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The Iterative Closest Point Algorithm: Supplementary Calculations

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1 Rigid registration

The goal of rigid registration is to align a source surface $\mathcal{X}$ to a target surface $\mathcal{Y}$. The alignment process involves iteratively transforming $\mathcal{X}$ closer and closer to $\mathcal{Y}$, such that $\mathcal{X} = \mathcal{Z}^0 \to \mathcal{Z}^1 \to \mathcal{Z}^2 \to ... \to \mathcal{Z}^T = \mathcal{Y}$. In other words, in each iteration we seek a rotation matrix $\tilde{\mathbf{R}}$ and a translation vector $\tilde{\mathbf{t}}$ such that $\mathcal{X}^{t+1} = \tilde{\mathbf{R}}(\mathbf{R}^t\mathcal{X} + \mathbf{t}^t) + \tilde{\mathbf{t}} = \tilde{\mathbf{R}}\mathcal{X}^t + \tilde{\mathbf{t}}$, which corresponds to minimizing the error function

$$w_1 \sum_{i=1}^{N} ||\Pi_{\mathcal{Y}}(\mathbf{z}_i^{t+1}) - \mathbf{z}_i^{t+1}||_2^2 + w_2 \sum_{i=1}^{N} ||\tilde{\mathbf{R}}\mathbf{x}_i^t + \tilde{\mathbf{t}} - \mathbf{z}_i^{t+1}||_2^2$$

$\mathbf{x}_1, \mathbf{x}_2 ... \mathbf{x}_n$ and $\mathbf{z}_1, \mathbf{z}_2 ... \mathbf{z}_n$ are sets of points sampled from the surfaces $\mathcal{X}$ and $\mathcal{Z}$, and $\Pi_{\mathcal{Y}}(\mathbf{z}_i)$ denotes the closest point on $\mathcal{Y}$ to the point $\mathbf{z}_i$. w_1 and w_2 are simple weighting factors. By applying two simplifications, we can trace the minimization problem back to solving a system of linear equations. First, there is nonlinearity in $\Pi_{\mathcal{Y}}(\mathbf{z}_i^{t+1})$, so we use the previous estimate $\Pi_{\mathcal{Y}}(\mathbf{z}_i^t)$ which is constant[1]. Second, there is also nonlinearity in $\tilde{\mathbf{R}}$. Assuming small rotations, we can linearize the rotation matrices using $\cos(\cdot) \approx 1$ and $\sin(\cdot) \approx 0$:

$$\tilde{\mathbf{R}} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

$$\tilde{\mathbf{R}} \approx \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix}$$

By introducing $\tilde{\mathbf{r}} = \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix}$, we can substitute $\tilde{\mathbf{R}}$ with $\mathbf{I} + \tilde{\mathbf{r}}\times$. Adding Tikhonov regularization might be beneficial in ensuring that the rotations remain small.

[1] For efficient closest point lookup, a k-d tree has to be built in advance.

1.1 Minimization calculations

We have to minimize the following function with respect to $\tilde{\mathbf{r}}$, $\tilde{\mathbf{t}}$, and $\mathbf{z}_j^{t+1}$:

$$\arg\min_{\tilde{\mathbf{r}},\tilde{\mathbf{t}},\mathbf{z}_j^{t+1}} w_1 \sum_{i=1}^{N} ||\Pi_{\mathcal{Y}}(\mathbf{z}_i^t) - \mathbf{z}_i^{t+1}||_2^2 + w_2 \sum_{i=1}^{N} ||\mathbf{x}_i^t + \tilde{\mathbf{r}} \times \mathbf{x}_i^t + \tilde{\mathbf{t}} - \mathbf{z}_i^{t+1}||_2^2$$

- Minimizing w.r.t. $\tilde{\mathbf{r}}$:

$$\frac{\partial}{\partial \tilde{\mathbf{r}}} \sum_{i=1}^{N} ||\mathbf{x}_i^t + \tilde{\mathbf{r}} \times \mathbf{x}_i^t + \tilde{\mathbf{t}} - \mathbf{z}_i^{t+1}||_2^2 = 0$$

$$\sum_{i=1}^{N} \mathbf{x}_i^t \times (\mathbf{x}_i^t + \tilde{\mathbf{r}} \times \mathbf{x}_i^t + \tilde{\mathbf{t}} - \mathbf{z}_i^{t+1}) = \mathbf{0}$$

$$-\sum_{i=1}^{N} \mathbf{X}_i^t \mathbf{X}_i^t \tilde{\mathbf{r}} + \sum_{i=1}^{N} \mathbf{X}_i^t \tilde{\mathbf{t}} - \sum_{i=1}^{N} \mathbf{X}_i^t \mathbf{z}_i^{t+1} = \mathbf{0}$$

- Minimizing w.r.t. $\tilde{\mathbf{t}}$:

$$\frac{\partial}{\partial \tilde{\mathbf{t}}} \sum_{i=1}^{N} ||\mathbf{x}_i^t + \tilde{\mathbf{r}} \times \mathbf{x}_i^t + \tilde{\mathbf{t}} - \mathbf{z}_i^{t+1}||_2^2 = 0$$

$$\sum_{i=1}^{N} \mathbf{x}_i^t + \tilde{\mathbf{r}} \times \mathbf{x}_i^t + \tilde{\mathbf{t}} - \mathbf{z}_i^{t+1} = \mathbf{0}$$

$$-\sum_{i=1}^{N} \mathbf{X}_i^t \tilde{\mathbf{r}} + N\mathbf{I}\tilde{\mathbf{t}} - \sum_{i=1}^{N} \mathbf{z}_i^{t+1} = -\sum_{i=1}^{N} \mathbf{x}_i^t$$

- Minimizing w.r.t. $\mathbf{z}_j^{t+1}$:

$$\frac{\partial}{\partial \mathbf{z}_j^{t+1}} (w_1 \sum_{i=1}^{N} ||\Pi_{\mathcal{Y}}(\mathbf{z}_i^t) - \mathbf{z}_i^{t+1}||_2^2 + w_2 \sum_{i=1}^{N} ||\mathbf{x}_i^t + \tilde{\mathbf{r}} \times \mathbf{x}_i^t + \tilde{\mathbf{t}} - \mathbf{z}_i^{t+1}||_2^2) = 0$$

$$\frac{w_1}{w_2}(\mathbf{z}_j^{t+1} - \Pi_{\mathcal{Y}}(\mathbf{z}_j^t)) + \mathbf{z}_j^{t+1} - \mathbf{x}_j^t - \tilde{\mathbf{r}} \times \mathbf{x}_j^t - \tilde{\mathbf{t}} = \mathbf{0}$$

$$\mathbf{X}_j^t \tilde{\mathbf{r}} - \tilde{\mathbf{t}} + \left(1 + \frac{w_1}{w_2}\right) \mathbf{z}_j^{t+1} = \frac{w_1}{w_2} \Pi_{\mathcal{Y}}(\mathbf{z}_j^t) + \mathbf{x}_j^t$$

The result is a system of 6 + 3N linear equations with the same number of unknowns.

2 As-rigid-as-possible registration

In each iteration we also compute the local rotation matrices $\tilde{\mathbf{R}}_i$:

$$w_1 \sum_{i=1}^{N} ||\Pi_{\mathcal{Y}}(\mathbf{z}_i^{t+1}) - \mathbf{z}_i^{t+1}||_2^2 + w_2 \sum_{i=1}^{N} ||\tilde{\mathbf{R}}\mathbf{x}_i^t + \tilde{\mathbf{t}} - \mathbf{z}_i^{t+1}||_2^2 + w_3 \sum_{i=1}^{N} \sum_{k \in \mathcal{N}(i)} ||\tilde{\mathbf{R}}_i(\mathbf{x}_k^t - \mathbf{x}_i^t) - (\mathbf{z}_k^{t+1} - \mathbf{z}_i^{t+1})||_2^2$$

We apply the same linearization to the local rotation matrices as we do to the global rotation matrix. The concept of connectivity has to be defined among the points of the source surface, resulting a graph (or a mesh specifically) with Laplacian $\mathbf{L}$.

2.1 Minimization calculations

We have to minimize the following function with respect to $\tilde{\mathbf{r}}$, $\tilde{\mathbf{t}}$, $\tilde{\mathbf{r}}_j$, and $\mathbf{z}_j^{t+1}$:

$$\underset{\tilde{\mathbf{r}},\tilde{\mathbf{t}},\tilde{\mathbf{r}}_j,\mathbf{z}_j^{t+1}}{\arg\min}\, w_1 \sum_{i=1}^{N} ||\Pi_{\mathcal{Y}}(\mathbf{z}_i^{t+1}) - \mathbf{z}_i^{t+1}||_2^2 + w_2 \sum_{i=1}^{N} ||\mathbf{x}_i^t + \tilde{\mathbf{r}} \times \mathbf{x}_i^t + \tilde{\mathbf{t}} - \mathbf{z}_i^{t+1}||_2^2 +$$

$$+ w_3 \sum_{i=1}^{N} \sum_{k \in \mathcal{N}(i)} ||(\mathbf{x}_k^t - \mathbf{x}_i^t) + \tilde{\mathbf{r}}_i \times (\mathbf{x}_k^t - \mathbf{x}_i^t) - (\mathbf{z}_k^{t+1} - \mathbf{z}_i^{t+1})||_2^2$$

- Minimizing w.r.t. $\tilde{\mathbf{r}}$:

$$\frac{\partial}{\partial \tilde{\mathbf{r}}} \sum_{i=1}^{N} ||\mathbf{x}_i^t + \tilde{\mathbf{r}} \times \mathbf{x}_i^t + \tilde{\mathbf{t}} - \mathbf{z}_i^{t+1}||_2^2 = 0$$

$$\sum_{i=1}^{N} \mathbf{x}_i^t \times (\mathbf{x}_i^t + \tilde{\mathbf{r}} \times \mathbf{x}_i^t + \tilde{\mathbf{t}} - \mathbf{z}_i^{t+1}) = \mathbf{0}$$

$$-\sum_{i=1}^{N} \mathbf{X}_i^t \mathbf{X}_i^t \tilde{\mathbf{r}} + \sum_{i=1}^{N} \mathbf{X}_i^t \tilde{\mathbf{t}} - \sum_{i=1}^{N} \mathbf{X}_i^t \mathbf{z}_i^{t+1} = \mathbf{0}$$

- Minimizing w.r.t. $\tilde{\mathbf{t}}$:

$$\frac{\partial}{\partial \tilde{\mathbf{t}}} \sum_{i=1}^{N} ||\mathbf{x}_i^t + \tilde{\mathbf{r}} \times \mathbf{x}_i^t + \tilde{\mathbf{t}} - \mathbf{z}_i^{t+1}||_2^2 = 0$$

$$\sum_{i=1}^{N} \mathbf{x}_i^t + \tilde{\mathbf{r}} \times \mathbf{x}_i^t + \tilde{\mathbf{t}} - \mathbf{z}_i^{t+1} = \mathbf{0}$$

$$-\sum_{i=1}^{N} \mathbf{X}_i^t \tilde{\mathbf{r}} + N\mathbf{I}\tilde{\mathbf{t}} - \sum_{i=1}^{N} \mathbf{z}_i^{t+1} = -\sum_{i=1}^{N} \mathbf{x}_i^t$$

- Minimizing w.r.t. $\tilde{\mathbf{r}}_j$:

$$\frac{\partial}{\partial \tilde{\mathbf{r}}_j} \sum_{i=1}^{N} \sum_{k \in \mathcal{N}(i)} ||(\mathbf{x}_k^t - \mathbf{x}_i^t) + \tilde{\mathbf{r}}_i \times (\mathbf{x}_k^t - \mathbf{x}_i^t) - (\mathbf{z}_k^{t+1} - \mathbf{z}_i^{t+1})||_2^2 = 0$$

$$\sum_{k \in \mathcal{N}(j)} (\mathbf{x}_k^t - \mathbf{x}_j^t) \times ((\mathbf{x}_k^t - \mathbf{x}_j^t) + \tilde{\mathbf{r}}_j \times (\mathbf{x}_k^t - \mathbf{x}_j^t) - (\mathbf{z}_k^{t+1} - \mathbf{z}_j^{t+1})) = \mathbf{0}$$

$$-\sum_{k \in \mathcal{N}(j)} (\mathbf{X}_k^t - \mathbf{X}_j^t)(\mathbf{X}_k^t - \mathbf{X}_j^t)\tilde{\mathbf{r}}_j + \sum_{k \in \mathcal{N}(j)} (\mathbf{X}_k^t - \mathbf{X}_j^t)(\mathbf{z}_j^{t+1} - \mathbf{z}_k^{t+1}) = \mathbf{0}$$

- Minimizing w.r.t. $\mathbf{z}_j^{t+1}$:

$$\frac{\partial}{\partial \mathbf{z}_j^{t+1}} (w_1 \sum_{i=1}^{N} ||\Pi_{\mathcal{Y}}(\mathbf{z}_i^t) - \mathbf{z}_i^{t+1}||_2^2 + w_2 \sum_{i=1}^{N} ||\mathbf{x}_i^t + \tilde{\mathbf{r}} \times \mathbf{x}_i^t + \tilde{\mathbf{t}} - \mathbf{z}_i^{t+1}||_2^2 +$$

$$+ w_3 \sum_{i=1}^{N} \sum_{k \in \mathcal{N}(i)} ||(\mathbf{x}_j^t - \mathbf{x}_i^t) + \tilde{\mathbf{r}}_i \times (\mathbf{x}_j^t - \mathbf{x}_i^t) - (\mathbf{z}_j^{t+1} - \mathbf{z}_i^{t+1})||_2^2) = 0$$

$$
\begin{aligned}
&w_1(\mathbf{z}_j^{t+1} - \Pi_{\mathcal{Y}}(\mathbf{z}_j^t)) + w_2(\mathbf{z}_j^{t+1} - \mathbf{x}_j^t - \tilde{\mathbf{r}} \times \mathbf{x}_j^t - \tilde{\mathbf{t}})+ \\
&+w_3 \sum_{k \in \mathcal{N}(j)} 2(\mathbf{x}_k^t - \mathbf{x}_j^t) + (\tilde{\mathbf{r}}_k + \tilde{\mathbf{r}}_j) \times (\mathbf{x}_k^t - \mathbf{x}_j^t) - 2(\mathbf{z}_k^{t+1} - \mathbf{z}_j^{t+1}) = \mathbf{0}
\end{aligned}
$$

$$
\begin{aligned}
&(w_1\mathbf{I} + w_2\mathbf{I} + 2w_3\mathbf{L})\mathbf{z}_j^{t+1} + w_2\mathbf{X}_j^t\tilde{\mathbf{r}} - w_2\tilde{\mathbf{t}} - w_3 \sum_{k \in \mathcal{N}(j)} (\mathbf{X}_k^t - \mathbf{X}_j^t)(\tilde{\mathbf{r}}_j + \tilde{\mathbf{r}}_k) = \\
&= w_1\Pi_{\mathcal{Y}}(\mathbf{z}_j^t) + w_2\mathbf{x}_j^t + 2w_3\mathbf{L}\mathbf{x}_j^t
\end{aligned}
$$

The result is a system of 6 + 3N linear equations with the same number of unknowns.