TECHNISCHE UNIVERSITÄT WIEN
Vienna | Austria

Basics of Quantum Information

States, Measurements, Operations, Entanglement

Felix Zilk, MSc

Classical and quantum bits

Classical bit is in 0 or 1 state

$$bit = 0, \text{or } bit = 1$$

Coin is either face up or down

$coin = heads, \text{or } coin = tails$

Classical and quantum bits

Classical bit is in 0 or 1 state

$$bit = |0\rangle, \text{ or } bit = |1\rangle$$

Coin is either face up or down

$$coin = |heads\rangle, \text{ or } coin = |tails\rangle$$

<u>Probabilistic state: Coin toss</u>

$$coin = \frac{1}{2}|heads\rangle + \frac{1}{2}|tails\rangle$$

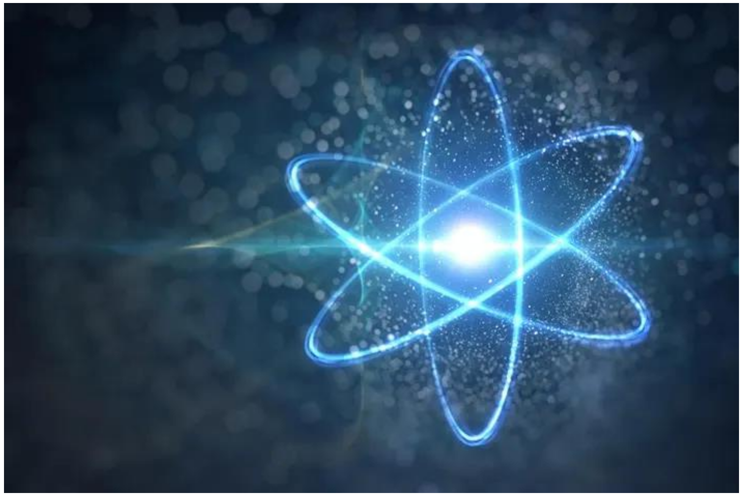
Classical and quantum bits

	Bits	Probabilistic Bits	Qubits
State of Single System	$bit \in \{ \|0\rangle, \|1\rangle \}$	$b = a\|0\rangle + b\|1\rangle$ $a + b = 1,$ $a, b \in \mathbb{R}_+$	

Probability of $|0\rangle$

Probability of $|1\rangle$

Classical and quantum bits



Quantum bit

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

with $|\alpha|^2 + |\beta|^2 = 1,\ \alpha, \beta \in \mathbb{C}$

<u>Superposition of $|0\rangle$ and $|1\rangle$</u>

Classical and quantum bits

Classical bit is in 0 or 1 state

$bit = |0\rangle$, or $bit = |1\rangle$

Coin is either face up or down

$coin = |heads\rangle$, or $coin = |tails\rangle$

<u>Probabilistic state: Coin toss</u>

$$coin = \frac{1}{2}|heads\rangle + \frac{1}{2}|tails\rangle$$

→ Result of incomplete knowledge!

Quantum bit

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

with $|\alpha|^2 + |\beta|^2 = 1,\ \alpha, \beta \in \mathbb{C}$

<u>Superposition of 0 and 1</u>

→ Complete knowledge about the system!

Classical and quantum bits

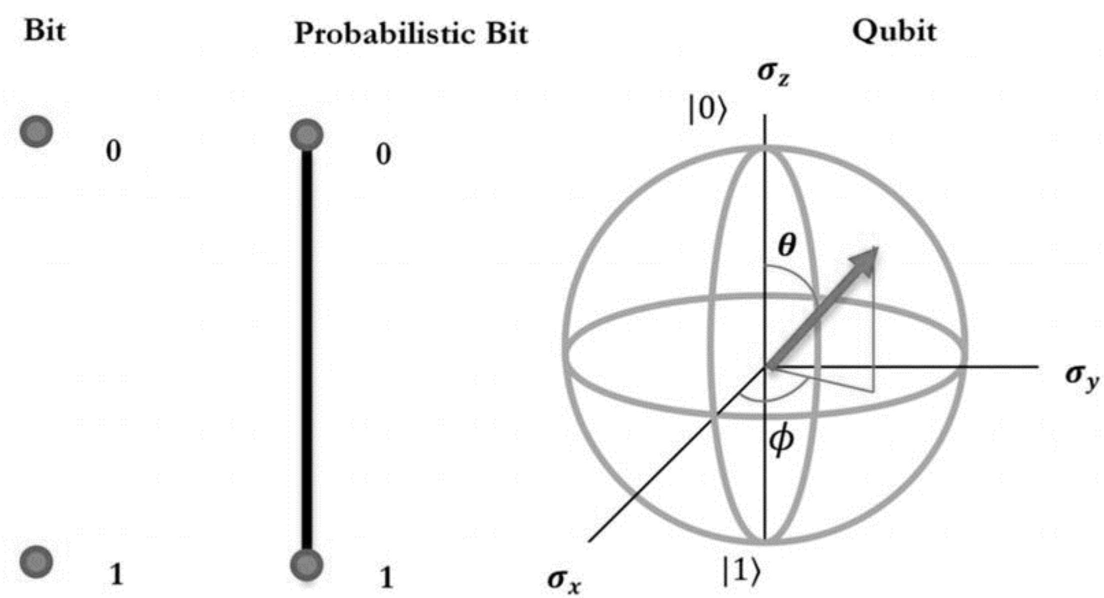
	Bits	Probabilistic Bits	Qubits
State of Single System	$bit \in \{\, \lvert 0\rangle, \lvert 1\rangle \,\}$	$b = a\lvert 0\rangle + b\lvert 1\rangle$ $a + b = 1,$ $a, b \in \mathbb{R}_+$	$\lvert\psi\rangle = \alpha\lvert 0\rangle + \beta\lvert 1\rangle$ $\lvert\alpha\rvert^2 + \lvert\beta\rvert^2 = 1,$ $\alpha, \beta \in \mathbb{C}$

Probability of $\lvert 0\rangle$

Probability of $\lvert 1\rangle$

Classical and quantum bits

	Bits	Probabilistic Bits	Qubits
State of Single System	$bit \in \{\, \|0\rangle, \|1\rangle \,\}$	$b = a\|0\rangle + b\|1\rangle$ $a + b = 1,$ $a, b \in \mathbb{R}_+$	$\|\psi\rangle = \alpha\|0\rangle + \beta\|1\rangle$ $\|\alpha\|^2 + \|\beta\|^2 = 1,$ $\alpha, \beta \in \mathbb{C}$



Graphic: "Getting Started with Liqui: > and Quantum Computing." TECHCOMMUNITY.MICROSOFT.COM, March 21, 2019.

Pure vs. mixed states

	Classical	Quantum
Well known state (Pure state)	$bit = \|0\rangle$, or $bit = \|1\rangle$ $coin = heads$, or $coin = tails$	$\|\psi\rangle = \alpha\|0\rangle + \beta\|1\rangle$
Statistical mix (Mixed state; ensemble)	$coin = \frac{1}{2}\|heads\rangle + \frac{1}{2}\|tails\rangle$	Density matrix ρ

Quantum two-level system

We can represent pure states as vectors in 2-dim. vector space over $\mathbb{C}$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \equiv \alpha\begin{bmatrix}1\\0\end{bmatrix} + \beta\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\alpha\\\beta\end{bmatrix}$$

$$p_0 = |\alpha|^2,\ \ p_1 = |\beta|^2$$

$|0\rangle \equiv \begin{bmatrix}1\\0\end{bmatrix}, |1\rangle \equiv \begin{bmatrix}0\\1\end{bmatrix} \in \mathcal{H} = \mathbb{C}^2$ Basis $\mathcal{B} = \{|0\rangle, |1\rangle\}$

$|\alpha|^2 + |\beta|^2 = 1,\ \alpha, \beta \in \mathbb{C} \rightarrow$ qubit states are complex vectors of length 1

Examples of pure states

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad \rightarrow p_0 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} = p_1$$

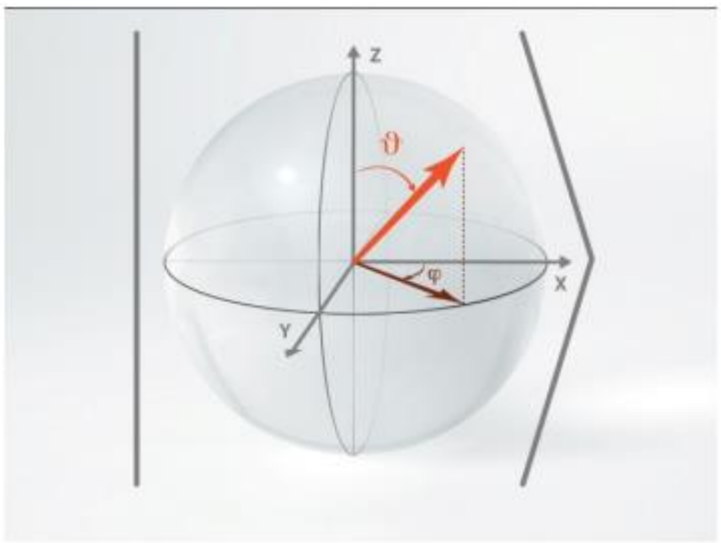
$$\frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle \qquad \rightarrow p_0 = \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3};\ p_1 = \left|\frac{\sqrt{2}}{\sqrt{3}}\right|^2 = \frac{2}{3}$$

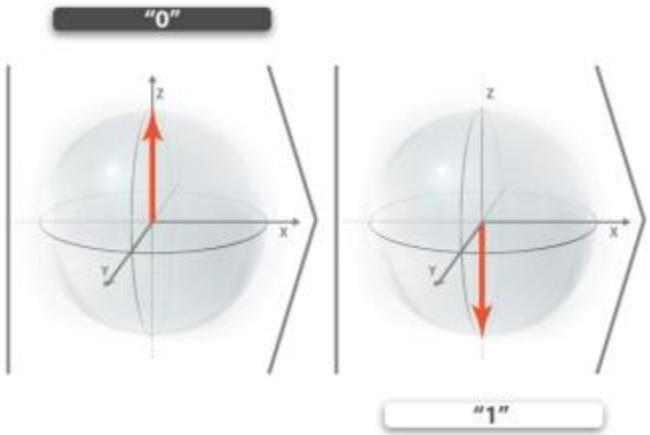
$$0|0\rangle + 1|1\rangle = |1\rangle \qquad \rightarrow p_0 = 0;\ p_1 = 1$$

You measure 100 Qubits in the state $\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$. What do you expect?

Result depends on the amplitudes! $\boldsymbol{p_0} = |\boldsymbol{\alpha}|^2 \quad \boldsymbol{p_1} = |\boldsymbol{\beta}|^2$

Visualization on the Bloch-sphere





$$|\psi\rangle = cos\left(\frac{\vartheta}{2}\right)|0\rangle + e^{i\varphi} sin\left(\frac{\vartheta}{2}\right)|1\rangle$$

W. Dür and S. Heusler, arXiv:1312.1463v1 **[physics.ed-ph]**

Unitary evolution describes qubit in isolation!

Single-qubit operations described by 2×2 unitary matrices U

$$U^\dagger U = \mathbb{I}\mathrm{d}$$

Example: Hadamard gate

$$H = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$
$$H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Realisation:

- Laser- or Microwavepulses, magnetic fields, birefringent crystals, etc.

Later:

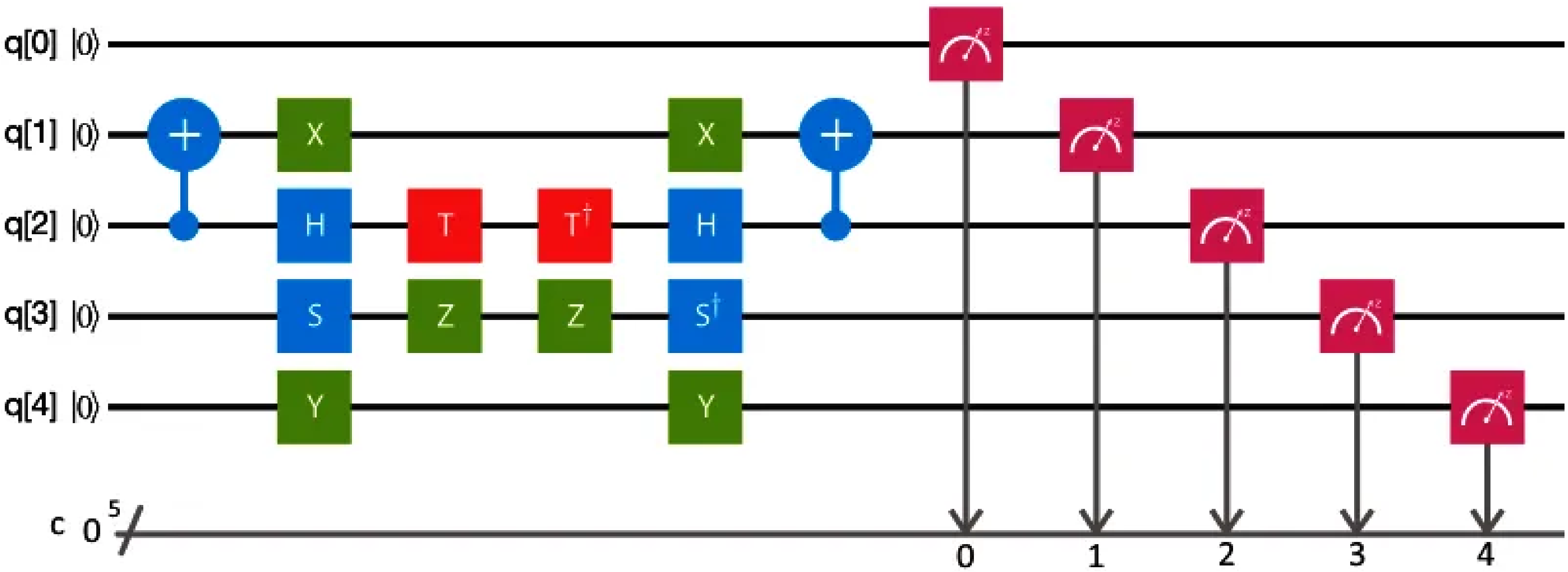
- Quantum circuits consist of multiple gates that perform unitary transformations on qubits

Commonly used Single-Qubit-Gates

Operator	Gate(s)	Matrix
Pauli-X (X)	X, $\oplus$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Pauli-Y (Y)	Y	$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
Pauli-Z (Z)	Z	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Hadamard (H)	H	$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
Phase (S, P)	S	$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
$\pi/8$ (T)	T	$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

Graphic taken from:
https://en.wikipedia.org/wiki/Quantum_logic_gate

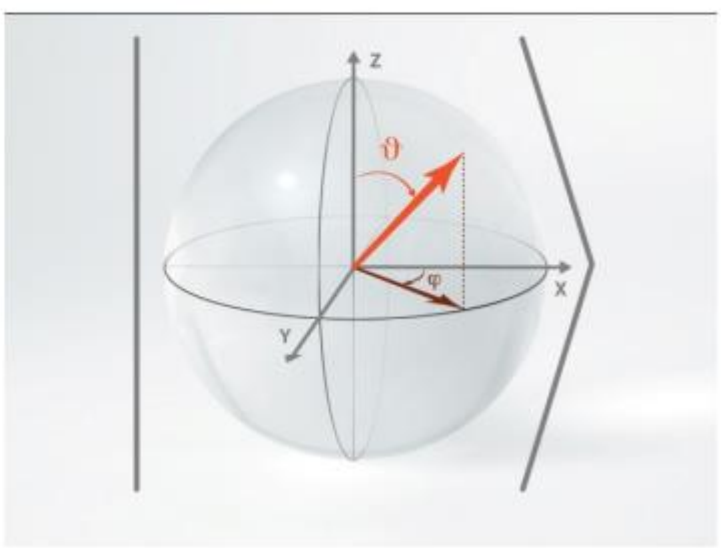
Quantum Circuits

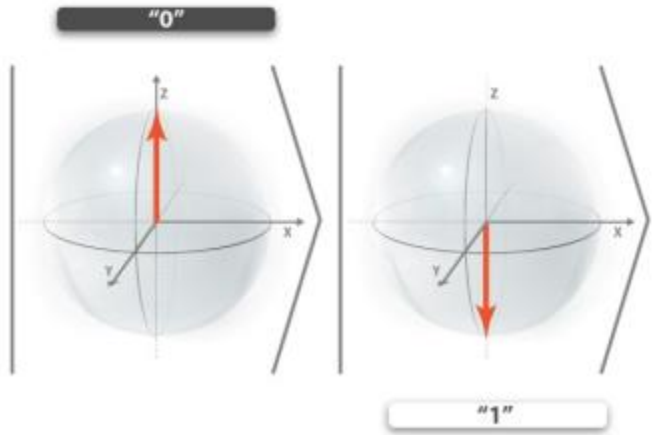


Graphic taken from:
https://towardsdatascience.com/what-is-a-quantum-circuit-transpiler-ba9a7853e6f9

Visualization on the Bloch-sphere

Visualization of different gate actions on Bloch-sphere





[Bloch sphere visualizer (kherb.io)](https://kherb.io)

W. Dür and S. Heusler, arXiv:1312.1463v1 **[physics.ed-ph]**

Classical and quantum bits

	Bits	Probabilistic Bits	Qubits
State of Single System	$bit \in \{ \|0\rangle, \|1\rangle \}$	$b = a\|0\rangle + b\|1\rangle$ $a + b = 1,$ $a, b \in \mathbb{R}_+$	$\|\psi\rangle = \alpha\|0\rangle + \beta\|1\rangle$ $\|\alpha\|^2 + \|\beta\|^2 = 1,$ $\alpha, \beta \in \mathbb{C}$
Operations	Boolean Logic	Stochastic Operations	Unitary Operations $U^\dagger U = \mathbb{I}\mathrm{d}$

Measuring Single-Qubit States

How would you measure the spinning coin?

You would just stop it.

What would happen?

$$coin = \frac{1}{2}|heads\rangle + \frac{1}{2}|tails\rangle$$

$$\Rightarrow p(heads) = \frac{1}{2} \quad p(tails) = \frac{1}{2}$$

Measuring Single-Qubit States

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \quad \rightarrow p_0 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} = p_1$$

$$\frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle \quad \rightarrow p_0 = \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3};\ p_1 = \left|\frac{\sqrt{2}}{\sqrt{3}}\right|^2 = \frac{2}{3}$$

$$0|0\rangle + 1|1\rangle = |1\rangle \quad \rightarrow p_0 = 0;\ p_1 = 1$$

This involves intrinsic randomness! **NOT** due to incomplete knowledge

We are not able to make any predictions on an individual event

However, the statistical behavior for repetitions of the experiment can be predicted

Observables

Mathematically, such a $\{|0\rangle, |1\rangle\}$ measurement is described by the observable σ_z

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Definition: An observable is a measurable physical quantity

(Math.) Definition: An observable is described by a hermitian operator A on Hilbert space (state space)

$$A^\dagger = A$$

Remarks:

- Hermitian operators have real eigenvalues
- Possible measurement outcomes upon measuring observable A of a system in the state $|\psi\rangle$ are the eigenvalues of A

Measuring Single-Qubit States

→ Quantum state only stable in isolation

→ If we want to read the state, we need to measure

→ Measurement destroys the initial superposition

→ We will NOT recover the initial superposition state!

→ We will recover one bit of information

→ Measurement result of an individual event is random!

→ Observable quantities are described by hermitian operators

→ Eigenvalues are possible measurement outcomes

Classical and quantum bits

	Bits (classical)	**Probabilistic Bits** (classical)	**Qubits** (quantum)
State of Single System	$bit \in \{ \|0\rangle, \|1\rangle \}$	$b = a\|0\rangle + b\|1\rangle$ $a + b = 1,$ $a, b \in \mathbb{R}_+$	$\|\psi\rangle = \alpha\|0\rangle + \beta\|1\rangle$ $\|\alpha\|^2 + \|\beta\|^2 = 1,$ $\alpha, \beta \in \mathbb{C}$
Operations	Boolean Logic	Stochastic Operations	Unitary Operations $U^\dagger U = \mathbb{I}\mathrm{d}$
Observable	Real-valued function on phase space	Real-valued function on phase space	Hermitian Operator on state space $\mathcal{H}$

Measuring Single-Qubit States

Expansion with respect to different basis (same state):

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \alpha'|+\rangle + \beta'|-\rangle$$

→ We can measure in the 0/1 basis and obtain the result

$|0\rangle$ with $p_0 = |\alpha|^2$, and result $|1\rangle$ with $p_1 = |\beta|^2$ respectively.

Corresponds to asking: Are you $|0\rangle$ or $|1\rangle$?

Measuring Single-Qubit States

Expansion with respect to different basis (same state):

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \alpha'|+\rangle + \beta'|-\rangle$$

→ If we measure in the +/- basis and obtain the result

$|+\rangle$ with $p_+ = |\alpha'|^2$, and result $|-\rangle$ with $p_- = |\beta'|^2$ respectively.

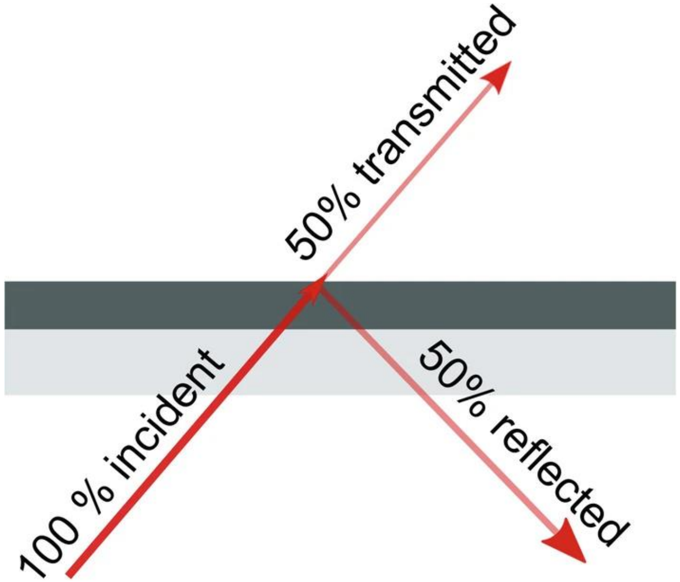
We can also ask the qubit: Are you $|+\rangle$ or $|-\rangle$?

→ This corresponds to a measurement of the observable σ_x

→ We can measure in an arbitrary basis (needs to be ONB!)

50-50 Beamsplitter

https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/SinglePhotonLab/SinglePhotonLab.html



A beam splitter reflects 50% of the incident light and transmits 50% of the incident light.

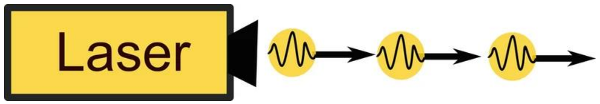
Hughes, C., Isaacson, J., Perry, A., Sun, R. F., & Turner, J. (2021). Quantum Computing for the Quantum Curious. Springer International Publishing. https://doi.org/10.1007/978-3-030-61601-4

Creating Superposition - Experiment

Weak Laser Light

https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/SinglePhotonLab/SinglePhotonLab.html

Low-intensity light is a stream of single photons.

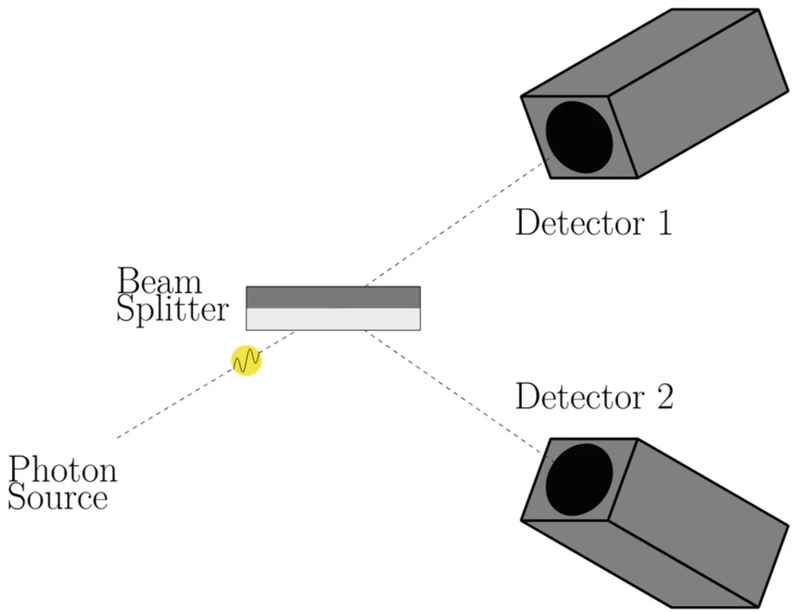


Hughes, C., Isaacson, J., Perry, A., Sun, R. F., & Turner, J. (2021). Quantum Computing for the Quantum Curious. Springer International Publishing. https://doi.org/10.1007/978-3-030-61601-4

Creating Superposition - Experiment

Single photon passing through Beamsplitter

https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/SinglePhotonLab/SinglePhotonLab.html

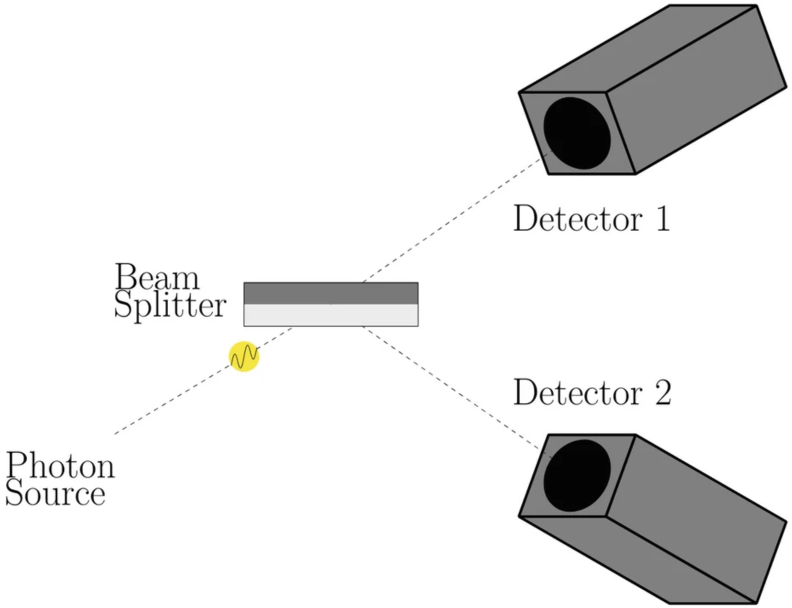


A single photon is sent at a beam splitter and the outcome is measured with detectors to see whether the beam splitter transmits or reflects.

Hughes, C., Isaacson, J., Perry, A., Sun, R. F., & Turner, J. (2021). Quantum Computing for the Quantum Curious. Springer International Publishing. https://doi.org/10.1007/978-3-030-61601-4

Single photon passing through Beamsplitter

https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/SinglePhotonLab/SinglePhotonLab.html



What would happen if a classical particle passes through the beamsplitter?

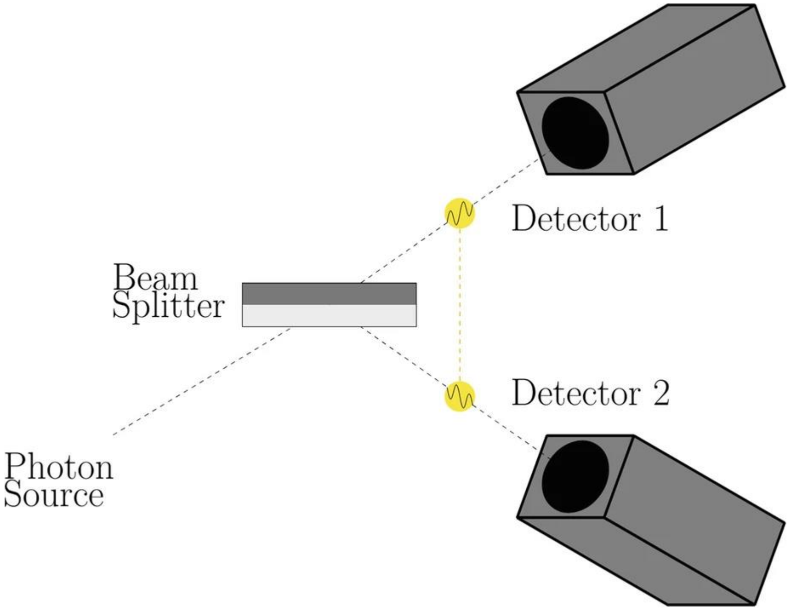
Let's fire some photons!

Which detectors are triggered?

Hughes, C., Isaacson, J., Perry, A., Sun, R. F., & Turner, J. (2021). Quantum Computing for the Quantum Curious. Springer International Publishing. https://doi.org/10.1007/978-3-030-61601-4

Single photon passing through Beamsplitter

https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/SinglePhotonLab/SinglePhotonLab.html



The beam splitter puts the photon into a superposition state.

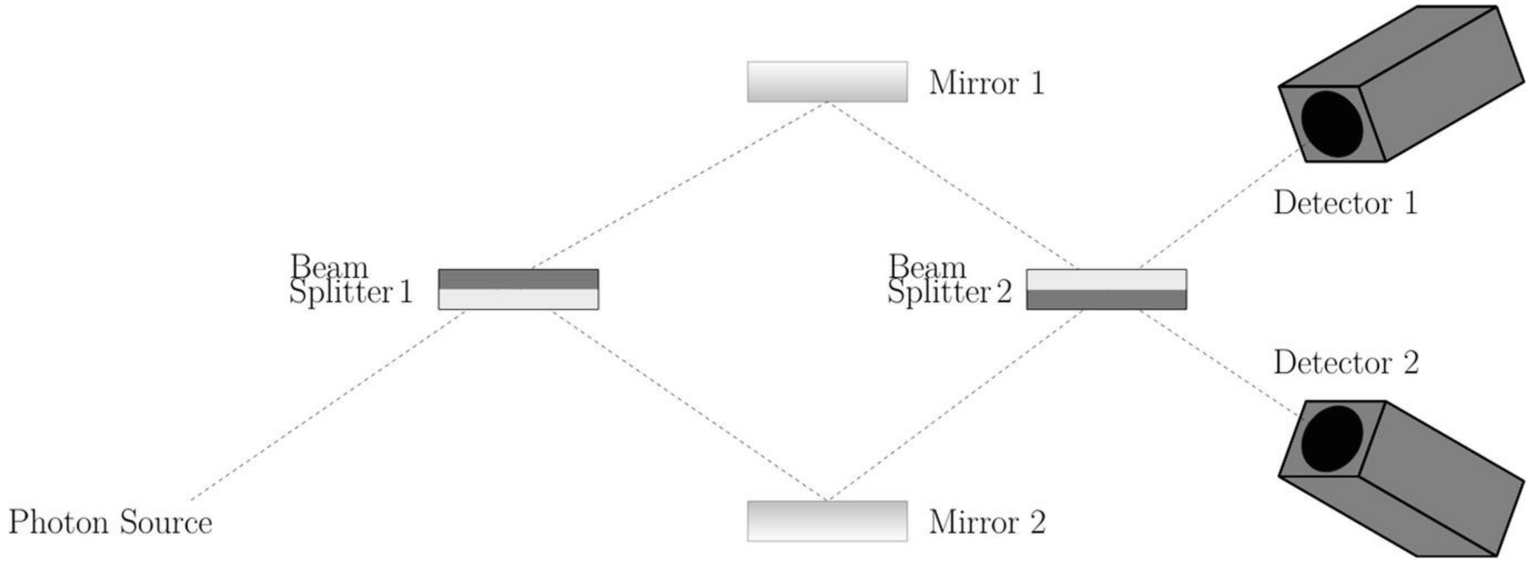
But how do we know that?

Hughes, C., Isaacson, J., Perry, A., Sun, R. F., & Turner, J. (2021). Quantum Computing for the Quantum Curious. Springer International Publishing. https://doi.org/10.1007/978-3-030-61601-4

Creating Superposition - Experiment

Single photon passing two Beamsplitters

https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/SinglePhotonLab/SinglePhotonLab.html

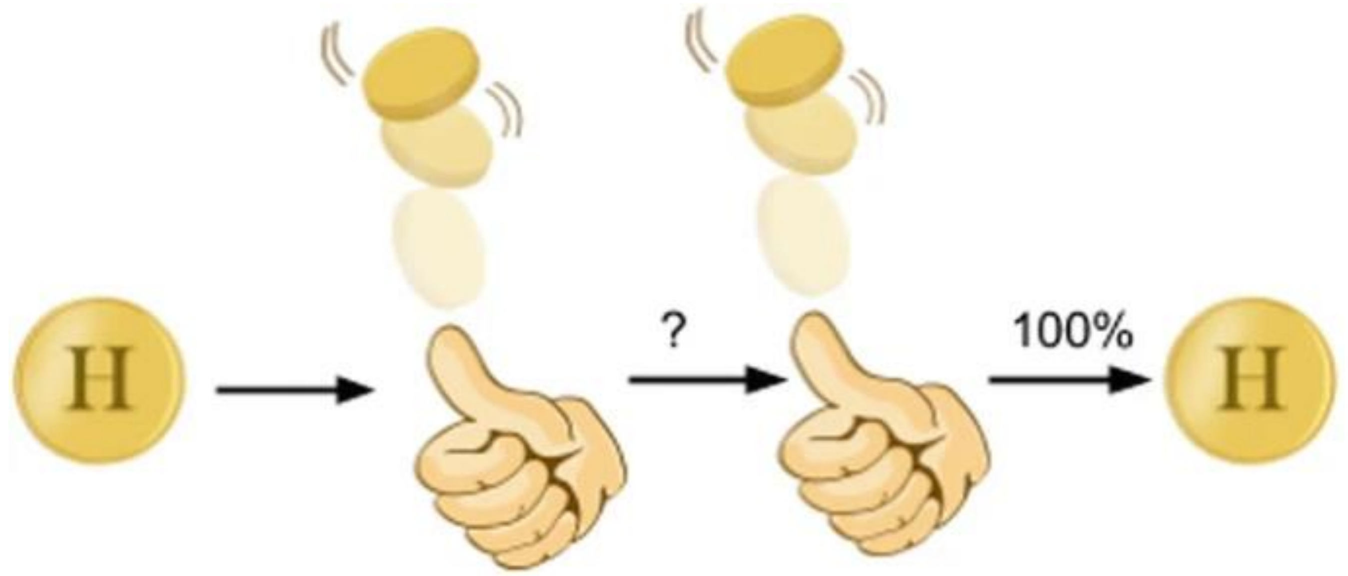


Hughes, C., Isaacson, J., Perry, A., Sun, R. F., & Turner, J. (2021). Quantum Computing for the Quantum Curious. Springer International Publishing. https://doi.org/10.1007/978-3-030-61601-4

Creating Superposition - Experiment

Single photon passing two Beamsplitters

https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/SinglePhotonLab/SinglePhotonLab.html



Hughes, C., Isaacson, J., Perry, A., Sun, R. F., & Turner, J. (2021). Quantum Computing for the Quantum Curious. Springer International Publishing. https://doi.org/10.1007/978-3-030-61601-4

Put multiple qubits together

Example: Two-qubits

$|\psi_1\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \qquad |\psi_2\rangle = \beta_0|0\rangle + \beta_1|1\rangle$

Joint state:

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle)$$
$$= \alpha_0\beta_0|0\rangle \otimes |0\rangle + \alpha_0\beta_1|0\rangle \otimes |1\rangle + \alpha_1\beta_0|1\rangle \otimes |0\rangle + \alpha_1\beta_1|1\rangle \otimes |1\rangle$$
$$= \gamma_{00}|0\rangle|0\rangle + \gamma_{01}|0\rangle|1\rangle + \gamma_{10}|1\rangle|0\rangle + \gamma_{11}|1\rangle|1\rangle$$
$$= \gamma_{00}|00\rangle + \gamma_{01}|01\rangle + \gamma_{10}|10\rangle + \gamma_{11}|11\rangle$$

$\Rightarrow$ $\gamma_{ij} = \alpha_i\beta_j$ result from amplitudes of both bits

Put multiple qubits together

Example: Two-qubits

$$|\psi_1\rangle = \alpha_1 \begin{bmatrix}1\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}0\\1\end{bmatrix} \qquad |\psi_2\rangle = \beta_1 \begin{bmatrix}1\\0\end{bmatrix} + \beta_2 \begin{bmatrix}0\\1\end{bmatrix}$$

Joint state:

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = (\alpha_1 \begin{bmatrix}1\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}0\\1\end{bmatrix}) \otimes (\beta_1 \begin{bmatrix}1\\0\end{bmatrix} + \beta_2 \begin{bmatrix}0\\1\end{bmatrix})$$

$$= \alpha_0\beta_0 \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} + \alpha_0\beta_1 \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} + \alpha_1\beta_0 \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} + \alpha_1\beta_1 \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix}$$

$$= \gamma_{00} \begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \gamma_{01} \begin{bmatrix}0\\1\\0\\0\end{bmatrix} + \gamma_{10} \begin{bmatrix}0\\0\\1\\0\end{bmatrix} + \gamma_{11} \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Put multiple qubits together

General case of N qubits

$$\dim \tilde{\mathcal{H}} = (\dim \mathbb{C}^2)^N = 2^N$$

$$\begin{aligned} |\psi\rangle &= |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_N\rangle \\ &= \gamma_1 |00 \ldots 0\rangle + \gamma_2 |00 \ldots 1\rangle + \cdots + \gamma_{2^N} |11 \ldots 1\rangle \\ &= \gamma_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \gamma_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \gamma_{2^N} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \in \tilde{\mathcal{H}} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \end{aligned}$$

Joint state of N qubit register can be in a superposition of all basis states!

Grows exponential in N!

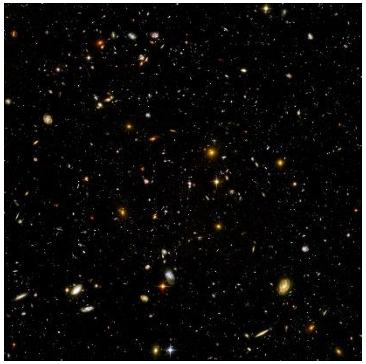
Quantum Register

Put multiple qubits together

General case of N qubits

$$\dim \widetilde{\mathcal{H}} = (\dim \mathbb{C}^2)^N = 2^N$$

$$\begin{aligned}
|\psi\rangle &= |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_N\rangle \\
&= \gamma_1|00\ldots0\rangle + \gamma_2|00\ldots1\rangle + \cdots + \gamma_{2^N}|11\ldots1\rangle \\
&= \gamma_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \gamma_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \gamma_{2^N} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \in \widetilde{\mathcal{H}} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2
\end{aligned}$$



Number of atoms in the universe: estimated $10^{78} - 10^{82}$

E.g. state of n=300 qubits, has $2^{300} = 10^{90}$ complex amplitudes

Consider Two Qubits

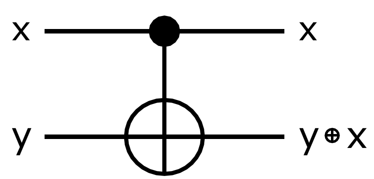
Operations on two qubits described by 4x4 unitary matrices

Example: CNOT

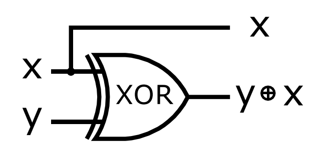
$$|00\rangle \rightarrow |00\rangle$$
$$|01\rangle \rightarrow |01\rangle$$
$$|10\rangle \rightarrow |11\rangle$$
$$|11\rangle \rightarrow |10\rangle$$

Changes amplitudes of $|01\rangle$ and $|11\rangle$

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



input		output	
x	y	x	y+x
$\vert 0\rangle$	$\vert 0\rangle$	$\vert 0\rangle$	$\vert 0\rangle$
$\vert 0\rangle$	$\vert 1\rangle$	$\vert 0\rangle$	$\vert 1\rangle$
$\vert 1\rangle$	$\vert 0\rangle$	$\vert 1\rangle$	$\vert 1\rangle$
$\vert 1\rangle$	$\vert 1\rangle$	$\vert 1\rangle$	$\vert 0\rangle$



input		output	
x	y	x	y+x
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

Consider Two Qubits

Operations on two qubits described by 4x4 unitary matrices

Another Example: Hadamard on both quits

$$\mathrm{H} \otimes \mathrm{H}|0\rangle \otimes |0\rangle = \mathrm{H}|0\rangle \otimes \mathrm{H}|0\rangle$$

$$= \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$= \frac{1}{2}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$$

$$= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

⇨ Operations „space by space“

Consider Two Qubits

Operations on two qubits described by 4x4 unitary matrices

Another Example: Hadamard on first of two quits

$$\mathrm{H} \otimes \mathrm{Id}\, |0\rangle \otimes |0\rangle = \mathrm{H}|0\rangle \otimes \mathrm{Id}|0\rangle$$

$$= \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle)$$

$$= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

Measurements on a Quantum Register

We measure the first qubit

$$|\psi\rangle = \gamma_{00}|0\rangle|0\rangle + \gamma_{01}|0\rangle|1\rangle + \gamma_{10}|1\rangle|0\rangle + \gamma_{11}|1\rangle|1\rangle$$

If the result is $|0\rangle$ then system state remains in *span* $\{|0\rangle|0\rangle, |0\rangle|1\rangle\}$

$$p(q_1 = 0) = |\gamma_{00}|^2 + |\gamma_{01}|^2$$

And the remaining state after measurement needs to be normalized

$$|\psi'\rangle = \frac{\gamma_{00}|0\rangle|0\rangle + \gamma_{01}|0\rangle|1\rangle}{\sqrt{|\gamma_{00}|^2 + |\gamma_{01}|^2}}$$

Measurements on a Quantum Register

We measure the first qubit

$$|\psi\rangle = \gamma_{00}|0\rangle|0\rangle + \gamma_{01}|0\rangle|1\rangle + \gamma_{10}|1\rangle|0\rangle + \gamma_{11}|1\rangle|1\rangle$$

If the result is $|1\rangle$ then system state remains in $span\ \{|1\rangle|0\rangle, |1\rangle|1\rangle\}$

$$p(q_1 = 1) = |\gamma_{10}|^2 + |\gamma_{11}|^2$$

And the remaining state after measurement needs to be normalized

$$|\psi'\rangle = \frac{\gamma_{10}|1\rangle|0\rangle + \gamma_{11}|1\rangle|1\rangle}{\sqrt{|\gamma_{10}|^2 + |\gamma_{11}|^2}}$$

We only retrieve the value of the measured qubit! NOT the amplitudes of the remaining nor initial state

Qubits, Operations, Measurements

Quantum states are represented by vectors in a complex vector space

→ state of an N-qubit register described by 2^N-dimensional vector

Operations on a quantum register are linear and can be described by unitary matrices

→ operations on N-qubits described by N × N unitary matrix

Measurements on a quantum register project the state onto a subspace of the larger vector space

Entanglement of two qubits

- Schrödinger 1937: „the characteristic feature of quantum mechanics“
- Consequence of the superposition principle
- Underlying feature of quantum teleportation and other protocols

Generate Entanglement

We apply an H-gate to the first qubit

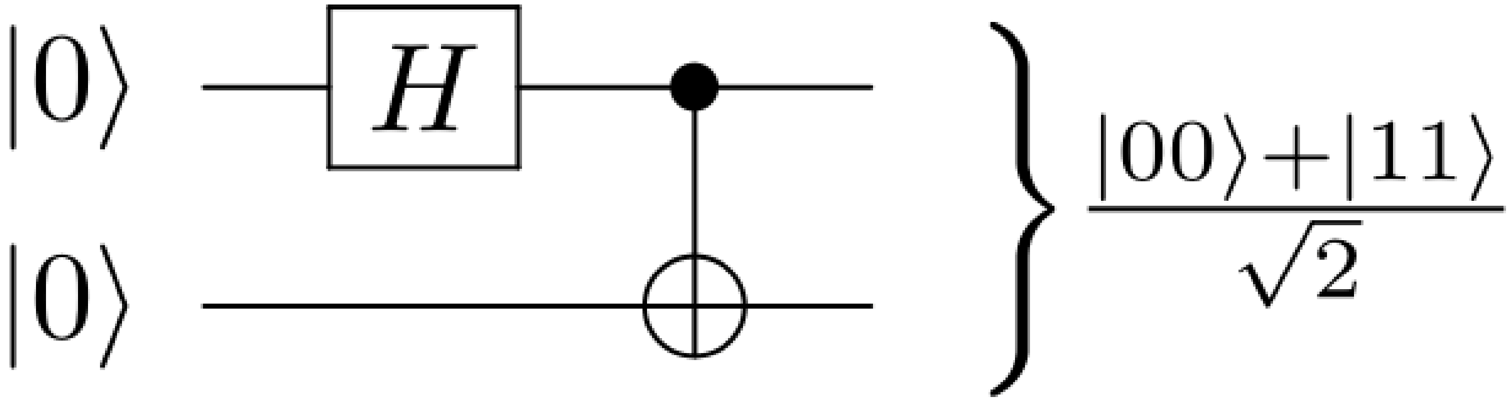
$$\mathrm{H} \otimes \mathrm{Id}\, |0\rangle|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$$

Now, we apply a CNOT-gate to the two-qubits

$$\begin{aligned} &\mathrm{CNOT}\, \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle \\ &= \mathrm{CNOT}\, \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \end{aligned}$$

Entanglement of two qubits

Circuit picture



Graphic taken from:
https://en.wikipedia.org/wiki/Bell_state

Entanglement of two qubits

Let‘s measure!

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$$

$$\rightarrow p(q_1 = 0) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} = p(q_1 = 1)$$

$$\rightarrow p(q_2 = 0) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} = p(q_2 = 1)$$

$\Rightarrow$ If $q_1 = |0\rangle$ we know that $q_2 = |0\rangle$ (analogue for $q_1 = |1\rangle$)

Outcomes are correlated, although invidual outcomes are undetermined!

Entanglement of two qubits

Bell states

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)$$

Remark: Bell states form an orthonormal basis

Entanglement and separability

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = |+\rangle \otimes |0\rangle$$

$$\mathrm{H} \otimes \mathrm{H}\, |1\rangle \otimes |1\rangle = \frac{1}{2}(|0\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$$
$$= |-\rangle \otimes |-\rangle$$

$\rightarrow$ states are separable

Definition: Separability. Any state that can be expressed as a tensor product of basis states is called separable

Definition: Entanglement. A state $|\Psi\rangle$ is called entangled if it is not separable, i.e.

$$|\Psi\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle$$

*in any basis!

Entanglement of two qubits

Degree of entanglement

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$$

$\rightarrow$ Maximally entangled!

$$\frac{1}{2}(|0\rangle \otimes |0\rangle + \frac{\sqrt{3}}{2}|1\rangle \otimes |1\rangle)$$

$\rightarrow$ Entangled but not as strongly correlated

$$|1\rangle \otimes |1\rangle$$

$\rightarrow$ Also perfectly correlated! But not entangled, outcomes are predetermined!

For maximally entangled states (like Bell states) the outcomes are correlated in infinitely many bases!

Correlations

Classical correlations depend on choosing one particular basis

→ basis-dependent, there is no correlation in other bases

Quantum correlations (entanglement) are basis-independent

$$
\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
&= \frac{1}{\sqrt{2}}\left[(\frac{1}{\sqrt{2}}|+\rangle + |-\rangle)(\frac{1}{\sqrt{2}}|+\rangle + |-\rangle) + (\frac{1}{\sqrt{2}}|+\rangle - |-\rangle)(\frac{1}{\sqrt{2}}|+\rangle - |-\rangle)\right] \\
&= \frac{1}{2\sqrt{2}}[|+\,+\rangle + \cancel{|+\,-\rangle} + \cancel{|-\,+\rangle} + |-\,-\rangle + |+\,+\rangle - \cancel{|+\,-\rangle} - \cancel{|-\,+\rangle} + |-\,-\rangle] \\
&= \frac{1}{\sqrt{2}}[|+\,+\rangle + |-\,-\rangle]
\end{aligned}
$$

Exact same type of correlations in new basis!

Outlook

This is not the end of the story!

- Philosophical aspects of quantum theory and entanglement
- Entanglement detection and quantification
- Noise and decoherence → requires treatment of quantum system with density matrices (error correction)
- We did not talk about: Heisenberg uncertainty
- Entanglement → Multipartite entanglement beyond Bell-states → How are 3 or more systems entangled?
- And more…