$$|JKM\rangle = (-1)^{M-K}\left[\frac{2J+1}{8\pi^2}\right]^{\frac{1}{2}} D^J_{-M-K}$$

$$\langle JKM| = \left[\frac{2J+1}{8\pi^2}\right]^{\frac{1}{2}} D^J_{MK}$$

$$\int D^{J_3}_{M_3'M_3} D^{J_2}_{M_2'M_2} D^{J_1}_{M_1'M_1} d\Omega =$$
$$8\pi^2 \begin{pmatrix} J_1 & J_2 & J_3 \\ M_1 & M_2 & M_3 \end{pmatrix} \begin{pmatrix} J_1 & J_2 & J_3 \\ K_1 & K_2 & K_3 \end{pmatrix}$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} c\phi\ c\theta\ c\chi - s\phi\ s\chi & -c\phi\ c\theta\ s\chi - s\phi\ c\chi & c\phi\ s\theta \\ s\phi\ c\theta\ c\chi + c\phi\ s\chi & -s\phi\ c\theta\ s\chi + c\phi\ c\chi & s\phi\ s\theta \\ -s\theta\ c\chi & s\theta\ s\chi & c\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Choose the external static field to be along the lab fixed Z axis. The dipole moment component along the Z axis is:

$$\mu_Z = -\sin(\theta)\cos(\chi)\mu_{Zx} + \sin(\theta)\sin(\chi)\mu_{Zy} + \cos(\theta)\mu_{Zz}$$

The static field Hamiltonian is:

$$H_s = -\mu \cdot \mathbf{E} = -E \cdot [-\sin(\theta)\cos(\chi)\mu_{Zx} + \sin(\theta)\sin(\chi)\mu_{Zy} + \cos(\theta)\mu_{Zz}]$$

All the trigonometric functions can be expressed as D matrixes using table 3.1 page 89 in Zare

$$D^J_{M'M} = e^{-i\phi M'} d^J_{M'M}(\theta) e^{-i\chi M}$$
$$cos(\theta) = D^1_{00}$$
$$\sin(\theta)\cos(\chi) = \frac{1}{\sqrt{2}}[D^1_{01} - D^1_{0-1}]$$
$$\sin(\theta)\sin(\chi) = \frac{1}{\sqrt{2}}[D^1_{0-1} + D^1_{01}]$$

Choose the laser electric field to be polarized along the Z axis.

$$H_{laser} = -\frac{1}{4}\epsilon(t)^2\left[\sin^2\theta(\alpha_{xx}\cos^2(\chi) + \alpha_{yy}\sin^2(\chi)) + \cos^2(\theta)\alpha_{zz}\right]$$

$$\sin^2(\theta)\cos^2(\chi) = \sqrt{\frac{1}{6}}(D^2_{02} + D^2_{0-2}) + \frac{1}{3}(1 - D^2_{00})$$

$$\sin^2(\theta)\sin^2(\chi) = -\sqrt{\frac{1}{6}}(D^2_{0-2} + D^2_{02}) + \frac{1}{3}(1 - D^2_{00})$$

$$\cos^2(\theta) = \frac{1}{3} + \frac{2}{3}D^2_{00}$$

Inserting in the Hamiltonian:

$$H_{laser} - \frac{1}{4}\epsilon^2\left[\sqrt{\frac{1}{6}}[D^2_{02} + D^2_{0-2}][\alpha_{xx} - \alpha_{yy}]\right.$$
$$\left.+\frac{1}{3}(\alpha_{xx} + \alpha_{yy} + \alpha_{zz}) + \frac{1}{3}D^2_{00}(2\alpha_{zz} - \alpha_{yy} - \alpha_{zz})\right]$$

We need to calculate

$$\int D^{J'}_{M'K'} D^2_{02} D^J_{MK} d\Omega = 8\pi^2 \begin{pmatrix} J & 2 & J' \\ -M & 0 & M' \end{pmatrix} \begin{pmatrix} J & 2 & J' \\ -K & 2 & K' \end{pmatrix}$$

$$\int D^{J'}_{M'K'} D^2_{0-2} D^J_{MK} d\Omega = 8\pi^2 \begin{pmatrix} J & 2 & J' \\ -M & 0 & M' \end{pmatrix} \begin{pmatrix} J & 2 & J' \\ -K & -2 & K' \end{pmatrix}$$

$$\int D^{J'}_{M'K'} D^2_{00} D^J_{MK} d\Omega = 8\pi^2 \begin{pmatrix} J & 2 & J' \\ -M & 0 & M' \end{pmatrix} \begin{pmatrix} J & 2 & J' \\ -K & 0 & K' \end{pmatrix}$$

First all M matrix elements: $J' = J$

$$\begin{pmatrix} J & 2 & J \\ -M & 0 & M \end{pmatrix} = (-1)^{J+M} 2A_2(2J+3)[3M^2 - J(J+1)]$$

$J' = J + 1$

$$\begin{pmatrix} J & 2 & J+1 \\ -M & 0 & M \end{pmatrix} = (-1)^{J+M+1} A_2(2J+4)2M[6(J-M+1)(J+M+1)]^{1/2}$$

$J' = J - 1$

$$\begin{pmatrix} J & 2 & J-1 \\ -M & 0 & M \end{pmatrix} = (-1)^{J+M} A_2(2J+2)2M[6(J+M)(J-M)]^{1/2}$$

$J' = J + 2$

$$\begin{pmatrix} J & 2 & J+2 \\ -M & 0 & M \end{pmatrix} = (-1)^{J+M} A_2(2J+5)[6(J-M+2)(J-M+1)(J+M+2)(J+M+1)]^{1/2}$$

$J' = J - 2$

$$\begin{pmatrix} J & 2 & J-2 \\ -M & 0 & M \end{pmatrix} = (-1)^{J+M} A_2(2J+1)[6(J+M)(J+M-1)(J-M)(J-M-1)]^{1/2}$$

Then the K elements +2 $J' = J$ $K' = K-2$

$$\begin{pmatrix} J & 2 & J \\ -K & 2 & K-2 \end{pmatrix} = (-1)^{J+K} A_2 (2J{+}3)[6(J{-}K{+}1)(J{-}K{+}2)(J{+}K{-}1)(J{+}K)]^{1/2}$$

$J' = J+1$

$$\begin{pmatrix} J & 2 & J+1 \\ -K & 2 & K-2 \end{pmatrix} = (-1)^{J+K+1} A_2 (2J{+}4)[4(J{-}K{+}1)(J{-}K{+}2)(J{-}K{+}3)(J{+}K)]^{1/2}$$

$J' = J-1$

$$\begin{pmatrix} J & 2 & J-1 \\ -K & 2 & K-2 \end{pmatrix} = (-1)^{J+K+1} A_2 (2J{+}2)[4(J{+}K{-}2)(J{+}K{-}1)(J{+}K)(J{-}K{+}1)]^{1/2}$$

$J' = J+2$

$$\begin{pmatrix} J & 2 & J+2 \\ -K & 2 & K-2 \end{pmatrix} = (-1)^{J+K} A_2 (2J{+}5)[(J{-}K{+}1)(J{-}K{+}2)(J{-}K{+}3)(J{-}K{+}4)]^{1/2}$$

$J' = J-2$

$$\begin{pmatrix} J & 2 & J-2 \\ -K & 2 & K-2 \end{pmatrix} = (-1)^{J+K} A_2 (2J{+}1)[(J{+}K{-}3)(J{+}K{-}2)(J{+}K{-}1)(J{+}K)]^{1/2}$$

Then the K elements -2 $J' = J$ $K' = K-2$

$$\begin{pmatrix} J & 2 & J \\ -K & -2 & K+2 \end{pmatrix} = (-1)^{J+K} A_2 (2J{+}3)[6(J{+}K{+}1)(J{+}K{+}2)(J{-}K{-}1)(J{-}K)]^{1/2}$$

$J' = J+1$

$$\begin{pmatrix} J & 2 & J+1 \\ -K & -2 & K+2 \end{pmatrix} = (-1)^{J+K} A_2 (2J{+}4)[4(J{+}K{+}1)(J{+}K{+}2)(J{+}K{+}3)(J{-}K)]^{1/2}$$

$J' = J-1$

$$\begin{pmatrix} J & 2 & J-1 \\ -K & -2 & K+2 \end{pmatrix} = (-1)^{J+K} A_2 (2J{+}2)[4(J{-}K{-}2)(J{-}K{-}1)(J{-}K)(J{+}K{+}1)]^{1/2}$$

$J' = J+2$

$$\begin{pmatrix} J & 2 & J+2 \\ -K & -2 & K+2 \end{pmatrix} = (-1)^{J+K} A_2 (2J{+}5)[(J{+}K{+}1)(J{+}K{+}2)(J{+}K{+}3)(J{+}K{+}4)]^{1/2}$$

$J' = J-2$

$$\begin{pmatrix} J & 2 & J-2 \\ -K & -2 & K+2 \end{pmatrix} = (-1)^{J+K} A_2 (2J{+}1)[(J{-}K{-}3)(J{-}K{-}2)(J{-}K{-}1)(J{-}K)]^{1/2}$$

1 D's

1.1 J'=J

$K' = K$

$$\begin{aligned}\begin{pmatrix} J & 2 & J \\ -M & 0 & M \end{pmatrix}\begin{pmatrix} J & 2 & J \\ -K & 0 & K \end{pmatrix} = \\ 4(-1)^{J+M} A_2 (2J+3)[3M^2 - J(J+1)]* \\ (-1)^{J+K} A_2 (2J+3)[3K^2 - J(J+1)]\end{aligned}$$

$K' = K - 2$

$$\begin{aligned}\begin{pmatrix} J & 2 & J \\ -M & 0 & M \end{pmatrix}\begin{pmatrix} J & 2 & J \\ -K & 2 & K-2 \end{pmatrix} = \\ (-1)^{J+M} A_2 (2J+3)[3M^2 - J(J+1)]* \\ (-1)^{J+K} A_2 (2J+3)[6(J-K+1)(J-K+2)(J+K-1)(J+K)]^{1/2}\end{aligned}$$

$K' = K + 2$

$$\begin{aligned}\begin{pmatrix} J & 2 & J \\ -M & 0 & M \end{pmatrix}\begin{pmatrix} J & 2 & J \\ -K & -2 & K+2 \end{pmatrix} = \\ (-1)^{J+M} A_2 (2J+3)[3M^2 - J(J+1)]* \\ (-1)^{J+K} A_2 (2J+3)[6(J+K+1)(J+K+2)(J-K-1)(J-K)]^{1/2}\end{aligned}$$

1.2 J'=J+1

$K' = K$

$$\begin{pmatrix} J & 2 & J+1 \\ -M & 0 & M \end{pmatrix} \begin{pmatrix} J & 2 & J+1 \\ -K & 0 & K \end{pmatrix} =$$
$$(-1)^{J+M+1} A_2 (2J+4) 2M [6(J-M+1)(J+M+1)]^{1/2} *$$
$$(-1)^{J+K+1} A_2 (2J+4) 2M [6(J-K+1)(J+K+1)]^{1/2}$$

$K' = K - 2$

$$\begin{pmatrix} J & 2 & J+1 \\ -M & 0 & M \end{pmatrix} \begin{pmatrix} J & 2 & J+1 \\ -K & 2 & K-2 \end{pmatrix} =$$
$$(-1)^{J+M+1} A_2 (2J+4) 2M [6(J-M+1)(J+M+1)]^{1/2} *$$
$$(-1)^{J+K+1} A_2 (2J+4) [4(J-K+1)(J-K+2)(J-K+3)(J+K)]^{1/2}$$

$K' = K + 2$

$$\begin{pmatrix} J & 2 & J+1 \\ -M & 0 & M \end{pmatrix} \begin{pmatrix} J & 2 & J+1 \\ -K & -2 & K+2 \end{pmatrix} =$$
$$(-1)^{J+M+1} A_2 (2J+4) 2M [6(J-M+1)(J+M+1)]^{1/2} *$$
$$(-1)^{J+K} A_2 (2J+4) [4(J+K+1)(J+K+2)(J+K+3)(J-K)]^{1/2}$$

1.3 J'=J-1

$K' = K$

$$\begin{pmatrix} J & 2 & J-1 \\ -M & 0 & M \end{pmatrix} \begin{pmatrix} J & 2 & J-1 \\ -K & 0 & K \end{pmatrix} =$$
$$(-1)^{J+M} A_2 (2J+2) 2M [6(J+M)(J-M)]^{1/2} *$$
$$(-1)^{J+K} A_2 (2J+2) 2K [6(J+K)(J-K)]^{1/2}$$

$K' = K - 2$

$$\begin{pmatrix} J & 2 & J-1 \\ -M & 0 & M \end{pmatrix} \begin{pmatrix} J & 2 & J-1 \\ -K & 2 & K-2 \end{pmatrix} =$$
$$(-1)^{J+M} A_2 (2J+2) 2M [6(J+M)(J-M)]^{1/2} *$$
$$(-1)^{J+K+1} A_2 (2J+2) [4(J+K-2)(J+K-1)(J+K)(J-K+1)]^{1/2}$$

$K' = K + 2$

$$\begin{pmatrix} J & 2 & J-1 \\ -M & 0 & M \end{pmatrix} \begin{pmatrix} J & 2 & J-1 \\ -K & -2 & K+2 \end{pmatrix} =$$
$$(-1)^{J+M} A_2 (2J+2) 2M [6(J+M)(J-M)]^{1/2} *$$
$$(-1)^{J+K} A_2 (2J+2) [4(J-K-2)(J-K-1)(J-K)(J+K+1)]^{1/2}$$

1.4 J'=J+2

$K' = K$

$$\begin{pmatrix} J & 2 & J+2 \\ -M & 0 & M \end{pmatrix} \begin{pmatrix} J & 2 & J+2 \\ -K & 0 & K \end{pmatrix} =$$

$$(-1)^{J+M} A_2 (2J+5)[6(J-M+2)(J-M+1)(J+M+2)(J+M+1)]^{1/2} *$$
$$(-1)^{J+K} A_2 (2J+5)[6(J-K+2)(J-K+1)(J+K+2)(J+K+1)]^{1/2}$$

$K' = K - 2$

$$\begin{pmatrix} J & 2 & J+2 \\ -M & 0 & M \end{pmatrix} \begin{pmatrix} J & 2 & J+2 \\ -K & 2 & K-2 \end{pmatrix} =$$

$$(-1)^{J+M} A_2 (2J+5)[6(J-M+2)(J-M+1)(J+M+2)(J+M+1)]^{1/2} *$$
$$(-1)^{J+K} A_2 (2J+5)[(J-K+1)(J-K+2)(J-K+3)(J-K+4)]^{1/2}$$

$K' = K + 2$

$$\begin{pmatrix} J & 2 & J+2 \\ -M & 0 & M \end{pmatrix} \begin{pmatrix} J & 2 & J+2 \\ -K & -2 & K+2 \end{pmatrix} =$$

$$(-1)^{J+M} A_2 (2J+5)[6(J-M+2)(J-M+1)(J+M+2)(J+M+1)]^{1/2} *$$
$$(-1)^{J+K} A_2 (2J+5)[(J+K+1)(J+K+2)(J+K+3)(J+K+4)]^{1/2}$$

$K' = K$

1.5 J'=J-2

$$\begin{pmatrix} J & 2 & J-2 \\ -M & 0 & M \end{pmatrix} \begin{pmatrix} J & 2 & J-2 \\ -K & 0 & K \end{pmatrix} =$$

$$(-1)^{J+M} A_2 (2J+1)[6(J+M)(J+M-1)(J-M)(J-M-1)]^{1/2} *$$
$$(-1)^{J+K} A_2 (2J+1)[6(J+K)(J+K-1)(J-K)(J-K-1)]^{1/2}$$

$K' = K - 2$

$$\begin{pmatrix} J & 2 & J-2 \\ -M & 0 & M \end{pmatrix} \begin{pmatrix} J & 2 & J-2 \\ -K & 2 & K-2 \end{pmatrix} =$$

$$(-1)^{J+M} A_2 (2J+1)[6(J+M)(J+M-1)(J-M)(J-M-1)]^{1/2} *$$
$$(-1)^{J+K} A_2 (2J+1)[(J+K-3)(J+K-2)(J+K-1)(J+K)]^{1/2}$$

$K' = K + 2$

$$\begin{pmatrix} J & 2 & J-2 \\ -M & 0 & M \end{pmatrix} \begin{pmatrix} J & 2 & J-2 \\ -K & -2 & K+2 \end{pmatrix} =$$

$$(-1)^{J+M} A_2 (2J+1)[6(J+M)(J+M-1)(J-M)(J-M-1)]^{1/2} *$$
$$(-1)^{J+K} A_2 (2J+1)[(J-K-3)(J-K-2)(J-K-1)(J-K)]^{1/2}$$

2 matrix elements

2.1 $J' = J$

$$\langle JKM| H_{laser} |JKM\rangle = \left[-\frac{1}{4}\epsilon^2\right] \cdot \left[\frac{1}{3}[\alpha_{xx} + \alpha_{yy} + \alpha_{zz}] + \frac{4}{3}[2J+1] \cdot [2\alpha_{zz} - \alpha_{xx} - \alpha_{yy}] \cdot A_2(2J+3)^2[3M^2 - J(J+1)] \cdot [3K^2 - J(J+1)]\right]$$

$$\langle JK \pm 2M| H_{laser} |JKM\rangle = -\frac{1}{4}\epsilon^2 \cdot [2J+1][\alpha_{xx} - \alpha_{yy}] \cdot A_2(2J+3)^2[3M^2 - J(J+1)] \cdot [6(J \pm K+1)(J \pm K+2)(J \mp K-1)(J \mp K)]^{1/2}$$

2.2 $J' = J + 1$

$$\langle J+1KM| H_{laser} |JKM\rangle = \left[-\frac{1}{4}\epsilon^2\right] \cdot \left[\sqrt{(2J+1)(2J+3)}\right] \left[\frac{1}{3}[2\alpha_{zz} - \alpha_{xx} - \alpha_{yy}] \cdot A_2(2J+4)^2 4MK[6(J-M+1)(J+M+1)]^{1/2} \cdot [6(J-K+1)(J+K+1)]^{1/2}\right]$$

$$\langle J+1K \pm 2M| H_{laser} |JKM\rangle = \left[\pm\frac{1}{4}\epsilon^2\right] \cdot \left[\sqrt{(2J+1)(2J+3)}\right] \cdot [\alpha_{xx} - \alpha_{yy}] \cdot A_2(2J+4)^2 2M[6(J-M+1)(J+M+1)]^{1/2} \cdot [4(J \pm K+1)(J \pm K+2)(J \pm K+3)(J \mp K)]^{1/2}$$

2.3 $J' = J + 2$

$$\langle J+2KM| H_{laser} |JKM\rangle = \left[-\frac{1}{4}\epsilon^2\right] \cdot \left[\sqrt{(2J+1)(2J+5)}\right] \left[\frac{1}{3}[2\alpha_{zz} - \alpha_{xx} - \alpha_{yy}] \cdot A_2(2J+5)^2 \cdot [6(J-M+2)(J-M+1)(J+M+2)(J+M+1)]^{1/2} \cdot [6(J-K+2)(J-K+1)(J+K+2)(J+K+1)]^{1/2}\right]$$

$$
\begin{aligned}
\langle J+2K\pm 2M|\, H_{laser}\, |JKM\rangle = \left[-\frac{1}{4}\epsilon^2\right]\cdot\left[\sqrt{(2J+1)(2J+5)}\right][\alpha_{xx}-\alpha_{yy}]\cdot \\
A_2(2J+5)^2[6(J-M+2)(J-M+1)(J+M+2)(J+M+1)]^{1/2}\cdot \\
[(J\pm K+1)(J\pm K+2)(J\pm K+3)(J\pm K+4)]^{1/2}
\end{aligned}
$$