A Course in Model Theory

An Introduction to Contemporary Mathematical Logic

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Bruno Poizat

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A Course in Model Theory

An Introduction to Contemporary Mathematical Logic

Translated by Moses Klein

نور المنطق والمعرفه

Springer

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السلامة

Preface to the English Edition

Can we reproduce the inimitable, or give a new life to what has been affected by the weariness of existence? Folks, what you have in your hands is a translation into English of a book that was first published in 1985 by its author, that is, myself, at the end of an editorial adventure about which you will find some details later. It was written in a dialect of Latin that is spoken as a native language in some parts of Europe, Canada, the U.S.A., the West Indies, and is used as a language of communication between several countries in Africa. It is also sometimes used as a language of communication between the members of a much more restricted community: mathematicians.

This translation is indeed quite a faithful rendering of the original: Only a final section, on the reals, has been added to Chapter 6, plus a few notes now and then. On the title page you see an inscription in Arabic letters, with a transcription in the Latin (some poorly informed people say English!) alphabet below; I designed the calligraphy myself. My first idea was to imitate the characters displayed at the top of a funeral tower located near the village of Resget, Māzandarān (I visited the spot twice, in 1968 and 1997), but finally I used a simpler inscription from the Masjid-e Djumeh of Zavareh (unfortunately, I have been there once only, in 1967), which was published in Athar-e, Iran, in 1936, nb. 1. "Nur al-mantiq wal-ma'rifah" means "Light of Logic and Knowledge." In my second book I had to explain that I did not wish that this name be applied to myself, but to the collection of books that I was launching.

On the back cover of the French edition you will find a short biography of the author, which translates as follows:

> Bruno Poizat spent his youth in the most prestigious schools and institutes, allowing him to obtain a modest position at Pierre and Marie Curie University. He is the best, and only, French specialist of modern Aramaic. He is a tolerable dancer of tango and swing. He is the owner of a reasonable collection of French and Belgian comic books, and also of some oriental rugs. His livelihood depends on mathematics, which explains why he has written this book.

Well, not so many things have changed in the meantime, except that I can no longer qualify my position as modest, since (thanks to my books!) I now hold the most pompous degree that a French university can provide: I am presently professor of exceptional class at Claude Bernard University.

When you open this book you will encounter five more lines in Arabic, expressing the standard of the collection. They are taken from a large terra-cotta plate from Afrāsīāb (in Samarqand), which is displayed at the entrance gate of the Islamic Antiquities Department at the Louvre (of course, I have read the inscription from the catalogue, not from the plate itself!). Here is the translation: "Science—its beginning has a bitter taste, but its end is sweeter than honey. Greetings!"

One feature of this book is that each chapter opens with a quotation the author of which is to be guessed by the reader. These quotations are reproduced in this edition. I assumed that the guesses were easy for a moderately educated French reader, and so in the French edition the solutions appeared only at the end. But I have been warned that they will be quite demanding for the prospective user of the English version. Therefore, I will simply tell you right here what they are, and toss in some clues pointing at their motivation, which is, I must admit, sometimes obscure.

The introduction opens with a poem of Charles Baudelaire: "I leave you these verses...." Baudelaire is known for his beautiful French version of the works of Edgar Allan Poe, which is remarkable if we consider his rudimentary knowledge of English. Being not so gifted, I will not risk a translation of Baudelaire's verses.

Chapter one begins with the first paragraph of *War and Peace* of Tolstoy (the novel begins in French—only four words in Russian in the first paragraph), which was applauded by my schoolteachers as the paradigm of excellent beginnings.

Chapter two starts with a play on language due to the late Pierre Culliford (Peyo), the designer of the Schtroumpfs.

Chapter three: a vigorous extract of a preface by Evariste Galois on the pedantry of mathematics written in symbols: a subject of meditation for the authors of our times.

Chapter four: a quotation from the original (in French) of Marco Polo's book, which can be summarized in one word: forward!

Chapter five quotes Molière. Alceste, the misanthrope, praises the simplicity of an old song.

Chapter six, old French again: Villehardouin reports on the siege of Constantinople, where limited means have produced tremendous results.

Chapter seven: in Greek, the question of Pilate on truth, from the Gospel of St. John.

Chapter eight: a quotation in Hebrew from the absolute bestseller. It poses some problems to the specialists (what is the true meaning of "Eden of the East"?), but the allusion to Cantor's paradise is, I hope, transparent.

Chapter nine: delicious, and morbid, definition of saturation by Littré in his celebrated *Dictionary of the French Language*.

Chapter ten: too difficult for me to explain; a pun on types, from a famous comic book scenarist (Michel Régnier = Greg).

Chapter eleven: from Victor Hugo's *Orientales*, an invitation to listen to a new story.

Chapter twelve: a verse of Arthur Rimbaud, "A mysterious song is falling from the golden stars." However poor the English rendering may be, the French original is very appealing, and so is the content of the chapter.

Chapter thirteen: a self-quotation. I was very proud of my fundamental order!

Chapter fourteen: for the addicts of Chuck Berry's music. Little Marie is the sequel of Memphis (Tennessee), as Chapter fourteen is the sequel of Chapter nine (phew!).

Chapter fifteen: Cicero (in Latin, of course), on the concept of deviation (forking, in this English version) according to Epicurus.

Chapter sixteen: in German, from Kummer's foundational text on modern algebra, introducing a kind of imaginary divisors, the ideal complex numbers.

Chapter seventeen: a rather long extract from *Democracy in America*, by Alexis de Tocqueville, a classic on social hierarchy.

Chapter eighteen: "Oh, I wish I was eighteen again," which I falsely attribute to Jerry Lee Lewis, who is only one of the interpreters of the song (and a fantastic one). Even now I don't know who the author is.

Chapter nineteen: on making things simple, a pun of Maurice Tilleux, again a well-known designer and writer of comic strips.

Chapter twenty: the beginning of Alistair Lachlan's article "Spectra of ω-Stable Theories," a sample of the English language that is remarkable for the three different ways by which the author refers to himself in a very short sentence.

Bibliography: the wife of Hector, who, according to Georges Brassens, is the only one among the wives of our friends who deserves our attention.

The book ends with a two-page historical note, introduced by a quotation of Bobby Lapointe, Le tube de toilette, which is absolutely untransferable to another language, but is nevertheless very appropriate. Here is the translation of the note itself:

History of a Publication

1978. "Classification Theory," by Saharon Shelah.

1979. "An introduction to forking," by Daniel Lascar and myself.

1978–81. I give several A.E.A. courses at Paris-6 on the foundations of stability.

1978–83. Working group on "Théories Stables," whose works are published by the I.H.P.

March 1981. Robert Königs, a Springer editor, offers spontaneously to publish the works of "Théories Stables." Since this is not possible, he encourages me to write an introductory book on the subject.

1981. The first ten chapters.

1982. The next six chapters.

April 1983. I offer my yet unfinished work for publication to North Holland, which rejects it, accepting only submissions in English.

December 1983. A critical reading of my manuscript by Wilfrid Hodges.

February 1984. I submit it to Springer, which admits its preference for English. I hold to my French.

June 1984. Final rejection by Springer, on linguistic basis.

July 1984. After completion of my manuscript, I bring it to Hermann; no answer, even after repeated calls.

October 1984. I offer it to Masson, which rejects it in December: too narrow commercial diffusion.

February 1985. I send it to the P.U.F: Immediate rejection!

March 1985. I offer it to Dunod, which rejects it for practically the same reasons as Masson. I deposit a file at the MIDIST—a government program in charge of assisting scientific editions in French—just in case.

April 1985. I offer it a second time to Springer for the *Lecture Notes* collection; rejected anew: Too thick, in French, in competition with other works on the same subject.

May 1985. I submit it to *Astérisque*. Rejected in July, in spite of an extremely favorable review by the referee.

July 1985. I decide to publish the "Course" myself. I sell the first copies at the European Meeting of the Association of Symbolic Logic, in Orsay. The 100 copies of this first edition are sold out in a couple of hours.

October 1985. Second edition, 100 copies.

January 1987. The MIDIST, an office under the Ministry of Research and Higher Education, which in the meantime has used all of its energy to transform itself into DIXIT, offers to provide funding for the printing costs of a new edition of 500 copies, subject to the condition that I sign a contract with a professional distributor.

Winter 1987. Experiences with various potential distributors.

June 1987. Third edition, distributed by OFFILIB.

I could have been more precise concerning the experiences of the winter of 1987: It was my first serious experience with racism. I was then assuming that my knowledge of Arabic would be appreciated as a kind of dandyism, an inclination toward exotic systems of spelling, like Sanskrit or Tibetan, something, after all, very natural for a logician. Then I discovered that a number of my fellow citizens did not consider this knowledge as an elevation of the mind, and that at least one director of collections in one of the most respectable publishing companies in Paris reacted strongly to the idea of having Arabic letters printed in his catalogue!

And now, I can add a few words on the sequel of the saga: In 1992 I was approached by Masson, which offered to publish an abridged version of my course. I worked hard to reduce my work by half, and suggested that it should be published under the title *Abrégé du Cours de Théorie des modèles*. The editor insisted on *Cours de Théorie des modèles*, which, as I explained patiently, was not possible, since the new book was appreciably different from the original. For some obscure reasons he rejected all the other titles that I suggested during a year of exchange by mail, and finally broke off bitterly.

Yes, intercourse with editors is an exhausting activity, and this is the reason why I published my second book, *Groupes Stables*, without approaching any of them. It had the happy consequence that I was able to sell the first copies exactly four weeks after the completion of the manuscript. This is also the reason why I am so grateful to Stephen Lempp, who decided as early as September 1995 that this English edition would see the light of day and has spent considerable energy in finding a publisher and a translator: His obstinacy and his pugnacity are indeed the only cause of its existence.

Since I have the opportunity to address, as I do today, an audience of English-speaking mathematicians, I will comment shortly on the fate of a mathematician who usually expresses himself in another language. (By the way, my translator, Moses Klein, should be discharged of any responsibility for this introduction, whose style will hurt the stomach of any genuine English speaker: I wrote it myself; yes, I did!)

As you can see from the story that has been summarized above, when I undertook to write this book I was fully confident in an academic tradition of tolerance, which considered as suitable for scientific exchange a reasonable variety of languages. But then, when the work was completed, all the international editors that I approached quite suddenly made the decision to publish exclusively in the language of the nation that ranks first in terms of economic and military power, and of the sexual vigor of its leader (but that shows an obvious backwardness when justice, politics, social welfare, and medical care are concerned). As for the French scientific publishers, they were all in a state of advanced coma at the moment. These international publishers sent me kind letters expressing their regrets, and explaining their decision by the harshness of economic constraints.

I find this reason very weak. And speaking of economy—of energy if not of money—I cannot help thinking that it would have been more efficient for the publisher of the present edition to have accepted the original version that was offered to him fifteen years ago.

Do not misunderstand me: Being myself a publisher, I understand that a responsible businessman, whose job is to sell books, is bound to sell as many copies as he can, and also that since he is risking his money, if not his life, in the venture, he has an absolute moral freedom (his physical freedom is not even in question!) to decide what to publish or not. What I mean is this: No matter how compelling for a publisher the economic facts are, they are surface reasons, of a local character; they are conditioned by deep and global factors of an ideological nature.

Do you realize that a book of this kind is printed in no more than a thousand copies and that it will be considered a success if half of them are sold during the first year? And that the high price you pay when you buy it is mainly for commercial expenses (think of a scientific bookseller who keeps thousands of books in his catalogue, and sells at most one or two copies of each every year)? In such a situation, the economic factors would weigh nothing against the linguistic ones should the editors and the publishers value the idea of keeping alive a certain number of scientific languages. The majority of them do not, and the same is true of a number of English-speaking mathematicians. On the contrary, they are explicitly willing to kill every language except one (I mean that they want English to be the only one in use for the things that really matter, other languages being relegated to folkloric practice), believing, like Marxists, in a progressive orientation of history, where mankind is walking triumphantly toward the paradise of uniformity.

In a word, the economic factor is recalled only when there are deeper reasons not to do the thing! Indeed, papers in mathematics are written in TeX more than in English; this word processor is universally praised for the fantastic possibilities it gives to combine hieroglyphic symbols with total freedom. This is a device of questionable utility, since symbolic script obscures rational arguments more often than it illuminates them, and also because, before compilation, TEX is composed linearly; nowadays, every mathematician can read TEX as text, so that beautifully disposed formulas are in fact a useless complication! Nevertheless, much energy has been spent for the elaboration of TEX, since formula drawing was considered to conform to the correct ideology. A part of this energy could have been spent to develop the interface between several scientific languages. The circumstances are very favorable, since never in the past have the study of languages and the tools attached to this study been so elaborate as they are now.

Even the most mercantile societies maintain individuals, such as priests or mathematicians, who are engaged in unproductive activities. These people influence the orientation of their society more than they are submitted to it, so that even if they make money, and big money (in America, the first category often does better than the second), their behavior is not doomed to the hard laws of business. Moreover, the economic aspect will soon disappear from the publication of textbooks in mathematics: In the near future it will be enough for the author (who is not expecting material profit from his work, since mathematicians belong to the fortunate category of people who are paid irrespective of the quantity of work they produce) to leave the text somewhere on the web. But it is likely that the current ideology will survive, and it is not clear that the new situation will profit languages other than English.

It is sad for the author of these lines to see the community of mathematicians transform itself into an extremely conservative and right-thinking society, rejecting those of its members who do not submit themselves to the norm. Worse than anything, this norm includes only a conventional amount of originality: Eccentric behavior is tolerated, and even expected, provided that it is expressed in English!

I suppose that a high percentage of American mathematicians, and of those of other nationalities who are fully Americanized in the exercise of their mathematical activity, will think that it is wise to spend some money for the comfort of a couple of bears and a few bison in the maintenance of a wildlife sanctuary, and possibly also for the preservation of the culture of the human aborigines who live within the borders of the sanctuary. They will also favor affirmative action discriminating in favor of sexual or racial features that have no relevance to the ability to do mathematics or to express mathematical ideas. Nevertheless, they seldom promote the idea of preserving, if not developing, scientific expression in a minority language.

The paradox is that the choice of language *has* effects, both on the elaboration (I do not discuss here this aspect of the thing; that would need a full book!) and on the expression of mathematics. It is not true that pure mathematical meaning is not affected by its linguistic expression (if it were so, why not in French?), because such a thing as "pure mathematical meaning" does not exist. Any paper contains a number of things for which it would be difficult to argue that they pertain to this pure mathematical meaning, whatever it may be. For instance, it is often written in English, demonstrating the domination of the nation that ranks first in terms of... (see above).

This linguistic imperialism causes irreversible damage to the diversity of languages, which is a share of the patrimony of humanity, and it denies the universal character of mathematics. The only material output of mathematics is in the form of papers, so that mathematicians produce nothing but literature. Who will dare to say that language does not affect literature? The author of a mathematical textbook is inhabited by a craving to capture the attention of his reader that is as strong as for the author of successful detective stories. It is, after all, a fragile and rare ability, and one should leave to him the choice of the language where he can do his best.

I know from personal experience that you have to fight for this natural freedom; that to give a lecture in a language different from the local dialect is accepted everywhere except in one country, where an attempt to do so transforms the audience into a hysterical mob; that you are submitted to blackmail by organizers of conferences, who tie their invitation to the unacceptable condition that you will speak English, or worse, withdraw it in panic when they realize that you will not do so!

You still find some enlightened journals that accept your papers. But as a whole, they are not treated as well as the ones written in English: They are quoted only if it is absolutely unavoidable, and even in this case they are often misquoted. A typical example is a paper quoting Theorem 12.17 of this book, and which does nothing but prove Theorem 12.18 by the same method!

On the whole, I have the feeling that this book has not been fairly treated by the community of logicians, who claim to prize highly the value of loyalty. It would have been asking too much from this community to salute the courageous enterprise of its author and publisher, but I expected at least that it would be often cited as a textbook, that it would be allowed to follow a normal course in the life of a book. I know that this book has been used by many people. It was indeed a commercial success, since I have sold no fewer than 620 copies in 14 years without spending a great amount of energy for advertising! When it is quoted, it is generally because you find in it a technical result that is found nowhere else, but it is seldom recommended as a textbook: The authors prefer to recommend as general textbooks for basic model theory totally obsolete books where in fact the fundamental results that they need are not found, for the only reason that these books do not hurt their feelings of linguistic correctness!

If you find a taste of bitterness to the conclusion of this preface, instead of the honey that you expected, it is because the young man that many years ago undertook the formidable task of providing students an introduction to the new methods in the theory of models had aspired to write a classic, a book that would take the place of the obsolete textbooks that were mentioned above. He did his best to achieve this aim and was convinced that he had succeeded. He was expecting the assistance of an open-minded community, tolerant of cultural diversity.

This young man, I am no more.

Roche Cocu
February 2000

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Introduction

Je te donne ces vers, afin que si mon nom
Aborde heureusement aux époques lointaines
Et fait rêver un soir les cervelles humaines
Vaisseau favorisé par un grand aquilon,
Ta mémoire ...

— C.B.

Each of the luxurious works that come to Europe from across the ocean is dressed in a book jacket on the back of which you can see the portrait of a smiling, well-dressed young man—the author—endowed with prestigious university degrees; we find a note extolling the merits and the universal import of the work, which is recommended for the whole world to buy, especially those interested in the book's subject.

I need to convince you, and even myself, of the usefulness of my work. First, let me specify its intended audience: *It is addressed to mathematicians.* I wrote it thinking of those of my colleagues who may one day have felt a trifle curious about mathematical logic, whether in a benevolent or a condescending way, and who have not been able to find in the existing, commercially available literature enough motivation to penetrate deeply into this field of study. I wanted them to recognize their favorite activity, proving theorems, from the opening pages of this volume; I needed to avoid obfuscatory openings, onslaughts of definitions whose motivations are only apparent much later, successions of lemmas that lack substantial mathematical content and that baffle the reader new to the subject matter, as plentiful as the more compact theorems. These all profoundly irritate the reader as soon as he realizes that he has been cracking his skull making sense of mere banalities. I tried to introduce definitions only as fast as the results demonstrating their purpose, in order to arrive as quickly as possible at the points where the most demanding reader will see the quality of the theorem, for example, Theorem 1.14, proved in a few pages, stating that two countable ∞-equivalent structures are isomorphic. Finally, I wanted to illustrate the methods of model theory as much as possible by examples involving the elaborate mathematical ideas (algebraically closed fields, etc.) that make up the everyday universe of today's mathematician, and that for her are more relevant than the trivial examples too often found in introductory logic texts. ("Trivial" here should be understood in the mathematical sense, which happens to coincide with the etymological sense.)

There is another sort of mathematician of whom I thought as I was drafting this text: beginning mathematicians, students at our universities, with a future before them but who have not yet been able to establish for themselves a niche on the discipline's stage. They may find certain hills a little steep to climb. What will reassure them that the difficulty they feel in reading a mathematical text, always full of gaps, in a field foreign to them, does not come from any intellectual shortcomings, is this: that established mathematicians have the same difficulties; that the stone-cold demeanor that they possess whenever attending conferences in their subject matter, where they do not understand the first word but show all the outward signs of complete comprehension, comes not from superior mental agility, but rather from their experience in attending colloquia and seminars.

Moreover, it is appropriate to address this work to students, for it deals with a course that was actually taught in part, on several occasions, at the beginning graduate level. That said, I do not believe that this course can be beneficial to a novice with absolutely no grounding in mathematical culture, unless he or she is pathologically gifted: The level assumed of the reader is at least that of a bachelor's or master's in mathematics. Indeed, the reader will need a certain familiarity with the conventions of mathematics, with the methods of proof commonly used in contemporary mathematics, which are used in this course as they are everywhere else, and with the examples used; it is equally important to have a clear enough sense of what a compact space is.

In this regard, I picture myself in one of my classes, my nose to the blackboard, declaring, "Let K be an algebraically closed field ..."; I turned to face the room: From the looks on my students' faces, I could see that the light emitted by this example had been blinding, as they had not understood what it was about. If a novice reader is lost by examples intended to guide her, she may skip them, or else consult the nearest algebra or topology manual, or get help from a more advanced person. However, if she has a sufficient level of familiarity with mathematical culture, no knowledge of specific subjects is needed to embark upon this course, which is primarily an introductory text, for beginning students of mathematical logic.

Finally, I considered a third category of readers, mathematicians specializing in logic. First of all, they may be teaching their specialty, and may see fit to recommend my book to their students, either as a guide if they are teaching a class on topics covered here, or as a reference if they are teaching more advanced material. But also—and here, I must admit, I am being more presumptuous—I believe that my book will be able to impart a new understanding even to professional logicians. Indeed, the second part of the book, consisting of Chapters 11–20, deals with what is often called stability theory, which specialists on the issue consider the heart of model theory, and that research area most likely to blossom outside the narrow field of mathematical logic, but which is a body of knowledge whose main points are not ten years old,[1] which has spread only in a limited fashion, and (as many believe, for various reasons) with great difficulty; it lacks an accessible text allowing a profound introduction, and it is this gap that the publication of this book aims to fill.

Moreover, these professional logicians are equally welcome to pore over the first chapters, for I have deliberately chosen an unusual approach to the foundations of logic: The idea that I take as primitive is that of the back-and-forth construction in the style of Fraïssé, rather than that of satisfaction of a formula. I have done this in order not to repeat yet again the first pages of previous textbooks (and often, alas, not just the first pages; I would readily upbraid my precursors, in general, for their lack of brevity in introducing their subject), and also because I think that this approach is much more satisfactory for a mathematician. (I will elaborate on this point later.) My logician reader will therefore learn nothing new from Chapters 1–10, which constitute what could be called "classical" model theory, but unless he has already been converted to Fraïssé's religion, he will certainly be surprised to see this model theory presented from a particular perspective, with different priorities; he will profit from comparing and evaluating the alternative approaches to his favorite discipline. This is certainly useful if he needs to teach the subject, and it will give him the chance to reflect on the nature of objects of study familiar to him, which will perhaps take a slightly different form under a new light.

Furthermore, it would be difficult to begin reading this work from Chapter 11, since I wanted to organize the book according to a coherent didactic plan which unfolds from start to finish following a few controlling ideas. (I leave to the reader alone the task of judging whether or not I have succeeded.) To this end, I noted that in the fashionable works described at the beginning of this introduction it is considered good taste to offer, between the preface and the table of contents, a graph, nonplanar if possible,

[1](Translator's note) Poizat wrote this in 1985.

indicating the order in which the various headings can or should be read; I refrained from drawing such a graph, which here would have been annoyingly linear, because the chapters of this book are meant to be read *in the natural order of their succession.* I may add that this is not a very unusual property for a book.

I now come to the contents of this work; as you have already discerned, it deals with an introduction to a subject. What is the subject? You will remark that the title is "A Course in Model Theory" and not "A Course in Mathematical Logic"; nevertheless, in my mind, these first chapters can be read by anyone wanting an introduction to any area of mathematical logic. This is because model theory has a position in logic somewhat like arithmetic for the rest of mathematics: Number theory is a prestigious and very specialized branch of contemporary mathematics; however, there could be no mathematics without numbers, and a mathematician who knew how to count to a hundred, and even beyond, without error, would never be considered a specialist in this theory. We can likewise say that basic model theory (here, Chapters 1–10) is to logic today what simple arithmetic is to mathematics; beyond that (Chapters 11–20), model theory becomes a specialty, like number theory. Model theory is thus the least "logical" (because least ghettoized) branch of mathematical logic, yet it is the branch whose foundations no logician can ignore; however, it does not follow that all logicians are equally familiar with the beginning notions of the first ten chapters of this book, or that they treat these notions as equally fundamental. We can be more or less liberal about which notions to call "fundamental": Some would include the density theorem for prime numbers as elementary arithmetic, while others would limit it to some simple properties of the gcd and lcm.

So far, I have often referred to models, and you may be wondering what they are. In applied mathematics, a model is an abstract representation of a material reality. In a sense, this is practically the opposite of how logicians use the word "model": For them, models are concrete (as much as any mathematical object can be) objects that illustrate or interpret certain abstract ideas. For example, you certainly know how to write down the axioms expressing that a binary operation defines a group. Logicians call a set of axioms a *theory*: here, the theory of groups. A *model* of a theory is a structure satisfying all the axioms of the theory. A model of the theory of groups (abstract representation) is simply a group (concrete object).

To write axioms, we use *formulas*, which are sequences of symbols, subject to certain rules of formation; everything relevant to the manipulation of formulas, the logician calls *syntax*. The notion of model comes from another domain, as it presupposes that we can interpret, in some definite way, the symbols appearing in the formulas (for example, we interpret a binary function symbol by some precisely specified group operation). It presupposes that we have given a *meaning* to the formulas, which translate as true or false statements: The logician calls this notion *semantics*.

Model theory involves very little syntax; the gist of its study is at the level of semantics. It is thus contrasted with theoretical computer science, of which an essential component is the algorithmic study of languages, which is principally syntactic. We see that it is natural to approach logic with model theory, as it is common practice in mathematics to give priority to substantial contents over formal developments. However, this is a rather new viewpoint; for our predecessors, the object of logic was to study not what was satisfied by a given structure, but more often what was *true* (the "model" of reference being the whole universe of our ideas, inaccessible though it may be), as well as the manner in which we can prove it true. They therefore needed to insist on the formalization of the language, giving the leading part to the implication symbol, which plays a central role in deductive systems; the axiomatizations, the formal rules of proof, the "effective" nature of certain processes, were all at the center of their preoccupations.

We realize that attacking head-on the problem of truth in mathematics presents some difficulty, and that it would be more valuable, at the start, to keep our ambitions more modest, and to begin by trying to classify and compare structures with the common property of being models of the same theory T: This is model theory. It has been observed that this study yields some interesting and entirely nontrivial results. Now, we hardly ever see logic anywhere else: As in all branches of mathematics, the initial motivation has faded from view, and a new area of knowledge developed in an autonomous fashion. We might add that in model theory we impose restrictions on the language (or rather on its semantics: It must be finitary and first-order); this is an indication of the autonomy of this discipline, since we have no *logical* reason to prefer this language to any other, and it would be fruitless to seek to justify this by an appeal to a more or less natural intuition. If we do it this way, it is ultimately for technical reasons, because the language considered is the only one, or almost the only one, allowing a presentable model theory. I said " or almost" because several years ago there were some who believed that everything provable in "ordinary" model theory had already been discovered, and that to get new theorems we needed to complicate the rules of the game by calling on different sorts of infinitary languages, bizarre quantifiers, etc. This is a direction against which I have always eagerly fought, to the fullest extent of my meager resources, because I believe that the most ordinary model theory has proved to be a source of profound results whose scope is so astoundingly general that more remains to be discovered, and that it is useless to complicate the subject artificially.

Whatever may be the current state of model theory, its past leaves unfortunately many residues in the opening pages of traditional textbooks, which profess to introduce this model theory by considerations that have nothing to do with the daily practice of model theorists: vague developments, fuzzy definitions, inadequate proofs, appeals to a supposedly natural intuition, reeking of the stale smell of metaphysics, that science for which the mathematician feels the most instinctive horror! So if I am writing for mathematicians, I must take a different path, in order to convince them that logic is mathematics rather than metaphysics; this is why in the first chapter I show how to compare two structures by means of local isomorphisms, in the manner of Roland Fraïssé; here, nothing alarming; everything happens in a mode of thought familiar to them. Even if we hide the title and the preface of this work from them, they will never doubt that I seek to lure them into the dirty chasm of an accursed science; their suspicions will be aroused in the second and third chapters, where I show that this back-and-forth construction is linked to certain statements in an appropriate language, but it will be too late to retreat, as they will have perceived that these linguistic considerations are based on important mathematical content. They will note with indulgence the haste and the shortcomings of the exposition, where I sometimes require them to fill in the gaps, where we do not go into the finest details with complete rigor; here, in a word, the language, the formulas, the statements, the syntax, are not considered as the principal object of study, but rather as an illustration, a sort of dressing for local isomorphisms. This is a way of making it convenient for us, for it spares us those lengthy developments of minutiae bordering on pedantry, without mathematical substance, that crowd the first pages of the textbooks that make syntax the keystone of their edifice.

The logician may be surprised at the author's foregone conclusion: Why so much digression about "intuition" from the outset? The first answer is that it is not such a strange thing for a contemporary mathematician: We are used to seeing rather overloaded definitions (just look at what a point or a curve is in an algebraic geometry textbook), which give us the pleasure of rediscovering, after some reflection, the formal translation of an "intuitive" (or rather informal) notion. Second, the author has not proven to be an intransigent Fraïsséan, as he is not loath to speak of formulas; we will return to a more traditional exposition in the second chapter. Finally, the very existence of more classical texts, as much as he finds fault with them, gives him the freedom to show a little originality (which is also necessary if he wants to interest professional logicians), since the reader who is resolutely allergic to local isomorphisms has the opportunity to get an introduction to logic elsewhere, before beginning this course.

In Chapter 4 you will learn that certain spaces are compact, and what the consequences of this compactness are. In Chapter 5, when the author brings together the results learned up to that point, he will convince you, if necessary, that back-and-forth constructions are not a bizarre and artificial method of introducing logic; on the contrary, they form a tool that model theorists use daily.

These five very brief chapters form a mini-treatise at the core of this course, containing the gist of the concepts comprising model theory. The rest will just be more or less subtle developments.

To illustrate your newly acquired knowledge, and to give some flesh to the skeleton that you will have been shown, Chapter 6 applies the classification principles of model theory to some examples from algebra. These are not always the most spectacular or most famous examples; some are included only to provide a source of examples and counterexamples for subsequent chapters.

Logic, which has been banished from the early chapters, will turn up in force in Chapter 7, which is devoted to arithmetic. (By that I mean the theory of the structure composed of the integers equipped with the operations of sum and product.) That chapter is the only one in the book to give information about the validity of proofs and of reasoning in mathematics; we might think it a little out of place in a work of pure model theory: If it is included, it is because the ideas introduced there are too classical, are fundamental to too many branches of logic, to be ignored completely. Moreover, we shall see that it is possible to approach them in a wholesome way (i.e., in a model-theoretic manner): Arithmetic is given here not as a list of axioms, but rather as the study of the structure formed by actual integers.

Logic is manifest in that chapter primarily in the definition of proof, obtained by arithmetizing Henkin's method, which has already been introduced in the chapter devoted to compactness. In opposition to a widespread but unfortunate practice, the author refuses to consider it informally, without reference to arithmetic coding, as with the notions of computability, decidability, computable axiomatization of theories, etc. He rebels just as much against the improper and artificial blending of computability and model theory. The crowning jewel of the chapter is clearly Gödel's famous theorem, of which everybody has heard chitchat, but few can state precisely: We will see what role coding has in it.

As far as the foundations of mathematics are concerned, what is most notably missing is the theory of sets. This may surprise some readers: In the first lines of the book, we speak of sets, of relations, and even of ordinals, as freely and as imprecisely as in a work of ordinary mathematics (meaning "not logic"). Perhaps it is the elucidation of these subjects that awaits a course in logic. One might think that it would be more satisfying to the spirit, more rigorous, to fix in advance the rules of the game, the outline of mathematical activity, and, as the custom of the past half-century or more has been to translate all mathematics into set-theoretic terms, we should begin every treatise with an exposition of set theory.

A little reflection will show how misleading this view is, and how fallacious the rigor that we would like to see. How do we ground a science before having practiced it? How do we justify a body of knowledge before having explored it? And yet nobody knows just what sets really are, least of all the logicians who are most inclined to make use of them. In most cases this attitude leads to developing an axiom system, to giving a list of properties on account of which we pretend that there is a general consensus among mathematicians. We can see that this gives the significance of dogma to something highly relative, and privileges the coding (the artifice) over the concept (the profound idea). The behavior of logicians is in general the opposite: We observe ordinary mathematics, such as algebra or analysis, and then wonder what set-theoretic framework (axiomatic or otherwise) could incorporate it.

Accordingly, I was not afraid to invoke set-theoretic ideas (essentially transfinite induction) in the opening chapters, knowing that they would be ignored by a certain number of readers; I figured that these readers would still be able to understand the gist of the arguments where these ideas are used, and would even be motivated to learn more about them. When we start using infinite enumerations and the arithmetic of ordinals very deeply, I introduce Chapter 8, which deals, in as informal a framework as possible, with a theory, or rather a "practice," of sets. I believe that this is the proper place for such a chapter, rather than at the introduction to the book. I know from experience that we can scare students away by starting a course with an unmotivated exposition of "set theory."

Let us add that set theory, properly speaking, is a highly specialized branch of logic, which is not approached in this course. In set theory we handle axiom systems and models with great virtuosity, and it has little connection to the exercises of the same name taught in elementary school, which are associated more with a technical than a theoretical knowledge. Much like little children, we are interested in sets in this book not to study them (or to ask whether they exist) but to handle them, as we live amidst them.

After this Chapter 8, which has no model theory, the reader learns in Chapter 9 (devoted to saturated models) how to realize types, and in Chapter 10 (devoted to prime models) how to omit them.

As I have already said, these first ten chapters form an exposition of "classical" or "elementary" model theory, i.e., without introducing the notion of stability. After having swallowed this first half, the reader will be able to approach any area of mathematical logic profitably. It will be clear that as the book progresses I take increasing liberty with the strictest formalism, at the expense of sacrosanct rigor. That is part of a deliberate plan, insofar as mathematics is more the science of abuse of language than of rigor, more of ellipsis than of clarity, and that the mind takes shape only through being bent out of shape. The goal of this course will be achieved when the reader is as comfortable with the linguistic twitches, subtexts, and conventions of model theorists as in his or her own specialty.

So, if you have decided to specialize in model theory, you will embark upon Chapter 11, which presents what we call the "Parisian approach to stability," using heirs. Pay careful attention to Chapter 12, which treats indiscernible sequences and everything revolving around them; you will find Chapter 13, devoted to the fundamental order, more relaxing.

As for Chapter 14, it is placed a little prematurely, to help you feel the importance of these hypotheses of stability, by showing you that they occur in an essential way in the construction of saturated models. That leads to a few of the theorems needing to be restated a little later in their definitive form, but will motivate you to swallow three more chapters of abstract developments: the first devoted to forking over an arbitrary set of parameters, the second to strong types, and the third to the various types of ranks that appear in these contexts.

This whole body of knowledge is applied in Chapter 18 to the construction of prime models, and in Chapters 19 and 20 to "dimension theory," that is to say, the classification of all the models of a totally transcendental dimensional theory. In these last two chapters, the reader will appreciate the power and the relevance of the theory developed up to that point. To be sure, much more could be said about these last two chapters; if I have not done so, it is because I wished to keep the (relatively) elementary character of this course, because I had to stop somewhere, and also because I had nothing original to say on this topic, and the reader who has followed me up to the end must be entirely capable of diving into the literature on this subject.

I did not systematically accompany each chapter with a series of exercises on applications; if a few are to be found at the beginning of the course, it is in order to leave some easy lemmas to the reader, who should have the slight challenge of proving them. The bibliographic references and historic notes are held over to a special section at the end of each chapter; that is due to the desire to provide the reader with a text with no discontinuity, but also to the conditions under which this course has been drafted, far removed from all documentation.

Indeed, I wrote the outline of this book while wandering across India, so that, in my mind, Henkin's method is inextricably linked to the droves of wild elephants that I met while crawling among the swamp plants of the preserves of Kerala; the elimination of imaginaries, to the gliding of the vultures above the high Himalayan peaks; and the theorem of the bound, to the naked bodies of the Mauryan women that the traveler saw on the bends of a jungle trail, before they had time to cover themselves. I dare hope only that this book will evoke similarly pleasant images in my reader; I wish only that it will be a pleasant companion for you, as it was for me.

1
Elementary Classes of Relations

– Eh bien, mon prince, Gênes et Lucques ne sont plus que des apanages, des поместья, de la famille Buonaparte. Non, je vous préviens, que si vous ne me dites pas que nous avons la guerre, si vous vous permettez encore de pallier toutes les infamies, toutes les atrocités de cet Antichrist (ma parole, j'y crois), je ne vous connais plus, vous n'êtes plus мой верный раб, *comme vous dites. ...*

Л. Н. Т.

1.1 Local Isomorphisms Between Relations

If E is a set and m a positive integer, we call a subset R of E^m an *m-ary relation* with *universe* E. If the m-tuple $\vec{a} = (a_1, \dots, a_m)$ in E belongs to R, we say that it *satisfies* the relation R; otherwise, $\vec{a}$ *does not satisfy* R. The integer m is called the *arity* of the relation.

Let R and R' be two m-ary relations (for the same m), with universes E and E' respectively. By an *isomorphism* from R to R' we mean a bijection s between E and E' such that for any m-tuple $\vec{a} = (a_1, \dots, a_m)$ in E, the tuple $s\vec{a} = (sa_1, \dots, sa_m)$ satisfies R' if and only if $\vec{a}$ satisfies R. If there is an isomorphism between R and R', we say that these relations are *isomorphic*. It is clear that the the inverse of an isomorphism and the composition of two isomorphisms are isomorphisms.

Let R be an m-ary relation with universe E, and E' a subset of E. The *restriction* of R to E' is defined as the m-ary relation R' with universe

E' consisting of the tuples taken from E' that satisfy R. To say that R' is a *restriction* of R is synonymous with saying that R is an *extension* of R'. By an *embedding* of R' into R we mean an isomorphism from R' to a restriction of R.

The *cardinality* of a relation is defined as the number of elements in its universe (not, as one might suppose, the number of tuples satisfying it). We will therefore say that a relation is *finite* if its universe is finite. The reader lacking knowledge about infinite cardinals and ordinals will pick it up at the beginning of Chapter 8; these subjects will not be broached until we need to do some fairly precise calculations on infinite numbers.

A *local isomorphism* from R to R' is defined as an isomorphism between a finite restriction of R and a finite restriction of R'. If s is a local isomorphism, we write $\mathrm{dom}(s)$ for its domain and $\mathrm{im}(s)$ for its image. If R and R' have the same arity, the set $S_0(R, R')$ of local isomorphisms from R to R' always contains the local isomorphism $\emptyset$, since for every $m > 0$ there is only one m-ary relation on an empty universe, namely the empty set. It is possible that this is the only member of $S_0(R, R')$, for example, if R is a reflexive binary (i.e., of arity 2) relation and R' is an antireflexive binary relation.

We now define, by induction on the integer p, the set $S_p(R, R')$ of p-isomorphisms (by implication, local) from R to R': The family $S_0(R, R')$ being given at the start, it is enough to explain how we construct $S_{p+1}(R, R')$ supposing we have already constructed $S_0(R, R'), \ldots, S_p(R, R')$. A local isomorphism s is in $S_{p+1}(R, R')$ if and only if it satisfies the following two conditions:

- *(forth condition)* For every a in the universe E of R, there is an extension t of s (i.e., $\mathrm{dom}(s) \subseteq \mathrm{dom}(t)$ and s is the restriction of t to $\mathrm{dom}(s)$), defined at a, that is in $S_p(R, R')$.

- *(back condition)* For every b in the universe E' of R', there is an extension t of s, whose image contains b, that is in $S_p(R, R')$.

As the sets S_p are defined by induction on p, their properties can be studied only by induction on this integer. We are going to show a few such properties.

First let us show that if $p < q$, then *every q-isomorphism is a p-isomorphism* (i.e., $S_p(R, R') \supseteq S_q(R, R')$).

Proof. It is enough to show that for every p, $S_p(R, R') \supseteq S_{p+1}(R, R')$; this is true for $p = 0$, as every 1-isomorphism is by definition a local isomorphism. So let $p = q+1$, and assume that the inductive hypothesis is true for q. If s is a $(p+1)$-isomorphism, then for every a in E there is an extension defined at a that is a p-isomorphism, and hence also a $(p-1)$-isomorphism by the inductive hypothesis. Similarly, the back condition is satisfied for every b in E'. Therefore, s is a p-isomorphism. □

Now let us check that *every restriction of a p-isomorphism is a p-isomorphism.*

Proof. Let s be a p-isomorphism, and s' a restriction of s. If $p = 0$, then s' is a local isomorphism, and therefore a 0-isomorphism. If $p = q+1$, then for every a in E there is a q-isomorphism t defined at a and extending s; this t also extends s'. The proof of the forth condition is similar, so s is a p-isomorphism. □

I leave it to the reader to check that the inverse of a p-isomorphism from R to R' is a p-isomorphism from R' to R, and that if s is a p-isomorphism from R to R' and t is a p-isomorphism from R' to R'', with $\mathrm{dom}(t) = \mathrm{im}(s)$, then the composition of s and t is a p-isomorphism from R to R''.

Note also that *if s is an isomorphism from R to R', then every finite restriction of s is a p-isomorphism from R to R' for every p.*

Proof. Let t be a finite restriction of s; for $p = 0$ it is clearly a local isomorphism, i.e., a 0-isomorphism. For $p = q+1$, for every a in E the restriction of s to $\mathrm{dom}(t) \cup \{a\}$ is a q-isomorphism by the inductive hypothesis. The same argument shows the back condition, so t is a p-isomorphism. □

Local isomorphisms that can be extended to isomorphisms are in a sense trivial examples of p-isomorphisms; we shall certainly see other examples, which allow us to compare nonisomorphic relations!

The sets $S_p(R, R')$ form a decreasing sequence:

$$S_0(R,R') \supseteq S_1(R,R') \supseteq \cdots \supseteq S_p(R,R') \supseteq S_{p+1}(R,R') \supseteq \cdots.$$

I leave to the reader the task of showing that if for some p, $S_p(R,R') = S_{p+1}(R,R')$, then for all $q > p$, $S_p(R,R') = S_q(R,R')$; in particular, this happens if $S_p(R,R')$ is empty!

By an *ω-isomorphism*, or *elementary local isomorphism*, from R to R' we mean a local isomorphism s between these two relations that is a p-isomorphism for every nonnegative integer p. We write $S_\omega(R,R')$ for the set of all ω-isomorphisms, i.e., the intersection of all $S_p(R,R')$. Here ω denotes the smallest denumerable ordinal, which comes immediately after all the integers; the reader who has not yet been enlightened about ordinals may consider this a mere convention of notation.

If $S_\omega(R,R')$ is empty, then for some p, $S_p(R,R')$ is empty; in fact, if there is a p-isomorphism for every p, then $S_\omega(R,R')$ contains at least the empty function.

There are two cases for a local isomorphism s. If for some integer p, $s \in S_p(R,R')$, $s \notin S_{p+1}(R,R')$, we say that its *Fraïssé rank* is p; otherwise, s is in every $S_p(R,R')$, and we say that its *Fraïssé rank* is greater than or equal to ω.

So far, we have introduced three synonymous expressions: "$s \in S_p(R,R')$," "s is a p-isomorphism," and "the Fraïssé rank of s is greater than or equal to p." We shall enrich our vocabulary even further: We say that the k-tuple $\vec{a} = (a_1, \ldots, a_k)$ in the universe of R, and the k-tuple $\vec{b} = (b_1, \ldots, b_k)$ taken from the universe of R', are *p-equivalent* if they correspond via a p-isomorphism from R to R'. In other words, they satisfy the same equations ($a_i = a_j$ iff $b_i = b_j$), and the function s defined by $sa_1 = b_1, \ldots, sa_k = b_k$ is a p-isomorphism from R to R'. This is an equivalence relation: reflexive, symmetric, and transitive. We write it $(\vec{a}, R) \sim_p (\vec{b}, R')$, the indication of the relation R being especially necessary if, for example, R' is an extension of R, so that $\vec{a}$ can also be considered to be from the universe of R'; when the relation is clear from the context, we can merely write $\vec{a} \sim_p \vec{b}$. If $\vec{a}$ and $\vec{b}$ are p-equivalent for every p, we say they are *ω-equivalent*, or that they *have the same type*.

If the empty function is a p-isomorphism from R to R' for every p (i.e., $S_\omega(R, R')$ in nonempty), we say that R and R' are *elementarily equivalent*; in symbols, $R \sim_\omega R'$. More generally, we can say that R and R' are *p-equivalent*, $R \sim_p R'$, if $\emptyset$ is a p-isomorphism from R to R'.

Now suppose that R' is an extension of R, and that the universe E of R is contained in the universe E' of R'; we call this extension *elementary* if for every $\vec{a}$ taken from E, $(\vec{a}, R)$ and $(\vec{a}, R')$ have the same type; in other words, the identity function restricted to a finite subset of E is a p-isomorphism from R to R' for every p. The assertion that R' is an elementary extension of R is written $R \prec R'$; in this case we also say that R is an *elementary restriction* of R'. An isomorphism between R and an elementary restriction of R' is called an *elementary embedding* of R into R'. Naturally, if R is elementarily embedded into R', then R and R' are elementarily equivalent.

Having reached this point, we can pause, as we have defined, in a few pages, the two notions at the heart of the theory of models, a theory whose principal activity is the study of the class of relations elementarily equivalent to a given relation R, and the elementary embeddings between the relations in this class.

Exercise 1.1. Show that if R and R' are elementarily equivalent, then for every k-tuple $\vec{a}$ in the universe of R and every nonnegative integer p, there is a k-tuple $\vec{b}$ in the universe of R' such that $(\vec{a}, R) \sim_p (\vec{b}, R')$.

Exercise 1.2. If $\vec{a}$ is a k-tuple and $\vec{b}$ is an h-tuple, we denote by $\vec{a}\frown\vec{b}$ the $(k+h)$-tuple "concatenated" from $\vec{a}$ and $\vec{b}$: If $\vec{a} = (a_1, \ldots, a_k), \vec{b} = (b_1, \ldots, b_h)$, then $\vec{a}\frown\vec{b} = (a_1, \ldots, a_k, b_1, \ldots, b_h)$. Show that if R' is an elementary extension of R, $\vec{a}$ is in the universe of R, and $\vec{b}$ is in the universe of R', then for every nonnegative integer p there exists $\vec{c}$ in the universe of R such that $(\vec{a}\frown\vec{c}, R) \sim_p (\vec{a}\frown\vec{b}, R')$.

1.2 Examples

The trivial example of elementary equivalence is isomorphism. This is the only possible case for finite relations, as shown by the following theorem:

Theorem 1.3. *If R is finite, defined on p elements, then every relation S that is $(p+1)$-equivalent to it is isomorphic to it.*

Proof. Let $a_1, \ldots, a_p$ be the p elements of the universe of R. As $\emptyset$ is a $(p+1)$-isomorphism between R and S, it has an extension s_1 to $\{a_1\}$ that is a p-isomorphism; this in turn has an extension s_2 to $\{a_1, a_2\}$ that is a $(p-1)$-isomorphism. Continuing in this way we get a 1-isomorphism s from R to S, defined everywhere on the universe of R. Therefore, s is an embedding of R into S, and if there is b in the universe of S outside the image of s, we cannot extend s^{-1} to a local isomorphism t defined at b, as $t(b)$ would have to be distinct from $a_1, \ldots, a_p$. Therefore, s is surjective, and hence an isomorphism between R and S. □

In particular, a finite relation has no elementary extensions other than itself.

Let us now consider an integer m and two empty m-ary relations R and R' with universes E and E', respectively. (This means that neither R nor R' is satisfied by any m-tuple in its universe.) R and R' are isomorphic if and only if E and E' have the same "number of elements." (This number can be finite or infinite; in technical language, we say that they have the same cardinality). According to Theorem 1.3, if they are elementarily equivalent, and one of them is finite, they are isomorphic; in contrast, we shall show that if both are infinite, then they are elementarily equivalent. (In this case they are not necessarily isomorphic, for example if E is denumerable, having as many elements as there are integers, and if E' has the power of the continuum, having as many elements as there are real numbers.) In fact, I claim that in this case every local isomorphism from R to R' is a p-isomorphism for every p (which implies that $\emptyset$ is an ω-isomorphism from R to R'!); for these relations, a local isomorphism s is nothing more than an injection from a finite subset of E into E'. We will show the step from p to $p+1$: if I add a to $\mathrm{dom}(s)$, then since E' is infinite, I can find b outside $\mathrm{im}(s)$, and extend s and t by letting $ta = b$. The "back" condition can be proved similarly.

The same result remains valid if we take R and R' to be the m-ary relations containing all m-tuples from their respective universes; as in the preceding case, this amounts to giving no structure to the sets E and E' other than that associated with the identity relation.

Now let us study the barely less trivial case of unary (i.e., of arity 1) relations. Associated with any such relation is the symbol (x, y), called its *character*, where x is the number of elements of E that satisfy R if that number is finite, and ∞ otherwise, and y is the number of elements of

E that do not satisfy R if that number is finite, and ∞ otherwise. For example, if ω_1 denotes the smallest uncountable cardinal, there are, up to isomorphism, three unary relations with character (∞, ∞), namely those corresponding to the partitions (ω, ω_1), (ω_1, ω), (ω_1, ω_1).

Theorem 1.4. *Two unary relations R and S are elementarily equivalent if and only if they have the same character; in this case, every local isomorphism from R to S is a p-isomorphism for every p.*

Proof. That s is a local isomorphism from R to S means that if $a \in R$, then $sa \in S$, and if $b \notin R$, then $sb \notin S$. Suppose that the character of R is (p, y) and that S is $(p+1)$-equivalent to R, and let $a_1, \ldots, a_p$ be the elements of the universe of R that satisfy R. As $\emptyset$ is a $(p+1)$-isomorphism, we can, in p stages, get a 1-isomorphism s defined on $a_1, \ldots, a_p$. Then $sa_1, \ldots, sa_p$ satisfy S, and there can be no other element b in the universe of S that satisfies S, as we would not be able to extend s^{-1} to b. The character of S is therefore of the form (p, z). In the same manner we can show that if R and S are $(q+1)$-equivalent, and the character of R is of the form (x, q), then so is that of S.

This proves that if R and S are elementarily equivalent, they have the same character. Conversely, if they have the same character, and if s is a local isomorphism from R to S, then if we add a to dom(s) which (for example) satisfies R, we always have a place to add to im(s) a b which satisfies S, from which the conclusion follows. □

Starting with arity 2, the problem of determining elementary equivalence classes of relations becomes as complex as the general problem, and here we can consider only a few simple cases of binary relations. A binary relation is called *reflexive* if every pair (a, a) from its universe satisfies it. If S is 1-equivalent to a reflexive relation R, it is also reflexive: If S does not contain the pair (b, b), then we cannot get any local isomorphism from S to R defined at b. The relation R is called *symmetric* if whenever it contains the pair (a, b), it also contains the pair (b, a); it is *antisymmetric* if it never contains both (a, b) and (b, a) when a is distinct from b. I leave it to the reader to check that a relation 2-equivalent to a symmetric (respectively antisymmetric) relation is itself symmetric (antisymmetric). Furthermore, R is called *transitive* if whenever (a, b) and (b, c) satisfy R, then (a, c) does also; a relation 3-equivalent to a transitive relation is also transitive. Finally, R is called *total* if for every a and b in its universe at least one of the two pairs (a, b) and (b, a) is in R; a relation 2-equivalent to a total relation is total.

From the tender days of childhood everyone knows that an *equivalence relation* is a reflexive, symmetric, and transitive binary relation, and that every equivalence relation with universe E is associated with a partition of the set E into pairwise disjoint classes. We have seen that a binary relation 3-equivalent to an equivalence relation is also an equivalence relation. It is easy enough to characterize equivalence relations up to elementary equivalence, thanks to a "characteristic function" associated with it; this is the object of Exercise 1.6. I am content to treat one particular case in detail:

Theorem 1.5. *If R and R' are equivalence relations with infinitely many equivalence classes, each having infinitely many elements, then they are elementarily equivalent, and every local isomorphism between R and R' is a p-isomorphism for every p; conversely, if R and R' are two elementarily equivalent equivalence relations, and if R has infinitely many classes, all infinite, then the same is true of R'.*

Proof. If s is a local isomorphism from R to R', and I add a to its domain, whether a is congruent modulo R to an element of $\mathrm{dom}(s)$ or a is in a new equivalence class, then since R' has infinitely many classes, and all of them are infinite, I can always extend s to a so as to have a local isomorphism. The conclusion follows.

Conversely, we can easily see that if R has at least p classes and if R' is p-equivalent to it, then R' also has at least p classes; and that if every class of R has at least p elements and R' is p-equivalent to it, then R' has the same property. □

Exercise 1.6. Associate with an equivalence relation R the function f_R that assigns to the natural number p the number of equivalence classes of R with p elements if this number is finite, and the symbol ∞ otherwise, and that assigns to the symbol ∞ the number of infinite classes of R if this number is finite, and ∞ otherwise.

1. Show that if $f_R(q) = p$ and if R' is $((p+1)q+1)$-equivalent to R, then $f_{R'}(q) = p$.
2. Show that if R has at least p equivalence classes with at least q elements, and if R' is pq-equivalent to R, then it has the same property.
3. Let R_p be the equivalence relation obtained by replacing every class of R with cardinality greater than p by a class with exactly p elements. Show that R and R_p are p-equivalent.
4. Show that R and R' are elementarily equivalent iff f_R and $f_{R'}$ are equal, except possibly on ∞ in the case where both take nonzero values infinitely often.
5. Let R and R' be elementarily equivalent, and let R' be an extension of R; show that this extension is elementary iff every finite class of R' with one element in the universe of R has all its elements in the universe of R and every infinite class of R' with at least one element in the domain of R has infinitely many such elements.

If the family of equivalence relations is easily disposed of, the problem of classifying *chains* (reflexive, antisymmetric, transitive, total binary relations; these are also called *total orderings* or *linear orderings*) up to elementary equivalence is much more complicated, and here we can give only some simple, albeit instructive, specific cases. A local isomorphism between chains is simply an increasing function: If $a < b$, then $s(a) < s(b)$.

A chain with at least two elements is called *dense* if between any two points there is always a third; we say that it *has no endpoints* if it has neither a greatest nor a least element.

Theorem 1.7. *A binary relation 3-equivalent to a dense chain without endpoints is also a dense chain without endpoints; conversely, any two such chains C and C' are elementarily equivalent, and every local isomorphism from C to C' is elementary.*

Proof. Let C be a dense chain without endpoints. We know that any binary relation C' 3-equivalent to C is a chain. If C' is not dense, there exist consecutive elements a and b in C'. Let s be a 1-isomorphism from C' to C defined at a and b, and let c be in C with $s(a) < c < s(b)$; we cannot extend s^{-1} to c, contradicting the hypothesis. If C' has a least element a, a 1-isomorphism s from C to C' whose image contains a cannot be extended to a point $b < s^{-1}(a)$, which is a contradiction; likewise, we see that C' cannot have a greatest element.

Conversely, let C and C' be dense chains without endpoints. A local isomorphism s from C to C' sends $a_1 < \cdots < a_k$ to $b_1 < \cdots < b_k$; if I add (for example) a to the left, I can respond by adding b to the corresponding segment on the right, which is never empty, as C' is dense and without endpoints. So every 0-isomorphism is a 1-isomorphism, which implies that every 0-isomorphism is a p-isomorphism for every p. □

For example, the chain $\mathbf{R}$ of reals is elementarily equivalent to the chain $\mathbf{Q}$ of rationals.

In opposition to dense chains we have *discrete* chains, which have no limit points: Every element, excepting the greatest if there is one, has a successor (b is called a *successor* of a, and a a *predecessor* of b, if $a < b$ and there are no elements between a and b); and every element, excepting the least if there is one, has a predecessor. We can easily see that a binary relation 3-equivalent to a discrete chain without endpoints is also a discrete chain without endpoints; we are going to show conversely that two discrete chains without endpoints (for example the chain $\mathbf{Z}$ of integers and the chain $\mathbf{Z}+\mathbf{Z}$ composed of two copies of $\mathbf{Z}$ placed end to end) are elementarily equivalent.

Let a and b be in such a chain, $a < b$, and define $d(a,b) = q$ if there are exactly q elements strictly between a and b ($d(a,b) = 0$ means that a and b are consecutive); $d(a,b) = \infty$ if there are infinitely many. The result is a consequence of the following theorem, which shows, among other things, that $\emptyset$ is a p-isomorphism for all p.

Theorem 1.8. *Let C and C' be two nonempty discrete chains without endpoints. Let $a_1 < \cdots < a_k$ be an increasing k-tuple from the former* *and $b_1 < \cdots < b_k$ an increasing k-tuple from the latter; for them to be p-equivalent it is necessary and sufficient that for every $i, 1 \leq i < k$, $d(a_i, a_{i+1})$ and $d(b_i, b_{i+1})$ and are equal or are both greater than or equal to $2^p - 1$.*

Proof. By induction on p. The case $p = 0$ is obvious, as the condition amounts to the two tuples being arranged in the same order (increasing). Let us show the step from p to $p + 1$. First, we show that if the two tuples satisfy the condition, they are $(p + 1)$-equivalent.

I add (for example) a to $\vec{a}$. Suppose first that $a < a_1$; if $d(a, a_1) = q$, I take $b < b_1$ such that $d(b, b_1) = q$; if $d(a, a_1) = \infty$, I take $b < b_1$ such that $d(b, b_1) = 2^p - 1$. We can treat the case $a > a_k$ in the same manner. Now suppose that $a_1 < a < a_{i+1}$, and distinguish two subcases:

1. $d(a_i, a_{i+1} < 2^{p+1} - 1$, in which case $d(a_i, a_{i+1}) = d(b_i, b_{i+1})$, and the response is b with $b_i < b < b_{i+1}$, $d(a_i, a) = d(b_i, b)$.

2. $d(a_i, a_{i+1}) \geq 2^{p+1} - 1 = (2^p - 1) + 1 + (2^p - 1)$. If $d(a_i, a) = q < 2^p - 1$, I respond with b such that $d(b_i, b) = q$, and since $d(b_i, b_{i+1} \geq 2^{p+1} - 1$, $d(b, b_{i+1}) \geq 2^p - 1$. Likewise, if $d(a, a_{i+1}) = q < 2^p - 1$, I respond with b such that $d(b, b_{i+1}) = q$, and if $d(a_i, a) \geq 2^p - 1$ and $d(a, a_{i+1}) \geq 2^p - 1$, I can respond with b such that $d(b_i, b) \geq 2^p - 1$ and $d(b, b_{i+1}) \geq 2^p - 1$.

Now let us show that the condition is necessary for two tuples to be $(p + 1)$-equivalent. We distinguish three cases:

1. $d(a_i, a_{i+1}) \geq 2^{p+1} - 1 = (2^p - 1) + 1 + (2^p - 1)$: Choose a such that $a_i < a < a_{i+1}, d(a_i, a) \geq 2^p - 1, d(a, a_{i+1}) \geq 2^p - 1$. There must be some b between b_i and b_{i+1} such that (a_i, a, a_{i+1}) and (b_i, b, b_{i+1}) are p-equivalent, so the induction hypothesis implies that $d(b_i, b) \geq 2^p - 1, d(b, b_{i+1})$, and $d(b_i, b_{i+1}) \geq 2^{p+1} - 1$.

2. $d(a_i, a_{i+1}) \leq 2^{p+1} - 3 = (2^p - 2) + 1 + (2^p - 2)$: If a_1 and a_{i+1} are consecutive, then b_i and b_{i+1} must likewise be; otherwise, we can find a between a_i and a_{i+1} such that $d(a_i, a)$ and $d(a, a_{i+1})$ are both strictly smaller than $2^p - 1$, and by the induction hypothesis there is b on the opposite side such that $d(a_i, a) = d(b_i, b), d(a, a_{i+1}) = d(b, b_{i+1})$, and $d(a_i, a_{i+1}) = d(b_i, b_{i+1})$.

3. $d(a_i, a_{i+1}) = 2^{p+1} - 2 = (2^p - 2) + 1 + (2^p - 1)$: Then there exists b between b_i and b_{i+1} such that $d(b_i, b) = 2^p - 2, d(b, b_{i+1}) \geq 2^p - 1$, and $d(b_i, b_{i+1}) \geq 2^{p+1} - 2$. Then we must have $d(b_i, b_{i+1}) = 2^{p+1} - 2$, by the first case going in the opposite direction (from b to a): If $d(b_i, b_{i+1}) \geq 2^{p+1} - 1$, the same is true of $d(a_i, a_{i+1})$.

□

We see, therefore, that an extension of discrete chains without endpoints is elementary if the values of d are preserved for pairs taken from the small chain. For example the chain $\mathbf{Z}$ of integers and the chain $\mathbf{Z}^*$ of nonzero integers are elementarily equivalent because they are isomorphic; however, $\mathbf{Z}^*$ is not an elementary restriction of $\mathbf{Z}$, since -1 and 1 are consecutive in $\mathbf{Z}^*$ but not in $\mathbf{Z}$.

In contrast, an extension of dense chains without endpoints, or of equivalence relations with infinitely many classes all of which are infinite, or of unary relations of the same character, is always elementary, since in these cases local isomorphisms are elementary.

Exercise 1.9. Show that two dense chains with least but not greatest element (respectively, with greatest but not least element, with least and greatest elements) are elementarily equivalent.

Exercise 1.10. Show that two discrete chains with least element (denoted by 0) but without greatest element (e.g., the chain ω of natural numbers) are elementarily equivalent. Classify all discrete chains up to elementary equivalence.

Exercise 1.11.

1. If C is a chain, C^- denotes the inverse chain: $a < b$ in the sense of C iff $b < a$ in the sense of C^-. Show that if C and C' are elementarily equivalent, so are C^- and C'^-.

2. If C and D are chains, assuming without loss of generality that their universes are disjoint (if not, we can replace C and D by isomorphic copies with disjoint universes), the chain $C+D$ is the common extension of C and D obtained by making each element of C smaller than each element of D. Show that if C and C' are elementarily equivalent, and D and D' likewise, then so are $C + D$ and $C' + D'$.

Exercise 1.12. Given two chains C and D, define the lexicographic product of C and D to be the chain defined on the Cartesian product of their universes such that $(a, b) < (c, d)$ in the sense of $C \times D$ if $b < d$ in the sense of D, or if $b = d$ and $a < c$ in the sense of C. If C and D are letters of the alphabet, this is the order in which two-letter words are arranged in an Arabic dictionary.

1. Show that the discrete chains without endpoints are those that can be written in the form $\mathbf{Z} \times C$, and that the discrete chains with least but not greatest element are those that can be written in the form $\omega + \mathbf{Z} \times C$.

2. Show that if C and C' are elementarily equivalent, and D and D' are elementarily equivalent, then so are $C \times D$ and $C' \times D'$.

1.3 Infinite Back-and-Forth

There is no reason to stop at ω: We can apply the back-and-forth condition one more time to define the notion of $(\omega+1)$-isomorphism, then that of $(\omega+2)$-isomorphism, etc., until we get to $\omega+\omega$, and still continue to define α-isomorphism inductively for all ordinals α. For now, we do not need to know very much about the ordinals, except that they are the objects whose function is to replace suspension points: They allow us to do inductive proofs and constructions that require more than finitely many steps. After all the natural numbers is the ordinal ω, then $\omega+1, \ldots, \omega+n, \ldots$; after all the $\omega+n$ comes $\omega+\omega=\omega\times 2, \ldots, \omega\times n, \ldots$, and after all the $\omega\times n$ comes $\omega\times\omega$, etc. We shall be satisfied for now to note that there are two types of ordinals: the *successors*, which are of the form $\alpha+1$, and the others, such as $0, \omega, \omega\times n, \omega\times\omega, \ldots$, which are called *limits*. By induction on α we define the family $S_\alpha(R, R')$ of α-isomorphisms from R to R' as follows:

- If α is a limit, then s is in $S_\alpha(R, R')$ iff it is in $S_\beta(R, R')$ for every ordinal β strictly less than α.
- If $\alpha=\beta+1$, then s is in $S_\alpha(R, R')$ iff when any element is added to its domain, or any element is added to its range, it can be extended to a β-isomorphism from R to R'.

We have no difficulty checking that an α-isomorphism is also a β-isomorphism for every β less than α, and therefore that there are two possible cases for local isomorphisms:

- There is some α such that $s \in S_\alpha(R, R'), s \notin S_{\alpha+1}(R, R')$, in which case we say that s has *Fraïssé rank* α.
- Otherwise, $s \in S_\alpha(R, R')$ for every ordinal α, and we say that the *Fraïssé rank* of s is undefined, or is ∞; we write $S_\infty(R, R')$ for the intersection of all the $S_\alpha(R, R')$.

The notions of α-equivalence and ∞-equivalence are defined as in the finite case.

As there are always more ordinals than subsets of $S_0(R, R')$, there must exist some α, depending on R and R', such that $S_\infty(R, R') = S_\alpha(R, R')$ (this set might well be empty), that is the first α for which $S_{\alpha+1}(R, R') = S_\alpha(R, R')$, so that an ∞-isomorphism has the following property: If any element is added to its domain, or any element added to its range, it can be extended to an ∞-isomorphism.

It is important not to confuse ∞-isomorphisms with (for example) ω-isomorphisms. If we adopt Ehrenfeucht's formulation of Fraïssé's back-and-forth method, then elementary equivalence can be characterized as follows: Consider two players, the first choosing an element in R or R' each round, the second replying with an element in the universe of the other relation. By definition, the second player wins the game in p rounds if, at the end of p choices, they have two (locally) isomorphic p-tuples; to say that two relations are elementarily equivalent is to say that for every p the second player has a strategy guaranteed to win the p-stage game, that is to say, a strategy, depending on p, that is effective, provided that he knows in advance that only p rounds will be played. On the other hand, in the case of ∞-equivalence, the second player has a uniform winning strategy, always the same, that makes him win no matter how many rounds are played.

Returning to the examples in Section 1.2, we can see that every local isomorphism between two elementarily equivalent unary relations, or between two equivalence relations each with infinitely many classes that are all infinite, or between two dense chains without endpoints, is an ∞-isomorphism. On the other hand, discrete nonempty chains without endpoints can be non-∞-equivalent: $\mathbf{Z}$ and $\mathbf{Z}+\mathbf{Z}$ are $(\omega+1)$-equivalent but not $(\omega+2)$-equivalent, since a pair (a, b) in the latter with $d(a, b) = \infty$ has no ω-equivalent counterpart in the former.

The principal object of model theory is the study of ω-isomorphisms, or elementary isomorphisms. However, it very useful to distinguish those among them that are ∞-isomorphisms, and we are going to define them from scratch without mentioning ordinals, in order to assuage the reader who is not yet familiar with them.

This hierarchy of local isomorphisms resembles that of Cantor–Bendixson, which historically motivated the introduction of the ordinals, and about which I am going to say a few words that I hope will help to clear up the situation.

Let E be a topological space, and let E_1 be the space E with its isolated points removed. The set E_1 is closed, possibly empty; we call it the *derivative* of E. Let E_2 be the derivative of E_1, etc. Let E_{n+1} be the derivative of E_n. It may happen that for some n, $E_{n+1} = E_n$, in which case the process stops, since we have found a set with no isolated points. However, if that does not happen, then we can continue by setting $E_\omega = \bigcap E_n$, then $E_{\omega+1} =$ the derivative of E_ω, etc. We thus define a decreasing sequence of subsets E_α of E by induction on the ordinal α; if α is a limit then E_α is the intersection of all the E_β for $\beta < \alpha$ (in particular, since there is no $\beta < 0$, we have $E_0 = E$); if $\alpha = \beta + 1$, then E_α is the derivative of E_β, i.e., E_β with its isolated points removed. Then E_∞ is by definition the intersection of all the E_α. The set E_∞ is actually E_α for α sufficiently large, depending on E, and it is the largest subset of E with no isolated points (naturally, E_∞ might be empty). However, there is another way to construct E_∞ that is simpler, or at least more elementary: Take the union of all subsets with no isolated points of E!

To return to our local isomorphisms, given two n-ary relations R and R', define a *Karpian family* to be a family K of local isomorphisms from R to R' with the following property: Given an element s of K, for any addition of one element to its domain, or any addition of one element to its image, s has an extension that is in K. It is clear that a union of Karpian families is a Karpian family, and that the union of all Karpian families is the set, possibly empty, of all ∞-isomorphisms from R to R'. In summary, to say that R and R' are ∞-equivalent is to say that $\emptyset$ is an ∞-isomorphism from R to R', or equivalently that there is a nonempty Karpian family of local isomorphisms from R to R'.

Exercise 1.13. Show that two equivalence relations are ∞-equivalent iff they have the same characteristic function.

Theorem 1.14. *Two ∞-equivalent denumerable relations are isomorphic.*

Proof. Let $a_1, \ldots, a_n, \ldots$ be an enumeration of the universe of R and let $b_1, \ldots, b_n, \ldots$ be an enumeration of the universe of R'. We can construct a sequence $s_0, s_1, \ldots, s_n, \ldots$ of ∞-isomorphisms from R to R', and a sequence $t_0, t_1, \ldots, t_n, \ldots$ of ∞-isomorphisms from R' to R, such that $s_0 = t_0 = \emptyset$, s_{n+1} is defined at a_{n+1} and extends t_n^{-1}, and t_{n+1} is defined at b_{n+1} and extends s_{n+1}^{-1}. Note that s_{n+1} is then defined at $a_1, \ldots, a_{n+1}$ and extends s_n. Let $s(a_n) = s_n(a_n)$; then $s(a_n) = s_m(a_n)$ for all s_m that are defined at a_n. Similarly, set $t(b_n) = t_n(b_n)$. It is clear that s and t are both isomorphisms between R and R' that are inverses of each other. $\square$

For example, two denumerable dense chains without endpoints are isomorphic. This result is no longer true for nondenumerable cardinalities: The chains $\mathbf{R}$ and $\mathbf{R} + \mathbf{Q}$ are ∞-equivalent without being isomorphic.

The philosophy of all this, is that two relations that are ∞-equivalent are much more similar than if they were merely elementarily equivalent.

1.4 Historic and Bibliographic Notes

One oratorical warning before beginning this first note: The mathematician's work is not the work of a historian of science, and these "historic and bibliographic notes" do not pretend to be complete, nor to reflect the gestation and contemporary trends in mathematical logic in any significant way. On the contrary, they are intended only to provide some references to the reader who desires to know from where all these ideas came, and to recommend a few works on those subjects proceeding beyond this course but that are barely touched upon here.

The treatment of logic by local isomorphisms, with back-and-forth conditions, is due to Fraïssé; the reference usually given is [Fra54b], as well as [Fra53], but the results are included in his thesis, written some time earlier. The most accessible Fraïsséan exposition is his textbook [Fra71], which I strongly recommend reading. What I call p-isomorphisms here are Fraïssé's (p,p)-isomorphisms; his notion of (k,p)-isomorphism is intended to capture not only the quantification rank of a formula (see Chapter 2), but also its "alternation rank."

References related to elementary equivalence and elementary extensions shall wait until the next chapter.

The examples of Section 1.2 are so rudimentary that they can be considered part of the folklore of model theory. I have refrained from any research to learn who first characterized them up to elementary equivalence. I shall indicate only that the case of unary relations is treated in [Sko19].

The presentation of Fraïssé's back-and-forth method in the form of a two-player game is due to Ehrenfeucht [Ehr61]; he did it several years after Fraïssé, and with an explicit reference to the latter's work. That fact has not prevented numerous logicians, both European and American, from crediting Ehrenfeucht alone with Fraïssé's back-and-forth method.

The notion of the derivative of a topological space is the jumping-off point for "set theory"; it was for that purpose that Cantor invented ordinals. For this, you can consult [Moo82], p.32ff.

The first systematic treatment of infinite languages, and the infinite back-and-forth techniques associated with them, is due to Carol Karp [Kar64]; thus the expression "Karpian family" coined by Fraïssé.

Theorem 1.14 is generally attributed to Dana Scott [Sco65]. Note, however, that it appeared eleven years earlier in [Fra54a]. A precursor can be found in [Can95] stating that, up to isomorphism there is only one denumerable dense chain without endpoints; that result was proved by a back-and-forth argument by Hausdorff [Hau14].

2

The Language Associated with a Relation

C'est pour qui, le bon lolo? Pour le Poupousse à sa Dadame? Oh, que c'est bon le miam-miam! L'avait faim le petit chat! Mais que tu es minime! Tu fais des ronrons et des doudouces! ...

Madame Durand, French teacher
P.C.

2.1 Formulas

As explained in the introduction, we are not introducing the language, or formulas, as primitive concepts. This is because the back-and-forth conditions introduced in Chapter 1 are enough to express the basic notions of model theory efficiently. A Fraïsséan extremist would find it pointless to speak of these formulas; at most, he would consider them a heuristic disguise for the study of local isomorphisms. We could compare him to a teacher who, in an elementary class on integral calculus, introduced Lebesgue's space L^1 as the completion of such-and-such a normed space, and remarked only at the end of the course that it happened to turn out that the elements of L^1 correspond to integrable functions. I shall not adopt such an extreme position, and am now ready to speak of formulas, which are quite handy in some cases for seeing whether relations are elementarily equivalent or not! But as I am not treating them as a principal notion, I shall allow myself to define them quickly, giving short shrift to those petty details of no mathematical content that can discourage the readers of the opening pages of a logic text.

To construct a language, we first need to specify an "alphabet," i.e., a list of symbols. A *word* is any finite sequence (which could be empty) of elements of the alphabet. In general, not all words are equally interesting, and we will distinguish some that are obtained by applying certain rules of formation.

Take a simple example: The alphabet consists of just two symbols, the open and close parentheses (and). By induction on integers n, define the sets $P_0, P_1, \dots, P_n, \dots$ of words over this alphabet as follows:

- P_0 consists of the empty word.
- P_{n+1} consists of words of the form (A) where $A \in P_n$, or of the form $(A)(B)$ where A and B are in the union of $P_0, \dots, P_n$, and at least one of them is in P_n.

The union P of all P_n is called the set of *bracketings*.

Theorem 2.1. *Every bracketing is empty, or can be written uniquely in the form (A), where A is a bracketing, or can be uniquely written in the form $(A)(B)$, where A and B are bracketings; furthermore, these three cases are mutually exclusive.*

Proof. By definition a bracketing can be written in one of these forms. We must show that the expression is unique, and that the three cases are mutually exclusive. This is clear for the empty bracketing.

Define the weight of a word to be the difference between the number of open parentheses and the number of close parentheses; we will by show, by induction on the *length* of the word (i.e., the number of symbols in the word) that a bracketing must have weight zero, and that every initial segment of a bracketing has nonnegative weight. Indeed, a bracketing A of nonzero length can be written in the form (B) or the form $(B)(C)$, B and C being bracketings of shorter length than A.

It follows that if A is a bracketing, then no nonempty initial segment of (A), except (A) itself, can be a bracketing, since such a word, because of the first open parenthesis, has strictly positive weight.

We can now prove the result. If $(A) = (B)$, it is clear that $A = B$, whether or not A is a bracketing. Suppose that $(A)(B) = (C)(D)$, where A, B, C, D are bracketings. If A has strictly shorter length than C, then the bracketing (A) is a proper initial segment of the bracketing (C), which is impossible. By symmetric reasoning, A and C must have equal length, and hence $A = C, B = D$. If $(A) = (B)(C)$, where A, B, C are all bracketings, then (B) is a proper initial segment of (A), which is impossible. $\square$

This theorem is a result on "uniqueness of readability"; it states that a bracketing can be formed only in one way, starting from the empty bracketing, by successive applications of the two rules, "Pass from A to (A)" and

"Pass from A, B to $(A)(B)$." We therefore can easily see that the sets P_n are pairwise disjoint. If $A \in P_n$, the integer n is called the *complexity* of A. This definition is unambiguous, and we can legitimately prove facts about bracketings by induction on their complexity.

Now we introduce somewhat richer expressions, those of the language associated with an m-ary relation. The alphabet consists of the following groups of symbols:

- ([open parenthesis];) [close parenthesis]; , [comma]
- r [symbol to denote the relation]; $=$ [symbol for identity]
- $\neg$ [negation, "not"]; $\wedge$ [conjunction, "and"]; $\vee$ [disjunction, "or"]. These three are called *connectives* or *Boolean symbols.*
- $\exists$ [existential quantifier, "there exists"], $\forall$ [universal quantifier, "for all"]
- and finally, a denumerably infinite list of symbols called *variables*: $v_0, v_1, \ldots, v_n, \ldots$

As in the previous example, we define the sets $F_0, \ldots, F_n$ by induction on n as follows:

- F_0, called the set of *atomic formulas* or *formulas of complexity* 0, consists of all words of the form $x_1 = x_2$, where x_1 and x_2 are variables, not necessarily distinct, and all words of the form $r(x_1, \ldots, x_m)$, where $x_1, \ldots, x_m$ are variables, not necessarily distinct. (Here the integer m is fixed: It corresponds to the arity of the relation that we wish to discuss.)
- F_{n+1}, called the set of *formulas of complexity* $n + 1$, consists of all words of the form $\neg(f)$, $(\exists x)(f)$, or $(\forall x)(f)$, where x is a variable and f is in F_n, as well as words of the form $(f) \wedge (g)$ or $(f) \vee (g)$, where f and g are in the union of $F_1, \ldots, F_n$, with at least one of them in F_n.

The union of all the F_n is called the set F of *formulas.* Note that if we suppress all but the parentheses from a formula we get a bracketing. From the nonambiguity theorem for bracketings (Theorem 2.1) we can easily deduce the uniqueness of formation of formulas: A formula is atomic, or can be written uniquely in exactly one of the forms $\neg(f)$, $(\exists x)(f)$, $(\forall x)(f)$, $(f) \wedge (g)$, or $(f) \vee (g)$, where f and g are formulas. Moreover, the sets F_n are pairwise disjoint, so a formula has a well-defined *complexity* on which we can do recursion.

For example, let us define the set $S(f)$ of *subformulas* of the formula f, by induction on the complexity of f:

- If f is atomic, then $S(f) = \{f\}$.

- If $f = \neg(g)$, or $f = (\exists x)(g)$, or $f = (\forall x)(g)$, where g is a formula, then g must have lower complexity than f, and we define $S(f) = S(g) \cup \{f\}$.
- If $f = (g) \wedge (h)$ or $f = (g) \vee (h)$, where g and h are formulas, then g and h must have lower complexity than f, and we define $S(f) = S(g) \cup S(h) \cup \{f\}$.

These rules unambiguously define the set $S(f)$ for every formula f, since they give the definition first for formulas of complexity 0, and then allow us to define $S(f)$ for every f of complexity $n+1$ assuming $S(g)$ given for each g of complexity at most n. This is an example of a construction by induction (also called recursion), and henceforth we shall be briefer when we give such a construction.

We can see that the subformulas of f are the formulas that occur in the formation of f starting from atomic formulas.

Let us likewise define, by induction on the complexity of the formulas, the *quantifier rank* of a formula:

- If f is atomic, then $QR(f) = 0$.
- If $f = \neg(g)$, then $QR(f) = QR(g)$.
- If $f = (g) \wedge (h)$ or $f = (g) \vee (h)$, then $QR(f) = \max\{QR(g), QR(h)\}$.
- If $f = (\exists x)(g)$ or $f = (\forall x)(g)$, then $QR(f) = QR(g) + 1$.

The formulas of quantifier rank 0, i.e., formulas that contain no quantifier, are called *Boolean formulas* or *quantifier-free formulas.*

Finally we define the set of *free variables* of a formula:

- If f is atomic, then $FV(f)$ consists of all the variables occurring in f.
- If $f = (g) \wedge (h)$ or $f = (g) \vee (h)$, then $FV(f) = FV(g) \cup FV(h)$.
- If $f = \neg(g)$, then $FV(f) = FV(g)$.
- If $f = (\exists x)(g)$ or $f = (\forall x)(g)$, then $FV(f) = FV(g)\backslash\{x\}$, i.e., the set $FV(g)$, except with x removed if it happens to be in $FV(g)$. (Note that when we write $(\exists x)(g)$ we do not assume that x occurs in g.)

If $FV(f) = \emptyset$, we call f a *closed* formula or *sentence.* A variable x that appears in the scope of a quantifier in f is called *bound*; one variable can have several occurrences, some free and some bound, in the same formula.

Note that in this text we use the open and close parentheses and comma in two ways: as elements of our alphabet, and in their everyday senses to express mathematical facts (for example, as in the expression $QR(f) = QR(g) + 1$, which is not a formula in our sense, but a way of expressing a fact about the formulas f and g). We hope that this will not confuse the reader!

Up to now, we have only constructed a series of expressions, without giving them any meaning or interpretation. We can develop a theory of formal languages, introducing "rules of formation," grammars more complex or more interesting than that of our formulas. Such a study has many applications, but we will avoid this approach because it is has little connection to the spirit of model theory. The study of the formal aspects of the language is called *syntax*.

The model theorist is more often interested in *semantics*, in giving meaning to formal expressions, and in studying the meanings rather than the expressions themselves. We shall see that the semantics are quite rudimentary: We need only assign a "truth value," true or false, to a formula. To do this, we must first specify what relation R the symbol r denotes, and also, in case the formula contains free variables, specify which elements of the universe of R are represented by those variables.

When we write a formula in the form $f(\vec{x})$, where $\vec{x}$ is an n-tuple of variables $(x_1, \ldots, x_n)$, we understand that all free variables of f are contained among $x_1, \ldots, x_n$, although not all of $x_1, \ldots, x_n$ need be in $FV(f)$. Let us consider an m-ary relation R, a formula $f(\vec{x})$, and an n-tuple $\vec{a} = (a_1, \ldots, a_n)$ in the universe of R; We will define, by induction on the complexity of f, what it means for R to *satisfy* $f(\vec{a})$, or equivalently for $f(\vec{a})$ to *be true* for R. We write

$$R \models f(\vec{a})$$

to mean R satisfies $f(\vec{a})$; $f(\vec{a})$ is not, strictly speaking, a formula of our language, but rather what we get from the formula $f(\vec{x})$ by replacing free occurrences of $x_1, \ldots, x_n$ by $a_1, \ldots, a_n$, respectively.

First suppose $f(\vec{x})$ is atomic:

- If f is of the form $x = y$, then $R \models a = b$ if and only if a and b are identical.

- If f is of the form $r(x_1, \ldots, x_n)$, then $R \models r(a_1, \ldots, a_n)$ if and only if $(a_1, \ldots, a_n) \in R$.

Now the inductive steps:

- $R \models \neg(f)(\vec{a})$ if and only if R does not satisfy $f(\vec{a})$.

- $R \models (f) \vee (g)(\vec{a})$ if and only if R satisfies $f(\vec{a})$ or R satisfies $g(\vec{a})$. ("Or" here is inclusive: R may satisfy both.)

- $R \models (f) \wedge (g)(\vec{a})$ if and only if R satisfies $f(\vec{a})$ and R satisfies $g(\vec{a})$.

- $R \models (\exists x)(f)(\vec{a}, x)$ if and only if there exists b in the universe of R such that R satisfies $f(\vec{a}, b)$.

- $R \models (\forall x)(f)(\vec{a}, x)$ if and only if for all b in the universe of R, R satisfies $f(\vec{a}, b)$.

One of the author's worst fears, ever since he has been teaching elementary logic, has been of portraying model theorists as simple-minded. "Not-f" is true if and only if f is not true, "there exists x such that f" is true if and only if there exists an x such that f, etc. ... Do we go through this complicated formalization just to duplicate common sense? Yet even the definition of truth necessarily involves such complexity; in particular, we have to convince ourselves that to determine the truth or falsity of a sentence, we need to consider subformulas that may not be sentences.

I have still not been pedantic enough to be thoroughly rigorous; several details still need to be elaborated. As I have said, when we write $f(\vec{x})$, where $\vec{x} = (x_1, \ldots, x_n)$, we do not assume that all of the variables $x_1, \ldots, x_n$ occur as free variables in f. We associate the variable x_1 with the element a_1, and so forth, but some of these associations may be irrelevant: Note that whether R satisfies $f(\vec{a})$ depends only on those a_i that actually correspond to free variables of f. (This must be proved by induction.)

If y is not one of the components of $\vec{x}$, then $(\exists y)(f(\vec{x}))$ is satisfied by $\vec{a}$ in R if and only if $f(\vec{x})$ is, and likewise for $(\forall y)(f(\vec{x}))$. Also, (a, b) satisfies $f(x) \wedge g(y)$ if and only if $f(a)$ and $g(b)$ are both satisfied, while a satisfies $f(x) \wedge g(x)$ if and only if $f(a)$ and $g(a)$ are both satisfied.

Let us hope that the reader will only temporarily have difficulty with these slightly confusing explanations and with all the underlying conventions of language and notation; his mathematical sense will be strengthened as soon he understands the link between satisfaction of formulas and local isomorphisms.

Two formulas f and g are said to be *equivalent* if they always have the same meaning, i.e., if for any n-tuple $\vec{a}$ and any relation R, $R \models f(\vec{a})$ if and only if $R \models g(\vec{a})$.

For example, from the definition of satisfaction, we can easily see the following equivalences:

1. $f, \neg(\neg(f))$
2. $(f) \vee (g), \neg((\neg(f)) \wedge (\neg(g)))$
3. $(f) \wedge (g), \neg((\neg(f)) \vee (\neg(g)))$
4. $((f) \vee (g)) \vee (h), (f) \vee ((g) \vee (h))$
5. $((f) \wedge (g)) \wedge (h), (f) \wedge ((g) \wedge (h))$
6. $(f) \vee (g), (g) \vee (f)$
7. $(f) \wedge (g), (g) \wedge (f)$
8. $f, (f) \vee (f)$
9. $f, (f) \wedge (f)$
10. $(f) \wedge ((g) \vee (h)), ((f) \wedge (g)) \vee ((f) \wedge (h))$
11. $(f) \vee ((g) \wedge (h)), ((f) \vee (g)) \wedge ((f) \vee (h))$

12. $(\exists x)(f), \neg((\forall x)(\neg(f)))$
13. $(\forall x)(f), \neg((\exists x)(\neg(f)))$
14. $(\exists x)((f) \vee (g)), ((\exists x)(f)) \vee (\exists x)(g))$
15. $(\forall x)((f) \wedge (g)), ((\forall x)(f)) \wedge (\forall x)(g))$
16. $(\exists x)((f) \wedge (g)), ((\exists x)(f)) \wedge (g)$ if x does not occur as a free variable of g
17. $(\forall x)((f) \vee (g)), ((\forall x)(f)) \vee (g)$ if x does not occur as a free variable of g
18. $(\exists y)(f(\vec{x}, y)), (\exists z)(f(\vec{x}, z))$ if neither y nor z is part of $\vec{x}$. [$f(\vec{x}, z)$ is obtained by replacing every free occurrence of y by z in $f(\vec{x}, y)$.]
19. $(\forall y)(f(\vec{x}, y)), (\forall z)(f(\vec{x}, z))$ if neither y nor z is part of $\vec{x}$

As it is not very useful to distinguish between two obviously equivalent formulas, we introduce a few conventions for abbreviating formulas to make them easier to read and write.

- We write $(f) \vee (g) \vee (h)$ for either of the equivalent forms in (4) (associativity of disjunction), as the order of operations is irrelevant to the satisfaction of the formula. The disjunction of several formulas $f_1, \ldots, f_n$ is written $\bigvee_{i=1}^{n}(f_i)$.
- Similarly, we use the notation $(f_1) \wedge \cdots \wedge (f_n)$ and $\bigwedge_{i=1}^{n}(f_i)$.
- $(f) \rightarrow (g)$ is an abbreviation for $(\neg(f)) \vee (g)$; $\rightarrow$ is read "implies." Thus $(f) \rightarrow (g)$ is satisfied if and only if either g is satisfied or f is not. Note that $(f) \rightarrow (g)$ is a formula that may be either true or false; in writing the formula we do not assume any particular connection between f and g. In particular, we do not assume that f actually implies g; that happens only when the formula is true! We need to resist the psychological consequences of common usage, which can lead us, in mathematics as elsewhere, to write only formulas that we consider true.
- The double implication symbol $(f) \leftrightarrow (g)$ is an abbreviation for $((f) \rightarrow (g)) \wedge ((g) \rightarrow (f))$.
- By equivalence rule (2) we could have defined $(f) \vee (g)$ as an abbreviation for $\neg(\neg(f) \wedge \neg(g))$, and by rule (13) $\forall$ could be considered an abbreviation for $\neg\exists\neg$. Or we could have taken the opposite approach and introduced $\rightarrow$ as a primitive symbol, adding appropriate rules for the satisfaction of formulas in which it appears. All these approaches are ultimately equivalent, and there is no point belaboring this issue.
- So far we have been weighed down by parentheses. We need to use a lot of them to prove the uniqueness of formation; however, in practice we omit superfluous parentheses. We divide the symbols into three groups as follows: first, $\neg$, $(\exists x)$, $(\forall x)$; second, $\wedge, \vee$;

third, $\rightarrow, \leftrightarrow$. We use order-of-operations rules as in elementary arithmetic: first group $-$; second group $+$; third group $\times$. The precise statement of the rules and the proof that they introduce no ambiguity are left to the reader.

Examples:

Abbreviated form	Long form
$\neg f \vee g$	$(\neg(f)) \vee g$
$\neg(f \vee g)$	$\neg((f) \vee (g))$
$(\forall x)(f \rightarrow g)$	$((\forall x)(f)) \rightarrow (g)$
$(\exists x)(f \leftrightarrow g)$	$(\exists x)((f) \leftrightarrow (g))$
$f \rightarrow g \wedge h$	$(f) \rightarrow ((g) \wedge (h))$
$(f \rightarrow g) \wedge h$	$((f) \rightarrow (g)) \wedge (h)$

- Sometimes we use $(\exists! x)$ for "there exists a unique x such that ...". $(\exists! x)$ is thus an abbreviation for $(\exists x) f(x) \wedge (\forall x)(\forall y)(f(x) \wedge f(y) \rightarrow x = y)$.
- Finally, there are conventions specific to binary relations. We often write xry instead of $r(x, y)$, and $x \not r y$ instead of $\neg r(x, y)$: $x = y, x \neq y, x \in y, x \notin y, x \sim y, x \not\sim y, x \leq y, x < y$, etc. When the relation symbol is written $\in$, we understand that the relation it represents resembles set membership (note that we often confuse the symbol r with the relation R that it represents); when the symbol is written $\sim$ or $\equiv$, we understand it to be an equivalence relation; when it is written $\leq$, it is a partial ordering, or at least a preorder. In this last case, the abbreviations $x \geq y$ for $y \leq x$, and $x < y$ for $x \leq y \wedge x \neq y$, should be familiar.

Most authors refuse to consider relations on an empty set, and say that two formulas are equivalent when they are satisfied by the same tuples for any relation on a nonempty set. There is undoubtedly much wisdom to this attitude, which can indeed avoid some untidy phrasing, but the present author does not allow himself to adopt it. Indeed, the null m-ary relation (for $m > 0$: we will see later what we need to do in order to deal with zero-ary relations!) and the null local isomorphism, the starting point for every back-and-forth construction, have already played too important a role to be consigned to the junkheap! We say that two formulas $f(\vec{x})$ and $g(\vec{x})$ are *almost equivalent* if for every $\vec{a}$ in the nonempty universe of a relation R, $R \models f(\vec{a})$ iff $R \models g(\vec{a})$. The issue of satisfaction of a formula by a tuple from the empty universe arises only for closed sentences: If f and g have free variables, they are equivalent iff they are almost equivalent. However, looking back at the rules defining satisfaction, we note that for the relation with empty universe, a statement beginning with $\forall$ is always true, while a statement beginning with $\exists$ is always false. The following pairs of almost equivalent sentences are therefore not equivalent:

20. $(\exists x)(f), f$ [where x is not a free variable of f]

21. $(\forall x)(f), f$ [where x is not a free variable of f]
Likewise with the following two pairs, which we can derive by combining (14/15) with (20/21):

22. $(\forall x)(f \wedge g), ((\forall x)f) \wedge g$ [where x is not a free variable of g]

23. $(\exists x)(f \vee g), ((\exists x)f) \vee g$ [where x is not a free variable of g]

Note the difference between these and (16/17), which are valid even for the relation on an empty universe.

One final definition to close this section: A formula is said to be *in prenex form* (or simply *prenex*) if all its quantifiers occur at the beginning. Thus if f and g are quantifier-free, $(\forall x)f \wedge (\exists y)g$ is not prenex, although $(\forall x)(\exists y)(f \wedge g)$ and $(\exists y)(\forall x)(f \wedge g)$ are. These three formulas are almost equivalent if x is not a free variable of g and y is not a free variable of f. To put a formula into prenex form is to find an almost-equivalent prenex formula. This is always possible, by using (14), (15), (22) and (23) to move occurrences of $\wedge$ or $\vee$ inside the scope of quantifiers, and using (12) and (13) to do likewise with $\neg$. In some cases we will need to change variables; by (18) and (19) this is always possible.

Therefore, by induction on the complexity of a formula, we can show that every formula has an almost-equivalent prenex form; in fact, it has infinitely many. However, the prenex form obtained by the method described above is often much harder to read than the original formula. Putting formulas into prenex form provides an inexhaustible source of silly exercises for novice logicians. (For example, put $(\exists y)r(x, y) \leftrightarrow (\forall x)r(x, x)$ into prenex form: The only reasonable first step is to replace $\leftrightarrow$ by its translation in terms of $\neg, \vee, \wedge$). Prenex form is especially important in the theory of arithmetic. (See Chapter 7.)

While we are on the subject of painstaking transformations, note that every formula actually has an *equivalent* prenex form, taking the same truth value even for the empty relation! This is trivial if the formula contains a free variable. For a closed sentence, add an extra quantifier in front of the almost-equivalent prenex form: $\forall$ if the sentence is true of the empty relation, $\exists$ otherwise.

The Boolean part of a formula is susceptible to just as rich an array of manipulations. For example, we can put it into "disjunctive form," using (1) and (11) to write it as a disjunction of conjunctions of atomic formulas and their negations.

2.2 Connections to the Back-and-Forth Technique

Some ideas about formulas stem from convention, but the essential fact, which we will reach no matter which definitions we choose, is the following:

Theorem 2.2 (Fraïssé's Theorem). *An n-tuple $\vec{a}$ from the universe of the m-ary relation R, and an n-tuple $\vec{b}$ from the universe of the m-ary relation S, are p-equivalent if and only if $\vec{a}$ in R and $\vec{b}$ in S satisfy the same formulas, in the language associated with one m-ary relation, with quantifier rank at most p.*

As an immediate consequence of this theorem, two m-ary relations are elementarily equivalent iff they satisfy the same sentences; if R is an extension of S, this extension is elementary iff an n-tuple from (the universe of) S satisfies the same sentences whether it is considered as being in S or in R; two n-tuples taken from m-ary relations have the same type iff they satisfy the same formulas.

Proof. [Proof, Part 1] First we show that if $\vec{a}$ and $\vec{b}$ are p-equivalent, they satisfy the same formulas up to quantifier rank p.

By definition, two n-tuples are 0-equivalent if they satisfy the same atomic formulas; they therefore satisfy the same quantifier-free formulas, Which are Boolean combinations of atomic formulas. (Indeed, the truth value of a Boolean combination depends only on the truth values taken by each of its terms.)

Now we show the step from p to $p+1$. Let $f(\vec{x}, y)$ be a formula of quantifier rank at most p, and suppose that $\vec{a}$ satisfies $(\exists y)f(\vec{x}, y)$. So there exists α such that $f(\vec{a}, \alpha)$ is true; moreover, there exists β on the other side such that $\vec{a}\frown\alpha$ and $\vec{b}\frown\beta$ are p-equivalent ($\vec{a}\frown\vec{b}$ denotes the *concatenation* of the n-tuple $\vec{a}$ and the n'-tuple $\vec{b}$, i.e., the $n+n'$-tuple $(a_1, \ldots, a_n, b_1, \ldots, b_{n'})$). By the induction hypothesis, $\vec{a}\frown\alpha$ and $\vec{b}\frown\beta$ satisfy the same formulas up to quantifier rank p, so $f(\vec{b}, \beta)$ is true, and hence also $(\exists y)f(\vec{b}, y)$. Thus $\vec{b}$ also satisfies $(\exists y)f(\vec{x}, y)$. By symmetry $\vec{a}$ and $\vec{b}$ satisfy the same $(\exists y)f(\vec{x}, y)$, where f has quantifier rank at most p, and also the same formulas of quantifier rank at most $p+1$, which are equivalent to Boolean combinations of them. □

To prove the converse, we need a lemma:

Lemma 2.3. *If the arity m of a relation, and the integers n and p, are fixed, there is only a finite number $C(n,p)$ of p-equivalence classes of n-tuples.*

Proof. By induction on p. There are only finitely many m-ary relations defined on n (not necessarily distinct) elements $a_1, \ldots, a_n$; therefore, the number of 0-equivalence classes is finite. We show the step from p to $p+1$: Two n-tuples are $(p+1)$-equivalent if whenever I add an element on one side, I can respond with an element on the other side of such a type as to have p-equivalence. In other words, the $(p+1)$-equivalence class of an n-tuple is determined by the set of all p-equivalence classes of $(n+1)$-tuples obtained by adding one element to it. Hence $C(n, p+1) \leq 2^{C(n+1,p)}$. □

Proof. [Proof of 2.2, Part 2] We now show, by induction on p, that we can associate with each p-equivalence class C a formula f_C of quantifier rank p such that the tuples in C are exactly those that satisfy f_C.

Given an n-tuple $\vec{a} = (a_1, \ldots, a_n)$ and n variables $x_1, \ldots, x_n$, there are only finitely many atomic formulas with variables $x_1, \ldots, x_n$. Let f_C be the conjunction of those satisfied by $\vec{a}$ and the negations of those not satisfied by $\vec{a}$; then it is clear that f_C characterizes the 0-equivalence class of $\vec{a}$.

To pass from p to $p+1$, we add to $\vec{x}$ a variable y, and list the formulas $f_1(\vec{x}, y), \ldots, f_k(\vec{x}, y)$ that characterize the p-equivalence classes on $n+1$ elements. (By the lemma, there are only finitely many.) Let f_C be the conjunction of all formulas of the form $(\exists y) f_i(\vec{x}, y)$ that are true at $\vec{a}$ and the negations of all those that are false at $\vec{a}$. Then f_C characterizes the p-equivalence class of $\vec{a}$. □

As a consequence of the finiteness of the number of p-equivalence classes, we see that up to equivalence there are only finitely many formulas of quantifier rank p and free variables $x_1, \ldots, x_n$: Any such formula is a disjunction of some of the f_C. What produces an infinity of formulas is redundancy (either Boolean or in the use of quantifiers) or the changing of names of bound variables.

The formula f_C that characterizes the class C in certainly not prenex; putting it into prenex form would considerably raise its quantifier rank. On the contrary, with Fraïssé's back-and-forth method we seek to put formulas in as nonprenex a form as possible.

2.3 Models and Theories

If the relation R satisfies the sentence f, we also say that R is a *model* of f; given a set A of sentences, R is a *model* of A if it is a model of every member of A. If A has a model, we say that A is *consistent.*

A *consequence* of A is a sentence f satisfied by every model of A. For example, if A is *inconsistent* (we also say *contradictory*), then any sentence is a consequence of A, as A has no model. In contrast, a consistent set A cannot have both f and $\neg f$ as consequences. The fact that f is a consequence of A is written $A \models f$; if $A = \{g\}$, we generally write $g \models f$. The symbol used here is the same as for satisfaction. We should never confuse $\models$ and $\rightarrow$; they are related in that $g \models f$ iff $\emptyset \models g \rightarrow f$.

The sentences that are consequences of $\emptyset$, i.e., those that are true for every relation, are called *theses* by some and *theorems* by others; an *antithesis* is the negation of a thesis, i.e., a sentence that is false for every relation.

By a *theory* we mean a consistent set of sentences containing all its own consequences (in the language of one m-ary relation for some fixed m): If A is a consistent set of sentences, the set T_A of consequences of A is a theory, the theory generated by A; we also call A a *set of axioms* for T_A

or an *axiomatization* of T_A. In common practice we confuse A and T_A; for example, we often say that A is complete—see below—to mean that T_A is complete. The elements of a theory T are called *axioms* of T or *theorems* of T, the distinction between axioms and theorems being of a psychological nature. (The list of axioms corresponds to what is given; a theorem is a proven sentence that is a consequence of the axioms.)

A maximal theory with respect to the inclusion relation among theories is called a *complete theory*. Every theory can be extended to a complete theory: Take a model M of T, and the set T_M of sentences satisfied by M. Then T is certainly a theory, the *theory of the relation* M, and is complete, since for every sentence f it contains either f or $\neg f$, so if $f \notin T_M$, then $\neg f \in T_M$, and hence $T_M \cup \{f\}$ is inconsistent. To say that a theory is complete is to say that any two of its models are elementarily equivalent, or equivalently that it is a consistent set A of sentences such that for every sentence f, either f or $\neg f$ is in A.

Like Monsieur Jourdain,[1] we have already, without knowing it, in Chapter 1.2 studied some complete theories, which we shall now axiomatize.

First consider the following axioms:

$$A_p \qquad (\exists x_1) \cdots (\exists x_n) \bigwedge_{1 \leq i < j \leq p} x_i \neq x_j .$$

Now let a relation R be defined on p elements $a_1, \ldots, a_p$, and let $f(\vec{x})$ be the conjunction of atomic formulas and negations of atomic formulas satisfied by the the tuple $(a_1, \ldots, a_p)$; the theory of R, which as we know characterizes R up to isomorphism, is axiomatized by the following single axiom:

$$((\exists x_1) \cdots (\exists x_p) f(x_1, \ldots, x_p)) \wedge \neg A_{p+1} .$$

To axiomatize the theory of the empty unary relation on infinitely many elements, we use $(\forall x) \neg R(x)$ and the infinite list of all the A_p.

I leave to the reader the task of axiomatizing the complete theories of unary relations.

Let us express that R is an equivalence relation:

$$\begin{array}{rl} (\forall x) & R(x,x), \\ (\forall x)(\forall y) & (R(x,y) \rightarrow R(y,x)), \\ (\forall x)(\forall y)(\forall z) & (R(x,y) \wedge R(y,z) \rightarrow R(x,z)). \end{array}$$

[1] Translator's Note: A character in Molière's *Le Bourgeois Gentilhomme* who was astonished to learn that he had been speaking prose all his life without knowing it.

To express that it has infinitely many classes, we need the following infinite list of axioms, one for each natural number n:

$$(\exists x_1)\cdots(\exists x_n)\bigwedge_{1\leq i<j\leq p}\neg R(x_i,x_j),$$

and likewise an infinite list of axioms to express that each class is infinite:

$$(\forall x)(\forall y_1)\cdots(\forall y_n)(\exists z)\big(R(x,z)\wedge\bigwedge_{1\leq i\leq n} y_i\neq z\big).$$

To express that the relation is a chain we need the following axioms:

$$\begin{aligned}
(\forall x)\quad & x\leq x,\\
(\forall x)(\forall y)\quad & (x\leq y\wedge y\leq x\rightarrow x=y),\\
(\forall x)(\forall y)(\forall z)\quad & (x\leq y\wedge y\leq z\rightarrow x\leq z),\\
(\forall x)(\forall y)\quad & (x\leq y\vee y\leq x).
\end{aligned}$$

We usually accept $x<y$ as an abbreviation for $x\leq y\wedge x\neq y$. To express that there is no greatest element: $(\forall x)(\exists y)x<y$; that there is no least element: $(\forall x)(\exists y)y<x$; that the chain is nonempty: $(\exists x)x=x$; that it is dense:

$$(\forall x)(\forall y)(\exists z)(x<y\rightarrow(x<z\wedge z<y)).$$

This finite list of axioms, which could be replaced by a single axiom (their conjunction) forms (or more precisely, axiomatizes) a complete theory, that of dense chains without endpoints.

We likewise obtain a complete theory by replacing the density axiom with the discreteness axiom:

$$\begin{aligned}
((\forall x)\quad & (\exists y)(\forall z)(x<y\wedge\neg(x<z\wedge z<y)))\\
& \wedge((\forall x)(\exists y)(\forall z)(y<x\wedge\neg(y<z\wedge z<x)))
\end{aligned}$$

Note: Those who do not allow relations on an empty universe consider $(\exists x)x=x$ and its consequences as theses; we, however, do not share this abhorrence, with so little logical ground, of a vacuum.

2.4 Elementary Extensions: Tarski's Test, Löwenheim's Theorem

If the relation S is an extension of the relation R, saying that S is an elementary extension of R means that every tuple from the domain of

R (the "smaller" relation) satisfies the same formulas in R as in S; this translates into the following very useful theorem, where we note that the satisfactions are relative to the "bigger" relation.

Theorem 2.4 (Tarski's Test). *If S is an extension of R, this extension is elementary if and only if for every $\vec{a}$ taken from the domain of R, and every formula $f(\vec{x}, y)$, if $S \models (\exists y) f(\vec{a}, y)$, then there exists b in the domain of R such that $S \models f(\vec{a}, b)$.*

Proof. The necessity is obvious; conversely, we show by induction on the quantifier rank p of f that if $\vec{a}$ is in the domain of R, then $S \models f(\vec{a})$ iff $R \models f(\vec{a})$. This is clear if $p = 0$, since by the very definition of extension $\vec{a}$ satisfies the same atomic formulas in R and in S. Suppose that f has the form $(\exists y) g(\vec{x}, y)$ where the quantifier rank of g is less than p. If $S \models (\exists y) g(\vec{a}, y)$, by hypothesis there exists b in the domain of R such that $S \models g(\vec{a}, b)$, and by the induction hypothesis $R \models g(\vec{a}, b)$, so $R \models (\exists y) g(\vec{a}, y)$. If $S \not\models (\exists y)(g\vec{a}, y)$, then for every b in the domain of S, and in particular for every b in the domain of R, $S \not\models g(\vec{a}, b)$. By the induction hypothesis $R \not\models g(\vec{a}, b)$, so $R \not\models (\exists y) g(\vec{a}, y)$. Finally, we note that the formulas of quantifier rank $p + 1$ are Boolean combinations of the $(\exists y) g(\vec{x}, y)$. □

Theorem 2.5 (Löwenheim's Theorem). *Every relation R has a finite or denumerable elementary restriction; more precisely, if A is an infinite subset of the domain of R, then we can find an elementary restriction of R whose domain contains A and has the same cardinality as A.*

Proof. We enumerate all formulas $f(\vec{a}, y)$ with parameters $\vec{a}$ in A: There are denumerably many formulas $f(\vec{x}, y)$, and in each such formula we need to replace $\vec{x}$ by an n-tuple from A; as A is infinite, there are $\text{card}(A)$ such n-tuples (because for any infinite A the set of finite subsets of A has the same cardinality as A). So there are $\omega \times \text{card}(A) = \text{card}(A)$ formulas with parameters in A. (We shall see the details of this "cardinal arithmetic" later.) For each of these formulas for which $R \models (\exists y) f(\vec{a}, y)$, add to A an element b of the domain of R such that $R \models f(\vec{a}, b)$; as there are at most $\text{card}(A)$ elements to add, we get a set A_1 containing A, with the same cardinality as A. Repeat the operation that replaced A by A_1, to get a set A_2, etc... Let B be the union of $A, A_1, A_2, \ldots, A_n, \ldots$. This is a set of cardinality $\omega \times \text{card}(A) = \text{card}(A)$, and the restriction of R to B satisfies the hypotheses of Tarski's test. □

In particular, we see that every consistent theory in the language of one m-ary relation has a finite or denumerable model.

Let I be a chain (or total order); by a *chain of extensions* indexed by I we mean a relation R_i with domain E_i for each $i \in I$ such that if $i < j$, then R_j is an extension of R_i. The (inductive!) *limit* of a chain of extensions is the unique common extension of all the R_i defined on the domain $E = \cup E_i$:

$\vec{a}$ satisfies R if for sufficiently large i (in fact, for any i large enough that $\vec{a}$ is in E_i) $\vec{a}$ satisfies R_i.

In practice I is most often a well-ordering (for example, the chain ω of integers, as in the proof of Theorem 2.5), and the extensions are constructed one after another; however, this assumption is not necessary to show the following theorem:

Theorem 2.6. *If $(R_i : i \in I)$ is a chain of elementary extensions (i.e., $R_i \prec R_j$ if $i < j$), then its limit R is an elementary extension of each R_i.*

Proof. By induction on the quantifier rank p of f we show that if $\vec{a}$ is in E_i, then $R \models f(\vec{a})$ iff $R_i \models f(\vec{a})$. This is obvious for $p = 0$. Suppose that $f(\vec{x}) = (\exists y)g(\vec{x}, y)$, where g has quantifier rank p. If $R \models f(\vec{a})$, then there exists b in E such that $R \models g(\vec{a}, b)$; this b is in some E_j, with $j \geq i$, and by the induction hypothesis $R_j \models g(\vec{a}, b)$. Therefore, $R_j \models (\exists y)g(\vec{a}, y)$, and as R_j is an elementary extension of R_i, $R_i \models (\exists y)g(\vec{a}, y)$. If $R \not\models f(\vec{a})$, then for every b in E, and in particular for every b in E_i, $R \not\models g(\vec{a}, b)$, from which, by the induction hypothesis, we get $R_i \not\models g(\vec{a}, b)$, and thus $R_i \not\models (\exists y)g(\vec{a}, y)$. $\square$

2.5 Historic and Bibliographic Notes

I shall not give any references relating to syntax, to anything involving the definition of formulas, as it really has to do with the development of contemporary mathematical symbolism, a phenomenon largely outside the scope of this work.

The inductive definition of satisfaction of a formula *in an arbitrary structure* by a tuple appeared quite late, in [TV56]. Some consider this article a turning point, as having determined the subsequent direction of logic; we should rather say that it was a matter of clarifying ideas that were in the air and that had been introduced, in substantial works several decades earlier, by such mathematicians as Skolem, Gödel, and even Tarski, ideas that indisputably belong to model theory. The notion of satisfaction had never previously been made explicit, since it was taken for granted.

For Fraïssé's theorem I direct you to the bibliographic notes for Chapter 1.

The terminology of Section 2.3 appeared for the most part in [TV56]; the notion of "contradictory set of sentences" was older, but originally referred to a set of sentences from which we can derive a contradiction in a formal system of inference rules. In this article, elementary equivalence and extensions were called "arithmetic," by analogy to Gödel's works, where formulas are coded into arithmetic (see Chapter 7). As this description was a little strange, it was later replaced by the term "elementary," which is scarcely better; Fraïssé tried unsuccessfully to introduce the phrases "logical equivalence" and "logical extension."

Classically, axioms were distinguished from postulates; axioms had a logical character, and concerned the validity of modes of reasoning, while postulates were a sort of primitive property of the specific objects (points, straight lines, etc.) under consideration. What we call "axioms" today are more often called "postulates."

Tarski's test and the theorem on chains of elementary extensions both come from [TV56]. As for Löwenheim's theorem, it is the forerunner of model theory; to be precise, it was shown in [Löw15] that a sentence with a model has one that is finite or denumerable; the case where A is denumerable appeared in the form given here in [Sko20].

3

Extensions of the Language: Structures

En second lieu, les deux mémoires sont courts et nullement proportionnés aux titres; et puis il y a au moins autant de français que d'algèbre à tel point que l'imprimeur, quand on lui a porté les manuscrits, a cru de bonne foi que c'était une introduction ... Il eut été si facile encore de substituer successivement toutes les lettres de l'alphabet dans chaque équation, en les numérotant par ordre pour pouvoir reconnaître à quelles combinaisons de lettres appartiennent les équations subséquentes; ce qui eut multiplié indéfiniment le nombre d'équations, si l'on réfléchit qu'après l'alphabet latin, il y a encore l'alphabet grec, que celui-ci épuisé, il reste les caractères allemands, que rien n'empêche de se servir des lettres syriaques, et au besoin des lettres chinoises! Il eut été si facile de transformer dix fois chaque phrase, en ayant soin de faire précéder chaque transformation du mot solennel théorème ...

E.G.

3.1 Multirelations, Relational Structures

Instead of considering just one relation, we could consider the structure formed by a finite set of relations $R_1, \ldots, R_k$, of arities $m_1, \ldots, m_k$, respectively, *all on the same universe E;* such a structure is called a *multirelation*; the sequence of arities $m_1, \ldots, m_k$ is called the *signature* (or *similarity* *type*) of the multirelation. Given a second multirelation $(S_1, \ldots, S_k)$ with universe F and the same signature, an *isomorphism* from $(R_1, \ldots, R_k)$ to $(S_1, \ldots, S_k)$ is a function s from E to F that is an isomorphism from R_1 to S_1, ..., from R_k to S_k. The notions of extension, embedding, local isomorphism, p-isomorphism, etc. are defined the same way as for relations. For the language associated with a multirelation, or more precisely with its signature, we now must introduce k relation symbols, instead of just one: one of arity m_1 to denote R_1, ..., one of arity m_k to denote R_k.

The conscientious reader will have no trouble verifying that *all* the theorems proved up to now for relations are also valid for multirelations.

This generalization is not really a generalization; we now cross a more delicate threshold, by considering *relational structures* $\mathcal{R}$, formed from a family (not necessarily finite) of relations R_i of arity m_i, all with the same universe E; the list m_i of arities is called the *signature*, or equivalently the *similarity type*, of the structure. The language associated with a similarity type consists of one symbol to denote each relation. As above, we define the notions of extension, restriction, isomorphism, p-isomorphism, etc. between relational structures of the same similarity type.

Here, however, there is an important change: If the first part of Fraïssé's theorem (2.2) remains valid, i.e., two p-equivalent tuples satisfy the same formulas of quantifier rank less than or equal to p, we cannot show the converse, for the simple reason that Lemma 2.3, stating that there are only finitely many p-equivalence classes, is false. If we wish to keep this converse, we must introduce infinitary conjunctions and disjunctions into our formulas, and thereby introduce infinitary languages. We shall not follow that path, but rather keep the finitary character, which is linked to those essential compactness properties that will appear in the next chapter.

We shall therefore say that two structures $\mathcal{R}$ and $\mathcal{S}$ of the same signature are *by definition* elementarily equivalent if they satisfy the same statements, and if $\mathcal{S}$ is an extension of $\mathcal{R}$, that the extension *by definition* is elementary if a tuple taken from the universe of $\mathcal{R}$ satisfies the same formulas in $\mathcal{R}$ as in $\mathcal{S}$.

How can we recover local isomorphisms if we are to keep the converse of Fraïssé's Theorem? We shall say that $\mathcal{R}_1$ is a *reduct* (not to be confused with a restriction) of $\mathcal{R}$ if it is a structure on the same universe as $\mathcal{R}$, obtained by omitting some of the relations occurring in $\mathcal{R}$; we also call $\mathcal{R}$ an *expansion* of $\mathcal{R}_1$. Expansion corresponds to an augmentation of the language; extension, to an augmentation of the universe. Since a formula can involve only finitely many elements of the signature, we see that $\mathcal{R}$ and $\mathcal{S}$ are elementarily equivalent iff for every finite subset i of the signature, their corresponding reducts $\mathcal{R}_i$ and $\mathcal{S}_i$ are ω-equivalent.

Note: In certain cases, it can be convenient to introduce 0-ary relations. As a function from X to Y associates to every element of X one and only one element of Y, there is always one function from $\emptyset$ to E, *whether or not E is empty.* The graph of this function is $\emptyset$; in other words, there is one 0-tuple of elements of E. Consequently, there are always two nullary relations with universe E, namely $\{\emptyset\}$ and $\emptyset$; the first can be called the *true*, the second the *false*. In contrast, if $m > 0$, there are no m-tuples of elements of $\emptyset$, and therefore only one m-ary relation with universe $\emptyset$, which is $\emptyset$. (We have often said that $\emptyset$ is always a local isomorphism between two m-ary relations for $m > 0$.)

A nullary relation symbol is therefore interpreted in a structure either by the true or by the false. These nullary relations are not very interesting in and of themselves, but occur naturally as a calculation tool: If a formula in n free variables represents an n-ary relation (the set of n-tuples that satisfy it), a statement represents a 0-ary relation, either the true or the false.

3.2 Functions

If, to take one example, we wish to speak of a group G as a relational structure, we can introduce a ternary relation symbol $R(x, y, z)$ to denote the set of triples (x, y, z) such that z is the product of x and y. This relation is functional in z; that is to say, it satisfies the formula $(\forall x)(\forall y)(\exists! z)R(x, y, z)$. A formula expressing that $x = y^2zy$, for example, is $(\exists u)(\exists v)(R(y, y, u) \wedge R(z, y, v) \wedge R(u, v, x))$: We can see that this notation is far removed from standard practice; moreover, it would be slightly abusive to introduce by a quantifier the unique z such that $R(x, y, z)$ holds.

It is therefore common to reserve a special treatment for functions, to avoid systematically replacing them by their graphs. (The graph of a function of m arguments is an $(m + 1)$-ary relation.) Now we can define the most general notion of *structure*: A *signature*, or *similarity type*, is a fixed set (possibly empty) consisting of *constant symbols* c_i (we also say *individual* instead of constant); *function symbols* f_j, f_j having arity (i.e., number of arguments) m_j; and *relation symbols* r_k, r_k having arity m_k. (We could also consider constants as functions of 0 arguments.)

A structure $\mathcal{S}$ of signature σ and universe E is the assignment to each constant symbol c_i of σ of an element of E interpreting it, the assignment to each function symbol f_j of a function from E^{m_j} to E, and the assignment to each relation symbol r_k of a subset of E^{m_k}. We often confuse a constant, function, or relation with the symbol representing it; where it is appropriate to specify the structure where it occurs, we can write $c_i^{\mathcal{S}}, f_j^{\mathcal{S}}, r_k^{\mathcal{S}}$, for "the interpretation of c_i in $\mathcal{S}$," etc.

Let us quickly describe the language associated with a given signature; before forming formulas, we need to form *terms*, which denote individuals, by the following rules:

- Terms of complexity 0: variables and constants.

- Terms of complexity $n+1$: words of the form $f(t_1, \dots, t_m)$, where f is a function symbol of arity m, and $t_1, \dots, t_m$ are terms of complexity at most n, of which at least one has complexity exactly n.
  Next, rules for forming formulas:

- Atomic formulas, or formulas of complexity 0: $t_1 = t_2$, $r(t_1, \dots, t_m)$, where $t_1, \dots, t_m$ are terms and r is an m-ary relation symbol.

- Formulas of complexity $n+1$: $\neg f$, $(\exists x)f$, $(\forall x)f$, where f is a formula of complexity n; $f \wedge g$, $f \vee g$, where f and g are formulas of complexity at most n, and at least one has complexity exactly n.

The set of subformulas, that of free variables, the quantifier rank of a formula, etc., are defined just as in the case of the language reduced to only a single relation symbol.

We leave it to the plucky reader to verify that there is a theorem on uniqueness of readability for terms and for formulas, on which the following depends.

Let us proceed to interpretation: A formula with n free variables is intended to denote a subset of E^n (the n-tuples that satisfy it); a term with n variables is intended to denote a function from E^n to E. Given a term $t(x_1, \dots, x_n)$ with variables $x_1, \dots, x_n$, and an n- tuple $(a_1, \dots, a_n)$ from the universe E of the structure $\mathcal{S}$, the element $t(a_1, \dots, a_n)$ of E is defined by induction on the complexity of t:

- If $t = c_i$, this element is $c_i^{\mathcal{S}}$; if $t = x_k$, this element is a_k.

- If $t(\vec{x})$ is $f(t_1(\vec{x}), \dots, t_m(\vec{x}))$, $t(\vec{a})$ is $f^{\mathcal{S}}(t_1(\vec{a}), \dots, t_m(\vec{a}))$.

We define the satisfaction of an atomic formula at a tuple $\vec{a}$ as follows:

- $\mathcal{S} \models t_1(\vec{a}) = t_2(\vec{a})$ if $t_1(\vec{a})$ is equal to $t_2(\vec{a})$;

- $\mathcal{S} \models r(t_1(\vec{a}), \dots, t_n(\vec{a}))$ if $(t_1(\vec{a}), \dots, t_n(\vec{a})) \in r^{\mathcal{S}}$.

For formulas of higher complexity, we proceed as in the case of a single relation.

We say that two structures are *elementarily equivalent* if they satisfy the same statements.

We call a structure $\mathcal{S}'$ a *restriction* or a *substructure* of $\mathcal{S}$ if:

- the universe E' of $\mathcal{S}'$ is contained in the universe E of $\mathcal{S}$;

- for each constant symbol c_i, $c_i^{\mathcal{S}} \in E'$ and $c_i^{\mathcal{S}'} = c_i^{\mathcal{S}}$;

- for each function symbol f_j, E' is closed under $f_j^{\mathcal{S}}$ and $f_j^{\mathcal{S}'}$ is the restriction of $f_j^{\mathcal{S}}$ to E'; and

- for each relation symbol r_k, $r_k^{\mathcal{S}'}$ is the restriction of $r_k^{\mathcal{S}}$ to E'.

An extension is *elementary* (in symbols $\mathcal{S}' \prec \mathcal{S}$) if every tuple of the small structure satisfies the same formulas in the small structure as in the large structure.

Beware of the following fact: If we axiomatize groups by introducing only the binary function of multiplication, a substructure of a group is not a subgroup, but merely a set closed under multiplication; if, however, we introduce a constant for the identity and a unary function for the inverse, we do not obtain a richer structure, since $x = 1$ is synonymous with $x = x^2$, and $x = y^{-1}$ with $xy = xyxy$, in the language with only product; however, we do change the notion of substructure.

Given a subset A of the universe of $\mathcal{S}$, the substructure of $\mathcal{S}$ *generated* by A is the restriction of $\mathcal{S}$ to the closure of A under the functions of $\mathcal{S}$ (which includes the constants, considered as functions of 0 arguments). A substructure is said to be *finitely generated* or of *finite type* if it has a finite generating set.

An *isomorphism* between two structures $\mathcal{S}$ and $\mathcal{S}'$, with the same signature and with universes E and E', respectively, is a bijection s between E and E' such that:

- $s(c^{\mathcal{S}}) = c^{\mathcal{S}'}$ for every constant c;
- $s(f^{\mathcal{S}}(s^{-1}(b_1), \ldots, s^{-1}(b_n))) = f^{\mathcal{S}'}(b_1, \ldots, b_n)$ for every function f; and
- $(a_1, \ldots, a_n) \in r^{\mathcal{S}}$ iff $(s(a_1), \ldots, s(a_n)) \in r^{\mathcal{S}'}$ for every relation r.

A *local isomorphism* from $\mathcal{S}$ to $\mathcal{S}'$ is an isomorphism between a finite–type restriction of $\mathcal{S}$ and a finite–type restriction of $\mathcal{S}'$. To say that $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are locally isomorphic, or 0-equivalent, is to say that the function sending a_1 to $b_1, \ldots, a_n$ to b_n, extends (uniquely) to an isomorphism between the substructures generated by these tuples; this amounts to saying that $\vec{a}$ and $\vec{b}$ satisfy the same atomic formulas.

As in the case of the reduced language of just one relation, we define the notion of p-isomorphism, where p is a finite number, or more generally an ordinal, and also the notions of Karpian family and ∞-isomorphism. We have no difficulty proving the first part of Fraïssé's theorem, that *two p-equivalent tuples satisfy the same formulas of quantifier rank less than or equal to p*. In particular two ω-equivalent tuples, and a fortiori two ∞-equivalent tuples, have the same type.

As for the converse, which depends on Lemma 2.3 ("there are only finitely many p-equivalence classes"), it is correct only if the signature can be reduced to a finite number of relations and and constants. In fact, *one single function* of nonzero arity allows the formation of infinitely many terms, and consequently infinitely many atomic formulas.

Therefore, if we want to reduce elementary equivalence completely to finite back-and-forth criteria (which may be necessary in a few delicate cases), we need to reduce it to finite signatures, and to replace each n-ary function by its graph, which is an $(n+1)$-ary relation. Doing this would preserve elementary equivalence, but change the notions of atomic formula, formula of quantifier rank p, p-isomorphism, etc.

But in most cases, the only part of Fraïssé's theorem that is really useful is the implication from back-and-forth to elementary equivalence, which is still valid, and most often we use only ∞-isomorphisms (see Chapter 5).

3.3 Löwenheim's Theorem Revisited

Now that we have introduced more general structures than the relations of Chapters 1 and 2, we need to see how to generalize the theorems proved in those chapters. We have already done this for Fraïssé's back-and-forth conditions. We can easily see that Theorem 1.14 remains true: Two ∞-equivalent structures with countable universes are isomorphic.

Equivalences 1–21 of Section 2.1 remain valid, as does the result on putting a formula into prenex form, which depends on them. The notions of model, consequence, consistent set of sentences, theory, and complete theory, from Section 2.3, also remain valid, although the notions of theory and complete theory are relative to a fixed language. Tarski's test (2.4) and the theorem on chains of elementary extensions (2.6) from Section 2.4 are still true; Löwenheim's theorem, however, needs to be fixed up slightly.

The *cardinality* of a language L is defined as the number of formulas of L: If L has finitely or denumerably many constant, function, and relation symbols, this number is denumerable ($\text{card}(L) = \omega$); if L has an infinite number κ of such symbols, then $\text{card}(L) = \kappa$. If T is a theory in a language L, we will often abusively say $\text{card}(T)$ instead of $\text{card}(L)$. (Note that $\text{card}(T)$ is therefore the number of formulas of the language of T, and not the number of axioms involved in an axiomatization of T; however, this usage is not quite that abusive, as $\text{card}(T)$ is indeed the number of theorems of T, since the number of theses of L is $\text{card}(L)$!)

In the proof of Löwenheim's theorem for a single relation, we have used the fact that the number of formulas without parameters is denumerable, so that the number of formulas with parameters in A is $\max(\text{card}(A), \omega)$; in the general case this number is $\max(\text{card}(A), \text{card}(L))$, and the same proof as before yields the following result:

Theorem 3.1. *[Löwenheim's Theorem] Let S be a structure in a language L, and let A be a subset of the universe of S; then there is an elementary restriction S' of S whose universe contains A and has cardinality at most* $\max(\text{card}(A), \text{card}(L))$.

In particular, we see that a theory T has a model of cardinality less than or equal to $\text{card}(T)$.

3.4 Historic and Bibliographic Notes

I shall not say much in regard to this short chapter, except that historians place the origins of structures in [Sch95]; the word "multirelation" comes from Fraïssé [Fra71]. For the rest I direct you back to the references in the two preceding chapters.

4
Compactness

> Et por l'achaison de celle bataille et de celle ghere nulo home ne pooit aler per chemin qui ne fust pris: et ce estoit deverç dont il estoient venu; mes avant pooient il bien aler. Et adonc les deus frers distroient entr'aus "puis que nos ne poons retorner a Gostantinople con notre mercandie, or alon por la voie dou levant ...
>
> M.P.

4.1 Ultraproducts

Every student of topology has heard of *filters*: If I is a nonempty set, a filter is a set F of subsets of I such that:

- $I \in F, \emptyset \notin F$;
- if $X, Y \in F$ then $X \cap Y \in F$;
- if $X \in F$ and $X \subset Y$, then $Y \in F$.

Examples of filters:

- if $A \subset I, A \neq \emptyset$, the set $F_{\{A\}}$ of subsets of I containing A;
- if I is infinite, the set of cofinite subsets of I, called the Fréchet filter.

A *filter prebase* B is a set of subsets of I contained in a filter; this means that the intersection of a finite number of elements of B is never empty.

The filter F_B consisting of subsets of I containing a finite intersection of elements of B is the smallest filter containing B; we call it the filter *generated* by B. If, in addition, the intersection of two elements of B is always in B, we call B a *filter base*; F_B then consists of all subsets of I containing an element of B.

Examples of filter bases:

- $\{A\}$ if $A \neq \emptyset, A \subset I$.
- Let J be a set, and I the set of finite subsets of J; for every $i \in I$, let $I_i = \{j : j \in I, j \supset i\}$, and let B be the set of all the I_i. Then $I_i \cap I_j = I_{i \cup j}, i \in I_i$; B is closed under finite intersections and does not contain $\emptyset$. It is therefore a filter base. The filter F_B, or one of its close relatives, often appears in model theory.

An *ultrafilter* is defined as a maximal filter.

Theorem 4.1. *A filter F of subsets of I is an ultrafilter if and only if for every subset A of I, either A or its complement $I - A$ is in F.*

Proof. Let F be an ultrafilter, and suppose $A \notin F$. By maximality of F, $F \cup \{A\}$ is not a filter prebase, so there exists B in F such that $A \cap B = \emptyset$; and since $B \subset I - A$, it follows that $I - A \in F$.

If F is a filter satisfying the hypothesis and $A \notin F$, then $F \cup \{A\}$ is not a filter prebase because $I - A \in F$, and $(I - A) \cap A = \emptyset$. Hence F is an ultrafilter. □

Theorem 4.2. *Let U be an ultrafilter of subsets of I. If I is covered by finitely many subsets $A_1, \ldots, A_n$ ($I = A_1 \cup \cdots \cup A_n$), then one of the A_i is in U; moreover, if the A_i are pairwise disjoint, exactly one of the A_i is in U.*

Proof. If $A_i \notin U$, then $I - A_i \in U$, and it is not possible that all the $I - A_i$ are in U, as their intersection is empty. If $A_i \in U$ and $A_j \in U$, then $A_i \cap A_j \neq \emptyset$. □

As an example of an ultrafilter we can really only give U_a, the set of subsets of I containing the element a of I: These ultrafilters are called *principal*. What characterizes them is that any intersection, even infinite, of elements of U is always in U. These ultrafilters are obviously of no interest. By Theorem 4.2, if an ultrafilter contains a finite set $A = \{a_1, \ldots, a_n\}$, it is principal: Consider the partition $I - A, \{a_1\}, \ldots, \{a_n\}$. A nonprincipal ultrafilter can exist only on an infinite set I, and in this case they are the ultrafilters containing the Fréchet filter.

The difficulty we have in "explicitly" giving an example of a nonprincipal ultrafilter will be understood only by the reader specializing in set theory; one of the axioms of that field, which is a weak version of the axiom of choice, is the following:

Axiom of Ultrafilters: Every filter F is contained in an ultrafilter.

We accept this axiom, which affirms the existence of many ultrafilters; it is used in "ordinary" mathematics. We will discuss its validity when we speak of the ordinals.

In model theory, ultrafilters are useful for allowing us to construct new structures out of given structures.

Let I be a nonempty set, let U be an ultrafilter of subsets of I, and let there be given for every i in I a structure S_i with nonempty universe E_i, all these structures having the same similarity type σ. We will define a structure S of similarity type σ that we will call the *ultraproduct of the* S_i *by the ultrafilter* U, which we will write $S = \Pi S_i/U$.

First let us define the universe E of S. Consider the following relation on the set product ΠE_i: We shall say that the I-tuple $a = (\ldots, a_i, \ldots)$ and the I-tuple $b = (\ldots, b_i, \ldots)$ are equal modulo U if $\{i : a_i = b_i\}$ is in U. This equality modulo U is clearly reflexive and symmetric; it is also transitive because $\{i : a_i = c_i\} \supset \{i : a_i = b_i\} \cap \{i : b_i = c_i\}$. This is therefore an equivalence relation, and E is defined to be its quotient space, i.e., its set of equivalence classes, written $E = \Pi E_i/U$.

An element α of E is therefore the equivalence class modulo U of an I-tuple $a = (\ldots, a_i, \ldots)$ of ΠE_i; in practice we identify α with a by a natural abuse of notation.

Now let us define the interpretations, in the sense of S, of the symbols of the signature σ:

- If c is a constant symbol, let c_i denote its interpretation in S_i; c^S is defined as the equivalence class of the I-tuple $(\ldots, c_i, \ldots)$.
- If f is an n-ary function symbol, let f_i denote its interpretation in S_i. Given $\alpha_1, \ldots, \alpha_n$ in E, we define $f(\alpha_1, \ldots, \alpha_n)$ in the sense of S: Choose representatives $a_1, \ldots, a_n$ of $\alpha_1, \ldots, \alpha_n, a_1 = (\ldots, a_{1,i}, \ldots), \ldots, a_n = (\ldots, a_{n,i}, \ldots)$, and take as the value of $f(\alpha_1, \ldots, \alpha_n)$ the class modulo U of the I-tuple $(\ldots, f_i(a_{1,i}, \ldots, a_{n,i}), \ldots)$. We need to check that this value is independent of the choice of representatives, that is to say that if a_1 and $b_1, \ldots, a_n$ and b_n are equal modulo U, then so are $(\ldots, f_i(a_{1,i}, \ldots, a_{n,i}), \ldots)$ and $(\ldots, f_i(b_{1,i}, \ldots, b_{n,i}), \ldots)$: This is left to the reader.
- If r is an n-ary relation symbol, let r_i denote its interpretation in S_i, and let us define its interpretation in S: If $\alpha_1, \ldots, \alpha_n$ are in E, choose representatives $a_1, \ldots, a_n$ for each, and say that $(\alpha_1, \ldots, \alpha_n)$ satisfies r^S if the set of i such that $(a_{1,i}, \ldots, a_{n,i})$ satisfies r_i is in U; again, we leave to the reader the task of verifying that this does not depend on the choice of representatives, that if a_1 and $b_1, \ldots, a_n$ and b_n are congruent modulo U, then $\{i : a_{1,i}, \ldots, a_{n,i} \in r_i\} \in U$ iff $\{i : b_{1,i}, \ldots, b_{n,i} \in r_i\} \in U$.

Note that this construction is uninteresting if U is principal: If U is the principal ultrafilter U_j generated by j, then $\Pi S_i/U_j$ is isomorphic to S_j.

The definition of ultraproduct is natural and, in short, easy to digest; what makes it a little confusing is the multiplicity of indices and notation that we need to introduce.

If you have trouble seeing what it all means, it might help you to consider the following example: The S_i are the groups G_i; let $G = \Pi G_i$, and let G_U be the set of elements of G equal to 1 modulo U; in other words, $(\ldots, a_i, \ldots)$ is in G_U if $\{i : a_i = 1\}$ is in U. Then G_U is a normal subgroup of G,and the ultraproduct of the G_i is isomorphic to G/G_U. Or let the S_i be the rings A_i, let $A = \Pi A_i$, and let A_U is the set of elements of A that are zero modulo U; then A_U is a two-sided ideal of A, and the ultraproduct of the A_i is A/A_U.

The motivation for ultraproducts is the following theorem. (Łoś is pronounced Wosh.)

Theorem 4.3 (Łoś's Theorem). *Let U be an ultrafilter of subsets of I, and let the structures S_i be indexed by I, all having the same signature σ. Let $f(\vec{x})$ be a formula of the language of σ, let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be a tuple taken from $\Pi S_i/U$, and let $a_1, \ldots, a_n$ be representatives in ΠS_i of $\alpha_1, \ldots, \alpha_n$. Then $\Pi S_i/U$ satisfies $f(\alpha_1, \ldots, \alpha_n)$ if and only if $\{i : S_i \models f(a_{1,i}, \ldots, a_{n,a})\}$ belongs to U.*

Proof. This is true by the definition of the ultraproduct if f is an atomic formula. We show the general case by induction on the complexity of f.

If $f(\vec{x}) = \neg g(\vec{x})$, then $\{i : S_i \models f(a_{1,i}, \ldots, a_{n,i})\}$ is the complement of $\{i : S_i \models g(a_{1,i}, \ldots, a_{n,i})\}$. If the first set is in U, its complement is not; by the induction hypothesis, $S \not\models g(\vec{\alpha})$, so $S \models f(\vec{\alpha})$. If it is not in U, its complement is, so by the induction hypothesis that $S \models g(\vec{\alpha})$, we get $S \not\models f(\vec{\alpha})$.

If $f(\vec{x}) = g(\vec{x}) \wedge h(\vec{x})$, then $\{i : S_i \models f(a_{1,i}, \ldots, a_{n,i})\} = \{i : S_i \models g(a_{1,i}, \ldots, a_{n,i})\} \cap \{i : S_i \models h(a_{1,i}, \ldots, a_{n,i})\}$; the intersection of these two sets is in the filter U iff both sets are in U, and, by the induction hypothesis, iff $S \models g(\vec{\alpha})$ and $S \models h(\vec{\alpha})$.

If $f(\vec{x}) = (\exists y) g(\vec{x}, y)$, first let us suppose that $S \models f(\vec{\alpha})$. Then there exists β in $\Pi S_i/U$ represented by the I-tuple $b = (\ldots, b_i, \ldots)$ such that $S \models g(\vec{\alpha}, \beta)$. By the induction hypothesis $\{i : S_i \models g(a_{1,i}, \ldots, a_{n,i}, b_i)\}$ belongs to U, as does its superset $\{i : S_i \models (\exists y) g(a_{1,i}, \ldots, a_{n,i}, y)\}$. Conversely, if the set $X = \{i : S_i \models (\exists y) g(a_{1,i}, \ldots, a_{n,i}, y)\}$ is in U, for every i in X we can find b_i such that $S_i \models g(a_{1,i}, \ldots, a_{n,i}, b_i)$; for i outside X take any element of E_i for b_i. (It was assumed in the definition of ultraproduct that all the structures S_i have nonempty universes E_i.) Let β be the class, modulo U, of the I-tuple $(\ldots, b_i, \ldots)$; by the induction hypothesis $S \models g(\vec{\alpha}, \beta)$, so $S \models (\exists y) g(\vec{\alpha}, y)$.

The cases for $\vee$ and $\forall$ are left to the reader; we can, however, systematically replace them by combinations of $\neg, \wedge, \exists$. $\square$

We see in particular that a sentence f is true in $\Pi S_i/U$ iff $\{i : S_i \models f\}$ belongs to U; in this case we often say that f is "true for almost all i modulo U," or that "the set of indices for which f is false is negligible modulo U."

Suppose that all the S_i are copies of the same structure S; their ultraproduct is called the *ultrapower* of S by U, denoted by S^U. If an element a of S is associated with the class modulo U of the constant I-tuple whose coordinates are all equal to a, we get an embedding of S into S^U that we call the *canonical diagonal embedding of S into S^U*; we most often identify S with its image S^U under this embedding.

An immediate consequence of Łoś's theorem is that *the canonical diagonal embedding is elementary*, that *S^U is an elementary extension of S*.

The reader has noticed that we have defined the ultraproduct only for structures with nonempty universes; if we wish not to exclude the possibility of empty universes, and to keep Theorem 4.3, we must proceed as follows:

- If the set J of indices i such that S_i has nonempty universe is in the ultrafilter U, let V be the ultrafilter of subsets of J that is the trace on J of U, and define $\Pi S_i/U$ to be equal to $\Pi S_i/V$.

- Otherwise, if it is the set of i such that S_i is empty that is in the ultrafilter, then $\Pi S_i/U$ has empty universe; the function symbols, and the relation symbols of positive arity, have only one possible interpretation on an empty base, and if r is a 0-ary relation symbol, it is interpreted as "true" if the set of i such that r_i is true is in the ultrafilter, and as "false" otherwise, in conformity with the preceding definition (for nonempty universes) of ultraproduct.

The reader will have no difficulty verifying that with these conventions, Łoś's theorem remains valid in its most general form.

4.2 Compactness, Löwenheim-Skolem Theorem, Theorem of Common Elementary Extensions

Let us fix a language L corresponding to a signature σ. We are going to equip the set $\mathcal{T}$ of complete theories in the language L with a topology: To each sentence f of this language we associate the set $\langle f \rangle$ of complete theories containing f; $T \in \langle f \rangle$ therefore means that $f \in T$, or equivalently that $T \models f$ (read "f is a consequence of T" or "T proves f"; we shall prefer the latter notation, so as not to overuse the symbol $\in$). As $\langle f \rangle \cap \langle g \rangle = \langle f \wedge g \rangle$, the $\langle f \rangle$ form an open base for a topology, the topology associated with $\mathcal{T}$.

The space $\mathcal{T}$ is a Hausdorff space; indeed, if $T \neq T'$, then since T and T' are *complete* theories, there is some sentence f such that $T \models f, T' \models \neg f$, and $\langle f \rangle$ and $\langle \neg f \rangle$ are disjoint neighborhoods of T and T', respectively.

Since the complement of $\langle f \rangle$ is $\langle \neg f \rangle$, the basic open sets are also closed: $\mathcal{T}$ is what is called a *0-dimensional* space (a space with a base of sets that are clopen, i.e., both open and closed); as a consequence, every connected subset of $\mathcal{T}$ either is empty or reduces to a point.

By definition, an *open set* of $\mathcal{T}$ is therefore a union of sets of the form $\langle f \rangle$, and a *closed set* is an intersection of sets of the form $\langle f \rangle$. The nonempty closed sets of $\mathcal{T}$ correspond exactly to incomplete theories in the language L: If F is the intersection of $\langle f_i \rangle$, we associate with it the theory Θ axiomatized by the f_i; F is exactly the set of complete theories containing Θ. (Note that f is a consequence of A iff every complete theory that contains A contains f.)

The following compactness theorem is one of the most essential to model theory; every effort has been made to preserve the finitary character of formulas, which guarantees its validity.

Theorem 4.4 (Compactness). *The space $\mathcal{T}$ of complete theories in a language L is a compact* 0*-dimensional space.*

Proof. We have seen that $\mathcal{T}$ is Hausdorff and 0-dimensional. It remains to be seen that every ultrafilter U of subsets of $\mathcal{T}$ is convergent, meaning that it is a refinement of the filter of neighborhoods of some point. For every T in $\mathcal{T}$, choose a model M_T of T, and let Θ be the theory of the model $M = \Pi M_T/U$. If A is a neighborhood of Θ, A contains some set $\langle f \rangle$, with$\Theta \models f$; by Łoś's theorem, $\{T : M_T \models f\} = \langle f \rangle$ is in U, as is A. □

A compact 0-dimensional space is often called a *Stone* space; it is the most natural generalization of the notion of a finite set with the discrete topology.

Another formulation, perhaps more telling, of Theorem 4.4 is the following:

Theorem 4.5 (Compactness). *For a set A of sentences to be consistent, it is sufficient that every finite subset of A be consistent.*

Proof. In fact, for a Hausdorff space, compactness is equivalent to the following property: If F_i is a family of closed sets all of whose finite subfamilies have nonempty intersection (we say that such a family of closed sets has the "finite intersection property"), then the intersection of all the F_i is nonempty, and we may naturally suppose that the F_i are basic sets, of the form $\langle f_i \rangle$. This is exactly what the theorem states.

Another, more direct, demonstration: Let I be the set of finite subsets of A; we know that for every $i \in I$, the $I_i = \{j : j \in I, i \subseteq j\}$ form a filter base, contained in some ultrafilter U. For every i in I, there is a model M_i of i: By Łoś's theorem, we easily see that $\Pi M_i/U$ is a model of A. □

Other equivalent forms:

Theorem 4.6 (Compactness).

1. *If A is a contradictory set of sentences, then there is a contradictory finite subset of A.*

2. *If the sentence f is a consequence of A, then there is a finite subset A' of A such that f is a consequence of A'.*

Proof. Assertion (1) is equivalent to the previous theorem; to deduce (2), note that f is a consequence of A iff $A \cup \{\neg f\}$ is contradictory. □

The compactness theorem is applied constantly, so to speak, in model theory; we begin here with some striking examples.

Theorem 4.7. *The clopen sets of $\mathcal{T}$ are the sets of the form $\langle f \rangle$, where f is a sentence.*

Proof. By definition of the topology of $\mathcal{T}$, $\langle f \rangle$ is both open and closed. Conversely, let A be a clopen subset of $\mathcal{T}$. As A is open, it is a union of a nonempty family of sets $\langle f_i \rangle$ (even if A is empty, since $\emptyset = \langle f \wedge \neg f \rangle$ for any sentence f); as it is closed, it is compact, and therefore the union of a finite number of them, $A = \langle f_1 \rangle \cup \cdots \cup \langle f_n \rangle = \langle f_1 \vee \cdots \vee f_n \rangle$. □

We say that a (not necessarily complete) theory is *finitely axiomatizable* if it has a finite system of axioms, which amounts to saying that it can be axiomatized by a single axiom (take the conjunction of the previous axiom system). We therefore see that a finitely axiomatizable theory corresponds to a clopen set of $\mathcal{T}$: To say that a complete theory T is finitely axiomatizable is to say that it is an *isolated point* of $\mathcal{T}$. (If f axiomatizes T, then T is the only point in $\langle f \rangle$.)

Examples of finitely axiomatizable complete theories:

- the theory of a finite structure in a finite language;
- in the language of one binary relation, the theory of dense chains without endpoints.

In fact, the lists of axioms given in Section 2.3 are finite.

One consequence of compactness is the following: *If Θ is finitely axiomatizable, then from any axiomatization of Θ we can extract a finite axiomatization of Θ.* Suppose that Θ is axiomatized on the one hand by f, and on the other by the set of g_i; f is then a consequence of the g_i, and so, by compactness, of a finite number of g_i, which therefore suffice to axiomatize Θ.

This observation allows us to see easily that some theories that we have already studied are not finitely axiomatizable; for example, if the language L contains no other symbols than the equality relation, consider the axioms

$$A_n : \qquad (\exists x_1) \cdots (\exists x_n) \bigwedge_{0<i<j\leq n} x_i \neq x_j;$$

the complete theories T_n axiomatized by $A_n \wedge \neg A_{n+1}$, and T_∞ containing all the A_n. Then T_∞ is not finitely axiomatizable, as a finite number $A_1, \ldots, A_n$ of its axioms can never imply A_{n+1}. The T_n converge to T_∞. (We can easily see that the complete theories in this language are $T_0, \ldots, T_n, \ldots$ and T_∞; $\mathcal{T}$ is obtained by adding a point at infinity T_∞ to compactify the discrete space $T_0, \ldots, T_n, \ldots$)

Exercise 4.8. Show that a unary relation is finitely axiomatizable iff it has a finite universe. Describe the space $\mathcal{T}$ associated with the language of one unary relation. What are the first three derived spaces of $\mathcal{T}$?

Exercise 4.9. Show that an equivalence relation is finitely axiomatizable (i.e., that its complete theory is finitely axiomatizable) iff its universe is finite.

Theorem 4.10 (Löwenheim–Skolem Theorem). *If T is a theory, not necessarily complete, that has an infinite model, or even finite models of arbitrarily large (finite) cardinality, then for every cardinal $\kappa \geq \mathrm{card}(T)$, T has a model of cardinality κ.*

Proof. Let L be the language of T, let L' be the language obtained by adding κ constant symbols c_i to L, and let T' be the set of sentences consisting of T together with all the sentences $c_i \neq c_j$ for every pair (c_i, c_j) of distinct new constant symbols. A finite fragment of T' can mention only finitely many $c_1, \ldots, c_n$ of the constants, and we can get a model from a model M_n of T containing at least n elements (such a model exists by the hypothesis) by interpreting $c_i, \ldots, c_n$ by distinct elements of M_n, and the other c_i by anything.

As a consequence, by compactness, T' has a model M'. (Note: T' is in general not a complete theory, even if T is complete in the language L.) If we take the reduct of the structure M' in the original language L, forgetting the additional constant symbols, we get a model M of T with at least κ elements. It might have more than κ, and to get a model of cardinality κ we use Löwenheim's theorem (Theorem 3.1). □

Now consider a model M of a complete theory T, in a language L; let $L(M)$ be the language obtained by adding to L a constant symbol to name each element of M, and let $T(M)$ be the set of sentences of this new language that are true in M. To say that $f(a_1, \ldots, a_n)$ belongs to T_M, where $f(\vec{x})$ is a formula of L, means that $\vec{a}$ in M satisfies $f(\vec{x})$, that $M \models f(\vec{a})$; $T(M)$ is called the *diagram* of M. What some writers call the diagram of M is the set of quantifier-free sentences of $L(M)$ that are true in M; we shall call that set the *"free diagram"* of M. We see that by definition, an elementary extension of M is just a model of $T(M)$. Thus the Löwenheim–Skolem theorem has as a consequence that if M is infinite, then for every cardinal κ greater than or equal to both $\mathrm{card}(M)$ and $\mathrm{card}(L)$, M has an elementary extension of cardinality κ.

All this shows that elementary equivalence can characterize only finite structures up to isomorphism; for example, a structure on a denumerable universe, in a denumerable language, has elementary extensions of every higher cardinality.

Lemma 4.11. *If M and N are two elementarily equivalent structures, then they have a common elementary extension: There is some structure P with elementary embeddings of M into P and of N into P.*

Proof. Form the theories $T(M)$ and $T(N)$ so that the names a_i of elements of M are all distinct from the names b_i of elements of N; a common elementary extension of M and N is a model of $T(M) \cup T(N)$. (Note that two distinct constant symbols might be interpreted by the same element!) We must show that this set of sentences is consistent. As we can replace a finite set of sentences by the conjunction of all of its elements, a finite fragment of $T(M) \cup T(N)$ is composed of one sentence $f(\vec{a})$ of $T(M)$ and one sentence $g(\vec{b})$ of $T(N)$. Since $N \models g(\vec{b})$, it follows that $N \models (\exists \vec{y}) g(\vec{y})$; the latter sentence is in the language L, and, by the definition of elementary equivalence, M and N satisfy the same sentences of that language. So it is also true in M, and there exists $\vec{b}'$ in M such that $N \models g(\vec{b}')$. Interpreting $\vec{b}$ by $\vec{b}'$ makes M into a model of $f(\vec{a}) \wedge g(\vec{b})$; consequently, by compactness, $T(M) \cup T(N)$ is consistent. □

The following lemma is a little more precise. Actually, these two lemmas are two versions of the same result; the proofs are slightly different, and the reader may choose whichever method best suits her taste.

Lemma 4.12. *If M and N are elementarily equivalent structures, then M can be elementarily embedded into an ultraproduct of N.*

Proof. Let I be the set of injections from finite subsets of M to N. If $f(\vec{a})$ is a formula with parameters $\vec{a}$ in M, $M \models f(\vec{a})$, let $I_{f(\vec{a})}$ denote the set of such injections s whose universe contains $\vec{a}$ and such that $N \models f(s\vec{a})$. The set $I_{f(\vec{a})}$ is never empty, as $M \models f(a_1, \ldots, a_n)$, so $M \models (\exists x_1) \cdots (\exists x_n) f(x_1, \ldots, x_n) \wedge D(x_1, \ldots, x_n)$, where D is the conjunction of the formulas $x_i = x_j$ if a_i is equal to a_j, and $x_i \neq x_j$ if a_i and a_j are different, and N also satisfies this formula. On the other hand, $I_{f(\vec{a})} \cap I_{g(\vec{b})} = I_{f(\vec{a}) \wedge g(\vec{b})}$, so the $I_{f(\vec{a})}$ form a filter base, which can be extended to an ultrafilter U.

Define a function S from M to N^U as follows: If $a \in M$, the ith coordinate of Sa is ia if i is defined at a, and any element of N otherwise. (We are excluding the case of empty universes, which is trivial.) Note that $\{i : i \text{ is defined at } a\} = I_{a=a}$, and that changing the coordinates outside of $I_{a=a}$ will not change Sa modulo U, so S is well-defined. If a is different from b, then $I_{a \neq b}$ is in the ultrafilter U, and we can easily infer that S is an injection; also, thanks to Łoś's theorem, the reader will have no trouble checking that S is an elementary embedding from M to N^U. □

In fact, a much stronger result is true: *If two structures M and N are elementarily equivalent, there is an ultrafilter U such that M^U and N^U are isomorphic.* This is, however, a difficult theorem, whose final stage required the intervention of the hand of Shelah; it requires extensive familiarity with model theory and with ultrafilters. Besides, it is almost useless for model theorists. Indeed, we could take as a *definition* of elementary equivalence the existence of isomorphic ultrapowers; this would please the mind of a mathematician of algebraic bent, and would avoid any need to speak (at least in the definition) of formulas, local isomorphisms, etc., but would make horribly complicated the proofs of the simplest properties of elementary equivalence! We shall therefore skip this result, although it can bring minor simplifications to some of the proofs that we will encounter later.

This theorem of Shelah answers the problem of finding an "algebraic" (i.e., without speaking of logic, formulas, or satisfaction, all notions whose origins seem suspect to some mathematicians) characterization of elementary equivalence. At a very different, and much simpler, level, local isomorphisms with Fraïssé rank give another such answer. But Fraïssé's answer, unlike Shelah's, gives an efficient and immediate method for proving that two structures are elementarily equivalent.

Exercise 4.13. Show that the ordering of the real numbers is not isomorphic to an ultrapower of the ordering of the rationals. (Hint: $\mathbf{Q}^U$ is not complete.)

To conclude this section:

Theorem 4.14. *If M_i is a set of structures, all elementarily equivalent, then they have a common elementary extension: There is a structure M with an embedding of M_i into M for each i.*

Proof. By iterating Lemma 4.11 we can prove the theorem if the set of the M_i is finite; this proves that $\cup T(M_i)$, where the sets of names for the elements of the M_i are assumed to be disjoint, is finitely satisfiable, and therefore, by compactness, consistent, so it has a model M. □

4.3 Henkin's Method

Although ultraproducts are powerful enough, we are going to give another proof of the compactness theorem, following the method attributed to Leon Henkin. It consists in reducing the consistency of a set of sentences in a language L to that of a set of quantifier-free sentences in an extended language L^H (in this method a sentence is replaced by an infinity of quantifier-free sentences of L^H), thus reducing the general compactness theorem to compactness for quantifier-free sentences, which is almost immediate. Its essential characteristic is the association to formulas of new constant symbols, called their *witnesses*, and the addition of axioms guaranteeing that if there is x such that $f(x)$ is true, the witness of f is one such x.

Here we will treat only the case of a denumerable language, which is the most useful case; but the method remains valid in the nondenumerable case, and the reader will be shown, in brief, how it can be generalized. This method has the advantage, if the language is denumerable, of introducing only finite or denumerable structures, unlike the ultraproduct method; because of this feature, we shall see that it lends itself to being coded into arithmetic (see Chapter 7). It is also quite useful for proving the omitting types theorem (see Chapter 10).

This method is, in fact, older than the definition of ultraproducts. Here is how our forefathers arrived at the compactness theorem: They formulated rules of inference allowing everyone to pass from one finite set of sentences (the "antecedents") to another sentence (the "consequent"). The statement f is said to be *provable* from A if there is a sequence $S_1, \ldots, S_n$ of derivations such that for every $i \leq n$ the antecedents of S_i are in A, or are consequents of some $S_j, j < i$, and such that f is the consequent of S_n. The set A is called *contradictory* if $f \wedge \neg f$ if provable from A for some f. As a proof can involve only finitely many elements of A, it is clear that A is noncontradictory iff every finite subset of A is noncontradictory. From a simple examination of the rules of proof, it was clear that a consistent set cannot be contradictory, and that if f is provable from A, it is a consequence of A (meaning, we recall, that every model of A is a model of f). The hard part was the converse: It was necessary to show that if A is noncontradictory, then it has a model. This could be shown by Henkin's method, which allows us to construct, step by step, a model of A, unless we reach a contradiction; all that will be revealed in greater detail in the chapter on arithmetic.

First, let us settle the quantifier-free case of the compactness theorem. Let us first consider a denumerable $E = \{a_0, \ldots, a_n, \ldots\}$ and, for some integer m, the set R of all m-ary relations with universe E; given a quantifier-free formula $f(\vec{x})$ in the language consisting of one m-ary relation symbol, and a tuple $\vec{a}$ from E, $\langle f(\vec{a})\rangle$ is defined as the set of relations r in R satisfying $f(\vec{a})$. These sets $\langle f(\vec{a})\rangle$ form a base of clopen sets for a topology on R called the *topology of simple convergence* of m-ary relations with universe E: a neighborhood base of the relation r is formed by the sets $O_n(r)$, $O_n(r)$ being the set of relations that have the same restriction as r to the set $\{a_0, \ldots, a_{n-1}\}$.

For every finite subset F of E, let R_F denote the set of m-ary relations with universe F, and let ρ_F denote the restriction map from R to R_F. If $F \subset F'$, let $\rho_{F',F}$ denote the restriction map from $R_{F'}$ to R_F: If we equip the sets R_F with the discrete topology, the $\rho_{F',F}$ are continuous, and R appears as the inverse limit of the directed system formed by the R_F, so *R is a 0-dimensional compact space.*

If you wish to get a more concrete grasp of this inverse limit, you can show the compactness of R thus: Let ϕ_i be a family of closed sets with the finite intersection property. You want to find a relation in all the ϕ_i. You can assume that each ϕ_i is of the form $\langle f_i(\vec{a})\rangle$, since every closed set is by definition an intersection of sets of that form. Let ψ_n denote the intersection of all these closed sets that are of the form $\langle f(a_0, \ldots, a_{n-1})\rangle$, i.e., those involving only the first n elements of E. As there are only finitely many, and the family has the finite intersection property, we can find a relation in each ψ_n; what we want is to find a relation in all of them. Note that $\psi_1 \supset \psi_2 \supset \cdots \supset \psi_n \supset \cdots$, and that a relation being in ψ_n depends only on its restriction to $\{a_0, \ldots, a_{n-1}\}$. As there are only finitely many m-ary relations defined on this set, we can construct, by induction on n, a sequence r_n of m-ary relations such that:

- the universe of r_n is $\{a_0, \ldots, a_{n-1}\}$;
- r_{n+1} is an extension of r_n;
- for every $n' \geq n$ there exists a relation from R in $\psi_{n'}$ whose restriction to $\{a_0, \ldots, a_{n-1}\}$ is r_n.

So the relation r in R whose restriction to each $\{a_0, \ldots, a_{n-1}\}$ is r_n provides the answer.

We can also suppose that the set E is not denumerable: The set of m-ary relations with universe E, equipped with the topology of simple convergence, remains compact, as it is always a projective limit of finite discrete spaces. We can also introduce many relation symbols (let the structure be a multirelation) instead of a single m-ary relation, even infinitely many: The space of relational structures with universe E, of a given signature, will still be compact for simple convergence; in order to express it as a projective limit of finite discrete spaces, we must consider finite restrictions of reducts of these multirelations to a finite language (since a formula never involves more than finitely many relation symbols).

One more preliminary remark: We have systematically interpreted the symbol $=$ by true equality; here, for the sake of the following proof, we shall provisionally consider it an ordinary binary relation symbol. (If it mattered, we would see fit to change its name.) Suppose that the language contains only relation and constant symbols. Equality satisfies the following axioms:

$$(\forall x)x = x,$$
$$(\forall x)(\forall y)(x = y \rightarrow y = x),$$
$$(\forall x)(\forall y)(\forall z)(x = y \wedge y = z \rightarrow x = z),$$

and, for every relation symbol r,

$$(\forall x_1) \cdots (\forall x_n)(\forall y_1) \cdots (\forall y_n)\Big(\bigwedge x_i = y_i \rightarrow (r(x_1, \ldots, x_n) \rightarrow r(y_1, \ldots, y_n))\Big).$$

Conversely, if a relation $=$ satisfies these axioms in a structure S, then every relation of S induces a relation in the quotient space, and $S/=$ becomes a structure where $=$ is interpreted, as it has always been up to now, by true equality.

Theorem 4.15 (Compactness Theorem for Quantifier-Free Sentences). *For a set A of quantifier-free sentences, in a language consisting of relation and constant symbols but no function symbols, to have a model (where the symbol $=$ is as usual interpreted by real equality), it is sufficient that every finite subset of A have a model.*

Proof. Let E be the set of constants of L; the closure of the space of multirelations with universe E and arities corresponding to those of the relational symbols of L, including the symbol $=$, defined by the formulas $a = a$, $a = b \to b = a$, $a = b \wedge b = c \to a = c$, $(a_1 = b_1 \wedge \cdots \wedge a_n = b_n) \to (r(a_1, \ldots, a_n) \to r(b_1, \ldots, b_n))$, where a, b, c range over E, and r is a relational symbol of L, is compact. □

Theorem 4.15 is obviously a consequence of Theorem 4.4; here, we consider it a stage in the proof of Theorem 4.4.

We now start Henkin's method: Consider a denumerable, purely relational language L (i.e. no function or constant symbols; we can always transform our language into such), and a set A of sentences of L whose consistency we wish to test.

We shall first add to this language L denumerably many constant symbols, each with two indices $a_{i,j}$, i and j ranging over the set ω of natural numbers; the set $\{a_{0,0}, a_{1,0}, \ldots, a_{i,0}, \ldots\}$ we call E_0. The formulas $f(\vec{a}, x)$ with one free variable x and parameters $\vec{a}$ from E_0, with $f(\vec{y}, x)$ in L, are denumerable; choose an enumeration $f_0, f_1, \ldots, f_i, \ldots$ of them, and put them in one-to-one correspondence with the elements of $E_1 = \{a_{0,1}, a_{1,1}, \ldots, a_{i,1}, \ldots\}$, which we shall call their *witnesses*: The witness of f_0 shall be $a_{0,1}, \ldots$, the witness of f_i shall be $a_{i,1}$. Next we enumerate the formulas with parameters in $E_0 \cup E_1$, and in the same manner associate each one with a witness in $E_2 = \{a_{0,2}, \ldots, a_{i,2}, \ldots\}$, and continue, so that every formula with one free variable and parameters in $E_0 \cup E_1 \cup \cdots \cup E_{n-1}$ is associated, in a one-to-one manner, with a witness, which is a constant symbol in E_n, the set of all $a_{i,n}$. It matters little how we enumerate the formulas; what we need is to choose an enumeration once, for all of them, in such a way that the assignments of witnesses are well-defined. In summary, every formula $f(\vec{a}, x)$ with parameters $\vec{a}$ in $E = E_0 \cup E_1 \cup \cdots \cup E_n \cdots$ has a witness in E.

Given an L-structure S with finite or denumerable universe, a *Henkin enumeration* of S is defined as a function from E to the universe of S such that (identifying $a_{i,j}$ with its image in S):

- every element of the universe of S is of the form $a_{i,j}$;

- if $S \models (\exists x)f(\vec{a}, x)$, then $S \models f(\vec{a}, a_{f(\vec{a},x)})$, where $a_{f(\vec{a},x)}$ is the witness of the formula $f(\vec{a}, x)$.

Lemma 4.16. *Every nonempty L-structure that is finite or denumerable has a Henkin enumeration.*

Proof. We begin by enumerating the universe of S as $a_{0,0}, \ldots, a_{i,0}, \ldots$ (if S is final, repeat the last element); then for $a_{i,1}$ we take an element satisfying $f_i(x)$ if there is one; otherwise, take anything. Then continue. □

Note that in the definition of Henkin enumeration we do not require each element to be of the form $a_{i,0}$!

In addition to the constants of E, we also add a relation symbol, which we write $f^H(\vec{x})$, for each formula $f(\vec{x})$, the arity of f^H being the number of free variables of f; if f is a sentence, f^H is therefore a nullary relation symbol (see Chapter 3). To simplify our life, we shall consider only formulas without the quantifier $\forall$; this has no effect on the power of our language, as we can replace $\forall$ everywhere by $\neg\exists\neg$. Let L^H denote the language obtained by adding to L the constants of E and the relation symbols f^H.

Let $T(H)$ be the following set of sentences of $L(H)$, all quantifier-free, where the $\vec{a}$ and the $\vec{b}$ refer to arbitrary tuples of E:

- if $f(\vec{x})$ is an atomic formula of L: $f(\vec{a}) \leftrightarrow f^H(\vec{a})$;
- if f and g are in L: $(f \wedge g)^H(\vec{a}) \leftrightarrow f^H(\vec{a}) \wedge g^H(\vec{a}), (f \vee g)^H(\vec{a}) \leftrightarrow f^H(\vec{a}) \vee g^H(\vec{a})$;
- if f is in L: $(\neg f)^H(\vec{a}) \leftrightarrow \neg f^H(\vec{a})$;
- if $f(\vec{y})$ is in L of the form $f(\vec{y}) = (\exists x)g(\vec{y}, x)$: $f^H(\vec{a}) \leftrightarrow g^H(\vec{a}, a_{g(\vec{a},x)})$, where $a_{g(\vec{a},x)}$ is the witness of the formula $g(\vec{a}, x)$;
- for each formula $f(\vec{y}, x)$ of L: $f^H(\vec{a}, b) \to f^H(\vec{a}, a_{f(\vec{a},x)})$, where $a_{f(\vec{a},x)}$ is the witness of $f(\vec{a}, x)$.

Theorem 4.17. *A set of sentences $\{\ldots, f_i, \ldots\}$ in the language L has a nonempty model if and only if $T(H) \cup \{\ldots, f_i, \ldots\}$ is consistent.*

Proof. First suppose that the set of f_i has a nonempty model; by Löwenheim's theorem (3.1), whose proof does not use compactness, it has a finite or denumerable model, since L is denumerable. Let us interpret the $a_{i,j}$ by a Henkin enumeration of this model M, and $f^H(\vec{x})$ by the set of tuples from M that satisfy $f(\vec{x})$. In particular, this means that if f is a sentence, f^H is interpreted by "true" if f is satisfied by M, and by "false" otherwise. We can easily see that we thus get a model of $T(H) \cup \{\ldots, f_i^H, \ldots\}$.

Conversely, consider a model N of $T(H) \cup \{\ldots, f_i^H, \ldots\}$; in this model, there may be elements that are not of the form $a_{i,j}$. Let M be the restriction of N to the set of $a_{i,j}$: M is also a model of $T(H) \cup \{\ldots, f_i^H, \ldots\}$, since the sentences are all quantifier-free. I leave it to the reader to check, by induction on the complexity of formulas, that for every formula $f(\vec{x})$ of L, $M \models (\forall \vec{x})(f(\vec{x}) \leftrightarrow f^H(\vec{x}))$; you will need only the facts that M is a model of $T(H)$, and that it is surjectively enumerated by the $a_{i,j}$; you will eventually see that the $a_{i,j}$ constitute a Henkin enumeration of M! In particular, $M \models f_i^H \leftrightarrow f_i$, and as M satisfies each f_i^H, the reduct of M to the language L is a (nonempty!) model of all the f_i. □

Theorem 4.17 was proved under the hypothesis that L was denumerable; to prove it if $|L| = \kappa > \omega$ it suffices to modify the Henkin construction by introducing κ constant symbols in each set $E_0, \ldots, E_n, \ldots$.

Proof. [Proof of the Compactness Theorem by Henkin's method] Now let A be a set of axioms in the language L, every finite subset of which has a model. Start by eliminating the possibility of empty models by praying that Heaven will no longer put this invention of the Devil in our way. To do this, we add a constant c to our language, and transform each sentence f of A as follows: We replace every quantifier $(\exists x)$ by $(\exists x)x \neq c\wedge$, and each quantifier $(\forall x)$ by $(\forall x)x = c\vee$; let f' be the sentence thus obtained from f. Then f' expresses, plainly and simply, the fact that the elements distinct from c form a structure satisfying f: A model of f' is obtained by adding a point c to a model of f, and a model of f by removing the point c from a model of f'. We see, therefore, that A has a model iff the set A' of all the f' has one, which is necessarily nonempty!

Another transformation whose profoundly metaphysical character will not escape the alert reader: We can replace the function symbols of L by their graphs, which are relation symbols, by adding an axiom for each expressing that it is the graph of a function; also, we can replace the constants of L, as well as c, by unary predicates, by adding an axiom for each expressing that it is satisfied by one and only one element! We thus obtain a set A'' of sentences, in a purely relational language, whose finite subfamilies all have nonempty models.

Applying Theorem 4.17, we see that every finite subfamily of $T(H) \cup A''^H$ has a model; by Theorem 4.15, it is therefore a consistent theory; Theorem 4.17 applied to A'' yields that A'', and hence also A, is consistent. □

4.4 Historic and Bibliographic Notes

The use of filters and ultrafilters in topology was introduced by Henri Cartan, [Car37b] and [Car37a]; the axiom of ultrafilters was shown, assuming the axiom of choice, by Tarski [Tar30].

The construction of ultraproducts in a general framework is due to Łoś [Łoś55]; there he proved the theorem that now bears his name. However, ultraproducts had been used sporadically before that date, in particular in

[Sko34], in which an ultraproduct limited to definable objects was used to construct a nonstandard model of arithmetic.

The theorem on isomorphic ultrapowers was first broached in [Kei64]; the final touch came in [She71a].

For an introduction to model theory starting with ultraproducts, I recommend [BS69].

The compactness theorem, in the forms of Theorems 4.5 and 4.6, is due to Gödel [Göd30]; in fact, as explained at the beginning of Section 4.3, the theorem was for Gödel a simple corollary (we could even say an unexpected corollary, a rather strange remark!) of his "completeness theorem" of logic, in which he showed that a finite system of rules of inference is sufficient to express the notion of consequence (see Chapter 7). It could also have been taken from [Her28] or [Gen34], in which results of the same sort were proven.

This unfortunate compactness theorem was brought in by the back door, and we might say that its original modesty still does it wrong in logic textbooks. In my opinion it is a much more essential and primordial result (and thus also less sophisticated) than Gödel's completeness theorem, which states that we can formalize deduction in a certain arithmetic way; it is an error of method to deduce it from the latter.

If we do it this way, it is by a very blind fidelity to the historic conditions that witnessed its birth. The weight of this tradition is apparent even in a work like [CK73], which was considered the bible of model theory in the 1970s; it begins with syntactic developments that have nothing to do with anything in the succeeding chapters. This approach—deducing Compactness from the possibility of axiomatizing the notion of deduction—once applied to the propositional calculus gives the strangest proof on record of the compactness of 2^ω!

It is undoubtedly more "logical," but it is inconvenient, to require the student to absorb a system of formal deduction, ultimately quite arbitrary, which can be justified only much later when we can show that it indeed represents the notion of semantic consequence. We should not lose sight of the fact that the formalisms have no raison d'être except insofar as they are adequate for representing notions of substance.

The Löwenheim–Skolem theorem (4.10) is very ill-named; in the complete form given here, it is in fact due to Tarski (see [TV56], which is undoubtedly not the oldest reference). As explained in the notes to Chapter 2, Löwenheim's theorems (2.5 and 3.1), could legitimately be styled the Löwenheim–Skolem theorems; they are often called the downward Löwenheim–Skolem theorems.

Tarski's contribution was to raise the cardinalities, to build arbitrarily large models. Hence the name upward Löwenheim–Skolem Theorem, which is sometimes applied to this result. Legend has it that Thoralf Skolem, up until the end of his life, was scandalized by the association of his name to a result of this type, which he considered an absurdity, nondenumerable sets being, for him, fictions without real existence.

The notion of diagram was used specifically by Abraham Robinson [Rob63].

Henkin's method, involving constructing models by naming constants and specifying at each stage the relations among them, appears in [Hen49].

5

The Back-and-Forth Method in ω-Saturated Models

Le méchant goût du siècle en cela me fait peur;
Nos pères, tous grossiers, l'avaient beaucoup meilleur
Et je prise bien moins tout ce que l'on admire
Qu'une vieille chanson que je m'en vais vous dire:

...

La rime n'est pas riche et le style en est vieux:
Mais ne voyez-vous pas que cela vaut bien mieux
Que ces colfichets ...

J. B. P.

5.1 Spaces of Types

Two elements a and b in L-structures (structures whose similarity type is that of L) are said to have the same type if they satisfy the same formulas $f(x)$ in L; the *type* of a is the set p_a of formulas $f(x)$ that a satisfies. What is a type but a complete theory in the language $L(x)$ obtained by adding to L a constant symbol x? For reasons of convenience, we write this constant symbol here as a variable.

Given the language L, the space S_0 of 0-types (implicitly, of the language L) is defined as the space $\mathcal{T}$ of complete theories in the language L; the space S_1 of 1-types is that of complete theories in the language $L \cup \{x\}$, an open base for which is the set of all $\langle f(x) \rangle$, the set of p in S_1 such that p

satisfies $f(x)$, where $f(x)$ is a formula in the language L with at most one free variable.

Likewise, we define the set S_n of *n-types*, or types of n-tuples, as the set of complete theories in $L \cup \{x_1, \ldots, x_n\}$; we can even define the set S_I of I-types, where I is an infinite set, by introducing what we call a "type variable": a new constant symbol x_i for each element of I. As we saw in Section 4.2, all these spaces are compact and 0-dimensional.

The convention is that when we say "types" without specifying anything else, we are referring to 1-types, and we are often satisfied with expressing results about 1-types that generalize without problem to n-types or I-types.

If T is a theory, we write $S_1(T), \ldots, S_n(T), \ldots$ to denote sets of types of 1-tuples,..., of n-tuples,... in models of T; as these types are the ones that satisfy the axioms of T, $S_n(T)$ is a closed subset of S_n, *and therefore compact.* This notation is, quite peculiarly, used when T is a complete theory; $S_0(T)$ is then reduced to a single element.

Given a complete theory T, a *set of parameters* is a subset A of a model M of T; what is relevant about this notion is not so much what the set A is, as what formulas its elements satisfy in M. Let us name every element of A by a constant symbol, and let $T(A)$ be the set of sentences, in the language of T thus augmented, that are true in M; the set of parameters is more accurately considered coming from $T(A)$ than just from A. It is apparent that if N is an elementary extension of M and A is a subset of M, then the set of parameters A, considered as a subset of M, can be identified with A, considered as a subset of N.

Given a formula $f(x, \vec{y})$ in the language L of T, and $\vec{a}$ in A, $f(x, \vec{a})$ is called a *formula with parameters in* A; this means nothing more than a formula of $L(A)$. A *type over* A is defined as a complete theory in the language $L(A) \cup \{x\}$ that contains $T(A)$; two elements have the same type over A if they satisfy the same formulas with parameters in A: They are, in a sense, similarly placed with respect to A.

$S_1(T(A))$, or simply $S_1(A)$ if the theory T is clear from the context, denotes the set of types over A; as usual, we equip it with the topology defined by the formulas for which an open base consists of the $\langle f(x, \vec{a}) \rangle = \{p : p \models f(x, \vec{a})\}$, and which is compact and 0-dimensional. We define the spaces $S_n(A)$ and $S_I(A)$ similarly. In this context, when T is complete, $S_n(T)$ becomes $S_n(\emptyset)$: We call these types *without parameters*, or *pure* types, or *absolute* types.

If A is a subset of M, where M is a model of the complete theory T, and if p belongs to $S_1(A)$, we say that p is *realized in* M, or that M *realizes* p, if there is an element of M with type p; otherwise, we say that M *omits* p. A type is consistent by definition, so it is realized by some element x of some model N of $T(A)$; as this is a complete theory in the language $L(A)$, by Lemma 4.11 the structures M and N have a common elementary extension for the language $L(A)$ (i.e., A is interpreted in the same way in all three structures). So we can conclude that *every type* p *of* $S_1(A)$ *is realized* *in some elementary extension of M*; however, if M is not rich enough in elements, it might be omitted in M.

In a slightly regrettable (because it can create some confusion) way, we also use the word "realized" in an absolute sense: We say that a type $p \in S_1(A)$ is *realized* if it is realized in A, i.e., if it is the type of an element a of A, if it contains the formula $x = a$.

Given a model M of the complete theory T and a subset A of M, how do we see that a set of formulas with parameters in A is a type—in other words, is complete and consistent? The latter property is especially delicate: *A set of formulas with parameters in A is consistent if and only if every finite fragment of p is realized by an element of M*. In fact, by compactness, it is sufficient to check that every finite subset $\{f_1(x, \vec{a}), \ldots, f_n(x, \vec{a})\}$ is consistent, that it is realized in some elementary extension of M; this means that $T(A)$, augmented by the sentence $(\exists x)(f_1(x, \vec{a}), \ldots, f_n(x, \vec{a}))$, is consistent, and as this last sentence is in $L(A)$, the "constant" x having been quantified, it is a consequence of $T(A)$, which is complete, so it is true in M, which is a model of $T(A)$, so there is x in M satisfying the formulas $f_1(x, \vec{a}), \ldots, f_n(x, \vec{a})$. From the topological viewpoint, this fact can be interpreted thus: *If T is a complete theory and A a set of parameters in a model M of T, the types of $S_1(A)$ that are realized in M form a dense subset of $S_1(A)$*; in fact, if $\langle f(x, \vec{a}) \rangle$ is a neighborhood of p, this formula is satisfied by an element of M whose type is in this neighborhood.

5.2 ω-Saturated Models

A model M of a complete theory T is called ω-saturated (we shall see later, in Chapter 9, the more general definition of a κ-saturated model) if for every finite subset $\vec{a}$ (which we shall write as a tuple) of M, every type in $S_1(\vec{a})$ is realized in M. The interest in ω-saturated models lies in the following two theorems:

Theorem 5.1. *Every structure has an ω-saturated elementary extension.*

Proof. For every p of $S_1(\vec{a})$, where $\vec{a}$ is a finite subset of M, there is an elementary extension M_p of M that realizes p; by the theorem on common elementary extensions, Theorem 4.14, applied to the theory $T(M)$ (so that the image of M in all the embeddings will be the same), all the M_p are elementarily embedded in one elementary extension M_1 of M, which therefore realizes all types over finite subsets of M. Repeating this process yields a sequence $M \prec M_1 \prec \cdots \prec M_n \prec M_{n+1} \prec \cdots$ of elementary extensions such that every type over a finite subset of M_n is realized in M_{n+1}. The limit N of this chain of elementary extensions is ω-saturated. $\square$

Theorem 5.2. *If two structures M and N are elementarily equivalent and ω-saturated, they are ∞-equivalent: More precisely, two tuples of the* *same type (over $\emptyset$), one in M and the other in N, can be matched up by an infinite back-and-forth construction.*

Proof. Let $\vec{a}$ in M and $\vec{b}$ in N have the same type. Add α to M, for example, and let p be the type of α over $\vec{a}$ (i.e., the type p in $S_1(\vec{a})$ that α realizes). Consider the set q of formulas with parameters in $\vec{b}$, obtained by replacing $\vec{a}$ by $\vec{b}$ in each formula $f(x,\vec{a})$ of p. Then q is consistent: In fact, if $f(x,\vec{a})$ is a finite fragment of p, then $M \models (\exists x) f(x,\vec{a})$, so $N \models (\exists x) f(x,\vec{b})$, since $\vec{a}$ and $\vec{b}$ have the same type, and every finite fragment of q is consistent. Also, for every formula $f(x,\vec{b})$, q contains $f(x,\vec{b})$ or $\neg f(x,\vec{b})$, and q is a type over $\vec{b}$. As N is ω-saturated, this type is realized by an element β of N, and $\vec{a}\frown\alpha$ and $\vec{b}\frown\alpha$, which satisfy the same formulas, have the same type over $\emptyset$. The addition of an element to $\vec{b}$ is treated the same way. □

Remarks.

1. If M and N are ∞-equivalent and M is ω-saturated, then N is also ω-saturated.
   In fact, for every $\vec{b}$ in N, there exists $\vec{a}$ in M such that $(M,\vec{a})$ and $(N,\vec{b})$ are ∞-equivalent, so $\vec{a}$ and $\vec{b}$ have the same type. Let q be in $S_1(\vec{b})$, and let p be the set of formulas obtained by replacing $\vec{b}$ by $\vec{a}$ in the formulas satisfied by q: By the same reasoning as in the previous proof, p is a type, which is therefore realized by an element α of M; there exists β in N such that $\vec{a}\frown\alpha$ and $\vec{b}\frown\beta$ are ∞-equivalent, and therefore have the same type over $\emptyset$: β realizes q.

2. If M is ω-saturated, then for every $\vec{a}$ of M and every p of $S_n(\vec{a})$, p is realized in M: It suffices to note that to realize the type of (a_1, a_2) is to realize the type of a_1 and then to realize the type of a_2 over a_1, and to repeat.
   An ω-saturated model therefore realizes all absolute n-types for all n. This condition, however, is not sufficient for a model to be ω-saturated. Example: Let T be the theory of discrete order without endpoints; we have no difficulty seeing that M is ω-saturated iff it has the form $\mathbf{Z} \times \mathbf{C}$ where $\mathbf{C}$ is a dense chain without endpoints, while it realizes all pure n-types iff it has the form $\mathbf{Z} \times \mathbf{C}$ where $\mathbf{C}$ is an infinite chain.

3. If T is a complete theory and M is an ω-saturated model of T, then every denumerable model N of T can be elementarily embedded in M. In fact, if $N = \{a_0, a_1, \ldots, a_n, \ldots\}$, we can successively realize, in M, the type of a_0, then the type of a_1 over $a_0, \ldots$ the type of a_{n+1} over $(a_0, \ldots, a_n), \ldots$.

4. By Theorem 1.14, two denumerable, elementarily equivalent, ω-saturated structures are isomorphic.

Under what conditions does a complete theory T have a (unique) ω-saturated denumerable model? That happens if and only if for every n, $S_n(T)$ is (finite or) denumerable. (Here, we do not assume that T is denumerable.)

In fact, this condition further implies that for every $\vec{a}$ in a model M of T, $S_1(\vec{a})$ is denumerable (because to say that b and c have the same type over $\vec{a}$ is to say that $\vec{a}\frown b$ and $\vec{a}\frown c$ have the same type over $\emptyset$). It is clearly necessary, because a denumerable model can realize only denumerably many n-types. To see that it is sufficient, repeat the proof of Theorem 5.1: Let A_1 be a denumerable subset of M that realizes all 1-types over $\emptyset$; then let A_2 be a denumerable subset of M that realizes all 1-types over finite subsets of A_1 (this is actually a denumerable family of types); etc. Let $A = \bigcup A_n$; A satisfies Tarski's test (Theorem 2.4), so it is an elementary submodel of M, and therefore a denumerable model of T, which is by construction ω-saturated.

Now we return to local isomorphisms, and find it convenient to take a pause to see how far we have come, to lay out a sort of balance sheet of what we have done. We introduced elementary equivalence between relations by Fraïssé's back-and-forth conditions; then we interpreted it in terms of satisfaction of formulas; we generalized it to structures; we proved its compactness properties; and now we return to the back-and-forth conditions. In model theory, a frequent (and altogether fundamental!) problem is presented thus: A consistent set T of sentences is provided, and we want to prove that T is a complete theory. Trying to determine the p-equivalence classes, as we did for discrete orders in Chapter 1, is often extremely complicated; in some cases we can find only a sufficient condition for p-equivalence (which may be enough to yield results, but is neither very simple nor very useful), and we need to impose restrictions on the language (no functions, finite language).

It is cleverer to proceed as follows: *We make a conjecture about what types must be*, then consider two models M and N, which we can take to be ω-saturated by Theorem 5.1. We take $\vec{a}$ in M and $\vec{b}$ in N satisfying the presumed equivalence condition, and seek to establish infinite back-and-forth conditions between $\vec{a}$ and $\vec{b}$: We add α to $\vec{a}$ and look for β in N such that $\vec{a}\frown\alpha$ and $\vec{b}\frown\beta$ satisfy our hypothesis. The fact that N is ω-saturated helps give us the existence of β, because we will be able to prove only that the condition we want is finitely satisfiable in N. If that works, we have succeeded; if not, we may have left out a condition for two tuples to have the same type, or else T may not be complete!

This *back-and-forth method* will be illustrated in the following chapters; naturally, in each particular case, an ingenious proof will be needed to establish the back-and-forth conditions, often drawing on facts of an algebraic nature: Model theory does not deal only with the obvious problems! But things unroll according to this plan surprisingly often: The back-and-forth conditions are not just a bizarre way of introducing logic, but rather an effective method that model theorists use daily.

5.3 Quantifier Elimination

With the back-and-forth method, very often the hypothesis to be verified is that two tuples have the same type iff they satisfy the same formulas from some set F; thus the theorem below is of interest:

Theorem 5.3. *Let T be a theory, not necessarily complete, and let F be a nonempty set of formulas $f(\vec{x})$ in the language L of T, having for free variables only $\vec{x} = (x_1, \ldots, x_n)$, such that two n-tuples from models of T have the same type whenever they satisfy the same formulas of F. Then for every formula $g(\vec{x})$ of L in these variables, there is some $f(\vec{x})$ that is a Boolean combination of elements of F such that $T \models (\forall \vec{x})(f(\vec{x}) \leftrightarrow g(\vec{x}))$.*

Remark. The converse is obvious.

Proof. Consider the clopen set $\langle g(\vec{x}) \rangle$ in $S_n(T)$. If $\langle g \rangle = \emptyset$, then $\langle g \rangle = \langle f \wedge \neg f \rangle$, and if $\langle g \rangle = S_n(T)$, then $\langle g \rangle = \langle f \vee \neg f \rangle$, where f is an arbitrary element of F, which is nonempty. These trivial cases having been eliminated, consider p in $\langle g \rangle$ and q outside of it. As p and q do not satisfy the same formulas of F, it follows that $p \models f_{p,q}(\vec{x}), q \models \neg f_{p,q}(\vec{x})$, where $f_{p,q}$ is some formula of the form ϕ or the form $\neg\phi$, with ϕ in F.

Keeping p fixed and varying q, all the $\langle f_{p,q} \rangle$ and $\neg\langle g \rangle$ form a family of closed sets whose intersection is empty; by compactness, one of its finite subfamilies must have empty intersection, meaning that for some $h_p = f_{p,q} \wedge \cdots \wedge f_{p,q_n}, p \in \langle h_p \rangle \subset \langle g \rangle$.

Now, when we vary p, $\langle g \rangle$ is a compact set that is covered by the open sets $\langle h_p \rangle$, so a finite number of them are enough to cover it; the disjunction of these h_p is equivalent, modulo T, to g. □

In the particular case where for every $n > 0$ we can take for F the quantifier-free formulas, we say that the theory T *eliminates quantifiers* or *admits quantifier elimination.* That means, therefore, that the following conditions are equivalent:

- For every formula $f(\vec{x})$, where $\vec{x} = (x_1, \ldots, x_n), n > 0$, there is some formula $g(\vec{x})$ without quantifiers such that $T \models (\forall \vec{x})(f(\vec{x}) \leftrightarrow g(\vec{x}))$.

- For every $n > 0$, two n-tuples from models of T that satisfy the same quantifier-free formulas have the same type.

Examples of theories admitting quantifier elimination: that of infinite sets; that of an equivalence relation with infinitely many infinite classes; that of a dense order without endpoints.

Some authors neglect the precaution of taking $n > 0$ in the definition of quantifier elimination, and demand in addition that every sentence be equivalent modulo T to a quantifier-free sentence; that requires, naturally, that the set of sentences without quantifiers be nonempty, meaning that the language involves constant symbols, or else nullary relation symbols. They therefore do not accept the three examples above, strictly speaking, as admitting quantifier elimination. Others acknowledge this elimination by pretending that a true sentence for these theories (which are complete) is equivalent to the formula $x = x$, which is always true no matter what the interpretation of x, and that a false sentence is equivalent to $x \neq x$; this means that they allow the equivalence of a sentence and a formula with one free variable! That has always caused discomfort (and what happens if all models of T are empty?) to the present author, who sticks to the definition above, *which is the one we must take for the theorems about quantifier elimination that we shall prove to be correct.*

In the three examples given above, we note that although there are no quantifier-free sentences, it is correct that two 0-tuples taken from models of T that satisfy the same quantifier-free sentences have the same type: In fact, the type of a 0-tuple taken from the structure M is nothing but the theory of M; T being complete, there is but a single 0-type! (See the oratorical precaution "F nonempty" in Section 5.3.)

Note also that if $n > 0$ (to avoid cracking our skulls!) and if two $(n+m)$-tuples that satisfy the same quantifier-free formulas have the same type, then the same is true of two n-tuples that satisfy the same quantifier-free formulas. Indeed, if we associate the n-tuple $\vec{a}$ with the $(n+m)$-tuple $\vec{a}'$ obtained by repeating its last element m times, we see, since the formula $x = y$ is quantifier-free, that $\vec{a}$ and $\vec{b}$ satisfy the same quantifier-free formulas iff $\vec{a}'$ and $\vec{b}'$ satisfy the same quantifier-free formulas, and that $\vec{a}$ and $\vec{b}$ have the same type iff $\vec{a}'$ and $\vec{b}'$ have the same type.

There is a canonical way to associate a theory T in a language L with a theory T' in an expanded language L' that admits quantifier elimination: In fact, we used it in applying Henkin's method in Section 4.3. The language L' is obtained by adding to L a relational symbol $f'(\vec{x})$ for every formula $f(\vec{x})$ of L; T' consists of T together with the axioms $(\forall\vec{x})(f(\vec{x}) \leftrightarrow f'(\vec{x}))$. We can see that the reduct to L of every model of T' is a model of T, and that a model of T extends uniquely to a model of T': Models of T and models of T' are the same things except for the language.

The theory T' admits quantifier elimination, because every formula $g(\vec{x})$ of L' is equivalent to a formula $f(\vec{x})$ of L (obtained by replacing every f' by its f), which is in turn equivalent to $f'(\vec{x})$, which is quantifier-free. We therefore see that an extension of models of T is elementary if and only if there is also an extension for the corresponding models of T'. As with all the canonical constructions, this one has only technical interest.

We say that a theory T is *model complete* if it has the following property: If M and N are models of T, and if M is an extension of N, then this extension is elementary. A model complete theory is not necessarily complete; two models of T that are not elementarily equivalent will simply not have any common extensions. If T admits quantifier elimination, it is certainly model complete, because the satisfaction of a quantifier-free formula is preserved by extensions!

We say that two theories T_1 and T_2, in the same language L, are *companions* if every model of one can be embedded (not necessarily elementarily!) into a model of the other; we see that this implies the following theorem:

Theorem 5.4. *Two theories are companions of each other if and only if they have the same universal consequences (a sentence being called universal if it is of the form* $(\forall x_1)\cdots(\forall x_n)f(x_1,\ldots,x_n)$*, with* f *quantifier-free).*

Proof. A universal sentence f that is true in a structure is also true in its substructures; if $T_1 \models f$, and if there is a model of T_2 that does not satisfy f, it cannot be extended to a model of T_1.

Conversely, suppose that T_1 and T_2 have the same universal consequences, and let M_1 be a model of T_1. We name each element of M_1 by a new constant symbol, and let $D(M_1)$ be the set of all *quantifier-free* sentences in the new language that are true in M_1. If $D(M_1) \models f(a_1,\ldots,a_n)$, then $M \models (\exists\vec{x})f(\vec{x})$, so $(\forall\vec{x})\neg f(\vec{x})$ is not a consequence of T_1, and therefore not of T_2, which has the same universal consequences. There is therefore some model M_2 of T_2 with $\vec{b}$ in M_2 such that $M \models f(\vec{b})$. By compactness, this means that $D(M_1) \cup T_2$ is consistent, in other words, that M_1 embeds into a model of T_2. □

A theory T therefore has a minimal companion, which we shall denote by $T_\forall$, and which is axiomatized by the universal consequences of T. This suggests that the terminology was badly chosen; next, it gets even worse.

We say that the theory T' is a *model companion* of T if it is a companion of T that is model complete.

Theorem 5.5. *A theory has at most one model companion.*

Proof. Let T_1 and T_2 be model companions of T. Therefore, T_1 and T_2 are companions (and even model companions!). Let M_1 be a model of T_1; it embeds into a model N_1 of T_2, which embeds into a model M_2 of T_1, etc. We get the chain $M_1 \subset N_1 \subset M_2 \subset N_2 \subset \cdots \subset M_n \subset N_n \subset M_{n+1} \subset \cdots$, whose limit we call P. As T_1 is model complete, the chain of the M_n is elementary, and P is an elementary extension of M_1; as T_2 is model complete, the chain of the N_n is elementary, and P is likewise an elementary extension of N_1, and hence a model of T_2. Therefore, M_1 is also a model of T_2; by symmetry we see that T_1 and T_2 have the same models, meaning that $T_1 = T_2$. □

We say that T' is a *model completion* of T if it is a model companion of T and also the following condition is satisfied: If M is a model of T, embedded on the one hand into a model M_1 of T' and on the other hand into a model M_2 of the same T', then a tuple $\vec{a}$ of M satisfies the same formulas in M_1 and in M_2.

Naturally, a model complete theory is its own model completion, and it is clear that a theory that admits quantifier elimination is the model completion of every one of its companions. A theory is the model completion of every one of its companions iff it is the model completion of the weakest of them all, $T_\forall$. Also, the following theorem holds:

Theorem 5.6. *The model completion of a universal theory (i.e., one that is axiomatized by universal sentences) admits quantifier elimination.*

Proof. Let $\vec{a}$ and $\vec{b}$, satisfying the same quantifier-free formulas, be in two models M_1 and M_2 of this theory T', and let N_1 and N_2 be the substructures of M_1 and M_2 generated by $\vec{a}$ and $\vec{b}$, respectively. They are then isomorphic models of $T_\forall$, i.e., of T, and we can consider them as two extensions of the same model N of T. By definition of model completion, $\vec{a}$ in M_1 and $\vec{b}$ in M_2 satisfy the same formulas, so they have the same type in the sense of T'. □

Note that the theory of dense orders without endpoints is, without any possible doubt, the model completion of the theory of chains, and that it must admit quantifier elimination!

Model completion is the ink bottle of garrulous model theorists, who see in it a fundamental contribution of model theory to algebra. They love to illustrate their presentations with arrows scattered throughout, like the people occupied with categories; they value the expression "quantifier elimination" as a magic spell. In their defense, let us say that systematic research into model companion, when it exists, can provide the subject for a presentable theory; here, we will content ourselves with a few theorems of practical interest, to be proven in the course of the text.

5.4 Historic and Bibliographic Notes

The notions of type and of κ-saturated model were worked out in the 1950s; the first systematic exposition is in [MV62]. The term "quantifier elimination" is due to Tarski [Tar35]. The notions of model completion and of model companion are from A. Robinson [Rob56b].

6

Examples Illustrating the Back-and-Forth Method

> Lors fu li conseils des barons telx que il se hebergeroient entre le palais de Blaquerne et le chastel Buimont, qui ere une abaîe close de murs. Et lors furent tendu li tref et li paveillon, et bien fu fiere chose a regarder: que de Costantinoble, qui tenoit III liues de front par devers la terre, ne pot tote l'ost assegier que l'une des portes. ... Et mult estoient perillousement, que onques par tant poi de gent ne furent assegié tant de gent en nulle ville.
>
> J. de V.

The five examples in this chapter illustrate the method described in the previous chapter; they will also serve to provide examples for the rest of the course. The author does not claim that these are the most important applications of model theory to algebra or to other branches of mathematics, or the most fruitful, and he recognizes that he has chosen cases where the back-and-forth method applies particularly well.

6.1 Algebraically Closed Fields

We associate a (commutative) field with the structure formed by its zero (written 0), its identity (written 1), its "negative" function ($x \mapsto -x$), and its two binary operations, addition and multiplication. The language of fields therefore contains two constant symbols, one unary function symbol, and two binary function symbols. I leave to the reader the task of axiomatizing the notion of field in this language, without forgetting the axiom $0 \neq 1$; note that since the function $x \to x^{-1}$ has not been included in the language, a substructure of a field is not a subfield, but rather a subring.

We can easily see that a term in the variables $x_1, \ldots, x_n$ corresponds to a polynomial $P(x_1, \ldots, x_n)$, with integer coefficients, which is written with those parentheses omitted that are made superfluous by the laws of associativity. An atomic formula has the form $P(\vec{x}) = Q(\vec{x})$, where P and Q are two polynomials; it will help to replace this by $P(\vec{x}) - Q(\vec{x}) = 0$. The atomic formulas are therefore of the form $P(\vec{x}) = 0$, so we call them *equations*; an *inequation* $P(\vec{x}) \neq 0$ is the negation of an atomic formula. A quantifier-free formula is a Boolean combination of equations, which can be put in the form of a disjunction of conjunctions of equations and inequations; a conjunction of equations and inequations is often called a *system* of equations and inequations.

As everyone knows, the map from the ring $\mathbf{Z}$ of integers to the field K that sends n to the element $1 + 1 + \cdots + 1$ of K, the sum being taken n times (we can denote this element by n; n is not strictly speaking part of the language of fields, but merely an abbreviation for $1 + \cdots + 1$) is a ring homomorphism. As its image is integral, its kernel is a prime ideal, which may be $\{0\}$, in which case we say K has *characteristic zero*, or may be $p\mathbf{Z}$, where p is a prime number, in which case we say K has *characteristic* p.

We express the fact that K has characteristic p by a single axiom, $p = 0$; we then know that K is an extension of the p-element field F_p. To express that K has characteristic zero, we need infinitely many axioms: $p \neq 0$ for every p; as a finite set of these axioms does not imply the others, the theory of fields of characteristic zero is not finitely axiomatizable (see Section 4.2). We remark in passing that compactness also yields that a sentence f of the theory of fields that has models of characteristic p for arbitrarily large p (or equivalently, for infinitely many p) has a model of characteristic zero. If K has characteristic zero, it is an extension of $\mathbf{Z}$, and therefore also of its field of fractions $\mathbf{Q}$.

Given a field k and two elements a and b in field extensions of k, we shall say that a and b are k-similar if the fields $k(a)$ and $k(b)$ that they generate are isomorphic by an isomorphism fixing k pointwise and sending a to b.

How do we see that two elements have this property? We consider the canonical surjection from $k[X]$, the ring of polynomials in one variable X with coefficients in k, onto $k[a]$, the ring generated by k and a, which consists in replacing the unknown by a in every polynomial, and we call its kernel $I_{a/k}$ the *ideal of equations of* a *over* k. As $k[a]$ is integral, this is a prime ideal, and conversely, if I is prime, it is the ideal of equations of the image of X in $k[X]/I$, embedded into its field of fractions. If two elements are similar, they satisfy the same equations, so $I_{a/k}$ determines both $k[a]$, which is isomorphic to $k[X]/I_{a/k}$, and $k(a)$, the field of fractions of $k[a]$.

A similarity class is therefore a prime ideal: Two elements are similar over k iff they satisfy the same equations with coefficients in k iff they satisfy the same quantifier-free equations with coefficients in k.

The ring $k[X]$ is Euclidean, and the description of its ideals is easy, as they are all principal:

- If $I_{a/k} = 0$, we say that a is *transcendental* over k; $k[a]$ is isomorphic to $k[X]$, and $k(a)$ is isomorphic to the field $k(X)$ of rational functions in one variable with coefficients in K.
- Otherwise, $I_{a/k}$ is generated by an irreducible polynomial $P_{a/k}$, which we call a *minimal polynomial* of a over k; in this case we say that a is *algebraic* over k, and we have $k[a] = k(a) = k[X]/P_{a/k}$.

We also know that a field k is called *algebraically closed* if every nonconstant polynomial in one variable with coefficients in k has a zero in k. To axiomatize this notion, we need the following infinite set of axioms (one axiom for each possible degree of polynomials):

$$A_n : \qquad (\forall a_0) \cdots (\forall a_{n-1})(\exists x) x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

(Naturally x^n is an abbreviation for $x \cdots x$, the product being taken n times; our language has no exponentiation symbol.)

Exercise 6.1.

1. For every p construct a field of characteristic p that is not algebraically closed and that satisfies $A_1, \ldots, A_n$. (Work in algebraic extensions of F_p, which are very familiar.)
2. Show that there is a field of characteristic zero with the same property.

Lemma 6.2. *An algebraically closed field is infinite.*

Proof. $(X - a_1) \cdots (X - a_n) + 1$ has no root among $a_1, \ldots, a_n$. □

Lemma 6.3. *Every field has an algebraically closed extension.*

Proof. Let k be a field, let K be an extension of k, and let P be a nonconstant polynomial of $k[X]$. If Q is an irreducible divisor of P in $K[X]$, then $K[X]/Q$ is an extension of K in which P has a zero.

That being said, we proceed as follows: Enumerate all polynomials P_α with coefficients in k; the confusing part for the logic novice is that if k is not denumerable, we cannot enumerate these polynomials by ω, but will need a larger ordinal. Let us pass over this inconvenience. We begin by adding a zero to P_0 to get a field K_0 extending k in which P_0 has a zero, then we construct an extension K_1 of K_0 in which P_1 has a zero, etc. We thus construct, step by step, an increasing sequence of fields K_α, the polynomials $P_0, \ldots, P_\alpha$ all having zeros in K_α. Let L_1 be the union of all the K_α. Then L_1 is a field extension of k in which every nonconstant polynomial with coefficients in k has a zero. If you are temporarily allergic to transfinite recursion, then for every finite set $F = \{P_1, \ldots, P_n\}$ of polynomials of $k[X]$ you can build an extension $L_{1,F}$ of k in which each has a zero; then you can use compactness to show the existence of L_1.

Repeating this process, you construct an increasing sequence $k = L_0 \subset L_1 \subset \cdots \subset L_n \subset L_{n+1} \subset \cdots$ of fields, every polynomial of L_n having a zero in L_{n+1}. The field $L = \bigcup L_n$ is algebraically closed. □

Theorem 6.4. *The theory of algebraically closed fields admits quantifier elimination: It is the model completion of the theory of fields, and also of the theory of (commutative) integral domains. The theory of algebraically closed fields of a given characteristic is complete.*

Proof. Let us consider $\vec{a}$ and $\vec{b}$ in algebraically closed fields K and L satisfying the same quantifier-free formulas; we want to show that $\vec{a}$ and $\vec{b}$ have the same type. We can assume that K and L are ω-saturated; if not, we can replace K and L by ω-saturated elementary extensions, which are themselves algebraically closed, since this notion is elementary.

We must therefore show that $\vec{a}$ and $\vec{b}$ are ∞-equivalent. Note that the hypothesis implies that K and L have the same characteristic, as the sentences $p = 0$ and $p \neq 0$ are quantifier-free. In fact, it implies that the rings generated by $\vec{a}$ and $\vec{b}$, and hence also the fields k and h generated by $\vec{a}$ and $\vec{b}$, are isomorphic (in general $\vec{a}$ and $\vec{b}$ satisfy the same quantifier-free formulas iff they generate isomorphic substructures!). So we add an element α to (for example) $\vec{a}$. If α is algebraic over k, with minimal polynomial P, we use the isomorphism sending $\vec{a}$ to $\vec{b}$ to transform this polynomial into an irreducible polynomial of $h[X]$, which, as L is algebraically closed, has a zero β in L: $k(\alpha)$ and $h(\beta)$ are isomorphic.

Now, if α is transcendental over k, we must show that there exists β in L transcendental over h. But a polynomial of degree n has at most n zeros in a field (because if a is a root of P, $X - a$ divides P), so if $Q_1(X), \ldots, Q_s(X) \in h[X]$, since L is infinite (Lemma 6.2), there is an element of L that is not a root of any of these polynomials. Therefore the set of all the formulas $Q(x) \neq 0$, for $Q \in h[X], Q \neq 0$, because it is finitely satisfiable in L, is consistent with the theory of L; since L is ω-saturated, and all the parameters of these formulas are in $\vec{b}$, it is realized by an element β of L.

We have therefore proved that the theory T of algebraically closed fields admits quantifier elimination; it is therefore the model completion of all its companions, and in particular of $T_\forall$, which is the theory of integral domains. Indeed, on the one hand all the axioms of this theory are universal; on the other hand, every integral domain can be embedded in an algebraically closed field. (The theory of fields, in the language $(0, 1, -, +, \cdot)$, is not universal: We need a $\forall\exists$ axiom to express the existence of inverses.)

For the last point, $\emptyset$ satisfies the same quantifier-free sentences in K and in L iff it generates isomorphic substructures in K and L; this happens iff K and L have the same characteristic. □

Remarks.

1. We have an example of an incomplete theory admitting quantifier elimination; this requires the presence in the language of either constant symbols or 0-ary relation symbols; otherwise, the tuples $\emptyset$ in two l-structures satisfy the same quantifier-free sentences (they generate isomorphic substructures).

2. As quantifier elimination is just a matter of language, when we discuss it we need to specify the language used to axiomatize the structures under consideration. Here, this is $(0, 1, -, +, \cdot)$; we see that the language $(+, \cdot)$ is enough to axiomatize the notion of algebraically closed field, 0, 1, and $-$ being defined from addition and multiplication by formulas, but not enough for quantifier elimination. If on the other hand we add to $(0, 1, -, +, \cdot)$ a unary function symbol i for the function sending 0 to 0, and sending every nonzero x to its inverse, we change the notion of substructure (which now means subfield rather than subring), but do not really introduce new quantifier-free formulas, since the relation $y = ix$ is defined by the following quantifier-free formula in the language $(0, 1, -, +, \cdot)$: $x = y = 0 \vee xy = 1$.

A fundamental theorem of algebraic geometry, which is an immediate corollary of Theorem 6.4, is the following:

Theorem 6.5 (Hilbert Nullstellensatz). *If a (finite) system S of equations and inequations in several unknowns $x_1, \ldots, x_n$, with coefficients in a field k, has a solution in a field extension K of k, then it has a solution in any algebraically closed extension of k.*

Proof. Let L be an algebraically closed field containing k and let K_1 be an algebraically closed extension of K. Then $K_1 \models (\exists \vec{x})S(\vec{x})$; this last sentence has parameters in k, so by model completeness $L \models (\exists \vec{x})S(\vec{x})$. □

Remark. The definition of an algebraically closed field is essentially the Nullstellensatz for systems in just one unknown.

It is well known that a field k has a minimal algebraically closed extension, its *algebraic closure*, which is uniquely determined up to k-isomorphism; we are going to prove this result generalized to a model-theoretic framework.

Let T be a complete theory, and let A be a set of parameters from a model M of T; an element a of M is called *algebraic* over A if there is a formula $f(\vec{x})$, with parameters in A, that is satisfied by a and that is satisfied by only a finite number n of elements of M. (We say that the formula f *algebraizes* a.) For example, if a is in A, it is algebraized over A by the formula $x = a$. If $A \subset B \subset M$, we say that B is *algebraic* over A if every element of B is algebraic over A; finally, if every element of M that is algebraic over A is in A, we say that A is *algebraically closed.* (Temporarily, we specify: relative to M. Be careful; some ill-inspired model theorists use the word in a completely different sense, related to model completeness.)

Lemma 6.6. *If $A \subset B \subset C \subset M$, B is algebraic over A, and C is algebraic over B, then C is algebraic over A.*

Proof. Let c be in C, let $f(x, b_1, \ldots, b_m)$ be a formula with parameters in B algebraizing c over B, realized by exactly n elements of M, and let $f_1(y), \ldots, f_m(y)$ be formulas with parameters in A algebraizing $b_1, \ldots, b_m$, respectively, over A. The formula $(\exists y_1) \cdots (\exists y_m)(f_1(y_1) \wedge \cdots \wedge f_m(y_m) \wedge f(x, y_1, \ldots, y_m) \wedge \neg((\exists x_0) \cdots (\exists x_n)(\bigwedge_{0 \leq i < j \leq n} x_i \neq x_j \wedge \bigwedge_{0 \leq i \leq n} f(x_i, y_1, \ldots, y_m))))$ algebraizes c over A. $\square$

One consequence of Lemma 6.6 is that the set $\bar{A}_M$ of the elements of M that are algebraic over A is algebraically closed: It is the smallest algebraically closed set containing A, and the only one that is algebraic over A. It is called the *algebraic closure* of A in M.

Indeed, the model M is irrelevant, a fact that enables us to define the notion of algebraic closure $\bar{A}$ of A without mentioning M: Suppose there is another model N of T containing A; then it is a model of $T(A)$. Then M and N, as models of $T(A)$, have a common elementary extension P; if f is an algebraizing formula, with parameters in A, then the fact that there are exactly n elements satisfying f is expressed by a sentence that is true in M, and therefore also in P. This implies that all its elements are already in M, that there cannot be any more in P. Hence $\bar{A}_M = \bar{A}_P$, and by the same reasoning $\bar{A}_P = \bar{A}_N$.

In other words, the elementary embeddings of M and N into P induce a bijection s between $\bar{A}_M$ and $\bar{A}_N$, which also preserves satisfaction of formulas: If $\vec{a} \in \bar{A}_M$, then $M \models g(\vec{a})$ iff $N \models g(s\vec{a})$.

Now suppose that T is the theory of algebraically closed fields of a given characteristic; one very specific consequence is that for every $A \subset M$ (M being a model of T, an algebraically closed field), $\bar{A}$ is itself a model of T, in other words an algebraically closed field. (Be careful of the wordplay here!)

In fact, $a + b, a, b, 0, 1, -a, a^{-1}$ are algebraic over $\{a, b\}$, so $\bar{A}$ is a field. Also, if P is a nonconstant polynomial of $\bar{A}[X]$, then it has a zero a in M, which is a model of T, and as a is algebraized over $\bar{A}$ by the formula $P(x) = 0$, it is in $\bar{A}$; $\bar{A}$ is therefore an algebraically closed field (in the algebraic, not the model-theoretic sense), in other words, a model of T.

The field $\bar{A}$ is what we call a *prime model* over A: For every model M of T containing A, there is an (elementary) embedding s of $\bar{A}$ into M. It also has the additional property that all embeddings of $\bar{A}$ into M have the same image $\bar{A}_M$. It is obviously the only model containing A having the property of being prime, for if $A \subset K \subsetneq \bar{A}$, then K cannot be an algebraically closed field, or else the extension $K \subset \bar{A}$ would be elementary, and $\bar{A}_K = \bar{A}$.

Since you know the basics of field theory, and perhaps even more, you have certainly been surprised at the directness with which the model theorist approaches a problem: We were able to capture the general picture of the theory of algebraically closed fields without speaking of degree, of transcendence degree, or of a transcendence basis of a field extension. Naturally, we will assume from now on that you are familiar with these concepts; if you have blank spots, consult the nearest elementary algebra textbook.

By the way, what are the spaces of types associated with the theory T of algebraically closed fields of a given characteristic? For the set of parameters, take a field K (if K is the field generated by A, $S_n(K)$ and $S_n(A)$ are identical). Therefore, $S_n(K)$, is the set of prime ideals of the ring $K[X_1, \ldots, X_n]$ of polynomials in n variables, equipped with the topology obtained by taking the sets $\langle P(\vec{x}) = 0\rangle = \{I : I \models P = 0\} = \{I : P \in I\}$ to be both open and closed. This is what geometers call the *constructible topology*.

Geometers, whose cunning is well known, work in the *Zariski topology* $Z_n(K)$, defined on the same set of prime ideals, for which a *closed* base is formed by the $\langle P(\vec{x}) = 0\rangle$, and an open base by the $\langle P(\vec{x}) \neq 0\rangle$. This space is nonseparable and Noetherian (every decreasing sequence of closed sets stabilizes), with some slightly unusual properties, and naturally, as the topology $Z_n(K)$ is weaker than $S_n(K)$, the theorems proved about $Z_n(K)$ are finer, and more specific, than those about $S_n(K)$.

Model theorists, who are simple folk, prefer to work in a compact 0-dimensional space, the nicest of all topological spaces after the finite discrete spaces.

Exercise 6.7. Formalize the theory T of algebraically closed fields using only the language $(0, 1, +, \cdot)$, and show that it admits quantifier elimination. What is the theory $T_\forall$?

6.2 Differentially Closed Fields

When Galois discussed the roots of an equation, he was thinking of complex numbers, and it was a long time after him until algebraists considered fields other than subfields of $\mathbf{C}$: Nature had very obligingly put an algebraically closed field at the disposal of mathematicians. (I do not mean to say that this field was given from the beginning to mathematicians, who did take some time to conceive of "imaginary solutions" to their equations, but I do mean that the field was known even before it was known what an algebraically closed field was.) But at the end of the last century, when the concern was to construct a theory analogous to that of Galois, but for differential equations, they got stuck on the following problem: In what domain do we need to be in order to have enough solutions to differential equations? It was an important contribution of model theory to algebra to answer this question with the notion of "differentially closed field," which is to differential equations what the notion of algebraically closed field is to algebraic equations, a domain where differential equations have as many solutions as we can reasonably hope for. There is no natural example of a differentially closed field.

Let A be a (commutative) ring; a *derivation* is a function d from A to A such that for all x and y in A, $d(x+y) = dx + dy \quad d(xy) = xdy + ydx.$

A *differential ring* is a ring equipped with a derivation; if this ring is a field, we speak of a *differential field.* The elements of A with derivative zero are called constants; they form a subring of A, which is a field if A is; in particular, $d1 = 0$. It is easy to check that if A is integral, then every derivation on A extends uniquely to its field of fractions by the rule $d(x/y) = ydx - xdy/y^2$.

The *language of differential fields* will be the language of fields, with an extra unary function symbol to denote the derivative: $(0, 1, -, d, +, \cdot)$. We normally write x' for dx, x'' for $d(dx), \ldots, x^{(n)}$ for $d(d\cdots(dx)\cdots)$, the derivation being applied n times.

Examples of differential fields:

- If K is a field, we can make it a field of constants by giving it the null derivation.
- If K is a field, we can give $K(X)$ the formal derivation $\partial R(X)/\partial X$ for rational functions.
- The field of formal power series $K((X))$ with its usual derivation.
- The field of meromorphic functions on an open connected subset of the complex plane.

From now on, we shall consider only differentially closed fields of characteristic zero.

Given a differential field K, define the differential ring $K[X]_d$ of differential polynomials in one variable X with coefficients in the field K. As a ring, it is the ring $K[X, X', X'', \ldots, X^{(n)}, \ldots]$ in denumerably many variables, which we write

$$X = X^{(0)}, \quad X' = X^{(1)}, \quad X'' = X^{(2)} \quad \ldots \quad X^{(n)}, \quad X^{(n+1)}, \quad \ldots.$$

Like every ring of polynomials with coefficients from a field, it is a *factorization domain*: Each of its elements decomposes uniquely as a product of irreducible elements. We give it the derivation extending that on K, such that the derivative of $X^{(n)}$ is $X^{(n+1)}$. In other words, if $P(X)$ is in $K[X]_d$, then

$$P(X)' = P^*(X) + \Sigma X^{(n+1)}\partial P/\partial X^{(n)}$$

where P^* denotes the polynomial obtained by deriving the coefficients of P (do not forget P^*: The elements of K are not necessarily constant!) and where $\partial P/\partial X^{(n)}$ denotes the partial derivative, in the usual sense, of P with respect to the variable $X^{(n)}$.

If P is a differential polynomial that is not in K (we would not dare to say nonconstant polynomial!), its *order* is by definition the greatest n such that $X^{(n)}$ appears in it: For example, a polynomial P of order 2 is a polynomial in X, X', X''; a polynomial of order 0 is one in X alone. Its *separant* is, still by definition, $S(P) = \partial P/\partial X^{(n)}$, where n is its order: *Because we are working in characteristic zero, $S(P)$ is never the zero polynomial.*

As we did in the case of fields, given a differential field K and an extension of K to a differential field L, with a in L, we will look for what determines, up to K-isomorphism, the differential field $K(a)_d$ generated by K and a. Consider the map from $K[a]_d$ to L consisting of replacing X by a: Its image is the differential ring $K[a]_d$ generated by a and K. Its kernel is the ideal I_a of differential equations in a over K: a prime ideal, since $K[a]_d$ is integral, and closed under derivation (if $P \in I_a$ then $P' \in I_a$); it is what we call a *prime differential ideal.* The ideal I_a determines $K(a)_d$ up to isomorphism, because the latter is just the field of fractions of the ring $K[a]_d$, equipped with the unique derivation extending that of $K[X]_d$; conversely, it is easy to see that every prime differential ideal I is associated with a simple extension of K, the field of fractions of $K[X]_d/I$. We therefore need to determine the prime differential ideals of $K[X]_d$: This is the object of the following lemmas.

We write (P) for the differential ideal generated by the polynomial P, which is also the ideal in the ordinary sense generated by P and its successive derivatives $P, P', \ldots, P^{(n)}, \ldots$; we write $I(P)$ for the set of differential polynomials Q such that for some large enough integer k, $S(P)^k Q$ is in (P): $I(P)$ is a differential ideal, since if $S(P)^k Q \in (P)$, then $S(P)^k Q' + kS(P)'S(P)^{k-1}Q \in (P)$ and $S(P)^{k+1}Q' \in P$.

Lemma 6.8. *If P is an irreducible polynomial of order n, then (P) contains no polynomial of order strictly less than n, and if Q in (P) has order n, then P divides Q. The same result holds for $I(P)$.*

Proof. Let Q in (P) have order less than or equal to n, $Q = A_0 P + A_1 P' + \cdots + A_k P^{(k)}$; deriving P k times, we get the following expression:

$$P^{(k)} = S(P)X^{(n+k)} + P_k(X, \ldots, X^{(n+k-1)}),$$

where P_k is a polynomial only in $X, \ldots, X^{(n+k-1)}$. If in the polynomial equation $Q = A_0 P + \cdots + A_k P^{(k)}$ we replace the unknown $X^{(n+k)}$ by the rational function $-P_k/S(P)$ ($S(P)$ is not zero!), we get an equation of rational functions. The left-hand side, since it does not contain $X^{(n+k)}$, is not altered, and if we multiply both sides by a suitable power of $S(P)$ to clear the denominators, we get a polynomial equation of the form

$$S(P)^h Q = B_0 P + \cdots + B_{k-1} P^{(k-1)}$$

Repeating this procedure k times, we get an expression $S(P)^m Q = AP$. But $S(P)$ has a smaller degree in $X^{(n)}$ than P, and P cannot divide $S(P)$; since it is irreducible, it must divide Q. In the case of $I(P)$, P divides $S(P)^m Q$ if and only if it divides Q. □

Exercise 6.9. Show that even if P is reducible of order n, (P) contains no polynomial of order strictly less than n.

Lemma 6.10. *If P is an irreducible polynomial of order n, then $I(P)$ is a differential prime ideal.*

Proof. How can we see that Q is in $I(P)$? If, as in the proof of Lemma 6.8, we replace $X^{(n+k)}$ by $-P_k/S(P)$, beginning with the highest derivative of X occurring in Q, then multiplying by a suitable power of $S(P)$ to make the formula integral and repeating, we ultimately obtain a polynomial Q_1 of order at most n such that for some h, Q_1 and $S(P)^h Q$ are congruent modulo (P).

Consequently, Q is in $I(P)$ if and only if for m large enough, $S(P)^m Q_1$ is in (P), which implies that P divides Q_1.

Therefore, if $UV \in I(P)$, we get that P divides $(UV)_1 = U_1 V_1$; hence P divides U_1 or P divides V_1. □

Lemma 6.11. *Every nonzero prime differential ideal has the form $I(P)$ where P is irreducible.*

Proof. Let I be a nonzero prime differential ideal; choose a nonzero P in I of minimal order n, then of minimal partial degree in $X^{(n)}$, and finally of minimal total degree. As I is prime, it is clear that P is irreducible. I will show that $I = I(P)$.

Let Q be in $I(P)$. Then $S(P)^m Q \in (P) \subset I$; $S(P)$, having smaller degree in $X^{(n)}$ than P, cannot be in I. But I is prime, so Q is in I.

Let Q be in I. Begin by using the method described above to form a polynomial Q_1 of order at most n, such that $S(P)^m Q$ and Q_1 are congruent modulo (P). Consider Q_1 and P as polynomials only in $X^{(n)}$, and apply Euclid's algorithm to divide Q_1 by P; to make integral the resulting equation of rational functions, we must multiply by a certain power of the leading coefficient M of P, which is a polynomial in $X, \ldots, X^{(n-1)}$, and we get a polynomial R_1, with a strictly lower degree in $X^{(n)}$ than P, such that $M^k Q_1 = AP + R_1$. As Q_1 and P are in I, so is R_1, and we must have $R_1 = 0$ by minimality of P. As P cannot divide its own leading coefficient, it divides Q_1, which implies that Q is in $I(P)$. □

We see, therefore, that a prime differential ideal is determined by a "minimal polynomial" P; $I(P)$ is not in general the ideal generated by P, nor even the (differential) radical ideal generated by P, and the tie that binds a prime ideal and its minimal polynomial is more subtle than in the case of ordinary fields. (The problem of finding a generating system for $I(P)$ as a differential radical ideal is known as "Ritt's problem"; only very fragmentary solutions are known.) To simplify the statement of theorems, we shall adopt the convention that the minimal polynomial of the null ideal is a mythical entity of order ω and separant 1. The *dimension rank* of the prime differential ideal $I(P)$ is defined as the order $\mathrm{RD}(I(P))$ of its minimal polynomial.

Lemma 6.12. *The transcendence degree of the field extension $K(a)_d/K$ is equal to* $\mathrm{RD}(I_{a/K})$.

Proof. If $I_{a/K} = 0$ (we then call a differentially transcendental over K), then a and its successive derivative form a transcendence base for $K(a)_d/K$. Otherwise, let n be the largest integer such that $a, \ldots, a^{(n-1)}$ are algebraically independent over K: Here n is the order of the minimal polynomial P of a over K. As $0 = P^{(k)}(a) = S(P)(a)a^{(n+k)} + P_k(a, \ldots, a^{(n+k-1)})$ and $S(P)(a) \neq 0$, $a^{(n+k)}$ can be expressed as a function of the derivatives of lower order, and $a, \ldots, a^{(n-1)}$ form a transcendence base for $K(a)_d/K$. □

Lemma 6.13. *Let P be irreducible of order n; then every prime differential ideal containing P other than $I(P)$ has* RD *strictly less than n.*

Proof. Let $I(Q)$ be a prime differential ideal containing P with dimension rank n. Then $S(Q)^m P \in (Q)$, which implies, by Lemma 6.8, that Q divides P; as these are irreducible polynomials, $P = Q$ (up to invertibles, i.e., elements of K^*). □

We will now examine what our ideals become when we take an extension $K \subset L$ of differential fields: A prime ideal J of $L[X]_d$ will be called a *son* of the prime ideal I of $K[X]_d$ if $I = J \cap K[X]_d$. As an irreducible polynomial P in $K[X]_d$ can also be considered as a polynomial in $L[X]_d$, we now must write $I(P, K)$ for the prime ideal of $K[X]_d$ of which it is the minimal polynomial.

A polynomial P that is irreducible in $K[X]_d$ can be decomposed into several factors P_i in $L[X]_d$; in fact, we get the ultimate irreducible factors of P_i when we go up to the algebraic closure $\bar{K}$ of K (you can check this, if you wish, although it will be a consequence of the next theorem, that the derivation of K extends uniquely to $\bar{K}$; characteristic zero is essential for this result). Indeed, by the Nullstellensatz (Theorem 6.5), a polynomial (in several variables!) that is irreducible over an algebraically closed field remains so in every field extension. An elementary argument from Galois theory shows that all its factors are simple and conjugate by K-automorphisms of $\bar{K}$; in fact, if P_i is one of them, let Q denote the product of its conjugates, which is invariant under any K-automorphism of $\bar{K}$: Since we are in characteristic zero, this is a polynomial with coefficients in K, which divides P, and therefore is equal to it.

Lemma 6.14. *Let $K \subset L$ be an extension of differential fields; let P be an irreducible polynomial in $K[X]_d$, and let P_1 be an irreducible factor of P in $L[X]_d$. Then $I(P_1, L)$ is a son of $I(P, K)$.*

Proof. Let $Q \in K[X]_d$. First, suppose Q is in $I(P, K)$. Then $S(P)^m Q \in (P)$. Let us go up to L: $P = AP_1$, where P_1 does not divide A, since it is a simple factor of P. Since $S(P) = AS(P_1) + \partial A/\partial X^{(n)} P_1$, we have $S(P_1)^m A^m Q \in (P_1)$, so $A^m Q \in I(P_1, L)$; Since A is not in $I(P_1, L)$ (Lemma 6.8), which is a prime ideal, we must have $Q \in I(P_1, L)$.

Now suppose that Q is in $I(P_1, L)$; staying in K, we can find a polynomial Q_1 of order at most n such that $S(P)^m Q$ and Q_1 are congruent modulo (P). Now go up to L: They are congruent modulo (P_1), so Q_1 is in $I(P_1, L)$. As it has order at most n, it must be divisible by P_1; as its coefficients are in K, it is also divisible by all conjugates of P_1 in $\bar{K}$, and also by their product, as they are irreducible polynomials. Therefore, Q_1 is divisible by P, and hence Q is in $I(P, K)$. □

In particular, we see that every prime differential ideal has sons, and these sons of the same RD, whose existence is evident by Lemma 6.14, are called *nonforking sons.* We can now start on the notion of differentially closed field; to simplify our lives, we shall consider an element of K^* to be a differential polynomial of order -1.

A differential field K of characteristic zero is called *differentially closed* if it has the following property: Any system formed from a differential equation of order $n \geq 0$ and a differential inequation of order m strictly less than n, in one variable x, with coefficients in K, has a solution in K. We express the fact that a differential field is differentially closed by the following infinite family of axioms:

$$(\forall a_1, \ldots, a_u)(\forall b_1, \ldots, b_v)(\exists x) P(x) = 0 \wedge Q(x) \neq 0,$$

where the a_i are the coefficients of the polynomial P of order n (one of them, inside the major, being set equal to 1 to guarantee that P is indeed of order n), and the b_j those of the polynomial Q of order $m < n$ (same stipulation). We need one axiom for each possible total degree of P and Q. Under the convention that we have adopted, we allow the systems $P(x) = 0 \wedge 1 \neq 0$, where P has order 0, so that in particular, a differentially closed field is also algebraically closed.

Theorem 6.15. *Every differential field embeds into a differentially closed field.*

Proof. Let K be a differential field, and let $(P_0, Q_0), \ldots, (P_\alpha, Q_\alpha), \ldots$ be an ordinal enumeration of the systems of two polynomials in $K[X]_d$, where the order of Q_α is strictly less than that of P_α.

Let U_0 be an irreducible divisor of P_0 in $K[X]_d$, of the same order as P_0, and let K_0 be an extension of K containing an element a_0 such that $I_{a_0/K} = I(U_0)$; by Lemma 6.8, Q_0 is not in $I(U_0)$, and a_0 is a solution of the system $P_0(x) = 0 \wedge Q_0(x) \neq 0$. Next, consider an irreducible divisor U_1 of P_1 in $K_0[X]_d$ of the same order as P_1, and add an element whose ideal of equations over K_0 is $I(U_1, K_0)$. Repeat the operation until we get an extension L_1 of K in which every system with coefficients in K has a solution; then construct a sequence $K \subset L_1 \subset \cdots \subset L_n \subset L_{n+1} \subset \cdots$ such that each system with coefficients in L_n has a solution in L_{n+1}. The limit of the L_n is a differentially closed extension of K. □

Theorem 6.16. *The theory of differentially closed fields (of characteristic zero) is complete and admits quantifier elimination.*

Proof. Let $\vec{a}$ and $\vec{b}$ be tuples of elements of differentially closed fields K and L, which we can assume to be ω-saturated, and let them generate isomorphic differential subfields k and h, respectively. We must show that $(K, \vec{a})$ and $(L, \vec{b})$ are ∞-equivalent. (Note: In our language, to satisfy the same quantifier-free formulas is to generate isomorphic differential subrings; this amounts to generating isomorphic differential subfields.)

Let us add, for example, α to $\vec{a}$. Let P be the minimal polynomial of α over k, and let Q be the differential polynomial obtained by taking P through the isomorphism onto k: Q is clearly irreducible. Given a finite set $Q_1, \ldots, Q_s$ of differential nonzero polynomials of order less than that of Q, since L is differentially closed, there is an element of L satisfying $Q = 0 \wedge Q_1 Q_2 \cdots Q_s \neq 0$, so by compactness it is consistent with the theory of L that there is an element β such that $Q(\beta) = 0$ and $Q_i(\beta) \neq 0$ for every polynomial $Q_i \neq 0$ for $h[X]_d$ of order less than that of Q. As L is ω- saturated, such a β exists in L. By Lemma 6.13, $I_{\beta/h} = I(Q, h)$, and $k(\alpha)_d$ and $h(\beta)_d$ are isomorphic. □

Theorem 6.17 (Differential Nullstellensatz). *Let K be a differential field (of characteristic zero), and let S be a (finite) system of differential equations and inequations in several unknowns that has a solution in some extension L of K. Then S has a solution in any differentially closed extension of K.*

Proof. Same as for Theorem 6.5. □

The next question is whether there is a *differential closure* associated with a differential field K, that is, a differentially closed field $\bar{K}$ such that every embedding of K into a differentially closed field extends to $\bar{K}$.

We are going to show that every possible K has such a closure: This proof is a particular case of the one proving the existence of prime models of totally transcendental theories, which will be studied later in much detail (see Chapter 18).

For the theory T of differentially closed fields, the space $S_1(K)$ is that of the prime differential ideals of $K[X]_d$, with the topology obtained by making the sets $\langle P = 0\rangle = \{I : I \in S_1(K), P \in I\}$ both open and closed. Under what conditions is a type *isolated* in this topology? (We also say that the associated ideal is *constrained.*) Lemma 6.13 states that $\langle P = 0\rangle$ isolates $I(P)$ from the types of higher RD, and therefore we get a neighborhood isolating $I(P)$ of the form $\langle P = 0 \wedge Q_1 \neq 0 \wedge \cdots \wedge Q_s \neq 0\rangle$, where the Q_i have lower order than P; equivalently, replacing the Q_i by their product Q, we obtain $\langle P = 0 \wedge Q \neq 0\rangle$. In other words, $I(P)$ is isolated if and only if there is a Q of order strictly less than that of P such that $I(P)$ is the only prime differential ideal containing P and not containing Q. We see that by definition—and this agrees with Section 5.1—a differentially closed extension field of K realizes every isolated type over K.

Lemma 6.18. *The isolated types of $S_1(K)$ form a dense set.*

Proof. Let f be a formula with parameters in K, defining a nonempty clopen set of $S_1(K)$, and let p be in $\langle f\rangle$ with minimal RD, having minimal polynomial P. A type satisfying $P = 0 \wedge f$ cannot have RD strictly less than that of P, and therefore must be equal to p, so this open set isolates p. $\square$

This lemma allows us to show that every differentially closed field has a differential closure. Indeed, let K be such a field, and let L be a differentially closed extension of K; if K is differentially closed, it is its own closure. Otherwise, there is a type $p \in S_1(K)$ that is isolated and not realized in K: A certain system $P = 0 \wedge Q \neq 0$ has no solution in K, and by Lemma 6.18 this system is satisfied by an isolated type over K. As L is differentially closed, it contains an element of this type, called a_0. Set $K_1 = K(a_0)_d$: If K_1 is differentially closed, stop; otherwise, find a_1 in $L - K_1$ whose type over K_1 is isolated, etc.

Proceeding this way, we construct a sequence a_α of elements of L, indexed by an ordinal, such that that if we set $K_\alpha = K(a_0, \ldots, a_\beta, \ldots)_{d\,\beta<\alpha}$, the type of a_α over K_α is isolated, and $a_\alpha \notin K_\alpha$.

After some (transfinite) time, this construction must stop, as it is all carried out in L, and the a_α do not repeat. At the end we get a differentially closed field $\bar{K}$, with an ordinal enumeration a_α of $\bar{K} - K$ such that the type of a_α over $K_\alpha = K(\ldots, a_\beta, \ldots)_{d\,\beta<\alpha}$ is always isolated.

I claim that $\bar{K}$ is a prime differentially closed field over K: Indeed, if L_1 is a differentially closed field extending K, the type of a_0 over K, being isolated, is realized by an element of L_1, so we can embed K_1 into L_1; then we realize the type a_1 over K_1 in L_1, etc. Iterating,we embed K_α into L_1, and at the following stage we can extend the embedding to a_α, since it has an isolated type over K_α.

Now that we have established the existence of the differential closure, the next question is about its uniqueness. It is correct that $\bar{K}$ is the only prime differentially closed field over K, but we are a long way from being able to prove this: It needs rather fine techniques from stability theory, and will not be proved until Chapter 18.

The differential closure has more pathological properties than the algebraic closure of a field. First, what is an algebraic type in the sense of the theory of differentially closed fields? If P is a differential polynomial of order at least 1, and Q a polynomial of lower order, we see that the system $P(x) = 0 \wedge Q(x) \neq 0 \wedge x \neq a_1 \wedge \cdots \wedge x \neq a_n$ has a solution in every differentially closed field containing its coefficients. Therefore, the only algebraic types are those whose minimal polynomials have order 0, meaning that the algebraic closure, in the model-theoretic sense, of a differentially closed field is nothing other than its algebraic closure as a field.

If, for example, K is a field of constants, we see that its differential closure is much larger than its algebraic closure, as we can easily check that every element that is algebraic over K is a constant. This implies that if L is a differentially closed extension of K, then the image of the embedding of $\bar{K}$ into L has no reason to be unique. Indeed, let p be an isolated type of $S_1(K)$ that is not algebraic (for example, whenever K is a field of constants, the type isolated by the formula $x' = 1$); as p is not algebraic, the theory, in a language with κ new constant symbols, expressing that these elements are distinct realizations of p is consistent. Consequently, K has a differentially closed extension L with κ realizations of p, where $\kappa > |K|$. But every one of them can be taken as the first element of an enumeration of a copy of $\bar{K}$, so each one is contained in the image of an embedding of $\bar{K}$; however, since K, by Löwenheim's theorem, has a differentially closed extension of the same cardinality, card(K) = card$(\bar{K})$, an embedding of $\bar{K}$ into L cannot contain all of them.

The field $\bar{K}$ has an even more baffling property whenever K is a field of constants: There are differentially closed fields strictly contained between K and $\bar{K}$. As they are also prime fields, they are naturally K-isomorphic to $\bar{K}$: There are nonsurjective K-embeddings from $\bar{K}$ to $\bar{K}$! (A prime model with this property is called *nonminimal.*) In addition to a little model theory, the proof of this fact uses a result from differential algebra that is not very simple.

6.3 Boolean Algebras

A *Boolean ring* is defined as a unitary associative ring in which $x^2 = x$ for all x; we then have $(x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$, but $(x + y)^2 = x + y$, from which it follows that $xy + yx = 0$ for any x and y. Then $x^2 + x^2 = 0$, and hence $x + x = 0$ for every x, so $x = -x$; a Boolean ring therefore has characteristic 2, and since $xy = -yx = yx$, it is commutative.

To axiomatize this notion, we introduce the language consisting of two constant symbols 0 and 1 and two binary operations $+$ and $\cdot$; the reader can write down the handful of universal axioms that express that A is a Boolean ring, without forgetting $0 \neq 1$.

In a Boolean ring we define two binary operations $\wedge$ and $\vee$, and one unary operation $\neg$, in the following way:

$$x \wedge y = x \cdot y, \qquad x \vee y = x + y + y \cdot x, \qquad \neg x = 1 + x.$$

The reader can check that the following properties are true for all x, y, z:

- (de Morgan's laws, or duality laws): $\neg(\neg x) = x, \neg(x \wedge y) = \neg x \vee \neg y, \neg(x \vee y) = \neg x \wedge \neg y$
- (associativity of $\wedge$): $(x \wedge y) \wedge z = x \wedge (y \wedge z)$
- (associativity of $\vee$): $(x \vee y) \vee z = x \vee (y \vee z)$
- (distributivity of $\wedge$ over $\vee$): $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
- (distributivity of $\vee$ over $\wedge$): $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
- (commutativity of $\wedge$ and $\vee$): $x \wedge y = y \wedge x, x \vee y = y \vee x$
- $x \wedge \neg x = 0$, $x \vee \neg x = 1$
- $x \wedge 0 = 0$, $x \vee 0 = x$, $x \wedge 1 = x$, $x \vee 1 = 1$
- $0 \neq 1$, $\neg 0 = 1$, $\neg 1 = 0$

A structure in the language $(0, 1, \neg, \wedge, \vee)$ that satisfies these universal axioms is called a *Boolean algebra.* The reader can further check that in a Boolean algebra the relation $x \wedge y = x$ is equivalent to the relation $x \vee y = y$, and that it is a (partial) order, which we shall write $x \leq y$; 0 is the least element and 1 the greatest; $x \vee y$ is the supremum (least upper bound) of x and y in this order, and $x \wedge y$ their infimum (greatest lower bound): The order therefore defines a lattice, indeed a *distributive* lattice (one satisfying the distributive laws of $\wedge$ over $\vee$ and of $\vee$ over $\wedge$); it is also equipped with a complementation operator $\neg$, which is an involution ($\neg(\neg x) = x$) that reverses the order (if $x \leq y$ then $\neg x \geq \neg y$) and that exchanges $\wedge$ and $\vee$; also, $\neg x$ is both the least element y such that $x \vee y = 1$ and the greatest such that $x \wedge y = 0$. As a consequence, we see that a Boolean algebra can also be considered as a partial order, called a Boolean order: All the operations $0, 1, \neg, \wedge, \vee$ can be defined from this order.

We have associated every Boolean ring A with a Boolean algebra $b(A)$; conversely—and once again I leave it to the reader to check this—if in a Boolean algebra we set $x \cdot y = x \wedge y, x + y = (x \vee y) \wedge (\neg x \vee \neg y)$, we get a Boolean ring $a(B)$; furthermore, $a(b(A)) = A, b(a(B)) = B$. We therefore see that Boolean rings and Boolean algebras are the same structures except for the language, that a Boolean ring can be transformed canonically into a Boolean algebra and vice versa, the translation being done in both cases through quantifier-free formulas.

Everything I have said obviously relies on simple, but somewhat long, verifications (especially to check that $+$ is associative); that is why I leave the pleasure to the reader.

As an example of a Boolean algebra we can give that of the subsets of a nonempty set E, with $0 = \emptyset$, $1 = E$, $\neg x$ the complement of x in E, $x \vee y$ the union of x and y, and $x \wedge y$ their intersection. Considered as a Boolean ring, this is nothing more than the product of copies of the two-element field $\mathbf{Z}/2\mathbf{Z}$ (addition table: $0 + 0 = 0, 0 + 1 = 1, 1 + 1 = 0$; multiplication table: $0 \cdot 0 = 0 \cdot 1 = 0, 1 \cdot 1 = 1$), which is itself the smallest Boolean algebra; each of its copies is indexed by an element of E: To see this, associate each subset x of E with its characteristic function, the function from E to the set $\{0, 1\}$ that sends e to 1 if $e \in x$ and to 0 if $e \notin x$. This Boolean algebra is therefore denoted by 2^E.

A subset I of the Boolean ring A will be called an *ideal* of A if it is a (nontrivial, i.e., not containing 1) ideal in the usual sense of rings:

- If x and y are in I, then $x + y$ is in I.
- If x is in I, then $x \cdot y$ is in I.
- $1 \notin I$.

When we switch to the language of Boolean algebras, this becomes, after several manipulations whose discovery is again left to the reader:

- If x and y are in I, then $x \vee y$ is in I.
- If x is in I and $y \leq x$, then y is in I.
- $1 \notin I$.

Everybody knows that the image of a ring A under a homomorphism f is isomorphic to the quotient of A by the kernel of f, which is an ideal, and that conversely, for every ideal I we can define the quotient ring A/I.

This means in particular that for Boolean rings and Boolean algebras, the algebra B/I will be the one associated with the quotient by I of the Boolean ring associated with B.

In the quotient A/I, the elements of I are those that are sent to 0. In the Boolean context, we often prefer to consider the elements that are sent to 1: We therefore define a *filter* as the set of elements of the form $\neg x$, where x belongs to an ideal. (Caution: A filter is not the complement of an ideal, but rather the set of elements whose complements are in an ideal!) A filter is therefore characterized by the following properties:

- If x and y are in F, then $x \wedge y$ is in F.
- If x is in F and $x \leq y$, then y is in F.
- $0 \notin F$.

We do not distinguish between the quotient of a Boolean algebra by a filter and its quotient by the associated ideal. We can note the following: If B is a Boolean algebra, then by setting

$$0' = 1,\ 1' = 0,\ \neg' x = \neg x,\ x \wedge' y = x \vee y,\ x \vee' y = x \wedge y$$

we can define another Boolean algebra structure B' on the set B, called the *dual* of B; the map $x \mapsto \neg x$ is an isomorphism; the filters of B' are the ideals of B, and some people even say "dual ideal" instead of filter.

The filters of subsets of I that we considered in Section 4.1 are the filters of the Boolean algebra 2^I.

In a Boolean algebra, an ideal is prime iff it is maximal, since the only integral Boolean ring is the two-element field $\{0, 1\}$: An ideal I of B is maximal iff $B/I = \{0, 1\}$, i.e., if for every x either x or $\neg x$ is in I. The filter associated with a maximal ideal is called an *ultrafilter*: Ultrafilters are exactly maximal filters, equivalently characterized as the filters with the property that for all x, either x or $\neg x$ belongs to F; therefore, the complement of an ultrafilter is the ideal associated with it.

If E is a compact 0-dimensional space (i.e., one with a base of open sets that are also closed), then the clopen sets of E form a Boolean subalgebra of 2^E. This is, in fact, the most general form of Boolean algebra, as the following representation theorem shows:

Theorem 6.19 (Stone's Theorem). *Every Boolean algebra B is isomorphic to the algebra of clopen sets of a compact* 0*-dimensional space E, which is uniquely determined up to isomorphism.*

Proof. Let E be the set of maximal ideals of B, and for every x of B set $\langle x \rangle = \{I : I \in E, x \in I\}$. Note that $\langle 0 \rangle = E, \langle 1 \rangle = \emptyset, \langle \neg x \rangle = E - \langle x \rangle$. Also $\langle x \wedge y \rangle = \langle x \rangle \cup \langle y \rangle$, since $x \wedge y = x \cdot y$ and these ideals are prime; $\langle x \vee y \rangle = \langle x \rangle \cap \langle y \rangle$, since an ideal containing x and y must contain $x \vee y$, and if it contains $x \vee y$ it also contains the lesser elements x and y.

The sets of the form $\langle x \rangle$ therefore form a subalgebra of 2^E and generate a topology $\mathcal{T}$ on E; let us show that E also becomes a compact space (which will be 0-dimensional because every open set of the base is also closed). The space E is Hausdorff, because if $I \neq J$, then there exists $x, x \in I, x \notin J$, so $\neg x \in J$; $\langle x \rangle$ and $\langle \neg x \rangle$ are therefore two open sets separating I and J. If $\langle x_i \rangle$ is a family of basic closed sets with the finite intersection property, then every finite set of the x_i is contained in a maximal ideal, so 1 does not belong to the ideal generated by the x_i; this implies that the ideal generated by the x_i is indeed a "proper" ideal (i.e., different from B), which is therefore contained in a maximal ideal (see Section 8.2, axiom of choice); this ideal is in the intersection of all the $\langle x_i \rangle$.

Note that the sets of the form $\langle x \rangle$ are the only clopen sets of E: In fact, an open set is the union of sets of the form $\langle x_i \rangle$, and if it is compact, it is the union of finitely many of them.

We have therefore defined a surjective homomorphism from B onto the dual of the algebra of clopen sets of E. What is its kernel? If x is in the kernel, $\langle x \rangle = E$ (the zero of the dual is E, not $\emptyset$); this means that $x = 0$, since if $x \neq 0$, there is a maximal ideal containing $\neg x$, which could not contain x.

This homomorphism is actually bijective. In general, we prefer to consider the space E' of ultrafilters of B, equipped with the topology defined by the $\langle x \rangle = \{F : F \in E', x \in F\} = \{F : F \in E', \neg x \notin F\}$. As B and B' are isomorphic, so are E and E', but this isomorphism is more practical, because we can directly obtain an isomorphism onto the algebra of clopen sets of E',

$$\langle 0 \rangle = \emptyset,\ \langle 1 \rangle = E',\ \langle x \wedge y \rangle = \langle x \rangle \cap \langle y \rangle,\ \langle x \vee y \rangle = \langle x \rangle \cup \langle y \rangle,$$

instead of an isomorphism onto its dual. This ultrafilter space of B is called the *Stone space* of B.

We know, therefore, that B can be represented as the algebra of clopen sets of its Stone space. To see uniqueness, if E is a compact 0-dimensional space, the ultrafilters of the algebra B of clopen sets of E, which will converge, correspond to the points of E, which is the Stone space of B. □

Note that the nonempty closed sets in the Stone space of B correspond to its filters: $F \in \bigcap \langle x_i \rangle$ iff F contains the filter generated by the x_i.

As a finite Boolean algebra has a finite Stone space, this space is discrete, and the algebra is isomorphic to 2^n, the Boolean algebra of subsets of the set $\{0, 1, \ldots, n-1\}$.

We meet Boolean algebras everywhere in model theory. The most obvious is the algebra of sentences associated with a language $\mathcal{L}$, considered modulo equivalence (two sentences are equivalent iff they have the same models): A theory is a filter in this algebra; its Stone space is therefore made up of the complete theories in the language. This algebra has a name: the *Tarski–Lindenbaum algebra* of $\mathcal{L}$. If we factor out a theory T, we get the Tarski–Lindenbaum algebra associated with T, which consists of the sentences of $\mathcal{L}$ up to equivalence modulo T (two sentences are equivalent modulo T if they have the same models among the models of T); if T is complete, this algebra reduces to two elements: 0, the class of sentences that are false for T ($\neg f \in T$), and 1, the class of sentences that are true for T ($f \in T$).

Along the same lines, we can consider the algebra of sentences in the language $\mathcal{L} \cup \{x\}$ or $\mathcal{L} \cup \{x_1, \ldots, x_n\}$; the Stone space of this algebra is simply the space of types, which are complete theories in this language.

We can approach the study of model theory in an algebraic fashion, considering Boolean algebras equipped with operations corresponding to the quantifiers. These are called *polyadic algebras*: They are the abstract equivalents of the algebras of formulas associated with languages; the compactness theorem can then be proved by means of a representation theorem for polyadic algebras, analogous to Theorem 6.19. This branch of logic, which we shall refrain from penetrating deeply, is *algebraic logic*.

We are now going to study some Boolean algebras as structures, that is to say, by doing model theory.

In a Boolean algebra B, a filter F generated by finitely many elements $x_1, \ldots, x_n$ is also generated by their infimum, and is therefore principal. If F is a principal ultrafilter, then its generator is a minimal nonzero element: Such an element is called an atom. We say that a Boolean algebra is *atomic* if every nonzero element y majorizes (bounds above) an atom: For example, 2^I, and in particular any finite Boolean algebra, is atomic. This implies that the isolated points of its Stone space, which are precisely the principal ultrafilters, form a dense set.

In an atomic Boolean algebra B, two elements are equal when they majorize the same atoms (for then $(\neg x \wedge y) \vee (\neg y \wedge x)$ is zero), and B appears as a subalgebra of the algebra of subsets of the set of its atoms.

In opposition to the atomic algebras, we have the *atomless algebras*, whose Stone spaces have no isolated points.

Saying that an algebra is atomic or atomless is easy with axioms. Since every Boolean algebra B embeds into 2^E, where E is its Stone space, the theory of atomic Boolean algebras is a companion of that of Boolean algebras.

The same is true of the theory of atomless Boolean algebras: Embed B into the algebra $B_1 = B \times B$ diagonally, associating each element x with the element (x, x); no element of B is an atom of B_1, since if $(x, x) \neq (0, 0)$, it is strictly minorized by $(x, 0)$; repeating this, we can construct a sequence $B \subset B_1 \subset \cdots \subset B_n \subset B_{n+1} \subset \cdots$ such that no element of B_n is an atom in B_{n+1}. The limit of the B_n is then atomless.

An atomic Boolean algebra with only finitely many atoms $a_1, \ldots, a_n$ is finite: If we associate an element x of this algebra with the set of atoms that x majorizes, we get an isomorphism from B onto 2^n.

Note also that a Boolean algebra finitely generated by $X_1, \ldots, X_n$ is finite; its atoms are the nonzero elements of the form $\epsilon_1 X_1 \wedge \cdots \wedge \epsilon_n X_n$, where ϵ_i is either nothing or $\neg$. Indeed, we can easily see that every nonzero element of the Boolean algebra generated by $X_1, \ldots, X_n$ is the supremum of elements of this form. (Use associativity, distributivity, and de Morgan's laws to show that the elements thus described do actually form a Boolean algebra.) In particular, we see that the free algebra generated by n elements has 2^n atoms, and therefore 2^{2^n} elements.

For every natural number n, consider the formula $A_n(x)$ stating that x majorizes at most n atoms.

Theorem 6.20. *The theory of infinite atomic Boolean algebras is complete and admits quantifier elimination in the language consisting of* $0, 1, \neg, \wedge, \vee, A_1, \ldots, A_n, \ldots$.

Proof. Consider, in two infinite atomic ω-saturated Boolean algebras, a tuple $\vec{a}$ in the first and a tuple $\vec{b}$ in the second, satisfying the same quantifier-free formulas in the language described. This means that if $\epsilon_1 a_1 \wedge \cdots \wedge \epsilon_n a_n$ majorizes exactly m atoms (i.e., satisfies $A_m(x) \wedge \neg A_{m+1}(x)$), the same is true of $\epsilon_i b_i \wedge \cdots \wedge \epsilon_n b_n$; this is true in particular for $m = 0$, which means, since the algebra is atomic, that if one of these elements is zero, so is the other. This condition is sufficient for $\vec{a}$ and $\vec{b}$ to generate isomorphic subalgebras, and if $\epsilon_1 a_1 \wedge \cdots \wedge \epsilon_n a_n$ majorizes infinitely many atoms, i.e., satisfies all the $A_m(x)$, the same is true of $\epsilon_1 b_1 \wedge \cdots \wedge \epsilon_n b_n$.

Let us add α to $\vec{a}$, and for every sequence $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ set $\epsilon\vec{a} = \epsilon_1 a_1 \wedge \cdots \wedge \epsilon_n a_n, \alpha_\epsilon = \alpha \wedge \epsilon\vec{a}, \alpha'_\epsilon = \neg\alpha \wedge \epsilon\vec{a}$. We distinguish the following cases:

- If $\epsilon\vec{a}$ majorizes exactly m atoms, then α_ϵ majorizes p, and α'_ϵ majorizes q, with $m = p + q$; by hypothesis, $\epsilon\vec{b}$ majorizes exactly m atoms, and we let β_ϵ denote the supremum of p of these, and β'_ϵ that of the remaining q. Then $\beta_\epsilon \wedge \beta'_\epsilon = 0, \beta_\epsilon \vee \beta'_\epsilon = \epsilon\vec{b}$.
- If $\epsilon\vec{a}$ majorizes infinitely many atoms, while α_ϵ majorizes only p, we take β_ϵ to be the supremum of the p atoms majorized by $\epsilon\vec{b}$, and β'_ϵ its relative complement; we do something analogous if it is α'_ϵ that majorizes only p atoms.
- If $\epsilon\vec{a}$, α_ϵ, and α'_ϵ each majorize infinitely many atoms, we can decompose $\epsilon\vec{b}$ into two disjoint pieces β_ϵ and β'_ϵ, each majorizing infinitely many atoms.

This is possible because in an ω-saturated Boolean algebra, if a majorizes infinitely many atoms, there exists b such that $a \wedge b$ and $a \wedge \neg b$ majorize infinitely many atoms: Indeed, if b_n is the supremum of n atoms above a, then $a \wedge b_n$ majorizes n atoms and $a \wedge \neg b_n$ majorizes infinitely many; the result then follows by compactness and ω-saturation.

Now, to find the β such that $\vec{a}^\frown\alpha$ and $\vec{b}^\frown\beta$ are ∞-equivalent, it suffices to collect the pieces, and to take β to be the supremum of the β_ϵ (whose complement is the supremum of the β'_ϵ). □

Theorem 6.21. *The theory of atomless Boolean algebras is complete and admits quantifier elimination (in the language* $0, 1, \neg, \wedge, \vee$*); it is the model completion of the theory of Boolean algebras.*

Proof. Let us show that in *any* two atomless Boolean algebras, two tuples $\vec{a}$ and $\vec{b}$ that satisfy the same quantifier-free formulas are ∞-equivalent. The Boolean algebra generated by $a_1, \ldots, a_n$ is determined, up to isomorphism, by the conditions $\epsilon\vec{a} = 0, \eta\vec{a} \neq 0$, which $\vec{a}$ satisfies. On the other hand, as there are no atoms, every nonzero element splits into two nonzero pieces.

If then I add α to $\vec{a}$, if $\alpha_\epsilon = 0$, I set $\beta_\epsilon = 0$; if $\alpha'_\epsilon = 0$, I set $\alpha_\epsilon = \epsilon\vec{b}$; and if neither α_ϵ nor α'_ϵ is zero, I split $\epsilon\vec{b}$ into two nonzero pieces: $\epsilon\vec{b} = \beta_\epsilon \vee \beta'_\epsilon, \beta_\epsilon \wedge \beta'_\epsilon = 0$. I now need only collect the β_ϵ. □

As two atomless Boolean algebras are always ∞-equivalent, we see that up to isomorphism there is just one denumerable atomless Boolean algebra (Theorem 1.14), which is the free algebra on denumerably many generators; by Stone duality (Theorem 6.19), this shows that there is only one compact 0-dimensional space without isolated points and having a countable open base (the Cantor space).

Theorem 6.22. *The theory of atomless Boolean rings is complete and admits quantifier elimination (in the language* $0, 1, +, \cdot$*); it is the model completion of the theory of Boolean rings.*

Proof. We have defined the Boolean ring operations from the Boolean algebra, and vice versa, by quantifier-free formulas. □

I remind you that a *Boolean order* is the order associated with a Boolean algebra; as the other operations are definable from the order, they are the models of the following theory: We state that there is a least and a greatest element, suprema, infima, and a complementation, and that these notions satisfy the axioms of Boolean algebras.

The atomless Boolean orders are the models of a complete theory, because we can recover the order from the Boolean operations. It is not model complete, because an embedding of a Boolean order does not preserve the notions of least element, supremum, etc. An extension of atomless Boolean orders is elementary exactly if the associated Boolean operations are preserved.

The theory of Boolean orders is a companion of that of (partial) orders: Indeed, we can embed the order I into the Boolean order of the algebra 2^I, associating a with the set $I_a = \{x : x \leq a\}$: $I_a \subset I_b$ iff $a \leq b$.

In particular, we see that every denumerable partial order embeds into the denumerable atomless Boolean algebra.

Finally, note that the theory of partial orders has a model completion: Consider a partial order defined on $n+1$ elements $x_1, \ldots, x_n, y$. Let $f(\vec{x}, y)$ denote the diagram of this order, i.e., the conjunction of the $u \leq v, \neg(u \leq v)$ that are true in it, and let $g(\vec{x})$ be the diagram of the restriction of this order to $x_1, \ldots, x_n$. We add to the theory of partial orders all of the following axioms, one axiom for every possible pair (f, g): $(\forall x_1) \cdots (\forall x_n)(\exists y)(g(\vec{x}) \rightarrow f(\vec{x}, y))$. This is certainly a consistent set, and in fact is a companion to the theory of order; indeed, if I is a partial order, A a finite subset of I, and b a new element, with a partial order on $A \cup \{b\}$ extending that on A, we define an order on $I \cup \{b\}$ extending both that on I and that on $A \cup \{b\}$ as follows: If $c \in I$, set $c \leq b$ if there exists a in A such that $c \leq a$ and $a \leq b$, and set $b \leq c$ if there exists a in A such that $b \leq a$ and $a \leq c$. It is not hard to check that this is indeed a partial order.

From this we deduce, by repeating ω times a type of construction that must now be familiar to the reader, that every partial order embeds into a model of this theory T. It is also fairly obvious that in two models of T, two tuples that satisfy the same quantifier-free formulas are ∞-equivalent. As two models of T are always ∞-equivalent, there is only one such denumerable model, up to isomorphism.

6.4 Ultrametric Spaces

Consider a total order (chain) I with least element 0, and define an *I-valued ultrametric space* as a set E equipped with a function d from E^2 to I that satisfies the following conditions:

- For every x and y in E, $d(x, y) = 0$ iff $x = y$.
- For every x and y in E, $d(x, y) = d(y, x)$.
- For every x, y, z in E, $d(x, y) \leq \max(d(x, y), d(y, z))$.

The function $d(x, y)$ is called the *distance* from x to y, and the third condition is called the *ultrametric inequality*; it means that the relations $d(x, y) \leq i$ and $d(x, y) < i$ are equivalence relations, and implies that given three points x, y, z, either they form an "equilateral" triangle ($d(x, y) = d(y, z) = d(z, x)$) or they form an "isosceles" triangle, with two equal sides and the third shorter. In passing we note the linguistic ambiguity of the term equilateral and its Greek translation isosceles; in order not to mix Latin and Greek, we adopt the convention here of calling a set of points all equidistant from one another an *isosceles polygon*.

In order to speak about this notion of ultrametric space, we introduce a unary predicate $I(u)$ to denote the universe of the chain I, a constant symbol for 0, a binary relation symbol $\leq$ for the order of the chain I, a unary symbol $E(u)$ to denote the space E itself, and a function symbol to denote the distance operation.

The universe of the structure associated with an ultrametric space is made up of the chain I on the one hand and the space E on the other, so that the following two conditions are satisfied:

$(\forall u)(I(u) \vee E(u))$, $\neg(\exists u)(I(u) \wedge E(u))$, and in order not to have to drag lots of I's and E's everywhere, we will distinguish two sorts of variables:

- variables such as $i, j, k, \ldots$ should be understood to represent *distances*, i.e., elements of I. $(\exists i)f(i), (\forall i)f(i)$ should be taken as abbreviations for $(\exists u)(I(u) \wedge f(u)), (\forall u)(I(u) \rightarrow f(u))$;
- variables such as $x, y, z, \ldots$ should be understood to represent *points*, i.e., elements of E. $(\exists x)f(x), (\forall x)f(x)$ are abbreviations for $(\exists u)(E(u) \wedge f(u)), (\forall u)(E(u) \rightarrow f(u))$.

We have no problems axiomatizing the notion of ultrametric space in this language, and by universal sentences. Consider the following list of axioms:

$$
\begin{array}{ll}
A_0: & (\exists x)x = x; \\
A_1: & (\forall i)(\forall x)(\exists y)d(x,y) = i; \\
A_2: & (\forall i)(\forall x_1)(\forall x_2)(\exists y)(d(x_1,x_2) = i \rightarrow d(x_1,y) = d(x_2,y) = i); \\
 & \\
A_n: & (\forall i)(\forall x_1)\cdots(\forall x_n)(\exists y) \bigwedge_{1\le\alpha<\beta<n} d(x_\alpha,x_\beta) = i \rightarrow \bigwedge_{1\le\alpha\le n} d(x_\alpha,y) = i; \\
 & \\
B_2: & \neg(\exists i)(\exists x_1)(\exists x_2)(\exists x_3)d(x_1,x_2) = d(x_2,x_3) = d(x_3,x_1) = i \neq 0; \\
 & \\
B_n: & \neg(\exists i)(\exists x_1)\cdots(\exists x_{n+1})(i \neq 0 \wedge \bigwedge_{1\le\alpha<\beta\le n+1} d(x_\alpha,x_\beta) = i);
\end{array}
$$

For $n \geq 2$, axiom A_n says that every isosceles n-gon can be extended to an isosceles $(n+1)$-gon; axiom B_n, on the contrary, says that there is no isosceles $(n+1)$-gon.

An ultrametric space will be called *rich* if it satisfies all the axioms $A_0, A_1, \ldots, A_n, \ldots$; if $n \geq 2$, an ultrametric space will be called n-rich if it satisfies axioms $A_0, A_1, \ldots, A_{n-1}, B_n$.

Lemma 6.23. *Every I-valued space embeds into a rich I-valued space; every I-valued space with no isosceles $(n+1)$-gons embeds into an n-rich I-valued space.*

Proof. Let E be an I-valued space. If E is empty, add a point a to it: The set $\{a\}$, with $d(a,a) = 0$, is obviously an ultrametric space, and axiom A_0 is satisfied.

Now suppose that there is a in E with no element at distance i from a. We then add a point b to E, and extend the distance function on E to $E \cup \{b\}$:

- $d(a,b) = i$.
- If $c \in E, c \neq a$, then necessarily $d(a,c) \neq i$. If $d(a,c) < i$, set $d(b,c) = i$, and if $d(a,c) > i$, set $d(b,c) = d(a,c)$.

We can easily check that we do get an ultrametric distance, and note that if $E \cup \{b\}$ contains an isosceles n-gon, so does E. (If the polygon passes through b, its side length is greater than i, and it does not pass through a, so we can replace b by a.)

Now suppose that we have n points $a_1, \ldots, a_n$ in E all at distance i from one another, but that this isosceles polygon cannot be extended. We then add a point b to E with the following distances:

- $d(a_1,b) = \cdots = d(a_n,b) = i$.

- If c is in E and distinct from $a_1, \ldots, a_n$, then for one of the a_m, $d(a_m, c) \neq i$; if $d(c, a_m) > i$, in this case all the $d(c, a_h)$ are equal, and we set $d(c, b) = d(c, a_h)$; if, on the other hand, $d(c, a_m) < i$, then $d(c, a_h) = i$ for all $h \neq m$, and we set $d(c, b) = i$.

The task of checking that this is actually an ultrametric distance is left to the reader. Note that i is the minimum of the distances from b to E.

Now we show that if E contains no isosceles $(n+2)$-gon, the same is true of $E \cup \{b\}$. Suppose that $c_1, \ldots, c_{n+1}, b$ form such a polygon, whose sides have length j. If $j > i$, all the $d(c_m, a_h)$ are equal to j, which gives us an isosceles $(n+2)$-gon in E. If $j = i$, then each c_h is associated with a unique a_k such that $d(c_h, a_k) < i$, by the maximality of the polygon $a_1, \ldots, a_n$; as there is at most one c_h, we have, for example, $d(c_1, a_1) < i, d(c_2, a_1) < i$, which, with $d(c_1, c_2)$, contradicts the ultrametric inequality.

That being noted, starting from an I-valued space E, and adding points one by one, we get an I-valued extension E_1 such that every point a of E is distance i from some point b in E_1, and such that every isosceles n-gon in E extends to an $(n+1)$-gon in E_1. Repeating this process, we get a sequence of I-valued spaces $E \subset E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots$ whose limit is rich.

Now, if E has no isosceles $(n+1)$-gons, we can do the same, but extending only isosceles polygons with fewer than n vertices, and without introducing any isosceles $(n+1)$-gons by our observation; thus we embed E into an n-rich space. □

The theory of a rich space includes, in particular, the theory of its order; we are going to see that the former can be reduced to the latter: If the chains I and J are elementarily equivalent, a rich I-valued space is elementarily equivalent to every rich J-valued space. Every tuple $(\vec{\imath}, \vec{x})$ formed by an n-tuple of elements of I and an m-tuple of elements of E, is associated with the $(n + m(m-1)/2)$-tuple formed by $\vec{\imath}$ and the distances among the elements of $\vec{x}$: This tuple will be called the *distance tuple* of $(\vec{\imath}, \vec{x})$.

Theorem 6.24. *Two rich ultrametric spaces are elementarily equivalent whenever their chains are, and an extension of rich spaces is elementary whenever the extension of their associated chains is elementary; more precisely, two tuples taken from such spaces have the same type iff their distance tuples have the same type in the sense of the theory of chains. The same result holds for n-rich spaces.*

Proof. Let two rich ultrametric spaces (I, E) and (J, F), assumed as usual to be ω-saturated, be given. Let us show that this implies that the chains I and J are ω-saturated (this is a particular case of a general result on one structure that can be interpreted in another; see Section 9.4). Indeed, if $\vec{a}$ is a finite subset of I, and if p is a type over $\vec{a}$ in the sense of the theory of the order I, it is realized by an element α of an ultrapower I^U of I. The element α also belongs to the ultrametric space $(I, E)^U$. Let q be a type over $\vec{a}$ in the sense of the theory of this space; since (I, E) is ω-saturated, q is realized by an element β of I that, if we restrict types to the language of the order I, realizes p.

That being the case, suppose further that I and J are elementarily equivalent, and consider $\vec{a}$ in (I, E) and $\vec{b}$ in (J, F) such that the associated distance tuples $\vec{a}'$ and $\vec{b}'$ have the same type in the language of chains. We must show that $\vec{a}$ and $\vec{b}$ are ∞-equivalent.

Add, for example, α to the left. If $\alpha \in I$, it is enough to answer it with a β in J whose type over $\vec{b}'$ corresponds to the type of α over $\vec{a}'$, which is possible since J is ω-saturated. Now suppose that α is in E. If $\vec{a}$ contains no element of E, we can answer it with any β in F; otherwise, let a be in $\vec{a}$ such that $d(\alpha, a)$ is minimal, and let b be the point in $\vec{b}$ corresponding to a. If $d(a, \alpha)$ is not in $\vec{a}'$, let j be an element of J whose type over $\vec{b}'$, in the sense of the theory of the order, corresponds to the type of $d(\alpha, a)$ over $\vec{a}'$. As F is rich, there is a point β in F such that $d(\beta, b) = j$. Then the ultrametric inequality determines all the distances from β to $\vec{b}$, and the tuples $\vec{a}^\frown\alpha$ and $\vec{b}^\frown\beta$ have distance tuples of the same type.

Now suppose that $d(a, \alpha)$ is in $\vec{a}'$. Let P be a maximal isosceles polygon contained in $\vec{a}$, with side $d(a, \alpha)$, passing through a. (The polygon P might be reduced to just $\{a\}$.) As $d(a, \alpha)$ is minimal, all distances from α to P are equal to $d(a, \alpha)$, and the other distances from α to $\vec{a}$ are determined by the ultrametric inequality. If Q is the isosceles polynomial of $\vec{b}$ corresponding to P, it is enough to answer with an element β such that $Q \cup \{\beta\}$ is isosceles, with $d(b, \beta)$ corresponding to $d(a, \alpha)$. (The precision is needed only if P has only one vertex.)

The same proof works for n-rich spaces: We simply never encounter isosceles polygons with more than n vertices. □

If we repeat the proof of Theorem 6.24, we see that in two rich I-valued spaces, two tuples $\vec{a}$ and $\vec{b}$ with the same associated distance tuples are ∞-equivalent, so two rich I-valued spaces are ∞-equivalent for a fixed I (i.e., the elements of I are kept fixed in the back-and-forth process); consequently, if I is denumerable, by Theorem 1.14, there is, up to isomorphism (I-isomorphism!) one and only one denumerable rich I-valued space. In this case, by Löwenheim's Theorem, every rich I-valued space has a denumerable I-valued elementary restriction, and this denumerable rich I-valued space is clearly the unique prime model, over I, of the theory of rich spaces. We shall see in Section 10.6 that for every chain I, even nondenumerable, there is a prime rich I-valued ultrametric space, but that this prime space is not always unique. The same results hold for n-rich spaces.

As a preview of the chapters on stability, we are going to describe the types of points (i.e., of elements of the space, rather than of the chain) over a model (I, E) of the theory of rich spaces corresponding to a given theory of chain. By Theorem 6.24, the type of x over (I, E) is determined by the types over I of the tuples made up of the $d(x,a)$, where a ranges over E; this allows us to classify types in four classes:

- *realized types*: the type of an element a of E, that is to say that there exists a in E such that $d(x,a) = 0$;

- *distance types*: there is an element a in E such that $d(a,x)$ is not in I (but is in an elementary extension of I!); the type of x is then entirely determined by a and by the type of $d(a,x)$ over I; in fact, if b is in E and $d(a,b) < d(a,x)$, then $d(b,x) = d(a,x)$, and if $d(a,b) > d(a,x)$, then $d(b,x) = d(a,b)$;

- *polygonal types*: all the $d(a,x)$ are in I and have a minimum i; then let a be such that $d(a,x) = i$, and let P be a maximal isosceles polygon in E with side length i passing through a (we can easily see that if Q is a second such polygon, there is a canonical bijection from P to Q, the one that associates the element b with the unique b' in Q such that $d(b,b') < i$). The type of x is entirely determined by a and i: If b is in P, then $d(b,x) \leq i$, and in fact $d(b,x) = i$ by minimality of i; if $b \notin P$, then there is c in P with $d(b,c) \neq i$, and the ultrametric inequality determines $d(b,x)$.

- *pseudo-limit types*: in the remaining case, all the $d(a,x)$ are in I, but there is no least element of the set A of elements i of I such that for some a_i in E, $d(x,a_i) = i$. Then A is actually a final segment of I: If $i \in A, i < j$, we take a_j such that $d(a_i,a_j) = j$, and then $d(x,a_j) = j$. The sequence a_i is what we call a pseudo-Cauchy sequence: If $i < j$, we have $d(a_i,a_j) = j$; a pseudo-limit for such a sequence is an element α (not necessarily unique if A does not have infimum 0!) such that $d(\alpha,a_i) = i$ for every i in A. We see that a pseudo-limit type is determined by the set A and a sequence a_i of elements of E that converges to x: This sequence must have no pseudo-limit a in E, as otherwise $d(x,a)$ would be less than every i in A.

We call a space *maximally complete* if every pseudo-Cauchy sequence in it has a pseudo-limit; we can easily see, by realizing the pseudo-limit types one by one, that every I-valued space embeds into a maximally complete I-valued space: Over such a space there are no more pseudo-limit types.

Note that only the distance types require us to extend the order I: The others can all be realized in an elementary extension (I,F) of (I,E).

Finally, the situation is more or less similar with n-rich spaces, except that there are no polygonal types (over a model of T).

6.5 Modules and Existentially Closed Modules

We can associate the following language with a left A-module M over a unitary ring A:

- a binary function symbol to denote the addition of M;
- a constant symbol to denote the zero of M;
- for every α in A, a unary function symbol to denote multiplication by α.

We can see that the ring A is not the structure under study; rather, it is incorporated into the language. The reader can write the (universal) axioms that A-modules satisfy. In the case where A is the ring of integers $\mathbf{Z}$, a $\mathbf{Z}$-module is nothing but an abelian group; then we can make do with just a symbol for addition and a symbol for opposite, since $nx = x + \cdots + x$.

For every complete theory T of an A-module, we are going to describe the types over models of T; for this purpose we will need a preliminary lemma, which we shall use later:

Lemma 6.25 (B.H. Neumann's Lemma). *Let G be a group, not necessarily abelian, and let $K_1, \ldots, K_n$ be finitely many right or left cosets of subgroups of G such that $G = K_1 \cup \cdots \cup K_n$; then G is covered by those among the K_i that are cosets of groups of finite index in G.*

Proof. As $Ha = aa^{-1}Ha$, the K_i all have the form aH_i for some subgroup H_i of G; the proof is by induction on the number m of these subgroups.

For $m = 1$ this is obvious: The only group involved must have finite index.

Let us show the step from m to $m+1$. So we are considering a minimal finite family of cosets covering G, the number of subgroups involved being $m+1$. If H is one of them, every class aH, if it is not in the family, is covered by the bH' in the family with $H \neq H'$. Translating everything by a^{-1}, and using the induction hypothesis, we see that the bH' for which $H \cap H'$ has finite index in H are enough to cover aH. We thus conclude that since the family is minimal, whenever we take two subgroups H and H' occurring in it, $H \cap H'$ has finite index in each of them.

Consequently, the intersection L of all the H_i has finite index in each of them, and every coset modulo H_i can be decomposed into finitely many cosets modulo L. Starting with the original covering, we get a covering of G by finitely many cosets modulo L; L must therefore have finite index, and hence so does every H_i. $\square$

We shall call a formula *primitive* if it has the form

$$(\exists y_1) \cdots (\exists y_n)\big(\bigwedge E_i(\vec{x}, \vec{y})\big),$$

where the $E_i(\vec{x}, \vec{y})$ are equations, i.e., formulas, of the form

$$\alpha_1 x_1 + \cdots + \alpha_n x_n + \beta_1 y_1 + \cdots + \beta_m y_m = 0,$$

where α_i and β_j are coefficients in A.

Note that the conjunction of two primitive formulas is equivalent to a primitive formula (put all the existential quantifiers at the front, after having made the sets of quantified variables disjoint).

If M is an A-module, and if $f(\vec{x})$ is a primitive formula in n free variables, then the tuples that satisfy it form a subgroup, which we shall call *primitive*, of M^n; call this group $G_{f(\vec{x})}$.

A formula $f(\vec{x}, \vec{a})$ obtained by replacing the tuple $\vec{x}'$ of a primitive formula without parameters $f(\vec{x}, \vec{x}')$ by a tuple of elements of M will be called a *primitive formula with parameters in* M. Note that the elements of M^n that satisfy $f(\vec{x}, \vec{a})$ form a coset modulo the primitive subgroup $G_{f(\vec{x},\vec{0})}$.

Theorem 6.26. *Let T be a complete theory of a module, and let M be a model of T. Then two tuples in an elementary extension of M have the same type over M iff they satisfy the same primitive formulas with parameters in M.*

Proof. Let N be an ω-saturated model of the theory $T(M)$, and let $\vec{a}$ and $\vec{b}$ be two tuples in N that satisfy the same primitive formulas with parameters in M; we need to show that $\vec{a}$ and $\vec{b}$ are ∞-equivalent.

Add an element c, for example, to $\vec{a}$. If $f(\vec{x}, y)$ is a primitive formula with parameters in M satisfied by $\vec{a}\frown c$, then $(\exists y) f(\vec{x}, y)$ is primitive and satisfied by $\vec{a}$, and hence also by $\vec{b}$. As the conjunction of finitely many primitive formulas is also primitive, the set of all formulas $f(\vec{b}, y)$, where $f(\vec{x}, y)$ is a primitive formula with parameters in M that is true for $\vec{a}\frown c$, is consistent, and by ω-saturation there is a d in N such that $\vec{b}\frown d$ satisfies all of them.

We therefore have d such that every primitive formula that is true for $\vec{a}\frown c$ is also true for $\vec{b}\frown d$. What we need is an e with this property, and also that every primitive formula false for $\vec{a}\frown c$ is also false for $\vec{b}\frown e$!

If $f(\vec{m}, \vec{x}, y)$ is a primitive formula, with parameters $\vec{m}$ in M, which is true for $\vec{a}\frown c$, we want e to be congruent to d modulo $G_{f(\vec{0},\vec{0},y)}$.

Now let $g(\vec{m}, \vec{x}, y)$ be such a formula that is false for $\vec{a}\frown c$; if there exists $f(\vec{m}, \vec{x}, y)$ true for $\vec{a}\frown c$ such that $N \models \neg(\exists y)(f(\vec{a}, y) \wedge g(\vec{a}, y))$, the same is true when we replace $\vec{a}$ by $\vec{b}$ that satisfies the same primitive formulas, and we have nothing to require of e: The simple fact of satisfying $f(\vec{b}, y)$ prevents it from satisfying $g(\vec{b}, y)$.

If this is not the case, but if $G_{g(\vec{0},\vec{0},y)} \cap G_{f(\vec{0},\vec{0},y)}$ has finite index n in $G_{f(\vec{0},\vec{0},y)}$, let us show that we need require nothing more of e. As the two subgroups of M in question are defined by a formula without parameters, this fact can be expressed in the theory T of the module M, and there exist $u_1, \ldots, u_n$ in M such that every element of $G_{f(\vec{0},\vec{0},y)}$ is congruent to u_1 or .

. . or to u_n modulo $G_f \cap G_g$: A coset of the former group partitions into n cosets modulo the latter. Then if c' is an element satisfying $g(\vec{a}, y) \wedge f(\vec{a}, y)$, c and c' are congruent modulo G_f, and there is a unique u_i such that $c+u_i$ and c' are congruent modulo G_g. There is therefore a unique u_i such that $\vec{a}\frown c$ satisfies the formula $g(\vec{x}, y+u_i)$, which is primitive and has parameters in M: If $\vec{a}\frown c$ does not satisfy $g(\vec{x}, y)$, then u_i is simply not in $G_{g(\vec{0},\vec{0},y)}$; as $\vec{b}\frown d$ satisfies the same formula, the simple fact of satisfying $g(\vec{x}, y+u_i)$ prevents it from satisfying $g(\vec{x}, y)$.

In the remaining case, there is an element d_g such that $\vec{b}\frown d_g$ satisfies $g(\vec{m}, \vec{x}, y)$, and we want e not to be congruent to d_g modulo the group $G_{g(\vec{0},\vec{0},y)}$.

To summarize the situation: We have a family G_f of groups, closed under finite intersections, and another family G_g of groups such that G_g never has finite index in any G_f; we want an e that is congruent to d modulo each G_f, and not congruent to d_g modulo G_g, for every G_g. By Lemma 6.25, a finite union of cosets modulo some of the G_g cannot cover any coset modulo G_f, and what we are looking for is consistent.

But in the language of $T(M)$, the condition that we wish to realize is expressed by the unique satisfaction of formulas with parameters in $\vec{b}$: Indeed, d and d_g are involved only through their cosets, and instead of saying "x is congruent to d modulo G_f," which requires using d as a parameter, we could say "x is in the coset modulo G_f of the elements that satisfy f," that is to say (to return to our starting point), "x satisfies $f(\vec{b}, x)$"! As a consequence, since N is ω-saturated, such an e does indeed exist in N. $\square$

Now let us look for a convenient way to describe the types in one variable over a model M of T. The type p of x over M is determined by the primitive formulas $f(x, \vec{a})$, with parameters $\vec{a}$ in M, that x satisfies; if $p \models f(x, \vec{a})$, then $M \models (\exists x)(f(x, \vec{a}))$, and there is an element a_f in M that satisfies this formula, and that is therefore congruent to x modulo $G_{f(x,\vec{0})}$. Changing the notation, this amounts to saying that the type p of x over M is determined by the congruences $x \sim a$ (modulo G) that it satisfies, where a is in M and G is a primitive subgroup of M. (As G is defined by a formula without parameters, $x - a \in G$ is indeed a formula in our language.)

We shall say that a family F of primitive subgroups of M is a *filter of groups* if the following conditions are satisfied:

- M is in F;
- if G and H are in F, then so is their intersection;
- if G is in F and $G \cap H$ has finite index in G, then H is in F.

To a type p of $S_1(M)$ we associate a filter F_p of primitive groups, consisting of all G such that there exists a_G in M such that p implies that x is congruent to a_G modulo G; we just noted that p is determined by F_p and the a_G (or more exactly, by their cosets modulo G).

Naturally, the a_G must satisfy the coherence condition; that is to say, if $G \subset G'$, then a_G and $a_{G'}$ must be congruent modulo G'; this, however, is the only constraint, and every filter F has the form F_p no matter which model M of T we are in. For by Lemma 6.25, it is consistent to say that x is congruent to 0 modulo each element of F, and not congruent to each element of M modulo each primitive subgroup that is not in F.

Likewise, if $p \in S_1(M)$ and $F_p \subset F'$, there is an elementary extension N of M and a son q of p over N such that $F_q = F'$: Let a realize p in some extension N of M, and take q to be the type stating that x is congruent to a modulo every G of F', and not congruent to any element of N for any primitive group outside of F'.

Finally, note that if p is in $S_1(M)$ and M is an elementary restriction of N, then p has one and only one son q over N such that $F_p = F_q$. For, as q adds no new cosets that we do not already know, there can be no choice. This is is called the *heir* or the *nonforking* son of p; the other sons, which increase the filter of groups, are its *forking* sons.

This second encounter with forking (we used this term before, in Section 6.2) heralds the study of stability in a general framework, which will be undertaken from Chapter 11 on. Before taking off toward such lofty heights, we still have earthly chores to do. For now, let us look for a condition to get a model companion to the theory of A-modules, which is universal in the language we have specified. To raise the stakes, we shall first develop a little theory on the quest for model companions, theory for which modules provide a good illustration.

Let M be a structure, and let N be an extension of M. We call M *existentially closed in* N if for every $\vec{a}$ in M and every quantifier-free formula $f(\vec{x},\vec{y})$ in the language of M, if $N \models (\exists\vec{y})f(\vec{a},\vec{y})$, then $M \models (\exists\vec{y})f(\vec{a},\vec{y})$.

Lemma 6.27. *M is existentially closed in N if and only if there is an elementary extension M_1 of M such that $M \subset N \subset M_1$.*

Proof. If $M \prec M_1$, then M is existentially closed in M_1, and a fortiori in N.

Conversely, if M is existentially closed in N, let us give a name to each element of N and consider the set of sentences $T(M) \cup D(N)$, where $T(M)$ is the set of all sentences true in M, and $D(N)$ the set of quantifier-free sentences true in N. This set is consistent, since for any of its finite fragments, we can interpret the constants of $N - M$ by elements of M; it therefore has a model, which is exactly what we are looking for. □

Lemma 6.28. *If M is existentially closed in N, and if M_1 is elementarily equivalent to M, then there is an existentially closed embedding of M_1 into a structure N_1 elementarily equivalent to N.*

Proof. 1st method: for some ultrafilter U we have $M_1 \prec M^U \subset N^U = N_1$; we can check that M^U is existentially closed in N^U, and therefore so is M_1.

2nd method: let T be the theory of N. Give a name to each element of M_1 and let T_1 be the set of universal sentences which are true for the tuples of M_1. The set $T \cup T_1$ is consistent, since, for each of its finite fragments, we can take N as a model with constants from M (if $M_1 \models (\forall \vec{y}) f(\vec{a}, \vec{y})$ then $M \models (\exists \vec{x})(\forall \vec{y}) f(\vec{x}, \vec{y})$). So this set of sentences has a model, which is exactly what we are looking for. □

Now we consider a universal theory $T_\forall$. A model of $T_\forall$ is called *existentially closed* (implicit: among the models of $T_\forall$) if it is existentially closed in every extension that is a model of $T_\forall$.

Lemma 6.29. *Every model of $T_\forall$ embeds into such an existentially closed model.*

Proof. Let M be a model of $T_\forall$. Enumerate the quantifier-free formulas $f(\vec{x}, \vec{a})$ with parameters in M. If f_0 is the first one, and there is a model of $T_\forall$ extending M in which this formula has a realization, choose one, say M^0; otherwise, let $M^0 = M$. Then if f_1 has a realization in an extension of M^0 (be careful: It may be that f_1 has a realization in an extension of M, but not in an extension of M^0!), choose one, say M^1. Iterating this procedure, we get an extension M_1 of M, a model of $T_\forall$ (since the satisfaction of universal sentences is preserved by taking limits), such that every quantifier-free formula with parameters in M that has a realization in an extension of M_1 has one in M_1.

We then construct a sequence $M \subset M_1 \subset \cdots \subset M_n \subset \cdots$ whose limit is existentially closed. □

Lemma 6.30. *An elementary restriction of an existentially closed model is existentially closed.*

Proof. If $M \prec M_1$ and M_1 is existentially closed, and if $M \subset N$, then there is a model of $T_\forall$ that is a common extension of N and M_1 (since every universal sentence with parameters in M that is true in N is also true in M, and therefore also in M_1; we can also embed M_1 into an ultrapower of M, keeping M fixed). So if $\vec{a} \in M, \vec{b} \in N$, if $f(\vec{x}, \vec{y})$ is a quantifier-free formula, and if $f(\vec{a}, \vec{b})$ is true (in N!), then it is true in this common extension, and consequently $M_1 \models (\exists \vec{y}) f(\vec{a}, \vec{y})$, so $M \models (\exists \vec{y}) f(\vec{a}, \vec{y})$. □

Lemma 6.31. *If every elementary extension of an existentially closed model is existentially closed, then every extension of existentially closed models is elementary.*

Proof. Let M and N be existentially closed, such that $M \subset N$. By Lemma 6.27, there is an elementary extension M_1 of M such that $M \subset N \subset M_1$; there is also an elementary extension N_1 of N such that $M \subset N \subset M_1 \subset N_1$. By hypothesis M_1 and N_1 are existentially closed, so we can repeat the process to construct two intertwined elementary chains $M \subset N \subset \cdots \subset M_n \subset N_n \subset \cdots$ whose common limit P is an elementary extension of both M and N. A sentence with parameters in M is true in M iff it is true in P iff it is true in N, so N is an elementary extension of M. □

Exercise 6.32. A $\forall\exists$ sentence is a sentence of the form $(\forall\vec{y})(\exists\vec{x})f(\vec{y},\vec{x})$, where f is quantifier-free. If T is a theory, $T_{\forall\exists}$ is the set of its $\forall\exists$ consequences.

Show that if M is a model of $T_{\forall\exists}$, then it is existentially closed in a model of T and has an elementary extension that is a limit of models of T. Show that T has a $\forall\exists$ axiomatization iff every limit of models of T is also a model of T.

One more definition: A class of $\mathcal{L}$-structures is called *elementary* if it consists exactly of the models of some theory T (not necessarily complete).

Theorem 6.33. *A universal theory $T_\forall$ has a model companion T iff the class of its existentially closed models is elementary; in this case, T is their theory.*

Proof. If the existentially closed models are exactly the models of a theory T, then by Lemma 6.29 T is a companion of $T_\forall$, and by Lemma 6.31 T is model complete.

Conversely, assume that $T_\forall$ has a model companion T. If M is a model of T, N a model of $T_\forall$, and $M \subset N$, then N embeds into a model M_1 of T, $M \subset N \subset M_1$; however, by the model completeness of T, the extension $M \subset M_1$ is elementary, so M is existentially closed in M_1, and a fortiori in N. So every model of T is existentially closed.

If M is existentially closed, it embeds (in an existentially closed way, of course) into a model N of T; then we have, by Lemma 6.27, $M \subset N \subset M_1$, where M_1 is an elementary extension of M; we do not yet know that M_1 is existentially closed, but we know, by Lemma 6.28, that it embeds in an existentially closed way into a model N_1 of T. We iterate to get a chain of models $M \subset N \subset M_1 \subset N_1 \subset \cdots \subset M_n \subset N_n \subset \cdots$ where the chain of the M_n is elementary; by the model completeness of T, so is the chain of the N_n. The common limit of these two sequences is therefore an elementary extension of both M and N, so M is a model of T. □

Let us apply this marvelous result to modules.

A module is said to be of *finite type* if it has a finite generating system (we also, more simply, say "finitely generated"): It is then an image of a free module A^n. It is further said to be *finitely presented* if it has the form A^n/R, where R is a submodule of A^n of finite type. A *presentation* of M is just the specification of a generating system $e_1, \ldots, e_n$ of M and a list $r_1, \ldots, r_k$ of relations among $e_1, \ldots, e_n$ generating R. Note that if M is presented, in the generating system $e_1, \ldots, e_n$, by the relations $r_1, \ldots, r_k$, and if $f_1, \ldots, f_m$ is another generating set of M, if we define $e'_1, \ldots, e'_m$ as the expressions defining $e_1, \ldots, e_n$ in terms of $f_1, \ldots, f_m$, we get a presentation of M by replacing $e_1, \ldots, e_n$ in $r_1, \ldots, r_k$ by $e'_1, \ldots, e'_n$ and adding the expressions defining $f_1, \ldots, f_m$ in terms of $e'_1, \ldots, e'_n$. We thus see that if M is finitely presented, then every finite generating system of M gives rise to a finite presentation.

A ring A is called (left) *coherent* if every (left) ideal of A of finite type has a finite presentation (recall that an ideal is just a submodule of A). For example, if A is (left) Noetherian, then it is coherent, since every submodule of A^n has finite type.

Theorem 6.34. *The theory of A-modules has a model companion if and only if the ring A is coherent, in which case it has a model completion that is complete and admits quantifier elimination.*

Proof. First, suppose that A is incoherent. Then there exist $\alpha_1, \ldots, \alpha_n$ in A generating a non-finitely-presented ideal; for every finite system R of relations satisfied by $\alpha_1, \ldots, \alpha_n$ there exist $a_1, \ldots, a_n$ in a module satisfying these relations that do not satisfy all of the relations on $\alpha_1, \ldots, \alpha_n$. Take the module A^n/R: This module can be embedded into an existentially closed module M^R. As $a_1, \ldots, a_n$ do not satisfy all the relations on $\alpha_1, \ldots, \alpha_n$, there cannot be an x such that $\alpha_1 x = a_1 \wedge \cdots \wedge a_n x = a_n$.

Consequently, the following list of axioms is consistent:

- The axioms of the theory T of existentially closed modules.
- For every relation $\lambda_1 \alpha_1 + \cdots + \lambda_n \alpha_n = 0$ on $\alpha_1, \ldots, \alpha_n$, the sentence $\lambda_1 a_1 + \cdots + \lambda_n a_n = 0$.
- $\neg(\exists x)(\alpha_1 x = a_1 \wedge \cdots \wedge \alpha_n x = a_n)$.

I claim that a model M of this theory is not existentially closed, which proves that existentially closed modules do not form an elementary class. Indeed, since $a_1, \ldots, a_n$ generate an image module of the ideal generated by $\alpha_1, \ldots, \alpha_n$, the module $M \oplus Ax$, with the relations $\alpha_1 x - a_1, \ldots, \alpha_n x - a_n$ factored out, is an extension of M.

Now suppose that A is coherent. We consider the following list of axioms consisting, in addition to the axioms of A-modules, of the following sentence for every ideal $\{\alpha_1, \ldots, \alpha_n\}$ of A of finite type, presented by the relations $r_1(\alpha_1, \ldots, \alpha_n), \ldots, r_k(\alpha_1, \ldots, \alpha_n)$, and every $\beta_1, \ldots, \beta_m$ that are not in the ideal generated by $\alpha_1, \ldots, \alpha_n$:

$$\begin{aligned}(\forall x_1) \quad & \cdots (\forall x_n)(\forall y_1) \cdots (\forall y_m)(\exists z)\Big(\bigwedge r_i(x_1, \ldots, x_n) \to \\ & (\alpha_1 z = x_1 \wedge \cdots \wedge \alpha_n z = x_n \wedge \beta_1 z \neq y_1 \wedge \cdots \wedge \beta_m z \neq y_m)\Big).\end{aligned}$$

In the case of the ideal 0, presented by 1, we get the following axiom: $(\forall y_1) \cdots (\forall y_m)(\exists z)(0z = 0 \wedge \beta_1 z \neq y_1 \wedge \cdots \wedge \beta_m z \neq y_m)$.

These axioms imply that if $a_1, \ldots, a_n$ satisfy the presenting relations $\alpha_1, \ldots, \alpha_n$, then the tuple $(a_1, \ldots, a_n)$ is divisible by the tuple $(\alpha_1, \ldots, \alpha_n)$, the quotient z being forced further to satisfy certain inequalities.

Take, for example, the ring $\mathbf{Z}$ of integers: Since an ideal $n\mathbf{Z}$ of $\mathbf{Z}$ is isomorphic, as a module, to $\mathbf{Z}$, it has a null presentation, and we get the following axioms, where $n_1, \dots, n_m$ are not divisible by n:

$$(\forall x)(\forall y_1) \cdots (\forall y_m)(\exists z)(x = nz \wedge n_1 z \neq y_1 \wedge \cdots \wedge n_m z \neq y_m).$$

We can easily see that the models of this theory are the divisible abelian groups with infinitely many elements of order p for every prime p.

This list of axioms is consistent, and a companion of the theory of modules; indeed, if $a_1, \dots, a_n$ satisfy the presentation R of $\alpha_1, \dots, \alpha_n$ in the module $M \oplus Az/(\alpha_1 z - a_1, \dots, \alpha_n z - a_n)$, then $\beta z \in M$ iff β is in the ideal generated by $\alpha_1, \dots, \alpha_n$; consequently, every existentially closed module is a model of T.

We will now show that T is complete and admits quantifier elimination. For this purpose, consider ω-saturated models M and N of T, with tuples $\vec{a}$ and $\vec{b}$ generating isomorphic submodules m and n; we need to show that $\vec{a}$ and $\vec{b}$ are ∞-equivalent.

Add c to $\vec{a}$. What determines the isomorphism type of the module generated by c and $\vec{a}$? Let us consider the ideal I of those α in A such that $\alpha c = a_\alpha \in m$: This module is isomorphic to $M \oplus Ac$ with the $\alpha c - a_\alpha$ factored out.

It is clear that if $\alpha_1, \dots, \alpha_n$ are in I, then $a_{\alpha_1}, \dots, a_{\alpha_n}$ satisfy the presentation of $\alpha_1, \dots, \alpha_n$. For every α in I, let b_α be the element of the module n corresponding to a_α; consider the following list of axioms:

- $\alpha z = b_\alpha$ for every α in I;
- $\beta z \neq b$ for every $\beta \notin I$ and every $b \in n$.

As the module N satisfies the axioms of T, every finite fragment of this list is satisfied by an element of N; by compactness and ω-saturation, there is an element d of N satisfying all of them, and $\vec{a}\frown c$ and $\vec{b}\frown d$ generate isomorphic submodules. □

6.6 Real Closed Fields (not in the original edition)

Although the name does not reflect the fact, the field of complex numbers is the prototype of an algebraically closed field; in contrast, the field of real numbers is indeed the typical example of a real closed field! We are going to determine its theory: It is an important example, from the historical perspective, since real geometry (see Euclid) can be translated exactly into this theory.

Unlike the complex numbers, the reals are equipped with a natural order that is compatible with the field operations. We should therefore first define the concept of *ordered field.* It amounts to a field with a distinguished subset, called the cone of positive (or null) elements, that must satisfy the following conditions:

1. -1 is not positive;
2. for any x, one of x and $-x$ is positive;
3. the sum of two positive elements is positive;
4. the product of two positive elements is positive.

These conditions can be translated immediately into universal axioms in the language of fields augmented by a unary predicate $\Pi(x)$ denoting the set of positive elements.

Let us develop a few immediate consequences of these axioms:

- 1 is positive by (1) and (2), and since -1 cannot be the sum of positive elements by (3), we are working in characteristic zero.
- By (2) and (4), every square is positive.
- If x is positive and nonzero, then $-x^{-1}$ cannot be, so x^{-1} is positive, and consequently $-x$ is not positive (we say that it is negative). In other words, if x is nonzero, then exactly one of x and $-x$ is positive; we write it as $|x|$ and call it the absolute value of x.
- It follows that the product of two negative elements, that is to say, two nonpositive elements, or equivalently negatives of positive numbers, is positive, while the product of a positive and a negative number is negative.

The binary relation "$x-y$ is positive" therefore defines a total order—this is easy to check—satisfying the following familiar conditions:

- if $x \leq y$, then $x + z \leq y + z$;
- if $x \leq y$ and $z \geq 0$, then $xz \leq yz$;
- if $x \leq y$ and $z \leq 0$, then $xz \geq yz$.

Conversely, if a total ordering $\leq$ satisfies these conditions, then the set Π of elements ≥ 0 forms a positive cone as above for which $\leq$ is the associated order. In other words, ordered fields can be described in two equivalent languages: that of the cone and that of the order. For our part, we prefer the latter, so that henceforth an ordered field will be a structure $(K; =, \leq, 0, 1, +, -, \cdot)$.

The ordering of an ordered field is always dense, since $(x+y)/2$ is always contained between x and y; there are no endpoints, since $x-1 < x < x+1$.

By discussing all possibilities for signs, we can show that $|x+y| \leq |x|+|y|$; $|xy| = |x| \cdot |y|$ is immediate.

We shall say that an ordered field K is *real closed* if it satisfies the intermediate value theorem for polynomials: If $P(x) \in K[x]$, $a, b \in K$, and

$P(a)$ and $P(b)$ have opposite signs, then P has a zero in K that is sandwiched between a and b. This notion is easily axiomatized in the language of ordered fields by introducing an axiom for every possible degree of the polynomial P.

In fact, it can be axiomatized only in the language of fields. Indeed, if K is real closed, then every positive element a is a square, since the polynomial $x^2 - a$ changes sign between 0 and $\max(1, a)$, so the ordering can be defined solely from the field operations: $x \leq y$ if and only if $y - x$ is a square. However, in order to eliminate quantifiers, we shall continue to consider these fields in the language of ordered fields.

A typical example: The field of reals is real closed. The goal of this section is to show that all real closed fields are elementarily equivalent; in other words, we are dealing with the theory of the reals.

We begin with a few lemmas. The derivative of a polynomial is defined formally, monomial by monomial. When we replace the unknown by $a+x$ in the polynomial $P(x)$, we get another polynomial; as we are in characteristic zero, this second polynomial can be expressed by Taylor's formula, $P(a + x) = P(a) + P'(a) \cdot x + \cdots + P^{(d)}(a) \cdot x^d/d!$, where d is the degree of P. It is sufficient to prove this for monomials x^n, which can be done by the binomial formula.

Lemma 6.35 (Bounding of Zeros). *If $P(x)$ is a unitary polynomial with coefficients in the ordered field K, $P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0$, and if m is the maximum of the absolute values of $a_{d-1}, \ldots, a_0$, then the zeros of P in K are contained between $-m-1$ and $m+1$.*

Proof. If $|x| \geq m+1$, then $|P(x) - x^d| \leq m \cdot (|x|^{d-1} + \cdots + 1) = m(|x|^d - 1)/(|x| - 1) \leq |x|^d - 1$; it is therefore impossible to have $P(x) = 0$. □

Lemma 6.36 (Continuity). *In an ordered field, if $P(a) > 0$, then there is some $\epsilon > 0$ such that $P(x)$ is always positive on the interval $[a - \epsilon, a + \epsilon]$.*

Proof. If m is the maximum of the absolute values of the coefficients of the polynomial $P(a + x)$, take ϵ to be the minimum of 1 and $P(a)/2md$. □

Lemma 6.37 (Local Monotonicity). *In an ordered field, if $P'(a) > 0$, then there is some $\epsilon > 0$ such that $P(x) > P(a)$ if $a < x \leq a + \epsilon$, and $P(x) < P(a)$ if $a - \epsilon \leq x < a$.*

Proof. Apply Lemma 6.36 to the polynomial $P(a + x) - P(x)/x$. □

Lemma 6.38. *In a real closed ordered field, if P is a nonconstant polynomial whose derivative P' is always positive or zero on the interval $[a, b]$, $a < b$, then $P(a) < P(b)$.*

Proof. We start with the case in which P' is strictly positive on $[a, b]$. If $P(a) \geq P(b)$, then by Lemma 6.37, we can find c and d with $a < c < d < b$ such that $P(c) > P(a)$ and $P(d) < P(a)$. Since we are working in a real closed field, the polynomial $P(x) - P(a)$ has a zero b_1 that is strictly between

c and d. By repeating, we find a second zero b_2 strictly between a and b_1, and we can repeat again, thus contradicting the fact that a nonzero polynomial can have only finitely many zeros.

In the general case, since P' has only finitely many zeros, we can do a partition into consecutive intervals to reduce to the case in which P' is strictly positive except at an endpoint of an interval. Suppose, for example, that $P'(a) = 0$, but that $P'(x) > 0$ for $a < x \leq b$. If $P(a) > P(b)$, then by Lemma 6.36 we can find c with $a < c < b$ such that $P(c) > P(b)$, contradicting what we just did. Therefore, $P(a) \leq P(b)$; however, since $P(a) \leq P((a+b)/2)$, the inequality is strict. □

Remark. Lemma 6.38 is true in all ordered fields, even those that are not real closed, but it is difficult to prove this fact.

Proposition 6.39. *Every ordered field can be embedded into a real closed field.*

Proof. Let K be ordered and not real closed, and let $P(x)$ be a polynomial with coefficients in K that changes sign between a and b without having a zero. By replacing P with $-P$, we can assume that $P(a) < 0, P(b) > 0, a < b$. If we take P of minimal degree, it is irreducible.

The field $K[x]/P(x)$ is therefore the ring of polynomials of degree strictly less than the degree d of P, with the usual addition and with multiplication modulo P. We are going to equip this field with an order extending that of K.

For that purpose, let A be the set of α in K bounded above by some α' in $[a, b]$ such that $P(\alpha') < 0$, and let B be its complement. The set A cannot have a greatest element α_0; otherwise, $P(\alpha_0) < 0$, and its existence contradicts Lemma 6.36. Similarly, there can be no least element β_0 in B, since $P(\beta_0) > 0$, as P does not vanish on $[a, b]$.

By the choice of a minimal P, a nonzero polynomial Q of degree strictly less than d has constant sign on each of the intervals bracketed by its zeros, of which there are only finitely many. There is therefore some α_Q in A and β_Q in B such that Q has constant sign on the interval (α_Q, β_Q). Let us adopt the convention that Q is called positive if it takes only strictly positive values on this interval. It is clear that the sum of two positive polynomials is positive, and that one and only one of Q and $-Q$ is positive, except when $Q = 0$. That leaves the product. If Q_1 and Q_2 are positive, we write $Q_1 \cdot Q_2 = Q \cdot P + R$, where R is the remainder from the Euclidean division of $Q_1 \cdot Q_2$ by P, that remainder being the product of Q_1 and Q_2 in the field under consideration. We note that the polynomial Q is itself of degree strictly less than d, so we can find α in A and β in B such that the four polynomials Q_1, Q_2, Q, and R all have constant sign on $[\alpha, \beta]$. On the other hand, P changes sign there, so $Q \cdot P$ is negative on some part of this interval: It is therefore impossible for R to be negative! We thus get an ordering of the field $K[x]/P(x)$, an order that extends that of K, which is included as the field of constant polynomials.

Therefore, if K is not real closed, we can find a proper ordered extension of it in the algebraic closure of K. Since this algebraic closure is a fixed universe, we can, by repeating the process, transfinitely often if need be, ultimately fall into a real closed field. (See Chapter 8; we could also apply the Kuratowski–Zorn lemma.) □

If k is a subfield of K, then the relative algebraic closure of k in K is defined as the set (in fact, the field) of elements of K that are algebraic over k.

Proposition 6.40. *Let k_1 and k_2 be isomorphic ordered fields embedded into the real closed fields K_1 and K_2. Let L_1 and L_2 be the relative algebraic closures of k_1 in K_1 and of k_2 in K_2. Then every isomorphism between k_1 and k_2 extends uniquely to an isomorphism (of ordered fields) between L_1 and L_2.*

Proof. We have no problem if $k_1 = L_1$ and $k_2 = L_2$. Otherwise, we can find α in $L_i - k_i$ of minimal degree d over k_i. By symmetry, we can assume that $i = 1$. A polynomial $Q(x)$ of degree less than d, with coefficients in k_1, therefore has all its sign changes in K_1 bounded by its zeros, which are all in k_1; the same is true with respect to K_2/k_2.

Let P_1 be the minimal polynomial of α over k_1. Since $P_1'(\alpha) \neq 0$, we can assume (by switching P with its negative) that $P_1'(\alpha) > 0$. I claim that there are a_1 and b_1 in k_1 such that $a_1 < \alpha < b_1$ and that $P_1'(x)$, which has degree $d - 1$, is always positive on the interval $[a_1, b_1]$ of K_1. This is clear if α is located between two consecutive zeros of P_1'; if one or the other is missing (for example, if P_1', having no zeros, has a constant sign), then we can use the sandwiching from Lemma 6.36. By Lemma 6.38, $P_1(a_1) < 0, P_1(b_1) > 0$.

The field $k_1(\alpha)$ is isomorphic to $k_1[x]/P_1(x)$. Since every polynomial in $k_1[x]$ of degree strictly less than d keeps its sign constant on an interval surrounding α with endpoints in k_1, the order defined on $k_1(\alpha)$ by restricting that on K_1 must be the same as that considered in Proposition 6.39.

Let P_2 be the polynomial in $k_2[x]$ corresponding to P_1 under the isomorphism in question, and let a_2 and b_2 be the homologues of a_1 and b_1. Since P_2 changes sign between a_2 and b_2, it has a zero β between these points in K_2, which is real closed. Indeed, it has only one, since $P_2(x)$ defines an increasing function between a_2 and b_2. For the same reason, the order on $k_2(\beta)$ is that described in Proposition 6.39, so that $k_1(\alpha)$ and $k_2(\beta)$ are isomorphic not only as fields, but even as ordered fields.

In other words, as long as $k_1 \neq L_1$ or $k_2 \neq L_2$, the isomorphism between k_1 and k_2 can be extended to larger fields; by a transfinite iteration, we will eventually get an isomorphism between L_1 and L_2.

Why is this isomorphism unique? Because α in L_1 is a zero of its own minimal polynomial P, but because there is an ordering, it is the smallest zero, or the second smallest, or the third, etc., and for its image in L_2 we must take the zero of P_2 of the same number. □

Proposition 6.39 provides us with a real closed field inside the algebraic closure of the ordered field k. By Proposition 6.40, this field is unique up to k-isomorphism, and can be found in any real closed extension field K of k: It is isomorphic to the relative algebraic closure of k in K. We shall call it the *real closure of* k.

If L is the real closure of k, then we can see that every α in L is definable from k as the nth zero of some polynomial $P(x)$ with coefficients in k. In other words, we can find a formula $\phi_\alpha(x)$, with parameters in k, that α alone satisfies in any real closed extension K of k.

Finally, these propositions give us an alternative characterization of real closed fields: An ordered field is real closed if and only if it has no proper algebraic ordered extension.

Always be careful about the following fact: If we forget the order, then the real closure of k is a subfield of its algebraic closure, but there are many ways to embed the real closure into the algebraic closure!

All this grind simply to prepare:

Theorem 6.41. *The theory of real closed fields is complete and admits quantifier elimination.*

Remark. For the elimination, it is essential to have the ordering in the language: restricted to the language of fields, we cannot characterize positivity without quantifiers.

Proof. Let K_1 and K_2 be ω-saturated real closed fields, and suppose that two tuples $\vec{a}$ in the former and $\vec{b}$ in the latter satisfy the same quantifier-free formulas. They therefore satisfy the same equations $P(\vec{x}) = 0$ and the same inequalities $Q(\vec{x}) > 0$ with integer coefficients, implying that $Q(\vec{a})$ and $Q(\vec{b})$ are isomorphic as ordered fields. Note that we are off to a good start whenever $\vec{a} = \vec{b} = \emptyset$, since the field of rationals can be ordered only in one way.

We need to show that if α is added to the left, then we can respond with β on the right in such a way that $\vec{a}^\frown\alpha$ and $\vec{b}^\frown\beta$ still satisfy the same quantifier-free formulas. Let L_1 and L_2 be the respective real closures of $Q(\vec{a})$ in K_1 and of $Q(\vec{b})$ in K_2, which we know to be isomorphic.

If α is in L_1, then we respond with the corresponding β in L_2. If not, let A_1 be the set of points in L_1 that are below α and let B_1 be the set of points in L_1 that are above it, and let A_2 and B_2 be the corresponding subsets of L_2.

First, suppose that neither A_1 nor B_1 is empty. Since L_2 is densely ordered, every condition of the form $c < x < d$, with $c \in A_2$ and $d \in B_2$, is satisfied by some x in L_2; since c and d are definable from $\vec{b}$, this condition translates into a formula (with quantifiers!) $\phi_{cd}(x, \vec{b})$. This family of formulas with parameters $\vec{b}$, being finitely satisfiable, is globally satisfiable in K_2 by ω-saturation. We thus find β such that $A_2 < \beta < B_2$.

This β is transcendental over L_2, since α is over L_1: $L_2(\beta)$ is isomorphic to the field of rational fractions $L_2(x)$. To order this field, it is enough to know the signs of the irreducible polynomials of $L_2[x]$. Those with degree greater than one keep their signs constant in every ordered extension of L_2, since the latter has no proper ordered algebraic extension: The signs are the signs of their respective values at 0. As for those of degree one, their signs are determined just by the cut of β over L_2! We can therefore see that $L_1(\alpha)$ and $L_2(\beta)$ are isomorphic as ordered fields.

The case where one of the classes of the cut is empty is treated in the same way, drawing on the fact that L_2 has neither least nor greatest element. □

According to the editor of the English edition of this work, a good textbook must contain a plethora of exercises; I have therefore made a special effort for this section.

Exercise 6.42. Eliminate the quantifiers in the following formulas:

$$\begin{array}{rl} \phi(x,y): & (\exists z) x \cdot z^{87} + y \cdot z^{44} + 1 = 0. \\ \psi(x,y,z): & (\forall u)(\exists v) x \cdot (u+v^2)^2 + y \cdot (u+v^2) + z = 0. \end{array}$$

1. for the field **C** of complex numbers in the language of fields
2. for the field **R** of real numbers in the language of ordered fields.

Exercise 6.43. Show that Lemma 6.38 is true for the field of rationals, or more generally for all ordered fields that are dense in their real closures.

Exercise 6.44. Consider an ordered field K, a polynomial $P(x) \in K[x]$, and an increasing sequence $a_0, \ldots, a_n$ of elements of K such that for $0 \leq i < n$, $P(a_i)$ and $P(a_{i+1})$ are nonzero and of opposite sign. Show that n is less than the degree of P.

Exercise 6.45. Using the fact that the field of complex numbers is algebraically closed, show that every irreducible polynomial with coefficients in a real closed field has degree one or two.

Exercise 6.46. We allow the following theorem of arithmetic, called the "four squares theorem": *Every natural number is the sum of four squares of natural numbers.*

1. For every natural number n, describe a formula $\phi_n(x)$ in the language of fields that states that the difference $x - n$ is a sum of four squares.
2. Show that this family of formulas is finitely satisfiable in the ring **Z** of integers.

3. Show that no subfield of $\mathbf{R}$ is ω-saturated.

Exercise 6.47.

1. Let K be a field such that (i) -1 is neither a square nor a sum of two squares, and (ii) for any a, either a or $-a$ is a square. Show that K can be ordered in exactly one way.

2. Suppose further that every polynomial of odd degree with coefficients in K has a root in K. Use your knowledge of Galois theory to show that every algebraic (not ordered) extension of K has degree a power of 2 and contains a square root of -1; then show that K is real closed.

3. From (1) and (2) deduce an axiomatization, in the language of fields, of the notion of real closed field.

Exercise 6.48.

1. Show that there are infinitely many prime numbers congruent to 3 modulo 4.

2. Show that there are infinitely many prime numbers p such that -1 is not a square in the field $F_p = \mathbf{Z}/p\mathbf{Z}$.

3. Deduce the existence of a field of characteristic zero such that (i) -1 is not a square, (ii) for every x, either x or $-x$ is a square, and (iii) -1 is a sum of two squares.

6.7 Historic and Bibliographic Notes

Quantifier elimination for algebraically closed fields was considered in its generality as a theorem of Tarski, published in [Tar51]. Let us simply say that this was the translation into modern jargon of the old method of solving systems of polynomial equations and inequations by successive elimination of the unknowns; we can find examples of this ever since mathematics first appeared in Babylonia, and its theoretical consequences have been developed since the Chinese middle ages (see [Hoe77]). In this work, Tarski also proved quantifier elimination for real closed fields, which are to real fields what algebraically closed fields are to fields, that is to say, those satisfying the analogue of the "Nullstellensatz"; understand that this elimination is a fundamental fact of "real geometry," the real analogue of algebraic geometry.

A more substantial success of model theory was the famous theorem of Ax and Kochen [AK65a], [AK65b], [AK66], which, under certain hypotheses, allows us to reduce the theory of a Henselian valued field to those of its residue field and its value group. This result is the one most often brandished by the standard-bearers of model theory. Indeed, it allows us asymptotically to resolve a conjecture of Artin. It was the first witness to the maturity of model theory, its first indisputable application outside the narrow scope of logic. Today, not a few researchers are occupied with extending the Ax–Kochen theorem.

Another significant application is the emergence of differentially closed fields. The algebra of differential fields was studied intensively by Ritt in the first half of this century, then by Ellis Kolchin, whose text [Kol73] should be consulted; it is remarkable that Kolchin had used isolated types under the name "constrained ideal"; however, neither he nor Ritt had the idea of establishing a differential analogue of an algebraic closure. That credit goes to Abraham Robinson, who, drawing on the work of Seidenberg [Sei56] on elimination in systems of differential equations and inequations, noticed that differentially closed fields of characteristic zero satisfied the necessary conditions that he had just established for the existence of a model completion; thus the notion of differentially closed field. The particular axiomatization of this notion given here is due to Lenore Blum [Blu68]; the actual tendency of logicians is to derive Seidenberg's elimination results, synonymous with the "differential Nullstellensatz," from the model complete character of the the theory of differentially closed fields, more often than the other way round.

The existence and uniqueness of the differential closure are consequences of general theorems on the total transcendence of the theory in question; the differential context does not simplify the proof of these facts, which takes some time, and which should also be considered a major contribution of model theory to algebra. The nonminimality of the differential closure was proved simultaneously by Kolchin [Kol74], Shelah [She73], and Rosenlicht [Ros74]; for this result we will can consult [Gra83] and the neighboring articles.

The model theory of differentially closed fields has not been well mastered, and the problem of classifying its denumerable models is still open.[1]

There is a theory of differentially closed fields of characteristic p, which is technically more complex; it is the model companion, but not the model completion, of the theory of differential fields of characteristic p, and was described by Carol Wood [Woo73].

Boolean algebras owe their name to the English mathematician George Boole (1815–1864); the representation theorem (6.19) is from [Sto36]. These algebras occur everywhere in logic; for an introduction to algebraic logic, I recommend the book of Helena Rasiowa [Ras74], and for their structural study, [Sik64]. In fact, we know how to describe all the complete theories of Boolean algebras.

Rich ultrametric spaces were my suggestion to Françoise Delon [Del84], to extract the essence of certain model-theoretic phenomena that she had

[1] Added in 1999: It is now known that they are too numerous to be classified.

observed in the theory of valued fields; they provide good examples and counterexamples to illustrate several points of this course.

Modules form a wide enough class of structures whose model theory is particularly simple. Lemma 6.25 is in [Neu52]. Theorem 6.26 is Walter Baur's [Bau76]; By doing a little algebra, we can actually show quantifier elimination by Boolean combinations of primitive formulas, even in the absence of the model M. This result had been shown several years earlier by Szmiliew [Szm55] for abelian groups; we can describe all complete theories of abelian groups by "Szmiliew invariants."

The study of existentially closed models is due to Abraham Robinson [Rob63]. For more detailed applications to algebraic contexts, you can consult [Mac77], and especially [Che76]; for the links between the model companion and "Robinson forcing," see [Rob71] and [HW75].

Theorem 6.34 is due to Paul Eklof and Gabriel Sabbagh, [ES70].

7
Arithmetic

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7.1 The Successor Function

In this chapter we study the structures formed by well-known relations or operations on the universe $\mathbf{N}$ of natural numbers, also denoted by ω. We begin with the study of the successor function, which sends the number x to $x+1$; the language will consist of a symbol s to denote this function, as well as a constant symbol to denote 0, the smallest natural number.

Let us use axioms to express that s is injective and that every element except 0 is a successor:

$$\begin{aligned}
&(\forall x)(\forall y)(sx = sy \rightarrow x = y);\\
&(\forall x)x \neq 0;\\
&(\forall x)(\exists y)(x = 0 \vee x = sy).
\end{aligned}$$

What are the models of the theory consisting of these three axioms like? They consist on the one hand of the orbit of 0 under the action of s: $0, s(0), \ldots, s^n(0), \ldots$, which is a copy of the successor function on the natural numbers, and possibly other orbits, which may be finite, and cyclic, or may be infinite, in which case they are copies of the successor function $(\mathbf{Z}, s)$ on the integers.

In any event our successor function has no orbits of order n, a fact expressed by the following axiom:

$$\neg(\exists x_1)\cdots(\exists x_n)(x_2 = sx_1 \wedge \cdots \wedge x_n = sx_{n-1} \wedge x_1 = sx_n).$$

Let S denote the theory axiomatized by this infinite set of axioms; we can easily see that it is not finitely axiomatizable: If we express the fact that there are no cycles of order $1, 2, \ldots, n$, we still allow cycles of order $n+1$. I claim that this theory is complete (that is to say, it is the theory of the successor function), and that it admits quantifier elimination in the language $s, 0$.

Indeed, every model of S contains a copy of $(\mathbf{N}, s)$, in which we can choose elements $a_1, \ldots, a_n$ at arbitrarily large distances from each other; it is therefore consistent with the theory of the model that there are $a_1, \ldots, a_n$ that do not satisfy any relation $a_i = s^k a_j$, and consequently that any ω-saturated model of S is formed by a copy of $(\mathbf{N}, s)$ and infinitely many copies of $(\mathbf{Z}, s)$; we can easily see that in two such models, two tuples that satisfy the same quantifier-free formulas are ω-equivalent.

We could also perform Fraïssé's back-and-forth process in the style of those in Chapter 1: We would need to prepare by replacing the successor function by its graph, the set of pairs $(n, n+1)$, which we call the "binary relation of consecutivity"; it is not very complicated to determine the p-equivalence classes.

We can also proceed a little differently, noting that, up to isomorphism, there are ω denumerable models of S, according to whether there are $0, 1, 2, \ldots, n, \ldots$ or ω copies of $(\mathbf{Z}, s)$, while for each larger cardinal λ there is only one model of that cardinality, since we have no option except λ copies of $(\mathbf{Z}, s)$. A theory that, like S, has only one model of cardinality λ is called *λ-categorical* or *categorical in λ*.

Theorem 7.1. *If a theory T has no finite models, and is categorical in some cardinal $\lambda \geq |T|$, then it is complete.*

Proof. Suppose that T has a model M that satisfies f, and another N that satisfies $\neg f$; M and N are therefore infinite, and by the Löwenheim–Skolem theorem they have elementarily equivalent models M_1 and N_1 of cardinality λ, which must be isomorphic: contradiction. □

This theorem thus also proves that S is complete; in practice, its range of applications is more limited than one might imagine, since we often cannot see that a theory is λ-categorical except by a back-and-forth argument, which yields completeness directly. For example, we could have used it to show the completeness of the theory of algebraically closed fields of a given characteristic, provided that we knew Steinitz's theorem on transcendence bases, and also did not use model theory to show the uniqueness of the algebraic closure!

To see quantifier elimination, knowing that S is complete, it would suffice to notice that in a nondenumerable model of S (and also in the ω-saturated denumerable model!) two tuples that satisfy the same quantifier-free formulas correspond by automorphism.

The conclusion of all this is that the theory of successor is especially simple, and the classification of its models immediate.

7.2 The Order

The order $(\mathbf{N}, \leq)$ of natural numbers was studied copiously in the first chapter: Its theory is that of discrete orders with a least element and no greatest element, and we know (see the description of the types in Section 1.2, Exercise 1.10) that it admits quantifier elimination if we add a symbol 0 to the language to represent the least element, and, for each natural number n, a symbol $d_n(x, y)$ to represent the relation "there are n elements between x and y."

Now that we are more knowledgeable about infinite back-and-forth constructions, we would sooner proceed as follows to see that we do indeed have a complete theory, and that the order and the "distances," together with the "distances from 0," determine the types:

- The models of the theory in question are those chains of the form $\mathbf{N} + \mathbf{Z} \times C$, where C is any chain. (Here, in contrast to the case of the successor, there is a relation among the different copies of $\mathbf{Z}$: They must be placed in some order.)
- In an ω-saturated model of this theory, the chain C is dense and without endpoints: If $a < b$ and $d(a, b) = \infty$, it is consistent with the theory of the model that there exists $x, a < x < b$, at infinite distance from both a and b, and that there exists y greater than b and at infinite distance from b.
- In two such models, increasing tuples with the same distances are ∞-equivalent.

The theory of the order on the natural numbers, which is finitely axiomatizable, does not impose very great difficulty: Its models are clearly described by the expression $\mathbf{N} + \mathbf{Z} \times C$ if we accept that the notion of a chain is clear enough.

This theory is more complex than the theory of the successor in the following technical sense: The relation $y = sx$ is *interpretable* by the formula $x < y \wedge (\forall z)(x \leq z \leq y \rightarrow x = z \vee z = y)$; we also say that the successor is *definable* from the order. Every sentence in the language of the successor can therefore be replaced by a sentence in the language of the order: Replace the successor function with its translation. Thus if we know the theory of the order, we know the theory of the successor.

Every model of the theory of the successor yields a model of the theory of the order: It is enough to order the different copies of $\mathbf{Z}$. This is a very peculiar situation.

7.3 The Sum

We shall now consider the structure formed by the binary function of sum on the universe of the integers; it allows the order to be interpreted, since the formula $x \leq y$ can be replaced by $(\exists z)(x + z = y)$. In this case, not every model of the order relation can come from a model of the sum: If the model contains an element x greater than all numbers of the form $0, 1, 1+1, \ldots, 1 + \cdots + 1, \ldots$ (x is then called *nonstandard*), we can easily see that $x + x$ has infinite distance from x; it is therefore necessary that the chain C not have a greatest element.

Exercise 7.2. A denumerable model of the order comes from a model of the sum iff it has the form ω or $\omega + \mathbf{Z} \times \mathbf{Q}$, where $\mathbf{Q}$ is the chain of the rationals.

Now we are going to axiomatize the theory of the sum, and describe its types; to facilitate these projects, we add two constant symbols to our language to denote 0 and 1. These elements are definable from the sum, $x = 0$ by $x + x = x$, and $x = 1$ by $x \neq 0 \wedge (\forall u)(\forall v)(u + v = x \rightarrow u = 0 \vee v = 0)$.

That being done, the sum satisfies the following axioms:

1. Commutativity: $(\forall x)(\forall y)x + y = y + x$
2. Associativity: $(\forall x)(\forall y)(\forall z)(x + y) + z = x + (y + z)$
3. $(\forall x)x + 0 = x$
4. $(\forall x)x + 1 \neq x$
5. $(\forall x)(\forall y)(x + y = 1 \rightarrow x = 0 \vee x = 1)$
6. $(\forall x)(\forall y)(\exists z)(x = y + z \vee y = x + z)$
7. $(\forall x)(\forall y)(\forall u)(\forall v)(x = y + u \wedge y = x + v \rightarrow x = y)$

These axioms express the fact that the relation $(\exists z)x + z = y$ is a total order (transitive by (2), reflexive by (3), antisymmetric by (7), total by (6)) which we write as $\leq$ from now on; 0 is its least element because of (3) (and (1)), and 1 its successor by (4) and (5).

They also imply that every nonzero x greater than 1 is of the form $y+1$, and that $x + 1$ is the successor of x: By (4) $x + 1 > x$, and if $y > x$, then $y = x + z$ with $z \neq 0$, so $y = x + 1 + t$.

Note that they also imply that if $x \leq y$, then $x + z \leq y + z$, and that if $x < y$, i.e., $x + 1 \leq y$, then $x + z < y + z$, and conversely; examining all the cases $(x < y, x = y, x > y)$, we see that they imply the following *simplifiability* properties: $x + z \leq y + z \rightarrow x \leq y, x + z = y + z \rightarrow x = y, x + z < y + z \rightarrow x < y$.

Now let us introduce some notational conventions: If n is a natural number, we write n, the number, for the term $1 + \cdots + 1$, the sum being taken n times; and we write nx for the term $x + \cdots + x$, the sum being taken n times. When using this convention, do not lose sight of the fact that the product of two numbers is not part of our language. In a model of axioms (1–7), the elements of the form $0, 1, \ldots, n, \ldots$ form an initial segment, on which the sum is isomorphic to the sum on the natural numbers; these elements are called *standard.* There is no inconvenience in identifying the standard part of the model with the corresponding natural numbers; we sometimes call them the *true* natural numbers of the model, in contrast with the *nonstandard* numbers of the model, which are all greater than all the standard numbers. The model itself is called *nonstandard* if it includes a nonstandard element; there is therefore just one standard model up to isomorphism, the model of the true natural numbers.

The sum also satisfies the Euclidean division property, which is expressed by the following infinite list of axioms (one axiom for very nonzero (standard!) natural number n):

$$(8_n) \qquad (\forall x)(\exists y)(x = ny \vee x = ny + 1 \vee \cdots \vee x = ny + (n-1))$$

which can also be written

$$(8'_n) \qquad (\forall x)(\exists y)(\exists r)(x = ny + r \wedge r < n).$$

We cannot replace the list $(8'_n)$ of axioms with a single axiom, as we might be tempted to do, by universally quantifying n: The product is not part of our language, as ny is just an abbreviation for $y + \cdots + y$. In fact, it is not very difficult to see that the theory formed by axioms $1, 2, \ldots, 8_1, \ldots, 8_n, \ldots$ is not finitely axiomatizable, that finitely many of the Euclidean division axioms are not sufficient to imply all of the others.

As a consequence of these axioms we get the uniqueness of the remainder and quotient of Euclidean division: If $ny + r = nz + s$, with r and s strictly less than n, then $y = z$ and $r = s$; indeed, if for example $z \geq y$, then $z = y + u$, so $ny + r = ny + nu + s$; simplifying, $r = nu + s$, so $nu \leq r < n$. The only number less than r that can be put in the form nu is 0, so $u = 0$.

We shall write $[x/n]$ for the *integer part* of the division of x by n, i.e., the unique y such that $x = ny + r$ with $r < n$; r is called the *remainder* of the division of x by n.

Theorem 7.3. *The set of axioms* $1, 2, \ldots, 8_1, \ldots, 8_n, \ldots$ *is a complete theory, the theory of the sum function on the integers; it admits quantifier elimination in the language* $0, 1, \leq, +, [/2], \ldots, [/n], \ldots$.

Proof. In the language under consideration, a substructure contains 0 and 1 and is closed under addition and taking integer parts.

Now suppose that tuples $\vec{a}$ and $\vec{b}$, in ω-saturated models M and M' of this theory, generate isomorphic substructures A and B; we must show that $\vec{a}$ and $\vec{b}$ are ∞-equivalent. (This will indeed show that the theory is complete, since the substructure generated by the empty set is always isomorphic to the addition of true natural numbers.)

Add, for example, an element α to A. I claim that the substructure generated by A and α consists of elements of the form $a + k[\alpha/n]$, where a is in A, and where k and n are natural numbers. The elements 0 and 1 have this form; How can we calculate the sum of $a+k[\alpha/n]$ and $b+h[\alpha/m]$? We obviously need to give them a common denominator:

$$\begin{aligned} \alpha &= n[\alpha/n] + r_n, \\ \alpha &= m[\alpha/m] + r_m, \\ \alpha &= nm[\alpha/nm] + r_{nm}, \\ r_{nm} &= np + r_n = mq + r_m, \end{aligned}$$

whence

$$[\alpha/n] = m[\alpha/nm] + p,$$

$$[\alpha/m] = n[\alpha/nm] + q,$$

and the sum is $a + b + p + q + (kn + hm)[\alpha/nm]$.

Note that the form of the expression depends only on the remainders of α modulo n, m, nm.

How do we divide $a+k[\alpha/n]$ by m? If $a = m[a/m]+r$, then $a+k[\alpha/n] = m([a/m]+k[\alpha/mn])+r+kp$, so the quotient is $[a/m]+k[\alpha/nm]+[r+kp/m]$, and again we see that the calculation depends only on the remainders of α.

Therefore, to determine the structure generated by A and α completely, up to isomorphism, we still need to specify under what conditions we have $a + k[\alpha/n] \leq b + h[\alpha/m]$ (which will give us, for free, the conditions under which $a + k[\alpha/n] = b + h[\alpha/m]$); that inequality can also be written $a + km[\alpha/nm] + kp \leq b + hn[\alpha/nm] + hq$. Next, multiplying both numbers by nm (the axioms imply $x \leq y \leftrightarrow kx \leq ky$), and adding $(km+hn)r_{nm}$ to both sides, we get $nma + nmkp + hnr_{nm} + km\alpha \leq nmb + nmhp + kmr_{nm} + hn\alpha$, which is an inequality of the form $a' + k'\alpha \leq b' + h'\alpha$, the procedure to obtain a', b', k', h' from the given parameters a, b, k, h, m, n depending only on the remainders of α modulo n, m, nm.

Now we further manipulate this inequality, from which we can suppress the $'$, and suppose that $k \geq h$: Replace k by $k - h$ to get $a + k\alpha \leq b$, or $k[a/k] + r + k\alpha \leq k[b/k] + s$. If $r \leq s$, this is equivalent to $[a/k] + \alpha \leq [b/k]$, and if $r > s$, to $[a/k] + \alpha < [b/k]$. The case $k \leq h$ is similar.

To summarize: Up to isomorphism, the substructure generated by A and α is determined by (1) the sequence of remainders of α modulo n; and (2) which inequalities of the form $a + \alpha \leq b, a \leq \alpha + b$ are satisfied by α, with a and b in A.

Let A_1 be the set consisting of all differences of two elements of A and let B_1 be the set consisting of all differences of two elements of B: The isomorphism from A onto B extends to an isomorphism from A_1 to B_1 that preserves the order (because $a - b \leq c - d \leftrightarrow a + d \leq c + b$; in fact, A_1 and B_1 are isomorphic *substructures*).

So we need to find β satisfying the same congruences as α and that defines the cut on B_1 corresponding to the one defined by α on A_1. We distinguish two cases:

- $\alpha \in A_1$. Then α has the form $a - b$; take β to be the corresponding difference of elements of B. The task of checking that β has the same remainder modulo n as α is left to the reader.

- $\alpha \notin A_1$. As A_1 is closed under successor and predecessor, if $a \in A_1$ and $a < \alpha$, then $a + 1 < \alpha$, and if $b \in A_1, \alpha < b$, then $\alpha < b - 1$. We want to find β in the corresponding cut satisfying the given congruences: We want to satisfy a list of sentences $\ldots, a \leq y, \ldots, y \leq b, \ldots, y \sim r_n(\text{modulo } n), \ldots$; since determining y modulo the product of $n_1, \ldots, n_s$ is sufficient to determine it modulo each factor (that is a consequence of the axioms!), a finite system taken from this list is equivalent to three conditions:

$$a \leq y, \quad y \leq b, \quad y \sim r_n(\text{modulo } n),$$

  which are satisfied by a, or by $a + 1$, or $a + 2, \ldots$, or by $a + (n - 1)$! These conditions are therefore consistent with the theory of M', and by compactness and ω-saturation there is a β in M' that satisfies all of them.

□

There is a more natural way to axiomatize the sum, which consists in introducing the property of induction, which characterizes the natural numbers: If a set of natural numbers contains 0, and if whenever it contains x it contains $x + 1$, then it is all of $\mathbf{N}$. If we introduce a variable X to speak of subsets of $\mathbf{N}$, this can be written

$$(\forall X)((0 \in X \wedge (\forall x)(x \in X \rightarrow x + 1 \in X)) \rightarrow (\forall x) x \in X).$$

The only problem is that this is not an axiom allowed in our language; we are allowed to quantify only over *individuals* in the universe of the structure that we are considering, not over its subsets. We can speak of sets, relations, and functions only if we introduce them in advance by a special symbol. This is why our language is called first order. Languages in which we also quantify over subsets, relations, etc., are called second order; they do not allow the development of a model theory similar to that for first-order languages, as the following exercise shows:

Exercise 7.4. Show that a second-order language does not have the compactness property (note that the theory of the sum, or even the successor, on the integers characterizes them up to isomorphism in the second-order language).

But although in our language we cannot speak in complete generality about a subset of $\mathbf{N}$, we can at least consider those that are defined by a formula $f(x)$ in the language of the sum, and say that if 0 satisfies f, and if whenever x satisfies f, so does $x+1$, then every x satisfies f. For this we need an axiom A_f for every formula f; however, as all these axioms have the same style, we sometimes call A_f an *axiom schema*: to get the axioms that describe the schema A_f we need to replace f by every formula in turn.

The following axioms are called the *Presburger axiomatization*:

$$\begin{array}{rl}
(\mathrm{I}): & (\forall x) x+1 \neq 0. \\
(\mathrm{II}): & (\forall x)(\forall y)(x+1=y+1 \rightarrow x=y). \\
(\mathrm{III}): & (\forall x) x+0=x. \\
(\mathrm{IV}): & (\forall x)(\forall y) x+(y+1)=(x+y)+1. \\
(\mathrm{V_f}): & (\forall x_1) \cdots (\forall x_n)((f(x_1, \ldots, x_n, 0) \wedge \\
& \quad (\forall x)(f(x_1, \ldots, x_n, x) \rightarrow f(x_1, \ldots, x_n, x+1))) \\
& \quad \rightarrow (\forall x)(f(x_1, \ldots, x_n, x)).
\end{array}$$

Induction (sometimes called recursion) is the motivating idea of this axiomatization, from which it is clear that the axioms are satisfied by the sum on the natural numbers. Axiom (IV) shows how we can calculate $x+y$ successively from $x+0, x+1, x+2, \ldots$. There is an axiom (V_f) for every formula f in $n+1$ free variables in the language $(+, 0, 1)$: This axiom is called the axiom of induction on the variable x in the formula $f(x_1, \ldots, x_n, x)$.

Theorem 7.5. *The Presburger axioms axiomatize the theory of the sum on the natural numbers.*

Proof. These axioms are clearly true of the sum on the natural numbers, and the best way to see that this is a complete theory is prove each axiom of the preceding list from them. Indeed, the Presburger axioms actually owe their elegance more to a metaphysical intuition than to an intuition about what the types are.

Let us do this rapidly here.

(2) Associativity: by induction on z:

$$x+(y+0) \quad = \quad x+y=(x+y)+0,$$

$$\begin{aligned} x + (y + (z + 1)) &= x + ((y + z) + 1) = (x + (y + z)) + 1 \\ &= ((x + y) + z) + 1 = (x + y) + (z + 1). \end{aligned}$$

(1) Commutativity: We proceed in three stages.

- induction on x: $0 + x = x$
- induction on x: $1 + x = x + 1$
- induction on y: $x + y = y + x$

(4) By induction on x: $0 + 1 \neq 0$; $x \neq x + 1$ implies $x + 1 \neq (x + 1) + 1$.

(5) By induction on x, show that every nonzero natural number has the form $u + 1$; if $(u + 1) + y = 1$, then $0 = u + y$, and neither u nor y has the form $v + 1$, so $u = 0$

Axioms (6), (7), (8_n) are left to the reader. □

Note that the proof of Theorem 7.5 used only a weak, although still infinite, subset of the axioms of induction.

Example of a model of the theory of the sum

Take the two ordered groups $\mathbf{Z}$ of integers and $\mathbf{Q}$ of rational numbers, and equip the product $\mathbf{Z} \times \mathbf{Q}$ with the lexicographic order. The elements greater than $(0, 0)$ form a model of the theory of the sum on the natural numbers.

Remark. For the theory of the sum, the number of types over $\emptyset$ is 2^ω: Indeed, there are 2^ω possible ways to choose the remainders modulo n (coherently!). This theory can therefore not have an ω-saturated denumerable model.

7.4 Sum and Product: Coding of Finite Sets

From now on, we shall be studying the structure formed by the natural numbers with their sum and their product; the theory of this structure is called *arithmetic*. We shall suppose that the language includes $0, 1, \leq, +, \cdot$, although $0, 1, \leq$ are not indispensable, since they are definable from the sum. Nevertheless, 0 and 1 are quite useful, as they allow us to name every (standard!) natural number by a term $n = 1 + \cdots + 1$; on the other hand, the order will be involved in such a fundamental way in the classification of formulas of arithmetic that it it is practically indispensable to give it a special symbol.

In the same way as for the sum, we call the model $\mathbf{N}$ of arithmetic formed by the true natural numbers the *standard model*. As every element of the standard model is named by a term, its diagram (set of sentences, not necessarily quantifier-free, satisfied by its elements) is part of arithmetic, and every *nonstandard model* is an elementary extension of the standard model, which is an initial segment of it.

In the preceding sections we studied partial structures of arithmetic, increasingly rich, each one interpreted in the next. We were able to axiomatize their theories and to describe some of their nonstandard models. This study became more and more delicate, and we had more trouble with the sum than with the order or the successor, but we pulled through, keenly, in the same way, the greatest difficulties being just technical.

However, in introducing the sum and the product, as we shall realize, we are making a qualitative leap, which takes us into a new domain, and we shall now attempt neither to axiomatize arithmetic, nor to construct nonstandard models, because we shall see that there are theoretical obstacles to these projects, that arithmetic can have neither an axiomatization nor nonstandard models of the same kind as those that we produced for the partial structures already studied. To be sure, we do need to say precisely what we understand to be the "kinds" of these objects.

We are not yet there, and what we are going to show in this section must be digested well, because it is completely essential for studying models of arithmetic and to assess the expressive power of this theory: that arithmetic codes *combinatorics*, the theory of finite sets. If we want to say "there is a finite sequence of 10 elements such that f," we have only to write $(\exists x_1)\cdots(\exists x_{10})f$, and we can do likewise for a sequence of a billion terms; but how do we say, in our first-order language, "there is a finite sequence with x elements," where x is a variable? Gödel's β function gives us the means.

The *Gödel function* $\beta(u,v,w)$ is the function sending three natural numbers u,v,w to the sequence of natural numbers $a_0,\ldots,a_{w-1}$, of length w (if $w=0$, then $\beta(u,v,0)$ is the empty sequence), defined thus: a_i is the remainder of the Euclidean division of u by $(i+1)v+1$.

Theorem 7.6. *The function $\beta(u,v,w)$ is a surjection from $\mathbf{N}^3$ onto the set of finite sequences of elements of $\mathbf{N}$.*

Proof. The natural numbers satisfy the Chinese remainder theorem: If $d_0,\ldots,d_{w-1}$ are pairwise relatively prime, then for any given $a_0,\ldots,a_{w-1}$ there is a natural number u congruent to a_0 modulo $d_0,\ldots$, congruent to a_{w-1} modulo d_{w-1}. (To prove this theorem, show that $\mathbf{Z}/d_0\cdots d_{w-1}\mathbf{Z} = \mathbf{Z}/d_0\mathbf{Z}\times\cdots\times\mathbf{Z}/d_{w-1}\mathbf{Z}$.)

We still need to find v greater than $a_0,\ldots,a_{w-1}$ (in order that all these numbers can be remainders modulo all the $(i+1)v+1$) such that if $i<j<w$, then $(i+1)v+1$ and $(j+1)v+1$ are relatively prime. Let $v=n!$ where n is greater than $w,a_0,\ldots,a_{w-1}$. Then if p is a common prime divisor of $(i+1)n!+1$ and $(j+1)n!+1$, it also divides $(j-i)n!$. It therefore divides either $j-i$ or $n!$, and as $j-i$ divides $n!$, it divides $n!$. But p also divides $(i+1)n!+1$, so it must divide 1, which is absurd. $\square$

The light has dawned on the reader: In order to say "there is a finite sequence whose ith term ..." we say "there are u, v, w such that the remainder of the division of u by $(i+1)v+1\ldots$"; that can all be said in the language of arithmetic, since we can express the fact that c is the remainder of the division of a by b thus: $(\exists y)a = yb + c \wedge c < b$.

We shall now, by way of illustration, define the exponential function x^y in arithmetic. First note that we get the sum by iterating the successor function, the product by iterating the sum, and the exponential by iterating the product: The sum is the unique function f such that $(\forall x)f(x,0) = x, (\forall x)(\forall y)f(x,y+1) = f(x,y)+1$; the product is the unique function f such that $(\forall x)f(x,0) = 0, (\forall x)(\forall y)f(x,y+1) = f(x,y)+x$; the exponential is the unique function f such that $(\forall x)f(x,0) = 1, (\forall x)(\forall y)f(x,y+1) = f(x,y)\cdot x$. But these definitions, which involve quantifications over functions ("the unique function"), are illegal. We might think that this hierarchy of structures gets more and more complex: successor; sum; sum and product; and the next term, sum, product, and exponential. It is not at all like that, since we will find a first-order formula, in the language of the sum and product, that defines the relation $z = x^y$, the graph of the exponential function.

Let us introduce an abbreviation: $r(u,v)$ will designate the remainder of the division of u by $v+1$. We have already noted that this function can be defined from the sum and product. Our formula $f(x,y,z)$, defining the relation $z = x^y$, is the following:

$$
\begin{aligned}
(\exists u) \qquad & (\exists v)(r(u,v) = 1 \wedge r(u,(y+1)v) = z \\
& \wedge(\forall i)(1 \leq i \leq y \to r(u,(i+1)v) = r(u,iv)\cdot x)).
\end{aligned}
$$

Why does this work? Suppose that x, y, z satisfy the formula. Consider the sequence $a_0 = r(u,v), \ldots, a_i = r(u,(i+1)v), \ldots, a_y = r(u,(y+1)v)$. Since $a_0 = 1$ and $a_{i+1} = a_i x$, we must have $a_y = x^y$, and the formula says that $x = a_y$!

Conversely, if $z = x^y$, the sequence $a_0 = 1, \ldots, a_i = x^i, \ldots, a_y = x^y$ has, by Theorem 7.6, the form $\beta(u,v,y+1)$, from which we get the existence of u and v, so the formula is satisfied.

The principle of this proof is that a recursive definition is transformed, thanks to the function β, into something expressible in the language of arithmetic; the reader is now invited to attempt to define any function from $\mathbf{N}$ to $\mathbf{N}$ that strikes her fancy (for example, the one sending x to the xth prime), in terms of the sum and product. If she runs aground in some cases, it is because she is not yet skillful enough at manipulating the function β, or that her imagination is perverse; otherwise, she will ascertain, with some surprise, that every "natural" function from $\mathbf{N}$ to $\mathbf{N}$ is definable in arithmetic.

While our work up to now has been that of artisans, we shall now bring in a more systematic method of investigation, consisting in introducing "a theory of finite sets," or combinatorics, inside of arithmetic.

Everybody, or at least every reader of a mathematics book, knows how to write a number in base ten, by means of ten digits, and those who have gone to primary school recently also know that we can write a number in base 2, by means of two digits 0 and 1: This consists in putting it in the form $\Sigma\epsilon_i 2^i$, with $\epsilon_i = 0$ or 1, the ϵ_i being all zero from some position on. We need some elementary lemmas about numeral systems.

Lemma 7.7. *Every number has a unique base-2 expansion.*

Proof. First let us show uniqueness. We need the formula $\Sigma_{0\leq i\leq k}2^i = 1 + 2 + \cdots + 2^k = 2^{k+1} - 1$, which we can easily show by induction on k. Suppose now that $\Sigma\epsilon_i 2^i = \Sigma\eta_i 2^i$ are two expansions of the same number. Since both the ϵ_i and the η_i are zero from some position on, there is a greatest index k for which $\epsilon_k \neq \eta_k$. Suppose, for example, that $\epsilon_k = 0, \eta_k = 1$. Drop the terms with indices above k to get $\Sigma_{0\leq i\leq k-1}\epsilon_i 2^i = \Sigma_{0\leq i\leq k-1}\eta_i 2^i + 2^k$, which is impossible, since the left-hand side is at most $2^k - 1$.

Now let us show that every x has such an expansion. We could do this by induction on x, but this would pose the complicated problem of addition with carry-overs. It is simpler to note that there are 2^k distinct expansions $\Sigma_{0\leq i\leq k-1}\epsilon_i 2^i$ whose digits are all zero from the kth place on; we have seen that they all represent distinct natural numbers that are less than or equal to $\Sigma_{0\leq i\leq k-1}\epsilon_i 2^i = 2^k - 1$. So these are all the integers from 0 to $2^k - 1$. □

Lemma 7.8. *The function $\gamma(x,i)$ that sends the pair (x,i) to the ith digit of the base two expansion of x, is definable from the sum and product.*

Proof. The relation $y = \gamma(x,i)$ is defined by the following formula, which can be written with only sum and product, since, as we have verified, the exponential is also definable from them:

$$
\begin{array}{rl}
(\exists u)(\exists v)(\exists w) & ((r(u,v) = 1 \vee r(u,v) = 0) \\
\wedge & ((\forall j)(1 \leq j \leq w \rightarrow (r(u,jv) = r(u,(j+1)v) \\
\vee & 2^j + r(u,jv) = r(u,(j+1)v)))) \\
\wedge & x = r(u,(w+1)v) \wedge (i > w \rightarrow y = 0) \wedge (i = 0 \rightarrow y = r(u,v)) \\
\wedge & (((1 \leq i \leq w) \wedge r(u,iv) = r(u,(i+1)v)) \rightarrow y = 0) \\
\wedge & (((1 \leq i \leq w) \wedge r(u,iv) \neq r(u,(i+1)v)) \rightarrow y = 1)).
\end{array}
$$

This works because the sequence $\beta(u,v,w+1)$ represented by the triple $u,v,w+1$ is necessarily the sequence of partial sums of the base-two expansion of x. □

Now we introduce a binary "membership" relation between natural numbers. We shall say that $x \in y$ if the xth digit of the base two expansion of

y is 1; if it is 0, then $x \notin y$. Because of Lemma 7.8, this relation $x \in y$ is definable in arithmetic by the formula $\gamma(y, x) = 1$.

This binary membership relation defines a model of the "theory of sets" with universe $\mathbf{N}$. To give a precise meaning to this assertion it would obviously be necessary to specify the axioms of this theory, something that the author of these lines is absolutely determined to avoid; it will be simpler for the reader to convince himself that natural numbers, equipped with this membership, do have the formal properties of the sets that he has manipulated since grade school and that have generally helped mathematicians to develop their discipline for a century.

For example, one principle of this theory of sets is the axiom of extensionality: "Two sets with the same elements are equal." This is indeed true here, since two numbers with the same base-two expansions are equal! We do indeed have an empty set, the number 0. Given two numbers a and b, we can form the pair $\{a, b\}$, which is 2^a, the singleton of a, if $a = b$, and which is $2^a + 2^b$ if a and b are distinct. More generally, given n distinct numbers $a_0, \ldots, a_{n-1}$, we can form the set $\{a_0, \ldots, a_{n-1}\}$, which is the number $2^{a_0} + \cdots + 2^{a_{n-1}}$.

Furthermore, since the elements of a are strictly less than a (if $x \in y$ then $x < y$ because $i < 2^i$; our membership relation is *well-founded*, which implies that we cannot have a sequence $x_1, \ldots, x_n$ with $x_1 \in x_2 \in \cdots \in x_n \in x_1$), we can see that if we have two sets a and b, we can form their union, which is the number $a + b$ if a and b are disjoint. Also true are the axiom of unions (we can form the union of the elements of a) and the axiom of the power set (we can form the set of all numbers that are subsets of a), as well as the axiom schema of substitution. This last, obscure expression is reserved for the knowledgeable; if you do not know it, but would like to shine in the community, consult a set theory textbook.

It will be understood that our model $(\mathbf{N}, \in)$ has only finite sets. At first, "finite" has meaning only outside the model: given the number a, the set of x such that $x \in a$ is finite; as we have already noted, any finite set of numbers has the form $\{x : x \in a\}$. In the previous phrase, the words "set" and "finite" are to be taken in a natural, formal, intuitive sense (if it is indeed true that sets are a natural thing, about which we can have intuitions!), rather than in the technical sense of the model $(\mathbf{N}, \epsilon)$. However, we can also show that this model satisfies the axiom "all sets are finite," or even "for every a there is a natural number n and a bijection between a and n."

How, by the techniques of set theory, do we represent the natural numbers? 0 is represented by the set $\emptyset$; 1 is represented by $\{\emptyset\}$, the singleton of the empty set; 2 by $\{\emptyset, \{\emptyset\}\}$; etc. If n^* designates the set representing the number n, then $n + 1$ is represented by $n^* \cup \{n^*\}$. We see that a natural number is thus identified with the set of natural numbers less than itself. We can see that a natural number n^* is *transitive* (if $x \in y$ and $y \in n^*$, then $x \in n^*$) and that the membership relation defines a total (strict) order on the elements of n^*. The reader can check that the elements of the model $(\mathbf{N}, \in)$ with these two properties are exactly those of the form n^*; they constitute the set $\mathbf{N}^*$ (a "set" not representable in the model $(\mathbf{N}, \in)$!) of natural numbers in the sense of the model $(\mathbf{N}, \in)$.

We have already noted that if x and y are disjoint in the sense of the relation $\in$, $x \cap y = \emptyset$ meaning that x and y have no common digit of 1, then $x \cup y = x + y$, and that $\{x\} = 2^x$. Furthermore, $x \notin x$ (because if $x \in y$, then $x < y$). Consequently, the map sending x to x^* is defined by the following induction: $0^* = 0, (x+1)^* = x + 2^{x^*}$. The reader will no longer be surprised to learn that the function $y = x^*$ is defined in arithmetic by the formula

$$\begin{aligned}(\exists u) \quad (\exists v) \quad & \big(r(u,v) = 0 \wedge r(u,(x+1)v) = y \\ \wedge \big(\forall i\big) \quad & (1 \leq i \leq x \rightarrow r(u,(i+1)v) = r(u,iv) + 2^{r(u,iv)})\big).\end{aligned}$$

As a consequence we can use the function $*$, which is definable, to carry the structure $\mathbf{N}$ over to its image $\mathbf{N}^*$. What have we achieved? The upshot is that $\mathbf{N}^*$ is now cloaked in combinatorics, that all talk of finite subsets of $\mathbf{N}^*$, of finite sets of finite subsets of $\mathbf{N}^*$, etc., can be translated in terms of the relation $\in$.

What is a pair (x, y)? It is the set $\{\{x\}, \{x, y\}\}$. What is the Cartesian product $x \times y$? The set of pairs whose first coordinate is an element of x and whose second coordinate is an element of y. What is a function from x to y? A subset of $x \times y$ with the property of being the graph of a function. What is a finite sequence of elements of x? A function, for some n^*, from n^* to x. In short, all set-theoretic concepts that do not take us outside the realm of finite sets, for example binary relations with universe x, groups with universe x, etc., are immediately translatable, thanks to the relation $\in$.

So if we need combinatorial concepts to show properties of natural numbers, rather than use the function β, which was useful as a technical step toward defining the relation $\in$, we can use our function $*$, work freely on $\mathbf{N}^*$ in our combinatorics $(\mathbf{N}, \epsilon)$, and return to $\mathbf{N}$ by the inverse function from $\mathbf{N}^*$ to $\mathbf{N}$.

For example, suppose that we want to define the function $y = p_x$ that sends x to the xth prime number. The reader knows enough to do this by the function β, as in the previous examples, but we can also go into $\mathbf{N}^*$ and say that y^* is the prime number (in the sense of $\mathbf{N}^*$) such that there is a bijection between x^* and the set of prime numbers strictly less than y^*.

The time saved here is not fantastic, but it will be if we need to introduce sets of sets, of functions, etc. It is a hefty burden to handle the function β to introduce codings of all these notions, whereas it can all be done naturally in $(\mathbf{N}, \in)$: All we need to do is translate our intuitions.

However, the reader whose intuition has not been well enough shaped, or misshapen, by the practice of set theory, and who therefore lacks confidence in this relation $\in$, would do well to sketch heavier but more concrete proofs from time to time using the function β; it is a thoroughly essential activity for really understanding what the coding of finite sets into arithmetic means.

We close this section with a few subtle considerations. For now we have worked in the standard model $\mathbf{N}$: What happens if we replace $\mathbf{N}$ with a nonstandard elementary extension N_1? The formula that helped us to define the relation $\in$ on $\mathbf{N}$ likewise defines a binary relation on N_1, which we denote the same way, and $(N_1, \in)$ is obviously an elementary extension of $(\mathbf{N}, \in)$, since every property of $\in$ can be translated in terms of the sum and product.

The subsets A of N_1 that correspond to sets in the sense of this combinatorics, i.e., the subsets A of N_1 that are associated with an a in N_1 such that for all x in N_1, $x \in a$ iff x is in A, will be called *"sets coded in the model N_1,"* or *"finite sets in the sense of N_1."* All truly finite subsets of N_1 are coded there, since the following sentence is a theorem of arithmetic:

$$(\forall x_1) \cdots (\forall x_n)(\exists y)(\forall x)(x \in y \leftrightarrow (x = x_1 \vee \cdots \vee x = x_n)).$$

But the following axiom, which expresses the fact that for every x the segment $[0, x)$ is coded, is equally true:

$$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow z < x).$$

Consequently, among these finite sets in the sense of N_1 are some whose number of elements x (or more exactly x^*) is nonstandard; that is to say, from outside the model they are infinite, but with respect to properties expressible in our language, they behave exactly like finite sets. Also, the model $(N_1, \in)$ satisfies that axiom "every set is finite," that is to say, "every set has a bijection to a natural number"!

The notion of standard or nonstandard number is perceptible only if we examine our model from the outside, or else in a more powerful language. Every model of arithmetic lives in the idea that it is made up of true natural numbers: Only an external observer can escape this illusion.

7.5 Coding of Formulas; Tarski's Theorem

As we have combinatorics at our disposal, we can code the formulas of the language $(0, 1, \leq, +, \cdot)$ (that is to say, represent them by numbers), and in such a way that formula manipulations can be translated as definable operations of arithmetic.

We therefore would like to associate each formula, in an injective manner, with its *code*; for this purpose we begin by enumerating all the symbols of the language, which are , ( ) $\neg \vee \wedge \exists \forall = 0\ 1 \leq + \cdot x_1\ x_2\ x_3 \cdots$, which are associated, in order, with the numbers 0 1 2 3 4 .... A word is a finite sequence $a_0, \ldots, a_n$ of these symbols; we can represent it, for example, by the number $p_0^{a_0+1} \cdots p_n^{a_n+1}$, where p_i denotes the ith prime number. Then the symbol found in the ith position is one less than the exponent on p_i. We can see easily enough that the set of codes of words is definable in arithmetic: It is the set of numbers x such that if p is a prime divisor of x, then every prime number less than p also divides x. It is more complicated to define the set of codes of formulas, but that can be done with the help of the function β.

This is why we are going to abandon this coding, and use our combinatorics $(\mathbf{N}, \in)$; indeed, a word is nothing but a finite sequence of symbols, i.e., a function from n^* to a finite subset of our list of symbols, which is $\mathbf{N}$ (we could also take it to be $\mathbf{N}^*$): This definition can be translated directly into the structure $(\mathbf{N}, \in)$. The words are therefore the finite sequences in the sense of $(\mathbf{N}, \in)$. How do we distinguish formulas from other words? By introducing their lists of subformulas: A word m is a formula if there is a finite sequence $m_1, \ldots, m_n$ of words such that the last one is m and every m_i is atomic, or else a conjunction or disjunction of two previous words in the sequence, or else the negation or quantification of a previous word. All of this can be translated directly into $(\mathbf{N}, \in)$ once we have noted that we can define the atomic formulas: They are the words with such-and-such a form.

All the notions related to formulas that do not take us out of combinatorics (i.e., that involve only finite sets) are definable in arithmetic. That can be seen uniformly in the model $(\mathbf{N}, \in)$; if we wished to avoid using it, we would need to manipulate the function β every time.

For example, the function that sends a formula (or, more precisely, its code) to its complexity, its quantification rank, the code of the set of its subformulas, that of the set of its free variables, etc. is definable. We can thus say, in arithmetic terms, that a formula is a sentence, i.e., that it has no free variables.

But how do we define the fact that the formula $f(\vec{x})$ is satisfied by the tuple $\vec{a}$? By induction on the complexity of f? Certainly, but the induction must be on the pair (f, A), where A is the set of tuples that satisfy f; since this set A has no reason to be finite, it escapes our combinatorics, and we cannot see how to translate this inductive definition. The impossibility of this translation is stated in the following theorem, which is placed this early because it indicates, very precisely, the limits on the expressive power of arithmetic.

Theorem 7.9 (Tarski's Theorem). *The set of codes of true sentences of arithmetic is not definable by an arithmetic formula.*

Proof. Suppose that there is a formula $V(x)$, in one free variable x, such that the numbers n satisfying it are exactly the codes of true sentences.

The set A of codes of formulas whose only free variable is x is definable, as is the function ϕ that sends the pair (n, m), where n is in A, to the code of the sentence obtained by replacing the variable x by the number m in the formula coded by n.

Now let $V(x, y)$ denote the formula $V(\phi(x, y))$; we can see that this formula is satisfied by those pairs (n, m) for which n is the code of a formula having x as its only free variable, such that the sentence obtained from it by replacing x with m is true.

Now let n_0 be the code of the formula $\neg V(x, x)$; we cannot avoid the following dilemma:

- If $V(n_0, n_0)$ is true, then $\neg V(n_0, n_0)$ is true.
- If $V(n_0, n_0)$ is false, then $\neg V(n_0, n_0)$ is false.

□

Remark. We needed to choose a well-defined coding system, but it is clear that the choice was for the most part arbitrary, and that we can modify it without changing anything essential; what we need of it is that it allow arithmetic translations of the "usual" operations on formulas (for example, substitution of a variable by a constant, or negation). If we had assigned even numbers to true sentences and odd numbers to false sentences, we would not have been able to prove Tarski's theorem.

7.6 The Hierarchy of Arithmetic Sets

We shall say that a subset of $\mathbf{N}^n$ is *arithmetic* if it is composed of the n-tuples that satisfy some formula $f(\vec{x})$ in the language of arithmetic; we shall also say that this subset is definable in arithmetic.

We are now going to introduce a classification scheme for arithmetic formulas, and the sets that they define, by the number of their quantifiers, once the formulas are put into prenex form. We have already discussed how to put formulas in general into prenex form, in Section 2.1; here we define a prenex form peculiar to arithmetic, involving the notion of "bounded quantifier."

In a formula of the form $(\exists y)(y \leq x \wedge f(x, y))$ or $(\forall y)(y \leq x \rightarrow f(x, y))$, in which the free variable is x, the quantifier is called bounded: To see whether the formula is true of n, we need only check values of y up to n. Once we know n, there are only finitely many trials to be done; we can well imagine that these sorts of quantifiers are by nature much simpler than unbounded quantifiers, and that it is useful to distinguish them. We therefore introduce the notation $(\exists y \leq x), (\forall y \leq x)$ as abbreviations for $(\exists y) y \leq x \wedge, (\forall y) y \leq x \rightarrow$; in the formulas $(\exists y \leq x) f, (\forall y \leq x) f$, *the variable x is free,* even if it is not free in f.

A formula is called Δ_0, or *with bounded quantifiers*, if all its quantifiers are bounded; just as a formula can be put into prenex form, it is easy to see that a Δ_0 formula is equivalent to a Δ_0 formula with all its quantifiers placed at the beginning.

We call a formula Σ_1 if it has the form $(\exists x)f$, where f is Δ_0; it is Π_1 if it has the form $(\forall x)f$, where f is Δ_0. For $n > 1$, we define the classes Σ_n and Π_n of formulas by induction on n: A formula is Σ_{n+1} if it has the form $(\exists x)f$, where f is Π_n; a formula is Π_{n+1} if it has the form $(\forall x)f$, where f is Σ_n.

The index n thus indicates that there are n quantifiers in front of the formula, alternately $\exists$ and $\forall$; Σ means that the formula begins with $\exists$; Π, that it begins with $\forall$.

Theorem 7.10. *In arithmetic, the conjunction or disjunction of two Σ_n formulas is equivalent to a Σ_n formula. A bounded quantifier or an existential quantifier applied to a Σ_n formula yields a formula equivalent to a Σ_n formula. The negation of a Σ_n formula is equivalent to a Π_n formula. The conjunction or disjunction of two Π_n formulas is equivalent to a Π_n formula. A bounded quantifier or a universal quantifier applied to a Π_n formula, yields a formula equivalent to a Π_n formula. The negation of a Π_n formula is equivalent to a Σ_n formula.*

Proof. By induction on n: A Σ_{n+1} formula has the form $(\exists x)f$, where f is Δ_0 or Π_n depending on the value of n. The formulas $(\exists x)f \wedge (\exists x)g, (\exists x)f \vee (\exists x)g$ are equivalent to $(\exists x)(\exists y)(f \wedge g), (\exists x)(\exists y)(f \vee g)$, respectively, provided that, in the first case, the dummy variables have had their names changed so that x is not free in g, nor y in f. The formula $(\exists y)(\exists x)f$ is equivalent to $(\exists z)(\exists y \leq z)(\exists x \leq z)f$; $(\exists y \leq u)(\exists x)f$ is equivalent to $(\exists z)(\exists y \leq u)(\exists x \leq z)f$. $(\forall y \leq u)(\exists x)f$ is equivalent to $(\exists z)(\forall y \leq u)(\exists x \leq z)f$: Indeed, each y less than u has an associated value for x, and as there are only finitely many such x, some z majorizes all of them. Finally, $\neg(\exists x)f$ is equivalent to $(\forall x)\neg f$. Proceed similarly for Π_{n+1}. $\square$

One consequence of Theorem 7.10 is that every formula is equivalent to a Σ_n or Π_n formula for some n. We tend to say that a formula "is" Σ_n when it is equivalent, in an obvious way, to a Σ_n formula. For example, we shall say that the negation of a Σ_n formula is Π_n, which is an abuse of language. We shall likewise say that a Σ_n formula is also Σ_{n+1} and Π_{n+1} (quantify a variable that does not appear in the formula).

We shall now say that a set $A \subset \mathbf{N}^k$ is Σ_n if it has a Σ_n definition, i.e., if it consists of all tuples satisfying some Σ_n formula, and that it is Π_n if it has a Π_n definition. Naturally, if a set is defined by a Σ_n formula, it is possible that this formula is equivalent to a formula much lower in the hierarchy.

We say that a set is Δ_n if it is both Σ_n and Π_n, that is to say, if both it and its complement are Σ_n. There is no such thing as a Δ_n formula: To specify a Δ_n set, we must give a Σ_n definition and a Π_n definition of the set that are equivalent in arithmetic.

As a consequence of Theorem 7.10, we get the following diagram of inclusions for the Δ, Σ, Π classes:

$$\begin{array}{ccccccccccccc}
 & & \Sigma_1 & & & & & & & & \Sigma_n & & \\
 & \subset & & \subset & & \subset & & \subset & & \subset & & \subset & \\
\Delta_0 \subset \Delta_1 & & & & \Delta_2 & & \cdots & & \Delta_n & & & & \Delta_{n+1} \cdots \\
 & \subset & & \subset & & \subset & & \subset & & \subset & & \subset & \\
 & & \Pi_1 & & & & & & & & \Pi_n & &
\end{array}$$

We shall see that every one of these inclusions is strict, that the hierarchy never stabilizes.

In summary, the Δ classes are Boolean algebras (closed under $\vee, \wedge, \neg$) closed under bounded quantification; the Σ and Π classes are closed under $\vee, \wedge$, and bounded quantification; the Σ classes are closed under $\exists$, and the Π classes under $\forall$; the complement of a Σ_n set is Π_n, and the complement of a Π_n set is Σ_n. We note in passing that in set-theoretic terms, an existential quantification translates as the projection of A from ω^{k+1} onto ω^k.

In fact, it turns out that the introduction of bounded quantifiers is not necessary to introduce this hierarchy. Let us call a set *Diophantine* if it is defined by a formula of the form $(\exists y_1) \cdots (\exists y_n) f$, in which the number of $\exists$ is not necessarily 1, where f is a conjunction of equations $P(\vec{x}, \vec{y}) = Q(\vec{x}, \vec{y})$, P and Q being polynomials with natural-number coefficients. As such an equation is a quantifier-free formula, it is clear that every Diophantine set is Σ_1, but it is also true that *every* Σ_1 *set is Diophantine.* That statement is Matijacević's theorem, which solved a famous Hilbert problem; I shall not here attempt to begin on this theorem.

Before continuing, we shall carry out some classic manipulations on Σ_n and Π_n sets.

We shall say that a function from ω^k to $\omega^{k'}$ is Σ_n if its graph is a Σ_n subset of $\omega^{k+k'}$. Here, function means *total function*: Every $\vec{x}$ in ω^k is associated with a unique $\vec{y}$ in $\omega^{k'}$.

If f is a function from ω^k to $\omega^{k'}$ and g is a function from ω^k to $\omega^{k''}$, the *juxtaposition* of f and g is the function from ω^k to $\omega^{k'+k''}$ that sends x to the concatenation $f(\vec{x})^\frown g(\vec{x})$; if f is a function from ω^k to $\omega^{k'}$, each of its *coordinates* is the composition of f with one of the k' canonical projections from $\omega^{k'}$ to ω.

Lemma 7.11. *The graph of a Σ_n function is also Π_n; the composition of two Σ_n functions is also a Σ_n function; the inverse of a Σ_n bijection is also Σ_n; the juxtaposition of two Σ_n functions is also Σ_n; finally, a function is Σ_n iff every one of its coordinates is.*

Proof. The relation $\vec{y} \neq f(\vec{x})$ is defined by the formula $(\exists \vec{z})(\vec{z} = f(\vec{x}) \wedge \vec{z} \neq \vec{y})$ (that works because f is a total function), which is Σ_n if the formula

$\vec{z} = f(\vec{x})$ is. If h is the composition of f and g, the relation $\vec{z} = h(\vec{x})$ is defined by $(\exists \vec{y})(\vec{z} = g(\vec{y}) \wedge \vec{y} = f(\vec{x}))$. A bijection and its inverse have the same graph (just exchange the roles of $\vec{x}$ and $\vec{y}$).

The graph of the juxtaposition h of f and g is defined by the formula $\vec{y}_1 = f(\vec{x}) \wedge \vec{y}_2 = g(\vec{x})$. The canonical projection from ω^k onto its ith coordinate is defined by a quantifier-free, and therefore Δ_0, formula $y = x_i$: If f is Σ_n, then so is its composition with a projection. Finally, f is obtained by juxtaposing its coordinates. □

Lemma 7.12. *Let $f(\vec{x})$ be a Σ_n function. If $\phi(\vec{y})$ is a Σ_n formula, then so is $\phi(f(\vec{x}))$; if $\phi(\vec{y})$ is a Π_n formula, then so is $\phi(f(\vec{x}))$.*

Proof. $\phi(f(\vec{x}))$ can be written by your choice of $(\exists \vec{y})(\vec{y} = f(\vec{x}) \wedge \phi(\vec{y}))$ or $(\forall \vec{y})(\vec{y} \neq f(\vec{x}) \vee \phi(\vec{y}))$! □

Another way to say the same thing:

Lemma 7.13. *The preimage of a Σ_n set under a Σ_n function is a Σ_n set; the preimage of a Π_n set under a Σ_n function is a Π_n set.*

Proof. $\vec{x} \in f^{-1}(A)$ can be written by your choice of $(\exists \vec{y})(\vec{y} = f(\vec{x}) \wedge \vec{y} \in A)$ or $(\forall \vec{y})(\vec{y} \neq f(\vec{x}) \vee \vec{y} \in A)$. □

Lemma 7.14. *The image of a Σ_n set under a Σ_n function is a Σ_n set.*

Proof. $\vec{x} \in f(A)$ can be written $(\exists \vec{y})(\vec{x} = f(\vec{y}) \wedge \vec{y} \in A)$. □

The characteristic function of a set A is the function sending $\vec{x}$ to 1 if $\vec{x} \in A$ and to 0 if $\vec{x} \notin A$.

Lemma 7.15. *A set is Δ_n iff its characteristic function is Σ_n.*

Proof. Let f be the characteristic function of A. If A is Δ_n, then the graph of f is defined by the Σ_n formula $(\vec{x} \in A \wedge y = 1) \vee (\vec{x} \notin A \wedge y = 0)$; if f is Σ_n, then A is defined by the formula $f(\vec{x}) = 1$, and its complement by the formula $f(\vec{x}) = 0$. □

Lemma 7.16. *For every k and every h, there is a Σ_1 bijection between ω^k and ω^h.*

Proof. Consider the bijection f between $\omega \times \omega$ and ω consisting of enumerating ω^2 by $x+y$ increasing, and then by y increasing: Cut the quarter-plane $\omega \times \omega$ into lines parallel to its transverse diagonal, which are then placed end to end and enumerated one after another. Once we have enumerated the first n segments corresponding to $x + y = 0, x + y = 1, \ldots, x + y = i, \ldots, x + y = n - 1$, we have enumerated $1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$ pairs; consequently, the inverse function g of f is defined by the formula $g(x, y) = \frac{1}{2}(x + y)(x + y + 1) + y$. The function g is a bijection from ω^2 onto ω (if you are not convinced by the geometry, you can always show this by induction on x and on y) that is Σ_1, as its graph is defined by the Δ_0 formula $2z = (x + y)(x + y + 1) + 2y$.

Now, the function that sends $(x_1, \ldots, x_k, x_{k+1})$ to $(x_1, \ldots, g(x_k, x_{k+1}))$ is a Σ_1 bijection from ω^{k+1} onto ω^k; the general result follows by compositions. □

As we can always reduce ω^k to ω by a Σ_1 bijection, which preserves the hierarchy, we generally speak, in this context, only of subsets of ω or of functions from ω to ω: Introducing ω^k gives us no more generality. As an example of the manipulation of a Σ_1 bijection from ω^2 to ω, let us prove the following lemma, which will be useful to us later:

Lemma 7.17 (Σ_n Choice Principle). *If $R(x, y)$ is a binary Σ_n relation between natural numbers such that for every x there is a y such that (x, y) satisfies R, then there is a Σ_n function f from ω to ω such that for every x, $(x, f(x))$ satisfies R.*

Proof. We would be tempted to send x to the least y such that $N \models R(x, y)$, but we can see that we would need a Π_n definition of R to express, as a Σ_1 statement, that for all $z < y, (x, z) \notin R$. Instead, we write R in the form $(\exists t)S(x, y, t)$, where S is Δ_0 or Π_{n-1} depending on the value of n, and take y corresponding to the pair (y, t) with the smallest number. In other words, we consider a canonical Σ_1 bijection π from ω^2 to ω. Let π_1 and π_2 be the two projections of the inverse of π, which are also Σ_1: For every x, $(\pi_1(x), \pi_2(x))$ is the unique pair such that $x = \pi(\pi_1(x), \pi_2(x))$. Now consider the function $y = f(x)$ whose graph is defined by the following Σ_n formula:

$$(\exists z)(S(x, \pi_1(z), \pi_2(z)) \wedge (\forall u < z)\neg S(x, \pi_1(u), \pi_2(u)) \wedge y = \pi_1(z)).$$

□

Lemma 7.18. *A nonempty subset of ω is Σ_n if and only if it is the image of a Σ_n function from ω to ω; it is Δ_n if and only if it is the image of an increasing Σ_n function from ω to ω.*

Proof. The whole set ω is Δ_0, defined by the formula $x = x$; therefore, by Lemma 7.14, the image of a Σ_n function is Σ_n. Conversely, suppose that A containing a is defined by a formula $(\exists y)f(x, y)$, where f is Π_{n-1} or Δ_0 depending on the value of n. Consider the function g from ω^2 to ω that sends (x, y) to x if $f(x, y)$ is satisfied, and to a otherwise: Its graph $z = g(x, y)$ is defined by the formula $(f(x, y) \wedge z = x) \vee (\neg f(x, y) \wedge z = a)$, which is Σ_n (and Π_n!), and the image of g is certainly A; g then needs only to be composed with a Σ_1 bijection from ω to ω^2.

Now suppose that f is an increasing Σ_n function from ω to ω: We have $f(x+1) \geq f(x)$ for all x. If its image is finite, in other words if f is constant from some point on, then it is Δ_0, since the set $\{a_0, \ldots, a_s\}$ is defined by the formula $x = a_0 \vee \cdots \vee x = a_s$; otherwise, the complement of the image of f, which is not bounded in ω, is defined by the following Σ_n formula: $x < f(0) \vee (\exists y)(f(y) < x < f(y+1))$; this image is thus Δ_n.

Any nonempty set of natural numbers can be associated with its enumeration function f: $f(0)$ is its smallest element, $f(1)$ its second element, $\cdots, f(n)$ its $(n+1)$th element (since $f(0)$ is the first, rather than the 0th, element!). If A is infinite, f is an increasing injection, a bijection between ω and A; if A is finite, let us adopt the convention that f repeats the greatest element of A. In the latter case, the graph of f is defined by the following Δ_0 formula:

$$(x = 0 \wedge y = a_0) \vee \cdots \vee (x = m-1 \wedge y = a_{m-1}) \vee (x \geq m \wedge y = a_m).$$

If A is infinite, then $y = f(x)$ is defined by the following formula, where the function β serves to code the sequence $f(0), f(1), \ldots, f(x)$:

$$\begin{array}{ll} (\exists u) & (\exists v)((r(u,v) \in A \wedge ((\forall t \leq v)(t < r(u,v) \rightarrow t \notin A)) \\ \wedge & (\forall i \leq x)(r(u,iv) \in A \wedge r(u,(i+1)v) \in A \wedge r(u,iv) < r(u,(i+1)v) \\ \wedge & ((\forall t \leq y)(r(u,iv) < t < r(u,(i+1)v) \rightarrow t \notin A))) \wedge r(u,(x+1)v) = y). \end{array}$$

This is certainly a Σ_n formula if A is Δ_n, since the relations $w = r(u,v)$ (which is equivalent to $w < v \wedge (\exists z \leq u) u = zv + w$), $t \in A$, $t \notin A$ are expressible by Σ_n formulas. □

Lemma 7.19. *If A is a Σ_n subset of ω and if $f_1, \ldots, f_s$ are Σ_n functions from ω^{m_i} to ω, then the closure of A under these functions is also Σ_n.*

Proof. An element of this closure is obtained from elements of A by iterating compositions of the f_i. If we fix such a way to get the element a, we get a finite sequence $a_0, \ldots, a_n$, whose last term is a, such that each a_j is in A, or else is f_i of previous elements. Code this sequence by means of the function β. □

In the last two lemmas, we coded recursive definitions with the help of the function β; we are going to see that such processes never take us out of the class of Σ_n functions (allowing us to see that all "usual" functions are Σ_1). More precisely, we shall say that the function f from ω^{k+1} to ω is defined from the function g from ω^k to ω (g has one fewer variable than f; if $k = 0$, it is a constant) and the function h from ω^{k+2} to ω (h has one more variable than f), by induction (we also say by recursion) on the variable y, if f is the (necessarily unique) function determined by the following conditions:

$$\begin{aligned} f(x_1, \ldots, x_k, 0) &= g(x_1, \ldots, x_k), \\ f(x_1, \ldots, x_k, y+1) &= h(x_1, \ldots, x_n, y)). \end{aligned}$$

The function g is the *given initial function*, equal to $f(\vec{x}, 0)$; the function h is the *transition function*, which allows us to calculate $f(\vec{x}, y+1)$ from $f(\vec{x}, y)$.

Lemma 7.20. *If both functions g and h are Σ_n, then the function f defined by induction from them is also Σ_n.*

Proof. To define the relation $z = f(x_1, \ldots, x_k, y)$, code the sequence $f(\vec{x}, 0), f(\vec{x}, 1), \ldots, f(\vec{x}, y)$ by means of the function β. □

The class Δ_0, which serves to start the hierarchy, has an artificial side, and its elements are difficult to characterize; not much would be changed by considering quantifications of the form $(\exists y \leq 2x), (\forall y \leq 2x)$, or even $(\exists y \leq 2^x), (\forall y \leq 2^x)$, yet these very simple quantifications take us out of the class Δ_0. On the other hand, we can see that if f is a Σ_n function and ϕ a formula, then $(\exists y \leq f(x))\phi$ can be written as either $(\exists z)(\exists y \leq z)(z = f(x) \wedge \phi)$ or $(\forall z)(\exists y \leq z)(z \neq f(x) \vee \phi)$, while $(\forall y \leq f(x))\phi$ can be written as either $(\exists z)(\forall y \leq z)(z = f(x) \wedge \phi)$ or $(\forall z)(\forall y \leq z)(z \neq f(x) \vee \phi)$. Hence *quantifications bounded by Σ_n functions do not take us out of the class Σ_n or the class Π_n*, nor, consequently, out of the class Δ_n. This solidity of the classes $\Delta_n, \Sigma_n, \Pi_n$ shows that they form a very natural hierarchy for arithmetic sets, and we shall see at the end of this section that a higher position in this hierarchy really means a greater complexity of definition.

The Σ_1 functions are also called *recursive functions* because definition by recursion is one of the principal processes allowing us to construct them; we also sometimes call them *(effectively) computable functions*, because they are the translations, into mathematical terms, of the slightly vague idea of "functions computable by a purely mechanical procedure" (we also say "computable by an algorithm"). The Σ_1 sets, which are their images, are called *recursively enumerable* or *computably enumerable*; as for the Δ_1 sets, whose characteristic functions are recursive, they are called *recursive*, or *computable*, or *decidable*.

By an algorithm to calculate the function f mechanically, we understand a specified program, or list of instructions, that an executor can carry out step by step, starting with the input x, with the following property: No matter what number x is given at the start, at the end of some time the executor receives the order to stop and to output the number $f(x)$. The executor, who could, for example, be what in the business is called a computer, has neither autonomy nor intellectual initiative—he/she/it does nothing that does not conform to the instructions on the list—but in this completely theoretical approach to computability, we do not concern ourselves with the practicability of the calculations by a mechanical apparatus; in particular, we ignore how much time and space are necessary to perform the calculation, provided that it does terminate.

First, let us see that a Σ_1 function f is indeed calculable by a being whose intellectual possibilities are all bounded, but who has unbounded aptitude for executing completely idiotic mechanical operations. The relation $y = f(x)$ has the form $(\exists z)\phi(x, y, z)$, where ϕ is a Δ_0 formula. But to see that a quantifier-free formula $\phi(\vec{x})$ is true of the tuple $\vec{a}$, is within the reach of an elementary-school student: It is enough to know how to do addition, multiplication, and subtraction to see whether $a \leq b$; given m, n, p, the truth or falsity of the sentence $\phi(m, n, p)$ can be decided by a modest method: Since all the quantifiers are bounded by $\max(m, n, p)$, there are only finitely many values to be tested for the bound variables, and we need only program our operator to try all values below $\max(m, n, p)$. This algorithm is obviously infeasible: The increased calculation time would bring defeat to a well-born number-cruncher, but it is perfectly suitable to us living in the paradise of objects existing only in theory. Here are the instructions we give to our operator: You start with the input x; list all pairs (y, z) of natural numbers by means of the canonical Σ_1 bijection from ω^2 to ω; for each pair (y, z) test the truth of the formula $\phi(x, y, z)$; stop when you find a true sentence; the value to output is the y of the corresponding pair.

How about the converse? For that we need to reflect a little on what a mechanical process can be. Our executor could be a number-cruncher, with pencil and eraser, or else, in a more contemporary fashion, a reading head in front of a ribbon composed of magnetized or demagnetized spaces; its program consists of a list of instructions $I_0, \ldots, I_n$ that are each of the following type: If you ascertain something (for example, a magnetized space in front of you), perform such-and-such an operation (for example, demagnetize the space, or else move yourself one space to the left or to the right) and then go to the ith instruction; otherwise (if the space in front of you is not magnetized), perform such-and-such an operation, and then go to the jth operation. Thanks to our combinatorics, we can code, as a number, the state of the sheet of paper, or the magnetic ribbon, in front of our number-cruncher, as well as the instruction to be executed next: This number, which we shall call the "computation stage," codes the situation that our operator has reached. The specification of a program involves nothing more than the "transition function" that allows us to pass from one stage to the next. It should be clear, from the totally rudimentary nature of the tests and the operations that we require our operator to perform, that this transition function will have a graph defined by a formula with bounded quantifiers by a somewhat miserable recursive function; in these conditions, by Lemma 7.20, the function sending the pair (x, z) to the code of the zth computation stage arrived at starting with the input x, with I_0 as the initial instruction, is also recursive. The function $y = f(x)$ computed by this process is defined by the Σ_1 formula "there is some time z such that the operator, starting with input x, receives after z stages the instruction to stop and to output y." We shall remember the interpretation of the unbounded existential quantifier in the formula: It is principally tied to the length of the computation.

The ideas developed above would become rigorous if we defined "mechanical computation procedures" mathematically; for this reason, the statement "a function is computable if and only if it is Σ_1" is known as *Church's thesis*: We call it a thesis, rather than a theorem, because without a precise definition of computability, it is just an estimate, whose plausibility can be justified only by the kind of arguments we have just outlined.

This thesis is strengthened by the fact that many formal approaches to computability (one of which is based on *Turing machines*, which can be considered as abstract equivalents of our computers; in fact, they are distant ancestors of computers, which should rather be considered as concrete realizations of Turing machines, as Turing machines were invented before anyone knew how to build a computer!) that have been proposed so far all define the same class of functions, that of Σ_1 functions. Moreover, this thesis is not ineffective: quite often, to find a Σ_1 formula defining the graph of a function, we code into arithmetic the natural procedure used to compute it.

Even Church's thesis is susceptible to the following strengthening: "Every function from $\mathbf{N}$ to $\mathbf{N}$ is Σ_1, unless it was specially constructed to be a counterexample"! That is a completely paradoxical claim, since there are only denumerably many Σ_1 functions (no more than there are formulas), and they therefore form a totally negligible set among all the functions from $\mathbf{N}$ to $\mathbf{N}$; however, experience shows that those functions and sets occurring "naturally" in a mathematical context are, apart from some very rare examples (for example the Hilbert problem, settled by Matijacević), Δ_1, and even easily seen to be so.

In fact, all the functions about which a normal mathematician might think belong to an even more restricted class, that of the primitive recursive functions, which are the functions that we get from the functions $x \to 0, x \to x, x \to x+1, \ldots, (x_1, \ldots, x_i, \ldots, x_n) \to x_i, \ldots$ by compositions, juxtapositions, and definitions by recursion; this is not a very natural class of functions, and while in general it is easy to see that a function is Σ_1 (with a little practice, we can get a Σ_1 formula for its graph), it is often painstaking to prove that it is primitive recursive: There is no other way to do this than to find a way of constructing it from the given starting functions. These functions have properties related to the axiomatization of arithmetic, and as an illustrative example, we are going to show that the function γ from Lemma 7.08 is primitive recursive:

Lemma 7.21. *The function $\gamma(x, y)$ is primitive recursive.*

Proof. [(1)]

min$(1, x)$ is primitive recursive: $\min(1, 0) = 0, \min(1, x+1) = 1$.

2. Let $x \dot{-} y = 0$ if $y > x$, $x \dot{-} y = x - y$ if $x \geq y$. Then $x \dot{-} 1$ is primitive recursive: $0 \dot{-} 1 = 0, (x+1) \dot{-} 1 = x$.

3. $x \dot{-} y$ is primitive recursive: $x \dot{-} 0 = x, x \dot{-} (y+1) = (x \dot{-} y) \dot{-} 1$.

4. Let $f(x, y)$ be the sum of the base-two expansion of x truncated at position y: If $x = \epsilon_0 + \epsilon_1 2 + \cdots + \epsilon_k 2^k$, then $f(x, y) = \epsilon_0 + \epsilon_1 2 + \cdots + \epsilon_y 2^y$. This function is primitive recursive: $f(0, y) = 0$; $f(x+1, y) = f(x, y) + 1$ if $f(x, y) + 1 < 2^{y+1}$, $f(x+1, y) = 0$ if $f(x, y) + 1 = 2^{y+1}$, and $f(x+1, y) = \min(1, 2^{y+1} \dot{-} (f(x, y) + 1))$. The function $f(x, y) + 1$ is

certainly primitive recursive, as is the transition function, since the sum, product and exponential all are.

5. $\gamma(x,0) = f(x,0), \gamma(x,y+1) = \min(1, f(x,y+1) \dot{-} f(x,y))$.

□

Having considered this example, the reader must now be convinced that all the functions introduced in Section 7.6 in regard to formulas, such as that sending a formula to its quantification rank, or the code of the set of its free variables, all these functions in whose definitions recursion played the principal role, are recursive, and even primitive recursive. Also, sets such as the set of sentences and the set of Σ_n sentences, are all Δ_1. That can only be justified by blind faith in the generalized Church's Thesis, or else by verifying each particular case, a verification that we leave up to the reader lacking in faith.

That being said, we are in a position to refine Tarski's theorem in the following way:

Theorem 7.22. *The set of codes of Σ_n sentences that are true in arithmetic is a Σ_n set that is not Π_n; the set of codes of Π_n sentences that are true in arithmetic is a Π_n set that is not Σ_n.*

Proof. First let us show the positive part of the theorem, and begin by showing that the set of codes of Δ_0 sentences true in arithmetic is Δ_1. Indeed, in the Δ_0 formula $f(a_1, \ldots, a_n)$ all bound variables are bounded by $\max(a_1, \ldots, a_n)$, and the truth of this sentence is defined by induction on the pairs consisting of a subformula g of f and the set of tuples bounded by $\max(a_1, \ldots, a_n)$ that satisfy g. As the set of tuples involved is finite, they correspond to objects coded in our combinatorics, and the statement "x is the code of a true Δ_0 sentence" can be translated by a Σ_1 formula, as can the statement "x is the code of a false Δ_0 sentence," because the transformation consisting in taking the negation of a formula corresponds to a recursive operation on the codes; the statement "x is not the code of a Δ_0 sentence, or else is the code of a false Δ_0 sentence" is therefore Σ_1, and defines the complement of our set.

And when is the Σ_1 sentence $(\exists x) f(x)$ true, where f is a Δ_0 formula? If there exists a such that the Δ_0 sentence $f(a)$ is true. Once we have noticed that it is a recursive function that sends the code of a Σ_1 formula in one free variable x, and a number a, to the code of the Δ_0 sentence obtained by substituting a for x in the formula, then, knowing that Δ_0 truth translates as a Σ_1 formula, we can see that "there exists a such that $f(a)$ is true" is Σ_1. Since the true Π_1 sentences correspond by negation to the false Σ_1 sentences, they form a Π_1 set.

Proceeding by induction on n, we can see in the same way that the true Σ_n sentences form a set defined by a formula $V\Sigma_n(x)$ that is Σ_n, and that the true Π_n sentences form a set defined by a formula $V\Pi_n(x)$ that is Π_n.

For the negative part, suppose that there is a Π_n formula $V(x)$ that defines the set of codes of true Σ_n sentences. Consider the formula $V(x,y) = V(\phi(x,y))$ defining the set of pairs whose first term is a Σ_n formula in one free variable such that the sentence obtained by replacing this variable by y is true. As the function ϕ to replacing one variable by a constant is Σ_1, $V(x,y)$ is a Π_n formula, so the formula $\neg V(x,x)$ is equivalent to a Σ_n formula with code n_0, and we get the same contradiction as in Tarski's theorem. □

This theorem proves that all the inclusions $\Delta_n \subset \Sigma_n \subset \Delta_{n+1}$ and $\Delta_n \subset \Pi_n \subset \Delta_{n+1}$ are strict.

We can see that every Σ_n set can simply be obtained from the "Σ_n truth" set defined by the formula $V\Sigma_n(x)$; indeed, to check that b belongs to the ath Σ_n formula (i.e., the set defined by the Σ_n formula with code a), we need to check that $\phi(a,b) \in V\Sigma_n$; for this reason, this set $V\Sigma_n$, which is the most complex of all the Σ_n sets, is called a "universal Σ_n set."

We can also relativize the hierarchy and define the notions of recursive, or Σ_1, or Σ_n, set with respect to a set A: Every Σ_n set is thus recursive in $V\Sigma_n$; if A and B are each recursive with respect to each other, we say that they have the same degree, and the study of these degrees, which we shall only touch upon, constitutes what is called recursion theory or computability theory.

7.7 Some Axioms, Models, and Fragments of Arithmetic

In this section we are going to examine a few axioms that are satisfied by the sum and product on the natural numbers, and we shall see to what extent the results of the previous sections remain valid for structures that are just models of some fragment of arithmetic.

By *minimal arithmetic* we will mean the following finite set A_0 of axioms:

- $(\forall x)x + 1 \neq 0$
- $(\forall x)0 \leq x$
- $(\forall x)(x = 0 \vee \neg(x \leq 0))$
- $(\forall x)x + 0 = x$
- $(\forall x)x \cdot 0 = 0$
- $(\forall x)(\forall y)(x + 1 = y + 1 \rightarrow x = y)$
- $(\forall x)(\forall y)(y \leq x + 1 \leftrightarrow y \leq x \vee y = x + 1)$
- $(\forall x)(\forall y)(x + 1 \leq y \leftrightarrow x \leq y \wedge y \neq x)$

- $(\forall x)(\forall y)x + (y + 1) = (x + y) + 1$
- $(\forall x)(\forall y)x \cdot (y + 1) = x \cdot y + x$

The axioms of A_0 settle the behavior of 0 vis-à-vis the arithmetic operations, as well as that of $x + 1$, assuming that we know that of x. The choice of such a system, consisting of finitely many universal axioms, is not made without discretion: We could, for example, switch the variables x and y in the axioms describing how the sum and product interact with the successor.

In a model M of A_0, the function that sends x to its successor, $s(x) = x + 1$, is injective; also, 0 is not in the image of s, but there might be other nonsuccessors. If n is a natural number, we will also write n for the nth successor of 0 in M; we can see that the axioms of A_0 force the elements less than or equal to n to be $0, \ldots, n-1, n$, while all elements of M except for $0, \ldots, n-1$ are greater than or equal to n. These successors of 0 will be called *standard* numbers (or elements) of M. This *standard part* of M is an initial segment of it, in that every standard number is less that every nonstandard number (this use of the phrase "initial segment" is naturally a little abusive, since we cannot deduce from A_0 that $\leq$ is an order). Moreover, we can easily see that this standard part is a restriction of M isomorphic to the true natural numbers, with their sum and product. That implies, and this is the principal raison d'être of A_0, that every Σ_1 sentence of arithmetic is a consequence of A_0.

However, beyond that, A_0 is an extremely weak theory, which allows nonstandard numbers to behave in the most fantastic ways. One of the rare non-Σ_1 constraints is that the successor function not have any cycles, since we can show in A_0 that for $n > 1, (\forall x)(x \leq x + n \land \neg(x + n \leq x))$. Here is a model of A_0: The universe consists of the standard part $0, 1, 2, \ldots$ together with $0', 1', 2', \ldots$, a second copy of ω. The relation $x \leq y$ is satisfied by the following pairs: In all cases $n \leq m'$; if n is less than m, then $n \leq m$ and $n' \leq m'$. Also, $n + m$ is the sum of n and m; $n' + m = (n + m)'$; $n' + m' = n + m' = m'$. $n \cdot m$ is the product of n and m; $n \cdot m' = (n \cdot m)'$; $n' \cdot m = n'$; $n' \cdot m' = n'$.

However, if we would let ourselves add the following second-order axiom to A_0;

$$(\forall X)((0 \in X \land (\forall x)(x \in X \to x + 1 \in X)) \to (\forall x)x \in X),$$

then we would be able to show that the model could be reduced to its standard part; that is to say, we would have characterized the structure formed by the sum and product on the true natural numbers up to isomorphism. That fact was noticed at the beginning of this century by the Italian mathematician Peano; for this reason, the name *Peano arithmetic* is given to the set of axioms consisting of A_0 and all of the following "induction axioms" (we need one axiom for each formula f):

$$(I(f)) \quad (\forall\vec{x})((f(\vec{x},0) \wedge (\forall y)(f(\vec{x},y) \rightarrow f(\vec{x},y+1))) \rightarrow (\forall y)f(\vec{x},y))$$

This list of axioms relies on the same metaphysics as that for Presburger's axiomatization (see Section 7.3) for the sum on the natural numbers: Since we do not have the right to express an induction axiom valid for all subsets of the model, we at least do it for those sets which are accessible in our language, that is, those that are definable by a formula of arithmetic.

If we add to A_0 only those induction axioms corresponding to $\Delta_0, \Sigma_n, \Pi_n$ formulas, we speak of the theories of Δ_0-, Σ_n-, Π_n-induction respectively. Note in passing that a Δ_0-induction axiom is both Σ_2 and Π_2, and that a Σ_n- or Π_n-induction axiom is Σ_{n+2} and Π_{n+2} (replace $f \rightarrow g$ by $\neg f \vee g$).

A_0 enriched by induction axioms reveals a more reasonable aspect: With $I(y = 0 \vee (\exists z \leq y)y = z + 1)$ we can show that every nonzero number is a successor; we can also see, simply by examining the proof of Theorem 7.5, that the theory of the sum is a consequence of Δ_0-induction (and even of finitely many Δ_0-induction axioms, since the properties of Euclidean division can be shown by a single induction axiom involving the product). We can similarly show by Δ_0-induction that $(\forall x)(\forall y)(x \leq y \leftrightarrow (\exists z)x + z = y)$, and consequently that the whole theory of the order can also be deduced from Δ_0-induction. We can also show by Δ_0-induction that the product is associative, commutative (and it is also correct that the theory of the product, that is to say, the set of sentences true of the natural numbers that involve only the product, is a consequence of Δ_0-induction), and distributive over addition.

Δ_0-induction is, however, still quite a weak theory. For example, we can see that one consequence of Δ_0-induction is that for every x there is a prime number that does not divide x: We show, by induction on the formula

$$\begin{aligned}
&(\forall x' \leq x)(x' \leq 1 \vee \\
&(\exists y \leq x)(\exists z \leq x)(y \geq 2 \wedge \\
&(\forall u \leq x)(\forall v \leq x)(y = uv \rightarrow u = 1 \vee v = 1) \\
&\vee x' = yz))
\end{aligned}$$

that every number has a prime factor; we then take a prime factor of $x+1$. However, it is an open problem today whether Δ_0-induction allows us to prove that for every x there is a prime above x, i.e., that the set of prime numbers is unbounded. The usual proof of this elementary fact is as follows: Consider the set of prime numbers less than x, and take a prime divisor of the successor of their product. To reproduce this proof, we need to prove the existence of a code for the set of prime numbers less than x, which Δ_0-induction does not allow.

As the membership relation, $x \in y$, is Δ_1, we should expect Δ_0-induction to tolerate pathological behavior in the combinatorics associated with arithmetic. For example, Δ_0-induction does not allow us to prove that every actually finite set, or even every singleton, is coded: Indeed, we can easily see that an initial segment of a model of arithmetic closed under sum and product is a model of Δ_0-induction; if this segment is not closed under the function $x \to 2^x$, there will be uncoded singletons. This method cannot be applied to the previous problem, because we know that there is always a prime between x and $2x$.

In contrast, it is a consequence of Σ_1-induction that

$$(\forall x_1)\cdots(\forall x_n)(\exists x)(\forall y)(y \in x \leftrightarrow (y = x_1 \vee \cdots \vee y = x_n)).$$

We can even prove that if we add an element to a coded set we get a coded set:

$$(\forall x)(\forall y)(\exists z)(\forall t)(t \in z \leftrightarrow t \in x \vee t = y).$$

That makes us think that Σ_1-induction is a strong enough theory not to mangle our intuitions about arithmetic, and the rest of this section will be dedicated to showing how reasonable this theory is. We therefore take it as the basis of the axiomatic approach to arithmetic, while indicating in passing that the specialists in these questions consider it too strong a theory: They are satisfied with adding to Δ_0-induction some axioms guaranteeing that exponentiation, and therefore the combinatorics, is well-behaved, and they can show that this system is enough to show all ordinary theorems of arithmetic, that is to say, all those that were not specially cooked up by logicians to serve as counterexamples.

However, before continuing, we have two remarks about the induction axioms. Beginning students are always terrified to discover that there are two kinds: a weak form, for which recursion starts with 0 and passes from x to $x+1$, which is the form we have been using; and a strong form, in which we pass from all z strictly less than y to y, that is to say,

$$(I'(f)): \quad (\forall \vec{x})(((\forall y)((\forall z < y)f(\vec{x}, z) \to f(\vec{x}, y))) \to (\forall y)f(\vec{x}, y)).$$

This is actually a stronger axiom than $I(f)$, seen as additions to A_0 with the axiom stating that every nonzero number is a successor: Indeed, if the premise of $I(f)$ is satisfied, so is that of $I'(f)$, since A_0 implies that $x < x + 1$. But in fact, as we have already seen in regard to the prime numbers, these two ways of presenting induction are equivalent, since $I'(f(\vec{x}, y))$ is a consequence of $I((\forall z \leq y)f(\vec{x}, z))$.

There is yet another way to introduce induction, namely to translate the fact that every nonempty set of natural numbers has a least element:

$$(I''(f)) \quad (\forall \vec{x})((\exists y)f(\vec{x}, y) \to (\exists y)(f(\vec{x}, y) \wedge (\forall z < y)\neg f(\vec{x}, z))).$$

We can see without difficulty that $I'(f)$ and $I''(\neg f)$ are equivalent.

Narrowly linked to the induction axioms are the *collection* axioms, which express the fact that the numbers satisfying some formula $f(t)$ that are less than a certain z form a finite set in the sense of the model:

$$(C(f)) \quad (\forall \vec{x})(\forall z)(\exists y)(\forall t)(t \in y \leftrightarrow t \leq z \wedge f(\vec{x}, t)).$$

When we study a model M of A_0, it is tempting to copy onto it everything that happens in the standard integers; for example, we may try to define an exponential function, taking the same defining formula that we used to define the graph of an exponential function on the real natural numbers. The problem is that there is no a priori reason to think that this formula defines the graph of a function in M. Also, we can positively doubt that for standard natural numbers n, the value of 2^n in M will be the same as in the standard model; even if that is true, there is still no reason for the usual properties of exponentiation to hold in M.

Let us consider a fragment T of arithmetic containing A_0, or even just a consistent theory containing A_0 (i.e., T might contain axioms that are false on the true natural numbers). Let f be a function from ω to ω whose graph is defined by an arithmetic formula $\phi(x, y)$. We shall say that f is *provably a total function in* T if it is a consequence of T that ϕ is the graph of a function: $T \models (\forall x)(\exists! y)\phi(x, y)$. In this case we shall call f *good for* T if, in addition, for all standard n, $T \models \phi(n, m)$ for the value $m = f(n)$. In this definition there is an obvious abuse of language: We are dealing with properties of the formula ϕ rather than the function f that it defines, as there is no reason why two definitions of f, which are equivalent on the true natural numbers, will remain so in all models of T.

Along the same lines, consider a subset A of ω defined by a formula $\phi(x)$. We shall call A *good for* T if, for all standard n, $T \models \phi(n)$ if $n \in A$, and $T \models \neg\phi(n)$ if $n \notin A$. If A is defined by a Σ_n formula ϕ and a Π_n formula ψ, we shall call A *provably* Δ_n *in* T if T ensures the equivalence of ϕ and ψ, $T \models (\forall x)(\phi(x) \leftrightarrow \psi(x))$.

When we work in a weakened arithmetic, we need to put up a railing in front of our intuitions, which can push us into extending some obvious properties of the true integers, without verification, to all models of the theory under consideration. If we want to translate the developments from the preceding sections, it will help to guarantee ourselves the goodness of all the sets and functions involved in the manipulations of codes of formulas; if our theory is too weak to guarantee their goodness, then we need to be conscientious and particularly vigilant. For example, we have said that quantification bounded by a recursive function does not take us out of the class Σ_1 or Π_1; that is to say, if ϕ is, say, Σ_1, then $(\forall y \leq f(x))\phi(x, y)$ is equivalent to a Σ_1 formula. For that to be true in T, we must prove in T that $y = f(x)$ defines a total function, so that we can replace $y \neq f(x)$ by a Σ_1 formula.

It is clear that the Δ_0 sets are good for minimal arithmetic A_0: for checking that $n \in A$, everything happens below n. The same is true for any set defined by a formula in which the quantifiers are all bounded by polynomials. Note, therefore, that for every theory T containing A_0, *every* *provably* Δ_1 *set is good*: Let the set A be defined in the model of the true natural numbers by both a Σ_1 formula $\phi(x)$ and a Π_1 formula $\psi(x)$, and let $T \models (\forall x)(\phi(x) \leftrightarrow \psi(x))$. Let M be a model of T. If n is a standard number, and if $n \in A$, then $\omega \models \phi(n)$, so $M \models \phi(n)$, since ϕ is Σ_1 and ω is an initial segment of M; likewise, if $n \notin A$, then $M \models \neg\psi(n)$, and as $T \models (\forall x)(\phi(x) \leftrightarrow \psi(x))$, $M \models \neg\phi(n)$. For example, a *provably recursive function* (i.e., a Σ_1 formula that can be proven in T to be the graph of a total function) *is good.*

What fragment of arithmetic guarantees that the usual functions, or the essential relations that appear in the coding of the combinatorics or of the language, are good? One reasonable fragment is Σ_1-induction; indeed, it allows us to prove the axiom expressing that if A is a set coded by the function β, then for all a, $A \cup \{a\}$ is also coded by this function β. This is a painstaking exercise involving the Chinese remainder theorem; it is left up to the reader. In any event, if it were not provable from Σ_1-induction, it would be urgently necessary to add it.

This property allows us to guarantee the goodness of constructions by simple recursion: looking at the definition of exponentiation in Section 7.4, we see that we can show, by Σ_1-induction (on y), that for every x and y there is a unique z such that $z = x^y$: What is essential is to be able to show the existence of a code for the sequence $x, x^2, \ldots, x^y, x^{y+1}$, assuming only one for $x, x^2, \ldots, x^y$. We will also be able to show the elementary properties of the exponential, such as $x^{u+v} = x^u \cdot x^v$, from which we can check that the properties correspond to recursion on Σ_1 formulas.

More generally, and for the same reason, we can see that *primitive recursive functions are provably total in* Σ_1*-induction*, and therefore provably recursive and good. As we have taken care to check in Lemma 7.21 that the function γ is primitive recursive, we can see that the combinatorics, our relation $\in$, is good for Σ_1-induction: In one of its models M, there will be "false" finite sets (i.e., finite sets of nonstandard cardinality), but we can be sure that the true finite sets of true numbers (i.e., sets of standard cardinality whose elements are all standard) get the same code in M as in ω. If, therefore, you are persuaded that the "usual" functions are primitive recursive, that the sets occurring "naturally" in a mathematical context have primitive recursive characteristic functions, you are now reassured that Σ_1-induction is trustworthy.

We can now compare the different forms of induction axioms that we have introduced:

Theorem 7.23. *The theories consisting of* Σ_1*-induction and* Σ_n*-collection, or* Π_n*-collection, are equivalent; in the presence of* Σ_1*-induction,* Σ_n*-collection implies* Σ_n*- and* Π_n*-induction, while* Σ_{n+1}*- or* Π_{n+1}*-induction implies* Σ_n*-collection.*

Proof. The collection axiom $C(f)$ is equivalent to the axiom $C(\neg f)$. For if a codes the set of all y such that $y \leq z \wedge f(z)$, then $\{y : y \leq z \wedge \neg f(y)\} =$

$\{y : y \leq z \wedge y \notin a\}$: The existence of the code of this last set is guaranteed by Σ_1-induction on z.

Σ_n-collection allows us to prove Σ_n- and Π_n-induction, by applying the principle that every finite set of natural numbers has a smallest element, which is provable from Σ_1-induction.

Finally, if we note that 2^{z+1} is the maximum of all codes of subsets of the segment $[0, z]$, we can see that the axiom of collection for f can be proved by induction on the formula $(\exists t \leq 2^{z+1})(\forall y \leq z)(y \in t \leftrightarrow f(y))$, which is both Σ_{n+1} and Π_{n+1} if f is either Σ_n or Π_n. □

In the model of Σ_1-induction, we therefore have a good set of formulas: The formulas that have a standard code represent true formulas, and in the same way as in the standard model. The standard formulas that are Σ_n in M are the same as in ω, and when the operation of replacing a variable by a standard constant is performed in M, the exact same operation is done in ω. The upshot of all this is the goodness of certain combinatorial manipulations, which we can check by showing the primitive recursive character of the functions that appear in the game.

As a consequence, Tarski's theorem can be generalized as follows:

Theorem 7.24 (Tarski). *Let M be a model of Σ_1-induction. Then there is no formula $V(x)$ such that for standard n, $M \models V(n)$ iff n is the code of a sentence that is true in M.*

Proof. Copy the proof of Theorem 7.9, using the goodness of the replacement function. □

Naturally, Tarski's theorem remains valid for a structure in, say, a finite language (if the language is denumerable, we need to give an "effective" enumeration) in which we can define a model of Σ_1-induction.

Moreover, Σ_1-induction is not absolutely necessary for Tarski's theorem. It can be proved for even weaker fragments of arithmetic, but if we work in a very weak theory, we need to anticipate a possible confusion with the codings and combinatorics.

The nonstandard formulas satisfy the formal laws that serve to define true formulas, but it is impossible to give them any meaning: This is what Tarski's theorem says. In contrast, we have defined a Σ_n predicate $V\Sigma_n(x)$ to translate the truth of Σ_n formulas, and we can easily show by Σ_1 induction, again because of the goodness of the function for replacing a variable by a constant, that for all standard m, $M \models V\Sigma_n(m)$ if and only if m is the code of a Σ_n sentence that is true in M. (That does not mean that $V\Sigma_n$ is good: The Σ_n sentences that are true in ω, and those that are true in M, are not necessarily the same!) We can therefore use this Σ_n truth predicate to define a notion of satisfaction for nonstandard Σ_n-formulas, where n is standard and fixed, even for those with a nonstandard number of free variables. But for Σ_x formulas with x nonstandard there is no hope: We can show, by the same method, that models of Σ_1-induction satisfy the refinement of Tarski's theorem (Theorem 7.22), preventing the existence of a Σ_n formula such that the standard numbers satisfying it are the codes of the Σ_n sentences that are false in the model.

7.8 Nonstandard Models with Arithmetic Definitions

One nonstandard model of the the theory of the ordering is $\omega + \mathbf{Z}$. It is easy to order the natural numbers, in an isomorphic way to $\omega + \mathbf{Z}$, by a Δ_1 binary relation: Represent the element n of ω by $3n$, the element $-n, n \neq 0$, of $\mathbf{Z}$ by $3n - 1$, and the element n of $\mathbf{Z}$ by $3n + 1$. Similarly, we can see that the model of the sum described at the end of Section 7.3 is isomorphic to a Δ_1 structure defined on the true natural numbers. If we believe that the Δ_1 sets are the "effectively describable" sets, we have constructed nonstandard models of the theory of the ordering and the theory of the sum in an "effective" way.

By the Löwenheim–Skolem theorem, there are denumerable nonstandard models of arithmetic, models whose universe can be taken to be ω. At the beginning of Section 7.4 I said that I was not going to try to construct one "effectively"; we are now going to see that it is not possible for the sum and product of such a model to be definable in arithmetic, and a fortiori they cannot be Δ_1. We will even be able to specify at what level a nonstandard model of A_0 definable in arithmetic necessarily lies, that is to say, satisfies a Σ_n or Π_n sentence that is false for the true integers (Theorem 7.25). Consequently, even if the identification of Δ_1 with "effective" seems questionable to you, you should grant that there is no nonstandard model of arithmetic with a definition as simple as the two structures described above.

So let us consider a model M of Σ_1-induction, defined on the universe ω of natural numbers, whose sum $x +_M y$ and product $x \cdot_M y$ have graphs defined by Σ_n formulas, and therefore also Π_n, since we are dealing with total functions. The other usual relations and functions attached to M will also have arithmetic definitions, which risk being slightly more complex than Δ_n: For example, the ordering $\leq_M$ on M, defined by the formula $(\exists z) x +_M z = y$, is Σ_n, but has no reason to be Δ_n; indeed, the quantifier in its definition cannot be bounded (in the sense of $\leq$, rather than of $\leq_M$), and if y is nonstandard, $\{z : z \leq_M y\}$ is an infinite, and therefore unbounded, subset of ω! Similarly, we can see that in the definition of the function γ, Lemma 7.8, the function $r_M(u, v)$ remains Σ_n, but the quantifier $(\forall j)$ cannot be bounded, so we can prove only that the function γ is Σ_{n+1} and that the relation $x \in_M y$ is Δ_{n+1}. In the following theorem we require this membership relation to be Σ_n; if we have a model for which we know the sum and product to be only Δ_n, we will need to replace n by $n + 1$.

Theorem 7.25. *Let M be a nonstandard model of Σ_1-induction and Π_n-collection, defined on the universe ω of natural numbers, whose sum, product, and membership are all Σ_n. Then there is a sentence that is Σ_n or Π_n and that is true in arithmetic (i.e., true in the standard model) and false in M.*

Proof. Assume that every Σ_n or Π_n sentence that is true in arithmetic is also true in M. If m is a standard number, let m_M denote the mth successor of 0_M in the sense of M. Since the formula $V\Pi_n(x)$ is Π_n, $M \models V\Pi_n(m_M)$ iff $\omega \models V\Pi_n(m)$ iff m is the code of a Π_n sentence that is true in both the structures.

If we let $V\Pi_{nM}(x)$ denote the formula obtained by substituting $+_M$ for $+$ and $\cdot_M$ for $\cdot$ in $V\Pi_n(x)$, then $M \models V\Pi_n(m_M)$ iff $\omega \models V\Pi_{nM}(m_M)$. Note a minor ambiguity with the two notations: To say that M satisfies the sentence $V\Pi_n(m)$, means that the formula $V\Pi_n(x)$ is satisfied by the element m_M; that is due to our tendency to confuse a constant with the symbol representing it in our language.

Let a be nonstandard in M. By Π_n-collection, there exists b that, in the sense of M, codes $\{x : x \leq a \wedge V\Pi_n(x)\}$; in other words, $x \in_M b \leftrightarrow x \leq_M a \wedge V\Pi_{nM}(x)$. Consequently, from inside M, to see that x is the code of a true Π_n formula, we need to see that $x_M \in_M b$. But the function that sends x to x_M, defined by the formula "there is a finite sequence $a_0, \ldots, a_x$ such that $a_0 = 0_M$, $a_{i+1} = a_i +_M 1_M$ for all i less than x, and $x_M = a_x$," is Σ_n. This formula provides a Σ_n definition of the set of Π_n codes that are true in arithmetic, contradicting Theorem 7.22. □

By being a little more careful, we can extend this result to models of Σ_1-induction: If N is a model of Σ_1-induction, and if M is a model of Σ_1-induction and Π_n-collection whose sum, product, and set membership are definable in N by Σ_n formulas, and if M and N satisfy the same Π_n sentences, then the function $x \to x_M$ from N to M is an isomorphism.

7.9 Arithmetic Translation of Henkin's Method

In the last section we saw that a sufficiently complex list of axioms (the Peano axioms augmented by the set of true Σ_n or Π_n formulas) cannot have too simple a nonstandard model. Here we consider the converse problem: Given a list of axioms, to find a method to determine whether or not it is consistent, and if so, to construct as simple a model as possible. For this purpose we are going to translate Henkin's method into arithmetic.

Let us consider a set A of axioms in a finite or denumerable language L. (A nondenumerable language clearly cannot be coded into arithmetic!) To apply Henkin's method, we first reduce the problem to the case in which the language has only relation symbols, a reduction that is immediate. We therefore have a list $r_1, \ldots, r_n, \ldots$ of relation symbols in the language L.

Assume that the function associating n with the arity of r_n is recursive. (This is truly a minimal requirement; if L is finite, there can be no problem with the presentation of the language.) By associating each symbol of the language with a numeral, we can code its formulas as we have already done for the language of arithmetic, in such a way that different "natural" manipulations on formulas translate into recursive functions.

Associated with the language L is the language L^H of the Henkin enumeration, which includes a list of constant symbols $a_{i,j}$ and a relation symbol f^H for every formula f of L, which serves to eliminate quantifiers; L^H is coded in the same way.

Now consider the list $T(H)$ of structural sentences of Henkin enumerations given in Section 4.3; as the whole process is bathed in the most straightforward type of effectiveness, the function sending a formula to its witness is recursive, and this set $T(H)$ is Δ_1.

The $a_{i,j}$ are indexed by $\omega \times \omega$, but since we have at our disposal a recursive bijection between $\omega \times \omega$ and ω, we can index them by ω: It will make sense to speak of the "first k of the $a_{i,j}$." The symbols f^H can likewise be indexed by ω, by considering the "enumeration function" of their set (Lemma 7.18).

The following partial order $\mathcal{H}$ is called the "Henkin tree": An element of $\mathcal{H}$ is, for a fixed natural number k called its height, a specification of the restrictions of the first k relations $f_0^H, \dots, f_{k-1}^H$ to the first k of the $a_{i,j}$ (it is thus a finite set of conditions of the form $\vec{a} \in f_i^H$ or $\vec{a} \notin f_i^H$, containing, for each example, either the condition or its negation, but not both, where $\vec{a}$ ranges over the set of tuples of the appropriate length taken from the first k of the $a_{i,j}$), which includes all the structural sentences involving only the first k of the f_i^H and the first k of the $a_{i,j}$. (Do not lose sight of the fact that we are dealing with quantifier-free sentences!) We order this set by inclusion: $p \subset q$ if every condition of p is in q, which implies that q has greater height than p.

It is clear that the set $\mathcal{H}$ is Δ_1, as is the order relation. Note also that there is a function $h(x)$ that is recursive and easy to compute (so the number theorists say) that bounds the (codes of) elements of $\mathcal{H}$ with height less than x.

Moreover, the function associating a sentence e of L with its quantifier-free translation e^H is recursive (it is defined by simple recursion). Consequently, if A is Σ_n, Π_n, or Δ_n, then so is its image A^H under this function. Recall that if e is a sentence, then e^H is just a nullary relation symbol.

The *Henkin tree associated with* A is the subset $\mathcal{H}(A)$ of $\mathcal{H}$ consisting of those elements that, if they have sufficient height, satisfy the elements of A^H. An element of height k is in $\mathcal{H}(A)$ if it satisfies all the e^H with index less than k; membership in $\mathcal{H}(A)$ is translated by the formula "$p \in \mathcal{H}$ and, for every symbol f^H of index less than the height of p, either $f^H \notin A^H$ or

$f(H)$ is involved in p." We can thus see that $\mathcal{H}$ is Π_n if A is Σ_n, Σ_n if A is Π_n, and Δ_n if A is Δ_n.

What will happen in this tree? Two cases:

- For some k, $\mathcal{H}(A)$ has no element of height k. That means, as we will see, that $\mathcal{H}(A)$ is finite. The construction of the Henkin tree stops; that is to say, if we follow all possible branches, we will ultimately reach a contradiction. That means, as we noted in Section 4.3, that the set A is contradictory. (More precisely, it has no nonempty model; the pleasure of finding a way to decide whether a set of sentences has an empty model is left to the reader!)

- If not, the construction will continue ad infinitum. We will see that the tree $\mathcal{H}(A)$ then has an infinite branch; a branch is nothing other than the successive specification of restrictions of a structure to the first k constants, then to $k+1$ constants, etc., that satisfies the Henkin sentences. At the end of the branch we get a Henkin enumeration of a model A.

These remarks will allow us to express, in arithmetic, the consistency of a Σ_n set of sentences, and even, when it is consistent, to give a model that is definable in arithmetic.

However, to justify them, we must first prove a lemma on trees. By *tree* we mean here a partially ordered set, with a smallest element called its *root*, such that the predecessors of every element x form a finite chain, whose length is called the *height* of x. A *branch* of a tree is a maximal chain contained in it. The tree is called *finitely branching* if every element of height k has only finitely many (possibly zero) successors of height $k+1$. The Henkin tree clearly has all of these properties (its root is the empty sequence).

Theorem 7.26 (Dènes König's Lemma). *Every infinite, finitely branching tree has an infinite branch; more precisely, if the tree is Π_n, if the height of an element y is given by a Σ_{n+1} function, and if the codes of the elements of height k are bounded by a Σ_{n+1} function of k, then it has an infinite Δ_{n+1} branch.*

Proof. Let $\mathcal{A}$ be our tree, and let $\mathcal{A}^*$ be the tree consisting of the elements of $\mathcal{A}$ with infinitely many successors. The tree $\mathcal{A}^*$ includes the root of $\mathcal{A}$, and as $\mathcal{A}$ is finitely branching, every element of $\mathcal{A}^*$ of height k (the height in $\mathcal{A}^*$ is the same as in $\mathcal{A}$!) has a successor in $\mathcal{A}^*$ of height $k+1$; such a branch of $\mathcal{A}^*$ is an infinite branch of $\mathcal{A}$.

The tree $\mathcal{A}^*$ is defined by the following formula: $p \in \mathcal{A}$, and for every k greater than the height of p, there is q in $\mathcal{A}$, of height k, that extends p. We can also define the branch of $\mathcal{A}^*$ obtained by always choosing the smallest code: $p \in \mathcal{A}^*$ and there is a sequence $p_0, \ldots, p_k$ whose length k is the height of p such that p_0 is the root of $\mathcal{A}^*$ and for all $i \leq k$, p_{i+1} is a successor of p_i in $\mathcal{A}^*$, with height $i+1$ (this bounds the quantification), and no element with code lower than that of p_{i+1} has this property, and finally, $p_k = p$.

Under the hypotheses of the second part of the theorem, quantifying over the elements of height less than that of p amounts to quantifying bounded by a Σ_{n+1} function, so that $\mathcal{A}^*$ has a Δ_1 definition, as does its branch. □

Theorem 7.27. *The statement that a Σ_n set of sentences is inconsistent translates as a Σ_n sentence; every consistent Σ_n set of sentences has a Δ_{n+1} model.*

Remark. Do not confuse "a Σ_n set of sentences," which refers to a set of codes of sentences each of which can be arbitrarily complex, and in a language that is not necessarily that of arithmetic, with "a Σ_n sentence" or "the set of Σ_n sentences."

Proof. Let A be our set of sentences. As $\mathcal{H}(A)$ is a subtree of $\mathcal{H}$, which is perfectly recursive, the height function in $\mathcal{H}$ is recursive, and quantifications on elements of height less than k are bounded by a recursive function.

To express the fact that A is inconsistent, we need to say that $\mathcal{H}(A)$ is finite, or equivalently that for some k, it contains no elements of height k: "There exists k such that for every element p of $\mathcal{H}$ with height k, $p \notin \mathcal{H}(A)$." If A is Σ_n, then $\mathcal{H}(A)$ is Π_n; its complement is thus Σ_n, as is this last sentence.

In the case where A is Σ_n and consistent the tree $\mathcal{H}(A)$ is Π_n, and its height satisfies the hypotheses of Lemma 7.26: It therefore has an infinite Δ_{n+1} branch. This branch builds a model of A on the set of constants $a_{i,j}$: To see that an atomic formula is satisfied, it is enough to see that it occurs on the branch exactly when it is expected; that defines a Δ_{n+1} sentence, modulo this fact: Equality is not interpreted by true identity, but rather by a Δ_{n+1} equivalence relation $\sim$. To get a model with the true identity relation, we restrict ourselves to those elements whose codes are smallest in their respective equivalence classes, a condition definable by the Δ_{n+1} formula $(\forall y < x) y \not\sim x$.

If we get a finite model, with m elements, we can transfer it to the first m natural numbers. Otherwise, we get an infinite model whose universe is Δ_{n+1}, as are the relations involved; we can transfer this infinite Δ_{n+1} set to the set ω of all natural numbers by means of its enumeration function (see Lemma 7.18) to get a Δ_{n+1} model with universe ω. □

A few comments on this last theorem: First, it is natural to look for models that are Δ_n; indeed, if A is a system of axioms in a language L, if we expand the language by adding a symbol r' for each relational predicate of the language of A, and add axioms $\neg r \leftrightarrow r'$, we get a set A' of axioms with the same complexity Σ_n, Π_n, or Δ_n as A, and a Σ_n or Π_n model of A' is the same thing as a Δ_n model of A!

In contrast, we might wonder why Theorem 7.27 gives a Δ_{n+1} model for a Σ_n or Δ_n theory, while it gives only a Δ_{n+2} model for a Π_n theory. The key to this asymmetry between Σ_n and Π_n is given by the following theorem, which will be well received by instructors and corporals, as well as by everyone whose principal task is to repeat the same things unrelentingly.

Theorem 7.28 (Pleonasm Theorem). *For every Σ_{n+1} set of sentences there is a set of sentences that is Δ_1 if $n = 0$, and Π_n otherwise, that has the same consequences.*

Proof. Let A be our set of sentences, defined by a formula $(\forall y) f(x, y)$, where f is Δ_0 or Π_n as appropriate. The idea is to replace the pair formed by the formula x and the number y by the formula obtained by putting y double negations in front of x. We thus get a set B that obviously has the same consequences as A. As a single x in A may be associated with several y, the formula x occurs several times in equivalent forms in B, hence the name "Pleonasm theorem."

$z \in B$ is defined by the formula "there is a subformula x of z and a natural number y less than the complexity of x, such that (x, y) satisfies f and such that z is obtained by putting $2y$ negation symbols in front of x"; as the quantifiers are bounded by recursive functions, B does have the indicated complexity. □

We can easily see that Henkin's method cannot give a better theorem than Theorem 7.27, since we cannot hope for a Π_n tree to have a Δ_n branch (imagine that the tree consists of just a single branch!); nevertheless, the counterexamples that we gave in Theorem 7.25 do not prove that Theorem 7.27 is the best possible; there are, however, Π_n theories that do not have Δ_n models, to wit:

- the language consists of the symbols $0, 1, \leq, +, \cdot, \in, a$;
- we include the Peano axioms (a Δ_1 list) and the axiom defining $\in$ in terms of the sum and product;
- we also include the Δ_1 list of axioms $a \geq n$ expressing that a is nonstandard;
- we add a Δ_n list of axioms, obtained by pleonasm, that is equivalent to the list of all true Σ_n sentences;
- finally, we include the list of all true Π_n sentences, which is Π_n.

To get a consistent Σ_n list of sentences with no Δ_n model, we need to use more subtle techniques, which we shall leave aside.

7.10 The Notion of Proof; Decidable Theories

If x and y are (codes of) sentences, to say that y is a consequence of x is to say that the set $\{x, \neg y\}$ is inconsistent. On the basis of what we saw in the last section, the set of pairs (x, y) of sentences such that y is a consequence of x is Σ_1, defined by a formula $\mathrm{Pr}(x, y)$ ("y is provable from x"). What, to us, is a proof? We determine the consistency of $\{x, \neg y\}$ by Henkin's method, and get the proof that y is a consequence of x as soon as we reach an integer k at which the branches of the tree all end at a contradiction; this integer k is the unbounded variable in the proof predicate.

I claim that this relation $\mathrm{Pr}(x, y)$, which is Σ_1, is not Π_1. For minimal arithmetic is a finite set A_0 of axioms, which can be replaced by a single one, their conjunction. If the provability predicate were Π_1, the sentences provable from A_0 would form a Π_1 set, as would the Σ_1 sentences provable from A_0. But every Σ_1 sentence that is true in arithmetic is a consequence of A_0, and since A_0 is true in the real integers, none of its consequences are false. Therefore, the Σ_1 sentences that are consequences of A_0 are exactly those that are true in arithmetic, and they form a Σ_1 set that is not Π_1.

This last result is reassuring for the future of the profession: As the provability predicate is not Δ_1, it is impossible to program a computer to be given a (finite) system of axioms x and a sentence y, and to answer, after some time, whether y is a consequence of x. Such a program would make the work of mathematicians, or at least those who believe in the virtues of the axiomatic method, useless!

If A is a Σ_n set of sentences, then the set of its consequences is Σ_n: We can say that Henkin's method shows $A \cup \{\neg x\}$ to be inconsistent, or equivalently that x is provable from the conjunction of finitely many elements of A. If A is Π_n, then its consequences form a Σ_{n+1} set; by the pleonasm theorem (7.28) we cannot improve on this.

We have long known (Tarski's theorem) that we cannot define the set of true sentences arithmetically. If we had earlier hoped to find an arithmetic axiomatization, we can now see that this hope was in vain: If a theory has a Σ_n axiomatization, it is Σ_n! When we announced that we had discovered axiomatizations for the ordering and the sum on the natural numbers, we produced Δ_1 lists of axioms; that is to say, we obtained especially simple axiomatizations. Arithmetic cannot have an axiomatization of this type, and the simplest definition we can give of arithmetic is just to say that it consists of those sentences that are true of the sum and product on the true natural numbers!

After having ascertained that Peano arithmetic was generally sufficient for the arithmetic needs of normal mathematicians, and being deluded by the underlying metaphysical inspiration (we keep what we can of second-order induction by translating it into the first-order language), we were for a moment able to believe that this was enough to axiomatize arithmetic. We now see that we were far off base, since this axiomatization is Δ_1. ("To be an induction axiom" is Δ_1.) By the mere fact that it is Σ_1, it cannot prove all true Π_1 sentences, and a model of Peano arithmetic can lie at the lowest level possible for a model of A_0, the level of Π_1. We can see that there is a vast difference between what is true (of the real natural numbers) and what is provable from the Peano axioms, or from any axiomatic system definable in arithmetic.

The provability predicate introduced in this section is perfectly adequate for the model theorist's needs; however, there is a branch of logic called "proof theory," devoted to the notion of proof, in which it is useful to analyze this notion in more detail. In general, as explained at the beginning of Section 4.3, a finite number of rules are given allowing one formula to be "inferred" from a finite set of other formulas; from these rules, it is obvious that if x can be inferred from A, it is a consequence of A. We then say that x is provable from A if we can obtain it from a finite subset of A by a finite succession of inference rules. The provability predicate is then clearly Σ_1, and we need to show, generally using a method much like Henkin's, that if x is a consequence of A, it is provable from A.

This result, expressing that a syntactic notion (that of provability) adequately captures a semantic notion (that of consequence), is called "Gödel's completeness theorem." Oddly enough, it was proved, by Gödel and others, before we had a clear understanding of the phenomenon of compactness; moreover, it was used to prove compactness, by the observation that a proof of a contradiction from A can involve only finitely many elements of A. This is because, in the context of that era, semantic notions like "x is a consequence of A," that is to say, "every model of A satisfies x," were suspect; logicians were looking for truth in the absolute sense in the universe of mathematics, rather than in a model, and were obsessed with the effective character of the notions that they introduced.

Out of inertia, or respect for history, textbooks continue this presentation, so at odds with the spirit of model theory, in which the fundamental notions are semantic. It is a tedious presentation if all the details are included, and incomprehensible if they are omitted; in both cases, it thoroughly discourages prospective logicians. Moreover, it is a methodological error, because compactness is a more fundamental property than the Σ_1 character of inference: All proofs of the adequacy of the notion of proof for that of consequence make use, albeit often hidden, of convergence of sequences in compact spaces.

Some call a theory T (i.e., a set of sentences that is consistent and closed under inferences) *axiomatizable* if it is Σ_1; since by the pleonasm theorem a Σ_1 theory is the same as a theory with a Δ_1 axiomatization, they mean that the theory has an effective axiomatization (effective $= \Delta_1$). Since every theory has an axiomatization, effective or not, it would be better to say *recursively axiomatizable* in this context. Finally, a theory T is called *decidable* if it is Δ_1 (the theory itself, not one of its axiomatizations!): That means that there is a mechanical procedure to determine whether or not a given sentence is in T. Naturally, all this assumes that the language of T is adequately coded into arithmetic, a condition that in general we will refrain from stating explicitly, especially if the language is finite. The following simple result is useful:

Lemma 7.29. *A complete Σ_n theory is Δ_n.*

Proof. If T is complete, then $x \notin T$ iff $\neg x \in T$, giving a Σ_n definition of the complement of T. □

As a consequence we get the decidability of the following complete Δ_1 theories, for which we have previously given effective axiomatizations: dense order without endpoints, equivalence relation with infinitely many equivalence classes that are all infinite, theory of the successor, order or sum on the natural numbers, algebraically closed fields of fixed characteristic, differentially closed fields of characteristic zero, etc.

Lemma 7.30. *The theory of algebraically closed fields is decidable.*

Proof. Let T be this theory, which is Σ_1: A sentence x does not belong to T if and only if $\neg x$ is a consequence of $T \cup \{2 \neq 0, 3 \neq 0, \ldots, p \neq 0\}$, or else there exists p such that $\neg x$ is a consequence of $T \cup \{p = 0\}$; this provides a Σ_1 definition of the complement of T. (Indeed, by compactness, if x is not in T, then there exists p such that $\neg x$ is a consequence of $T \cup \{p = 0\}$.) □

An example of a recursively axiomatizable but undecidable theory: A_0, and for the same reason (the set of its Σ_1 consequences is not decidable) any Σ_1 theory containing A_0. Recall that A_0 is finitely axiomatizable.

There is no shortage of goodly souls opposed to the proofs of quantifier elimination in Chapter 6 on account of their "noneffective" character: They use compactness arguments, which cannot be formalized in arithmetic. These prefer methods that are laborious and, above all, scarcely reliable, consisting of showing step by step that a formula f can be replaced by a quantifier-free formula that is theoretically given as an effective function of f. The following result shows them that it is useless to give themselves headaches constructing an object (an elimination algorithm under the circumstances) that can very easily be seen to exist: In fact, *every Σ_1 theory that admits quantifier elimination, eliminates quantifiers effectively.*

Lemma 7.31. *If T is a Σ_n theory that admits quantifier elimination, then there is a Σ_n function that sends every formula f to a quantifier-free formula equivalent to it modulo T.*

Proof. Apply the Σ_n choice principle (Lemma 7.17) to the Σ_n binary relation consisting of pairs $(f(\vec{x}), g(\vec{x}))$ of formulas, with g quantifier-free, such that $(\forall \vec{x})(f(\vec{x}) \leftrightarrow g(\vec{x}))$ is provable in T. □

Naturally, we could object to the algorithms suggested by the results above on account of their monstrously impracticable character: If we have a complete recursively axiomatizable theory, the proposed decision algorithm for knowing whether or not f is in T is to list all consequences of T until we reach either f or $\neg f$, this enumeration resting on a systematic pursuit of contradictory Henkin trees! Similarly, with the algorithm for eliminating quantifiers, we wait until a quantifier-free formula equivalent to $f(\vec{x})$ appears. This type of algorithm, in which we wait to see some phenomenon happen, without being able to measure the waiting time, is absolutely unrealistic; the execution time for such a program can reach such dizzying heights as to make a number-cruncher's hair stand on end! And these algorithms can require considerable quantities not only of time, but also of space, in order to carry out the computations.

In the case of algebraically closed fields, a more efficient algorithm is available, but even it rapidly becomes impracticable: successively eliminating unknowns in polynomial equations and inequations, according to a method known since antiquity.

For a model theorist, the effective construction of a good elimination algorithm is of interest only if we actually intend to use it, which is never the case. Consequently, it is a rather pointless exercise to seek to evaluate precisely the amount of time and space (calculated as a function of the length of the formula f under consideration) needed to test membership in a decidable theory T. However, if we are looking for improvements in time and space, Fraïssé's back-and-forth method still gives the best results, since it allows us to reduce the satisfaction of a sentence f in a structure S to that of a sentence f' in a finite structure S'.

Suppose we want to determine the validity of a sentence f, purely relational, with quantifier rank k, in the structure S. Let C_k be a finite set of representatives of the $(k-1)$-equivalence classes on elements of S. Let C_{k-1} be a finite set such that for every a in C_k, each $(k-2)$-equivalence class (a, x) has a representative (a, b) with b in C_{k-1}, and so forth. We thus define finite sets $C_k \subset C_{k-1} \subset \cdots \subset C_1$.

We shall say that a quantifier in f has rank i if i is the quantifier rank of the formula in its scope. From the definition of i-equivalence, it is clear that f is true in S if and only if the sentence f', obtained by relativizing each quantifier of rank i to C_i, is true there (i.e., by replacing $(\exists x)$ by $(\exists x \in C_i)$ and $(\forall x)$ by $(\forall x \in C_i)$). As a consequence, if a very simple description of the k-equivalence classes is available, such as that given in Section 1.2 for discrete orders, we can reduce the satisfaction of a sentence to the satisfaction of sentences in finite structures of easily manageable size, which will give a good decision algorithm in the favorable cases.

The ultimate goodness of this algorithm is still of only weak interest: The decidable or undecidable character of a theory has no connection to the structural properties of its models, which are the object of model theory, and it brings nothing to the study of the recursiveness or the complexity of algorithms, which appear in these subjects only through routine techniques. If we too often meet model theorists who, by some sort of overbearing faithfulness to the logical origins of model theory, believe they are required to observe that such a theory—of groups, of rings, etc.—is decidable, that is largely because model theory has not known how to coin its own language to formulate its results: They want to say that the theory under consideration is *simple*, that they know how to describe its types, and perhaps to classify its models. This structural simplicity in principle has nothing to do with recursiveness, which in a certain way measures the complexity of axiom systems, not of a class of structures. This complexity is also sensitive to all sorts of linguistic manipulations with no structural implications; in practice, however, we often ascertain that an easily axiomatizable structure is structurally simple, and vice versa, although it is child's play to construct counterexamples.

Along the same lines, we should note that the notion of provability to which we are bound does not capture the actual thought processes of mathematicians: To prove something, they never systematically search along a Henkin tree for a contradiction! Using this method to prove Theorem 7.5, in which we show the equivalence of two systems for axiomatizing the sum on the natural numbers (appealing to arguments considered admissible by every mathematician to convince the reader that every model of one system is a model of the other), would fill an entire volume.

In conclusion, note that an elimination algorithm is not necessarily a decision algorithm: A sentence f is replaced by a quantifier-free sentence g, which is equivalent to it modulo T, but the question of knowing whether or not g belongs to T is another problem.[1]

7.11 Gödel's Theorem

By Tarski's theorem we know that if A is a Σ_n fragment of arithmetic, then there is a Π_n sentence of arithmetic that is not a consequence of A; indeed, the Π_n consequences of A form a Σ_n set, which could not be the set of all true Π_n sentences.

Gödel's theorem gives an example of such a sentence. We know that there is a Σ_n formula, which we write $\mathrm{Pr}_A(x, y)$, which is satisfied iff x is the code of a formula with one free variable and the sentence obtained by substituting y for this variable is a consequence of A; let $G_A(x)$ denote the Π_n formula $\neg\mathrm{Pr}_A(x, x)$.

[1] **Note (1999):** Well, I would not be so aggressive now against fast quantifier elimination algorithms, since their existence, whence formulas have been replaced by circuits, is the model-theoretic equivalent of the famous question P=?NP. The most efficient way to clarify your ideas on this important question is to learn French and read my third book, B. Poizat, *Les petits cailloux*, Aléas, 1985.

Theorem 7.32 (Gödel's First Theorem). *If A is a Σ_n fragment of arithmetic, and if a is the code of the formula $G_A(x)$, then the Π_n sentence $G_A(a)$ is true in the standard model and is not a consequence of A.*

Proof. Suppose that $A \models G_A(a)$; as a consequence, in the real world, $G_A(a)$ is provable from A, and $\mathrm{Pr}_A(a, a)$ is true in the true natural numbers, so $\neg G_A(a)$ is true, and $G_A(a)$ is false. As the true natural numbers satisfy A, everything that is provable from A is true in the standard model, so we have a contradiction.

Consequently, $A \not\models G_A(a)$; $\mathrm{Pr}_A(a, a)$ is false in the true integers, so $G_A(a)$ is true. □

In the case of Σ_1 fragments of arithmetic, Gödel had the blessed idea of finding a much more troubling form of his theorem. We know that the consistency of a Σ_n fragment A of arithmetic can be expressed by a Π_n sentence, which we call $\mathrm{Cons}(A)$: Indeed, it is enough to express that $0 \neq 0$ is not a consequence of A, or else to express that the Henkin tree associated with A is infinite.

Theorem 7.33 (Gödel's Second Theorem). *If A is a Σ_1 fragment of arithmetic containing Σ_1-induction (or some reasonable approximation to Σ_1-induction),* $\mathrm{Cons}(A)$ *is true in the standard model and is not a consequence of A.*

Proof. $\mathrm{Cons}(A)$ is true of the true natural numbers because A has a model (the true natural numbers!) and is therefore *really* consistent.

We are going to show that $G_A(a)$ is a consequence of $A \cup \{\mathrm{Cons}(A)\}$, which, together with Gödel's first theorem, will prove that $\mathrm{Cons}(A)$ cannot be a consequence of A.

Suppose, by way of contradiction, that we have a model N of $A \cup \{\mathrm{Cons}(A)\}$ that satisfies $\neg G_A(a)$. Since N satisfies $\mathrm{Cons}(A)$, it can build (by Henkin's method) a Δ_2 structure M that is a model of A, or rather of the Henkin structural sentences associated with A (which are quantifier-free). Since N satisfies $\neg G_A(a)$, and therefore also $\mathrm{Pr}_A(a, a)$, it proves, still by Henkin's method, that $G_A(a)$ is a consequence of A, and in particular that the model M satisfies $G_A(a)$.

But M is a model of A_0, and we can define, in N, the function mapping x to the element x_M, which is the xth successor of zero in the sense of M. (Do not forget that the model N thinks that it is the true natural numbers; we, from our external vantage, do not share this point of view.) The image N' of N under this map is an initial segment of M, which N thinks is the "standard part" of M. But M satisfies $G_A(a)$, which is a Π_1 sentence; so, therefore, does its initial segment N', and also N, contradicting the hypothesis.

Where in this argument did we use the hypothesis that A includes Σ_1-induction? First, to prove the existence of the model M, that is to say, the existence of an infinite branch of a finitely branching infinite tree; later, for the relevance to the real world of certain statements about the model N: for example, when N claims that the (standard) sentence $G_A(a)$ is satisfied by M, it must really be satisfied, that the replacement of variables by constants be carried out correctly. It is also necessary that the Henkin sentences associated with $G_A(a)$ in N behave correctly, and really imply $G_A(a)$ for M. □

We can see that this theorem of Gödel is rather subtle, even if it does not involve very complicated techniques: The first theorem is proved in the real world, that is, the true natural numbers, while the second consists of repeating the proof of the first in the model N. In time, the apprentice logician will come to appreciate its beauty every day; it is one of those rare mathematical results that gives us a new pleasure every time we repeat its proof (if we get there!), once we have forgotten it.

There is therefore a model of Peano arithmetic that contains a proof of the inconsistency of Peano arithmetic! Naturally, this "proof" is nonstandard, and corresponds to nothing in the real world: The level at which all the branches of the Henkin tree end in a contradiction is a nonstandard number, which is finite only when seen from inside the model in question. Logic is saved, but it is still troubling!

In order to have a version of the second theorem in the case of a Σ_n fragment A of arithmetic, we would need to add something to destroy the aesthetic: Define Σ_n-truth to be the Σ_n set of Σ_n sentences true in arithmetic, the set defined by the Σ_n formula $V\Sigma_n$. We can then show that the Π_n sentence asserting the consistency of $A + \Sigma_n$-truth is not provable in A. Naturally, the model N does not satisfy the same Σ_n sentences as the standard model: Cons($A + \Sigma_n$-truth) implies that N can construct, by Henkin's method, a model M of A that satisfies the same Σ_n sentences as N itself! If we do not need this precaution in the Σ_1 case, it is because we were able to translate into N the fact that A_0 implies Σ_1-truth. We also need to assume that A contains something like Σ_n-induction in order to have our branch in the tree, and so that satisfaction of standard Σ_n sentences in N will correspond to something sensible.

Gödel's theorem extends naturally to theories T that allow the interpretation of a model of arithmetic, or at least a model of A_0 that is not too crazy.[2]

[2]**Note (1999):** In the original edition you find here an unfortunate definition of the concept of ω-consistency; I made the case even worse in an attempt to correct it by a note at the end of the third edition. The problem is a matter of fidelity to history: I prove Gödel's theorem for true theories, i.e., for sets of sentences that are satisfied in the standard model of arithmetic, although Gödel's original statement concerns a slightly broader class, the ω-consistent theories. It was extended late by Rosser to consistent theories. Since those Byzantine subtleties are relevant only to the trivial direction of the proof of the theorem, the case of true theories is indeed the only one that matters, and

Although it is more than forty years old,[3] Gödel's theorem is still very much in style where it is spoken of; to be sure, amateur epistemologists see scarcely more than an aphorism along the lines of "nothing can prove its own consistency by itself" or "nothing can prove its own existence by itself," and can only be surprised at the technical relentlessness with which mathematicians so laboriously prove what falls under common sense: an intuition thoroughly divided, since the same people generally accept the Cartesian axiom "I think, therefore I am"! This cheapens the theorem: first, because it really speaks not of proofs or of formulas, but of codes (whatever logicians may playfully say of it); second, because the theory in question must be Σ_1 (must we believe that only "axiomatizable" theories exist?), or at worst Σ_n. However, the statement "the consistency of arithmetic is not provable in arithmetic" makes no sense, since, by Tarski's theorem, the consistency of arithmetic does not correspond to any sentence in the language of arithmetic. Also, any epistemological interpretation of this theorem must rest on the perilous assertion of its validity, and of the validity of the syntax and semantics of model theory, outside of the broad field of mathematical logic. What is certain is that those who would philosophize about this theorem would do well first know know its precise statement and, if possible, its proof; it is, for example, unreasonable to teach it to students who have not mastered the decomposition of a number into its prime factors.

It is, however, correct that Gödel's theorem, like Tarski's, relies, up to coding, on a simple argument: the liar paradox, which, from Saint Paul on down to Bertrand Russell, has served as the grounding of countless more or less well crafted mathematical recreations; we sometimes wonder whether these theorems are due to Gödel, to Tarski, or to Berloquin! In the case of Tarski, we look for the formula that says "I am lying"; in the case of Gödel, which is distinctly more subtle, we look for an argument that proves its own contradiction.

If we want to sound erudite, when we do this type of reasoning in mathematics, we say that we are doing a "diagonalization argument." This is the type of argument that Cantor used to show that there is no bijection between one set and the set of its subsets (see Chapter 8); these arguments are involved in an essential way in the study of the Σ_n and Π_n hierarchies of sets definable in arithmetic.

As this process is very general, Gödel's diagonal argument can be repeated for any language (first order or otherwise), *provided that we can define a provability relation*: This characteristic of having a Σ_1 provabil-

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you can forget about ω-consistency: It is the product of a difficult mind that could not face the truth, and certainly not that of a model theorist!

[3]**Note (1999):** Now more than 60 years old.

ity predicate is, together with compactness, a fundamental property of the syntax and semantics studied in model theory.

7.12 A Little Mathematical Fiction

Arithmetic, unlike a theory such as that of algebraically closed fields, has implications for us: We can code our language and our thought processes into it. Unfortunately, we need to guard our roles as external and impartial observers, and we experience a delightful thrill at playing this clever game, which consists in pretending to be mere automata manipulated by our model.

Its study certainly teaches us of the limits on our capabilities, or more precisely, the capabilities of the axiomatic method: For example, we know that we can axiomatize arithmetic in a way that is codable in arithmetic. Ever since Euclid, continuing through Leibniz, to Whitehead, Russell, and Hilbert, mathematicians have dreamed of perfecting an axiom system that would definitively be sufficient for their needs: We would once give, for all time, one list of the primitives that we take to be true, and another of the procedures accepted in proofs, and from then on we would have to develop mathematics without leaving the "paradise" of our system. The a priori specification of all the rules of the game was, for many mathematicians at the turn of the century, the only guarantee of rigor. It would, however, be a completely relative rigor, since it relies on the adequacy of the fundamental notions at the foundations of the chosen axiomatic system: We speak of sets to small children, we give them examples of sets, but who has ever told them what a set is? This attitude only pushes the foundational problems in mathematics back toward the fundamental notions; without resolving them, it nevertheless has the merit of better isolating them.

Gödel's theorem makes us doubt the advantages of such a method, since it clearly indicates that whatever axiom system may be chosen, it will be necessary to leave it. One obvious weakness in the axiomatic method is that if T is a (recursively enumerable—but then who has proposed any other kind of axiom system?) theory that contains minimal arithmetic, we cannot prove the consistency of T in T. But there is another, more catastrophic for its partisans: Gödel's theorem allows us to construct theories that prove their own inconsistency and that are consistent! In fact, Peano+$\neg$Cons(Peano) is certainly a consistent theory, and as it proves Peano inconsistent, a fortiori it proves Peano+$\neg$Cons(Peano) inconsistent! So if you put yourself inside a given axiom system, and chance to prove inside that system that the system is inconsistent, it is not clear that you have really proved the inconsistency of your system. Perhaps you have merely included an axiom that is false for the true natural numbers.

Gödel's theorem was rightly presented by its author as a definitive defeat for what was known as "Hilbert's program," the hopes at the beginning of the twentieth century for the possibilities of the axiomatic method. In that era some mathematicians doubted the validity of infinite methods, reasoning by transfinite induction, and the existence of the "actual infinite", which at the very least should be handled only with care. Hilbert thought that, not being able to settle the metaphysical question of the actual existence of the infinite, we could at least prove its consistency. Everyone would be happy that way; even if you did not believe in that actual infinite, you should at least acknowledge that its introduction into mathematical proofs would not lead to a contradiction. (He also wanted, to reassure his critics, to show that anything that could be proved by using the actual infinite, and that did not directly concern it, could be proved without it.) However, in order for the argument to be completely convincing, he needed this noncontradiction to be proved by entirely finitistic means. They therefore sought, if we may express the project in an anachronistic way, to prove the consistency of set theory by purely arithmetic methods, or more precisely, to prove the consistency of an axiomatizable set theory in an axiomatizable arithmetic. But set theory allows us to define a model of arithmetic, the set of natural numbers; if you are not suffering from serious schizophrenia, you need to choose a set theory that incorporates the arithmetic that you have admitted.

So the consistency of your set theory proves that of your arithmetic, and consequently cannot be proven in your arithmetic!

It is hardly surprising to see the author of a textbook yield to the ease of the axiomatic method, prefixing the work with a Chapter 0, a so-called "set theory", fixing the rules of the game for the rest of the book; this is yielding to the ease of an a priori, dogmatic, and noncritical exposition of the contents of a discipline; it reassures students, who love to be hit with absolute truths, even if they understand none of them. What is surprising, however, is to see confirmed mathematicians very rigidly stick to this viewpoint, without drawing the conclusions of the well-known, and already classical, theorem of Gödel. The most characteristic is Nicolas Bourbaki, who undertook to write a treatise developing mathematics "from first principles": an anachronistic dream encyclopedia, in the opinion of this author, who considers the encyclopedia no longer to be a suitable form of expression for the scientific thought of our time. Some are astonished to see Bourbaki, or at least some of his disciples, putting the foundations of mathematics so high and logicians so low, but it is on the whole a logical attitude, for if the axioms are oracles, and if rigor is the object of a cult, it is sacrilege to conduct a positive study of them, and it is fitting to excommunicate, without hesitation, all those who would try.

The theorems of Gödel and Tarski are indeed very negative results on the possibilities of the axiomatic approach. What do the people (mathematicians) demand? They are not counting on logicians for deep insights into arithmetic, but what they can hope for is that when a number theorist proves some profound arithmetic result, the logician will immediately know whether this result is a consequence of an axiomatic system like that of Peano. That is the logician's work! Work of despair, since a model of Peano arithmetic can lie at the lowest level, namely Π_1; the only answer the logician can make is that to be certain that a result is a consequence of Peano arithmetic, it must be proved in Peano arithmetic!

Why, then, are Peano's axioms so cherished? Because we were able to believe, in a bygone day, that they formed a complete axiomatization of arithmetic? We have now repented of that youthful folly. On what grounds are we convinced that every ordinary theorem of arithmetic can be proved in Peano arithmetic, that every natural function is well behaved in all models of Peano arithmetic, a conviction that is derailed only when made explicit? Tarski's theorem tells us that this sentiment rests on nothing, that it is just a vague principle, which also serves to justify the "generalized Church's thesis," to know that the most frequent objects are the least probable. (Can you find actually discontinuous functions in a compendium of exercises for freshman calculus? Or antiderivatives not computable by elementary functions?)

As for the idea that a malicious God put us in a nonstandard model of arithmetic, which could not even be the real arithmetic, and which we could naturally be aware of, it can only be the product of a foggy brain. It is to lose sight of the fact that the language familiar to model theorists is principally used by them only for technical purposes, because it allows a practicable model theory (compactness, definability of the notion of proof, etc.) and that it is only by a deformation of the mind that we end up believing it to be the natural frame for mathematical thought: In fact, mathematicians easily break loose from it.

As I have already had occasion to say, logicians, or at least model theorists, are simple folk: to introduce elementary equivalence through local isomorphisms is to avoid an approach to satisfaction contaminated by the stale smell of metaphysics. They believe that the natural numbers at their disposal are the true numbers, and that the set of sentences that are true in arithmetic does exist, even if it turns out not to have a description as simple as that of the set of even numbers. Gödel's theorem, which they set forth without bypassing the underlying coding, is to them well explained as a property of "false integers," and in no way shakes their faith in actual arithmetic. In a word, their bliss is in a slightly silly philosophy, which can be summed up in one sentence: "Mathematical objects exist, their descriptions are provisional."

To end on a positive note, we are going to enter a work of fiction. One famous theorem is that of Fermat; it might have been proved by its author, but he left nothing to enlighten posterity, and since then the theorem has resisted the efforts of thousands of mathematicians. It is stated as follows:

$$(\forall x)(\forall y)(\forall z)(\forall t)(x^t + y^t = z^t \to t \leq 2 \vee x \cdot y = 0).$$

If we replace exponentiation with its definition in terms of sum and product, we can see that this is a Π_1 sentence. Consequently, if it is false, its negation is a consequence of A_0 (we have a *true* counterexample, and it suffices to perform the operations!), the uninteresting case. Now suppose that it is true, and we can imagine that if nobody has been able to prove it up to now, it might be because it is not a consequence of the Peano axioms.[4] Logicians, whose means are more limited than those of number theorists, have a more modest ambition: Instead of seeking to prove Fermat's last theorem (FLT) true, they seek only to prove it consistent.

In what system of axioms are they going to carry out this proof? Naturally, in Peano arithmetic, which is our system of reference. But if a model N of Peano arithmetic thinks that FLT is consistent with A_0, it constructs a model M of FLT, with an initial segment N' isomorphic to N (see the proof of Gödel's second theorem), and just by the fact that FLT is Π_1, it is also true in N' and in N: To prove in Peano arithmetic that FLT is consistent with Peano arithmetic is to prove in Peano arithmetic that FLT is true. Whatever axiom system A Little Nicolas admits, it cannot prove, staying within the system, that FLT is consistent without proving that it is true (i.e., a consequence of A). All this to say that logic cannot advance Fermat's last theorem by a single step.

7.13 Historic and Bibliographic Notes

Theorem 7.1, on the completeness of categorical theories, is from [Łoś54] and [Vau54].

Presburger's axioms, Theorem 7.5, are described in [Pre30]; for the theory of the product on the natural numbers, see [Ceg81].

Although for us it is a direct consequence of compactness, the existence of nonstandard models of arithmetic was a somewhat disconcerting phenomenon for our forefathers; one of its first appearances was in [Sko34]. The coding of combinatorics and of the language of arithmetic is due to [Göd31]; our relation $\in$ was defined by [Ack37].

Tarski's theorem is in his famous article on truth [Tar35].

The definition of the hierarchy of arithmetic sets, which parallels the Borel and analytic hierarchies introduced by the Poles and other Slavs

[4] **Note (1999):** This was written well before Andrew Wiles announced his proof of Fermat's last theorem. The comments, however, apply equally well to some remaining open problems, such as the Goldbach conjecture, another famous Π_1 conjecture about arithmetic.

in descriptive set theory, is due to [Kle36]; primitive recursive functions were defined by Gödel [Göd31], and general recursive functions by Kleene [Kle36]. Matijacević's theorem was published in [Mat70]; it was the final touch in a long sequence of works on a problem posed by Hilbert, which asked whether there is an algorithm to decide whether a polynomial in several variables, with integer coefficients, has an integer zero. The answer is no, according to Matijacević's theorem, which recursively transforms a Σ_1 sentence into an equivalent Diophantine equation.

Church's thesis was formulated in [Chu36]; Turing machines are due to [Pos36] and [Tur36]; degrees of computability and Σ_n-complete degrees appeared in [Pos48]. As a good introduction to recursion theory, I recommend [Sho71].

The term "Peano arithmetic" pays homage to Giuseppe Peano's *Formulaire de mathematiques* [Pea95]. This formulary, which had an important influence on contemporary mathematical symbolism, is more a systematic presentation of the basic notions of mathematics than an "axiomatization" in the technical sense that we understand today. In that era, nobody had any idea of the considerable distance separating the expressive power of a second-order language from that of a first-order language.

The constructions of models in Sections 7.8 and 7.9 are part of the folklore of the logic of the 1950s; for example, you might consult [Sho60]. The nonexistence of a recursive nonstandard model of Peano arithmetic is due to [Ten59]; the pleonasm theorem originated in [Fef57].

The combinatorial lemma of Denes König on trees is obviously much older: [Kön27].

You can find material, as much for an initiation into the modern theory of models of arithmetic as to deepen your knowledge of it, in [CBR82].

The adequacy of a Σ_1 notion of proof for the semantics of first-order logic was shown by [Her28], [Göd30], and [Gen34], following acutely different approaches; this is not the place to examine the influence that they had on contemporary "proof theory."

For a detailed knowledge of the algorithmic complexity of some logical theories, the exceptionally determined reader could consult [FR79].

The two theorems of Gödel are in [Göd31].

8
Ordinals and Cardinals

ויטע יהוה אלהים גן בעדן
מקדם וישם שם את-האדם
אשר יצר

בראשית

This chapter includes no model theory. It is written for the reader's convenience: Every time there has been a reference to "ordinals" or "transfinite recursion" up to this point, you have been asked to accept the validity of the proofs offered by analogy to the denumerable case, and promised a more extensive exposition later. The time has come for this exposition, since we are going to need some quite precise results on "cardinal arithmetic" beginning in the next chapter.

8.1 Well-Ordered Sets

A *well-ordering* is a total ordering every nonempty subset of which has a least element; we say that a set is *well-ordered* if it is equipped with a well-ordering.

For example, the empty order, a finite chain, and the chain ω of natural numbers are well-orderings. If A and B are well-orderings, so are their sum $A+B$ and their product $A \times B$ (as an exercise, check this); every restriction of a well-ordering is a well-ordering.

Recall that an *initial segment* of a chain A is a subset B of A such that if a is in B, every predecessor of a is also in B. If A is a well-ordering, there are two types of initial segments: A itself is one type; otherwise, the complement of the initial segment has a least element a, and the initial segment is obviously the set A_a of elements of A strictly less than a. We can thus see that a well-ordering is isomorphic to the set of its proper initial segments ordered by inclusion.

Lemma 8.1. *Let A and B be well-orderings, and f and g two isomorphisms from A to initial segments of B. Then $f = g$.*

Proof. Suppose f and g are different. Let a be the least element of A such that $f(a) \neq g(a)$; for example, in B, $f(a) < g(a)$: In this case $f(a)$ cannot be in the image of g, which is therefore not an initial segment of B. □

This lemma states, in particular, that if A is a well-ordering, the identity is the only isomorphism of A onto an initial segment of A; A is never isomorphic to one of its proper initial segments.

Lemma 8.2. *Let A and B be two well-orderings. Then there exists an isomorphism from one onto an initial segment of the other.*

Proof. First, suppose that for all a in A, A_a is isomorphic to a proper initial segment of B. By the preceding lemma, this initial segment is unique, and we denote it by $B_{f(a)}$; we can then easily see that the function f is an isomorphism from A onto an initial segment of B.

Otherwise, let a be the least element of A such that A_a is not isomorphic to a proper initial segment of B: For all $b < a$, A_b is isomorphic to a proper initial segment $B_{f(b)}$ of B. We can see that f is an isomorphism from A_a onto an initial segment of B, which, by definition of a, cannot be proper. Therefore, f is an isomorphism between A_a and B. □

Given two well-orderings A and B, we shall say that *the ordinal of A is less than or equal to the ordinal of B*, in symbols $\mathrm{ord}(A) \leq \mathrm{ord}(B)$, if A is isomorphic to an initial segment of B. For now, do not try to give a precise meaning to the word "ordinal"; we are considering $\mathrm{ord}(A) \leq \mathrm{ord}(B)$ as a unified expression, much as we can define the notion of an n-dimensional subspace without knowing what a dimension is!

This relation is clearly reflexive, transitive, and antisymmetric in the following sense: If $\mathrm{ord}(A) \leq \mathrm{ord}(B)$ and $\mathrm{ord}(B) \leq \mathrm{ord}(A)$ (we then say that A and B *have the same ordinal*), then A and B are isomorphic; indeed, if f is an isomorphism from A onto an initial segment of B, and g is an isomorphism from B onto an initial segment of A, then $f \circ g$ and $g \circ f$ are identity functions. Moreover, by Lemma 8.2, two well-orderings are always comparable.

We can therefore define a "total ordering on the ordinals": Given a well-ordering A, the ordinals strictly less that that of A are the ordinals of proper initial segments of A, which correspond to points of A. We thus see that an ordinal can be identified with the set of ordinals strictly less than it, equipped with their natural order relation.

If we wish, we can think of the ordinal of a well-ordered set A as the class of well-orderings isomorphic to it. However, we generally do not take this approach, since there is a better technique, in set theory, of choosing a particular element in this class, its *von Neumann ordinal* (named for one of the fathers of the bomb; not the same as the namesake of Lemma 6.25), which I am going to explain.

A set is called *transitive* if each of its elements is a subset of it: In other words, a is transitive if whenever $x \in a$ and $y \in x$, then $y \in a$. For example, $\emptyset$ is transitive, as is $\{\emptyset\}$. More generally, if a is transitive, so is $a \cup \{a\}$. We call a set a a *von Neumann ordinal* if it is transitive and the membership relation $x \in y$ is the strict ordering associated with a well-ordering on a: $x \leq y$ means $x \in y \vee x = y$; $x \in y$ means $x \leq y \wedge x \neq y$, or, equivalently, $x < y$. For example, $\emptyset$ is a von Neumann ordinal, which, in this context, is usually called 0; so are $\{\emptyset\}$, usually called 1, and $\{\emptyset, \{\emptyset\}\}$, called 2. More generally, if a is a von Neumann ordinal, so is $a \cup \{a\}$.

Note that an element b of a V.N. ordinal a is itself a V.N. ordinal: b is transitive, since the membership relation is transitive on elements of a, and $(b, \in)$ is a well-ordering, since it is a restriction of $(a, \in)$. The initial segments of a V.N. ordinal are itself and its proper initial segments, which are also its elements! (It is not possible that $a \in a$, since the relation $x \in y$ is a *strict* order on a.)

Lemma 8.3. *Any isomorphism between two von Neumann ordinals is the identity: If two von Neumann ordinals are isomorphic (i.e., their associated chains are isomorphic), then they are equal.*

Proof. Let a and b be our two V.N. ordinals, and let f be an isomorphism between a and b. Suppose f is not the identity map from a to a, and let c be the least element of a such that $f(c) \neq c$. Then f sends the initial segment of a consisting of elements strictly less than c to the initial segment of b consisting of elements strictly less than $f(c)$. Consequently, c and $f(c)$ are two sets with the same elements, which are therefore equal: contradiction. □

Lemma 8.4. *Every well-ordering is isomorphic to a von Neumann ordinal.*

Proof. Let A be a well-ordered set. Suppose that there exists x in A such that A_x is not isomorphic to a V.N. ordinal, and let a be the least such element. Then for all $x < a$ there is an isomorphism, unique by Lemma 8.3, between A_x and a von Neumann ordinal x'. If $x < y < a$, then x' must be the initial segment of y' that is the image of x under the isomorphism between y and y'; in other words $x' \in y'$. We can thus see that the set of all x' is a V.N. ordinal that is isomorphic to A_a: contradiction.

We have seen that every proper initial segment of a well-ordering is isomorphic to a V.N. ordinal; in conclusion, it is sufficient to note that A is a proper initial segment of $A + 1$, the ordering obtained by adding another point to it on the right. □

We therefore call the unique von Neumann ordinal isomorphic to any well-ordered set A the *ordinal* of A. We meet some students who are allergic to ordinals as "well-ordering types" and who find the notion of von Neumann ordinals easier to digest; that is a singular consequence of dogmatic teaching, which confuses formalism with rigor, and which favors technical craft to the detriment of the fundamental idea: It takes a strangely warped mind to find the notion of a transitive set natural!

A von Neumann ordinal *is* the set of (V.N.—I will not drag this name everywhere any longer) ordinals less than it; its elements are also its proper initial segments.

The natural ordering, that is, membership or equality, is a sort of well-ordering on the ordinals. (I say "sort of" because the ordinals form too large a class—at least that is the explanation usually given—to constitute a set; otherwise, they would form an ordinal, which would belong to itself, and therefore be strictly less than itself, which is impossible.) In other words, we have the following induction principle:

If there is an ordinal that satisfies property P, then there is a least one.

To see this, let a be an ordinal satisfying P. If it is the least, we have succeeded. Otherwise, the set of elements of a that satisfy P is nonempty, and has a least element b, which is the least $b < a$ satisfying P.

We can also express this principle in the following form:

If for every ordinal x the fact that all $y < x$ satisfy the property P implies that x also satisfies P, then every ordinal satisfies P.

If an ordinal has a greatest element a, so it has the form $a \cup \{a\}$, it is called a *successor*; otherwise, it is called a *limit*. An ordinal is a limit iff it is the union, and also the supremum, of the strictly smaller ordinals. (Note: Any ordinal, limit or successor, is always the *set* of strictly smaller ordinals.)

In arguments by induction, it often happens that we consider two cases, according to whether the ordinal in question is a successor or a limit. Because of the presence of limit ordinals, induction cannot be reduced, as in arithmetic, to the step from one ordinal to its successor. (By way of reference to the notation of Section 7.7, the forms I' and I'' are the only suitable forms of the induction axioms; the form I is inadequate.)

An ordinal is called *finite* if all nonzero ordinals less than or equal to it are successors; for example, $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, etc. We can convince ourselves that the notion of finite ordinal is the translation, in set-theoretic terms, of the informal notion of a natural number. The smallest infinite ordinal, which is also the smallest nonzero limit ordinal, and whose order is therefore isomorphic to that of the natural numbers, is called ω or $\aleph_0$; the isomorphism in question is, moreover, the identity, since, under von

Neumann's convention, ω *is* the set of natural numbers and its ordering *is* that of the natural numbers.

Another use of induction is in *construction by transfinite recursion*: Given an ordinal x, we can describe a procedure allowing us to construct a well-defined object M_x, assuming that M_y has been constructed for all $y < x$. It is clear that we can construct a unique object M_x for every x: Consider the least ordinal for which this does not work.

Sometimes, we are satisfied to do the construction on those x less than a fixed ordinal a. As examples of such constructions by transfinite recursion, we have seen in Section 1.3 that of the derivatives of a topological space, and that of α-isomorphisms.

We are going to use transfinite recursion to prove the following lemma:

Lemma 8.5. *Let A be a well-ordering, and let B be a restriction (not necessarily an initial segment!) of A. Then* $\operatorname{ord}(B) \leq \operatorname{ord}(A)$.

Proof. Let β be the ordinal of B, with which it can be identified. Consider the function f from β to A defined recursively as follows: $f(x)$ is the least element of A strictly greater than all $f(y)$ for $y < x$ if this exists; otherwise, $f(x)$ is the least element of A. We can easily see, by induction, that $f(x) \leq x$, so that $f(x)$ is always the least element of A strictly greater than all $f(y)$ for $y < x$. (The second case was merely an oratorical precaution to ensure that the function f was well-defined.) The function f is an isomorphism from B onto an initial segment of A (which could be A itself even if B is a proper subset of A). □

8.2 Axiom of Choice

Theorem 8.6 (Zermelo's Theorem). *Every set can be well-ordered.*

Proof. If A is empty, it is well-ordered; if not, it has an element, say a_0. If it does not consist solely of a_0, then it has another element a_1; continuing this process by induction, we get an element a_α distinct from all previous elements for every ordinal α for which the set of a_β, for $\beta < \alpha$; is not all of A. The construction must stop, since the ordinals of well-orderings that can be defined on subsets of A form a set, while the class of all ordinals do not form a set. When we are finished, we have well-ordered the set A. □

The proof of this theorem is definitely a fraud, because everyone knows that the theorem is an equivalent form of the *axiom of choice*, and that the axiom of choice is not a consequence of the "other" axioms of set theory. This last assertion is only really meaningful if we specify the other axioms, in other words, if we work in a formal theory of sets, such as Zermelo–Fraenkel set theory (ZF).

This poses a methodological problem here. In the last chapter we could have put ourselves in an axiomatic setting, the list of Peano's axioms to give an example, and worked solely in that system. We did not do that, because we always have our eyes on a natural model of arithmetic, that of the *true* natural numbers, and we believe in the existence of the set of sentences that are true of the true natural numbers, even after realizing that such a set has a rather complicated definition. Then, after a little work, we saw that the Peano axioms formed only a very weak approximation to this true arithmetic.

The natural approach to arithmetic is this: "We have the numbers; let us try to describe them." But what, then, are sets?

If the situation is different for sets, it is because *there does not exist a natural model of set theory.* Just what the true sets are, at rock bottom, nobody is really sure. So we are in a fairly embarrassing situation: We need to give a list of axioms that are reasonable in the sense that if true sets exist, they probably satisfy these axioms (an obviously very subjective claim), and then to develop the consequences of this axiom system; however, the problem is that this set theory matters more to us than arithmetic, because we can define all of our mathematics in it (not just combinatorics), and that at a less formal level, we use the same set theory to study set theory! The translation of every mathematical notion is even quite direct, quite natural (at least for a mathematician of this century!), so that it provides a trap for novice logicians, who start to confuse the notions internal to the model with those external to it. When they study arithmetic, where the translations are done by rather artificial coding, it is easier to guard against this confusion.

We can give an argument explaining the impossibility of an absolutely convincing natural model of the theory of sets: If it existed, it would be a set M, equipped with a binary membership relation $\in$, that would be a sort of elementary restriction of the universe in which we live. But, since the satisfaction of a sentence in M is easily defined in set theory, we would have a definition of truth inside the theory, contradicting Tarski's theorem.

This argument is not very strong: Applied to arithmetic, it simply proves that the set of natural numbers is infinite, a fact that is generally accepted. We imagine that this natural "model" of the theory of sets should be of a higher type, a sort of limit of things definable in the theory. (ZF has a model of this sort, the "cumulative hierarchy," provided that we can assure ourselves of the consistency of ZF, although naturally that cannot be proven in ZF.)

But since nobody even has a clear idea of a model of this species, able to serve as an absolute guidepost for mathematical thought, the question of knowing whether the axiom of choice is true does not make a lot of sense. (In contrast, we do not hesitate to assert that Cons(Peano) is true, even though it cannot be proven in Peano arithmetic.) The only reasonable question is to ask whether or not it is a consequence of a given set of axioms, or whether, together with this system, it is noncontradictory. (In fact, by Gödel's theorem, the systems considered cannot prove their own consistency, but they can prove the *relative* consistency of the axiom of choice:

we can show *in* ZF that *if* ZF is consistent, then ZF + AC is consistent, as is ZF together with the negation of the Axiom of Choice.) This problem, and others of the same type, have given rise to a fertile and very specific branch of logic, called "set theory," which lies completely beyond the scope of this book; moreover, the reader could not be too strongly advised to embark on its study only after having acquired a solid understanding of model theory, which is the foundation of logic. Nevertheless, as we have been immersed in sets since Cantor, we cannot avoid them, and we are going to master those properties of these hypothetical sets that are necessary for our purposes, always remaining as informal as possible.

This "informal theory"—paradoxical object if ever there was one—is always strongly influenced by the Zermelo–Fraenkel axiomatization, whose underlying metaphysical idea is that sets are "small" in comparison to the universe. For example, the complement of a set cannot be a set. Set theory is usually developed in this axiomatization. To call it the natural framework for mathematical thought is very bold; however, it is correct to say that it has increasingly become just that, as in mathematical practice the everyday set-theoretic concepts of mathematicians are more and more influenced by ZF.

We might also think that the deep theorems in set theory, which are proved in ZF, remain valid in other systems; among these other systems, the ones that have been relatively successful are BG (Bernays-Gödel), which resembles ZF, and NF (new foundations, due to Quine), which is based on a very different idea.

We are now going to prove the equivalence of various classical forms of the axiom of choice. To make this proof completely sound, we ought to specify what axioms are allowed, something we wish to avoid: You can convince yourself that the properties of sets used here informally are reasonable.

1. *Axiom of Choice.* For every set A, there is a function f from the set of nonempty subsets of A to A that maps every nonempty subset of A to one of its elements. (We shall call f a *choice function* on A.)

2. *Axiom of Choice* (2nd form). Let R be an equivalence relation with universe A. Then there is a subset B of A containing exactly one element in each class of the partition associated with R.

3. *Axiom of Choice* (3rd form). A product of sets, all nonempty, is nonempty.

4. *Zermelo's Axiom.* Every set can be well-ordered.

5. *Hausdorff's Axiom.* In a partial order, every chain (i.e., every subset totally ordered by this partial ordering) is contained in a maximal chain.

6. *Kuratowski's Axiom.* (Recall that a partial order is called *inductive* if every chain in it has an upper bound.) In an nonempty inductive partial order there is at least one maximal element.

7. *Kuratowski's Axiom* (2nd form). In an inductive partial order every element is bounded above by a maximal element.

Kuratowski's axiom is better known as *Zorn's lemma*, although Zorn stated it more that twenty years after Kuratowski. This error in attribution is due to enthusiasm for the axiom on the part of Bourbaki, who used it (not always in a natural way) to avoid transfinite recursion.

Proof. [Proof (redundant)]

- $1 \leftrightarrow 2$
  $1 \to 2$ because a choice function f on A gives a representative for each equivalence class: Take B to be the image under f of the set of equivalence classes of R.
  To show $2 \to 1$, consider the subset of $P(A) \times A$ consisting of pairs (X, x) such that $x \in X$ ($P(A)$ denotes the set of subsets of A) and the equivalence relation R of "having the same first coordinate" on this set. Let B be a set of representatives of classes modulo R given by axiom 2. The following function f is a choice function on A: Every nonempty subset X of A is associated with the class, modulo R, consisting of all pairs with first coordinate X, and $f(X)$ is taken to be the second coordinate of the unique element of this class in B.

- $1 \leftrightarrow 3$
  To say that there exists a choice function is to say that the Cartesian product of all nonempty subsets of A is nonempty. If $\{\dots, A_i, \dots\}$ is a set of nonempty sets, the restriction of a choice function on $\bigcup A_i$ to the set of the A_i is an element of their product.

- $1 \leftrightarrow 4$
  If A is well-ordered, we get a choice function by mapping every nonempty subset of A to its least element in the sense of the well-ordering.
  For the converse, we repeat the proof of Theorem 8.6, where the flaw was our failure to describe a_α, when all a_β, $\beta < \alpha$, are known. We now have a choice function on A at our disposal, allowing us to define a_α unambiguously as a function of the a_β: If $\{\dots, a_\beta, \dots\}_{\beta<\alpha} = A$, we stop; otherwise, a_α is the image under f of the complement of this set in A.

- $6 \leftrightarrow 7$
  In an inductive order, the upper bounds of an element form a nonempty inductive order.

- $5 \leftrightarrow 7$

The chains of a partial order, ordered by inclusion, form an inductive set.
Conversely if, in an inductive set we consider a maximal chain C extending the chain consisting of only the element a, then this chain has an upper bound b, which can only be its greatest element, and which is maximal.

- $6 \to 4$
  Let A be a set and let B be the set of well-orderings of subsets of A. This set is not empty, since it includes the well-ordering $\emptyset$. Order B as follows: $a \leq b$ if a is an initial segment of b. We can see without difficulty that this is an inductive order: A chain is bounded above by the common extension of each of its elements, defined on the union of their universes. (This common extension is indeed a well-ordering, since each one is an initial segment.) Let a be a maximal element of B. I claim that a is defined on all of A. If not, if there is an element x of A outside the universe of a, then we can extend a by making x greater than every element of (the universe of) a, contradicting the maximality of a.

- $1 \to 7$
  By induction on the ordinal α define a sequence a_α of elements of the inductive order A, assuming that A is equipped with a choice function f.
  $a_0 = a$. If $\alpha = \beta + 1$ and a_β is maximal, stop; otherwise, or if α is a limit, the chain $\{a_0, \ldots, a_\beta, \ldots\}_{\beta<\alpha}$ has a nonempty set M of strict upper bounds (in the limit case it has no greatest element), so let $a_\alpha = f(M)$.
  The construction must stop, because we cannot map the class of all ordinals injectively into a set. (That is a consequence of the fact that the ordinals do not form a set, and of the axiom schema of replacement, for those who know ZF.) When it stops, we have found our maximal upper bound for a.

□

The axiom of choice makes mathematicians uneasy. They prefer to use it in the Kuratowski–Zorn form, for example to show the existence of maximal ideals in a unitary ring, since this seems more tangible to them: With the axiom of choice, they do not see what they need to show. As they loathe constructions by transfinite recursion, they use this Kuratowski–Zorn axiom everywhere, often being forced to acrobatics: How many algebra textbooks, in order to show that two bases of an infinite-dimensional vector space have the same number of elements, pass through finite-dimensional subspaces, instead of proving a version of the exchange lemma (Theorem 19.11) directly!

The axiom of choice is often misunderstood: We do not need it to find an element in a nonempty set! Indeed, the set being nonempty means that it has at least one element! The problem is the following: If we have a set of sets, each of which we know individually to be nonempty, nothing assures us of the existence of a choice function, a sort of uniform global procedure to show them all to be nonempty; this is the fine line between "each" and "all"! If you are allergic to this metaphysics, you would do well to get down to set theory and to study the (difficult) proof of the independence of the axiom of choice from the other axioms.

To clarify these ideas for the reader, let us show, without the axiom of choice, that a product of finitely many nonempty sets in nonempty: This is proved by induction on the number n of sets. By the induction hypothesis, we have a function f with domain $[0, n)$ that sends i to an element of A_i; since A_n is nonempty, it has an element a, and we extend f to n by setting $f(n) = a$!

The finite axiom of choice is not an axiom, but rather a theorem that can be proved from the other axioms. In contrast, there are weak forms of the axiom of choice that are not provable. One example is the *axiom of countable choice*, which states that if $A_0, A_1, \ldots, A_n, \ldots$ form a denumerable set of nonempty sets, their product is nonempty. Another is the *axiom of dependent choice*, which is as follows: Let A be a nonempty set, and let R be a binary relation with universe A, such that for all a in A, there exists b in A such that (a, b) satisfies R. Then there is a sequence a_n of elements of A (i.e., a function from ω to A) such that for all n, (a_n, a_{n+1}) is in R. Compare this sentence with Lemma 7.17.

This last axiom is easy to show if there is a choice function on A available, for example if A is well-ordered; it is therefore a consequence of the axiom of choice. It allows the axiom of countable choice to be proved: Consider the binary relation R defined on the union A of the A_n such that (a, b) satisfies R if for some integer n, a is in A_n and b is in A_{n+1}. In set theory it is proved (this is difficult!) that this axiom is strictly stronger than the axiom of countable choice.

The axiom of countable choice is constantly used in analysis; it is often hidden in order not to sow confusion in the minds of the students (who are inclined to accept anything desired) or of the professors (who do not like to shake the foundations of the discipline). For example, to show that a union of denumerably many sets of measure zero also has measure zero: Each A_n is contained in an open O_n of measure less than $\epsilon/2^{n+1}$, so $\mu(\bigcup A_n) \leq \Sigma\mu(O_n) \leq \Sigma\epsilon/2^{n+1} = \epsilon$; fine, but we need to *choose* an O_n for every n from all the possibilities.

Let us have no illusions: Without the axiom of choice, we cannot prove that the product of denumerably many sets A_n, each with two elements, is nonempty! But if $A = \{0, 1\}$, then A^ω is nonempty, since it contains the null sequence; since each A_n is in bijection with A, A^ω is in bijection with ΠA_n. The error: To construct such a bijection, we need to choose, for each n, one of the two possible bijections between A and A_n. As Bertrand Russell so excellently put it, it is the difference between a family of pairs of socks and a family of pairs of shoes: The latter has a choice function, consisting of always taking the left shoe.

The existence of a well-ordering on the set of real numbers can be guaranteed only by an axiom: It is an act of faith, we cannot find a "natural" one. It has some very annoying consequences for analysis, since it builds some bizarre sets that we would have no chance of ever meeting, such as a basis for $\mathbf{R}$ as a $\mathbf{Q}$-vector space, or a set that is not measurable in the sense of Lebesgue.

A good choice (if I can say that) for the analyst is therefore to accept the axiom of countable choice, or of dependent choice, and to reject the general axiom of choice, replacing it with more amenable axioms that contradict it strongly. This will keep a human face on analysis (we never see any interest in a theory of sets of real numbers in which it is false that a union of a denumerable family of measure-zero sets has measure zero) without introducing exotic sets. This choice is completely pragmatic, since the metaphysical reasons that can be given to accept or reject the axiom of countable choice are also valid for the axiom of choice in general.

In contrast, in algebra, and also in model theory, which is close to algebra, *we accept the axiom of choice.* Why? *Because it simplifies our lives.* All vector spaces are going to have bases; fields, transcendence bases; unitary rings, maximal ideals. The axiom of choice is absolutely necessary to the first chapters of this course: To prove Löwenheim's theorem (where there was a choice of a_i, subtly hidden by the author), for the compactness theorem (in using ultrafilters, or Henkin's method), for the definition of the notion of proof, etc. In short, it is indispensable at every level of our work. If we did not accept it, we would need to weigh down our theorems by specifying every time that certain sets are well-ordered, or else defined in a particular manner: a totally useless complication, with no relevance to the object of our study.

For example, we have used the *ultrafilter axiom*: every filter of subsets of A extends to an ultrafilter. As filters, ordered by inclusion, form an inductive set (the union of a chain of filters is a filter, since it does not contain $\emptyset$), the assertion is an immediate consequence of the axiom of choice. In set theory it is shown to be strictly weaker than the axiom of choice.

The ultrafilter axiom is enough to prove the *Tykhonoff theorem*, which states that every product of compact spaces is compact. (A French compact space satisfies the Hausdorff separation condition.)

In the constructions by recursion that we have done, we have often behaved typically for those who accept the axiom of choice: We define a_α as an element of a certain set A, without specifying which one, merely because we understand that we take what the choice function gives, and it is pointless to repeat that every time.

In a word, in this book, the axiom of choice is considered true.

8.3 Cardinals

Theorem 8.7 (Cantor's Theorem). *If there is an injection f from A into B and an injection g from B into A then, there is a bijection between A and B.*

Proof. We shall say that an element a of A or an element b of B is of the first type if it has the following property: If, for example, the element is a in A, it is in the image of g, $g^{-1}(a)$ is in the image of f, $f^{-1}(g^{-1}(a))$ is in the image of g, etc. Starting from the element, and successively taking preimages under f and g alternately, we never have to stop.

We shall say that it is of the second type if when we apply this process, we reach an element of A that is not in the image of g, and that it is of the third type if we reach an element of B that is not in the image of f.

Consider the functions ϕ from A to B, and ψ from B to A, defined as follows:

- If a is of the first or second type, then $\phi(a) = f(a)$.
- If a is of the third type, then $\phi(a) = g^{-1}(a)$.
- If b is of the first or third type, then $\psi(b) = f^{-1}(b)$.
- If b is of the second type, then $\psi(b) = g(b)$.

As $\phi \circ \psi = \mathrm{Id}_B$ and $\psi \circ \phi = \mathrm{Id}_A$, ϕ and ψ are bijections that are inverses to each other. □

We write 2^A for the set of subsets of A; it can be identified with the set of functions from A to the set $2 = \{0, 1\}$ (identify a subset of A with its characteristic function).

Theorem 8.8 (Bernstein's Theorem). *If A is any set, then there is no surjection from A onto 2^A.*

Proof. Let f be a surjection from A onto 2^A and set $X = \{a : a \notin f(a)\}$. As X is in the image of f, $X = f(a_0)$; if $a_0 \notin X$, then $a_0 \in X$, and if $a_0 \in X$, then $a_0 \notin X$: contradiction. □

Note that both these theorems are proved without using the axiom of choice.

We say that the *cardinality* of the set A is less than that of the set B, in symbols $\mathrm{card}(A) \leq \mathrm{card}(B)$, if there is an injection from A to B (considering $\mathrm{card}(A) \leq \mathrm{card}(B)$ as a unified expression for now). This is clearly transitive: $\mathrm{card}(A) \leq \mathrm{card}(B)$ and $\mathrm{card}(B) \leq \mathrm{card}(C)$ implies $\mathrm{card}(A) \leq \mathrm{card}(C)$. If $\mathrm{card}(A) \leq \mathrm{card}(B)$ and $\mathrm{card}(B) \leq \mathrm{card}(A)$ (we then say that A and B have the same cardinality) that means, by Theorem 8.7, that there is a bijection between A and B. By Theorem 8.8, the cardinality of 2^A is strictly greater than that of A, since there is an injection from A to 2^A sending every element of A to its own singleton.

Note that if $\text{card}(A) \leq \text{card}(B)$ and if A is not empty, then there is a surjection g from B onto A: Let f be an injection from A to B and let a be an element of A; then define $g(b) = f^{-1}(b)$ if b is in the image of f, $g(b) = a$ otherwise.

This is about all we can say about cardinals in the absence of the axiom of choice. In particular, we cannot prove that two cardinals are always comparable, nor that if there is a surjection from A onto B, then $\text{card}(A) \geq \text{card}(B)$. From here on, we shall work with the axiom of choice.

If f is a surjection from A onto B, then by means of a choice function on A we can map every b in B to an element of the set $f^{-1}(b)$, thus defining an injection from B into A, so $\text{card}(A) \geq \text{card}(B)$.

If A and B are any sets, they can be well-ordered, the first by an ordinal α and the second by an order β. If α is an initial segment of β, we can thus derive an injection from A to B; if β is an initial segment of α, we can derive an injection from B to A. As two ordinals are always comparable, we always have $\text{card}(A) \leq \text{card}(B)$ or $\text{card}(B) \leq \text{card}(A)$.

The *cardinality of the set* A is defined as the smallest ordinal α such that there is a well-ordering of type α on the universe A. The cardinality of A is written $\text{card}(A)$ or $|A|$ equivalently. A cardinal is a (von Neumann) ordinal that cannot be put into bijection (not necessarily increasing!) with any of its own proper initial segments; for this reason, cardinals are sometimes called *initial ordinals.*

For example, $0, 1, 2, \ldots$ and all the finite ordinals are cardinals; ω is a cardinal; $\omega + 1$ is not a cardinal, since it has a bijection to ω (send ω to 0 and n to $n + 1$). A set of cardinality ω is called *denumerable* or *countable*; however, sometimes, depending on the context, "countable" means "finite or denumerable." The least nondenumerable ordinal is a cardinal. As $\text{card}(\omega + 1) = \omega$, and as all infinite ordinals begin with ω, note that an infinite cardinal is a limit ordinal: Adding one point to an infinite set does not increase its cardinality.

If κ is a cardinal, then κ^+ denotes the smallest cardinal strictly greater than κ. Often, κ^+ is called the *successor cardinal* of κ: Be careful, because if κ is infinite, this is not the successor of κ as an ordinal. By Theorem 8.8, $\kappa^+ \leq 2^\kappa$, where 2^κ denotes the cardinality of the set of subsets of a cardinal κ.

The *continuum hypothesis* (CH) is the following axiom: $\omega^+ = 2^\omega$. The generalized continuum hypothesis (GCH) is the statement that for all infinite cardinals κ, $\kappa^+ = 2^\kappa$: For every infinite set A, if A is mapped injectively into B and B is mapped injectively into 2^A, then either A can be mapped bijectively to B or B can be mapped bijectively to 2^A. GCH (in this last form) implies the axiom of choice: That result is due to Sierpinski. However, CH is independent of ZF + AC by results of Gödel and Cohen. In model theory, the custom is not to assume the continuum hypothesis, which overly simplifies cardinal arithmetic: We endeavor to prove theorems without it, and if we ever use it, we specify this explicitly.

We can define the sequence of the $\aleph_\alpha$ ($\aleph$, aleph, is the first letter of the Hebrew alphabet) by induction on the ordinal α as follows: $\aleph_\alpha$ is the smallest infinite cardinal strictly greater than all the $\aleph_\beta$ for $\beta < \alpha$. Thus $\aleph_0 = \omega$, $\aleph_1 = \omega^+$ (we also sometimes call this ω_1), etc. $\aleph_\alpha$ is the "αth infinite cardinal." We thus get an indexing of all infinite cardinals by the ordinals: It is easy to see that the cardinal κ is less than or equal to $\aleph_\kappa$, so it has the form $\aleph_\alpha$ for some $\alpha \leq \kappa$.

We can also define the sequence of the $\beth_\alpha$ ($\beth$, beth, is the second letter of the Hebrew alphabet):

- If $\alpha = \beta + 1$, then $\beth_\alpha = 2^{\beth_\beta}$.
- $\beth_0 = \omega$.
- If α is a nonzero limit ordinal, then $\beth_\alpha$ is the supremum (whether as an ordinal or as cardinal, it does not matter) of all $\beth_\beta$ for $\beta < \alpha$.

The generalized continuum hypothesis states that $\beth_\alpha = \aleph_\alpha$ for all α; if, on the contrary, this is false, then the sequence of the $\beth$ omits some cardinals.

Theorem 8.9. *If A is an infinite set, then A and $A \times A$ have the same cardinality.*

Proof. Let κ be the cardinality of A. Define an ordering $\leq'$ on the set $\kappa \times \kappa$ in the following way (do not forget that the elements of κ are ordinals, equipped with their natural order):

$$
\begin{array}{lll}
(\alpha, \beta) <' (\alpha_1, \beta_1) & \text{if} & \max(\alpha, \beta) < \max(\alpha_1, \beta_1) \\
& \text{or if} & \max(\alpha, \beta) = \max(\alpha_1, \beta_1) \text{ and } \alpha < \alpha_1 \\
& \text{or if} & \max(\alpha, \beta) = \max(\alpha_1, \beta_1),\ \alpha = \alpha_1, \text{ and } \beta < \beta_1.
\end{array}
$$

The task of checking that this is a total order is left to the reader. Let us show that it is a well-ordering. Let A be a nonempty subset of $\kappa \times \kappa$; let B be the set of those elements (α, β) of A such that $\max(\alpha, \beta)$ has the smallest possible value (in the ordinal κ); let C be the set of elements of B such that α has the smallest possible value; finally, let a be the element of C with β minimal. Then a is the least element of A for the order $\leq'$.

We now proceed by induction on κ, or, equivalently, induction on its index as an $\aleph$.

First note that $\kappa \leq \operatorname{card}(\kappa \times \kappa)$, since the function sending α to $(\alpha, 0)$ is an injection from κ to $\kappa \times \kappa$.

If $\kappa = \aleph_0$ and if $(\alpha, \beta) \in \omega \times \omega$, then the (α_1, β_1) that are less than (α, β) in the order $\leq'$ are such that α_1 and β_1 are bounded above by $\max(\alpha, \beta)$, which is a finite number. There are therefore only finitely many such pairs. The order $\leq'$ is an infinite well-ordering, whose initial segments are all proper: Its ordinal is therefore ω. We thus get a bijection between ω and

$\omega \times \omega$, a little different (although still primitive recursive) from the one we used in arithmetic (Lemma 7.16): Here we group the numbers in bunches with $\max(x, y) =$ constant instead of $x + y =$ constant.

In the other cases, if $(\alpha_1, \beta_1) \leq' (\alpha, \beta)$, then α_1 and β_1 are bounded above by $\max(\alpha, \beta)$, which is an ordinal strictly less than κ; its cardinal λ is therefore strictly less than κ, and the number of (α_1, β_1) is therefore at most $\lambda \times \lambda$, which is equal to λ by the induction hypothesis. Therefore, the ordinal of the well-ordering $\leq'$ has cardinality at most κ, and all its proper initial segments have cardinality strictly less than κ: It can be only the ordinal κ. We have thus established a bijection between κ and $\kappa \times \kappa$. □

The proof of the last theorem might, at first glance, seem a little complicated to you: The axiom of choice, allowing us to well-order A, is essential here.

We can see, as a consequence of this theorem, that the "arithmetic of infinite cardinals" is especially simple. If κ and λ are cardinals, we define $\kappa + \lambda$ (careful: This is not their ordinal sum) to be the cardinality of $A \cup B$, where A has cardinality κ, B has cardinality λ, and A and B are disjoint. If κ and λ are finite, this is their sum as natural numbers. If one of them is infinite, then $\kappa + \lambda = \max(\kappa, \lambda)$, since, taking (for example) $\kappa = \max(\kappa, \lambda)$,

$$\kappa \leq \kappa + \lambda \leq \kappa + \kappa \leq 2\kappa \leq \kappa \times \kappa = \kappa.$$

Likewise, we define the product of two cardinals κ and λ to be the cardinality of the Cartesian product $A \times B$, where A has cardinality κ and B has cardinality λ (careful: This is not their lexicographic product as ordinals). If κ and λ are finite, this is their product as natural numbers. If one is infinite and the other nonzero ($\kappa \times 0 = 0$), then $\kappa \times \lambda = \max(\kappa, \lambda)$, since, if $\kappa = \max(\kappa, \lambda)$,

$$\kappa = \kappa \times 1 \leq \kappa \times \lambda \leq \kappa \times \kappa = \kappa.$$

Define κ^λ to be the cardinality of the set A^B of functions from B into A, where A has cardinality κ and B has cardinality λ. If B and C are disjoint, then a function from $B \cup C$ to A is determined by its restrictions to B and to C, which can be chosen however we wish, so that $A^{B \cup C}$ is in bijection with $A^B \times A^C$, and $\kappa^{\lambda + \mu} = \kappa^\lambda \times \kappa^\mu$.

Specifying a function from C into A^B sending the element c to the function f_c is the same as specifying the function from $B \times C$ into A that sends (b, c) to $f_c(b)$: $(\kappa^\lambda)^\mu = \kappa^{\lambda \times \mu}$.

If κ and λ are finite, then κ^λ is the exponential in the sense of arithmetic.

Now suppose that κ is infinite. First, $\kappa^0 = 1$; if $\lambda = n$ is finite and nonzero, then, since $\kappa^2 = \kappa$, we can show by induction on n that $\kappa^n = \kappa$. If λ is infinite and less than κ, we can give only upper and lower bounds. First, note that if κ is infinite, then $\kappa^\kappa = 2^\kappa$: Clearly, $2^\kappa \leq \kappa^\kappa$. Conversely, specifying a function from κ to κ is the same as specifying its graph, which is a subset of $\kappa \times \kappa$, so $\kappa^\kappa \leq 2^{\kappa\times\kappa} = 2^\kappa$. Consequently, if κ and λ are infinite and $\lambda \leq \kappa$, then $\kappa \leq \kappa^\lambda \leq 2^\kappa$, since $\kappa = \kappa^1 \leq \kappa^\lambda \leq \kappa^\kappa = 2^\kappa$.

If κ is infinite and $\lambda \geq \kappa$, then $\kappa^\lambda = 2^\lambda$, since $2^\lambda \leq \kappa^\lambda \leq \lambda^\lambda = 2^\lambda$.

The remaining case is where λ is infinite and κ finite: $0^0 = 1$, and if $\lambda \neq 0$, then $0^\lambda = 0$; $1^\lambda = 1$; for $n \geq 2, n^\lambda = 2^\lambda$, since $2^\lambda \leq n^\lambda \leq \lambda^\lambda = 2^\lambda$.

Now define $\kappa^{<\omega}$ to be the set of finite sequences of terms in κ, that is to say, the set of functions from $[0, n)$ to κ for some n. Equivalently, $\kappa^{<\omega} = \bigcup \kappa^n$. If κ is infinite, since $\kappa^n = \kappa$, $\kappa^{<\omega} = \omega \times \kappa = \kappa$. Therefore, if κ is infinite, *the set of finite sequences of terms in* κ *has cardinality* κ. The same is true of the set of finite subsets of κ, which are the images of these sequences.

All this allows us to justify completely the evaluations of the cardinals that we used in the proof of the Löwenheim–Skolem theorem: if the language has only finitely or denumerably many relation, function, and constant symbols, the number of basic symbols in the language is ω. A word is a finite sequence of symbols, so there are $\omega^{<\omega} = \omega$ of them. There are not more than ω formulas, and there are clearly at least ω, for example all those of the form $x_i = x_i$, so $|T| = \omega$. We can likewise see that if the number of symbols is κ, then $|T| = \kappa^{<\omega} = \kappa$.

In Löwenheim's theorem (2.5, 3.1), we evaluated the number of formulas with parameters in a set A of cardinality κ: That amounts to counting the formulas of the language $L(A)$ obtained by adding a name for every element of A, so $|T(A)| = (|L| + |A|)^{<\omega} = \max(|T|, |A|)$.

After having reassured the reader of the validity of past theorems, I add, as a preview of the future, two theorems on counting.

Theorem 8.10. *For every infinite cardinal* κ*, there is a chain of cardinality* κ *having at least* κ^+ *cuts (or, equivalently, there is a chain of cardinality* κ^+ *having a dense subchain of size* κ*).*

Proof. Let λ be the smallest cardinal such that $\kappa^\lambda > \kappa$; as $\kappa^\kappa = 2^\kappa, \lambda \leq \kappa$. Let A be the set κ^λ of functions from λ to κ (its cardinality is at least κ^+), equipped with the following order relation, called "lexicographic order": Let f and g be two distinct elements of A. Then there is a least α in λ such that $f(\alpha) \neq g(\alpha)$. If, for that α, $f(\alpha) < g(\alpha)$ (in the order of the ordinal κ), then set $f < g$; if, on the other hand, $g(\alpha) < f(\alpha)$, then set $g < f$.

We can see without difficulty that this is a total order, and that the set B of functions from λ to κ that are constantly 0 from some point on is a dense subchain.

For every α in λ, let B_α denote the set of elements of A that are 0 for all $\beta \geq \alpha$; there are as many of them as there are functions from $[0, \alpha)$ to κ. As α is a strictly smaller ordinal than λ, its cardinality is also strictly less than λ, and by definition of λ, $|B_\alpha| \leq \kappa$. (In fact, $|B_\alpha| = \kappa$ except for $\alpha = 0$. As $B = \bigcup_{\alpha<\lambda} B_\alpha$, its cardinality is at most $\lambda \times \kappa \leq \kappa \times \kappa = \kappa$. $\square$

Given an infinite cardinal κ, let $\mathrm{Ded}(\kappa)$ be the least cardinal λ such that no chain of cardinality κ can have λ distinct cuts. (Ded stands for Dedekind.) If $\mathrm{Ded}(\kappa)$ is a successor cardinal, let $\mathrm{ded}(\kappa)$ denote its predecessor: $\mathrm{ded}(\kappa)$ is therefore the largest number of cuts possible for a chain of cardinality κ. There is no reason why ded should exist for arbitrary κ.

As a cut of A is determined by a subset of A, its lower part, it follows that $\mathrm{Ded}(\kappa) \leq (2^\kappa)^+$, and if it exists, then $\mathrm{ded}(\kappa) \leq 2^\kappa$. Theorem 8.10 therefore gives the following bounds: $(\kappa^+)^+ \leq \mathrm{Ded}(\kappa) \leq (2^\kappa)^+$, $\kappa^+ \leq \mathrm{ded}(\kappa) \leq 2^\kappa$.

For $\kappa = \omega$, $\mathrm{ded}(\kappa)$ does exist, and is equal to 2^ω, since the chain of rationals has 2^ω cuts. Naturally, if we accept the continuum hypothesis, $\mathrm{ded}(\kappa)$ always exists and is equal to 2^κ, but the behavior of the function Ded, in the absence of this hypothesis, causes many worries to set theorists.

Theorem 8.11 (Hausdorff's Theorem). *If A is an infinite set of cardinality κ, then there are $2^{(2^\kappa)}$ distinct ultrafilters of subsets of A.*

Proof. As an ultrafilter is a set of subsets of A, there cannot be more than $2^{(2^\kappa)}$ of them; we must show that there are at least that many.

We shall say that a family F of subsets of A is *Boolean-independent* if it forms a basis for a free Boolean algebra; this means that if $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are distinct elements of F, the intersection of the X_i and the complements in A of the Y_j is never empty.

If F is a Boolean-independent family, then every subset G of F is associated with the filter base formed by elements of G and the complements of the elements of F that are not in G. This filter base extends to an ultrafilter U_G. If $G \neq G'$, then necessarily $U_G \neq U_{G'}$: If for example $X \in G, X \notin G'$, then X is in U_G, while the complement of X is in $U_{G'}$. Consequently, it is sufficient to prove that there is a Boolean-independent family, of cardinality 2^κ, of subsets of A.

We therefore consider our set A of cardinality κ, and a set B disjoint from A consisting of elements $b(F; P_1, \ldots, P_n)$, indexed on the one hand by a finite subset F of A, and on the other hand by a (finite!) set $P_1, \ldots, P_n$ of subsets of F. As there are only κ finite subsets of A, each of which has only finitely many subsets, B also has cardinality κ.

Every subset X of A is associated, in an injective manner, with a subset X' of $A \cup B$ defined as follows:

- If a is in A, then $a \in X'$ iff $a \in X$.
- If b is in B, $b = b(F; P_1, \ldots, P_n)$, then $b \in X'$ iff $X \cap F$ is one of the P_i.

I claim that the family of the X' is free. Consider distinct subsets $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ of A. If two sets are distinct, that means that there is a point belonging to one and not to the other; there is therefore a finite subset F of A such that the traces $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ of these sets on

T are pairwise distinct, and the element $b(F; P_1, \ldots, P_n)$ belongs to the X_i' and not to the Y_i'.

We have therefore constructed a family of 2^κ independent subsets of $A \cup B$; as this last set has cardinality κ, we need only carry everything over to A by a bijection to have the desired family. $\square$

8.4 Cofinality

A *cofinal subset* B of a chain A is simply an unbounded subset: For every a in A, there exists b in B greater than or equal to A. A chain B is called *cofinal* in the chain A if it is isomorphic to a cofinal subchain of A. This relation is clearly transitive: If A is cofinal in B and B in C, then A is cofinal in C.

Lemma 8.12. *Every chain has a cofinal subset that is well-ordered (by the restriction of the chain!).*

Proof. Let A be our chain. We define an increasing sequence of elements of A, by induction on α, as follows: If the chain of the $\{\ldots, a_\beta, \ldots\}_{\beta<\alpha}$ is unbounded in A, we stop with a_α undefined. Otherwise, a_α is a strict upper bound of all the a_β for $\beta < \alpha$. When we stop, we have constructed a well-ordered subchain that is unbounded in A. $\square$

The *cofinality* of A, written $\text{cof}(A)$, is therefore defined to be the least ordinal that is cofinal in A. By transitivity, an ordinal $\alpha < \text{cof}(A)$ cannot be cofinal in $\text{cof}(A)$, and $\text{cof}(A)$ is equal to its own cofinality: An ordinal with this property is called *regular.*

Lemma 8.13. *Every regular ordinal is a cardinal.*

Proof. It is sufficient to prove that every well-ordering A has a final segment whose ordinal is less than or equal to $\text{card}(A) = \kappa$. We can index A by κ, which is in bijection with A: The elements of A thus have the form $a_\alpha, \alpha \in \kappa$. We have two well-orderings on A: the ordering of the elements of A, and that of their indices. We can thus repeat the proof of the previous lemma, constructing a sequence b_α by induction on α:

- If the sequence of $b_\beta, \beta < \alpha$, is unbounded in A, we stop without defining b_α.
- Otherwise, let b_α be the a_γ of minimal index γ such that a_γ is greater than all the $b_\beta, \beta < \alpha$, and greater than or equal to a_α.

If we stop before κ, we have a cofinal subset of ordinal $\alpha < \kappa$; otherwise, we have a cofinal subset of ordinal κ. $\square$

We thus distinguish two types of cardinals: the *regular cardinals*, and the others, which are called *singular cardinals.*

For example, the finite regular cardinals are 0 (the cofinality of the empty sequence) and 1 (the cofinality of any chain with a greatest element); ω is regular.

Theorem 8.14. *A cardinal $\kappa \neq 2$ is singular if and only if there is a family of strictly fewer than κ sets, each of cardinality strictly less than κ, whose union has cardinality κ.*

Proof. If κ is singular, it has a cofinal subset A of smaller cardinality. For every α in κ, let $I_\alpha = \{\beta : \beta \leq \alpha\}$ (in fact, $I_\alpha = \alpha + 1$!). As $I_\alpha = \{\beta : \beta < \alpha + 1\}$, every I_α is a proper initial segment of κ, which therefore has cardinality strictly less than κ, unless κ is finite and α is its greatest element. If $\kappa = 0$ or $\kappa = 1$, it is regular, and the condition is satisfied; if $\kappa = n$ is finite and strictly greater than 2, $n = (n-1) + 1$; if κ is infinite, $\kappa = \bigcup_{\alpha \in A} I_\alpha$.

Now suppose that $A = \bigcup_{i \in I} A_i$, with $\kappa = |A|, |A_i| < \kappa, |I| < \kappa$, such that the cardinality λ is the smallest with this property. Then A cannot be expressed as the union of fewer than λ of its subsets all with cardinalities less than κ.

If we give I the well-ordering λ, that amounts to indexing the A_i by λ. We can assume that the A_α are disjoint: Otherwise, replace each A_α by $A_\alpha - \bigcup_{\beta < \alpha} A_\beta$. We equip each A_α with a well-ordering, for example, that of its cardinal.

Consider the well-ordering on A given by the "sum" of those of the A_α: $a < b$ if $a \in A_\alpha, b \in A_\beta$, with $\alpha < \beta$, or else if $a, b \in A_\alpha$ and $a < b$ in the sense of A_α.

If λ is finite, that means that A is finite, and of cardinality at least 2 (in fact, $\lambda = 2$). If λ is infinite, it is a limit ordinal, and as it is minimal for the property in question, every proper initial segment of the well-order sum of the A_α has cardinality strictly less than κ: It must therefore be the well-ordering κ. The sequence of the smallest elements of each A_α, whose ordinal is λ, is cofinal in κ. $\square$

The proof of the last theorem shows that in fact, for every infinite cardinal κ, $\mathrm{cof}(\kappa)$ is the least cardinal λ such that κ can be expressed as a sum of λ cardinals all strictly less than κ.

One characterization of the regular cardinals is the following: Let A_α be an increasing sequence of sets indexed by κ: If $\alpha < \beta$, then $A_\alpha \subseteq A_\beta$. Then every subset of $\bigcup_{\alpha < \kappa} A_\alpha$ of cardinality strictly less than κ is contained in one of the A_α.

For example, we have often used the regularity of ω in the following form: If for every n, $A_n \subseteq A_{n+1}$, then every finite subset of $\bigcup A_n$ is contained in one of the A_n.

More generally, if A_α is an increasing sequence indexed by the ordinal β, then every subset B of $\bigcup_{\alpha < \beta} A_\alpha$ of cardinality strictly less than $\mathrm{cof}(\beta)$ is contained in one of the A_α: If this were not the case, then the set of α

such that A_α contains an element of B not in any of the earlier sets (this set of indices has cardinality at most that of B) would be cofinal in β.

Every infinite successor cardinal is regular: If $\kappa = \lambda^+$, the union of a family of λ sets of cardinality λ has cardinality $\lambda \times \lambda = \lambda$.

If κ is an infinite regular nonsuccessor cardinal different from $\aleph_0$, then $\kappa = \aleph_\kappa$: If α is a nonzero limit, then the sequence of $\aleph_\beta, \beta < \alpha$ is cofinal in $\aleph_\alpha$.

If we accept the generalized continuum hypothesis, then $2^\kappa = \kappa^+$, and 2^κ is always regular. That is not true in general, but we are going to show that the cofinality of 2^κ is strictly greater than κ.

Theorem 8.15 (Julius König's Theorem; the father of the namesake of Theorem 7.26, on trees). *Let A_i and B_i be two families of sets, each indexed by the same set I, such that for all i, the cardinality of A_i is strictly less than that of B_i. Then the cardinality of the union of the A_i is strictly less than the cardinality of the product of the B_i.*

Proof. This theorem is an equivalent of the axiom of choice: setting $A_i = \emptyset$, we can see that it says that a product of nonempty sets is nonempty. We certainly need the axiom of choice to prove it.

As there is an injection from A_i into B_i, we can assume that $A_i \subset B_i$. First we show that $|\bigcup A_i| \leq |\Pi B_i|$. For this purpose, we choose an element b_i in each B_i that is not in A_i; we can obviously assume that the A_i are disjoint. Define an injection from $\bigcup A_i$ to ΠB_i as follows: The element a of A_i is mapped to the I-tuple whose ith coordinate is a, and whose jth coordinate, for $j \neq i$, is b_j.

Now we show that there is no surjection s from $\bigcup A_i$ onto ΠB_i: Let s be a function from $\bigcup A_i$ to ΠB_i, and let C_i be the subset of the B_i consisting of the ith projections of the elements of $s(A_i)$. As $|C_i| \leq |A_i| < |B_i|$, there is an element b_i of B_i that is not in C_i, and the tuple $(\ldots, b_i, \ldots)$ is not in the image of s. □

Corollary 8.16. *For every infinite cardinal κ, $\kappa < \kappa^{\mathrm{cof}(\kappa)}$.*

Proof. The cardinal κ can be expressed as a union of the sets $A_\alpha, \alpha < \mathrm{cof}(\kappa), |A_\alpha| < \kappa$. Setting $B_\alpha = \kappa$, we can see that the cardinality of the sum of the A_α, which is κ, is strictly less than that of the product of the B_α, which is $\kappa^{\mathrm{cof}(\kappa)}$. □

As $(2^\kappa)^\kappa = 2^{\kappa\times\kappa} = 2^\kappa$, the cofinality of 2^κ must be greater than κ. For example, the cofinality of 2^ω is greater than ω, so 2^ω cannot be $\aleph_\omega$; however, apart from this constraint, set theorists know how to prove that 2^ω can be located just about anywhere in the hierarchy of the $\aleph$.

Exercise 8.17. Under the generalized continuum hypothesis, show that if κ is an infinite cardinal, and if λ and μ are cardinals strictly less than κ, then $\lambda^\mu \leq \kappa$. Deduce that $\kappa^\lambda > \kappa$ if and only if $\lambda \geq \mathrm{cof}(\kappa)$.

8.5 Historic and Bibliographic Notes

Considering the utilitarian character of this chapter, it is useless to give references for the results mentioned here, all of which are quite classical; it would be a labor unconnected to my project, since an account of the evolution of "set theory" since the end of the last century would actually be a description of the birth of the modern conception of mathematics. For all of that, I am satisfied with referring you to [Moo82]. I make one exception in favor of [Jec77], which I recommend to the reader looking for some ideas about the axiom of choice.

9
Saturated Models

SATURABILITÉ sf. T. de chimie. Qualité de ce qui est saturable.

SATURABLE adj. Qui est susceptible de saturation.

SATURANT, ANTE adj. Qui a la propriété de saturer.

SATURATION (sa-tu-ra-sion) sf. T. de chimie

(1) Le terme où, les affinités réciproques des deux principes d'un corps binaire étant satisfaites, aucun des deux principes n'est plus susceptible de s'unir avec une nouvelle quantité de l'autre ...

(2) Saturation du sol des cimetières, condition qui provient de ce que, des cadavres nouveaux y étant sans cesse inhumés, avant que les cadavres plus anciens aient eu le temps de se consumer, le sol devient impropre à opérer les changements qui constituent la putréfaction.

E.L.

We are now embarking on what can properly be called model theory. First, let us define some notation in general usage. Consider a theory T, in a language L, which shall be complete unless otherwise stated. We let $|T|$ or $|L|$ denote the cardinality of the language, that is to say, the number of formulas: It is equal to ω if L has finitely or denumerably many relation, function, and constant symbols; it is equal to κ if L has $\kappa > \omega$ of them. If T is complete and has a finite model M, that is its only model up to isomorphism; as this is not a very interesting case, we generally suppose that all the models of T are infinite.

Let $M, N, \ldots$ denote models of T. A set of parameters is a subset A of a model of T; $L(A)$ denotes the language obtained by adding to L a constant symbol to name each element of A, and $T(A)$ denotes the theory consisting of the sentences of $L(A)$ that the elements of A satisfy in the model M of T of which they are part. We must understand that the specification of a set of parameters is not just the specification of the set A, but rather that of $T(A)$. If $A \subset M, M \prec N$ (N is an elementary extension of M), then A, considered as a subset of M, is identified, as a set of parameters, with A, considered as a subset of N, since $T(A)$ is preserved by elementary extension.

If M is a model of T, a model of $T(M)$ is nothing but an elementary extension of M (in which every element of M has a name).

If $A \subset M$, then two elements x_1 and x_2 are said to have the same type over A if they satisfy the same "formulas with parameters in A," i.e., the same formulas of $L(A)$. A type is therefore a complete theory, containing $T(A)$, in the language $L(A \cup \{x\})$; every type is eventually realized in some elementary extension of M. The set of types over A is written $S_1(A)$; it is equipped with the topology defined by the formulas: It is a compact 0-dimensional space.

Recall that the types that are realized in any model containing A (i.e., in a model of $T(A)$) form a dense subset of $S_1(A)$ (see Section 5.1) and that if A is included in B, the restriction function from $S_1(B)$ to $S_1(A)$ is continuous.

We can likewise introduce the spaces $S_n(A)$ of types of n-tuples, or n-types, by using n variables $x_1, \ldots, x_n$ instead of a single x; we can even define the space $S_\alpha(A)$ of α-types, where α is an ordinal, by introducing α-tuples of variables, i.e., sequences $\{\ldots x_\beta \ldots\}_{\beta<\alpha}$ of variables indexed by α.

Some use "type" to mean a theory, not necessarily complete, in $L(A \cup \{x\})$, and call the elements of $S_1(A)$ or $S_n(A)$ "complete types." For us, a type will always be complete, except when we specify "incomplete type"; an incomplete type is a closed set in $S_1(A)$.

If $A = \emptyset$, we write $S_1(T), \ldots, S_n(T), \ldots$ rather than $S_1(\emptyset), \ldots, S_n(\emptyset), \ldots$; we then say that the type is over $\emptyset$, or without parameters, or pure, or absolute, as the mood strikes us.

We sometimes speak of a "model" without saying what theory it is a model of; after all, a model is just a model of its own theory!

Letters such as $\alpha, \beta, \gamma, \ldots$ denote ordinals; $\kappa, \lambda, \mu, \ldots$, inifinite cardinals, that is to say, initial ordinals; card(A) or $|A|$ is the cardinality of A.

9.1 Svenonius's Theorem

If two elements a and b of a model M correspond, by an automorphism s of M ($sa = b$), or else by an automorphism of an elementary extension of M, it is clear that a and b have the same type. The following theorem states the converse:

Theorem 9.1. *If two elements a and b of a model M have the same type, then there is an elementary extension N of M and an automorphism s of N such that $s(a) = b$.*

Proof. We need to show the consistency of the set of sentences, in the language $L(M) \cup \{s\}$, consisting of $T(M)$, $s(a) = b$, the axiom saying that s is a bijection, and those saying that s preserves the relations, functions, and constants of the structure M.

By compactness, it is enough to show that every one of its finite fragments is consistent, so we can reduce the problem to the case in which M is a finite or denumerable structure, in a finite language, which, as always, we can assume to be purely relational.

So let us introduce a new symbol $E_p(x_1, \dots, x_p, y_1, \dots, y_p)$ for every integer $p \geq 1$, and the list of axioms consisting of $T(M)$, the axioms saying that $E_p(\vec{x}, \vec{y})$ defines an equivalence relation on p-tuples, the axioms $(\forall x)(\exists y)E_1(x, y), \dots, (\forall x)(\forall y)(\forall u)(\exists v)E_p(\vec{x}, \vec{y}) \to E_{p+1}(\vec{x}, u, \vec{y}, v), \dots$, and finally, the axioms $(\forall \vec{x})(\forall \vec{y})(E_p(\vec{x}, \vec{y}) \to (f(a, \vec{x}) \leftrightarrow f(b, \vec{y})))$ for every atomic formula $f(t, \vec{x})$ of the original language.

This family of axioms is consistent: Indeed, any of its finite fragments can mention only p symbols $E_1, \dots, E_p$, and we get a model with universe M by interpreting $E_i(\vec{x}, \vec{y})$ by the set of $2i$-tuples $(\vec{x}, \vec{y})$ of elements of M such that $a\frown\vec{x}$ and $b\frown\vec{y}$ are $(p-i)$-equivalent (i.e., satisfy the same formulas of L up of quantification rank up to $p - i$). That obviously works, because a and b have the same type, that is to say, are p-equivalent for all p.

This denumerable theory therefore has a (finite or) denumerable model N that, if we consider only the underlying L-structure, is an elementary extension of M, because it is a model of $T(M)$. Also, if $\vec{c}$ and $\vec{d}$ are p-tuples of N satisfying $E_p(\vec{x}, \vec{y})$, in the sense of the L-structure, then $a\frown\vec{c}$ and $b\frown\vec{d}$ are ∞-equivalent.

As a consequence, a and b are ∞-equivalent in the L-structure N; by Theorem 1.14, since N is denumerable, that means that (N, a) and (N, b) are isomorphic, and hence there is an automorphism s of N such that $s(a) = b$. $\square$

If A is a subset of M, an automorphism of M that fixes A pointwise is called an *A-automorphism*. Replacing T by $T(A)$ in Theorem 9.1, we can see that two elements of M have the same type over A if and only if there is an elementary extension of M having an A-automorphism that sends one to the other.

Moreover, the same proof works for n-types and, by compactness, for types of α-tuples.

This theorem, which clarifies the notion of type, also allows us to prove some results on "definability" that are set forth below.

Let M be an L-structure, and let r be an n-ary relation on the same universe. The relation r is said to be *definable* (or *interpretable*) in M if there is a formula $f(\vec{x})$ of L such that r consists of the tuples of M that satisfy f; in the language $L(r)$, the structure (M, r) satisfies $(\forall \vec{x})(r(\vec{x}) \leftrightarrow f(\vec{x}))$. Here, the formula f is *without parameters* in M; if the formula f involves parameters in M, we say *definable with parameters.*

If (M', r') is an elementary extension of (M, r), then the relation r' is also defined in M' by the formula f, and it is clear that every automorphism of the structure M' is also an automorphism of r'. The converse is equally true:

Theorem 9.2 (Svenonius's Theorem). *If the relation r with universe M is not definable in the structure M, then there is an elementary extension (M', r') of (M, r) and an automorphism s of M' that does not preserve r'.*

Proof. First let us show that there is an elementary extension (M_1, r_1) of (M, r), with two n-tuples $\vec{a}$ and $\vec{b}$ that have the same type in the sense of M_1, with one in r_1, the other in $\neg r_1$. If that were not the case, it would mean that in the sense of the theory $T(r)$ the fact that a type of n-tuples satisfies r would depend only on its reduct to the language L of M.

So let $p \in S_n(T, r)$ and assume that $p \models r(\vec{x})$. Let π be the set of formulas in the language L satisfied by p. By hypothesis $\pi \models r(\vec{x})$, and by compactness a finite fragment of π, which can be replaced by the conjunction of the finitely many formulas involved, is sufficient to imply r. So there is a formula $f_p(\vec{x})$ in the language L such that $p \models f_p(\vec{x}), T(r) \models (\forall \vec{x})(f_p(\vec{x}) \rightarrow r(\vec{x}))$. We have covered the clopen set $\langle r \rangle$ by (cl)open sets $\langle f_p \rangle$; by compactness, finitely many of them, corresponding to the formulas $f_1, \ldots, f_m$, are sufficient, and $\langle r \rangle = \langle f_1 \vee \cdots \vee f_m \rangle$. $T(r) \models (\forall \vec{x})(r(\vec{x}) \leftrightarrow f_1(\vec{x}) \vee \cdots \vee f_m(\vec{x}))$; that means precisely that r is definable in M.

We can therefore show, in a manner completely parallel to the proof of Theorem 9.1, that there is an elementary extension (M', r') of (M, r) (and not only an elementary extension M' of M!) such that M' has an automorphism s sending $\vec{a}$ to $\vec{b}$: s is not an automorphism for r'. □

Let T be a theory, not necessarily complete, in a language $L(r)$ obtained by adding a *new* relational predicate r to L. We say that r is *explicitly defined* in T if there is a formula f of L such that $T \models (\forall \vec{x})(f(\vec{x}) \leftrightarrow r(\vec{x}))$, and that r is *implicitly defined* in T if, for every model M of T_L, the set of sentences of T in only the language of L (do not forget that T is closed under deduction), there is at most one relation r with universe M such that (M, r) is a model of T. Explicit definition clearly implies implicit definition:

In the case of explicit definition, every model of T_L can be transformed into a model of T by interpreting r as $f(\vec{x})$, and that is the only possibility.

Theorem 9.3 (Beth's Theorem). *The theory T, in the language $L(r)$, defines r explicitly by a formula of L if and only if it defines it implicitly.*

Proof. First, suppose that there are formulas $r_1(\vec{x}), \ldots, r_m(\vec{x})$ of L such that $T \models (\forall\vec{x})(r(\vec{x}) \leftrightarrow r_1(\vec{x})) \vee \cdots \vee (\forall\vec{x})(r(\vec{x}) \leftrightarrow r_m(\vec{x}))$. If $m = 1$, then r is defined explicitly. So assume that $m \geq 2$ and that r is defined implicitly. Consider the sets of axioms $T_1, \ldots, T_m$, in the language L, obtained by replacing r everywhere in T by $r_1, \ldots, r_m$, respectively.

By implicit definition of r, we can see that $T_1 \cup T_2 \models (\forall\vec{x})(r_1(\vec{x} \leftrightarrow r_2(\vec{x}))$; therefore, by compactness, there is a formula $f(r)$ of T such that $f(r_1) \wedge f(r_2) \models (\forall\vec{x})(r_1(\vec{x}) \leftrightarrow r_2(\vec{x}))$, where $f(r_1)$ and $f(r_2)$ denote the results of replacing r in $f(r)$ by r_1 and r_2, respectively.

Repeating the argument for the other pairs, we finally get a formula $g(r)$ of T such that:

- in every model of T, $g(r_1) \vee \cdots \vee g(r_n)$ is true;
- if $g(r_i)$ is true, the only possible interpretation of r is r_i: $(\forall\vec{x})(r \leftrightarrow r_i)$ is true;
- obviously, if we have $g(r_i) \wedge g(r_j)$, we also have $(\forall\vec{x})(r_i \leftrightarrow r_j)$, and in this case we see that $T \models (\forall\vec{x})(r(\vec{x}) \leftrightarrow \bigwedge_{1\leq i\leq n}(g(r_i) \to r_i(\vec{x}))$, which is an explicit definition of r.

In the other case, by compactness, there is a model (M, r) of T where r is not defined in M by a formula of L. By Svenonius's theorem, there is an elementary extension (M', r') of (M, r) with an automorphism s of M' that changes r'. Since (M', r') and (M', sr') are both models of T, r cannot be defined implicitly. □

If (M, r) is a finite structure, with no proper elementary extension, then Svenonius's theorem says that r is preserved by all automorphisms of M iff it is definable (without parameters!) in M. That is a particular case of the fundamental theorem of the "abstract Galois theory" of Marc Krasner, which establishes a duality between sets of relations closed under "logical operations" and their automorphism groups; in the case of a set with an infinite universe, infinitary languages need to be used.

Note also that if E is a finite set with n elements, and if G is any subgroup of the permutation group of E, then G is the automorphism group of the n-ary relation $G_{\vec{a}} = \{\vec{b} : \vec{b} = s\vec{a}, s \in G\}$, the G-orbit of an n-tuple $\vec{a}$ that bijectively enumerates E.

9.2 Compact, Saturated, Homogeneous, and Universal Models

We shall now introduce a battery of definitions. A model M of T is called:

- *κ-compact* if for every set of cardinality strictly less than κ consisting of formulas $f(\vec{a}, x)$ of $L(M \cup x)$ that is consistent with $T(M)$, there exists x in M such that (M, x) is a model of it; in other words, if every incomplete (but possibly complete!) type with parameters in M that is axiomatized modulo $T(M)$ by fewer than κ formulas is realized in M.
  By definition, every model is ω-compact: If $\{f_1(\vec{a}_1, x), \ldots, f_n(\vec{a}_n, x)\}$ is consistent with $T(M)$, then $M \models (\exists x)(f_1(\vec{a}_1, x) \wedge \cdots \wedge f_n(\vec{a}_n, x))$.

- *κ-saturated* if for every subset A of M of cardinality strictly less than κ, every type of $S_1(A)$ is realized in M.
  We again meet the definition of ω-saturated model from Chapter 5. A κ-saturated model is κ-compact: If π is an incomplete type of cardinality less than κ, then the parameters involved in it form a set A of cardinality strictly less that κ, and π can be completed as a type p of $S_1(A)$, which is realized in M by an element that a fortiori realizes π. Conversely, if $\kappa \geq |T|^+$, then a κ-compact model is κ-saturated, since the number of formulas of $L(A \cup \{x\})$ is $\max(|A|, |T|)$. The notion of a κ-saturated model is much more interesting than that of a κ-compact model.

- *saturated* if it is finite, or else of cardinality κ and κ-saturated.
  Note that a model of cardinality κ cannot be κ^+-compact, since the set of formulas $x \neq a$, with a ranging over M, is consistent. A saturated model therefore has the strongest saturation property that its cardinality allows.

- *κ-universal* if every model of T of cardinality less than *or equal to* κ is isomorphic to an elementary submodel of M. We sometimes call a model *universal* if it is κ-universal and of cardinality κ.

- *(weakly) κ-homogeneous* if, given two subsets A and B of M, of the same cardinality strictly less than κ, and of the same type (A and B are enumerated by the same ordinal strictly less than κ, and if we replace every a_α in $T(A)$ by the b_α of the same index, we get $T(B)$), then for every a in M there exists b in M such that $A \cup \{a\}$ and $B \cup \{b\}$ have the same type.
  The term (weakly) *homogeneous* is often used for ω-homogeneous; as this does not match the parallel conventions for saturated models, we shall avoid it.

- *strongly κ-homogeneous* if, given two subsets (or rather two α-tuples) of M, A and B, of the same type, of cardinality strictly less than κ, there is an automorphism s of M such that $sA = B$.
  If M is strongly κ-homogeneous, then it is weakly κ-homogeneous, since $A \cup \{a\}$ and $sA \cup \{sa\}$ have the same type.

We can therefore see that a finite structure has all the properties we could want of saturation, universality, and, by Theorem 9.1, homogeneity.

Note that if M is κ-compact, then for any ordinal α strictly less than κ, every incomplete α-type axiomatized by fewer than κ formulas is realized in M. Indeed, let F be a set of at most κ formulas of the form $f(\vec{a}, x_{\alpha_1}, \ldots, x_{\alpha_n})$, consistent with $T(M)$, with $\alpha_1, \ldots, \alpha_n < \kappa$. If it is finite, we can take the conjunction of all formulas of F, and if it is infinite, we can take its closure under finite conjunctions, which will not increase the cardinality. Now let F_0 be the set obtained by existentially quantifying all the x_α except x_0 in every formula of F (which is assumed to be closed under finite conjunctions). By κ-compactness, this set is realized by an element b_0 of M; let F_1 be the set obtained by replacing x_0 by b_0 in F, then existentially quantifying all the x_α except x_1. This is certainly a consistent set of fewer than κ formulas of $L(M) \cup \{x_1\}$, which is therefore realized by an element b_1 of M. Continue the process, successively realizing all the x_α by elements b_α.

For a similar reason, if M is κ-saturated, $A \subset M$, and $|A| < \kappa$, then every type of $S_\kappa(A)$ (and not just every type of $S_\alpha(A)$ for each $\alpha < \kappa$) is realized in M: We can realize the type of x_0 over A, then the type of x_1 over $A \cup \{x_0\}$, etc. realize the type of x_α over $A \cup \{x_0, x_1, \ldots, x_\beta, \ldots\}_{\beta<\alpha}$, which is indeed a set of cardinality strictly less than κ.

Furthermore, if M is weakly κ-homogeneous, and if A and B are subsets of M of the same type and of cardinality strictly less than κ, and if C is a subset of M of cardinality less than or equal to κ, then there exists $D \subset M$ such that $A \cup C$ and $B \cup D$ have the same type. (Note that two α-tuples have the same type if all corresponding finite subtuples have the same type: We can realize the variables of the type of D one by one.)

Now let us prove a few theorems.

Theorem 9.4. *Two weakly ω-homogeneous models are ∞-equivalent if and only if they realize the same (pure) types of $S_n(T)$ for every n.*

Proof. If M and N are ∞-equivalent, then for every $\vec{a}$ in M there is a $\vec{b}$ in N such that $(M, \vec{a})$ and $(N, \vec{b})$ are ∞-equivalent; $\vec{a}$ and $\vec{b}$ therefore have the same type, and M and N realize the same pure types.

Conversely, let M and N be weakly ω-homogeneous and realize the same pure types. Let $\vec{a}$ and $\vec{b}$ be of the same type, the former in M, the latter in N. (As M and N are models of the complete theory T, we could take $\vec{a} = \vec{b} = \emptyset$.) Add c to $\vec{a}$. Then there exist $\vec{b}'$ and d' in N such that $\vec{a}\frown c$ and

$\vec{b}^\frown d'$ have the same type. By homogeneity of N, there also exists d such that $\vec{b}^\frown d$ has the same type as $\vec{a}^\frown c$. □

In particular, we see that two denumerable, weakly ω-homogeneous models are isomorphic whenever they realize the same pure types.

Corollary 9.5. *A denumerable model that is weakly ω-homogeneous is strongly ω-homogeneous.*

Proof. If $\vec{a}$ and $\vec{b}$ have the same pure type in M, then the structures $(M, \vec{a})$ and $(M, \vec{b})$ are denumerable and weakly ω-homogeneous, and realize the same pure types, since a type in $(M, \vec{a})$ is in fact nothing other than a type of $S_n(\vec{a})$; they are therefore isomorphic. □

Theorem 9.6. *If $\kappa \geq |T|$, then a model M of T is κ-saturated if and only if it is weakly κ-homogeneous and κ-universal.*

Proof. A κ-saturated model M is κ-universal, since it realizes all types of κ-tuples. Let us check its homogeneity. Consider two subsets A and B of M, of the same type, with $|A| = |B| < \kappa$, and a in M, of type p over A. Let q be what we get by replacing the parameters of A by their counterparts in B in every formula of p. As A and B have the same type, if $p \models f(x, \vec{a})$, then $M \models (\exists x)f(x, \vec{a})$, and $M \models (\exists x)f(x, \vec{b})$. Thus q is itself a consistent set. It is a type over B THAT, by saturation, is realized by an element b of M. From this we get homogeneity.

Now let M be κ-homogeneous and κ-universal. Let A be a subset of M, $|A| < \kappa$, and let p be a type over A, realized by an element a of an elementary extension of M. By the Löwenheim–Skolem theorem, we can realize, by $A'^\frown a'$ (by which we mean, using a natural abuse of notation, an enumeration of A' followed by a'; it is a more or less equivalent notation—the distinction is psychological—to $A' \cup \{a'\}$, which we used earlier), the type of $A^\frown a$ over $\emptyset$ in a model N of cardinality at most κ, which embeds elementarily into M by κ-universality. By κ-homogeneity, there exists a_1 in M such that $A'^\frown a'$ and $A^\frown a_1$ have the same type; a_1 realizes p. □

Theorem 9.7. *Two saturated models of the complete theory T, of the same cardinality κ, are isomorphic.*

Proof. Let $M = \{\ldots, a_\alpha, \ldots\}_{\alpha<\kappa}$ and $N = \{\ldots, b_\alpha, \ldots\}_{\alpha<\kappa}$ be our two models. We are going to construct, by induction on $\alpha \leq \kappa$, two sequences of injections, f_α from a subset of M into N, and g_α from a subset of N into M, such that the domain of f_α contains $\{\ldots, a_\beta, \ldots\}_{\beta<\alpha}$ and has cardinality at most $2 \cdot |\alpha|$, and that the domain of g_α contains $\{\ldots, b_\beta, \ldots\}_{\beta<\alpha}$ and has cardinality at most $2 \cdot |\alpha|$, and such that if $\beta < \alpha$, then f_α extends f_β and g_α extends g_β, and such that f_κ and g_κ are two isomorphisms, inverses of each other, from M to N and from N to M, respectively. We proceed as follows:

- If α is a limit, then f_α is the limit of the f_β for $\beta < \alpha$, that is to say, the common extension of the f_β on the union of their domains, and g_α is the limit of the g_β; in particular, $f_0 = g_0 = \emptyset$.

- If $\alpha = \beta + 1$, we have already constructed f_β and g_β. The domain of f_α is the image of g_β, augmented by a_α if it is not already in there. If $a \in \mathrm{im}(g_\beta)$, then $f_\alpha(a) = g_\beta^{-1}(a)$; we take $f_\alpha(a_\alpha)$ to be an element b of N such that $\mathrm{im}(g_\beta) \cup \{a_\alpha\}$ and $\mathrm{dom}(g_\beta) \cup \{b\}$ have the same type. We then construct g_α, whose domain is $\mathrm{im}(f_\alpha) \cup \{b_\alpha\}$, by setting $g(b) = f^1(b)$ if $b \in \mathrm{im}(f_\alpha)$, and taking $g_\alpha(b_\alpha)$ to be an element a of M such that $\mathrm{im}(f_\alpha) \cup \{b_\alpha\}$ and $\mathrm{dom}(f_\alpha) \cup \{a\}$ have the same type.

Everything has been done to ensure that $f_\kappa \circ g_\kappa = Id_N$ and $g_\kappa \circ f_\kappa = Id_M$; since these functions preserve types, they conserve, a fortiori, satisfaction of quantifier-free formulas. □

We can see, as a consequence of this theorem, that a saturated model of cardinality κ (i.e., κ-saturated of cardinality κ) is strongly κ-homogeneous; indeed, if $A, B \subset M, |A| = |B| < \kappa$, and A and B have the same type, then (M, A) and (M, B) are two saturated models of the same complete theory with the same cardinality; they are therefore isomorphic.

Theorem 9.7 (and also Corollary 9.5) is actually a special case of a more subtle result, generalizing Theorem 9.4, concerning homogeneous models; everything depends on the following result, which, if you think hard about it, is quite bewildering.

Theorem 9.8. *A model M is weakly κ-homogeneous if and only if it is weakly ω-homogeneous and has the following saturation property: If A is a subset of M, $|A| < \kappa$, and p is a type over A all of whose restrictions to finite subsets of A are realized in M, then M realizes p.*

Proof. First, let us suppose that M is κ-homogeneous and prove, by induction on the cardinality of A, that it does have the property in question. This is obvious if A is finite. Express A, by indexing it by its own cardinal, as a union of a continuous increasing sequence A_α, with $|A_\alpha| < |A|$ (A_α consists of the elements of A of index *strictly* less than the ordinal α). By the induction hypothesis, the restriction p_α of p to A_α is realized by an element a_α of M. By induction on α, we construct an increasing sequence A'_α of subsets of M such that $a_0{}^\frown A'_\alpha$ has the same type over $\emptyset$ as $a_\alpha{}^\frown A_\alpha$, and thus the same as $a_\beta{}^\frown A_\alpha$, for $\beta > \alpha$. At limit stages, everything is done, and at successor stages, we use κ-homogeneity to realize the type over $a_0{}^\frown A'_\alpha$ corresponding to that of $A_{\alpha+1}$ over $a_{\alpha+1}{}^\frown A_\alpha$. At the end we obtain A' of the same type over $\emptyset$ as A, such that a_0 realizes the type p' corresponding to p. By κ-homogeneity, p must be realized by an element a of M.

For the converse, I consider A and A' in M, of the same type over $\emptyset$, $|A| = |A'| < \kappa$, and an element a of M, of type p over A. Let p' be the type over A' corresponding to p. If $\vec{b}$ is a finite subset of A, corresponding to $\vec{b}'$ in A' (we are really more interested in tuples than in subsets!), then, by ω-homogeneity, the restriction of p' to $\vec{b}'$ is realized in M. Considering the property of M, p' is therefore also realized in M. □

Corollary 9.9. *If M is weakly κ-homogeneous, it realizes a κ-type over $\emptyset$ whenever it realizes each of its fragments involving only finitely many variables.*

Proof. List the variables $x_0, \ldots, x_\alpha, \ldots, \alpha < \kappa$ of our κ-type. Suppose we have succeeded in realizing the type of the first α of the $x_0, \ldots, x_\beta, \ldots, \beta < \alpha$, in M, by A_α. In order to be able to continue, and to realize the type p_α of x_α over A_α, it is sufficient, by Theorem 9.8, to check that each of its finite restrictions is realized in M. That is a consequence of ω-homogeneity and of the fact that the type of $(x_{\beta_0}, \ldots, x_{\beta_n}, x_\alpha)$ is realized somewhere in M! □

Corollary 9.10. *A model M is κ-saturated if and only if it is weakly κ-homogeneous and realizes all types of n-tuples.*

Proof. Consequence of Theorems 9.8 and 9.6 (for the direction needed, we do not need the hypothesis $\kappa \geq |T|$). □

Corollary 9.11. *Two weakly κ-homogeneous models of T (which is complete) of cardinality κ are isomorphic whenever they realize the same types of n-tuples; in fact, they are strongly κ-homogeneous.*

Proof. Repeat the back-and-forth argument of Theorem 9.7, noting that the two models realize the same types of κ-tuples. For strong homogeneity, if A and B are two subsets of M of the same type, with $|A| = |B| < \kappa$, then note that (M, A) and (M, B) are two κ-homogeneous structures realizing the same types of n-tuples, and are therefore isomorphic. □

We approach the question of the existence of saturated models in the next section (Theorem 9.15); it will not really be cleared up until Chapter 14, when we know what a stable theory is. Nevertheless, we can now prove a simple result:

Theorem 9.12. *A theory T has a denumerable saturated model if and only if for every natural number n, $S_n(T)$ is finite or denumerable.*

Proof. As a denumerable model can realize only denumerably many types, the condition is necessary.

Conversely, we know from Chapter 5 that T has an ω-saturated model N. Note that for every tuple $\vec{a}$ of parameters, $S_n(\vec{a})$ is denumerable, since the type of $\vec{b}$ over $\vec{a}$ is the same thing as the type of $\vec{a}\frown\vec{b}$ over $\emptyset$.

There is therefore a denumerable subset A_0 of N in which all pure types are realized. For every finite subset $\vec{a}$ of A_0, $S_n(\vec{a})$ is denumerable, and there are only denumerably many finite subsets of A_0. We can therefore realize all types over finite subsets of A_0 in a denumerable subset A_1 of N.

Continuing, we construct a sequence $A_0 \subset A_1 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots$ of denumerable subsets of N, such that every type over a finite subset of A_n is realized in A_{n+1}; their union M is an elementary submodel of N (it passes Tarski's test) and is clearly an ω-saturated, denumerable model of T.

Note that we have not made any assumptions about the cardinality of T: See Lemma 9.18. □

9.3 Resplendent Models

Consider a model M of T and a sentence $f(r_1, \ldots, r_k)$ in a language consisting of the symbols of the language L of T, the elements of M, and new symbols, for example all relations. (We can reduce to this case, by introducing functions through their graphs.) We shall say that this sentence is consistent with $T(M)$ if $T(M) \cup \{f(r_1, \ldots, r_k)\}$ is consistent, in other words, if there is an elementary extension N of M on whose domain we can interpret $r_1, \ldots, r_n$ by relations is such a way as to satisfy $f(r_1, \ldots, r_n)$.

We shall call the model M *resplendent* if, for every sentence $f(r_1, \ldots, r_k)$ consistent with $T(M)$, there are interpretations of $r_1, \ldots, r_k$ on the domain of M satisfying f. For example, if the language L includes only finitely many relation, function and constant symbols, a resplendent model is strongly ω-homogeneous, since, by Theorem 9.1, if $\vec{a}$ and $\vec{b}$ have the same type, it is consistent to say that there is an automorphism s sending one to the other, a property that can be expressed by a sentence f.

The following result will help to show the existence of resplendent models.

Theorem 9.13 (Disjoint Consistency Lemma). *Let T_i be a family of theories, each in a language L_i. Suppose that the common intersection L of all these languages is also the intersection of any two of them, and let $T_{i,L}$ be the set of consequences of T_i in just the language L. Then, if the union of the $T_{i,L}$ is consistent, the union of the T_i is also consistent.*

Proof. First, assume that there are only two theories T_1 and T_2, in languages L_1 and L_2 with intersection L. Note that if T is a (consistent!) theory, in only the language L, which includes $T_{1,L}$, then $T_1 \cup T$ is consistent. For if that were not the case, we could find f in T_1 and g in T by compactness such that $\{f, g\}$ is inconsistent; in other words, every model of f is a model of $\neg g$, so $f \to \neg g$; but as g is in the language L, $\neg g \in T_{1,L}$, and $T_{1,L}$ cannot be included in T.

So if we take T to be a complete theory, in the language L, containing $T_{1,L} \cup T_{2,L}$, we get a model M of T_1 and a model N of T_2, whose underlying L-structures, being models of the complete theory T, are elementarily equivalent.

If we were allowed to use Shelah's theorem (see the remark after Lemma 4.12) stating that in this case the two L-structures have a common ultrapower, that M^U and N^U are isomorphic L-structures for some ultrafilter U, then we could immediately construct a model of $T_1 \cup T_2$. But as we have not considered how to prove this theorem, we are going to proceed in a less elegant manner. By Lemma 4.12, the underlying L-structure M_L of M can be embedded elementarily in the L-structure of an ultrapower $N_1 = N^U$ of N, and the L-structure of N_1 can in turn be embedded in that of an elementary extension of M_1. Repeating this process ω times, we get the following three elementary chains, the first with respect to the language L, the second to L_1, and the third to L_2:

$$M_L \prec N_{1,L} \prec M_{1,L} \prec \cdots \prec N_{n,L} \prec M_{n,L} \prec N_{n+1,L} \prec \cdots;$$

$$M \prec M_1 \prec \cdots \prec M_n \prec \cdots; \quad N \prec N_1 \prec \cdots < N_n \prec \cdots.$$

The limit M_ω of the M_n and the limit N_ω of the N_n have the same underlying L-structure; we thus get a model of $T_1 \cup T_2$.

If there are only finitely many theories T_i, we can show, step by step, that their union is consistent. The general case follows by compactness. □

This result lets us clear up a little detail that might have bothered an overly perspicacious reader: If the formula $f(r_1, \ldots, r_n)$ mentions only constants in some subset A of M, then in order for it to be consistent with $T(M)$, it need only be consistent with $T(A)$. Also:

Theorem 9.14. *If $\kappa \geq |T|$, and if M is a model of T of cardinality κ, then it has resplendent elementary extension of the same cardinality.*

Proof. Consider a copy of each sentence $f(r_1, \ldots, r_n)$, with parameters in M, that is consistent with $T(M)$; by "a copy" I mean that we avoid repeating formulas that differ only by virtue of the names of their new symbols. As the new symbols can for now be taken as part of a fixed denumerable supply, the language L of T has cardinality at most κ, and there are only κ elements of M, it follows that the number of sentences to consider is κ. Now, make the languages of each of these sentences disjoint, in such a way that no two of them have any common new symbols, other than those of $T(M)$: For that purpose we need a language L_1 extending L, but still of cardinality κ, and with these sentences we get a set T_1 containing $T(M)$ that is consistent by the disjoint consistency lemma.

By Löwenheim's theorem, this theory has a model M_1 of cardinality κ; as an L-structure, this is a model of $T(M)$, that is, an elementary extension of M, and every sentence consistent with $T(M)$ has a realization there.

We can then apply the same process to the structure M_1 and find an L_2-structure M_2 that, as an L_1-structure, is an elementary extension of M_1, and allows the interpretation of each copy of a sentence consistent with $T(M_1)$; then we repeat the process.

We thus obtain a sequence of structures M_n of cardinality κ in a language L_n that increases every time, always remaining of cardinality κ; seen as L-structures, this chain is elementary (it is, in fact, elementary in L_n from n

on), and its limit M_ω is an L-structure that is a resplendent extension of M of cardinality κ; it is even resplendent as an L_ω-structure. □

More generally, we shall say that a model is κ-resplendent if for every theory Θ in a language comprising as many symbols of the language L of T as we want, but strictly fewer than κ constant symbols taken from M, and strictly fewer than κ new (relation, function, or constant) symbols, then it is possible to interpret the new symbols on the domain of M in such a way as to have a model of Θ. For example, a κ-resplendent model is κ-saturated, since a type over A is a theory in $L(A, x)$; by Theorem 9.1, it is strongly κ-homogeneous.

Theorem 9.15. *If $|T| \leq \kappa$, then every model of T of cardinality less than or equal to 2^κ embeds elementarily into a κ^+-resplendent model of cardinality 2^κ.*

Proof. This is essentially the same proof as for the previous theorem. If the theories whose new vocabularies are disjoint are consistent with $T(M)$, then so is their union. As there are $(2^\kappa)^\kappa = 2^{\kappa\times\kappa} = 2^\kappa$ subsets of 2^κ of cardinality κ, once we have fixed a subset A of M of cardinality κ and added κ new relational symbols of arity n for each n (which makes altogether $\kappa \times \omega = \kappa$ new symbols!), this new language has κ formulas and yields no more than 2^κ theories consistent with $T(M)$. We therefore need consider only $2^\kappa \times 2^\kappa = 2^\kappa$ theories, whose vocabularies are disjoint, giving a language L_1 of cardinality 2^κ; by the Löwenheim–Skolem theorem, we get a model M_1 of the theory T_1 under consideration.

In the second stage, we do not completely apply the same construction to M_1, since L_1 has become too big, and we would be required to consider more than 2^κ new theories. It is enough to examine those that involve only symbols of the language L (not L_1), fewer than κ constants in M_1, and fewer than κ new symbols. We get a theory T_2 in a language L_2, whose intersection with L_1 reduces to L; in these conditions, given that the vocabularies are disjoint, the consistency of T_2 with the theory of the L-structure M_1 implies the consistency of T_2 with the theory of the L_1-structure M_1. We thus get an $L_1 \cup L_2$-structure M_2 that as an L_2-structure is a model of T_2, as an L_1- structure is an elementary extension of M_1 (we do not need to lose the results of the first stage!), and has cardinality 2^κ.

We then repeat this construction κ^+ times, the model constructed at each stage being an elementary extension for all the languages introduced up to that point, but allowing the interpretation of the new theories only with respect to the language L; at the end we get a κ-resplendent model of cardinality $\kappa^+ \times 2^\kappa = 2^\kappa$. □

Exercise 9.16. Show that if $|T| \leq \kappa$, every model of T of cardinality κ has a strongly ω-homogeneous elementary extension of the same cardinality.

If $\lambda < \kappa$, a κ-saturated model is also λ-saturated; Theorem 9.15 therefore tells us that every model has a λ-saturated extension (and we can even bound the cardinal; this is left as an exercise for the reader).

We therefore have κ^+-saturated models of cardinality 2^κ; as there can be 2^κ types over a set of parameters of cardinality κ, we cannot do better. If we accept the generalized continuum hypothesis, then $\kappa^+ = 2^\kappa$, and we get a saturated model (unique up to isomorphism) of cardinality 2^κ. But without set-theoretic hypotheses, it is impossible to prove the existence of a saturated model for an arbitrary theory T; however, there is one if the theory T has certain stability properties, which we shall see in Chapter 14.

The resplendency properties of a model are often delicate to verify; the only generally applicable fact is that a saturated model is resplendent. More precisely:

Theorem 9.17. *A saturated model M of T of cardinality κ (i.e. κ-saturated of cardinality κ) is κ-resplendent.*

Proof. First, suppose that $\kappa > |T|$. Let Θ be a theory consistent with $T(M)$ in a language L' that involves only a set A, $|A| < \kappa$, of parameters in M, and that adds strictly fewer than κ new symbols to the language L of T. We enumerate M by its cardinal, $M = \{a_0, \ldots, a_\alpha, \ldots\}, \alpha < \kappa$, and construct, by induction on $\alpha < \kappa$, an elementary chain N_α of models of Θ, with $|N_\alpha| \leq \max(|L'|, |\alpha|) < \kappa$, whose underlying L-structure is elementarily embedded into M.

We start with a model N_0 of Θ of cardinality $|L'|$, given by the Löwenheim–Skolem theorem, whose L-structure, by κ-universality, embeds into M.

At the other limit stages α, we take N_α to be the union of the $N_\beta, \beta < \alpha$, already constructed; $|N_\alpha| \leq \Sigma_{\beta<\alpha} \max(|L'|, |\beta|) \leq |\alpha| \times \max(|L'|, |\alpha|) = \max(|L', |\alpha|)$, so the hypothesis on the cardinality of N_α is indeed satisfied.

At a successor stage, note that the type of a_α over N_α, *in the sense of* L, is realized in an extension $N_{\alpha+1}$ of N_α of cardinality $\max(|L'|, |\alpha|)$, an extension whose L-structure can be embedded into M by κ-universality; I mean that we can realize, in M, the type, in the sense of L, of $N_{\alpha+1}$ over $N_\alpha \cup \{a_\alpha\}$.

If we define N_κ as the limit of the N_α, which is a model of Θ, we can note that it contains every a_α, that its L-structure is all of M: We have succeeded in turning M into a model of Θ.

For the case in which $\kappa = |T|$, we proceed in somewhat the same way, taking one minor precaution: as it is essential that the N_α be of cardinality strictly less than κ, we introduce the language L' (in fact, the language L) only step by step; that is to say, we take N_α to be, not a model of Θ, but a set of parameters (in the sense of this theory) that satisfies Tarski's test for the first α formulas. That works even in the denumerable case, in which, thanks to a renumbering analogous to what we did for Henkin's method, we can arrange to construct our model N_ω, step by step, as a limit of finite sequences.

Finally, if $\kappa < |T|$, the following lemma shows that there is a subset L_1 of L, of cardinality κ, that is sufficient to interpret every formula of T: we can reduce the problem to the case $\kappa = |T|$. □

Lemma 9.18. *If T has a saturated model of cardinality κ, then there is a fragment L_1 of the language L of T, of cardinality at most κ, such that every formula of L is equivalent, modulo T, to a formula of L_1.*

Proof. As a κ-saturated model realizes all types of n-tuples, $|S_n(T)| \leq \kappa$. For each pair p, q of distinct types in $S_n(T)$, we choose a clopen set $O_{p,q}$ that contains p and does not contain q: There are no more than κ such sets $O_{p,q}$.

Let us show that a clopen set O can be expressed as a Boolean combination of finitely many of the $O_{p,q}$. Let p be in O. The complement of O is covered by the $O_{q,p}$, where q ranges over the $\neg O$, and, by compactness, finitely many of them are sufficient for this covering. Taking the intersection of their complements, we get a clopen set O_p containing p and included in O; again by compactness, O is a union of finitely many of the O_p.

As a consequence, every formula in n free variables is a Boolean combination of formulas defining the $O_{p,q}$, from which the result follows.

Topologically, this means nothing more than that an infinite 0-dimensional compact space does not have more clopen sets than points, and, by the Stone representation theorem (6.19), that a Boolean algebra of cardinality κ has at least κ ultrafilters. □

9.4 Properties Preserved Under Interpretation

A structure N is called *interpretable*, or *definable*, in a structure M if the following conditions are satisfied:

- the universe of N is composed of a definable subset A of M^n for some n (i.e., the elements of A are tuples satisfying a certain formula of L), modulo an equivalence relation E definable in M (equivalence becoming equality in N);
- every m-ary relation r of N is definable in M in the following way: There is a formula $f(\vec{x}_1, \ldots, \vec{x}_m)$ of L such that $M \models f(\vec{a}_1, \ldots, \vec{a}_m)$ iff the tuples $\vec{a}_1, \ldots, \vec{a}_m$ are in A and their equivalence classes $c_1, \ldots, c_m$ form a tuple of N satisfying r;
- every function and every constant of N has a graph defined as above.

As an example, a definable subgroup of a group A, or a quotient of G by a definable normal subgroup, is interpretable in G. We have already had the occasion to consider such examples of interpretation in arithmetic.

It was much simpler then, because, aside from finite structures, we could always assume that the universe of N was all of ω, and that the relation E was true equality: We have a definable bijection between ω and ω^n at our disposal; if A is infinite and definable, its "enumeration function" is a definable bijection between A and ω; and finally, in arithmetic, every definable equivalence relation has a definable choice function, obtained by taking the element of each class with the lowest code.

If T is the theory of M, and if N is interpretable in M, we say, by way of abuse, that the theory T' is interpretable in T.

If the formulas that serve to define the structure N have parameters in M, we specify (at least if we have not adopted any contrary convention) *"interpretable with parameters in M"*.

Define a *"good model"* of T' to be a model that yields a model of T in the same way that N yields M. Not every model of T' is good, but they all have good elementary extensions, since an ultrapower of a good model is good. The good models of T' form what is called a *"pseudo-elementary class,"* that is, the class of reducts to a language L of models of a theory Θ in a language L' extending L.

Lemma 9.19. *Let N be interpretable (even with parameters) in M; let Θ' be a theory that is consistent with $T'(N)$, and let Θ be the theory obtained by replacing the symbols of the language of N by their interpretations in M and relativizing all quantifiers to the universe of N. (We can represent a class modulo E by one of its elements: Replace $(\exists x)$ by $(\exists x \in A)$ and $(\forall x)$ by $(\forall x \in A)$; we also need to replace equality by E.) Then Θ is consistent with $T(M)$.*

Proof. We can assume that the language of $T(M)$ and the language of $T'(N)$ are disjoint. So let T_1 be the theory, in the common language, which consists of $T(M), T'(N)$, and the interpretation of N in M. By the disjoint consistency lemma, $T_1 \cup \Theta'$ has a model; this model is also a model of Θ!

By repeating the proof of the disjoint consistency lemma, we can also sandwich the models of $T'(N) \cup \Theta'$ with good models of $T'(N)$. □

As a consequence, if κ is greater than the cardinality of the set of parameters used in the interpretation of N, and if M is κ-resplendent, κ-saturated, or κ-universal, then so is N, since a theory consistent with $T'(N)$ corresponds to a theory consistent with $T(M)$, and a type in the sense of T' to an incomplete type in the sense of T.

In contrast, we have no hope of seeing homogeneity preserved under interpretation, since two elements of the same type in N may have different types in M, which is richer; for example, if we introduce a unary predicate $A(x)$, and consider the theory T saying that A and its complement are both infinite, then every model of T is strongly ω-homogeneous; if we take T' to be the theory of the equivalence relation $A(x) \leftrightarrow A(y)$, then a model of T' is strongly ω-homogeneous only if both its classes have the same cardinality.

Note that if N is a κ-resplendent model of T', where $\kappa > \max(|T|, |T'|)$, it is good. This is obvious if N is finite, since it is the only model of T'; if it is infinite, it is consistent to assume that there exists a model M_1 of T on the universe of N whose associated model N_1 is isomorphic to N.

Finally, note that if the languages comprise only finitely many relation and function symbols, and if T is finitely axiomatizable, the same argument proves that the resplendency of N implies its goodness.

9.5 Recursively Saturated Models

The notion of recursively saturated model is an incongruous mix of recursion theory and model theory; if it is discussed here it is because, in the case of a denumerable model, this notion coincides with that of a resplendent model, which consequently has the same shortcoming, in spite of first appearances. Moreover, the proof of this fact is a good exercise in the construction of models.

Consider a *finite* language L, i.e., one consisting of only finitely many relation, function, and constant symbols; this restriction to finite languages is essential for the theorems below.

Now let M be an L-structure, and let $\vec{a} = \{a_1, \dots, a_n\}$ be a tuple from M. We can associate each primitive symbol of the language $L(a_1, \dots, a_n, x)$ with a natural number, and code the sentences of the language by natural numbers, as we did in Chapter 7 for the sentences of the language of arithmetic. *We shall say that the model M is recursively saturated* if for every tuple $\vec{a}$ from M, and every recursive set of formulas $f(\vec{a}, x)$ that is consistent with $T(M)$, there is an element x of M that satisfies all these formulas. By "recursive set of formulas" we really mean "set of formulas whose set of codes is recursive"; we can see that changing the codes of $a_1, \dots, a_n$, for example by permuting them, amounts to performing a recursive operation on the codes of the formulas, and that does not change the recursive or nonrecursive character of the set of formulas considered. Note also that by the pleonasm theorem (7.28), it does not matter whether we take recursive or recursively enumerable sets of formulas.

This notion of recursively saturated model is more treacherous than it seems: The fact that some recursive set of formulas with parameters in M is or is not consistent often allows the model M to code information generally overflowing the limits of recursiveness. We will not forget that the recursive types realized are *incomplete* types with parameters in $\vec{a}$: In the hypothesis of the definition, it is said that the set of the $f(\vec{a}, x)$ is recursive and consistent with $T(\vec{a})$, not that it is complete.

Lemma 9.20. *A recursively saturated model is weakly ω-homogeneous.*

Proof. Let $\vec{a}$ and $\vec{b}$ have the same type in M, which is assumed to be recursively saturated; let c be in M. As $\vec{a}$ and $\vec{b}$ have the same type, they satisfy the same formulas of the form $(\exists z)f(\vec{y}, z)$. Consequently, the set of formulas $f(\vec{a}, c) \to f(\vec{b}, x)$, which is recursive, is consistent; it is therefore satisfied by an element d of M, and $\vec{a}\frown c$ and $\vec{b}\frown d$ have the same type. $\square$

Theorem 9.21. *For a finite language L, the infinite recursively saturated L-structures form a pseudo-elementary class; more precisely, there is a sentence $f(R)$, in a language $L \cup \{R\}$, such that M is recursively saturated if and only if R can be interpreted in it in such a way as to satisfy $f(R)$.*

Proof. Since in this context finitely many relations $r_1, \ldots, r_m$ of arities $n_1, \ldots, n_m$ respectively, can be replaced by a single one, the product $R = r_1 \times \cdots \times r_n$, of arity $n_1 + \cdots + n_m$, we will allow ourselves to add finitely many symbols to L, namely:

- three unary relational symbols $E(x), A(x), U(x)$;
- two unary functions $y = i(x), y = \lambda(x)$;
- four binary functions $z = x + y, z = x \cdot y, z = c(x, y), z = t(x, y)$;
- a binary relation symbol $\mathrm{Sat}(x, y)$.

The sentence $f(R)$ is obtained by taking the conjunction of the following sentences, which I shall describe informally:

1. We state that E, A, and U are disjoint, and that i is a bijection from M onto E. This manipulation is not indispensable; its object is to clarify the construction. Here E is for "element," A for "arithmetic," U for "tuple" ["uple" in French].

2. We express the fact that the sum and product, restricted to A, form a model of minimal arithmetic, augmented by finitely many axioms guaranteeing the goodness of the relation $\in$, and (primitive recursive) functions that occur in the coding of formulas (it is enough to say that a certain number of functions are total).

3. The function λ goes from U to A, sending a tuple to its "length." The function c goes from $A \times U$ to E; if n is in A and strictly less than the length of u, then $c(n, u)$ will be called the "nth coordinate of u." We state that two tuples of the same length and the same coordinates are equal, that the (empty) 0-tuple exists, and that for every tuple u and every element e of E, the tuple $u\frown e$ exists. This last axiom guarantees that all the *true* n-tuples of elements of E, where n is *standard*, are indeed represented in U.

4. As the language L is finite, we can code its formulas in our arithmetic. We have at our disposal the notion of a formula $f(\vec{x})$, without parameters, and of everything that revolves around it. The predicate Sat, called satisfaction, is a relation between a formula f and a tuple u, the length of u corresponding to the number of free variables of f.

We add the following axioms, where f and g represent formulas in just the language L (it is essential that they involve neither Sat nor t):

- for every primitive symbol r of L (there are only finitely many), $(\forall e_1)\cdots(\forall e_n)(\mathrm{Sat}(r(\vec{x}),(e_1,\ldots,e_n)) \leftrightarrow r(i^{-1}(e_1),\ldots,i^{-1}(e_n)))$
- $(\forall f \in L)(\forall u)(\mathrm{Sat}(f,u) \leftrightarrow \neg\mathrm{Sat}(\neg f,u))$
- $(\forall f \in L)(\forall g \in L)(\forall u)(\forall v)(\mathrm{Sat}(f \wedge g, u\frown v) \leftrightarrow (\mathrm{Sat}(f,u) \wedge \mathrm{Sat}(g,v)))$
- $(\forall f \in L)(\forall g \in L)(\forall u)(\forall v)(\mathrm{Sat}(f \vee g, u\frown v) \leftrightarrow (\mathrm{Sat}(f,u) \vee \mathrm{Sat}(g,v)))$
- $(\forall f \in L)(\forall u)(\mathrm{Sat}((\exists x)f,u) \leftrightarrow (\exists e)\mathrm{Sat}(f,u\frown e))$
- $(\forall f \in L)(\forall u)(\mathrm{Sat}((\forall x)f,u) \leftrightarrow (\forall e)\mathrm{Sat}(f,u\frown e))$

We can see that these finitely many axioms ensure that for every *standard* formula $f(\vec{x})$ and every *true* tuple $\vec{a}$ of E, $\mathrm{Sat}(f(\vec{x}),\vec{a})$ is true if and only if $f(\vec{a})$ is true in the structure that is the image of M under the map i onto E.

5. The pleonasm theorem, Theorem 7.28, authorizes the replacement of a Σ_1 list of axioms by a Δ_1 list, which is even a set defined by an arithmetic formula with quantifiers bounded by two or three functions whose goodness our axioms guarantee. By coding the arithmetic formulas, we can enumerate these sets, say $F_0,\ldots,F_n,\ldots$; *this list continues beyond the standard integers.* The last axiom determines the behavior of the "witness function" t: It states that for every n, every m, and every tuple $\vec{a}$, if there is a b such that the tuple $\vec{a}\frown b$ satisfies the first m formulas $f(\vec{x},y)$ of F_n of appropriate arity, then $t(n,\vec{a})$ is one such b. It is important that m not occur in the definition of the witness function.

Our list of axioms is now finished. We can easily see that a recursively saturated model M can be turned into a model of $f(R)$: Begin by taking a bijection i from M to a subset E of M, such that $|M| = |M - E|$; put the *standard* model of arithmetic on a denumerable subset A of the complement of E, and associate every remaining point, in a bijective manner, with an n-tuple of elements of E; the length and coordinate functions are interpreted in the standard way.

Next we interpret the predicate Sat by true satisfaction.

For the witness, we distinguish two cases:

- The set F_n, with $\vec{a}$ as parameters, defines a consistent type. As it is recursive, it is realized in E by an element that we take as the witness $t(n,\vec{a})$.

- Otherwise, there is a greatest integer m such that the conjunction of the first m formulas of F_n, with $\vec{a}$ as parameters, can be realized by an element of E. We then take such an element as our witness.

Conversely, suppose that M can be turned into a model of $f(R)$. We have already noted that as far as formulas of standard complexity and tuples of standard length are concerned, our axiomatization forces the predicate Sat to represent *true* satisfaction; furthermore, if n and m are standard, then our model can think that $m \in F_n$ or that $m \notin F_n$ only if that is actually true, since F_n is defined by a formula whose quantifiers are bounded by functions whose goodness $f(R)$ guarantees, and our arithmetic cannot lie at this level.

So let $\vec{a}$ by a real tuple taken from E, and n a standard natural number, and consider the set π of formulas $f(\vec{a}, y)$, where $f(\vec{x}, y)$ ranges over F_n. Suppose that π is consistent in the real world. We distinguish two cases:

- The model thinks that for every m, there is y satisfying the first m formulas of π. In that case, $t(n, \vec{a})$ is a realization of π.
- The model thinks that π is inconsistent, that is to say, that there is a greatest m for which there exists y satisfying the first m formulas of π. As π is actually consistent, m must be nonstandard, and $t(n, \vec{a})$ is again a realization of π.

Consequently, M does realize all consistent types associated with F_n; by pleonasm, it is recursively saturated. □

Exercise 9.22. Consider a theory T, not necessarily complete, in a finite language L, with no finite models. Show that T is recursively axiomatizable if and only if there is a sentence $f(R)$ such that the models of T are the reducts, to L, of the models of $f(R)$.

Corollary 9.23. *A resplendent model is recursively saturated; moreover, if $f(R)$ is consistent with $T(M)$, then R can be interpreted on M in such a way that $f(R)$ is satisfied and that (M, R) is recursively saturated.*

Proof. If M is resplendent, then the sentence $f(R)$ of Theorem 9.21 is consistent with $T(M)$, if only because M has an ω-saturated extension, and $f(R)$ can be interpreted on it. For the second part, if $f(R)$ is consistent with $T(M)$, then so is $f(R) \wedge g(R, R')$, where $g(R, R')$ is the sentence guaranteeing the recursive saturation of $L \cup \{R\}$-structures. □

In contrast to Theorem 9.21, the resplendent models do not always form a pseudo-elementary class. Let us take T to be the theory of an equivalence relation that for every n has exactly one n-element class. The type stating that the class of a is infinite is recursive, and therefore realized in any recursively saturated model. It is consistent to assume that there is a bijection between the class of a and the model M; it is also consistent to assume the existence of a set A, in bijection with M, consisting of pairwise inequivalent elements. It is easy enough to see that a recursively saturated model must have infinitely many infinite classes.

Consequently, in a resplendent model M of cardinality κ, all the infinite classes have cardinality κ, and there must be κ of them, so M can be only the saturated model of cardinality κ. The reader will have no difficulty checking that a model is recursively saturated if and only if it contains infinitely many infinite classes.

So take M to be saturated of cardinality $\kappa > 2^\omega$ and U a nonprincipal ultrafilter of subsets of ω. In M^U there are classes of cardinality 2^ω, coming from the ultrapower of finite classes: It is therefore not saturated. But an ultraproduct can clearly not take us out of a pseudo-elementary class.

A model M is sometimes called *chronically resplendent* if whenever $f(R)$ is consistent with $T(M)$, we can interpret R so that $f(R)$ is satisfied and (M, R) is resplendent; the model constructed in Theorem 9.14 is chronically resplendent. One consequence of the last theorem of this section, due to Jean-Pierre Ressayre, is that a denumerable recursively saturated model is chronically resplendent; however, we know nothing general about chronic resplendency.

We can see that in Corollary 9.23, the finite character of the language L is absolutely necessary, since it allows us to express the adequacy of the satisfaction predicate by finitely many conditions. If we assume, for example, that the language contains denumerably many constant symbols $a_0, \ldots, a_n, \ldots$, and we consider the complete theory T stating that the a_n are all distinct, then we can see that every model of T is resplendent. Indeed, a sentence with supplementary symbols can involve only the first n of the a_i, and the reduct of any model of T to the language that mentions only the first n of the a_i is saturated; however, the prime model of T omits the recursive type stating that x is distinct from every a_n.

This theorem has a converse in the denumerable case:

Theorem 9.24. *If L is a language comprising only finitely many relation, function, and constant symbols, then every denumerable recursively saturated L-structure is resplendent.*

Proof. Let M be our denumerable recursively saturated model, which we enumerate: $M = \{a_0, a_1, \ldots, a_n, \ldots\}$. Let $f(\vec{a}, r_1, \ldots, r_m)$ be a sentence involving a tuple $\vec{a}$ of elements of M and new symbols $r_1, \ldots, r_m$, and that is consistent with $T(M)$, that is to say, with $T(\vec{a})$, by the disjoint consistency lemma (9.10).

We are going to carry out Henkin's construction for this sentence, always staying inside the model M. For that purpose, we add new constant symbols $b_{i,j}$ to the language $L(\vec{a}, r_1, \ldots, r_m)$, and consider an assignment of witnesses giving every formula whose parameters have their second index less than k a witness whose second index is k. We consider the theory consisting of $f(\vec{a}, r_1, \ldots, r_m)$, of the sentences identifying the n elements of $\vec{a}$ with the first n of the $b_{i,0}$, and of all sentences of the form $(\exists x)(g(\vec{b}, x) \to g(\vec{b}, b_{g(\vec{b},x)}))$, where $b_{g(\vec{b},x)}$ is the witness for $g(\vec{b}, x)$.

Note that this is a recursive set of formulas, and that it is consistent with $T(\vec{a})$. Take a Henkin enumeration of a denumerable model of $T(\vec{a}) \cup$

$\{f(\vec{a}, r_1, \ldots, r_m)\}$. As usual we change the enumeration of the $b_{i,j}$ by means of a recursive bijection from $\omega \times \omega$ onto ω, and they become $b_0, \ldots, b_n, \ldots$.

The set of the consequences of this recursive family of sentences is Σ_1, as is the set of its consequences that can be expressed in just the language $L(\vec{a}, b_0)$. As it is a recursively enumerable incomplete type, and is consistent with $T(\vec{a})$, it is realized by an element b'_0 of M. We can therefore choose the type of b_0 over $\vec{a}$ in the language $L' = L(r_1, \ldots, r_m)$ in such a way that its restriction to the language L is the type of b'_0 over a. We may assume that b_0 is b'_0, so that amounts to putting an L'-structure on a certain elementary extension of M. Once b_0 is interpreted as b'_0, the set of sentences that we need to satisfy, $f(\vec{a}, r_1, \ldots, r_m)$, and the Henkin sentences, is always recursive and *consistent with* $T(\vec{a}\frown b'_0)$.

We can therefore repeat the operation. At the nth stage, we have chosen $b'_0, \ldots, b'_{n-1}$ in M, so that the set consisting of $f(\vec{a}, r_1, \ldots, r_m)$ and the Henkin sentences is consistent with $T(\vec{a}, b'_0, \ldots, b'_{n-1})$. The set of consequences of these sentences in the language $L(\vec{a}, b'_0, \ldots, b'_{n-1}, b'_n)$ is recursively enumerable and consistent with $T(a, b'_0, \ldots, b'_{n-1})$, a fact that allows us to interpret b_n by an element b'_n of M.

Moreover, if b_n has the form $b_{i,0}$, it can be interpreted as any element of M; indeed, the type of a over $\vec{a} \cup (b'_0, \ldots, b'_{n-1})$ is consistent with $T(\vec{a}, b'_0, \ldots, b'_{n-1}) \cup \{f(a, r_1, \ldots, r_m)\}$, since, if we quantify a existentially in a finite fragment of this type, we get a consequence of $T(\vec{a}, b'_0, \ldots, b'_{n-1})$. As the Henkin sentences impose nothing on $b_{i,0}$, we can take a Henkin enumeration of a model of $f(\vec{a}, r_1, \ldots, r_n) \cup T(\vec{a}, b'_0, \ldots, b'_{n-1}, a)$, chosen so that $b_{i,0} = a$.

As a consequence, the elements $b_{i,0}$ will be interpreted as a_i during the process. In our Henkin enumeration, we have therefore finally interpreted all the $b_{i,j}$ by the elements of M, and we have left no element of M aside; that means that we have given M an L'-structure, which is a model of the sentence $f(\vec{a}, r_1, \ldots, r_m)$. □

9.6 Historic and Bibliographic Notes

Beth's theorem appeared in [Bet53] and Svenonius' theorem in [Sve59a]; this type of result had considered attraction for logicians in the 1950s. Abstract Galois theory saw the light of day in [Kra38], in which, in addition, Krasner described a gigantic algorithm for the elimination of quantifiers for algebraically closed fields in an infinite language, before the term was introduced.

The history of the notion of saturated model is rather obscure. Its prehistory is in [Hau14], in which saturated dense chains are described. The notions of κ-homogeneity and κ-universality seem to belong to [Jon56] and [Jon60], and also, for the denumerable case, to [Fra54a]. In any case, the uniqueness (assuming existence) and the homogeneity of the saturated model were well-established facts by the time of [MV62]; homogeneous models, and their characterization by the types of n-tuples that they realize, were studied by [KM67].

The term "resplendent model" was introduced by [BS76], but the notion was known before then, in particular from [CK73], in which the resplendency of the saturated model is left as an exercise for the reader! We sometimes say "relation-universal" for ω-resplendent. Lemma 9.13, on disjoint consistency, comes from [Rob56a].

Theorem 9.23 is folklore. The first proof of Theorem 9.24 is in [Res72]. As for Exercise 9.22, it comes straight out of [CV58].

For the unhealthy side of recursive saturation, consult [Poi84].

10
Prime Models

– Rzkd sdsd, bd sxod!
– Bah! On s'y habitue ...

M.R.

10.1 Omitting Types Theorem

In the preceding chapter we constructed models that were as rich as possible in types realized: In brief, it was a matter of amalgamating structures while controlling their cardinalities by the Löwenheim–Skolem theorem.

However, we understand that to construct different models, we need to find some omitting certain types, and to see under what condition we can realize one type while omitting another. For that purpose, we primarily use two sorts of techniques. In one case, in which the hypotheses of stability are necessary, we can completely master the situation; the study of stable theories will begin in the following chapter and continue until the end of the book. The other case, which we are studying here, and which relies on a topological property, is more generally relevant, but applies only if everything is *denumerable.*

Recall that a subset A of a topological space E is called *dense* if every nonempty open set of E contains a point of A. We can easily see that the intersection of two open dense sets, and therefore also of any finite number of open dense sets, is open and dense; we say that E has the *property of Baire* if for every denumerable family $O_1, \dots, O_n, \dots$ of open dense sets, the intersection of the O_n is still dense; in particular, it is nonempty if E is nonempty.

A subset A of such a space is called *fat*, or *comeager*, if it contains a denumerable intersection of open dense sets; it is called *meager* if its complement is fat, i.e., if it is contained in the denumerable union of closed sets with empty interior. A denumerable intersection of fat sets is fat, and a denumerable union of meager sets is meager.

When we want to show that there exist x with the property $P(x)$, we can construct an object satisfying P, or else show that the set of x that satisfy P is not empty: One possibility is to show that this set is a fat set in a topological space having the property of Baire.

Theorem 10.1. *Compact and locally compact (every point has a neighborhood base of compact sets) Hausdorff spaces have the property of Baire.*

Proof. First, let us show that a compact set E is locally compact. Let a be a point in E and let O be an open neighborhood of a. Then for every point b that is not in O, since the space is Hausdorff, there is an open neighborhood U_b of a and an open neighborhood V_b of b such that $U_b \cap V_b = \emptyset$. As we can cover the complement of O, which is closed and therefore compact, with open sets, finitely many of those open sets $V_1, \ldots, V_n$ are sufficient for the covering; the complement F of $V_1 \cup \cdots \cup V_n$ is a closed set contained in O, and is a neighborhood of a, since it contains the open set $U_1 \cap \cdots \cap U_n$.

Now let E be locally compact, let $O_1, \ldots, O_n, \ldots$ be a denumerable sequence of open dense subsets of E, and let U be a nonempty open subset of E. We need to find a point that is in U and in every O_n. Let a_1 be a point in $U \cap O_1$, which is a neighborhood of it. Then a_1 has a compact neighborhood K_1 that is contained in $U \cap O_1$, and K_1 contains an open set U_1 that contains a_1. Let a_2 be a point in $U_1 \cap O_2$, which has a compact neighborhood $K_2 \subset U_1 \cap O_2$; K_2 contains an open neighborhood U_2 of a_2, etc. At the $(n+1)$th stage, we choose a_{n+1} in $U_n \cap O_{n+1}$ containing a compact neighborhood K_{n+1} of a_{n+1} that contains an open neighborhood U_{n+1}.

The intersection of the decreasing family K_n of nonempty compact sets is nonempty and is included in U and in every O_n. □

Now we consider a theory T, *not necessarily complete, in a denumerable language.* The space $H(T)$ of Henkin enumerations of models of T is defined to be the following subspace of $S_\omega(T)$, or rather of $S_{\omega\times\omega}(T)$: As in Section 4.3, we consider variable symbols $a_{i,j}$ (written as a rather than x in order to stay in the same style), and an enumeration of formulas $f(\vec{a}, x)$ such that, if $\vec{a} \subset E_0 \cup \cdots \cup E_n$, where $E_k = \{a_{0,k}, \ldots, a_{n,k}, \ldots\}$, then the formula $f(\vec{a}, x)$ has a witness $a_{f(\vec{a},x)}$ in E_{n+1}. This assignment of witnesses is fixed once and for all. The Henkin space is then the *closed* subspace of

$S_{\omega\times\omega}(T)$ defined by the formulas $(\exists x)f(\vec{a},x) \to f(\vec{a}, a_{f(\vec{a},x)})$; if the formula $f(\vec{a},x)$ is true for some x, it is true for its witness.

As we can see, a point in this compact (because closed) subspace of $S_{\omega\times\omega}(T)$ corresponds to a Henkin enumeration of a finite or denumerable model of T; we no longer need to eliminate quantifiers by introducing the f^H because we know now the compactness theorem!

By induction on j we define the *package* associated with $a_{i,j}$ as follows:

- if $j = 0$ then $P(a_{i,0}) = \{a_{i,0}\}$;
- if $j \neq 0$ then $a_{i,j}$ is the witness of a unique formula $f(\vec{a},x)$ in which all the elements of $\vec{a}$ have second index strictly less than j; the package of $a_{i,j}$ is by definition the union of the packages of elements of $\vec{a}$ together with $a_{i,j}$.

We can see that $P(a_{i,j})$ is always a finite set.

Lemma 10.2. *Let F be a closed subset of $S_n(T)$ with empty interior, defined by the formulas $f_u(\vec{x})$. Then, no matter what the tuple $\vec{a}$ of $a_{i,j}$'s is, the formulas $f_u(\vec{a})$ define a closed subset of $H(T)$ with empty interior.*

Proof. Let $\langle g(\vec{b})\rangle$ be a nonempty clopen set in $H(T)$. We must prove that there is a point of $H(T)$ in $\langle g(\vec{b})\rangle \cap (\bigcup\langle\neg f_u(\vec{a})\rangle)$.

Consider the union of the packages of elements that figure in $\vec{a}$ and $\vec{b}$, and let $\vec{c}$ be the tuple of $a_{i,j}$'s that must be added to $\vec{a}$ and $\vec{b}$ to get this union. Let $F(\vec{a},\vec{b},\vec{c})$ be the conjunction of the formulas $g(\vec{b})$ and $(\exists x)f(\vec{\alpha},x) \to f(\vec{\alpha}, a_{f(\vec{\alpha},x)})$ for every witness $a_{f(\vec{\alpha},x)}$ that figures in $\vec{a}\frown\vec{b}\frown\vec{c}$; it is a formula that uses only $\vec{a}\frown\vec{b}\frown\vec{c}$ as parameters.

As $\langle g(\vec{b})\rangle \cap H(T)$ is nonempty, we can take an element from it; that constitutes a Henkin enumeration of a model M of T that is finite or denumerable and *that satisfies each of the the formulas* $(\exists x)f(\vec{\alpha},x) \to f(\vec{\alpha}, a_{f(\vec{\alpha},x)})$; consequently $M \models F(\vec{a},\vec{b},\vec{c})$, $M \models (\exists\vec{y})(\exists\vec{z})F(\vec{a},\vec{y},\vec{z})$, and the formula $(\exists\vec{y})(\exists\vec{z})F(\vec{a},\vec{y},\vec{z})$ defines a nonempty clopen set in $S_n(T)$.

This open set can therefore not be included in $\bigcap\langle f_u(\vec{x})\rangle$, and there is a denumerable model N of T, with a tuple $\vec{a}'$ satisfying $(\exists\vec{y})(\exists\vec{z})F(\vec{a}',\vec{y},\vec{z})$ and not satisfying all the f_u. Let $\vec{b}'$ and $\vec{c}'$ be in N such that $N \models F(\vec{a}',\vec{b}',\vec{c}')$. We can therefore take a Henkin enumeration of N such that the elements of $\vec{a}'\frown\vec{b}'\frown\vec{c}'$ have the same indices i,j as their counterparts in $\vec{a}\frown\vec{b}\frown\vec{c}$; that is possible because the formula F states precisely that we can take $a_{i,j}$ as the witness of the necessary formula.

This Henkin enumeration is a point in $H(T)$ that satisfies $g(\vec{b})$ and does not satisfy all the $f_u(\vec{a})$. □

Theorem 10.3 (Omitting Types Theorem). *Let T be a theory, not necessarily complete, in a denumerable language, and let A_n be a meager subset of $S_n(T)$ for every natural number n. Then there exists a model of T that omits every type of every A_n.*

Proof. Each A_n is contained in a denumerable union of closed sets $F_{n,m}(\vec{x})$ in $S_n(T)$ with empty interior. By the preceding lemma, whenever we replace $\vec{x}$ in $F_{n,m}$ by a tuple of variables as in the Henkin enumeration, we get a closed subset of $H(T)$ with empty interior. All the $F_{n,m}(\vec{a})$ form a denumerable family of closed sets with empty interior, and their union is meager. As $H(T)$ is compact and nonempty, the complement of this union is dense, and therefore nonempty, and a point in this complement is a Henkin enumeration of a model of T that omits every type of every A_n. □

One example of a closed set with empty interior is the set consisting of one nonisolated point: If T is denumerable, there is therefore a model of T that omits this type. On the other hand, if T is complete and if p is an isolated point, then it is the only one to satisfy some formula $f(\vec{x})$, and since $T \models (\exists \vec{x}) f(\vec{x})$, every model realizes p. Do not forget that the n-types realized in a model M of T form a dense subset of $S_n(T)$ *if T is complete.*

In the omitting types theorem, everything is denumerable, the language as well as the model constructed. That is due to the very nature of the property of Baire. Note that if T is the theory of algebraically closed fields of characteristic zero, the type "x is transcendental over $\mathbf{Q}$," which is the only non-isolated type of $S_1(\emptyset)$ (the others are all isolated by their minimal equations over $\mathbf{Q}$), is omitted in only one model of T, namely the algebraic closure of $\mathbf{Q}$; all the nondenumerable models realize it.

10.2 Prime Models, Atomic Models: The Denumerable Case

Here we consider a complete theory T.

A subset A of a model M of T is called *atomic* (understood: over $\emptyset$) if every n-tuple from A has an isolated type in $S_n(T)$; if $A \subset B \subset M$, we say that B is *atomic over A* if B is atomic in the sense of $T(A)$, i.e. if every tuple $\vec{b}$ from B has isolated type over $\vec{a}$. Note that A is atomic over A.

Lemma 10.4. *The type of $\vec{a}\frown\vec{b}$ is isolated if and only if the type of $\vec{a}$ is isolated and the type of $\vec{b}$ over $\vec{a}$ is isolated.*

Proof. Let $f(\vec{x}, \vec{y})$ be a formula isolating the type of $\vec{a}\frown\vec{b}$. Then the type of $\vec{a}$ is isolated by the formula $(\exists \vec{y}) f(\vec{x}, \vec{y})$, and the type of $\vec{b}$ over $\vec{a}$ by the formula $f(\vec{a}, \vec{y})$.

Conversely, let $g(\vec{x})$ be a formula isolating the type of $\vec{a}$ and let $h(\vec{a}, \vec{y})$ be a formula with parameters $\vec{a}$ isolating the type of $\vec{b}$ over $\vec{a}$. Then the type of $\vec{a}\frown\vec{b}$ is isolated by $g(\vec{x}) \wedge h(\vec{x}, \vec{y})$. □

Remark. The driving force behind the lemma is the fact that the function sending the type of $\vec{a}\frown\vec{b}$ to that of $\vec{a}$ is a continuous surjection from $S_{n+m}(T)$ onto $S_n(T)$ that is also open.

Lemma 10.5. *If A is atomic, then for every finite subset $\vec{a}$ of A, A is atomic over $\vec{a}$.*

Proof. If the type of $\vec{a}\frown\vec{b}$ is isolated, then the type of $\vec{b}$ over $\vec{a}$ is isolated. $\square$

Lemma 10.6. *If $A \subset B \subset C$, and if B is atomic over A and C is atomic over B, then C is atomic over A.*

Proof. Let $\vec{c}$ be in C and let $f(\vec{b}, \vec{x})$ be a formula with parameters is B isolating the type of $\vec{c}$ over B. Let $g(\vec{a}, \vec{y})$ be a formula with parameters in A isolating the type of $\vec{b}$ over A. Then the type of $\vec{c}$ over A is isolated by the formula $(\exists \vec{y})g(\vec{a}, \vec{y}) \wedge f(\vec{y}, \vec{x})$. $\square$

If $A \subset B \subset C$, if C is atomic over A, and if $B - A$ is infinite, then C is not necessarily atomic over B; for example, a model M of the theory of infinite sets (the language consists only of equality) is atomic only over its finite subsets.

Theorem 10.7. *An atomic model is weakly ω-homogeneous; two atomic models of the same complete theory are ∞-equivalent.*

Proof. Let $\vec{a}$ and $\vec{b}$ be of the same type in an atomic model. Let us add α to $\vec{a}$ and let $f(\vec{x}, y)$ be a formula isolating the type of $\vec{a}\frown\alpha$. Then $\vec{a}$ satisfies $(\exists y)f(\vec{x}, y)$. So does $\vec{b}$, which has the same type as $\vec{a}$, and we can find β in the model such that $\vec{a}\frown\alpha$ and $\vec{b}\frown\beta$ have the same type.

As T is complete, every model of T realizes all its isolated types (since if $\langle f(\vec{x})\rangle$ is nonempty, then $T \models (\exists \vec{x})f(\vec{x})$), and two atomic models of T realize the same pure types; they are therefore ∞-equivalent by Theorem 9.4. $\square$

We call a model *prime* if it embeds elementarily into every model of T. If $A \subset M$ and M is a prime model of $T(A)$, then we say it is *prime over* A.

Theorem 10.8. *A denumerable complete theory T has a prime model if and only if it has an atomic model; in this case it has, up to isomorphism, only one prime model M, which is also its unique denumerable atomic model; this model M is strongly ω-homogeneous, and it is prime over every one of its finite subsets.*

Proof. If M realizes a nonisolated type p then, by the omitting types theorem, there is a model N of T that omits p, and M cannot be embedded elementarily into N: M is not prime.

So if M is prime, it is atomic, and denumerable, since T has denumerable models. If T has an atomic model, then by Löwenheim's Theorem it has a denumerable atomic model; let us show that this model M is prime.

Consider an enumeration of M *of type* ω: $M = \{a_0, \ldots, a_n, \ldots\}$. Let N be any model of T. As the type of a_0 is isolated, we can realize it in N.

Having done this, since the type of a_1 over a_0 is isolated (Lemma 10.4), we can also realize it in N, etc., successively realizing the type of each a_{n+1} over $\{a_0, \ldots, a_n\}$, to get an elementary embedding of M into N.

Two denumerable atomic models, being ∞-equivalent, are isomorphic; they are also strongly ω-homogeneous (Corollary 9.5). If M is atomic and denumerable, it is also atomic and denumerable, and therefore prime, over each of its finite subsets. □

It is possible for a prime model M of T to have proper elementary submodels. Such a submodel N of M is obviously itself a prime model of T; in the denumerable case, in which the prime model is unique if it exists, N is isomorphic to M.

If A is an infinite subset of the prime model M, then M is not necessarily prime over A: first, because there might not be a prime model over A; also, because if there is a prime model N, then N embeds elementarily into M, so N is also a prime model of T, but if there is a unique prime model, M and N can be isomorphic without being A-isomorphic.

Theorem 10.9. *A denumerable complete theory T has a prime model if and only if for every n, the isolated types of $S_n(T)$ form a dense set.*

Proof. If M is atomic, the types realized in M, which are isolated types, form a dense set. Conversely, if the isolated types form a dense set, their complement is a closed set of empty interior, which can be omitted by the omitting types theorem. □

Theorem 10.10. *If T is a denumerable complete theory such that for all n, $S_n(T)$ is finite or denumerable, then for every finite set with parameters $\vec{a}$, T has a prime model over $\vec{a}$.*

Proof. As the type of $\vec{b}$ over $\vec{a}$ is determined by the type of $\vec{a}\frown\vec{b}$, each $S_n(\vec{a})$ is also denumerable. It is therefore sufficient to see that in a denumerable compact space, the isolated points form a dense set. However, the set of nonisolated points is the union of denumerably many closed sets with empty interiors; by Theorem 10.1, its complement is dense. □

10.3 Theories with Finitely Many Denumerable Models

A denumerable complete theory T is called *ω-categorical* (or *$\aleph_0$-categorical*) if it has only one denumerable model, up to isomorphism; it is therefore clear, by Löwenheim's theorem, that this denumerable model is prime.

Theorem 10.11 (Ryll-Nardzewski Theorem). *A denumerable complete theory T is ω-categorical iff $S_n(T)$ is finite for every n.*

Proof. If $S_n(T)$ is infinite for some n, then by compactness it cannot be made up entirely of isolated points. It therefore contains a nonisolated type p; T has one denumerable model realizing p and another omitting p, so it is not ω-categorical.

Conversely, if $S_n(T)$ is finite for all n, then it is discrete, and every model of T is atomic. □

As the type of $\vec{a}\frown\vec{b}$ is determined by the type of $\vec{a}$ and the type of $\vec{b}$ over $\vec{a}$, the condition "S_n is finite for all n" is equivalent to "$S_1(\vec{a})$ is finite for every tuple $\vec{a}$ of parameters."

This condition can also be formulated as follows: For all n, the number of formulas $f(\vec{x})$ in n free variables $x_1, \ldots, x_n$, considered up to equivalence modulo T, is finite. Indeed, the sets of the form $\langle f(\vec{x})\rangle$ are finite in number iff $S_n(T)$ is finite.

Note that any two models of T are ∞-equivalent iff T is ω-categorical; in this case, every model of T is ω-saturated.

Examples of ω-categorical theories: that of a finite structure, those of infinite sets, of equivalence relations with infinitely many classes all of which are infinite, of dense orderings without endpoints, and of atomless boolean algebras.

As a consequence of Theorem 10.11, we have the following result, which is a little unexpected.

Theorem 10.12. *Let N be a structure interpretable in an ω-categorical structure (i.e., one whose theory is ω-categorical). Then N is also ω-categorical.*

Proof. The type of a tuple in the sense of N is determined by the type of the tuple that represents it in M (see Section 9.4). Since M has only finitely many n-types for each n, the same is true of N. □

In this last theorem, "interpretable" means "interpretable without parameters." If we want the same theorem allowing parameters, the total number of parameters involved in interpreting N in M must be finite. For example, this happens all the time if the language of N is finite. (If M is ω-categorical, so is $(M, \vec{a})$!)

It is not possible for a denumerable complete theory to have exactly two denumerable models up to isomorphism, as the next theorem states:

Theorem 10.13. *A denumerable complete theory that is not ω-categorical has at least three pairwise nonisomorphic denumerable models.*

Proof. If $S_n(T)$ is not denumerable for some n, then, since every type is realized in some denumerable model, and each such model can realize only denumerably many types, T has a nondenumerable infinity of denumerable models.

Thus, if $S_n(T)$ is denumerable for all n, T has a prime model M_1 by Theorem 10.10, and a saturated denumerable model M_2 by Theorem 9.9.

If T is not ω-categorical, then $S_n(T)$ is infinite for some n, and contains a nonisolated point p, which is omitted in M_1 and realized in M_2. Therefore, M_1 and M_2 are not isomorphic.

Let $\vec{a}$ be a realization of p. As $S_n(T)$ is infinite and every type over $\emptyset$ has an extension over $\vec{a}$, $S_n(\vec{a})$ is also infinite, and contains a nonisolated point. By Theorem 10.10 again, there is a model that is prime over $\vec{a}$, that is not prime because it realizes p, and that is not saturated because it is atomic over $\vec{a}$: This is our third model M_3. □

Example of a complete theory with three denumerable models: The language consists of a binary predicate $\leq$ and a denumerable list of constants $a_0, a_1, \ldots, a_n, \ldots$. The axioms say that $\leq$ is a dense chain without endpoints and that $a_0 < a_1 < \cdots < a_n < a_{n+1} < \cdots$. We can easily see that this theory T is complete and admits quantifier elimination; the only nonisolated type p of $S_1(T)$ is the one containing all formulas of the form $x > a_n$.

In the model M_1, the sequence a_n is cofinal. In M_2, it is bounded above, but has no least upper bound. In M_3, it has a least upper bound a: This is the prime model over a.

Starting with this example, it is easy enough to construct a complete theory with exactly n denumerable models up to isomorphism for every finite $n \geq 3$.

Note that if the language L is denumerable, then there are only 2^ω L-structures with universe ω. For an n-ary relation is a subset of ω^n (which is denumerable), so there are only 2^ω of them; we need to assign a relation to each symbol of the language, giving us a function from ω into 2^ω, and $(2^\omega)^\omega = 2^{\omega \times \omega} = 2^\omega$. Up to isomorphism, a denumerable complete theory cannot have more than 2^ω denumerable models, and we know one that attains this maximum (the theory of discrete order without endpoints). There are also some (the theory of algebraically closed fields of a given characteristic; the theory of the successor on the natural numbers) that have exactly ω.

Vaught's conjecture is that a denumerable complete theory with infinitely many denumerable models up to isomorphism, has either ω or 2^ω. If we assume the continuum hypothesis, then $2^\omega = \omega^+$, and the conjecture loses interest!

One important argument in its favor is a theorem of Morley, which states that such a theory has $\aleph_0$, $\aleph_1$, of $2^{\aleph_0}$ models; although it remains only to eliminate $\aleph_1$, in the case in which it is not $2^{\aleph_0}$, Vaught's conjecture is still an open problem.

10.4 Constructed Models

We say that a set of parameters A is *constructed* if there is an ordinal enumeration of A, $A = \{\dots, a_\alpha, \dots\}$, such that if A_α denotes $\{\dots, a_\beta, \dots\}_{\beta<\alpha}$, the type of a_α over A_α is isolated for all α. Such an enumeration, which is not necessarily injective, is called a *construction* of A; we say that B is constructed over A if it is constructed in the sense of $T(A)$, i.e., if there is an ordinal enumeration $\{\dots, b_\alpha, \dots\}$ of B such that the type of b_α is isolated over $A \cup \{\dots, b_\beta, \dots\}_{\beta<\alpha}$ for all α.

In this last case, if $A \subset B$, since the elements of A are all named in the language, they are virtually present in every set of parameters. Adding them will not change types, and they are atomic over every set of parameters. For this reason, if we want to construct B over A, we are most often satisfied with enumerating $B-A$. If we wish, we can insert the points of A anywhere, and thus get a construction in the previous sense.

Along the same lines, we can see that if a_α has already been obtained, i.e., if it is in A_α, we can skip it, and get another construction: If we keep only the first occurrence of each element of A in a construction, we thus get an injective construction.

Note that if A is constructed, *it is atomic.* To see this, we can show by induction on α that A_α is atomic. If α is a limit, $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$, and any tuple from A_α is contained in some A_β. For the successor case $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$; since A_α is atomic and $A_{\alpha+1}$ is atomic over A_α, the conclusion follows from Lemma 10.6

Lemma 10.14. *If A is constructed and $\vec{a}$ is a tuple from A, then A is also constructed over $\vec{a}$; in fact, every construction of A over $\emptyset$ is also a construction of A over $\vec{a}$.*

Proof. As A is constructed over A_α, it is atomic over A_α, and the type of $\vec{a}\frown a_\alpha$ over A_α is isolated; this implies that the type of a_α over $A_\alpha \cup \{\vec{a}\}$ is also isolated (Lemma 10.4). □

Let A be a constructed set. We choose a construction of it and also, for every α, choose a formula $f_\alpha(\vec{b}_\alpha, x)$ that isolates the type of a_α over A_α. Then $\vec{b}_\alpha$ comes from A_α, consisting of a_β of index strictly less than α. Having made this choice, we define the *package* of a_α, by induction on α, as the union of the packages of elements of $\vec{b}_\alpha$ together with a_α. We can see that this package P_α of a_α is a finite set, and that for every a_β in P_α, the formula $f_\beta(\vec{b}_\beta, x)$ has parameters in P_α.

We say that a subset C of A is *closed* if, whenever a_α is in C, its package P_α is included in C; equivalently, C is a union of packages.

Lemma 10.15. *Let C be a closed subset of the constructed set A. Then C is constructed.*

Proof. Let us show that if a_α is in C, then its type over $C_\alpha = A_\alpha \cap C$ is isolated. Its type over A_α is isolated by the formula $f_\alpha(\vec{b}_\alpha, x)$: Two elements that satisfy this formula have the same type over A_α, and a fortiori over every subset of A_α that contains $\vec{b}_\alpha$. As C_α contains $\vec{b}_\alpha$, which consists of elements of C of index strictly less than α, this formula also isolates the type of a_α over C_α.

The indices α of the a_α in C form a well-ordered set; it is sufficient to reenumerate it by its own ordinal to get a construction of C. □

Lemma 10.16. *If C is a closed subset of the constructed set A, then the type of C (over $\emptyset$) is determined by the formulas $f_\alpha(\vec{b}_\alpha, a_\alpha)$ for a_α in C.*

Proof. Note that each $C_\alpha = A_\alpha \cap C$ is closed. We show by induction that the $f_\beta(\vec{b}_\beta, a_\beta)$, for $\beta < \alpha$, determine a complete type. If $\alpha = 0$, then $C_0 = \emptyset$, and its type is complete because T is complete. If α is a nonzero limit, let $g(\vec{c})$ be a formula with parameters in C_α. All elements of $\vec{c}$ are therefore contained in some $C_\beta, \beta < \alpha$; by the induction hypothesis, either $g(\vec{c})$ or $\neg g(\vec{c})$ is a consequence of the $f_\gamma(\vec{b}_\gamma, a_\gamma), \gamma < \beta$. If α is a successor, then $\alpha = \beta + 1$, the type of C_β is determined by the $f_\gamma(\vec{b}_\gamma, a_\gamma), \gamma < \beta$, and the type of a_β (β is the greatest element of α!) is determined by the formula $f_\beta(\vec{b}_\beta, a_\beta)$. □

Lemma 10.17. *If C is a closed subset of the constructed set A (relative to a fixed construction of A and a fixed choice of the isolating formulas), then A is constructed over C; also, every enumeration $A = \{\dots, b_\alpha \dots,\}$ of A such that for every limit α, $B_\alpha = \{\dots, b_\beta, \dots\}_{\beta<\alpha}$ is closed (relative to the construction chosen at the start) is a construction of A over C.*

Proof. First, let us show that A is atomic over C. Let $\vec{a}$ be a finite subset of A, and let $\vec{a}\frown\vec{b}$ be the union of the packages of the elements of $\vec{a}$. Then the set $C \cup \{\vec{a}, \vec{b}\}$ is closed, and, by the last lemma, the type of $\vec{a}\frown\vec{b}$ over C is isolated by the conjunction of the formulas $f_\alpha(\vec{b}_\alpha, a_\alpha)$, where a_α is in $\vec{a}$ or $\vec{b}$. Since the type of $\vec{a}\frown\vec{b}$ is isolated over C, the type of $\vec{a}$ over C is isolated.

Now let us show that every enumeration like that in the statement of the theorem is a construction of A over C. Every ordinal has the form $\alpha + n$, where α is a limit. (Consider the least ordinal not of this form: It could be neither a limit nor a successor.) By hypothesis, for limit α, B_α is closed, as is $B_\alpha \cup C$; consequently, the type of $(b_\alpha, b_{\alpha+1}, \dots, b_{\alpha+n})$ over $B_\alpha \cup C$ is isolated, which implies that the type of $b_{\alpha+n}$ over $B_\alpha \cup \{b_\alpha, \dots, b_{\alpha+n-1}) \cup C = B_{\alpha+n} \cup C$ is isolated.

It is therefore clear that the original construction is a construction of A over C. □

The upshot is that we get another construction of A if we renumber it without dislocating the little packages. In particular, we can see that if A is constructed, and if κ is its cardinality, then it has a construction of type κ:

If it is finite, every enumeration of it is a construction, and if it is infinite, it has κ packages, which we can enumerate one after another.

Some call a *constructed model* a *strictly prime model.* A constructed model is indeed prime: If N is any model of T, then since the type of a_α over A_α is always isolated, we can successively realize all these types in N. By Löwenheim's theorem, a constructed model (and for the same reason, any constructed set of parameters) has cardinality less than or equal to $|T|$; a model that is constructed over A is prime over A and of cardinality less than or equal to $\max(|A|, |T|)$.

Note that every enumeration, of type ω, of a denumerable atomic model, or more generally of a denumerable atomic set of parameters, is a construction of it.

The following theorem states the uniqueness of the constructed model, if it exists, with no hypothesis on the cardinality of the theory; we prove it by a back-and-forth argument that proceeds, not element by element as before, but rather package by package, in such a way as to preserve the atomic character of the models at each stage.

Theorem 10.18 (Ressayre's Theorem). *If a complete theory T has a constructed model, then it has only one such model up to isomorphism; this model is strongly ω-homogeneous.*

Proof. Let M and N be two constructed models of T. As both are prime, they can be embedded into each other, and they both have cardinality κ. We have noted that we can find constructions of length κ, by enumerating the packages without dislocating them.

So $M = \{\ldots, a_\alpha, \ldots\}_{\alpha<\kappa} = \bigcup_{\alpha<\kappa} A_\alpha$ and $N = \{\ldots, b_\alpha, \ldots,\}_{\alpha<\kappa} = \bigcup_{\alpha<\kappa} B_\alpha$.

We therefore construct, by induction on α, a sequence f_α of isomorphisms between submodels of M and submodels of N, and a subsequence g_α of isomorphisms between submodels of N and submodels of M. such that:

- if $\beta < \alpha$, then f_β is a restriction of f_α and g_β is a restriction of g_α;
- g_α is an extension of f_α^{-1} and $f_{\alpha+1}$ is an extension of g_α^{-1};
- $\mathrm{dom}(f_\alpha), \mathrm{im}(f_\alpha), \mathrm{dom}(g_\alpha)$ and $\mathrm{im}(g_\alpha)$ are closed sets;
- $\mathrm{dom}(f_\alpha)$ and $\mathrm{im}(f_\alpha)$ have the same type, and $\mathrm{dom}(g_\alpha)$ and $\mathrm{im}(g_\alpha)$ have the same type;
- if α is a limit, then f_α is the limit of the $f_\beta, \beta < \alpha$, and g_α is the limit of the $g_\beta, \beta < \alpha$;
- if $\beta < \alpha$, then $a_\beta \in \mathrm{dom}(f_\alpha)$ and $b_\beta \in \mathrm{dom}(g_\alpha)$.

For this purpose, we must take $f_0 = g_0 = \emptyset$, and more generally, at the limit stages, the limit of the partial isomorphisms already constructed.

If $\alpha = \beta + 1$, we proceed thus. First we construct f_α in the following manner. We begin by adding the package P_α of a_α to the image of g_β, which is closed. Since M is atomic over $\mathrm{im}(g_\beta)$, P_α has isolated type over that set, and we can find a tuple P'_α in the model N such that $\mathrm{im}(g_\beta) \cup P_\alpha$ and $\mathrm{dom}(g_\beta) \cup P'_\alpha$ have the same type, and extend g_β^{-1} to f^0_α such that $f^0_\alpha(P_\alpha) = P'_\alpha$. The problem is that P'_α is not closed in N; to close it, we need to add a finite set $P'_{\alpha,1}$; since the model N is atomic over $\mathrm{im}(f_\beta)$, the type of $P'_{\alpha,1}$ over $\mathrm{im}(f_\beta) \cup P'_\alpha$ is isolated, and we can find $P_{\alpha,1}$ in M such that $\mathrm{dom}(f_\beta) \cup P_\alpha \cup P_{\alpha,1}$ and $\mathrm{im}(f_\beta) \cup P'_\alpha \cup P'_{\alpha,1}$ have the same type. We then extend f^0_α to f^1_α in such a way that $f^1_\alpha(P_{\alpha,1}) = P'_{\alpha,1}$. Now we must close $\mathrm{dom}(f^1_\alpha)$ by adding finitely many elements to it, and thus extend the isomorphisms, taking the closure first of the domain, then of the image, proceeding ω times by a back-and-forth construction. This is possible because at each stage we add only finitely many parameters to close it, and the models are atomic, the one over $\mathrm{dom}(f_\beta)$, the other over $\mathrm{im}(f_\beta)$. And naturally, for $f_{\beta+1} = f_\alpha$, we take the limit of the f^n_α.

Next, we construct g_α by taking f_α, then adding b_β to its domain, and finally closing the images and the domains by a back-and-forth construction of type ω.

The functions f_κ and g_κ that we get at the end are inverse isomorphisms to each other.

To see homogeneity, note that if M is constructed, and if $\vec{a}$ and $\vec{b}$ have the same type in M, then $(M, \vec{a})$ and $(M, \vec{b})$ are constructed models of the same theory; they are therefore isomorphic. □

Note that the differential closure which we constructed in Section 6.2 is a constructed model. We just proved its ω-homogeneity, and its uniqueness as a constructed model, but for the time being we are unable to prove its uniqueness as a prime model (in case K, the field whose differential closure we are taking, is not denumerable).

There are no known examples of (nondenumerable!) theories that have prime models but no constructed models; since two prime models are elementarily embeddable in each other, all known examples of prime models are atomic.

10.5 Minimal Models

A model is called *minimal* if it has no proper elementary submodel; it is *minimal* over A if it is a minimal model of $T(A)$. If a complete theory T has a prime model and a minimal model, then, since the former embeds into the latter, T has only one prime model, which is its unique minimal model. Beyond that, however, we cannot say very much about minimal models.

One very specific case of the minimal prime model was already studied in Section 6.1, since the algebraic closure of a set of parameters is a model. An even more extreme case is that what could be called the *rational closure* of the set of parameters A, consisting of those elements a of its algebraic closure with unique conjugates (so that a satisfies a formula $f(x)$, with parameters in A, such that $T(A) \models (\exists! x) f(x))$ is itself a model.

That fact is illustrated in arithmetic. In that theory, every formula $\phi(\vec{x}, y)$ is associated with the definable function $f_\phi(\vec{x})$ that sends $\vec{x}$ to the least y satisfying $\phi(\vec{x}, y)$ if it exists, and to 0 otherwise. It is therefore clear that if A is a subset of a model of arithmetic (or even of any complete theory containing Peano arithmetic), then the closure $\bar{A}$ of A under all functions definable (without parameters) in arithmetic satisfies Tarski's test: $\bar{A}$, which is the rational closure of A, is an elementary submodel of M, and is the minimal prime model over A. We can see that this model is uniquely determined, up to A-isomorphism, since relations of the form $f(\vec{a}) = g(\vec{a}), f(\vec{a}) + g(\vec{a}) = h(\vec{a}), \ldots$ are entirely determined by the type of A.

For arithmetic, we therefore have a minimal prime model over any set of parameters; the prime model over $\emptyset$ is obviously the standard model.

Note that if a is algebraic over A, its type over A is isolated. For we can take a formula $f(x)$, with parameters in A, satisfied by n points including a, such that n is minimal. We cannot distinguish the types of these n points, and this formula isolates the type of a over A. If $A \subset B$ and if a is algebraic over A, it is also algebraic over B; it therefore follows that every ordinal enumeration of the algebraic closure of A is a construction of it over A.

We can see that the theory of the order on the (positive and negative) integers, as well as the theory of the successor function on the (positive and negative) integers, has a minimal prime model that is not the algebraic closure of the empty set; in fact, in this case, there are no algebraic elements over $\emptyset$.

It is also easy to construct theories with minimal models but no prime models. Consider the theory T_1, in the language of one unary function symbol s and one unary relational predicate $A(x)$, consisting of the axioms of the theory of the successor on the integers (s is a bijection with no cycles).

Let us define a "block" to be a copy of $\mathbf{Z}$ equipped with a unary relation; the models of T_1 are formed by juxtaposing blocks.

For every sequence $\vec{\epsilon} = \{\epsilon_0, \ldots, \epsilon_n\}$ of symbols, each of which is either nothing or $\neg$, we add the axioms $(\exists x)(\epsilon_0 A(x) \wedge \epsilon_1 A(sx) \wedge \cdots \wedge \epsilon_n A(s^n x))$ to T_1 to get a theory T.

We can then see that in a model of T every sequence $\vec{\epsilon}$ is realized by consecutive elements; since the sequences $\vec{\epsilon}, \vec{\epsilon}^\frown\vec{\epsilon}, \vec{\epsilon}^\frown \cdots {}^\frown\vec{\epsilon}, \ldots$ must also be realized, each $\vec{\epsilon}$ actually appears infinitely often, and necessarily at arbitrarily large (i.e. either infinite or arbitrarily large finite) distances from a given point. From this we deduce, by compactness, that an ω-saturated model of T contains infinitely many copies of each block, and therefore that two such models are ∞-equivalent, and T is a complete theory admitting quantifier elimination.

If in a model of T we suppress a block B that is not a model of T, we get another model of T. If block B contains no occurrence of the sequence $\vec{\eta}$, then for every sequence $\vec{\epsilon}$, $\vec{\epsilon}^\frown\vec{\eta}$ is realized somewhere else. A minimal model of T is therefore a block that is a model of T. We can easily see that there exist 2^ω pairwise nonisomorphic such models. To see this, first take a copy of the consecutivity relation on the negative integers, with a unary predicate A such that -1 and -2 are in A, and all the sequences $\vec{\epsilon}$ appear there. For the construction it is sufficient to enumerate these sequences $\vec{\epsilon}_0, \vec{\epsilon}_1, \ldots, \vec{\epsilon}_n, \ldots$ and to put them end to end. Every sequence X from ω is thus associated with a block B_X that has this last distribution on the negative integers, and such that for $x \geq 0$, we have $x \in A$ iff $x = 2y + 1$ with $y \in X$. Since -1 and -2 form the highest occurrence in B_X of two consecutive elements satisfying A, 0 is marked there, and B_X and B_Y are isomorphic only if $X = Y$.

Since there are many minimal models, there can be no prime model. Moreover, if we define "periodic block" to mean a block obtained by repeating the same sequence $\vec{\epsilon}$, we can see that the model of T consisting of a copy of each periodic block has no minimal submodel.

Exercise 10.19. Consider a compact 0-dimensional topological space E together with a function f that assigns every isolated point of E an integer $n \geq 1$, or else the symbol ∞. Every clopen set A of E is associated with a unary relational symbol $R_A(x)$, and we consider the following theory $T_{E,f}$:

- $(\forall x)\neg R_\emptyset(x)$, and for each $A \neq \emptyset, (\exists x)R_A(x)$;
- $(\forall x)(R_A(x) \leftrightarrow \neg R_{\neg A}(x)), (\forall x)(R_{A\cap B}(x) \leftrightarrow (R_A(x) \wedge R_B(x)))$
- if A is an atom, i.e., if A isolates a point p of E, we express the fact that there are exactly $f(p)$ elements that satisfy R_A if $f(p)$ is finite, and that there are infinitely many if $f(p)$ is ∞.

1. Show that a type for $T_{E,f}$ corresponds to an ultrafilter of clopen sets of E, that is to say, by compactness, to a point of E. What are the ω-saturated models of this theory?
2. Show that $T_{E,f}$ is complete and admits quantifier elimination, and that $S_1(T_{E,f}) = E$.
3. Show that up to interpretation of the primitive symbols of the language, $T_{E,f}$ is the general case of the theory of a structure whose language includes only unary relational symbols (see Theorem 1.4).
4. Show that $T_{E,f}$ has a prime model if and only if the isolated points of E form a dense subset.
5. Show that if we remove an element of nonisolated type from a model of $T_{E,f}$, we get an elementary submodel. Show that this theory has a minimal model iff it has a prime model, and that if in addition $f(p)$

is finite for every p, then the minimal model is the algebraic closure of $\emptyset$.

10.6 Nonuniqueness of the Prime Model

In this section we consider a chain I with a least element called 0. We write I^+ for the set of nonzero elements of I. We are going to study rich prime I-valued spaces (see Section 6.4). We note in passing that every rich I-valued space is atomic over I: If I is denumerable, the denumerable rich I-valued space is the unique rich prime I-valued space.

Define $E(I)$ to be the space consisting of all functions from I^+ into ω that take the value 0 at all but finitely many points, equipped with the following distance: if $a = (\ldots, a_i, \ldots), b = (\ldots, b_i, \ldots), a \neq b$, then $d(a, b)$ is the greatest index i such that $a_i \neq b_i$. In other words, if $d(a, b) \leq i$, then the two sequences take the same values for all $j > i$.

We see immediately that $E(I)$ is a rich I-valued space.

If $a = (\ldots, a_i, \ldots)$ is a point in $E(I)$, we define the *package* of a to be the set $P(a)$ of all $b = (\ldots, b_i, \ldots)$ such that $b_i = 0$ if $a_i = 0$. As the sequence a_i takes nonzero values only finitely often, *$P(a)$ is a denumerable set* (or of cardinality 1, if all a_i are 0).

We shall say that a subset A of $E(I)$ is *closed* if it is a union of packages, i.e., if for every a in A, $P(a)$ is contained in A.

Lemma 10.20. *If A is a closed subset of $E(I)$, then $E(I)$ is atomic over $I \cup A$.*

Proof. First, let us show that every point a in $E(I)$ has isolated type over $I \cup A$. If A is empty, we already know this.

If not, then A contains an element of $P(a)$, for example the zero sequence (which is in every package), and there is one whose distance from a is minimal. Indeed, the distance from a to an element of $P(a)$ is 0, or else one of the indices i such that $a_i \neq 0$. So let b be an element of $P(a)$ with minimal distance from a.

If $d(a, b) = 0$, then $a = b$ and its type is isolated over A, since it is an element of A. On the other hand, if $d(a, b) = i \neq 0$, that implies that $a_i \neq 0$ and $b_i = 0$ (otherwise, we could get an element b' of $P(b) \cap P(a)$ closer to a by replacing b_i with a_i in b). I claim that the formula $d(x, b) = i$ isolates the type of a over $A \cup I$.

To see this, we need to show that this condition determines the value of $d(x, c)$ for every c in A. If $d(b, c) = j, j < i$, then $d(x, c) = i$; if $d(b, c) = j, j > i$, then $d(x, c) = j$. Is it possible that $d(b, c) = i$? If that were the case, then, since $b_i = 0$, c_i would be nonzero, and the element b' defined by $b'_j = a_j$ for $j \geq i$, and $b'_j = 0$ for $j < i$, would be in the package of c', and therefore in A, and of lesser distance to a than b is.

Now, if $\vec{a}$ is a tuple of points, each one has a type over $A \cup I$ isolated by a formula in the style of the last paragraph, and in order to isolate the type of the n-tuple it is sufficient to specify further what the distances among these elements are. $\square$

Lemma 10.21. *$E(I)$ is constructed over I.*

Proof. First let us enumerate all the packages as $A_0, \dots, A_\alpha, \dots, \alpha < \kappa$. Next, take an enumeration of A_0 (which is denumerable) *of type* ω, etc. Doing this gives us an enumeration of $E(I)$ of type $\omega \times \kappa$ that is a construction of it by the preceding lemma, since for every limit α, the elements of index less than α form a closed set. $\square$

We therefore have a unique rich I-valued space constructed over I for every I. We are now going to determine under what condition this space is the unique prime rich space over I. Note that $E(I)$ cannot be minimal, since it is fairly obvious that if we remove one point from a rich I-valued space, we still have a rich I-valued space. The prime I-valued models are obviously to be found among the subspaces of $E(I)$.

Lemma 10.22. *If I is well-ordered and if A is a subset of $E(I)$ with no infinite isosceles polygons, then $E(I)$ is atomic over $A \cup I$.*

Proof. In order for the type of $\vec{a} = \{a_1, \dots, a_n\}$ to be isolated over $A \cup I$ it is sufficient for the type of each a_i to be isolated over $A \cup I$: Adding the formulas expressing the distances among the a_i to the conjunction of the isolating formulas of each a_i will then suffice to isolate the type of $\vec{a}$.

So let a be in $E(I)$. Since I is well-ordered, there is an element b of A at minimal distance from a. If $d(a,b) = 0$, then $a \in A$ and its type over $A \cup I$ is isolated. If $d(a,b) \neq 0$, consider a maximal isosceles polygon $\{b = b_0, b_1, \dots, b_m\}$ of side length i passing through b and included in A. (We know that all such polygons have the same number of elements; they might possibly consist only of b.) Then the type of a over $A \cup I$ is isolated by the formula $d(x, b_0) = i \wedge \cdots \wedge d(x, b_m) = i$; indeed, if c is in A, then for some h, $d(c, b_h) \neq i$, and $d(x, c)$ is determined by the ultrametric inequality. $\square$

We can therefore see that if I is well-ordered, then every subset A of $E(I)$ is constructed over I; it is sufficient to enumerate it without ever introducing an infinite isosceles polygon until we have finished, for example, by successively enumerating $A \cap 1^{I^+}, \dots, A \cap n^{I^+}, \dots$ (recall that $n = \{0, \dots, n-1\}$), since the isosceles polygons of $E(I) \cap n^{I^+}$ have at most n elements. In that case, $E(I)$ is the unique prime rich space over I: Indeed, every prime model, being embeddable into $E(I)$, is constructed. As an exercise, the reader can check that $E(I)$ is prime over $A \cup I$ if and only if every maximal isosceles polygon of A is either finite or maximal in $E(I)$.

To generalize this last lemma, we need an easy but nevertheless subtle result from model theory whose depth will be more clearly seen later:

Theorem 10.23. *Let M be a model of a complete theory T, let B be a subset of M, and let $\vec{a}$ and $\vec{b}$ be tuples taken from M. If the type of $\vec{a}$ over B has only one extension to $B \cup \{\vec{b}\}$ (i.e., if the type of $\vec{a}$ over $B \cup \{\vec{b}\}$ is determined by its restriction to B), then the type of $\vec{b}$ over B has only one extension to $B \cup \{\vec{a}\}$; if, in addition, the type of $\vec{b}$ over $B \cup \{\vec{a}\}$ is isolated, then the type of $\vec{b}$ over B is isolated.*

Proof. The hypothesis of the theorem means that every formula $f(\vec{a}, \vec{y})$ with parameters in B that is satisfied by $\vec{b}$ is implied, modulo $T(B \cup \{\vec{a}\})$, by the type of $\vec{b}$ over B. That further implies that the type of $\vec{a}\frown\vec{b}$ over B is axiomatized by the type of $\vec{a}$ over B and the type of $\vec{b}$ over B, which is clearly a symmetric condition on $\vec{a}$ and $\vec{b}$.

If we assume that the type of $\vec{b}$ over $B \cup \{\vec{a}\}$ is isolated by a formula $f(\vec{a}, \vec{y})$, then we can find a formula $g(\vec{y})$, satisfied by $\vec{b}$, that implies it modulo $T(B \cup \{\vec{a}\})$. By compactness, only a finite fragment of the type of $\vec{a}$ over B is involved; in other words, we can find a formula $h(\vec{x})$, satisfied by $\vec{a}$, such that $T(B) \models (\forall \vec{y})(g(\vec{y}) \to (\forall \vec{x})(h(\vec{x}) \to f(\vec{x}, \vec{y})))$.

Under these conditions, $g(\vec{y})$ isolates the type of $\vec{b}$ over B. For if $\vec{b}'$ is a tuple from an elementary extension of M that satisfies $g(\vec{y})$, then $\vec{a}\frown\vec{b}'$ satisfies $f(\vec{x}, \vec{y})$, so $\vec{b}$ and $\vec{b}'$ have the same type over $B \cup \{\vec{a}\}$, and also, a fortiori, over B. □

Lemma 10.24. *If every anti-well-ordered subset of I is finite or denumerable (i.e., I does not contain a strictly decreasing sequence indexed by $\aleph_1$), then for every a in $E(I)$ and every $A \subset E(I)$, the type of a over $A \cup I$ is determined (i.e., is axiomatized modulo $T(A \cup I)$) by a finite or denumerable family of formulas.*

Proof. This is clear if A is empty, since $E(I)$ is atomic over I (and, moreover, all its points have the same type over I). If $a \in A$, its type is isolated by $x = a$. If not, and if there exists b in A of minimal distance i from a, consider a maximal isosceles polygon B of side length i in A passing through b. Since $E(I)$ contains only denumerable isosceles polygons, B is finite or denumerable, $B = \{b = b_0, \ldots, b_n, \ldots\}$, and the conditions $d(x, b_n) = i$ determine the type of a over $A \cup I$. Otherwise, consider the nonempty set of j in I such that there exists b in A with $d(a, b) = j$. This set has no smallest element, and since I contains no descending sequence of type $\aleph_1$, the coinitiality of J, i.e., the cofinality of its inverse order, is ω. So let $i_0 > i_1 > \cdots > i_n > \cdots$ be a coinitial sequence in J and let $b_0, \ldots, b_n, \ldots$ be a sequence of elements of A such that $d(a, b_n) = i_n$. I claim that the conditions $d(x, b_n) = i_n$ determine the type of a over $A \cup I$: Indeed, for every c in A, there must be some n such that $d(c, b_n) > i_n$; otherwise, $d(a, c)$ would be less than every element of J. □

Theorem 10.25. *If I contains no strictly decreasing sequence indexed by $\aleph_1$, then $E(I)$ is the unique prime rich space over I. More precisely, every subset B of $E(I)$ is constructed over I.*

Proof. Every element a of $E(I)$ is associated with its package $P(a)$ (for the notion of package introduced at the beginning of this section; we could also use the notion of package associated with a construction of $E(I)$). Every package is denumerable, and $E(I)$ is atomic over any closed set, i.e., over any set that is a union of packages.

We are going to associate every element b of B with a denumerable subset $Q_1(b)$ of $E(I)$, containing b, which is closed, such that the type of each element of $Q_1(b)$ over B is determined by its restriction to $Q(b) = B \cap Q_1(b)$.

For that purpose, we start with b. Take its package $P(b)$, and for every tuple $\vec{a}$ of this package, add a denumerable subset $B_{\vec{a}}$ of B such that the type of $\vec{a}$ over $B \cup I$ is the unique extension of its restriction to $B_{\vec{a}} \cup I$. All in all, we add only denumerably many elements. Then we take its closure, then add parameters in B, and repeat ω times.

I claim that B is atomic over every set C that is the union of $Q(b)$. Let C_1 be the union of the corresponding $Q_1(b)$. It is sufficient to show that every element b of B has isolated type over $C \cup I$; indeed, as we have often noted, a tuple has isolated type over $C \cup I$ whenever each of its elements has isolated type over $C \cup I$, since all the distances among the elements of this tuple are in I. Since C_1 is closed, b has a type over $C_1 \cup I$ that is isolated by a formula of the form $d(x, a) = i$, with a single parameter a. The type of b over $I \cup C \cup \{a\}$ is a fortiori isolated by the same formula. But the type of a over $B \cup I$ is determined by its restriction to $C \cup I$; a fortiori, the type of a over $C \cup \{b\} \cup I$ is the unique extension of its restriction to $C \cup I$. By symmetry (Theorem 10.23) the type of b over $I \cup C \cup \{a\}$ is the unique extension of its restriction to $I \cup C$, and since the former is isolated, the latter is also.

It is therefore sufficient to put enumerations of $Q(b)$ of type ω end to end to get a construction of B. $\square$

This theorem has a more general significance, since the only facts about $E(I)$ used are Theorems 10.23 and 10.24. In fact, we can show that *if M is a constructed model of T, and if for every a in M and every $A \subset M$, there is a denumerable subset A' of M such that the type of a over A' determines the type of a over A, then M is the unique prime model of T.*

The only difference with the case of rich ultrametric spaces is that to isolate the type of a tuple taken from M, it is no longer sufficient to isolate that of each of its elements. We can thus show, as in Theorem 10.25, that every subset B of M is constructed in the following way: We consider an ordinal enumeration $\ldots, b_\alpha, \ldots$ of B and an increasing sequence C_α of subsets of M such that

- every C_α is closed;

- b_α is in $B_{\alpha+1} = C_{\alpha+1} \cap B$;
- the type of C_α over B is determined by its restriction to $B_\alpha = C_\alpha \cap B$;
- $C_{\alpha+1} - C_\alpha$ is denumerable;
- for α limit, C_α is the union of the $C_\beta, \beta < \alpha$.

To construct this sequence C_α, we proceed as follows: There is no problem if α is a limit. To get $C_{\alpha+1}$ from C_α we start by adding b_α. Then we take the closure, which adds a finite set of points. For every tuple from this set, we add a denumerable subset B' of B such that the type this tuple over $C_\alpha \cup B$ is determined by its restriction to $C_\alpha \cup B'$. Next we take the closure, add parameters in such a way as to determine the types over $C_\alpha \cup B$, and repeat ω times. The type over B of a tuple taken from $C_{\alpha+1}$ is determined by its restriction to $B_{\alpha+1}$; indeed, this tuple consists of a tuple $\vec{a}$ taken from C_α and a tuple $\vec{a}'$ consisting of new elements, and the type of $\vec{a}\frown\vec{a}'$ over B is determined by the type of $\vec{a}$ over B and the type of $\vec{a}'$ over $B \cup \{\vec{a}\}$.

We can therefore show that B is atomic over every B_α. For if $\vec{b}$ is a tuple from B, then its type over C_α, which is closed, is isolated by the formula $f(\vec{x}, \vec{a})$; by Theorem 10.23, the type of $\vec{b}$ over B has only a single extension to $B \cup \{\vec{a}\}$, and is isolated, since its unique extension is isolated.

We thus obtain a construction of B by putting enumerations of $B_{\alpha+1} - B_\alpha$ of type ω end to end.

Theorem 10.26. *If I contains a decreasing sequence indexed by $\aleph_1$, then there are infinitely many pairwise nonisomorphic prime rich subspaces over I.*

Proof. If $i_0 > \cdots > i_\alpha > \cdots$ is a decreasing sequence in I indexed by $\aleph_1$, we define a Cauchy pseudo-$\aleph_1$-sequence to be a sequence of elements $b_0, \ldots, b_\alpha, \ldots$ of an I-valued space such that, if $\alpha > \beta$, then $d(b_\alpha, b_\beta) \leq i_\beta$. We call b a pseudo-limit of this sequence if $d(b, b_\alpha) \leq i_\alpha$ for every α; we say pseudo-limit because this "limit" is not unique if the sequence of the i is not coinitial in I^+.

Let us show that in $E(I)$ every Cauchy $\aleph_1$-sequence is pseudo-convergent (the same would be true for any Cauchy κ-sequence, for $\mathrm{cof}(\kappa) > \omega$). For this purpose, consider the element c of ω^{I^+} such that $c_i = (b_\alpha)_i$ if $i > i_\alpha$ (and consequently $c_i = (b_\beta)_i$ for $\beta > \alpha$), and such that $c_i = 0$ if i is less than all the i_α. I claim that in fact c_i takes nonzero values only finitely often; that is to say, c is in $E(I)$. If not, there would be at least a denumerable infinity of indices j_n, each one obviously greater than one i_α, at which it takes nonzero values. Since $\aleph_1$ does not have denumerable cofinality, there exists α such that $i_\alpha < j_n$ for every n. As b_α takes the same values as c above i_α, that contradicts the fact that b_α only finitely often takes nonzero values. Therefore, c is in $E(I)$, and is a pseudo-limit of the Cauchy pseudo-sequence in question.

It therefore remains to be verified (without any difficulty) that if we consider a Cauchy $\aleph_1$-sequence in $E(I)$ none of whose elements is a pseudo-limit (such a sequence does exist), and if we remove all the pseudo-limits from $E(I)$, we get a model M_1, i.e., a rich I-valued space, that is prime and nonisomorphic to $E(I)$: there is a Cauchy sequence in M_1 without a limit. The task of defining equivalence for two Cauchy sequences is left to the reader: In M_1 there is only a single class of Cauchy $\aleph_1$-pseudo-sequences without limits; we get more prime models $M_2, M_3, \ldots, M_n, \ldots$ by removing $2, 3, \ldots, n, \ldots$ pseudo-limits. □

All these theorems remain valid when we replace the theory of rich spaces with the theory of n-rich spaces. The constructed n-rich I-valued space $E_n(I)$ consists of the functions from I^+ into n that only finitely often take nonzero values. This time, the packages $P(a)$ are finite.

Note that if I is well-ordered, then by the analogue of Lemma 10.22, every enumeration of $E_n(I)$ is a construction of it over I. That implies that $E_n(I)$ is minimal over I, since, if A is a proper subset, it is not a model because there are isolated types over A that are not realized in A. In this case, $E_n(I)$ is the only n-rich I-valued space; indeed, since it is prime over I, it embeds into every n-rich I-valued space, and we can easily see that, in a proper elementary extension of $E_n(I)$ there are distances that are not in I.

If I is finite, with $m + 1$ elements (including 0), then $E_n(I)$ has n^m elements: An I-valued space with no isosceles $(n + 1)$-element polygons, which must be embeddable in this space, therefore has at most n^m elements.

In all other cases, $E_n(I)$ is not minimal, since we can remove the limits of Cauchy ω-pseudo-sequences. This is not the only n-rich I-valued space up to isomorphism, since there are non-pseudo-convergent Cauchy ω-pseudo-sequences, while we can construct n-rich maximally complete I-valued spaces in which all sequences converge.

10.7 Historic and Bibliographic Notes

The omitting types theorem appears in a special form in [Hen54] and [Ore56]; a more general form is in [GRN61], although the connection to the property of Baire (a classic from the beginning of the century!) is still not made explicit there. Denumerable atomic models are studied in [Vau77]. Although Theorem 10.11, on the characterization of ω-categorical theories, is generally attributed to [RN59], it appeared simultaneously in [Eng59] and [Sve59b]. Theorem 10.13, on the three models, is in [Vau77].

It was also in [Vau77], in which Vaught's conjecture was made; the important contribution of Michael Morley to this question is in [Mor70].

Theorem 10.18, on the uniqueness of the constructed model, is due to Jean-Pierre Ressayre; it was published by Saharon Shelah: See [She78], p. 175, Conclusions 3–9, and p. 507 for the reference to Ressayre.

Prime models for ultrametric spaces are studied in [Del84]; Theorem 10.25, on uniqueness, copies a proof of the uniqueness of the prime model from [She79] in the stable context, while Theorem 10.26, on nonuniqueness, is merely the systematization of an example from the same article, the earliest example of a nonunique prime model.

11
Heirs

Venez, vous dont l'oeil étincelle,
Pour entendre une histoire encore.
Approchez: je vous dirai celle
De doña Padilla del Flor.
Elle était d'Alanje, où s'entassent
Les collines et les halliers.
Enfants, ...

V.H.

11.1 Heirs

In the next three chapters we shall principally consider complete types over *models* of a complete theory T. Recall that if $M \prec N$, if q is in $S_1(N)$ and has p as its restriction to M, we call q an *extension* or a *son* of p. The whole study of stability, which we are now beginning, depends on the search for particular sons of a type p.

Now let $M \prec N$, let p be in $S_1(M)$, and let q be a son of p over N. We call q an *heir* of p if for every formula $f(x, \vec{y}, \vec{z})$ in the language L of T, every $\vec{a}$ in M, and every $\vec{b}$ in N, if $q \models f(x, \vec{a}, \vec{b})$, then there exists $\vec{b}'$ in M such that $p \models f(x, \vec{a}, \vec{b}')$.

That means that every finite situation, with parameters in M, exhibited by q over N, already has an example proffered by p over M.

A few easy lemmas to begin. First, note that if q is a son of p and r a son of q, and if q is an heir of p and r an heir of q, then r is an heir of p. Also, if r is an heir of p, then q is an heir of p, although there is no reason r should be an heir of q.

Heirs exist, as the following theorem shows.

Theorem 11.1. *Let M be a model of T, let p be in $S_1(M)$, let N be an elementary extension of M, and let π be an incomplete type over N extending p such that if $\pi \models f(x, \vec{a}, \vec{b}), \vec{a} \in M, \vec{b} \in N$, then there exists $b' \in N$ such that $p \models f(x, \vec{a}, \vec{b}')$. Then π can be completed as an heir q of p over N.*

Proof. Consider the set of formulas consisting of π (in which we implicitly include $T(N)$) and all formulas $\neg g(x, \vec{a}, \vec{b})$ for which there is no $\vec{b}'$ in M such that $p \models g(x, \vec{a}, \vec{b}')$. I claim that this set of formulas is consistent. Indeed, any of its finite fragments can mention only a finite fragment $f(x, \vec{a}, \vec{b})$ of π and finitely many formulas $\neg g_1(x, \vec{a}, \vec{b}), \ldots, \neg g_n(x, \vec{a}, \vec{b})$. We can get a model of this finite fragment by taking x to be a realization of p (in an elementary extension of M) and interpreting $\vec{b}$ by a $\vec{b}'$ such that $p \models f(x, \vec{a}, \vec{b}')$; it is, in fact, possible that $p \models g_i(x, \vec{a}, \vec{b}')$.

Then by compactness, the set is consistent, and every complete type in the nonempty closed set defined by its formulas is an heir of p. □

We can even improve the formulation of this theorem, in such a way as to reassure those readers who would be ruffled by changes in sets of parameters:

Theorem 11.2. *Let p be in $S_1(M), M \prec N$, and let A be a set of parameters, $M \subset A \subset N$. Let π be an incomplete type (possibly complete!) over A, such that whenever $\pi \models f(x, \vec{a}, \vec{b}), \vec{a} \in M, \vec{b} \in A$, there is a $\vec{b}'$ in M such that $p \models f(x, \vec{a}, \vec{b}')$. Then π can be completed as an heir of p over N.*

Proof. Now we are considering π as an incomplete type over N. Suppose that $\pi \models f(x, \vec{a}, \vec{c})$, with $\vec{a} \in M, \vec{c} \in N$. Then $f(x, \vec{a}, \vec{c})$ is a consequence of π and of $T(N)$. By compactness, that means that there is a finite fragment $g(\vec{b}, \vec{c})$ of the type of $\vec{c}$ over A such that $\pi \models (\exists \vec{y}) g(\vec{b}, \vec{y}) \wedge ((\forall \vec{y})(g(\vec{b}, \vec{y}) \rightarrow f(x, \vec{a}, \vec{y})))$. We can therefore find $\vec{b}'$ in M such that $p \models (\exists \vec{y}) g(\vec{b}', \vec{y}) \wedge ((\forall \vec{y})(g(\vec{b}', \vec{y}) \rightarrow f(x, \vec{a}, \vec{y})))$. Since $(\exists \vec{y}) g(\vec{b}', \vec{y})$ does not contain x, this means that this formula is simply true in M, that we can find $\vec{c}'$ such that $M \models g(\vec{b}', \vec{c}')$. Consequently, there is $\vec{c}'$ in M such that $p \models f(x, \vec{a}, \vec{c}')$, and we can apply the preceding theorem. □

Corollary 11.3. *If p is in $S_1(M)$ and $M \prec N$, then p has a least heir over N; more generally, if $M \prec N \prec P, p \in S_1(M)$, and q is an heir of p over N, then q has a son r over P that inherits p.*

Proof. Consider p as an incomplete type over M and apply the preceding theorem. For the second part, consider q as an incomplete type over P, or else take an heir of q over P. □

Another definition: Let p be in $S_1(M)$ and let F be a set of sentences in the language L, or even $L(M)$, augmented by constant symbols x_i ("type variables") and y_j ("parameter variables"). We say that F is *consistent with* p if the set of sentences obtained by adding to F sentences saying that all the x_i satisfy p is consistent. It is clear that if q is a son of p and if F is consistent with q, then it is also consistent with p, which is a weaker theory; conversely, if q is an heir of p, and F is consistent with p, it is also consistent with q, since every finite fragment of q can be interpreted in p. This is the reason why "heirs" are so named: They are going to inherit all those properties of their fathers that can be translated as consistency with some set of formulas.

One example of an heir is given by ultrapowers of types. Consider an ultrafilter U of subsets of I, and a type p_i over a model M_i of T for every i in I. We define the ultraproduct $\Pi p_i/U$ of the p_i as the type $q \in S_1(\Pi M_i/U)$ such that $q \models f(x, \vec{a})$, where $\vec{a} = (\ldots, \vec{a_i}, \ldots)$ if $p_i \models f(x, \vec{a_i})$ for almost all i modulo U. We can easily see that this is indeed a complete and consistent set of formulas: A realization of p is provided by an ultraproduct modulo U of realizations of the p_i.

If all the p_i are equal to the same p in $S_1(M)$, we shall refer to the *ultrapower* p^U of p. The canonical diagonal embedding of M into M^U makes p^U a son of p that is clearly an heir.

To those who prefer ultraproducts to more or less direct reasoning by compactness, the following theorem will recall Lemma 4.12.

Theorem 11.4. *If $M \prec N$, $p \in S_1(M)$, and q is an heir of p over N, then there is an M-elementary embedding (i.e., elementary in $L(M)$) of N into an ultrapower M^U of M that makes p^U a son of q.*

Proof. Let I be the set of injections from a finite subset of N into M. For every formula $f(x, \vec{a}, \vec{b}), \vec{a} \in M, \vec{b} \in N$, such that $q \models f(x, \vec{a}, \vec{b})$, let $I_{f(x,\vec{a},\vec{b})}$ be the set of elements i of I, defined in $\vec{a}$ and $\vec{b}$, that fix $\vec{a}$ pointwise, and such that $p \models f(x, \vec{a}, i\vec{b})$. These sets form a filter base; let U be an ultrafilter containing them.

We thus get a function s from N into M^U by setting $(sb)_i = ib$ if i is defined in b, and anything otherwise. This is an elementary embedding: Consider those formulas $f(\vec{a}, \vec{b})$ that do not contain x such that $q \models f(\vec{a}, \vec{b})$, that is to say, such that $N \models f(\vec{a}, \vec{b})$! If we thus embed N into M^U, then q is a restriction of p^U. □

Ultrapowers are a rather particular sort of heir. Every type p over a model M of T can be associated with the following structure (M, dp), with universe M, in the language L of T augmented by a relational symbol $df(\vec{y})$

for every formula $f(x,\vec{y})$ of L: Interpret $df(\vec{y})$ on M by the set of $\vec{a}$ in M such that $p \models f(x,\vec{a})$.

We shall say that a son q of p over N is a "strong heir" of p if (N, dq) is an elementary extension of (M, dp); an ultrapower is a strong heir.

Lemma 11.5. *A strong heir is an heir, and if $p \in S_1(M)$, then every elementary extension of (M, dp) is of the form (N, dq), where q is a strong heir of p.*

Proof. Suppose that $(M, dp) \prec (N, dq)$. If $q \models f(x,\vec{a},\vec{b}), f \in L, \vec{a} \in M, \vec{b} \in N$, then $(N,dq) \models df(\vec{a},\vec{b}), (N,dq) \models (\exists\vec{y})df(\vec{a},\vec{y})$, as does (M,dp). Therefore, there exists $\vec{b}'$ in M such that $(M,dp) \models df(\vec{a},\vec{b}'), p \models f(x,\vec{a},\vec{b}')$.

As p is a (consistent) type over M, each of its finite fragments is satisfied by an element of M; consequently, in (M, dp), the following axioms are true:

$$(\forall\vec{y}_1)\cdots(\forall\vec{y}_n)(\exists x)(df_1(\vec{y}_1) \wedge \cdots \wedge df_n(\vec{y}_n) \rightarrow f_1(x,\vec{y}_1) \wedge \cdots \wedge f_n(x,\vec{y}_n)).$$

They are also true in every structure elementarily equivalent to (M, dp): In such a structure, the set of formulas $f(x,\vec{a})$ for all $\vec{a}$ satisfying $df(\vec{a})$ is consistent and complete (relative to the language L); this set is a type q, in the sense of the original theory T, and this model has the form (N, dq). □

Every heir of p has a son that is a strong heir of p, for example an ultrapower of p. (Alternatively, we can show that $T(N)$, (M, dp), and the formulas $df(\vec{a})$ for which $q \models f(x,\vec{a}), \vec{a} \in N$, form a consistent set.) However, if $p \in S_1(M), M \prec N$, then p in general does not have a strong heir over N; the model N must be sufficiently resplendent. The advantage of heirs is that they always exist.

Example. Let T be the theory of dense orders without endpoints, let M be a model of T, and let N be an elementary extension of M. Then N can always add points at every cut of M, as well as at the extremities of M. We know, from Chapter 1, that T admits quantifier elimination, so a type p is determined by the inequalities $a \leq x, a \in M$ that it satisfies. We therefore distinguish six sorts of types over M:

- realized type: $p \models x = a$. Then p has only one extension over N, the type of a over N, which is therefore its unique heir.
- the type a^+: p is not realized, $p \models x < b$ for every b in M such that $b > a$, and $p \models x > b$ for every b in M such that $b \leq a$. Let q be a heir of p over N. It is impossible to have $q \models x = c$, since there is no c' in M such that $p \models x = c'$. Therefore, q is not realized; this is likewise true of every heir of a nonrealized type (that is the least we can say!). Let c be in M, with $a < c$ and $c < b$ for every b in M such that $a < b$, so that c is a realization of p. It is impossible that $q \models x > c$, since then $q \models x > c \wedge c > a$, and there is no c' in M such

that $p \models x > c' \wedge c' > a$. The only possibility for q is therefore to be the a^+ type over N, which is consequently the unique heir of p.

- the type a^-: defined symmetrically. For a similar reason, its unique heir is the a^- type over N.

- irrational types: p is not realized, p is bounded above and below by elements of M, and is neither of the form a^+ nor of the form a^-. That means that p constitutes a proper cut of M with neither a least upper bound nor a greatest lower bound.
  Let us show that in this case, every nonrealized son of p is its heir. If $q \models f(x, \vec{a}, \vec{b})$, with $\vec{a} \in M, \vec{b} \in N$, then the formula $f(x, \vec{y}, \vec{z})$, by quantifier elimination, is equivalent to a Boolean combination of $u = v, u > v$; it can be put into the form of a disjunction of conjunctions of the formulas of the form $x > c_i, x < c_i, c_i = c_j, c_i > c_j, x = c_i$, where c_i and c_j are taken from $\vec{a}$ and $\vec{b}$. The type q satisfies one of the terms $g(x, \vec{a}, \vec{b})$ of this disjunction; since q is not realized, equalities $x = c_i$ do not appear in g, but only inequalities, which do not contradict the fact that x satisfies p, since q is a son of p. Furthermore, g places the elements of $\vec{b}$ in certain positions relative to $\vec{a}$: From the fact that the cut of p is irrational, we can see that we can reproduce that cut in M, and find $\vec{b}'$ there such that $p \models g(x, \vec{a}, \vec{b}')$; a fortiori $p \models f(x, \vec{a}, \vec{b}')$.
  Note also that the strong heirs of p are its irrational sons: (M, dp) is defined from the formula $df(y)$, where f is $x \geq y$. Indeed, all other formulas g are Boolean combinations of $f(x, y_i)$, so dg will be the corresponding Boolean combination of the $df(y_i)$, and we hope the reader now finds obvious the elementary equivalence of two dense chains without endpoints, with an additional unary relation symbol A in the language, defining a nonempty initial segment without greatest element whose complement is nonempty and without least element.
  We therefore see that if M is the chain of rationals and N the chain of reals, an irrational type over M has no strong heir over N: There are no irrational types over the chain $\mathbf{R}$. Also, if q is an heir of the form a^+ of an irrational type p, and if r is a son of q that is a strong heir of p, then r is not an heir of q.

- the type $+\infty$: $p \models x > a$ for every a in M. Then if q is an heir of p over N, it is impossible that $q \models x \leq b$, since there is no b' in M such that $p \models x \leq b'$; therefore, q must be the type $+\infty$ over N.

- the type $-\infty$: defined symmetrically. Its unique heir is the type $-\infty$ of N.

Note that the classification of types into six "sorts" corresponds exactly to the six possible theories for (M, dp).

To conclude this section, an analogue of Löwenheim's theorem:

Theorem 11.6. *If p is in $S_1(M)$, then there exists an elementary submodel M_0 of M, $|M_0| \leq |T|$, such that p inherits (even strongly) its restriction to M_0.*

Proof. Apply Löwenheim's theorem to the structure (M, dp). □

11.2 Definable Types

Let p be a type over M. We say that p is *definable* if for every formula $f(x, \vec{y})$ in the language L of T, there is a formula $df(\vec{y})$ *(possibly) with parameters in M* such that, for every $\vec{a}$ in M, $p \models f(x, \vec{a})$ iff $\vec{a}$ satisfies $df(\vec{y})$. In other words, the structure (M, dp) is definable, with parameters, in the structure M.

Be careful of a novice's error: This does not mean that $p \models (\forall \vec{y})(f(x, \vec{y}) \leftrightarrow df(\vec{y}))$; $p \models f(x, \vec{a}) \leftrightarrow df(\vec{a})$ only for the $\vec{a}$ in M, not for those in a model in which p is realized!

The function d that sends f to df is called a *definition* of p. A definable type p has essentially (i.e. modulo $T(M)$) only one definition: If d' is another definition of p, then $T(M) \models (\forall \vec{y})(df(\vec{y}) \leftrightarrow d'f(\vec{y}))$. Note also that $T(M) \models (\forall \vec{y})(d\neg f(\vec{y}) \leftrightarrow \neg df(\vec{y})), (\forall \vec{y})(d(f \vee g)(\vec{y}) \leftrightarrow df(\vec{y}) \vee dg(\vec{y})), (\forall \vec{y})(d(f \wedge g)(\vec{y}) \leftrightarrow df(\vec{y}) \wedge dg(\vec{y}))$. As a consequence, if every formula of L is a Boolean combination, modulo T, of formulas taken in some set F (for example, if T admits quantifier elimination and F is the set of quantifier-free formulas), then we need only define df for the formulas f of F, since if g is a Boolean combination of $f_1, \ldots, f_n$, then dg will be the corresponding Boolean combination of $df_1, \ldots, df_n$.

Theorem 11.7. *A type p over a model M of T is definable if and only if for every elementary extension N of M, it has only one heir over N.*

Proof. Let p be definable, and let q be an heir of p over N. If $q \models f(x, \vec{b})$, then $N \models df(\vec{b})$, as there is no $\vec{b}'$ in N such that $p \models f(x, \vec{b}') \wedge \neg df(\vec{b}')$. Since $(\forall \vec{y})(df(\vec{y}) \vee d\neg f(\vec{y}))$ is true in M, and therefore also in N, that leaves only one possibility for q.

Conversely, if p is not definable, then the structure (M, dp) is not definable, even with parameters, in the structure M. By Svenonius's theorem (Theorem 9.2; it needs to be generalized very lightly, since here we are adding several relations instead of just one), there is an elementary extension of (M, dp), which by Lemma 11.5 has the form (N, dq), with an automorphism s of the $L(M)$-structure of (N, dq), that is to say, an automorphism of N that fixes M pointwise and which moves (N, dq). Then q and sq are two distinct heirs of p. □

Theorem 11.7 shows that the heir q of a definable type p is obtained by using the definition of p over the model N. Therefore, q is also definable.

Furthermore, q is a strong heir of p: Since (N, dq) is interpretable in N the same way as (M, dp) is interpretable in M, and since $M \prec N$, we must have $(M, dp) \prec (N, dq)$!

It is possible for a definable type q to inherit an undefinable type p: In the theory of dense orders without endpoints, all types are definable except those of the irrational sort, and an irrational type has heirs of the a^+ and a^- sorts. In this case, q is not a strong heir of p: For every formula $f(x, \vec{y})$, there is a formula $g(\vec{y}, \vec{z})$ such that $(N, dq) \models (\exists \vec{y})(\forall \vec{z})(df(\vec{y}) \leftrightarrow g(\vec{y}, \vec{z}))$. So if (N, dq) were an elementary extension of (M, dp), or even just elementarily equivalent to (M, dp), the same would be true in (M, dp), and p would be definable.

Finally, note that for definable types, the transitivity of heritage can no longer fail: If $p \subset q \subset r$, and if p is definable, then r is the heir of p if and only if q is the heir of p and r is the heir of q. The only thing to check is that if r inherits p, it inherits q: That is due to the fact that q and r are both definable, with the same definition, which is the definition of p.

11.3 End Extension Types in Arithmetic

Definable types were not originally invented for questions of stability, but rather to construct models of arithmetic: We shall briefly explain how they helped in that context.

T is the theory of arithmetic (it could also be any complete theory containing the Peano axioms). Let F be the set of functions $y = \phi(\vec{x})$ definable in T by a formula without parameters. We have already noted, in Section 10.5, that if A is a set of parameters, then the closure $F(A)$ of A under all functions of F is the minimal prime model over A.

An *end extension* of a model M of T is an elementary extension N of M such that every element of $N - M$ is greater than every element of M; the standard model has only end extensions, but every other model M has nonend extensions: By compactness, we can add a nonstandard element less than all its nonstandard elements.

A type p in $S_1(M)$ is called an *end extension type* if it is realized in an end extension of M: That means that if a realizes p, the prime model over $M \cup \{a\}$ is an end extension of M. The uninteresting case of end extension types is where p is realized in M; in all other cases, p is obviously *unbounded* in M, meaning that $p \models x > a$ for every a in M.

Lemma 11.8. *A type p over a model M of arithmetic is an end extension type if and only if for every formula $f(x, \vec{y})$ and every a in M, the set of $\vec{b}$ in M such that $p \models f(x, \vec{b}) \wedge \vec{b} \leq a$ is coded in M (i.e., is a finite set in the sense of M).*

Proof. Suppose that p is an end extension type, and let N be an end extension of M that realizes p as x. In N, $\{\vec{b} : \vec{b} \leq a \wedge f(x, \vec{b})\}$ is a finite set in the sense of N. It therefore has a code c in N, which is less than the maximum of the codes of sets of n-tuples bounded above by a. If we let d denote this last number, then since $M \prec N$, d is the same in M and in N. Since N is an end extension of M and $c \leq d$, c is also in M, and consequently the set in question is indeed coded in M.

Conversely, suppose that p has the property described. Let x be a realization of p, $\vec{b}$ a tuple from M, and $z = \phi(x, \vec{y})$ a function definable in arithmetic. We must prove that $\phi(x, \vec{b})$ either is in M or else is greater than every element of M. So suppose that $\phi(x, \vec{b})$ is bounded above by some element a of M. Let A be the set of all elements of M less than $\phi(x, \vec{b})$. Since A is bounded above by a, A is coded in M. It is a finite set in the sense of M, and therefore has a greatest element c. As $c + 1$ is also in M, we must have $c = \phi(x, \vec{b})$. □

We can see that a definable type is an end extension type: indeed, the set of $\vec{b}$ in M such that $\vec{b} \leq a$ and $p \models f(x, \vec{b})$ is definable and bounded in M, so (by collection axioms) it is a finite set in the sense of M.

We are now going to show the abundance of definable types:

Theorem 11.9. *Let M be a model of arithmetic, and let $f(x, \vec{a})$ be a formula with parameters $\vec{a}$ in M, satisfied by arbitrarily large elements of M. Then there is an unbounded definable type over M that satisfies $f(x, \vec{a})$.*

Proof. In M we have an enumeration, by the elements i of M, of all formulas $f_i(x, \vec{a_i})$ in one variable x with parameters $\vec{a_i}$ in M. It is tempting to define $g_i(x, \vec{a_i})$ by induction as $f_i(x, \vec{a_i})$ if $f(x, \vec{a}) \wedge g_0(x, \vec{a_i}) \wedge \cdots \wedge f_{i-1}(x, \vec{a_i}) \wedge f_i(x, \vec{a_i})$ is satisfied by arbitrarily large elements of M, and as $\neg f_i(x, \vec{a_i})$ otherwise, and to take p as the set of the g_i. The problem is that we would need an arithmetic definition of satisfaction in M, which Tarski's theorem prevents. So we proceed as follows:

Let F_0 be the set consisting of $f(x, \vec{a})$ and of all formulas $x > b, b \in M$. Every fragment of F_0 *that is finite in the sense of M* is satisfied by an element of M; indeed, the b that appear in this fragment form a bounded set.

Now we enumerate all Σ_1 formulas with parameters in M, in the free variable x, as $f_0(x), f_1(x), \ldots, f_i(x), \ldots$ and define the formula g_i by induction on i as f_i if $g_0(x) \wedge \cdots \wedge g_{i-1}(x) \wedge f_i(x)$ (if i is nonstandard, then the conjunction is nonstandard finite), together with any finite fragment of F_0, is satisfiable by an element of M: Since the formulas are of bounded Σ_n class (here, Σ_1) this definition does have an arithmetic character, and can be translated into M. Otherwise, g_i is $\neg f_i$. Let F_i be the (definable) set of all the g_i: It is clear that in M, every finite subset of $F_0 \cup F_1$ is satisfied by an element of M. That implies that seen from outside, the standard part of $F_0 \cup F_1$ is a really consistent set of formulas, since the true finite subsets of M, of standard cardinality, are coded in M.

Continue: The nth stage, for some *standard* number n, consists in enumerating the Σ_n formulas and thus defining the set F_n: Its ith element g_i is f_i if $g_0 \wedge \cdots \wedge g_{i-1} \wedge f_i$, together with an arbitrary finite fragment of $F_0 \cup F_1 \cup \cdots \cup F_n$, is satisfiable by an element of M, and $\neg f_i$ otherwise. Every F_n is a definable set of formulas in M.

Then let p be the set of formulas *of standard complexity* (i.e., of *real* formulas) in $\bigcup F_n$. Then P is consistent, since each of its finite fragments is satisfiable in M; it is complete, since either f_i or $\neg f_i$ appears in p; it is indeed unbounded and satisfies the original formula $f(x, \vec{a})$, since it contains F_0; and it is a definable type, since if $g(x, \vec{y})$ is a (real) Σ_n formula, then to see whether $p \models g(x, \vec{a})$, we need to determine whether $g(x, \vec{a}) \in F_n$. □

As a corollary, we can see that every model of arithmetic has proper end extensions. Refining the previous theorem a little, we can even see that if $|M| = \kappa$, there are at least κ^ω unbounded definable types, and also κ^ω pairwise non-M-isomorphic end extensions.

As I have already remarked, these results remain valid for any complete extension of Peano arithmetic, since if a model of such an arithmetic thinks that some sentence is true, that means it actually does satisfy the sentence.

11.4 Stable Types and Theories

A type p over a model M of T is called *stable* if all its sons, over any elementary extension of M, are definable. A theory T is called *stable* if every type over every model of T is stable, or equivalently, if every type over every model of T is definable.

Theorem 11.10. *Let p be a stable type, $p \in S_1(M)$, and let N be an elementary extension of M of cardinality λ. Then the number of sons of p over N is at most $\lambda^{|T|}$. If T is stable, and if A is a set of parameters of cardinality less than or equal to λ, then $|S_1(A)| \leq \lambda^{|T|}$.*

Proof. Every son q of p over N is definable, and therefore is the unique heir of its restriction q_1 to some elementary submodel N_1 of N of cardinality $|T|$ (Theorem 11.6). There are $\lambda^{|T|}$ possible choices for N_1 and at most $2^{|T|}$ types over N_1, making altogether $\lambda^{|T|} \times 2^{|T|} = \lambda^{|T|}$ possibilities for q_1, i.e., for q. Or alternatively: There are not more q than there are definitions, and each definition is a function from the set of formulas $f(x, \vec{y})$ to the set of formulas with parameters in N, which has size $\max(\lambda, |T|)$; the number of possible definitions is therefore at most $\max(\lambda^{|T|}, |T|^{|T|}) = \lambda^{|T|}$.

For the second part, A embeds into a model M of T of cardinality $\max(\lambda, |T|)$. Every type over A extends to a type over M, so there cannot be more types over A than over M; the number of types over M is at most the number of possible definitions, i.e. $\lambda^{|T|}$. □

Theorem 11.11. *Let p be an unstable type, $p \in S_1(M)$. Then for every cardinal $\lambda \geq \max(|T|, |M|)$ there is an elementary extension N of M, of cardinality λ, over which p has at least λ^+ sons. If T is unstable (i.e. not stable), then for every infinite cardinal λ, there is a set of parameters A, of cardinality λ, on which there are at least λ^+ types.*

Proof. As p is unstable, it has an undefinable son p_0 over an elementary extension M_0 of M. By possibly replacing (M_0, dp_0) by one of its elementary submodels, we can assume that $|M_0| = \lambda$. Therefore, there is an elementary extension N of M_0 over which there are two distinct heirs q_1 and q_2 of p_0. Then $q_1 \models f(x, \vec{a})$ and $q_2 \models \neg f(x, \vec{a})$ for some formula $f(x, \vec{y})$ and some $\vec{a}$ in N. Note therefore that for every heir p' of p_0 over a model M', there is an extension N' of M' in which there are two distinct sons q_1' and q_2' of p' that are heirs of p_0 (and not of p', which might be definable!), with $q_1' \models f(x, \vec{a}')$ and $q_2' \models \neg f(x, \vec{a}')$ for some $\vec{a}'$ in N. Indeed, since p' has an ultrapower p_0^U of p_0 as a son, we can take $q_1' = q_1^U, q_2' = q_2^U$.

We then construct an increasing ordinal sequence of models M_α of T, with, for every sequence $\sigma \in 2^\alpha$, tuples $\vec{a}_\sigma$ of parameters, $\vec{a}_\sigma \in M_{\alpha+1}$, and distinct heirs p_σ of p_0, $p_\sigma \in S_1(M_\alpha)$, such that if τ is a restriction of σ, then p_σ is a son of p_τ. The construction is as follows:

- If α is a limit, M_α is the limit of the M_β for $\beta < \alpha$, and if $\sigma \in 2^\alpha$, p_σ is the limit of the p_τ, where τ ranges over the set of restrictions of σ to the $\beta < \alpha$.

- If α is a successor, $\alpha = \beta + 1$, we consider $\sigma \in 2^\beta$ and the type p_σ over M_β. The sequence σ has two extensions to $\beta + 1$: The first, written $\sigma\frown 0$, gives β (which is the greatest element of $\beta + 1$) the value 0; the second, written $\sigma\frown 1$, gives it the value 1. Since p_σ is, by the induction hypothesis, an heir of p_0, there is an extension N_σ of M_β, with $\vec{a}_\sigma$ in N_σ, and two heirs $q_{\sigma\frown 0}$ and $q_{\sigma\frown 1}$ of p_0 over N_σ, which are sons of p, such that $q_{\sigma\frown 0} \models f(x, \vec{a}_\sigma), q_{\sigma\frown 1} \models \neg f(x, \vec{a}_\sigma)$. All the N_σ have a common elementary extension M_α. We take $p_{\sigma\frown 0}$ to be an heir of $q_{\sigma\frown 0}$ over M_α and $p_{\sigma\frown 1}$ to be an heir of $q_{\sigma\frown 1}$ over M_α.

We continue this construction up to the smallest cardinal μ such that $2^\mu > \lambda$; we know that $\mu \leq \lambda$. We thus get 2^μ sons of p, all distinct heirs of p_0, over the model M_μ. The $\vec{a}_\sigma$ for $\sigma \in 2^\alpha, \alpha < \mu$, are sufficient to distinguish them: if $|\alpha| < \mu, |2^\alpha| \leq \lambda$, then the number of the $\vec{a}_\sigma$ is at most $\mu \times \lambda = \lambda$. By the Löwenheim–Skolem theorem, there is a model N of cardinality λ containing M and all the $\vec{a}_\sigma$: The restrictions of the p_σ to N for $\sigma \in 2^\mu$ form a family of at least λ^+ pairwise distinct sons of p.

For the second part, we do the same construction starting from an unstable type, and take A to be the set of the $\vec{a}_\sigma, \sigma \in 2^\alpha, \alpha < \mu$. The restrictions of the p_σ to A, for $\sigma \in 2^\mu$, are then pairwise distinct. $\square$

We call a theory T *λ-stable* or *stable in* λ, where λ is an infinite cardinal, if it has at most λ types over any given set of parameters of cardinality at most λ. We can therefore see, by the last two theorems, that if T is unstable, it is unstable in every λ, while if it is stable, it is stable in every cardinal λ such that $\lambda = \lambda^{|T|}$, i.e., every cardinal of the form $\lambda^{|T|}$ (since $(\lambda^{|T|})^{|T|} = \lambda^{|T|\times|T|} = \lambda^{|T|}$). In particular, T is stable if and only if it is $2^{|T|}$-stable.

Later, we will determine the possible "stability spectra" for theories T, i.e., the class of cardinals in which T is stable.

Note that if T is stable, then $T(A)$ is also stable for every set A of parameters, and that T and $T(A)$ are stable in the same cardinals $\lambda \geq |A|$.

Furthermore, we can see that if T is stable in λ, and if $|A| \leq \lambda$, then $|S_n(A)| \leq \lambda$. Indeed, realizing the type of $(a_1, \ldots, a_n)$ over A is the same as first realizing the type of a_1 over A, then that of a_2 over $A \cup \{a_1\}, \ldots$, then the type of a_n over $A \cup \{a_1, \ldots, a_{n-1}\}$; since we have only λ possibilities each time, we have at most $\lambda^n = \lambda$ possibilities in all.

The notions of heir, definable types, and stable types, which we have presented for 1-types in order not to carry lots of indices around, can be defined in the same way for n-types, and the analogues of all the theorems which we have proved remain true. We can thus see that *if T is stable* (i.e., all 1-types are definable), *then every n-type over every model of T is definable.* (Otherwise, we would have instability for n-types in every λ, by the analogue of Theorem 11.11.) Moreover, since a formula can involve only finitely many type variables, all α-types are definable. We can see that if T is stable and $|A| \leq \lambda$, then $|S_\alpha(A)| \leq \lambda^{\max(|\alpha|,|T|)}$, which bounds the number of possible definitions of α-types over a model of cardinality $\max(|\alpha|, |T|)$ containing A.

Now consider a formula $f(\vec{x}, \vec{y})$, with parameters $\vec{a}_\sigma$ indexed by the $\sigma \in 2^\alpha$, where α is a fixed ordinal, and $\vec{b}_\tau$, where $\tau \in 2^\beta$, for every $\beta < \alpha$. Let D_α be the set of formulas consisting of the $f(\vec{a}_\sigma, \vec{b}_\tau)$, where $\sigma \in 2^\alpha, \sigma|\beta = \tau, \sigma(\beta) = 0$, together with the $\neg f(\vec{a}_\sigma, \vec{b}_\tau)$, where $\sigma \in 2^\alpha, \sigma|\beta = \tau, \sigma(\beta) = 1$.

The formula $f(\vec{x}, \vec{y})$ is said to have *the dichotomy property* in T if D_ω is consistent with T. Note that every finite fragment of D_α can be interpreted in D_n, for a large enough n. Hence if f has the dichotomy property, then every D_α is consistent, and for f to have the dichotomy property it is (necessary and) sufficient that every D_n be consistent, that is to say, that T contain, for every n, the axiom obtained by existentially quantifying the $\vec{a}_\sigma$ and the $\vec{b}_\tau$ in front of the conjunction of the formulas of D_n.

Theorem 11.12. *T is unstable if and only if some formula $f(x, \vec{y})$ (where x is reduced to a single variable) in it has the dichotomy property.*

Proof. If T is unstable, then the tree of types constructed in the proof of Theorem 11.11 exhibits the dichotomy for a formula $f(x, \vec{y})$.

Suppose that f has this property. Let λ be an infinite cardinal and let μ be the least cardinal such that $\lambda^\mu > \lambda$. Then there is a realization of D_μ in a model of T: If A is the set of the $\vec{b}_\tau$, then $|A| \leq \lambda$, and the a_σ, which number at least λ^+, all have distinct types over A. There is therefore instability in every λ: T is unstable. □

We can see therefore that if there is a formula $f(\vec{x}, \vec{y})$ with the dichotomy property, then there is a formula $g(x, \vec{y})$ with the same property.

Similarly, we can see that p is unstable if and only if for some formula $f(x, \vec{y})$, it is consistent with the set D_ω associated with f. (We add to D_ω the sentences stating that the a_σ are realizations of p.)

If p is stable, then its heir is stable, as are all of its sons. Conversely, if p is unstable, then every heir q of p is unstable. For p then has a son r that has two distinct heirs r_1 and r_2, and for some ultrafilter U, q has p^U as a son, which has r^U as a son, which has two distinct heirs r_1^U and r_2^U. Alternatively: q inherits its father's dichotomy property.

One last remark: Consider a model M of T, an elementary extension N of M, and a formula $f(\vec{x}, \vec{b})$ with parameters $\vec{b}$ in N. The set A of n-tuples of M that satisfy this formula has no reason to be definable with parameters in M; nevertheless, that happens if T is stable, so we can find a formula $g(\vec{x}, \vec{a})$ with parameters $\vec{a}$ in M such that the $\vec{x}$ in M that satisfy $f(\vec{x}, \vec{b})$ are the same as those that satisfy $g(\vec{x}, \vec{a})$. (In no way does that mean that these two formulas are equivalent: That is no longer true for the $\vec{x}$ in N!) Indeed, this condition (for every $M, N, f(\vec{x}, \vec{b})$) is equivalent to stability, since it implies that the type of $\vec{b}$ over M is definable.

11.5 Historic and Bibliographic Notes

The study of stable theories, which we are now beginning, is clearly branded with the personality of a single man, Saharon Shelah. The impressive body of results that he obtained on constructions of models, for stable or unstable theories, has been condensed into a monumental work [She78], whose second edition, with complementary sections, is eagerly awaited. By "monumental" I do not mean that the book is especially thick, but rather that it is a particularly enticing work by virtue of the power of invention, the extraordinary technical virtuosity, and the visionary inspiration of which it is the tangible manifestation.

It could hardly be recommended to a novice, even a gifted one, since its author's trying style makes it almost totally incomprehensible even to an experienced logician. Exegeses of Shelah's work argue about the importance of the hypotheses and conclusions of the theorems appearing there; examination of the proofs is not very helpful for understanding, because they are particularly abstruse, or else absent altogether, the expression "left to the reader" coming most frequently from Shelah's pen. By way of illustration, pages 520 and 521 of the book, "added in proof," remain a mystery to the author of these lines, who does not understand to what part of this work Shelah is so kind as to allude.

When someone wants to give bibliographic references on this subject, one solution, which combines ease with safety, would be to declare that everything can be found, in explicit or implicit form, in Shelah's book. Myself, I would really like to make this proclamation as a precaution, but for several reasons I am not satisfied with it.

The first is that it is the easy solution for the author, but not for the reader!

The second is that Shelah boarded a moving train, that he was not involved in the beginnings of stability theory, and that it is excessive to reckon that everything done before him was just a prelude to his work. One who cannot be considered a mere precursor to Shelah is Michael Morley; his pivotal article [Mor65] introduced, from square one, the gist of the techniques that were subsequently developed. Morley, who, by proving the first theorem on structure, took model theory out of the "general nonsense" where it had been sinking, is the founder of stability theory, that is to say, of contemporary model theory.

Shelah's intrusion into this domain discouraged a number of good mathematicians, who left it; there were others who chose to continue; still others—the unwary ones—wanted to begin and, instead of rivaling Shelah in his favorite domain, at least to discuss aspects that Shelah had not considered.

One example is provided by the notion of heir. Daniel Lascar used the term "heir" or "eldest son" to refer to the son of a definable type obtained by using the same definition [Las73], [Las75]; the general notion of heir appeared in [Poi77] and was set forth in [LP79], with the characterization of definable types by the uniqueness of the heir.

Some trustworthy witnesses attest that the notion of definable type was introduced in 1968, not by Shelah, but by Haim Gaifman, in order to construct end extensions of models of arithmetic; see [Gai76]. In Gaifman's work, the expression "end extension type" had a more restricted sense than here.

The term "stable in λ" is due to [Row64], and the expression "stable theory" appeared in [She69]. If the former can be justified by the fact that the cardinality does not increase when we go from parameters to types, the latter is not very fortunate; it is now consecrated by common usage. The equivalence between stability and the definable character of types is in [She71b]; there we also find the dichotomy property, although with no special name attached.

Finally, let us point out to the lovers of curiosities that some techniques of stability, analogous to those that we are going to use, are useful in the study of Banach spaces, and that the combinatorial contents of some of Shelah's lemmas had long been known by Banach space theorists! For all that, you can consult [Ray83].

12

Special Sons, Morley Sequences

Un chant mystérieux tombe des astres d'or.

A.R.

12.1 Special Sons

Let M be a model of T and p a type over M. Let N be an elementary extension of M that realizes all types of $S_n(M)$ for all n. A son q of p over N is called *special* if for every formula $f(x, \vec{y})$, if $\vec{a}$ and $\vec{b}$ are in N and have the same type over M, and if $q \models f(x, \vec{a})$, then $q \models f(x, \vec{b})$. In other words, the fact that $q \models f(x, \vec{a})$ depends only on the type of $\vec{a}$ over M. We also call q *M-special* to say that it is a special son of its restriction to M; in this case, the function that sends a formula $f(x, \vec{y})$ to the set of all types over M of tuples $\vec{a}$ of N such that $q \models f(x, \vec{a})$ is called an *infinitary definition of q over M*.

More generally, if N is any elementary extension of M (or even simply a set of parameters containing M), q in $S_1(N)$ will be called a *special son* of p if there is a son q_1 of q over an elementary extension N_1 of N realizing all n-types over M for every n, q_1 being special in the sense of the last paragraph. Obviously, there can be many possibilities for q_1, and we cannot speak of an infinitary definition of q, since not all types of tuples of parameters have appeared in N yet.

We can see that it would be unwise to repeat the definition of M-special sons as is in the case where the model N omits some types of $S_n(M)$. Let us take a model M of the theory T of infinite sets (the language consists only of equality), and let p be the nonrealized type in M. Let N be an elementary extension of M obtained by adding a single point a to it. The son q of p realized by a does have the property that for every formula $f(x,\vec{y})$ and every $\vec{b}$ in N, the fact that $q \models f(x,\vec{b})$ depends only on the type of $\vec{b}$ over M; indeed, by quantifier elimination, the only formulas to consider are $x = y, x \neq y$, and a is the only realization of p in N. However, q is not a special son of p: When we add a second point b, then a and b have the same type over M, and the unique son q_1 of q no longer has this property.

That being said, let us begin with a few lemmas:

Lemma 12.1. *Every type p is a special son of itself.*

Proof. If p is in $S_1(M)$ and N is an elementary extension of M, I am going to show that there exists a son q of p over N such that the fact that $q \models f(x,\vec{a})$ depends only on the type of $\vec{a}$ over M; taking N to be sufficiently saturated will yield the desired result.

Consider the set of formulas consisting of p, $T(N)$, and the axioms $f(x,\vec{a}) \leftrightarrow f(x,\vec{b})$ whenever $\vec{a}$ and $\vec{b}$ have the same type over M. A finite fragment of this family mentions only a finite fragment of p, which is satisfied by some element α of M. If we interpret x as α, then all the other axioms are satisfied! Consequently, this set is consistent, and if we complete it we get the desired type q. $\square$

Lemma 12.2. *Let $p \in S_1(M)$, let $q \in S_1(N)$ be a special son of p, and let P be an elementary extension of N. Then q has a son r over P that is M-special; moreover, if N realizes all types of n-tuples over M, this M-special son is unique.*

Proof. Let N_1 be an extension of N, realizing all types of n-tuples over M, over which q has an M-special son q_1. We need only use the infinitary definition of q_1 to produce a type over P. The set of formulas $f(x,\vec{a})$, where $\vec{a} \in P$ and is such that there exists $\vec{a}'$ in N_1 of the same type over M as $\vec{a}$ and such that $q_1 \models f(x,\vec{a}')$, is consistent, because every finite set of parameters of P has a type over M that can be copied in N_1. These formulas define a type r over P, which is obviously M-special and a son of q.

If N realizes all types of n-tuples over M, we have no further choices for the infinitary definition of r. $\square$

We can thus survey the landscape: If the model N is not sufficiently saturated, then the M-special types can be ramified, but once N realizes all types of $S_n(M)$, the M-special extension is unique. I shall speak of the number of special sons of a type, thinking of these "definitive" special sons.

Lemma 12.3. *If $|M| = \lambda \geq |T|$, then the number of M-special types is at most $2^{(2^\lambda)}$.*

Proof. There are not more M-special sons than there are infinitary definitions. An infinitary definition is a function which sends a formula to a set of types over M; there are $|T|$ formulas and at most 2^λ types over M, so the number of infinitary definitions is at most $(2^{(2^\lambda)})^{|T|} = 2^{(2^\lambda \times |T|)} = 2^{(2^\lambda)}$. □

Now, a little lemma of model theory:

Lemma 12.4. *Let A be a set of parameters, and let $\vec{b}$ be a tuple of parameters, from a model M of T. If the satisfaction of a formula $f(x, \vec{b})$ (for x in any elementary extension of M) depends only on the type of x over A, then this formula is equivalent to a formula $g(x, \vec{a})$, with parameters $\vec{a}$ in A.*

Proof. Consider the restriction function from $S_1(A \cup \{\vec{b}\})$ to $S_1(A)$. By continuity, the images of the compact sets $\langle f(x, \vec{b})\rangle$ and $\langle \neg f(x, \vec{b})\rangle$ are compact, and therefore closed, in $S_1(A)$; their union is still all of $S_1(A)$. The hypothesis tells us that these two images are disjoint, that is to say, they are complements of each other, and therefore both clopen in $S_1(A)$. □

Theorem 12.5. *Let p be in $S_1(M)$ and let q be a special son of p over some elementary extension N of M with the following property: For every $\vec{a}$ in N, every type over $M \cup \{\vec{a}\}$ is realized in N. Then if q is definable, p is definable and q is its heir.*

Proof. Consider a formula $f(x, \vec{y})$ and the (finitary!) definition of q for this formula. Then $df(\vec{y})$ uses parameters $\vec{a}$ in N, but, since every type of n-tuple over $M \cup \{\vec{a}\}$ is realized in N, and since q is M-special, then whether $\vec{y}$ satisfies $df(\vec{y})$ depends only on the type of $\vec{y}$ over M. By the previous lemma, $df(\vec{y})$ can be replaced by a formula with parameters in M, which will therefore also be a definition for p. □

Corollary 12.6. *A stable type has only one special son, its heir.*

Proof. Let p be a stable type over M, let q be a special son of p over N, and let q_1 be an M-special son of q over a model N_1 with the saturation property described in the hypothesis of the previous theorem. Since q_1 is definable, it is the heir of p over N_1, and q is the heir of p over N. □

The heir of a definable type is obviously special. However, a nondefinable type having as many heirs as we want over a large enough model (see the proof of Theorem 11.11) will have some of them nonspecial. Again, by Theorem 11.7, it has a strong heir that can be moved by an M-automorphism; this heir is obviously not M-special.

Note also that a special son of an unstable type is unstable. Indeed, if p has a stable special son q, then q is definable, as are all its sons, so q is the unique heir of p, so we know that p is stable. (An heir of an unstable type is unstable: See the end of Section 11.4.)

The goal of the end of this section is to clarify the notion of special son; it contains nothing essential for succeeding chapters. We adopt the following convention: If $M \subset A$, then a type q of $S_1(A)$ is called M-special if it has an M-special (in the previous sense) son over a model N containing A.

Lemma 12.7. *Let M be a model of T, let $\lambda > |M|$, and let N be a λ-saturated, strongly λ-homogeneous elementary extension of M. Let A be a subset of N, $M \subset A \subset N, |A| < \lambda$, and let p and q be two distinct types of $S_1(A)$ with the same restriction to M, with p M-special. Let X be a subset of N containing all realizations of p in N and no realizations of q, and let Y be a subset of N containing all realizations of q in N and no realizations of p. Then X and Y each have at least λ distinct images under M-automorphisms of N.*

Proof. Choose an infinitary definition d of an M-special son of p over a sufficiently saturated model. We construct a sequence a_α of elements of N and a sequence A_α of subsets of N by induction on $\alpha < \lambda$ as follows: $A_0 = A$ and a_0 is a realization of q. The sequence a_α realizes the type over $\bigcup_{\beta<\alpha}$ obtained by using the definition d. Since the type of a_α over M is the restriction of q to M, there exists A_α, of the same type over M as A, such that the type of a_α over A_α is the type corresponding to q (i.e., $A_\alpha\frown a_\alpha$ and $A_0\frown a_0$ have the same type over M). Since N is sufficiently saturated, this construction can be done in N, and since it is homogeneous, there is an M-automorphism s_α of N that carries A onto A_α. If $\beta < \alpha$, then $a_\alpha \in s_\beta x, a_\alpha \notin s_\alpha x, a_\alpha \in s_\alpha y, a_\alpha \notin s_\beta y$. □

Theorem 12.8. *Let M be a model of T, $\lambda > |M|$, and N a λ-saturated strongly λ-homogeneous elementary extension of M. Let X be a subset of N that can be moved by an M-automorphism of N (i.e., membership in X does not depend only on the type of x over M). Then X has at least λ distinct images under these M-automorphisms of N.*

Proof. First case: There is a set A, $M \subset A \subset N, |A| < \lambda$, such that membership in X depends only on type over A. Since X is not fixed under all M-automorphisms, this membership does not depend only on type over M, and there exist p and q in $S_1(A)$ with the same restriction to M, such that X contains all realizations of p and none of q. We can obviously assume that one of the two is special, and apply Lemma 12.7.

Second case: Otherwise, we can construct two sequences $a_\alpha, b_\alpha, \alpha < \lambda$, with a_α and b_α having the same type over $M \cup \{\ldots, a_\beta, \ldots\}_{\beta<\alpha}$, with $a_\alpha \in X, b_\alpha \notin X$. By homogeneity, there is an automorphism s of N, fixing M and all a_β for $\beta < \alpha$, and sending a_α to b_α. For $\beta < \alpha$, $a_\beta \notin s_\beta x, a_\beta \in s_\alpha x$. □

Theorem 12.9. *Let M be a model of T, $\lambda > |M|$, and N a λ-saturated, strongly λ-homogeneous elementary extension of N. If a type p of $S_1(N)$ is not M-special, then it has at least λ distinct conjugates under the M-automorphisms of N.*

Proof. By generalizing Theorem 12.8 to subsets X of N^n. □

It is obvious that a special son cannot be moved by an M-automorphism. Theorem 12.9 states the converse of this property: p in $S_1(N)$ is M-special if and only if it has a son over every elementary extension of N that cannot be moved by M-automorphisms; also, in the case where it is not M-special, we can find an elementary extension of N in which every son of p has as many conjugates under M-automorphisms as we want.

12.2 Coheirs

We know that a type p of $S_1(M)$ is finitely satisfiable in M: Whenever $p \models f(x, \vec{a})$, where $\vec{a} \in M$, then, since $T(M) \models (\exists x)f(x, \vec{a})$, there exists b in M such that $M \models f(b, \vec{a})$. That means that the types of elements of M form a dense set in $S_1(M)$. (Moreover, this property characterizes the sets of parameters that are models, since it is equivalent to Tarski's test: See Section 5.1).

Now, if q is a type over an elementary extension N of M, we call q a *coheir* of its restriction p to M if it is finitely satisfiable in M, that is to say if it belongs to the closure in $S_1(N)$ of the set of types of elements of M; in other words, whenever $q \models f(x, \vec{a}), \vec{a} \in N$, there exists b in M such that $N \models f(b, \vec{a})$.

Why do we say coheir? We are generalizing the notion of heir a little: If $M \subset A$ and $p \in S_1(A)$, we shall say, even if A is not a model (although M must be!), that p inherits its restriction to M if whenever $p \models f(x, \vec{a}, \vec{b}), \vec{a} \in M, \vec{b} \in A$, then there exists $\vec{b}'$ in M such that $p \models f(x, \vec{a}, \vec{b}')$. By Theorem 11.2, that means that p extends to an heir in the usual sense over every model containing p. Likewise, we generalize the notion of coheir, saying that $p \in S_1(A)$ coinherits its restriction to M if it is finitely satisfiable in M; we shall see very soon that that similarly means that it extends to a coheir of its restriction to M over any model containing A. Also, these notions are meaningful not only for 1-types, but also for n-types and even α-types.

That being said, note that to say that the type of $\vec{a}$ over $M \cup \{\vec{b}\}$ inherits its restriction to M is to say that the type of $\vec{b}$ over $M \cup \{\vec{a}\}$ coinherits its restriction to M. In both cases, that really means that, whenever $f(\vec{a}, \vec{b})$ is true, where $f(\vec{x}, \vec{y})$ is a formula with parameters in M, then there exists $\vec{b}'$ in M such that $f(\vec{a}, \vec{b}')$ is true. Similarly, if there are infinitely many type variables, to say that the type of A over $M \cup B$ inherits its restriction to M is to say that the type of B over $M \cup A$ coinherits its restriction to M.

The notion of coheir is in some way dual to that of heir: We exchange the roles of the type variables and the parameter variables. This duality allows us to economize in a few lemmas to follow, since every property of heirs has a translation in terms of coheirs. For example, the existence of an heir implies that of a coheir, since if the types of A and B over M are given, then to place $A^\frown B$ in relation to M (i.e., to complete the union of the types of A and B over M into one type over M) in such a way that the type of A over $M \cup B$ inherits its restriction to M amounts to placing A and B in relation to each other over M in such a way that the type of B over $M \cup A$ coinherits its restriction to M! But as this is a somewhat Byzantine way to proceed, and nothing about the redundancy will frighten us, we are going to prove these lemmas:

Lemma 12.10. *Let N be an elementary extension of M and let π be an incomplete type over N (i.e., a closed set of $S_1(N)$) that is finitely satisfiable in M. Then π can be completed as a type p over N that coinherits its restriction to M.*

Proof. Consider the closure $\bar{M}$ in $S_1(N)$ of the set of types of elements of M, and the clopen sets defined by finite fragments of π. By hypothesis, this family of closed sets has the finite intersection property; by compactness, its intersection is nonempty. □

Lemma 12.11. *Every type (over a model of T) has coheirs. More precisely, if p is in $S_1(M)$, q is a coheir of p over an elementary extension N of M, and P is an elementary extension of N, then q has a son in $S_1(P)$ that is a coheir of p.*

Proof. Consider q as an incomplete type over P and apply the preceding lemma. □

As p is its own coheir, the lemma implies that it has coheirs over every elementary extension of its own model. Another obvious result: If $p \subset q \subset r$, and if r is a coheir of p, then r is a coheir of q and q is a coheir of p.

Lemma 12.12. *If $|M| = \lambda$, then the number of types that coinherit their restrictions to M is at most $2^{(2^\lambda)}$.*

Proof. In $S_1(N)$, a point of $\bar{M}$ is determined by the trace over M of its filter of neighborhoods, and there are at most $2^{(2^\lambda)}$ possibilities for this set of subsets of M. □

And now what you have all been waiting for. . .

Theorem 12.13. *A coheir is special.*

Proof. Let p be in $S_1(M)$, let q be a coheir of p over N, and let N_1 be an elementary extension of N that realizes all types of n-tuples over M. By Lemma 12.11, q has a son q_1 over N_1 that is a coheir of p. Also, if $\vec{a}$ and $\vec{b}$ are in N_1 and have the same type over M, there is no x in M that satisfies $f(x, \vec{a}) \wedge \neg f(x, \vec{b})$. Consequently, q_1 is M-special, as is q. □

Note that the special type exhibited in Section 12.1 is a coheir.

Let us return to our example in which T is the theory of dense orders without endpoints. If p is realized, then it has only one son over N, which is its unique heir and its unique special son. If p is an a^+ type, then its two special sons are its heir, which is the a^+ type over N, and its coheir, which is an upper bound for all realizations of p in N. (The other sons q of p are not special, since we can find two realizations b_1 and b_2 of p in N such that $q \models b_1 < x < b_2$.) Similarly, if p is an a^- type, then it has two special sons: its heir, which bounds all points of N realizing p from above, and its coheir, which bounds them all from below. An irrational type p also has two special sons, which are both coheirs (and heirs) of p: one, stuck onto the left of M, which bounds all realizations of p from below, and the other, stuck onto the right, which bounds them from above. The type $+\infty$ also has two special sons, its heir, and its unique coheir, which bounds all realizations of p in M from below; likewise the type $-\infty$ has two special sons, its heir and its coheir.

To have a special son that is neither an heir nor a coheir, consider the following 2-type, obtained by putting two elements into an a^+ type: $p \models b < x_1 < x_2 < c$ for all b, c in M, $b \leq a, a < c$. The type p has three special sons: p_1, for which all realizations of p in N are bounded below by x_2, is the heir of p; p_2, for which all realizations of p in N are bounded above by x_1, is the unique coheir of p; and finally p_3, for which all realizations of p are between x_1 and x_2. This type p_3, an heir from the side of x_1 and a coheir from the side of x_2, is neither an heir nor a coheir of p. We can easily transform this example of a 2-type into an example of a 1-type (consider the structure consisting of the chain $\mathbf{Q} \times \mathbf{Q}$, with the two projections in the language).

Since a coheir is special, by Theorem 12.6, *a stable type has only one coheir, which is its heir.* If the type of $\vec{a}$ over M is stable, then the type of $\vec{a}$ over $M \cup \{\vec{b}\}$ inherits its restriction to M if and only if it coinherits it, i.e. if and only if the type of $\vec{b}$ over $M \cup \{\vec{a}\}$ inherits (or coinherits!) its restriction to M.

That is a thoroughly fundamental property associated with stability, called *"symmetry of forking"* exhibited here over a model M, although it will later be generalized to an arbitrary set of parameters. If T is stable, we say that $\vec{a}$ and $\vec{b}$ are *independently placed*, or simply *independent over* M, if the type of $\vec{a}$ over $M \cup \{\vec{b}\}$ inherits its restriction to M, or equivalently if the type of $\vec{b}$ over $M \cup \{\vec{a}\}$ inherits its restriction to M. This notion of independence, essential to stability, is therefore symmetric in $\vec{a}$ and $\vec{b}$.

Finally, note that it is close to obvious that in a stable theory we have uniqueness of the coheir: To say that the type of A over M has only one coheir over $M \cup B$ amounts to saying that the type of B over M has only one heir over $M \cup A$.

Alternatively: A type p of $S_1(M)$ is determined by the traces over M of its neighborhoods $\langle f(x, \vec{a}) \rangle$, which are sets definable with parameters in M. If T is unstable, and if we extend the model M to N and if $g(x, \vec{b})$ is a formula with parameters in N, then its trace over M might be a new set, and the neighborhood filter of p can be refined into two new filters, one containing $\langle g(x, \vec{b})\rangle$, and the other containing $\langle \neg g(x, \vec{b})\rangle$, from which we get the existence of several coheirs for p. But that is not possible if T is stable, since, as we have noted, the type of $\vec{b}$ over M being definable, the trace over M of $g(x, \vec{b})$ is definable in M: A type p of $S_1(M)$ has only one coheir, since we cannot refine the filter of the traces of its neighborhoods over M.

However, it remains to be proved that in the stable case, this unique coheir of p is its heir.

12.3 Morley Sequences

Consider a sequence a_i of elements of a model M of T, indexed by a totally ordered set I. (Strictly speaking a, "sequence" is a function from ω to M; here we use this term in a slightly more general way, an I-sequence being a function from I to M.) Note that the order $\leq$ on I has nothing to do with the language of T.

We shall say that this sequence is *indiscernible* (in the order) if for every natural number n, whenever $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$ are two strictly increasing n-tuples of I, the n-tuples $(a_{i_1}, \ldots, a_{i_n})$ and $(a_{j_1}, \ldots, a_{j_n})$ have the same type; in other words, this sequence satisfies the axioms $f(a_{i_1}, \ldots, a_{i_n}) \leftrightarrow f(a_{j_1}, \ldots, a_{j_n})$ whenever $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$.

If in addition these n-tuples have the same type over the set of parameters A, i.e., if the sequence is indiscernible for the theory $T(A)$, then we call it *indiscernible over* A or A-*indiscernible.*

Define the *Ehrenfeucht set* of the indiscernible sequence s as the set of formulas $f(x_1, \ldots, x_n)$ satisfied by the tuples of s whose sequence of indices is increasing. If we say that the I-sequence s is indiscernible, and we specify its Ehrenfeucht set, then we completely determine the type of s (over $\emptyset$; or over A if the indiscernibility is over A, and naturally, the Ehrenfeucht set contains those formulas with parameters in A that are satisfied by the increasing tuples of the sequence). We shall call two indiscernible sequences *similar* if they have the same Ehrenfeucht set.

Note that if there is an infinite indiscernible sequence with Ehrenfeucht set E, or even just arbitrarily long finite indiscernible sequences with Ehrenfeucht set E, then for every chain I, there is an indiscernible I-sequence with Ehrenfeucht set E. For we can write the axioms stating that the a_i form an indiscernible sequence with Ehrenfeucht set E. A finite fragment of this set of sentences mentions only finitely many indices $i_1 < \cdots < i_n$, and it is interpretable in any indiscernible sequence with Ehrenfeucht set E and length at least n. Consequently, these axioms do have a model. We will never lose sight of the fact that stating the existence of an indiscernible sequence in a model of T amounts to stating the consistency of a certain theory, a certain type of I-tuple.

Similarly, we can see that if s is an infinite indiscernible I-sequence, and if J is a chain extension of I, then s can be extended to an indiscernible J-sequence t, which is obviously similar to s. Indeed, there is an indiscernible J-sequence $t' = (\ldots, a'_j, \ldots)$ similar to s, since I is infinite. The sequence s and the sequence $s' = (\ldots, a'_1, \ldots)_{i\in I}$ have the same type, since they are similar and have the same set of indices I; hence $\bar{t}$ exists.

We shall call the sequence s *totally indiscernible* if it is indiscernible no matter how its indices are ordered; in other words, whenever $i_1, \ldots, i_n$ are distinct, as are $j_1, \ldots, j_n$, the sequence satisfies the axioms $f(a_{i_1}, \ldots, a_{i_n}) \leftrightarrow f(a_{j_1}, \ldots, a_{j_n})$. That means that if the Ehrenfeucht set of s contains $f(x_1, \ldots, x_n)$, then it also contains every formula obtained from $f(x_1, \ldots, x_n)$ by permuting the x_i; a sequence that is similar to a totally indiscernible sequence is itself totally indiscernible.

With the exception of the trivial case in which the sequence repeats the same element, the a_i of an indiscernible sequence are pairwise distinct. In the case of total indiscernibility, we shall rather speak of a (totally) *indiscernible set*, since the definition of total indiscernibility does not involve a particular enumeration.

Finally, we are satisfied with defining indiscernible sequences of elements; we could define indiscernible sequences of n-tuples, or even of α-tuples, in the same way.

These indiscernible sequences play a particularly important role in the construction of models of a stable theory. This role appeared in Michael Morley's first article, which began the study of stability. Thus the name *Morley sequence* is given to the indiscernible sequence associated with a special type in the following way:

Consider a model M of T, an extension N of M realizing all types of n-tuples over M, and an M-special type p over N. We are also given a set A of parameters containing M, A and N being located in a large model of T. The *Morley sequence of* (p, M) *over* A is defined as the sequence (or, more precisely, the type of sequence over A) constructed as follows: Let a_0 realize the unique M-special son of p over $N \cup A \cup \{a_0\}$, then let a_1 realize the unique M-special son of p over $N \cup A \cup \{a_0\}, \ldots,$ let a_{n+1} realize the unique M-special son of p over $N \cup A \cup \{a_0, \ldots, a_n\}, \ldots$.

If we find the model N too cumbersome, we can economize, since p occurs only through its *infinitary definition* over M: At stage $n+1$ we let a_{n+1} realize the type given by using this infinitary definition over $A \cup \{a_0, \ldots, a_n\}$. Then a_{n+1} satisfies the formula $f(x, \vec{a}, a_0, \ldots, a_n)$, where $\vec{a} \in A$, iff the tuple $(\vec{a}, a_0, \ldots, a_n)$ has a type over M appropriate to the infinitary definition of p over M.

For example, if p in $S(M)$ is definable, the Morley sequence of its heir will be obtained by using the (finitary!) definition of p: Let a_0 realize the unique heir of p over A, then let a_1 realize the unique heir of p over $A \cup \{a_0\}, \ldots$, let a_{n+1} realize the unique heir of p over $A \cup \{a_0, \ldots, a_n\}), \ldots$.

If p is a stable type, we shall speak of the *Morley sequence of p*, instead of the Morley sequence of the heir of p, since p has only one special son, which is its heir. However, when a type has several special sons, it gives rise to as many Morley sequences as it has special sons, and it will be necessary to specify which one we are speaking of.

The uninteresting case of Morley sequence is when the M-special type is that of an element of M: The sequence only repeats this element. If not, the sequence consists of distinct elements, since a special son of a nonrealized type is not realized. (If p is a nonrealized type over M, then we can find an elementary extension N of M in which it has as many realizations as we want, and if $a \neq b$, we cannot have $q \models x = a$ and $q \models x = b$!)

Theorem 12.14. *The Morley sequence of a special type over A is A-indiscernible.*

Proof. Let us show by induction on n that if $m_0 < \cdots < m_n$, then $(a_0, \ldots, a_n)$ and $(a_{m_0}, \ldots, a_{m_n})$ have the same type over A. If $f(\vec{a}, a_0, \ldots, a_n)$ is true, that means that $(\vec{a}, a_0, \ldots, a_{n-1})$ has a type over M appropriate to the infinitary definition of p. As $(\vec{a}, a_0, \ldots, a_{n-1})$ and $(\vec{a}, a_{m_0}, \ldots, a_{m_{n-1}})$ have the same type over M by the induction hypothesis, $f(\vec{a}, a_{m_0}, \ldots, a_{m_n})$ is also true. □

Special sons thus allow us to construct indiscernible sequences. For example, if $f(x)$ is a formula that is satisfied by infinitely many elements of a model M of T, then there is an infinite indiscernible sequence in an elementary extension of M consisting of (distinct!) elements, all satisfying $f(x)$: Take the Morley sequence of an M-special son, satisfying $f(x)$, of a nonrealized type over M.

Theorem 12.15. *The Morley sequence over A of a stable type is totally indiscernible over A.*

Proof. First let us show that for all i, $(a_0, \ldots, a_{i-1}, a_i, a_{i+1})$ and $(a_0, \ldots, a_{i-1}, a_{i+1}, a_i)$ (we transpose the last two elements) have the same type over A. For that purpose, consider a model N containing $A \cup \{a_0, \ldots, a_{i-1}\}$, with a'_i realizing the heir q of p over N and a'_{i+1} realizing the heir of q (and thus of p) over $N \cup \{a'_i\}$. It is clear that $(a_0, \ldots, a_{i-1}, a'_i, a'_{i+1})$ and $(a_0, \ldots, a_{i-1}, a_i, a_{i+1})$ have the same type over A: Look at the example of definitions of types. However, the type of (a'_i, a'_{i+1}) over N is obtained by realizing first q, then its heir, while the type of (a'_{i+1}, a'_i) is obtained by realizing first q, then its coheir. Since heirs and coheirs are the same thing for stable types, (a'_i, a'_{i+1}) and (a'_{i+1}, a'_i) have the same type over N, from which the conclusion follows.

Now let us show by induction on $n > i + 1$ that $(a_0, \ldots, a_i, a_{i+1}, \ldots, a_n)$ and $(a_0, \ldots, a_{i+1}, a_i, \ldots, a_n)$ have the same type over A. If a_n satisfies

$f(a_0, \ldots, a_i, a_{i+1}, \ldots, a_{n-1}, x)$, it also satisfies $f(a_0, \ldots, a_{i+1}, a_i, \ldots, a_{n-1}, x)$, since the sets of parameters of these formulas have the same type over A.

The theorem therefore follows from the fact that transpositions of two consecutive elements are sufficient to generate the group of all permutations of n elements. □

We have defined only the Morley ω-sequence of a special type, but we can naturally continue the construction and define the Morley α-sequence of this type for every ordinal α, and even its Morley I-sequence for any chain I. These are indiscernible sequences similar to the Morley ω-sequence.

One way to build Morley sequences more closely resembling what Morley did is the following: Consider a model M of T, $M \subset A$, and an $|A|^+$-saturated elementary extension N of M, $M \subset A \subset N$. Then let p be an M-special type in $S_1(N)$. We get a copy in N of the Morley sequence of p over A by letting a_0 realize the restriction of p to A, then letting a_1 realize the restriction of p to $A \cup \{a_0\}, \ldots$, letting a_{n+1} realize the restriction of p to $A \cup \{a_1, \ldots, a_n\}, \ldots$.

12.4 The Independence Property

Let T be a complete theory as usual. A formula $f(\vec{x}, \vec{y})$ is said to have the *independence property* in T if the following axioms I_n are all consequences of T, where the $\vec{x}$ are indexed by $n = \{0, \ldots, n-1\}$ and the $\vec{y}$ by the set of all 2^n subsets of n:

$$(I_n): \qquad (\exists \vec{x}_0) \cdots (\exists \vec{x}_i) \cdots (\exists \vec{x}_{n-1})(\exists \vec{y}_\emptyset) \cdots (\exists \vec{y}_W) \cdots (\exists \vec{y}_{2^n})$$
$$\bigwedge_{i \in W} f(\vec{x}_i, \vec{y}_W) \wedge \bigwedge_{i \notin W} \neg f(\vec{x}_i, \vec{y}_W)$$

By compactness, that means that if we index the $\vec{a}_i$ by the cardinal λ and the $\vec{b}_W$ by the set 2^λ of subsets of λ, then the set I_λ, consisting of the formulas $f(\vec{a}_i, \vec{b}_W)$ if $i \in W$ and $\neg f(\vec{a}_i, \vec{b}_W)$ if $i \notin W$, is consistent with T. We can see that the $\vec{a}_i$ can be split as desired by the formula f; thus the name "independence property."

For example, in arithmetic, the formula "x divides y" has the independence property: It is sufficient to take the x_i pairwise relatively prime. In every theory T of infinite Boolean algebras, the formula $x \leq y$ has the independence property, since we can easily see that we can find $x_0, \ldots, x_n$ in such an algebra, all nonzero and with pairwise intersection zero, then take y_W as the supremum of the $x_i, i \in W$.

Lemma 12.16. *The independence property is symmetric in $\vec{x}$ and $\vec{y}$: If $f(\vec{x}, \vec{y})$ has the independence property, then so does the formula $g(\vec{x}_1, \vec{y}_1) = f(\vec{y}_1, \vec{x}_1)$.*

Proof. Let us show that $I_{2^n}(f)$ implies $I_n(g)$. In I_{2^n}, the $\vec{x}$ are indexed by the subsets W of n, and the $\vec{y}$ by the subsets W of 2^n. We therefore have

$$\bigwedge_{w \in W} f(\vec{x}_w, \vec{y}_W) \wedge \bigwedge_{w \notin W} \neg f(\vec{x}_w, \vec{y}_W).$$

It is sufficient to keep only those W that are ultrafilters, since if W_i is the (principal!) ultrafilter generated by i, the statement that $w \in W_i$ is equivalent to $i \in w$. □

Another definition: Let $s = (\ldots, \vec{a}_i, \ldots)$ be an I-sequence of n-tuples from a model M of T, where I is a chain. We shall call this sequence *divisible* if there exists $\vec{b}$ in an elementary extension of M and a formula $f(\vec{x}, \vec{y})$ such that the indices of the $\vec{a}_i$ that satisfy $f(\vec{x}, \vec{b})$ form a cofinal set in I, as do the indices of the $\vec{a}_i$ that satisfy $\neg f(\vec{x}, \vec{b})$. In order that the sequence be divisible, the chain I must be infinite, and without greatest element.

If the sequence s is *indivisible*, and if A is a subset of an elementary extension of M, then the *limit type of s over A* is the set $t(s/A)$ of formulas $f(\vec{x}, \vec{b})$ with parameters $\vec{b}$ in A that are satisfied by all the $\vec{a}_i$ from some index on. This type is consistent, because it is finitely satisfiable in s, and complete, because s is indivisible. If there is an element a in s of greatest index, this is simply the type of a over A: the uninteresting case. Otherwise, if I has no greatest element, and if the $\vec{a}_i$ are distinct, then the limit type of s over a set A is not realized in A.

Note that the divisible or indivisible character of a sequence is part of its type over $\emptyset$.

The link to the independence property is the following:

Theorem 12.17. *A formula $f(\vec{x}, \vec{y})$ has the independence property in T if and only if there is a divisible indiscernible sequence, consisting of tuples $\vec{a}_i$ of the same length as $\vec{x}$, that is split into two cofinal subsets by a formula $f(\vec{x}, \vec{b})$.*

Proof. Suppose that f has the independence property and let M be a model of T containing a realization of I_ω: $A = \{\vec{a}_0, \ldots, \vec{a}_n, \ldots\}_{n \in \omega}$, $B = \{\vec{b}_\emptyset, \ldots, \vec{b}_w, \ldots\}_{w \in 2^\omega}$, $M \models f(\vec{a}_n, \vec{b}_w)$ if $n \in w$, $M \models \neg f(\vec{a}_n, \vec{b}_w)$ if $n \notin w$.

Let U be a nonprincipal ultrafilter of subsets of ω and let N be a sufficiently saturated extension of M. Let p_U denote the type over N defined as follows: $p_U \models g(\vec{x}, \vec{c})$, where $\vec{c} \in N$, if $\{n : N \models f(\vec{a}_n, \vec{c})\} \in U$. We can see that this type is M-special, and since it is finitely satisfiable in A, it coinherits its restriction to M.

Next, let us construct the Morley sequence $\vec{a}'_0, \ldots, \vec{a}'_n \ldots$ of p_U over M. By definition $\vec{a}'_{n+1}$ satisfies a certain formula with parameters in $M \cup \{\vec{a}'_0, \ldots, \vec{a}'_n\}$ iff this formula is satisfied by almost all of the $\vec{a}_i$ modulo U. Note that by the fact that U is nonprincipal, all the $\vec{a}'_i$ are distinct. We know that this sequence of the $\vec{a}'_n$ is indiscernible, and we are going to show that it is divisible in the way indicated in the hypothesis of the theorem.

I claim that every formula $g(\vec{x}_0, \ldots, \vec{x}_n)$ satisfied by $(\vec{a}'_0, \ldots, \vec{a}'_n)$ is satisfied by an $(n+1)$-tuple of $\vec{a}_i$, and even of pairwise distinct $\vec{a}_i$, since we can incorporate into this formula the fact that $\vec{x}_0, \ldots, \vec{x}_n$ are distinct. Indeed, there exist $\vec{a}_{i_n}$ such that $\vec{a}'_0{}^\frown \cdots {}^\frown \vec{a}'_{n-1}{}^\frown \vec{a}_{i_n}$ satisfies g, then $\vec{a}_{i_{n-1}}$ such that $\vec{a}'_0{}^\frown \cdots {}^\frown \vec{a}'_{n-2}{}^\frown \vec{a}_{i_{n-1}}{}^\frown \vec{a}_{i_n}$ satisfies g, etc. That implies that the tuple $(\vec{a}'_0, \vec{a}'_1, \ldots, \vec{a}'_{2n}, \vec{a}'_{2n+1})$ satisfies the formula

$$(\exists \vec{y}) \bigwedge f(\vec{x}_{2i}, \vec{y}) \wedge \bigwedge \neg f(\vec{x}_{2i+1}, \vec{y}).$$

This holds, since, all tuples consisting of $2n+2$ distinct tuples of A satisfy this formula: take $\vec{y}$ to be a suitable b_w, so that our tuple cannot satisfy the negation of this formula. As a consequence, by compactness, there exist $\vec{b}$ such that $f(\vec{a}'_n, \vec{b})$ is true if n is even and false if n is odd, and the sequence does have the indicated divisibility property.

Conversely, suppose that we have a divisible indiscernible sequence $\vec{a}_i$, this sequence being split by $f(\vec{x}, \vec{b})$ into two cofinal parts. Starting from an element $\vec{a}_0$ which satisfies $f(\vec{x}, \vec{b})$, we choose $\vec{a}_1$ of greater index that satisfies $\neg f(\vec{x}, \vec{b})$, then $\vec{a}_2$ of greater index satisfying $f(\vec{x}, \vec{b})$, etc. Having done this, we take a subsequence $\vec{a}_0, \vec{a}_1, \ldots, \vec{a}_n, \ldots$, which is obviously also indiscernible, such that $f(\vec{a}_n, \vec{b})$ is true iff n is even. I claim now that for every subset w of $[0, n]$ the following formula is satisfied:

$$(\exists \vec{y}_w) \bigwedge_{i \in w} f(\vec{a}_i, \vec{y}_w) \wedge \bigwedge_{i \notin w} \neg f(\vec{a}_i, \vec{y}_w).$$

Indeed, we choose an increasing sequence of indices $m_0, \ldots, m_i, \ldots, m_n$ such that m_i is even if and only if $i \in w$. Then the tuple $\vec{a}_{m_0}{}^\frown \cdots {}^\frown \vec{a}_{m_n}$ satisfies this formula: take $\vec{y}_w = \vec{b}$. Therefore, the tuple $\vec{a}_0{}^\frown \cdots {}^\frown \vec{a}_n$, which has the same type, also satisfies it. Consequently the formula $f(\vec{x}, \vec{y})$ does have the independence property. □

We will carry over a principle that was used in the proof of the first part of this theorem, to know that the Morley sequence of an M-coheir has finitely satisfiable type in M.

Theorem 12.18. *If there is a formula $f(\vec{x}, \vec{y})$ with the independence property in T, then there is one in which $\vec{x}$ consists of just one variable.*

Proof. Let $f(x_1, \vec{x}, \vec{y})$ be a formula with the independence property. By symmetry (Lemma 12.16) and the preceding theorem, we can find an indiscernible sequence of tuples $\vec{a}_0, \ldots, \vec{a}_n, \ldots$ and a tuple $b^\frown\vec{c}$ such that $f(b, \vec{c}, \vec{a}_n)$ is true for even n and false for odd n. As we are seeking to construct a formula like that in which $\vec{x}$ has minimal length, we can assume that no formula $g(\vec{c}, \vec{z})$, no matter what the length k of $\vec{z}$ is, can split an indiscernible sequence of k-tuples into two cofinal subsets; otherwise, we would be able to find a formula with the independence property having a shorter $\vec{x}$, and repeat the process.

I claim that under these conditions, we can further assume that the $\vec{a}_n$ form an indiscernible sequence over $\vec{c}$. To prove that, I am going to use induction on m to show the consistency of the theory consisting of the following axioms:

- the axioms $g(\vec{a}_0, \ldots, \vec{a}_n) \leftrightarrow g(\vec{a}_{i_0}, \ldots, \vec{a}_{i_n})$, where $i_0 < \cdots < i_n$, expressing the fact that the sequence $\vec{a}_n$ is indiscernible over $\emptyset$;
- the axioms $f(b, \vec{c}, \vec{a}_{2n})$, $\neg f(b, \vec{c}, \vec{a}_{2n+1})$;
- for every $k \leq m$, the axioms stating that a k-tuple of $\vec{a}_i$ with increasing indices has the same type over $\vec{c}$ as $\vec{a}_0{}^\frown \cdots {}^\frown \vec{a}_{k-1}$.

I thus assume that at stage m, I have a model of these axioms available. With the help of the $\vec{a}_i$ I will be able to interpret every finite fragment of the system of axioms that I get at stage $m+1$. From this the consistency of this set of axioms follows. I proceed as follows:

If we fix $i_0, \ldots, i_{m-1}$, arranged in increasing order by assumption, then the sequence of the $\vec{a}_{i_0}{}^\frown \cdots {}^\frown \vec{a}_{i_{m-1}}{}^\frown \vec{a}_n$ is indiscernible, for $n > i_{m-1}$. It cannot be split in two cofinal subsets by a formula with parameters $\vec{c}$: For large enough n, either all the $\vec{a}_n$ satisfy $g(\vec{c}, \vec{a}_{i_0}, \ldots, \vec{a}_{i_{m-1}}, \vec{y})$, or they all satisfy its negation.

It is consistent to assume that that happens for $n > i_{m-1}$. Indeed, it is sufficient to take a subsequence in the following way: Start with $A_0 = \{\vec{a}_0, \ldots, \vec{a}_{m-1}\}$. Add an $\vec{a}_n$ to A_0, where n is of the same parity as m, from which the truth value of $g(\vec{c}, \vec{a}_0, \ldots, \vec{a}_{i_{m-1}}, \vec{y})$ remains constant, thus forming a set A_1. Then add to A_1 an $\vec{a}_n$ of greater index than anything already in, of the same parity as $m+1$, and such that we have this constancy for all formulas g corresponding to m-tuples from A_1, etc. When we are done, it is sufficient to renumber the sequence thus obtained.

This manipulation, which we have performed for a single formula g, can just as well be done when we have finitely many $g_1, \ldots, g_k$; therefore, by compactness it is consistent to assume that we can assign every increasing m-tuple $s = (i_0, \ldots, i_{m-1})$ a type p_s over $\vec{c}$ that is realized by all the $\vec{a}_s{}^\frown \vec{a}_n$, whenever $n > i_{m-1}$, where $\vec{a}_s$ denotes the concatenation $\vec{a}_{i_0}{}^\frown \vec{a}_{i_1}{}^\frown \cdots {}^\frown \vec{a}_{i_{m-1}}$.

Furthermore, we can assume that there exists $\vec{a}_\omega$ such that the sequence $\vec{a}_0, \ldots, \vec{a}_n, \ldots, \vec{a}_\omega$ is indiscernible (over $\emptyset$) and such that for every increasing m-tuple s of indices, $\vec{a}_s{}^\frown \vec{a}$ realizes p_s. Indeed, in a finite fragment of the theory stating that, $\vec{a}_\omega$ can be interpreted as some $\vec{a}_n$ for n large enough.

Given a formula $h(\vec{c}, \vec{y}_0, \ldots, \vec{y}_m)$, I claim that there is an integer N such that either every p_s corresponding to a sequence s whose least element is greater than N contains h, or else they all contain $\neg h$. Indeed, if this were not the case, we could find a sequence $s_0 < s_1 < \cdots < s_n < \cdots$, where $s_i < s_j$ means that the least element of s_j is strictly greater than the greatest element of s_i, such that the type p_{s_n} contains h if n is even and $\neg h$ if n is odd. But the sequence of the $\vec{a}_{s_n}{}^\frown\vec{a}_\omega$ is indiscernible; it is split into two cofinal pieces by the formula h, contradicting the hypothesis.

Consequently, by dropping an even number of elements from the beginning of the sequence, and renumbering the sequence, we can see that we can assume that the p_s either all contain h or else all contain $\neg h$. We can do likewise for finitely many formulas $h_1, \ldots, h_k$; by compactness, we can assume that all types p_s are equal; this is precisely what we wanted.

We can therefore assume that the sequence $\vec{a}_0, \ldots, \vec{a}_n, \ldots$ is indiscernible over $\vec{c}$, i.e., that the sequence $\vec{c}^\frown\vec{a}_0, \ldots, \vec{c}^\frown\vec{a}_n, \ldots$ is indiscernible. It is split into two cofinal pieces by the formula $f(b, \vec{x}, \vec{y})$, and it is sufficient to move $\vec{x}$ to the side of the $\vec{y}$ to have a formula in just the variable x_1 with the independence property. □

We shall say that T *has the independence property* in T if one of its formulas has this property. By the previous theorem, that means equivalently that there is a formula $f(x, \vec{y})$, in which x has just one element, that has the independence property in T. To paraphrase, T does not have the independence property if and only if it satisfies one of the following two equivalent conditions:

- Every indiscernible sequence of elements is indivisible.
- For every n, every indiscernible sequence of n-tuples is indivisible.

If T has the independence property, it is unstable. For we can let I_λ realize the dual formula of f, with the a indexed by 2^λ and the $\vec{b}$ by λ. Then the set B of all $\vec{b}$ has cardinality λ, while the a all have distinct types over B. We therefore get instability in every λ; the instability is even maximal, is some sense, since if $\lambda \geq |T|$, then there cannot be more than 2^λ types over a set of cardinality λ.

Note in passing that T and $T(A)$ either both have or both do not have the independence property, and that if T' that has the independence property is interpretable in T, then T also has this property.

Having said that, we are going to examine some properties of divisible, and especially indivisible, indiscernible sequences.

First, note that if s is a divisible indiscernible sequence, then every similar indiscernible sequence *without greatest index* is divisible. Indeed, if we split the set J of indices of this sequence t into two cofinal subsets X and Y, then it is consistent that there exists $\vec{b}$ satisfying $f(a_i, \vec{b})$ if $i \in X$ and satisfying $\neg f(a_i, \vec{b})$ if $i \in Y$, since every finite fragment of the corresponding theory can be interpreted by means of the sequence s. If we exclude the trivial case of an index set with a greatest element, we see that the divisible or indivisible character of an indiscernible sequence depends only on its Ehrenfeucht set.

Let us consider an indivisible indiscernible ω-sequence s. (Just as in the course of our exposition we considered only 1-types, in order not to complicate the notation unnecessarily, from now on we shall limit ourselves to sequences of indiscernible elements, and not tuples.) If $f(x, \vec{b})$ is a formula with parameters $\vec{b}$ in an elementary extension of the model in which s lives, then from some level n on, either all the a_n satisfy $f(x, \vec{b})$, or else they all satisfy $\neg f(x, \vec{b})$, so this formula splits the sequence s into finitely many segments $s_1, \ldots, s_k$ such that the elements of each segment give the same truth value to $f(x, \vec{b})$, while consecutive segments give it opposite truth values. This number k will be called the *alternation number* of $f(x, \vec{b})$ over s.

Let us show that the alternation number has an upper bound independent of $\vec{b}$. If for every natural number k we could find $\vec{b}_k$ such that $f(x, \vec{b}_k)$ alternates at least k times over s, we would show the consistency with the type of s of the set of sentences $f(a_{2n}, \vec{b})$, $\neg f(a_{2n+1}, \vec{b})$, that is to say of the divisibility of s. The maximum of all alternation numbers $f(x, \vec{b})$ over s will be called the *alternation number* of $f(x, \vec{y})$ over s.

If $f(x, \vec{y})$ does not have the independence property, this alternation number can be bounded by a function only of $f(x, \vec{y})$ independent of s. If that were not the case, we would show the consistency with T of the theory saying that s is an indiscernible sequence that is divisible by $f(x, \vec{y})$.

Naturally, the alternation number of s depends only on its Ehrenfeucht set, provided that s is infinite: $f(x, \vec{y})$ will not alternate any more over a different but similar sequence.

What will happen now if the sequence s is totally indiscernible and indivisible? I claim then that for every formula $f(x, \vec{y})$, there is a number n such that for every $\vec{b}$, either all but at most n of the elements of s satisfy $f(x, \vec{b})$, or else all but at most n of them satisfy $\neg f(x, \vec{b})$. Indeed, if there are k elements of s that satisfy $f(x, \vec{b})$ and k elements of s that satisfy $\neg f(x, \vec{b})$, then by renumbering the sequence (which does not change its type as far as total indiscernibility is concerned) we can make $f(x, \vec{b})$ alternate at least $2k$ times.

We thus see that if s is an indivisible totally indiscernible sequence, then its limit type over A is the set of formulas with parameters in A that are satisfied by all but finitely many elements of s. Some people therefore speak of the *mean type* of s over A rather than the limit type of s over A.

Lemma 12.19. *If the model M contains the indivisible totally indiscernible sequence s (or even if M contains only an infinite subset of s!), then the mean type of s over M is definable.*

Proof. Let n be a natural number such that for every $\vec{b}$, either all but at most n elements of s satisfy $f(x, \vec{b})$, or else all but at most n of them satisfy $\neg f(x, \vec{b})$. Then the formula

$$g(\vec{y}, a_0, \ldots, a_{2n}) = \bigvee_{i_0 < \cdots < i_n \leq 2n} f(a_{i_0}, \vec{y}) \wedge \cdots \wedge f(a_{i_n}, \vec{y})$$

is a definition $df(\vec{y})$ for the mean type of s over M. For $f(x, \vec{b})$ is true for all but at most n of the a_i, since it is true for $n+1$ of the first $2n+1$ of the a_i. □

12.5 Indivisible Morley Sequences

Theorem 12.20. *If the Morley sequence of the special son q of p is totally indiscernible and indivisible, then p is definable and q is its heir.*

Proof. Let p be in $S_1(M)$ and let q be a special son of p over a model N realizing all types of parameters over M. Let a_0 realize q in a model N_0 realizing all types over N. Repeat this ω times: a_{n+1} realizes the unique M-special extension of q over N_n, in a model N_{n+1} that realizes all types over N_n. The sequence $a_0, \ldots, a_n, \ldots$ is a copy of the Morley sequence of q. The model N_ω, the limit of the N_n, has the property of realizing all types over $M \cup \{\vec{b}\}$, for every $\vec{b}$ in N; moreover, the unique M-special son of q over N_ω is obviously the limit type of the sequence a_n over N_ω. By Lemma 12.19, this type is definable, and the conclusion follows from Theorem 12.5. □

Corollary 12.21. *If T is unstable but without the independence property, then there is an infinite sequence of elements that is indiscernible and not totally indiscernible.*

Proof. If T is unstable, there there is a type (a 1-type) p that is not definable, so the Morley sequence of a special son of p, which is indivisible, cannot be totally indiscernible. □

There exist unstable theories for which every indiscernible sequence of elements is totally indiscernible (they therefore have the independence property). Consider the following example, due to Shelah. The language consists of a unary relational symbol X and a binary relational symbol $\in$. Consider the universal axiom $(\forall u)(\forall v)(u \in v \rightarrow X(u) \wedge \neg X(v))$, and take its model completion T. I leave to the reader the task of verifying that T does exist, and that it is axiomatized by the following axioms:

$$(\forall u_1) \quad \cdots (\forall u_n)(\forall v_1) \cdots (\forall v_n)(\exists y)$$
$$\left(\bigwedge u_i \neq v_j \wedge \bigwedge X(u_i) \wedge \bigwedge X(v_j) \rightarrow \bigwedge u_i \in y \wedge \bigwedge v_j \notin y\right)$$

$$(\forall u_1) \quad \cdots (\forall u_n)(\forall v_1) \cdots (\forall v_n)(\exists x)$$
$$\left(\bigwedge u_i \neq v_j \wedge \bigwedge \neg X(u_i) \wedge \bigwedge \neg X(v_j) \rightarrow \bigwedge x \in u_i \wedge \bigwedge x \notin v_j\right)$$

This is an ω-categorical theory with quantifier elimination. We can see that over every set with parameters A an indiscernible sequence of elements is totally indiscernible. If the elements satisfy X, then all we can say is that they are all different from each other, that they are all different from elements of A, and that they all belong to the same elements of A. If they satisfy $\neg X$, we can say only that they are different from each other, different from elements of A, and that they all contain the same elements of A.

In T, the formula $x \in y$ has the independence property, and even in a strong sense, since every set of elements satisfying X can be split as desired by this formula. Note the following symmetry: From one model of T we get another by interchanging X and $\neg X$ and changing the direction of the relation $\in$.

Lemma 12.22. *The Morley sequence of a stable type is totally indiscernible and indivisible.*

Proof. We saw in Theorem 12.15 that it is totally indiscernible. If it were divisible by some formula $f(x, \vec{y})$, then for every λ this type p would be consistent with the set $I_\lambda(f)$ (see Section 12.4) and we would have instability for the sons of p in every λ.

We can also proceed as follows: We let a_0 realize p in a model M_0, then let a_1 realize the heir of p over M_0 in a model M_1, etc. In an elementary extension M_{n+1} of M_n, a_{n+1} realizes the heir of p over $M_n, \ldots$. The heir q of p over the limit M_ω of the M_n is the limit type of the sequence $a_0, \ldots, a_n, \ldots$, which is a realization of the Morley sequence of p. Suppose that this sequence is divisible. Then there exists $\vec{b}$ in an elementary extension N of M_ω, and a formula $f(x, \vec{b})$ such that both it and its negation are true cofinally often in this sequence. Consequently, $q \wedge f(x, \vec{b})$ and $q \wedge \neg f(x, \vec{b})$ are both finitely satisfiable in M; by Lemma 12.10, that means that q has at least two coheirs over N, contradicting its stable character. □

Lemma 12.23. *If T is stable, then every infinite indiscernible sequence has the same type (over $\emptyset$) as the Morley sequence of some type over a model of T.*

Proof. Let M be a model containing the indiscernible sequence $s = (a_0, \ldots, a_n, \ldots)$, which is indivisible because T does not have the independence property, and let p be the limit type of s over M. If N is an elementary extension of M, then the limit type of s over N is a son of p that is finitely satisfiable in M. It is therefore both the coheir and the heir of p.

So let $b_0, \ldots, b_n, \ldots$ be the Morley sequence of p over M. Since this is the Morley sequence of a coheir, we have already noted that its type is finitely satisfiable in M. More precisely, if $f(b_0, \ldots, b_n)$ is true, then we can find i_n such that $f(b_0, \ldots, b_{n-1}, a_{i_n})$ is true, then $i_{n-1} > i_n$ such that $f(b_0, \ldots, b_{n-2}, a_{i_{n-1}}, a_{i_n})$ is true, etc. Finally, we can find $i_n < i_{n-1} < \cdots < i_0$ such that $f(a_{i_0}, \ldots, a_{i_n})$ is true. If the formula f does not contain parameters in M, then, since the sequence s is indiscernible over $\emptyset$, that means that $f(a_n, \ldots, a_0)$ is true. Therefore $(b_0, \ldots, b_n)$ has the same type as $(a_n, \ldots, a_0)$. However, since the sequence of the b_n is totally indiscernible,

$(b_0, \ldots, b_n)$ has the same type as $(b_n, \ldots, b_0)$, so that the sequence of the b_n and that of the a_n are similar indiscernible sequences, and therefore of the same type. (Note that $a_0, \ldots, a_n, \ldots, b_0, \ldots, b_n, \ldots)$ is in fact an indiscernible set.) □

Corollary 12.24. *A theory T is stable if and only if every infinite indiscernible sequence consisting of elements from a model M of T is totally indiscernible and indivisible.*

Proof. By Corollary 12.21, if T is unstable, it has a Morley sequence that is either divisible or not totally indiscernible. If T is stable, then by the last two lemmas, an infinite indiscernible sequence is isomorphic to a Morley sequence, and is both totally indiscernible and indivisible.

It would also be easy enough to show that if we had a sequence that is indiscernible but not totally, then for every cardinal λ we could build a set A of parameters of that cardinality, producing at least as many types as there are cuts of an ordering of cardinality λ. We would therefore get instability in every cardinal λ. □

Theorem 12.25 (Uniform Definability Theorem). *Let T be a stable theory, let $f(x, \vec{y})$ be a formula in the language of T, and let n be the least natural number such that for every indiscernible set A and every $\vec{b}$, either all but at most n of the elements of A satisfy $f(x, \vec{b})$, or else all but at most n satisfy $\neg f(x, \vec{b})$. Let $g(\vec{y}, z_0, \ldots, z_{2n})$ be the formula $\bigvee_{i_0 < \cdots < i_n \leq 2n} f(z_{i_0}, \vec{y}) \wedge \cdots \wedge f(z_{i_n}, \vec{y})$. Then for every type p, over a model M of T, satisfying a formula $h(x, \vec{b})$, the definition $df(\vec{y})$ associated with p can be taken to be of the form $g(\vec{y}, \vec{a})$, for some $\vec{a}$ in M, whose elements all satisfy $h(x, \vec{b})$.*

Proof. Let a_0 realize p in an elementary extension M_0 of M, then let a_1 realize the heir of p over M_0 in an elementary extension M_1 of M_0, etc. Let N be the limit of the M_n: the heir q of p over N is the limit type of the sequence a_n, which is totally indiscernible and indivisible as we have seen, and $g(\vec{y}, a_0, \ldots, a_{2n})$ is a definition of q (see Lemma 12.19). Consequently, $(N, dq) \models (\exists z_1) \cdots (\exists z_{2n})(h(z_1, \vec{b}) \wedge \cdots \wedge h(z_{2n}, \vec{b}) \wedge (\forall \vec{y})(df(\vec{y}) \leftrightarrow g(\vec{y}, \vec{z})))$. The same is therefore true of (M, dp), of which (N, dq) is an elementary extension. □

The preceding theorem says in particular that in a stable theory (and in fact, we can easily see that it is sufficient for p to be stable), if $p \models h(x, \vec{b})$, then the definition of p really uses only parameters from M that satisfy that formula; in contrast, if T is the theory of dense orders without endpoints, the type a^+ cannot be defined with parameters satisfying $x > a$.

Lemma 12.26. *Let p be a type over M, and let p_1 and p_2 be two special sons of p whose Morley sequences are indivisible and have the same type over M. Then $p_1 = p_2$.*

Proof. Let N be a sufficiently saturated elementary extension of M over which p_1 and p_2 are defined. Let $a_0, \ldots, a_n, \ldots$ be the Morley sequence of p_1 over N, and $b_0, \ldots, b_n, \ldots$ that of p_2. I construct a third sequence $c_0, \ldots, c_n, \ldots$ over N by using the infinitary definitions of p_1 and p_2 alternately. Then c_{2n+1} realizes the unique M-special son of p_1 over $N \cup \{c_0, \ldots, c_{2n}\}$, and c_{2n+2} realizes the unique M-special son of p_2 over $N \cup \{c_0, \ldots, c_{2n+1}\}$.

I claim that these three sequences have the same type over M. To show this, I proceed by induction on n. I therefore assume that $(c_0, \ldots, c_{2n}), (a_0, \ldots, a_{2n}), (b_0, \ldots, b_{2n})$ have the same type over M (for the last two, that is the hypothesis of the lemma). If $f(c_0, \ldots, c_{2n}, c_{2n+1})$ is true, where f is a formula with parameters in M, then for the infinitary definition of p_1 the tuple $(c_0, \ldots, c_{2n})$ has an appropriate type over M. Since $(a_0, \ldots, a_{2n})$ has the same type over M as $(c_0, \ldots, c_{2n})$, that means that $f(a_0, \ldots, a_{2n})$ is also true. Therefore $(c_0, \ldots, c_{2n+1})$ has the same type over M as $(a_0, \ldots, a_{2n+1})$, and therefore also as $(b_0, \ldots, b_{2n+1})$, which, by hypothesis, has the same type over M as $(a_0, \ldots, a_{2n+1})$. At even stages, we proceed the same way, this time using the infinitary definition of p_2.

Consequently, the sequence $c_0, \ldots, c_n, \ldots$ is itself indivisible (indivisibility is part of the type of the sequence over $\emptyset$), and since it alternately realizes p_1 and p_2, we must have $p_1 = p_2$. □

Corollary 12.27. *If T does not have the independence property, then the number of M-special types over any elementary extension of M is at most $|S_\omega(M)|$.*

Proof. By Lemma 12.26, an M-special type is determined by the type over M of its Morley sequence. □

Theorem 12.28. *If T has the independence property, then for every cardinal $\lambda \geq |T|$ there is a type p over a model M of T, $|M| = \lambda$, that has $2^{(2^\lambda)}$ coheirs. If T does not have the independence property, then for every cardinal $\lambda \geq |T|$ the number of coheirs (and even of special sons) of a type p over a model M of cardinality λ is at most 2^λ.*

Proof. Suppose that T has the independence property. Let us realize I_λ. We then have a_α for $\alpha \in \lambda$ and $\vec{b}_w$ for $w \subset \lambda$ such that $f(a_\alpha, \vec{b}_w)$ is true if and only if $\alpha \in w$. By the Löwenheim–Skolem theorem, there is a model M of cardinality λ that contains all the a_α. Let N be an elementary extension of M that in addition contains all the $\vec{b}_w$. (The cardinality of N is at least 2^λ.)

Every ultrafilter U of subsets of λ can be associated with a type p_U over N defined as follows: $p_U \models g(x, \vec{c})$, where $\vec{c} \in N$, if $\{\alpha : N \models g(a_\alpha, \vec{c})\} \in U$. Every p_U, being finitely satisfiable in M, coinherits its restriction to M.

Also, if $U \neq V$, then $p_U \neq p_V$, since if $w \in U, w \notin V, then p_U \models f(x, \vec{b}_w), p_V \models \neg f(x, \vec{b}_w)$. Since there are $2^{(2^\lambda)}$ ultrafilters of subsets of

λ (see Theorem 8.11), we thus get $2^{(2^\lambda)}$ types that coinherit their respective restrictions to M. The number of types over M is at most 2^λ, and since 2^λ is strictly less than the cofinality of $2^{(2^\lambda)}$ (see Corollary 8.16), $2^{(2^\lambda)}$ of the p_U must have the same restriction p to M.

Conversely, if T does not have the independence property, then the number of special sons of p is at most $|S_\omega(M)|$ by Lemma 12.26. Over a set of cardinality λ, there are at most 2^λ 1-types; to choose an ω-type, we need to choose a 1-type ω times, so $|S_\omega(M)| \leq (2^\lambda)^\omega = 2^{\lambda\times\omega} = 2^\lambda$. □

Theorem 12.28 gives a very effective (in practice) criterion for determining whether or not a theory T has the independence property. Indeed, experience shows that it is easy to determine coheirs in concrete examples. It comes down to counting closures of sets of types: If T has the independence property, then we can find $X \subset S_1(M), |X| = \lambda$, whose closure has cardinality $2^{(2^\lambda)}$; if not, the cardinality of the closure of such X cannot exceed 2^λ. (This is left as an exercise for the reader: Prove that we can assume that every type of X is realized, and then apply Corollary 12.27.)

This criterion is more effective than a criterion due to Keisler and Shelah, which consists in counting the types that can produce a set of cardinality λ, this number being significant only if, for some λ, $\mathrm{ded}(\lambda) \neq 2^\lambda$.

The reader who found the proof of Theorem 12.18 abnormally complicated can try to apply this criterion in this case. Unfortunately, it is ineffective, since we do not see how to reduce the coinheritance properties of n-types to those of cascades of 1-types. The author of this Theorem 12.18 is Saharon Shelah, and the few individuals who have read his works have learned, with a shiver of horror, that Shelah did not really show the truth of Theorem 12.18, but only its consistency! Naturally, if you are expert in set theory, you will immediately see that in this case it amounts to the same thing. Just as to see that a Π_1 sentence is true in arithmetic it is sufficient to show that it is consistent with minimal arithmetic A_0 (see 7.1), so, thanks to a construction of models by forcing, to prove the consistency of a sentence such as Theorem 12.18 is to prove that it is true. This is nevertheless a strange way to proceed, and the reader will be more convinced by the "elementary" character of the proof that has been presented here; the possibility of finding one much simpler has not been ruled out.

Theorem 12.29. *If T is unstable, then there is a model M of T and p in $S_1(M)$ with at least two coheirs.*

Proof. If T has the independence property, then we already have as many coheirs as we want. So suppose otherwise, and consider a sequence of elements a_i that is indiscernible but not totally (for example, a Morley sequence of a special son of a nondefinable type), indexed by a chain I of cardinality $\lambda \geq |T|$, and having at least λ^+ cuts (see Theorem 8.10). Every subsequence of this sequence is indivisible. Let M be a model of T of cardinality λ containing all the a_i. If (A, B) is a cut of I, A and B being nonempty, let p^- denote the limit type over M of the sequence of those a_i indexed by A, and p^+ the limit type over M of the sequence of those a_j indexed by B ordered in the reverse sense, i.e., the set of formulas with parameters in M that are true for a_j when j is a small enough element of B.

If $p^- \neq p^+$, then there are i in A, j in B, $\vec{c}$ in M, and a formula $f(x, \vec{y})$ such that A consists of all indices less than i and those indices k less than j such that $f(a_k, \vec{c})$ is true. As there are only λ possible choices for $i, j, f(x, \vec{c})$, that is possible for only at most λ cuts. We can therefore find one for which $p^- = p^+ = p$.

Let N be an elementary extension of M that realizes p by an element c. Note that the sequence obtained by inserting c into the cut (A, B) is still indiscernible. As our sequence $s = (\ldots, a_i, \ldots)$ is not totally indiscernible, there is some formula f such that $f(x_1, \ldots, x_n, u, v, y_1, \ldots, y_m) \wedge \neg f(x_1, \ldots, x_n, v, u, y_1, \ldots, y_m)$ is satisfied by every increasing tuple of elements of s. Note that A has no greatest element and B no least element. They are therefore infinite, since they are not empty, and we can choose $a_1, \ldots, a_n$ of increasing indices in A and $b_1, \ldots, b_m$ of increasing indices in B. The incomplete type $p \cup \{f(a_1, \ldots, a_n, x, c, b_1, \ldots, b_m) \wedge \neg f(a_1, \ldots, a_n, c, x, b_1, \ldots, b_m)\}$ is finitely satisfiable in A, while the incomplete type $p \cup \{f(a_1, \ldots, a_n, c, x, , b_1, \ldots, b_m) \wedge \neg f(a_1, \ldots, a_n, x, c, b_1, \ldots, b_m)\}$ is finitely satisfiable in B. By Theorem 12.10, each of these types can be completed as an heir of p over N, which therefore has at least two. □

This section may have seemed a little hard to digest for the reader new to the subject, who wanted only to skim it. For that reader, the essential part is to remember its consequences for stable theories, which I shall summarize here.

The following four conditions are equivalent, and all mean that T is stable:

1. *Every 1-type over any model of T is definable, i.e. has only one heir* (by definition of stability).
2. *Every 1-type, over any model of T, has only one coheir* (Corollary 12.6, Theorems 12.13,12.29).
3. *Every 1-type, over any model of T, has only one special son (ibid.).*
4. *Every infinite indiscernible sequence consisting of elements of a model of T is totally indiscernible and indivisible* (Corollary 12.24).

Naturally, stability is also equivalent to the fact that each n-type, for any $n > 0$, has a unique heir, a unique coheir, a unique special son, and that infinite indiscernible sequences of n-tuples are totally indiscernible and indivisible.

By symmetry, it is immediate that the uniqueness of coheirs of n-tuples for every n is equivalent to the uniqueness of heirs; however, what is remarkable is that the uniqueness of coheirs for 1-types is sufficient to imply stability.

We conclude this section with one last result, whose proof uses Morley sequences.

Theorem 12.30. *Let T be a stable theory, let p and q be two types over a model M of T both of which satisfy some formula $f(x,\vec{a})$ with parameters in M, and let A be the set of elements of M that satisfy this formula. Then if p and q have the same restriction to A, they are equal.*

Proof. Let $g(\vec{x},\vec{y})$ be a formula and let $d_1g(\vec{y})$ and $d_2g(\vec{y})$ be the definitions, of p and q respectively, for g, which, by Theorem 12.25, can be taken to have parameters in A. The hypothesis says that these two definitions have the same effect on A, that is to say that

$$\begin{aligned} M \models \quad & (\forall y_1)\cdots(\forall y_n)(f(y_1,\vec{a}) \wedge \cdots \wedge f(y_n,\vec{a}) \\ & \to (d_1g(y_1,\ldots,y_n) \leftrightarrow d_2g(y_1,\ldots,y_n))), \end{aligned}$$

and this axiom remains true in every elementary extension of M.

Construct the Morley sequence $a_1,\ldots,a_n,\ldots$ of p over M, then the Morley sequence $b_0,\ldots,b_n,\ldots$ of q over M, and finally the mixed sequence $c_0,\ldots,c_n,\ldots$ obtained by using the definition of p at even stages and that of q at odd stages. As both these definitions have parameters in A, and have the same effects on parameters that satisfy $f(x,\vec{a})$, we show, in a manner analogous to the proof of Theorem 12.25, that these three sequences have the same type over A. Consequently, since the sequence $c_0,\ldots,c_n,\ldots$ must be indivisible, we must have $p = q$. □

Corollary 12.31 (Parameter Separation Theorem). *Let M be a model of a stable theory T, $f(x)$ a formula without parameters, and $g(x,\vec{b})$ a formula with parameters in M. Then there is a formula $h(x,\vec{a})$ whose parameters all satisfy $f(x)$ and that is equivalent to $f(x) \wedge g(x,\vec{b})$.*

Proof. By the preceding theorem, the fact that a type p of $S_1(M)$ satisfies $f(x) \wedge g(x,\vec{b})$ depends only on the restriction of p to the set A of elements of M that satisfy $f(x)$. The conclusion then follows from Lemma 12.4. □

If in Corollary 12.31 the formula $f(x)$ involved parameters $\vec{c}$, we could still apply the corollary to the theory $T(\vec{c})$, which is also stable. The theorem also remains valid if f has a tuple of free variables instead of just one.

As an example of an application, note that the theory of differentially closed fields of characteristic zero (see Section 6.2) is stable in every cardinal λ, as there are not more types than there are possible minimal polynomials. Take $f(x)$ to be the formula $x' = 0$. If C is the field of constants of the differentially closed field K, then everything we can define over C with parameters in K can be so defined with parameters in C; we can define nothing more over C than we can in the theory of ordinary algebraically closed fields, with parameters in C.

There are theories without the independence property that do not satisfy this parameter separation theorem. Consider the theory of chains, and add a unary relation symbol $R(x)$ to the language. To get more of a picture, we shall call elements satisfying R white, and those satisfying $\neg R$ black. Let T be the model completion. The reader will easily recognize the fact that it exists, and that it is the theory of dense colored chains without endpoints in which between any two distinct points there is always a white point and a black point. We met this theory earlier, at the end of Section 10.1. Like every theory of colored chains, T does not have the independence property, as we will see in the next section.

If b is a white point in a model M of T, then the set of black points less than b cannot be defined with only black parameters, and the black b^- and b^+ types, which are distinct, have the same restrictions to the black points of the model M: They cannot be defined with only black parameters.

12.6 An Example: The Theories of Chains

By way of illustrating Theorem 12.28, we are going to show that a chain never has the independence property. Note that the theory of an infinite chain is unstable: Since the elements of an indiscernible sequence must be ordered (by an order that can be only that on their indices, or else its inverse), it cannot be totally indiscernible.

For this purpose we need a theorem that contains the essentials of model theory for chains. Given a sequence $f_1(x), \dots, f_n(x)$ of formulas in one free variable x, in the language of chains, and two points $a < b$ of a chain C, we shall say that this sequence of formulas is realized between a and b if there are $c_1, \dots, c_n$ in C with c_1 satisfying $f_1(x), \dots, c_n$ satisfying $f_n(x)$, and such that $a < c_1 < \cdots < c_n < b$.

Theorem 12.32 (Rubin's theorem). *For two increasing n-tuples, $a_1 < \cdots < a_n$ in the chain C and $b_1 < \cdots < b_n$ in the chain D, to have the same type, it is necessary and sufficient that they satisfy the following conditions:*

- *for every $i \leq n$, a_i and b_i satisfy the same formulas;*
- *for every $i < n$, the same finite sequences of formulas are realized between a_i and a_{i+1} as between b_i and b_{i+1}.*

Proof. The condition is obviously necessary (and note that it implies that the chains C and D are elementarily equivalent). For the converse, we assume that the chains C and D are ω-saturated, and seek to establish an infinite back-and-forth construction between the two tuples. I will be content to add α to the left.

If α is less than a_1, I consider formulas $f_1(x), \ldots, f_k(x)$ satisfied by α, sequences of formulas $s_1, \ldots, s_k$ realized between α and a_1, and sequences of formula $t_1, \ldots, t_k$ omitted between α and a_1. One formula in the type of a_1 says that there is a y less than a_1 satisfying the f_i, such that the s_i are realized between y and a_1, and the t_i omitted between y and a_1. By hypothesis b_1 satisfies the same formula, and by compactness and the ω-saturation of D, there is a β in D of the same type as α, $\beta < b_1$, such that the same sequences of formulas are realized between α and a_1 as between β and b_1.

We proceed the same way if α is greater than a_n, and the only remaining case is where α is between two consecutive a_i, for example a_1 and a_2.

To satisfy finitely many formulas is to satisfy their conjunction, and to realize finitely many sequences is to realize just one obtained by mixing them. Consequently, a finite fragment of the situation presented by α between a_1 and a_2, which we want to duplicate on the other side, appears as follows:

- a formula $f(x)$ satisfied by α;
- a sequence s of formulas realized between a_1 and α, and a sequence t of formulas realized between α and a_2; s or t might possibly be the empty sequence;
- a finite set $u_1, \ldots, u_n$, possibly empty, of sequences omitted between a_1 and α, and a finite set $v_1, \ldots, v_m$, possibly empty, of sequences omitted between α and a_2.

We can further assume that every sequence strictly contained in a u_i is realized between a_1 and α, and even appears as a subsequence of s; otherwise, we replace u_i by a shorter sequence. Likewise, we assume that every subsequence of a v_j appears in t.

Finally, assume in addition that every $u_i{}^\frown f^\frown t$ is realized between a_1 and a_2. If that were not the case, by hypothesis the situation would be the same between b_1 and b_2, and no realization of $f^\frown t$ between b_1 and b_2 could have u_i to its left. We could thus drop u_i from the list, since it would only be expressing a useless condition. Likewise, we can assume that every $s^\frown f^\frown v_j$ is realized between a_1 and a_2.

First, suppose that there exist neither u_i nor v_j. The sequence $s^\frown f^\frown t$ is therefore realized between b_1 and b_2, and the realization β corresponding to f satisfies what we want.

Next, suppose that there are u_i but no v_j. Let A be the segment of the ordering C consisting of the $y > a_1$ such that none of $u_1, \ldots, u_n$ are realized between a_1 and y. Then there is a realization $\vec{c}^\frown a$ of $s^\frown f$ in A. The segment A can be defined uniquely with a_1 as a parameter, as can the segment A' on the other side obtained the same way as A using b_1 as the parameter. Since a_1 and b_1 have the same type, there is a realization $\vec{c}'^\frown\beta$ of $s^\frown f$ in A'. As the sequence $u_1{}^\frown f^\frown t$ is realized between b_1 and b_2, there is a realization of t located entirely to the right of A' (A' is bounded above by b_2!). Consequently, the element β satisfies our need. The case in which there are v but no u is treated symmetrically.

Finally, suppose that there exist both u and v, and define A and A' as before, as well as B and B' in a symmetric manner: B is the set of the $y < a_2$ such that none of $v_1, \ldots, v_m$ are realized between y and a_2, while B' is the set of the $y < b_2$ such that none of $v_1, \ldots, v_m$ are realized between y and b_2. Indeed, for some v (which we may assume to be the first) B is the set of the $y < a_2$ such that v_1 is omitted between y and a_2; the same is true of B', since a_2 and b_2 have the same type. Let E be the intersection of A and B, and E' the intersection of A' and B'.

Since there is a realization of $u_1{}^\frown f^\frown t$ between b_1 and b_2, we can find a realization of t there that strictly bounds A' above; likewise, since $s^\frown f^\frown v_1$ is realized there, we can find a realization of s there that strictly bounds B' below. It is therefore sufficient to show the existence of an element β of E' satisfying f.

For the same reason there is a realization of t between a_1 and a_2 located entirely to the right of A.

The segment $A - B$ is nonempty, since $f^\frown v_1$ is realized between a_1 and a_2. An element of $A - B$ has a realization of v_1 to its right, and the first term of this realization must be in A. If g then denotes the first formula of v_1, this element has a realization of g in A to its right. Conversely, if an element of A is bounded above by a realization of g in A, then there is a realization of v_1 between this element and a_2; indeed, there is a realization of t between A and a_2, and t contains the sequence w obtained by dropping the first term from v_1. Consequently, E is definable with a_1 as its only parameter: E is the set of y in A that are not strictly bounded above by any element of A satisfying g. Let E'' be the corresponding set associated with b_1.

As there is a realization of t, which contains w, between A' and b_2, it is clear that an element of A' that is not in E'' is not in B': E' is a subset of E''. If this inclusion were proper, we could find a realization of v_1 entirely between A' and b_2; consequently, for some u_i, $u_i{}^\frown v_1$ would be realized between b_1 and b_2, and therefore also between a_1 and a_2, which is absurd, since every element between a_1 and a_2, in particular α, would have u_i to its left and v_1 to its right.

Consequently, $E'' = E'$. Since there is an element α in E that satisfies $f(x)$, since a_1 and b_1 have the same type, and since E'' is defined from b_1 the same way E is defined from a_1, there is also a β in E'' that satisfies $f(x)$.

The rest is just a matter of ω-saturation. □

Rubin's theorem gives a simple description of the types p over a model M of a theory T of chains (the language of T consists just of a symbol denoting the order). If we exclude the trivial case in which p is realized, i.e., $p \models x = a$ for some a in M (in which case p is completely determined by a), then p splits the model M into two classes $A = \{a : a \in M \text{ and } p \models x > a\}$ and $B = \{b : b \in M \text{ and } p \models x < b\}$. This partition (A, B) of M will be called the *cut* defined by p. Since p is a type over a model, it must be finitely satisfiable in this model. If every finite fragment of p is satisfied by an element of A, we shall say that p is *satisfiable on the left*; if every finite fragment of p is satisfied by an element of B, we shall say that p is *satisfiable on the right.* In general, a type is satisfiable on both sides; that does not happen only if its cut is *definable*, that is to say if there is a formula with parameters in M that is satisfied exactly by those elements of M that are in A. In that case, it follows from the fact that p satisfies either the definition of A or that of its complement B, that it will be satisfiable either on the left or on the right.

One consequence of Rubin is that a nonrealized type p of $S_1(M)$ is completely determined by its *absolute type* (i.e., the restriction of p to the empty set of parameters), *its cut, and its side of satisfiability* in the case in which its cut is definable. Indeed, given a in M and a sequence s of formulas, it is sufficient to see that these conditions are enough to determine whether or not s is realized between a and x. If, for example, p is satisfiable on the left, that happens if s is realized between a and b for every sufficiently large b in A.

The first consequence of that fact is that for a type to be definable, it is sufficient (and certainly necessary!) for its cut to be definable. Indeed, suppose that the cut of p in $S_1(M)$ is definable and let q be an heir of p over a very saturated elementary extension N of M. Since q is a son of p, its restriction to $\emptyset$ is that of p. The cut of q is then forced, since it must be the cut (A', B') over N having the same definition as the cut (A, B) of p over M; so is its side of satisfiability, since if, for example, p is satisfiable in A, it satisfies the negation of the formula defining B, and its son q can be satisfied only in A'.

Next, we see that a definable type p has only one coheir. For if we assume that p is definable and satisfiable on the left, then we have no choice of cut for one of its coheirs q over N, or for its side of satisfaction: Since q must be satisfiable in M, it must also be satisfiable in A, and its cut can be only the leftmost one, which bounds all realizations in N of the cut (A, B) from below. We can see that this unique coheir of p is distinct from its unique heir, which bounds all realizations of p in N from above.

A nondefinable type has exactly two coheirs: one coheir on the left, which is finitely satisfiable in A, and whose cut bounds all realizations of (A, B) in N from below, and one coheir on the right, which is finitely satisfiable in B, and whose cut in contrast is located as far to the right as possible, bounding all others from above.

Consequently *a theory of chains can never have the independence property*, since a 1-type can have only 2 coheirs.

We can even see that the heir and the coheir of a definable type, and the two coheirs of a nondefinable type, are the only cases of special sons: For any other son q of p over N, we can find two realizations a and b of p such that $q \models a < x < b$. Indeed, there are a' and b' in N, satisfying the formula defining the appropriate class of the cut of p if it is definable, such that $q \models a' < x < b'$. So N, being sufficiently saturated, contains a realization a of the special son on the left of p over $M \cup \{a'\}$, and a realization b of the special son on the right of p over $M \cup \{b'\}$.

We can also very easily deduce from Rubin's theorem that in order for the type of $(a_1, \ldots, a_n)$ over N to be a coheir, or a special son, of its restriction to M, it is sufficient (and necessary!) that the same be true of the types of every a_i. We see therefore that a type of n-tuples never has more than 2^n special sons; moreover, the special sons correspond to a mixture of heirs and coheirs of 1-types. We can also see that every nonrealized son of a nondefinable 1-type is one of its heirs.

We can therefore see that the especially simple description that we have obtained for heirs, coheirs, and special sons in the case of the theory T of dense chains without endpoints relies on properties that are common to all chains. Moreover, Rubin's theorem, as well as all its consequences, is immediately generalized to *colored chains*, that is to say structures consisting of chains and some number (finite or infinite) of unary relations.

Having treated the case of chains, we are now going to determine the coheirs for rich ultrametric spaces, which are intimately connected to chains. Consider a theory T' of chains with a least element, called 0, and the corresponding theory T of rich valued spaces $M = (I, E)$, where the I are models of T'. The types corresponding to a distance introduce nothing new: By Theorem 6.24 their heirs, coheirs, and special sons are those that we have already seen in the sense of T'; henceforth, we shall consider only types of "points," i.e., of elements of E.

So let us consider such a type p over a model $M = (I, E)$ of T and a coheir q of p over a sufficiently saturated extension $N = (J, F)$ of M.

Let $I(p)$ be the set of those elements i of I for which $p \models d(x, a) = i$ for some a of E. The set $I(p)$ is a final segment (possibly empty) of I, since if $i \in I(p), i < j$, then since the space is rich, there is a b in E such that $d(a, b) = j$, and $p \models d(x, b) = j$ by the ultrametric inequality. Then $J(q)$ must be the final segment of J generated by $I(p)$; indeed, if $q \models d(x, b) = j$ with $b \in F, j \in J$, then there exists a in M, that is to say in E, such that $d(a, b) = j$, and by the ultrametric inequality $j \geq d(x, a) \in I(p)$.

That having been noted, with the classification of types that we established in Section 6.3, an easy analysis, whose details are as usual left to the reader, leads to the following description of coheirs:

- p is realized: Its unique son is its unique coheir.
- p is a distance type: Then there exists a in M such that $d(x, a)$ is not in I. We know that p is therefore determined by a and by the type $d(x, a)$ over I, the upper class of the cut being $I(p)$. Therefore q is a coheir of p if and only if it is a distance type, attached to the same a

as p, and such that the type of $d(x, a)$ over J coinherits its restriction to I.

- p is polygonal, which means that it is not a distance type and that $I(p)$ has a least element i. If $p \models d(x, a) = i$, then this condition completely determines p among those polygonal types for which the least element of $I(p)$ is i. Also, p has only one coheir, which is also its unique heir: the polygonal type over N associated with the same a and the same i.

- p is a pseudo-limit, that is to say it is not a distance type and $I(p)$ has no least element. If we choose an a_i for every i in $I(p)$ such that $p \models d(x, a_i) = i$, then these conditions determine p, which is the pseudo-limit of the non-pseudo-convergent Cauchy pseudo-sequence of the a_i. If an extension N' of M contains no pseudo-limit to this sequence, then p has only one son over N'; however, since we have already taken N to be a sufficiently saturated model, N has such a limit, and we can fix one, say b. We can therefore see that q is an heir of p if and only if it is a distance type, attached to b, such that the type of $d(x, b)$ over J has $J(q)$ as its upper class, and *is finitely satisfiable in* $I(p)$. We therefore see that the cut of the type of $d(x, b)$ over J is uniquely determined, as is its side of satisfiability, but that *its type over* $\emptyset$ *is not determined* (except, naturally, if for every $f(x)$ without parameters, either all sufficiently small elements of $I(p)$ satisfy f or they all satisfy $\neg f$). That is due to the fact that $d(x, b)$ is defined by a parameter b in the large model N (b is any limit of the sequence a_i), and that nothing can be hung onto the type of p.

We therefore see that *a theory of rich valued spaces never has the independence property.* Indeed, a realized or polygonal type has only one coheir; a distance type has one or two; and a pseudo-limit type has at most 2^ω, no more than there are pure types for a distance. The same results are also valid for n-rich spaces, except that they have no polygonal types (over models).

In these examples of unstable theories T without the independence property, the number of coheirs of a type p of $S_1(M)$ can be bounded independently of the cardinality λ of the model M. To obtain a theory without the independence property in which that is not the case, we are going to enrich the structure I by adding new relation symbols to its order so as to have a theory T' without the independence property, but that allows us to define more than 2^ω types in a given cut, with the side of satisfaction fixed. A parallel analysis will remain valid for the theory T of rich I-valued spaces, which will not have the independence property, since in the pseudo-limit case, the type of $d(x, b)$ over J coinherits its restriction to I.

For example, take the model completion of the theory of two chains $\leq_1$ and $\leq_2$, defined over the same set; it is left to the reader to figure out what that is. It admits quantifier elimination, and a nonrealized type over a model I of this theory T' is determined by its cut (A_1, B_1) with respect to $\leq_1$ and its cut (A_2, B_2) with respect to $\leq_2$; these cuts can be chosen independently of each other. The theory T' does not have the independence property, since a type has at most four coheirs; moreover, every nonrealized type has exactly four special sons. For the theory T', take $\leq_1$ as the order of distances and $\leq_2$ as a supplementary structure. We can therefore see that over a model M of cardinality λ, a pseudo-limit type of $S_1(M)$ can have up to $\mathrm{ded}(\lambda)$ coheirs.

12.7 Special Sequences

In this section we assume that T does not have the independence property. We know that an M-special type is therefore determined by the type of its Morley sequence over M. We need to answer the following question: Under what conditions is an indiscernible sequence over M the Morley sequence of a special type? And how can we recover this type, i.e., its infinitary definition, when we know only its Morley sequence?

An infinite indiscernible sequence s over M (or rather, a *type* of sequence over M) will be called *M-special* if it has the following property: Whenever it is realized by distinct copies $\dots, s_i, \dots$, we can find an element a such that each of the sequences $s_i{}^\frown a$ remains indiscernible over M, where $s_i{}^\frown a$ denotes the sequence obtained by adding a to the end of s_i.

By compactness, that means equivalently that if finitely many finite fragments $s'_1, \dots, s'_n$ of s are realized, we can find a such that each of the extended sequences $s'_1{}^\frown a, \dots, s'_n{}^\frown$ is still indiscernible over M. We can therefore see that this property of being M-special depends only on the Ehrenfeucht set of s over M (i.e., the formulas with parameters in M that are satisfied by increasing tuples from the sequence). As the sequences $s_i{}^\frown a$ are all similar to s, we can in turn find a_1, then $a_2, \dots$ then $a_n, \dots$ such that the sequence $a, a_1, \dots, a_n, \dots$ indiscernibly extends each of the s_i.

Theorem 12.33. *If T does not have the independence property, then an indiscernible ω-sequence over M is the Morley sequence of an M-special type if and only if it is M-special.*

Proof. The Morley sequence of a special type p is special: It is sufficient to have a realize the type given by the infinitary definition of p over the set consisting of M and the union of the s_i.

Conversely, consider a special sequence s over M. Let N be an M^+-saturated elementary extension of M and let a be an element such that for every copy s' of s in N, $s'{}^\frown a$ is indiscernible over M. I claim that the type p of a over N is M-special.

So let $f(x,\vec{b})$ be a formula with parameters in N, and let s_1 be a copy of s in N such that $f(x,\vec{b})$ alternates as much as possible. As it cannot alternate one more time over $s_1{}^\frown a$, it follows that $p \models f(x,\vec{b})$ iff all elements of s_1 from some point on satisfy $f(x,\vec{b})$. Indeed, all types of copies of s over $M \cup \{\vec{b}\}$ are realized in N. This "final" truth value that $f(x,\vec{b})$ takes over a copy of s where it alternates as much as possible (which is always the same because p is consistent!) depends only on the type of $\vec{b}$ over M.

Now let us show that the Morley sequence of p is indeed s. We embed $N \cup \{a\}$ into an $|M|^+$-saturated model N_1, and consider a point a_1 that extends all copies of s in N_1 to M-indiscernible sequences, then repeat ω times. The sequence $a, a_1, a_2, \ldots, a_n, \ldots$ is the Morley sequence of p over N; as it extends every copy of s in N, it is a similar sequence to s over M, and has the same type as s over M. □

Recall the method allowing us to recover the infinitary definition of p from s. Given a formula $f(x,\vec{b})$, we locate s in relation to $\vec{b}$ so that it alternates as much as possible, and take the final truth value.

Let us return to our canonical example, in which T is the theory of dense orders without endpoints. The Morley sequence of the heir of the type a^+ is a decreasing sequence $a_0 > a_1 > \cdots > a_n > \cdots$ located in the cut a^+. Let c be in the cut a^+ in an elementary extension of M. If c bounds a copy of this sequence either below or above, then the formula $x < c$ does not alternate. In order for it to alternate maximally, c must be placed between a_n and a_{n+1} for some n; in that case the final truth value does indeed give $x < c$ as the heir of the type a^+.

In order to produce the unique M-special son of p over A as the limit of the Morley sequence of p, we need to locate this sequence in relation to A. However, the stable types have a thoroughly remarkable additional property: However the Morley sequence s of p is positioned in relation to A, the mean type of s over A is always the heir of p!

Theorem 12.34. *Let p be a stable type in $S_1(M)$. Let A be a set of parameters containing M and let s be a realization of the Morley sequence of p over M. Then the mean type of s over A is the heir of p. In particular, we see that two copies s_1 and s_2 of the Morley sequence of p always give the same mean type over A, independently of their types with respect to A.*

Proof. We begin with an auxiliary construction: We let a'_0 realize p in an extension M'_0 of M, then let a'_1 realize the heir of p over M'_0 in a model M'_1, and repeat ω times. Since the sequence $a'_0, \ldots, a'_n, \ldots$ has the same type over M as the given Morley sequence $a_0, \ldots, a_n, \ldots$ of p, there are also models $M_n, M \prec M_0 \prec M_1 \prec \cdots \prec M_n \prec \cdots$ for this sequence such that a_n is in M_n and realizes the heir of p over M_{n-1}. Let M_ω be the limit of the M_n and let N be a model containing M_ω and A. The heir q of p over M_ω is the mean type of the sequence $a_0, \ldots, a_n, \ldots$ there. Consequently, the mean type of this sequence over N, which is the coheir of q, is also the heir of p, as well as its restriction to A. □

Remark. We should not fallaciously conclude from Lemma 12.23 and Theorem 12.33 that if T is stable, and if s_1 and s_2 are two infinite indiscernible sequences that have the same type over $\emptyset$, then they have the same mean type over every set A of parameters. For that, it is necessary in addition that there exist a model M such that s_1 and s_2 have the same type over M. That means, as we shall see in Chapter 16, that s_1 and s_2 have the same *strong type* over $\emptyset$, and the condition is positively not sufficient (see Theorem 16.3).

Corollary 12.35. *If T is stable and if M is a $|T|^+$-saturated model of T, then every p in $S_1(M)$ is the mean type of a (totally) indiscernible ω-sequence of elements of M.*

Proof. If p is in $S_1(M)$, then there is an elementary submodel N of M, $|N| = |T|$, such that p inherits its restriction q to N (Theorem 11.6). By saturation, M contains a copy $a_0, \ldots, a_n, \ldots$ of the Morley sequence of q, of which p is the limit. □

12.8 Instability and Order

Now we shall examine some properties of formulas that have a combinatorial nature and are equivalent to stability, or rather to instability.

Consider a formula $f(\vec{x}, \vec{y})$, where the tuples of variables $\vec{x}$ and $\vec{y}$ have the same length n. Let A be a set of n-tuples $\vec{a}_i$ from a model M of T, indexed by a chain I. We say that the formula $f(\vec{x}, \vec{y})$ *orders* the set A if $M \models f(\vec{a}_i, \vec{a}_j)$ when $i \leq j$ and $M \models \neg f(\vec{a}_i, \vec{a}_j)$ when $i > j$. If f orders an infinite set, or even arbitrarily large finite sets, then we can see by compactness that it orders sets indexed by any chain I.

Lemma 12.36. *If f orders an infinite set of n-tuples, then it orders an infinite indiscernible set of n-tuples.*

Proof. Let $\vec{a}_0, \ldots, \vec{a}_n, \ldots$ be ordered by f and let U be a nonprincipal ultrafilter of subsets of ω. Let $\vec{b}_0, \ldots, \vec{b}_n, \ldots$ realize the Morley sequence of coheirs associated with this ultrafilter (see Theorem 12.17). It is indiscernible, and if $n < m$ and $g(\vec{b}_n, \vec{b}_m)$ is satisfied, then there exist n' and m', $n' > m'$, such that $g(\vec{a}_{n'}, \vec{a}_{m'})$ is true. Consequently, the formula f orders the inverse of this indiscernible sequence. □

Theorem 12.37. *T is unstable if and only if for some n, there is a formula (without parameters) that orders an infinite set of n-tuples.*

Proof. An indiscernible sequence ordered by f cannot be totally indiscernible. Therefore, if f orders an infinite set, then T is unstable. (We could also note that with a formula ordering an infinite set, we can construct a set of parameters of any cardinality λ that has at least $\mathrm{ded}(\lambda)$ types, thus yielding instability in every cardinal.)

Conversely, assume that T is unstable, and first suppose that it has the independence property. Let us realize I_ω; we therefore have $\ldots, a_n, \ldots$ and $\ldots, \vec{b}_w, \ldots$, of which we shall only keep the $\vec{b}_m$ associated with sets $m = \{0, \ldots, m-1\}$. Then $f(a_n, \vec{b}_m)$ is true if and only if $n < m$, which implies that the formula $g(x_1, \vec{x}_2; y_1, \vec{y}_2) = f(x_1, \vec{y}_2)$ orders the set of tuples $a_n\frown\vec{b}_{n+1}$.

If T is unstable and does not have the independence property, then there is a sequence $a_0, \ldots, a_n, \ldots$ that is indiscernible but not totally. Consequently, there exists a formula f such that the increasing tuples of this sequence satisfy $f(x_0, \ldots, x_i, x_{i+1}, \ldots, x_n) \wedge \neg f(x_0, \ldots, x_{i+1}, x_i, \ldots, x_n)$. Extend this sequence beyond ω and set

$$g(x, y, \vec{a}) = (x = y) \vee f(a_0, \ldots, a_{i-1}, x, y, a_\omega, \ldots, a_{\omega+n-i-2}).$$

This formula, which uses the tuple $\vec{a}$ of parameters, orders the infinite set $a_i, \ldots, a_{i+m}, \ldots$; that means that the formula $h(x_1, \vec{x}_2; y_1, \vec{y}_2) = g(x_1, y_1, \vec{y}_2)$ orders the set of the $a_{i+m}\frown\vec{a}$. □

Note that we cannot take $n = 1$ in this theorem: In the example of an unstable theory given after Corollary 12.21, no set of elements is ordered by a formula, or even by a formula with parameters.

When a formula orders an infinite set A, this set has no reason to be definable, and we absolutely cannot control what this formula does outside A. There is no reason for this formula to define an order on all n-tuples of the model M; nothing prevents there being cycles.

We shall say that the formula $f(\vec{x}, \vec{y})$ *defines a partial preorder* or *partially (pre)orders* the (n-tuples of) models of T if it defines a preorder on M^n:

$$T \models (\forall \vec{x}) f(\vec{x}, \vec{x}) \wedge (\forall \vec{x})(\forall \vec{y})(\forall \vec{z})(f(\vec{x}, \vec{y}) \wedge f(\vec{y}, \vec{z}) \rightarrow f(\vec{x}, \vec{z})).$$

If we pass to the quotient modulo the equivalence relation $f(\vec{x}, \vec{y}) \wedge f(\vec{y}, \vec{x})$, we get an order defined on the entire model. If for some model this order has an infinite chain, or if for every model there are arbitrarily long finite chains (which amounts to the same thing), then it is clear that the formula $f(\vec{x}, \vec{y})$ orders an infinite set; however, that is a much stronger property.

The case in which the formula f merely preorders, with infinite chains, a subset A *definable by a formula* without parameters can be reduced to the case of a formula defined on the whole model by modifying f so that every element outside A is incomparable to everything except itself.

Theorem 12.38. *If T is unstable and without the independence property, then for some n there is a formula $f(\vec{x}, \vec{y})$ that preorders the n-tuples of models of T with infinite chains.*

Proof. In Theorem 12.37 we obtained an infinite indiscernible sequence $s = a_0, \ldots, a_n, \ldots$ ordered by a formula $f(x, y, \vec{a})$ using parameters $\vec{a}$ from an indiscernible sequence extending s. Therefore, s remains indiscernible over $\vec{a}$.

As this sequence is indivisible, then the following sentence is satisfied for some n (the parameter $\vec{a}$ has been omitted for clarity):

$$\begin{aligned}\neg(\exists x)(f(x, a_0) \quad &\wedge \quad \neg f(x, a_1) \wedge \cdots \wedge f(x, a_{2i}) \wedge \neg f(x, a_{2i+1}) \\ &\wedge \quad \cdots \wedge f(x, a_{2n}) \wedge \neg f(x, a_{2n+1})).\end{aligned}$$

In contrast,

$$(\exists x)(f(x, a_0) \wedge \cdots \wedge f(x, a_n) \wedge \neg f(x, a_{n+1}) \wedge \cdots \wedge \neg f(x, a_{2n+1})).$$

Indeed, that is true with $x = a_{n+1}$ for the tuple $a_0, \ldots, a_n, a_{n+2}, \ldots, a_{2n+2}$, which has the same type over $\vec{a}$ that $x = a_{n+1}$ has for the tuple $a_0, \ldots, a_n, a_{n+1}, \ldots, a_{2n+1}$.

As every permutation of $2n + 2$ elements is generated by transpositions of consecutive elements, we can find a sequence $\epsilon_0, \ldots, \epsilon_{2n+1}$ of symbols that are either $\neg$ or nothing, of which $n + 1$ are negation symbols, and an index i such that

$$\begin{aligned}(\exists x) \qquad &(\epsilon_0 f(x, a_0) \wedge \cdots \wedge \epsilon_i f(x, a_i) \wedge \\ &\epsilon_{i+1} f(x, a_{i+1}) \wedge \cdots \wedge \epsilon_{2n+1} f(x, a_{2n+1})),\end{aligned}$$

$$\begin{aligned}\neg(\exists x) \qquad &(\epsilon_0 f(x, a_0) \wedge \cdots \wedge \epsilon_{i+1} f(x, a_i) \wedge \\ &\epsilon_i f(x, a_{i+1}) \wedge \cdots \wedge \epsilon_{2n+1} f(x, a_{2n+1})),\end{aligned}$$

and it is certainly necessary for ϵ_{i+1} to be the opposite of ϵ_i.

By extending the indiscernible sequence, and moving $a_0, \ldots, a_{i-1}$ to the left and $a_{i+2}, \ldots, a_{2n+1}$ to the right, to the ends of this extension, in a manner analogous to what we did in Theorem 12.37, we get a formula $g(x, y, \vec{b})$, with parameters $\vec{b}$ that include $\vec{a}$, such that if $i < j$, then $(\exists x)(\neg g(x, a_i, \vec{b}) \wedge g(x, a_j, \vec{b}))$, and if $i > j$, then $\neg(\exists x)(g(x, a_i, \vec{b}) \wedge \neg g(x, a_j, \vec{b}))$. That means that the set of x satisfying $g(x, a_i, \vec{b})$ is strictly contained in the set of x satisfying $g(x, a_j, \vec{b})$. Consequently, the preorder defined by the formula $(\forall x)(g(x, y_1, \vec{y_2}) \to g(x, z_1, \vec{z_2}))$ has infinite subchains. □

Let us finish with two definitions of a technical character, which the reader will encounter from Shelah's pen.

The formula $f(\vec{x}, \vec{y})$ is said to have the *order property* if we can find $\vec{a}_n$ and $\vec{b}_m$ in a model M of T, $n, m \in \omega$, such that $f(\vec{a}_n, \vec{b}_m)$ is true if $n \leq m$ and false if $n > m$. As we can see, this means that the formula $g(\vec{x}_1, \vec{x}_2, \vec{y}_1, \vec{y}_2) = f(\vec{x}_1, \vec{y}_2)$ orders an infinite set.

The formula $f(\vec{x}, \vec{y})$ is said to have the *strict order property* if we can find $\vec{b}_n$ in a model M of T, $n, m \in \omega$, such that $(\exists \vec{x})(\neg f(\vec{x}, \vec{b}_n) \wedge f(\vec{x}, \vec{b}_m))$ is true if $n < m$ and false if $n > m$. That means that the preorder defined by $(\forall \vec{x})(f(\vec{x}, \vec{y}) \rightarrow f(\vec{x}, \vec{z}))$ has infinite chains.

This is the order property that appeared in Theorem 12.37 and the strict order property that appeared in Theorem 12.38; both these theorems, translated into Shelah's language, come out as follows:

T is unstable if and only if we can find a formula $f(x, \vec{y}))$ with the order property if and only if we can find a formula $f(x, \vec{y})$ with either the strict order property or the independence property.

In this result, as inspection of the proofs of Theorems 12.34 and 12.38 shows, this is a formula for which x is reduced to a single variable, which is made apparent; we can show, much more easily than in the case of the independence property, that if a formula $f(\vec{x}, \vec{y})$ has the strict order property, then there is one for which $\vec{x}$ is just one term.

An infinite chain has the strict order property but not the independence property, while the example given after Corollary 12.21 has the independence property but not the strict order property.

12.9 Appendix: Ramsey's Theorem

I am speaking here of a theorem that is traditionally used in model theory to build indiscernible sequences. It can quite well be skipped, and is not used anywhere in this text. However, the reader risks meeting it in the course of his readings, when he is assured of the existence of indiscernible sequences of a certain form with no other explanation than that "it follows from Ramsey's Theorem." He therefore must know what this is all about; besides, it is a fundamental theorem of combinatorics, which no well-bred person can ignore.

Consider a set E and a partition P of the set of n-element subsets of E into finitely many classes, which we shall call "colors", to make the theorem more beautiful. A subset A of E is called *homogeneous* for P if all its n-element subsets have the same color. It is stressed that P partitions the n-element subsets, rather than the n-tuples, of E.

Theorem 12.39 (Finite Ramsey Theorem). *Given three finite numbers n, m, k there is an integer $R(n, m, k)$ such that if the set E has at least $R(n, m, k)$ elements, then for every partition of the n-element subsets of E into k colors, there is a homogeneous subset of E with at least m elements.*

Proof. We associate a structure with such a partition by introducing n-ary relation symbols $R_1, \ldots, R_k$ for each color. Let T be the incomplete theory saying that each R_i is symmetric $((\forall x_1) \cdots (\forall x_n)(R_i(x_1, \ldots, x_n) \leftrightarrow R_i(x_{s1}, \ldots, x_{sn}))$, for each permutation s of the indices) and is satisfied only by injective n-tuples, and that $R_1, \ldots, R_k$ form a partition of the set of injective n-tuples.

The existence of a homogeneous subset of a model of T with n elements is expressed by the satisfaction of an axiom H_m.

If the theorem were false, we would have arbitrarily large finite models of $T \cup \{\neg H_m\}$, and therefore also an infinite model M. Let p be a nonrealized type in $S_1(M)$ and let $a_0, \ldots, a_i, \ldots$ be a Morley sequence of a special son of p in an elementary extension N of M. As this sequence is indiscernible (in this order), all its n-element subsets must have the same color, contradicting the fact that $N \models \neg H_m$. □

Theorem 12.40 (Infinite Ramsey Theorem). *If E is an infinite set, then for every partition of the n-element subsets of E into finitely many colors, there is an infinite homogeneous subset of E.*

Proof. By induction on n: It is obvious if $n = 1$; let us prove it for $n + 1$, assuming that it has been proven for n.

Take a_0 in E. We get a partition of the n-element subsets w of $E - \{a_0\}$ by giving w the color of $w \cup \{a_0\}$; by the induction hypothesis there is an infinite homogeneous subset A_0 of this infinite set, whose color we write c_0.

We then take a_1 in A_0, give an n-element subset w of $A_0 - \{a_1\}$ the color of $w \cup \{a_1\}$, and get an infinite homogeneous subset A_1 of color c_1.

Repeating this ω times, we get an infinite sequence $c_0, \ldots, c_i, \ldots$ of colors and a denumerably infinite subset $A = \{a_0, \ldots, a_i, \ldots\}$ of E, such that every $n+1$-element subset of A whose smallest index element is a_i receives the color c_i. As there are only finitely many colors, some color is repeated infinitely often, and the a_i of indices corresponding to this color form a homogeneous subset of E. □

The infinite Ramsey theorem is "stronger" than the finite Ramsey theorem; indeed, the infinite version of the finite Ramsey's theorem only states the existence of arbitrarily large finite sets.

The finite Ramsey theorem can be proved by compactness, either as we have already done or else from the infinite Ramsey theorem. We can also take our cue from the proof of the latter theorem and prove the finite Ramsey theorem by induction on n; when the details are worked out, we will realize that this theorem, which speaks only of finite sets, and therefore of combinatorics, is provable in Peano arithmetic.

Examples of applications of the (infinite) Ramsey theorem: An infinite graph has either an infinite subset consisting of pairwise connected elements or else pairwise nonconnected elements. An infinite ultrametric space contains either an infinite isosceles polygon or else an infinite subset with no isosceles triangles.

An example of an application of the finite Ramsey theorem: A graph that has at least 6 elements contains three points that are either all connected or else all nonconnected (i.e., $R(2,2,3) = 6$). The precise computations of $R(n, m, k)$, even for small values of n, are puzzles for combinatoricists.

Here is how some people use Ramsey's theorem to construct indiscernible sequences. Let $f(x)$ be a formula satisfied by infinitely many elements. We add $a_0, \ldots, a_n, \ldots$ to the language, and we want to show the consistency of the set of sentences consisting of the $f(a_n)$ and the $g(a_0, \ldots, a_n) \leftrightarrow g(a_{i_0}, \ldots, a_{i_n})$ whenever $i_0 < \cdots < i_n$.

For that purpose, it is enough by compactness to look at finitely many formulas $g_1, \ldots, g_k$, of arities $n_1, \ldots, n_k$, respectively. We therefore consider an infinite set $A = \{a_0, \ldots, a_n, \ldots\}$ of realizations of $f(x)$ in a model of T. We partition the n_1-element subsets of A into two classes: For those of the first class, when we put their indices in increasing order we get n_1-tuples that satisfy g_1, while for those of the second class, we get n_1-tuples that satisfy $\neg g_1$. By the infinite Ramsey theorem, A has an infinite homogeneous subset A_1. Repeating, we get an infinite subset of A_1 that is also homogeneous for g_2, etc. The set A_k that we get at the end is homogeneous for $g_1, \ldots, g_k$; it is enough to renumber its elements in increasing order to get what we are looking for.

We have used the infinite Ramsey theorem, which is easier to handle, but it is clear that, insofar as we are concerned with the *consistency* of something, the finite version suffices.

We can see that it is also quite simple and direct to take a Morley sequence of a coheir associated with an ultrafilter.

12.10 Historic and Bibliographic Notes

"Special sons" were called "nonsplitting extensions" by Shelah; the contents of Section 12.1 were set forth in [Poi81c], in particular the quick proof of the uniqueness of the nonsplitting extension for a stable type, Corollary 12.6. Coheirs appeared in [Poi77], following a suggestion of D. Lascar.

Indiscernible sequences come from [EM56]. An equivalent notion, that of a "chainable relation," was previously studied by [Fra54a]: It involves a relation R whose universe can be ordered as a chain C such that R can be defined from C by a quantifier-free formula. That amounts to saying that the universe of R is an indiscernible sequence for R, if we limit ourselves to quantifier-free formulas. Fraïssé used Ramsey's theorem to show the equivalence of chainability and monomorphy, which means that for every natural number n all restrictions of R to n elements are isomorphic. Frasnay showed that monomorphy can be expressed by finitely many axioms [Fra65]. His theory of "permuted chains" suggests the following question: How many ways can I be ordered as an indiscernible sequence? It was resolved in [WHS77].

Beginning with [Mor65] indiscernible sets appear everywhere in constructions of models of stable theories, hence the name "Morley sequence" given to the indiscernible sequence associated with an M-special type.

The independence property was defined in [She71b]; it was used in [Kei76] to determine the function that for a fixed T sends a cardinal λ to the maximum number of types that a set of parameters of that cardinality can have. The proof of Theorem 12.18 is original (see Section 12.5).

The characterization of stability by indiscernible sequences is the basis of Shelah's treatment of forking [She78]. Theorem 12.28, characterizing the independence property by the number of coheirs, was proved in [Poi81c] by means of a combinatorial lemma on ultrafilters; the argument given here, which consists in determining a special type by its Morley sequence, appears in the "added in proof" of [She80], several months before the article to which he referred!

The "parameter separation theorem" (12.31) was emphasized in [Poi77].

Matatyahu Rubin's Theorem is in [Rub74]. The absence of the independence property for chains was noted in [Poi81c], but the exact number of coheirs was determined only in [GS84]. The analysis of the coheirs in the case of ultrametric spaces has its origin in [Del81], where it was shown, under the applicability conditions of the Ax–Kochen theorem, that a valued field can have the independence property only if its quotient field or its group of values does. The group was eliminated by [GS84], who showed that a totally ordered abelian group does not have the independence property; their argument rests on an elimination theorem of Yuri Gurevič, who reduced the theory of an ordered group to that of ultrametric spaces. By [Par82], a tree cannot have the independence property.

Special sequences come from [Poi83d].

All the combinatorial properties of formulas from Section 12.8 were results in [She71b]; the various equivalents of instability are established here by methods noticeably different from Shelah's. It was [Lac75a] who showed that for the strict order property, $\vec{x}$ could be replaced by x.

One direct method of proving the equivalence of the order property and the dichotomy property can be found in [Hod81].

Another important combinatorial property of a formula, which is not studied in this course, is the "finite covering property" defined by [Kei67], studied by [She71b] and [BK80]; in [Poi83c] we see what it means for definability of types of a stable theory.

Ramsey's theorem, in its finite version as much as its infinite version, is the key to the vault of innumerable combinatorial constructions; it was originally proved by F. Ramsey in 1929–1930 to give an algorithm to decide the truth of $\forall\exists$ formulas!

13
The Fundamental Order

> L'ordre fondamental est un moyen de comparer les types sur les modèles d'une théorie complète ... il est surtout efficace dans le cas d'une théorie stable.
>
> B.P.

13.1 The Fundamental Order

The fundamental order is a means of comparing types over a model of a complete theory. It measures how far a type is from being realized, and is especially effective in the case of a stable theory.

We say that a type p over the model M of T *represents* the formula (without parameters) $f(x, \vec{y})$ if there exists $\vec{a}$ in M such that $p \models f(x, \vec{a})$; otherwise, we say it omits $f(x, \vec{y})$. Warning: If $p \models (\exists \vec{y}) f(x, \vec{y})$, that means that in every model where p is realized by x, there will be a $\vec{y}$ satisfying $f(x, \vec{y})$. It does not mean that there already exists such $\vec{y}$ in the model M. For example, every type satisfies $(\exists y) x = y$, while those that represent $x = y$ are the realized types.

The *class* of p (for the fundamental order) is the set of all formulas that it represents, and the *fundamental order* is the inverse inclusion order on classes of types: $p \geq q$ means that the class of p is contained in that of q, that every formula represented by p is represented by q, that whenever $p \models f(x, \vec{a})$ for $\vec{a}$ in M there is $\vec{a}'$ in N such that $q \models f(x, \vec{a}')$.

We can see that the cardinality of the fundamental order is at most $2^{|T|}$; that if q is a son of p, then $p \geq q$. Also, if q is the heir of p, then p and q have the same class, in which case we write $p \sim q$: Indeed, an heir of p belonging to $S_1(M)$ is by definition a son that is equivalent to it, that is to say not strictly lesser, in the fundamental order of $T(M)$.

Note also that if p and q are comparable in the fundamental order, then they have the same restriction to the empty set of parameters; indeed, if $f(x)$ is a formula with no parameters $\vec{y}$, then to say that p represents $f(x)$ is to say that p satisfies $f(x)$.

Lemma 13.1. *If p is a realized type, then q is equivalent to p in the fundamental order if and only if q is realized and has the same restriction to $\emptyset$ as p.*

Proof. Suppose that p is realized by a in M. If $q \leq p$, then q is realized, since it represents $x = y$, and we have seen that p and q have the same restriction to $\emptyset$. Conversely, suppose that q is realized by b in N, with a and b having the same type over $\emptyset$. If p represents $f(x, \vec{y})$, then $M \models (\exists \vec{y}) f(a, \vec{y})$, so $N \models (\exists \vec{y}) f(b, \vec{y})$, and there is indeed a $\vec{y}$ in N satisfying $f(b, \vec{y})$. □

As every type has a realized son, we see that the realized types are exactly the minimal types in the fundamental order, that every type bounds a unique minimal type. Two types have the same restriction to $\emptyset$ if and only if they bound the same minimal type in the fundamental order.

If we return to our example of the theory T of dense chains without endpoints, we get a fundamental order with four elements:

1. The least element, which is the class of realized types.
2. The class of the types a^+, a^-, and irrationals, which are all equivalent: This class is above the previous one.
3. The class of $+\infty$, which is above the previous one.
4. The class of $-\infty$, which is above (2) and is incomparable to (3).

Theorem 13.2. *The fundamental order is complete in the following sense: Every chain in it has a least upper bound (its supremum) and a greatest lower bound (its infimum).*

Proof. Let $p_i \in S_1(M_i)$ be a chain in the fundamental order, indexed by the chain I: if $i < j$, then $p_i < p_j$. Let F_i be the class of p_i, i.e., the set of all formulas that it represents.

I augment the language of T by adding a unary relational symbol $M(y)$ (which will more conveniently be written $y \in M$) and a constant symbol x. Consider the list of axioms saying that M is an elementary substructure of the universe (in other words, it satisfies Tarski's test), $(\forall \vec{y})((\vec{y} \in M \wedge (\exists u) f(u, \vec{y})) \rightarrow (\exists u)(u \in M \wedge f(u, \vec{y})))$; that the type of x over M represents all formulas f in $\cap F_i$, $(\exists \vec{y})(\vec{y} \in M \wedge f(x, \vec{y}))$; and that it omits all formulas g outside of $\cap F_i$, $(\forall \vec{y})(\vec{y} \in M \rightarrow \neg g(x, \vec{y}))$.

I claim that this set is consistent. Indeed, a finite fragment of it mentions only finitely many formulas to omit, and we can get a model by interpreting M as M_i, and x by a realization x_i of p_i in an elementary extension of M_i, for small enough i. Consequently, if we take a model of this theory, the class of the type p of x over M is $\cup F_i$, which is obviously the least upper bound of the F_i.

Let us do the same construction with $\cup F_i$ instead of $\cap F_i$. We still have a consistent set, since a finite fragment of the set of sentences being considered mentions only finitely many formulas to realize, and we get a model with a realization p_i over M_i for small enough i. Consequently, $\cup F_i$ is also a class of types, which is the greatest lower bound of the chain. □

Define an *atom* of the fundamental order to be a type, or rather a class of types, that is not realized, but that has only one element, its realization, strictly below it.

Lemma 13.3. *Every nonrealized (i.e., nonminimal) type bounds an atom.*

Proof. Let p be such a type and let C be a maximal chain of nonrealized types including p. As the lower bound of C omits $x = y$ (see the proof of the preceding theorem), it is not realized, and it is an atom. □

Lemma 13.4. *Every chain of the fundamental order has a dense family of cuts of cardinality less than or equal to that of T. In particular, a well-ordered or anti-well-ordered subset of the fundamental order has cardinality less than or equal to $|T|$.*

Proof. Let C be a chain of the fundamental order. Every formula $f(x, \vec{y})$ is associated with the cut whose upper class consists of the elements of C that omit f and whose lower class consists of those that represent it. This family of cuts separates the points of C.

The number of cuts of a well-ordering is equal to its cardinality, since all upper classes have a least element; the same is true for the inverse chains. □

We see therefore that the fundamental order of a theory T of cardinality λ is subject to some constraints. Here is one more: The fundamental order cannot be reduced to infinitely many minimal elements, since in that case T could not be the theory of a finite structure, and would have to have a nonrealized element. It seems unlikely that the conditions listed here are sufficient for a partial order to be fundamental, but nothing is known about this. There are a few additional constraints that the fundamental orders of stable theories must satisfy, but we are still very far from being able to determine them.

Let us briefly examine what happens with ultrapowers of types:

Theorem 13.5. *If p is in $S_1(M)$ and q in $S_1(N)$, then $p \geq q$ if and only if there is an ultrafilter U and an embedding of M into N^U that makes q^U a son of p.*

Proof. Since q^U is an heir of q, then $q^U \sim q$, so if q^U is a son of p, then $p \geq q^U, p \geq q$, and the condition is sufficient.

Conversely, suppose that $p \geq q$. Let I be the set of injections from finite subsets of M into N. If $p \models f(x, \vec{a}), \vec{a} \in M$, we define $I_{f(x,\vec{a})}$ as the set of all i in I that are defined at $\vec{a}$ such that $q \models f(x, i\vec{a})$. Since $p \geq q$, all these sets are nonempty and form a filter base that is contained in an ultrafilter U of subsets of I.

Now consider the function s from M to N^U obtained by setting $(sa)_i = ia$ if i is defined at a, and anything otherwise. This is an elementary injection from M into N^U, since if the formula f contains only the variable x, then saying that $p \models f(\vec{a})$ amounts simply to saying that $f(\vec{a})$ is true in M. Therefore, q^U is a son of p modulo this embedding. □

Theorem 11.4 can be proved from this result; it is instructive to compare the two. If $M \prec N, p \in S_1(M), q \in S_1(N)$, and q is an heir of p, that means that $q \geq p$ for the fundamental order of $T(M)$. Consequently, by Theorem 13.5 there exists U and an embedding *fixing* M (every element of which is named in the language), i.e., an embedding extending the canonical diagonal embedding, from N into M^U, which makes p^U a son of q.

Now we move on to a thoroughly remarkable, and essential, property of stable types. In general, for q to be an heir of p, it is not sufficient that it be a son that is equivalent in the fundamental order of T, since the parameters of the model M of p must be taken into consideration; this is not the case for stable types.

Theorem 13.6. *A stable type over a model of T has only one son equivalent to it in the fundamental order, namely its heir.*

Proof. If p has two equivalent sons p_1 and p_2, $p_1 \models f(x, \vec{a}), p_2 \models \neg f(x, \vec{a})$, then every type q equivalent to p will have p^U, and hence p_1^U and p_2^U, as sons. The type q can also be split into two sons equivalent to p, and we can construct a tree of types, yielding the dichotomy property, as in the proof of Theorem 11.11. That contradicts the stability of p. □

Theorem 13.7. *If $p \geq q$ and p is stable, then q is stable. In particular, stability is a property of classes of types: Types of the same class are either all stable or all unstable.*

Proof. Let us show that if q is unstable, then so is p. We observe that p has a son q^U that is an heir of q and therefore unstable. Alternatively, q has the dichotomy property for some formula; therefore, so does p, every finite fragment of which can be interpreted in q. □

If (M, dp) and (N, dp) are elementarily equivalent, then p and q represent the same formulas; indeed, if $(M, dp) \models (\exists \vec{y}) df(\vec{y})$, then so does (N, dq). The converse is true for stable types:

Theorem 13.8. *If p in $S_1(M)$ and q in $S_1(N)$ are stable and equivalent in the fundamental order, then (M, dp) and (N, dq) are elementarily equivalent.*

Proof. Since $p \geq q$, p has a son q^U. Since q^U is the heir of q, then p and q^U are equivalent, and by Theorem 13.6, q^U is the heir of p. Since an heir is the same as a strong heir for a stable (or even just definable) type, (N^U, dq^U) is a common elementary extension of (N, dq) and (M, dp), which are therefore elementarily equivalent. □

If we know Shelah's result stating that two elementarily equivalent structures have isomorphic ultrapowers, we can see that two stable types of the same class have a common ultrapower.

We shall see in Chapter 15, as a consequence of the theorem of the bound (15.6), that there is another constraint on the fundamental order of a stable theory: Over a minimal element corresponding to realizations of a given type over $\emptyset$, there is a maximal class, which bounds all classes containing this type over $\emptyset$. The fundamental order of the theory of dense orders without endpoints does not have this property.

Note that two stable types have the same class in the fundamental order if and only if their Morley sequences have the same type over $\emptyset$. Indeed, if p and q are equivalent, then they have a common heir up to isomorphism. Also, if $p \in S_1(M), q \in S_1(M)$, where M and N are such that p and q have the same Morley sequence s, then we know that the mean type of s over a model containing M and N is the heir both of p and of q.

In the case in which T is stable, there is therefore a bijection between the classes of the fundamental order and the types of indiscernible ω-sequences over $\emptyset$, which can all be considered as Morley sequences. More precisely, a sequence s can be associated with the class of the mean type p_s of s over a model M containing s: The Morley sequence of p_s extends s; that is to say, it has the same type as s.

What order do we define this way on the types of indiscernible sequences? If $p_s \geq p_t$, that means that up to isomorphism p_t can be considered as a son of p_s. It is therefore necessary that we be able to put s and t with respect to each other in such a way that the mean type of t over s is also the mean type of s over s, and this condition is sufficient for $p_s \geq p_t$. The reader will easily recognize this upon seeing the theorem of the bound in Chapter 15 and understanding that p_s is the unique nonforking son of its restriction to s.

13.2 Stability Spectrum

We know that an unstable theory is unstable in every infinite cardinal λ. We are going to show that the class of cardinals in which a stable theory T is stable, is determined by two invariants of T: $\lambda_0(T)$, *the smallest infinite* *cardinal in which T is stable*, and $\kappa(T)$, *the smallest infinite cardinal such that there is no strictly decreasing ordinal sequence indexed by $\kappa(T)$ in the fundamental order of T* (in other words, every anti-well-ordered subset of the fundamental order has cardinality strictly less than $\kappa(T)$).

Lemma 13.9. *If $\lambda^\mu > \lambda$ for some $\mu < \kappa(T)$, then T is unstable in λ.*

Proof. We can obviously assume that T is stable. Let μ be the smallest cardinal such that $\lambda^\mu > \lambda$. There is a decreasing sequence $p_0 > p_1 > \cdots > p_\alpha > \cdots, \alpha \in \mu$ in the fundamental order of T, and we can assume, since we can always add the infima, that if α is a limit, then the class of p_α is the infimum of the $p_\beta, \beta < \alpha$. Let $f_\alpha(x, \vec{y})$ be a formula represented by $p_{\alpha+1}$ and omitted by p_α.

I am going to use induction on $\alpha \leq \mu$ to construct an increasing sequence of models N_α of T, and types q_σ, for $\sigma \in \lambda^\alpha$, such that:

- if α is a limit, then N_α is the limit of the $N_\beta, \beta < \alpha$;
- if $\sigma \in \lambda^\alpha$, then q_σ is a type over N_α whose class is that of p_α, and if α is a limit, q_σ is the limit of the $q_{\sigma|\beta}$ for $\beta < \alpha$;
- if σ is an extension of τ, then q_σ is a son of q_τ;
- for every sequence $\sigma \in \lambda^\alpha$, there are tuples $\vec{a}_{\sigma\frown u}$ in the model $N_{\alpha+1}$ for $u \in \lambda$ such that $q_{\sigma\frown u} \models f_{\alpha+1}(x, \vec{a}_{\sigma\frown u})$, while if $v \neq u$, then $q_{\sigma\frown u} \models \neg f_{\alpha+1}(x, \vec{a}_{\sigma\frown v})$.

For the empty sequence I take the type $p_0 = q_\emptyset$ over its model $M_0 = N_0$. Since at limit stages I just need to take limits, I need only explain what I do at successor stages.

I therefore have my model N_α, and, for every $\sigma \in \lambda^\alpha$, my type q_σ of $S_1(N_\alpha)$, whose class is that of p_α. The type q_σ has a son r equivalent to $p_{\alpha+1}$, for example an ultrapower of $p_{\alpha+1}$, over an extension P of N_α. Let b be a realization of r; then there exists $\vec{a}$ in P such that $f_{\alpha+1}(b, \vec{a})$ is satisfied.

We therefore realize the Morley λ-sequence of the type of $P \cup \{b\}$ over N_α: It is a type in infinitely many variables, but that will still work out well. We thus get a sequence $P_0 \cup \{b_0\}, \ldots, P_u \cup \{b_u\}, \ldots, u \in \lambda$ that is (totally) indiscernible over N_α. Each P_u contains a copy $\vec{a}_u$ of $\vec{a}$ for which we have $f_{\alpha+1}(b_u, \vec{a}_u)$. On the other hand, if $u \neq v$, then, by the construction of the Morley sequence if $v < u$ and by total indiscernibility (or equality of the coheir and the heir) if $u < v$, the type of $P_u \cup \{b_u\}$ over $P_v \cup \{b_v\}$ inherits its restriction to N_α. Therefore, the type of b_u over P_v inherits its restriction to N_α, which is q_σ, and which does not satisfy $f_{\alpha+1}(x, \vec{y})$; consequently, it is $\neg f_{\alpha+1}(b_u, \vec{a}_v)$ that is satisfied.

We do that for all σ in λ^α. We therefore have $P_{\sigma\frown u}$, $b_{\sigma\frown u}$, and $\vec{a}_{\sigma\frown u}$. Include all the $P_{\sigma\frown u}$ in one elementary extension $N_{\alpha+1}$ of N_α, which therefore contains all the $a_{\sigma\frown u}$, and take $q_{\sigma\frown u}$ to be the heir over $N_{\alpha+1}$ of the type of $b_{\sigma\frown u}$ over $P_{\sigma\frown u}$, which is indeed the class of the $p_{\alpha+1}$.

Once we have done that μ times, we get λ^μ types $q_\sigma, \sigma \in \lambda^\mu$ over the model N_μ. To distinguish these types pairwise, it is sufficient to restrict them to the parameters $\vec{a}_\tau$ for $\tau \in \lambda^\alpha, \alpha < \mu$. By the definition of μ, if $\alpha < \mu$, then $|\lambda^\alpha| = \lambda$, the number of these parameters is $\mu \times \lambda = \lambda$, and we have instability in λ. □

Lemma 13.10. *If $\lambda \geq \lambda_0(T)$, then for every subset A, of cardinality λ, of a model N of T there is an elementary restriction M of N of cardinality λ containing A.*

Proof. Let $A = A_0$. Consider a set $A_1 \subset N$ that contains a realization of every type over a finite subset of A_0 that is realized in N. There are λ finite subsets of A, each of which can produce at most $\lambda_0(T)$ types. That leaves only λ types to realize, and we can get A_1 of cardinality λ. We start again, replacing A_0 by A_1, and repeat ω times; the union M of the A_n has cardinality λ and satisfies Tarski's test, so it is an elementary submodel of N. □

Theorem 13.11 (Spectrum of Stability Theorem). *If T is stable, then T is stable in λ if and only if $\lambda \geq \lambda_0(T)$ and $\lambda^\mu = \lambda$ for all $\mu < \kappa(T)$.*

Proof. The condition is necessary by Lemma 13.9. Conversely, suppose that it is satisfied; we must show stability in λ. By Lemma 13.10, every set of parameters of cardinality λ embeds into a model of the same cardinality; we can be satisfied with counting the types over a model of cardinality λ.

As T is stable in $\lambda_0 = \lambda_0(T)$, it follows that $\lambda_0 = \lambda_0^{<\kappa(T)}$ (this notation denotes the cardinal supremum of the $\lambda^\alpha, \alpha < \kappa(T)$) and $\kappa(T) \leq \lambda_0$. Consider the functions s from an ordinal $\alpha < \kappa(T)$ into M. By induction on the length α of s we can construct an elementary submodel M_s of M of cardinality λ_0, containing the image of s, such that if s is a restriction of t, then $M_s \prec M_t$. We therefore take a fixed family of M_s with this property; since $\lambda^{<\kappa(T)} = \Sigma_{\alpha<\kappa(T)}\lambda^\alpha = \kappa(T) \times \lambda = \lambda$, there are λ models in the family of the M_s.

Now let p be in $S_1(M)$. Let p_0 be the restriction of p to the model $M_\emptyset$. If p is not the heir of p_0, then for some formula, $p \models f(x, \vec{a}_1)$, whereas p_0 omits $f(x, \vec{y})$. Let M_1 be the model associated with the sequence s_1 obtained by enumerating $\vec{a}_1$. If p is not the heir of its restriction p_1 to M_1, that is because of a parameter $\vec{a}_2$. We therefore consider the sequence s_2 obtained by adding an enumeration of $\vec{a}_2$ to the end of s_1 and the restriction p_2 of p to $M_2 = M_{s_2}$. We continue: At limit stages, consider the limit s_α of $s_\beta, \beta < \alpha$, the model $M_\alpha = M_{s_\alpha}$, and the restriction p_α of p to M_α. At successor stages, if p is not the heir of p_α, it is because of a tuple $\vec{a}_{\alpha+1}$ of parameters, which we add to the end of s_α to get $s_{\alpha+1}$, and we define $p_{\alpha+1}$ as the restriction of p to the model $M_{\alpha+1}$ associated with this last sequence.

Since $p_\alpha > p_{\alpha+1}$ for all α, we are forced to stop before $\kappa(T)$.

When we stop, we have found M_s such that p is the heir of its restriction p_s to M_s. There are λ possible choices for M_s, and once we have chosen it, there are λ_0 choices for p_s, since $|M_s| = \lambda_0$, making altogether $\lambda \times \lambda_0 = \lambda$ possible choices for p. □

If $\kappa(T) = \omega$, that is to say, if the fundamental order is well-founded (contains no strictly decreasing ω-sequence), T is called *superstable*. That means that it is stable in every cardinal greater than $\lambda_0(T)$. A theory that is stable in every cardinal from some point on is superstable, since there are arbitrarily large cardinals for which $\lambda^\omega > \lambda$; for example, by König's lemma (8.15, 8.16), all those of denumerable cofinality.

Note also that if T is stable in λ, it is stable in λ^+. Indeed, if $\lambda^\mu = \lambda$, then $(\lambda^+)^\mu = \lambda^+$, since $\mu < \lambda$, and since λ^+ is regular, every function from μ into λ^+ is bounded by some $\alpha < \lambda^+$; there are λ^+ possible choices for α, and, for fixed α, since $|\alpha| \leq \lambda$, we have $|\alpha^\mu| \leq \lambda^\mu = \lambda$, so all in all there are $\lambda^+ \times \lambda = \lambda^+$ choices.

We see therefore that the stability spectrum of T, that is to say the class of all cardinals in which T is stable, is determined by $\lambda_0(T)$ and $\kappa(T)$. Conversely, the stability spectrum certainly allows us to recover $\lambda_0(T)$, but also $\kappa(T)$. Let us take a cardinal $\beth_\alpha > \lambda_0(T)$ and consider $\lambda = \beth_{\alpha+\beta^+}$. The cofinality of λ is β^+, which is regular, so by König's Lemma $\lambda^{\beta^+} > \lambda$. On the other hand, since a function from β to λ is bounded by some $\beth_{\alpha+\gamma}$, we have $\lambda^\beta = \lambda$. In other words, T is stable in λ if and only if $\beta^+ < \kappa(T)$, and any cardinal is clearly determined by the list of successor cardinals strictly below it.

That has an important consequence: We define $\kappa_n(T)$ as the cardinal analogous to $\kappa(T)$ associated with the fundamental order on n-tuples; this is therefore the smallest infinite cardinal such that there is no descending ordinal sequence indexed by $\kappa_n(T)$ in the fundamental order on the n-tuples of T.

Lemma 13.12. *If T is stable, then for every natural number n, $\kappa(T) = \kappa_n(T)$.*

Proof. As on the one hand we can associate the type of a with the type of the n-tuple $(a, a, \ldots, a)$ over A, and on the other hand the type of $(a_1, \ldots, a_n)$ over A is given by the type of a_1 over A, then the type of a_2 over $A \cup \{a_1\}, \ldots$ then the type of a_n over $\{a_1, \ldots, a_n\}$, T has the same stability spectrum for its singletons as for its n-tuples. □

We know from Lemma 13.4 that $\kappa(T) \leq |T|^+$; furthermore, we have noted that $\kappa(T) \leq \lambda_0(T)$, and we also know (see Theorem 11.10) that $\lambda_0(T) \leq 2^{|T|}$.

There is another constraint: If $\lambda_0(T) > |T|$, then $\lambda_0(T) \geq 2^\omega$. That is due to the following purely topological lemma applied to the spaces $S_1(A), |A| = |T|$.

Lemma 13.13. *A compact* 0*-dimensional space whose cardinality is strictly greater than the number of its clopen sets has at least* 2^ω *points.*

Proof. Recall the definitions of the Cantor rank and the derived spaces of the space E (see Section 1.3). Define E_α by induction on α: If α is a limit, then $E_\alpha = \cap E_\beta$ for $\beta < \alpha$; in the successor case $E_{\alpha+1}$ is E_α with its isolated points removed. Set $R(a) = \alpha$ if $a \in E_\alpha - E_{\alpha+1}$ and $R(a) = \infty$ if a is in the intersection E_∞ of all the E_α.

If $E_\infty = \emptyset$, then every point has an ordinal rank, and each one has a clopen neighborhood in which it is the unique point of maximal rank; in this case, E does not have more points than clopen sets.

If $E_\infty \neq \emptyset$, it is a nonempty Stone space without isolated points, and it has at least 2^ω points. For every nonempty clopen set in E_∞ can be split into two nonempty clopen sets. We thus proceed by dichotomy: Divide E_∞ in two, then divide each piece in two, etc. Repeat ω times. Each of the 2^ω sequences of nested compact sets that we have constructed converges. □

As a consequence, if $|T| = \omega$, then $\lambda_0(T) = \omega$ or 2^ω and $\kappa(T) = \omega$ or ω_1. Thus the only possible stability spectra for a denumerable theory are the following four:

- T is *unstable*: T is unstable in every λ.
- T is *stable but not superstable*: T is stable in all λ^ω and unstable in all other cardinals.
- T is *superstable but not ω-stable*: T is stable in all $\lambda \geq 2^\omega$ and unstable in all other cardinals.
- T is *ω-stable* (also called *totally transcendental*): T is stable in all λ.

13.3 Some Examples

Did you find the developments in the last three chapters too abstract? The time has come to illustrate them with a few examples and to lay out the stability properties of the theories that have appeared in the course of this work. As usual, the details are left to the reader.

Example 1: Theory of a Finite Structure

ω-stable. Since every type is realized, the fundamental order consists of finitely many incomparable points, and the Morley sequences can only repeat the same element.

Example 2: Theory of an Infinite Set

ω-stable. The fundamental order includes two points—realized and nonrealized—one above the other.

Example 3: Complete Theory in a Language Involving Only Unary Relations

Superstable. A type over $\emptyset$ merely specifies, for each symbol R, whether $p \models R(x)$ or $p \models \neg R(x)$. If p has only finitely many realizations, then it yields only one point in the fundamental order; otherwise, it gives two, realized and nonrealized, the one over the other. $\lambda_0(T) = |S_1(\emptyset)|$.

Example 4: Equivalence Relation with Infinitely Many Infinite Classes

ω-stable. The fundamental order consists of three points lined up above each other: (i) realized; (ii) omits $x = y$ but represents $x \sim y$; (iii) omits $x \sim y$.

Example 5: Dense Order Without Endpoints

Unstable, without the independence property (like all chains). The fundamental order has already been described.

Example 6: Discrete Order Without Endpoints

Unstable, without the independence property. The fundamental order consists of two points, realized and nonrealized. Every nonrealized son of a nonrealized type is an heir of it.

Example 7: Algebraically Closed Field of a Given Characteristic

ω-stable. Some types over $\emptyset$ are algebraic over the prime subfield: ω points of the fundamental order that are incomparable to all others. Others are transcendental over the prime subfield: two points, realized and nonrealized, one above the other. The Morley sequence of a nonrealized type over M is a set of elements that are algebraically independent over M.

Example 8: Differentially Closed Field of Characteristic Zero

ω-stable, since the types correspond to irreducible differential polynomials. The fundamental order is rather complicated, and our knowledge of it is very incomplete.

Example 9: Theory of an Infinite Boolean Algebra

Unstable, with the independence property and the strict order property (both for the formula $x \leq y$).

Example 10: Rich or n-Rich Ultrametric Spaces

Unstable, but without the independence property (see Section 12.5).

Example 11: Theory of the Successor on the Natural Numbers

ω-stable. The fundamental order contains ω points incomparable to the others, corresponding to the types of $0, 1, \ldots, n, \ldots$, and two other points, one above the other: realized nonstandard type and nonrealized nonstandard type.

Example 12: Theory of the Order on the Natural Numbers

Unstable, without the independence property. The fundamental order is the same as in the preceding example.

Example 13: Theory of the Sum on the Natural Numbers

Unstable. It can be shown to lack the independence property.

Example 14: Arithmetic

Unstable, with the independence property (for the divisibility formula or the membership formula), and also the strict order property.

Example 15: One Theory Interpretable in Another

Note, following the principle set forth in Section 9.4, that if N is interpretable in a structure M whose theory T is λ-stable, with λ greater than the number of parameters involved in this interpretation, then the theory T' of N is also λ-stable. We see therefore that $\kappa(T') \leq \kappa(T)$.

We can also see that if N is interpretable in M, and its theory T' has the independence property, or else the strict order property, then the theory T of M has the corresponding property: Replace the elements of N with the tuples of M that represent them, and the formulas of N with their translations in M.

Example 16: Modules

If we spoke so much of modules in Section 6.5, it was chiefly with a view to the following result:

Theorem 13.14. *Every theory of modules, in particular every theory of abelian groups, is stable.*

Proof. A type p in $S_1(M)$ is determined by the formulas $x \sim a \pmod G$ that it satisfies, where a is in M and where G is a primitive subgroup of M, by Theorem 6.26. If p omits the formula $x \sim y(\text{mod } G)$, we take $y \neq y$ as the associated definition, and if $p \models x \sim a(\text{mod } G)$ for some a, we take $y \sim a(\text{mod } G)$ as the definition. In conclusion, every type is definable. □

In Section 6.5 we associated a type p of $S_1(M)$ with the *filter of primitive groups* corresponding to the formulas $x \sim y \pmod G$ that p represents. We know that at least one type over M corresponds to each filter of primitive groups. Note that if q is the heir of p over $N \prec M$, then it cannot represent any more formulas. Therefore, q is the only son of p with the same filter of groups, since we have already noted that this son is unique.

Consequently, if we start with a type p_0 over a model M_0 and take successive sons $p_1, \ldots, p_n, \ldots$ over models $M_1, \ldots, M_n, \ldots$, then a decrease with respect to the fundamental order corresponds exactly to an increase in the filters of primitive groups $F_0, F_1, \ldots, F_n, \ldots$ associated with $p_0, p_1, \ldots, p_n, \ldots$, respectively.

We can therefore see that the fragment of the fundamental order consisting of those classes below that of p_0 is nothing other than the inverse inclusion relation on filters of primitive groups containing F_0. Nevertheless, the fundamental order is not just the inverse of the inclusion order on filters of groups, since in the fundamental order we must take the restrictions of types to $\emptyset$ into consideration. Note that if p is the type of x over M, and if a is in M, then the type $a+p$ of $a+x$ over M has the same filter as p. This inverse inclusion of filters is what we call the *stratified order* obtained by considering the fundamental order "up to translation by an element of M." We shall say nothing about the stratified order in this work, except that they provide a very effective tool for the investigation of stable groups.

Consider, for example, a coherent ring A and the theory T of its existentially closed modules. If p is in $S_1(M)$, then (M, dp) is determined by the following two ideals: $I(p) = \{\alpha : p \models \alpha x \in M\}$, $I_0(p) = \{\alpha : p \models \alpha x = 0\}$. The ideal $I(p)$ determines the filter of groups of p; $I_0(p)$, its type over $\emptyset$. The fundamental order is as follows: $p \geq q$ if and only if $I_0(p) = I_0(q)$ and $I(p) \subseteq I(q)$. As every ideal is of the form $I(p)$, T is superstable if and only if A is Noetherian. In this case it is $|A|$-stable, since a type is determined by finitely many equations corresponding to the generators of $I(p)$.

More generally, the stability spectrum of a module M, or more precisely of its theory, depends only on its primitive subgroups:

Theorem 13.15. *A module M is superstable if and only if it has no infinite decreasing chain $G_0 \supset G_1 \supset \cdots \supset G_n \supset \cdots$ of primitive subgroups, with each G_{n+1} having infinite index in G_n. If the ring A is denumerable, then it is ω-stable if and only if it has no strictly decreasing infinite sequence of primitive subgroups.*

Proof. The first condition is trivially equivalent to superstability, since it means that there is no infinite strictly increasing sequence of filters of primitive groups.

If the second condition is true, then every filter F has a minimal element G, and a type p in the filter F is determined by a condition $x \sim a \pmod G$ (and by all the $x \not\sim b \pmod H$ for $H \notin F$). We therefore have stability in every λ. If, on the other hand, there is a strictly decreasing sequence $G_0 \supset \cdots \supset G_n \supset \cdots$ of such subgroups, then every class modulo G_n can be divided into at least two classes modulo G_{n+1}, and in some model of T we can find a_s indexed by $2^n, n \in \omega$, and b_σ indexed by 2^ω, such that if s is a restriction of σ, then $a_s \sim b_\sigma (\mathrm{mod}\ G_n)$ if $\sigma(n) = 1$, while $a_s \not\sim b_\sigma (\mathrm{mod}\ G_n)$ if $\sigma(n) = 0$. There are therefore 2^ω types over the denumerable set of the a_s. □

For example, if Z_p denotes the cyclic group with p elements, where p is a prime number, the sum $Z_p^{(\omega)}$ of ω copies of Z_p is ω-stable. Indeed, it can be considered as a vector space over the p-element field, and an infinite vector space over a field K is an existentially closed K-module.

The subgroup $n\mathbf{Z}$ of $\mathbf{Z}$, defined by the formula $(\exists y)(x = ny)$, is primitive. As all subgroups of $\mathbf{Z}$ are of the form $n\mathbf{Z}$, those are its primitive subgroups. We can therefore see that $\mathbf{Z}$ is superstable and not ω-stable. It has two filters of primitive groups: that consisting of all primitive groups, which corresponds to the realized types, and that of all nonzero primitive groups, which corresponds to the nonrealized types.

Likewise, the groups $\mathbf{Z}^m$ are superstable, since they are interpretable in $\mathbf{Z}$; we can see that they are not ω-stable by looking at the $n\mathbf{Z}^m$. In contrast, the group $\mathbf{Z}^{(\omega)}$ is not superstable.

13.4 Historic and Bibliographic Notes

The fundamental order is the key to the "Parisian approach to stability" [Poi77], [LP79]. To reassure those who would find that the author lacked modesty in describing one of his own creations as "fundamental," I might point out that it was named thus because Lascar's rank U is the *foundation* rank of this fundamental order.

Once Chapters 15, 16, and 17 of this course have been digested, the reader will find more information on the fundamental orders of superstable theories in [Bue86] and [Poi].

The stability spectrum was almost exclusively determined by Shelah; see [She78]. The terms "totally transcendental" and "superstable" appeared in [Mor65] and [She69], respectively.

The reduction of stability properties of modules to the presence or absence of chains of primitive subgroups originally comes from [Gar79]; for the stratified order, see [Poi81b] and [Poi83b].

14

Stability and Saturated Models

Yes, oh yes, Long Distance, I accept the charge, I'll pay
Which loved one is calling me? I didn't hear you say
Both are deep within my heart, her Mum and my Marie,
It's so good to hear your voice from Memphis (Tennessee)

C.E.B.

14.1 Existence Theorem

In the general case, we are not able to prove the existence of saturated (i.e., λ-saturated of cardinality λ) models; we are going to see that the situation is different for a stable theory.

We begin with a remark: If T is stable, and if $\ldots, a_i, \ldots$ is an indiscernible sequence, which is therefore totally indiscernible and indivisible, then for every parameter $\vec{b}$, all but at most $|T|$ of the a_1 realize the mean type of the sequence over $\vec{b}$. Indeed, a formula $f(x, \vec{b})$ eliminates only finitely many of the a_i. The following lemma states that for this property, $|T|^+$ can be replaced by $\kappa(T)$.

Lemma 14.1. *If T is stable, if s is an indiscernible sequence, and if $\vec{b}$ is an n-tuple of parameters, then the number of elements of s that do not realize the mean type of s over $\vec{b}$ is strictly less than $\kappa(T)$.*

Proof. We know that the sequence $s = \{a_0, \ldots, a_\alpha, \ldots\}$ has the same type over $\emptyset$ as a Morley sequence (Lemma 12.33). Consequently, there is an increasing sequence M_α of models and a type p over M_0 such that a_α realizes, in $M_{\alpha+1}$, the heir of p over M_α.

Let q_α be the type of $\vec{b}$ over M_α. If the type of a_α over $M_0 \cup \{\vec{b}\}$ is not the heir of p, then neither is the type of a_α over $M_\alpha \cup \{\vec{b}\}$. Since, on the other hand, the type of a_α over M_α is the heir of p, the type of a_α over $M_\alpha \cup \{\vec{b}\}$ does not inherit its restriction to M_α. By symmetry, the type of $\vec{b}$ over $M_\alpha \cup \{a_\alpha\}$ does not inherit its restriction to M_α, and $q_{\alpha+1}$ is not the heir of q_α. Since $\kappa_n(T) = \kappa(T)$ (see Lemma 13.12), that happens fewer than $\kappa(T)$ times. □

Theorem 14.2. *If T is stable in λ, then it has a saturated model of cardinality λ.*

Proof. By the stability spectrum theorem (13.11), since T is stable in λ, $\lambda \geq \lambda_0(T)$ and $\operatorname{cof}(\lambda) \geq \kappa(T)$. For by König's lemma, $\lambda^{\operatorname{cof}(\lambda)}$ is strictly greater than λ.

By Theorem 13.10, we know that T has a model M_0 of cardinality λ. As T is stable in λ, there are only λ types over this model, and, again by Theorem 13.10, we can realize all of them in an extension M_1 of M_0 of cardinality λ. Repeat this λ times: We thus get an increasing sequence $M_\alpha, \alpha < \lambda$, of models of T, all of cardinality λ, such that $M_{\alpha+1}$ realizes all types of $S_1(M_\alpha)$.

I claim that the model M, the limit of the M_α, which has cardinality $\lambda \times \lambda = \lambda$, is saturated. Let A be a subset of M of cardinality $\mu < \lambda$ and let p be a type over A. We need to see that p is realized in M.

Let q be any son of p over M and let q_α be the restriction of q to M_α. In the fundamental order, we have $q_0 \geq q_1 \geq \cdots \geq q_\alpha \geq \cdots$, and as there is no strictly decreasing ordinal sequence cofinal in λ in this fundamental sequence, from some point α on, all the $q_\beta, \beta \geq \alpha$ are equivalent to q_α. These q_β are therefore heirs of p, and so is their limit q.

The number of $\beta > \alpha$ is obviously λ, so M contains a copy $s = \{a_{\alpha+1}, a_{\alpha+2}, \ldots, a_{\alpha+\beta}, \ldots\}$ of the Morley λ-sequence of q_α, and q is nothing other than the mean type over M of this sequence. Therefore, p is the mean type of s over λ.

If $\lambda > \kappa(T)$, then by Lemma 14.1, all but at most $\mu \times \kappa(T) < \lambda$ elements of this sequence realize p. Also if $\lambda = \kappa(T)$ then $\operatorname{cof}(\lambda) = \kappa(T)$, and λ is regular. But if λ is regular, then we do not need to break our skulls, since A is then contained in some M_α for α large enough. □

14.2 Nonexistence Theorems

Theorem 14.3. *An unstable theory has no saturated model of singular cardinality.*

Proof. Suppose that T is unstable and has a saturated model M of singular cardinality λ. We can then write M in the form $\bigcup_{\alpha<\mathrm{cof}(\alpha)} A_\alpha$, where the A_α form an increasing sequence of subsets of M, each of cardinality strictly less than λ, indexed by $\mathrm{cof}(\lambda)$.

An unstable type has as many unstable sons as we wish: For example, take the heirs of one of its nondefinable sons. Consequently, it has an unstable son that is not one of its coheirs. Also, if T is unstable, we can then construct arbitrarily long ordinal sequences of types $p_0 \subset p_1 \subset \cdots \subset p_\alpha \subset p_{\alpha+1} \subset \cdots$ such that $p_{\alpha+1}$ is never a coheir of p_α. (This is false for heirs: If T is the theory of discrete orders without endpoints, then every nonrealized son of a nonrealized type is an heir.)

There is therefore an increasing sequence of models $N_0 \prec N_1 \prec \cdots \prec N_\alpha \prec \cdots$, $\alpha < \mathrm{cof}(\lambda)$, with limit N, with a type $p \in S_1(N)$, whose restriction to N_α we call p_α, such that $p_{\alpha+1}$ is not a coheir of p_α for any α. There is a formula $f_\alpha(x, \vec{a}_\alpha)$, with parameters in $N_{\alpha+1}$, that is satisfied by p and that is not satisfied by any element of N_α.

We can assume that the language of the theory T is denumerable: If not, we could replace it with its reduct to a denumerable sublanguage, keeping a formula with the dichotomy property in order to keep instability. In fact, a model that is saturated for T is a fortiori saturated for its reduct.

If T, then, is denumerable, we can replace every N_α with one of its elementary submodels containing all the $\vec{a}_\beta, \beta < \alpha$, which, by the Löwenheim–Skolem theorem, can be taken to be of cardinality $\max(|\alpha|, \omega)$. Under these conditions, the cardinality of N is $\mathrm{cof}(\lambda)$, and is strictly less than λ. We can likewise assume that if α is a limit, then N_α is the limit of the $N_\beta, \beta < \alpha$.

Since M is λ-saturated and $|N_0| < \lambda$, it realizes the type of N_0 over $\emptyset$, and we can consider that it contains N_0. The type of N_1 over N_0 has an heir over $N_0 \cup A_0$, which we can realize in M. We have now placed N_1 there. Continue: At the αth stage, N_α has already been placed in M, and we realize, in M, an heir over $N_\alpha \cup A_\alpha$ of the type of $N_{\alpha+1}$ over N_α. That is always possible, since $|N_\alpha \cup A_\alpha| < \lambda$ and the type in question has fewer than λ variables.

When we are done, we have located a copy of the sequence N_α in M in such a way that for every α the type of $N_{\alpha+1}$ over $N_\alpha \cup A_\alpha$ is the heir of its restriction to N_α; in other words, the type of A_α over $N_{\alpha+1}$ is a coheir of its restriction to N_α. Consequently, an element of A_α has a type over $N_{\alpha+1}$ that coinherits its restriction to N_α and cannot realize p. Thus p is omitted in M, which is a contradiction. □

Theorem 14.4. *If T is stable, λ is singular, and $\mathrm{cof}(\lambda) < \kappa(T)$, then T does not have a saturated model of cardinality λ.*

Proof. The proof is almost the same as for the previous theorem. Let M be our saturated model of cardinality λ.

By hypothesis, there is a decreasing sequence, indexed by $\mathrm{cof}(\lambda)$, in the fundamental order. In other words, we can find a sequence $N_0 \prec \cdots \prec N_\alpha \prec \cdots$, $\alpha < \mathrm{cof}(\lambda)$ of models, and a type p over $N = \bigcup N_\alpha$ whose restriction to N_α is p_α, such that for every α, $p_{\alpha+1}$ is not the heir (which is the same as the coheir) of p_α.

We can assume that $|T| = \mathrm{cof}(\lambda)$, by replacing T with one of its reducts of that cardinality, keeping the formulas necessary for the existence of the sequence p_α. In that case, we can assume that $|N| = \mathrm{cof}(\lambda)$, and conclude as in Theorem 14.3. □

If we accept the generalized continuum hypothesis, then $\lambda^{<\kappa} = \lambda$ if and only if $\kappa \leq \mathrm{cof}(\lambda)$. We can therefore see, by combining Theorems 14.2, 14.4, and 13.11, that if T is stable and $\lambda \geq \lambda_0(T)$, then T has a saturated model of cardinality λ iff T is stable in λ. With a little work, we can dispense with the continuum hypothesis:

Theorem 14.5. *If T is stable and if $\lambda \geq \lambda_0(T)$, then T is stable in λ if and only if T has a saturated model of cardinality λ.*

Proof. One direction is given by Theorem 14.2. Conversely, suppose that T has a saturated model of cardinality $\lambda \geq \lambda_0(T)$. We must show that T is stable in λ. This is clear if T is superstable or if $\lambda = \lambda_0(T)$. Suppose, therefore, that $\lambda > \lambda_0(T)$ and that $\kappa(T) > \omega$. By Theorem 14.4, $\mathrm{cof}(\lambda) \geq \kappa(T)$.

Note, therefore, that just by the fact that M is λ-saturated and $\lambda > \lambda_0(T) \geq \kappa(T)$, every type over M is the mean type of an indiscernible ω-sequence of elements of M. (In fact, we shall see in Chapter 16 that the hypothesis $\lambda \geq \kappa(T)$ is sufficient for that.) Indeed, as in the proof of Theorem 13.11, every function s from α to M, where $\alpha < \kappa(T)$, can be associated with an elementary submodel M_s of M of cardinality $\max(|\alpha|, \lambda_0(T))$, and in such a way that M_t is an extension of M_s if t is an extension of s. Under these conditions, since there is no descending ordinal sequence indexed by $\kappa(T)$ in the fundamental order of T, every type p of $S_1(M)$ must be the heir of its restriction p_s to M_s for some s. Since $|M_s| < \lambda$, M contains a copy of the Morley ω-sequence of p_s, whose mean type is p.

We can therefore see that there are no more than λ^ω types over M. Since every set of parameters of cardinality at most λ can be embedded in M, if we assume that T is unstable in λ, we must have $\lambda^\omega > \lambda$.

Since ω is not cofinal in λ, a function from ω to λ has bounded image: It is a function from ω to some $\alpha < \lambda$. So $\lambda^\omega = \sum_{\alpha<\lambda} \alpha^\omega$. We therefore must have $\mu^\omega > \lambda$ for some $\mu < \lambda$; otherwise, $\lambda^\omega \leq \lambda \times \lambda = \lambda$. Since $\mu^n = \mu$ for $n \in \omega$, and there is a decreasing ω-sequence in the fundamental order, we can see that there is a set A of parameters of cardinality μ such that $S_1(A)$ contains at least μ^ω types (see the proof of Lemma 13.9).

However, A has a type over $\emptyset$ that is realizable in M, so we can consider it to be a subset of M, which can realize only λ types over A. Then M is not saturated, a contradiction. $\square$

We shall see that we can improve this theorem slightly by specifying what happens between $\lambda_0(T)$ and $\kappa(T)$ (Theorem 16.10). Recall that Theorem 14.4 states that if T has a saturated model of cardinality $\lambda \leq \kappa(T)$, this cardinal is regular.

14.3 Resplendent Models

We begin by highlighting a result that helped us in the proof of Theorem 14.5. This lemma is provisional, since we shall see in Section 16.3 that the result remains true just under the hypothesis of $\kappa(T)$-saturation if T is not superstable.

Lemma 14.6. *If T is stable and $\lambda > |T|$, where $\lambda > \lambda_0(T)$, then every type p over a λ-saturated model M of T is the mean type of an infinite indiscernible sequence of elements of M.*

Proof. As in Theorem 14.5, we construct models M_s. λ is large enough that every M_s has cardinality strictly less than λ (by the Löwenheim theorem if $\lambda > |T|$, by Lemma 13.10 if $\lambda > \lambda_0(T)$). The type p is the heir of its restriction p_s to some M_s, and M contains a copy of the Morley sequence of p_s. $\square$

The following theorem shows that the resplendent models of a stable theory have a tendency to be firmly saturated. We shall improve this theorem in Section 16.3, by replacing the hypothesis of $|T|^+$- or $\lambda_0(T)^+$-saturation with that of $\kappa(T)$-saturation.

Theorem 14.7. *If T is stable, then a model of T that is ω_1-resplendent and $|T|^+$- or $\lambda_0(T)^+$-saturated is saturated.*

Proof. Let M be such a model of cardinality λ. Let A be a subset of M of cardinality $\mu < \lambda$ and let p be a type over A. We need to show that it is realized in A. So let q be a son of p over M. As M is sufficiently saturated, q is the mean type over M of an indiscernible sequence $a_0, a_1, \ldots, a_n, \ldots$ of elements of M.

I claim, therefore, that since M is ω_1-resplendent, this indiscernible set can be extended in M to an indiscernible set of cardinality λ. For we can add a unary relation symbol $R(x)$ and a unary function symbol $f(x)$ to the language. Consider the axioms saying that f is a bijection between the model and those of its elements that satisfy R, along with the following axioms, saying that R is an indiscernible set extending the sequence

$a_0, a_1, \ldots, a_n, \ldots$:

$$(\forall x_1) \cdots (\forall x_n) \qquad (\bigwedge R(x_i) \wedge \bigwedge x_i \neq x_j \rightarrow$$
$$(f(a_1, \ldots, a_m, x_1, \ldots, x_n) \leftrightarrow f(a_1, \ldots, a_m, a_{m+1}, \ldots, a_{m+n})))$$

This set of sentences uses only ω parameters in the model M, and is indeed consistent. Consider a saturated elementary extension of M. By the ω_1-resplendency of M, it is realizable in M. All but at most $\mu \times \kappa(T) < \lambda$ elements of this indiscernible set of cardinality λ realize p. □

Lemma 14.8. *Let T be a complete theory, λ an infinite cardinal, and $M_0 \prec M_1 \prec \cdots \prec M_\alpha \prec \cdots$, $\alpha < \lambda$, an increasing sequence, indexed by λ, of models of T that are λ-saturated of cardinality λ (and therefore all isomorphic). Then their limit M is also saturated of cardinality λ.*

Proof. [Incomplete proof] Let $A \subset M, |A| = \mu < \lambda$, and let p be a type over A: We must show that it is realized in M. Note that this is obvious if λ is regular, since then A is contained in one of the M_α. So let us assume that λ is singular; since T has a saturated model of cardinality λ, this requires T to be stable by Theorem 14.3, and requires $\lambda > \mathrm{cof}(\lambda) \geq \kappa(T)$ by Theorem 14.4.

Let q be any son of p over M. For some α, q must be the heir of its restriction q_α to M_α; otherwise, we would have decreasing sequences of length $\kappa(T)$ in the fundamental order of T.

So let us assume that $\lambda > |T|$ or $\lambda > \lambda_0(T)$. Under these conditions, q_α is the mean type of an indiscernible set of cardinality λ consisting of elements of M_α. All but at most $\mu \times \kappa(T)$ of these elements realize p. □

In order to have the complete proof of Lemma 14.8 we will need to wait for the definitive version of Lemma 14.6.

14.4 Sufficiently Saturated Extensions of a Given Model

Theorem 14.9. *Let T be a theory of modules and $\lambda > |T|$. Then every model of T that is $|T|^+$-saturated and is an elementary extension of a λ-saturated model is itself λ-saturated. If in addition T is superstable, then every extension of a λ-saturated model of T is itself λ-saturated.*

Proof. Let M_0 be a λ-saturated model of T, let N be an elementary extension of M_0, which is $|T|^+$-saturated in the nonsuperstable case, let M be an elementary submodel of N of cardinality strictly less than λ, and let p be a type over M. We need to show that p is realized in N. We know (see Sections 6.5 and 13.3) that p is associated with a filter F of primitive groups and an element a_G of M for every $G \in F$, and that p is axiomatized by the following formulas: $x \sim a_G($ (mod G)) for $G \in F$, $x \not\sim b($ (mod H)) for $b \in M, H \notin F$.

If N is $|T|^+$-saturated, then there exists a in N that satisfies all the congruences $a \sim a_G($ (mod G)). In the superstable case, this hypothesis of $|T|^+$-saturation is not necessary, since there is G_0 in F such that for every G in F, $G \cap G_0$ has finite index in G_0: All classes of $G_0/G \cap G_0$ have representatives in the model M_0, which is λ-saturated, so each consistent combination of classes, modulo the G in F, for an element of G_0, is realized in M_0. There is therefore an element b in M_0 such that $a = a_{G_0} + b$ satisfies all the $a \sim a_G(\text{mod } G)$.

I now claim that the model M_0 contains λ points b_i that are in every G in F and that are pairwise incongruent modulo every primitive subgroup H that is not in F. Indeed, that corresponds to something consistent, since, by B.H. Neumann's lemma (6.25), the intersection of finitely many of the G_i cannot be covered by finitely many classes modulo the H_j.

As a consequence, given a formula $x \not\sim b$ (mod H)), all but at most one of the $a + b_i$ satisfy it. Since there are at least λ formulas to satisfy, there are some $a + b_i$ left over that realize p. □

We can therefore see that a superstable module is λ-saturated, since it has a λ-saturated elementary submodel. This property has been called *non-multidimensionality* by Shelah; we shall, more simply, call it *dimensionality*, since it actually means that every sufficiently ($|T|^+$ is generally enough) saturated model of T is determined by specifying a certain fixed number of cardinals, called its dimensions. We shall see all that in Chapter 20.

This property implies superstability:

Theorem 14.10. *If T is not superstable, then every model of T has an elementary extension that is not ω_1-saturated. More precisely, if T is unstable, or if T is stable and $\lambda < \kappa(T)$, then every model of T has a* $\operatorname{cof}(\lambda)$*-resplendent extension that is not λ^+-saturated.*

Proof. We have a sequence of models $M_0 \prec M_1 \prec \cdots \prec M_\alpha \prec \cdots, \alpha < \lambda$ of T available to us, and a type p over $\bigcup M_\alpha$ whose restriction to $M_{\alpha+1}$, does not coinherit its restriction to M_α for any α.

Let $A \subset \bigcup M_\alpha, |A| = \lambda$ be a set containing, for every α, a tuple $\vec{a}_\alpha$ of $M_{\alpha+1}$ that prevents its coheritage.

Let N_0 be any model of T. We realize the type of M_0 over $\emptyset$ in an elementary extension N_1 of N_0. Having thus interpreted M_0 in N_1, we realize an heir over N_1 of the type of M_1 over M_0 in an extension N_2 of N_1, and iterate λ times. We thus get a model $N = N_\lambda$ containing a copy of the sequence of the M_α such that the type of $M_{\alpha+1}$ over $N_{\alpha+1}$ inherits its restriction to M_α, which is included in $N_{\alpha+1}$, such that the type of $N_{\alpha+1}$ over $M_{\alpha+1}$ coinherits its restriction to M_α. The model N is not λ^+-saturated, since it omits the restriction of p to A.

As we require nothing of the N_α except that they realize something, their construction can be combined with a step-by-step saturation process, as in the proof of Theorem 9.12, which guarantees that their limit N is $\mathrm{cof}(\lambda)$-resplendent. □

In particular, we see that in the case of an unstable theory, λ-resplendency can never imply saturation. In the extreme case of instability, the same type of argument allows us easily to build many sufficiently saturated models of the given cardinality. Recall that if $\lambda \geq |T|$, then T has, up to isomorphism, at most 2^λ models of cardinality λ.

Theorem 14.11. *Consider a theory T with the independence property and a cardinal $\lambda \geq |T|$ of the form $\lambda = 2^\kappa$. Given a model M of T of cardinality at most λ and a regular cardinal $\mu \leq \kappa$, T has 2^λ pairwise nonisomorphic models of cardinality λ that are moreover μ-resplendent and not μ^+-saturated and that are all elementary extensions of M.*

Proof. There is a formula $f(x, \vec{y})$ that has the independence property.

If U is an ultrafilter of subsets of κ, we shall say that the sequence $a_0, \ldots, a_i, \ldots, a_\kappa, a_{\kappa+1}, \ldots, a_{\kappa+n}, \ldots$ of length $\kappa + \omega$, taken from a model N of T, codes U in N if the following two conditions are satisfied:

- For every subset w of κ there exists $\vec{b}_w$ in N such that for $i < \kappa$, $N \models f(a_i, \vec{b}_w)$ if and only if $i \in w$.
- For every formula $g(x, \vec{c})$ with parameters $\vec{c}$ in N there is an integer m such that for all $n > m$, $g(a_{\kappa+n}, \vec{c})$ is true if and only if $\{i : i < \kappa \text{ and } N \models g(a_i, \vec{c})\}$ is in the ultrafilter U.

Note that by virtue of the existence of the $\vec{b}_w$, a sequence can code at most one ultrafilter.

For every ultrafilter U, we build a model N_U in which it is coded as follows: We consider a model N_0 of cardinality $\lambda = 2^\kappa$ that contains M, and $a_i, i < \kappa$, and $\vec{b}_w, w \subset \kappa$, such that $N_0 \models f(a_i, \vec{b}_w)$ iff $i \in w$. We consider the type p_U over N_0 corresponding to the ultrafilter U, that is to say the set of formulas with parameters in N_0 that are true for almost all the a_i modulo U. Let a_κ realize this type in a model $N_1 \succ N_0$ of cardinality λ; then let $a_{\kappa+1}$ realize the limit type of U over N_1 in an extension N_2 of N_1 of cardinality λ; repeat ω times.

As we require nothing of the N_n except that they realize something and not have too large a cardinality, we can combine their construction with a saturation process at each stage that will guarantee that their limit N_U is ω-resplendent. This limit N_U cannot be ω_1-saturated; indeed, since $a_\kappa, a_{\kappa+1}, \ldots, a_{\kappa+n}, \ldots$ is a Morley sequence of coheirs, its type is finitely satisfiable with the $a_i, i < \kappa$. Consequently, the set of formulas $f(a_{\kappa+2n}, \vec{y}), \neg f(a_{\kappa+2n+1}, \vec{y})$ forms a consistent (incomplete) type that is omitted in N_U.

It remains to be seen that among our models N_U there are 2^λ that are not isomorphic. A $(\kappa+\omega)$-sequence can code only one ultrafilter. Since $\lambda^{\kappa+\omega} = \lambda^\kappa = (2^\kappa)^\kappa = \lambda$, a model of cardinality $\lambda = 2^\kappa$ can code at most λ ultrafilters. The conclusion follows from the fact that there are $2^\lambda = 2^{(2^\kappa)}$ ultrafilters of subsets of κ.

We have treated the case $\mu = \omega$. For the general case, code ultrafilters by sequences of length $\kappa + \mu$. □

The restriction $\lambda = 2^\kappa$ is a great weakness of this theorem; however, what strengthens it is its versatility and its simplicity: The models constructed are not saturated simply because they realize something! The only argument that we need to limit their size to 2^κ, which guarantees that there are lots of them, is the Löwenheim–Skolem theorem.

You need to know that Shelah has shown that every non-superstable theory T has 2^λ models of every cardinality $\lambda > |T|$; moreover, if T is unstable and μ reasonably small relative to λ, it has 2^λ μ-saturated models of cardinality λ.

His method, especially in the stable nonsuperstable case, is obviously much more complicated. He constructs a class of prime models around a motif, a "skeleton" (for example, in the unstable case, the skeleton is an indiscernible but not totally indiscernible chain) such that the properties of the skeleton can be recovered sufficiently from the model that cloaks it, with the models corresponding to very different skeletons not being isomorphic. We shall not attempt this in the present work, although it is such an important theorem; the problem is that I cannot give it any sensible simplification, nor present it with any originality, so the reader, after finishing Chapter 18, can embark on Shelah's works directly.

14.5 Historic and Bibliographic Notes

Theorem 14.2, on the existence of saturated models for a stable theory, is due to Victor Harnik, [Har75]; the nonexistence theorems are due to Shelah, [She78], p. 423, as is the curious Lemma 14.8, on chains of saturated models, [She78], p.8, conclusions 1–13, in which it is used to show the splendidness of a saturated model.

It was Steven Buechler who noted that the resplendent models of a stable theory tend to be saturated [Bue84]. The material in Section 14.4 is included in [Poi83a].

For the number of models of an unstable theory: After reading Chapter 18, consult [She71c], which is still understandable enough; for the nonsuperstable case, which requires more courage, consult [She78], Chapter VIII.

15
Forking

> Deinde ibidem homo acutus, cum illud occurreret, si omnia deorsum e regione ferrentur et, ut dixi, ad lineam, nunquam fore ut atomus altera alteram posset attingere, itaque attulit rem commenticiam: declinare dixit atomum per paulum quo nihil posset fieri minus; ita effici ...
>
> M.T.C.

In the preceding chapters we defined heirs, special sons, coheirs, Morley sequences, etc. for a type p defined over a *model* of the theory T. It is essential for the existence of these notions that we start with a set of parameters M that is a model of T: We continually used the fact that if A is any set of parameters containing M, then every finite situation exhibited by the elements of A can be copied inside M.

For a type p defined over a model M of T it is easy enough on the whole to distinguish sons that have interesting properties, whose usefulness we saw in Chapter 14. However, the reader will have noticed the acrobatic character of some of the proofs. This is because, with the tools available to us, we needed systematically to cover our sets of parameters with models of T (the models M_s of Theorem 13.10 and Lemma 14.6, the M_α in Lemma 14.1, etc.) in a rather artificial way. In some cases it was not only the elegance or the relevance of our method that came into play, but simply its effectiveness: We were unable to prove what we wanted in 14.3, when we were dealing with cardinals too small to construct models of those cardinalities.

The time has come to reveal the real power of stability: the notion of *nonforking son* of a type p over an arbitrary set A of parameters, which generalizes that of heir. If A is a model of T, the only nonforking son of p is its heir (by hypothesis the theory T is stable). If Chapter 14 was placed where it was, prematurely in some sense, it was to convince the reader of the importance of the hypotheses of stability or of instability, which appeared in a fundamental way in the constructions of models, before making you swallow three more chapters of theoretic developments on types.

That could be done without any major inconvenience, since the methods for constructing saturated models are indeed the ones we have described. The reader who has the definitive version of Lemma 14.6 will immediately have the definitive versions of its consequences. In contrast, for subtler constructions, such as those of prime models, we shall need the whole arsenal of forking.

15.1 The Theorem of the Bound

Lemma 15.1. *Let M be a model of T and let F be a nonempty closed set in $S_1(M)$. Then there is at least one type in F that is maximal, in the fundamental order, among the types of F.*

Proof. Let C be a maximal chain in the fundamental order consisting of types p_i in F. Consider the set of formulas, in the language $L(M)$ augmented by a unary relation symbol $\mathcal{M}(y)$ and a constant symbol x, affirming $T(M)$, that $\mathcal{M}$ is an elementary submodel of the universe containing every element of M, that x satisfies every formula of F, and that its type over $\mathcal{M}$ omits every formula of L omitted by one of the p_i. We can see, as in Theorem 13.2, that this set of sentences is consistent.

A realization of this set provides a type q over an elementary extension N of M that satisfies all the formulas defining F and that is greater than all the p_i in the fundamental order. If p is the restriction of q to M, then $p \geq q$, and p is a maximal type in F. □

From the proof of the lemma it is clear that the types of the closed set F form an inductive set for the fundamental order.

If $p \in S_1(A)$ and M is a model containing A (i.e., a model of $T(A)$), then the sons of p over M form a closed set in $S_1(M)$. The preceding lemma states that this closed set contains maximal types. The class of such a maximal son of p over M in the fundamental order will be called a *bound of p over M*.

Lemma 15.2. *If p is in $S_1(A)$, and if M and N are two models containing A, then the bounds of p over M are the same as the bounds of p over N.*

Proof. Let β be a bound of p over M. Then p has a son q with class β that is maximal among the sons of p over M. Let P be a common elementary extension (in the sense of $L(A)$!) of M and N, and let q_1 be an heir of q

over P, with restriction q_2 to N. Since the class of q_1 is that of q, the class of q_2 is greater than or equal to β.

In fact, they are identical, and q_2 is maximal among the sons of p over N. Otherwise, there would exist a type q_3 over N with class strictly greater than β. The same would be true of the class of an heir q_4 of q_3 over P, and of the class of its restriction q_5 to M; that contradicts the fact that β is a bound of p over M. □

As there is no point specifying the model, we shall henceforth speak of *bounds* of p.

If p is in $S_1(A)$, $A \subset B$, we shall define a *maximal son* of p over B to be a son q of p over B that is the restriction of a son of p over a model M containing B and whose class in the fundamental order is a bound of p. By the preceding lemma, if B is a model of T, that means that the class of q is a bound of p. It is also easy enough to see that that means, for any model M containing B, q has a son over M whose class is a bound of p.

Every type p is obviously a maximal son of itself, and maximal sons exist by the following lemma:

Lemma 15.3. *Let $A \subset B \subset C$, let p be in $S_1(A)$, and let q be a maximal son of p over B. Then q has a son r over C that is a maximal son of p.*

Proof. Let M be a model containing B and let q_1 be a son of q over M whose class is a bound of p. Let N be a model containing both M and C. Take r to be the restriction to C of an heir of q over N. □

Now we can dualize the notion of maximal son, following the gymnastics that have become familiar to the reader, now a master in the manipulation of heirs and coheirs. We shall say that the type of $\vec{a}$ over $A \cup \{\vec{b}\}$ is a *comaximal* extension of the type of $\vec{a}$ over A if the type of $\vec{b}$ over $A \cup \{\vec{a}\}$ is a maximal extension of the type of $\vec{b}$ over A. In order to understand the definition, it is important to comprehend this: Given A, all that matters about the set of parameters $A \cup \{\vec{b}\}$ containing A is the type of $\vec{b}$ over A. Also, given a type p over A, realized by $\vec{a}$, choosing a son of p over $A \cup \{\vec{b}\}$ is the same as completing the theory including the type of $\vec{a}$ over A and the type of $\vec{b}$ over A in such a way as to have the type of $\vec{a}\vec{b}$ over A. That amounts to choosing a son over $A \cup \{\vec{a}\}$ of the type of $\vec{b}$ over A, by the principle that placing $\vec{a}$ in relation to $\vec{b}$ over A is the same as placing $\vec{b}$ in relation to $\vec{a}$.

Where do we get these definitions by symmetry? Naturally, to define the notion of a comaximal son of 1-types, we need to consider maximal sons of sets of parameters, which may have infinitely many variables; this does not present any difficulty.

From the mere fact that maximal sons exist, comaximal sons do too: to place $\vec{a}$ and $\vec{b}$ so that the type of $\vec{a}$ over $A \cup \{\vec{b}\}$ is a comaximal extension of its restriction to A is to place them so that the type of $\vec{b}$ over $A \cup \{\vec{a}\}$ is a maximal extension of its restriction to A.

Lemma 15.4. *Suppose that T is stable. Let p be in $S_1(A)$, let M be a model of T containing A, and let q be a comaximal son of p over M. Then q is greater than every son of p over every model of T, in the fundamental orders of both T and $T(A)$.*

Proof. Let r be a son of p, over a model N containing A, realized by a. Since the type of a over A is p, there is a model M', A-isomorphic to M, such that the type of a over M' is the type q' corresponding to q. This condition determines only the type of M' over $A \cup \{a\}$, which is the maximal son of its restriction to A; by Lemma 15.3, this type extends to a type over $N \cup \{a\}$ with the same property: We realize this situation.

Since the type of M' over $N \cup \{a\}$ is a maximal son of its restriction to A, it is also a maximal son of its restriction to N. Consequently, by Theorem 13.6, the type of M' over $N \cup \{a\}$ is the heir of the type of M' over N, and by symmetry the type of a over $N \cup M'$ is the heir of the type of a over N. That means that if P is a model containing both N and M', then the heir of r over P is a son of q', so $r \leq q'$ in the fundamental orders of both T and $T(A)$. □

Theorem 15.5 (Duality of Forking). *If T is stable and p belongs to $S_1(A)$, then the notions of maximal son of p and of comaximal son of p coincide, and have the same sense in T as in $T(A)$.*

Proof. The preceding lemma states that a comaximal son is maximal; however, by duality, that implies that a maximal son is comaximal. Indeed, if the type of $\vec{a}$ over $A \cup \{\vec{b}\}$ is a maximal son of the type of $\vec{a}$ over A, that means that the type of $\vec{b}$ over $A \cup \{\vec{a}\}$ is a comaximal, and therefore maximal, son of its restriction to A, and that the type of $\vec{a}$ over $A \cup \{\vec{b}\}$ is the comaximal son of its restriction to A. This duality holds for T as well as for $T(A)$. It is obvious that a maximal son in the sense of $T(A)$ is also maximal in the sense of T; Lemma 15.4 states also that a comaximal son in the sense of T is maximal in the sense of $T(A)$. □

Corollary 15.6 (Theorem of the Bound). *If T is stable, then every type has only one bound.*

Proof. If p is in $S_1(A)$, then the bound of p is the class of a maximal son of p over a model containing A; they are all equivalent by Lemma 15.4. □

We can see that in the theory of dense orders without endpoints, the unique type over $\emptyset$ has two bounds, the classes of $+\infty$ and $-\infty$. As we have already declared, the fundamental order of a stable theory has an additional constraint: Given a type p over $\emptyset$, there is a maximal class in the fundamental order whose restriction to $\emptyset$ is p, namely that of the bound of

p. The types over models of T whose restrictions to $\emptyset$ are p are therefore contained between the bound of p and the realization of p.

15.2 Forking and Nonforking Sons

We now assume that T is stable. In that case, the notion of "maximal son" that appeared in the last section is so important that we are going to give it a more practical name: We shall call these sons *nonforking sons.*

So if p is in $S_1(A)$, $A \subset B$, and q is a son of p over B, then the bound of q can only be less than or equal to that of p. If the bound of q is strictly less than that of p, then q is a *forking son* of p; if not, p and q have the same bound, and q is a *nonforking son* of p. We also say that q *forks over* A if it is a forking son of its restriction to A, and that q *does not fork over* A if it is a nonforking son of its restriction to A.

As the noteworthy sons of p are generally its nonforking sons, this manner of expression often yields phrases congested with double negations. Shelah's fingerprints are in evidence. We must be careful not to get lost among the various uses of the word *over*: If $A \subset B$ and $p \in S_1(A)$, then p is a type *over* A; it has a son q *over* B, which may or may not fork *over* A. Do not forget that the terms forking and nonforking are used only in the stable case.

Let me gather together the fundamental properties of this forking, which we have already proved, and which it is thoroughly essential to remember:

- (transitivity) *If q is a son of p and r a son of q, then r is a nonforking son of p if and only if r is a nonforking son of q and q is a nonforking son of p* (follow the bounds).
- (symmetry) *The type of $\vec{a}$ over $A \cup \{\vec{b}\}$ does not fork over A if and only if the type of $\vec{b}$ over $A \cup \{\vec{a}\}$ does not fork over A* (Theorem 15.5).
- *If p is a type over a model M of T, then its only nonforking son is its heir* (Theorem 13.5)
- *If we consider types over sets of parameters containing A, then forking has the same sense in T as in $T(A)$* (Theorem 15.5).

Another important property of forking is its local character: If a type forks, it is because of a formula. More precisely:

Theorem 15.7. *If p is in $S_1(A)$, $A \subset B$, and q is a forking son of p over B, then for some $\vec{b}$ in B and some formula $f(x, \vec{y})$ omitted by the bound of p, $q \models f(x, \vec{b})$.*

Proof. Let N be a model containing B. To say that q forks is to say that the set of formulas containing q, $T(N)$, and all formulas $\neg g(x, \vec{a})$, where $\vec{a}$ is in N and g is omitted by the bound of p, is inconsistent. It therefore has an inconsistent finite fragment, which involves only finitely many elements $\vec{a}$ of N, a finite fragment $h(\vec{z}, \vec{b})$ of their type over B, and finitely many formulas $g_1, \ldots, g_k$ omitted by the bound of p, so q satisfies

$$f(x, \vec{b}) = (\exists \vec{z}) h(\vec{z}, \vec{b}) \wedge (\forall \vec{z})(h(\vec{z}, \vec{b}) \to g_1(x, \vec{z}) \vee \cdots \vee g_k(x, \vec{z})).$$

The formula $f(x, \vec{y})$ is clearly omitted by the bound of p. □

Corollary 15.8. *For every p in $S_1(A)$ there is a subset B of A, of strictly smaller cardinality than $\kappa(T)$, such that p does not fork over B.*

Proof. If p forks over $B_0 = \emptyset$, that means that it satisfies a formula $f_1(x, \vec{a}_1)$, where f_1 is omitted by the bound of $p|B_0$. If p forks over B_1, it is because of a formula $f_2(x, \vec{a}_2)$, where f_2 is omitted by the bound of $p|B_1$, etc. The bounds decrease strictly every time, so we will be forced to stop before having repeated the operation $\kappa(T)$ times. □

Another very important result to remember is the following one, which reduces the notion of forking for 2-types to that for 1-types.

Theorem 15.9. *If $A \subset B$, then the type of $\vec{a}^\frown\vec{b}$ over B does not fork over A if and only if the type of $\vec{a}$ over B does not fork over A and the type of $\vec{b}$ over $B \cup \{\vec{a}\}$ does not fork over $A \cup \{\vec{a}\}$*

Proof. If we dualize over A, the nonforking over A of the type of $\vec{a}^\frown\vec{b}$ over B is equivalent to the nonforking over A of the type of B over $A \cup \{\vec{a}, \vec{b}\}$. By transitivity, that is equivalent to the nonforking over $A \cup \{\vec{a}\}$ of the type of B over $A \cup \{\vec{a}, \vec{b}\}$ and the nonforking over A of the type of B over $A \cup \{\vec{a}\}$; dualizing again, over $A \cup \{\vec{a}\}$ for the first clause and over A for the second, we get the desired result. □

With this result we can better understand why $\kappa_2(T) = \kappa(T)$ (Lemma 13.12): If we have an ordinal sequence of forking 2-types, the first variable forks fewer than $\kappa(T)$ times, and once it has stopped forking, the second variable forks fewer than $\kappa(T)$ times over the first.

Forking allows us to define a notion of independence: $\vec{a}$ and $\vec{b}$ are called *independent over A* if the type of $\vec{a}$ over $A \cup \{\vec{b}\}$ does not fork over A; this relation is symmetric in $\vec{a}$ and $\vec{b}$. We shall say that the family of the $\vec{a}_i$ is *independent* (or *free*) over A if for all i, the type of $\vec{a}_i$ over $A \cup \{\ldots \vec{a}_j \ldots\}_{j \neq i}$ does not fork over A. Since forking happens formula by formula, we can see that an infinite family is independent whenever all of its finite subfamilies are independent.

As an example, if we consider the theory T of an infinite vector space over a field K, i.e., the theory of existentially closed K-modules, then nonzero elements form a free family over $\emptyset$ if and only if they are linearly independent. For a theory of algebraically closed fields, elements that are nonalgebraic over the prime field form a free family if and only if they are algebraically independent over the prime field. So we can understand that this notion of independence, which generalizes some very classic constructions in algebra, will have great importance for the construction and classification of the models of a stable theory.

Like Monsieur Jourdain, we have been handling independent sets for a long time without knowing it: In fact, Morley sequences are independent, as the following result shows. (It would also be a consequence of their total indiscernibility.)

Theorem 15.10. *If the $\vec{a}_i$ are indexed by a totally ordered set I, then for them to form an independent set over A, it is sufficient that for every i the type of $\vec{a}_i$ over $A_i = A \cup \{\ldots \vec{a}_j \ldots\}_{j<i}$ not fork over A.*

Proof. It is enough to see this for a finite set $\vec{a}_0, \ldots, \vec{a}_m, \ldots, \vec{a}_n$. We can show by induction on k that the type of $\vec{a}_m$ over $B_k = A \cup \{\vec{a}_0, \ldots, \vec{a}_{m-1}, \vec{a}_{m+1}, \ldots, \vec{a}_{m+k}\}$ does not fork over A. That is true by hypothesis for $k = 0$; let us prove the step from k to $k+1$. The type of a_{m+k+1} over A_{m+k+1} does not fork over A, so it also does not fork over B_k. By duality over B_k, the type of a_m over $B_{k+1} = B_k \cup \{a_{m+k+1}\}$ does not fork over B_k; by the induction hypothesis the type of a_m over B_k does not fork over A, so by transitivity the type of a_m over B_{k+1} does not fork over A.
□

These arguments by "symmetry and transitivity" are currency in stability, and will very quickly become familiar to the reader of this work.

15.3 Multiplicity

We are still assuming that T is stable. If p is in $S_1(A)$, we define the *multiplicity* of p to be the number of its nonforking sons over a model M containing A; this multiplicity does not depend on the choice of model M, since if N is another model containing A, we can establish a bijection between the nonforking sons of p over M and the nonforking sons of p over N as follows: Consider a common extension P of M and N, and associate a nonforking son q of p over M with the restriction to N of the heir of q over P.

The multiplicity of a nonforking son of p is obviously less than or equal to that of its father. So every multiplicity is less than $2^{|T|}$; indeed, by Corollary 15.8, every type p is a nonforking son of a type p_1 defined over a set A_1 of cardinality less than or equal to $|T|$. This set A_1 embeds into a model M_1 of T of cardinality $|T|$, which can yield only $2^{|T|}$ types.

If the multiplicity of p is 1, that is to say it never has more than one nonforking son, we call p *stationary*. For example, every type over a model is stationary.

We shall say that two types over M are *$f(x,\vec{y})$-distinct* if for some $\vec{a}$ in M, one yields $f(x,\vec{a})$ while the other gives $\neg f(x,\vec{a})$. The *$f(x,\vec{y})$-type* of q is the set of formulas $f(x,\vec{a}), \neg f(x,\vec{a}')$ satisfied by q.

Theorem 15.11. *If p is in $S_1(A)$ and if M is a model containing A, then for every formula $f(x,\vec{y})$, the $f(x,\vec{y})$-types of nonforking sons of p over M are finite in number.*

Proof. Suppose otherwise. Let us add a unary symbol $M(y)$ and constants $a_i, \vec{b}_{i,j}, i,j \in \lambda$, to the language. It would be consistent that M is an elementary submodel of the universe, with the $\vec{b}_{i,j}$ in M and the a_i satisfying p, omitting every formula omitted by the bound of p over M, and $f(a_i,\vec{b}_{i,j}) \leftrightarrow \neg f(a_j,\vec{b}_{i,j})$. That would mean that we could get as many nonforking sons of p as we wanted over a large enough M, contradicting the existence of the multiplicity. $\square$

Given any set A of parameters, we can define the notion of *definable type* as we did when A is a model: Every formula $f(x,\vec{y})$ is associated with a formula with parameters in A such that for all $\vec{a}$ in A, $p \models f(x,\vec{a})$ iff $df(\vec{a})$ is satisfied. In contrast to the situation for a model, a type might have many definitions: $d_1 f$ and $d_2 f$ are satisfied by the same tuples in A, but since A is not a model, that does not prove that they are equivalent. Moreover, if $A \subset B$, a definition df of p over A is not certain to yield anything consistent over B. It is easy to find examples by noticing that if A is finite, then every type over A is definable!

In the stable case, however, every type is not only definable, but has a definition allowing us to define the closed set (i.e., incomplete type) of nonforking sons over any model containing A. This definition, which shall be called a *good definition* of p, is obviously unique, since its interpretation over a model is determined.

Theorem 15.12. *If T is stable, then every type p over A has a definition, that is to say a function that sends each formula $f(x,\vec{y})$ to a formula $df(\vec{y})$ with parameters in A, with the following additional property: If M is a model of T containing A, then $M \models df(\vec{a})$ iff $f(x,\vec{a})$ is satisfied by all nonforking extensions of p.*

Proof. By Theorem 15.11, there are only finitely many definitions $d_1 f, \ldots, d_n f$ of $f(x,\vec{y})$-types of non-forking extensions of p. The formula $df(\vec{y}) = d_1 f(\vec{y}) \wedge \cdots \wedge d_n f(\vec{y})$ does have the required property, and since its satisfaction depends only on the type of $\vec{y}$ over A, it is equivalent to a formula with parameters in A (Lemma 12.4). $\square$

Let p be in $S_1(A)$ and let M be a $|T(A)|^+$-saturated strongly $|T(A)|^+$-homogeneous model containing A. Let q_1 and q_2 be two nonforking sons of p over M. Since q_1 and q_2 are equivalent in the fundamental order of $T(A)$, the two structures (M, dq_1) and (M, dq_2) are elementarily equivalent in the language in which the elements of A have names. As they are both interpretable in M, they are also $|T(A)|^+$-saturated and strongly homogeneous.

If now q_1 has a definition $df(\vec{y}) = g(\vec{y}, \vec{a}_1)$ for f, where $\vec{a}_1$ has a certain type over A, then by elementary equivalence and saturation q_2 will also have a definition of the form $g(\vec{y}, \vec{a}_2)$ for f, with $\vec{a}_2$ having the same type over A as $\vec{a}_1$. We can therefore see, by using homogeneity, that q_1 and q_2 are conjugates via an A-automorphism of M: If the model is sufficiently saturated and homogeneous, *all nonforking sons of p have the same orbit under the actions of A-automorphisms of the model.*

Note that the nonforking sons of p form a closed set: Indeed, they are the sons of p that satisfy all the $\neg f(x, \vec{a})$ for every formula $f(x, \vec{y})$ omitted by the bound of p. It is clear that an A-automorphism of M induces a continuous bijection of this closed set onto itself; all points of this closed set, being conjugate, either are all isolated or are all accumulation points. As we are in a compact Stone space, in the first case, that means that the multiplicity of p is finite, and in the second case, by Lemma 13.13, it is at least 2^ω.

We shall keep in mind in particular that for a *denumerable* theory T, the multiplicity of a type is either finite or at least 2^ω.

15.4 Stable Types in an Unstable Theory

We developed the preceding sections on the assumption that the whole theory T is stable; in fact, they remain valid for stable types in an unstable theory. I shall briefly explain why, leaving the details to the reader.

We shall say that p in $S_1(A)$ is *stable* if all its sons over any model M of T containing A are stable, or equivalently are definable. An example of a stable type, albeit not very interesting, is a realized type; for some theory T there might be no others.

Stability can be characterized by the number of sons of p over $B \supset A, |B| = \lambda$, as in Theorems 11.10 and 11.11.

One rather remarkable fact is the following: For the type of $(\ldots, a_i, \ldots)$ over A to be stable, it is sufficient (and naturally necessary) that for every i the type of a_i over A be stable. Indeed, as definability is done formula by formula, it is enough to count the types of n-tuples $(a_1, \ldots, a_n)$; since realizing the type of $(a_1, \ldots, a_n)$ over B is the same as realizing the type of a_1 over B, then the type of a_2 over $B \cup \{a_1\}$, etc., the stability in λ of every a_i implies that of their n-tuple.

Consequently, if M is a model containing A, its *stable subset over A* is defined as the set $M_{\mathrm{Stab}/A}$ of elements of M whose type over A is stable. We just saw that the type of $M_{\mathrm{Stab}/A}$ over A is stable!

Every time we used the principle of duality in the preceding sections, we needed the stability of types of tuples of parameters; all this machinery can be adapted here, because if M is sufficiently saturated, a type p in $S_1(M)$ whose restriction to A is stable depends only on its restriction to $M_{\mathrm{Stab}/A}$. Indeed, we can prove the following infinitary analogue of Theorem 12.30:

Lemma 15.13. *Let p be in $S_1(A)$, let M be a $|T(A)|^+$-saturated model containing A, let B be the set of realizations of p in M, and let q_1 and q_2 be two stable sons of p over M. Then, if q_1 and q_2 have the same restriction to B, they are equal.*

Proof. Let N be an elementary submodel of M, containing A, of cardinality $|T(A)|$, such that q_1 and q_2 inherit their respective restrictions to N. By saturation, we can construct in M the Morley sequence of the restriction of q_1 to N, whose mean type is q_1. That proves that q_1 is definable with parameters in B (see Lemma 12.19); the same is true of q_2.

Also, still by the saturation of M, the definitions of q_1 and q_2 have the same effect on the parameters that realize p. We can therefore show that the Morley sequence of q_1, that of q_2, and the sequence obtained by alternating the definitions of q_1 and q_2 (see Theorem 12.30) have the same type over B; since the last one must be indivisible, we must have $q_1 = q_2$. □

Consequently, in order to prove the theorem of the bound *("a stable type has only one bound")*, from which the rest follows, we proceed as in Theorem 15.5, taking a $|T(A)|^+$-saturated model M, *and we need only consider the stable part $M_{\mathrm{Stab}/A}$ of M*: Since a has been placed relative to $M_{\mathrm{Stab}/A}$, it is automatically placed relative to M!

15.5 Historic and Bibliographic Notes

The story is told (by Wilfrid Hodges) that Shelah engaged in a pantomime in front of Chang in order to ask him what English term was best suited to what he had in mind; the response was to fork.

All the results on forking are properly attributed to Shelah, but the Parisian approach, which is guiding us here, and which is centered on the theorem of the bound (15.6), is much gentler than Shelah's: With the original definition, [She78], p. 84, it was not even clear that p was a nonforking son of itself! The term "multiplicity" is used here in a slightly different sense than in [She78]. The treatment of unstable types in a stable theory, based on separation of parameters, is from [Poi77].

16

Strong Types

... und zwar durch Einführung einer eigenthümlichen Art imaginärer Divisoren, welche ich ideale complexe Zahlen nenne ...

E.E.K.

16.1 The Finite Equivalence Relation Theorem

Lemma 16.1. *In a stable theory T, let p be a type in $S_1(A)$ and let M be a model of T containing A. Then for every formula $f(x, \vec{y})$ there is an equivalence relation $E_f(\vec{y}, \vec{z})$, with finitely many classes, defined by a formula with parameters in A, such that two tuples $\vec{a}$ and $\vec{b}$ in M are congruent modulo E_f if and only if all nonforking sons of p satisfy $f(x, \vec{a}) \leftrightarrow f(x, \vec{b})$.*

Proof. The nonforking sons of p have only finitely many $f(x, \vec{y})$-types, corresponding to definitions $d_1 f, \dots, d_n f$. It is therefore clear that the equivalence relation E_f has at most 2^n classes, and that it is defined by the formula $(d_1 f(\vec{y}) \leftrightarrow d_1 f(\vec{z})) \wedge \cdots \wedge (d_n f(\vec{y}) \leftrightarrow d_n f(\vec{z}))$. As the satisfaction of this formula depends only on the type of $\vec{y}\,\hat{}\,\vec{z}$ over A, it is equivalent to a formula with parameters in A. $\square$

If a and b are congruent modulo all equivalence relations with finitely many classes (which for the sake of abbreviation we shall call "finite equivalence relations") definable with parameters in A, then a and b are said to have the same *strong type* over A. If a and b have the same strong type over

A, they have the same type over A: A formula $f(x, \vec{c})$ with parameters $\vec{c}$ in A can be associated with an equivalence relation with two classes, one consisting of all elements that satisfy $f(x, \vec{c})$, and the other of all elements that satisfy its negation.

Conversely, if A is a model M of T, then two elements with the same strong type over M have the same type over M. Indeed, every class of a finite equivalence relation E must have a representative in M, and the type over M says whether or not x is congruent to a modulo E.

However, in the general case, a type over A corresponds to several strong types. If we wish, we can consider a strong type as an incomplete type over a model M of T containing A, consisting solely of formulas of the form $x \sim a$ modulo E, where a is in M and E is a definable finite equivalence relation on A. The set $SF_1(A)$ is defined as the set of strong types in one variable over A; the sets $SF_n(A)$ of strong types in n variables and $SF_\alpha(A)$ in α variables are defined analogously. We can see that the definition of $SF_1(A)$ does not depend on the model M chosen to enclose A: If N is another model containing A, then M and N have a common elementary extension P that identifies the classes, modulo E, over M, N, P.

If A is contained in B, then by restricting a strong type over B to formulas involving only finite equivalence relations defined over A, we get a strong type over A. Moreover, since the strong type determines the type, if M is a model containing A, we have the canonical surjections $S_1(M) = SF(M) \to SF_1(A) \to S_1(A)$. By Svenonius's theorem, if the model M is sufficiently homogeneous and saturated, we can see that two strong types over A that correspond to the same type over A are conjugate by an A-automorphism of M.

As a consequence, if T is stable and p is in $S_1(M)$, then since a nonforking son of p over M must have a strong type over A extending p, we see that every strong type extending p contains at least one nonforking son of p. The following theorem states that it contains only one, that is to say, that *nonforking extensions of p correspond exactly to strong types over A that extend p.*

Theorem 16.2 (Finite Equivalence Relation Theorem). *Let T be a stable theory, let p be in $S_1(A)$, and let q_1 and q_2 be two distinct nonforking sons of p over a model M containing A. Then there is a finite equivalence relation E, definable with parameters in A, such that q_1 and q_2 are incongruent modulo E (i.e., $q_1 \models x_1 \sim a_1(\pmod{E}), q_2 \models x_2 \sim a_2 \pmod{E})$, with a_1 and a_2 incongruent modulo E).*

Proof. Let b_1 realize q_1 and let b_2 realize the heir of q_2 over $M \cup \{b_1\}$. By Theorem 15.9, the type of $b_1{}^\frown b_2$ over M does not fork over A.

Since $q_1 \neq q_2$, for some formula $f(\vec{c}, x)$ with parameters $\vec{c}$ in M, we have $f(\vec{c}, b_1) \wedge \neg f(\vec{c}, b_2)$. By duality, the type of $\vec{c}$ over $A \cup \{b_1, b_2\}$ does not fork over A, and by Lemma 16.1, b_1 and b_2 are incongruent modulo some finite equivalence relation E that is definable by a formula with parameters in A.

As the class of x_1 modulo E is determined by q_1, and that of x_2 by q_2, and we have proved that it is consistent with $q_1 \wedge q_2$ that x_1 and x_2 be incongruent modulo E, we have the result sought. $\square$

Theorem 16.3. *If the theory T is stable, then a and b have the same strong type over A if and only if there is a model M containing A such that a and b have the same type over M.*

Proof. If a and b have the same type over a model containing A, they have the same strong type over A, since all classes of an equivalence relation definable over A have representatives in M.

Conversely, suppose that a and b have the same strong type over A. Consider a type of model over A, for which we take a nonforking son over $A \cup \{a, b\}$, which is realized in M. By symmetry, the type of $a\frown b$ over M does not fork over A, so the type of a over M does not fork over A, nor does the type of b over M. Since these types are in the same strong type over A, they are equal by Theorem 16.2. $\square$

A few comments to clarify this theorem a little: That there is a model M containing A such that a and b have the same type over A means that whenever $(\exists \vec{y}) g(\vec{y})$ is true, where $g(\vec{y})$ is a formula with parameters in A, all sentences of the form $(\exists \vec{y})(g(\vec{y}) \wedge (f(a, \vec{y}) \leftrightarrow f(b, \vec{y})))$ are satisfied. Indeed, on the one hand every model containing A contains some $\vec{y}$ satisfying g; moreover, if all the sentences are satisfied, and if M is a model containing A, the set of sentences consisting of $T(M)$, of the type of $a\frown b$ over A, and of all sentences $f(a, \vec{c}) \leftrightarrow f(b, \vec{c}), \vec{c} \in M$ is consistent.

We can therefore see that this relation "a and b have the same type over a model containing A" depends only on the type of $a\frown b$ over A, and that it corresponds to the satisfaction of infinitely many formulas: It is a closed set in $S_2(A)$. In the stable case, this relation is especially simple, since this closed set is nothing but the (infinite) conjunction of all finite equivalence relations. However, that is no longer true if T is unstable, and it is even possible that this relation "having the same type over a model" is not transitive, that it is not an equivalence relation! For example, if T is the theory of infinite atomic Boolean algebras, and if a and b bound infinitely many atoms, and their respective complements do, too (so that by Theorem 6.20, a and b have the same type over $\emptyset$), they have the same type over a model if and only if either $a \wedge b$ or $\neg a \wedge \neg b$ bounds infinitely many atoms; that relation is not transitive.

By using Lemma 15.13 to eliminate unstable parameters, the reader will easily check that the three theorems of this section, as well as those that follow (perhaps with slight modifications) remain valid for stable types of an unstable theory.

As examples of strong types, we have those that correspond to algebraic elements over A: If p is an algebraic type over A with only finitely many realizations $a_1, \ldots, a_n$ in a model M containing A, then every element a_i corresponds to a strong type extending p. If $f(x)$ is a formula with parameters in A that isolates p, consider the equivalence relation, with $n+1$ classes, defined by the formula $(\neg f(x) \wedge \neg f(y)) \vee x = y$.

The manipulation of strong types can confound the beginner. The principal danger is the following: We never stop using an obvious principle, which reduces 2-types to 1-types, to know that that specifying the type of $a\frown b$ over A is the same as specifying the type of a over A, then the type of b over $A \cup \{a\}$ (or else the type of b over A, then the type of a over $A \cup \{b\}$). *This principle does not apply to strong types*, since the strong type of $a\frown b$ over A does not determine the strong type of b over $A \cup \{a\}$.

For example, take the theory T of algebraically closed fields of characteristic zero, $A = \emptyset$, a transcendental over $\mathbf{Q}$, $b = a^2$. The type p of $a\frown b$ over $\emptyset$ is stationary, since it has only one nonforking son over a model M of T, corresponding to a being transcendental over M. There is therefore only one strong type containing p. The type of a over $\emptyset$ is also stationary, as is the type of b over a; the latter is even rational, i.e., satisfied by a single element, since $b = a^2$.

However, that does not work in the other direction: The type of b over $\emptyset$ is stationary, but the type of a over b has multiplicity 2: If we have a model around b, it has two sons (which are nonforking), each corresponding to a square root of b. The strong type of $a\frown b$ over $\emptyset$ is of no help in deciding which one to take.

16.2 Spaces of Strong Types; Open Mapping Theorem

We give the set $SF_1(A)$ the topology with an open base consisting of the $\langle xEa \rangle$, the set of strong types implying that x is congruent to a modulo E, where a is in M and E is a finite equivalence relation definable over A. These sets are indeed closed under finite intersections: To know the equivalence class of x modulo E and modulo F is to know it modulo $E \wedge F$, which is again a finite equivalence relation. These open sets are also closed, since the complement of $\langle xEa \rangle$ has the form $\langle xEa_1 \rangle \vee \cdots \vee \langle xEa_n \rangle$.

By the compactness theorem (Section 4.2), in order for a set of formulas xE_ia_i to be consistent it is necessary that every one of its finite fragments be consistent. Therefore, $SF_1(A)$ is a compact 0-dimensional space.

We can see that the model M that serves to define it is only an accessory: If we change the model, we still define the same space.

If A is included in B, we consider the restriction map from $SF_1(B)$ to $SF_1(A)$, which consists in keeping only the finite equivalence relations definable over A. If M is a model that contains B, $A \subset B \subset M$, we have the following commutative diagram, where the arrows represent the canonical restrictions:

$$\begin{tikzcd} S_1(M) \arrow[r, phantom, "="] & SF_1(M) \arrow[r] & SF_1(B) \arrow[r] \arrow[dr] & SF_1(A) \arrow[dr] & \\ & & & S_1(B) \arrow[r] & S_1(A) \end{tikzcd}$$

Each of the arrows is a *continuous* surjection. For example, to see that $SF_1(A) \to S_1(A)$ is continuous, consider a clopen set $\langle f(x, \vec{a})\rangle$ in $S_1(A)$, the two-class relation E associated with this formula, and an element b in M satisfying the formula: The inverse image of $\langle f(x, \vec{a})\rangle$ is $\langle xEb\rangle$.

Lemma 16.4. *For every A, the restriction map from $SF_1(A)$ to $S_1(A)$ is open.*

Proof. By Svenonius's theorem, two strong types corresponding to the same type over A are conjugate via an A-automorphism of a sufficiently saturated and homogeneous model M containing A. Consequently, there is a group G of homeomorphisms of $SF_1(A)$ such that two elements of $SF_1(A)$ are conjugate under the action of G if and only if they have the same restriction to $S_1(A)$; this implies the result.

Let O be an open set of $SF_1(A)$ and let F be its complement. We need to show that the image of O under the restriction map ρ is open. I claim that the intersection of the $\rho(\sigma F)$ is the complement of $\rho(O)$, where σ ranges over G. Indeed, every point of $S_1(A)$ is of the form ρp, and $\rho p = \rho q$ if and only if q is of the form σp. Consequently, to say that $\rho p \notin \rho(O)$ is to say that every σp is in F. Since every σF is compact, the same is true of their projections, which are closed; as a consequence, $\rho(O)$ is indeed open. $\square$

We can understand that the notion of strong type is especially useful in the stable case:

Lemma 16.5. *If T is stable and $A \subset B$, then the set of types in $S_1(B)$ that do not fork over A form a closed set in $S_1(B)$.*

Proof. We will show rather that the types that fork form an open set. So let p, which is realized by a, be a type over B that forks over A. If it forks, it is because of some formula, and there is $\vec{b}$ in B such that the type of a over $A \cup \{\vec{b}\}$ forks over A. By symmetry, the type of $\vec{b}$ over $A \cup \{a\}$ forks over A, which means, by Theorem 15.7, that a formula $f(\vec{b}, \vec{c}, a)$ is satisfied, where $\vec{c} \in A$, so that $f(\vec{x}, \vec{y}, z)$ is omitted by the bound of the type of $\vec{b}$ over A. Finally, $\langle f(\vec{b}, \vec{c}, x)\rangle$ is a neighborhood of p, and every type that satisfies this formula forks over A. $\square$

Lemma 16.6. *If T is stable and if $A \subset M$, where M is a model of T, then the restriction map from $S_1(M)$ onto $SF_1(A)$ induces a homeomorphism between the closed set of types over M that do not fork over A and $SF_1(A)$.*

Proof. Let X be the set of types over M that do not fork over A. X is compact by Lemma 16.5, and the restriction induces a continuous bijection from X onto $SF_1(A)$ by 16.2. Since we are dealing with compact spaces, its inverse is also continuous. (A continuous function from one compact space to another is closed; if it is bijective, it is also open, so its inverse is continuous.) □

In the case where A is a model M_0 of T, we can see that the heir function is continuous and brings the restriction back up. It is a continuous splitting of the restriction function, establishing a homeomorphism between $S_1(M_0)$ and the set of types over M that inherit or coinherit their restrictions to M_0.

Corollary 16.7 (Open Mapping Theorem). *If T is stable and $A \subset B$, then the restriction map induces a continuous open surjection (i.e., the image of an open set is open) from the closed set X consisting of the types that do not fork over A onto $S_1(A)$.*

Proof. Let M be a model containing B and let Y be the closed subset of $S_1(M)$ consisting of all types that do not fork over A. If O is an open subset of X, then by continuity its inverse image is an open subset O' of Y. By Lemmas 16.4 and 16.6, the image of O' under the restriction map to $S_1(A)$ is open. □

One consequence of the open mapping theorem that is very frequently used in constructing models is the following:

Theorem 16.8. *If T is stable, $A \subset B$, $p \in S_1(B)$, p does not fork over A, and p is isolated in $S_1(B)$, then the restriction of p to A is isolated in $S_1(A)$.*

Proof. The set $\{p\}$ open in $S_1(B)$. Apply Corollary 16.7. □

16.3 Morley Sequences for Strong Types; Saturated Models Revisited

If T is a stable theory and p is a stationary type in $S_1(A)$ (or more generally if p is a stable stationary type in an unstable theory), we can define the Morley sequence of p over A as we have been doing for a while when A is a model: Let a_0 realize p, then let a_1 realize the unique nonforking son of p over $A \cup \{a_0\}$, etc. For the same reason, this is an indiscernible sequence over A, which is totally indiscernible and indivisible, like every infinite indiscernible sequence in a stable theory.

More generally let us consider any type p in $S_1(A)$ and a strong type p^* in $SF_1(A)$ extending p. Assume that A embeds into a large, very saturated model, so that the notion of realization of a strong type makes sense. We define the *Morley sequence of the strong type* p^* as follows: Let a_0 realize p^*, then let a_1 realize the unique nonforking extension of p over $A \cup \{a_0\}$ that is in the strong type $p^*, \ldots$, let a_{n+1} realize the unique nonforking extension of p over $A \cup \{a_0, \ldots, a_n\}$ that is in the strong type $p^*, \ldots$.

Note that once the first element a_0 has been chosen, this amounts to constructing the Morley sequence of a stationary type, that of a_1 over $A \cup \{a_0\}$: We realize the unique nonforking extension of p over $A \cup \{a_0, \ldots, a_n\}$ that is congruent to a_0 modulo every finite equivalence relation definable over A.

This sequence is naturally totally indiscernible and indivisible, and if $A \subset B$, then *its mean type over B is the nonforking extension of p corresponding to the strong type p^**. Indeed, let s be this sequence and let M be a model of T containing A, with M and s placed independently over A. The type of s over M does not fork over A, implying by Theorem 15.9 that for every n the type of a_n over $M \cup \{a_0, \ldots, a_{n-1}\}$ does not fork over $A \cup \{a_0, \ldots, a_{n-1}\}$, nor therefore over A, by transitivity. Once M and s have been placed, s becomes the Morley sequence of the unique nonforking extension of p over M corresponding to the strong type p^*; the conclusion follows by Theorem 12.33.

We can see that saying that the sequence $s = \{a_0, \ldots, a_n, \ldots\}$ is the Morley sequence of a strong type over A amounts to saying that it is an *independent and indiscernible* set over A. Indeed, if s is indiscernible, then its elements are either all congruent pairwise or all incongruent pairwise modulo an equivalence relation E that is definable with parameters in A. If E has only finitely many classes, then they are all congruent modulo E. They are therefore all in the same strong type. The mean type of such a sequence over B therefore depends only on the type of B over A, and not on the type of B over $A \cup s$.

We can also see that every infinite indiscernible sequence can be considered as a Morley sequence over a suitably chosen set A: Extend the sequence $\kappa(T)$ times. Since the bound of the type p_α of a_α over $A_\alpha = \{\ldots, a_\beta, \ldots\}_{\beta<\alpha}$ cannot decrease $\kappa(T)$ times in the fundamental order, it never decreases after some α, and it is a Morley sequence over A_α. We should not fallaciously conclude that two infinite indiscernible sequences s and t have the same mean type over every set just because they have the same strong type over $\emptyset$: They also need to keep the same strong type over a set of parameters where each becomes a Morley sequence. For example, if T is the theory of an infinite set, there is only one strong type over $\emptyset$; as a consequence, the sequence s obtained by repeating the element a and the sequence t obtained by repeating the element b have the same strong type, but they have the same mean type only if $a = b$.

With this notion of Morley sequence, we can finally give the definitive version of Lemma 14.6 and its corollaries.

Lemma 16.9. *If T is stable, and if M is a $\kappa(T)$-saturated and ω_1-saturated model of T, then every type over M is the mean type of an infinite totally indiscernible sequence of elements of M.*

Proof. If p is in $S_1(M)$, then by Theorem 15.8 there is a subset A of M of cardinality strictly less than $\kappa(T)$ such that p does not fork over A. In M, let a_0 realize the restriction of p to A, then let a_1 realize the restriction of p to $A \cup \{a_0\}$, etc. Continue $\omega + \omega$ times to get a sequence $a_0, \ldots, a_n, \ldots, a_\omega, \ldots, a_{\omega+n}, \ldots$ in the model M, where a_i realizes the restriction of p to $A \cup \{\ldots, a_j \ldots\}$, $j < i$ for every $i < \omega + \omega$.

If E is an equivalence relation, definable with parameters in A, with n classes, and if $p \models x \not\sim a_0(\text{mod } E)$, then a_1, and all a_i for $i > 0$, are incongruent to a_0 modulo E. If a_1 is also not congruent to x modulo E, then none of the $a_i, i > 1$, is in either the class of a_0 or the class of a_1. That can happen only n times, and we can be certain that all $a_i, i \geq n$, are in the same class of E, that $p \models x \sim a_n(\text{mod } E)$.

Consequently, the element a_ω and its successors are all in the strong type of p. We can see that the restriction of p to $A \cup \{a_\omega\}$ is stationary, and the sequence $a_\omega, \ldots, a_{\omega+n}, \ldots$ is a Morley sequence whose mean type is p. $\square$

In the superstable case, the hypothesis of $\kappa(T)$-saturation, i.e., ω-saturation, is not sufficient for Lemma 16.9. The adjusted hypothesis, weaker than ω_1-saturation, which we could call $(\omega + \epsilon)$-saturation, or $\aleph_\epsilon$ saturation, is the following: M is $(\omega+\epsilon)$-saturated if for every tuple $\vec{a}$ in M, every strong type over $\vec{a}$ is realized in M. If all types have finite multiplicity, we can see that $(\omega + \epsilon)$-saturation is equivalent to ω-saturation.

Let us reflect on the following example: The language of T consists of denumerably many binary relation symbols $E_0, E_1, \ldots, E_n, \ldots$; the axioms of T say that each E_n is an equivalence relation, with E_0 trivial, and each E_{n+1} being a refinement of E_n, with every class modulo E_n partitioned into exactly two classes modulo E_{n+1} (so that there are 2^n classes for E_n). We have no difficulty seeing that T is a complete theory with quantifier elimination, and that it is a superstable theory, with its fundamental order consisting of only two points, one realized and one nonrealized. Two elements have the same strong type over $\emptyset$ if they are congruent modulo all the E_n. There are 2^ω strong types, and a model M of T is ω-saturated if and only if for every a in M there are infinitely many elements in M with the same strong type as a. We can see that there is such a model that is denumerable, while the smallest $(\omega + \epsilon)$-saturated model has cardinality 2^ω.

Here is the definitive version of Theorem 14.5.

Theorem 16.10. *If T is stable and $\lambda \geq \kappa(T), \lambda > \omega$, then T has a saturated model of cardinality λ if and only if it is stable in λ.*

Proof. We know, by Theorem 14.2, that if T is stable in λ, it has a saturated model of cardinality λ.

For the converse, first suppose that T is superstable. Let M be a saturated model of cardinality λ. If $p \in S_1(M)$, then there exists finite $\vec{a}$ in M such that p does not fork over $\vec{a}$, and we saw, when we proved Lemma 16.9, that M, being ω_1-saturated, is $(\omega + \epsilon)$-saturated, that we can have a_ω realize the strong type of p over $\vec{a}$. We can see, therefore, that the restriction of p to $\vec{a}\frown a_\omega$ is stationary. Consequently, every type p over M is the unique nonforking extension of its restriction to some finite set $\vec{b}$ of parameters of M. Since all types over $\vec{b}$ are realized in M, there are not more than λ. There are therefore λ choices for $\vec{b}$ and λ choices for the restriction of p to $\vec{b}$, so altogether $\lambda \times \lambda$ choices for p. Since every set of parameters of cardinality at most λ embeds in M, we have stability in λ.

Now assume that T is not superstable, $\lambda \geq \kappa(T) > \omega$. Let M be the saturated model of cardinality λ. By Lemma 16.9, there are not more than λ^ω types over M; if we had instability in λ, we would need $\lambda^\omega > \lambda$, and we can finish as in Theorem 14.5. □

We thus have a threefold lesson to keep in mind from Theorems 14.5 and 16.10, for a stable theory T:

- If T has a saturated model of cardinality $\lambda \leq \kappa(T)$, then this cardinal is regular.
- T does not have a saturated model of cardinality $\lambda \geq \omega_1, \kappa(T) \leq \lambda < \lambda_0(T)$.
- For $\lambda \geq \lambda_0(T)$, T has a saturated model of cardinality λ if and only if it is stable in λ, i.e., if $\lambda^{<\kappa(T)} = \lambda$.

We can thus establish that the class of cardinals in which a theory T has a saturated model allows us to see whether it is stable (it then has a saturated model of singular cardinality), and almost allows us to determine its stability spectrum. For we can find $\kappa(T)$ by looking at large values, and $\lambda_0(T)$ is either ω or else the smallest cardinal greater than $\kappa(T)$ in which T has a saturated model.

If $\omega_1 < 2^\omega$, we can in this way distinguish the ω-stable theories from the others, which do not have saturated models of cardinality ω_1. (If T is not ω-stable, we can find a denumerable set of parameters that yields 2^ω types). However, under the continuum hypothesis, $\omega_1 = 2^\omega$, and the theory T given as an example after Lemma 16.9 has saturated models of every cardinality; it is, however, not ω-stable, since the 2^ω strong types over $\emptyset$ yield 2^ω types over any denumerable model. What distinguishes the ω-stable theories among the superstable theories with denumerable saturated models is the existence of denumerable $(\omega + \epsilon)$-saturated models.

Now let us look at what happens to Theorem 14.7:

Theorem 16.11. *If T is stable, then a model of T that is $\kappa(T)$-saturated and ω_1-resplendent is saturated. If T is superstable, then a model of T which* *is $(\omega+\epsilon)$-saturated (for example, if it is ω_1-saturated) and ω-resplendent is saturated.*

Proof. For the stable case, M is $\max(\kappa(T), \omega_1)$-saturated, so Lemma 16.9 applies, and every type over M is the mean type of an indiscernible ω-sequence of elements of M. We can then proceed as in Theorem 14.7.

In the superstable case, given the hypotheses, every type p over M is the unique nonforking extension of one of its stationary restrictions q to a finite set $\vec{a}$ of elements of M. For q to have a Morley sequence of the same cardinality as M, the hypothesis of ω-resplendency is sufficient, since the Ehrenfeucht set of this sequence uses only finitely many parameters. □

As to Lemma 14.8, it has now been completely proved: The remaining case is that in which λ is singular and strictly greater than $\kappa(T)$. In that case Lemma 16.9 applies, and every type over a λ-saturated model M is the mean type of an indiscernible λ-sequence of elements of M.

We can even specify what happens with chains of sufficiently saturated models of a stable theory:

Theorem 16.12. *Let T be a stable theory, and let $\ldots, M_i, \ldots$ be an elementary chain, indexed by a chain I of cofinality greater than or equal to $\kappa(T)$, consisting of λ-saturated models of T. Then the limit M of the M_i is also λ-saturated.*

Proof. This is obvious if $\lambda \leq \kappa(T)$, since then every subset A of M of cardinality strictly less than λ is contained in some M_i. So assume that $\lambda > \kappa(T)$. Let $A \subset M, |A| < \lambda, p \in S_1(A)$, and let q be any son of p over M. Since every cofinal subset of I has at least $\kappa(T)$ elements, there is i such that q is the heir of its restriction q_i to M_i. Since $\lambda > \kappa(T), it follows that \lambda > \omega$, and we can apply Lemma 16.9. Then q_i is the mean type of an indiscernible sequence of length λ consisting of elements of M_i, and the same is true of its heir q; there are at most $|A| \times \kappa(T) < \lambda$ elements of this sequence that do not realize p. □

16.4 Imaginary Elements

Here I set forth a construction due to Shelah, consisting of making elements that are virtually present appear in a structure M.

Let M be a structure, with theory T, in a language L. The language L^{eq} is obtained by adding a relational predicate M_E to L for every equivalence relation $E(\vec{x}, \vec{y})$ between n-tuples of M that is defined without parameters by a formula of L, as well as a function symbol f_E from $(M_=)^n$ into M_E.

The structure M^{eq} associated with M is defined as the L^{eq}-structure for which M_E consists of the set of classes of n-tuples of M modulo E (if $E \neq E'$, then M_E and $M_{E'}$ are disjoint) and f_E is the function from $(M_=)^n$ to M_E, that is to say, the function from M^n to M_E, that sends each n-tuple to its class modulo E. As for the other symbols of L^{eq}, those of L, they are interpreted over $M_=$ as they are over M; we thus identify M and $M_=$, so that M appears as a definable substructure of M^{eq}.

M^{eq} is not quite definable in M, in the sense of Section 9.4, since its universe does not consist of a quotient of any M^m with m fixed. If we were considering only finitely many equivalence relations $E_1, \ldots, E_k$, between n_1-tuples, $\ldots, n_k$-tuples, respectively, we could define the corresponding structure as a quotient of M^{m+k+1}, where m is the maximum of $n_1, \ldots, n_k$ (see the proof of Lemma 16.13). Thus M^{eq} appears as a sort of limit of structures definable in M.

It is easy to see, by a back-and-forth argument, that if M and N are elementarily equivalent, then so are M^{eq} and N^{eq}, and that if N is an elementary extension of M, then N^{eq} is an elementary extension of M^{eq}.

Consequently, a complete theory T can be associated with the complete theory T^{eq} of all M^{eq}, where M ranges over the class of models of T.

What are the other models of T^{eq}? As the M_E are pairwise disjoint and infinite in number, the set of formulas $\ldots, x \notin M_E, \ldots$ is consistent, and we can easily see that it is a complete type, which we shall call the point at infinity p_∞. If N is a model of T^{eq}, then its $M_=$ is a model M of T, and M^{eq} is an elementary substructure of N: N is simply obtained by adding some number of realizations of p_∞ to M^{eq}; they form a totally indiscernible class, with no connections to M^{eq}, its elements having no relations among themselves other than that of being distinct. Over an arbitrary set A of parameters, p_∞ has its realized sons and a unique nonrealized son.

We can see that the correspondence between M and M^{eq} is a bijection between the models of T and the models of T^{eq} omitting the point at infinity. If we wish, we can limit ourselves to considering models of that sort, but in any event this supplementary, neutral and parasitic set of realizations of p_∞ is no great inconvenience.

Naturally, every property of M translates into a property of M^{eq}: The transition from M to M^{eq} preserves "model-theoretic" properties. For example, two n-tuples of $M_=$ have the same type in M^{eq} iff they have the same type in M; if M is λ-homogeneous, then so is M^{eq}. If M is λ-saturated, then M^{eq} has the following saturation property: for every $A \subset M^{\mathrm{eq}}$, $|A| < \lambda$, and for every $p \in S_1(A), p_\infty \not\subset p$, p is realized in M^{eq}. Indeed, a type satisfying M_E has no connection to the parameters of type p_∞ and comes from a type of n-tuples in the sense of M.

Likewise, T^{eq} has properties similar to those of T: *T is stable in λ* iff *T^{eq} is stable in λ*; for on the one hand T is interpretable in T^{eq}, and on the other, if A is a set of parameters of a model of T^{eq}, the types of $S_1(A)$ distinct from the point at infinity correspond to the types (of n-tuples) in the sense of T.

It must also be noted that the step from T to T^{eq} is neutral for the forking properties of the types that satisfy $M_=$: their fundamental order in T^{eq} is the fundamental order of T (since representing a formula of L^{eq} is equivalent to representing its translation in L), and if T is stable, $A \subset B \subset M, p \in S_1(B), p \models M_=(x)$, then p does not fork over A in the sense of T iff it does not fork over A in the sense of T^{eq}.

The interest in and the usefulness of T^{eq}, lie in the fact that as we shall see, it gives an intrinsic character to the parameters of a formula $f(\vec{x}, \vec{a})$ and associates a definable type with a minimal and intrinsic set of parameters necessary to its definition.

First, let us show the following lemma, which proves that it is useless to repeat the construction eq on T^{eq}, to the extent that the point at infinity is not under consideration.

Lemma 16.13. *Let F be an equivalence relation, definable in T^{eq}, on the set $(M_{E_1} \cup \cdots \cup M_{E_k})^n$. Then there is a definable function f_F from $(M_{E_1} \cup \cdots \cup M_{E_k})^n$ to M^{eq} such that two n-tuples have the same image under f_F if and only if they are congruent modulo F.*

Proof. To simplify the notation, assume that $n = 1$. E_1 is defined on the n_1-tuples, $\ldots, E_k$ on the n_k-tuples. Let m be the maximum of $n_1, \ldots, n_k$.

Consider the set A of the $(m+k+1)$-tuples of M, $(a_0, \ldots, a_k, b_1, \ldots, b_m)$, that satisfy one and only one equation of the type $a_0 = a_i$:

$$\bigwedge_{0<i<j} a_0 \neq a_i \vee a_0 \neq a_j \wedge \bigvee_{0<i<j} a_0 = a_i.$$

Consider the equivalence relation E defined on the $(m + k + 1)$-tuples for which the complement of A forms one class and whose trace on A is the following: $(a_0, \ldots, a_k, b_1, \ldots, b_m)$ is equivalent modulo E to $(a'_0, \ldots, a'_k, b'_1, \ldots, b'_m)$ if whenever $a_0 = a'_j$ and $a'_0 = a'_j$, the class modulo E_i of the n_i-tuple $(b_1, \ldots, b_{n_i})$ is equivalent modulo F to the class modulo E_j of the n_j-tuple $(b'_1, \ldots, b'_{n_j})$.

This relation E is defined by a formula of L^{eq}; however, as the $(m+k+1)$-tuples of $M_=$ have the same type whenever they satisfy the same formulas of L, by compactness (see Section 5.3!) E is definable by a formula of L. Consequently it corresponds to some set of classes M_E contained in M^{eq}.

So consider the function f_F from $M_{E_1} \cup \cdots \cup M_{E_k}$ to M_E that associates an element α of M_{E_i} with the class modulo E of a tuple (or all tuples) $(a_0, \ldots, a_k, b_1, \ldots, b_m)$ with $a_0 = a_i$ and $(b_1, \ldots, b_{n_i})$ being a representative of α. $\square$

This is what happens in T^{eq}: *Every formula $f(\vec{x}, \vec{a})$ of L, with parameters $\vec{a}$ in M, is associated with a formula $g(\vec{x}, y)$ of L^{eq} such that there is one and only one element b of M^{eq} with $f(\vec{x}, \vec{a})$ equivalent to $g(\vec{x}, b)$.*

Furthermore, this is almost obvious. Consider the equivalence relation E defined by the formula, without parameters, $(\forall \vec{x})(f(\vec{x}, \vec{y}) \leftrightarrow f(\vec{x}, \vec{z}))$. Take b to be the class of the tuple $\vec{a}$ modulo E and take $g(\vec{x}, y)$ to be the formula saying that y is in M_E and that $f(\vec{x}, \vec{z})$ is true for some (or for every) representative $\vec{z}$ of y.

Likewise, Lemma 16.13 says that all that is true not only for the formulas of L with parameters in $M_=$, but also for the formulas of L^{eq} for which no variable or parameter has type p_∞.

As an illustration of this principle, let us show that in T^{eq}, $SF_1(A) = S_1(A)$ when A is algebraically closed, a condition that is not at all sufficient in general. Consider a type over A. If it is a son of p_∞, then it is stable and stationary, since there is only one nonrealized type of point at infinity, and consequently it determines a strong type. For a type that satisfies M_{E_i}, consider an equivalence relation definable with parameters in A that partitions M_{E_i} into finitely many classes $C_1, \ldots, C_n$. Let $c_1, \ldots, c_n$ be the parameters canonically associated with the formulas $x \in C_1, \ldots, x \in C_n$ respectively. Since the type of c_i over A ensures that there be only finitely many possibilities for the formula $g(\vec{x}, c_i)$, every c_i must be algebraic over A. Every c_i is therefore in A, so every C_i is definable with parameters in A, and every type yields a strong type.

16.5 Elimination of Imaginaries

For some theories, the imaginary elements are already present in the model M and it is unnecessary to add them; in that case, we say that there is "elimination of imaginaries," a notion which we shall now make precise.

Recall (see Section 6.1) that an element a is said to be *rational* over A if it is the only element satisfying some particular formula with parameters in A, and that it is *algebraic* over A if it satisfies a formula with parameters in A that is satisfiable by only finitely many elements.

The *rational closure* of A is the set of elements that are rational over A; it is the smallest set of parameters containing A that is rationally closed, i.e., that contains all rational elements over itself. The *algebraic closure* of A is the set of elements that are algebraic over A, and is the smallest superset of A that is algebraically closed, i.e., that contains all elements that are algebraic over itself. All these notions depend only on the type of A.

If we are looking for minimal sets of parameters to define something, it is reasonable to consider only rationally closed sets; moreover, a type over A has only one extension to the rational closure of A: A type over A is the same thing as a type over its rational closure.

We shall say that T *(strongly) eliminates imaginaries* if for every formula $f(\vec{x}, \vec{a})$ with parameters $\vec{a}$ in a model M of T there is a tuple $\vec{b}$ of parameters with the following property: If s is an automorphism of an elementary extension of M, it preserves the formula $f(\vec{x}, \vec{a})$ (i.e., for every $\vec{x}$, $f(\vec{x}, \vec{a}) \leftrightarrow f(s\vec{x}, \vec{a})$) if and only if it fixes every element of $\vec{b}$.

We shall say that T *weakly eliminates imaginaries* if for every formula $f(\vec{x}, \vec{a})$ with parameters in M there is a smallest algebraically closed set

$A \subset M$ such that $f(\vec{x}, \vec{a})$ is equivalent to a formula with parameters in A. (We shall say that $f(\vec{x}, \vec{a})$ is definable over A.)

In the first definition, we will not go so far as to insist, as is the case for T^{eq}, that the tuple $\vec{b}$ associated with $f(\vec{x}, \vec{a})$ consist of just a single element; it is enough to consider only the automorphisms of an ω-saturated ω-homogeneous extension of M, since everything depends only on types.

Note that strong elimination entails weak elimination: For if the formula $f(\vec{x}, \vec{a})$ is definable with $\vec{c}$ as parameters, then every automorphism which that $\vec{c}$ pointwise fixes $f(\vec{x}, \vec{a})$, and therefore also fixes $\vec{b}$ pointwise. By Svenonius's theorem, every element of $\vec{b}$ is rational over $\vec{c}$; furthermore, since the satisfaction of $f(\vec{x}, \vec{a})$ depends only on the type of $\vec{x}$ over $\vec{b}$, this formula is equivalent to a formula with parameters $\vec{b}$. We can therefore see that the rational closure of $\vec{b}$ is the smallest rationally closed set of definition for this formula, and that its algebraic closure is the smallest algebraically closed such set.

We can also see that weak elimination does not imply strong elimination: If we consider the formula $x = a \vee x = b$ in the theory T of infinite sets, where a and b are distinct, its smallest defining set is $\{a, b\}$, and an automorphism that permutes a and b preserves the formula.

The construction eq is a canonical way to eliminate imaginaries, at least for formulas without infinitary parameters. For by Lemma 16.13, fixing such a formula is the same as fixing a class of the associated equivalence relation E, and the same as fixing the image of this class under the function f_E that defines the equivalence E. If we allow infinitary parameters, then we have only weak elimination.

We could introduce an intermediate notion between strong and weak elimination, one in which every formula would have a minimal rationally closed defining set; this would hardly be of use, since it would have no characterization in the style of Theorems 16.14 and 16.15 below. Let us note in passing that when we go into T^{eq} in the example above, a and b become algebraic and conjugates over the element c of M^{eq} that is the class of (a, b) in the equivalence relation $(y = y' \wedge z = z') \vee (y = z' \wedge z = y')$.

Note that if T eliminates, or weakly eliminates, imaginaries, then the same is true of $T(A)$.

Also, elimination of imaginaries assigns every definable type a minimal set of definition: Fixing p in $S_1(M)$ is the same is fixing the tuple canonically associated with the formula (with parameters) $df(\vec{y})$ given by the definition of p, for every formula (without parameters!) $f(\vec{x}, \vec{y})$. In the case of weak elimination, there is only a minimal algebraically closed defining set. We can therefore understand that this elimination of imaginaries will have especially interesting applications for stable theories.

The following two theorems clarify these notions: Strong elimination canonically associates a tuple of parameters to a formula; weak elimination canonically associates a finite set of tuples of parameters to a formula.

Theorem 16.14. *T (strongly) eliminates imaginaries if and only if there is a formula $g(\vec{x}, \vec{z})$ associated with every formula $f(\vec{x}, \vec{a}), \vec{a} \in M$, such that there is a unique tuple $\vec{b}$ in M with $f(\vec{x}, \vec{a})$ equivalent to $g(\vec{x}, \vec{b})$.*

Proof. If this property is true, then there is elimination, since to fix f is to fix the $\vec{b}$ associated with it.

Conversely, assume elimination, and let $\vec{b}$ be the tuple of parameters associated with f. Since the satisfaction of f depends only on the type of $\vec{x}$ over $\vec{b}$, f is equivalent to a formula $g'(\vec{x}, \vec{b})$. I claim that there is then a formula $h(\vec{z})$, without parameters, satisfied by $\vec{b}$, such that $(\forall \vec{z})((h(\vec{z}) \wedge ((\forall \vec{x})(g'(x, \vec{z}) \leftrightarrow g'(\vec{x}, \vec{b})))) \to \vec{z} = \vec{b})$. Otherwise, by compactness we could find $\vec{b}'$ of the same type as $\vec{b}$, $\vec{b}' \neq \vec{b}$, with $g'(\vec{x}, \vec{b})$ equivalent to $g'(\vec{x}, \vec{b}')$, and an automorphism sending $\vec{b}$ to $\vec{b}'$ preserving f without fixing $\vec{b}$. We could then take $h(\vec{z}) \wedge g'(\vec{x}, \vec{z})$ as $g(\vec{x}, \vec{z})$. □

Theorem 16.15. *T weakly eliminates imaginaries if and only if there is a formula $g(\vec{x}, \vec{z})$ associated with every formula $f(\vec{x}, \vec{a})$ such that there are finitely many (and at least one) parameters $\vec{b}_1, \dots, \vec{b}_n$ for which $f(\vec{x}, \vec{a})$ is equivalent to $g(\vec{x}, \vec{b}_1), \dots, g(\vec{x}, \vec{b}_n)$.*

Proof. If we have this property, then we have weak elimination, since an automorphism that preserves the formula permutes the $\vec{b}_i$, which have only finitely many conjugates under these automorphisms, and which are therefore algebraic over every set of definition for f.

Suppose that we have weak elimination. Let B be the minimal algebraically closed defining set for f and let $\vec{b}$ be a tuple of parameters of B such that $f(\vec{x}, \vec{a})$ is equivalent to a formula $g'(\vec{x}, \vec{b})$. If we can find an arbitrarily large finite number of $\vec{b}_i$, with the same type as $\vec{b}$, such that $g'(\vec{x}, \vec{b}_i)$ is equivalent to $g'(\vec{x}, \vec{b})$, then by compactness we can find as many as we want. We can therefore find $\vec{b}_1$, with the same type as $\vec{b}$, such that $g'(\vec{x}, \vec{b}_1) \leftrightarrow g'(\vec{x}, \vec{b})$ and $\vec{b}_1$ is not algebraic over $\vec{b}$. On the other hand, by the minimality of B, $\vec{b}$ must be algebraic over $\vec{b}_1$. Since $\vec{b}$ and $\vec{b}_1$ have the same type, there is $\vec{b}_2$ algebraic over $\vec{b}$, with $\vec{b}$ not algebraic over $\vec{b}_2$, such that $g'(\vec{x}, \vec{b})$ and $g'(\vec{x}, \vec{b}_2)$ are equivalent. That contradicts the minimality of B, since the algebraic closure of $\vec{b}_2$ is strictly contained in B, yet allows us to define f.

Consequently, there is a finite fragment $h(\vec{z})$ of the type of $\vec{b}$ such that there are at most n elements satisfying $h(\vec{z})$ and $(\forall \vec{x})(g'(\vec{x}, \vec{z}) \leftrightarrow g'(\vec{x}, \vec{b}))$; set $g(\vec{x}, \vec{z}) = h(\vec{z}) \wedge g'(\vec{x}, \vec{z})$. □

If every equivalence relation E definable (without parameters) on M^n is of the form $f_E(y) = f_E(z)$, where f_E is a definable function from M^n to M^m, then we have (strong) elimination of imaginaries. Indeed, we can associate $f(\vec{x}, \vec{a})$ with the relation E: $(\forall \vec{x})(f(\vec{x}, \vec{y}) \leftrightarrow f(\vec{x}, \vec{z}))$. Then fixing $f(\vec{x}, \vec{a})$ is the same as fixing the class of $\vec{a}$ modulo E, and the same as fixing the tuple $f_E(\vec{a})$.

The converse is almost right: In the following theorem the two constants 0 and 1 cannot be avoided, since both classes of the equivalence relation $(y_1 = y_2 \wedge z_1 = z_2) \vee (y_1 \neq y_2 \wedge z_1 \neq z_2)$ are definable without parameters.

Theorem 16.16. *If T eliminates imaginaries and allows at least two constants to be defined (i.e., the rational closure of $\emptyset$ contains at least two elements), then every equivalence relation $E(\vec{y}, \vec{z})$ definable on M^n is of the form $f_E(\vec{y}) = f_E(\vec{z})$, where f_E is a definable function from M^n to M^m.*

Proof. Let 0 and 1 denote the two definable constants. First let us show that in Lemma 16.14 the formula $g(\vec{x}, \vec{z})$ can be chosen independently of $\vec{a}$, as a function only of $f(\vec{x}, \vec{y})$. Indeed, for every tuple $\vec{a}_i$ we can find a formula g_i such that the following sentence is satisfied: $(\exists! \vec{z})(\forall \vec{x})(f(\vec{x}, \vec{a}_i) \leftrightarrow g_i(\vec{x}, \vec{z}))$. By compactness we can take the g_i from some finite set $g_1, \ldots, g_k$ of formulas. We can further assume that all the $\vec{z}$ have the same length m, since we could add 0 to the end of the ones that are too short. So let us consider the formula $g(\vec{x}, \vec{u}, \vec{z})$, where $\vec{u} = (u_1, \ldots, u_k)$ has length k and $\vec{z}$ has length m, which is the conjunction of the following formulas:

- that saying that all the u_i are equal to zero except for one that is equal to 1;
- that saying that if $u_i = 1$, then for every $j < i$ there are either zero or at least two $\vec{t}$ such that $(\forall \vec{x})(g_j(\vec{x}, \vec{t}) \leftrightarrow g_i(\vec{x}, \vec{z}))$, while $(\forall \vec{x})(g_i(\vec{x}, \vec{t}) \leftrightarrow g_i(\vec{x}, \vec{z}))$ implies $\vec{t} = \vec{z}$.

From the definition of g it is clear that for every $\vec{a}$ there is a unique tuple $\vec{b}$ such that $(\forall \vec{x})(f(\vec{x}, \vec{a}) \leftrightarrow g(\vec{x}, \vec{b}))$.

So let E be a definable equivalence relation. The formula $\vec{x} E \vec{y}$ is associated with a formula $g(\vec{x}, \vec{y})$ such that for every $\vec{a}$ there is a unique $\vec{b}$ such that $\vec{x} E \vec{a}$ is equivalent to $g(\vec{x}, \vec{b})$. The function f_E which sends $\vec{a}$ to $\vec{b}$ is definable, and satisfies our need. □

The following lemma can be useful for showing weak elimination of imaginaries in some cases, for example for the theory of the infinite set:

Lemma 16.17. *Suppose that T satisfies the following two conditions:*

- *There is no strictly decreasing sequence $A_0 \supsetneq A_1 \supsetneq \cdots \supsetneq A_n \supsetneq \cdots$, where each A_n is the algebraic closure of a finite set of parameters.*
- *If A and B are algebraic closures of finite sets of parameters and M is a very homogeneous and saturated model containing A and B, then the group of $(A \cap B)$-automorphisms of M is generated by the groups of its A-automorphisms and its B-automorphisms.*

Then T weakly eliminates imaginaries.

Proof. The second condition implies that if $f(\vec{x}, \vec{a})$ is definable over A and over B, its satisfaction depends only on the type over $A \cap B$, and it is definable over $A \cap B$. The first condition implies that every formula has minimal algebraically closed sets of definition. $\square$

One very striking example of elimination of imaginaries is that of algebraically closed fields. We could illustrate this by using a technique inspired by Lemma 16.17, but it is simpler to deduce it as a consequence of a few very elementary lemmas of algebra.

Lemma 16.18. *Let K be a field and I an ideal of polynomials in as many variables as desired. The ideal I is associated with a subfield k of K, called the field of definition of I, such that for every automorphism s of K, $sI = I$ if and only if s fixes every point in k.*

Proof. $I \subset K[\ldots, X_i, \ldots]$. Given a sequence $u = (\ldots, n_i, \ldots)$ of natural numbers of which all but finitely many are zero, let X^u denote the monomial that is the product of the $X_i^{n_1}$. By definition s acts on the polynomials, fixing these monomials. The module $K[\ldots, X_i, \ldots]/I$ is a vector space, generated by the images of these monomials, for which we can choose a basis $B = \{\ldots, b_j, \ldots\}$ of monomials from among the images of these X^u. Each monomial can therefore be written uniquely in the form $X^u = \Sigma a_{j,u} b_j$ (modulo I), where the $a_{j,u}$ are coefficients in K. Let k be the field generated by the $a_{j,u}$. Note that the polynomials $X^u - \Sigma a_{j,u} b_j$ are in I and that they form a generating system for it: Indeed, since the b_i are free modulo I, a polynomial is in I if and only if when every X^u is replaced by its expression as a function of the b_j, we get the zero polynomial.

If s fixes I, then it also fixes all of the $a_{j,u}$, which are defined intrinsically from I. If s fixes k, then it fixes I, since it fixes a system of generators of I. $\square$

Given a finite family $I_1, \ldots, I_m$ of ideals, we shall say that an automorphism s of K permutes them if the set $\{I_1, \ldots, I_m\}$ is identical to the set $\{sI_1, \ldots, sI_m\}$.

Lemma 16.19. *If $I_1, \ldots, I_m$ are prime ideals of $K[\ldots, X_i, \ldots]$, none of which is a subset of another, then there is a subfield k of K such that an automorphism s of K fixes every point of k if and only if it permutes $I_1, \ldots, I_m$.*

Proof. Note that if J is a prime ideal that contains $J_1 \cap \cdots \cap J_m$, then it contains one of the J_i: This is the ideal version of Gauss's lemma, stating that if a prime number divides a product, it divides one of the factors of the product! Indeed, if J does not contain some element P_1 of $J_1, \ldots$, some element P_m of J_m, then it does not contain their product, which is in $J_1 \cap \cdots \cap J_m$.

We can therefore see that the field of definition k of the ideal $I_1 \cap \cdots \cap I_m$ fits, since $I_1, \ldots, I_m$ are the minimal prime ideals containing this ideal. $\square$

We now need a little inroad into the notion of the "dimension" of an ideal or of an algebraic variety. This will all become clearer, and take a very general character, when we discuss "Morley ranks" in the next chapter.

We defined the *Cantor rank* of a point in a topological space in Section 1.3 and revisited it in Section 11.4. Here we note that for every *closed* set F in a compact space E we can define the Cantor rank of F as the maximum of the Cantor ranks of elements of F. This maximum does exist: This poses no problem if F has a point of rank ∞; otherwise, there is a least ordinal α such that F contains no point of rank greater than or equal to α. It is impossible for α to be a limit, since then F would contain a point p_β of rank at least β for every $\beta < \alpha$; by compactness they would accumulate to a point that would necessarily have rank at least α. So α is a successor, and its predecessor is the maximum of the ranks of points of F.

Note that, if the Cantor rank of F is $\alpha < \infty$, then F *contains only finitely many points of rank* α. Otherwise, those points would have an accumulation point that would have rank at least $\alpha + 1$.

All that is helpful for ω-stable theories:

Lemma 16.20. *If T is ω-stable, then every type p in $S_1(M)$ has a Cantor rank that is an ordinal (i.e., $\neq \infty$).*

Proof. If not, the set X of elements of $S_1(M)$ of Cantor rank ∞ form a Stone space with no isolated points. Every nonempty clopen set can be divided into two nonempty clopen sets. Since the clopen sets correspond to formulas, we can proceed by dichotomy, as in Lemma 13.13, to construct a denumerable set of parameters with 2^ω types. □

We are now equipped for the following theorem:

Theorem 16.21. *The theory of algebraically closed fields of any characteristic, and the theory of differentially closed fields of characteristic zero, (strongly) eliminate imaginaries.*

Proof. Let $f(\vec{x}, \vec{a})$ be a formula of the theory T of algebraically closed fields of a specified characteristic, which is ω-stable. We shall work in an ω-saturated ω-homogeneous model M of T containing A (under the circumstances, this is nothing other than an algebraically closed field of infinite transcendence degree over its prime field). As usual, consider the equivalence relation E, defined by the formula without parameters $(\forall \vec{x})(f(\vec{x}, \vec{y}) \leftrightarrow f(\vec{x}, \vec{z}))$. We consider the class $\vec{y} E \vec{a}$ of $\vec{a}$ modulo E, and take a type p of maximal Cantor rank in this formula. Since we have quantifier elimination, a type p is just a prime ideal I of $M[\vec{X}]$.

We shall say that two ideals I and J are conjugate if $I = sJ$ for some automorphism s of M. Note that if B is a basis of monomials modulo I, it is also one modulo sI. Consequently, if $I \subset sI$, then the unique expression of X^u as a function of B modulo sI is necessarily the same as that modulo

I: I contains a generating system of sI, so $I = sI$. In other words, two conjugate ideals are either equal or incomparable with respect to inclusion.

Since a conjugate of I has the same Cantor rank, the clopen set $\vec{y}E\vec{a}$ contains only finitely many conjugates $I_1, \ldots, I_m$ of our type I of maximal Cantor rank.

For an automorphism s, preserving the formula $f(\vec{x}, \vec{a})$ is the same as preserving the class of $\vec{a}$ modulo E: If an automorphism preserves this class, it permutes $I_1, \ldots, I_m$; if it permutes $I_1, \ldots, I_m$, then because E is definable without parameters, it permutes the classes of E without dissociating them (i.e., $\vec{x}E\vec{y}$ iff $s\vec{x}Es\vec{y}$), and it can only preserve the class common to $I_1, \ldots, I_m$.

Consequently, by Lemma 16.19, s preserves the formula $f(\vec{x}, \vec{a})$ if and only if it fixes every element of the field of definition k of the ideal $I_1 \cap \cdots \cap I_m$. Since M is ω-saturated and ω-homogeneous, that implies that k is contained in the rational closure of $\vec{a}$; since every n-type over $\vec{a}$, and therefore also every n-type over k, is realized in M, that means that the satisfaction of $f(\vec{x}, \vec{a})$ depends only on the type of $\vec{x}$ over k, that $f(\vec{x}, \vec{a})$ is equivalent to a formula $g(\vec{x}, \vec{b})$ with $\vec{b} \in k$. In fact, k is the rational closure of $\vec{b}$, since an automorphism s of M fixes $f(\vec{x}, \vec{a})$ iff it fixes $\vec{b}$.

For differential fields, replace ideals everywhere with differential ideals. $\square$

Remark. We used a very simple property of ω-stable theories, namely, that a formula contains only finitely many types of maximal Cantor rank. In the case of algebraically closed fields, the Cantor rank of a type $I \subset M[X_1, \ldots, X_n]$ is in fact the dimension, in the geometric sense, of the associated prime ideal, that is to say, the transcendence degree over M of the field of fractions of the ring $M[X]/I$; these types of maximal dimension correspond to prime ideals that are minimal among those satisfying the formula.

We could also have used a result that is more specific but that is completely banal for an algebraist, namely, that the ring $M[X_1, \ldots, X_n]$ is Noetherian; indeed, a consequence of that is that a formula contains finitely many (but not zero) prime ideals that are minimal in it.

By the Noetherian property, every prime ideal of the formula contains a minimal prime ideal of that formula; by quantifier elimination it follows that every formula f is the disjunction of finitely many formulas of the type $P_1(\vec{X}) = 0 \wedge \cdots \wedge P_k(\vec{X}) = 0 \wedge Q(\vec{X}) \neq 0$, and the minimal ideals of f are to be sought among those of each term of the disjunction. A prime ideal that is minimal among them that contains $P_1, \ldots, P_k$ and does not contain Q, provided that the formula is not contradictory, is a minimal prime ideal among those that contain $P_1, \ldots, P_k$. There are only finitely many of the latter: If there were infinitely many $I_0, I_1, \ldots, I_n, \ldots$, then the sequence $I_0, I_0 \cap I_1, \ldots, I_0 \cap \cdots \cap I_n, \ldots$ would constitute a strictly decreasing sequence of ideals.

If we wanted, we could then skip Cantor rank by appealing to a result from algebra that is not very hard. We could also do it for differential fields (of characteristic zero), although the ring $M[X_1, \ldots, X_n]_d$ does not satisfy the Noetherian condition for differential ideals; it satisfies it only for radical differential ideals, that is to say, for those which are intersections of prime differential ideals. That is sufficient to establish the result we want, but this theorem, attributed to Ritt–Radenbusch, is not so easy to prove.

16.6 A Galois Theory for Strong Types

If $E(x, y)$ is a finite equivalence relation definable over A, then an A-automorphism of a very homogeneous and saturated model M containing A induces a permutation on the classes $C_1, \ldots, C_n$ of E; let $G(E/A)$ denote the group of permutations thus obtained on the classes of E.

We can characterize the permutations σ of $C_1, \ldots, C_n$ that are in $G(E/A)$ as follows: We choose an element a_1 in $C_1, \ldots$, an element a_n in C_n; $\sigma \in C(E/A)$ iff we can find a'_1 in $\sigma C_1, \ldots, a'_n$ in σC_n such that $(a_1, \ldots, a_n)$ and $(a'_1, \ldots, a'_n)$ have the same type over A.

Similarly, the A-automorphisms of M operate on the strong types over A: We thus get a group, which we call $G(SF_1(A)/A)$. The action of an A-automorphism on the classes of E is determined by its action on the strong types, so that by restriction to the classes of E we get a surjective homomorphism from $G(SF_1(A)/A)$ onto $G(E/A)$, which we shall call canonical.

Furthermore, if the relation E is finer, i.e. has more classes, than the relation E', we also have a canonical restriction homomorphism $G(E/A) \to G(E'/A)$. If E is finer than E' which is in turn finer than E'', the following diagram commutes:

$$\begin{tikzcd} & G(E'/A) \arrow[dr] & \\ G(E/A) \arrow[ur] \arrow[rr] & & G(E''/A) \end{tikzcd}$$

The $G(E/A)$ form a directed system, since

$$\begin{tikzcd} & G(E/A) \\ G(E^{\wedge}F/A) \arrow[ur] \arrow[dr] & \\ & G(F/A) \end{tikzcd}$$

Since an element of $G(SF_1(A)/A)$ is completely determined by all its restrictions to the $G(E/A)$, $G(SF_1(A)/A)$ appears as a subgroup of the *projective limit* (if you do not know what that is, consult the nearest algebra textbook) of the directed system of the $G(E/A)$. By compactness, it is easy to see that in fact it consists of all of this projective limit.

Like every profinite group, $G(SF_1(A)/A)$ becomes a zero-dimensional compact group for the projective limit topology of the discrete topologies on the $G(E/A)$. The clopen subsets of this group are the sets of the following form: Consider an equivalence relation E, a subset X of $G(E/A)$, and the set of elements of $G(SF_1(A)/A)$ whose projection to $G(E/A)$ is in X. Its open subgroups are, by compactness, the same as its closed subgroups of finite index; they are the sets consisting of all elements that project into Γ, where Γ is some subgroup of some $G(E/A)$. Note that a closed group is the intersection of open groups.

Consider a finite equivalence relation E, whose classes we shall denote by $C_1, \ldots, C_n$, and a subgroup Γ of $G(E/A)$. Let r_Γ denote the n-ary relation defined by the following formula, which has parameters in M: $\bigvee_{\sigma \in \Gamma} x_1 \in \sigma C_1 \wedge \cdots \wedge x_n \in \sigma C_n$.

As this relation is a Boolean combination of relations of the form $y \in C_i$, it has only finitely many conjugates under A-automorphisms of M. The action on r_Γ of an automorphism depends only on its action on the classes of E, so it makes sense to speak of the action of an element of $G(SF_1(A)/A)$ on r_Γ, and we can see that *the subgroup of $G(SF_1(A)/A)$ consisting of the elements that preserve r_Γ is exactly the open subgroup defined by* Γ.

We could develop a "Galois theory" between the closed subgroups of $G(SF_1(A)/A)$ and some sets of relations of the same form as r_Γ; we shall do this only when we have elimination of imaginaries.

First, note that $G(SF_1(A)/A)$ acts on the algebraic closure A_{alg} of A, since an algebraic element is nothing but a certain sort of strong type. We thus obtain, by restriction, a surjective homomorphism, called canonical, from $G(SF_1(A)/A)$ onto $G(A_{\text{alg}}/A)$, where $G(A_{\text{alg}}/A)$ is the group of permutations of A_{alg} that are induced by that of the A-automorphisms of M (i.e., the group of permutations s of A_{alg} such that A_{alg} and sA_{alg} have the same type over A).

$G(A_{\text{alg}}/A)$ is itself equipped with a natural profinite group structure: Consider the subsets B of A_{alg} that are obtained by adding finitely many elements to A_{alg} and which contain, along with some element b, all conjugates of b under A-automorphisms (i.e., the elements with the same type over A as b; there are only finitely many). If we let $G(B/A)$ denote the set of permutations of B induced by the A-automorphisms of M (it is a particular $G(E/A)$), we can see that $G(A_{\text{alg}}/A)$ is the projective limit of the $G(B/A)$.

We can also see that the canonical surjection from $G(SF_1(A)/A)$ onto $G(A_{\text{alg}}/A)$ is continuous.

Lemma 16.22. *If T (strongly!) eliminates imaginaries, then the canonical surjective homomorphism from $G(SF_1(A)/A)$ onto $G(A_{\text{alg}}/A)$ is bijective and bicontinuous.*

Proof. Let σ be in the kernel of this homomorphism; it is induced by an A-automorphism s of M that is the identity on A_{alg}. Let C be a class of E and let $\vec{c}$ be the tuple of parameters associated with the formula $x \in C$ by elimination of imaginaries. As this formula has only finitely many conjugates under A-automorphisms of M, as does $\vec{c}$, $\vec{c}$ is algebraic over A, so it is fixed by s, as is C. Consequently σ is the identity of $G(SF_1(A)/A)$.

This bijection between compact sets is continuous and therefore bicontinuous. □

From this we come to Galois theory: We associate a set B, $A \subset B \subset A_{\text{alg}}$, with the group $G(A_{\text{alg}}/B)$ consisting of the elements of $G(A_{\text{alg}}/A)$ which fix every point of B; we associate a subgroup H of $G(A_{\text{alg}}/A)$ with the set B_H of its fixed points, i.e., of the elements of A_{alg} that are fixed by every element of H. This double correspondence is called the *Galois correspondence.*

Theorem 16.23. *If T (strongly) eliminates imaginaries, then the Galois correspondence establishes a bijection between the closed subgroups of $G(A_{\text{alg}}/A)$ and the rationally closed sets of parameters contained between A and A_{alg}.*

Proof. Since the stabilizer of a point b of A_{alg} is open, if $A \subset B \subset A_{\text{alg}}$, then the group $G(A_{\text{alg}}/B)$ is the intersection of open groups, and therefore closed. It is also clear that the fixed points of a subgroup of $G(A_{\text{alg}}/A)$ form a rationally closed set.

If b is not rational over B, then it can be moved by a B-automorphism, so B is exactly the set of fixed points of $G(A_{\text{alg}}/B)$.

It remains to be seen that if H is a closed subgroup of $G(A_{\text{alg}}/A)$, it is of the form $G(A_{\text{alg}}/B)$. But H is the intersection of open subgroups H_Γ, where H_Γ is the stabilizer of the relation r_Γ. Since r_Γ has only finitely many conjugates under A-automorphisms, it follows that the tuple $\vec{c}$ that is canonically associated with it by elimination of imaginaries is in A_{alg}; H_Γ is nothing other than $G(A_{\text{alg}}/A \cup \{\vec{c}\})$, and if B denotes the union of the $\vec{c}$, then $H = G(A_{\text{alg}}/B)$. □

In the case of the theory of algebraically closed fields, the algebraic closures are those known to algebraists, and it is easy to see that an irreducible polynomial that has only one root is of the form $x^{p^n} - a$, where p is the characteristic, so the rationally closed sets of parameters are the perfect fields, those that are closed under pth roots. We can therefore see that as a particular case of Theorem 16.23 we get the very classical theorem of Galois theory (of fields), which establishes a bijective correspondence between the perfect fields intermediate between K and K_{alg}, and the closed subgroups of $G(K_{\text{alg}}/K)$.

This result is not given here as one to be considered in parallel to Theorem 16.23, but rather as a particular case *that we have effectively proved.* For that we needed in Section 6.1 to establish quantifier elimination, that is to say the Hilbert Nullstellensatz, which we did by starting from very general considerations about ω-saturated models and some rudimentary lemmas on prime ideals of polynomials in *one* variable; then we needed to establish elimination of imaginaries in the last section, starting from general considerations about ω-stable theories and an easy lemma establishing the existence of the field of definition of an ideal. We can see that Galois theory relies on very general principles, and is not very closely tied to the context of field extensions of finite degree, in spite of what contemporary algebra textbooks might make us think. The method we used to approach it is in fact close to that of Galois, who did not speak of field extensions, nor even of fields (which had not yet been invented!), but rather used the method of "Lagrange resolvents" to reduce the preservation of a relation similar to r_Γ to the fixing of a parameter.

If in Theorem 16.23 elimination is obtained by the process of T^{eq}, we get a version of the *primitive element theorem*, that an open group is the stabilizer of some element. This theorem turns out to be true for fields, because of an appropriate algebraic situation, but cannot be deduced from Theorem 16.23 in its full generality.

16.7 Historic and Bibliographic Notes

Strong types, the finite equivalence relation theorem, and imaginary elements, are all from Shelah [She78]. Theorem 16.7, which is so important in applications of forking, was christened the "open mapping theorem" by Lascar. Elimination of imaginaries and their Galois theory are in [Poi83e]. The field of definition of an ideal, Lemma 16.18, is well known to geometers; it is a construction of André Weil.

17
Notions of Rank

Comme, dans les sociétés aristocratiques, tous les citoyens sont placés à poste fixe, les uns au-dessus des autres, il en résulte encore que chacun aperçoit toujours plus haut que lui un homme dont la protection lui est nécessaire, et plus bas il en découvre un autre dont il peut réclamer le concours. ... Dans les siècles démocratiques, au contraire, ... de nouvelles familles sortent sans cesse du néant, d'autres y retombent sans cesse, et toutes celles qui demeurent changent de face; la trame des temps se rompt à tout moment, et la vestige des générations s'efface ... L'aristocratie avait fait de tous les citoyens une longue chaîne qui remontait du paysan au roi; la démocratie brise la chaîne et met chaque anneau à part ...

A. de T.

17.1 Lascar Rank

In keeping with our custom, we are considering only the 1-types of a complete theory T, but the notions which we are going to introduce are equally meaningful for n-types, and have the same properties.The *rank in the sense of Lascar*, or simply the *rank*, is defined as a function R that sends every type p in $S_1(A)$ to an ordinal or to the symbol ∞, where A is a subset of a model M of T, such that the following four properties are satisfied:

1. *(heredity)* The rank of a father is greater than or equal to the ranks of its sons.
2. *(extension)* If A is a subset of B and p is in $S_1(A)$, then p has a son over B with the same rank.
3. *(isomorphism)* If s is an isomorphism from M onto N, if p is in $S_1(M)$, and if q is the type sp corresponding to p via s, then $R(p) = R(q)$.
4. *(bounded multiplicity)* If p is in $S_1(A)$ and $R(p) < \infty$, then there is a cardinal λ such that for any B containing A, p has no more than λ sons over B with the same rank.

If $R(p)$ is an ordinal, and not the symbol ∞, then p is said to be *ranked* by R.

Note that the extension condition can be replaced by the following: If $A \subset B$, $p \in S_1(A)$, and $R(p) \geq \alpha$, then p has a son q over B such that $R(q) \geq \alpha$.

Lemma 17.1. *If p is ranked by the rank R, then p is stable.*

Proof. If p is unstable, then since p is in $S_1(A)$, for every cardinal λ we can find B containing A over which p has at least λ pairwise distinct unstable sons (see the proof of Theorem 11.11). We can then show by induction on α, thanks to the bounded multiplicity condition, that $R(p) \geq \alpha$. □

Lemma 17.2. *If p is ranked by the rank R, then its sons with the same rank are exactly its nonforking sons.*

Proof. Let $A \subset B$, let p be in $S_1(A)$, and let q be a forking son of p over B. Let λ be a cardinal limiting the number of sons of p with the same rank. If we take a sufficiently saturated and homogeneous model M containing B, then we know (see Section 15.3) that all the forking sons of p will have more than λ conjugates under A-automorphisms of M, so that none can have the same rank as p. Since q must have a son of the same rank over M, $R(q)$ must be strictly less than $R(p)$.

Let us still consider a model M containing B that is sufficiently saturated and homogeneous, so that all nonforking sons of p are conjugate under A-automorphisms of M. Since p must have a son of the same rank, one of its nonforking sons must have the same rank as it; they all have the same rank, since they are conjugate. Therefore, if q in $S_1(B)$ is a nonforking son of p, it has a nonforking son r over M. By the heredity condition $R(p) \geq R(q) \geq R(r)$, and in fact we get equality. □

We shall insist on this slightly unexpected property: Whatever the notion R of rank may be, the sons with the same rank are always the same for the ranked types: They are the nonforking sons. We can see a posteriori that in Axiom 4, the cardinal λ can be taken to be equal to the multiplicity of the type in question; it would change nothing to take $\lambda = 2^{|T|}$ in the general case and $\lambda = 1$ if only types over models are considered.

Lemma 17.3. *Let R be a notion of rank, let p be in $S_1(M)$, and let q be in $S_1(N)$. If p and q are equivalent in the fundamental order, then $R(p) = R(q)$; if $p < q$ and $R(p) \neq \infty$, then $R(p) < R(q)$.*

Proof. If p and q are equivalent and one is unstable, we know that the other is too; in the stable case we know that, up to isomorphism, they have a common heir, so the ranks are equal.

If p is stable and $p < q$, we know that, up to isomorphism, there is an heir of p that is a son but not an heir of q. □

Likewise, we see that the rank of a stable type depends only on its bound. As the fundamental order has at most $2^{|T|}$ elements, there are only $2^{|T|}$ possible values for a notion R of rank, so that there is always a (least!) ordinal α_R such that $R(p) \geq \alpha_R$ if and only if $R(p) = \infty$.

A rank is said to be *jump-free* if the ordinal values of this rank R form an initial segment of the class of ordinals; in other words, if $R(p) \geq \alpha$ implies $R(p) \geq \alpha + 1$ for every p, then $R(p) \geq \alpha$ implies $R(p) = \infty$. For such a rank, α_R is less than or equal to $(2^{|T|})^+$. All natural ranks are jump-free, but this condition is not ensured by the axioms for ranks.

Recall the definition of *foundation rank* in a partial order: The expression $\mathrm{RF}(x) \geq \alpha$ is defined by induction on the ordinal α in the following way:

- for α limit, $\mathrm{RF}(x) \geq \alpha$ if $\mathrm{RF}(x) \geq \beta$ for every $\beta \leq \alpha$;
- $\mathrm{RF}(x) \geq \alpha + 1$ if we can find y strictly less than x with $\mathrm{RF}(y) \geq \alpha$.

We have no difficulty seeing that if there is an α such that $\mathrm{RF}(x) \geq \alpha$ but $\mathrm{RF}(x) \not\geq \alpha + 1$, this α is unique, and we set $\mathrm{RF}(x) = \alpha$; otherwise, $\mathrm{RF}(x) \geq \alpha$ for every α, and we set $\mathrm{RF}(x) = \infty$.

For example, $\mathrm{RF}(x) = 0$ means that x is minimal, $\mathrm{RF}(x) = 1$ means that x strictly bounds only minimal elements, etc. Saying that $\mathrm{RF}(x)$ is an ordinal is the same as saying that the order is *well-founded* below x, that is to say that there is no strictly decreasing ω-sequence below x.

Lemma 17.4. *Whatever the rank R and the stable type p may be, $R(p)$ is greater than or equal to the foundation rank of the bound of p in the fundamental order.*

Proof. Since the rank depends only on the bound, we can assume that p is over a model M. Let us show by induction that $\mathrm{RF}(p) \geq \alpha$ implies $R(p) \geq \alpha$. This is obvious for α a limit. If $\mathrm{RF}(p) \geq \alpha + 1$, then p has a forking son q such that $\mathrm{RF}(q) \geq \alpha$; by the inductive hypothesis $R(q) \geq \alpha$, and since $R(q)$ must be ∞ or strictly less than $R(p)$, $R(p) \geq \alpha + 1$. □

Now we shall define the smallest of all the ranks, the *Lascar rank*, which was christened the "rang U" by its inventor. We do this by induction as follows:

- for α limit, $\mathrm{RU}(p) \geq \alpha$ if $\mathrm{RU}(p) \geq \beta$ for all $\beta < \alpha$;

- if p is in $S_1(Q)$, then $\mathrm{RU}(p) \geq \alpha + 1$ if for every cardinal λ there exists B containing A over which p has at least λ distinct sons q with $\mathrm{RU}(q) \geq \alpha$.

We have no difficulty seeing, by induction on α, that if $\beta < \alpha$ and $\mathrm{RU}(p) \geq \alpha$, then $\mathrm{RU}(p) \geq \beta$. Consequently, the following two cases can happen:

- For some α, necessarily unique, $\mathrm{RU}(p) \geq \alpha, \mathrm{RU}(p) \not\geq \alpha + 1$, in which case we set $\mathrm{RU}(p) = \alpha$.

- $\mathrm{RU}(p) \geq \alpha$ for every α, and we set $\mathrm{RU}(p) = \infty$.

Theorem 17.5. RU *is a notion of rank; more precisely, if p is unstable, then* $\mathrm{RU}(p) = \infty$, *and if p is stable, then* $\mathrm{RU}(p)$ *is the foundation rank of the bound of p in the fundamental order. As a consequence, if* $\mathrm{RU}(p) \geq |T|^+$, *then* $\mathrm{RU}(p) = \infty$.

Proof. If p is unstable, then we have already noted that it has as many unstable sons as we want, and $\mathrm{RU}(p) = \infty$.

So let us assume that p is stable. If $\mathrm{RU}(p) \geq \alpha + 1$, then by taking λ to be a cardinal greater than the multiplicity of p, we can see that by the definition of RU, p has a forking son q with $\mathrm{RU}(q) \geq \alpha$. From this we can deduce without difficulty, by induction on α, that $\mathrm{RU}(p) \geq \alpha$ implies $\mathrm{RF}(\beta(p)) \geq \alpha$.

Now we show by induction that $\mathrm{RF}(\beta(p)) \geq \alpha$ implies $\mathrm{RU}(p) \geq \alpha$; there is no problem if α is a limit. Suppose that $\mathrm{RF}(\beta(p)) \geq \alpha + 1$. If p is over A, then for every cardinal λ we can find a model M containing A over which p has λ sons q such that $\mathrm{RF}(q) \geq \alpha$. By the induction hypothesis $\mathrm{RU}(q) \geq \alpha$; therefore, by definition of RU, $\mathrm{RU}(p) \geq \alpha + 1$.

The last point is not a direct consequence of Lemma 13.4, since the U rank of p is not the supremum of the foundation ranks of the chains that are below p in the fundamental order (given an arbitrary ordinal α, it is easy to build a partial order, every subchain of which is finite, with a maximum point of rank α).

We must proceed as follows: Assume that p is a type over a model of T and that $|T|^+ \leq \mathrm{RU}(p) < \infty$. Then for every ordinal $\alpha < |T|^+$, we can find a forking son $p_{1,\alpha}$ of p with U rank α, representing a formula $f_{1,\alpha}$ that is omitted by p; the same formula $f_1 = f_{1,\alpha}$ is used for α in a subset A_1 of $|T|^+$ of the same cardinality. Since A_1 is cofinal in $|T|^+$, every α is strictly less that a certain $\phi_1(\alpha)$ in A_1, so that $p_{1,\phi_1(\alpha)}$ has a forking son $p_{2,\alpha}$ of rank α. For α in a subset A_2 of $|T|^+$ of the same cardinality, this forking is witnessed by the representation of some formula f_2. Proceeding this way, we end up building infinite sequences of formulas $f_1, f_2, \ldots, f_n, \ldots$, of sets $A_1, A_2, \ldots, A_n, \ldots$, of functions $f_1, f_2, \ldots, f_n, \ldots$, and of types $p_{1,\alpha}, p_{2,\alpha}, \ldots, p_{n,\alpha}, \ldots$ in such a way that if $\alpha \in A_{n+1}$, then $p_{n+1,\alpha}$ represents the formula f_{n+1} that is omitted by its father $p_{n,\phi_n(\alpha)}$.

Add to the language of T a constant x and an infinite list of unary predicates $M_0, M_1, \ldots, M_n, \ldots$. Consider the set of axioms expressing that the M_n form an increasing sequence of elementary restrictions of the universe, which is a model of T, that the type of x over M_0 has the same bound as p, and that the type of x over M_n represents f_n and omits f_{n+1}. This is a consistent theory, since each finite fragment of it is; it allows us to build a decreasing ω-sequence in the fundamental order below p, contradicting the fact that $\mathrm{RU}(p) < \infty$. □

Remark. If we want to check that RU is indeed a rank directly from the definition, then properties (1), (3), and (4) pose no problem; it is easy enough to see that if p is over A, A is included in B, and $\mathrm{RU}(p) \geq n$, then p has a son q over B with $\mathrm{RU}(q) \geq n$, but we cannot see how the argument can carry over naturally at the limit stages. No direct proof is known of the extension property of the rank U, in all its generality, which does not appeal to the theory of forking.

The rank U is therefore the lowest of all ranks. It is the *only* rank with the following property: Whenever $\mathrm{RU}(p) = \alpha$ and $\beta < \alpha$, then p has a son q with $\mathrm{RU}(q) = \beta$.

A type is said to be *superstable* if it is ranked by at least one notion of rank, that is to say, if it is ranked by the rank U. A theory is called *superstable* if all its types are superstable, that is to say, if it is stable and its fundamental order is well-founded ($\kappa(T) = \omega$).

The types of U-rank zero are obviously the realized types; note that Lemma 13.3 states that in a stable theory there are always types of U-rank 1.

The notions of rank and of U rank are defined for n-types by analogy to the corresponding notions for 1-types. We have checked (Lemma 13.12) that for a stable theory, the fundamental orders of the 1-types and of the n-types have the same $\kappa(T)$, so that if T is superstable, the fundamental order of the n-types is also well-founded. We shall also see, in Section 19.2, how to calculate the RU of an n-type from the RU of 1-types.

These notions can obviously be extended to types in infinitely many variables, but that has little interest, since with the exception of the trivial case of a finite structure, the fundamental order of the ω-types is always ill-founded (introduce ω independent points and realize them one by one).

17.2 Shelah Rank

The rank U has an intrinsic definition and, as we shall see, interesting additivity properties. It is not, however, the only rank that can be associated with the types of a superstable theory in a natural way. The shortcoming of the rank U is that it allows us to rank only complete types, while we often want to rank incomplete types or formulas.

We shall say that the rank R is *continuous* if it has the following property: For every A and every α, the types of $S_1(A)$ with rank greater than or equal to to α form a closed set; equivalently, if p is an accumulation point, in $S_1(A)$, of types with ranks greater than or equal to α, then $R(p) \geq \alpha$.

If R is a continuous rank and if $f(x, \vec{a})$ is a formula with parameters in A, then there are types that satisfy this formula and that have maximal rank. To see this, let α be the least ordinal (possibly ∞) such that $\langle f(x, \vec{a}) \rangle$ does not contain types of rank at least α. Let us show that α cannot be a limit. Otherwise, for every $\beta < \alpha$ there is a p_β satisfying $f(x, \vec{a})$ with rank at least β. By compactness, these p_β accumulate at a type p that, by the continuity of R, has rank greater than or equal to α, contradicting the assumption.

We can therefore define the *rank of the formula* $f(x, \vec{a})$ to be the maximum of the ranks of the types that satisfy that formula. As the rank of a son is less than that of its father, note that if A is included in B, then the rank of $f(x, \vec{a})$ considered as a formula with parameters in A is the same as the rank of $f(x, \vec{a})$ considered as a formula with parameters in B. The maximum of all these ranks, which is the rank of the tautological formula $x = x$, will be denoted by $R(T)$.

In fact, we can define the rank of a *closed* subset of the space of types, that is to say, of an incomplete type.

The rank RC (C for "continuous") is defined by induction as follows:

- if α is a limit, then $\mathrm{RC}(p) \geq \alpha$ if $\mathrm{RC}(p) \geq \beta$ for every $\beta < \alpha$ (in particular, $\mathrm{RC}(p) \geq 0$ is always true);
- if p is in $S_1(A)$, then $\mathrm{RC}(p) \geq \alpha + 1$ if for every cardinal λ and every formula $f(x, \vec{a})$ satisfied by p, with $\vec{a} \in A$, I can find B containing A such that there are at least λ distinct types in $S_1(B)$ satisfying $f(x, \vec{a})$, all with $\mathrm{RC} \geq \alpha$.

We have no difficulty seeing, as usual, that if $\beta < \alpha$ and $\mathrm{RC}(p) \geq \alpha$, then $\mathrm{RC}(p) \geq \beta$. So we can get two cases:

- There is a unique α such that $\mathrm{RC}(p) \geq \alpha$ and $\mathrm{RC}(p) \not\geq \alpha + 1$; in that case, we set $\mathrm{RC}(p) = \alpha$.
- $\mathrm{RC}(p) \geq \alpha$ for every α, in which case we set $\mathrm{RC}(p) = \infty$.

Lemma 17.6. $\mathrm{RC}(p)$ *is a continuous rank.*

Proof. It is clear that RC is continuous from the formula-by-formula definition of RC.

We can easily see by induction that if q is a son of p and $RC(q) \geq \alpha$, then $RC(p) \geq \alpha$.

Suppose that p is in $S_1(A)$, with A included in B, and $\mathrm{RC}(p) \geq \alpha$. Let us show by induction on α that p has a son q over B with $\mathrm{RC}(q) \geq \alpha$. If α is a limit, then by the induction hypothesis, for every $\beta < \alpha$, p has a son q_β with $\mathrm{RC}(q_\beta) \geq \beta$. By compactness, these q_β accumulate at some q, and $\mathrm{RC}(q) \geq \alpha$ by the continuity of RC. If $\mathrm{RC}(p) \geq \alpha + 1$, we consider a formula $f(x, \vec{a})$ satisfied by p. For every λ, I can find C_λ containing A over which there are at least λ types, with RC at least α, satisfying $f(x, \vec{a})$. If B_λ contains both B and C_λ, then by the induction hypothesis these types all have sons over B_λ of rank at least α. Since λ is greater than $|S_1(B)|$, λ of them must have the same restriction q_λ to B. Moreover, it is the same q that is taken as q_λ for λ ranging over a cofinal subset of the class of all cardinals: For every λ, q has at least λ distinct sons with $\mathrm{RC} \geq \alpha$ and $RC(q) \geq \alpha + 1$.

Consequently, every $f(x, \vec{a})$ that is true for p is satisfied by some q in $S_1(B)$ with RC greater than or equal to $\alpha + 1$: That means that the closed set consisting of the types of $\langle f(x, \vec{a}) \rangle$ with $\mathrm{RC} \geq \alpha + 1$ is never empty. By compactness, the intersection of this family of compact sets is not empty, and p has a son over B with $\mathrm{RC} \geq \alpha + 1$.

The isomorphism property is clear, since an isomorphism between models induces a bicontinuous bijection between the spaces of types over those models.

As for the property of bounded multiplicity, it is an immediate consequence of the definition. □

Note that the proof of the extension property for RC actually gives us the following lemma:

Lemma 17.7. $\mathrm{RC}(p) \geq \alpha + 1$ *iff every formula* $f(x, \vec{a})$ *satisfied by* p *is satisfied by a type* q *that for all* λ *has at least* λ *sons of* $\mathrm{RC} \geq \alpha$.

This rank RC is often called the *Shelah rank* or *Shelah degree* (a very awkward term), since it is the favorite rank of that mathematician.

Lemma 17.8. *The rank* RC *is the smallest continuous rank.*

Proof. Let R be a continuous rank. Let us show by induction that if $\mathrm{RC}(p) \geq \alpha$, then $R(p) \geq \alpha$; there is no problem if α is a limit. If $\mathrm{RC}(p) \geq \alpha + 1$, then by Lemma 17.7 we can find a q in every neighborhood of p with as many sons as we want with RC greater than or equal to α. By the induction hypothesis, these sons have R greater than or equal to α, and by the bounded multiplicity property of R, $R(q) \geq \alpha + 1$. By continuity of R, $R(p) \geq \alpha + 1$. □

Remark. We can strengthen Lemma 17.7 by noting that $\mathrm{RC}(p) \geq \alpha + 1$ whenever, for every formula $f(x, \vec{a})$ satisfied by p, there is a type q with at most $2^{|T|}$ sons of RC at least α; one among them must be forking.

Likewise, if we return to the definition of RC, in order for $\mathrm{RC}(p) \geq \alpha + 1$ it is sufficient that for every formula $f(x, \vec{a})$ satisfied by p, we can find a set B extending A over which there are at most $2^{|T|}$ types satisfying $f(x, \vec{a})$. That is clear if A has cardinality at most $|T|$, since then the types over B satisfying $f(x, \vec{a})$ with rank at least α must include some that fork over A.

For the general case, replace p by its restriction to a set A' of cardinality at most $|T|$, over which p does not fork. Naturally, to see that this condition is sufficient, we did need to prove first that RC is indeed a rank!

Now we examine the connections between the ranks RU and RC. First, a corollary of Lemma 17.7:

Lemma 17.9. *For every finite ordinal n, $\mathrm{RC}(p) \geq n$ if and only if p is a cluster point of the set of types of $\mathrm{RU} \geq n$.*

Proof. If every neighborhood of p has a point q with $\mathrm{RU}(q) \geq n$, then $\mathrm{RC}(q) \geq n$, since $\mathrm{RC} \geq \mathrm{RU}$, and $\mathrm{RC}(p) \geq n$ by continuity.

The converse is obvious if p is a cluster point of unstable types, since then $\mathrm{RC}(p) = \infty$. So assume that p is isolated from the unstable types by a formula $f(x, \vec{a})$. We then have no difficulty checking that all the topological theorems on forking, in particular the open mapping theorem (16.07), which were proved on the hypothesis of the stability of T, remain valid if we consider only the types satisfying $f(x, \vec{a})$. We therefore show this converse by induction. Assume that $\mathrm{RC}(p) \geq n + 1$ and consider a formula $g(x, \vec{a}_1)$ satisfied by p. By Lemma 17.7 there is a type q in $\langle f(x, \vec{a}) \wedge g(x, \vec{a}_1)\rangle$ that has many sons as we want with $\mathrm{RC} \geq n$. Some of them are therefore forking: We can find B containing A and a forking son r of q over B with $\mathrm{RC}(r) \geq n$. By analogy with Lemma 16.5, the types in $S_1(B)$ that satisfy $f(x, \vec{a})$ and fork over A form an open set; consequently, there is a neighborhood $\langle f(x, \vec{a}) \wedge g(x, \vec{a}_1) \wedge h(x, \vec{b})\rangle$ of r consisting entirely of types that fork over A. By the induction hypothesis, this neighborhood contains a type r_1 of $\mathrm{RU} \geq n$; since r_1 forks, its restriction q_1 to A has $\mathrm{RU} \geq n + 1$. Since q_1 is in $\langle g(x, \vec{a}_1)\rangle$, the result is proven. □

We can therefore see that if T is a superstable theory in which the greatest $\mathrm{RU}(p)$ is n, then n will also be the maximum of the $\mathrm{RC}(p)$. But the argument by induction in the proof of Lemma 17.9 does not carry over the limit stages, and in fact we can construct superstable theories with all the $\mathrm{RU}(p)$ finite but with the RC as large as we want. They are limited only by $|T|^+$, which is a consequence of the following theorem:

Theorem 17.10. $\mathrm{RC}(p) = \infty$ *if and only if* $\mathrm{RC}(p) \geq |T|^+$ *if and only if p is a cluster point of types of infinite (∞)* RU. *As a consequence, if T is superstable,* $\mathrm{RC}(T) < |T|^+$.

Proof. It is sufficient to show that if $\mathrm{RC}(p) \geq |T|^+$, then there is a type in each neighborhood of p that is not ranked by the rank U. Let p be a type over M and let $f(x, \vec{a})$ be a formula satisfied by p. We can assume that f isolates p from the unstable types.

By induction on n I am going to define:

- a sequence of formulas $f_1(x, \vec{y}_1), \ldots, f_n(x, \vec{y}_n), \ldots$;
- a decreasing sequence $X_1, \ldots, X_n, \ldots$ of cofinal subsets of $|T|^+$;

- for every α in X_n, an n-tuple $M^\alpha_{1,n}, \ldots, M^\alpha_{n,n}$ of models of T with $M = M^\alpha_{0,n} \prec M^\alpha_{1,n} \prec \cdots \prec M^\alpha_{n,n}$;
- for every pair (i, α) with $i \leq n$ and $\alpha \in X_n$, a tuple of elements $\vec{a}^\alpha_{i,n}$ of elements of $M^\alpha_{i,n}$ such that the formula $f_i(x, \vec{a}^\alpha_{i,n})$ is not satisfied by any element of $M^\alpha_{i-1,n}$; moreover, for α in X_n, the formula $f(x, \vec{a}) \wedge f(x, \vec{a}^\alpha_{1,n}) \wedge \cdots \wedge f_n(x, \vec{a}^\alpha_{n,n})$ must have RC at least α.

Assuming that the first n stages have been defined, I proceed as follows for the $(n+1)$th: considering α in X_n, I define β as the least ordinal strictly greater than α that is in X_n, and I set $M^\alpha_{1,n+1} = M^{\beta\cdot}_{1,n}, \ldots, M^\alpha_{n,n+1} = M^\beta_{n,n}$ and $\vec{a}^\alpha_{1,n+1} = \vec{a}^\beta_{1,n}, \ldots, \vec{a}^\alpha_{n,n+1} = \vec{a}^\beta_{n,n}$. By hypothesis, the formula $f(x, \vec{a}) \wedge f_1(x, \vec{a}^\alpha_{1,n+1}) \wedge \cdots \wedge f_n(x, \vec{a}^\alpha_{n,n+1})$ has RC at least $\alpha+1$. It is therefore satisfied by a type over $M^\alpha_{n,n+1}$ with a forking son over an extension $M^\alpha_{n+1,n+1}$ of that model with RC at least α (Lemma 17.1). That type is isolated from $M^\alpha_{n,N+1}$ by a formula $f^\alpha_{n+1}(x, \vec{a}^\alpha_{n+1,n+1})$. A formula f_{n+1} is chosen over a cofinal subset X_{n+1} of X_n.

Having done that, by the compactness of the appropriate theory, we can construct an increasing sequence of models $M \prec M_1 \prec \cdots \prec M_n \prec \cdots$, with a tuple $\vec{a}_n$ in M_n, such that:

- for every n, no element of M_n satisfies $f_{n+1}(x, \vec{a}_{n+1})$;
- there is a type q over the union M_ω of the M_n that satisfies $f(x, \vec{a})$ as well as every $f_n(x, \vec{a}_n)$.

The restriction of q to M is not superstable and satisfies $f(x, \vec{a})$. $\square$

We can therefore see that if T is superstable, that is to say if RU ranks all types, then RC also ranks all types; to study such a theory, we therefore have the choice between the two ranks; let me recall the advantages and characteristics of each.

- The rank U is the only rank for which if $\mathrm{RU}(p) = \alpha > \beta$, then p has a son q with $\mathrm{RU}(q) = \beta$; it has additional additivity properties, which we shall see in Section 19.2, which in the good cases allow us to reduce the calculation of the RU of an n-type to a cascade of calculations of RU of 1-types.
- The rank C is continuous, so that we can speak of the RC of a formula, or more generally of a closed set of types, as the maximum of the RC of the types found in it; if T is superstable, then there are types of maximum RC; this maximum, called $\mathrm{RC}(T)$, is strictly less than $|T|^+$.
- The rank U is not continuous, and in general there are no types of maximum RU for a superstable theory; however, note that in such a theory all the RU are bounded above by $\mathrm{RC}(T) < |T|^+$.

In general, these two ranks do not coincide, and it is indeed necessary to distinguish them, and not carelessly to attribute the additivity properties to RC or the continuity properties to RU, properties that are not present in the general case.

17.3 Morley Rank

We shall say that a rank is *Cantorian* if it has the following property: Every type p that is an accumulation point of types of rank at least α itself has rank greater than or equal to $\alpha + 1$. Such a rank is obviously continuous.

We define the *Morley rank* by induction as follows:

- $\mathrm{RM}(a) \geq \alpha$, for α limit, if $\mathrm{RM}(p) \geq \beta$ for all $\beta < \alpha$;
- $\mathrm{RM}(p) \geq \alpha + 1$ if p has a son that is an accumulation point of types with RM greater than or equal to α.

The idea underlying this definition is clear: we would like to take the rank of p to be its Cantor rank in the topological space $S_1(A)$ (see Section 1.3). Unfortunately, that "rank" does not have the extension property, even if A is a model, so we do what we need to do to correct this defect.

Proposition 17.11. *The Morley rank is indeed a notion of rank; it is the smallest Cantorian rank; a type ranked by the Morley rank has finite multiplicity.*

Proof. The isomorphism property is clear from the definition, and it is easy to see by induction that the rank of a father is greater than those of its sons. If P is in $S_1(A)$, A is included in B, and $\mathrm{RM}(p) = \alpha$, then p cannot have infinitely many sons over B with the same RM; if it did, then by compactness they would have an accumulation point q, which would be a son of p with $\mathrm{RM}(q) \geq \alpha + 1$ by the definition of RM.

The extension property remains to be shown. Suppose we are given p in $S_1(A)$, $A \subset B$, and let us show by induction on α that if $\mathrm{RM}(p) \geq \alpha$, then p has a son q over B with $\mathrm{RM}(q) \geq \alpha$. If α is a limit, that is a consequence of the compactness of $S_1(B)$ and the continuity of RM. So suppose that $\mathrm{RM}(p) \geq \alpha + 1$. Let C be a set of parameters over which p has a son q that is an accumulation point of q_i, with $\mathrm{RM}(q_i) \geq \alpha$, and let D be a set of parameters containing B and C. By the induction hypothesis, each q_i has a son r_i over D with $\mathrm{RM}(r_i) \geq \alpha$, and by compactness these r_i have an accumulation point r that is a son of p. By the definition of RM, $\mathrm{RM}(r) \geq \alpha + 1$, and the restriction of r to B is the desired son of p.

RM is therefore indeed a rank, and it is clear from its definition and from the axioms for ranks that it bounds every Cantorian rank from below. □

By continuity, the *Morley rank of a formula* is defined as the maximum of the Morley ranks of the types that satisfy it. If $f(x, \vec{a})$ is ranked by this rank, $\mathrm{RM}(f(x, \vec{a})) = \alpha < \infty$, and if M is a model containing A, the *Morley degree* of this formula is defined as the number of types p in $S_1(M)$

of Morley rank α that satisfy it. This degree is necessarily finite; if it were not, these types would accumulate at a type of RM $\geq \alpha + 1$; it does not depend on the model M, since if N is an elementary extension of M, a type of $S_1(N)$ with RM $\geq \alpha$ that satisfies $f(x, \vec{a})$ must be the heir of its restriction to M. The Morley degree generalizes the notion of the degree of a polynomial equation: To say that $f(x, \vec{a})$ has Morley rank zero is to say that it is algebraic, that it is satisfied by only a finite number n of elements of M, and this number n is precisely its Morley degree.

The Morley degree is also the maximum number of pairwise contradictory formulas of RM α into which the formula $f(x, \vec{a})$ can be divided, so we could also define the Morley rank of a formula directly by induction as follows:

- RM$(f(x, \vec{a})) \geq \alpha$ limit if RM$(f(x, \vec{a})) \geq \beta$ for all $\beta < \alpha$ (in particular, RM$(f(x, \vec{a})) \geq 0$ for every consistent formula);
- RM$(f(x, \vec{a})) \geq \alpha + 1$ if for every integer n I can find B containing A, and formulas $g_1(x, \vec{b}_1), \ldots, g_n(x, \vec{b}_n)$, with parameters in B, all implying $f(x, \vec{a})$, all pairwise contradictory, and all with RM $\geq \alpha$.

We can thus find the Morley rank of a type by decreeing it to be the minimum of the Morley ranks of the formulas that it satisfies.

For a type, to be ranked by the Morley rank is a much stronger property than to be ranked by the rank U, or even the rank C. (In particular, p must have finite multiplicity.) A theory is said to be *totally transcendental* if all its 1-types are ranked by the Morley rank. Thus, RM(T) is defined as the maximum of these Morley ranks, which is also that of the formula $x = x$, and α_{RM} (also often called α_T) as the first ordinal that is not a rank, that is to say $\alpha_T = \mathrm{RM}(T) + 1$.

Proposition 17.12. *If T is totally transcendental, it is stable in λ for every λ greater than or equal to $|T|$. If T is denumerable, it is totally transcendental if and only if it is ω-stable.*

Proof. Let M be a model of T of cardinality λ and let p be a type in $S_1(M)$. If RM$(p) = \alpha$, then by the definition of RM there is a formula $f(x, \vec{a})$ satisfied by p that isolates it from the other types with RM greater than or equal to α. The link between p and $f(x, \vec{a})$ is as follows: p is the unique type of maximum rank in the formula $f(x, \vec{a})$. Consequently, there cannot be more types over M than there are formulas with parameters in M; hence stability in λ.

If T is denumerable and totally transcendental, it is ω-stable. If it is not totally transcendental, we can see that, by the definition of RM, every formula of RM ∞ can be divided into two contradictory formulas of RM ∞ (because of the existence of the ordinal α_{RM}: If RM$(f) \geq \alpha_{\mathrm{RM}}$, then RM$(f) \geq \alpha_{\mathrm{RM}} + 1$!) That allows us to construct 2^ω types with only ω parameters, as at the end of Section 13.2. □

Proposition 17.13. *If* $\mathrm{RM}(p) \geq |T|^+$, *then* $\mathrm{RM}(p) = \infty$. *Consequently, if* T *is totally transcendental, then* $\mathrm{RM}(T) < |T|^+$.

Proof. The Morley rank is "jump-free," and there are no more Morley ranks than there are formulas $f(x, \vec{a})$. For $f(x, \vec{y})$, there are only $|T|$ possibilities, the same as for the type of $\vec{a}$ over $\emptyset$, by stability in $|T|$. □

Lemma 17.14. *If* T *is totally transcendental, then the fundamental order of* T *has at most* $|T|$ *elements.*

Proof. T, being stable in $|T|$, has a saturated model M of that cardinality. Every class of type is represented in $S_1(M)$: indeed, if p is in $S_1(N)$, it is a nonforking extension of its restriction p_1 to an elementary submodel N_1 of N with $|N_1| \leq |T|$. We can embed N_1 elementarily in M, then take the heir q over M of the type corresponding to p_1: p and q are equivalent. The conclusion follows by stability in $|T|$.

We can also note that if $f(x, \vec{a})$ isolates p among the types of the same Morley rank, then specifying $f(x, \vec{a})$, that is to say specifying $f(x, \vec{y})$ and the type of $\vec{a}$ over $\emptyset$, determines the class of p in the fundamental order. □

A very simple use of Morley rank proves the following altogether fundamental theorem:

Theorem 17.15. *If* T *is totally transcendental, then for every set* A *of parameters the isolated types form a dense set in* $S_1(A)$.

Proof. Let $f(x, \vec{a})$ be a consistent formula with parameters in A, and let p be a type of minimal Morley rank α satisfying this formula. Let $g(x, \vec{b})$ be a formula with parameters in A isolating p among the types of RM α: the formula $f(x, \vec{a}) \wedge g(x, \vec{b})$ isolates p in $S_1(A)$. □

Corollary 17.16. *If* T *is totally transcendental, then it has a prime model over every set* A *of parameters.*

Proof. A consequence of Theorem 17.15. Let M be a model containing A. Construct a sequence a_α of distinct elements of M, with the type of a_α over $A_\alpha = A \cup \{\ldots, a_\beta, \ldots\}_{\beta<\alpha}$ isolated, continuing as far as possible: The procedure stops because it is all done inside M. When we are done, by the density of the isolated types, the set obtained is a model, and an elementary submodel of M. (We used this argument already, in Section 6.2, in regard to differentially closed fields.) □

Proposition 17.17. *If* M *is* ω*-saturated and* p *is in* $S_1(M)$, *then the Morley rank of* p *is its Cantor rank in the topological space* $S_1(M)$

Proof. Instead of reasoning about types, we are going to reason about formulas, which define clopen sets in $S_1(M)$; as a Cantor rank, we give each of them the maximum of the Cantor ranks of the points which it contains.

It is sufficient to show by induction that if $\mathrm{RM}(f(x,\vec{a})) \geq \alpha$ implies $\mathrm{RCantor}(f(x,\vec{a})) \geq \alpha$. This poses no problems if α is a limit. If $\mathrm{RM}(f(x,\vec{a})) \geq \alpha + 1$, then for every n I can find pairwise contradictory formulas $g_1(x,\vec{b}_1),\ldots,g_n(x,\vec{b}_n)$, with parameters in an elementary extension of M, each one implying $f(x,\vec{a})$ and with RM at least α. Since M is ω-saturated, I can have $\vec{a}_1{}^\frown\cdots{}^\frown\cdots{}^\frown\vec{a}_n$ realize in it the type of $\vec{b}_1{}^\frown\cdots{}^\frown\cdots{}^\frown\vec{b}_n$ over $\vec{a}$, and I get formulas $g_1(x,\vec{a}_1),\ldots,g_n(x,\vec{a}_n)$, with parameters in M, that have the same properties. By the induction hypothesis, the Cantor rank of the clopen set defined by each of them is at least α. Since this constitutes a partition of the clopen set defined by $f(x,\vec{a})$, and since n is arbitrarily large, $\mathrm{RCantor}(f(x,\vec{a})) \geq \alpha + 1$. □

Corollary 17.18. *T is totally transcendental if and only if no $S_1(A)$ contains a perfect (closed, nonempty, without isolated points) subset.*

Proof. Indeed, this condition means that every point of $S_1(A)$ is ranked by the Cantor rank. □

Corollary 17.19. *If T is totally transcendental, then for every n,* RM *ranks all the n-tuples.*

Proof. If T is denumerable, then total transcendence is equivalent to ω-stability, and ω-stability for n-types is equivalent to ω-stability for 1-types.

In the general case, if RM does not rank all types, we construct a binary tree of formulas in n variables, all of RM ∞, each one divided in two at the stage after it appears. Since this tree uses only denumerably many formulas, we get a reduct T' of T to a denumerable sublanguage that is not ω-stable, and therefore not totally transcendental, preventing T from being totally transcendental. □

We could also stay in topology and show, without bringing ω-stability into play, that the existence of a perfect set in $S_n(A)$ ensures the existence of a perfect set in some $S_1(B)$.

One easy consequence of Corollary 17.19, whose details are left to the reader, is that *a theory interpretable in a totally transcendental theory is itself totally transcendental*; likewise, for a similar reason, *a theory interpretable in a superstable theory is itself superstable.* This is all reminiscent of Section 13.3, Example 15.

In conclusion, we have three ranks, $\mathrm{RU}(p) \leq \mathrm{RC}(p) \leq \mathrm{RM}(p)$ at our disposal for the study of the types of a totally transcendental theory; in some cases, there are other ranks, continuous or otherwise, Cantorian or otherwise, which can be introduced naturally in the context. For example, in the case of differentially closed fields (Section 6.2) the rank RD, which is defined only for 1-types, is a Cantorian rank which we can prove strictly greater than the Morley rank for some types.

We have discussed the respective advantages of RU and RC. Those of RM are obvious: It allows us to treat properties of types "formula by formula," and when we study a type p in a totally transcendental theory we very often start by choosing a formula that isolates types of the same Morley rank. However, we shall take care not to confuse it with either RC or RU, and we shall not unduly attribute the additivity properties of RU to it.

In some cases, however, these three ranks are equal:

Theorem 17.20. *If T is a denumerable, ω-categorical, superstable theory, then it is totally transcendental, and for every type p,* $\mathrm{RU}(p) = \mathrm{RC}(p) = \mathrm{RM}(p)$.

Proof. Let p be a type over a model M of T. Since T is superstable, we can find $\vec{a}$ in M such that p is a nonforking extension of its restriction p_1 to $\vec{a}$. By the ω-categoricity criterion (Theorem 10.11), there are only finitely many finite equivalence relations definable with parameters in $\vec{a}$, so there are only finitely many strong types over $\vec{a}$, and they are all realized in M. So let b realize the strong type of p over $\vec{a}$. By the finite equivalence relation theorem, p is the unique nonforking extension of its restriction p_2 to $\vec{a}\frown b$; by ω-categoricity, p_2 is isolated by a formula $f(x, \vec{a}\frown b)$. When we go up to M, we can see that this formula, which guarantees sons of p_2, isolates p among the types of the same RU.

We can therefore deduce, by an easy induction, that RU is a Cantorian rank, and therefore that RU = RM. □

17.4 Local Ranks

Ranks obviously allow us to study only superstable theories, for which they are defined. Here, we introduce "formula by formula" ranks, which allow us to apprehend stable theories. These are not precisely ranks (they do not satisfy the bounded multiplicity property, and the second does not even have the extension property), but rather approximations to notions of rank.

In the following two definitions of induction we introduce incomplete types; this is an absolute necessity for defining the dichotomy rank.

Given a formula $f(x, \vec{y})$ and two incomplete types π_1 and π_2 with parameters in A, we shall say that π_1 and π_2 are *explicitly f-contradictory* if one implies $f(x, \vec{a})$ and the other implies $\neg f(x, \vec{a})$ for some $\vec{a}$. A minor subtlety: Such $\vec{a}$ might not be in A, but can in any event be found in any model containing A.

We shall say that π_1 and π_2 are *implicitly f-contradictory* if they cannot be extended to complete types over a model M containing A with the same f-type. By compactness, that means that there are $\vec{a}_1, \ldots, \vec{a}_n$ in M such that

$$\pi_1(x_1) \wedge \pi_2(x_2) \text{ implies } \bigvee f(x_1, \vec{a}_i) \leftrightarrow \neg f(x_2, \vec{a}_i)$$

We define the *infinite f-branching rank* by induction as follows:

- For limit α, $R(\pi, f, \omega) \geq \alpha$ if $r(\pi, f, \omega) \geq \beta$ for every $\beta < \alpha$ (and $R(\pi, f, \omega) \geq 0$ when π is consistent);
- $R(\pi, f, \omega) \geq \alpha + 1$ if for all n we can find extensions $\pi_1, \ldots, \pi_n$ of π whose parameters are all in a set extending that of the parameters of π, all of them have $R(\quad , f, \omega) \geq \alpha$, and that are pairwise explicitly f-contradictory.

Likewise, we introduce the *f-twofold branching rank*, or *f-dichotomy rank* in the following way:

- For limit α, $R(\pi, f, 2) \geq \alpha$ if $R(\pi, f, 2) \geq \beta$. for all $\beta < \alpha$;
•$R(\pi, f, \alpha + 1) \geq \alpha + 1$ if π has two explicitly f-contradictory extensions π_1 and π_2 with $R(\quad , f, 2) \geq \alpha$.

As usual, $R(\pi, f, \omega) = \alpha$ if $R(\pi, f, \omega) \geq \alpha$ and $R(\pi, f, \omega) \not\geq \alpha + 1$; $R(\pi, f, \omega) = \infty$ if $R(\pi, f, \omega) \geq \alpha$ for all α. The same is true for $R(\pi, f, 2)$.

Now we introduce two kinds of trees of formulas adapted for these ranks.

The f-dichotomy tree of height n is the following set of formulas: We introduce a type variable x_s for each s in 2^n, which is—remember this—the set of functions from $\{0, \ldots, n-1\}$ to $\{0, 1\}$, and a tuple of parameter variables $\vec{y}_u$ for every u in $2^{<n} = 2^0 \cup \cdots \cup 2^{n-1}$. Whenever s is an extension of $u\frown 0$, we include the formula $f(x_s, \vec{y}_s)$, and whenever s is an extension of $u\frown 1$, we include the formula $\neg f(x_s, \vec{y}_u)$. The dichotomy tree of height ω is defined analogously, except that this time the s are indexed by the s of 2^ω and the y by the u of $2^{<\omega}$.

For the infinite f-branching tree of height n, we need to introduce x_s for s in ω^n, and $\vec{y}_{u,i,j}$ where u is in $\omega^{<n}$ and i and j are natural numbers with $i < j$. This is the set of formulas $f(x_s, \vec{y}_{u,i,j}) \leftrightarrow \neg f(x_t, \vec{y}_{u,i,j})$, where s is an extension of $u\frown i$ and t an extension of $u\frown j$. For the infinite f-branching tree of height ω, the s range over ω^ω and the u over $\omega^{<\omega}$.

We shall say that such a tree is *consistent with* π if by adding to the tree the fact that all the x_s satisfy π we get a consistent set of formulas.

Lemma 17.21. *$R(\pi, f, 2) \geq n$ iff π is consistent with the f-dichotomy tree of height n; $R(\pi, f, \omega) \geq n$ iff π is consistent with the infinite f-branching tree of height n.*

Proof. This is more or less obvious: The trees were constructed for this purpose. □

Lemma 17.22. *$R(\pi, f, 2) \geq \omega$ iff $R(\pi, f, 2) = \infty$ iff $R(\pi, f, \omega) \geq \omega$ iff $R(\pi, f, \omega) = \infty$ iff π is consistent with the f-dichotomy tree of height ω iff π is consistent with the infinite f-branching tree of height ω.*

Proof. Since every infinite fragment of the dichotomy tree of height ω can be interpreted in a dichotomy tree of finite height, if $R(\pi, f, 2) \geq \omega$, then π is consistent with the dichotomy tree of height ω. We can then easily show, by induction on α, that $R(\pi, f, 2) \geq \alpha$. Indeed, π splits into two explicitly

f-contradictory (incomplete) types, each with a dichotomy tree of height ω above it. We can also see that $R(\pi, f, \omega) = \infty$, since by climbing this tree to level n, we split π into 2^n types, each with an infinite dichotomy tree above it; that does imply that the infinite branching tree of height ω, each of whose finite fragments can be interpreted in an infinite branching tree of finite height, is consistent with π. (We can also note that every finite fragment of the infinite branching tree of height ω is interpretable in the dichotomy tree of height ω.) The rest follows from the obvious fact that $R(\pi, f, \omega) \leq R(\pi, f, 2)$. □

Remarks.

1. In the definition of $R(\pi, f, \omega)$, we could have set $R(\pi, f, \omega) \geq \alpha + 1$ if for every λ, π can be split into λ pairwise explicitly f-contradictory extensions with $R(\quad, f, \omega) \geq \alpha$. Indeed, by compactness, the consistency with π of a tree that branches λ times is equivalent to that of a tree that branches ω times.
   If we wish: $R(\pi, f, \omega) = R(\pi, f, \lambda) = R(\pi, f, \infty)$.

2. The complete types of $R(\pi, f, 2) \geq n, R(\pi, f, 2) = \infty, R(\pi, f, \omega) \geq n, R(\pi, f, \omega) = \infty$ always form closed sets, since these expressions only express the consistency of sets of formulas.

3. For the same reason, if p and q are complete types over models of T, and if $p \geq q$ in the fundamental order, then $R(p, f, 2) \geq R(q, f, 2), R(p, f, \omega) \geq R(q, f, \omega)$, since every finite fragment of p is interpreted in q.

4. If $R(\pi, f, 2) = n$, then there is a finite fragment π_1 of π, i.e., a formula that is a consequence of π, such that $R(\pi, f, 2) = n$; indeed, if π is not consistent with the dichotomy tree of height $n+1$, this is because of one of its finite fragments. The same is true of $R(\pi, f, \omega)$.

Lemma 17.23. *If π is an incomplete type with parameters in A, we can complete it as a type p in $S_1(A)$ with $R(p, f, \omega) = R(\pi, f, \omega)$; if p is in $S_1(A)$ and A is contained in B, then p has a son over B with the same $R(\quad, f, \omega)$.*

Proof. As the only question is of consistency, it is sufficient to show that for every π and every formula $g(x, \vec{a})$, if $R(\pi, f, \omega) \geq n$, then $R(\pi \wedge g(x, \vec{a}), f, \omega) \geq n$ or $R(\pi \wedge \neg g(x, \vec{a}), f, \omega) \geq n$. That can be done through an easy induction on n, by noting that if we split arbitrarily large finite sets into two parts, one of the two parts will always be arbitrarily large.

For the second claim, consider p as an incomplete type over B. □

One consequence of the preceding lemma is that we could have stayed within the complete types to define their $R(\quad, f, \omega)$: $R(p, f, \omega) \geq \alpha + 1$

iff for every n, p has at least n pairwise f-contradictory (implicitly or explicitly!) sons with $R(\ \ ,f,\omega) \geq \alpha$.

It is, however, more practical to do as we did, and *in the case of the dichotomy rank*, which does not satisfy Lemma 17.23, *there is no other way.*

Lemma 17.24. *If p is in $S_1(A)$, then p is stable if and only if for every formula f, $R(p,f,2) < \infty$ if and only if for every formula f, $R(p,f,\omega) < \infty$.*

Proof. Lemma 17.22 states the equivalence of assertions about the two ranks. If $R(p,f,2) = \infty$, then p is consistent with the dichotomy tree of height ω. For every cardinal λ it is also consistent with the f-dichotomy tree of height μ, where μ is the smallest cardinal such that 2^μ is a strictly larger cardinal than λ: This tree allows us to construct 2^μ sons of p with only λ parameters. We therefore have instability for the sons of p in every λ, that is to say instability for p.

If $R(p,f,2) = n$, let $g(x,\vec{b})$ be a finite fragment of p of the same $R(\ \ ,f,2)$. If p is in $S_1(A)$ and $\vec{a}$ is in A, then only one of $g(x,\vec{b}) \wedge f(x,\vec{a})$ and $g(x,\vec{b}) \wedge \neg f(x,\vec{a})$ has dichotomy rank n, and that is the one satisfied by p! However, a formula can express that $g(x,\vec{b}) \wedge f(x,\vec{a})$ has $R(\ \ ,f,2)$ greater than or equal to n: Existentially quantify everything that appears in the dichotomy tree of height n, which is finite. This formula $h(\vec{a},\vec{b})$ is therefore an f-definition for p. Moreover, every son of p is also definable for f, since it also has finite $R(\ \ ,f,2)$.

If that happens for every formula, then every son of p is definable, which means that p is stable. □

Proposition 17.25. *If p is stable and if q is a son of p, then q is a nonforking son of p if and only if for every formula f, $R(p,f,\omega) = R(q,f,\omega)$.*

Proof. If p is in $S_1(A)$ and q is in $S_1(B)$, let M be a model containing B and let q_1 be a nonforking son of q over M. Let r be a son of p over M, with the same $R(\ \ ,f,\omega)$, which does exist by Lemma 17.23. If q is a nonforking son of p, then by the theorem of the bound, q_1 is greater than or equal to r in the fundamental order, every finite fragment of q_1 can be interpreted in r, and, as we have noted, $R(r,f,\omega) \leq R(q,f,\omega)$. Therefore, we actually have $R(q_1,f,\omega) = R(q,f,\omega) = R(p,f,\omega)$.

Now suppose that q is a forking son of p. We just saw that the $R(\ \ ,f,\omega)$ of a stable type depends only on its bound, so that we can assume that A and B are models of T, which we can call M and N. Let $f(x,\vec{a})$ be a formula, with parameters in N, such that the heir of p satisfies $f(x,\vec{a})$, while $q \models \neg f(x,\vec{a})$. Let b realize q and consider the Morley sequence $\vec{a}_0{}^\frown\vec{b}_0, \vec{a}_1{}^\frown\vec{b}_1, \ldots, \vec{a}_n{}^\frown\vec{b}_n, \ldots$ associated with a special son of the type of $\vec{a}^\frown\vec{b}$ over M. Let q_n be the type of b_n over $M \cup \{\vec{a}_0, \ldots, \vec{a}_n, \ldots,\}$. By definition

$q_n \models \neg f(x, a_n)$, while if $m \neq n$, $q_m \models f(x, \vec{a}_n)$, since the restriction of q_m to $M \cup \{\vec{a}_n\}$ is the heir of p. These types are therefore pairwise f-distinct. Furthermore, q_n is a nonforking extension of its restriction to $M \cup \{\vec{a}_n\}$, which is isomorphic to, and therefore has the same local rank as, the restriction of q to $M \cup \{\vec{a}\}$. We have $R(q_n, f, \omega) = R(q, f, \omega)$, so the latter is strictly less that $R(p, f, \omega)$. $\square$

We can therefore see that it is possible to define not only stability, but also forking, in terms of the local rank $R(p, f, \omega)$. It is a painful approach, much more laborious and much less conceptual than that of the fundamental order: Just try to prove the existence of nonforking extensions of a stable type p (i.e., of sons q of p with $R(p, f, \omega) = R(q, f, \omega)$ for *every* formula f) directly, or else the duality for forking!

One last word: For a finite set Δ of formulas $\{f_1(x, \vec{y}_1), \ldots, f_n(x, \vec{y}_n)\}$, explicit Δ-contradiction means that for some index i, one of the types gives $f_i(x, \vec{a})$ and the other gives $\neg f_i(x, \vec{a})$ for some $\vec{a}$. We can define the local ranks $R(\pi, \Delta, \omega)$ and $R(\pi, \Delta, 2)$ analogously and show, by means of a little fixing up, that they have the same properties as when the set Δ consists of only one formula.

Indeed, at least when the theory T is that of an infinite structure, this generalization is only apparent, since every finite set Δ of formulas can be associated with one formula f such that Δ-contradiction is the same as f-contradiction, so that the local ranks associated with Δ are the same as those associated with f. For example, if $\Delta = \{f_1(x, \vec{y}_1), f_2(x, \vec{y}_2)\}$, we take f to be the following formula $f(x, \vec{y}_1, \vec{y}_2, z, z_1, z_2)$:

$$(z = z_1 \leftrightarrow f_1(x, \vec{y}_1)) \wedge (z = z_2 \leftrightarrow f_2(x, \vec{y}_2)).$$

17.5 Historic and Bibliographic Notes

The credit for having disengaged the conditions on ranks, so that the notion of a son of the same rank is independent of the rank chosen, and of having defined the smallest of all ranks, goes to Lascar, [Las73], [Las76]. The Shelah rank does come from Shelah, as do the local ranks, to which he gives a central role in his treatment of forking [She71b], [She78]. The connections between the Lascar rank and the Shelah rank, Lemma 17.9 and Theorem 17.10, were established in [Poi77].

The Morley rank is an essential tool of Morley's original work, [Mor65]; we had to wait until [Lac72] to know how to prove that a complete type over a model has only one son of the same Morley rank, because it was so painful for logicians to conceive of forking free from any reference to rank.

For the ranks in differential fields, see [Poi78a].

With respect to Theorem 17.20, no example of an ω-categorical, stable, nonsuperstable theory is known.[1] The structure of ω-categorical superstable theories was elucidated in the magisterial study of [GCL85], which was inspired by earlier work of Boris Zil'ber, and which represented one of the major successes of model theory in recent years. These theories all have finite fundamental orders, and are not finitely axiomatizable.

[1] **Note (1999):** Hrushovski has constructed such an example.

18
Stability and Prime Models

I talk to the young folks,
They don't understand
The things this old man has to say
Oh I wish I was eighteen again!

J.L.L.

18.1 Uniqueness Theorem

Proposition 18.1. *If T is stable and $\kappa(T) \leq \omega_1$, then for every set A of parameters, if B is constructed over A, then every subset of B containing A is also constructed over A.*

Proof. Let $\ldots, a_\alpha, \ldots$ be a construction of B over A. We know that there is a notion of closure corresponding to it (see Section 10.4). Given a subset C of B containing A, I define an increasing sequence B_α of subsets of B containing A by induction on α such that:

- B_α is closed and contains all a_β for $\beta < \alpha$;
- if we set $C_\alpha = B_\alpha \cap C$, then the type of B_α over C does not fork over C_α;
- for every α, $C_{\alpha+1} - C_\alpha$ is denumerable.

For that purpose I set $B_0 = A$ and at limit stages I set $B_\alpha = \cup B_\beta$ for $\beta < \alpha$. The nonforking condition is satisfied, since for every tuple $\vec{b}$ from B_α the type of $\vec{b}$ over C does not fork over C_α.

Now I define $B_{\alpha+1}$ assuming that B_α has already been defined. As $\kappa(T) \leq \omega_1$, I can find a denumerable subset D^1 of C such that the type of a_α over $B_\alpha \cup C$ does not fork over $B_\alpha \cup D^1$; I set $B^1_\alpha = B_\alpha \cup \{a_\alpha\} \cup D^1$. The type of B_α over C does not fork over $C_\alpha \cup D^1$, and the type of a_α over $B_\alpha \cup C$ does not fork over $C_\alpha \cup D^1 \cup B_\alpha$, so the type of $B_\alpha \cup \{a_\alpha\}$ over C does not fork over $C_\alpha \cup D^1$, and even the type of $B_\alpha \cup \{a_\alpha\} \cup D^1$ over C does not fork over $C_\alpha \cup D^1$. Letting $C^1_\alpha = B^1_\alpha \cap C$, we see that the type of B^1_α over C does not fork over C^1_α.

The problem is that B^1_α is not closed. However, it is obtained by adding denumerably many points to B_α, which is closed, so to close it, it is sufficient to add denumerably many small packages, or equivalently an ω-tuple A^2. We then add an ω-tuple D^2 of elements of C such that the type of A^2 over $B^1_\alpha \cup C$ does not fork over $B^1_\alpha \cup D^2$. We thus obtain B^2_α, which is no longer closed, but whose type over C does not fork over $C^2_\alpha = B^2_\alpha \cap C$. Repeat ω times, and set $B_{\alpha+1} = \cup B^n_\alpha$, which satisfies the conditions required.

Having done that, note that $B = \cup B_\alpha, C = \cup C_\alpha$. I claim therefore that for every α, C is atomic over C_α. Indeed, let $\vec{c}$ be in C. Since B_α is closed, the type of $\vec{c}$ over B_α is isolated. The type of B_α over C does not fork over C_α, so the type of B_α over $C_\alpha \cup \{\vec{c}\}$ does not fork over C_α, and by symmetry the type of $\vec{c}$ over B_α does not fork over C_α. By the open mapping theorem (16.17 and 16.18), the type of $\vec{c}$ over C_α is also isolated.

It is therefore clear that we get a construction of C by putting enumerations of the $C_{\alpha+1} - C_\alpha$ with type ω end to end. □

Corollary 18.2. *If A is a set of parameters of a stable theory T with $\kappa(T) \leq \omega_1$ (for example if T is denumerable or superstable), and if there is a model constructed over A, then it is the unique prime model over A up to isomorphism.*

Proof. Let M be the model constructed over A. If N is prime over A, it embeds elementarily into M and, by Proposition 18.1, it is constructed over A. The result follows from the uniqueness of the constructed model (Theorem 10.18). □

The simplest of theories T with $\kappa(T) > \omega_1$, that of ω_1 nested equivalence relations, provides a counterexample to Corollary 18.2. The language of T consists of a binary relation symbol E_α for every denumerable ordinal α; its axioms say that every E_α is an equivalence relation with infinitely many infinite classes, and that whenever $\alpha < \beta$, E_β is finer than E_α, with every equivalence class modulo E_α divided into infinitely many classes modulo E_β.

If we define the distance between two distinct elements a and b of a model M of T to be the smallest ordinal α such that a and b are not congruent modulo E_α, we can see that a model of T is the same as a rich I-valued ultrametric space, where I is the inverse chain of $\omega_1 + 2$ (the minimal element corresponds to zero distance). The result follows from the existence of a constructed I-valued space that is not the unique prime I-valued space (Lemma 10.21, Theorem 10.26).

18.2 Prime Models of a Totally Transcendental Theory

If T is totally transcendental, then by Corollary 17.16, for every A there is a prime model over A that is unique by Corollary 18.2. Theorem 18.5 below is in fact another proof of the uniqueness of the prime model in this case.

Lemma 18.3. *(T totally transcendental.) The prime model over A contains no nondenumerable indiscernible set over A.*

Proof. Consider a sequence $\ldots, a_\alpha, \ldots$, of length ω_1, consisting of elements of this model M, that is indiscernible over A. As usual, let us write p_α for the type of a_α over $A_\alpha = A \cup \{\ldots, a_\beta, \ldots\}_{\beta<\alpha}$. By superstability, $p_{\alpha+1}$ can be a forking extension of p_α only finitely often, and because the sequence is indiscernible, there is a natural number n such that it never forks after p_n. As the sequence of the $a_\alpha, \alpha \geq n$, is indiscernible over $A \cup \{a_0, \ldots, a_{n-1}\}$, these a_α are all congruent modulo every finite equivalence relation definable with parameters in this set. Consequently, p_{n+1} is stationary, and the sequence $a_{n+1}, \ldots, a_\alpha, \ldots$ is the Morley sequence of p_{n+1} over $A \cup \{a_0, \ldots, a_n\}$. We can therefore see that by adding a finite set $\vec{a}$ to A (and we know that M is still prime over $A \cup \{\vec{a}\}$) we can assume that our indiscernible set is the Morley sequence s, of length ω_1, of a stationary type p over A.

Let us begin the construction of a model with the first ω elements of s, $a_0, a_1, \ldots, a_n, \ldots$ (we do indeed have an atomic sequence) and continue as long as there are isolated types to be realized. At the end we get a model M_1 that is A-isomorphic to M. By construction, M_1 is atomic over $A \cup \{a_0, \ldots, a_n, \ldots\}$, and it omits the type p_ω, which, being finitely satisfiable in the sequence $a_0, \ldots .a_n, \ldots$, is not isolated. Consequently, this set $a_0, \ldots, a_n, \ldots$ is a maximal indiscernible set over A in M.

Since M_1 and M are A-isomorphic, there is also a denumerable sequence t that is a maximal realization in M of the Morley sequence of p. But we know that a finite set $\vec{b}$ of parameters can make only finitely many elements of the independent set s fork: If $t_p(a_\alpha/A \cup \{\vec{b}\})$ forks over A, then $t_p(a_\alpha/A \cup \{\vec{b}\} \cup \{\ldots, a_\beta, \ldots\}_{\beta<\alpha})$ forks over $A \cup \{\ldots, a_\beta, \ldots\}_{\beta<\alpha}$, and that can happen only finitely many times by the superstability of the type of $\vec{b}$

over A. Therefore, all but denumerably many of the elements of s realize the nonforking son of p over $A \cup t$, contradicting the maximality of t. $\square$

If A and B are two subsets of a model M of T, we shall say that B is *A-normal* if whenever b is in B, every element b' of M with the same type as b over A is also in B; in particular, that is the case for a subset of M defined by a formula with parameters in A.

Lemma 18.4. *(T totally transcendental) If M is atomic over A and B is an A-normal subset of M, then M is atomic over B.*

Proof. Let $\vec{c}$ be in M and let $f(\vec{x}, \vec{b})$ be a formula with parameters in $B \cup A$, satisfied by $\vec{c}$, with minimal Morley rank and Morley degree. If $\mathrm{RM}(t_p(\vec{c}/B \cup A)) = \alpha$, then $t_p(\vec{c}/B \cup A)$ is the only type over $B \cup A$ with Morley rank greater than or equal to α that satisfies $f(x, \vec{b})$.

The type of $\vec{b}\frown\vec{c}$ over A is isolated by a formula $g(\vec{c}, \vec{b})$ with parameters in A. I claim that the formula $g(\vec{x}, \vec{b})$ isolates the type of $\vec{c}$ over $B \cup A$.

If it did not, then in any model containing $B \cup A$, and in particular in M, we could find a tuple $\vec{c}_1$, satisfying that formula, that does not have the same type as $\vec{c}$ over $B \cup A$. Since $\vec{c}$ and $\vec{c}_1$ have the same type over $A \cup \{\vec{b}\}$, the type of $\vec{c}_1$ over $B \cup A$ forks over $A \cup \{\vec{b}\}$, and this is due to some formula $h(\vec{x}, \vec{d}, \vec{b})$, with parameters in $A \cup B$, which we can assume to be strong enough to isolate the type of $\vec{d}$ over $A \cup \{\vec{b}\}$.

However, since $\vec{c}$ and $\vec{c}_1$ have the same type over $A \cup \{\vec{b}\}$, $\vec{c}$ satisfies $(\exists \vec{y}) h(\vec{x}, \vec{y}, \vec{b})$, and there is $\vec{d}_1$ in M satisfying $h(\vec{c}, \vec{d}_1, \vec{b})$. Since B is normal, $\vec{d}_1$ must be in B, and since $\vec{d}_1$ has the same type as $\vec{d}$ over $A \cup \{\vec{b}\}$, this formula makes the type of $\vec{c}$ fork: a contradiction. $\square$

Theorem 18.5. *If T is totally transcendental, then the prime model over A is the only atomic model over A that contains no nondenumerable indiscernible set over A.*

Proof. We have seen that the prime model has these properties. We consider an atomic model M over A without a nondenumerable indiscernible set, and we are going to show that M is constructed over A. It is worthwhile to note first that if $A \subset B \subset M$, if we have succeeded in constructing B, *and if M is atomic over B*, then it satisfies the same hypotheses relative to B, namely, it is atomic and without any nondenumerable indiscernible sets over B.

What we shall prove, by induction on the Morley rank α of the formula $f(x, \vec{a})$ with parameters $\vec{a}$ in A, is that we can construct the set f^M of elements of M that satisfy this formula. We will have constructed all of our model when we arrive at the formula $x = x$; we therefore assume that under the hypotheses of the theorem, we know how to construct the set defined by any formula of Morley rank strictly less than α.

If the formula $f(x, \vec{a})$ has Morley degree 1, then it is satisfied by a unique stationary type of RM α, which has a maximal Morley sequence that is constructed in M: Let a_0 realize p if possible, then let a_1 realize the heir of p if possible, etc., and continue as far as possible. By hypothesis this Morley sequence is denumerable, and we can renumber it as $a_0, \ldots, a_n, \ldots$. In the general case, we carry out this same construction for each of the strong types extending the types over A satisfying this formula that have RM α: Their number is the Morley degree of $f(x, \vec{a})$, and we thus get a finite set of finite (possibly empty!) or denumerable indiscernible sequences. Let $\vec{a}_0$ denote the tuple consisting of the first elements of these sequences, $\ldots, \vec{a}_n$ the tuple consisting of the nth elements of these sequences, ...

As these sequences cannot be extended by elements of M, it follows that for every a in f^M there is a natural number n such that a satisfies a formula with Morley rank strictly less than α with parameters in $A \cup \{\vec{a}_0, \ldots, \vec{a}_n\}$.

Having done that, we begin by enumerating all formulas g_i that imply f, have parameters in A, and have RM strictly less than α. By hypothesis, we can construct every g_i^M, and by putting the construction of each of these over the previous end to end, we get a construction of the union B_0 of these g_i^M, since the union of g_i^M is normal over A, and, by Lemma 18.4, the model M remains atomic over this set.

Next, we add $\vec{a}_0$ to B_0: M is indeed atomic over $A \cup B_0 \cup \{\vec{a}_0\}$. This time we enumerate the formulas with parameters in that set that imply f and that have RM strictly less than α, and we also construct their union B_1. We repeat this ω times: At the end we get a construction of f^M. $\square$

As I announced, this Theorem 18.5 is a second proof of the uniqueness of the prime model in the totally transcendental case, since every prime model, being embeddable in the constructed model, must satisfy the hypotheses of this theorem.

We are now going to study the minimality properties of the prime model, still in the totally transcendental case.

Let B be a set of parameters containing A that is atomic over A, and let p in $S_1(B)$ be realized by a. We shall say, for lack of a better term, that p *is atomic over* A if $B \cup \{a\}$ is atomic over A, that is to say if for every $\vec{b}$ in B the type of $\vec{b}\frown a$ over A is isolated. By transitivity of atomicity, the prime model over $B \cup \{a\}$ is also atomic over A.

Lemma 18.6. *If N is an elementary extension of M, with M and N atomic over A, and if p in $S_1(M)$ is atomic over A, then so is its heir q over N.*

Proof. Let a be a realization of q and let $\vec{b}$ be in N. We need to prove that the type of $a\frown\vec{b}$ over A is isolated.

So let $\vec{c}$ be in M such that p and q are both the unique nonforking extensions of their respective restrictions to $\vec{c}$: We can take $\vec{c}$ to be the parameters of a formula $f(x, \vec{c})$ isolating p from the types of higher or equal Morley rank. Since the type of $\vec{b}$ over $A \cup \{\vec{c}\}$ is isolated, and M is a model, it is realized by $\vec{b}'$ in M. By hypothesis the type of $\vec{c}^\frown\vec{b}'^\frown a$ over A is isolated, and this type is the same as that of $\vec{c}^\frown\vec{b}^\frown a$. □

Theorem 18.7. *(T totally transcendental) The prime model over A is minimal if and only if it contains no infinite indiscernible set over A. If it is minimal, it is the unique atomic model over A; if not, there are atomic models over A of arbitrarily large cardinality.*

Proof. Assume that the prime model M contains an infinite (denumerable!) set $a_0, a_1, \ldots, a_n, \ldots$ that is indiscernible over A. Every formula of the type of a_0 over $A \cup \{a_1, \ldots, a_n, \ldots\}$ is satisfied by all but finitely many of the a_n, so this type is not isolated, and a_0 cannot belong to the prime model over $A \cup \{a_1, \ldots, a_n, \ldots\}$. We thus obtain a nonsurjective elementary A-embedding of M into itself.

If M is not minimal, or even if there exist other atomic models, then it has a proper atomic extension N. Let p be the type over M of an element of N that is not in M. By Lemma 18.6, the Morley sequence $a_0, \ldots, a_n, \ldots$ of p is atomic over A, and must have a copy in M. Moreover, if we extend it, we get an indiscernible sequence of length λ that is atomic over A, as is the prime model over this sequence, by transitivity of atomicity. □

Lemma 18.8. *If $A \subset B \subset C$, C is atomic over A, p is in $S_1(B)$, and p is atomic over A, then p has a son q over C that is atomic over A.*

Proof. We can replace C by the prime model N over C, which is atomic over A. This model N contains a copy of the prime model M over B. By Lemma 18.6, it is sufficient to show that p has an atomic son over M; we shall then take its heir.

To see this, let a realize p, and consider the prime model P over $B \cup \{a\}$, which, by transitivity, is atomic over A, and which contains a copy of M: The type of a over M is atomic over A (and in general it is a highly forking extension of p!). □

Corollary 18.9. *If $A \subset B$, with B atomic over A, and if the p_i are all types in $S_1(B)$ that are atomic over A, then there is a superset C of B that is atomic over A and that realizes all of them.*

Proof. We realize the first, say p_0, in C_0; then we take an atomic extension of the second over C_0, say p_1, which we realize in C_1, etc. □

One way to translate Corollary 18.9 is as follows: if the M_i are atomic models over A, then they have a common elementary extension M that is atomic over A. Compare this result to Theorem 4.14.

We can see that in the case of a totally transcendental theory, we can develop a whole model theory restricted to atomic models over A, the interesting case obviously being that in which the prime model is not minimal.

For example, we can define the notion of *κ-atomically saturated* model: It is a model M that is atomic over A such that for very B with $A \subset B \subset M, |B| < \kappa$, every A-atomic type over B is realized in M. By Corollary 18.9, every atomic model embeds in a κ-atomically saturated model; we can easily see, by a back-and-forth argument, that such a model is (weakly) κ-homogeneous. *Atomically saturated* means atomically saturated in its own cardinality: For every $\lambda \geq |T(A)|$, we shall show the existence and uniqueness of the A-atomically saturated model of cardinality λ by copying the proofs of existence (14.2) and uniqueness (9.8) of the saturated models.

We are now going to study the sort of kernel that we get by taking the intersections of all A-embeddings of the prime model over A into itself. We shall say that a type p over A, or an element a that realizes this type, is *atomically algebraic over A* if p is isolated and if a is not the beginning of any infinite indiscernible atomic sequence over A.

These atomically algebraic elements have properties in relation to the class of atomic models that parallel those of the algebraic elements in relation to the class of all models. For example, if we take a large and very saturated model N, and if M is the prime model over A, then the intersection of all A-embeddings of M into N consists of the algebraic elements over A. If N is atomic, even nonatomically saturated, then this intersection consists of atomically algebraic elements: Indeed, if a is not one, then we can find an indiscernible atomic sequence $a, a_1, \ldots, a_n, \ldots$ in the model, and the prime model over $A \cup \{a_1, \ldots, a_n, \ldots\}$, which contains a copy of M, avoids a. If it is possible to embed M into N avoiding a, as we have already noted, the Morley sequence of the type of a over M is atomic over A.

In summary, *the intersection of all A-embeddings from the prime model M over A into itself, or into any atomic model over A, consists of the atomically algebraic elements; we shall define the minimal closure of A* to be the structure formed by the atomically algebraic elements over A; in any model N that is atomic over A, the atomically algebraic elements form an isomorphic structure to this minimal closure.

Lemma 18.10. *If $A \subset B \subset M$, if M is atomic over A and over B, and if B' is obtained by adding to B or removing from B some atomically algebraic elements over A, then M is still atomic over $A \cup B'$.*

Proof. I begin by adding all the atomically algebraic elements to B. I get a set B_1 that is normal over A; by Lemma 18.4, M remains atomic over B_1.

Then to get B' I remove some A-atomically algebraic elements from B_1. Let $\vec{a}$ be a tuple in M whose type over B_1 is isolated by a formula $f(\vec{x}, \vec{b})$. Since the prime model over $A \cup B'$ must contain the atomically algebraic elements over A, the tuple $\vec{b}$ has a type over $A \cup B'$ that is isolated by a formula $g(\vec{y})$. (Moreover, this property characterizes the atomically algebraic elements: They are those whose type remains isolated over any set of parameters of an atomic model.) Consequently the formula $(\exists \vec{y})(f(\vec{x},\vec{y}) \wedge g(\vec{y}))$ isolates the type of $\vec{a}$ over $A \cup B'$. $\square$

One immediate consequence of Lemma 18.10 is that if we rearrange the atomically algebraic elements in a construction of the prime model M over A, we get another construction of M. We can therefore begin the construction by enumerating, in any way, the minimal closure $A_{\min}$ of A, so that *M remains the prime model over $A_{\min}$, as well as over any subset of $A_{\min}$*. Consequently, *the minimal closure of $A_{\min}$, or of any subset contained between A and $A_{\min}$, is still $A_{\min}$*: Indeed, if a is not in $A_{\min}$, then we can find a copy of the prime model over A, which is also the prime model over $A_{\min}$, that avoids it. That fact justifies the term "closure."

From the uniqueness of the prime model, we can see that $A_{\min}$ has the following homogeneity property: *Two κ-tuples in $A_{\min}$ correspond via an A-automorphism of the prime model M over A.* (If $\kappa > |T(A)|$, then these κ-tuples will obviously involve many repetitions!) In contrast, the prime model, which is ω-homogeneous (since it remains prime over $A \cup \{\vec{b}\}$ where $\vec{b}$ is a finite tuple) is not ω_1-homogeneous if it is not minimal, since a maximal infinite A-indiscernible set cannot be conjugate to one of its infinite proper subsets by an A-automorphism.

18.3 Galois Theory of Differential Equations

I am going to illustrate the foregoing developments here with an example borrowed from algebra. It is a very old example, since it involves the "Galois group," which some mathematicians of the last century, such as Liouville, Picard, and Vessiot, associated with a linear differential equation: an algebraic group of matrices acting on the solutions of the equation. The solvability of the connected component of this group means that it is possible to solve the equation by a succession of simple processes (antiderivatives, exponential antiderivatives, algebraic operations). It parallels the famous theorem of Galois on the solvability of algebraic equations.

Naturally, there was no question of fields for Galois, nor of differential fields for Liouville (who was the editor of Galois's papers); for Galois a root of a polynomial was a complex number, and for Liouville a solution to a differential equation was an analytic function. We, however, are looking for the solutions to a differential equation with coefficients in K, inside the differential closure of K.

This theory was generalized by Kolchin to certain types, rather restricted to be sure, of nonlinear differential equations; the Galois groups obtained are then the most general algebraic groups, and not only affine groups.

The framework of all this is the theory T of differentially closed fields of characteristic zero (see Section 6.2). As we have noted, this is a totally transcendental theory, since RD (which is defined only for 1-types) is a Cantorian rank; alternatively, there are no more 1-types over K than there are irreducible differential polynomials, from which we get stability in every λ. It is easy enough to see that RD is not the Morley rank.

As a consequence, every differential field K (of characteristic zero: We shall not repeat this any more) has a prime model over K, which we write K_{dc}, and which we call its *differential closure.* Inside this differential closure is the *algebraic closure* K_{alg} of K, which is the same in the model-theoretic sense as in the algebraic sense, as well as its *minimal closure* K_{min}, and we have $K \subset K_{\mathrm{alg}} \subset K_{\mathrm{min}} \subset K_{\mathrm{dc}}$.

If $K = C$ is a field of constants (i.e., the derivation is zero on K), then we can see that *the differential closure of C is not minimal.* Indeed, we can show by a purely algebraic argument, which I shall not reproduce here, that pairwise distinct solutions of the equation $x' = x^3 - x^2$, say $a_0, \ldots, a_n, \ldots$, in a differential field L, are always algebraically independent over the constants of this field. Consequently, there is an infinite, atomic, indiscernible set over C that is the Morley sequence of the type of the minimal equation $x' = x^3 - x^2$. The equation $x' + x'/x = 1$, or equivalently $x'(x+1) - x = 0$, has the same property, as does almost every first-order equation with constant coefficients.

To simplify our lives, we assume henceforth that we are working over a differential field K *whose field of constants C is algebraically closed.* Since the type of a transcendental constant over K is not isolated (it is finitely satisfied in C), the differential closure K_{dc} contains no constants except those in C.

We shall call a formula $f(x)$, in one free variable, with parameters in K, *Kolchin* if it has the following property: There is a differentially closed field M containing K such that for every a satisfying f, where a is in some extension of M, the differential field $M(a)_{\mathrm{d}}$ is generated over M by constants. That means further that every a satisfying f satisfies a sentence of the form $(\exists \vec{c})x = \phi(\vec{a}, \vec{c})$, where ϕ is a function definable in T (in this case, a rational fraction), where $\vec{a}$ is a tuple of parameters in M, and where $\vec{c}$ is a tuple of constants. By the compactness of the set of types $S_1(M)$ we need involve only finitely many functions $\phi_1, \ldots, \phi_n$ definable with parameters in M, so there is $\vec{a}$ in M such that every solution of f satisfies

$$\begin{array}{ll}(\exists c_1, \ldots, c_n) & (c_1' = 0 \wedge \cdots \wedge c_n' = 0 \wedge (x = \phi_1(\vec{a}, \vec{c}) \\ \vee & x = \phi_2(\vec{a}, \vec{c}) \vee \cdots \vee x = \phi_n(\vec{a}, \vec{c}))).\end{array}$$

Define the *fundamental system* for the formula $f(x)$ to be such a tuple $\vec{a}$ that allows us to define rationally, over K, the elements that satisfy f (we shall say the "solutions" of f) with the help of constant parameters. We can see, by quantifying the formula above, that the existence of a fundamental system is translated by the satisfaction of a sentence, so that if $f(x)$ is

Kolchin, then it has a fundamental system in any model (i.e., differentially closed field) containing K.

Lemma 18.11. *If $f(x)$ is Kolchin, then it has a fundamental system consisting of elements satisfying f.*

Proof. Let M be a differentially closed field containing K (for example, its differential closure), and let L be the differential field generated by K, the constants of M, and the solutions of f in M. Let a be a solution of f in an extension of M, and let $\vec{c}$ be a tuple of constants such that a is rational over $M \cup \{\vec{c}\}$. By an absolutely identical proof to that of the separation of parameters theorem (Corollary 12.31), we can see that the type of $\vec{c}^\frown a$ over M is determined by its restriction to L. Consequently, the type of a over $L \cup \{\vec{c}\}$ has only one son over $M \cup \{\vec{c}\}$; since the latter is rational, so is the former. As a consequence, $a = \phi(\vec{c}, \vec{c}_1, \vec{b})$, where ϕ is a K-definable function, $\vec{c}_1$ is a tuple of constants from L, and $\vec{b}$ is a tuple of solutions of f in M (i.e., in L). The rest is just a question of compactness. □

Lemma 18.12. *Let p be a type in $S_1(K)$ satisfying a Kolchin formula $f(x)$, and let a be a realization of p. Then p is isolated if and only if the differential field $K(a)_{\mathrm{d}}$ generated over K by a contains no constants other than those of K; in that case it is a subfield of the minimal closure of K. (Recall that by hypothesis the field of constants of K is algebraically closed.)*

Proof. If p is isolated, then it is realized by an element a of the differential closure K_{dc} of K. The latter contains no new constants, and neither does $K(a)_{\mathrm{d}}$. Furthermore, if L is a differentially closed field contained between K and K_{dc}, then it contains a fundamental system $\vec{b}$ for $f(x)$, and a can be expressed rationally from $\vec{b}$ and constant parameters, which are all in K; consequently, a is in L.

Conversely, if $K(a)_{\mathrm{d}}$ contains no new constants, then neither does its differential closure L. Embed K_{dc} into L: Then K_{dc} contains a fundamental system $\vec{b}$ for $f(x)$, and a can be expressed rationally from $\vec{b}$ and constant parameters, which are in K. Therefore, a must be in K_{dc}, and its type p over K is isolated. □

With the previous lemma, we can easily see that no inequation $p(x) \neq 0$ is Kolchin; indeed, if a is differentially transcendental (i.e., satisfies no nontrivial equation) over K, then its type over K is not isolated, and a simple calculation of the derivative shows that every constant of $K(a)_{\mathrm{d}}$ is in K. We shall see a profound reason for this phenomenon in the following chapter when we study the notion of orthogonality.

The most classical example of Kolchin formula is given by linear differential equations. To embark on this example, we begin with a small lemma. By definition the *Wronskian* or *Wronski determinant* of $x_0, \ldots, x_n$ is the determinant $W(x_0, \ldots, x_n)$ whose rows are $(x_i, x_i', x_i'', \ldots, x_i^{(n)})$.

Lemma 18.13. *$W(a_0, \ldots, a_n)$ is zero if and only if $a_0, \ldots, a_n$ satisfy a nontrivial linear dependence relation with constant coefficients.*

Proof. If $c_0a_0 + \cdots + c_na_n = 0$, then by differentiating this equation n times we get a nontrivial relation on the rows of the Wronskian, which is therefore zero.

Conversely, the result is clear if $a_0, \ldots, a_n$ are all zero. If not, we can find an index i such that $W(a_0, \ldots, a_i) \neq 0, W(a_0, \ldots, a_{i+1} = 0)$, so the last row of the latter determinant is a linear combination of the others:

$$\begin{aligned} a_{i+1} &= b_0a_0 + \cdots + b_ia_i, \\ a'_{i+1} &= b_0a'_0 + \cdots + b_ia'_i, \\ &\cdots \\ a_{i+1}^{(i+1)} &= b_0a_0^{(i+1)} + \cdots + b_ia_i^{(i+1)}. \end{aligned}$$

By differentiating each of these equations and subtracting the next, we can see that each b'_j is zero, since $W(a_0, \ldots, a_i)$ is zero. □

One small comment on the importance of the lemma: The fact that $a_0, \ldots, a_n$ satisfies a linear relation with constant coefficients can be expressed by a sentence. As far as satisfaction of this sentence in a differentially closed field M containing $a_0, \ldots, a_n$ is concerned, we know by quantifier elimination that it is equivalent to a quantifier-free sentence, which we have even determined: It is $W(a_0, \ldots, a_n) = 0$! Naturally, satisfaction of this sentence does not depend on the differential closed field in which we are working; Lemma 8.13 states that it also does not depend on the differentially closed field K (containing $a_0, \ldots, a_n$, of course) under consideration, and that in every case this sentence is equivalent to $W(a_0, \ldots, a_n) = 0$. Essentially, that is due to the fact that every field is "linearly closed" in each of its extensions, that every finite system of linear equations with coefficients in K that has a solution in an extension of K already has a solution in K: Remember the "Cramer formulas" of your adolescence!

Proposition 18.14. *Every homogeneous linear equation $P(x) = x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_0x = 0$ is Kolchin; more precisely, the solutions of this equation are all linear combinations, with constant coefficients, of n among them.*

Proof. I leave it to the reader to check that the solutions of this equation form a vector space E over the constants. (Be careful: The coefficients a_i are not necessarily constant!) If we consider $n+1$ solutions $b_0, \ldots, b_n$ of this equation, we can see that there is a relation among the columns of their Wronskian, and $W(b_0, \ldots, b_n) = 0$. Consequently, $b_0, \ldots, b_n$ are linearly dependent over the constants, and the dimension of E cannot exceed n.

It is in fact equal to n, and we can find a basis of E, which therefore constitutes a fundamental system for $P(x) = 0$, in every differentially closed field containing the coefficients of P. Indeed, in a system $P(x) = 0 \wedge W(b_0, \ldots, b_i, x) \neq 0, i < n-1$, the order of the inequation is strictly less than that of the equation. $\square$

A nonhomogeneous linear equation of the form $x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_0 x = b$ is also Kolchin: Differentiate and combine the two linear equations obtained to have a homogeneous linear equation of order $n+1$ that is a consequence of the first one. For example, an equation $x' = b$, which corresponds to taking an antiderivative, is Kolchin.

Another famous but less trivial example of a Kolchin formula, of a very different nature from that of linear equations, is given by the equation $x'^2 = x^3 + px + q$, where p and q are constants, of the Weierstrass $\mathcal{P}$ function.

The *Kolchin extension* of K associated with the Kolchin formula $f(x)$ is defined as the subfield L of $K_{\min}$ generated by the solutions of f in K_{dc}; it is also the differential subfield generated by the solutions of f in any differentially closed extension of K that contains no new constants. Kolchin introduced these extensions under the name "strongly normal extensions" long before anyone knew what a differentially closed field was, in order to help with an outline of a Galois theory for differential equations, for which, as far as linear equations are concerned, the essentials have been known since the end of the last century; we are going to reinterpret this Galois theory in light of model theory.

So let L/K be a Kolchin extension. The group of automorphisms of L that fix every element of K will be written $G(L/K)$ and called the *Galois group* of L/K. In view of the homogeneity property of $K_{\min}$, that is the same as the group of restrictions to L of K-automorphisms of K_{dc}.

The first stage consists in remarking that $G(L/K)$ is a definable group in the theory of differentially closed fields. By Lemma 18.11, the Kolchin formula $f(x)$ has a fundamental system $\vec{b}$ that is in L. Let $g(\vec{y})$ be a formula isolating the type of $\vec{b}$ over K. An automorphism s of L/K is entirely determined by its action on $\vec{b}$, i.e., by its image $s\vec{b}$, since every solution of f in L can be expressed in terms of $\vec{b}$ and constants, which are all in K.

Using $\vec{b}$ as parameters, we can thus represent s by its image $\vec{y} = s\vec{b}$, which is any element of L satisfying $g(\vec{y})$. To define our group, we still need to define the composition law, $\vec{y}_1 \circ \vec{y}_2 = \vec{z}$, which means that the automorphism composed of the one sending $\vec{b}$ to $\vec{y}_2$ and the one sending $\vec{b}$ to $\vec{y}_1$ is the one that sends $\vec{b}$ to $\vec{z}$. But we know that every tuple consisting of solutions of f can be expressed rationally from $\vec{b}$ and constant parameters; that is to say, it satisfies a formula

$$(\exists \vec{c})(\vec{c}' = 0 \wedge (\vec{y} = R_1(\vec{b}, \vec{c}) \vee \cdots \vee \vec{y} = R_m(\vec{b}, \vec{c}))),$$

where $R_1, \ldots, R_m$ are tuples of rational fractions with coefficients in K. Since all constants of L are in K, to express that $\vec{y}_1 \circ \vec{y}_2 = \vec{z}$, it is sufficient to say that if $\vec{y}_1 = R_i(\vec{b}, \vec{c})$, then $\vec{z} = R_i(\vec{y}_2, \vec{c})$, which can be done by a formula.

Proposition 18.15. *If L/K is Kolchin, then there is a Galois correspondence between the differential fields intermediate between K and L and the subgroups of $G(L/K)$ definable with parameters in L.*

Proof. If $K \subset L_1 \subset L$, then we need to prove that $G(L/L_1)$ is a definable subgroup of $G(L/K)$. Consider an element a of L, which is of the form $a = R(\vec{b})$, where R is a differential rational fraction with coefficients in K. In $G(K/L)$, the stabilizer of a is defined by the formula $R(\vec{y}) = R(\vec{b})$. The group $G(L/L_1)$ is the intersection of all the $G(L/K(a)_{\mathrm{d}})$, with a in L_1, which are consequently all definable. I claim that in fact $G(L/L_1)$ is the intersection of finitely many of them, which implies that it is itself also definable. That is due to the fact that $G(L/K)$, being definable in a totally transcendental structure, is itself totally transcendental: We can easily see that if G and H are two definable subgroups of such a group, with H strictly contained in G, then the Morley rank of H is strictly less than that of G, or the Morley degree of H is strictly less than that of G (since as all right cosets of G modulo H can be matched pairwise via definable bijections, they all have the same rank and the same Morley degree). Consequently, in a totally transcendental group every decreasing sequence $G_0 \supset G_1 \supset \cdots \supset G_n \supset G_{n+1} \supset \cdots$ of definable subgroups stabilizes at some point, and every intersection of a family of definable subgroups is the intersection of finitely many of them.

We have shown in passing that L_1/K is finitely generated, since it is clear, by the homogeneity of K_{min}, that the set of invariants of $G(L/K(a_0, \ldots, a_n)_{\mathrm{d}})$ is the field $K(a_0, \ldots, a_n)_{\mathrm{d}}$ itself.

Conversely, let H be a subgroup of $G(L/K)$ defined by the formula (with parameters) $h(\vec{y})$. Obviously, H is the group of automorphisms of L/K that preserve this formula; since, by Theorem 16.21 we have elimination of imaginaries, it is the Galois group of $L/K(\vec{a})_{\mathrm{d}}$, where $\vec{a}$ is the canonical tuple of parameters of h. $\square$

The reader will note that if $G(L/L_1)$ is normal in $G(L/K)$, then L_1/K is Kolchin, and its Galois group is $G(L/K)/G(L/L_1)$.

Let us describe the nature of $G(L/K)$ and that of its definable subgroups. We have used the parameter $\vec{b}$ to define $G(L/K)$ as a group of permutations of L, by taking the elements that satisfy $g(\vec{y})$ as its support. To define $G(L/K)$ as an abstract group, we are first going change the support, so as not to be manipulating more than the constants.

We know that every $\vec{y}$ can be written rationally as a function of a tuple of constants $\vec{y} = R_1(\vec{b}, \vec{c}) \vee \cdots \vee \vec{y} = R_m(\vec{b}, \vec{c})$. Consider the definable set A of $(n+1)$-tuples of constants $\vec{c}\,\hat{}\,i$, with $i = 1, \ldots,$ or $, \ldots i = m$ (i.e. $i = 1 + \cdots + 1$ m times). Consider the equivalence relation E defined on that set as follows: $\vec{c}\,\hat{}\,i \; E \; \vec{c_2}\,\hat{}\,j$ if $R_i(\vec{b}, \vec{c_1}) = R_j(\vec{b}, \vec{c_2})$. Since the quotient

A/E is in bijection with the interpretation of $g(\vec{y})$ in L, we can carry over the group structure of $G(L/K)$.

However, this time, A is composed only of constants, so that *by the parameter separation theorem* (Corollary 12.31), the equivalence relation E, as the group law defined on A/E, which we defined by using $\vec{b}$ as parameters, and possibly other parameters in K, can be defined by using only constant parameters, which are in K_{dc}, i.e. in K. *Therefore, $G(L/K)$ appears as a group definable in the field C of the constants of K, and its subgroups definable with parameters in L are actually definable with parameters in C!*

An object definable in the theory of algebraically closed fields is called *constructible* by geometers; the algebraic varieties are particular constructible objects, but in the case of groups, constructible groups and algebraic groups are the same thing, up to constructible isomorphism. So, when L/K is Kolchin, *$G(L/K)$ is actually an algebraic group over the field C of constants of K, and the differentially closed fields intermediate between K and L correspond to the algebraic subgroups of $G(L/K)$ definable with parameters in C.*

If the formula $f(x)$ is a linear equation, then the associated Kolchin extension is called Picard–Vessiot. In that case, its Galois group is a group of matrices, that is to say an affine algebraic group, and we can show the converse: If $G(K/L)$ is affine, then K/L is Picard–Vessiot. In contrast, in the case of the equation of the Weierstrass function, the Galois group is an abelian variety.

As it is clear that the disjunction of two Kolchin formulas is also Kolchin, we can see that the differential field generated by two Kolchin extensions L_1/K and L_2/K, both located inside the differential closure K_{dc} of K, is again a Kolchin extension of K. It is therefore reasonable to define the *Kolchin envelope* as the subfield of K_{dc} generated over K by the solutions of Kolchin formulas; that is the limit of the directed family of Kolchin extensions of K.

We can also see that the field generated by two Picard-Vessiot extensions, both inside K_{dc}, is itself Picard–Vessiot, and we define the *Picard–Vessiot envelope* of K as the differential subfield of K_{dc} generated by the solutions of linear differential equations with coefficients in K.

We therefore have the following inclusions:

$$K \subset K_{\text{alg}} \subset K_{\text{Picard-Vessiot}} \subset K_{\text{Kolchin}} \subset K_{\text{min}} \subset K_{\text{dc}}$$

We say Kolchin envelope, or Picard–Vessiot envelope, rather than closure, since these notions are not transitive. If K is a field of constants, its Kolchin envelope is not even algebraically closed: It contains the antiderivatives of 1, the solutions of $x' = 1$, but not their square roots, the solutions of $2xx' = 1$.

I add, to conclude this section, an observation that the driving force behind Galois theory has little to do with the differential context: Any formula of Morley rank and degree 1 (such an f is often called *strongly* *minimal*), or even having a different Morley rank and degree, can play the role of the constants. We shall define the most general analogue of Kolchin formulas and still get definable automorphism groups; for Galois theory, it obviously might be necessary to appeal to imaginary elements.

18.4 Prime $|T|^+$-Saturated Models

The theorems on the existence and the uniqueness of the prime model that have appeared in the course of this book (Chapters 10, 18, and Section 17.3) actually have a much more general reach, which extends beyond the framework in which they have been presented.

Let us examine what are actually the moving forces:

- a notion of isolation, of atomicity, with the transitivity of atomicity (Lemma 10.6);
- a density theorem for isolated types (Theorem 17.15);
- the possibility of defining a notion of closure by small packages, which are not even necessarily finite, in such a way as to guarantee the uniqueness of the model constructed (Theorem 10.18, Section 10.6);
- the possibility of determining types by some of their restrictions to sets that are sufficiently small in relation to the sizes of their packages, so that Theorem 18.1, guaranteeing that every subset of a constructed set is constructed, remains true.

Note that in Section 10.6, in regard to ultrametric spaces, we were able to carry out this program in the framework of an unstable theory. Note also that for the proof of uniqueness, it was Corollary 18.2 that we sought to generalize, and not Theorem 18.5, which relies too heavily on the existence of a rank.

If A is a set of parameters, a κ-saturated model M containing A is said to be *prime over* A (understood: *among the κ-saturated models containing A*) if M embeds elementarily into every κ-saturated model of $T(A)$. We are shortly going to show *the existence and the uniqueness of the prime κ-saturated model for every set A of parameters in a stable theory T, where κ is a regular cardinal strictly larger than $|T|$*. The proof consists in copying that of the existence and uniqueness of the prime model of a totally transcendental theory; we shall recall, as a generally valid principle, that the constructions that we carried out relative to the class of all models of a totally transcendental theory can also be done in the case of a stable theory if we restrict ourselves to $|T|^+$-saturated models.

If p is in $S_1(A)$, it is called *κ-isolated* if it is determined by fewer than κ formulas $f_\alpha(x, \vec{a})$ with parameters in A (i.e., it is the only type to satisfy all of them). A subset X of $S_1(A)$ is called *κ-dense* if it has the following property: Every nonempty subset of $S_1(A)$ that is the intersection of fewer than κ clopen sets contains a point in X.

If M is a κ-saturated model of $T(A)$ (or even only κ-compact; but we are only considering sufficiently large cardinals, so that κ-compactness is equivalent to κ-saturation) then it realizes all κ-isolated types of $T(A)$. We shall say that a κ-saturated model M of $T(A)$ is *κ-constructed* if it is possible to enumerate $M = \{\ldots, a_\alpha, \ldots\}$ ordinally so that for every α the type of a_α over $A_\alpha = A \cup \{\ldots, a_\beta, \ldots\}_{\beta<\alpha}$ is κ-isolated. It is clear that a κ-constructed model is κ-prime (i.e., prime among the κ-saturated models).

The analogue of Theorem 17.15 is sufficient to prove the existence of such a model:

Theorem 18.16. *If T is stable and $\kappa \geq |T|^+$, then for every A the κ-isolated types are κ-dense in $S_1(A)$.*

Proof. Let F be a closed set in $S_1(A)$ defined by fewer than κ formulas $g_i(x, \vec{a}_i)$ with parameters in A. We enumerate all the formulas $f_\alpha(x, \vec{y})$ of the language of T without parameters, and by induction construct a sequence of formulas $h_\alpha(x, \vec{a}_\alpha)$ with parameters in A as follows: Take a type p_α of $S_1(A)$ in the conjunction of all the g_i and the $h_\beta, \beta < \alpha$, with $R(p, f_\alpha, 2)$ minimal, and a formula $h_\alpha(x, \vec{a}_\alpha)$ of the same $R(p, f_\alpha, 2)$ and satisfied by this type. The construction will not get jammed because it does construct a consistent set of formulas at every stage: At the limit stages, this is a consequence of compactness.

By the definition of $R(p, f_\alpha, 2)$, for every $\vec{a}$ in A, one of $f_\alpha(x, \vec{a})$ and $\neg f_\alpha(x, \vec{a})$ is incompatible with the conjunction of the g_i and the $h_\beta, \beta < \alpha$; that is to say, the other one is implied by this conjunction. In other words, the conjunction of the g_i and the $h_\beta, \beta < \alpha$, implies a complete f_α-type; the conjunction of the g_i and of all the h_α κ-isolates a complete type. □

If $A \subset B$, we shall call B *κ-atomic* over A if every tuple of length less than κ taken from B has a κ-isolated type over A. Here is the analogue of Lemma 10.6:

Lemma 18.17. *If κ is regular, then the notion of κ-atomicity is transitive: If $A \subset B \subset C$, B is κ-atomic over A, and C is κ-atomic over B, then C is κ-atomic over A.*

Proof. Let $\vec{a}$ be a λ-tuple in C, $\lambda < \kappa$. Its type over B is κ-isolated by formulas $f_i(\vec{x}, \vec{b}_i)$, where the $\vec{b}_i$ are finite tuples. We can assume that this set of formulas is closed under finite conjunctions. Every $\vec{b}_i$ has a type over A that is κ-isolated by formulas $g_{ij}(\vec{y})$ with parameters in A, which we can likewise assume to form a family closed under finite conjunctions. The type of $\vec{a}$ is determined by the formulas $(\exists \vec{y})(f_i(\vec{x}, \vec{y}) \wedge g_{ij}(\vec{y}))$; indeed, if $\vec{a}'$ satisfies all that, then it is consistent that there exists B' of the same type as B over A, such that the type of $\vec{a}'$ over B' is that corresponding to the type of $\vec{a}$ over B. If κ is regular, then this set of formulas is indeed of cardinality strictly smaller than κ. □

As a consequence the κ-constructed model is κ-atomic. To show the uniqueness of the constructed model, it remains only to define a notion of closure by making packages that are no longer finite, but of strictly smaller cardinality than κ, and to proceed by a back-and-forth argument between closed sets.

The rules of the game are understood: What was possible with finite sets becomes possible with sets of cardinality strictly less than κ, and what was possible with denumerable sets becomes possible with sets of cardinality κ. In particular, every enumeration, of type κ, of a κ-atomic set, is a construction of it.

As everyone knows, every type of a stable theory is a nonforking extension of one of its restrictions to a set of parameters of cardinality $|T|$ (Corollary 15.8); moreover, if $A \subset B, p \in S_1(A), q \in S_1(B)$, and q is a nonforking κ-isolated son of p, then p is itself κ-isolated: That is a simple consequence of the open mapping theorem (Corollary 16.7). That is enough to adapt the proof of Proposition 18.1, so the uniqueness of the κ-saturated prime model is proved.

There is a rather impressive volume of varieties of prime models entering into this scheme, all conceived by Shelah. For most of them, we manage to prove the existence and the uniqueness of the prime model by following the schema developed above, but that does not always work out without any technical problems.

The notion adapted to superstable theories is that of an $\aleph_\epsilon$-*saturated prime model*: A model M is called $\aleph_\epsilon$-saturated if for every (finite) tuple $\vec{a}$ in M, every *strong type* over $\vec{a}$ is realized in M. If every type has finite multiplicity, this is the same notion as that of $\aleph_0$-saturation.

18.5 Ehrenfeucht Models

The prime models that we have considered have an intrinsic characterization, altogether satisfying to the spirit: They are the models that envelop A by adding as few things to it as possible. To show the existence of such models, and again to prove their uniqueness, we needed strong hypotheses of stability, which allowed us to control their construction entirely.

When these conditions are absent, we have to be satisfied with more approximate notions; if there are no prime models, we can at least attempt to construct models that do not add too much to A.

One extreme case of existence of prime models occurs, as in the case of arithmetic, when T has *Skolem functions*; the prime model over A is then nothing other than the rational closure of A. The function $\phi(\vec{y})$ is called a Skolem function for the formula $f(x, \vec{y})$ (without parameters) if the following axiom belongs to T: $(\forall \vec{y})((\exists x) f(x, \vec{y}) \to f(\phi(\vec{y}), \vec{y}))$.

If the theory T does not have Skolem functions, we can add them; we augment the language of T by introducing a new function symbol $\phi_f(\vec{y})$ for every formula $f(x, \vec{y})$ of the language of T. Note that the ϕ_f can be interpreted over every model of T in such a way as to satisfy every axiom $(\forall \vec{y})((\exists x) f(x, \vec{y}) \to f(\phi_f(\vec{y}), \vec{y}))$; for that end, interpret $\phi_f(\vec{b})$ as any element x satisfying $f(x, \vec{b})$ if there is one, and anything otherwise.

We can thus consider a complete theory T^*, in the augmented language, containing T and all these new axioms; such a theory is called a *Skolemization* of T. It is this choice of a completion that has no intrinsic character; moreover, the stability properties and other properties of T will not carry over to T^*.

Truth be told, T^* Skolemizes only the formulas of the language of T, which is enough for what we are going to do; to have a theory with Skolem functions, the construction must be repeated ω times.

To construct types of prime models, we do the following: We start with a set A of parameters from a model of T; we can consider it as a set of parameters for T^* (albeit not in a canonical manner), since every model of T has an elementary extension that can be enriched to give a model of T^*. We then take the closure of A under the Skolem functions in the language of T^* (which is called the *Skolem envelope* of A), then forget the added symbols. We thus obtain a structure $M(A)$, for the language of T, which is a model of T, since it satisfies Tarski's test.

We can see that in the construction of $M(A)$ there were at least two entirely noncanonical stages: the choice of the Skolemization T^*, and the completion of the theory of A in the new language.

When the set A is an indiscernible sequence in the order *for the theory* T^*, $M(A)$ is called an *Ehrenfeucht model*; such a model is therefore determined by three things:

- the choice of T^*;
- the Ehrenfeucht set, in the sense of T^*, of the indiscernible sequence A (only sets allowing infinite sequences are considered);
- the isomorphism type of the order I of the indices of the elements of A.

In practice, we assume that the first two have been done, so that we are satisfied to let $M(I)$ denote such a model. I can be any chain.

It is clear that every automorphism of I extends to an automorphism of $M(I)$, that every isomorphism between I and J extends to an isomorphism between $M(I)$ and $M(J)$, and that every order-preserving embedding of I into J extends to an elementary embedding of $M(I)$ into $M(J)$.

The original goal of this construction was to produce models of T having many automorphisms. It was used by Shelah, with the hand of the master, to prove the following theorem:

Theorem 18.18. *If T is unstable, $\lambda \geq |T|$, λ nondenumerable, then T has 2^λ pairwise nonisomorphic models of cardinality λ.*

To prove this, Shelah considered Ehrenfeucht models associated with indiscernible sequences (of tuples) *ordered by a formula in the language of T*: That is possible by Theorem 12.36. He showed that if the chains I and J are different enough, then the models $M(I)$ and $M(J)$ cannot be isomorphic; that gave him the maximum number of models possible, namely 2^λ.

Moreover, this same Shelah succeeded in extending his result to non-superstable theories, at the cost of a proof of frightful complexity, by constructing Ehrenfeucht models around "indiscernible" sets indexed, not by chains, but by trees filled by formulas of T.

Theorem 18.19. *If T is not superstable and $\lambda > |T|$, then T has 2^λ pairwise nonisomorphic models of cardinality λ.*

I shall not undertake to prove these theorems, not because I downplay their importance (they should be etched into the memory of every model theorist, and even of every algebraist), but because the proofs draw us into subtle combinatorial technicalities, strangers to the spirit of this course, and because moreover I am incapable of bringing any sensible simplifications to Shelah's arguments, to which I refer. I will, however, note that in Theorem 14.11 we obtained a particular case of Theorem 18.18 by a totally different method.

We shall not use Ehrenfeucht models except for a result that will be exploited in the next section:

Lemma 18.20. *If $M(I)$ is an Ehrenfeucht model associated with a well-ordering I, and if A is a subset of $M(I)$, then there are at most $|A| + |T|$ types of $S_1(A)$ that are realized in $M(I)$.*

Proof. We are going to count the types over A, in the sense of T^*, that are realized in $M(I)$. Since every element a of A has the form $\phi(b_1, \ldots, b_n)$, where the b_i are in I, we can find a subset B of I, either finite or of the same cardinality as A, whose rational closure contains A. We count the types over B, since two elements that have the same type over B will have the same type over A.

An element c of $M(I)$ is of the form $c = \phi(d_1, \ldots, d_m)$ with the d_j in I; its type is determined by specifying ϕ and the type of $\vec{d}$ over B. There are $|T|$ possible choices for the function ϕ, which is a composite of Skolem functions; by the indiscernibility of I in the sense of T^*, the type of $\vec{d}$ over B is determined by those cuts that every d_j splits over B; since B is a well-ordering, there are only $|B| + 1$ cuts over B. $\square$

Recall that a theory T is called *λ-categorical* if it has only one model of cardinality λ up to isomorphism.

Corollary 18.21. *If T is denumerable and categorical in some nondenumerable λ, then it is totally transcendental.*

Proof. If T is not ω-stable, then it has a model M, of cardinality λ, containing a denumerable set A such that M realizes at least $\aleph_1$ types of $S_1(A)$; by Lemma 18.20, this model cannot be isomorphic to an Ehrenfeucht model $M(\lambda)$. □

18.6 Two-Cardinal Theorem; $\aleph_1$-Categorical Theories

In this section we assume that we are dealing with a *totally transcendental* theory; indeed, the useful phenomenon is stability, and after a few technical modifications, the results generalize to the stable case.

Let M be a model of T, let $f(x,\vec{b})$ be a formula with parameters in M, and let p be a type over M. We shall say that p is *orthogonal* to the formula $f(x,\vec{b})$ if, when a realizes p, the prime model N over $M \cup \{a\}$ contains no elements satisfying $f(x,\vec{b})$ that are not already in M. We exclude realized types p, which are orthogonal to every formula, from the discussion.

Lemma 18.22. *The fact that p is orthogonal to $f(x,\vec{b})$ depends only on its class in the fundamental order of $T(\vec{b})$; if q is the heir of p over an elementary extension N of M, then q is orthogonal to $f(x,\vec{b})$ if and only if p is.*

Proof. If p is not orthogonal to $f(x,\vec{b})$, we can find c in the prime model over $M \cup \{a\}$, c not in M, satisfying $f(x,\vec{b})$. The type of c over $M \cup \{a\}$ is isolated by a formula $g(x,a,\vec{d})$; therefore, there exists $\vec{d}$ in M such that p satisfies $(\exists y)(f(y,\vec{b}) \wedge g(y,x,\vec{d}))$, while p does not satisfy $f(y,\vec{b}) \wedge g(y,x,\vec{d})$ for any y in M. That can be translated as the satisfaction of a sentence in the model (M,dp) with $\vec{b}$ as parameter. □

Define a *Vaught pair* as the specification of a model M, a formula $f(x,\vec{b})$ with parameters in M, and a proper elementary extension N of M that contains no new elements satisfying $f(x,\vec{b})$. The trivial case, where $f(x,\vec{b})$ is satisfied by only finitely many elements, is excluded from the discussion.

If we have a model M, of cardinality λ, for some denumerable theory, with a formula $f(x,\vec{b})$ satisfied by κ elements of N, κ strictly less than λ, then we get a Vaught pair by considering an elementary submodel M of N, of cardinality κ, that contains $\vec{b}$ and all of its elements.

Proposition 18.23 (Two-Cardinal Theorem for a Totally Transcendental Theory). *If T is totally transcendental and has a Vaught pair associated* *with $f(x, \vec{b})$, then for all cardinals κ, λ with $|T| \leq \kappa \leq \lambda$, there is a model M of $T(\vec{b})$, of cardinality λ, in which there are exactly κ elements that satisfy $f(x, \vec{b})$.*

Proof. When there is a Vaught pair (N, M), we can find a nonrealized type p in $S_1(M)$ that is orthogonal to $f(x, \vec{b})$: Take the type of any element of $N - M$. By Lemma 18.22 we can find such p over a model M, containing $\vec{b}$, of the same cardinality as T. Begin by extending M to a model M_0 of cardinality κ containing κ elements satisfying $f(x, \vec{b})$; this is very simple to do by the Löwenheim–Skolem theorem. Let p_0 be the heir of p over M_0, and let a_0 realize it: In the prime model M_1 over $M_0 \cup \{a_0\}$ there are no new elements satisfying $f(x, \vec{b})$. Next, let a_1 realize the heir p_1 of p over M_1: In the model M_2, which is prime over $M_1 \cup \{a_1\}$, there are still no new elements satisfying $f(x, \vec{b})$. Repeat this λ times, taking limits at the limit stages. The model M_λ obtained at the end, which is what we are looking for, is in fact prime over M_0 and the Morley sequence of p_0 (see Section 20.1). □

Proposition 18.24. *Let T be totally transcendental and let λ be a cardinal strictly greater than that of T. Then T is λ-categorical if and only if it has no Vaught pair.*

Proof. If T has a Vaught pair for the formula $f(x, \vec{b})$, then by Proposition 18.23 we can construct a model M of T, of cardinality λ, containing only $|T|$ elements satisfying $f(x, \vec{b})$; this model is not isomorphic to the saturated model of cardinality λ.

Define a *strongly minimal formula* to be a formula $f(x, \vec{a})$ of Morley rank 1 and Morley degree 1. That means that for any formula $g(x, \vec{b})$ one (and not the other!) of $g(x, \vec{b}) \wedge f(x, \vec{a})$ and $\neg g(x, \vec{b}) \wedge f(x, \vec{a})$ is satisfied by finitely many elements. As there must be types of Morley rank 1 somewhere, there do indeed exist such formulas. Let us first show that the absence of Vaught pairs guarantees the existence of a strongly minimal formula with parameters in the prime model M_0 of T.

Note that every formula $f(x, \vec{y})$ can be associated with an integer n such that for any $\vec{a}$, if there are n elements satisfying $f(x, \vec{a})$, then there are infinitely many. (In other words, we can express the fact that the formula $f(x, \vec{a})$ is nonalgebraic by a sentence with parameter $\vec{a}$. If this were not the case, then for every n we could find $\vec{a}_n$ in some model M_n of T such that $f(x, \vec{a}_n)$ would be satisfied by finitely many, but more than n, elements of M_n. A proper extension N_n of M_n could obviously contain no new elements satisfying this algebraic formula; an ultraproduct would provide a formula $f(x, \vec{a})$ with a magnificent Vaught pair (N, M).

So consider a formula $f(x, \vec{a})$, of Cantor rank 1 and Cantor degree 1, over the prime model M_0. If this formula had Morley rank or Morley degree greater than 1, we could split it into two nonalgebraic formulas $g(x, \vec{b})$ and

$h(x, \vec{c})$ with parameters $\vec{b}$ and $\vec{c}$ in an elementary extension of M_0. As we can express the nonalgebraicity of these formulas by a sentence over their parameters, we could find $\vec{b}'$ and $\vec{c}'$ in M_0 with the same property. That contradicts the definition of Cantor rank. Consequently, $f(x, \vec{a})$ is indeed strongly minimal.

Let us now fix a strongly minimal formula $f(x, \vec{a})$ with parameters in the prime model M_0. Let M be a model of cardinality λ. Since the type of $\vec{a}$ over $\emptyset$ is isolated, it is realized in M by a tuple which we denote the same way. If we define A as the set of all elements of M which satisfy $f(x, \vec{a})$, then the absence of a Vaught pair for $f(x, \vec{a})$ forces M to be prime, and even minimal, over $\vec{a}^\frown A$. Let p be the type over $\vec{a}$ of Morley rank 1 that satisfies $f(x, \vec{a})$, and let B be a maximal independent set of realizations of p: Since A is contained in the algebraic closure of $\vec{a}^\frown B$, M is prime over this set. For reasons of cardinality, B must be the Morley sequence of p of length λ, which entirely determines the type of $\vec{a}^\frown B$, and therefore also M.

□

Corollary 18.25 (Morley's Theorem). *If T is denumerable and categorical in some nondenumerable cardinal, then it is categorical in every nondenumerable cardinal.*

Proof. By Corollary 18.21, T is ω-stable. □

These theories are simply called $\aleph_1$-*categorical* or ω_1-*categorical*. We shall indeed take care that a theory interpretable in an $\aleph_1$-categorical theory is not in general itself $\aleph_1$-categorical (in contrast to the situation with ω-categoricity). For example, take the structure whose language includes a unary predicate $A(x)$ satisfied by infinitely many elements, and a bijection s between A and its complement. It is ω_1-categorical; however, if you forget about s, you can find three models of cardinality $\aleph_1$.

As I have often had occasion to do in this course, I proved Theorem 18.25 prematurely: It is actually a corollary of a more precise theorem from Section 20.2. If I placed it too early, this is first because it does not require too complicated a technical arsenal, such as the one that the reader is invited to digest in the next two chapters, and also out of respect for its history. This proof is not very far removed from Michael Morley's original proof, which is the departure point for the study of stability, that is to say, the study of contemporary model theory.

18.7 Historic and Bibliographic Notes

The beautiful proof of the uniqueness of the prime model, in Corollary 18.2, comes from [She79]. The second, Theorem 18.5, which involved the lengths of the indiscernible sequences presented in the model, is older: [She72]. The end of 18.2 is taken from [Poi81a].

The examination of the Galois–Kolchin theory in light of model theory was done in [Poi83e]. For references to differential fields, see the notes to Chapters 6 and 10.

Prime $|T|^+$-saturated models are a particular case of the five principal varieties of prime models of which Shelah managed to conceive in [She78]; the situation has since been developed further.

Skolem functions go back to [Sko20], and Ehrenfeucht models to [EM56], with the proof of Lemma 18.20. They allowed Shelah to construct many models for an unstable theory, [She71c], [She78]. For a different approach to the problem, see [CP83]. Theorem 18.23 is from [Mor65].

Vaught pairs appeared in [Vau77]. Model theorists seem subsequently to have been infected with a kind of mania to attach their names to "two-cardinal theorems" (one for the model, one for the formula); thus an unbelievable variety of theorems of this sort blossomed at the end of the 1960s.

As I have said, and repeat, Morley's theorem (18.25) was our second birth [Mor65]: It solved a conjecture of Łoś. The hypothesis "T denumerable" can be eliminated (i.e., if T is categorical in any cardinal strictly greater than that of T, it is categorical in all of them); that actually had several stages: [Row64], [Res69], [She69]; see [She78], p. 469 and p. 512.

An ω_1-categorical theory has finite Morley rank, and whatever the model under consideration, the Morley rank of a type is equal to its Cantor rank [Bal73]; it is also equal to its Lascar rank [Poi78b].

Morley's theorem was treated in [Sac72], an already outdated expository work, but very useful in its own time, and whose reading will be considered as pleasant relaxation by all who want to start on the two remaining chapters.

19

Stability, Indiscernible Sequences and Weights

Si vous voulez simplifier, évitez de compliquer, car ce n'est pas en compliquant que vous simplifierez.

(Essai sur la vulgarisation)

M.T.

19.1 Indiscernible Sequences

Indiscernible sequences were covered at length in Chapter 12, and also in Section 16.3; I am now going to survey their properties, the gist of which we already know, under the hypothesis that the theory T is *stable.* As usual, I consider only sequences of elements, but there is no problem generalizing the situation to sequences of tuples.

We consider a sequence $a_0, a_1, \ldots, a_\alpha, \ldots$ indexed by an ordinal (we could also consider sequences indexed by arbitrary chains) consisting of elements of a model M of T; we study some properties of this sequence relative to a set A of parameters, which could be empty, and which we shall refrain from mentioning systematically. We define $A_\alpha = A \cup \{\ldots, a_\beta, \ldots\}_{\beta<\alpha}$ and let p_α be the type of a_α over A_α.

We shall call the sequence

- *increasing* if p_α is a son of p_β whenever $\beta > \alpha$; in this case we write p to denote the limit of the p_α;
- *indiscernible* if, whenever $\alpha_0 < \alpha_1 < \cdots < \alpha_n$, $(a_{\alpha_0}, a_{\alpha_1}, \ldots, a_{\alpha_n})$ and $(a_0, a_1, \ldots, a_n)$ have the same type over A;

- *independent* if for every α, p_α does not fork over A;

- *Morley* if it is independent and all the a_α have the same strong type over A.

An indiscernible sequence is obviously increasing. If it is infinite, then we know, in light of stability, that it is *totally indiscernible*, that $(a_0, \ldots, a_n)$ and $(a_{\alpha_0}, \ldots, a_{\alpha_n})$ have the same type over A whenever $\alpha_0, \ldots, \alpha_n$ are pairwise distinct; also, that it is *indivisible* (a property not involving A), that for every formula $f(x, \vec{b})$ there are either only finitely many a_α that satisfy $f(x, \vec{b})$, or else only finitely many a_α that satisfy $\neg f(x, \vec{b})$; moreover, these finite numbers are bounded by a function depending only on $f(x, \vec{y})$. We therefore define the *mean type* or *limit type* of the sequence over B as the type consisting of all formulas with parameters in B that are satisfied by all but finitely many of the a_α. You can return to Chapter 12, from Section 12.3 to 12.7, to review all of this. Recall that if the sequence $a_0, \ldots, a_\alpha, \ldots$ is independent, then the set $\{a_0, \ldots, a_\alpha, \ldots\}$ is independent, which means that for all α the type of a_α over $A \cup \{\ldots, a_\beta, \ldots\}_{\beta \neq \alpha}$ does not fork over A. Considering the finitistic character of forking, it is sufficient to verify this for finite sequences $a_0, \ldots, a_n$. By the symmetry and transitivity of forking, we can see that if we exchange two consecutive elements in the sequence, we get another independent sequence; by finitely many transpositions we can move any a_i to the last place, proving what we wanted.

It is therefore clear that what we are calling *Morley sequences* here are indeed Morley sequences of the strong types that we defined in Section 16.3; they are indiscernible.

A long enough increasing sequence, or rather an increasing sequence of large enough cofinality, will have a strong tendency to become Morley from some point on; more precisely, if the sequence has length α, and if the cofinality of α is at least $\kappa(T)$, then the $p_\beta, \beta < \alpha$ cannot cofinally often be forking extensions of each other, so from some β_0 on, the sequence becomes independent over A_{β_0}. It may not likewise be Morley, since the $a_\beta, \beta \geq \beta_0$, do not necessarily have the same strong type over A_{β_0}. Consider an equivalence relation E, with n equivalence classes, defined by a formula with parameters in this set; if a_{β_0} is not congruent to a_{β_0+1} modulo E, then it is not congruent to any later elements, because the sequence is increasing. Likewise, if a_{β_0+1} is not congruent to a_{β_0+2}, then it is not congruent to any later elements, etc. Since E has only n classes, all the elements must be in the same class from some a_{β_0+n-1} on. Therefore, if the multiplicity of p_{β_0} is finite, at the end of finitely many steps we find a sequence that is Morley over A_{β_0+n}. Otherwise, it is possible that ω stages will be needed to choose a strong type, but no more, regardless of how many E need to be considered, and in any event the sequence is Morley over $A_{\beta_0+\omega}$, provided that there remain elements farther out!

In the case in which the sequence is also indiscernible, a_{β_0} and a_{β_0+1} are congruent modulo E; if not, E would have infinitely many classes. Consequently, in this case, the sequence stabilizes at the stage after β_0.

Since Chapter 14 we have been using the fact that the majority of the elements of a long indiscernible sequence realize the mean type over $\vec{b}$ of this sequence (see Lemma 14.1); we generalize this phenomenon by showing that a tuple $\vec{b}$ cannot make too many elements of an independent sequence fork.

Lemma 19.1. *Let $\vec{a}_0, \ldots, \vec{a}_\alpha, \ldots$ be a sequence of tuples that is independent over A. Given $\vec{b}$, the number of indices α such that the type of a_α over $A_\alpha \cup \{\vec{b}\}$ forks over A is strictly less than $\kappa(T)$; it is finite if $tp(\vec{b}/A)$ is superstable.*

Proof. Since $tp(\vec{a}_\alpha/A_\alpha)$ does not fork over A, then $tp(\vec{a}_\alpha/A_\alpha \cup \{\vec{b}\})$ forks over A if and only if it forks over A_α, if and only if, by symmetry, $tp(\vec{b}/A_\alpha \cup \{\vec{a}_\alpha\}) = tp(\vec{b}/A_{\alpha_1})$ forks over A_α. That happens fewer than $\kappa(T)$ times, and in fact fewer times than the cardinal limiting descending ordinal chains below the bound of $tp(\vec{b}/A)$ in the fundamental order. $\square$

Therefore, if we remove those a_α, limited in number, whose type over $A_\alpha \cup \{\vec{b}\}$ forks over A, from our independent set, *we can see that the set consisting of the $\vec{b}$* (which we put first) *and the other $\vec{a}_\beta$ is independent over A.*

One immediate consequence of the lemma is also the following: in the independent sequence, *there are strictly fewer than $\kappa(T)$ of the $\vec{a}_\alpha$ such that $tp(\vec{a}_\alpha/A \cup \{\vec{b}\})$ forks over A.*

19.2 Lascar Inequalities

We begin this section with a little ordinal arithmetic: I remind you that the ordinal *sum* of two ordinals α and β is obtained by putting a copy of α and a copy of β end to end, every element of the former being less than every element of the latter. The sum can also be defined by induction on β:

- $\alpha + 0 = 0$;
- if β is a nonzero limit, then $\alpha + \beta = \sup\{\alpha + \gamma : \gamma < \beta\}$;
- $\alpha + (\beta + 1) = (\alpha + \beta) + 1$.

The sum is associative, but not commutative: $1 + \omega = \omega \neq \omega + 1$. It cancels on the left, since $\alpha + \beta$ has only one initial segment isomorphic to α, but not on the right.

The product $\alpha \cdot \beta$ is obtained by ordering the Cartesian product $\alpha \times \beta$ lexicographically *giving priority to the second coordinate*: $\alpha \cdot \beta$ is the sum of β copies of α. It can also be defined by induction:

- if β is a limit, then $\alpha \cdot \beta = \sup\{\alpha \cdot \gamma : \gamma < \alpha\}$ (in particular $\alpha \cdot 0 = 0$);
- $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$.

The product is associative but not commutative: $2 \cdot \omega = \omega \neq \omega \cdot 2 = \omega + \omega$. It distributes over addition on the left, but not on the right: $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.

Now we define the exponential α^β by induction on β:

- $\alpha^0 = 1$;
- $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$.

There will be no difficulty showing, by induction on γ, that $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$. We define the product as we do in order to have this identity.

Ordinal exponentiation will not be confused with cardinal exponentiation: From the cardinal perspective, ω^ω denotes the set of functions from ω to ω, whose cardinality is 2^ω; from the ordinal perspective, $2^\omega = \sup\ 2^n = \omega$ and $\omega^\omega = \sup\ \omega^n$. ω^ω is therefore a denumerable ordinal; as the contexts in which these two notions appear are sufficiently disjoint, we shall not bother to introduce two types of notation.

Lemma 19.2. *The ordinals of the form ω^α are the ordinals β such that for every $\gamma < \beta$, $\gamma + \beta = \beta$.*

Proof. Suppose $\beta = \omega^\alpha$, and let $\gamma < \beta$. If α is a limit, then $\gamma < \omega^{\alpha'}$ for some $\alpha' < \alpha$; by the induction hypothesis $\gamma + \omega^{\alpha'} = \omega^{\alpha'}$, from which the conclusion follows, since $\omega^{\alpha'}$ is an initial segment of ω^α. If $\beta = \omega^{\alpha+1}$, then for some n, $\omega^\alpha \cdot n \leq \gamma < \omega^\alpha \cdot (n+1)$ and $\gamma = \omega^\alpha \cdot n + \gamma'$ with $\gamma' < \omega^\alpha$. By the induction hypothesis $\gamma + \omega^{\alpha+1} = \omega^\alpha \cdot n + \gamma' + \omega^\alpha \cdot \omega = \omega^\alpha \cdot n + \gamma' + \omega^\alpha + \omega^\alpha \cdot \omega = \omega^\alpha \cdot (n + 1 + \omega) = \omega^\alpha \omega = \omega^{\alpha+1}$.

Now suppose that for all $\gamma < \beta$, $\gamma + \beta = \beta$. Let α be the least ordinal such that $\beta \leq \omega^\alpha$ (which does exist since $\beta \leq \omega^\beta$). If α is a limit, then $\beta = \omega^\alpha$; if $\alpha = \delta + 1$, then $\omega^\delta + \beta = \beta, \omega^\delta + \omega^\delta + \beta = \beta$, etc., so that $\omega^\delta \cdot n < \beta$ for every n, so $\beta \geq \omega^{\delta+1}$ and $\beta = \omega^\alpha$. □

Everybody knows how to write an integer in base ten; those who attended primary school not long ago also know that it can be written in base two. For ordinals, an arbitrary base can be chosen, but we shall focus our attention on base ω.

Lemma 19.3. *Every nonzero ordinal α can be written uniquely in the form $\alpha = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_m} \cdot n_m$, with $\alpha_1 > \alpha_2 > \cdots > \alpha_m, n_1, n_2, \ldots, n_m$ being positive integers.*

Proof. First let us show the existence of the expression. Let α'_1 be the least ordinal such that $\omega^{\alpha'_1} > \alpha$. This ordinal cannot be a limit, so $\alpha'_1 = \alpha_1 + 1$, and there is n_1 nonzero such that $\omega^{\alpha_1} \cdot n_1 \leq \alpha < \omega^{\alpha_1} \cdot (n_1 \cdot 1)$. Therefore, $\alpha = \omega^{\alpha_1} \cdot n_1 + \beta_1$, with $\beta_1 < \omega^{\alpha_1}$. If $\beta_1 = 0$, then we have what we are looking for; otherwise, repeat the operation, and $\beta_1 = \omega^{\alpha_2} \cdot n_2 + \beta_2$ with $\beta_2 < \omega^{\alpha_2}$. The process must stop at the end of finitely many stages, since the sequence $\alpha_1, \alpha_2, \ldots, \alpha_m$ is a decreasing sequence of ordinals.

For uniqueness, note that α_1 and n_1 are characterized by the bounds $\omega^{\alpha_1} \cdot n_1 \leq \alpha < \omega^{\alpha_1} \cdot (n_1 + 1)$, and we can use the fact that the sum cancels on the left. □

The *Cantor normal form* of the ordinal α is defined as its expression in the form $\Sigma\omega^{\alpha_i} \cdot n_i$, where all but finitely many of the n_i are zero; the preceding lemma states the existence and uniqueness of this expression.

How can we compare two ordinals α and β knowing their Cantor normal forms $\Sigma\omega^{\alpha_i} \cdot n_i$ and $\Sigma\omega^{\beta_i} \cdot m_i$? First look at their respective highest terms, i.e., the highest nonzero coefficient: $\alpha = \omega^{\alpha_1} \cdot n_1 + \cdots, \beta = \omega^{\beta_1} \cdot m_1 + \cdots$. If $\alpha_1 > \beta_1$, then $\alpha > \beta$; if $\alpha_1 = \beta_1$ and $n_1 > m_1$, then $\alpha > \beta$; if $\alpha_1 = \beta_1$ and $n_1 = m_1$, then we need to compare the remaining parts of each expression, which are strictly less than ω^{α_1}. We can therefore see that the well-ordering of the ordinals can be expressed as a lexicographic ordering of the finite sequences of nonzero integers indexed by ordinals, giving priority to terms of higher index. (We can also define ω^α as the set of functions from α to ω that take the value 0 on all but finitely many points, ordered according to this principle.)

How do we find the Cantor normal form of the sum $\alpha + \beta$? By Lemma 19.2, the term $\omega^{\beta_1} \cdot m_1$ is going to absorb all the $\omega^{\alpha_p} \cdot n_p$ with $\alpha_p < \beta_1$. So if $\alpha_q = \beta_1$, with the associated n_q possibly zero, then we have $\alpha + \beta = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\beta_1} \cdot (n_q + m_1) + \cdots + \omega^{\beta_s} \cdot m_s$. For example, if $\alpha = \omega^\omega + \omega^3 \cdot 6 + \omega^2 + 4$ and $\beta = \omega^3 + \omega \cdot 6$, then $\alpha + \beta = \omega^\omega + \omega^3 \cdot 7 + \omega \cdot 6$.

We now introduce a new function, the *natural sum* $\alpha \oplus \beta$ of two ordinals, which is the ordinal obtained by adding the Cantor normal forms of α and β term by term: If $\alpha = \Sigma\omega^{\alpha_i} \cdot n_i$ and $\beta = \Sigma\omega^{\alpha_i} \cdot m_i$, then $\alpha \oplus \beta = \Sigma\omega^{\alpha_i} \cdot (n_i + m_i)$. This natural sum is commutative, associative, and cancellable: these are all consequences of the uniqueness of the Cantor normal form. It allows us to do recursion on two variables at once: *If $\alpha \leq \alpha'$ and $\beta \leq \beta'$ and if one of these inequalities is strict, then $\alpha \oplus \beta < \alpha' \oplus \beta'$.*

In order to speak about unranked types, we add the symbol ∞ to the ordinals, with the conventions $\alpha + \infty = \infty + \alpha = \alpha \oplus \infty = \infty$.

All this arithmetic will help us in the following theorem, which allows us to bound, and in some cases to calculate, the U rank of an n-tuple as a function of the U ranks of the types of its elements. *We must insist on the fact that it is Lascar's U rank rather than any other rank that is involved.*

Let $tp(\vec{a}/A)$ denote the type of $\vec{a}$ over A, and let $\mathrm{RU}(\vec{a}/A)$ denote its Lascar rank.

Theorem 19.4 (Lascar Inequalities). $\mathrm{RU}(\vec{b}/A \cup \{\vec{a}\}) + \mathrm{RU}(\vec{a}/A) \leq \mathrm{RU}(\vec{a}^\frown\vec{b}/A) \leq \mathrm{RU}(\vec{b}/A \cup \{\vec{a}\}) \oplus \mathrm{RU}(\vec{a}/A)$.

Proof. We start with the first inequality, and show by induction on α that if $\mathrm{RU}(\vec{a}/A) \geq \alpha$, then $\mathrm{RU}(\vec{b}/A \cup \{\vec{a}\}) + \alpha \leq \mathrm{RU}(\vec{a}\frown\vec{b}/A)$.

If $\alpha = 0$ we show by induction on β that $\mathrm{RU}(\vec{b}/A \cup \{\vec{a}\}) \geq \beta$ implies $\mathrm{RU}(\vec{a}\frown\vec{b}/A) \geq \beta$. This is no problem if β is a limit. If $\mathrm{RU}(\vec{b}/A\cup\{\vec{a}\}) \geq \beta+1$, and if $tp(\vec{b}/A \cup \{\vec{a}\})$ is unstable, then we can see by counting types that $tp(\vec{a}\frown\vec{b})$ is unstable. Otherwise, $tp(\vec{b}/A \cup \{\vec{a}\})$ has a forking son with RU at least β, and we can find $B \supset A, \mathrm{RU}(\vec{b}/B \cup \{\vec{a}\}) \geq \beta$ with $tp(\vec{b}/B \cup \{\vec{a}\})$ forking over A. By Theorem 15.9 $tp(\vec{a}\frown\vec{b})$ is either unstable or forks over A, and by the induction hypothesis it has RU at least β, from which the conclusion follows.

The case of α a nonzero limit poses no problem.

If $\mathrm{RU}(\vec{a}/A) \geq \alpha + 1$, then $tp(\vec{a}/A)$ is unstable, and so is $tp(\vec{a}\frown\vec{b}/A)$, or else it has a forking son of RU at least α. We can therefore find $B \supset A, \mathrm{RU}(\vec{a}/B) \geq \alpha$, with $tp(\vec{a}/B)$ forking over A. Place B relative to $\vec{b}$ in such a way that the type of $\vec{b}$ over $B \cup \{\vec{a}\}$ does not fork over $A \cup \{\vec{a}\}$. (We can assume that the type of $\vec{b}$ over $A \cup \{\vec{a}\}$ is stable; if not, the type of $\vec{a}\frown\vec{b}$ over A would be unstable.) By the induction hypothesis $\mathrm{RU}(\vec{b}/B \cup \{\vec{a}\}) + \alpha \leq \mathrm{RU}(\vec{a}\frown\vec{b}/B)$; the conclusion follows from the facts that $\mathrm{RU}(\vec{b}/B \cup \{\vec{a}\}) = \mathrm{RU}(\vec{b}/A \cup \{\vec{a}\})$ and that $tp(\vec{a}\frown\vec{b})$ forks over A, by Theorem 15.9.

Now let us show the second inequality. It is obvious if one part of the right side is infinite. If $tp(\vec{a}\frown\vec{b})$ is unstable, then we can see by counting types that, as noted previously, either $tp(\vec{a}/A)$ or $tp(\vec{b}/A\cup\{\vec{a}\})$ is unstable. In the other cases, we can show by induction on α that if $\mathrm{RU}(\vec{a}\frown\vec{b}/A)$ is greater than or equal to α, then so is the second part. This is clear if α is a limit. If $\mathrm{RU}(\vec{a}\frown\vec{b}/A) \geq \alpha + 1$, then there is $B \supset A$ with $\mathrm{RU}(\vec{a}\frown\vec{b}/B) \geq \alpha$ and $tp(\vec{a}\frown\vec{b}/B)$ forking over A. By Theorem 15.9, either $tp(\vec{a}/B)$ forks over A or else $tp(\vec{b}/B \cup \{\vec{a}\})$ forks over $A \cup \{\vec{a}\}$. By the induction hypothesis $\alpha \leq \mathrm{RU}(\vec{b}/B\cup\{\vec{a}\}) \oplus \mathrm{RU}(\vec{a}/B)$; the conclusion follows from the facts that $\mathrm{RU}(\vec{a}/B) \leq \mathrm{RU}(\vec{a}/A), \mathrm{RU}(\vec{b}/B\cup\{\vec{a}\}) \leq \mathrm{RU}(\vec{b}/A\cup\vec{a})$, and that one of these inequalities is strict. □

The Lascar inequalities become equalities in the case of finite RU, or more generally all RU in the form $\omega^\alpha \cdot n$ for the same α. As for the general case, we shall contemplate the following example: T is the theory of differentially closed fields of characteristic zero, a is differentially transcendental over A, and b is the derivative of a.

$$\begin{aligned}
\mathrm{RU}(b/A \cup \{a\}) + \mathrm{RU}(a/A) &= 0 + \omega = \omega; \\
\mathrm{RU}(a/A \cup \{b\}) + \mathrm{RU}(b/A) &= 1 + \omega = \omega; \\
\mathrm{RU}(a\frown b/A) &= \omega; \\
\mathrm{RU}(b/A \cup \{a\}) \oplus \mathrm{RU}(a/A) &= 0 \oplus \omega = \omega; \\
\mathrm{RU}(a/A \cup \{b\}) \oplus \mathrm{RU}(b/A) &= 1 \oplus \omega = \omega + 1.
\end{aligned}$$

Theorem 19.5. *If tp($\vec{a}/A$) and tp($\vec{b}/A$) are stable, and if $\vec{a}$ and $\vec{b}$ are independent over A, then* $\mathrm{RU}(\vec{a}\frown\vec{b}/A) = \mathrm{RU}(\vec{a}/A) \oplus \mathrm{RU}(\vec{b}/A)$.

Proof. Since $\mathrm{RU}(\vec{b}/A\cup\{\vec{a}\}) = \mathrm{RU}(\vec{b}/A)$, it is enough to see that $\mathrm{RU}(\vec{a}/A)\oplus \mathrm{RU}(\vec{b}/A) \leq \mathrm{RU}(\vec{a}\frown\vec{b}/A)$. By Theorem 19.4, this is true if one of the RU is ∞; in the other case, we can show by induction on α that if $\mathrm{RU}(\vec{a}/A) \geq \alpha$ and $\mathrm{RU}(\vec{b}/A) \geq \beta$, then $\mathrm{RU}(\vec{a}\frown\vec{b}/A) \geq \alpha\oplus\beta$. Indeed, let us consider $\alpha' < \alpha$. Then there exists $B \supset A$ such that $\mathrm{RU}(\vec{a}/B) = \alpha'$. Moreover, we can place B so that the type of $\vec{b}$ over $B\cup\{\vec{a}\}$ does not fork over $A\cup\{\vec{a}\}$, that is to say over A, so $\vec{a}$ and $\vec{b}$ are independent over B, and by the induction hypothesis $\mathrm{RU}(\vec{a}\frown\vec{b}) \geq \alpha' \oplus \beta$. Since *tp*($\vec{a}/B$) forks over A, *tp*($\vec{a}\frown\vec{b}/B$) forks over A; therefore, $\mathrm{RU}(\vec{a}\frown\vec{b}/A) > \alpha' \oplus \beta$, and consequently $\mathrm{RU}(\vec{a}\frown\vec{b}/A) \geq \alpha \oplus \beta$. □

Theorem 19.6. *If tp($\vec{a}/A \cup \{\vec{b}\}$) is superstable, and if* $\mathrm{RU}(\vec{a}/A) \geq \mathrm{RU}(\vec{a}/A \cup \{\vec{b}\}) \oplus \alpha$, *then* $\mathrm{RU}(\vec{b}/A) \geq \mathrm{RU}(\vec{b}/A \cup \{\alpha\}) + \alpha$.

Proof. By induction on $\mathrm{RU}(\vec{a}/A \cup \{\vec{b}\}) \oplus \alpha$. Given $\beta < \alpha$, we choose B extending A such that *tp*($\vec{a}/B$) forks over A and $\mathrm{RU}(\vec{a}/B) \geq \mathrm{RU}(\vec{a}/A \cup \{\vec{b}\}) \oplus \beta$. Indeed, by Theorem 12.33 and Lemma 15.13, we can take β to be a fragment of the Morley sequence of a son of *tp*($\vec{a}/A$), and therefore the type of B over A will be stable. We place B so that *tp*($\vec{b}/B\cup\{\vec{a}\}$) does not fork over $A \cup \{\vec{a}\}$, and we distinguish two cases:

1. *tp*($\vec{b}/B$) forks over A. $\mathrm{RU}(\vec{b}/A)$ is infinite or strictly greater than $\mathrm{RU}(\vec{b}/B)$; by the induction hypothesis $\mathrm{RU}(\vec{b}/B) \geq \mathrm{RU}(\vec{b}/B \cup \{\vec{a}\}) + \beta = \mathrm{RU}(\vec{b}/A \cup \{\vec{a}\}) + \beta$.
2. *tp*($\vec{b}/B$) does not fork over A. By duality the type of B over $A \cup \{\vec{b}\}$ does not fork over A, but on the other hand, the type of B over $A \cup \{\vec{a}\}$ forks over A. The type of B over $A \cup \{\vec{a}, \vec{b})$ must therefore (follow the bounds!) fork over $A\cup\{\vec{b}\}$, that is to say $\mathrm{RU}(\vec{a}/B\cup\{\vec{b}) < \mathrm{RU}(\vec{a}/A\cup\{\vec{b})$. Hence $\mathrm{RU}(\vec{a}/B) \geq \mathrm{RU}(\vec{a}/B\cup\{\vec{b}\})\oplus(\beta+1)$, and by the induction hypothesis $\mathrm{RU}(\vec{b}/A) = \mathrm{RU}(\vec{b}/B) \geq \mathrm{RU}(\vec{b}/B\cup\{\vec{a}\})+\beta+1 = \mathrm{RU}(\vec{b}/A \cup \{\vec{a}\}) + \beta + 1$.

In both cases $\mathrm{RU}(\vec{b}/A) \geq \mathrm{RU}(\vec{b}/A \cup \{\vec{a}\}) + \beta + 1$, from which the result follows by the right-continuity of the sum. □

Corollary 19.7 (Lascar's Symmetry Lemma). *If tp($\vec{a}/A \cup \{\vec{b}\}$) is superstable, and if* $\mathrm{RU}(\vec{a}/A) \geq \mathrm{RU}(\vec{a}/A \cup \{\vec{b}\}) + \omega^\alpha \cdot n$, *then* $\mathrm{RU}(\vec{b}/A) \geq \mathrm{RU}(\vec{b}/A \cup \{\vec{a}\}) + \omega^\alpha \cdot n$.

Proof. If $\alpha = 0$, then $\mathrm{RU}(\vec{a}/A\cup\{\vec{b}\})+n = \mathrm{RU}(\vec{a}/A\cup\{\vec{b}\})\oplus n$. Otherwise, $\omega^\alpha \cdot n$ is a limit, and for all β strictly less than it, $\mathrm{RU}(\vec{a}/A\cup\{\vec{b}\})+\omega^\alpha \cdot n > \mathrm{RU}(\vec{a}/A \cup \{\vec{b}\}) \oplus \beta$. The result follows from the preceding theorem. □

It is convenient to interpret Lascar's lemma as a strengthening of the symmetry property of forking: if $\vec{b}$ makes the type of $\vec{a}$ fork at least $\omega^\alpha \cdot n$ times, then $\vec{a}$ makes the type of $\vec{b}$ fork at least $\omega^\alpha \cdot n$ times.

19.3 Weight of a Superstable Type

Consider a set X of elements, or of tuples, in a model M of the stable theory T. In light of the finitistic character of forking, there are maximal independent sets (over $\emptyset$, or else over a fixed set of parameters A) in X; call such a set a *basis* of X. For example, if X consists of realizations of a given type p in the model M, and if we can prove that two bases always have the same number of elements, we will thus get an invariant associated with M, a cardinal number that we shall call the *dimension* of X. The problem is that two distinct bases in general have no reason to have the same cardinality; in order to prove it, we would need the analogue of the *exchange lemma* for bases of vector spaces, in which one element of one basis is replaced by a single element of another.

That is where the notion of *weight* is involved. In the case of vector spaces, the elements, or rather their types, have weight one; if an element has a weight of type n, in the adapted version of the exchange lemma it will be possible to replace it by up to n elements. For example, if T is the theory of infinite sets, and $X = \{(a,b),(a,c),(b,d)\}$, with a,b,c,d distinct, we have $\{(a,b)\}$ and $\{(a,c),(b,d)\}$ as bases: That is due to the fact that the type of (a,b) has weight two (and these pairs become elements in T^{eq}). The weights of types are what prevent cardinalities of bases from varying beyond certain bounds.

So let p be in $S_1(A)$. We shall say that *the weight of p is strictly less than κ* if, for every B containing A, every independent sequence s over B, and every a realizing a nonforking extension of p over B, we can remove fewer than κ elements of s so as to have an independent sequence s' over $B \cup \{a\}$. The *weight* of p will be the maximum, if it exists, of all cardinals κ for which the weight of p is not strictly less than κ. With this convention, it is possible for a type to have weight strictly less than ω without having finite weight; in applications, we will mainly be considering types of finite weight.

Weight can also be characterized as follows:

Lemma 19.8. *The weight of p is less than κ if and only if for every B extending A, every independent sequence s over B, and every a realizing a nonforking extension of p over B, there are fewer than κ elements of s whose type over $B \cup \{a\}$ forks over B.*

Proof. If the weight of p is less than κ, then the criterion is indeed satisfied, since all these elements must be removed from s to form s'.

Conversely, consider B, a, s as in the definition of weight, and let s' be a maximal sequence from s which is independent over $B \cup \{a\}$. Set $B' = B \cup s'$. Then the type of a over B' does not fork over B, and for every b in $s - s'$, the type of b over $B' \cup \{a\}$ forks over B'. □

An element of weight 0, which can never make anything fork, even itself, is algebraic: It can be repeated as often as desired in an independent sequence. For this trivial reason, when we count bases, we exclude these elements from consideration.

By Lemma 19.2 we can see that the weight of p is strictly less than $\kappa(T)$, and even less than the cardinal that bounds the lengths of decreasing ordinal sequences in the fundamental order below p. Indeed, if $a_0, \ldots, a_\alpha, \ldots$ denotes the independent sequence s, then the number of a_α for which the type of a_α over $B_\alpha \cup \{a\}$ forks over B_α is less than $\kappa(T)$, and if we remove these a_α, we do get an independent sequence over $B \cup \{a\}$. Nevertheless, we must not conclude that the weight is the maximum number of times that a can make p_α fork; this number is in general much greater than the weight. For example, take the theory T of differentially closed fields of characteristic zero, with a_0 a differentially transcendental element and a_1 a transcendental constant. We can easily see, by calculating the transcendence degree over $\mathbf{Q}$ of $\mathbf{Q}(a_0, a_1)_{\mathrm{d}}$, that a_0 and a_1 are automatically independent over $\emptyset$; this is a consequence of a general result proved in Theorem 19.15. Set $a = a_0 + a_1$: Then a makes p_0 fork, since $a' = a_0'$, and also makes p_1, which is not realized, fork; most importantly, we shall see that the type of a, with rank ω, has weight one, and that the exchange lemma works: a and a_0 are independent, as are a and a_1.

If a type is of RU n, it cannot make p_α fork more than n times, and its weight is at most n; however, if its RU is ω, it can make p_α fork any arbitrarily large finite number of times, as we can easily see by modifying the example above. All these remarks are so that the next theorem does not seem to be a triviality.

Theorem 19.9. *In a superstable theory, every type (of an element or of a tuple) a has finite weight; if the Cantor normal form of the rank U of p is* $RU(p) = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_m} \cdot n_m$, *then the weight of p is at most* $n_1 + \cdots + n_m$.

Proof. Consider an independent sequence $s = (a_0, \ldots, a_\alpha, \ldots)$ over B and a realization a of a nonforking extension of p over B. Consider an index α such that the type of a_α over $B_\alpha \cup \{a\}$ forks at least ω^{α_1} times over B_α. That means that $RU(a/B_\alpha)$ has a nonzero coefficient for ω^{α_1} in its Cantor normal form, and that the coefficient of ω^{α_1} in $RU(a/B_{\alpha+1})$ is strictly less than it. Consequently, there are not more than n_1 such elements.

Let us now eliminate all those elements from the sequence s. We thus get a sequence $s' = (a_0', \ldots, a_\alpha', \ldots)$; the type of a_α' over $B_\alpha = B \cup \{\ldots, a_\beta, \ldots\}_{\beta<\alpha}$ does not fork over $B_\alpha' = B \cup \{\ldots, a_\beta', \ldots\}_{\beta<\alpha}$. The U

rank of the type of a'_α over $B'_\alpha \cup \{a\}$ is greater than that of the type of a'_α over $B_\alpha \cup \{a\}$: a makes p'_α fork less than p_α! Since, on account of Lascar's symmetry lemma, a cannot make the type of a'_α fork over B_α more than ω^{α_1} times, it also does not make the type of a'_α fork over B'_α any more often. Also, when we consider the indices for which this type forks at least ω^{α_2} times, we can see that there cannot be more than n_2 of them, again by symmetry: Indeed, in the U rank of the type of a, the leading coefficient is not modified, while the next one, that of ω^{α_2}, decreases by at least one every time.

We can therefore eliminate all those elements, and repeat the operation m times. □

Note that this proof uses not only the superstability of p, but also that of the elements of the independent sequence.

The following lemma should be compared to Theorem 19.5:

Lemma 19.10. *If a and b are independent over A, the weight of $tp(a\frown b/A)$ is the sum of that of $tp(a/A)$ and that of $tp(b/A)$.*

Proof. When we have an independent sequence, we remove fewer elements than the weight of $tp(a/A)$ to get an independent sequence over $A \cup \{a\}$, then fewer elements than the weight of $tp(b/A)$ to have an independent sequence over $A \cup \{a, b\}$.

For the converse, we use the second definition of weight, given by Lemma 19.8: We can find as many independent a_α as the weight of $tp(a)$, where a makes the type of each one fork. Likewise, we can find as many independent b_β as the weight of $tp(b)$, where b makes the type of each one fork. Then we can place $\{a, \dots, a_\alpha, \dots\}$ and $\{b, \dots, b_\beta, \dots\}$ independently. □

Theorem 19.11 (Exchange Lemma). *Let $A \subset M$, and let X be a subset of M (or a set of tuples in M) consisting of elements whose types over A all have weight one. Then two bases (i.e., maximal independent sets) of X over A always have the same cardinality.*

Proof. Let B and C be two bases of X with $|B| \leq |C|$. Enumerate B, $B = \{b_0, \dots, b_\alpha, \dots\}$, and set $B_\alpha = \{\dots, b_\beta, \dots\}_{\beta<\alpha}$. I construct a decreasing sequence C^α of subsets of C by induction so that $B_\alpha \cup C^\alpha$ are free, for every limit α, $C^\alpha = \bigcap_{\beta<\alpha} C^\beta$, and $C^{\alpha+1}$ is obtained by removing at most one point from C^α.

I start from $C = C^0$. Since $tp(b_0/A)$ has weight one, I get C^1 by removing at most one element from C^0 so that $C^1 \cup \{b_0\}$ is independent over A. I can remove an element from C^1 so as to obtain C^2 such that $C^2 \cup \{b_0, b_1\}$ is independent over A, etc. I thus know what to do at successor stages. At a limit stage α, this works simply because every finite subset of $B_\alpha \cup C^\alpha$ can be found in some $B_\beta \cup C^\beta$ with $\beta < \alpha$.

When I have finished, I must have exhausted all elements of C, since otherwise B would not be a maximal independent set; therefore, $|B| = |C|$. □

Theorem 19.12. *Let $A \subset M$, and let X be a subset of M consisting of nonalgebraic elements over A whose types over A all have weight strictly less than κ, where κ is a regular cardinal. Then if X has a basis of cardinality greater than or equal to κ, all its bases have the same cardinality.*

Proof. We do the same proof as in Theorem 19.11, this time by removing strictly fewer than κ elements from C at every stage. If $|C| = \kappa$, then the conclusion follows by the regularity of κ; if $|C| > \kappa$, we can even remove κ elements at each stage. □

For a very similar reason, if all the elements of X have weight at most n, and if B and C are two bases, then $\frac{1}{n} \cdot |B| \leq |C| \leq n \cdot |B|$.

To conclude, dimensions are well-defined for any stable theory whenever they are greater than $|T|^+$; they are well-defined for a superstable theory whenever they are infinite; and if they are finite and weights are bounded, they can fluctuate only within certain limits.

To illustrate this notion of weight, we are going to prove the following theorem:

Theorem 19.13. *If T is denumerable, superstable, and not ω-categorical, then it has infinitely many pairwise nonisomorphic denumerable models.*

Proof. If T has few (i.e., fewer than 2^ω) denumerable models, it has a prime model over every n-tuple of parameters (see Theorem 10.10 and Section 10.3). If T is not ω-categorical, then there is a tuple $\vec{a}$ whose type p over $\emptyset$ is not isolated. Let k denote its weight and let M_n be the prime model over a tuple $\vec{a}_n$ consisting of $(k+1)^n$ independent realizations of p.

The M_n are pairwise nonisomorphic, since M_{n+1} cannot be elementarily embedded in M_n. Otherwise, every basic tuple of $\vec{a}_{n+1}$ would have type that is isolated over a_n and not isolated over $\emptyset$; by the open mapping theorem, this type would fork, which is impossible, since the weight of the type of $\vec{a}_n$ over $\emptyset$ is $k(k+1)^n$, which is strictly less than $(k+1)^{n+1}$. □

19.4 Independence and Domination

If p and q are two types over A, we shall say that they are *weakly orthogonal* if whenever a realizes p and b realizes q, a and b are independent over A. We shall say that they are *orthogonal* if whenever B contains A and p' and q' are nonforking sons of p and q over B, p' and q' are weakly orthogonal.

When A is a model of T (which is assumed to be stable), weak orthogonality means further that the conjunction of $p(x_1)$ and $q(x_2)$ defines a unique complete type in two variables over M; in the case of an arbitrary set of parameters, that is true if one of the types is stationary. This definition is also valid when p is a type of n-tuple and q is a type of m-tuple.

Lemma 19.14. *Let $M \prec N$ be two models of T, let p and q be two types over M, and let p' and q' be their heirs over N. If p' and q' are weakly orthogonal, then so are p and q; if M is $|T|^+$-saturated and p and q are weakly orthogonal, then so are p' and q'.*

Proof. If a realizing p and b realizing q are not independent over M, place N such that the type of $a\hat{\ }b$ over N does not fork over M. Then a and b are not independent over N, and realize p' and q' respectively, which are not orthogonal.

Conversely, if p' and q' are not (weakly) orthogonal, then we can let a realize p' and b realize q', with some $\vec{c}$ in N such that some formula $f(a, b, \vec{c})$ is satisfied without being represented by either p' or q'. If M is $|T|^+$-saturated, we can find $\vec{c}'$ in M with the same type as $\vec{c}$ over the sets of parameters that define p and q, which are therefore not weakly orthogonal. $\square$

We can therefore see that over a sufficiently saturated model, orthogonality and weak orthogonality are equivalent notions.

In Section 18.4 we introduced the notion of a type p orthogonal to a formula f; we shall see in Section 20.1 that that means that p is orthogonal to every type satisfying f.

Weak orthogonality is actually a very weak notion when types over arbitrary sets of parameters are considered. For example, if T is the theory of differentially closed fields of characteristic zero, and C is a field of constants, then the type p of the minimal equation $x' = 0$ is weakly orthogonal to the type q of the minimal equation $x' = 1$; they are no longer weakly orthogonal once the base field contains an antiderivative of 1.

It is a more significant notion when types over models of T are considered. For T totally transcendental, we shall show that it passes to heirs; that is to say, it is equivalent to orthogonality in that case.

However, that is not the general case. Here is a very simple counterexample: T is the theory of the group $\mathbf{Z}$; the only subgroups of $\mathbf{Z}$ are those of the form $n\mathbf{Z}$, which are also the only definable subgroups, and there are only two possible filters of subgroups (see Section 13.3 Example 16): that of the realized types, which contains 0, and that of all $n\mathbf{Z}$ for $n \neq 0$. Consequently the U rank of every nonrealized type is equal to one and is determined by the specification of its class modulo $n\mathbf{Z}$ for every n, i.e., by its strong type over $\emptyset$; there are 2^ω nonrealized types over every model of T.

A type of rank one can fork only by being algebraized. Therefore, if p and q are two types over M, and if p is not realized in the algebraic closure of $M \cup \{b\}$, where b realizes q, then p and q are weakly orthogonal. Consequently, if M is denumerable, then p is weakly orthogonal to all but finitely many of the nonrealized types. Note in passing that if T is a theory with U-rank equal to one, then every algebraically closed set containing a model of T satisfies Tarski's test, and is therefore itself a model of T.

Now suppose that M is a model of T realizing all strong types over $\emptyset$, for example an ω_1-saturated model. If p is not realized, and if a realizes its strong type over $\emptyset$ in M, then $x-a$ is the nonrealized type divisible by every n over such a model: Two nonrealized types correspond via a translation by an element of M, and are not orthogonal.

Lascar's symmetry lemma allows us to show the orthogonality of some superstable types:

Theorem 19.15. *If p and q are superstable and not orthogonal, then they have a common index a_i corresponding to a nonzero coefficient of the Cantor normal form of their rank U; in particular, a type of U rank ω^α is orthogonal to every type of strictly smaller U-rank.*

Proof. If p and q are not orthogonal, then either they or some of their nonforking extensions can be realized by a and b that are not independent over a set A. Let ω^α be the greatest power of ω such that the type of a over $A \cup \{b\}$ forks at least ω^α times over A. When we go from the Cantor normal form of $\mathrm{RU}(a/A)$ to that of $\mathrm{RU}(a/A \cup \{b\})$, the term corresponding to ω^α must decrease by at least one, while the terms of order greater than ω^α are unchanged. By Lascar's symmetry lemma, the same phenomenon happens when we go from $\mathrm{RU}(b/A)$ to $\mathrm{RU}(b/A \cup \{a\})$; consequently, ω^α has a nonzero coefficient in both $\mathrm{RU}(a/A)$ and $\mathrm{RU}(b/A)$. □

We shall call a type p over A *regular* if it has the following property: For every B extending A, every nonforking son of p is orthogonal to every forking son of p. The preceding theorem therefore states that a type of rank ω^α is regular. For such a type p, we can therefore see that if $a_0, \ldots, a_\alpha, \ldots$ is a sequence of independent realizations of p, and if a realizes p, then it can make p_α fork only once. The definition therefore implies that p in some sense has weight one with respect to its own Morley sequence; we are going to see that p does in fact have weight one.

Lemma 19.16. *Let p be a stationary type over A and p' a nonforking extension of p over B. Then p is regular if and only if p' is regular.*

Proof. By definition of regularity, if p is regular, then so is p'; if p is not regular, we can find an extension C of A with $\vec{a}$ realizing a nonforking extension of p over C, and $\vec{a}$ and $\vec{b}$ dependent over C. Place B so that $tp(\vec{a}\vec{b}/B \cup C)$ does not fork over C. □

The following result is much less obvious:

Theorem 19.17. *A regular type has weight one.*

Proof. Let $\vec{a}$ realize the type p in $S_m(M)$, with $(\vec{b}, \vec{c})$ independent over M, while $(\vec{a}, \vec{b})$ and $(\vec{a}, \vec{c})$ are not. We must show that p is not regular.

Let q be the strong type of $\vec{a}$ over $M \cup \{\vec{b}\}$. Let $\vec{a}_1$ realize the nonforking extension of q over $M \cup \{\vec{a}, \vec{b}, \vec{c}\}$, then let $\vec{a}_2$ realize the nonforking extension of q over $M \cup \{\vec{a}, \vec{a}_1, \vec{b}, \vec{c}\}$, etc. $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n, \ldots$ is the Morley sequence of q over $M \cup \{\vec{a}, \vec{b}, \vec{c}\}$, while $\vec{a}, \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n, \ldots$ is the Morley sequence of q over $M \cup \{\vec{b}\}$.

There is a natural number n, and therefore also a smallest natural number n, such that the sequence $\vec{a}, \vec{a}_1, \ldots, \vec{a}_n$ is not independent over M; if there were none, then $\vec{a}, \vec{a}_1, \ldots, \vec{a}_n, \ldots$ would be the Morley sequence of p over M, and its mean type over $M \cup \{\vec{b}\}$, which is q, would be the heir of p, which it is not.

By construction, $tp(\vec{a}_1 \frown \cdots \frown \vec{a}_n / M \cup \{\vec{a}, \vec{b}, \vec{c}\})$ does not fork over $M \cup \{\vec{b}\}$, nor does $tp(\vec{a}_1 \frown \cdots \frown \vec{a}_n / M \cup \{\vec{b}, \vec{c}\})$. By duality, $tp(\vec{c}/M \cup \{\vec{a}_1, \ldots, \vec{a}_n, \vec{b}\})$ does not fork over $M \cup \{\vec{b}\}$, and also does not fork over M, since $\vec{b}$ and $\vec{c}$ are independent over M. In brief, $\vec{c}$ and $\vec{a}_1, \ldots, \vec{a}_n$ are independent over M.

By the minimality of the choice of n, and the indiscernibility of $\vec{a}, \vec{a}_1, \ldots, \vec{a}_n$ over $M \cup \{\vec{b}\}$, the sequence $\vec{a}_1, \ldots, \vec{a}_n$ is independent over M; it is the beginning of the Morley sequence of p over M. As it does not fork, it is also the beginning of the Morley sequence of the heir of p over $M \cup \{\vec{c}\}$.

So $tp(\vec{a}_1/M \cup \{\vec{a}_2, \ldots, \vec{a}_n, \vec{c}\})$ is the heir of p. $tp(\vec{a}/M \cup \{\vec{a}_2, \ldots, \vec{a}_n, \vec{c}\})$ is not the heir of p, since $\vec{a}$ and $\vec{c}$ are dependent over M; $tp(\vec{a}_1/M \cup \{\vec{a}, \vec{a}_2, \ldots, \vec{a}_n, \vec{c}\})$ is not the heir of p, since $\vec{a}, \vec{a}_1, \ldots, \vec{a}_n$ are not independent over M, and $tp(\vec{a}_1/M \cup \{\vec{a}, \vec{a}_2, \ldots, \vec{a}_n\})$ is not the heir of p. In other words, $\vec{a}$ and $\vec{a}_1$ are not independent over $M \cup \{\vec{a}_2, \ldots, \vec{a}_n, \vec{c}\}$, and we have found a forking son of p that is not orthogonal to the heir of p. □

When p and q are orthogonal, the subsets of the models associated with p and q respectively can be constructed autonomously in some sense: What we add on the side of p has no influence on anything relevant to q. We now introduce the opposite notion, which means that everything that comes from q is determined by what comes from p.

We consider $A, \vec{a}, \vec{b}$; I say that $\vec{a}$ *weakly dominates* $\vec{b}$ over A if for every $\vec{c}$ such that $tp(\vec{c}/A \cup \{\vec{b}\})$ forks over A, $tp(\vec{c}/A \cup \{\vec{a}\})$ also forks over A; in other words, everything made to fork by $\vec{b}$ is also made to fork by $\vec{a}$. I say that $\vec{a}$ *dominates* $\vec{b}$ over A if for every B containing A, if $tp(\vec{a} \frown \vec{b}/B)$ does not fork over A, then $\vec{a}$ weakly dominates $\vec{b}$ over B.

It is fairly clear that domination implies weak domination, that if $tp(\vec{a} \frown \vec{b})$ does not fork over A and $\vec{a}$ does not dominate $\vec{b}$ over A it also does not dominate it over B: By considering $\vec{c}$ dependent on $\vec{b}$ but independent of $\vec{a}$ over A, place it without forking, etc.

As for the following lemma, it clarifies an irritating problem about heirs.

Lemma 19.18. *Over a model M that is $|T|^+$-saturated, domination and weak domination are equivalent; if N is an elementary extension of M, if*

$tp(\vec{a}\frown\vec{b}/N)$ does not fork over M, then $\vec{a}$ dominates $\vec{b}$ over M if and only if it does the same over N.

Proof. Only the nontrivial direction remains to be seen. If $\vec{a}$ does not dominate $\vec{b}$ over N, then we can find $\vec{c}$ with $\vec{a}$ and $\vec{c}$ independent but $\vec{b}$ and $\vec{c}$ dependent over N. Let A be a subset of M of cardinality at most $|T|$ such that $tp(\vec{a}\frown\vec{b}/N)$ is the unique nonforking extension of its restriction to M. Let C be a subset of N of cardinality $|T|$ such that $tp(\vec{c}/N)$ is the unique nonforking extension of its restriction to C. In light of the hypothesis of saturation, we can let C' in M realize the type of C over A, and by transferring the definition of $tp(\vec{c}/C)$, we get a tuple $\vec{c}'$, independent of $\vec{a}$, and dependent on $\vec{b}$, over M. □

That being said, we shall say that p *has higher power than* q, or *D-majorizes* q, in symbols $p \geq_D q$ (D for domination), if it is possible for p to be realized by $\vec{a}$ and q by $\vec{b}$ so that $\vec{a}$ dominates $\vec{b}$. We have done all this to ensure that this notion is preserved by heirs and that two types can be compared by passing to a sufficiently saturated model.

Note that this notion is indeed transitive: If p majorizes q and q majorizes r, then p majorizes r; to see this, take heirs in a very saturated model, realizing p, q, r by $\vec{a}, \vec{b}$, and $\vec{c}$, with $\vec{a}$ dominating $\vec{b}$ and $\vec{b}$ dominating $\vec{c}$. Then everything that forks because of $\vec{c}$, forks because of $\vec{b}$, and therefore also because of $\vec{a}$. Power thus defines a preorder, and if $p \geq_D q$ and $q \geq_D p$, we shall say that p and q *have the same power* or are *D-equivalent.*

We leave algebraic types out of the discussion (especially when we speak of D-minimal types), as they can never make anything fork, and are therefore both independent from everything and dominated by everything. Apart from this case, if p dominates q, then p and q are not orthogonal, since q, when it is realized, makes itself fork.

The counterexample given for orthogonality shows that we cannot do without the reference to sufficiently saturated models to define power, since two weakly orthogonal types can have D-equivalent heirs.

As with orthogonality and independence, we should not confuse power with domination. Orthogonality and power are properties of types, while independence and domination are properties of their realizations. When we are working over a sufficiently saturated model, p and q are orthogonal if they can be realized only in an independent manner; p has higher power than q if we can realize them so that the realization of p dominates that of q.

Lemma 19.19. *If p has higher power than q, its weight is greater than or equal to that of q; if p is of strictly greater power than q, then* $\text{weight}(p) \geq \text{weight}(q) + 1$.

Proof. We must show that in a sufficiently saturated model M, we can realize p by $\vec{a}$ and q by $\vec{b}$ so that $\vec{a}$ dominates $\vec{b}$. We can find $\vec{b}_1, \ldots, \vec{b}_k$, equal in number to the weight of $tp(\vec{b}/M)$, all independent over M, with $\vec{b}$ making every $\vec{b}_i$ fork (Lemma 19.8); for every i, $\vec{a}$ and $\vec{b}_i$ are not independent over M.

If, moreover, $\vec{b}$ does not dominate $\vec{a}$, then we can find $\vec{c}$ dependent on $\vec{a}$ and independent of $\vec{b}$ over M. We therefore place $\vec{b}_1, \ldots, \vec{b}_k$, chosen as before, so that $tp(\vec{b}_1{}^\frown \cdots {}^\frown\vec{b}_k/M \cup \{\vec{a}, \vec{b}, \vec{c}\})$ does not fork over $M \cup \{\vec{b}\}$. The type of $\vec{c}$ over $M \cup \{\vec{b}, \vec{b}_1, \ldots, \vec{b}_k\}$ does not fork over $M \cup \{\vec{b}\}$, that is to say over M, so that $\vec{c}$ and $\vec{b}_1{}^\frown \cdots {}^\frown\vec{b}_k$ are independent over M. Since $\vec{a}$ makes the type of each fork, the weight of p is at least $k+1$. □

We have actually proved more than we claimed in the statement of the lemma: If we are working over a very saturated model M, if $\vec{a}$ dominates $\vec{b}$, and if the weights of $tp(\vec{a}/M)$ and $tp(\vec{b}/M)$ are finite and equal, then $\vec{b}$ dominates $\vec{a}$.

Lemma 19.20. *Two types of weight one are either orthogonal or D-equivalent; a type of weight one is minimal for the order D and D-minorizes every type that is not orthogonal to it.*

Proof. Work in a very saturated model over M. Let p, with weight one, be realized by $\vec{a}$, and let q be realized by $\vec{b}$, with $\vec{a}$ and $\vec{b}$ dependent over M. If the type of $\vec{c}$ over $M \cup \{\vec{a}\}$ forks over M, then the type of $\vec{c}$ over $M \cup \{\vec{b}\}$ must also fork over M; otherwise, $\vec{b}$ and $\vec{c}$ would be independent over M, and $\vec{a}$ would make both fork. Therefore, p would have weight at least two. □

Lemma 19.21. *Let M be a model of T and let $\vec{a}$ and $\vec{b}$ not be independent over M. There is therefore an elementary extension N of M such that $tp(\vec{a}/N)$ does not fork over M, $\vec{b}$ is not in N, and $\vec{a}$ dominates $\vec{b}$ over N.*

Proof. If $\vec{a}$ dominates $\vec{b}$ over M, this is perfect. If not, we can go up to a sufficiently saturated model M_0 without forking, and find $\vec{c}_1$ such that $\vec{a}$ and $\vec{c}_1$ are independent over M_0 and $\vec{b}$ and $\vec{c}_1$ are independent over M_0.

So let M_1 be a sufficiently saturated model containing M_0 and $\vec{c}_1$, placed so that the type of $\vec{a}{}^\frown\vec{b}$ over M_1 does not fork over $M_0 \cup \{\vec{c}_1\}$. Note that $tp(\vec{a}/M_1)$ is the heir of $tp(\vec{a}/M)$ and that $tp(\vec{b}/M_1)$ forks over M. The type of $\vec{a}$ over $M_1 \cup \{\vec{b}\}$ forks over M, since the type of $\vec{a}$ over $M \cup \{\vec{b}\}$ already forks over M. Consequently, $\vec{a}$ and $\vec{b}$ are not independent over M_1, which implies that $\vec{b}$ is not in M_1.

If $\vec{a}$ dominates $\vec{b}$ over M_1, stop. Otherwise, proceed by introducing $\vec{c}_2$ and M_2, etc. and continue by taking limits at limit stages. We must stop before $\kappa(T)$, since the type of $\vec{b}$ forks every time. □

Corollary 19.22. *If p and q are not orthogonal, then q has a nonrealized son of lesser power than p.*

Proof. By simply translating the previous lemma. □

Corollary 19.23. *If T is superstable, then every (nonrealized) type over a sufficiently saturated model has higher power than a regular type (and even higher than a type orthogonal to all types of lesser* RU*).*

Proof. Let q be a nonrealized type of minimum U rank that is weaker than p. By Corollary 19.22 and the fact that we are working over a sufficiently saturated model, which allows us to transfer what we get over an elementary extension of M to merely over M, q is orthogonal to every type of strictly lower RU. It is therefore regular. □

If we jump into the imaginary world we can be even more precise (by Theorem 19.12 a type of U rank ω^α is regular):

Theorem 19.24 (Buechler's Lemma). *If T is superstable and* $\mathrm{RU}(p) = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot n_k$, *with* $\alpha_1 > \cdots > \alpha_k, n_k \neq 0$, *then p is nonorthogonal to an imaginary type of* RU ω^{α_k}.

Proof. Let p be a type over M. Then it has an extension to some model M_1 with RU $\omega^{\alpha_1} n_1 + \cdots + \omega^{\alpha_k}(n_k - 1)$, with $\alpha_1 > \cdots > \alpha_k$, which we can let a realize. Let A be the canonical set of definition of $tp(a/M_1)$.

Since A makes p fork, we must have $\mathrm{RU}(a/M) \geq \omega^{\alpha_k}$, and we can find an extension N of M such that $\mathrm{RU}(A/N) = \omega^{\alpha_k}$. (Indeed, $tp(a/M)$ does have a U rank, since A is the algebraic closure of a finite set of parameters.)

Place N and a independently over $M^\frown A$. The type of a over $N^\frown A$ does not fork over $M^\frown A$; since A is canonical for $tp(a/N^\frown A)$, and is not contained in N, $tp(a/N^\frown A)$ forks over N. By Lascar's symmetry lemma, this forking must alter the coefficient of ω^{α_k} in $\mathrm{RU}(a)$ by one, $\mathrm{RU}(a/N) \geq \omega^{\alpha_1} n_1 + \cdots + \omega^{\alpha_k} n_k$, and since $\mathrm{RU}(a/N) \leq \mathrm{RU}(a/M)$, they are actually equal; $tp(a/N)$ is an heir of $tp(a/M)$ and is not orthogonal to $tp(a/N)$. □

Corollary 19.25. *If T is superstable, then the following four conditions are equivalent:*

1. *p has weight one.*

2. *p is D-minimal.*

3. *p is D-equivalent to a regular type.*

4. *p is D-equivalent to an imaginary type (i.e., a type in the sense of T^{eq}) with U rank of the form ω^α.*

Proof. All equivalences are immediate consequences of the foregoing.
time,

□

Given types $p_1, \ldots, p_n$ over the model M, we shall call the type of $\vec{a}_1{}^\frown \cdots {}^\frown \vec{a}_n$ the *product* of these types, where $\vec{a}_1, \ldots, \vec{a}_n$ are realizations of

$p_1, \dots, p_n$, respectively, that are independent over M. By forking symmetry, this product is "commutative," at least up to D-equivalence, and associative.

Theorem 19.26. *Let T be a superstable theory and let M be a model of T. Every type of $S_m(M)$ is D-equivalent to a product of finitely many D-minimal types, their number being equal to the weight of the type in question; furthermore, this decomposition is unique up to permutation of factors: To describe the order D of the types of finite tuples over M, we form all commutative monomials in as many variables as there are classes of D-minimal types over M and define the monomial μ to be less than the monomial μ' if every partial degree of μ is less than the corresponding partial degree of μ'.*

Proof. Let $\vec{b}$ realize the type q, and consider a sequence $\vec{a}_1, \dots, \vec{a}_k$ of maximal length realizing a product $p_1 \cdots p_k$ of D-minimal types, with each $\vec{a}_i$ dependent on $\vec{b}$ over M. Then, k is majorized by the weight of q.

As we are working over a sufficiently saturated model, for verifying domination it is sufficient to do it with D-minimal types (Corollary 19.23). So let us show that $\vec{a}_1{}^\frown \cdots {}^\frown \vec{a}_n$ dominates $\vec{b}$ over M. If it did not, we could find $\vec{a}_{k+1}$ dependent on $\vec{b}$ and independent of $\vec{a}_1{}^\frown \cdots {}^\frown \vec{a}_n$ with $tp(\vec{a}_{k+1}/M)$ minimal: This contradicts the maximality of k.

As a consequence, by Lemma 19.19, k is actually the weight of q, and $\vec{b}$ also dominates $\vec{a}_1{}^\frown \cdots {}^\frown \vec{a}_k$; q is D-equivalent to the product $p_1 \cdots p_k$.

By grouping all equivalent p_i we get an expression for q in the form $p_1^{n_1} \cdots p_m^{n_m}$, which is unique. Indeed, by using the pairwise orthogonality of the p_i, which is preserved by their heirs, we can easily see that $p_i^{n_1}$ is the highest power of p_i that is weaker than q.

The description of the order D follows immediately. □

We shall show one last result, which shows that the order D has the meaning that we attributed to it:

Theorem 19.27. *Let M be a $|T|^+$-saturated model of T and let p and q be two types over M. Then p has greater power than q if and only if every $|T|^+$-saturated extension of M that realizes p also realizes q.*

Proof. First, assume that p has greater power than q. Let $\vec{a}$ realize the former and $\vec{b}$ the latter, so that $\vec{a}$ dominates $\vec{b}$ over M. If M_0 is a submodel of M of cardinality $|T|$ such that $tp(\vec{a}{}^\frown\vec{b})$ does not fork over M_0, we know that $\vec{a}$ dominates, and therefore weakly dominates, $\vec{b}$ over M_0. Now, if N is a $|T|^+$-saturated model containing M and $\vec{a}$, then it realizes, as $\vec{b}'$, the type of $\vec{b}$ over $M_0 \cup \{\vec{a}\}$. Since $\vec{a}$ and M are independent over M_0, and $\vec{a}$ dominates $\vec{b}'$ over that model, $\vec{b}'$ and M are independent over M_0, and $\vec{b}'$ realizes the heir of $tp(\vec{b}'/M_0)$, i.e., q, over M.

Conversely, let $\vec{b}$ realize q in the $|T|^+$-saturated prime model over M and over a realization $\vec{a}$ of p. That means that the type of $\vec{b}$ over $M \cup$

$\{\vec{a}\}$ is $|T|^+$-isolated by $|T|$ formulas $f_i(\vec{b},\vec{a})$ with parameters in M. Let us show that that implies that $\vec{a}$ dominates $\vec{b}$. If it did not, we could find $\vec{c}$ independent of $\vec{a}$ over M satisfying a formula $g(\vec{c},\vec{b})$ making it fork, i.e., isolating it from M. We have that $\vec{c}$ satisfies $(\exists\vec{y})(g(\vec{c},\vec{y}) \wedge f_i(\vec{y},\vec{a}))$. Let M_0 be a model of small cardinality allowing the type of $\vec{a}$ over M to be defined, and let $\vec{c}'$ realize $tp(\vec{c}/M_0)$ in M. Then $\vec{c}'$ also satisfies every one of these formulas; consequently, we can find $\vec{b}'$ somewhere such that $g(\vec{c}',\vec{b}')$ and all the $f_i(\vec{b}',\vec{a})$ are true. Then $\vec{b}'$ has the same type as $\vec{b}$ over $M \cup \{\vec{a}\}$, and we get a contradiction with the fact that no tuple of elements of M can satisfy $g(\vec{x},\vec{b}')$. □

Beware of the fact that even if the model M of reference is very saturated, if $\vec{a}$ dominates $\vec{b}$, then the type of $\vec{b}$ over $M \cup \{\vec{a}\}$ is not necessarily $|T|^+$-isolated; this is only because the notion of domination is transitive, while that of isolation is not: If $\vec{b}$ is $|T|^+$-isolated over $\vec{a}$, and likewise $\vec{c}$ over $\vec{b}$, then $\vec{c}$ is not necessarily $|T|^+$-isolated over $\vec{a}$. That does not conflict with the transitivity of atomicity, which states that if $tp(\vec{b}/\vec{a})$ and $tp(\vec{c}/\vec{a}^\frown\vec{b})$ are both isolated, then $tp(\vec{c}/\vec{a})$ is isolated. For example, if T is the theory of differentially closed fields of characteristic zero, and if a_1 and a_2 are differentially transcendental over M and have the same derivative a, then they each dominate each other, and the type of a over $M \cup \{a_1\}$ is isolated, as is its type over $M \cup \{a_2\}$, and as are the types of a_1 and a_2 over $M \cup \{a\}$. In contrast, the type of a_1 over $M \cup \{a_2\}$ is isolated, or even $|T|^+$-isolated if M is $|T|^+$-saturated, only if the constant $a_1 - a_2$ is in M.

19.5 Historic and Bibliographic Notes

Lascar's Inequalities, and his symmetry lemma, come from [Las76]. There, he majorized the weight of a superstable type by the product of the coefficients of the Cantor normal form of its U-rank; he replaced the product by the sum in [Las84].

The notions of weight, regularity, orthogonality, etc. are from Shelah, [She78].

Theorem 19.13, on the denumerable models of a superstable theory, is from [Lac73], but the proof given here is Lascar's; it is not known whether the result remains true assuming only stability.

It was also Lascar who introduced the notions of domination and of the order D in [Las82], to which section 19.4 owes much, especially for Lemma 19.21, Corollary 19.25, and Theorem 19.26; the exposition, however, has been simplified by the very fast introduction of the weight. The imaginary characterization of a regular type, Corollary 19.25 (4), is from [Las84].

The product of types is the Lascarian version of forking, [Las73].

20

Dimension in Models of a Totally Transcendental Theory

In fact, after the appearance of the author's abstract (6), SHELAH informed me that he had been familiar with our theorem for a long time and pointed out the terse statement in (16) where he announces that ...

A.H.L.

20.1 Rudin–Keisler Order

Throughout this chapter we will be considering a *totally transcendental* theory T; sometimes, this will be only to simplify our lives, as the arguments that we will advance remain valid in the case of a stable or superstable theory, provided that we consider only sufficiently saturated models. In other cases, in contrast, our life will be more complicated, since we will be doing more ambitious proofs: When we adopt the rather strong hypothesis of total transcendence, we are going to attempt to classify all models of T, if possible, instead of only the sufficiently saturated ones. The tools that will help us in this task will not always behave as simply as we might hope.

A typical case of good behavior is the theorem on decomposition into regular types (19.26), which relates to the order D; Theorem 19.27 clearly shows that this order, with such delightful properties, is adequate when we want to construct or examine sufficiently saturated, or $\aleph_\epsilon$-saturated, models—that is, ω-saturated in the totally transcendental case.

We now introduce the order R, also called RK, which is the analogue of the order D without saturation, applying to all models, and which is sufficiently similar to it without being identical. Just as the order D (for a superstable theory) is based on regular types, so this order R will be based on strongly regular types (which exist by total transcendence), but I must say that the situation will not be so fine through and through. Let me add at the onset that the reader who is frugal with his effort, who finds the ideas developed in this section and in Section 19.4 a little Byzantine, can only skim the text here, giving most of his attention to the strongly regular types: They are the useful ones for constructing and classifying models.

In this section we principally consider complete types over models of T.

If p is in $S_n(M)$ and q is in $S_m(M)$ we say that *p is greater than q in the order R* (R for realization) *over M*, and write $p \geq_R q$, if every elementary extension of M that realizes p also realizes q. In light of the existence, for all $\vec{a}$, of the prime model $M(\vec{a})$ over $M \cup \{\vec{a}\}$, that means that it is possible to have $\vec{a}$ realizing p and $\vec{b}$ realizing q in such a way that the type of $\vec{b}$ over $M \cup \{\vec{a}\}$ is isolated. It is clear that we are dealing with a partial order (or more precisely a preorder); if $p \geq_R q$ and $q \geq_R p$, we shall call p and q *R-equivalent.*

The order R is sometimes called the *Rudin–Keisler order.*

As in the last chapter, we are excluding realized types from consideration, especially when we talk about R-minimal types.

We can therefore see that we have an order R for each model M of T. When we change models, the various R orders that appear are not unrelated, since, as we shall see, various properties are inherited. Under the conventions that we have adopted, the order D involves types over ω-saturated models: To compare two types, we actually compare their heirs over a sufficiently saturated model. The order D is the analogue of the order R for ω-saturated models, and we are going to see that $p \geq_R q$ implies $p \geq_D q$, *but the converse is false even if M is very saturated.*

It is important to keep an eye on the gist of the following counterexample. The language of T consists of a unary function symbol s and a binary relation E; its axioms say that s is an acyclic bijection and that E is an equivalence relation with infinitely many classes, each class of which is closed under s. We can see that every E-class consists of some number of copies of the integers equipped with their successor function. Since every class in an ω-saturated model of T must contain infinitely many of these copies, we can easily see by a back-and-forth argument that T is complete and admits quantifier elimination. Let us take p to be the type stating that the class of x has no representative in M, and q to be the type in two variables stating that x_1 and x_2 are congruent modulo E, that their class is not represented in M, and that neither one is obtained from the other by iterating the successor function. If I add a new class to M to get an ω-saturated model, I will need to have infinitely many copies of $\mathbf{Z}$ in this class, so p and q are D-equivalent; we have two types of weight one, which are even regular ($\mathrm{RU}(p) = 1, \mathrm{RU}(q) = 2$). On the other hand, even if M is very saturated, p is strictly below q in the order R, since I can simply get a model of T by adding one class with just one copy of $\mathbf{Z}$.

Lemma 20.1. *Let N be an elementary extension of M, let p and q be two types over M, and p' and q' their respective heirs over N. If $p \geq_R q$, then $p' \geq_R q'$; the converse is true if M is ω-saturated.*

Proof. Let $\vec{a}$ realize p and p' and let $\vec{b}$ realize q, with the type of $\vec{b}$ over $M \cup \{\vec{a}\}$ isolated by a formula $f(\vec{x}, \vec{a}, \vec{c})$ for some $\vec{c}$ in M. I claim that this formula also isolates a type over N, a type that can only be the heir q' of q ($\vec{b}$ is placed with no forking). Otherwise, we could find $\vec{d}$ in N and a formula $g(\vec{x}, \vec{a}, \vec{d})$ such that $g(\vec{x}, \vec{a}, \vec{d}) \wedge f(\vec{x}, \vec{a}, \vec{c})$ and $\neg g(\vec{x}, \vec{a}, \vec{d}) \wedge f(\vec{x}, \vec{a}, \vec{c})$ are both consistent. In other words,

$$p' \models (\exists \vec{y})(f(\vec{y}, \vec{x}, \vec{c}) \wedge g(\vec{y}, \vec{x}, \vec{d})) \wedge (\exists \vec{z})(f(\vec{z}, \vec{x}, \vec{c}) \wedge g(\vec{z}, \vec{x}, \vec{d})).$$

Since p' is the heir of p, we can actually find $\vec{d_1}$ in M such that this remains true when we replace $\vec{d}$ by $\vec{d_1}$, contradicting the isolating character of $f(\vec{x}, \vec{a}, \vec{c})$.

For the converse, we can assume that N is ω-saturated and consider $\vec{a}$ realizing p', $\vec{b}$ realizing q', with the type of $\vec{b}$ over $N \cup \{\vec{a}\}$ isolated by $f(\vec{x}, \vec{a}, \vec{c}')$. We know that p and q can be defined over finite subsets A and B of M; since the latter is ω-saturated, we can find $\vec{c}$ in M realizing the same type as $\vec{c}'$ over $A \cup B$. It is therefore clear that $f(\vec{x}, \vec{a}, \vec{c})$ isolates a type over $M \cup \{\vec{a}\}$ whose restriction to M is q. □

One problem that is still open is to know whether we can drop the hypothesis "M is ω-saturated" from the converse of Lemma 20.1.

Lemma 20.2. *Let M be a model of T. If the type of $\vec{b}$ over $M \cup \{\vec{a}\}$ is isolated, then $\vec{a}$ weakly dominates $\vec{b}$ over M. (If $\vec{b}$ and $\vec{c}$ are not independent over M, then neither are $\vec{a}$ and $\vec{c}$.)*

Proof. Let $\vec{c}$ be dependent on $\vec{b}$ over M. There is therefore a formula $f(\vec{x}, \vec{b})$, with additional parameters in M, that isolates $\vec{c}$ from the elements of M. Let $g(\vec{y}, \vec{a})$ be a formula isolating the type of $\vec{b}$ over $M \cup \{\vec{a}\}$.

The type of $\vec{c}$ over $M \cup \{\vec{a}\}$ therefore satisfies $(\exists \vec{y})(f(\vec{x}, \vec{y}) \wedge g(\vec{y}, \vec{a}))$. If it did not fork, this formula would be satisfiable in M, and we could find $\vec{c}'$ in M, and $\vec{b}'$ somewhere, such that $f(\vec{c}', \vec{b}') \wedge g(\vec{b}', \vec{a})$ is true. Since $\vec{b}$ and $\vec{b}'$ have the same type over $M \cup \{\vec{a}\}$, $f(\vec{c}', \vec{b})$ is also true, contradicting the fact that $f(\vec{x}, \vec{b})$ is not satisfiable in M. □

Note the following somewhat strange situation: Assume that $tp(\vec{a}\frown\vec{b}/N)$ does not fork over M. If $tp(\vec{b}/M \cup \{\vec{a}\})$ is isolated, then $tp(\vec{b}/N \cup \{\vec{a}\})$ is isolated; if $\vec{a}$ weakly dominates $\vec{b}$ over N, then it also does so over M. In fact, because of Lemma 20.1, we get strong domination under the hypotheses of Lemma 20.2.

Corollary 20.3. *If p dominates q in the order R, then it also dominates it in the order D.*

Proof. We can climb to a sufficiently saturated model by Lemma 20.1, then use Lemma 20.2. □

Corollary 20.4. *If p and q are (weakly) orthogonal, then they have no common R-minorants.*

Proof. If p and q have a common (nonrealized!) R-minorant, we can have them realized by $\vec{a}$ and $\vec{b}$ so that there is some $\vec{c}$ whose types over both $M \cup \{\vec{a}\}$ and $M \cup \{\vec{b}\}$ are isolated. We therefore know that $\vec{a}$ dominates $\vec{c}$ over M. Since $\vec{b}$ and $\vec{c}$ are not independent, neither are $\vec{a}$ and $\vec{b}$. □

Corollary 20.5. *If $p \geq_R q$ and p is (weakly or strongly) orthogonal to r, then r and q are (likewise) orthogonal; (weak or strong) orthogonality is preserved under R-equivalence.*

Proof. Let $\vec{a}$ realize p and $\vec{b}$ realize q, with the type of $\vec{b}$ over $M \cup \{\vec{a}\}$ isolated. No realization $\vec{c}$ of r can be made to fork by $\vec{a}$, nor therefore by $\vec{b}$, since $\vec{a}$ dominates $\vec{b}$; this yields the results for weak orthogonality. Use Lemma 20.1 for strong orthogonality. □

Corollary 20.6. *If $\vec{a}$ and $\vec{b}$ are independent over M, and if N_1 and N_2 are atomic over $M \cup \{\vec{a}\}$ and $M \cup \{\vec{b}\}$ respectively, then they are independent over M; in particular, $N_1 \cap N_2 = M$.*

Proof. By Lemma 20.2, N_1 and $\vec{b}$ are independent, hence N_1 and N_2 too. For the last point, if c is not in M, then $tp(c/M \cup \{c\})$ forks over M. □

Corollary 20.7. *Let $\vec{a}_0, \ldots, \vec{a}_\alpha, \ldots$ be an independent sequence over M, and let M_α be an increasing sequence of models constructed as follows:*

- *$M_0 = M$, and for α limit, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$*
- *$M_{\alpha+1}$ is prime over $M_\alpha \cup \{\vec{a}_\alpha\}$.*

Then for every α, M_α is prime over $M \cup \{\vec{a}_0, \ldots, \vec{a}_\beta, \ldots\}_{\beta<\alpha}$.

Proof. It is sufficient to embed each M_β, $\beta \leq \alpha$, successively into the prime model N_α over $M \cup \{\vec{a}_0, \ldots, \vec{a}_\beta, \ldots\}_{\beta<\alpha}$. At limit stages, this is automatic. Once we have embedded M_β, note that the type of $\vec{a}_\beta$ over M_β is necessarily the heir of its restriction to M; consequently, the submodel of N_α that is prime over $M_\beta \cup \{\vec{a}_\beta\}$ is necessarily isomorphic to $M_{\beta+1}$. □

One brief comment on Corollary 20.7: If I first take the prime model M_1 over $M \cup \{\vec{a}\}$, and then the prime model M_2 over $M_1 \cup \{\vec{b}\}$, I do indeed get something that embeds in, and is therefore M-isomorphic to, the prime model N over $M \cup \{\vec{a}, \vec{b}\}$. The problem is that there is nothing canonical about this construction: there are many ways to embed M_1 into N, and we will not always get the same type for $\vec{b}$ over N_1. It is, on the other hand, canonical if $\vec{a}$ and $\vec{b}$ are independent over M.

I now recall the definition of the notion of a type p in $S_n(M)$ being orthogonal to a formula $f(\vec{x}, \vec{b})$ with parameters $\vec{b}$ in M: *p is orthogonal to f* if in the prime model $M(\vec{a})$ over $M \cup \{\vec{a}\}$, where $\vec{a}$ realizes p, every tuple that satisfies $f(\vec{x}, \vec{b})$ is in M. We saw in Lemma 18.12 that this property is preserved by heirs; to be sure, we saw it only for 1-types and formulas in one variable, which is generally the useful case, and which, by a simple trick of writing, can be transformed into the general case.

Lemma 20.8. *p is orthogonal to $f(\vec{x}, \vec{a})$ if and only if it is orthogonal to every (nonrealized!) q in $S_m(M)$ satisfying that formula.*

Proof. Suppose that p is orthogonal to f. We let $\vec{a}$ realize p, and let $\vec{b}$ realize a type q satisfying f. If $g(\vec{x}, \vec{c})$ is a formula of the type of $\vec{b}$ over $M(\vec{a})$, then we can find a tuple in $M(\vec{a})$ that satisfies $g(\vec{x}, \vec{c}) \wedge f(\vec{x})$; this tuple is necessarily in M, and the type of $\vec{b}$ over $M(\vec{a})$ coinherits its restriction to M. That makes p and q weakly orthogonal, but also strongly orthogonal, since the heir of p remains orthogonal to f.

If, conversely, p is not orthogonal to f, then it R-majorizes a (nonrealized!) type satisfying f, to which it is not orthogonal. $\square$

Given a formula $f(\vec{x}, \vec{b})$ with parameters $\vec{b}$ in M, a type p in $S_1(M)$ is called *f-regular* if p satisfies f and if it is (weakly) orthogonal to every type in $S_1(M)$ that satisfies f distinct from itself. For example, if p is orthogonal to all types of strictly lower Morley rank, then it is f-regular for every f that isolates it from types of the same Morley rank. A type is sometimes called *strongly regular* if it is f-regular for some formula f; we can define strongly regular types of n-tuples the same way, but that does not help much, at least up to R-equivalence, since, as we are going to see, the strongly regular types correspond to the R-minimal classes.

Lemma 20.9. *p is f-regular if and only if in the model $M(a)$ that is prime over M and a realization a of p, all elements that satisfy f and that are not in M have type p.*

Proof. All the c in $M(a)$ that are not in M have types over $M \cup \{a\}$ that fork over M (by the open mapping theorem, or else because a dominates them!); their types over M are therefore not orthogonal to p. Therefore, if p is f-regular, we do indeed have the property.

Conversely, if p is not orthogonal to q, or $q \neq p$ and q satisfies f, then p is nonorthogonal to every formula $f(x, \vec{b}) \wedge g(x, \vec{c})$, where g is satisfied by q and not by p. Consequently, this formula must grow when we pass from M to $M(a)$. $\square$

Lemma 20.10. *Let N be an elementary extension of M and let f be a formula with parameters in M. Let p be a type over M and p' its heir over N. Then p is f-regular if and only if p' is f-regular.*

Proof. Assume that p is not f-regular. We can then find an element of $M(a)$ that is not in M, whose type over $M \cup \{a\}$ is isolated by a formula $g(x, a, \vec{b})$ (which implies f), and that is contradictory with p. We thus find a formula h such that $p \models (\exists y)(g(y, x, \vec{b}) \wedge f(y) \wedge h(x, \vec{b}) \wedge \neg h(y, \vec{b}))$, while for no c in M does $p \models g(c, x, \vec{b})$. This situation translates into the satisfaction by $\vec{b}$ of a formula in the model (M, dp), using as parameters those that occur in f. □

One consequence of Lemma 20.10 is that a strongly regular type is indeed regular.

Proposition 20.11. *An f-regular type is R-minimal and R-minorizes every type that is not weakly orthogonal to it. Every type majorizes some strongly regular type in the order R: the R-minimal types are the types that are R-equivalent to f-regular types, and even to types orthogonal to every type of strictly smaller Morley rank.*

Proof. Let p be f-regular, and let q be R-less than p, or even just nonorthogonal to p. If $\vec{b}$ realizes q in a manner dependent on a, then I claim that p must be realized in $M(\vec{b})$. Indeed, let F be the set of elements of $M(\vec{b})$ satisfying f. We know that the type of a over $M(\vec{b})$ is the unique nonforking extension of its restriction to $M \cup F$, which therefore forks over M. Since p is strongly orthogonal to every other type satisfying f, F must contain a realization of p.

Now let p be arbitrary, and consider an element b of $M(\vec{a}) - M$ whose type over M has minimal Morley rank. A formula g, with parameters in M, whose Morley rank is strictly less than that of the type q of b over M, cannot grow from M to $M(\vec{b})$, which is included in $M(\vec{a})$. We have that q is orthogonal to g, and orthogonal to every type of strictly smaller Morley rank; it is f-regular for every formula f that isolates it from the types of the same Morley rank. □

Proposition 20.12. *Two R-minimal types p and q are orthogonal or R-equivalent; for two such types, weak orthogonality is preserved by heirs, just like R-equivalence.*

Proof. Choose p_1 to be R-equivalent to p and q_1 R-equivalent to q, both of minimal Morley rank. They are each orthogonal to every type of lower RM. Suppose, for example, that $\mathrm{RM}(p_1) \leq \mathrm{RM}(q_1)$, and let f be a formula isolating q_1 from the types of the same Morley rank. If p_1 and q_1 are not weakly orthogonal, then p_1, when realized, makes f grow; it can do this only when a realization of q is added, so p and q are indeed R-equivalent.

We have that p_1 and q_1 are weakly orthogonal under the same conditions as p and q (Corollary 20.5); that happens only if p_1 is orthogonal to f, a property that is preserved by heirs. □

Lemma 20.13 (Pillay's Lemma). *If $tp(a/M)$ is f-regular, $\vec{b} \notin M$, and $tp(\vec{b}/M \cup \{a\})$ is isolated, then $tp(a/M \cup \{\vec{b}\})$ is isolated.*

Proof. We know that $tp(\vec{b}/M \cup \{a\})$ forks over M, so let $g(x, \vec{b})$ be a formula isolating a from the elements of M. I claim that every element that satisfies this formula has type p over M. If not, we could find a formula $h(x)$, with parameters in M, such that $p \models h(x)$ and $(\exists x)(h(x) \wedge g(x, \vec{b}))$ is true. We could therefore find such x in $M(\vec{b})$, contradicting the fact that since $M(\vec{b})$ is contained in $M(a)$, all elements of $M(\vec{b})$ that satisfy f and are not in M have type p.

Under these conditions, the formula $g(x, \vec{b}) \wedge k(x, \vec{b})$ isolates the type of a over $M \cup \{\vec{b}\}$, where the formula $k(a, \vec{y})$ isolates the type of $\vec{b}$ over $M \cup \{a\}$. Indeed, if a' satisfies this formula, then a' and a have the same type over M, so $a'^\frown\vec{b}$ and $a^\frown\vec{b}$ have the same type over M. □

Corollary 20.14. *N is of the form $M(a)$, with $tp(a/M)$ R-minimal, if and only if every $N' \neq M, M \prec N' \prec N$, is M-isomorphic to N.*

Proof. If N has the property indicated, consider a in $N - M$, with $tp(a/M)$ of minimal Morley rank. Then $tp(a/M)$ is strongly regular, and N is isomorphic to $M(a)$.

For the converse, note first that $M(a)$ contains no indiscernible sequence $b_0, \ldots, b_n, \ldots, b_\alpha, \ldots$ over M of longer than denumerable length; otherwise, from some n on, $b_{n+1}, \ldots, b_\alpha, \ldots$ would become a Morley sequence over $M \cup \{a, b_0, \ldots, b_n\}$, contradicting the fact that $M(a)$, also being prime over this set, can contain no nondenumerable sequence that is indiscernible over the latter (Lemma 18.3). So let b be in $N - N'$; by Pillay's lemma, N is atomic over $M \cup \{b\}$, as is N'. Since they contain no nondenumerable sequences that are indiscernible over $M \cup \{b\}$, these are two copies of the prime model over $M \cup \{b\}$. □

Corollary 20.15. *If $tp(\vec{a}/M)$ and $tp(\vec{b}/M)$ are R-minimal, they are R-equivalent if and only if $M(\vec{a})$ and $M(\vec{b})$ are M-isomorphic.*

Proof. $M(\vec{a})$ and $M(\vec{b})$ can be embedded into each other; the conclusion follows from Corollary 20.14.

It is not known whether Corollary 20.15 remains valid without assuming the types to be R-minimal.

Corollary 20.16. *Let N be an elementary extension of M, p a type over M, and p' its heir over N. Then p is R-minimal if and only if p' is.*

Proof. If p is R-minimal, then it is R-equivalent to a strongly regular type, a property preserved by heirs.

If p' is R-minimal, let $\vec{a}$ realize it. $N(\vec{a})$ contains a copy of $M(\vec{a})$, and, by Lemma 20.2, $tp(M(\vec{a})/N)$ inherits its restriction to M. Take any b in $M(\vec{a}) - M$; since it does not fork, b is not in N, and by Pillay's lemma, the type of $\vec{a}$ over $N \cup \{b\}$ is isolated. As $tp(\vec{a}^\frown b/N)$ does not fork over M, $tp(\vec{a}/N \cup \{b\})$ is a nonforking extension of $tp(\vec{a}/M \cup \{b\})$, and the latter is isolated by the open mapping theorem. That proves that $tp(\vec{a}/M)$ and $tp(b/M)$ are R-equivalent, and that $tp(\vec{a}/M)$ is R-equivalent to every type R-less than it; in other words, it is R-minimal. □

Now that the properties of R-minimal types have been made clear, we are going to attempt to give at least a fragmentary description of the order R by means of these minimal types. We begin with an easy lemma:

Lemma 20.17. *A type p in $S_n(M)$ majorizes only finitely many (pairwise inequivalent!) R-minimal types.*

Proof. If q majorizes $p_1, \ldots, p_n$, which are R-minimal, let $\vec{b}$ realize q. In $M(\vec{b})$ there are realizations $a_1, \ldots, a_n$ of these types that, in light of the strong orthogonality of the p_i, are independent over M. Since $\vec{b}$ makes the type of each a_i fork, there are not more of them than the weight of q. □

Remark. A careful reader may have noticed something bizarre in the proof of Lemma 20.17: We have as a hypothesis that the p_i are pairwise strongly orthogonal, and use the fact that each one is orthogonal to the set of all the others. This is not faulty reasoning, but rather a consequence of the preservation of strong orthogonality by inheritance.

Now we introduce a kind of decomposition of a type into regular types, that we shall call the *Lascar decomposition*, defined as follows: Let $\vec{b}$ realize q, and consider the prime model $M(\vec{b})$ over $M \cup \{\vec{b}\}$. In this model, take an element a_1 of R-minimal type p_1 over M, and embed the model $M_1 = M(a_1)$ into $M(\vec{b})$. If $\vec{b}$ is in M_1, stop; otherwise, consider a_2 of R-minimal type p_2 over M_1, and embed $M_2 = M_1(a_2)$ into $M(\vec{b})$, etc. However we choose it, this construction must stop after finitely many steps, since the type of $\vec{b}$ forks at every stage. It is this sequence of types $p_1, \ldots, p_n$ which we call the Lascar decomposition.

If we want to choose the a_i independent over M, then by Corollary 20.7, $M(a_1, \ldots, a_n)$ will also be the model $M(\vec{b})$, the type of b will be R-equivalent to the product of strongly regular types $p_1, \ldots, p_n$, and its weight will be n. However, this decomposition is not in general of the same nature as that given by Theorem 19.26. First, because the a_i are not required to be independent, even when that is possible, there is nothing canonical in the choice of the embedding of M_1 into $M(\vec{b})$, and in general the number n will be greater than the weight of q. Second, because it is not in general possible to choose the a_i independent, it is not possible to choose p_2 nonforking over M, as the example given at the beginning of this section clearly shows.

As we have defined it, this decomposition also imposes no requirement for the type of $\vec{b}$ over M to be R-equivalent to that of $a_1\frown\cdots\frown a_n$ over M; for that to happen, it is sufficient to take $\vec{a}_n = \vec{b}$ as the last choice!

To avoid unpleasant surprises, we are going to strengthen this decomposition by replacing $a_1, \ldots, a_n$ by tuples $\vec{a}_1, \ldots, \vec{a}_n$ containing them, such that the type of $\vec{a}_i$ over M_{i-1} does not fork over $M \cup \{\vec{a}_{i-1}\}$. For that, we start from the end, setting $\vec{b}\frown a_n = \vec{a}_n$, taking $\vec{a}_{n-1}$ in M_{n-1} containing a_{n-1}, such that the type of $\vec{a}_n$ over M_{n-1} is the unique nonforking extension of its restriction to $M \cup \{\vec{a}_{n-1}\}$; then we take $\vec{a}_{n-2}$ in M_{n-2}, containing a_{n-2}, such that the type of $\vec{a}_{n-1}$ over M_{n-2} is the unique nonforking extension of the restriction to $M \cup \{\vec{a}_{n-2}\}$, etc. Having done that, we get a cascade of minimal types, since the types of $\vec{a}_i$ and a_i over M_{i-1} are equivalent.

This decomposition is useful for proving the following theorem:

Theorem 20.18. *In a totally transcendental theory T, two types over a model of T are orthogonal whenever they are weakly orthogonal: Weak orthogonality is preserved by heirs.*

Proof. Let p and q be two weakly orthogonal types over M; at first, let us assume that p is R-minimal. We introduce a Lascar decomposition $\vec{a}_1\frown\cdots\frown\vec{a}_n$ of q, with $\vec{a}_n$ realizing q, strengthened so that the type of $\vec{a}_{i+1}$ over M_i does not fork over $M \cup \{\vec{a}_1, \ldots, \vec{a}_i\}$. Therefore, q is R-equivalent to the type of $\vec{a}_1\frown\cdots\frown\vec{a}_n$ over M, and the respective heirs of these types over an elementary extension N of M are also R-equivalent.

We realize the heir of the latter in the following way: Let $\vec{a'}_1$ be a realization of the heir of $tp(\vec{a}_1/M)$. Let N_1 be the model $N(\vec{a'}_1)$ that contains $M_1 = M(\vec{a'}_1)$. Let $\vec{a'}_2$ realize the heir over N_1 of $tp(\vec{a}_2/M_1)$, which is unambiguous, regardless of the embedding of M_1 into N_1, since we are dealing with the unique nonforking extension of $tp(\vec{a}_2/M \cup \{\vec{a}_1\})$, and continue.

Since p and q are weakly orthogonal, p, when realized, cannot make $tp(\vec{a}_1\frown\cdots\frown\vec{a}_n/M)$ fork. Therefore, p is weakly orthogonal to $tp(\vec{a}_1/M)$. Next, the heir of p over M_1, which is also its only extension to this model, is weakly orthogonal to $tp(\vec{a}_2/M_1)$, etc. The heir of p over M_i is always weakly orthogonal to $tp(\vec{a}_{i+1}/M_i)$. By Proposition 19.12, since we are dealing with R-minimal types, these weak orthogonalities are preserved by the heirs, and are therefore strong orthogonalities.

So let $\vec{a}$ realize the heir of p over N. Therefore, $\vec{a}$ and $\vec{a}_1$ are independent over N, as are $\vec{a}$ and N_1 (Corollary 20.6): $tp(\vec{a}/N_1)$ is the heir of p. Similarly, $\vec{a}$ and $\vec{a}_2$ are independent over N_1, and $tp(\vec{a}/N_2)$ is the heir of p, etc. Therefore, $tp(\vec{a}/N)$ and $tp(\vec{a}_1\frown\cdots\frown\vec{a}_n/N)$ are weakly orthogonal, from which the conclusion follows.

The theorem is therefore true if one of the types is R-minimal. To prove the general case, we do an analogous proof, decomposing q according to the same method. □

20.2 Dimensional Types and Theories

We now consider an R-minimal type p over a model M of T; since everything we are going to say will be preserved by R-equivalence, we can just as well take p to be strongly regular and consider a type of element instead of a tuple.

This type p is definable with the help of a finite set $\vec{a}$ of parameters. For example, we can take $\vec{a}$ to be the tuple of parameters of a formula $f(x, \vec{a})$ that isolates it among the types of the same Morley rank: Clearly, whether p satisfies $g(x, \vec{b})$ depends only on the type of $\vec{b}$ over $\vec{a}$ (that happens if and only if $\mathrm{RM}(p) = \mathrm{RM}(g(x, \vec{b}) \wedge f(x, \vec{a}))$), and $dg(\vec{y})$ can be chosen with parameters in $\vec{a}$. If desired, everything can be done canonically, and we can go to T^{eq}, where we find the *canonical parameter* (which, to be sure, is canonical only up to rationality) that allows p to be defined.

To emphasize this parameter, we let p_a denote our type, and if a' is an element (of T^{eq} here) that realizes the same type over $\emptyset$ as a, we let $p_{a'}$ denote the type obtained by replacing a with a' in the definition of p.

We shall call p_a (which, we recall, is R-minimal) *dimensional* if, whatever model M may be, if a and a' realize the same strong type over $\emptyset$, then p_a and $p_{a'}$ are R-equivalent.

NOTE: When we say that p_a and q_b are orthogonal, or, to the contrary, R-equivalent, we shall refrain from specifying the model M of reference for the order R; we can take the prime model over $a^\frown b$, or else any model containing a and b, since orthogonality is preserved by heirs.

Lemma 20.19. *If p_a is not dimensional, and if $a_0, a_1, \ldots, a_\alpha, \ldots$ form an independent and indiscernible sequence of realizations of the type of a over $\emptyset$, then the p_{a_α} are orthogonal.*

Proof. By indiscernibility, the p_{a_α} are either all R-equivalent or else pairwise orthogonal (i.e., each one is orthogonal to the product of the others!). Furthermore, it is clear that if p_a is dimensional, since all the a_α have the same strong type (the strong type whose Morley sequence they constitute), all the p_{a_α} are R-equivalent.

Assume now that p_a is not dimensional, and consider b_1 and b_2 of the same type as a, both having the same strong type, such that p_{b_1} and p_{b_2} are orthogonal (i.e., not R-equivalent). Consider, therefore, a sequence $a'_0, \ldots, a'_\alpha, \ldots$ of realizations of the type of a that are in this strong type and whose type over $\{b_1, b_2\}$ does not fork over $\emptyset$. Then $b_1, a'_0, a'_1, \ldots, a'_\alpha, \ldots$ and $b_2, a'_0, a'_1, \ldots, a'_\alpha, \ldots$ are both copies of the Morley sequence of this strong type, and are indiscernible. Since the $p_{a'_\alpha}$ cannot be simultaneously R-equivalent to p_{b_1} and p_{b_2}, which are not R-equivalent to each other, they must be orthogonal. The same is true of the p_{a_α}, since these sequences of parameters have the same type. $\square$

In a totally transcendental theory, every type has finite multiplicity, and extends to only finitely many strong types. Thus if p_a is dimensional it has only finitely many conjugates that are pairwise orthogonal; if p_a is not dimensional, it has as many as desired.

Dimensionality can also be characterized in the following way: *p is dimensional if and only if, over any model M of T, there are only finitely many pairwise orthogonal types equivalent to p in the fundamental order.* Indeed, a type equivalent to p_a is of the form $p_{a'}$, since, in T^{eq}, the element a can be characterized as the only one satisfying a sentence of the model (M^{eq}, dp); two types are thus equivalent in the fundamental order if and only if they have canonical parameters of the same type and definitions of the same form. We note in passing that whether p is R-minimal depends only on its class in the fundamental order, and also that since this characterization does not actually involve the parameter a, we could have defined dimensionality starting from any $\vec{a}$ allowing us to define p, without having to be canonical (such $\vec{a}$ can be found in any (M, dp') with p' equivalent to p in the fundamental order, provided that it is sufficiently saturated). We could also define a notion of dimensionality for arbitrary types: If we restrict ourselves to R-minimal types, it is because they are the only types that we use.

Returning to R-minimal types, it is clear that if p_a is R-equivalent to q_b, then every $p_{a'}$ is equivalent to some $q_{b'}$, so this notion of dimensionality is preserved by R-equivalence.

We shall call T *dimensional* if all its (R-minimal) types are dimensional. The number of R-classes of R-minimal types thus obtained (in principle, over an ω-saturated model; however, see Lemma 20.31 below) will be called the *number of dimensions* of T.

For example, an ω_1-categorical theory T is dimensional, with only one dimension; indeed, as there are no Vaught pairs, no type can be orthogonal to any formula, and all R-minimal types are therefore equivalent.

Likewise, all (totally transcendental) modules are dimensional, and even have an especially simple dimensionality property: Every type is R-equivalent to a type definable without parameters. Indeed, in a totally transcendental group there are no strictly decreasing infinite sequences of definable subgroups (since we would have to decrease either the Morley rank or the Morley degree of the group at every stage; in fact, all classes modulo H have the same Morley rank and the same Morley degree). As a consequence, every filter of primitive groups (see Section 13.3, example 16) is principal: Every p in $S_1(M)$ can be associated with a primitive group G and an element a of M such that p is axiomatized by the formulas $x - a \in G$ and $x - b \notin H$ for every b in M and every primitive group strictly contained in G; the type q of $x - a$, which is R-equivalent to p, is definable without parameters.

In contrast, the theory of an equivalence relation with infinitely many classes, all infinite, is not dimensional: If p_a is the nonrealized type stating that x is in the equivalence class of a (the canonical parameter is therefore the class of a), we can see that two distinct p_a are orthogonal. Note that this theory can be interpreted in that of an infinite set: Take the pairs (a, b) equipped with the relation "having the same first coordinate." Also, the theory of an infinite set can be interpreted in every theory (with no finite models). Consequently, this property of dimensionality is not preserved in either direction by interpretation.

Theorem 20.20. *If $p_{\vec{a}}$ is nondimensional, if $\vec{a}$ and A are independent (over $\emptyset$), and if M is a model containing them, then $p_{\vec{a}}$ is orthogonal to every type over M that does not fork over A.*

Proof. Begin by extending A to an ω-saturated model N placed independently to $\vec{a}$ over $\emptyset$. This may force us to extend the model M, but that does not matter, since orthogonality is preserved by heirs.

Assume therefore that $q_{\vec{b}} \geq_R p_{\vec{a}}$, with $\vec{b}$ in N. Since N is sufficiently saturated, we can find $\vec{a}_1$ with the same strong type as $\vec{a}$ over $\vec{b}$, and again because N is ω-saturated, $q'_{\vec{b}} \geq_R p'_{\vec{a}_1}$. These two types specify the restrictions of $q_{\vec{b}}$ and $p_{\vec{a}_1}$, respectively, to N, with the order R considered as that of N. Similarly, we can find $\vec{a}_2$ in N realizing the same strong type as $\vec{a}$ over $\vec{b}\frown\vec{a}_1$, and $q'_{\vec{b}} \geq_R p'_{\vec{a}_2}$, and so forth. Therefore, $q'_{\vec{b}}$ majorizes $p'_{\vec{a}_1}, \ldots, p'_{\vec{a}_n}, \ldots$; this is impossible, since, the $p_{\vec{a}_n}$ are pairwise orthogonal, because $p_{\vec{a}}$ is not dimensional, and $\vec{a}_1, \ldots, \vec{a}_n, \ldots$ form a Morley sequence (over $\emptyset$). □

In particular, we can see that if $p_{\vec{a}}$ is (R-minimal and) nondimensional, and $\vec{a}_1$ and $\vec{a}_2$ are independent, then $p_{\vec{a}_1}$ and $p_{\vec{a}_2}$ are orthogonal.

The scrupulous reader might be bothered by the fact that in Lemma 20.19 and Theorem 20.20 it is indeed independence over $\emptyset$ that we need. We can naturally add a set A of parameters, and state these theorems relative to the theory $T(A)$; however, it is worth remarking that "dimensional in the sense of T" does not mean the same thing as "dimensional in the sense of $T(A)$"!

We now consider a strongly regular type $p_{\vec{a}}$ over a model M and a subset A of M containing $\vec{a}$. A *basis of $p_{\vec{a}}$ over A in M* is defined as a maximal set of realizations B in M of the restriction of $p_{\vec{a}}$ to A that is independent over A. (Do not lose sight of the fact that $p_{\vec{a}}$ denotes a type over a model, which we shall often neglect to mention, which is the unique nonforking extension of its restriction to $\vec{a}$.) Since $p_{\vec{a}}$ is regular, and therefore of weight one, all these bases have the same cardinality, which we shall call the *dimension of $p_{\vec{a}}$ over A in M.*

Henceforth, when we say that $p_{\vec{a}}$ is strongly regular, we assume that $p_{\vec{a}}$ is f-regular for some formula f with parameters in $\vec{a}$; if, moreover, $p_{\vec{a}}$ is orthogonal to all types of strictly lesser Morley rank, we can take $\vec{a}$ to be its canonical parameter.

Lemma 20.21. *If $p_{\vec{a}}$ is strongly regular, and if B is a basis of $p_{\vec{a}}$ over A in M, and if b is a realization of the nonforking extension of $p_{\vec{a}}|A$ to $A \cup B$, then the type of b over M does not fork over A.*

Proof. Let f be a formula with parameters in A such that $p_{\vec{a}}$ is f-regular. Since b satisfies f, we know that its type over M does not fork over $A \cup F$, where F is the set of all elements of M that satisfy f. Moreover, if $\vec{c}$ is a tuple consisting of elements of F that are not in B, then each element of $\vec{c}$ realizes something other than the nonforking extension of the restriction of $p_{\vec{a}}$ to $A \cup B$, and by f-regularity cannot make the type of b fork. □

Theorem 20.22. *If $p_{\vec{a}}$ is strongly regular, $\vec{a} \subset A \subset M \prec N$, if B_1 is a basis of $p_{\vec{a}}$ over A in M, and B_2 is a basis of $p_{\vec{a}}$ over M in N, then $B_1 \cup B_2$ is a basis of $p_{\vec{a}}$ over A in N; the dimension of $p_{\vec{a}}$ over A in N is the sum of that of $p_{\vec{a}}$ over A in M and that of $p_{\vec{a}}$ over M in N.*

Proof. To extend B_1 to a basis of $p_{\vec{a}}$ over A in N, we must first add an element b_0 realizing, over $A \cup B_1$, the nonforking extension of the restriction of $p_{\vec{a}}$ to A. By Lemma 20.21, this element realizes $p_{\vec{a}}$ over M. Therefore, $B_1 \cup \{b_0\}$ is a basis in the prime model $M(b_0)$ over M and b_0. We then must add an element that realizes the heir of $p_{\vec{a}}$ over $M(b_0)$, and continue as far as possible. □

Theorem 20.23. *Assume that $p_{\vec{a}}$ is strongly regular and nondimensional. If A and $\vec{a}$ are independent over $\emptyset$, and if M is the prime model over $A \cup \{\vec{a}\}$, then the dimension of $p_{\vec{a}}$ over $\vec{a}$ in M is at most denumerable: in fact, it is the dimension of the prime model over $\vec{a}$.*

Proof. Enumerate $A = \{a_0, \ldots, a_\alpha, \ldots\}, A_\alpha = \{\ldots, a_\beta, \ldots\}_{\beta<\alpha}$. Let M_0 be the prime model over $\vec{a}$, whose dimension is finite or denumerable, since it contains no nondenumerable indiscernible sequence over $\vec{a}$ (Lemma 18.3, regardless of the cardinality of T). By induction on α, we construct a sequence of models M_α, with M_α containing A_α, that does not increase the dimension of M_0. At limit stages, that happens automatically. Assuming that we have constructed M_α, place a_α so that its type over M_α does not fork over $A_\alpha \cup \{\vec{a}\}$. Since A and $\vec{a}$ are independent, it also does not fork over A_α, and by Theorem 20.20 this type is orthogonal to the heir of $p_{\vec{a}}$ over M_α. Consequently, the prime model $M_{\alpha+1}$ over $M_\alpha \cup \{\vec{a}_\alpha\}$ does not realize this heir, and by Theorem 20.22, the dimension of $p_{\vec{a}}$ over $\vec{a}$ is the same in M_α and in $M_{\alpha+1}$. □

Corollary 20.24. *If T is not dimensional, then every model M of T has an elementary extension that is not ω_1-saturated.*

Proof. Let $p_{\vec{a}}$ be strongly regular and nondimensional. If M and $\vec{a}$ are placed independently, then by the preceding theorem the prime model over $M \cup \{\vec{a}\}$ is not ω_1-saturated. □

If p_a and q_b are strongly regular types over M that are R-equivalent, then it is clear that they have the same dimension over M in every elementary extension of M. What happens when we have not yet reached a given model relies on two delicate lemmas.

Lemma 20.25. *Let M be a model containing $\vec{a}$ and $\vec{b}$, and assume that p_a and q_b are strongly regular and R-equivalent. Then the dimensions in M of $p_{\vec{a}}$ over $\vec{a}$ and of $q_{\vec{b}}$ over $\vec{b}$ are either both finite or else equal.*

Proof. First, assume that the dimension of $p_{\vec{a}}$ over $\vec{a}$ in $M(\vec{a})$ is finite. By Theorem 20.22, since $tp(\vec{c}/M(\vec{a}))$ has finite weight, then the dimension of $p_{\vec{a}}$ over $\vec{a}$ in any $M(\vec{a}^\frown\vec{c})$ is also finite. Take $\vec{c}$ to be a parameter allowing us to isolate $p_{\vec{a}}$ over $q_{\vec{b}}$, and conversely: We can see that in the model $M(\vec{a}^\frown\vec{b}^\frown\vec{c})$ that embeds into M, $p_{\vec{a}}$ and $q_{\vec{b}}$ have the same dimension over $\vec{a}^\frown\vec{b}^\frown\vec{c}$. Consequently, the dimension of $q_{\vec{b}}$ over $\vec{b}$ in $M(\vec{a}^\frown\vec{b}^\frown\vec{c})$ is necessarily finite (otherwise $\vec{a}^\frown\vec{c}$ would make infinitely many elements of an independent sequence fork), and the conclusion follows from Theorem 20.22.

If, on the other hand, the dimension of $p_{\vec{a}}$ over $\vec{a}$ in $M(\vec{a})$ is denumerable, the same must be true of the dimensions of $p_{\vec{a}}$ over $\vec{a}^\frown\vec{b}^\frown\vec{c}$ in $M(\vec{a}^\frown\vec{b}^\frown\vec{c})$, of $q_{\vec{b}}$ over $\vec{a}^\frown\vec{b}^\frown\vec{c}$ in $M(\vec{a}^\frown\vec{b}^\frown\vec{c})$, and of $q_{\vec{b}}$ over $\vec{b}$ in $M(\vec{b})$; the conclusion then follows from Theorem 20.22. □

In light of Lemma 20.25, we shall say that the (strongly regular) type $p_{\vec{a}}$ is of the *first kind* if the dimension of $p_{\vec{a}}$ over $\vec{a}$ in $M(\vec{a})$ is finite; otherwise, it is of the *second kind.* We can see that this definition does not depend on the tuple that serves to determine the type: If the dimension of $p_{\vec{a}}$ over $\vec{a}$ in $M(\vec{a})$ is finite, then so is that of $p_{\vec{a}}$ over $\vec{a}^\frown\vec{b}$ in $M(\vec{a}^\frown\vec{b}^\frown\vec{c})$! We have seen that R-equivalence preserves kind; it is also clear that if p and p' are conjugate, then they are of the same kind.

Lemma 20.26. *Let M be a model of T, containing $\vec{a}_1$ and $\vec{a}_2$ of the same strong type over $\emptyset$. If $p_{\vec{a}}$ is strongly regular and dimensional, then the dimension of $p_{\vec{a}_1}$ over $\vec{a}_1$ in M is equal to the dimension of $p_{\vec{a}_2}$ over $\vec{a}_2$ in M.*

Proof. We know that under these conditions $p_{\vec{a}_1}$ and $p_{\vec{a}_2}$ are R-equivalent. First, assume that $\vec{a}_1$ and $\vec{a}_2$ are independent. Since they form the beginning of a Morley sequence, $\vec{a}_1{}^\frown\vec{a}_2$ has the same type as $\vec{a}_2{}^\frown\vec{a}_1$, and there is an automorphism of the prime model M_0 over $\vec{a}_1{}^\frown\vec{a}_2$ that transposes them. The theorem is therefore true for this model M_0, which embeds into M; we can then apply Theorem 20.22.

Now let us go to the general case. Let $\vec{a}$ have the same strong type as $\vec{a}_1$ and $\vec{a}_2$, whose type over $\vec{a}_1{}^\frown\vec{a}_2$ does not fork over $\emptyset$, and let N be the prime model over $M \cup \{\vec{a}\}$. By what we have seen, the dimensions of $p_{\vec{a}_1}$ over $\vec{a}_1$ and of $p_{\vec{a}_2}$ over $\vec{a}_2$ in N are equal to that of $p_{\vec{a}}$ over $\vec{a}$. Since $p_{\vec{a}_1}$ and $p_{\vec{a}_2}$ are R-equivalent, their dimensions over M in N are equal, and *are* *finite*, majorized by the weight of the type of $\vec{a}$ over M. The conclusion then follows, again by Theorem 20.22. □

Theorem 20.27. *If T is totally transcendental and nondimensional, then T has at least $|\omega+\alpha|^{|\alpha|}$ distinct models of cardinality $\aleph_\alpha$ for every $\aleph_\alpha$ greater than or equal to the cardinality of T.*

Proof. Consider a function f from α to $\omega + \alpha + 1$ (which has the same cardinality as $\omega + \alpha$), that sends every ordinal $\beta < \alpha$ to an ordinal $f(\beta) \leq \omega + \alpha$. Let $p_{\vec{a}}$ be a strongly regular nondimensional type. We are going to construct a model M_f of cardinality $\aleph_\alpha$ enjoying the following property: Fix $\beta < \alpha$, and consider the realizations $\vec{a}_1$ of the type of $\vec{a}$ in M such that the dimension of $p_{\vec{a}_i}$ over $\vec{a}_i$ in M is at most denumerable if $\beta = 0$, and equal to $\aleph_\beta$ otherwise. By Lemma 20.25, if $p_{\vec{a}_j}$ is R-equivalent to $p_{\vec{a}_i}$, then it will also have this property; the number of such $p_{\vec{a}_i}$, up to R-equivalence, will be exactly n if $f(\beta) = n$, and will be $\aleph_\gamma$ if $f(\beta) = \omega + \gamma$.

It is clear that the function f is an invariant of the model, that two distinct functions will produce two nonisomorphic models.

For that result, consider $\aleph_\alpha$ independent realizations $\vec{a}_i$ of the type of $\vec{a}$ over $\emptyset$. We therefore know that the $p_{\vec{a}_i}$ are orthogonal and that, in the prime model M_0 over all that, the dimension of each $p_{\vec{a}_i}$ over $\vec{a}_i$ is finite or denumerable (Theorem 20.23).

We therefore regroup these $\vec{a}_i$ into packages, with the βth package containing n if $f(\beta) = n$, and $\aleph_\gamma$ if $f(\beta) = \omega + \gamma$. Having done this, we might have to leave some number of $\vec{a}_i$ outside of the packages (if the sum of the cardinals considered is too small). Given the orthogonality of the $p_{\vec{a}_i}$, we can increase the dimension of one of them without affecting that of the others. We thus obtain a model M_1 such that the dimension of $p_{\vec{a}_i}$ over $\vec{a}_i$ in M_1 is $\aleph_\beta$ if $\vec{a}_i$ is in the βth package, and $\aleph_\alpha$ if $\vec{a}_i$ is left over. In M_1, some new $p_{\vec{a}}$ may appear that are orthogonal to all the $p_{\vec{a}_i}$; to destroy them, we extend M_1 to a model M_2 that gives them dimension $\aleph_\alpha$, without affecting the dimensions of the $p_{\vec{a}_i}$, and repeat ω times. □

We are now going to attempt to classify the models of T, an effort that we shall be able to carry through to the end, in the next section, for dimensional theories. Let M be a model of T. Then we choose a strongly regular representative $p_{\vec{a}}$ of each R-minimal class of types over M, and define a *foundation* of M to be a set A containing the parameters $\vec{a}$ for every $p_{\vec{a}}$. (Once $p_{\vec{a}}$ has been chosen, we can be more canonical, by working in T^{eq} and by taking A to be the union of the canonical parameters; however, the choice of the $p_{\vec{a}}$ in their respective R-classes remains tainted with arbitrariness.) We define a *basis* of M over A to be a union B of bases of $p_{\vec{a}}$ over A in M for each $p_{\vec{a}}$. In light of the orthogonality of the $p_{\vec{a}}$, the basis is an independent set over A, and the type of B over A is completely determined by specifying the dimensions of each $p_{\vec{a}}$ over A.

Lemma 20.28. *If A is a foundation of M and b is a basis of M over A, then M is prime, and even minimal, over $A \cup B$.*

Proof. We embed the prime model M_0 over $A \cup B$ into M. If $M_0 \neq M$, then M contains an element b whose type q over M_0 is R-minimal. Then its heir over M is also R-minimal, and is R-equivalent to one of the $p_{\vec{a}}$. We have that q is R-equivalent to the restriction of $p_{\vec{a}}$ to M_0, since in that case R-equivalence is nothing other than nonorthogonality, which commutes with inheritance. Therefore, M realizes the restriction of $p_{\vec{a}}$ to M_0, contradicting the maximality of B. □

This lemma is useful only if the model has a relatively small foundation; in fact, it allows us to classify, i.e., to determine up to isomorphism, all models of a dimensional theory, since we shall show that they all have the same foundation. For now, we end this section by proving the converse of Theorem 20.24, which provides us a characterization of dimensionality; we shall compare it to Theorems 14.9 and 14.10.

Theorem 20.29. *If T is (totally transcendental and) dimensional, and if $\lambda > |T|$, then every elementary extension of a λ-saturated model of T is itself λ-saturated.*

Proof. Let M_0 be the saturated model of cardinality $|T|$. It embeds into all the models that we are considering, and, as T is dimensional, constitutes a foundation, since it realizes all strong types over $\emptyset$. There are many ways to embed M_0 into M, and corresponding bases may have different dimensions; in any case, for the saturated model of cardinality λ, we have no choice: All dimensions must be equal to λ, regardless of the embedding of M_0. Likewise, if M is λ-saturated, then all the dimensions must be at least λ, regardless of the embedding of M_0 into M.

Conversely, I am going to prove that if all the dimensions are at least λ for *one* embedding of M_0 in M, then M is λ-saturated. That will prove the theorem, since the dimensions can only increase under elementary extensions.

So consider A in M of cardinality less than λ. Let N be an elementary submodel of M, of cardinality λ, containing A, M_0, and λ elements in each subset of the basis corresponding to some $p_{\vec{a}}$ (there are no more than $|T|$ such $p_{\vec{a}}$, not more than there are types over M_0). Each of the dimensions of this model N over M_0 is at least λ, and not more than λ, which is its cardinality. Therefore, N is the saturated model of cardinality λ, which realizes all types over A. □

20.3 Classification of the Models of a Dimensional Theory

For more precise results, we need to settle our foundation. Indeed, if T is dimensional, then the prime model M_0 can serve as a foundation for all models: That is a consequence of the next two lemmas:

Lemma 20.30. *A strongly regular type p is dimensional if and only if it is not orthogonal to some type q that does not fork over $\emptyset$.*

Proof. If q does not fork over $\emptyset$, then its canonical set of definition in T^{eq} can have only finitely many conjugates, since $q/\emptyset$ has finite multiplicity. That means that q is definable over the algebraic closure of $\emptyset$ in M^{eq}. If $p_{\vec{a}}$ is R-less than q, then so are all the $p_{\vec{a}_i}$, where the $\vec{a}_i$ realize a Morley sequence of the strong type of $\vec{a}$ over $\emptyset$. Indeed, in T^{eq}, to have the same strong type over $\emptyset$ means to have the same type over the algebraic closure of $\emptyset$. Since q has a (finite) weight, the $p_{\vec{a}_i}$ cannot be orthogonal, and $p_{\vec{a}}$ is dimensional.

Conversely, assume that p in $S_1(M)$ is dimensional. Consider M' of the same strong type as M over $\emptyset$, with M and M' independent. Let p' be the type over M' corresponding to p: We know that when we go up to a model containing M and M', p and p' will have R-equivalent heirs. So let N be the prime model over $M \cup M' \cup \{b'\}$, where b' realizes the heir of p over $M \cup M'$, that contains a copy N_0 of the prime model over $M \cup M'$. Under these conditions N has a b realizing the heir of p over N_0. By construction, the type of $M'^\frown b'$ over M does not fork over $\emptyset$. Since b is in N, b and N are not independent over M, meaning that b and $M'^\frown b'$ are not independent over M, by Corollary 20.5, since N is prime over $M \cup M' \cup \{b'\}$. As a consequence, p is nonorthogonal to the type over M of a finite tuple from $M' \cup \{b'\}$, a type that does not fork over $\emptyset$. □

Lemma 20.31. *If $M_0 \prec M$ and if p is a dimensional R-minimal type of $S_1(M)$, then it is R-equivalent to a type that does not fork over M_0.*

Proof. By replacing p with one of its heirs if necessary, we can assume that the model M is very saturated. Consider the prime model $M(a)$ over $M \cup \{a\}$, where a realizes p. Let b_1 be a point in $M(a) - M$ chosen so that $\mathrm{RM}(b_1/M_0)$ is minimal. Let us choose a formula $f(x)$, with parameters in M_0, that isolates $tp(b_1/M_0)$ from all types of the same Morley rank. Note that two elements of $M(a) - M$ that satisfy f have the same type over M_0.

Note that $q_1 = tp(b_1/M)$ is an R-minimal type that is R-equivalent to p. Let M_1 be a model of small cardinality, intermediate between M_0 and M, such that q_1 is the heir of its restriction to M_1.

Since M is very saturated, we have space to realize the Morley sequence $M_1, M_2, \ldots, M_n, \ldots$ of the type of M_1 over M; there is a corresponding sequence $q_1, q_2, \ldots, q_n, \ldots$ of types that are conjugates of q_1 over M_0.

By Lemma 20.30, we know that p is R-majorized by a type r that does not fork over $\emptyset$. It also R-majorizes q_1, and since it is definable with parameters in M_0, it R-majorizes all the q_n. Since r has finite weight, it is not possible for the q_n to be pairwise orthogonal; they are therefore all R-equivalent.

Now, if M_3 denotes the prime model over $M_1 \cup M_2$, we can find a realization of the heir over M_3 of the restriction of q_2 to M_2, with a type isolated over $M_3 \cup \{b_1\}$ by some formula $\phi(y, b_1, \vec{d})$. Its type over $M_1 \cup M_2 \cup \{b_1\}$ is obviously isolated by the formula $(\exists \vec{z})(\phi(y, b_1, \vec{z}) \wedge \psi(\vec{z}))$, where $\psi(\vec{z})$ isolates the type of $\vec{d}$ over $M_1 \cup M_2$, which also isolates it over $M_1 \cup M_2 \cup \{b_1\}$, since b_1 and M_2 are independent over M_1 (Corollary 20.6). Consequently, by the open mapping theorem, the type of b_2 over $M_1 \cup M_2 \cup \{b_1\}$ forks over M_2, and as the situation is symmetrical, b_1 and $M_2{}^\frown b_2$ are not independent over M_1. We can find a true sentence $g(\vec{c}_1, b_1; \vec{c}_2, b_2)$, with $\vec{c}_1$ in M_1 and $\vec{c}_2$ in M_2, while $g(\vec{c}_1, b_1; \vec{c}, b)$ is always false for $\vec{c}$ and b in M_1. The same is true if $\vec{c}$ and b are in M, since $tp(b_1/M)$ inherits $tp(b_1/M_1)$.

We are going to show that in fact q_1 does not fork over M_0, that is to say, $q_1 = q_2$, which solves our problem. If that were not the case, we could find parameters in M_2, which can be incorporated into our $\vec{c}_2$, and a sentence $h(b_2, \vec{c}_2)$ that is true for b_2, with the formula $h(x, \vec{y})$ being omitted by $tp(b_2/M_0)$, which is nothing other than $tp(b_1/M_0)$.

We therefore have:

$$\begin{array}{lll} f(b_2)\wedge & g(\vec{c}_1, b_1; \vec{c}_2, b_2) & \wedge h(b_2, \vec{c}_2), \\ (\exists y)(f(y)\wedge & g(\vec{c}_1, b_1; \vec{c}_2, y) & \wedge h(y, \vec{c}_2)). \end{array}$$

However, by construction, $\vec{c}_1{}^\frown b_1$ and $\vec{c}_2$ are independent over M_0, and we can find $\vec{c}$ in M_0 such that

$$(\exists y)(f(y) \wedge g(\vec{c}_1, b_1; \vec{c}, y) \wedge h(y, \vec{c})).$$

Such a y must therefore exist in $M(a) = M(b_1)$. Since it satisfies $g(\vec{c}_1, b_1; \vec{c}, y)$, it cannot be in M'; since it satisfies $h(y, \vec{c})$, it does not have the same type over M_0 as b_1, contradicting the definition of f. □

One unresolved question is whether a dimensional R-minimal type is always R-equivalent to some type that does not fork over $\emptyset$; that would greatly simplify our job below, since, if T is dimensional, that would mean that provided we worked in T^{eq}, the algebraic closure of $\emptyset$ could serve as a foundation for any model. This foundation would be entirely canonical, since it corresponds to a set that is uniquely determined in every model M. In the dimensional case, Lemma 20.31 allows us only to take as our foundation the prime model M_0, which does indeed embed into every model M of T, but many times over, unfortunately.

From now on, we assume that T is dimensional. We therefore know that every R-minimal class has a representative $p_{\vec{a}}$ in $S_1(M_0)$, that is to say with an $\vec{a}$ whose type over $\emptyset$ is isolated. We choose such a $p_{\vec{a}}$ for each R-minimal class; moreover, what we are going to say will not depend on this choice.

The automorphisms of the prime model M_0 induce a group G of permutations on the R-minimal classes, called the *orientation group*. As the class of $p_{\vec{a}}$ can have only a finite number conjugates under G, at most the number of strong types extending $tp(\vec{a}/\emptyset)$, every orbit of G is finite, and G is contained in the projective limit of its restrictions to the finite unions of its orbits; we can easily see, by compactness, that G is equal to this whole projective limit, that G is a profinite group (in fact, an image of the automorphism group of the algebraic closure of $\emptyset$ in M^{eq}, since the class of the image of $p_{\vec{a}}$ does not depend on what the strong type of $\vec{a}$ becomes!).

We now define an invariant, associated with every model M, called its *motive*. To do this, we begin by choosing an embedding of M_0 into M, and define the motive of M relative to this embedding: It is a function μ that sends R-minimal classes to cardinals. By abuse of language, if $p_{\vec{a}}$ is R-minimal in $S_1(M_0)$, let $\mu(p_{\vec{a}})$ be the value of its class under μ. It is defined as follows for every $p_{\vec{a}}$ in $S_1(M_0)$:

- $\mu(p_{\vec{a}})$ is the dimension of $p_{\vec{a}}$ over $\vec{a}$ in M if this dimension is infinite.
- Otherwise, $p_{\vec{a}}$ is of the first kind (see the definition after Lemma 20.25), and we take $\mu(p_{\vec{a}})$ to be the difference between the dimension of $p_{\vec{a}}$ over $\vec{a}$ in M, and that of $p_{\vec{a}}$ over $\vec{a}$ in M_0, or alternatively the dimension of $p_{\vec{a}}$ over M_0 in M.

In other words, $\mu(p_{\vec{a}})$ is always the dimension of $p_{\vec{a}}$ over M_0 in M, except when $p_{\vec{a}}$ is of the second kind and the latter expression is finite, in which case we take $\mu(p_{\vec{a}}) = \omega$.

We can thus see that if $p_{\vec{a}}$ and $q_{\vec{b}}$ are in $S_1(M)$ and are R-equivalent, then they give the same value to the motive function.

Now, let us see what happens if we take a different embedding M_0' of M_0 into M. There is an automorphism s of M_0' such that M_0 and M_0' have the same strong type: The best way to see this is to go to M^{eq} and to consider the algebraic closure A of $\emptyset$ in M_0, which as a set is the same as the A' of M_0'. Next, permute A' so that A and A' have the same strong type: We know that every automorphism of A' extends to M_0'. Having done that, we subject the motive associated with M_0 to a permutation from the orientation group; once that has been done, the two motives coincide by Lemma 20.26, since if $p_{\vec{a'}}$ corresponds to $p_{\vec{a}}$, then $\vec{a}$ and $\vec{a'}$ have the same strong type.

In other words, we see that *the motive of M is well-defined up to orientation*, independently of the embedding of M_0 into M; we therefore do get an invariant of the model.

What are the possible motives? It is clear, in light of the orthogonality of the $p_{\vec{a}}$, that the $\mu(p_{\vec{a}})$ can be chosen independently of each other, the only constraint being that $\mu(p_{\vec{a}}) \geq \omega$ if $p_{\vec{a}}$ is of the second kind. Indeed, by taking M prime over M_0, and sequences of appropriate length, we get a model M with the desired motive.

There are therefore at least as many models as motives, a fact that allows us to count them:

Theorem 20.32. *If T is dimensional, with an infinite number δ of dimensions, and if $\aleph_\alpha$ is a nondenumerable cardinal that is greater than or equal to the cardinality of T, then T has exactly $|\alpha+1|^\delta$ pairwise nonisomorphic models of cardinality $\aleph_\alpha$.*

Proof. A model M of T, considered as an extension of M_0, is determined by specifying the dimensions of every $p_{\vec{a}}$ over M_0 (Lemma 20.28). If M has cardinality $\aleph_\alpha$, then every dimension is at most $\aleph_\alpha$, leaving only $|\omega+\alpha+1|^\delta$ possibilities. If α is finite, then $|\omega+\alpha+1|^\delta = \omega^\delta \leq \delta^\delta \leq 2^{\delta\times\delta} = 2^\delta$. If α is infinite, then $|\omega+\alpha+1| = |\alpha+1| = |\alpha|$; we therefore have $|\alpha+1|^\delta$ pairwise nonisomorphic extensions of M_0 of cardinality $\aleph_\alpha$. There can be many ways to embed M_0 into M, but in any event, we do not have more than $|\alpha+1|^\delta$ models of cardinality $\aleph_\alpha$.

To get at least $|\alpha+1|^\delta$, consider those motives that are invariant under the orientation group and that take only infinite dimensions less than $\aleph_\alpha$, at least one being $\aleph_\alpha$. Each of these motives corresponds to a model. Since each R-minimal class has only finitely many conjugates, there are δ finite packages of conjugate classes, and a dimension must be chosen for each of these packages in a manner independent of the dimensions chosen for the other packages: The number of these packages is therefore $|\alpha+1|^\delta$. □

With a little more work we can classify the models completely, instead of merely counting them. We note in what follows that there is no assumption about the cardinality of the language T, nor about its number of dimensions.

Lemma 20.33. *If T is dimensional, then its prime model M_0 has an isomorphic elementary extension M_1 such that for every $p_{\vec{a}}$ of the second kind in $S_1(M_0)$, the dimension of $p_{\vec{a}}$ over M_0 in M_1 is infinite (denumerable).*

Proof. Suppose that $p_{\vec{a}}$ is of the second kind, and let $a_0, a_1, \ldots, a_n, \ldots$ be a sequence of independent realizations of the restriction of $P_{\vec{a}}$ to $\vec{a}$. Let M_0' be the prime model over $\{\vec{a}, a_1, \ldots, a_n, \ldots\}$ that is contained in M_0. I claim that the type of a_0 over M_0' does not fork over $\vec{a}$. Indeed, let $\vec{b}$ be in M_0'. Its type over $\{\vec{a}, a_1, \ldots, a_n, \ldots\}$ is isolated by some formula with parameters in some $\{\vec{a}, a_1, \ldots, a_N\}$. By indiscernibility of the a_n, $n > N$, over this set, all the a_n must have the same type over $\vec{b}$, which is also that of a_0, as the satisfaction of $f(\vec{b}, a_n)$ can depend only on the type of a_n over $\{\vec{a}, a_1, \ldots, a_N\}$. As a consequence a_0 does realize the mean type over M_0' of the sequence $a_1, \ldots, a_n, \ldots$.

If we take things in the other direction, and let a_0 realize $p_{\vec{a}}$, we see that the prime model over $M_0 \cup \{a_0\}$ is contained in a copy of the prime model, which is therefore isomorphic to it. We can repeat this operation and get $M_0(a_0, \ldots, a_n)$ isomorphic to M_0, with $a_0, \ldots, a_n$ being independent realizations of types of the second kind. Let us choose a representative of each R-equivalence class of types of the second kind and realize ω independent copies of each of them over M_0. Then the prime model M_1 over all this is atomic. It does not contain indiscernible sequences of length $\aleph_1$: If it did, we could find in it a Morley sequence of a regular type of length $\aleph_1$ over some $M_0(\vec{a})$. By Theorem 18.5, it is isomorphic to M_0.

□

Corollary 20.34. *If T is dimensional, then every model M of T is characterized up to isomorphism by its motive, which is defined up to orientation; the number of models is the number of these motives.*

Proof. Suppose the motive of M is given, up to orientation, and let us consider an arbitrary embedding of M_0 into M. By applying an automorphism to it we can give the model the right orientation. Having done that, replace M_0 with one of its elementary submodels M_0' such that for every type p of the second kind, the dimension of p over M_0' in M_0 is infinite. Naturally, since the prime model remains prime over a finite set of parameters, the dimensions of types of the first kind are unchanged. Therefore, for every R-minimal p in $S_1(M_0')$, its dimension over M_0' in M is the number given by the motive: That determines M completely up to M_0'-isomorphism, and therefore also up to isomorphism! □

With the characterization of Corollary 20.34, we can even specify under what conditions M can be embedded elementarily into N: i It is necessary and sufficient that we can choose orientations of the motives of M and of N so that each dimension provided by the former is less than the corresponding dimension provided by the latter.

We can also specify under what conditions a model M has proper isomorphic elementary submodels: That happens whenever one of its dimensions is infinite, and in particular whenever there are types of the second kind (which means that the prime model is not minimal). If, on the other hand, all R-minimal types are of the first kind, then a model whose motive takes only finite values cannot be elementarily embedded into itself in a strict way.

We are going to collect all the results of this section for a *denumerable* ω-stable theory T.

First case: T is dimensional, with only one dimension (we call it *one-dimensional*). Then T is ω_1-categorical, since the dimension in a model of cardinality $\lambda \geq \omega_1$ can only be λ.

If its minimal type is of the second kind, then it is ω-categorical; if it is of the first kind, then it has denumerably many models $M_0, M_1, \ldots, M_n, \ldots, M_\omega$ corresponding to the various possible dimensions.

Second case: T is dimensional, with finitely many dimensions (we call it *finite-dimensional*). Then the motives corresponding to models of cardinality λ are those whose dimensions are at most λ, with one being λ if $\lambda > \omega$. If T is ω-categorical, then all its types are of the second kind, and only infinite dimensions are allowed in the motives. There are only finitely many, but more than one, even if motives are counted up to orientation, corresponding to the models of cardinality $\aleph_n$; this number is easily calculated as a function of the orientation group. For example, in the most unfavorable case of two dimensions permuted by the action of G, there are $n+1$ models of cardinality $\aleph_n$. If $\alpha \geq \omega$, then there are $|\alpha|$ models of cardinality $\aleph_\alpha$.

If, on the other hand, T is not ω-categorical, then there is at least one type of the first kind allowing a finite dimension, and there are $|\omega + \alpha|$ models of cardinality $\aleph_\alpha$ (including the case of $\aleph_0$).

THIRD CASE: T is dimensional, with ω dimensions. Then it is possible, although difficult, to see that T cannot be ω-categorical, that it always has at least one type of the first kind. The number of models of cardinality $\aleph_\alpha, \alpha > 0$, is $|\alpha + 1|^\omega$; the number of denumerable models is ω if there are only finitely many classes of the first kind, and 2^ω otherwise.

FOURTH CASE: T is nondimensional. Its number of models of cardinality $\aleph_\alpha$ is then greater than or equal to $|\omega+\alpha|^\alpha$ (Theorem 20.27); this statement is correct, but loses its significance, when $\alpha = 0$. For the theory T of an equivalence relation with infinitely many classes all of which are infinite, the number of models of cardinality $\aleph_\alpha$ is exactly $|\omega + \alpha|^\alpha$.

Note that if T is totally transcendental (which, in this context, is a trifle), is not ω_1-categorical, and has only finitely many models of cardinality λ, then this λ is an $\aleph_n$ and T is ω-categorical.

To end this section, we can say that it is more difficult really to classify models by invariants than just to count their number. Many of the lemmas of the last two sections guided the reader up some particularly slippery slopes; furthermore, for the most delicate of them, there are no known examples where they are really necessary, and it is not impossible, although it is improbable, that some day they will be replaced by structural statements that are both simpler and more powerful, as I indicated in passing. My reader will not be surprised to learn that they are the product of an especially artful mind, that of Daniel Lascar.

20.4 The Dope

For a dimensional theory, everything is taken care of. In the other cases, we can attempt to decompose constructions of models into cascades of realizations of regular types. In some cases that is hopeless, since regular types sprout everywhere: We then say that the theory T has the dope.

Before explaining what this is in more detail, let me give an example of a theory with the dope. Consider two equivalence relations E and F;

the classes of the former are called "rows"; those of the latter, "columns." The axioms of T state that there are infinitely many rows and infinitely many columns and that each row meets each column in infinitely many points. It is clear that T is complete and admits quantifier elimination. It is interpretable in the theory of equality by means of triples (a, b, c): say that they have the same row if they have the same first coordinate, and the same column if they have the same second coordinate. (This proves that the dope is not preserved in either direction by interpretation.)

T is ω-categorical, and its models are easy to classify: It is enough to specify what is at the intersection of each row and column (once the numbers of rows and of columns has been determined). There are the maximum possible number of models, of every nondenumerable cardinality λ, that is to say 2^λ, as many as there are nonisomorphic binary relations between two sets of cardinality λ (play around with denumerable and nondenumerable intersections).

We now come to the definition: Consider a model M of T, and an extension N of M, containing two independent tuples $\vec{a}$ and $\vec{b}$ over M, and a strongly regular type $p_{\vec{a}\frown\vec{b}}$ in $S_1(N)$. We know then that the prime models M_1 and M_2 over $M \cup \{\vec{a}\}$ and $M \cup \{\vec{b}\}$ respectively, which are embedded in N, are independent over M. We say then that T *has the dope* if this situation can be realized so that $p_{\vec{a}\frown\vec{b}}$ is orthogonal to every type that does not fork over M_1 and to every type that does not fork over M_2.

If we then replace M_1 by an elementary extension N that is as saturated as desired, so that N_1 and M_2 are independent over M, then $p_{\vec{a}\frown\vec{b}}$ will also be orthogonal to every type that is definable over N_1: The reader is invited to consider that a technical detail.

So if we take $\vec{a}'\frown\vec{b}'$ independent of $\vec{a}\frown\vec{b}$ over M, then $\vec{a}\frown\vec{b}$ and $\vec{a}'\frown\vec{b}'$ have the same type. We can therefore add $\vec{b}'$ on the side of N_1, and we can see that $p_{\vec{a}\frown\vec{b}}$ *and* $p_{\vec{a}'\frown\vec{b}'}$ *are orthogonal.* For a similar reason, $p_{\vec{a}\frown\vec{b}}$ and $p_{\vec{a}'\frown\vec{b}'}$ are orthogonal.

The situation is therefore as follows: If $\dots, \vec{a}_i\frown\vec{b}_i, \dots$ is a Morley sequence of the type of $\vec{a}\frown\vec{b}$, then the $\vec{a}_i$ and the $\vec{b}_j$ are all independent, and all the $\vec{a}\frown\vec{b}$ have the same type: Each pair corresponds to a type p_{ij}, and p_{ij} *is orthogonal to* p_{hk} *when* $i \neq h$ *or* $j \neq k$.

The p_{ij} are obviously nondimensional; consequently, by Theorem 20.23, the dimension of p_{ij} over $\vec{a}_i\frown\vec{b}_j$ in the prime model N over the set of all the $\vec{a}_{i'}$ and the $\vec{b}_{j'}$ is the same as that in the prime model over $\vec{a}_i\frown\vec{b}_j$, that is to say, at most denumerable. (We can assume that $tp(\vec{a}\frown\vec{b}/M)$ does not fork over $\emptyset$, since we can always add a finite tuple to the language.) Since the p_{ij} are orthogonal, we can increase the dimension of one of them as much as we want without touching the others. If we then order the index set I arbitrarily, we can construct a model N in which the dimension of p_{ij} is $\aleph_1$ for $i > j$ and $\aleph_0$ otherwise. That makes it possible to define the analogue of the order property for the $\vec{a}_i\frown\vec{b}_j$ by playing with the dimensions, not just by means of a formula: That explains the initials D.O.P., an acronym for "dimensional order property," from which we get the word "dope."

If T has the dope, then it has 2^λ models of cardinality λ for every nondenumerable λ greater than or equal to $|T|$. This result would be available to us if we had undertaken to show that an unstable theory has the same property: Quite simply, the proof that in the unstable case Ehrenfeucht models constructed over sufficiently different chains cannot be isomorphic also works for the models described in the preceding paragraph. This is therefore a result that my readers will attempt only if they decide to proceed outside this course.

It is possible, however, within the means of this course, to prove this true for cardinals of the form 2^λ, since the dope also allows us to define the analogue of the independence property by means of dimensions; I leave as an exercise the task of coding ultrafilters by sequences, as in Section 14.4, Theorem 14.11.

Given an ultrafilter U of subsets of λ, you will need to consider $\vec{a}_\alpha, \alpha < \lambda$, and $\vec{a}_0, \ldots, \vec{a}_n, \ldots$, all independent, and to construct a model M around them, of cardinality 2^λ, coding the ultrafilter, i.e., a model such that:

- For every subset w of λ there is a $\vec{b}_w$ in M that is independent of the $\vec{a}_\alpha$ and of the right strong type, such that the dimension of $p_{\vec{a}_\alpha{}^\frown\vec{b}_w}$ in M is $\aleph_1$ if $\alpha \in w$ and $\aleph_0$ if $\alpha \notin w$.
- For every $\vec{b}$ in M independent of the $\vec{a}_i$ and of the right strong type, if U includes the set of all α such that the dimension of $p_{\vec{a}_\alpha{}^\frown\vec{b}}$ in M is nondenumerable, then for large enough n, the dimension of $p_{\vec{a}_n{}^\frown\vec{b}}$ in M is $\aleph_1$; on the other hand, if this set is not in the ultrafilter, then for large enough n the dimension of $p_{\vec{a}_n{}^\frown\vec{b}}$ in M is denumerable.

20.5 Depth and the Main Gap

When the dope is there, we really have too many models to be able to hope to classify them in a refined and general way. Suppose now that T does not have the dope, allowing us to hope that cascades of regular types will have a civilized character. We can then define the depth of a strongly regular type inductively as follows:

- If α is a limit, then $\mathrm{prof}(p) \geq \alpha$ if $\mathrm{prof}(p) \geq \beta$ for every $\beta < \alpha$ (in particular, $\mathrm{prof}(p)$ is always ≥ 0.
- $\mathrm{prof}(p) \geq \alpha + 1$, where $p \in S_1(M)$, if we can find an R-minimal q in $S_1(N)$, where N is an elementary extension of M in which a realizes p, such that $\mathrm{prof}(q) \geq \alpha$, q is orthogonal to every type that does not fork over M, and q is dimensional in $T(M \cup \{a\})$, which is the theory obtained by adding the elements of M and a to the language of T.

Following a firmly established tradition, $\text{prof}(p) = \alpha$ if $\text{prof}(p) \geq \alpha$ and $\text{prof}(p) \not\geq \alpha + 1$; $\text{prof}(p) = \infty$ if $\text{prof}(p) \geq \alpha$ for every α.

If there is a type of depth ∞, then we call the theory *deep*, and show—this is laborious enough—that there are 2^λ models of every cardinality λ greater than that of T.

Now let us consider a non–deep theory without the dope. To characterize a model M of this theory T, we begin by embedding the prime model M into it. We then choose representatives $\ldots, a_i, \ldots$ of the classes of R-minimal types over M_0 that are realized in M, and take the prime model M_1 over all of that. At the second stage, if we take a in M with $tp(a/M_1)$ R-minimal, since there is no dope, this type is R-equivalent to a type that does not fork over an $M_0(a_i)$: The lack of dope allows us to anticipate where the new regular types will be coming out from! Since the depth is an ordinal, the process eventually stops, and we get a kind of tree of types more or less canonically associated with the model.

Shelah called the distinction between the nondeep theories without dope and the other theories the "main gap": Below the main gap, models can be classified; above, there are too many for that to be possible. Others prefer to put this main gap between the theories with dope and the theories without dope.

My reader will be happy to learn that we know how to count models below the main gap: We need to distinguish six cases for a (totally transcendental) denumerable theory. His bliss will be unbounded when he knows that the parallel problem for superstable theories is also solved. As the solution is quite horrid, I shall refrain from delving into it here.

At some point I do need to end this course, which is intended to give the groundwork for contemporary model theory. Moreover, I would have nothing original to say about this main gap.

My reader, whose hand I have held up to this point, can now address God rather than the saints, and knows enough to read the Torah without a Targum, the Vedas without Upanishads, and the Koran without Tafsir; she will ask Saffe why Shelah is so happy.

20.6 Historic and Bibliographic Notes

The canonical warning applies here more than anywhere else: *Everything in this chapter was done by Shelah.* The study of dimensional theories, which he calls nonmultidimensional, is in [She78]; that of the other cases, in [She82a] and [She82b].

However, the dimensional case was essentially known to Lachlan, who expounded it in [Lac78]; for the epigraph to this chapter, I have quoted from this work a sentence remarkable for the three different ways by which the author referred to himself. For the denumerable models of these theories, see also [Mor67], [BL71], and [Lac75b].

Lachlan relied considerably on the successive choices of formulas of minimal Morley rank, and ultimately counted models instead of actually classifying them. The classification, as comes from fair Shelah, is not so simple to determine; if the drafting of a coherent exposition of this subject is desired, many technical details, sometimes delicate, remain to be settled, and this chapter owes much to [Las85], whose general outline it follows, particularly for its very practical definition of dimensional types.

The Rudin–Keisler ordering was christened thus by [Las75], because of an analogy to an ordering that Rudin and Keisler defined on ultrafilters; as this is rather far from our subject, it is a little abusive thus to call the ordering R.

Lemma 20.13 is in [Pil82]; the Lascar decompositions, with the equivalence (Theorem 20.18) between strong and weak orthogonality, in [Las82].

To the best of my knowledge, the dope first appeared in [She73], without being named; a short proof of the existence of many models when the dope is present is found in [Saf82]. It was also Saffe who counted models below the main gap [Saf83]; some good tips about denumerable models can be found in [Bou83] and [BL83].

For expository works on the classification of models, I must point you beyond [Las86] and [Pil83]; Baldwin's and Hodges' books are currently in preparation.[1]

[1]**Note (1999):** [Las86]; [Hod93]

Bibliography

Ne jetons pas les morceaux
De nos coeurs aux porceaux
Perdons pas notre latin
Au profit des pantins
Parlons pas la langue des Dieux
Pour les balourds, les fesse-mathieu
Les paltoquets ni les bobèches
Les foutriquets ni les pimbêches
Ni pour la femme $\cdots$
G.B.

My bibliographic research was greatly simplified by "A thousand papers in Model Theory and Algebra", a list compiled by Wilfrid Augustine Hodges; I shall take the opportunity to thank him for the critical reading which he gave the manuscript of this text.

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Index of Notation

Not seeing anything better to do, I have arranged all symbols in order of their appearance in the text.

$\alpha, \beta, \gamma, \ldots$ ordinals 9.1
$\kappa, \lambda, \mu, \ldots$ cardinals 9.1

$H(T)$ space of Henkin enumerations 10.1
A_α first α terms of a construction 10.4
$E(I)$ rich space constructed over I 10.6

$\Pi p_i/U, p^U$ ultraproduct, ultrapower of types 11.1
(M, dp) structure of "definition" of a type 11.1, 13.2

I_n, I_λ axioms of the independence property 12.4

$p \geq q, p \sim q$ fundamental order 13.1
$\lambda_0(T)$ minimum of the spectrum 13.2
$\kappa(T), \kappa_n(T)$ supremum of decreasing ordinal sequences in the fundamental order 13.2

$\beta(p)$ bound of a type 15.1

$SF_1(A), SF_n(A), SF_\alpha(A)$ space of strong types 16.1
$T^{\mathrm{eq}}, L^{\mathrm{eq}}, M^{\mathrm{eq}}$ jump to the imaginary 16.1
$G(SF_1(A)/A)$ Galois group of strong types 16.6
$G(A_{alg}/A)$ Galois group of the algebraic closure 16.6

$R(p)$ rank 17.1
α_R ordinal supremum of a rank 17.1
RU Lascar's U rank 17.1
RC Shelah's continuous rank 17.2
$R(T)$ maximum of a continuous rank 17.2
$\mathrm{RM}(p), \mathrm{RM}(f)$ Morley rank 17.3
$\alpha_T = \alpha_{RM}$ α of the Morley rank 17.3
$R(\ \ , f, \omega), R(\ \ , \Delta, 2)$ infinite division rank 17.4
$R(\ \ , f, 2), R(\ \ , \Delta, 2)$ dichotomy rank 17.4

$\mathrm{tp}(\vec{a}/A)$ type of $\vec{a}$ over A 18.2
$R(\mathrm{tp}(\vec{a}/A)), R(\vec{a}/A)$ rank of the type of $\vec{a}/A$ 18.2
f^M elements of M satisfying f 18.2
$K_{\mathrm{alg}}, K_{\mathrm{min}}, K_{\mathrm{dc}}$ algebraic, minimal, and differential closures 18.3
$M(I)$ Ehrenfeucht model 18.5

p_α in a sequence, the type of a_α over A_α 19.1
α^β ordinal exponentiation 19.2
$\alpha + \beta$ natural sum 19.2
$p \geq_D q$ domination order D 19.4

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