Implicit vs. explicit ODE

Fredrik Magnusson[1] **Karl Berntorp**[2]
Christan & Claus?

[1]Department of Automatic Control
Lund University, Sweden

[2]Mitsubishi Electric Research Laboratories
Cambridge, MA

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Review

Last time:

- Compared simulation results for implicit DAE and explicit ODE with IDA and Radau5
- Implicit DAE formulation needs 10-20 times more steps
- Suppressing algebraic variables behaves very similarly to explicit ODE
- Pacejka Magic Formula (algebraic equation) stiff
- Surprisingly large relative errors with ODE formulation

Preview

This time:

- Revisit the error comparisons
- Discuss how all of this affects optimization

Partial DAE

- Last time considered eliminating all algebraics except the 4 nominal forces (Magic formula)
- I tested this as well, and call it DAE par.

Simulation results

Radau5 tolerance chosen to get reference solution, the rest to get similar accuracy

Setup	tol	Time [s]	Steps [1000]	Evals [1000]
Radau5 DAE	1e-12	69	17	640
IDA DAE	1e-6	59	26	930
IDA DAE sup. alg.	1e-8	2.8	2.9	42
IDA DAE par.	1e-6	6.3	8	102
IDA ODE	1e-8	0.9	3.4	15

Radau5 DAE

```
Final Run Statistics: ---

 Number of steps                                 : 16701
 Number of function evaluations                  : 231074
 Number of Jacobian evaluations                  : 12338
 Number of function eval. due to Jacobian eval.  : 407154
 Number of error test failures                   : 1040
 Number of LU decompositions                     : 21266

Solver options:

 Solver                     : Radau5 (implicit)
 Tolerances (absolute)      : [  1.00000000e-12]
 Tolerances (relative)      : 1e-12
```

IDA DAE

```
Final Run Statistics: ---

 Number of steps                                 : 26361
 Number of function evaluations                  : 67823
 Number of Jacobian evaluations                  : 25991
 Number of function eval. due to Jacobian eval.  : 857703
 Number of error test failures                   : 12046
 Number of nonlinear iterations                  : 67823
 Number of nonlinear convergence failures        : 0

Solver options:

 Solver                       : IDA (BDF)
 Maximal order                : 5
 Suppressed algebr. variables : False
 Tolerances (absolute)        : 1e-06
 Tolerances (relative)        : 1e-06
```

IDA DAE sup. alg.

```
Final Run Statistics: ---

 Number of steps                                   : 2923
 Number of function evaluations                    : 7072
 Number of Jacobian evaluations                    : 1065
 Number of function eval. due to Jacobian eval.    : 35145
 Number of error test failures                     : 424
 Number of nonlinear iterations                    : 7072
 Number of nonlinear convergence failures          : 0

Solver options:

 Solver                        : IDA (BDF)
 Maximal order                 : 5
 Suppressed algebr. variables  : True
 Tolerances (absolute)         : 1e-08
 Tolerances (relative)         : 1e-08
```

IDA DAE par.

```
Final Run Statistics: ---

 Number of steps                                   : 8489
 Number of function evaluations                    : 18467
 Number of Jacobian evaluations                    : 5952
 Number of function eval. due to Jacobian eval.    : 83328
 Number of error test failures                     : 2867
 Number of nonlinear iterations                    : 18467
 Number of nonlinear convergence failures          : 0

Solver options:

 Solver                       : IDA (BDF)
 Maximal order                : 5
 Suppressed algebr. variables : False
 Tolerances (absolute)        : 1e-06
 Tolerances (relative)        : 1e-06
```

IDA ODE

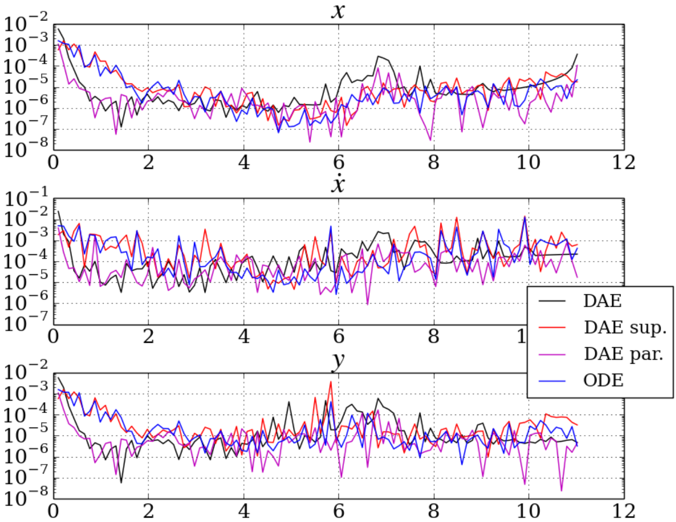
```
Final Run Statistics: ---

 Number of steps                                 : 3428
 Number of function evaluations                  : 5524
 Number of Jacobian evaluations                  : 919
 Number of function eval. due to Jacobian eval.  : 9190
 Number of error test failures                   : 477
 Number of nonlinear iterations                  : 5524
 Number of nonlinear convergence failures        : 0

Solver options:

 Solver                       : IDA (BDF)
 Maximal order                : 5
 Suppressed algebr. variables : False
 Tolerances (absolute)        : 1e-08
 Tolerances (relative)        : 1e-08
```

Error

- Reference solution v from Radau5 with `ATOL=RTOL=1e-12`
- Compare with the various IDA solutions $\hat{v}$, all with `ATOL=RTOL=1e-6`
- Compute relative error as function of time for each variable kind:

$$e_v(t) = \left\| \frac{v(t) - \hat{v}(t)}{v(t) + \epsilon_{\text{mach}}} \right\|_\infty, \quad \forall v \in \{\dot{x}, x, y\}$$

Variable kind errors



Variable kind errors

Can we conclude that ODE formulation is superior for simulation?

Commutativity

- Last time: Does BLT and fixed-step collocation commute?
- Consider single integration step of

$$\begin{aligned} \dot{x} &= f(x, y, u) \\ y &= g(x, u), \end{aligned} \tag{1}$$

and

$$\dot{x} = f(x, g(x, u), u) \tag{2}$$

Commutativity

Explicit DAE:

$$\dot{x}_k = f(x_k, y_k, u_k), \tag{3a}$$

$$y_k = g(x_k, u_k), \tag{3b}$$

$$\dot{x}_k = \frac{1}{h} \cdot \sum_{n=0}^{n_c} \alpha_{n,k} x_n, \tag{3c}$$

$$\forall k \in [1..n_c] \tag{3d}$$

Explicit ODE:

$$\dot{x}_k = f(x_k, g(x_k, u_k), u_k), \tag{4a}$$

$$\dot{x}_k = \frac{1}{h} \cdot \sum_{n=0}^{n_c} \alpha_{n,k} x_n, \tag{4b}$$

$$\forall k \in [1..n_c] \tag{4c}$$

Clearly commutative?

Dynamic optimization

$$
\begin{aligned}
&\text{minimize} && \phi(t_0, t_f, \boldsymbol{z}_T, \boldsymbol{p}) + \int_{t_0}^{t_f} L(t, \boldsymbol{z}(t), \boldsymbol{z}_T, \boldsymbol{p})\,\mathrm{d}t && && \text{(5a)}\\
&\text{with respect to} && \boldsymbol{x} : [t_0, t_f] \to \mathbb{R}^{n_x}, \quad \boldsymbol{y} : [t_0, t_f] \to \mathbb{R}^{n_y}, \quad \boldsymbol{u} : [t_0, t_f] \to \mathbb{R}^{n_u} && && \\
& && t_0 \in \mathbb{R}, \quad t_f \in \mathbb{R}, \quad \boldsymbol{p} \in \mathbb{R}^{n_p} && && \\
&\text{subject to} && \boldsymbol{F}(t, \boldsymbol{z}(t), \boldsymbol{p}) = \boldsymbol{0}, && \boldsymbol{F}_0(t_0, \boldsymbol{z}(t_0), \boldsymbol{p}) = \boldsymbol{0} && \text{(5b)}\\
& && \boldsymbol{z}_L \leq \boldsymbol{z}(t) \leq \boldsymbol{z}_U, && \boldsymbol{p}_L \leq \boldsymbol{p} \leq \boldsymbol{p}_U && \text{(5c)}\\
& && \boldsymbol{g}_e(t_0, t_f, t, \boldsymbol{z}(t), \boldsymbol{z}_T, \boldsymbol{p}) = \boldsymbol{0}, && \boldsymbol{g}_i(t_0, t_f, t, \boldsymbol{z}(t), \boldsymbol{z}_T, \boldsymbol{p}) \leq \boldsymbol{0} && \text{(5d)}\\
& && \boldsymbol{G}_e(t_0, t_f, \boldsymbol{z}_T, \boldsymbol{p}) = \boldsymbol{0}, && \boldsymbol{G}_i(t_0, t_f, \boldsymbol{z}_T, \boldsymbol{p}) \leq \boldsymbol{0} && \text{(5e)}\\
& && \forall t \in [t_0, t_f] && &&
\end{aligned}
$$