Exercise sheet 7: Markov chains

Exercise 1 - Up-to-date or Behind

Alex is taking a bioinformatics class and in each week he can be either up-to-date or he may have fallen behind. If he is up-to-date in a given week, the probability that he will be up-to-date in the next week is 0.75. If he is behind in the given week, the probability that he will be up-to-date in the next week is 0.5.

If we assume that these probabilities do not depend on whether he was up-to-date or behind in previous weeks, we can model the problem using a Markov chain.

1a)

Draw a Markov chain that models the states of being Up-to-date or behind

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Solution

1b)

Assume Alex is up-to-date in the first class; what is the probability that he is up-to-date two classes later?

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Hint : Formulae

$$\pi(0) : initial\ probabilities \tag{1}$$
$$P : transition\ matrix \tag{2}$$
$$\pi(t) = \pi(0) * P^t \tag{3}$$

Solution The Probability is 0.6875

$$\pi(0) = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{4}$$

$$\tag{5}$$

$$P = \begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{pmatrix} \tag{6}$$

$$\tag{7}$$

$$\pi(2) = \pi(0) \times P^2 \tag{8}$$

$$= \begin{pmatrix} 0.6875 & 0.3125 \end{pmatrix} \tag{9}$$

1c)

What is the expected probability that he is behind after an infinitely long semester?

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Hint : Formulae

$$\pi(0) : \textit{initial probabilities} \tag{10}$$

$$P : \textit{transition matrix} \tag{11}$$

$$\lim_{t \to \infty} \pi(t) = \pi(0) * P^t \tag{12}$$

Solution The Probability is 2/3

$$\lim_{t \to \infty} \pi(t) = \pi(0) * P^t = \begin{pmatrix} 2/3 & 1/3 \end{pmatrix}$$

1d)

What is the transition probability matrix product for limit of P^t as t approaches infinity?

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Solution

$$\lim_{t \to \infty} P^t = \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix} \tag{13}$$

Exercise 2 - Stationary distribution

Consider a three-state Markov chain having the following transition probability matrix:

$$\begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}$$

2a)

In the long run, what proportion of time is the process in each of the three states?

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Hint: Formulae See Question 1B

Correct Answer

$$\lim_{t\to\infty} P^t = \begin{pmatrix} 0.339 & 0.371 & 0.290 \\ 0.339 & 0.371 & 0.290 \\ 0.339 & 0.371 & 0.290 \end{pmatrix} \tag{14}$$

$$\tag{15}$$

$$\lim_{t\to\infty} \pi(t) = \begin{pmatrix} 0.339 & 0.371 & 0.290 \end{pmatrix} \tag{16}$$

rix} \end{align}

Note

$$\lim_{t\to\infty} \pi(t)$$

is independent of $\pi(0)$ as long as P does not contain disconnected subgraphs and only if the limit exists.

Exercise 3 - Reversibility

Consider a three-state Markov chain having the following transition probability matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 \end{pmatrix}$$

3a)

Draw the Markov chain for this problem

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Solution

3b)

Given the stationary distribution $\begin{pmatrix} \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{pmatrix}$, is this Markov chain reversible and what does this property tell you?

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Hint A markov chain is reversible if:

$$\pi_i^* P_{i,j} = \pi_j^* P_{j,i}$$

Easiest way is to calculate it for all pairs of i and j

Solution Because $\pi_i^* P_{i,j} = \pi_j^* P_{j,i}\ \forall\ i, j$ the Markov chain is reversible

$$\begin{aligned}
\pi_1^* P_{1,2} = &\quad \frac{1}{6} \times 1 = &\quad \frac{1}{6} = &\quad \frac{1}{2} \times \frac{1}{3} = &\quad \pi_2^* P_{2,1} \qquad (17)\\
\pi_1^* P_{1,3} = &\quad \frac{1}{6} \times 0 = &\quad 0 = &\quad \frac{1}{3} \times 0 = &\quad \pi_3^* P_{3,1} \qquad (18)\\
\pi_2^* P_{2,3} = &\quad \frac{1}{2} \times \frac{2}{3} = &\quad \frac{1}{3} = &\quad \frac{1}{3} \times 1 = &\quad \pi_3^* P_{3,2} \qquad (19)
\end{aligned}$$

Exercise 4 - Markov chain representation

4a)

Decide which of the following figures represents a valid Markov Chain

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Solution

- ☒ i
- ☐ ii - initial probabilities add up to 0.2
- ☐ iii - transition probabilities for states A and B do not add up to 1
- ☐ iv - duplicate state A
- ☐ v - initial probabilities add up to 1.1
- ☒ vi
- ☐ vii missing transition probabilites (0.1) for state C

4b)

Which of these statements about Markov Chains are valid?

Statements

- ☐ In the graph representation of Markov chains, a single state cannot have more than 3 outgoing edges.
- ☐ In the matrix representation of Markov chains, values in each row have to add up to 1.
- ☐ In the matrix representation of Markov chains, values in each column have to add up to 1.
- ☐ The diagonal entries of the Markov chain matrix represent the transition probability of remaining in the current state.
- ☐ In the graph representation of Markov chains, a single state cannot have more than 3 ingoing edges.
- ☐ The graph representation of Markov chains is directed and acyclic by definition.

Solution

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