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STRANGE ACTIONS OF GROUPS ON SPHERES

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A theme in topology is that certain group actions may be made geometric by a change of coordinates. In this paper geometric means conformal. In the mid 1970's F. Gehring and B. Palka expressed hope that a uniformly quasiconformal action $G \times S^n \to S^n$ is conjugate by a quasiconformal homeomorphism to a conformal action [11]. This was proved to be true by D. Sullivan [20] and P. Tukia [21] when $n = 2$.

Let F_r denote a free group of rank r and $F_r \rtimes \mathbf{Z}_{2r}$ a certain semidirect product (defined precisely later). One of our two main results is (see §3): For r sufficiently large[1] there is a discrete, smooth, uniformly quasiconformal action ψ: $(F_r \rtimes \mathbf{Z}_{2r}) \times S^2 \to S^3$ which is not conjugate (by any homeomorphism) to a conformal action.

There has been interesting earlier work in this direction. Tukia [22] for $n > 2$ constructed a uniformly quasiconformal action $G \times S^n \to S^n$ of a connected solvable Lie group G, where G does not embed in the Möbius group of S^n. Our example differs from Tukia's in that our action is discrete and smooth ($= C^\infty$). Recently, G. Martin [15] has constructed a discrete (but not smooth), uniformly quasiconformal action on S^n, $n \geqslant 3$, which is not quasiconformally conjugate to a conformal action but is topologically conjugate to a conformal action.

The failure of the higher dimensional Smith conjecture is relevant. It was long known to topologists that for each $n \geqslant 4$ there are smooth, finite cyclic actions on S^n whose fixed point sets are nontrivially knotted $(n-2)$-spheres [12]. These, of course, could not be topologically conjugate to elliptic (conformal) groups which after a further conjugation are linear. In fact, the action ψ

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[1] The minimal r suitable in our constructions seems to be more than ten and less than 100.

can be thought of as a counterexample to a natural three-dimensional generalization of the Smith conjecture where the compactness of the group is replaced by the compactness condition: uniform quasiconformality.

At the other extreme, we produce, for any $r \geqslant 2$, an action ϕ of F_r on S^3 which is continuous but is not topologically conjugate to any group of uniformly quasiconformal transformations. What is novel here is that each element of F_r is *individually* conjugate to a conformal (actually hyperbolic) transformation; so that the wildness arises from interplay of the generators, not the dynamics of any one singly.

Our examples are drawn from a class we call *admissible actions* (defined below). Within this class are Schottky groups, ϕ, and ψ which respect the conformal structure to varying degrees. Like Schottky actions, admissible actions have limit sets homeomorphic to Cantor sets. The embedding of this Cantor set in S^n is key to our investigation.

Cantor sets imbedded in S^n are of two types. A Cantor set $\mathscr{C} \subset S^n$ is *tame* if there is a homeomorphism $h\colon S^n \to S^n$ such that $h(\mathscr{C})$ lies on a smoothly embedded arc; otherwise, $\mathscr{C}$ is *wild*. It is well known that a Cantor set $\mathscr{C} \subset S^n$ is tame if and only if for all $\varepsilon > 0$ there are disjoint n-balls $B_1, \cdots, B_k \subset S^n$ such that each is of diameter less than ε and $\mathscr{C} \subset \bigcup \mathring{B}_k$. Hence, the Schottky actions have tame Cantor set limit sets.

Both ϕ and ψ have wild Cantor set limit sets. It is an attractive idea that purely topological properties of a Cantor limit set would be obstructions to the action being compatible with various structures (e.g., conformal, uniformly quasiconformal, $\mathbf{C}^n$, Lipshitz, and Hölder). However, contrary to an earlier conjecture of ours, Bestvina and Cooper ([1] and [2]) have devised conformal actions on S^3 whose limit set is a wild Cantor set. Thus any constraint that the topology of Λ imposes on the compatible structures is subtle.

In §1 we review the Schottky group, define admissible action, and recall the definition of a uniformly quasiconformal action. The action ϕ is constructed in §2, and it is proved that ϕ is not conjugate to a uniformly quasiconformal action. The action ψ is constructed in §3, and it is shown that ψ is smooth and uniformly quasiconformal but not conjugate to a conformal action. In an earlier paper [10] the first author showed that extension from S^3 to S^4 of admissible actions by free groups is equivalent to the topological surgery conjecture. With this background in mind we consider the extension question for ϕ and ψ. In §4 it is shown that both ϕ and ψ extend to admissible actions on S^4. We also give an example of an admissible action (by a nonfree group) which does not have an extension. In §5 we describe two techniques which yield admissible actions in higher dimensions. We also show that these actions extend to admissible actions on the next higher dimensional sphere.