0$$

$$\frac{A_\phi}{A_\theta} = \frac{2\lambda^2 + 4\omega_0^2}{\omega_0\lambda}\,, \qquad \lambda^2 = \left(-\frac{4}{3} \pm \sqrt{\frac{16}{9} - \frac{2}{3}}\right)\omega_0^2 = \begin{cases} -0.2792\,\omega_0^2 \\ -2.3874\,\omega_0^2 \end{cases}$$

1st Mode: $\omega_1 = 0.5284\,\omega_0\,, \quad \dfrac{A_{\phi 1}}{A_{\theta 1}} = \dfrac{3.4415}{i\,0.5284} = -i\,6.5127$

3rd Mode: $\omega_3 = 1.5451\,\omega_0\,, \quad \dfrac{A_{\phi 3}}{A_{\theta 3}} = \dfrac{-0.7749}{i\,1.5451} = i\,0.5015$

9-24. $I_t = 10\,I_a$

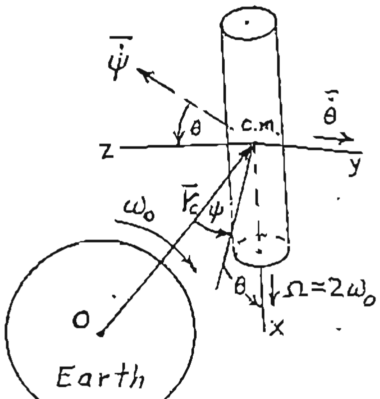
The differential equations of (8-315) are

$$I_t\ddot{\psi} + 3\omega_0^2(I_t - I_a)\psi - I_a\Omega\dot{\theta} = 0$$
$$I_t\ddot{\theta} + \omega_0^2(4I_t - 3I_a)\theta + I_a\Omega\dot{\psi} = I_a\Omega\omega_0$$

or

$$10\ddot{\psi} + 27\omega_0^2\psi - 2\omega_0\dot{\theta} = 0$$
$$10\ddot{\theta} + 37\omega_0^2\theta + 2\omega_0\dot{\psi} = 2\omega_0^2$$

Assuming solutions of the form $A_j C e^{\lambda t}$,



9-24. (cont'd.) the characteristic equation is

$$\begin{vmatrix} (10\lambda^2 + 27\omega_0^2) & -2\omega_0\lambda \\ 2\omega_0\lambda & (10\lambda^2 + 37\omega_0^2) \end{vmatrix} = 100\lambda^4 + 644\omega_0^2\lambda^2 + 999\omega_0^4 = 0$$

$$\lambda_{1,2}^2 = -2.6049\,\omega_0^2\,, \; -3.8351\,\omega_0^2$$

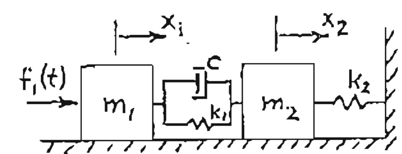
$$\frac{A_\theta}{A_\psi} = \frac{10\lambda^2 + 27\omega_0^2}{2\omega_0\lambda}$$

1st Mode: $\omega_1 = 1.6140\,\omega_0\,, \quad \dfrac{A_{\theta 1}}{A_{\psi 1}} = \dfrac{0.9514}{i\,3.2279} = -i\,0.2947$

2nd Mode: $\omega_2 = 1.9584\,\omega_0\,, \quad \dfrac{A_{\theta 2}}{A_{\psi 2}} = \dfrac{-11.351}{i\,3.9167} = i\,2.8982$

The average value of θ is its steady-state solution. From the θ equation, $\theta_{av} = \frac{2}{37}$ rad $= 3.097°$

9-25. Considering the forces acting on the blocks, the equations of motion are



$$m_1\ddot{x}_1 + c(\dot{x}_1 - \dot{x}_2) + k_1(x_1 - x_2) = f_1(t)$$
$$m_2\ddot{x}_2 + c(\dot{x}_2 - \dot{x}_1) + k_1(x_2 - x_1) + k_2x_2 = 0$$

$m_1 = 4$ kg, $k_1 = 1000$ N/m

$m_2 = 1$ kg, $k_2 = 1000$ N/m

$c = 50$ N·sec/m

$f_1(t) = 100\cos 50t$ N

or

$$4\ddot{x}_1 + 50(\dot{x}_1 - \dot{x}_2) + 1000(x_1 - x_2) = 100\cos 50t$$
$$\ddot{x}_2 + 50(\dot{x}_2 - \dot{x}_1) + 2000x_2 - 1000x_1 = 0$$

Let $f_1(t) = F_0 e^{i\omega t} = 100\,e^{i50t}$

$x_j = B_j e^{i\omega t}$ where B_j is complex

Then, from the equations of motion,

$$[(1000 - 4\omega^2) + i\,50\omega]B_1 - (1000 + i\,50\omega)B_2 = F_0$$
$$-(1000 + i\,50\omega)B_1 + [(2000 - \omega^2) + i\,50\omega]B_2 = 0$$

or

$$(-9000 + i\,2500)B_1 - (1000 + i\,2500)B_2 = 100$$
$$-(1000 + i\,2500)B_1 + (-500 + i\,2500)B_2 = 0$$

3-25. (cont'd.) $B_1 = (0.7931 + i\,0.5172)B_2 = 0.9469\angle 33.11^\circ B_2$

resulting in $(-.9431 - i\,.5172)B_2 = 100$

$$B_2 = 0.009297\angle 151.26^\circ, \quad B_1 = 0.008803\angle 184.37^\circ$$

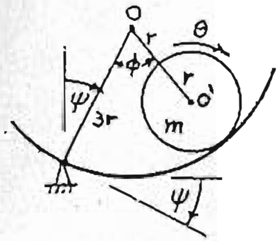
The steady-state solutions are

$$x_1 = 0.8803\ \cos(50t + 184.37^\circ)\ \text{cm}$$

$$x_2 = 0.9297\ \cos(50t + 151.26^\circ)\ \text{cm}$$

9-26, θ = rotation angle of solid cylinder relative to the shell.

$2r\dot\phi = r\dot\theta$, $\dot\phi = \frac{1}{2}\dot\theta$, $\phi = \frac{1}{2}\theta$

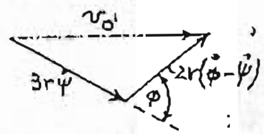


Assume small motion.

$$v_{O'}^2 = r^2\left[9\dot\psi^2 + 4(\dot\phi-\dot\psi)^2 + 12\dot\psi(\dot\phi-\dot\psi)\cos\phi\right]$$

$$\cong r^2(\dot\psi + 2\dot\phi)^2 = r^2(\dot\psi+\dot\theta)^2$$

$$T = \frac{1}{2}m v_{O'}^2 + \frac{mr^2}{4}(\dot\psi+\dot\theta)^2 = \frac{3}{4}mr^2(\dot\psi+\dot\theta)^2$$



$$V = mg\left[3r\cos\psi - 2r\cos(\tfrac{1}{2}\theta - \psi)\right]$$

$$\psi = 0.05\cos\omega t = 0.05\cos\sqrt{\frac{g}{r}}\,t$$

$$\ddot\psi = \frac{-0.05g}{r}\cos\sqrt{\frac{g}{r}}\,t$$

Use $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} + \frac{\partial V}{\partial\theta} = 0$

$$\frac{\partial T}{\partial\dot\theta} = \frac{3}{2}mr^2(\dot\theta+\dot\psi), \quad \frac{\partial T}{\partial\theta} = 0$$

$$\frac{\partial V}{\partial\theta} = mgr\sin(\tfrac{\theta}{2}-\psi) \cong mgr(\tfrac{\theta}{2}-\psi)$$

θ equation: $\frac{3}{2}mr^2(\ddot\theta+\ddot\psi) + mgr(\frac{\theta}{2}-\psi) = 0$

or $\frac{3}{2}mr^2\ddot\theta + \frac{1}{2}mgr\theta = 0.125\,mgr\cos\sqrt{\frac{g}{r}}\,t$

Assume a steady-state solution $\theta_s = Be^{i\omega t}$.

Then $(-\frac{3}{2}mr^2\omega^2 + \frac{1}{2}mgr)B = \frac{1}{8}mgr$, $B = -\frac{1}{8}$

The steady-state solution is $\theta_s = -\frac{1}{8}\cos\sqrt{\frac{g}{r}}\,t$

9-25. (cont'd.) $B_1 = (0.7931 + i\,0.5172)B_2 = 0.9469\angle 33.11\, B_2$

resulting in $(-.9431 - i\,.5172)B_2 = 100$

$$B_2 = 0.009297\angle 151.26^\circ, \quad B_1 = 0.008803\angle 184.37^\circ$$

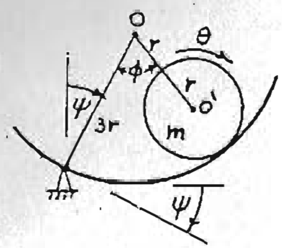
The steady-state solutions are

$$x_1 = 0.8803 \cos(50t + 184.37^\circ)\ \text{cm}$$

$$\underline{x_2 = 0.9297 \cos(50t + 151.26^\circ)\ \text{cm}}$$

9-26. θ = rotation angle of solid cylinder relative to the shell.

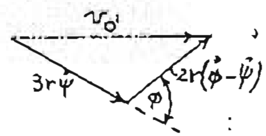
$2r\dot\phi = r\dot\theta, \quad \dot\phi = \frac{1}{2}\dot\theta, \quad \phi = \frac{1}{2}\theta$



Assume small motion.

$$v_{o'}^2 = r^2\left[9\dot\psi^2 + 4(\dot\phi - \dot\psi)^2 + 12\dot\psi(\dot\phi - \dot\psi)\cos\phi\right]$$

$$\cong r^2(\dot\psi + 2\dot\phi)^2 = r^2(\dot\psi + \dot\theta)^2$$



$$T = \frac{1}{2}m v_{o'}^2 + \frac{mr^2}{4}(\dot\psi + \dot\theta)^2 = \frac{3}{4}mr^2(\dot\psi + \dot\theta)^2$$

$$V = mg\left[3r\cos\psi - 2r\cos\left(\tfrac{1}{2}\theta - \psi\right)\right]$$

$\psi = 0.05\cos\omega t$

$= 0.05\cos\sqrt{\frac{g}{r}}\,t$

$\ddot\psi = \frac{-0.05g}{r}\cos\sqrt{\frac{g}{r}}\,t$

Use $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} + \frac{\partial V}{\partial\theta} = 0$

$$\frac{\partial T}{\partial\dot\theta} = \frac{3}{2}mr^2(\dot\theta + \dot\psi), \quad \frac{\partial T}{\partial\theta} = 0$$

$$\frac{\partial V}{\partial\theta} = mgr\sin\left(\frac{\theta}{2} - \psi\right) \cong mgr\left(\frac{\theta}{2} - \psi\right)$$

θ equation: $\frac{3}{2}mr^2(\ddot\theta + \ddot\psi) + mgr\left(\frac{\theta}{2} - \psi\right) = 0$

or $\frac{3}{2}mr^2\ddot\theta + \frac{1}{2}mgr\theta = 0.125\, mgr\cos\sqrt{\frac{g}{r}}\,t$

Assume a steady-state solution $\theta_s = Be^{i\omega t}$.

Then $\left(-\frac{3}{2}mr^2\omega^2 + \frac{1}{2}mgr\right)B = \frac{1}{8}mgr, \quad B = -\frac{1}{8}$

The steady-state solution is $\underline{\theta_s = -\frac{1}{8}\cos\sqrt{\frac{g}{r}}\,t}$