23.3.21

13. 即判断这些子集是否关于 $M_n(\mathbb{R})$上的加法和取逆封闭,
由矩阵论知识,易知
(1)(2)(4)构成子群,
(3)不构成.

15. (1) $\{e\}$

(2) $\{e, a\}$, $aa = e$.

(3) $\{e, a, a^2, a^3\}$, $a^4 = e$.

16. $\Leftarrow$ 只需证 $\forall x, y \in H_1H_2$, $xy^{-1} \in H_1H_2$.

设 $x = h_1h_2$, $y = h_1'h_2'$,
其中 $h_1, h_1' \in H_1$, $h_2, h_2' \in H_2$.

$xy^{-1} = h_1h_2h_2'^{-1}h_1'^{-1}$.

由于 $H_1H_2 = H_2H_1$, $h_2h_2'^{-1}h_1'^{-1} \in H_2H_1$,
$\exists h_1^* \in H_1$, $h_2^* \in H_2$, $h_1^*h_2^* = h_2h_2'^{-1}h_1'^{-1}$.

故 $xy^{-1} = h_1h_1^*h_2^* \in H_1H_2$.

$\Rightarrow$ $\forall x, y \in H_1H_2$, $xy \in H_1H_2$.

设 $x = h_1h_2$, $y = h_1'h_2'$,
其中 $h_1, h_1' \in H_1$, $h_2, h_2' \in H_2$.

$xy = h_1h_2h_1'h_2' \in H_1H_2$.

由任意性,可知 $H_1H_2H_1H_2 \subseteq H_1H_2$.

又显然 $H_1H_2 \subseteq H_1H_2H_1H_2$, 故 $H_1H_2H_1H_2 = H_1H_2$.

设 $xy = h_1^*h_2^*$, $h_1^* \in H_1$, $h_2^* \in H_2$.

$(xy)^{-1} = h_2^{*-1}h_1^{*-1} \in H_2H_1$.

同理,由 x, y 的任意性, $H_1H_2H_1H_2 = H_2H_1$.

故 $H_1H_2 = H_2H_1$.

19. (1) $a, a^2, a^4, a^7, a^8,$
a^{11}, a^{13}, a^{14}

(2) $\{a^k \mid k = 0, 1, 2, \cdots, 14\} = \langle a \rangle$

$\{e\} = \langle e \rangle$

$\{a^k \mid k = 0, 3, 6, 9, 12\} = \langle a^3 \rangle$

$\{a^k \mid k = 0, 5, 10\} = \langle a^5 \rangle$

$$
\begin{tikzcd}
 & \langle a \rangle \arrow[dl, dash] \arrow[dr, dash] & \\
\langle a^3 \rangle \arrow[dr, dash] & & \langle a^5 \rangle \arrow[dl, dash] \\
 & \langle e \rangle &
\end{tikzcd}
$$

20. 反之,若 $\exists c = a^m = b^n$, $0 < m < p$

由于 $(m, p) = 1$, $\exists x, y$, $xm + yp = 1$.

$a = a^{xm+yp} = (a^m)^x = b^{xn} \in \langle b \rangle$,
矛盾.

故 $\langle a \rangle \cap \langle b \rangle = \{e\}$.