HW1 24.3.20

1. (a) $P_{jk}=\begin{cases} p, & k=j+1, \\ 1-p, & k=j-1, \\ 0, & 否则. \end{cases}$

$j,k\in\mathbb{Z}.$

(b) $P_{jk}=\begin{cases} (N-j)^2/N^2, & k=j+1, \\ j^2/N^2, & k=j-1, \\ 2j(N-j)/N^2, & k=j, \\ 0, & 否则. \end{cases}$

$j,k\in\{0,\cdots,N\}.$

(c)

$$P=\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

2. (a) $P_{ij}=\begin{cases} qi/N, & j=i-1, \\ p(N-i)/N, & j=i+1, \\ (pi+q(N-i))/N, & j=i, \\ 0, & 否则. \end{cases}$

$i,j\in\{0,\cdots,N\}.$

(b) $P_{ij}=\begin{cases} p(N-i)/N, & j=i-1, \\ qi/N, & j=i+1, \\ (pi+q(N-i))/N, & j=i, \\ 0, & 否则. \end{cases}$

$i,j\in\{0,\cdots,N\}.$

该题结果不合理，当罐中已空，无法以正概率从中抽球.

(c) $P_{ij}=\begin{cases} i(N-i)/N^2, & j=i\pm1, \\ (i^2+(N-i)^2)/N^2, & j=i, \\ 0, & 否则. \end{cases}$

$i,j\in\{0,\cdots,N\}.$

(d) (a)中为 $\{0,\cdots,N\}$

(b)(c)中为 $\{0\},\{1,\cdots,N-1\},\{N\}.$

3. (a) $P_{ij}=\begin{cases}\pi, & j=i,\\ (1-\pi)i/N, & j=i-1,\\ (1-\pi)(N-i)/N, & j=i+1,\\ 0, & \text{否则}.\end{cases}$

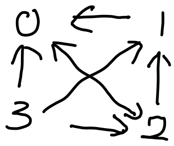
$i,j\in\{0,\cdots,N\}$

(b)
$$P=\begin{pmatrix}1-c & \frac{c}{2} & \frac{c}{2}\\ 1-\pi_1 & \pi_1 & 0\\ 1-\pi_2 & 0 & \pi_2\end{pmatrix}$$

4. (a) 等价类：$\{0,1,2\},\{3\}$.

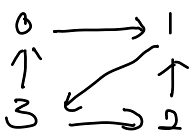
周期：$d(0)=d(1)=d(2)=1$,

$d(3)=0$.



(b) 等价类：$\{0,1,2,3\}$.

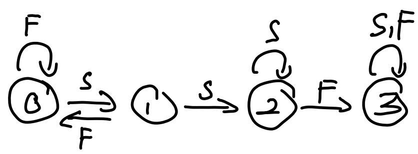
周期：$d(0)=d(1)=d(2)=d(3)=3$.



5. 下面都假设 $0<p<1$，否则是平凡的.

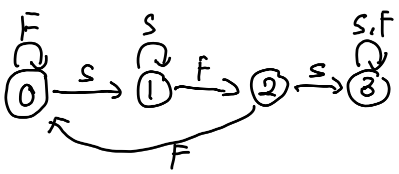
$$P(SF=k)=\sum_{i=0}^{k-2}q^{i+1}p^{k-i-1}$$
$$=p^k\sum_{i=1}^{k-1}\left(\frac{q}{p}\right)^i$$
$$=\begin{cases}(p^nq-pq^n)/(p-q), & p\neq\frac{1}{2},\\ (k-1)/2^k, & p=\frac{1}{2}\end{cases}$$

$$P_{SSF}=\begin{pmatrix}q & p & 0 & 0\\ q & 0 & p & 0\\ 0 & 0 & p & q\\ 0 & 0 & 0 & 1\end{pmatrix}$$



$P(SSF=k)$ 为该Markov链中的 f_{03}^{k}.

$$P_{SFS}=\begin{pmatrix}q & p & 0 & 0\\ 0 & p & q & 0\\ q & 0 & 0 & p\\ 0 & 0 & 0 & 1\end{pmatrix}$$



$P(SFS=k)$ 为该Markov链中的 f_{03}^{k}.

6. (a) $P_{i,j}^{n,n+1}=\begin{cases}(N-i)/N, & j=i+1,\\ i/N, & j=0,\\ 0, & \text{否则}\end{cases}$

$i,j\in N$.

由该形式知 X_n 是成功游程 Markov 链.

$p_n=n/N,\ q_n=(N-n)/N$.

(b) $P(T=t\mid X_0=0)=\prod_{i=1}^{t-2}\frac{N-i}{N}\cdot\frac{t-1}{N}$

$=\frac{t-1}{N^{t-1}}\cdot\prod_{i=1}^{t-2}(N-i)$

$=\frac{(t-1)N!}{N^t(N-t+1)!}$.

7. (a) $p_i=a_i/\sum_{k=i}^{\infty}a_k$.

$q_i=1-p_i=\sum_{k=i+1}^{\infty}a_k/\sum_{k=i}^{\infty}a_k$.

(b) $E(T^*)=\sum_{k=1}^{N-1}ka_k+N\cdot\sum_{k=N}^{\infty}a_k$

$=N-\sum_{k=1}^{N-1}(N-k)a_k$.

8. 中文版翻译漏了"首次发现带病者时".

对于 $n\geq 0$，记 $T=\min\{t\mid 2^t\geq n\}$.

$P(X=n)=\sum_{t=T}^{\infty}\left(\prod_{i=1}^{t-1}p^{2^{i-1}}\cdot C_{2^{t-1}}^{n/2}p^{\frac{n}{2}}(1-p)^{2^{t-1}-\frac{n}{2}}\right)$

$=\sum_{t=T}^{\infty}C_{2^{t-1}}^{n/2}p^{2^{t-1}+\frac{n}{2}-1}(1-p)^{2^{t-1}-\frac{n}{2}}$.

n 为非负偶数.

下为以已感染而未被发现的患者数为状态的 Markov 链的转移矩阵:

$P_{ij}=\begin{cases}C_i^{j/2}p^{\frac{j}{2}}(1-p)^{i-\frac{j}{2}}, & 0\leq j\leq 2i,\ 2\mid j,\\ 0, & \text{否则}\end{cases}$

$i,j\in N$.