Ecole Polytechnique Fédérale de Lausanne

Summary in

Signal Processing for Communications

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

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Contents

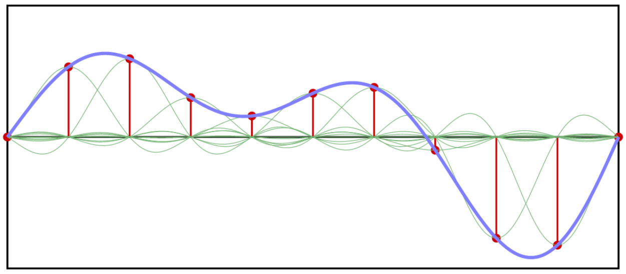


Figure 1: Visualization of the sampling theorem

1 Introduction

Signals Describe the evolution of a real life phenomenon.

Sampling Instead of considering *continuous* time signals (temperature,...), it might be easier to **sample** them and consider it as *discrete*

Sampling Theorem See Figure 1 and equation 1

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]sinc\left(\frac{t - nT_s}{T_s}\right) \quad (1)$$

Discrete signal Sequence of **complex** numbers. Notation: $x[n]$. n is "a-dimensional". Analysis $\sim$ periodic measurements and Synthesis $\sim$ stream of generated samples.

Delta signal $x[n] = \delta[n]$. 1 when $n = 0$, 0 elsewhere.

Unit step $x[n] = u[n]$. 1 when $n \geq 0$, 0 elsewhere.

Exponential decay $x[n] = |a|^n u[n]$ with $|a| < 1$

Signal classes Finite-length, infinite-length, periodic, finite-support

Finite-length Notation: $x[n], n = 0, 1, \ldots, N-1$. Vector: $\mathbf{x} = [x_0, x_1, \ldots, x_{N-1}]^T$. Good for practice.

Infinite-length Notation: $x[n], n \in \mathbb{Z}$. Abstraction $\rightarrow$ good for theory.

Periodic N-periodic sequence $\tilde{x}[n] = \tilde{x}[n + kN], \quad k, n, N \in \mathbb{Z}$

Finite-support $\bar{x}[n] = \begin{cases} x[n] & \text{if} 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$

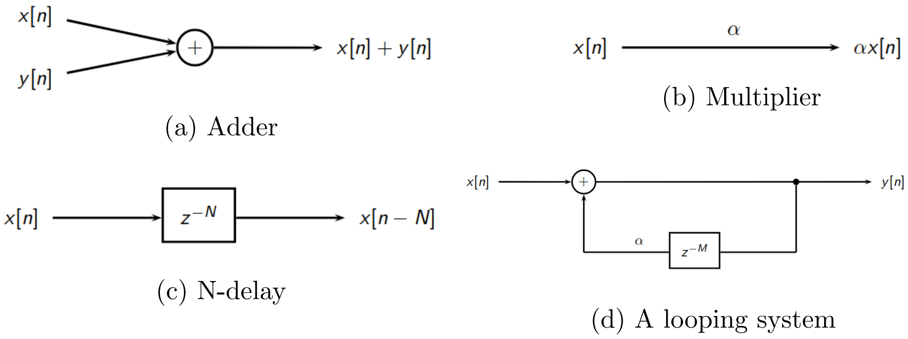
Operators Scaling: $< y[n] = \alpha x[n]$. Sum: $y[n] = x[n] + z[n]$. Product: $y[n] = x[n] \cdot z[n]$. Shift by k (delay): $y[n] = x[n - k]$

Finite-length shift We must chose either to use *finite-support* (0's outside of the interval, shifting "creates" 0's) or *periodic extension* (leaving on a sides makes entering on the other).

Energy

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 \quad (2)$$

Infinite for periodic signals



(a) Adder (b) Multiplier (c) N-delay (d) A looping system

Figure 2: Fundamental building blocks

Power For periodic signals: $P_{\tilde{x}} \equiv \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}[n]|^2$

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \tag{3}$$

Legos DPS is composed of fundamental building blocks. See figure 2.

Averages Simple average: $m = \frac{a+b}{2}$. Moving average: take a "local" average

$$y[n] = \frac{x[n] + x[n-1]}{2} \tag{4}$$

Loops When feeding the output of a system to the input, we obtain a loop, of the type $y[n] = \alpha y[n-M] + x[n]$. This is a powerful concept! Figure 2d shows an example. The parameters we can tweak: M (size of delay), α (decay factor), $\overline{x}[n]$ (input signal)

Karplus-Strong

2 Vector spaces

Signal model We work in $\mathbb{C}^N$: vector space of ordered tuples of N complex values. N can be ∞. We need more than a vector space, we need a *Hilbert space*.

Some spaces $\ell_2(\mathbb{Z})$: space of square-summable infinite sequences. $L_2([a,b])$: space of square-integrable functions over an interval

Vector spaces Ingredients: the set of vectors V, and a set of scalars (say $\mathbb{C}$). We need at least to be able to: resize vectors (multiply vector by scalar) and combine vectors together (sum them).

Formal Properties For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha, \beta \in \mathbb{C}$:

- $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
- $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{x} + \mathbf{y})$
- $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$
- $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$
- $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$
- $\exists 0 \in V | \mathbf{x} + 0 = 0 + \mathbf{x} = \mathbf{x}$
- $\forall \mathbf{x} \in V \exists (-\mathbf{x}) | x + (-\mathbf{x}) = 0$

Dot Product We also need something to measure and compare: **inner product** (or **dot product**). Notation:

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$$

Measures similarity between vectors. If 0, then vectors are completely orthogonal.

Formal Properties The dot product has several interesting properties. For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha \in \mathbb{C}$:

- $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
- $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle^*$
- $\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha^* \langle \mathbf{x}, \mathbf{y} \rangle$
  $\langle \mathbf{x}, \alpha\mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$
- $\langle \mathbf{x}, \mathbf{x} \rangle = ||\mathbf{x}||^2 \geq 0$
- $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \iff \mathbf{x} = \mathbf{0}$
- If $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ and $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal

Examples In $\mathbb{R}^2$, the norm is simply $x_0y_0 + x_1y_1 = ||\mathbf{x}||\ ||\mathbf{y}|| \cos\alpha$. Another more interesting example, is $L_2[a, b]$ In this case, the inner product is defined as $\int_a^b x(t)y(t)\,\mathrm{d}t$

Distance Inner product defines a norm: $||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ while norm defines a distance: $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||$. In L_2, the distance corresponds to the Mean Square Error

For signals the inner product is defined as following:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=0}^{N-1} x^*[n]y[n] \tag{5}$$

It is well defined for all finite-length vectors in $\mathbb{C}^N$. Careful: if $N = \infty$, then the sum may explode! We require the sequences to be *square-summable*, i.e. $\sum |x[n]| < \infty$. That is the space $\ell_2(\mathbb{Z})$.

2.1 Basis

Basis Vectors can be linearly combined in vector space: $\mathbf{g} = \alpha\mathbf{x} + \beta\mathbf{y}$. A basis is a set of vectors $\{\mathbf{w}^{(k)}\}$ that lets us write any vector as a linear combination of those vectors. Alternatively, it is a set $\{\mathbf{w}^{(k)}\}$ such as there exists (unique) α_1, α_2 such as for any $\mathbf{x}$, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \alpha_1\mathbf{w}^{(1)} + \alpha_2\mathbf{w}^{(2)} + \dots \alpha_k\mathbf{w}^{(k)} = \sum_{k=0}^{N} \alpha_k\mathbf{w}^{(k)}, \quad \alpha_k \in \mathbb{C} \tag{6}$$

Example The canonical $\mathbb{R}^2$ basis is as follows: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. But this is not the *only* base of $\mathbb{R}^2$! For example $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is another valid base. Oppositely, $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$ is not a valid base as we can't express any vector $\mathbf{x}$ with them (e.g. no vector with $x_2 \neq 0$ can be expressed)

Ortho* basis Ortho**gonal** basis: All vectors are orthogonal with one another:

$$\langle \mathbf{w}^{(k)}, \mathbf{w}^{(n)} \rangle = 0, \text{ for } k \neq n$$

Ortho**normal** basis same as orthogonal, but vectors are normalized; thus all are orthogonal and vectors have unit length:

$$\langle \mathbf{w}^{(k)}, \mathbf{w}^{(n)} \rangle = \delta[n-k]$$

Basis expansion Given a basis and a vector, finding the α_k might be hard. With orthonormal basis, it is easy:

$$\alpha_k = \langle \mathbf{w}^{(k)}, \mathbf{x} \rangle \tag{7}$$

Basis change We want to easily change between our basis and a given other basis:

$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)} = \sum_{k=0}^{K-1} \beta_k \mathbf{v}^{(k)}$$

We look for the β_k using $\alpha_k, \mathbf{v}^{(k)}, \mathbf{w}^{(k)}$. Simply:

$$\beta_h = \sum_{k=0}^{K-1} \alpha_k \langle \mathbf{v}^{(h)}, \mathbf{w}^{(k)} \rangle = \sum_{k=0}^{K-1} \alpha_k c_{hk} = \begin{bmatrix} c_{00} & \cdots & c_{0(K-1)} \\ & & \vdots \\ c_{(K-1)0} & \cdots & c_{(K-1)(K-1)} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{K-1} \end{bmatrix} \tag{8}$$

Energy $||\mathbf{x}|| = \langle \mathbf{x}, \mathbf{x} \rangle = \sum_{k=0}^{K-1} |x_k|^2$

Parseval "*Energy is conserved across a change of basis*"

$$||\mathbf{x}||^2 = \sum |\alpha_k|^2 \tag{9}$$

2.2 Subspaces and approximation

Subspace A vector subspace is a subset of vectors *closed* under addition and scalar multiplication.

Approximation For a vector $\mathbf{x} \in V$ and a subspace $S \subseteq V$ then we can approximate $\mathbf{x}$ with $\hat{\mathbf{x}} \in S$.

LS Least-square approximation. Given an orthonormal basis for the subspace S: $\{\mathbf{s}^{(k)}\}_{k=0,1,\ldots,K-1}$ Then the orthogonal projection is the "best" approximation over S. Best because it has the minimum-norm error:

$$\arg\min_{\mathbf{y} \in S} ||\mathbf{x} - \mathbf{y}|| = \hat{\mathbf{x}}$$

Beside, the error is orthogonal to approximation: $\langle \mathbf{x} - \hat{\mathbf{x}}, \mathbf{x} \rangle = 0$

Gram-Schmidt Used to build an orthonormal $\{\mathbf{u}^{(k)}\}$ set from any set $\{\mathbf{s}^{(k)}\}$. The algorithmic procedure:

1. $\mathbf{p}^{(k)} = \mathbf{s}^{(k)} - \sum_{n=0}^{k-1} \langle \mathbf{u}^{(n)}, \mathbf{s}^{(k)} \rangle \mathbf{u}^{(n)}$
2. $\mathbf{u}^{(k)} = \frac{\mathbf{p}^{(k)}}{||\mathbf{p}^{(k)}||}$

Legendre Legendre polynomials are a better (orthonormal) base than classical polynomials base. When approximating sinusoid with polynomials, Legendre polynomials yield a smaller error than regular polynomials base.

2.3 Hilbert space

Ingredients For a Hilbert space, we need a vector space $H(V, \mathbb{C})$, an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ and completeness

Completeness Limiting operations must yield elements in the vector space.

3 Fourier

3.1 Introduction

Time domain Discrete signals are expressed as linear combinations of “atomic” time units

$$x[n] = \sum_{k=0}^{N-1} x[k]\delta[n-k] \iff \mathbf{x} = \sum_{k=0}^{N-1} x_k \boldsymbol{\delta}^k$$

Where $\{\boldsymbol{\delta}\}$ is a canonical basis for $\mathbb{C}^N$, e.g. $\boldsymbol{\delta}^{(2)} = [0\ 0\ 1\ 0\ \cdots\ 0]$

Frequency domain Fourier analysis: express a signal as combination of periodic oscillations:

$$\mathbf{x} = \sum_{k=0}^{N-1} X_k \mathbf{w}^{(k)} \tag{10}$$

with $\mathbf{w}^{(k)}$ the Fourier basis. The Fourier transform is a change of basis in the space of discrete time signals.

Analysis/Synthesis *Fourier analysis*: time domain $\to$ frequency domain, to find contribution of different frequencies.
Fourier synthesis: frequency domain $\to$ time domain, to create signals with known frequency content.

Math reminders $e^{j\alpha} = \cos\alpha + j\sin\alpha \simeq$ point on the unit circle, at angle α. See Figure3. Rotations of an angle β (centre at origin) are made by multiplying by $e^{j\beta}$. To represent discrete-time oscillatory, we need a frequency ω, an initial phase ϕ and an amplitude A:

$$x[n] = Ae^{j(\omega n+\phi)} = A[\cos(\omega n + \phi) + j\sin(\omega n + \phi)]$$

Periodicity Consider the signal $x[n] = e^{j\omega n}$, then $x[n+1] = e^{j\omega n}x[n]$. In some cases, this is periodic. The condition for $e^{j\omega n}$ to be periodic in n, is to have $\omega = \frac{M}{N}2\pi$ with $M, N \in \mathbb{Z}$. So if the frequency is a (rational) multiple of 2π, the signal is periodic.

Max Frequency The higher we chose ω, the ‘less points” we will have between each loop. But once we reached $\omega = \pi$, we only have 2 points (± 1). Going at speed $\pi + \alpha$ is similar as going at speed $-(\pi - \alpha)$

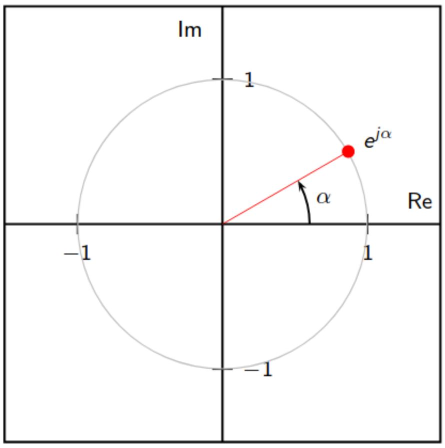


Figure 3: The trigonometric circle

Digit./Physic. freq In discrete time, n is a-dimensional, just a counter. Periodicity is the number of samples before pattern repeats. But in real world, periodicity is the number of *seconds* before pattern repeats; it's measured in Hz (s^{-1}). Now, set T_s seconds between samples, and a periodicity of M samples (that is a periodicity of MT_s seconds). Then the real-world frequency is $\frac{1}{MT_s}$

3.2 Fourier Basis

Basis The set of N signals in $\mathbb{C}^N$ represented in eq. 11 is an orthogonal basis in C^N. The proof won't be presented here. Note that the vectors are not orthonormal. The normalization factor would be $1/\sqrt{N}$

$$w_k[n] = e^{j\frac{2\pi}{N}nk}, \ \ n,k = 0,1,\dots,N-1 \sim \{\mathbf{w}^{(k)}\}_{k=0,1,\dots,N-1} \text{ with } w_n^{(k)} = e^{j\frac{2\pi}{N}nk} \tag{11}$$

3.3 Discrete Fourier Transform

Basis expansion Following Equation 7, the *analysis* formula (respectively the *synthesis* formula) is

$$X_k = \langle \mathbf{w}^{(k)}, \mathbf{x} \rangle \qquad \mathbf{x} = \frac{1}{N}\sum_{k=0}^{N-1} X_k \mathbf{w}^{(k)} \tag{12}$$

Change of basis We try to define the matrix of basis change (as in Equation 8). First we define $W_N = e^{-j\frac{2\pi}{n}}$ (or W when N is evident). Then the change of basis matrix $\mathbf{W}$ with $\mathbf{W}[n,m] = W_N^{nm}$:

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \cdots & W^{2(N-1)} \\ & & & \vdots & & \\ 1 & W^{N-1} & W^{2(N-1)} & W^{3(N-1)} & \cdots & W^{(N-1)^2} \end{bmatrix} \tag{13}$$

This lets us redefine the analysis and synthesis formula:

$$\mathbf{X} = \mathbf{W}\mathbf{x} \qquad \mathbf{x} = \frac{1}{N}\mathbf{W}^H\mathbf{X} \tag{14}$$

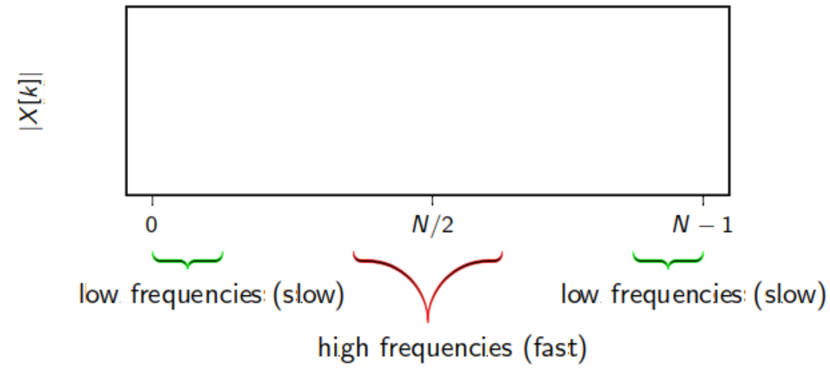


Figure 4: Relation between frequencies and the plot

DFT Matrix We can simplify many elements in this matrix, because $W_N^m = W_N^{(m \mod N)}$. Thus, for example, $W_8^{11} = W_8^3$. Furthermore, with $\mathbf{W}_6$, we should have elements up to W^{25}, but with this theorem we will only need to compute elements up to W^5

Linearity Obviously, DFT is linear:

$$DFT\{\alpha x[n] + \beta y[n]\} = \alpha DFT\{x[n]\} + \beta DFT\{y[n]\}$$

Examples

Example 1. Let's find the DFT of $x[n] = 3\cos(2\pi/16)$ in $\mathbb{C}^{64}$. We easily transform $x[n] = \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} + e^{j\frac{2\pi}{64}60n}\right] = \frac{3}{2}(w_4[n] + w_{60}[n])$. Then we just apply the dot product to change the basis, use linearity to obtain the result

Example 2. The above transformation was made easier because we could transform $2\pi/16$ into the form of $2\pi/N \cdot C$. So it was easy because 16 divides 64. But if it is not the case (that is, if the constant multiplying 2π) does not divide N, we don't have a "fast" result and must use the formal definition (the sum) to obtain each element.

Phase wrapping As the phase is an angle, we can make the result more appealing by adding multiples of 2π without changing the result.

DFT plot Once we plotted the DFT, it's important to know what to do with it. Mostly, interpret it. On figure 4 we see an explanation of how to interpret the plot. Note that between 0 and $N/2$, we are dealing with frequencies $< \pi$ (counterclockwise), while between $N/2$ and $N-1$ are frequencies $> \pi$ (clockwise).

Energy distribution Recall Parseval's theorem (equation 9). Then obviously:

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 \tag{15}$$

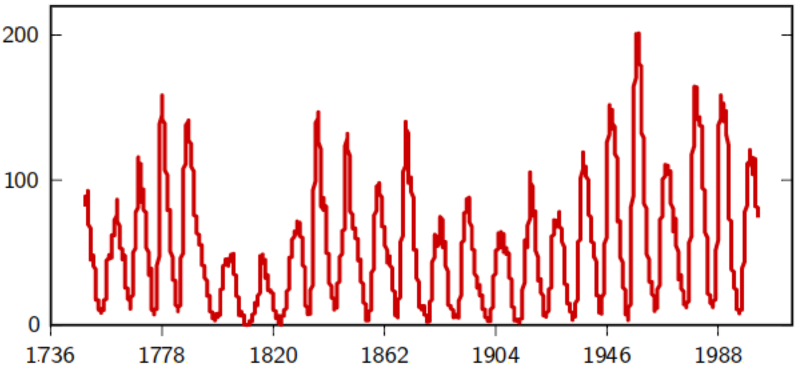
The square magnitude of k^{th} DFT coefficient are proportional to the signal's energy at frequency $\omega = (2\pi/N)k$

Symmetry For real signals, the DFT is symmetric in magnitude: $|X[k]| = |X[N-k]|$ for $k = 1, 2, \ldots, \lfloor N/2 \rfloor$. Thus, we magnitude plots need only $\lfloor N/2 \rfloor + 1$ points

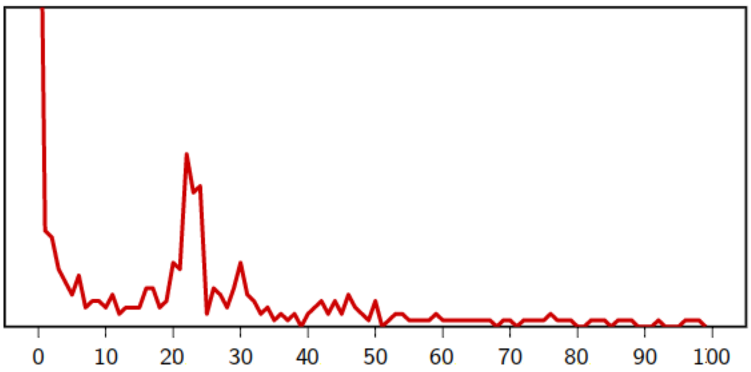
3.4 DFT as an analysis tool

We will here use some examples to explain how to use DFT to analyse signals.

3.4.1 Solar spots



(a) Solar spots throughout the years



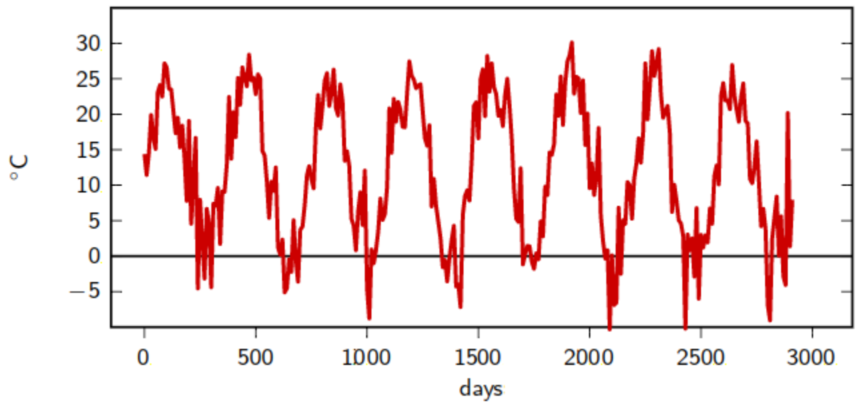
(b) DFT of the signal

Figure 5: First example: solar spots

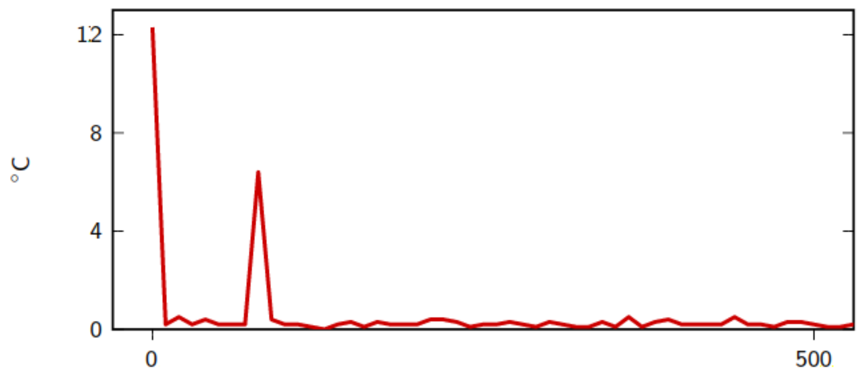
On the first example, presented in Figure 5, we see the sunspots throughout the years, and its DFT. The DFT shows a clear peak at $k = 22$. This means that the frequency is of 22 cycles. Our data is spread across 2904 months, and thus we have a period of $\frac{2904}{22} \simeq 11$ years.

3.4.2 Temperature

Here we need to observe 2 points: at $k = 0$, we have the **average** temperature (12.3 °C). At $k = 8$ there is a new peak. Same computation as before: we have 2920 days, and thus 8 cycles represent a period of $\frac{2920}{8} = 365$. The peak goes to $6.4°C$, and thus the temperature ranges at $12.3 \pm 12.8°C$



(a) Temperatures throughout the years



(b) DFT of the signal

Figure 6: First example: solar spots

About the 12.8°:

$$DFT\left\{A\cos\left(\frac{2\pi}{N}Mn\right)\right\}[k] = \begin{cases} \frac{A}{2}N & k = M, N-M \\ 0 & \text{otherwise} \end{cases} \quad (16)$$

3.4.3 Labeling the frequency axis

We need to know the "clock" of the system T_s, or its frequency $F_s = 1/T_s$. Then, let's reason this way: the fastest frequency is $\omega = \pi$; it then takes 2 samples to do a full revolution. Then the real-world period for the fastest sinusoid is $2T_s$, or frequency $F_s/2Hz$.

This means that on our plot, the highest frequency (that is the middle of the plot, remember figure 4) corresponds to $F_s/2$. From here, we only apply a linear mapping to find the frequency corresponding to the sample numbers.

3.5 DFT as a synthesis tool

Sinusoidal generator Let's consider the signal $w_k[n] = A_k e^{j(\frac{2\pi}{N}kn+\phi_k)}$ By choosing appropriate A_k and ϕ_k, we can generate any sinusoid by summing these elements. By initializing the machine with $A_k = |X[k]|/N$ and $\phi_k = \angle X[k]$ we can find back our original signal.

Time? If we run our algorithm for too long, we obtain the interesting result: $x[n+N] = x[n]$, that is the output is $N-$periodic!

DFS Therefore, if we remind our synthesis (respectively analysis) formula produces a N-point signal in the time (frequency) domain:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}nk}, \quad n = 0, 1, \ldots, N-1 \tag{17}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}nk}, \quad k = 0, 1, \ldots, N-1 \tag{18}$$

Then we can extend this to the concept of **DFS**[1] that produce a N-periodic signal in the time (frequency) domain:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}nk}, \quad n \in \mathbb{Z} \tag{19}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}nk}, \quad k \in \mathbb{Z} \tag{20}$$

Basically, DFS is just DFT with periodicity explicit.

Properties The DFS maps N-periodic signal onto an N-periodic sequence of Fourier coefficients. The inverse DFS does exactly the opposite. And obviously the DFS is equivalent to the DFT of one period.

The DFS then

A Fourier summary

(I)DFT (Inverse) Discrete Fourier Transform

(I)DFS (Inverse) Discrete Fourier Series: DFT with explicit periodicity

[1] Discrete Fourier Series