Ecole Polytechnique Fédérale de Lausanne

Summary in

# Signal Processing for Communications

ÉCOLE POLYTECHNIQUE
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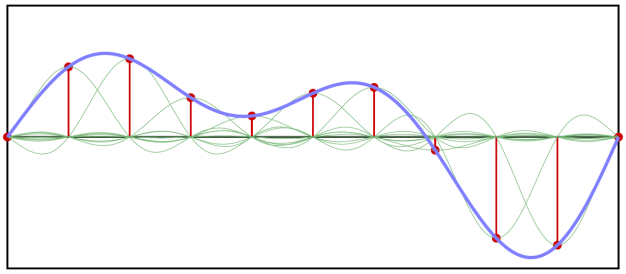


Figure 1: Visualization of the sampling theorem

# 1 Introduction

**Signals** Describe the evolution of a real life phenomenon.

**Sampling** Instead of considering *continuous* time signals (temperature,...), it might be easier to **sample** them and consider it as *discrete*

**Sampling Theorem** See Figure 1 and equation 1

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] sinc\left(\frac{t - nT_s}{T_s}\right) \quad (1)$$

**Discrete signal** Sequence of **complex** numbers. Notation: $x[n]$. $n$ is "a-dimensional". Analysis $\sim$ periodic measurements and Synthesis $\sim$ stream of generated samples.

**Delta signal** $x[n] = \delta[n]$. 1 when $n = 0$, 0 elsewhere.

**Unit step** $x[n] = u[n]$. 1 when $n \geq 0$, 0 elsewhere.

**Exponential decay** $x[n] = |a|^n u[n]$ with $|a| < 1$

**Signal classes** Finite-length, infinite-length, periodic, finite-support

**Finite-length** Notation: $x[n], n = 0, 1, ..., N-1$. Vector: $\mathbf{x} = [x_0, x_1, ..., x_{N-1}]^T$. Good for practice.

**Infinite-length** Notation: $x[n], n \in \mathbb{Z}$. Abstraction $\rightarrow$ good for theory.

**Periodic** N-periodic sequence $\tilde{x}[n] = \tilde{x}[n + kN], \quad k, n, N \in \mathbb{Z}$

**Finite-support** $\bar{x}[n] = \begin{cases} x[n] & \text{if} 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$

**Operators** Scaling:$<$ $y[n] = \alpha x[n]$. Sum: $y[n] = x[n] + z[n]$. Product: $y[n] = x[n] \cdot z[n]$. Shift by $k$ (delay): $y[n] = x[n - k]$

**Finite-length shift** We must chose either to use *finite-support* (0's outside of the interval, shifting "creates" 0's) or *periodic extension* (leaving on a sides makes entering on the other).

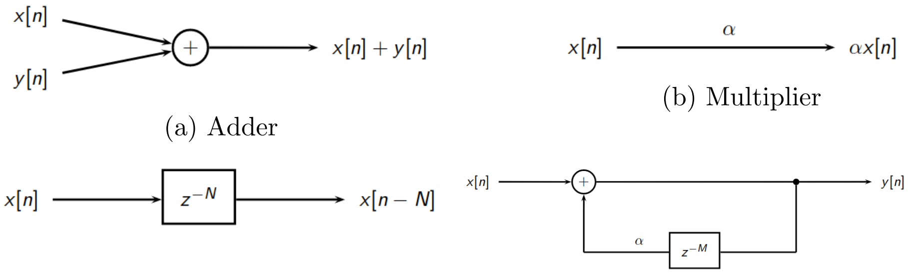
**Energy**

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 \quad (2)$$

Infinite for periodic signals

**Power**

$$P_x = \lim_{N\to\infty} \frac{1}{2N + 1} \sum_{n=-N}^{N} |x[n]|^2 \quad (3)$$




(a) Adder (b) Multiplier (c) N-delay (d) A looping system

Figure 2: Fundamental building blocks

For periodic signals: $P_{\tilde{x}} \equiv \frac{1}{N}\sum_{n=0}^{N-1} |\tilde{x}[n]|^2$

**Legos** DPS is composed of fundamental building blocks. See figure 2.

**Averages** Simple average: $m = \frac{a+b}{2}$. Moving average: take a “local” average

$$y[n] = \frac{x[n] + x[n-1]}{2} \tag{4}$$

**Loops** When feeding the output of a system to the input, we obtain a loop, of the type $y[n] = \alpha y[n-M] + x[n]$. This is a powerful concept ! Figure 2d shows an example. The parameters we can tweak: $M$ (size of delay), $\alpha$ (decay factor), $\overline{x}[n]$ (input signal)

**Karplus-Strong**

## 2 Vector spaces

**Signal model** We work in $\mathbb{C}^N$: vector space of ordered tuples of $N$ complex values. $N$ can be $\infty$. We need more than a vector space, we need a *Hilbert space*.

**Some spaces** $\ell_2(\mathbb{Z})$: space of square-summable infinite sequences. $L_2([a,b])$: space of square-integrable functions over an interval

**Vector spaces** Ingredients: the set of vectors $V$, and a set of scalars (say $\mathbb{C}$ ). We need at least to be able to: resize vectors (multiply vector by scalar) and combine vectors together (sum them).

**Formal Properties** For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha, \beta \in \mathbb{C}$:

- $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
- $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{x} + \mathbf{y})$
- $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$
- $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$
- $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$
- $\exists 0 \in V | \mathbf{x} + 0 = 0 + \mathbf{x} = \mathbf{x}$
- $\forall \mathbf{x} \in V \exists (-\mathbf{x}) | x + (-\mathbf{x}) = 0$

**Dot Product** We also need something to measure and compare: **inner product** (or **dot product**). Notation:

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$$

Measures similarity between vectors. If 0, then vectors are completely orthogonal.

**Formal Properties** The dot product has several interesting properties. For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha \in \mathbb{C}$ :

- $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
- $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle^*$
- $\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha^* \langle \mathbf{x}, \mathbf{y} \rangle$
  $\langle \mathbf{x}, \alpha\mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$
- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$
- $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \iff \mathbf{x} = \mathbf{0}$
- If $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ and $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal
- $\langle \mathbf{x}, \mathbf{x} \rangle = ||\mathbf{x}||^2$