Under which conditions one may be lucky enough to not get infected?

Let us denote a small perturbation, ε, in the SIR compartments as:

$$
\begin{aligned}
s &= \bar{s} + \varepsilon s \\
i &= \bar{i} + \varepsilon i \\
r &= \bar{r} + \varepsilon r
\end{aligned}
$$

Consider a system which was initially **at rest** (no infections: $\bar{s} \neq 0, \bar{i} = 0, \bar{r} = 1 - \bar{s}$).

One day, a couple of infectious patients, εi, have returned from holidays.

What are the conditions to prevent the epidemy from growth?

In other words, what proportion of the population should be immunized to eradicate a disease?

To answer this question, we have to insert the small perturbation into the (nondimensional) system of SIR equations :

$$
\begin{aligned}
\frac{\partial}{\partial \tau} s &= -R_0 s i \\
\frac{\partial}{\partial \tau} i &= R_0 s i - i \\
\frac{\partial}{\partial \tau} r &= i
\end{aligned}
$$

Expanding,

$$
\begin{aligned}
\frac{\partial}{\partial \tau} s &= -R_0(\bar{s} + \varepsilon s)\varepsilon i = -R_0 \bar{s} \varepsilon i - R_0 \bar{s} \varepsilon^2 i \\
\frac{\partial}{\partial \tau} i &= R_0(\bar{s} + \varepsilon s)\varepsilon i - \varepsilon i = R_0 \bar{s} \varepsilon i + R_0 \bar{s} \varepsilon^2 i - \varepsilon i \\
\frac{\partial}{\partial \tau} r &= \varepsilon i
\end{aligned}
$$

Eliminating the higher order terms, ε^2, the condition of decay of the epidemy is:

$$
\begin{aligned}
\frac{\partial}{\partial \tau} i < 0 &\Leftrightarrow (R_0 \bar{s} - 1)\varepsilon i < 0 \\
&\Leftrightarrow R_0 \bar{s} < 1
\end{aligned}
$$

Since $\bar{s} = 1 - \bar{r}$, the condition can be expressed as:

$$
\begin{aligned}
\frac{\partial}{\partial \tau} i < 0 &\Leftrightarrow R_0(1 - \bar{r}) < 1 \\
&\Leftrightarrow \bar{r} > 1 - \frac{1}{R_0}
\end{aligned}
$$

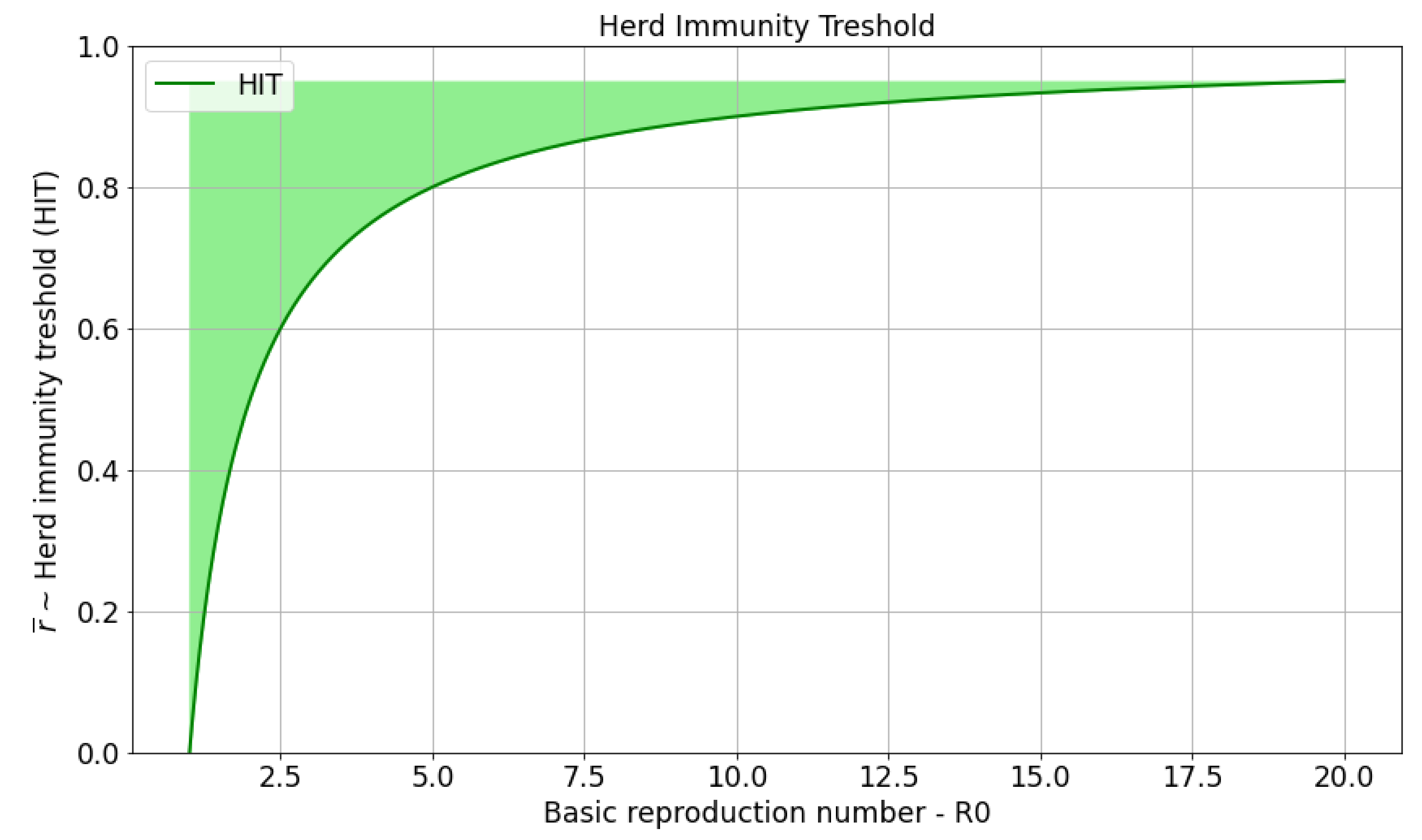
In [1]:

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.pylab as pylab

R0 = np.linspace(1, 20, 1000)

params = {'legend.fontsize': 'xx-large',
           'figure.figsize': (14, 8),
          'axes.labelsize': 'xx-large',
          'axes.titlesize':'xx-large',
          'xtick.labelsize':'xx-large',
          'ytick.labelsize':'xx-large'}
pylab.rcParams.update(params)
axes = plt.gca()
y =  1 - 1/R0
plt.plot(R0, y,
         color="green", marker="", markevery=1, markersize=15, linestyle="-", linewidth=2,
         label='HIT')

plt.fill_between(R0, y,  np.max(y), color='lightgreen')
plt.xlabel('Basic reproduction number - R0')
plt.ylabel(r'$\overline{r} $ ~ Herd immunity treshold (HIT)')
axes.set_ylim([0,1])
plt.title('Herd Immunity Treshold')
plt.legend()
plt.grid()
fig = plt.gcf()
fig.savefig("Herd_Immunity_Treshold", bbox_inches='tight')
plt.show()
```



Herd immunity vs R0

And the plot from wikipedia

Herd_immunity_threshold_vs_r0

Stability of linear systems

The perturbed, linearized, 0D SIR system can be expressed in matrix form as:

$ \underbrace{ \begin{bmatrix} \dot s \ \dot i \ \dot r

\end{bmatrix}}_{\boldsymbol{\dot x}}

\underbrace{

$$
\begin{bmatrix}
0 & -R_0 \bar{s} & 0 \\
0 & R_0 \bar{s} - 1 & 0 \\
0 & 1 & 0
\end{bmatrix}
$$

}{\mathbb{A}} \underbrace{

$$
\begin{bmatrix}
\varepsilon s \\
\varepsilon i \\
\varepsilon r
\end{bmatrix}
$$

}{\boldsymbol{x}} $

where $\dot{s}$ denotes $\frac{\partial}{\partial \tau} s$.

The stability condition for the continous system is $Re(eigenvalues(\mathbb{A})) < \mathbf{0}$ (all real parts of the eigenvalues of $\mathbb{A}$ are negative).

Intuition: for ODE, $\dot{x} = \lambda x \rightarrow x = Ce^{\lambda t}$

Again, calculating $eigenvalues(\mathbb{A})$ we obtain condition $R_0 \bar{s} - 1 < 0 \Leftrightarrow \bar{r} > 1 - \frac{1}{R_0}$.

The matrix notation would be even more useful if one would like to consider additional couplings to the SIR system (like loss of the immunity of the recovered).

In [2]:

```
import sympy as sy
R0,s = sy.symbols('R0 s')

A = sy.Matrix([[0, -R0*s, 0],
               [0, R0*s-1, 0],
               [0, 1,0]])

A.eigenvals() #returns eigenvalues and their algebraic multiplicity
```

Out[2]:

```
{R0*s - 1: 1, 0: 2}
```

Inspirations

Herd Immunity Treshold

- https://en.wikipedia.org/wiki/Basic_reproduction_number

Stability of linear systems

- https://en.wikipedia.org/wiki/Stability_theory
- https://www.youtube.com/watch?v=nyqJJdhReiA&list=PLMrJAkhIeNNR20Mz-VpzgfQs5zrYi085m&index=2&ab_channel=SteveBrunton
- https://www.youtube.com/watch?v=h7nJ6ZL4Lf0&list=PLMrJAkhIeNNR20Mz-VpzgfQs5zrYi085m&index=3&ab_channel=SteveBrunton