COT 5600 Quantum Computing

Spring 2019

Homework 1

Problem 1 (Eigenvalues of Pauli operators)

Let $B = \{|\psi_1\rangle, \ldots, |\psi_d\rangle\}$ and $B' = \{|\psi'_1\rangle, \ldots, |\psi'_d\rangle\}$ be two orthonormal bases of $\mathbb{C}^d$.

The ONBs B and B' are called mutually unbiased if

$$|\langle\psi_i|\psi'_j\rangle|^2 = \frac{1}{d}$$

for all $1 \leq i, j \leq d$.

Show that the eigenbases of the Pauli operators

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are mutually unbiased. Implement Python methods that compute the eigenbases of the Pauli operators and check that they form mutually unbiased bases.

1. Find the eigenbases

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_x - \lambda I = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}$$

$$det(\sigma_x - \lambda I) = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

$\lambda = 1$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & -0 & 0 \end{bmatrix}$$

$$x_1 - x_2 = 0$$

$$x_1 = x_2$$

$$eigenvector for \lambda = 1 is \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda = -1$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -0 & 0 \end{bmatrix}$$

$$x_1 = x_2 = 0$$

$$x_1 = -x_2$$

$$eigenvector for \lambda = 1 is \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Using the method above, we can repeat the calculation for the other eigenbases (non-normalized)

$$\beta_x = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \beta_y = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}, \beta_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2. Check if mutually unbiased

Normalized eigenbases:

$$\beta_x = \{ \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \ , \ \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \}$$

$$\beta_y = \{ \frac{1}{\sqrt{2}}(i|0\rangle + |1\rangle) \ , \ \frac{1}{\sqrt{2}}(-i|0\rangle - |1\rangle) \}$$

$$\beta_z = \{ |0\rangle \ , \ |1\rangle \}$$

$i = 1, j = 1$

$$\left| \frac{1}{\sqrt{2}}\langle \ (|0\rangle + |1\rangle) \mid \frac{1}{\sqrt{2}} \ (i|0\rangle + |1\rangle) \ \rangle \right|^2 =$$

$$= \left| \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right|^2$$

$$= abs\frac{i}{2} + \frac{1}{2}^2$$

$$= \sqrt{(\frac{1}{2} + \frac{i}{2})(\frac{1}{2} - \frac{i}{2})}^2$$

$$(\frac{1}{2} + \frac{i}{2})(\frac{1}{2} - \frac{i}{2})$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Repeating that calculationg fo rthe pther combinations we find that each one results in a value of $\frac{1}{2}$, meaning that the eigenbases of the Pauli operators are mutually unbiased.

Problem 2 (Trace inner product)

For $A, B \in \mathbb{C}^{d\times d}$, define

$$\langle A|B\rangle_{\mathrm{Tr}} = \mathrm{Tr}(A^\dagger B)\,.$$

Prove that the above map defines an inner product on the vector space $\mathbb{C}^{d\times d}$. (In the literature, this inner product is called the trace inner product or Hilbert-Schmidt inner product.)

Problem 3 (Unitary error basis)
Define the matrices $X, Z \in \mathbb{C}^{d\times d}$ as follows

$$\begin{aligned} X &= \sum_{k=0}^{d} |k+1\rangle\langle k| & (1) \\ Z &= \sum_{\ell=0}^{d-1} \omega^{\ell} |\ell\rangle\langle \ell| & (2) \end{aligned}$$

where the addition is modulo $k+1$ and $\omega = e^{2\pi i/d}$ is a primitive dth root of unity. Show that the d^2 matrices

$$M^{(a,b)} = X^a Z^b$$

where $a, b \in \{0, \ldots, d-1\}$ form an orthonormal basis with respect to the trace inner product. Implement methods in Python that construct these matrices and compute the trace inner product for all pairs. (The above collection of matrices is called a unitary error basis in the literature and is used, for instance, in the theory of quantum error correcting codes and quantum channels for qudit systems. It is a generalization of the Pauli basis for a qubit system to a qudit system.)