Info Theory HW 1

24.03.26

> Calculate the differential entropy of $X \sim N(0, \sigma^2)$.

First, we prove for two continuous random variables X, Y satisfying $Y = kX, k > 0$, their differential entropies satisfy $h(Y) = h(X) + \log_2 k$.

For any given t, $\Pr(X \leq t) = \Pr(Y \leq kt)$, so we have

$$\int_{-\infty}^{t} f_X(x)\, \mathrm{d}y = \int_{-\infty}^{kt} f_Y(y)\, \mathrm{d}y.$$

Derive with respect to t, we obtain

$$f_X(t) = k f_Y(kt),$$

which shows that

$$f_Y(y) = \frac{1}{k} f_X(\frac{y}{k}).$$

Thus the differential entropy of Y is

$$\begin{aligned} h(Y) &= \int_{-\infty}^{\infty} f_Y(y) \log_2 \frac{1}{f_Y(y)}\, \mathrm{d}y \\ &= \int_{-\infty}^{\infty} f_X(\frac{y}{k}) \log_2 \frac{k}{f_X(y/k)}\, \mathrm{d}\frac{y}{k} \\ &= \int_{-\infty}^{\infty} f_X(t) \log_2 \frac{1}{f_X(t)}\, \mathrm{d}t + \log_2 k \int_{-\infty}^{\infty} f_X(t)\, \mathrm{d}t \\ &= h(X) + \log_2 k. \end{aligned}$$

Next, we calculate the differential entropy of $Z \sim N(0, 1)$. We know its probability density function is $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$. Using Gauss integral $\int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} = \sqrt{2\pi}$, we obtain

$$\begin{aligned} h(Z) &= \int_{-\infty}^{\infty} f_Z(z) \log_2 \frac{1}{f_Z(z)}\, \mathrm{d}z \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \log_2(\sqrt{2\pi} e^{\frac{z^2}{2}})\, \mathrm{d}z \\ &= \frac{\log_2 \sqrt{2\pi}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\, \mathrm{d}z + \frac{\log_2 e}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}}\, \mathrm{d}z \\ &= \log_2 \sqrt{2\pi} + \frac{1}{2} \log_2 e \\ &= \frac{1}{2} \log_2(2\pi e). \end{aligned}$$

Finally, for $X \sim N(0, \sigma^2)$, we have

$$\begin{aligned} h(X) &= h(\sigma Z) \\ &= h(Z) + \log_2 \sigma \\ &= \frac{1}{2} \log_2(2\pi e \sigma^2). \end{aligned}$$