Overall_Plan

- Compare all on three tasks:
- MNIST, single variable function interpolation, multivariable function interpolation
- https://en.wikipedia.org/wiki/Classical_orthogonal_polynomials
- Chebyshev:
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- https://en.wikipedia.org/wiki/Chebyshev_polynomials#Definitions

The **Chebyshev polynomials of the first kind** are obtained from the recurrence relation:

$$
\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_{n+1}(x) &= 2x\,T_n(x) - T_{n-1}(x).
\end{aligned}
$$

The recurrence also allows to represent them explicitly as the determinant of a tridiagonal matrix of size $k \times k$:

$$
T_k(x) = \det \begin{bmatrix} x & 1 & 0 & \cdots & 0 \\ 1 & 2x & 1 & \ddots & \vdots \\ 0 & 1 & 2x & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 2x \end{bmatrix}
$$

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The **Chebyshev polynomials of the second kind** are defined by the recurrence relation:

$$
\begin{aligned}
U_0(x) &= 1 \\
U_1(x) &= 2x \\
U_{n+1}(x) &= 2x\,U_n(x) - U_{n-1}(x).
\end{aligned}
$$

Notice that the two sets of recurrence relations are identical, except for $T_1(x) = x$ vs. $U_1(x) = 2x$.

The **Chebyshev polynomials** are two sequences of polynomials related to the cosine and sine functions, notated as $T_n(x)$ and $U_n(x)$. They can be defined in several equivalent ways, one of which starts with trigonometric functions:

The **Chebyshev polynomials of the first kind** T_n are defined by:

$$T_n(\cos\theta) = \cos(n\theta).$$

Similarly, the **Chebyshev polynomials of the second kind** U_n are defined by:

$$U_n(\cos\theta)\sin\theta = \sin\big((n+1)\theta\big).$$

- Legendre:
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- https://en.wikipedia.org/wiki/Legendre_polynomials#Recurrence_relations

As discussed above, the Legendre polynomials obey the three-term recurrence relation known as Bonnet's recursion formula given by

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

and

$$\frac{x^2-1}{n}\frac{d}{dx}P_n(x) = xP_n(x) - P_{n-1}(x)$$

or, with the alternative expression, which also holds at the endpoints

$$\frac{d}{dx}P_{n+1}(x) = (n+1)P_n(x) + x\frac{d}{dx}P_n(x)\,.$$

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An especially compact expression for the Legendre polynomials is given by Rodrigues' formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n\,.$$

This formula enables derivation of a large number of properties of the P_n's. Among these are explicit representations such as

$$\begin{aligned}
P_n(x) &= [t^n]\frac{\big((t+x)^2-1\big)^n}{2^n} = [t^n]\frac{(t+x+1)^n(t+x-1)^n}{2^n},\\
P_n(x) &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2(x-1)^{n-k}(x+1)^k,\\
P_n(x) &= \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k,\\
P_n(x) &= \frac{1}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{k}\binom{2n-2k}{n}x^{n-2k},\\
P_n(x) &= 2^n\sum_{k=0}^{n}x^k\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}.
\end{aligned}$$

- Bessel:

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- https://en.wikipedia.org/wiki/Bessel_polynomials

The Bessel polynomial may also be defined by a recursion formula:

$$y_0(x) = 1$$
$$y_1(x) = x + 1$$
$$y_n(x) = (2n-1)x\, y_{n-1}(x) + y_{n-2}(x)$$

and

$$\theta_0(x) = 1$$
$$\theta_1(x) = x + 1$$
$$\theta_n(x) = (2n-1)\theta_{n-1}(x) + x^2\theta_{n-2}(x)$$

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$$y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!} \left(\frac{x}{2}\right)^k.$$

Another definition, favored by electrical engineers, is sometimes known as the **reverse Bessel polynomials**

$$\theta_n(x) = x^n\, y_n(1/x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!} \frac{x^{n-k}}{2^k}.$$

- Hermite:
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- https://en.wikipedia.org/wiki/Hermite_polynomials

The physicist's Hermite polynomials can be written explicitly as

$$H_n(x) = \begin{cases} n! \sum_{l=0}^{\frac{n}{2}} \frac{(-1)^{\frac{n}{2}-l}}{(2l)!\left(\frac{n}{2}-l\right)!}(2x)^{2l} & \text{for even } n, \\ n! \sum_{l=0}^{\frac{n-1}{2}} \frac{(-1)^{\frac{n-1}{2}-l}}{(2l+1)!\left(\frac{n-1}{2}-l\right)!}(2x)^{2l+1} & \text{for odd } n. \end{cases}$$

These two equations may be combined into one using the floor function:

$$H_n(x) = n! \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^m}{m!(n-2m)!}(2x)^{n-2m}.$$

- $$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$
- Jacobi;
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- https://en.wikipedia.org/wiki/Jacobi_polynomials
-

Recurrence relations [edit]

The recurrence relation for the Jacobi polynomials of fixed α, β is:[1]

$$\begin{aligned}&2n(n+\alpha+\beta)(2n+\alpha+\beta-2)P_n^{(\alpha,\beta)}(z)\\&\qquad=(2n+\alpha+\beta-1)\Big\{(2n+\alpha+\beta)(2n+\alpha+\beta-2)z+\alpha^2-\beta^2\Big\}P_{n-1}^{(\alpha,\beta)}(z)-2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)P_{n-2}^{(\alpha,\beta)}(z),\end{aligned}$$

for $n=2,3,\ldots$. Writing for brevity $a:=n+\alpha$, $b:=n+\beta$ and $c:=a+b=2n+\alpha+\beta$, this becomes in terms of a,b,c

$$2n(c-n)(c-2)P_n^{(\alpha,\beta)}(z)=(c-1)\Big\{c(c-2)z+(a-b)(c-2n)\Big\}P_{n-1}^{(\alpha,\beta)}(z)-2(a-1)(b-1)c\,P_{n-2}^{(\alpha,\beta)}(z).$$

Since the Jacobi polynomials can be described in terms of the hypergeometric function, recurrences of the hypergeometric function give equivalent recurrences of the Jacobi polynomials. In particular, Gauss' contiguous relations correspond to the identities

$$\begin{aligned}(z-1)\frac{d}{dz}P_n^{(\alpha,\beta)}(z)&=\frac{1}{2}(z-1)(1+\alpha+\beta+n)P_{n-1}^{(\alpha+1,\beta+1)}\\&=nP_n^{(\alpha,\beta)}-(\alpha+n)P_{n-1}^{(\alpha,\beta+1)}\\&=(1+\alpha+\beta+n)\left(P_n^{(\alpha,\beta+1)}-P_n^{(\alpha,\beta)}\right)\\&=(\alpha+n)P_n^{(\alpha-1,\beta+1)}-\alpha P_n^{(\alpha,\beta)}\\&=\frac{2(n+1)P_{n+1}^{(\alpha,\beta-1)}-\left(z(1+\alpha+\beta+n)+\alpha+1+n-\beta\right)P_n^{(\alpha,\beta)}}{1+z}\\&=\frac{(2\beta+n+nz)P_n^{(\alpha,\beta)}-2(\beta+n)P_n^{(\alpha,\beta-1)}}{1+z}\\&=\frac{1-z}{1+z}\left(\beta P_n^{(\alpha,\beta)}-(\beta+n)P_n^{(\alpha+1,\beta-1)}\right).\end{aligned}$$

Generating function [edit]

The generating function of the Jacobi polynomials is given by

$$\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(z)t^n=2^{\alpha+\beta}R^{-1}(1-t+R)^{-\alpha}(1+t+R)^{-\beta},$$

where

$$R=R(z,t)=\left(1-2zt+t^2\right)^{\frac{1}{2}},$$

and the branch of square root is chosen so that $R(z,0)=1$.[1]

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$$P_n^{(\alpha,\beta)}(z)=\frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+\beta+n+1)}\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)}\left(\frac{z-1}{2}\right)^m$$

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Alternate expression for real argument [edit]

For real x the Jacobi polynomial can alternatively be written as

$$P_n^{(\alpha,\beta)}(x) = \sum_{s=0}^{n} \binom{n+\alpha}{n-s}\binom{n+\beta}{s}\left(\frac{x-1}{2}\right)^{s}\left(\frac{x+1}{2}\right)^{n-s}$$

and for integer n

$$\binom{z}{n} = \begin{cases} \frac{\Gamma(z+1)}{\Gamma(n+1)\Gamma(z-n+1)} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

where $\Gamma(z)$ is the gamma function.

In the special case that the four quantities n, $n+\alpha$, $n+\beta$, $n+\alpha+\beta$ are nonnegative integers, the Jacobi polynomial can be written as

$$P_n^{(\alpha,\beta)}(x) = (n+\alpha)!(n+\beta)! \sum_{s=0}^{n} \frac{1}{s!(n+\alpha-s)!(\beta+s)!(n-s)!}\left(\frac{x-1}{2}\right)^{n-s}\left(\frac{x+1}{2}\right)^{s}.$$

The sum extends over all integer values of s for which the arguments of the factorials are nonnegative.

Special cases [edit]

$$P_0^{(\alpha,\beta)}(z) = 1,$$
$$P_1^{(\alpha,\beta)}(z) = (\alpha+1) + (\alpha+\beta+2)\frac{z-1}{2},$$
$$P_2^{(\alpha,\beta)}(z) = \frac{(\alpha+1)(\alpha+2)}{2} + (\alpha+2)(\alpha+\beta+3)\frac{z-1}{2} + \frac{(\alpha+\beta+3)(\alpha+\beta+4)}{2}\left(\frac{z-1}{2}\right)^{2}.$$

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- Gegenbauer:
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- https://en.wikipedia.org/wiki/Gegenbauer_polynomials
-

- The polynomials satisfy the recurrence relation (Suetin 2001):

$$C_0^{(\alpha)}(x) = 1$$
$$C_1^{(\alpha)}(x) = 2\alpha x$$
$$(n+1)C_{n+1}^{(\alpha)}(x) = 2(n+\alpha)xC_n^{(\alpha)}(x) - (n+2\alpha-1)C_{n-1}^{(\alpha)}(x)$$

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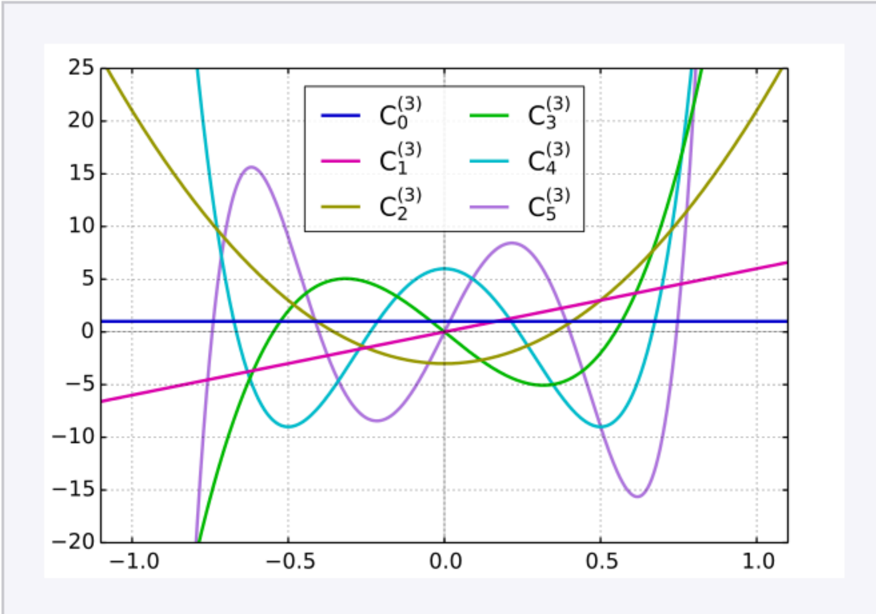

Gegenbauer polynomials with α=3

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- Fibonacci: (Not orthogonal!)
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- https://en.wikipedia.org/wiki/Fibonacci_polynomials
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These Fibonacci polynomials are defined by a recurrence relation:[1]

$$F_n(x) = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ xF_{n-1}(x) + F_{n-2}(x), & \text{if } n \geq 2 \end{cases}$$

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-

The Lucas polynomials use the same recurrence with different starting values:[2]

$$L_n(x) = \begin{cases} 2, & \text{if } n = 0 \\ x, & \text{if } n = 1 \\ xL_{n-1}(x) + L_{n-2}(x), & \text{if } n \geq 2. \end{cases}$$

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- Romaovski:
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- https://en.wikipedia.org/wiki/Romanovski_polynomials#Recurrence_relations
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Recurrence relations between the infinite orthogonal series of Romanovski polynomials with the parameters in the above equations (14) follow from the generating function,[18]

$$\nu\big(\nu+1-2(\beta_n+n)\big)R_{\nu-1}^{(\alpha_n,\beta_n+n-\nu+1)}(x)+\frac{\mathrm{d}}{\mathrm{d}x}R_{\nu}^{(\alpha_n,\beta_n+n-\nu)}(x)=0, \tag{21}$$

and

$$R_{\nu+1}^{(\alpha_n,\beta_n+n-\nu-1)}(x)=\big(\alpha_n-2x(-\beta_n-n+\nu+1)\big)R_{\nu}^{(\alpha_n,\beta_n+n-\nu)}-\nu\left(1+x^2\right)\big(2(-\beta_n-n)+\nu+1\big)R_{\nu-1}^{(\alpha_n,\beta_n+n-\nu+1)}, \tag{22}$$

as Equations (10) and (23) of Weber (2007)[18] respectively.

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$$\alpha\to\alpha_n=\frac{2b}{n+1+a},\quad \beta\to\beta_n=-(a+n+1)+1,\quad n=0,1,2,\ldots,\infty. \tag{14}$$

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