牛顿法

公式

$x_{n+1} = \phi(x_n) = x_n - \frac{f(x_n)}{f^1(x_n)}$

收敛速度判断

由于

$$\begin{aligned}\phi(x) &= x - \frac{f(x)}{f^1(x)} \\ \phi^1(x) &= 1 - \frac{f^1(x)^2 - f(x)f^2(x)}{f^1(x)^2} \\ &= \frac{f(x)f^2(x)}{f^1(x)^2}\end{aligned}$$

且在零点 x^* 处

$$f(x^*) = 0$$

得

$$\phi^1(x^*) = 0$$

根据泰勒展开，将$\phi(x_n)$在零点x^*展开

$$\phi(x_n) = \phi(x^*) + \phi^1(x^*)(x_n - x^*) + \phi^2(x^*)(x_n - x^*)^2 + \ldots$$

且

$$\begin{aligned}x_{n+1} &= \phi(x_n) \\ x^* &= \phi(x^*)\end{aligned}$$

得

$$\begin{aligned}x_{n+1} - x^* &= \phi(x_n) - \phi(x^*) \\ &\approx \phi^2(x^*)(x_n - x^*)^2 \\ &\approx \phi^2(x^*)(\phi^2(x_{n-1})(x_{n-1} - x^*)^2 - x^*)^2 \\ &= \ldots\ldots \\ &\approx (x_0 - x^*)^{2^n}\end{aligned}$$

即

$$x_{n+1} - x^* \approx (x_0 - x^*)^{2^n}$$

所以，牛顿法满足2阶收敛，收敛速度较快