Generalized Hypergeometric Functions

The generalized hypergeometric series is a power series having rational coefficients.

The generalized hypergeometric series has the form

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_k}{\prod_{j=1}^{q}(b_j)_k k!} z^k $$

where the symbols (a1)n ... (ap)n and (b1)n ... (bq)n represent the so-called Pochhammer symbols (also called rising factorials) and the argument z can be real or complex.

The generalized hypergeometric series is remarkably versatile because many well-known functions of pure and applied mathematics are special cases of the generalized hypergeometric series. For example, some special cases of the generalized hypergeometric series include various mathematical constants, elementary transcendental functions, Bessel functions, the incomplete gamma function, confluent hypergemetric functions, generalized Legendre functions, the dilogarithm function, various polynomials, and (infinitely) many more. In particular, the hypergeometric function 1F1(a ; b ; x) is also called the confluent hypergeometric function of the first kind M(a, b, x), and the lower incomplete gamma function is a special case thereof.

For more information, see

http://en.wikipedia.org/wiki/Generalized_hypergeometric_function

The generalized hypergeometric series can be extended to include non-rational coefficients. This results in a series that can be successfully numerically calculated in certain parameter ranges. Generalized hypergeometric series having irrational coefficients, however, do not usually correspond to any known special functions. We will call generalized hypergeometric series with irrational coefficients generalized hypergeometric functions.
TBD: Chris : verify this paragraph.

The Boost.Math library does not intend to provide support for all generalized hypergeometric functions with unlimited numbers of parameters and ranges of parameters and argument. But rather, the Boost.Math library supports selected special cases of generalized hypergeometric functions that frequently arise in science and engineering. These are described in the following sections.

The Hypergeometric Function 0F1(; b ; x)

Synopsis

```
template <class T1, class T2, class Policy>
calculated-result-type hypergeometric_0f1(T1 b, T2 z, const Policy&);

template <class T1, class T2>
calculated-result-type hypergeometric_0f1(T1 b, T2 z);
```

Description

The hypergeometric function 0F1(b ; x) is defined by the following power series

$$ {}_0F_1(;b;z) = \sum_{k=0}^{\infty} \frac{1}{(b)_k k!} z^k $$

For negative integer b parameter pole error is raised.

0F1 is also represented in terms of Bessel functions:

$$ {}_0F_1 = \Gamma(b) z^{\frac{1-b}{2}} I_{b-1}(2\sqrt{z})) $$

and

$$ {}_0F_1 = \Gamma(b) (-z)^{\frac{1-b}{2}} J_{b-1}(2\sqrt{-z})) $$

The Hypergeometric Function 1F0(a ; ; x)

Synopsis

```
template <class T1, class T2, class Policy>
calculated-result-type hypergeometric_1f0(T1 a, T2 z, const Policy&);

template <class T1, class T2>
calculated-result-type hypergeometric_1f0(T1 a, T2 z);
```

Description

The hypergeometric function 1F0(a ; x) is defined by the following power series

$$ {}_0F_1(;b;z) = \sum_{k=0}^{\infty} \frac{1}{(b)_k k!} z^k $$

For negative integer b parameter pole error is raised.

1F0 is also represented by Binomial series:

$$ {}_1F_0 = \sum_{k=0}^{\infty} \binom{k+a-1}{k} z^k = (1-z)^{-a} $$

The Hypergeometric Function 1F1(a ; b ; x)

Synopsis

```
template <class T1, class T2, class T3, class Policy>
calculated-result-type hypergeometric_1f1(T1 a, T2 b, T3 z, const Policy&);

template <class T1, class T2, class T3>
calculated-result-type hypergeometric_1f1(T1 a, T2 b, T3 z);
```

Description

The hypergeometric function 1F1(a ; b ; x) is defined by the following power series

$$ {}_1F_1(a;b;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k k!} z^k $$

Function is calculated for real-valued coefficients and real-valued argument using a variety of numerical methods in various convergence regions.

When calculating 1F1(a ; b ; x), certain tests are applied to the parameters a, b and the argument x in order to decide which algorithm is expected to have the best convergence properties. The corresponding algorithm is subsequently selected and used for the calculation of 1F1(a ; b ; x).

For small parameters a and small argument x, a Taylor series is used. For example, the calculation of 1F1(1/2, 2/3, 0.1234) uses a Taylor series.

For small parameters a, b and large argument x, a divergent asymptotic series is used. Here, the series is terminated as soon as the desired precision has been reached yet before the series begins to diverge. For example, the calculation of 1F1(1/2, 2/3, 1234.5) uses a divergent asymptotic series.

For small parameters a, b and intermediate argument x, various methods are used. These include recurrence relations with series of Bessel functions, recurrence relations with series of shifted Chebyshev polynomials, continued fractions, Pade approximations, and rational approximations.
The references for these numerical methods include:
TBD: list the references.
TBD: Provide an example of a calculation in this intermediate parameter range.

For some parameter ranges, there is no known convergent computational method for fixed-precision floating-point types. In these cases, an out-of-range exception is thrown. One such case is if the coefficient a is much larger than the coefficient b and the argument x is in the region of the coefficient a. For example, the attempted computation of 1F1(1234, 1/4, 1000) throws an out-of-range exception for double precision.

The following parameter ranges are supported for 1F1(a ; b ; x):
TBD: List the supported parameter ranges.

The Hypergeometric Function 2F0(a1, a2 ; ; x)

Synopsis

```
template <class T1, class T2, class T3, class Policy>
calculated-result-type hypergeometric_2f0(T1 a1, T2 a2, T3 z, const Policy&);

template <class T1, class T2, class T3>
calculated-result-type hypergeometric_2f0(T1 a1, T2 a2, T3 z);
```

Description

The hypergeometric function 2F0(a1 , a2 ; ; x) is defined by the following power series

$$ {}_2F_0(a_1, a_2; ; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{k!} z^k $$

For negative integer b parameter pole error is raised.

When a1 and a2 parameters are integral, we may use Laguerre function relation:

$$ {}_2F_0(a_1, a_2; ; z) = z^n n! L_n^m(-\frac{1}{z}); \qquad n = -a_1, m = -a_2 - n $$