Ecole Polytechnique Fédérale de Lausanne

Summary in

Signal Processing for Communications

Olivier Cloux

Spring 2017

Contents

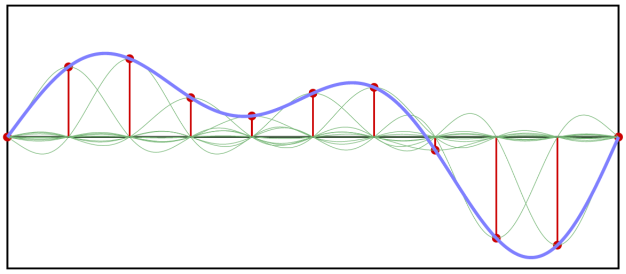


Figure 1: Visualization of the sampling theorem

1 Introduction

Signals Describe the evolution of a real life phenomenon.

Sampling Instead of considering *continuous* time signals (temperature,...), it might be easier to **sample** them and consider it as *discrete*

Sampling Theorem See Figure 1 and equation 1

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] sinc\left(\frac{t - nT_s}{T_s}\right) \tag{1}$$

Discrete signal Sequence of **complex** numbers. Notation: $x[n]$. n is "a-dimensional". Analysis $\sim$ periodic measurements and Synthesis $\sim$ stream of generated samples.

Delta signal $x[n] = \delta[n]$. 1 when $n = 0$, 0 elsewhere.

Unit step $x[n] = u[n]$. 1 when $n \geq 0$, 0 elsewhere.

Exponential decay $x[n] = |a|^n u[n]$ with $|a| < 1$

Signal classes Finite-length, infinite-length, periodic, finite-support

Finite-length Notation: $x[n], n = 0, 1, \ldots, N-1$. Vector: $\mathbf{x} = [x_0, x_1, \ldots, x_{N-1}]^T$. Good for practice.

Infinite-length Notation: $x[n], n \in \mathbb{Z}$. Abstraction $\rightarrow$ good for theory.

Periodic N-periodic sequence $\tilde{x}[n] = \tilde{x}[n + kN], \quad k, n, N \in \mathbb{Z}$

Finite-support $\bar{x}[n] = \begin{cases} x[n] & \text{if} 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$

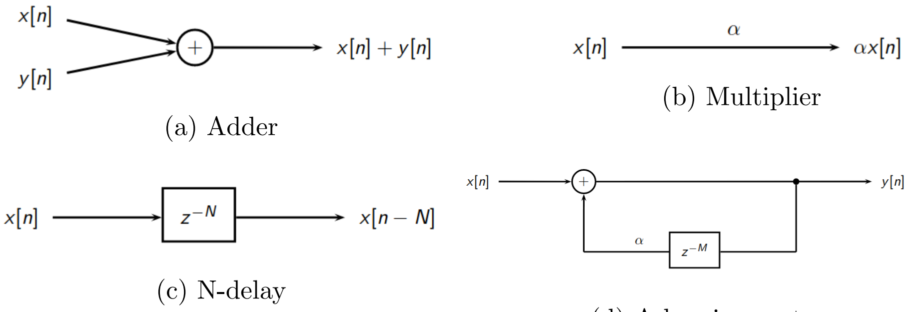
Operators Scaling: $< y[n] = \alpha x[n]$. Sum: $y[n] = x[n] + z[n]$. Product: $y[n] = x[n] \cdot z[n]$. Shift by k (delay): $y[n] = x[n-k]$

Finite-length shift We must chose either to use *finite-support* (0's outside of the interval, shifting "creates" 0's) or *periodic extension* (leaving on a sides makes entering on the other).

Energy

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 \tag{2}$$

Infinite for periodic signals



(a) Adder (b) Multiplier (c) N-delay (d) A looping system

Figure 2: Fundamental building blocks

Power For periodic signals: $P_{\tilde{x}} \equiv \frac{1}{N}\sum_{n=0}^{N-1} |\tilde{x}[n]|^2$

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \tag{3}$$

Legos DPS is composed of fundamental building blocks. See figure 2.

Averages Simple average: $m = \frac{a+b}{2}$. Moving average: take a "local" average

$$y[n] = \frac{x[n] + x[n-1]}{2} \tag{4}$$

Loops When feeding the output of a system to the input, we obtain a loop, of the type $y[n] = \alpha y[n-M] + x[n]$. This is a powerful concept! Figure 2d shows an example. The parameters we can tweak: M (size of delay), α (decay factor), $\overline{x}[n]$ (input signal)

Karplus-Strong

2 Vector spaces

Signal model We work in $\mathbb{C}^N$: vector space of ordered tuples of N complex values. N can be ∞. We need more than a vector space, we need a *Hilbert space*.

Some spaces $\ell_2(\mathbb{Z})$: space of square-summable infinite sequences. $L_2([a,b])$: space of square-integrable functions over an interval

Vector spaces Ingredients: the set of vectors V, and a set of scalars (say $\mathbb{C}$). We need at least to be able to: resize vectors (multiply vector by scalar) and combine vectors together (sum them).

Formal Properties For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha, \beta \in \mathbb{C}$:

- $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
- $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{x} + \mathbf{y})$
- $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$
- $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$
- $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$
- $\exists 0 \in V | \mathbf{x} + 0 = 0 + \mathbf{x} = \mathbf{x}$
- $\forall \mathbf{x} \in V \exists (-\mathbf{x}) | x + (-\mathbf{x}) = 0$

Dot Product We also need something to measure and compare: **inner product** (or **dot product**). Notation:

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$$

Measures similarity between vectors. If 0, then vectors are completely orthogonal.

Formal Properties The dot product has several interesting properties. For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha \in \mathbb{C}$:

- $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
- $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle^*$
- $\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha^* \langle \mathbf{x}, \mathbf{y} \rangle$
  $\langle \mathbf{x}, \alpha\mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$
- $\langle \mathbf{x}, \mathbf{x} \rangle = ||\mathbf{x}||^2 \geq 0$
- $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \iff \mathbf{x} = \mathbf{0}$
- If $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ and $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal

Examples In $\mathbb{R}^2$, the norm is simply $x_0 y_0 + x_1 y_1 = ||\mathbf{x}|| \, ||\mathbf{y}|| \cos\alpha$. Another more interesting example, is $L_2[a, b]$ In this case, the inner product is defined as $\int_a^b x(t)y(t)\,\mathrm{d}t$

Distance Inner product defines a norm: $||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ while norm defines a distance: $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||$. In L_2, the distance corresponds to the Mean Square Error

For signals the inner product is defined as following:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=0}^{N-1} x^*[n]y[n] \tag{5}$$

It is well defined for all finite-length vectors in $\mathbb{C}^N$. Careful: if $N = \infty$, then the sum may explode! We require the sequences to be *square-summable*, i.e. $\sum |x[n]| < \infty$. That is the space $\ell_2(\mathbb{Z})$.

2.1 Basis

Basis Vectors can be linearly combined in vector space: $\mathbf{g} = \alpha\mathbf{x} + \beta\mathbf{y}$. A basis is a set of vectors $\{\mathbf{w}^{(k)}\}$ that lets us write any vector as a linear combination of those vectors. Alternatively, it is a set $\{\mathbf{w}^{(k)}\}$ such as there exists (unique) α_1, α_2 such as for any $\mathbf{x}$, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \alpha_1 \mathbf{w}^{(1)} + \alpha_2 \mathbf{w}^{(2)} + \ldots \alpha_k \mathbf{w}^{(k)} = \sum_{k=0}^{N} \alpha_k \mathbf{w}^{(k)}, \quad \alpha_k \in \mathbb{C} \tag{6}$$

Example The canonical $\mathbb{R}^2$ basis is as follows: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. But this is not the *only* base of $\mathbb{R}^2$! For example $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is another valid base. Oppositely, $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$ is not a valid base as we can't express any vector $\mathbf{x}$ with them (e.g. no vector with $x_2 \neq 0$ can be expressed)

Ortho* basis Ortho**gonal** basis: All vectors are orthogonal with one another:

$$\langle \mathbf{w}^{(k)}, \mathbf{w}^{(n)} \rangle = 0, \text{ for } k \neq n$$

Ortho**normal** basis same as orthogonal, but vectors are normalized; thus all are orthogonal and vectors have unit length:

$$\langle \mathbf{w}^{(k)}, \mathbf{w}^{(n)} \rangle = \delta[n-k]$$

Basis expansion Given a basis and a vector, finding the α_k might be hard. With orthonormal basis, it is easy:

$$\alpha_k = \langle \mathbf{w}^{(k)}, \mathbf{x} \rangle \tag{7}$$

Basis change We want to easily change between our basis and a given other basis:

$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)} = \sum_{k=0}^{K-1} \beta_k \mathbf{v}^{(k)}$$

We look for the β_k using $\alpha_k, \mathbf{v}^{(k)}, \mathbf{w}^{(k)}$. Simply:

$$\beta_h = \sum_{k=0}^{K-1} \alpha_k \langle \mathbf{v}^{(h)}, \mathbf{w}^{(k)} \rangle = \sum_{k=0}^{K-1} \alpha_k c_{hk} = \begin{bmatrix} c_{00} & \cdots & c_{0(K-1)} \\ & & \vdots \\ c_{(K-1)0} & \cdots & c_{(K-1)(K-1)} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{K-1} \end{bmatrix} \tag{8}$$

Energy $||\mathbf{x}|| = \langle \mathbf{x}, \mathbf{x} \rangle = \sum_{k=0}^{K-1} |x_k|^2$

Parseval "*Energy is conserved across a change of basis*

2.2 Subspaces and approximation

Subspace A vector subspace is a subset of vectors *closed* under addition and scalar multiplicat0ion.

Approximation For a vector $\mathbf{x} \in V$ and a subspace $S \subseteq V$ then we can approximate $\mathbf{x}$ with $\hat{\mathbf{x}} \in S$.

LS Least-square approximation. Given an orthonormal basis for S: $\{\mathbf{s}^{(k)}\}_{k=0,1,\ldots,K-1}$. Then the orthogonal projection is the "best" approximation over S. Best because it has the minimum-norm error: $\arg\min_{\mathbf{y} \in S} ||\mathbf{x} - \mathbf{y}|| = \hat{\mathbf{x}}$. Beside, the error is orthogonal to approximation: $\langle \mathbf{x} - \hat{\mathbf{x}}, \mathbf{x} \rangle = 0$

Gram-Schmidt Used to build an orthonormal $\{\mathbf{u}^{(k)}\}$ set from any set $\{\mathbf{s}^{(k)}\}$. The algorithmic procedure:

1. $\mathbf{p}^{(k)} = \mathbf{s}^{(k)} - \sum_{n=0}^{k-1} \langle \mathbf{u}^{(n)}, \mathbf{s}^{(k)} \rangle \mathbf{u}^{(n)}$
2. $\mathbf{u}^{(k)} = \frac{\mathbf{p}^{(k)}}{||\mathbf{p}^{(k)}||}$

Legendre Legendre polynomials are a better (orthonormal) base than classical polynomials base. When approximating sinusoid with polynomials, Legendre polynomials yield a smaller error than regular polynomials base.

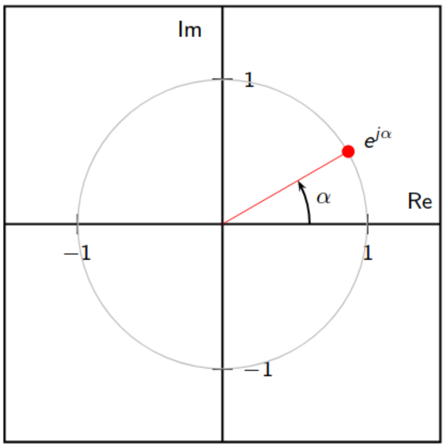


Figure 3: The trigonometric circle

2.3 Hilbert space

Ingredients For a Hilbert space, we need a vector space $H(V, \mathbb{C})$, an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ and completeness

Completeness Limiting operations must yield elements in the vector space.

3 Fourier

3.1 Introduction

Time domain Discrete signals are expressed as linear combinations of "atomic" time units

$$x[n] = \sum_{k=0}^{N-1} x[k]\delta[n-k] \iff \mathbf{x} = \sum_{k=0}^{N-1} x_k \boldsymbol{\delta}^k$$

Where $\{\boldsymbol{\delta}\}$ is a canonical basis for $\mathbb{C}^N$, e.g. $\boldsymbol{\delta}^{(2)} = [0\ 0\ 1\ 0\ \cdots\ 0]$

Frequency domain Fourier analysis: express a signal as combination of periodic oscillations:

$$\mathbf{x} = \sum_{k=0}^{N-1} X_k \mathbf{w}^{(k)} \tag{9}$$

with $\mathbf{w}^{(k)}$ the Fourier basis. The Fourier transform is a change of basis in the space of discrete time signals.

Analysis/Synthesis *Fourier analysis*: time domain $\to$ frequency domain, to find contribution of different frequencies.
Fourier synthesis: frequency domain $\to$ time domain, to create signals with known frequency content.

Math reminders $e^{j\alpha} = \cos\alpha + j\sin\alpha \simeq$ point on the unit circle, at angle α. See Figure3. Rotations of an angle β (centre at origin) are made by multiplying by $e^{j\beta}$. To represent discrete-time oscillatory, we need a frequency ω, an initial phase ϕ

and an amplitude A:

$$x[n] = Ae^{j(\omega n+\phi)} = A[\cos(\omega n + \phi) + j\sin(\omega n + \phi)]$$

Periodicity Consider the signal $x[n] = e^{j\omega n}$, then $x[n+1] = e^{j\omega n}x[n]$. In some cases, this is periodic. The condition for $e^{j\omega n}$ to be periodic in n, is to have $\omega = \frac{M}{N}2\pi$ with $M, N \in \mathbb{Z}$. So if the frequency is a (rational) multiple of 2π, the signal is periodic.

Max Frequency The higher we chose ω, the 'less points" we will have between each loop. But once we reached $\omega = \pi$, we only have 2 points (± 1). Going at speed $\pi + \alpha$ is similar as going at speed $-(\pi - \alpha)$

Digit./Physic. freq In discrete time, n is a-dimensional, just a counter. Periodicity is the number of samples before pattern repeats. But in real world, periodicity is the number of *seconds* before pattern repeats; it's measured in Hz (s^{-1}). Now, set T_s seconds between samples, and a periodicity of M samples (that is a periodicity of MT_s seconds). Then the real-world frequency is $\frac{1}{MT_s}$

3.2 Fourier Basis

Basis The set of N signals in $\mathbb{C}^N$ represented in eq. 10 is an orthogonal basis in C^N. The proof won't be presented here. Note that the vectors are not orthonormal. The normalization factor would be $1/\sqrt{N}$

$$w_k[n] = e^{j\frac{2\pi}{N}nk}, \; n, k = 0, 1, \ldots, N-1 \sim \{\mathbf{w}^{(k)}\}_{k=0,1,\ldots,N-1} \text{ with } w_n^{(k)} = e^{j\frac{2\pi}{N}nk} \tag{10}$$

3.3 Discrete Fourier Transform

Basis expansion Following Equation 7, the *analysis* formula (respectively the *synthesis* formula) is

$$X_k = \langle \mathbf{w}^{(k)}, \mathbf{x} \rangle \qquad \mathbf{x} = \frac{1}{N}\sum_{k=0}^{N-1} X_k \mathbf{w}^{(k)} \tag{11}$$

Change of basis We try to define the matrix of basis change (as in Equation 8). First we define $W_N = e^{-j\frac{2\pi}{n}}$ (or W when N is evident). Then the change of basis matrix $\mathbf{W}$ with $\mathbf{W}[n,m] = W_N^{nm}$:

$$W = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \cdots & W^{2(N-1)} \\ & & & \vdots & & \\ 1 & W^{N-1} & W^{2(N-1)} & W^{3(N-1)} & \cdots & W^{(N-1)^2} \end{bmatrix} \tag{12}$$