HW3 24.5.15

1. (a) $P(N=1) = P(X_1 < X_2)$

$$= \int_0^\infty \lambda_1 e^{-\lambda_1 x_1} P(X_2 > x_1)\, dx_1$$

$$= \int_0^\infty \lambda_1 e^{-\lambda_1 x_1} e^{-\lambda_2 x_1}\, dx_1$$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2}.$$

$$P(N=2) = 1 - P(N=1) = \frac{\lambda_2}{\lambda_1+\lambda_2}.$$

(b) $P(U>t) = P(X_1>t, X_2>t)$

$$= P(X_1>t)P(X_2>t)$$

$$= e^{-(\lambda_1+\lambda_2)t}$$

(c) $P(U>t \mid N=1) = P(U>t \mid X_1<X_2)$

$$= P(X_1>t \mid X_1<X_2)$$

$$= \frac{P(t<X_1<X_2)}{P(X_1<X_2)}$$

$$= \frac{\lambda_1+\lambda_2}{\lambda_1} \cdot \int_t^\infty \lambda_1 e^{-\lambda_1 x_1} P(X_2>x_1)\, dx_1$$

$$= e^{-(\lambda_1+\lambda_2)t}$$

$$= P(U>t)$$

同理 $P(U>t \mid N=2) = P(U>t)$.

由定义知 U, N 独立.

(d) $P(W>t \mid N=1) = P(X_2 > X_1+t \mid X_1<X_2)$

$$= \frac{P(X_2>X_1+t)}{P(X_2>X_1)}$$

$$= \frac{\lambda_1+\lambda_2}{\lambda_1} \int_0^\infty \lambda_1 e^{-\lambda_1 x_1} P(X_2 > x_1+t)\, dx_1$$

$$= (\lambda_1+\lambda_2) \int_0^\infty e^{-\lambda_1 x_1} e^{-\lambda_2 (x_1+t)}\, dx_1$$

$$= e^{-\lambda_2 t}$$

同理 $P(W>t \mid N=2) = e^{-\lambda_1 t}$

(e) $X_1<X_2$时, $P(W>t, U>u)=P(X_1>u, X_2>X_1+t)$

$$=\int_u^{\infty}\lambda_1 e^{-\lambda_1 x_1}P(X_2>x_1+t)dx_1$$

$$=e^{-\lambda_1 u}e^{-\lambda_2(u+t)}$$

$$=P(U>u)P(W>t).$$

同理$X_1\geq X_2$时也成立 $P(W>t, U>u)=P(W>t)P(U>u)$.

由定义知W, U独立

2. 由于震动次数$\sim P(\lambda)$, 知震动间隔$\sim Exp(\lambda)$

k个独立$Exp(\lambda)$的和$\sim\Gamma(k,\lambda)$, 即$T\sim\Gamma(k,\lambda)$.

$$f_T(t)=\begin{cases}\frac{1}{\Gamma(k)}\lambda^k t^{k-1}e^{-\lambda t}, & t>0,\\ 0, & t\leq 0.\end{cases}$$

3. $P(Y(t)=y)=\sum_{x=y}^{\infty}C_x^y p^y(1-p)^{x-y}P(X(t)=x)$

$$=\sum_{x=y}^{\infty}C_x^y p^y(1-p)^{x-y}\frac{(\lambda t)^x}{x!}e^{-\lambda t}$$

$$=\sum_{x=y}^{\infty}p^y(1-p)^{x-y}\frac{(\lambda t)^x}{y!(x-y)!}e^{-\lambda t}$$

$$=p^y\frac{(\lambda t)^y}{y!}e^{-\lambda t}\sum_{x=y}^{\infty}\frac{((1-p)\lambda t)^{x-y}}{(x-y)!}$$

$$=p^y\frac{(\lambda t)^y}{y!}e^{-\lambda t}e^{(1-p)\lambda t}$$

$$=\frac{(p\lambda t)^y}{y!}e^{-p\lambda t}$$

故$Y(t)$是参数为$p\lambda$的泊松过程.

4 $P(Z_1(t)=z)=\sum_{x=0}^{z}P(X(t)=x)P(Y(t)=z-x)$

$$=\sum_{x=0}^{z}\frac{(\lambda_1 t)^x}{x!}e^{-\lambda_1 t}\frac{(\lambda_2 t)^{z-x}}{(z-x)!}e^{-\lambda_2 t}$$

$$=\frac{((\lambda_1+\lambda_2)t)^z}{z!}e^{-(\lambda_1+\lambda_2)t}\sum_{x=0}^{z}C_z^x\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{z-x}$$

$$=\frac{((\lambda_1+\lambda_2)t)^z}{z!}e^{-(\lambda_1+\lambda_2)t}$$

故$Z_1(t)$是泊松过程，参数为$\lambda_1+\lambda_2$.

$Z_2(t)$一定不是泊松过程，因为$Z_2(t)<0$不是零概的.

$Z_3(t)$一定不是泊松过程，因为$\lim_{h\to 0}\frac{P(Z_3(h)=1)}{h}$不是常数.

5. 设$X(t)$为参数$\lambda=3$的泊松过程.

(a) $P(X(4)=0)=\frac{(3\times 4)^0}{0!}e^{-3\times 4}=e^{-12}$

(b) 设T为首达时刻（从12时开始）.

$P(T\le t)=P(X(t)>0)=1-P(X(t)=0)=1-e^{-3t}$.

$f_T(t)=P'_t(T\le t)=3e^{-3t}$.

6 $\mathrm{cov}(X(t),X(t+\tau))=E(X(t)X(t+\tau))-E(X(t))E(X(t+\tau))$

$$=E(X(t)X(t+\tau))-\lambda t\cdot\lambda(t+\tau)$$

$E(X(t)X(t+\tau)\mid X(t)=n)=nE(X(t+\tau)\mid X(t)=n)$

$$=nE(n+X(\tau))$$
$$=n(n+\lambda\tau)$$
$$=n^2+n\lambda\tau.$$

$E(X(t)X(t+\tau))=E(E(X(t)X(t+\tau))\mid X(t))$

$$=\sum_{n=0}^{\infty}(n^2+n\lambda\tau)P(X(t)=n)$$

$$=\sum_{n=0}^{\infty}(n^2+n\lambda\tau)\frac{(\lambda t)^n}{n!}e^{-\lambda t}$$

$$=e^{-\lambda t}\lambda t\sum_{n=0}^{\infty}(n+\lambda\tau+1)\frac{(\lambda t)^n}{n!}$$

$$=\lambda t(\lambda t+\lambda\tau+1).$$

故$\mathrm{cov}(X(t),X(t+\tau))=\lambda t$.

7. $P_{00}(t+h)=P_{00}(t)P_{00}(h)+P_{01}(t)P_{10}(h)$.

令$h\to 0$，$P_{00}(t+h)=P_{00}(t)(1-\lambda h)+P_{01}(t)\mu h$.

即 $P_{00}'(t)=-\lambda P_{00}(t)+\mu P_{01}(t)$

$$=-\lambda P_{00}(t)+\mu(1-P_{00}(t))$$

$$=\mu-(\lambda+\mu)P_{00}(t).$$

$$\left(e^{(\lambda+\mu)t}P_{00}(t)\right)'=e^{(\lambda+\mu)t}\left(P_{00}'(t)+(\lambda+\mu)P_{00}(t)\right)$$

$$=\mu e^{(\lambda+\mu)t}.$$

$$P_{00}(t)=e^{-(\lambda+\mu)t}\int\mu e^{(\lambda+\mu)t}dt$$

$$=\frac{\mu}{\lambda+\mu}+Ce^{-(\lambda+\mu)t}$$

$$=\frac{\mu}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}\quad(P_{00}(0)=1)$$

8. 由于状态改变概率与所处状态无关，

每次改变时间$\sim Exp(\lambda)$，i.i.d.

故$X(t)$为参数λ的泊松过程，即$P_n(t)=\frac{(\lambda t)^n}{n!}e^{-\lambda t}$.

9. $P_1(t+h)=P_1(t)(1-\lambda_1 h)+P_2(t)\lambda_2 h$，$h\to 0$.

$\Rightarrow P_1'(t)=-\lambda_1P_1(t)+\lambda_2P_2(t)$

$$=-\lambda_1P_1(t)+\lambda_2(1-P_1(t))$$

$$=\lambda_2-(\lambda_1+\lambda_2)P_1(t).$$

同题7，易得$P_1(t)=\frac{\lambda_2}{\lambda_1+\lambda_2}+Ce^{-(\lambda_1+\lambda_2)t}$

$$=\frac{\lambda_2}{\lambda_1+\lambda_2}-\frac{\lambda_2}{\lambda_1+\lambda_2}e^{-(\lambda_1+\lambda_2)t}\quad(P_1(0)=0)$$

$$P_2(t)=1-P_1(t)$$

$$=\frac{\lambda_1}{\lambda_1+\lambda_2}+\frac{\lambda_2}{\lambda_1+\lambda_2}e^{-(\lambda_1+\lambda_2)t}.$$

10. $P_{00}'(t) = -\lambda_2 P_{00}(t)$

$P_{01}'(t) = \lambda_2 P_{00}(t) - \lambda_1 P_{01}(t)$

$P_{02}'(t) = \lambda_1 P_{01}(t) - \lambda_2 P_{02}(t)$

$\vdots$

求和，有 $\sum_{i=0}^{\infty} i P_{0i}'(t) = \sum_{i奇} \lambda_1 P_{0i}(t) + \sum_{i偶} \lambda_2 P_{0i}(t)$

$$= \lambda_1 P_1(t) + \lambda_2 P_2(t)$$

$$= \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\left(1-e^{-(\lambda_1+\lambda_2)t}\right) + \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} + \frac{\lambda_2^2}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)t}$$

$$= \frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2} + \frac{\lambda_2(\lambda_2-\lambda_1)}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)t}$$

$$\triangleq f(t).$$

$E(X(t)) = \sum_{i=0}^{\infty} i P_{0i}'(t)$

$$= \int f(t)\,dt$$

$$= \frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2} t + \frac{\lambda_2(\lambda_1-\lambda_2)}{(\lambda_1+\lambda_2)^2} e^{-(\lambda_1+\lambda_2)t} + \frac{\lambda_2(\lambda_2-\lambda_1)}{(\lambda_1+\lambda_2)^2} \quad (E(X(0))=0).$$

11 $F(t+h) = F(t) + (1-F(t))g(t)h,\ h\to 0.$

$\Rightarrow F'(t) = (1-F(t))g(t).$

设 $G(x) = \int_0^t g(x)dx$

$$\left(e^{G(t)}F(t)\right)' = e^{G(t)}g(t)$$

$$\Rightarrow F(t) = e^{-G(t)}\int e^{G(t)}g(t)\,dt$$

$$= e^{-G(t)}\cdot\left(e^{G(t)}+C\right)$$

$$= 1 - e^{-G(t)}. \quad (F(0)=0)$$

12 (a) 设 $f(s)=P(E(s)=0)$

$f(s+h)=f(s)\cdot(1-\lambda(s)h),\ h\to 0.$

$\Rightarrow f'(s)=-\lambda(s)f(s)$

$\Rightarrow (\ln f(s))'=-\lambda(s)$

$\Rightarrow f(s)=e^{-\int_0^s \lambda(t)dt}\quad (f(0)=0).$

(b) $P_k(s+h)=P_{k-1}(s)\lambda(s)h+P_k(s)(1-\lambda(s)h),\ h\to 0, k>0.$

$\Rightarrow P_k'(s)=P_{k-1}(s)\lambda(s)-P_k(s)\lambda(s)$

设 $G(x)=\int_0^s \lambda(t)dt.$

$(e^{G(s)}P_k(s))'=e^{G(s)}P_{k-1}(s)\lambda(s)$

$P_k(s)=e^{-G(s)}\int_0^s e^{G(t)}P_{k-1}(t)\lambda(t)dt,\ k>0.$

$P_0(s)=f(s)=e^{-G(s)}.$

$P_1(s)=e^{-G(s)}\int_0^s \lambda(t)dt$

$=G(s)e^{-G(s)}.$

$P_2(s)=e^{-G(s)}\int_0^s G(t)\lambda(t)dt$

$=\frac{1}{2}G^2(s)e^{-G(s)}.$

$\vdots$

归纳易证 $P_k(s)=\frac{1}{k!}G^k(s)e^{-G(s)},\ k\geq 0.$

13. 先考虑一条电缆，设 $f(t)$ 为其在 t 时刻工作的概率.

$f(t+h)=f(t)(1-\lambda h)+(1-f(t))\mu h$，$h\to 0$.

$\Rightarrow f'(t)=\mu-(\lambda+\mu)f(t)$

同题7可得 $f(t)=\frac{1}{\lambda+\mu}(\mu+\lambda e^{-(\lambda+\mu)t})$ $(f(0)=1)$.

由于两条电缆独立，所求即为

$f^2(t)=\frac{1}{(\lambda+\mu)^2}(\mu+\lambda e^{-(\lambda+\mu)t})^2$.

14. $P(\text{首工时数目为}k)=(\frac{\lambda}{\lambda+\mu})^{k-1}\frac{\mu}{\lambda+\mu}$.

15 $\lambda_i=i\lambda+a$，$\mu_i=i\mu$.

$\pi_j=\frac{\lambda_0\cdots\lambda_{j-1}}{\mu_1\cdots\mu_j}=\frac{1}{j!\mu^j}\prod_{i=0}^{j-1}(i\lambda+a)=C_{j-1+a/\lambda}^{j}(\frac{\lambda}{\mu})^j$.

$\sum_{j=0}^{\infty}\pi_j=\sum_{j=0}^{\infty}C_{j+a/\lambda-1}^{j}(\frac{\lambda}{\mu})^j=(1-\frac{\lambda}{\mu})^{-\frac{a}{\lambda}}$. $\left(\sum_{x=0}^{\infty}C_{x+r-1}^{x}p^r(1-p)^x=1\right)$

$\sum_{j=0}^{\infty}\pi_j<\infty$，故 $p_n=\pi_n/\sum_{j=0}^{\infty}\pi_j=C_{n+a/\lambda-1}^{n}(\frac{\lambda}{\mu})^n(1-\frac{\lambda}{\mu})^{\frac{a}{\lambda}}$

16. 显然这是生灭过程.

$\lambda_i=\begin{cases}\lambda, & 0\le i<m,\\ 0, & i\ge m.\end{cases}$

$\mu_i=\begin{cases}i\mu, & 1\le i\le m,\\ 0, & i=0 \vee i>m.\end{cases}$

$\pi_j=\frac{\lambda_0\cdots\lambda_{j-1}}{\mu_1\cdots\mu_j}=\frac{\lambda^j}{j!\mu^j}$，$0\le j\le m$　$\pi_j=0$，$j>m$.

$\sum_{j=0}^{\infty}\pi_j=\sum_{j=0}^{m}\frac{\lambda^j}{j!\mu^j}<\infty$.

$p_n=\pi_n/\sum_{j=0}^{m}\pi_j=\frac{\lambda^n}{n!\mu^n}\Big/\sum_{j=0}^{m}\frac{\lambda^j}{j!\mu^j}$.

17. 取微元h.

$$P(X_{t+h}-X_t=1\mid X_t=m)=C_{N-m}^{1}(\lambda h+o(h))(1-\lambda h+o(h))^{N-m-1}$$

$$=(N-m)(\lambda h+o(h))(1+o(1))$$

$$=(N-m)\lambda h+o(h)$$

$$\approx N\lambda h+o(h)$$

$$P(X_{t+h}-X_t=0\mid X_t=m)=(1-\lambda h+o(h))^{N-m}$$

$$=1-(N-m)\lambda h+o(h)$$

$$\approx 1-N\lambda h+o(h).$$

根据定义，X_t 近似于参数为 $N\lambda$ 的泊松过程.

18. (i) 不具有. t较大时不能将 $N-m$ 近似为 N

(ii) 否 同(i).

(iii) 有. 转移概率与t无关

(iv) 是. 未来与过去无关.

19. $\lim\limits_{i\to\infty}\prod_{j=1}^{i}\frac{\mu_j}{\lambda_j}=\infty$，故必定灭亡.

$$\rho_i=\prod_{j=1}^{i-1}\lambda_j\Big/\prod_{j=1}^{i}\mu_j=\begin{cases}\frac{1}{\lambda_i}\left(\frac{\lambda}{\mu}\right)^i, & 1\le i\le k,\\ 0, & i>k\end{cases}$$

由于 $\sum_{i=1}^{\infty}\rho_i<\infty$，故平均灭亡时间为

$$\sum_{i=1}^{\infty}\rho_i=\sum_{i=1}^{k}\frac{1}{\lambda_i}\left(\frac{\lambda}{\mu}\right)^i$$

20. 这是一个有限状态连续时间Markov链，

具有三个状态：工作，失灵一，失灵二.

无限小矩阵 $A=\begin{pmatrix}-(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2\\ \mu_1 & -\mu_1 & 0\\ \mu_2 & 0 & -\mu_2\end{pmatrix}$.

在 $P(0)=I$ 下有 $P(t)=e^{At}$. $P(t)$ 的 $(0,0)$ 元即为所求.

21. 排队长度和服务忙碌期相同，因为显然与服务顺序无关。
等候时间不同，因为显然取决于服务顺序。

22 设 $X(t)$ 为 t 时刻的队长
易知 $X(t)$ 是生灭过程，参数为

$$\lambda_i = p\lambda,\ i \geq 0,$$

$$\mu_i = \begin{cases} \mu, & i > 0, \\ 0, & i = 0. \end{cases}$$

23 由题21知队长与服务顺序无关，故 $X(t)$ 是生灭过程，
参数为 $\lambda_i = \lambda,\ i \geq 0,$

$$\mu_i = \begin{cases} \mu, & i > 0, \\ 0, & i = 0. \end{cases}$$

24. $P_n(t) = e^{-\beta t}(1-e^{-\beta t})^{n-1} \sim GE(e^{-\beta t}),\ n \geq 1.$
我们知道对于 $X \sim GE(p)$，$EX = \frac{1}{p}$，$Var(x) = \frac{1-p}{p^2}$.
故 $E(X(t)) = e^{\beta t}$，$Var(X(t)) = e^{2\beta t}(1-e^{-\beta t})$