22.9.20

P53: 1
P54: 6, 7(1), 9, 11(2,4,5),12

1. $\langle a,b,c\rangle = \langle\langle a,b\rangle, c\rangle$

$= \{\{\langle a,b\rangle\}, \{\langle a,b\rangle, c\}\}$

$= \{\{\{\{a\},\{a,b\}\}\}, \{\{\{a\},\{a,b\}\}, c\}\}$

6. (1) $\langle x,y\rangle \in (A\times C)\cup(B\times D)$

$\Leftrightarrow \langle x,y\rangle \in A\times C \vee \langle x,y\rangle \in B\times D$

$\Leftrightarrow (x\in A\wedge y\in C)\vee(x\in B\wedge y\in D)$

$\Rightarrow (x\in A\cup B)\wedge(y\in C\cup D)$

$\Leftrightarrow \langle x,y\rangle \in (A\cup B)\times(C\cup D)$

(2) $\langle x,y\rangle \in (A-B)\times(C-D)$

$\Leftrightarrow (x\in A-B)\wedge(y\in C-D)$

$\Leftrightarrow (x\in A\wedge x\notin B)\wedge(y\in C\wedge y\notin D)$

$\Leftrightarrow (x\in A\wedge y\in C)\wedge(x\notin B\wedge y\notin D)$

$\Rightarrow (x\in A\wedge y\in C)\wedge(x\notin B\vee y\notin D)$

$\Leftrightarrow (x\in A\wedge y\in C)\wedge\neg(x\in B\wedge y\in D)$

$\Leftrightarrow \langle x,y\rangle \in A\times C\wedge\neg(\langle x,y\rangle\in B\times D)$

$\Leftrightarrow \langle x,y\rangle \in (A\times C)-(B\times D)$

7.(1) $\langle x,y\rangle \in (A-B)\times C$

$\Leftrightarrow (x\in A-B)\wedge y\in C$

$\Leftrightarrow x\in A\wedge x\notin B\wedge y\in C$

$\Leftrightarrow (x\in A\wedge y\in C)\wedge\neg(x\in B\wedge y\in C)$

$\Leftrightarrow (\langle x,y\rangle\in A\times C)\wedge\neg(\langle x,y\rangle\in B\times C)$

$\Leftrightarrow \langle x,y\rangle \in A\times C - B\times C$

9. 2^{mn}

A到B: (1) $\emptyset$ (2) $\{\langle a,1\rangle\}$ (3) $\{\langle b,1\rangle\}$ (4) $\{\langle c,1\rangle\}$ (5) $\{\langle a,1\rangle,\langle b,1\rangle\}$ (6) $\{\langle a,1\rangle,\langle c,1\rangle\}$ (7) $\{\langle b,1\rangle,\langle c,1\rangle\}$ (8) $\{\langle a,1\rangle,\langle b,1\rangle,\langle c,1\rangle\}$

B到A: (1) $\emptyset$ (2) $\{\langle 1,a\rangle\}$ (3) $\{\langle 1,b\rangle\}$ (4) $\{\langle 1,c\rangle\}$ (5) $\{\langle 1,a\rangle,\langle 1,b\rangle\}$ (6) $\{\langle 1,a\rangle,\langle 1,c\rangle\}$ (7) $\{\langle 1,b\rangle,\langle 1,c\rangle\}$ (8) $\{\langle 1,a\rangle,\langle 1,b\rangle,\langle 1,c\rangle\}$

11.(2) $\mathrm{dom}R_1 = \{a,b,c\}$

$\mathrm{dom}R_2 = \{a,b,d\}$

$\mathrm{dom}(R_1\cup R_2) = \{a,b,c,d\}$

(4) $R_1\upharpoonright A = \{\langle a,b\rangle,\langle c,c\rangle,\langle c,d\rangle\}$

$R_1\upharpoonright\{c\} = \{\langle c,c\rangle,\langle c,d\rangle\}$

$(R_1\cup R_2)\upharpoonright A = \{\langle a,b\rangle,\langle a,c\rangle,\langle c,c\rangle,\langle c,d\rangle\}$

$R_2\upharpoonright A = \{\langle a,c\rangle\}$

(5) $R_1[A] = \{b,c,d\}$

$R_2[A] = \{c\}$

$(R_1\cap R_2)[A] = \emptyset$

12. 记 $a=\phi$, $b=\{\phi\}$, $c=\{\phi,\{\phi\}\}$, 则

$R=\{\langle a,c\rangle,\langle b,a\rangle,\langle a,a\rangle\}$

(1) $R^{-1}=\{\langle c,a\rangle,\langle a,b\rangle,\langle a,a\rangle\}$

(2) $R\circ R=\{\langle a,a\rangle,\langle a,c\rangle,\langle b,a\rangle,\langle b,c\rangle\}$

(3) $R\upharpoonright\phi=\phi$

$R\upharpoonright\{\phi\}=\{\langle a,c\rangle,\langle a,a\rangle\}$

$R\upharpoonright\{\{\phi\}\}=\{\langle b,a\rangle\}$

$R\upharpoonright\{\phi,\{\phi\}\}=\{\langle a,c\rangle,\langle b,a\rangle,\langle a,a\rangle\}$

(4) $R[\phi]=\phi$

$R[\{\phi\}]=\{a,c\}$

$R[\{\{\phi\}\}]=\{a\}$

$R[\{\phi,\{\phi\}\}]=\{a,c\}$

(5) $\mathrm{dom}\,R=\{a,b\}$

$\mathrm{ran}\,R=\{a,c\}$

$\mathrm{fld}\,R=\{a,b,c\}$

22.9.27

作业

- P54: 16, 17, 19,
- P55: 21, 22, 27，28

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16. (1) $R=\{\langle 0,10\rangle, \langle 1,9\rangle, \cdots, \langle 10,0\rangle\}$

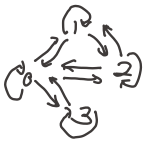
$S=\{\langle 0,4\rangle, \langle 3,3\rangle, \langle 6,2\rangle, \langle 9,1\rangle, \langle 12,0\rangle\}$

(2) R: 对称性

S: 反对称性

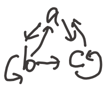
17. $M(R)=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

$G(R)$:



R具有自反性,对称性.

19. (1) $R_1=\{\langle a,b\rangle, \langle a,c\rangle, \langle b,a\rangle, \langle b,b\rangle, \langle b,c\rangle, \langle c,a\rangle, \langle c,c\rangle\}$

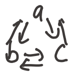


R_1具有自反性

(2) $R_2=\{\langle a,a\rangle, \langle a,b\rangle, \langle b,b\rangle, \langle c,a\rangle, \langle c,b\rangle\}$

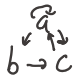
R_2具有反对称性,传递性

(3) $R_3=\{\langle a,b\rangle, \langle a,c\rangle, \langle b,a\rangle, \langle b,c\rangle, \langle c,a\rangle, \langle c,b\rangle\}$



R_3具有反自反性、对称性

(4) $R_4=\{\langle a,a\rangle, \langle a,b\rangle, \langle a,c\rangle, \langle b,c\rangle, \langle c,a\rangle\}$



R_4不具有这五种性质.

21. $M(R_1)=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $M(R_2)=\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

$M(R_1)\cdot M(R_2)=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$R_2\circ R_1=\{\langle 1,\beta\rangle\}$.

22. $\Rightarrow$ 由传递性知 $R\circ R\subseteq R$, 只需证 $R\subseteq R\circ R$.

$\forall\langle x,y\rangle\in R$, 注意到 $\langle x,x\rangle\in R$,

从而由定义 $\langle x,y\rangle\in R\circ R$, 故 $R\subseteq R\circ R$.

这就说明 $R\circ R=R$.

$\Leftarrow$ 反例为 $A=\{a\}$, $R=\varnothing$.

显然 $R\circ R=\varnothing=R$, 但R不自反.

27. $\Leftarrow$ $\langle x,y\rangle \in R_1^m \cup R_2^m$

$\Leftrightarrow \langle x,y\rangle \in R_1^m \vee \langle x,y\rangle \in R_2^m$

$\Leftrightarrow \exists t_1 \cdots \exists t_{m-1} (\langle x,t_1\rangle \in R_1 \wedge \cdots \wedge \langle t_{m-1},y\rangle \in R_1)$

$\vee \exists t_1 \cdots \exists t_{m-1} (\langle x,t_1\rangle \in R_2 \wedge \cdots \wedge \langle t_{m-1},y\rangle \in R_2)$

(*) $\Rightarrow \exists t_1 \cdots \exists t_{m-1} (\langle x,t_1\rangle \in R_1\cup R_2 \wedge \cdots \wedge \langle t_{m-1},y\rangle \in R_1\cup R_2)$

$\Leftrightarrow \langle x,y\rangle \in (R_1\cup R_2)^m$

$\Rightarrow$ 只需证 (*) 其实也成立"$\Leftarrow$".

为了方便，记 $t_0=x$, $t_m=y$.

不妨设 $\langle t_0,t_1\rangle \in R_1$ (若 $\langle t_0,t_1\rangle \in R_2$，证明完全相同).

反之，若"$\Leftarrow$"不成立，则 $\exists j\in\{1,\cdots,m-1\}$, $\langle t_j,t_{j+1}\rangle \in R_2$.

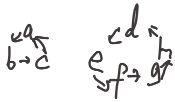
我们断言，$\exists i\in\{1,\cdots,m-1\}$, $\langle t_{i-1},t_i\rangle \in R_1 \wedge \langle t_i,t_{i+1}\rangle \in R_2$.

事实上，可以依次考察 $i=1,2,\cdots,m-1$，直到发现第一个 $\langle t_i,t_{i+1}\rangle \in R_2$. 显然这样的 i 存在.

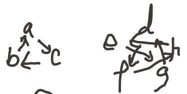
于是，$t_i \in fld(R_1)\cap fld(R_2)$，矛盾.

\#

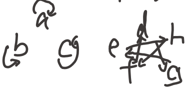
28. $G(R)$:



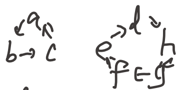
$G(R^2)$:



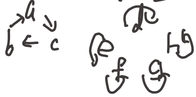
$G(R^3)$:



$G(R^4)$:



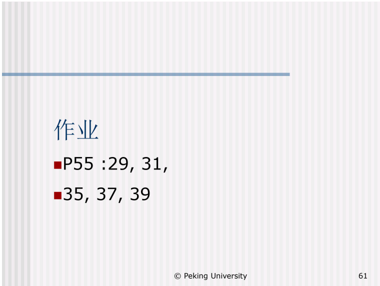
$G(R^5)$:

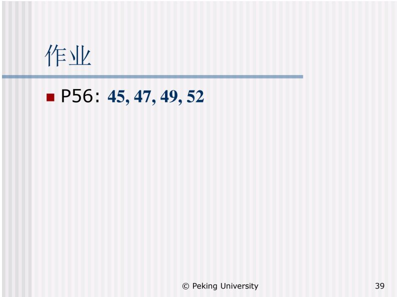


继续下去，可以发现

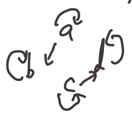
$m=0$, $n=15$ 即为所求.

22.9.29

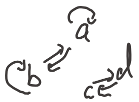




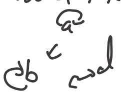
29. (1) $r(R) = R \cup \{\langle c,c\rangle, \langle d,d\rangle\}$.



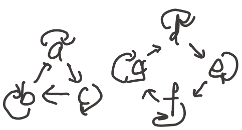
(2) $s(R) = R \cup \{\langle b,a\rangle, \langle d,c\rangle\}$



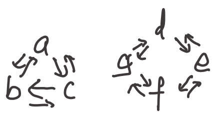
(3) $t(R) = R$



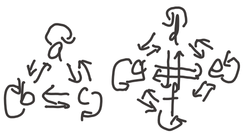
31. $r(R)$:



$s(R)$:



$t(R)$:



35. 只需证R是对称的、传递的.

$\forall x,y \in A$ 且 xRy, 由(1)知 xRx, 于是由(2)有 yRx, 即R是对称的.

$\forall x,y,z \in A$ 且 xRy, yRz, 由对称性有 yRx, 从而由(2)知 xRz, 即R是传递的.

#

37. 自反性: $\forall x \in A$, $x \equiv x \pmod 5$

对称性, $\forall x,y \in A$, $x \equiv y \pmod 5 \Rightarrow y \equiv x \pmod 5$

传递性: $\forall x,y,z \in A$, $x \equiv y \pmod 5$, $y \equiv z \pmod 5$
$\Rightarrow x \equiv z \pmod 5$

因此R是等价关系.

$A/R = \{\{1,6,11,16\}, \{2,7,12,17\},$
$\{3,8,13,15\}, \{4,9,14,19\},$
$\{5,10,15,20\}\}$.

39. (1) $R_\pi = \{\langle1,2\rangle, \langle2,1\rangle, \langle1,3\rangle, \langle3,1\rangle, \langle2,3\rangle, \langle3,1\rangle, \langle1,1\rangle, \langle2,2\rangle, \langle3,3\rangle, \langle4,4\rangle\}$.

$A/R_\pi = \{\{1,2,3\},\{4\}\}$.

(2) ① $R_1 = R_\pi \setminus \{\langle1,2\rangle,\langle2,1\rangle,\langle1,3\rangle,\langle3,1\rangle\}$

$A/R_1 = \{\{1\},\{2,3\},\{4\}\}$

② $R_2 = R_\pi\setminus\{\langle1,2\rangle,\langle2,1\rangle,\langle2,3\rangle,\langle3,2\rangle\}$

$A/R_2=\{\{1,3\},\{2\},\{4\}\}$

③ $R_3 = R_\pi \setminus\{\langle2,3\rangle,\langle3,2\rangle,\langle1,3\rangle,\langle3,1\rangle\}$

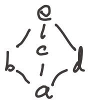
$A/R_3 = \{\{1,2\},\{3\},\{4\}\}$

④ $R_4 = \{\langle1,1\rangle,\langle2,2\rangle,\langle3,3\rangle,\langle4,4\rangle\}$

$A/R_4 = \{\{1\},\{2\},\{3\},\{4\}\}$.

⑤ $R_4 = R_\pi$, $A/R_4 = A/R_\pi$.

45. (1)



最大元: e　极大元: e

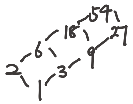
最小元: a　极小元: a

(2)

最大元: 无　极大元: a, d, e

最小元: 无　极小元: a, b, c, e

47. $A=\{1,2,3,6,9,18,27,54\}$



最长链有4条

至少可以划分为5个互不相交的反链

至多可以划分为8个互不相交的反链

49. 反自反性:

$\forall\langle x,y\rangle\in A\times B$,

$\langle y,y\rangle\notin R_2$, $\langle x,x\rangle\notin R_1$,

故 $\langle x,y\rangle \not R \langle x,y\rangle$.

传递性:

$\forall\langle x_1,y_1\rangle, \langle x_2,y_2\rangle, \langle x_3,y_3\rangle\in A\times B$,

因 $\langle x_1,y_1\rangle R\langle x_2,y_2\rangle$, $\langle x_2,y_2\rangle R\langle x_3,y_3\rangle$.

① $y_1=y_2$, $y_2=y_3$

那么 $x_1R_1x_2$, $x_2R_1x_3$, 从而 $x_1R_1x_3$,

故 $\langle x_1,y_1\rangle R\langle x_3,y_3\rangle$.

② $y_1=y_2$, $y_2\neq y_3$

那么 $y_2R_2y_3$, 从而 $y_1R_2y_3$,

故 $\langle x_1,y_1\rangle R\langle x_3,y_3\rangle$.

③ $y_1\neq y_2$, $y_2=y_3$

那么 $y_1R_2y_2$, 从而 $y_1R_2y_3$,

故 $\langle x_1,y_1\rangle R\langle x_3,y_3\rangle$.

④ $y_1\neq y_2$, $y_2\neq y_3$

那么 $y_1R_2y_2$, $y_2R_2y_3$, 从而 $y_1R_2y_3$,

故 $\langle x_1,y_1\rangle R\langle x_3,y_3\rangle$.

综上, R 是 $A\times B$ 上的拟序关系.

52. 设 $A=\{a,b,c\}$.

A上的反自反的二元关系共 $2^6=64$ 个,

其中传递的共19个, 即偏序关系共19个.

其关系图列举如下: