

# Gaussian approximation with moment matching, aka proj() operator in Expectation Propagation

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## Overview

Moment matching is a technique for approximating a function  $p(x)$  with a Gaussian distribution  $\tilde{p}(x) \sim \mathcal{N}(m, v)$  by matching expectations  $E_p[x]$  and  $E_p[x^2]$ , where

$$E_p[x] = E_{\tilde{p}}[x] \quad (1)$$

$$E_p[x^2] = E_{\tilde{p}}[x^2] \quad (2)$$

Then the mean and the variance of Gaussian approximation  $\tilde{p}(x)$  are defined by

$$m_{\tilde{p}} = E_p[x] \quad (3)$$

$$v_{\tilde{p}} = E_p[x^2] - E_p[x]^2 \quad (4)$$

Following Thomas Minka [1] and Kevin P. Murphy [2], in a specific case of a function  $p(x) = \frac{f(x)q(x)}{Z(m, v)}$ , where  $q(x) \sim \mathcal{N}(m, v)$  and  $Z(m, v) = \int f(x)q(x)dx$ , it can be shown that

$$E_p[x] = m + v \nabla_m \log Z \quad (5)$$

$$E_p[x^2] = 2v^2 \nabla_v \log Z + v + m^2 + 2vm \nabla_m \log Z \quad (6)$$

$$m_{\tilde{p}} = E_p[x] = m + v \nabla_m \log Z \quad (7)$$

$$v_{\tilde{p}} = E_p[x^2] - E_p[x]^2 = v - v(\nabla_m^2 \log Z - 2\nabla_v \log Z)v \quad (8)$$

More generally, for any member of exponential family [3]  $p(x|\eta) = h(x)g(\eta)\exp\{\eta^T u(x)\}$ , moments can be computed by differentiating the log partition function  $A(\eta) = -\log g(\eta)$

## Example

Consider a function  $p(\theta) = \frac{q(\theta)f(x|\theta)}{\int q(\theta)f(x|\theta)d\theta}$ , borrowed from a Clutter Problem [1], where

$$q(\theta) \sim \mathcal{N}(\theta|m, v)$$

$$f(x|\theta) = (1 - w)\mathcal{N}(x|\theta, 1) + w\mathcal{N}(0, a)$$

In order to compute Gaussian approximation  $\tilde{p}(\theta)$  to function  $p(\theta)$ , evaluated at the value of  $x = 3, m = 15, v = 100, w = 0.4, a = 10$ , we compute normalisation constant  $Z(m, v)$  and derivatives of  $\log(Z)$  with respect to mean and variance.

$$Z(m, v) = (1 - w)\mathcal{N}(x|m, v + 1) + w\mathcal{N}(x|0, a)$$

$$\nabla_m \log Z = (1 - w) \frac{1}{Z} \nabla_m \mathcal{N}(x|m, v + 1)$$

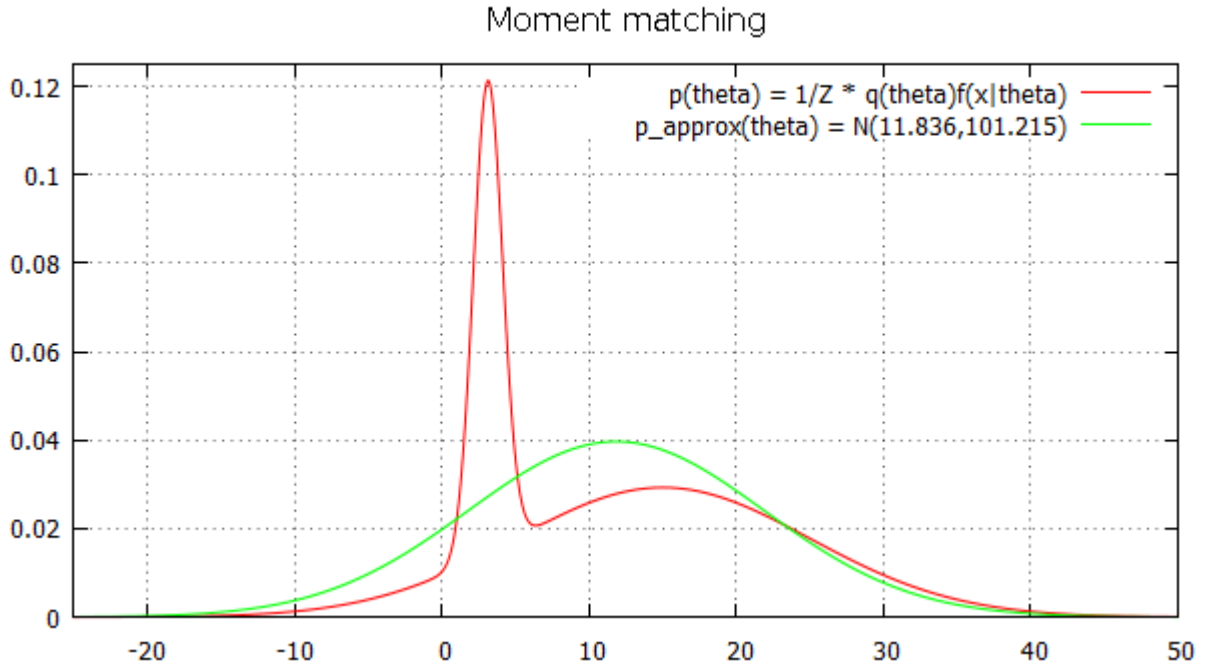
$$\nabla_v \log Z = (1 - w) \frac{1}{Z} \nabla_v \mathcal{N}(x|m, v + 1)$$

Then, we can calculate Gaussian approximation  $\tilde{p}(\theta)$  with equations 7 and 8

$$m_{\tilde{p}} = 11.8364$$

$$v_{\tilde{p}} = 101.21589$$

The following chart presents both  $p(\theta)$  distribution and its Gaussian approximation  $\tilde{p}(\theta)$ .



Bayes-Scala toolbox [4] provides example implementation of Moment Matching for a Clutter Problem.

## References

- [1] Thomas P Minka. A family of algorithms for approximate Bayesian inference, 2001
- [2] Kevin P. Murphy. From Belief Propagation to Expectation Propagation , 2001
- [3] Exponential Family. [http://en.wikipedia.org/wiki/Exponential\\_family](http://en.wikipedia.org/wiki/Exponential_family)
- [4] Daniel Korzekwa. Bayes-Scala tool, MomentMatchingTest.scala