Time-Varying Parameter Models –
Efficient Bayesian Estimation using Shrinkage Priors

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Time-varying parameter (TVP) models

TVP models very popular in recent work in applied econometrics:

- Primiceri (2005): time-varying structural VARs in a monetary policy application
- Dangl and Halling (2012): TVP model for equity return prediction
- Lopes et al. (2012): time-varying multivariate SV model for high-dimensional financial time series

Time-varying parameter (TVP) models

- Belmonte et al. (2014): TVP model with many predictors to forecast EU-area inflation

- Eisenstat et al. (2014): Time-Varying Parameter VARs with an application to analyze the response of macro variables to fiscal shocks

. . . and many others

Why should we care about shrinkage?

- Time-varying parameter models are very popular because of their flexibility, however ...

- Allowing flexibility where none is required may lead to
  - **inefficient** estimation for parameters of interest
  - **poor predictions**

- Shrinkage: allow flexibility where needed, but simplify wherever possible.

How to achieve shrinkage?

How shrinkage is introduced:

- Non-Bayesian: use regularization

- Bayesian: substitute "usual priors" by shrinkage priors that introduce sparsity.

There is a close connection between regularization and shrinkage priors.

Regularization and Log Posteriors in Regression Analysis

Regression model: $y_i = \mathbf{x}_i\boldsymbol{\alpha} + \varepsilon_i, \quad \varepsilon_i \sim \mathcal{N}\left(0, \sigma_\varepsilon^2\right)$.

LASSO (Tibshirani, 1996) - penalized sum of squared errors:

$$\sum_{i=1}^{N}(y_i - \mathbf{x}_i\boldsymbol{\alpha})^2 + \lambda \sum_{j=1}^{d} |\alpha_j|$$

Bayesian LASSO (Park and Casella, 2008) - log posterior $\log p(\boldsymbol{\alpha}|\mathbf{y}, \sigma_\varepsilon^2, \tau)$ under prior $\alpha_j \sim \mathrm{Lap}\left(0, \sigma_\varepsilon^2\tau\right)$:

$$-\frac{0.5}{\sigma_\varepsilon^2}\left(\sum_{i=1}^{N}(y_i - \mathbf{x}_i\boldsymbol{\alpha})^2 + \frac{1}{\tau}\sum_{j=1}^{d}|\alpha_j|\right) + \text{constant}$$

Regularization and Log Posteriors

- Any penalty $\psi(\boldsymbol{\alpha})$ in likelihood estimation translates into a prior $p(\boldsymbol{\alpha})$ and vice versa; see Fahrmeir et al. (2010).

- No messing around with the likelihood $p(\mathbf{y}|\boldsymbol{\alpha})$; sparsity introduced through the prior (penalty).

- **Principled Bayesian framework** to introduce sparsity into latent variable models: random coefficient models (Frühwirth-Schnatter and Tüchler, 2008), fixed-effects modeling of Gaussian and non-Gaussian panel data (Frühwirth-Schnatter and Wagner, 2011; Wagner and Duller, 2012), unobserved components models (Frühwirth-Schnatter and Wagner, 2010), . . .

Sparsity/shrinkage/variable selection for TVP models

Let y_t, $t = 1, \ldots, T$, be time series observations, which are supposed to be driven by latent variables, summarized in a state vector $\boldsymbol{\beta}_t$ we are unable to observe.

Observation equation:

$$y_t = \mathbf{x}_t \boldsymbol{\beta}_t + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}\left(0, \sigma_\varepsilon^2\right).$$

State equation for the time-varying regression parameter:

$$\boldsymbol{\beta}_t = \boldsymbol{\beta}_{t-1} + \mathbf{w}_t, \quad \mathbf{w}_t \sim \mathcal{N}\left(\mathbf{0}, \mathbf{Q}\right).$$

Sparsity/shrinkage/variable selection for TVP models

Assume that $\mathbf{Q} = \mathrm{Diag}\,(\theta_1, \ldots, \theta_r)$, i.e.:

$$\beta_{tj} = \beta_{t-1,j} + w_{jt}, \qquad w_{jt} \sim \mathcal{N}\left(0, \theta_j\right), \quad j = 1, \ldots, r,$$
$$y_t = \mathbf{x}_t \boldsymbol{\beta}_t + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{N}\left(0, \sigma^2_\varepsilon\right),$$

with initial values $\beta_{0,j} \sim \mathcal{N}\left(\beta_j, \theta_j P_{0j}\right)$.

- Why introduce shrinkage?
- How to introduce shrinkage?

Sparsity/shrinkage/variable selection for TVP models

Shrinkage priors should allow to discriminate between the following cases:

- $\beta_j \neq 0$ and $\theta_j \neq 0$, in which case $\beta_{tj} \neq \beta_{t-1,j}$ for all $t = 1, \ldots, T$
  $\Rightarrow$ **Coefficient dynamic (time-varying)**

- $\beta_j \neq 0$ and $\theta_j = 0$, in which case $\beta_{tj} \equiv \beta_j$ for all $t = 1, \ldots, T$
  $\Rightarrow$ **Coefficient significant, but static**

- $\beta_j = 0$ and $\theta_j = 0$, in which case $\beta_{tj} = 0$ for all $t = 1, \ldots, T$
  $\Rightarrow$ **Coefficient insignificant**

Shrinkage priors for variances in state space models

Shrinkage priors should allow the posterior to concentrate over the reduced model, if the model is overfitting:

- The standard choice for the prior of the variances is the conditional conjugate prior, i.e. $\theta_j \sim \mathcal{G}^{-1}(c_0, C_0)$. Bounds the posterior away from 0.

- Useful shrinkage priors: normal prior on the signed square root, $\pm\sqrt{\theta_j} \sim \mathcal{N}(0, B_0)$ (Frühwirth-Schnatter, 2004; Frühwirth-Schnatter and Wagner, 2010); equivalent to:

$$\theta_j \sim \mathcal{G}(1/2, 1/(2B_0)) = B_0 \cdot \chi^2_1.$$

Unobserved component models

UC model with a local level:

$$\mu_t = \mu_{t-1} + \omega_{1t}, \qquad \omega_{1t} \sim \mathcal{N}\left(0, \theta_1\right),$$
$$y_t = \mu_t + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}\left(0, \sigma^2_\varepsilon\right)$$

UC model with a local trend:

$$\mu_t = \mu_{t-1} + a_{t-1} + \omega_{1t}, \qquad \omega_{1t} \sim \mathcal{N}\left(0, \theta_1\right)$$
$$a_t = a_{t-1} + \omega_{2t}, \qquad \omega_{2t} \sim \mathcal{N}\left(0, \theta_2\right),$$
$$y_t = \mu_t + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}\left(0, \sigma^2_\varepsilon\right)$$

UC model with a local level

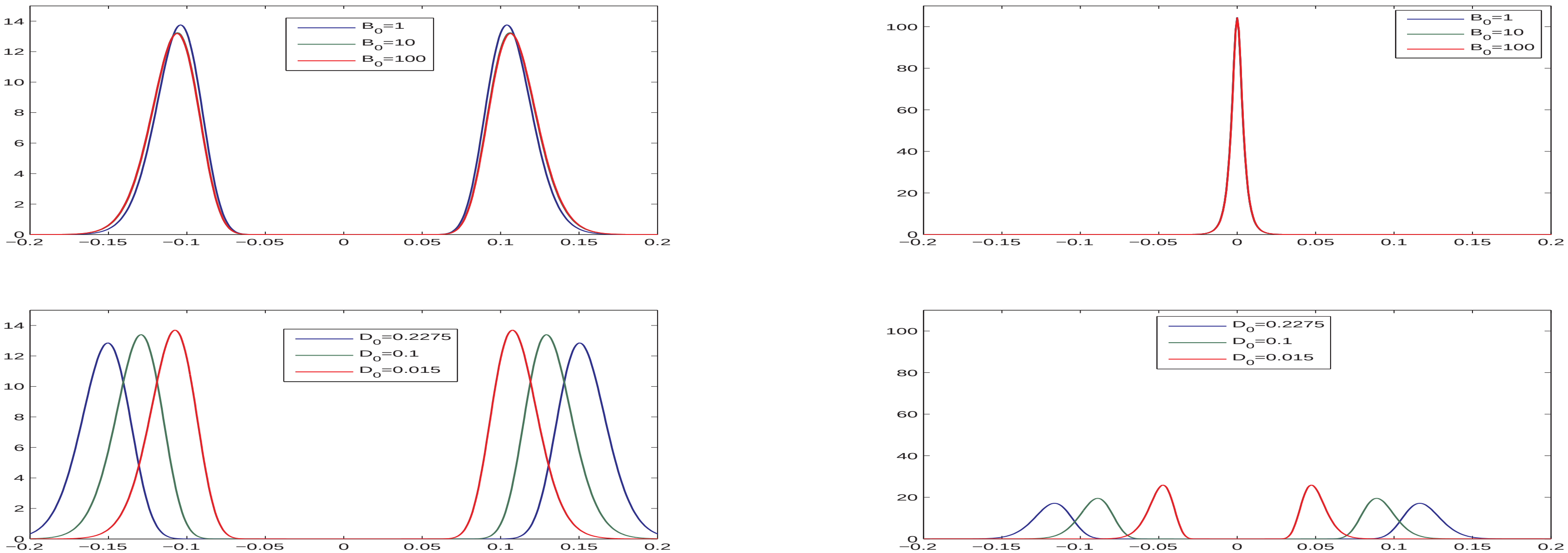


Figure 1: Posterior density for $\pm\sqrt{\theta}$ (data simulated from an UC model with local level, $\sigma_\varepsilon^2 = 1$, $T = 100$); left: $\theta = 0.01$; right: $\theta = 0$; top: prior $\theta \sim \mathcal{G}(0.5, 0.5/B_0)$, bottom: prior $\theta \sim \mathcal{G}^{-1}(0.5, C_0)$

UC model with a local trend

If the true variances θ_1^{true} and θ_2^{true} are positive, then the likelihood function concentrates around four modes.

Figure 2: Contour and surface plots of the (scaled) profile likelihood $l(\sqrt{\theta_1}, \sqrt{\theta_2}) = p(\mathbf{y}|\sqrt{\theta_1}, \sqrt{\theta_2}, \sigma_\varepsilon^{2,\text{true}}, \mu_0^{\text{true}}, a_0^{\text{true}})$ for simulated data $(T = 1000)$ with $(\theta_1^{\text{true}}, \theta_2^{\text{true}}) = (0.15, 0.02)$

UC model with a local trend

If one the true variances θ_1^{true} and θ_2^{true} is equal to 0 while the other is positive, two of those modes collapse and the likelihood is bimodal with an increasing number of observations T.

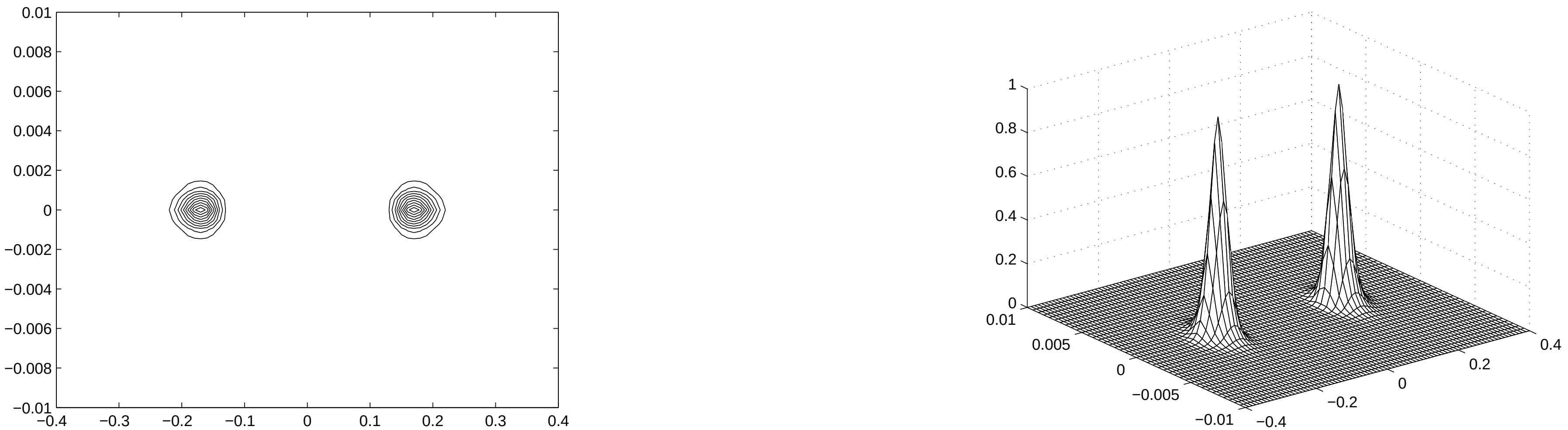
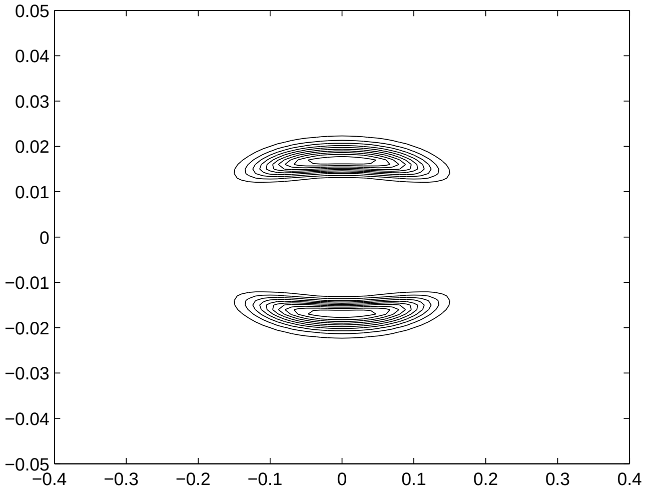


Figure 3: Contour and surface plots of the (scaled) profile likelihood for simulated data ($T = 1000$) with $(\theta_1^{\text{true}}, \theta_2^{\text{true}}) = (0.15, 0)$

UC model with a local trend



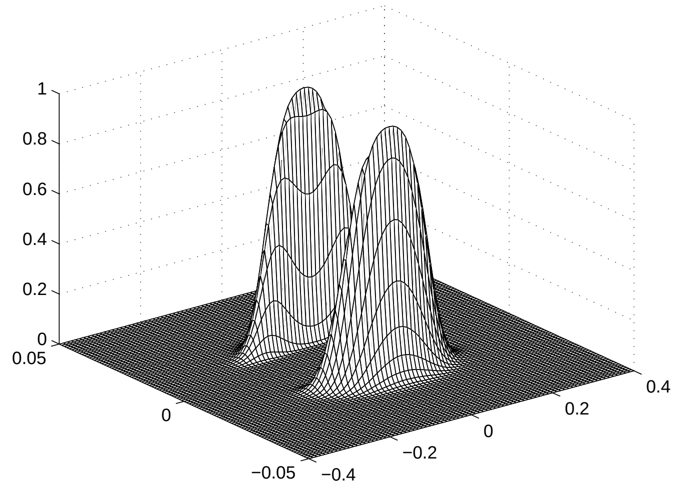


Figure 4: Contour and surface plots of the (scaled) profile likelihood for simulated data $(T = 1000)$ with $(\theta_1^{\mathrm{true}}, \theta_2^{\mathrm{true}}) = (0, 0.02)$

UC model with a local trend

If both variances θ_1^{true} and θ_2^{true} are equal to zero, then the likelihood function will be unimodal with an increasing number of observations T.

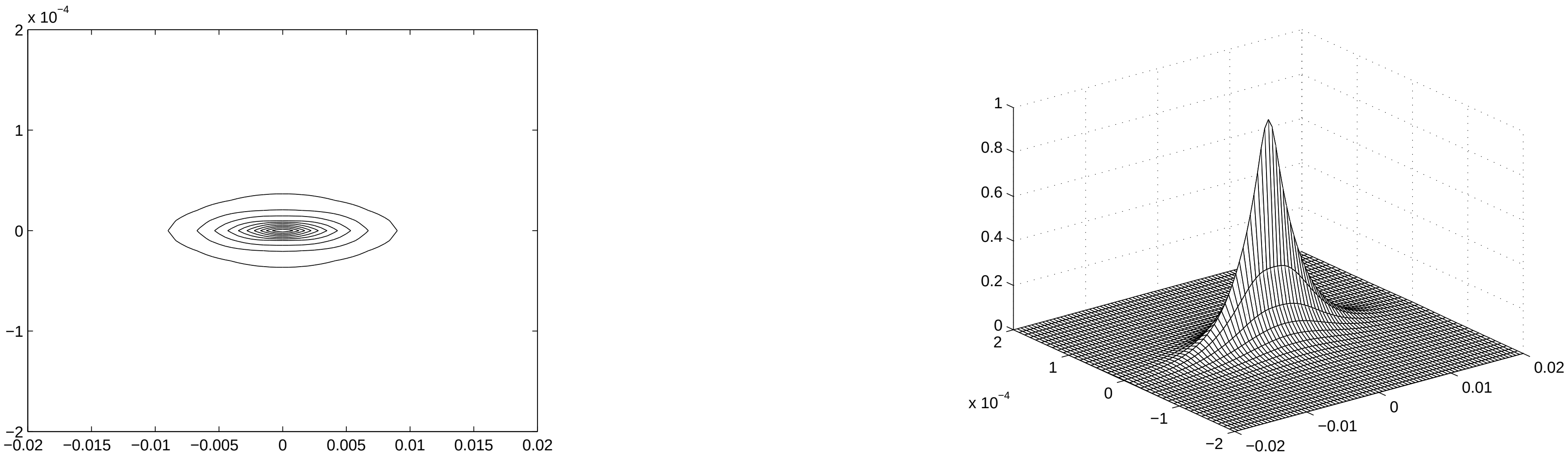


Figure 5: Contour and surface plots of the (scaled) profile likelihood for simulated data $(T = 1000)$ with $(\theta_1^{\text{true}}, \theta_2^{\text{true}}) = (0, 0)$

Choosing shrinkage priors for process variances

Take-away message from these figures:

- The information is contained in the likelihood function
- Don't destroy it by using the wrong prior
- For each process variance, a shrinkage prior should put positive prior probability to shrinking neighborhoods of 0 (Castillo and van der Vaart, 2012)

The non-centered state space model

The TVP model with initial value $\beta_{0,j} \sim \mathcal{N}\left(\beta_j, \theta_j P_{0j}\right)$,

$$\beta_{tj} = \beta_{t-1,j} + w_{jt}, \qquad w_{jt} \sim \mathcal{N}\left(0, \theta_j\right), \quad j = 1, \ldots, r,$$
$$y_t = \mathbf{x}_t \boldsymbol{\beta}_t + \varepsilon_t,$$

could be reparameterized as a **non-centered state space model**:

$$\tilde{\beta}_{jt} = \tilde{\beta}_{j,t-1} + \tilde{\omega}_{jt}, \quad \tilde{\omega}_{jt} \sim \mathcal{N}\left(0, 1\right), \quad j = 1, \ldots, r,$$
$$y_t = \mathbf{x}_t \boldsymbol{\beta} + \mathbf{x}_t \mathrm{Diag}\left(\sqrt{\theta_1}, \ldots, \sqrt{\theta_d}\right) \tilde{\boldsymbol{\beta}}_t + \varepsilon_t, \qquad (1)$$

with $\tilde{\beta}_{j0} \sim \mathcal{N}\left(0, P_{0j}\right)$ (Frühwirth-Schnatter and Wagner, 2010).

Dirac-spike-slab-priors for TVP models

Preferred choice: **Dirac-spike-and-slab prior** for β_j and $\sqrt{\theta_j}$:

$$\beta_j \sim (1-\omega^\delta)\Delta_0(\beta_j) + \omega^\delta \mathcal{N}\left(0, \tau_j^2\right),$$
$$\sqrt{\theta_j} \sim (1-\omega^\gamma)\Delta_0(\sqrt{\theta_j}) + \omega^\gamma \mathcal{N}\left(0, \xi_j^2\right),$$

i.e.:

- $\Pr(\beta_j = 0) = 1 - \omega^\delta$;
- $\Pr(\theta_j = 0) = 1 - \omega^\gamma$.

Choosing $\tau_j^2 \sim \mathcal{G}^{-1}(\nu_1, \nu_2)$ and $\xi_j^2 \sim \mathcal{G}^{-1}(\nu_3, \nu_4)$ leads to Student-t distributions in the slab (Castillo and van der Vaart, 2012).

Dirac-spike-slab-priors for TVP models

Representation involving binary indicators δ_j and γ_j with $\Pr(\delta_j = 1) = \omega^\delta$ and $\Pr(\gamma_j = 1) = \omega^\gamma$:

$$\beta_j|(\delta_j = 0) \equiv 0, \quad \beta_j|(\delta_j = 1) \sim \mathcal{N}\left(0, \tau_j^2\right),$$
$$\sqrt{\theta}_j|(\gamma_j = 0) \equiv 0, \quad \sqrt{\theta}_j|(\gamma_j = 1) \sim \mathcal{N}\left(0, \xi_j^2\right).$$

Classification of the coefficients based on the posterior probabilities $\Pr(\delta_j = 0, \gamma_j = 0|\mathbf{y})$, $\Pr(\delta_i = 1, \gamma_j = 0|\mathbf{y})$, and $\Pr(\delta_i = 1, \gamma_j = 1|\mathbf{y})$.

Dirac-spike-slab-priors for TVP models

- Perform MCMC based search in the model space defined by all 2^{2r} combinations of δ_j and γ_j (model search conditional on the latent variables $\tilde{\beta}_{jt}$).
- Use the posterior draws of δ_j and γ_j for model specification search.
- Allows intrinsic classification of each parameter into the three categories (insignificant, significant but fixed, dynamic).

MCMC for the Dirac-spike-and-slab prior

(a) Joint sampling of the indicators $\boldsymbol{\delta}, \boldsymbol{\gamma}$ from $p(\boldsymbol{\delta}, \boldsymbol{\gamma}|\mathbf{x}, \mathbf{y})$ using the marginal likelihood (closed form expression) of regression model (1) conditional on the state vector $\mathbf{x} = (\tilde{\boldsymbol{\beta}}_1, \ldots, \tilde{\boldsymbol{\beta}}_T)$

(b) Joint sampling of $\boldsymbol{\beta}, \sqrt{\theta_1}, \ldots, \sqrt{\theta_d}, \sigma^2$ conditional on $\tilde{\boldsymbol{\beta}}_1, \ldots, \tilde{\boldsymbol{\beta}}_T$ (inverted Gamma for σ^2, multivariate normal conditional on σ^2).

(c) Sample the states $\tilde{\boldsymbol{\beta}}_1, \ldots, \tilde{\boldsymbol{\beta}}_T$ in the NC state space model jointly conditional on $\boldsymbol{\beta}$ and $\boldsymbol{\theta}$: FFBS or AWOL for the **dynamic coefficients**, if $\theta_j = 0$ then simply sampled from the prior.

Sampling the state process

- **FFBS** (Frühwirth-Schnatter, 1994; Carter and Kohn, 1994; De Jong and Shephard, 1995; Durbin and Koopman, 2002)

- **A**ll **W**ith**O**ut a **L**oop (Kastner and Frühwirth-Schnatter, 2014): $\mathbf{x} = vec(\tilde{\boldsymbol{\beta}}_1, \ldots, \tilde{\boldsymbol{\beta}}_T) \sim \mathcal{N}\left(\Omega^{-1}c, \Omega^{-1}\right)$ where Ω is a very sparse band precision matrix (Rue, 2001; McCausland et al., 2011)
  - **No need to invert Ω**; instead fast band back-substitution
  - Cholesky decomposition of $\Omega = LL'$; solve $c = Lb$ for b
  - draw Td univariate standard normals, i.e. $\boldsymbol{\varepsilon} \sim \mathcal{N}\left(0, I_{Td}\right)$
  - solve $L'\mathbf{x} = b + \boldsymbol{\varepsilon}$ for $\mathbf{x}$.

Closely related to Chan and Jeliazkov (2009) (information filtering)

Dirac-spike-and-slab prior in practice

- Dirac-spike-and-slab priors for SSM and TVP model have been applied successfully in various papers (Frühwirth-Schnatter and Wagner, 2010; Proietti and Grassi, 2011; Grassi and Proietti, 2014).

- d large: single move MH of the indicators δ_j, γ_j for $j = 1, \ldots, d$ by sampling from $p(\delta_j, \gamma_j | (\boldsymbol{\delta}, \boldsymbol{\gamma})_{-j}, \mathbf{x}, \mathbf{y})$ may lead to convergence problems.

Need for alternative shrinkage priors.

Hierarchical shrinkage priors

Well-known for regression models ...

- **Bayesian Lasso prior** (Park and Casella, 2008):

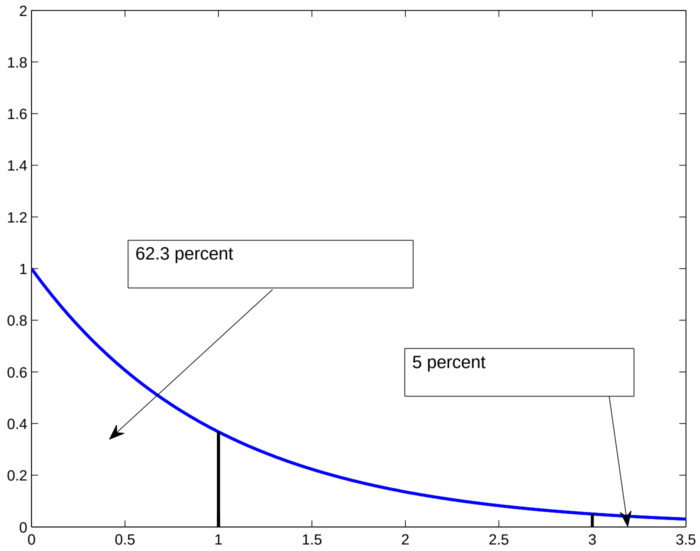
$$\beta_j|\psi_j \sim \mathcal{N}\left(0, 2/\lambda^2\psi_j\right), \qquad \psi_j \sim \mathcal{E}\left(1\right).$$

- **Normal-Gamma prior** (Griffin and Brown, 2010):

$$\beta_j|\psi_j \sim \mathcal{N}\left(0, 2/\lambda^2\psi_j\right), \qquad \psi_j \sim \mathcal{G}\left(a, a\right), \quad a < 1.$$

Shrink globally ($2/\lambda^2$), act locally ($\psi_j < 1$: more, $\psi_j > 1$: less shrinkage) (Polson and Scott, 2011)

Hierarchical shrinkage priors



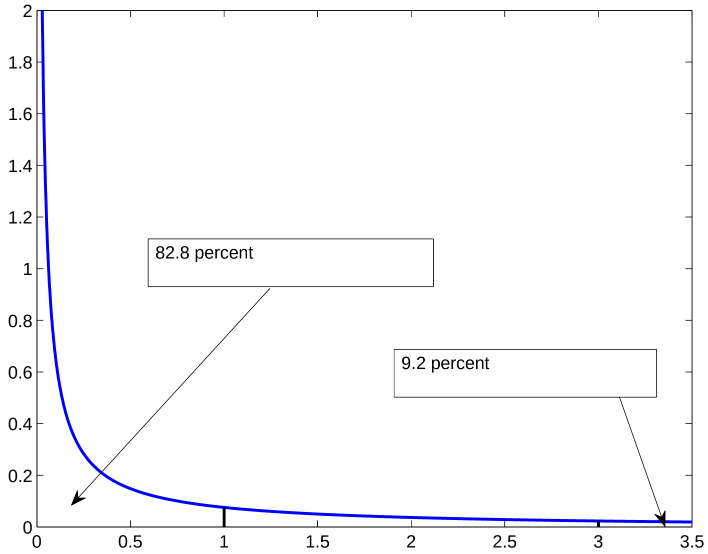


Figure 6: Shrinkage prior for ψ_i based on the exponential distribution (left) and the Gamma prior with $a = 0.1$ (right)

Hierarchical shrinkage priors for TVP models

Belmonte et al. (2014) use the **Bayesian Lasso** priors as shrinkage priors for the initial value β_j and the process variance $\sqrt{\theta_j}$.

Hierarchical representation:

$$\begin{aligned} \beta_j|\tau_j^2 &\sim \mathcal{N}\left(0, \tau_j^2\right), & \tau_j^2 &\sim \mathcal{E}\left(\lambda^2/2\right), \\ \sqrt{\theta_j}|\xi_j^2 &\sim \mathcal{N}\left(0, \xi_j^2\right), & \xi_j^2 &\sim \mathcal{E}\left(\kappa^2/2\right), \end{aligned} \tag{2}$$

A hierarchical prior is used for the level of overall shrinkage, i.e. $\lambda^2 \sim \mathcal{G}\left(\nu_1, \nu_2\right)$ and $\kappa^2 \sim \mathcal{G}\left(\nu_3, \nu_4\right)$.

The LASSO prior is known for overshrinking significant effects

More general regularization priors for TVP models

PhD project of Angela Bitto: more flexibility is obtained by using a **normal-Gamma prior** both for β_j and $\sqrt{\theta_j}$.
This prior has a hierarchical representation as in (2):

$$\begin{aligned} \beta_j|\tau_j^2 &\sim \mathcal{N}\left(0,\tau_j^2\right), \qquad & \tau_j^2 &\sim \mathcal{G}\left(a^\tau, a^\tau \lambda^2/2\right), \\ \sqrt{\theta_j}|\xi_j^2 &\sim \mathcal{N}\left(0,\xi_j^2\right), \qquad & \xi_j^2 &\sim \mathcal{G}\left(a^\xi, a^\xi \kappa^2/2\right). \end{aligned}$$

- Special case: $a^\tau = 1$ and $a^\xi = 1$ leads to Lasso hierarchical shrinkage prior.
- $a^\tau < 1$ and $a^\xi < 1$ introduces more sparsity without overshrinking.

MCMC for hierarchical shrinkage priors

(a) Sample the state $\tilde{\boldsymbol{\beta}}_1, \ldots, \tilde{\boldsymbol{\beta}}_T$ in the NC state space model jointly conditional on $\boldsymbol{\beta}$ and $\sqrt{\theta_1}, \ldots, \sqrt{\theta_d}$.

(b) Joint sampling of $(\boldsymbol{\beta}, \sqrt{\theta_1}, \ldots, \sqrt{\theta_d})$ **conditional on the prior variances τ_j^2 and ξ_j^2** and $\tilde{\boldsymbol{\beta}}_1, \ldots, \tilde{\boldsymbol{\beta}}_T$. The corresponding distribution is a multivariate normal distribution.

(c) Sample $\sigma^2|\cdot$ from an inverted Gamma distribution.

(d) Sample the prior variances τ_j^2 and ξ_j^2 conditional on $(\boldsymbol{\beta}, \sqrt{\theta_1}, \ldots, \sqrt{\theta_d})$ from independent **generalized inverse Gaussian (GIG) distributions**.

The generalized inverse Gaussian (GIG) distribution

The conditionally normal prior $\beta_j \sim \mathcal{N}\left(0, \tau_j^2\right)$ leads to a posterior for $\tau_j^2|\beta_j$, where the likelihood is the **kernel of an inverted Gamma density** in τ_j^2 which is combined with the **Gamma prior** $\tau_j^2 \sim \mathcal{G}\left(a^\tau, a^\tau \lambda^2/2\right)$:

$$\begin{aligned} p(\tau_j^2|\beta_j) &\propto (\tau_j^2)^{-1/2} \exp\left(-\frac{\beta_j^2}{2\tau_j^2}\right) \times (\tau_j^2)^{a^\tau - 1} \exp\left(-\frac{a^\tau \lambda^2 \tau_j^2}{2}\right) \\ &\propto (\tau_j^2)^{(a^\tau - 1/2) - 1} \exp\left(-\frac{a^\tau \lambda^2 \tau_j^2}{2}\right) \exp\left(-\frac{\beta_j^2}{2\tau_j^2}\right). \end{aligned}$$

The generalized inverse Gaussian (GIG) distribution

This leads to the **generalized inverse Gaussian (GIG) distribution** both for $\tau_j^2|\beta_j$ and $\xi_j^2|\theta_j$:

$$\tau_j^2|\beta_j \sim \mathcal{GIG}\left(a^\tau - 1/2, a^\tau \lambda^2, \beta_j^2\right),$$
$$\xi_j^2|\theta_j \sim \mathcal{GIG}\left(a^\xi - 1/2, a^\xi \kappa^2, \theta_j\right).$$

The inverse Gaussian distribution, $\mathcal{GIG}(p, a, b)$ is a three-parameter family with density ($a > 0$, $b > 0$ and p is a real parameter):

$$f(y) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} y^{p-1} e^{-(a/2)y} e^{-b/(2y)},$$

where $K_p(z)$ is the modified Bessel function of the second kind.

Sampling from the GIG distribution

A **very stable generator** is implemented in the recent R-package `GIGrvg` by J. Leydold (Hörmann and Leydold, 2013), see http://cran.r-project.org/web/packages/GIGrvg/index.html:

- Devroye (2013) shows that $Z = \log X$, where $X \sim \mathcal{GIG}\left(p, \sqrt{ab}, \sqrt{ab}\right)$ with $p > 0$, has a log-concave density.

- If $p > 0$, then $Y = \sqrt{b/a}e^{Z}$ follows $\mathcal{GIG}\left(p, a, b\right)$; if $p < 0$, then $Y = \sqrt{b/a}e^{-Z}$ follows $\mathcal{GIG}\left(-p, a, b\right)$.

MATLAB (`randraw`) or older R-packages fail with sampling from $\mathcal{GIG}\left(p, a, b\right)$, if b is close to 0 (natural under shrinkage priors!)

EU-aera inflation modelling

TVP version of the generalized Phillips curve (Belmonte et al., 2014):

$$y_{t+h} = c_t + \sum_{j=0}^{p-1} \phi_{j,t} \cdot y_{t-j} + \boldsymbol{\alpha}_t \mathbf{x}_t + s_t + \varepsilon_{t+h},$$

where inflation depends on

- lags of inflation ($p = 12$) and 11 monthly seasonal dummies
- 13 other predictors $\mathbf{x}_t$ (1-month and 1-year Euribor, change in industrial production index, change in loans, change in monetary aggregate M3, unemployment rate, oil price, . . .)

EU-aera inflation modelling

- 37 possibly time-varying coefficients, each following $\beta_{tj} = \beta_{t-1,j} + w_{jt}, w_{jt} \sim \mathcal{N}(0, \theta_j)$ with unknown initial value β_j and unknown variance θ_j.

- Monthly data from February 1994 to November 2010

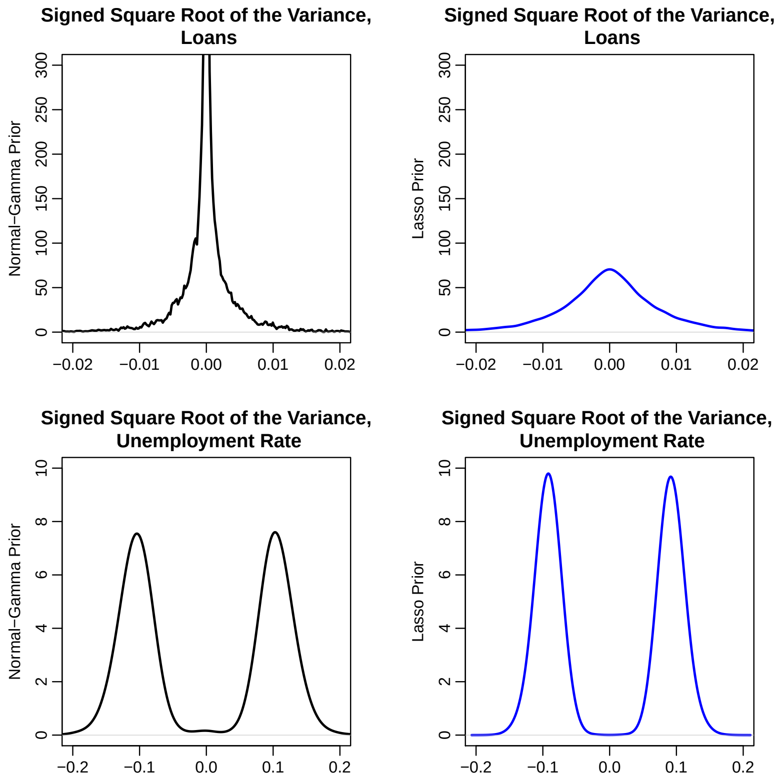
- LASSO priors (vary $\nu_1 = \nu_2 = \nu_3 = \nu_4 = \nu$)

- Normal-gamma priors (vary $a^\tau < 1$ and $a^\xi < 1$, vary $\nu_1 = \nu_2 = \nu_3 = \nu_4 = \nu$)

- Dirac-spike-and-slab priors lead to different solutions, depending on the starting values.

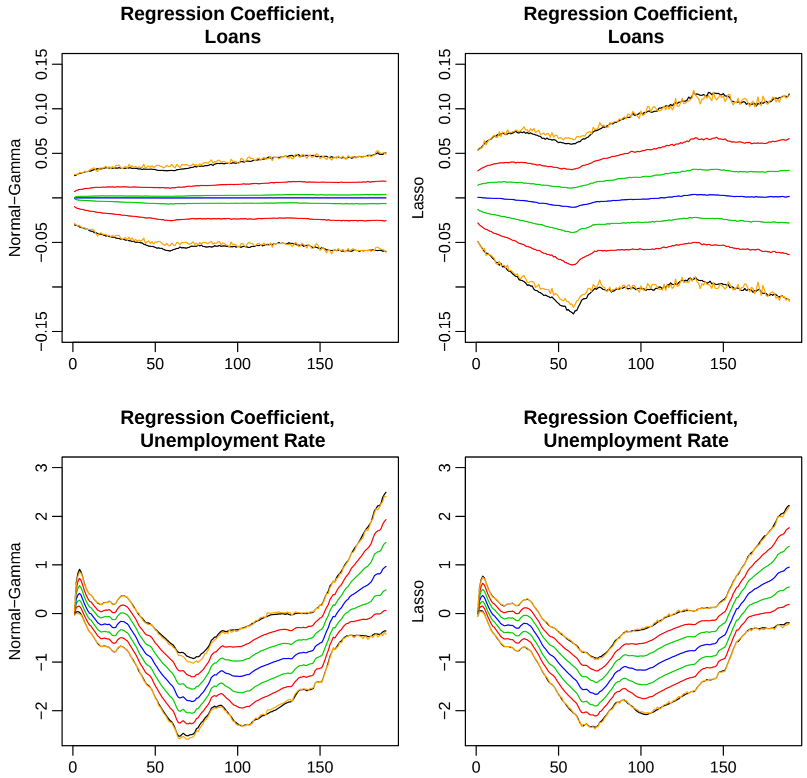
Posterior exploration for hierarchical shrinkage priors

Visual inspection of the posterior densities of $\sqrt{\theta_j}$ allows „classification" between static/dynamic:



Posterior exploration for hierarchical shrinkage priors

Visual inspection of the posterior paths of $\beta_{tj}, t = 1, \ldots, 190$ (20,000 draws)



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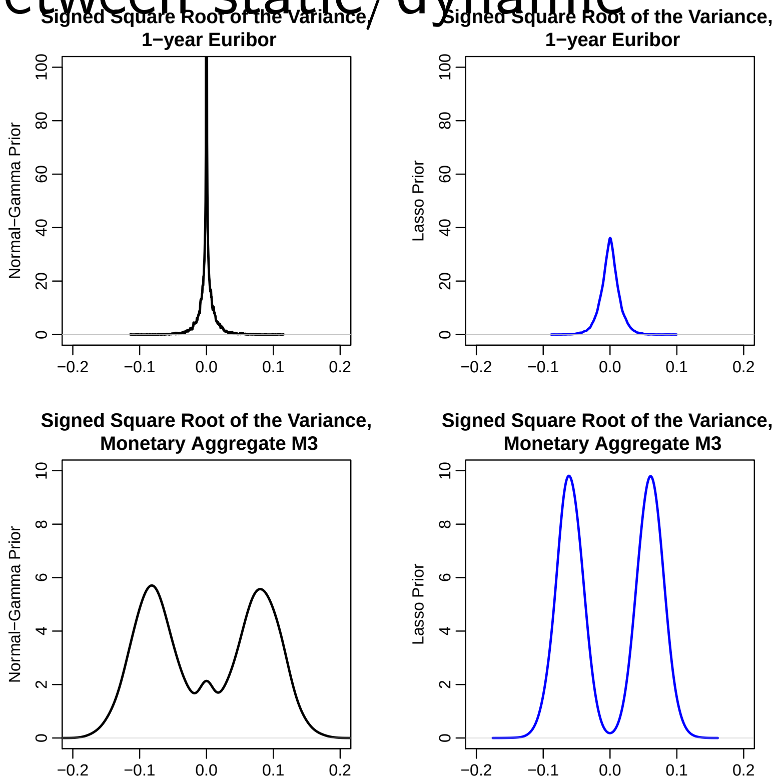

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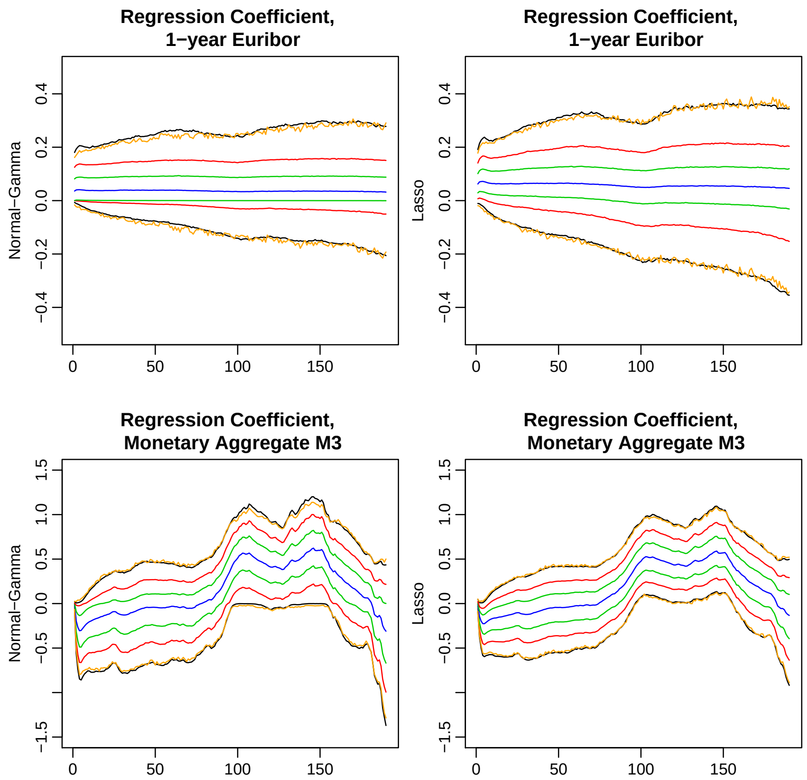


Predictive evaluation

Predictive evaluation for various shrinkage priors $\mathcal{M}$ over the last 60 months (i.e. $t_0 = 134$):

$$\log p(y_{t_0+1}, \ldots, y_T | \mathbf{y}^{\text{tr}}, \mathcal{M}) = \sum_{t=t_0+1}^{T} \log p(y_t | y_1, \ldots, y_{t-1}, \mathcal{M}).$$

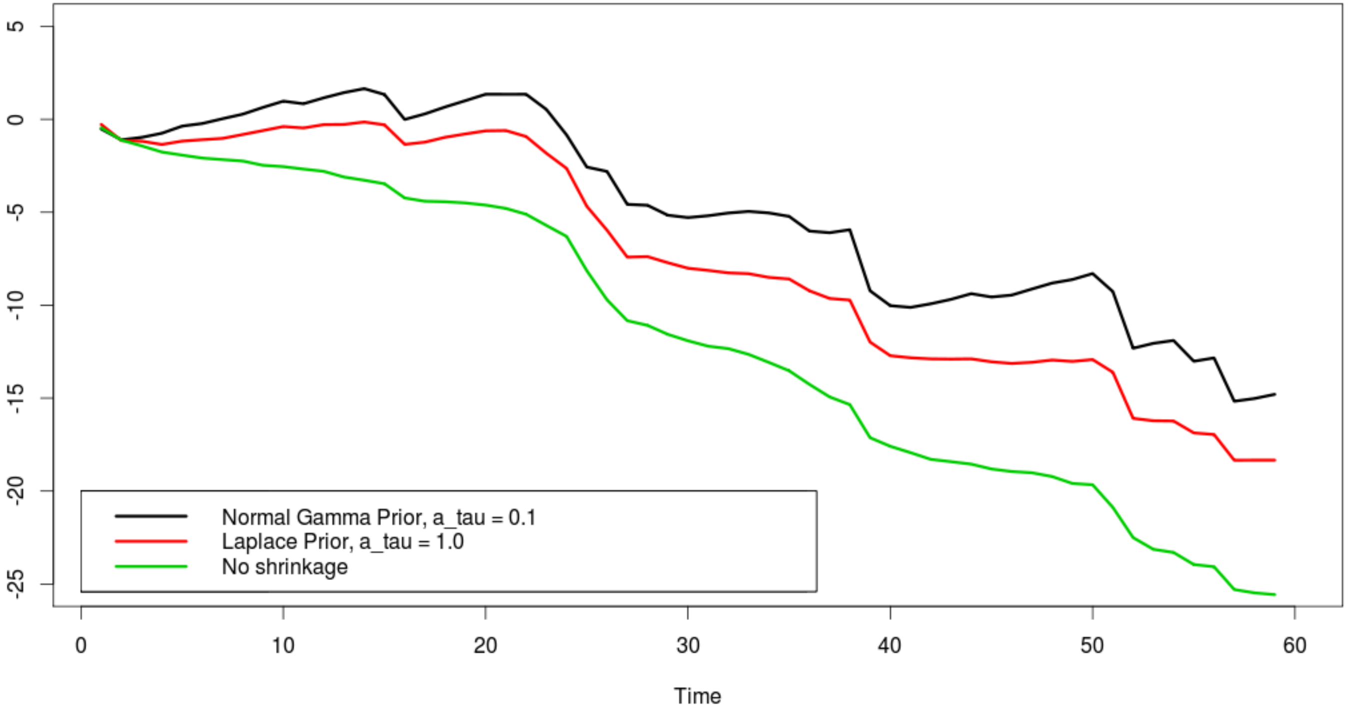
- Interpretation as training sample marginal likelihood for shrinkage prior $\mathcal{M}$.
- For each $t = t_0 + 1, \ldots, T$ independent MCMC runs on local cluster (500 cores) to determine $p(y_t | y_1, \ldots, y_{t-1}, \mathcal{M})$ for each $\mathcal{M}$.

Predictive evaluation for EU-aera inflation modelling

What is a good hierarchical shrinkage prior?

- First results suggest that **spike-slab priors (if they converge)** beat hierarchical shrinkage priors
- The normal-Gamma with small $a^{\tau} < 1$ and $a^{\xi} < 1$ beats the LASSO prior considered in Belmonte et al. (2014)
- All priors dominate models without shrinkage.

Predictive evaluation for EU-aera inflation modelling



Extensions of the standard TVP model

- **Stochastic volatility** using the R-package `stochvol` (Kastner and Frühwirth-Schnatter, 2014)

- Extend $\mathbf{Q} = \mathrm{Var}(\boldsymbol{\beta}_t | \boldsymbol{\beta}_{t-1})$ to a **full covariance matrix** as for covariance selection in random effects models (Chen and Dunson, 2003; Frühwirth-Schnatter and Tüchler, 2008)

- **Non-Gaussian** TVP models using auxiliary mixture sampling (Frühwirth-Schnatter and Wagner, 2010)

- **Multivariate** TVP models (financial application)

Cholesky SV Modelling - Multivariate TVP models

A multivariate time series $\mathbf{y}_t \sim \mathcal{N}_r(\mathbf{0}, \mathbf{\Sigma}_t)$ with time-varying covariance matrix may be represented as a set of time-varying regressions (Lopes et al., 2012), e.g. for $r = 3$:

$$
\begin{aligned}
y_{1t} &= \varepsilon_{1t}, \qquad \varepsilon_{1t} \sim \mathcal{N}\left(0, \sigma^2_{1,t}\right) \\
y_{2t} &= \beta_{21t} y_{1t} + \varepsilon_{2t}, \qquad \varepsilon_{2t} \sim \mathcal{N}\left(0, \sigma^2_{2,t}\right) \\
y_{3t} &= \beta_{31t} y_{1t} + \beta_{32t} y_{2t} + \varepsilon_{3t}, \qquad \varepsilon_{3t} \sim \mathcal{N}\left(0, \sigma^2_{3,t}\right)
\end{aligned}
$$

with state equation:

$$
\beta_{ij,t} = \beta_{ij,t-1} + w_{ijt}, \qquad w_{ijt} \sim \mathcal{N}\left(0, \theta_{ij}\right).
$$

More details on Cholesky SV modelling

Lopes et al. (2012) consider the Cholesky decomposition of $\mathbf{\Sigma}_t$:

$$\mathbf{\Sigma}_t = \boldsymbol{A}_t \boldsymbol{D}_t \boldsymbol{A}_t^{'},$$

where $\boldsymbol{A}_t$ is a lower triangular (LT) with ones on the main diagonal and $\boldsymbol{D}_t$ is a diagonal matrix. Then

$$\boldsymbol{A}_t^{-1} \mathbf{y}_t \sim \mathcal{N}_r \left(\mathbf{0}, \boldsymbol{A}_t^{-1} \mathbf{\Sigma}_t (\boldsymbol{A}_t^{'})^{-1} \right) = \mathcal{N}_r \left(\mathbf{0}, \boldsymbol{D}_t \right). \tag{3}$$

Since $\boldsymbol{A}_t^{-1}$ is also a LT with ones on the main diagonal, (3) may be rewritten with $\mathbf{B}_t = -(\boldsymbol{A}_t^{-1} - \mathbf{I}_r)$ is LT with zero main diagonal:

$$\mathbf{y}_t = \mathbf{B}_t \mathbf{y}_t + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t \sim \mathcal{N}_r \left(\mathbf{0}, \boldsymbol{D}_t \right).$$

Application to DAX data

Daily stock returns indices from the DAX from September 4th 2001 to August 31st, 2011

$T = 2500$, $r = 29$ series

- Filtered using univariate SV-models to reduce conditional heteroscedasticity (Frühwirth-Schnatter and Lopes, 2012)
- Unfiltered returns using the TVP-SV model

Model search with SSVS

3 (filtered) stock returns from the DAX: BMW/Daimler/Volkswagen (Car industry)

Indicators highest probability model (frequency equals 0.54):

$\beta_{21,0}$	$\beta_{31,0}$	$\beta_{32,0}$
0	**1**	**1**
θ_{21}	θ_{31}	θ_{32}
0	0	**1**

Indicators median probability model (Scott and Berger, 2006):

$\beta_{21,0}$	$\beta_{31,0}$	$\beta_{32,0}$
0	**1.0**	**1.0**
θ_{21}	θ_{31}	θ_{32}
0.	0.45	**0.99**

Posterior exploration

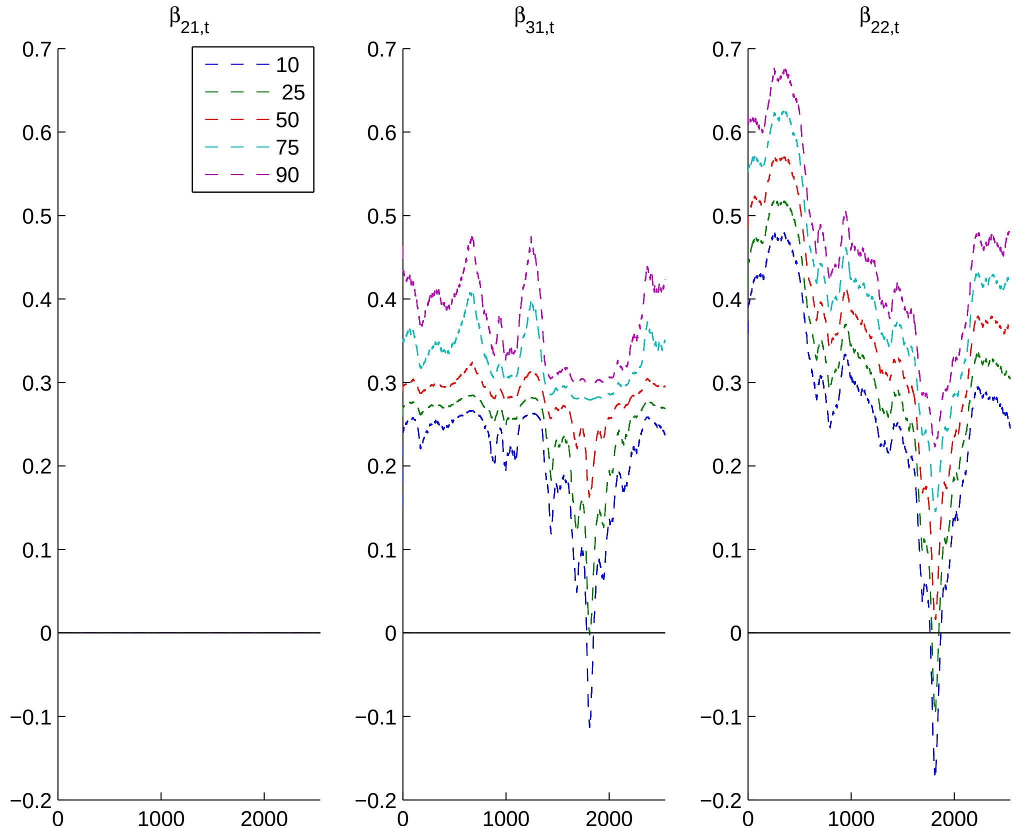


Figure 7: Posterior simulations for $\beta_{t,21}$, $\beta_{t,31}$ and $\beta_{t,32}$

Posterior exploration

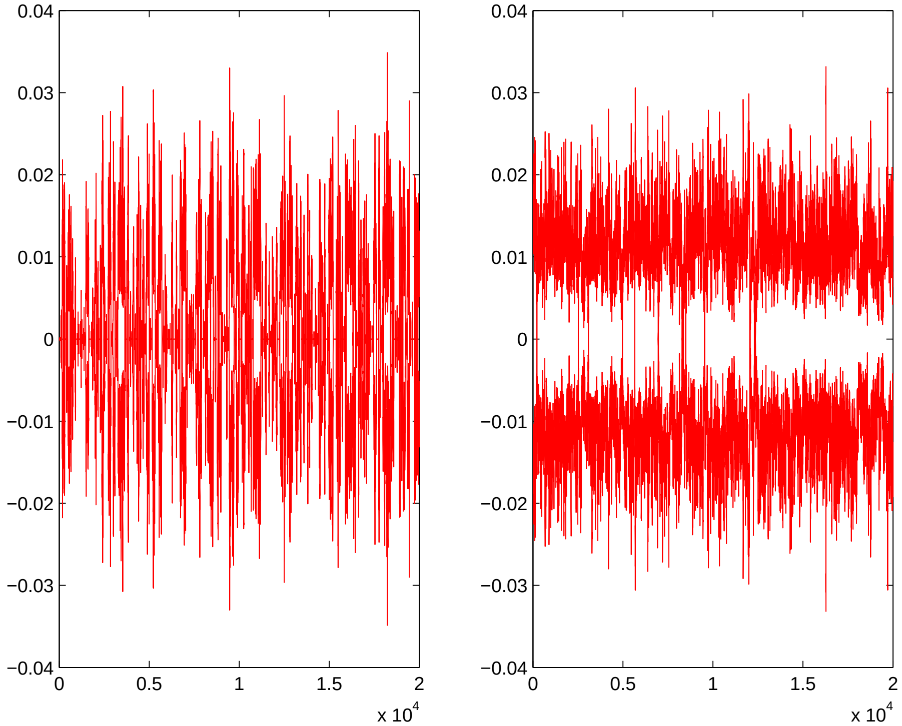


Figure 8: Posterior simulations for $\sqrt{\theta_{t,31}}$ and $\sqrt{\theta_{t,32}}$

Posterior exploration

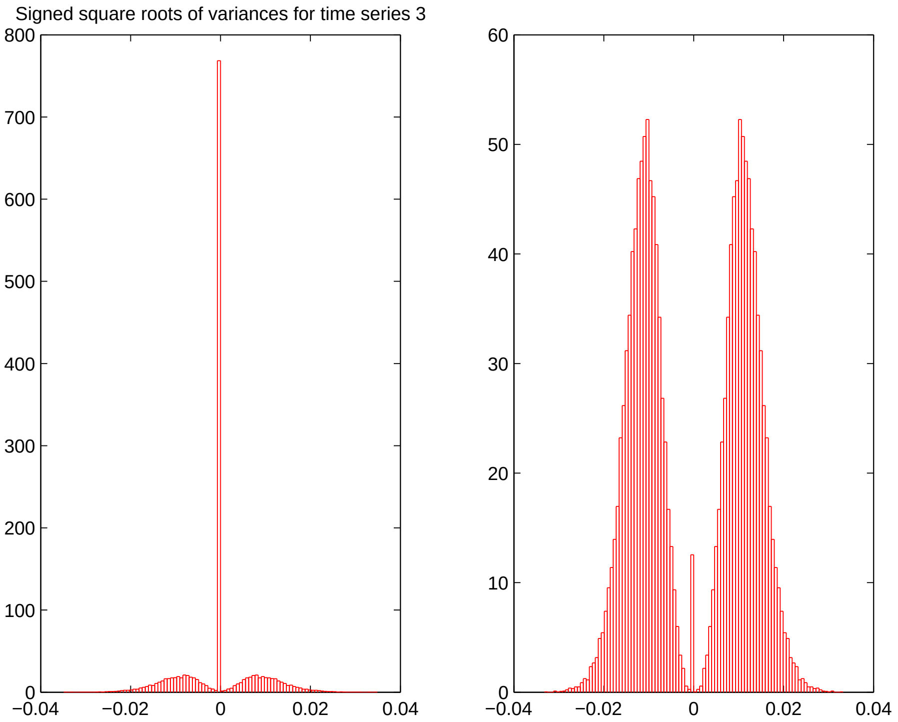


Figure 9: Posterior distributions for $\sqrt{\theta_{t,31}}$ and $\sqrt{\theta_{t,32}}$

Global minimum variance portfolio

Time-varying weights w_{t1}, w_{t2}, and w_{t3} for the global minimum variance portfolio $P_t = \mathbf{w}_t'\mathbf{y}_t$

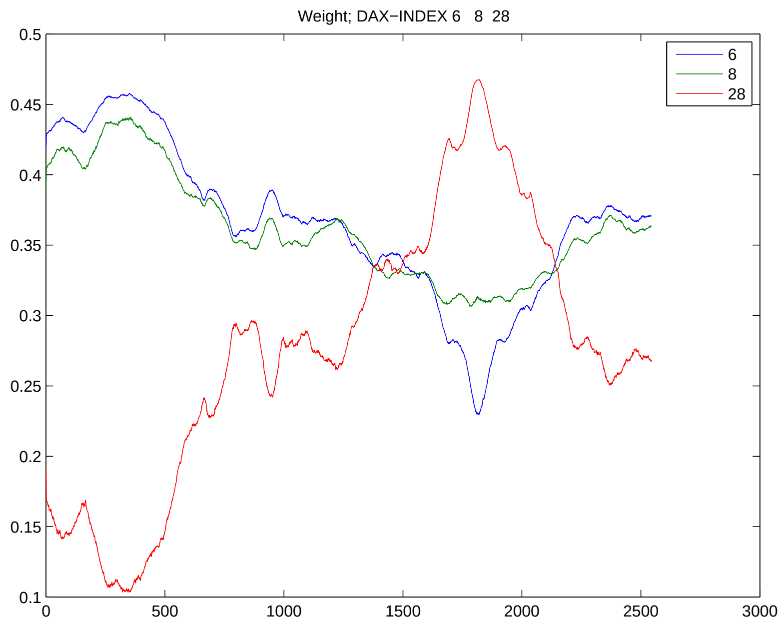


Figure 10: 6 BMW; 8 Daimler; 28 Volkswagen

What difference does it make?

Fully dynamic model (green) versus parsimonious model using spike-slab-priors (red)

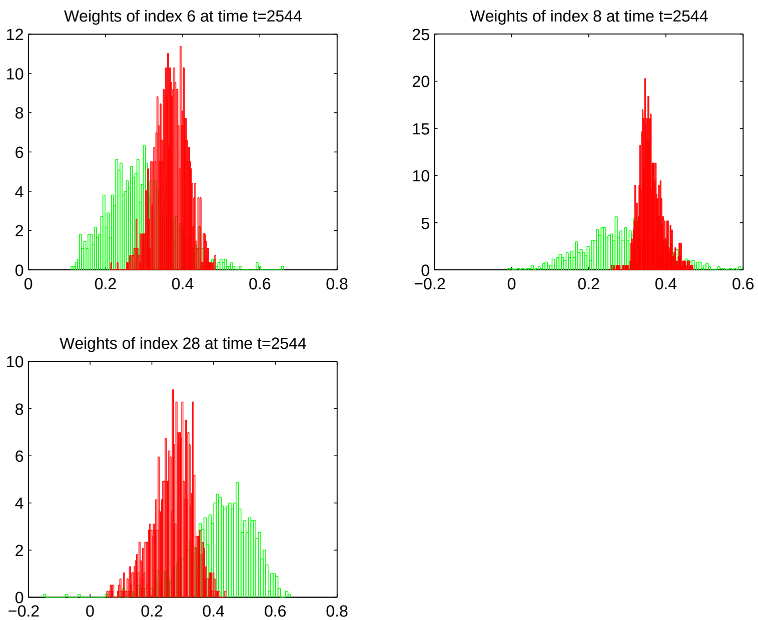


Figure 11: Weights w_{t1}, w_{t2}, and w_{t3} for the global minimum variance portfolio $P_t = \mathbf{w}_t'\mathbf{y}_t$ at $t = T$

All 29 (filtered) DAX returns

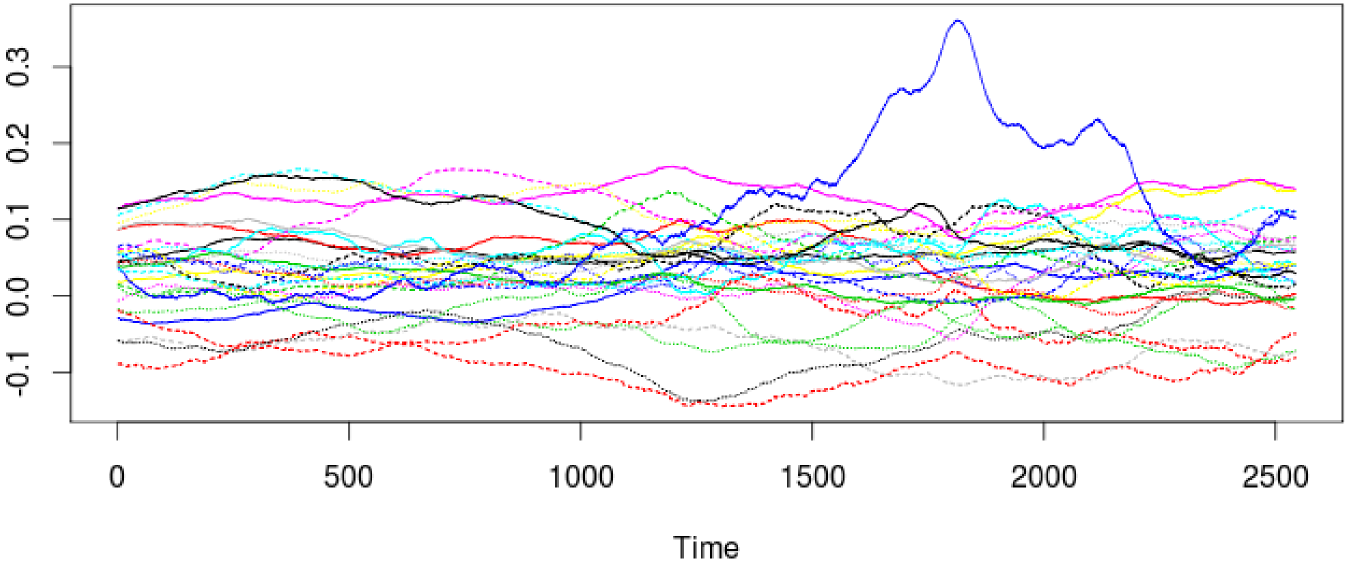
- Leads to $29 \cdot 28/4 = 406$ (!) dynamic coefficients $\beta_{ij,t}$

- Variable selection among $3^{406} = 5 \cdot 10^{193}$ models $\Rightarrow$ great challenge for the Dirac-spike-and-slab prior

- LASSO priors (vary $\nu_1 = \nu_2 = \nu_3 = \nu_4 = \nu$)

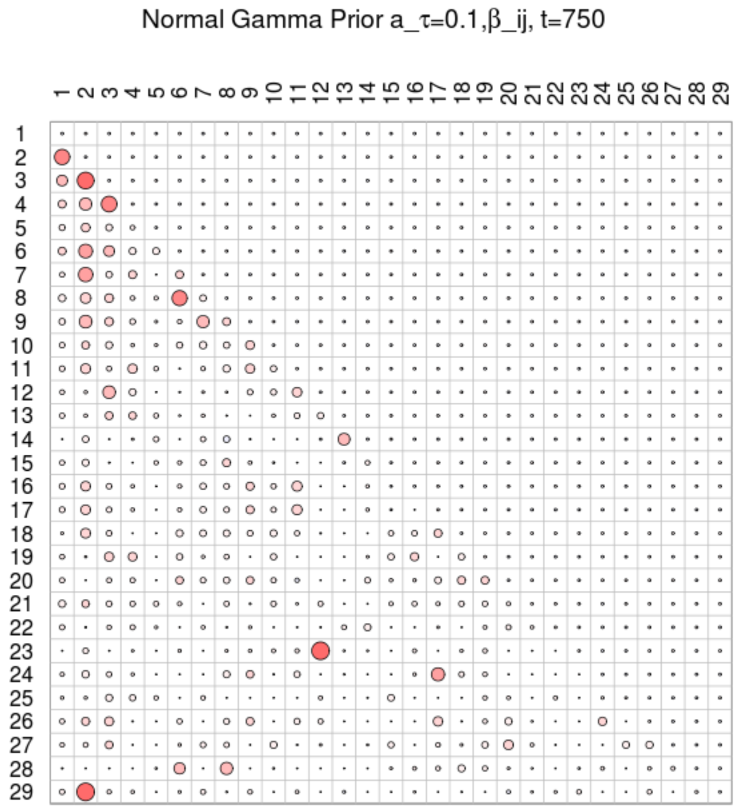
- Normal-gamma priors (vary $a^\tau < 1$ and $a^\xi < 1$, vary $\nu_1 = \nu_2 = \nu_3 = \nu_4 = \nu$)

All 29 (filtered) DAX returns



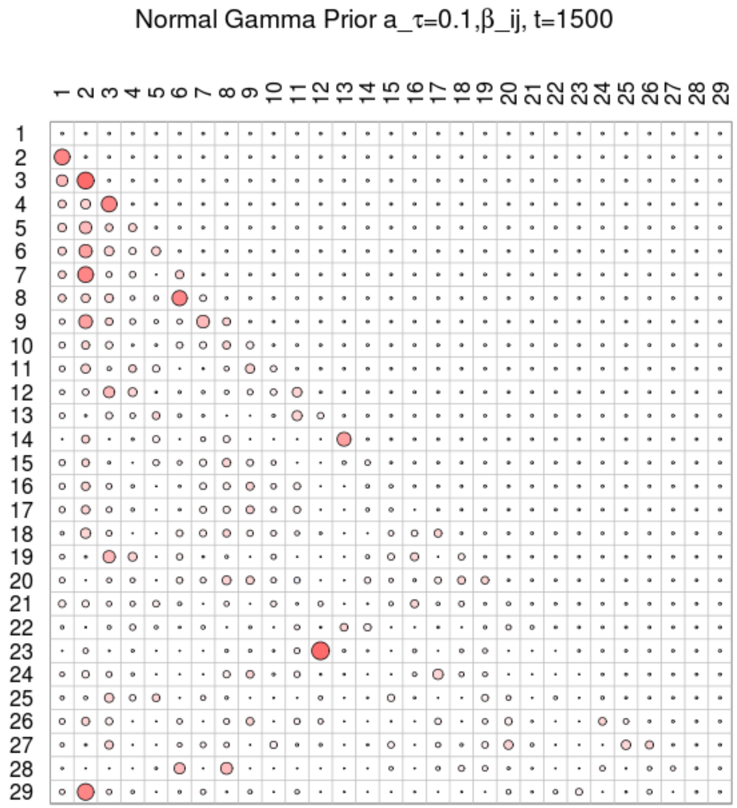
Time-varying portfolio weights (normal Gamma Prior, $a_\tau = 0.1$)

Sparsity



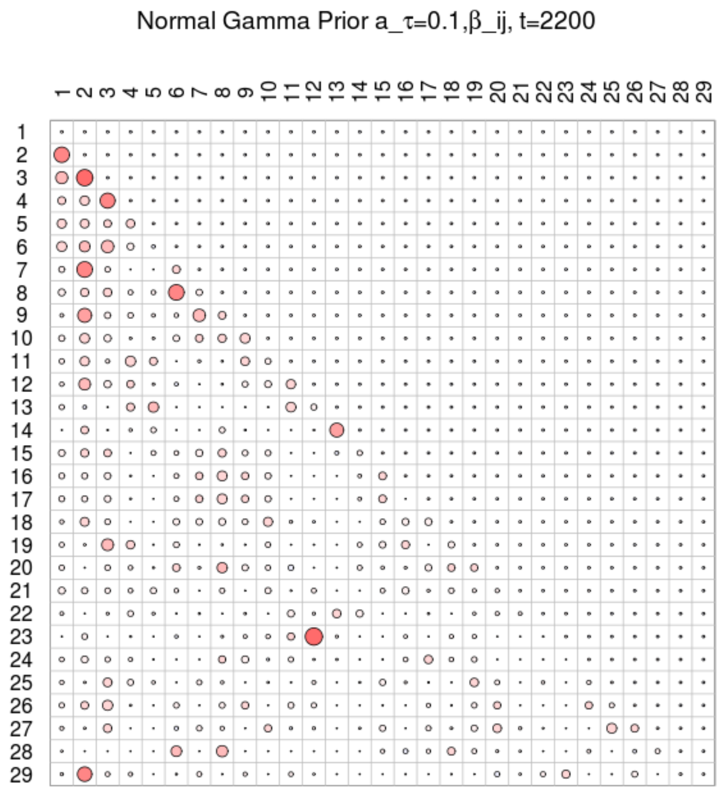
Cholesky factors $\beta_{ij,t}$ at $t = 750$

Sparsity



Cholesky factors $\beta_{ij,t}$ at $t = 1500$

Sparsity



Cholesky factors $\beta_{ij,t}$ at $t = 2200$

Summary

- Well-known ideas of shrinkage and variable selection for the standard regression model can be easily extended to TVP models using a non-centered version of the state space model

- Hierarchical representation allows efficient MCMC sampling

- Normal gamma hierarchical shrinkage priors seem preferable to the LASSO; strongly outperforms standard TVP without shrinkage

- Spike-and-slab priors allow model search/variable selection, but are unstable for higher dimensional models

Outlook

- More theoretical work in the spirit of Castillo and van der Vaart (2012) needed for TVP models (model consistency, oracle property)
- Incorporate recent work on time-varying sparsity (Nakajima and West, 2013; Kalli and Griffin, 2014)
- Apply to (high-dimensional) data sets from

  Economics (VAR models with TVP coefficients)
  Finance (returns and other financial time series)
  and
  Business (marketing research)

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