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Low-rank optimal transport

▶ Setting

The goal of this project is to combine low-rank approximations and the Sinkhorn algorithm for matrix scaling to accelerate the (approximate) solution of optimal transport problems of the form: Find a transport plan $P \in \mathbb{R}^{n \times n}$ such that $\sum_{i,j} P_{i,j} C_{i,j}$ is minimized for a given cost matrix $C \in \mathbb{R}^{n \times n}$ subject to the constraint that $\sum_i P_{i,j} = b_j$ for all j and $\sum_j P_{i,j} = a_i$ for all i, where $a, b \in \mathbb{R}^n$ are prescribed. One application of optimal transport is the transfer of color between images. In our numerical experiments, we study how low-rank approximations affect the Sinkhorn algorithm in the context of color transfer.

▶ Tasks

1. By adding entropic regularization [2, Eq. (4.2)] we can compute approximate solutions to the original optimal transport problem [2, Eq. (2.11)] by solving matrix scaling problem. Explain the relationship of the regularized problem and matrix scaling (see [2, Proposition 4.3]) in your own words. You do not need to discuss the relationship between the original problem and the regularized problem (see [2, Proposition 4.1]).

2. The Sinkhorn algorithm is essentially a matrix scaling algorithm that repeatedly evaluates marginals of a given matrix K. Explain how a low-rank approximation of this matrix might help to accelerate the computation.

3. Optimal transport can be used to transfer color from one source image to a target image [3]. For this purpose we enumerate the n pixels of each image. Let $x_i \in \mathbb{R}^3$ denote the RGB value (treated as $\mathbb{R}^3$) of the ith pixel in the source image. Let $y_j \in \mathbb{R}^3$ analogously denote the RGB value of the jth pixel in the target image. We define the entries of the cost matrix as $C_{i,j} = ||x_i - y_j||_2^2$ and use the marginals $m_1 = 1/n \cdot \mathbf{1}$, $m_2 = 1/n \cdot \mathbf{1}$, where $\mathbf{1} \in \mathbb{R}^n$ denotes the vector of all ones. We now compute the transport plan P for this optimal transport problem using the Sinkhorn algorithm with regularization parameter $\eta = 1$. From this matrix we define the new color of the jth pixel of the target image as $y_j^* = \sum_{i=1}^n P_{i,j} x_i$. Implement the color transfer by solving the optimal transport problem using the Sinkhorn algorithm. Plot the target image, the source image and the image after transferring the color.

4. Implement a second version of the Sinkhorn algorithm, in which you replace the matrix K by a low-rank approximation (obtained using a truncated SVD or randomized SVD). Study how the rank of the approximation affects the solution. Study how much the color transfer is accelerated depending on the rank of the low-rank approximation.

5. State and prove the error bound in [1, Theorem 5] in your own words. You may use any Lemmas in the appendix of [1] and any results cited in [1] directly without proofing them further. Compare your numerical results to the error bound in [1, Theorem 5].

▶ References

[1] Altschuler, J., Bach, F., Rudi, A., Niles-Weed, J. Massively scalable Sinkhorn distances via the Nyström method. arXiv preprint arXiv:1812.05189 (2019). `https://arxiv.org/abs/1812.05189`

[2] Peyré, G., Cuturi, M. Computational Optimal Transport. arXiv preprint arXiv:1803.00567 (2020). `https://arxiv.org/abs/1803.00567`

[3] Rabin et at. Adaptive color transfer with relaxed optimal transport. IEEE Int. Conf. Imag. Process. (2014). `https://doi.org/10.1109/ICIP.2014.7025983`

Remark. Note that the the paper [1] denotes regularization parameter by η whereas [2] uses $\frac{1}{\varepsilon}$. Use either of these notations, but do not mix them in your report.