Reliability theory

Exponential law of reliability

Introduction

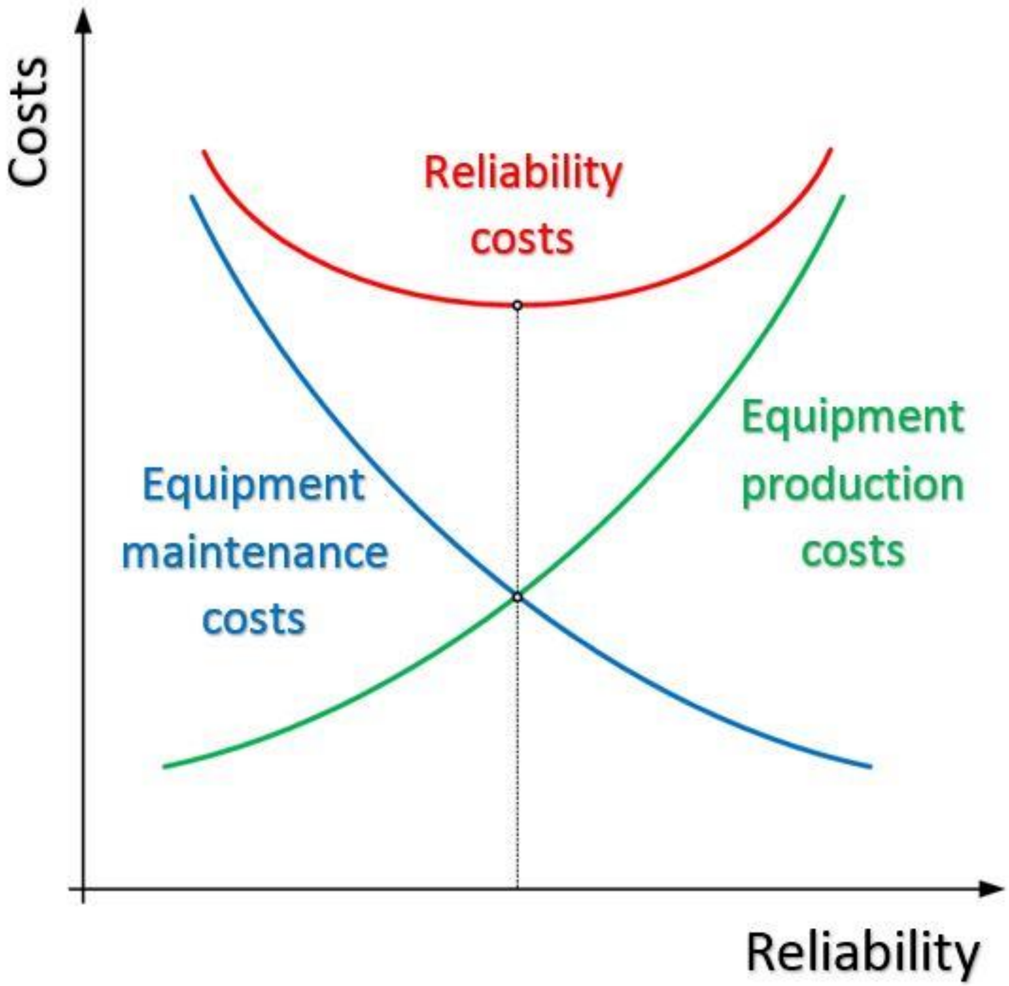
- With the increasing complexity of technical systems and the increasing costs of their development, construction, operation and maintenance, reliability and availability of the systems are becoming important factors of the quality of the entire process
- This is especially important for the systems whose potential malfunction can cause great economic losses and endanger the safety of people
- Both reliability and availability are directly related to the failure-free operation of the systems and each of its components
- Malfunctions, ie. the reduction or loss of working capability, reduce both reliability and availability, as well as reduce the quality and competitiveness of the technical system
- It is crucial that each technical system operates reliably, without interruptions, with as less failures as possible, that the components are out of operation as shortly as possible, and that their availability during life-time is maximized
- Increasing the reliability is an imperative for all participants in the process of system development and operation: designers, engineers, workers, managers, maintainers, quality control ...

Component and system reliability

The reliability of the component or a system is a mathematical probability that it will work satisfactorily under given environmental and operational conditions for an anticipated time period.

- component - one unit (component) or set of units
- mathematical probability - probability theory
- satisfactory operation - failure-free operation, unsatisfactory values of individual quantities (eg. voltage or frequency in electrical devices) are treated as an unsatisfactory operation
- operating conditions - constant or variable, the influence of the working environment on the component failure rate
- time - reliability is not instantaneous, but a continuous quantity that can be determined only after a certain amount of time has passed

Costs and reliability



Analytical expression for a component reliability

- Let's determine the analytical expression based on the experimental data of component failures
- There are **N** identical components which start to work at time $\mathbf{t = 0}$
- As the time passes, components start to fail and the number of components which are still operable at time $\mathbf{t = t_1}$ is recorded
- Let's observe one such component and determine that it failed at $\mathbf{t = \tau}$
- $\boldsymbol{\tau}$ is time period during which the component worked without failure, ***time to failure***
- $\boldsymbol{\tau}$ is a ***random variable*** (continuous random variable; **infinite** possible values)

➢ We assume that we know its ***cumulative distribution function***

Discrete random variable

Random variable is a quantity whose values are determined according to laws of probability.

1) **Distribution** of discrete random variable **X**:
   - **Set of all pairs {x_i, $p(x_i)$}**
2) **Probability function** of discrete random variable **X**:
   - **$p(x_i)$ = probability that random variable X will assume value equal to x_i**
3) **Cumulative distribution function (CDF)** of discrete random variable **X** :
   - **probability that random variable X will assume a value less than or equal to x_k:**

$$Q(x_k) = \sum_{i=1}^{k} p(x_i) = P(X \le x_k)$$

Mathematical reliability model of a component – for continuous variable

- **Distribution function (failure) → unreliability function**

$$Q(t) = P\{\tau \le t\}$$ (CDF) cumulative distribution function

- function $Q(t)$ is a probability that a component is going to fail before time t

- **Failure probability density function (PDF)**

$$q(t) = \frac{dQ(t)}{dt} = Q'(t)$$

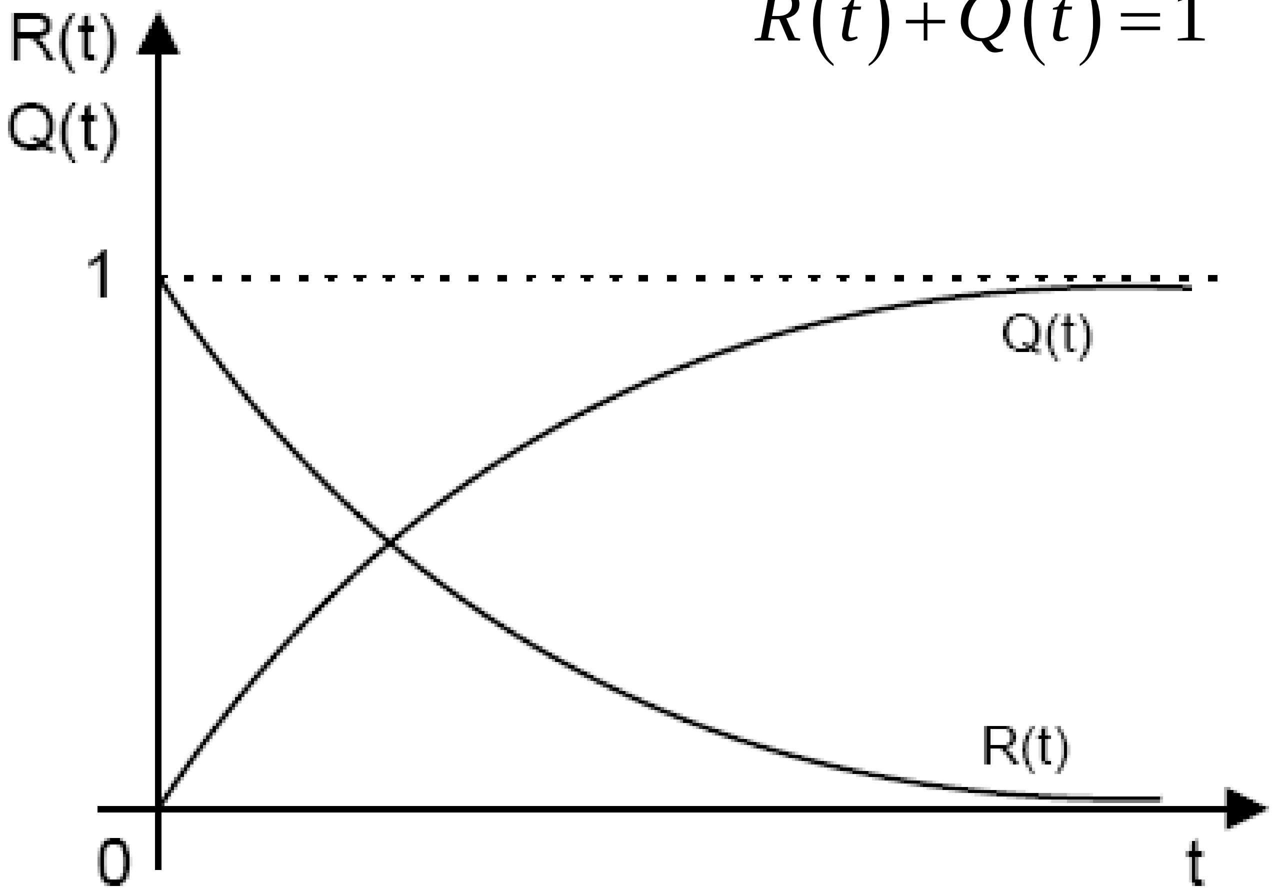
- **Reliability function**

$$R(t) = 1 - Q(t) = 1 - P\{\tau \le t\} = P\{\tau \ge t\}$$

τ – time to failure

Reliability and unreliability functions

$$R(t)+Q(t)=1$$



R(t) = ?

- **N** identical components with the same failure distribution function **Q(t)** (the same unreliability)
- Components fail **independently** with probability **Q(t) = 1 - R(t)**, where R(t) is a probability for correct operation
- At the end of the experiment (limited time), **n** out of **N** components work correctly and the rest failed in time interval **[0,t]**

➢ **Let's determine probabilities that n is equal to 0,1,2..., N**

Discrete random variable *M(t)*

- **Random variable M(t)**:

  Number of components working correctly at time **t** (eg. at the end of the experiment)

  - that can be any number (n, N, etc.)!
- Let's derive probability function of that random variable: probability that **M(t)** assumes values **0,1,2…N**: **P[M(t)=n]**
- experiment: sequence of **N** independent statements
- In each statement there are only two outcomes: component failed or works properly
- One of possible outcomes:

  In the first **n** statements the component did not fail and in the other **N-n** outcomes it failed

Probability that M(t) assumes values 0,1,2…N: P[M(t)=n]

- Since the failure and proper operation are independent events, the probability of such outcome is:

$$R(t)^{n}\left[1-R(t)\right]^{N-n}$$

- The probability of **n** components working properly at the end of the experiment is:

$$\binom{N}{n}R(t)^{n}\left[1-R(t)\right]^{N-n}$$

N choose n

Binomial distribution of the random variable M(t)

- Probability that random variable will exactly assume value **n** is:

Factorial of N

$$P(M(t) = n) = \frac{N!}{n!\,(N-n)!} R(t)^{\,n}[1 - R(t)]^{N-n}$$

$$n = 0,1,2, \dots, N$$

- **Mathematical expectation of random variable M(t)**

$$\left(E(x) = \sum_i x_i p(x_i) \right)$$

Exact value

Any value

$$n(t) = E(M(t)) = \sum_{n=0}^{N} n \frac{N!}{n!(N-n)!} R(t)^n \left[1 - R(t)\right]^{N-n}$$

Mathematical expectation of random variable M(t)

$$= NR(t)\sum_{n=1}^{N}\frac{(N-1)!}{(n-1)!\,(N-n)!}R(t)^{\,n-1}[1-R(t)]^{N-n}$$

$$\frac{n}{n!}=\frac{1}{(n-1)!} \qquad\qquad s=n-1$$

$$n(t)\equiv E\big(M(t)\big)=N\cdot R(t)\cdot\sum_{s=0}^{N-1}\frac{(N-1)!}{s!(N-1-s)!}R(t)^{s}\left[1-R(t)\right]^{N-1-s}=$$

$$=N\cdot R(t)\cdot\sum_{s=0}^{N-1}\binom{N-1}{s}R(t)^{s}\left[1-R(t)\right]^{N-1-s}=N\cdot R(t)\cdot\left[1-R(t)+R(t)\right]^{N-1}=$$

$$=N\cdot R(t)$$

Experimental derivation of reliability function

[1-R(t) = a; R(t) = b]

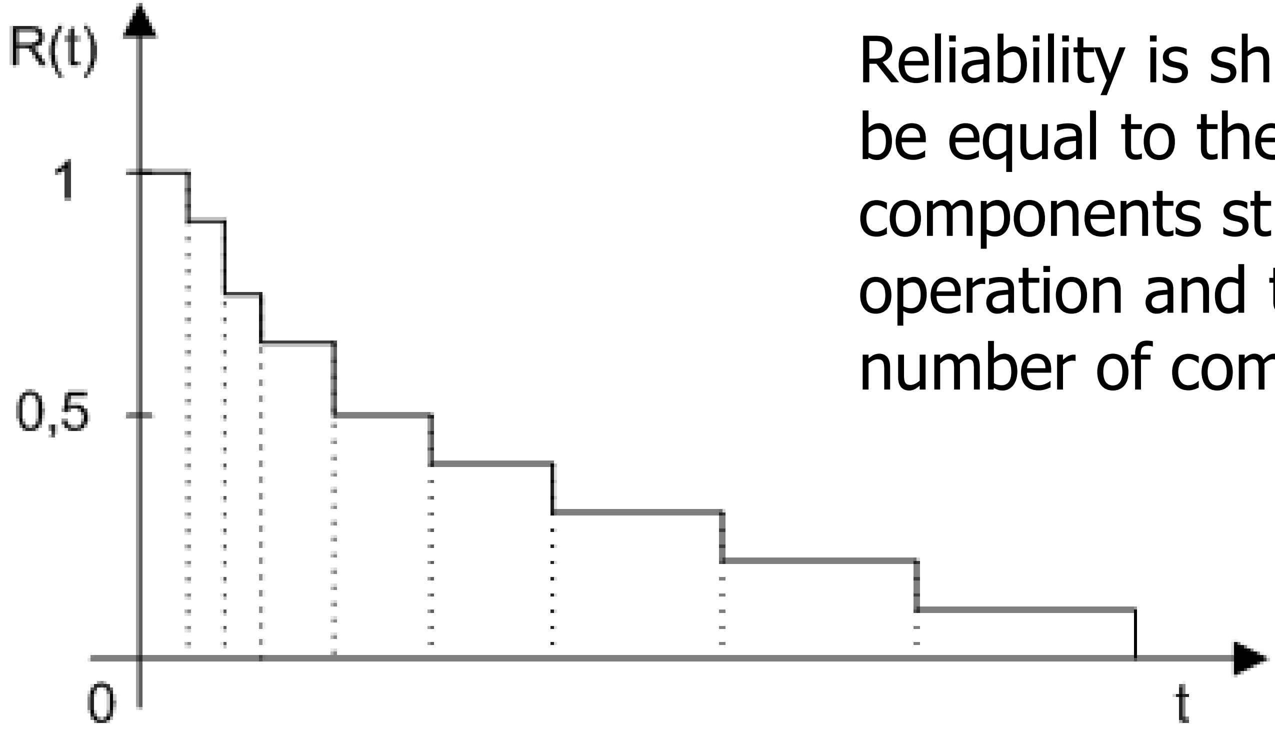
$$\sum_{s=0}^{N-1} \binom{N-1}{s} a^{N-1-s} b^{s} = (a+b)^{N-1}$$

Binomial theorem

$$R(t) = \frac{n(t)}{N}$$

Experimental derivation of reliability function

$$R(t) = \frac{n(t)}{N}$$



Reliability is shown to be equal to the ratio of components still in operation and the total number of components

Mean time to failure (MTTF) T_0

- Mean time to failure (T_0) is a mathematical expectation of continuous random variable τ (time to failure)
- **Mean time of correct operation (operation without failure)**

$$discrete\ variable: E(X) = \sum_i x_i p(x_i)$$

$$T_0 \equiv E(\tau) \equiv \int_0^\infty tq(t)dt \qquad \left(\int udv = uv - \int vdu\right)$$

$$\int_0^\infty tq(t)dt = \left|t \cdot Q(t)\right|_0^\infty - \int_0^\infty Q(t)dt = \left|t - tR(t)\right|_0^\infty - \int_0^\infty (1 - R(t))dt =$$

$$= \left|-tR(t)\right|_0^\infty + \int_0^\infty R(t)dt = \int_0^\infty R(t)dt$$

Experimental mean time to failure determination

- We are testing the components (N components) until the last fails. Let $\tau_1, \tau_2, \dots, \tau_N$ be the life-times of the components. The empirical mean life-time is:

$$\overline{\tau_N} = \frac{\tau_1 + \tau_2 + \cdots + \tau_N}{N} = \frac{\sum_{i=1}^{N} \tau_i}{N}$$

- According to the law of large numbers $\overline{\tau_N} \rightarrow T_0$ when $N \rightarrow \infty$
- For large number of components N we can use the approximation $\overline{\tau_N} \approx T_0$

Random failures

- Why the mean value, mathematical expectation, random event?
- We cannot exactly know how long a component will work properly before it fails, that is, we do not know when the failure will occur, but we can therefore calculate the average time of proper operation of a component
  - in a set of multiple equal components working simultaneously, the times of their failures will vary
  - random failures occur at random intervals, irregularly and unexpectedly
  - We can determine the probability of random failure occurrence and subsequently other important numerical quantities such as MTTF using numerical methods of probability theory and statistics, as well as collecting data on failures (experiment, experience)
- In a set of multiple components, we will still not exactly know which component will fail, but we will be able to estimate (calculate) the number of components that will fail after a certain period of time

Experimental derivation of failure probability density function

$$Q(t) = 1 - R(t) = 1 - \frac{n(t)}{N} = \frac{N - n(t)}{N}$$

$$q(t) = \frac{dQ(t)}{dt} = -\frac{1}{N}\frac{dn(t)}{dt} = \lim_{\Delta t \to 0} \frac{n(t) - n(t + \Delta t)}{N \Delta t}$$

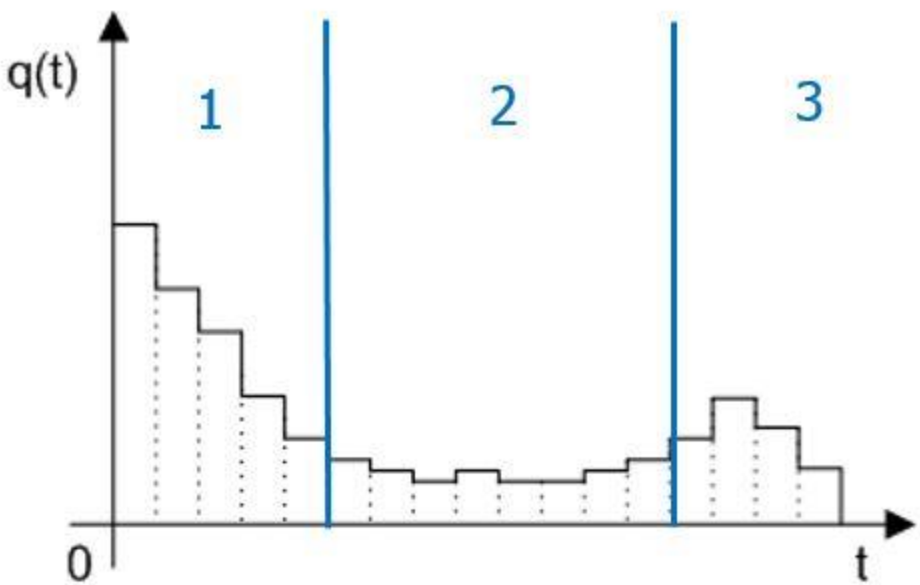
$$q(t) = \frac{\Delta n}{N \Delta t}$$

Δn is a number of failures in time interval Δt

$$\Delta n = n(t) - n(t + \Delta t)$$

Experimental derivation of failure probability density function

- If we observe time interval Δt, the product $q\Delta t$ presents the probability of failure in time interval Δt component



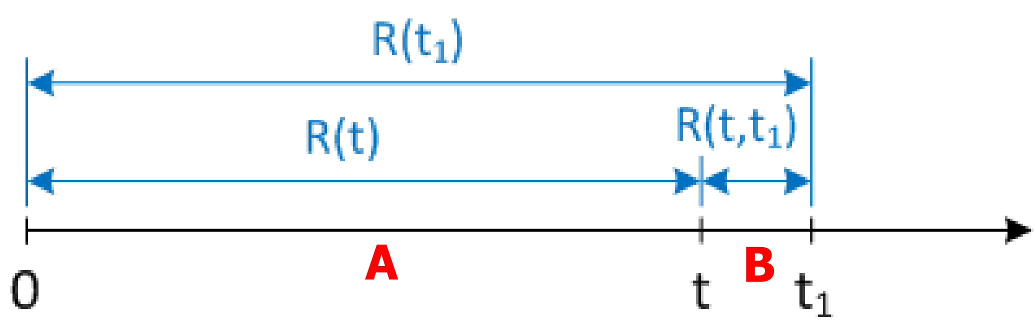
- 1 – period of early failures
- 2 – period of standard operation (random failures)
- 3 – period of deterioration (aging)

$$q(t) = \frac{\Delta n}{N \Delta t}$$

Failure rate function

Let's assume that component worked properly until time **t** and we want to know what is the probability that component failed in the interval **(t, t_1]**

- The probability that it will not fail is **$R(t,t_1)$**
- It is a **conditional probability**
- The event **A** means that component did not fail in the interval [**0, t**], and event **B** that component did not fail in the interval **(t, t_1]**



Failure rate function

- Then we are looking for a conditional probability, probability of the occurrence of event B if event A occurred first:

$$R(t, t_1) = P(B|A) = \frac{P(A \cap B)}{P(A)}$$

- Because the event **AB** presents proper operation of component during interval [$\mathbf{0, t_1}$], it means that:

$$\mathbf{P(AB) = R(t_1)}, \text{ and } \mathbf{P(A) = R(t)},$$

because **A** means proper operation in the interval [**0, t**], we get:

Failure rate function

$$R(t, t_1) = \frac{R(t_1)}{R(t)}$$

- Probability that component failed in the interval $[\boldsymbol{t}, \boldsymbol{t_1}]$ (the opposite event)

$$Q(t, t_1) = 1 - R(t, t_1) = \frac{R(t) - R(t_1)}{R(t)}$$

- If we assume $t_1 = t + \Delta t$, and constraint Δt to be sufficiently small:

$$Q(t, t + dt) = \frac{R(t) - R(t + dt)}{R(t)} = \frac{\frac{R(t) - R(t + dt)}{dt} dt}{R(t)} =$$

Failure rate function

$$= \frac{-\frac{dR(t)}{dt}}{R(t)} dt = -\frac{R'(t)}{R(t)} dt = Q(t, t + dt)$$

- We introduce designation:

$$z(t) = -\frac{R'(t)}{R(t)} = \frac{q(t)}{R(t)} \qquad \left(\frac{dQ(t)}{dt} = q(t); \ \frac{dR(t)}{dt} = -q(t)\right)$$

- For small Δt:

$$Q(t, t + \Delta t) \approx z(t)\Delta t$$

- Function $z(t)$ is called ***the failure rate function***, and the product $z(t)\Delta t$ is the probability that the component, which had worked properly until time t, failed in the time interval $(t, t + \Delta t)$ if that interval is sufficiently small.

Failure rate function

- **function $z(t)$ is called the *failure rate***

- Differential equation

$$z(t) = -\frac{R'(t)}{R(t)} = \frac{-dR(t)}{R(t)dt}$$

is easily solved and for the reliability function we get:

$$\int_0^t z(u)du = -\int_1^{R(t)} \frac{dR(t)}{R(t)} = -\left|\ln R(t)\right|_1^{R(t)} = -\ln R(t)$$

Reliability function

$$\int_0^t z(u)du = -lnR(t)$$

$$R(t) = e^{-\int_0^t z(u)du}$$

- During time interval (t_1, t_2) the probability for operation without failure is:

$$R(t) = e^{-\int_{t_1}^{t_2} z(t)dt}$$

Experimental derivation of failure rate function

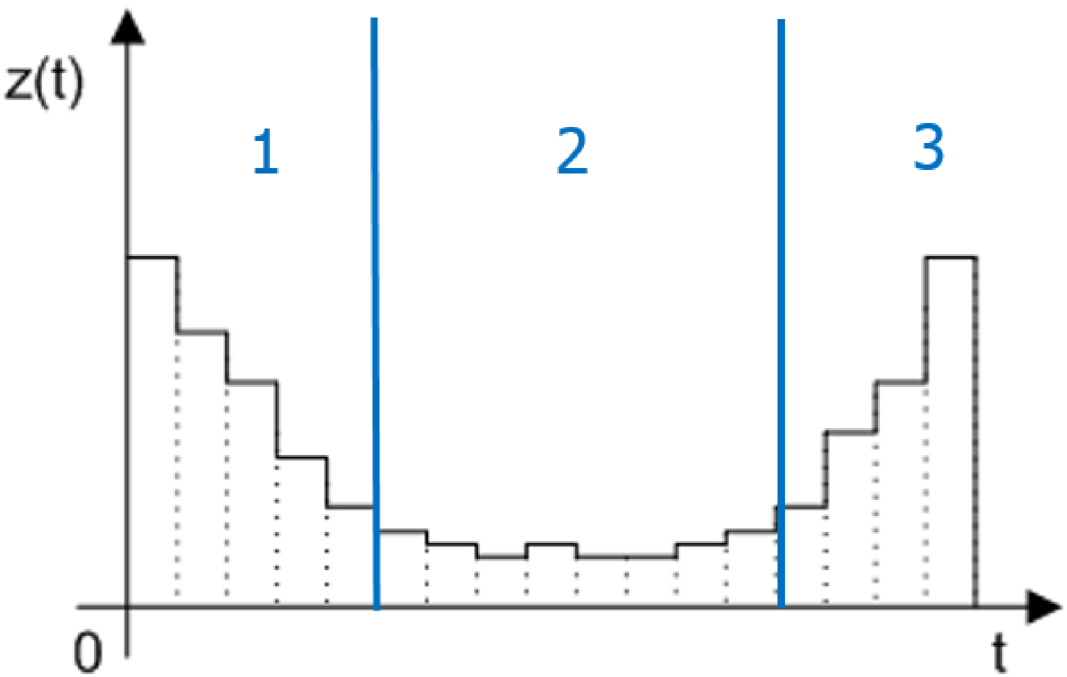
$$z(t) = -\frac{R'(t)}{R(t)} = \frac{q(t)}{R(t)}$$

$$R(t) = \frac{n(t)}{N}$$

$$q(t) = \frac{\Delta n}{N\Delta t}$$

$$z(t) = \frac{\Delta n}{n(t)\Delta t}$$

Empirical failure rate function



- 1 – period of early failures
- 2 – period of normal operation (random failures)
- 3 – period of deterioration (aging)

$$z(t) = \frac{\Delta n}{n(t)\Delta t}$$

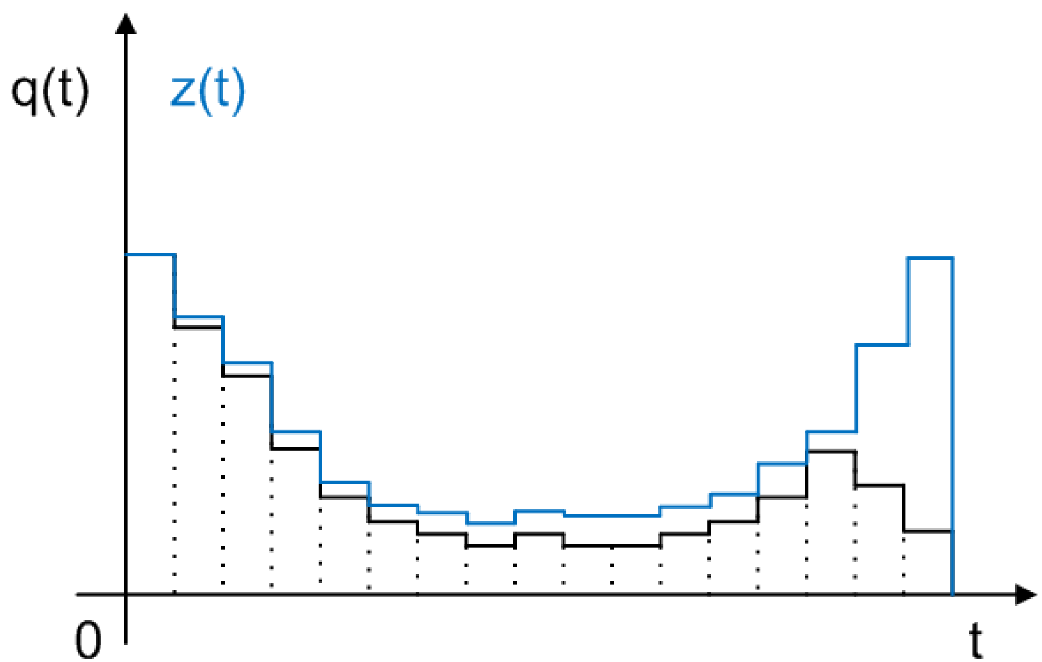
Comparison between PDF and failure rate functions

- The measure of the total failure rate

$$q(t) = \frac{\Delta n}{N\Delta t}$$

- The measure of the current failure rate

$$z(t) = \frac{\Delta n}{n(t)\Delta t}$$



Component failure model and exponential law of reliability

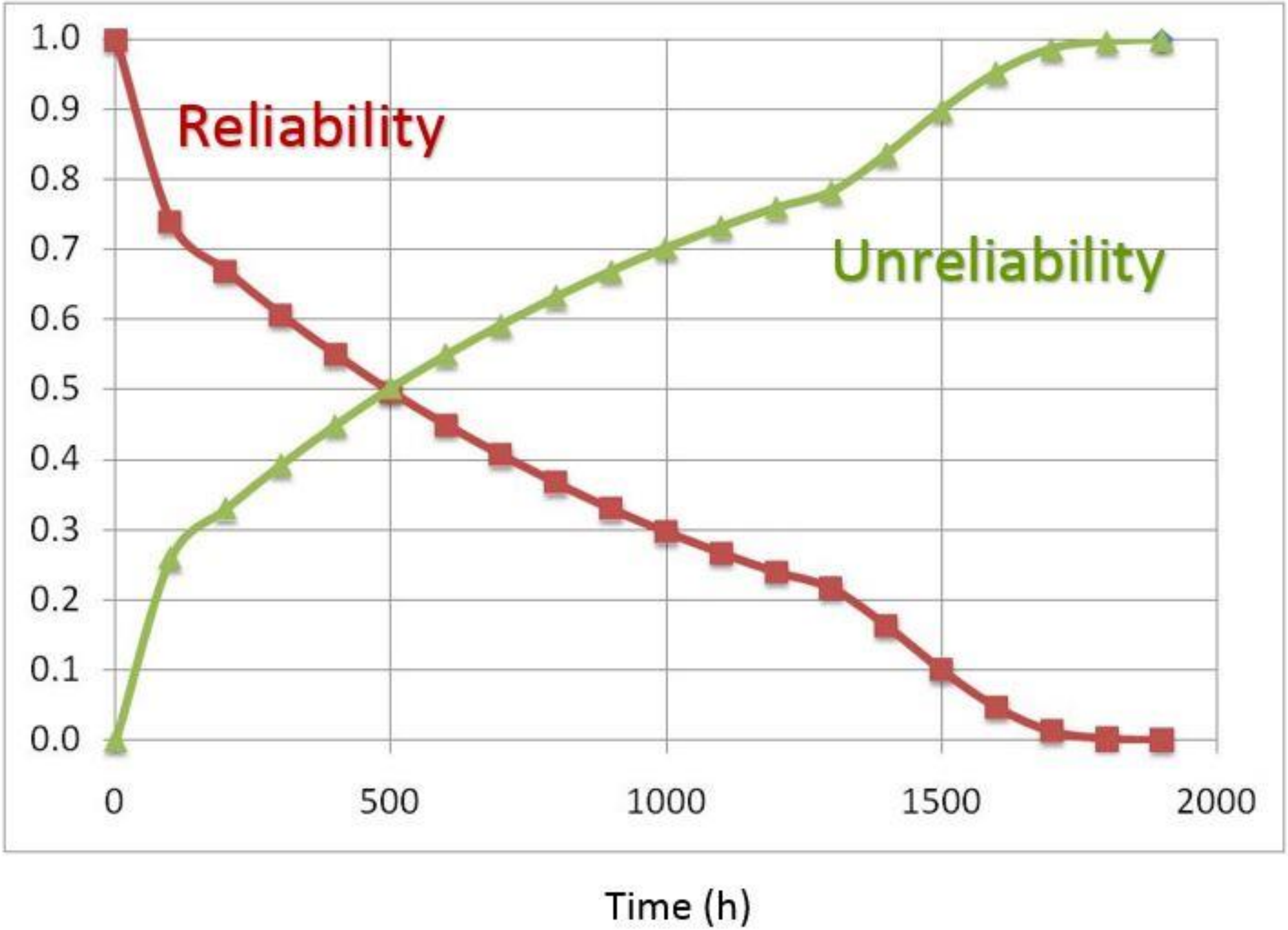
Numerous empirical observations show that, for the majority of components, functions **q(t)** and **z(t)** have characteristic shape and it is reasonable to divide time axis in **three distinct regions:**

1. **Period of early failures**
2. **Period of normal, standard operation** (random failures)
3. **Period of deterioration** (failures due to aging and deterioration of the materials)

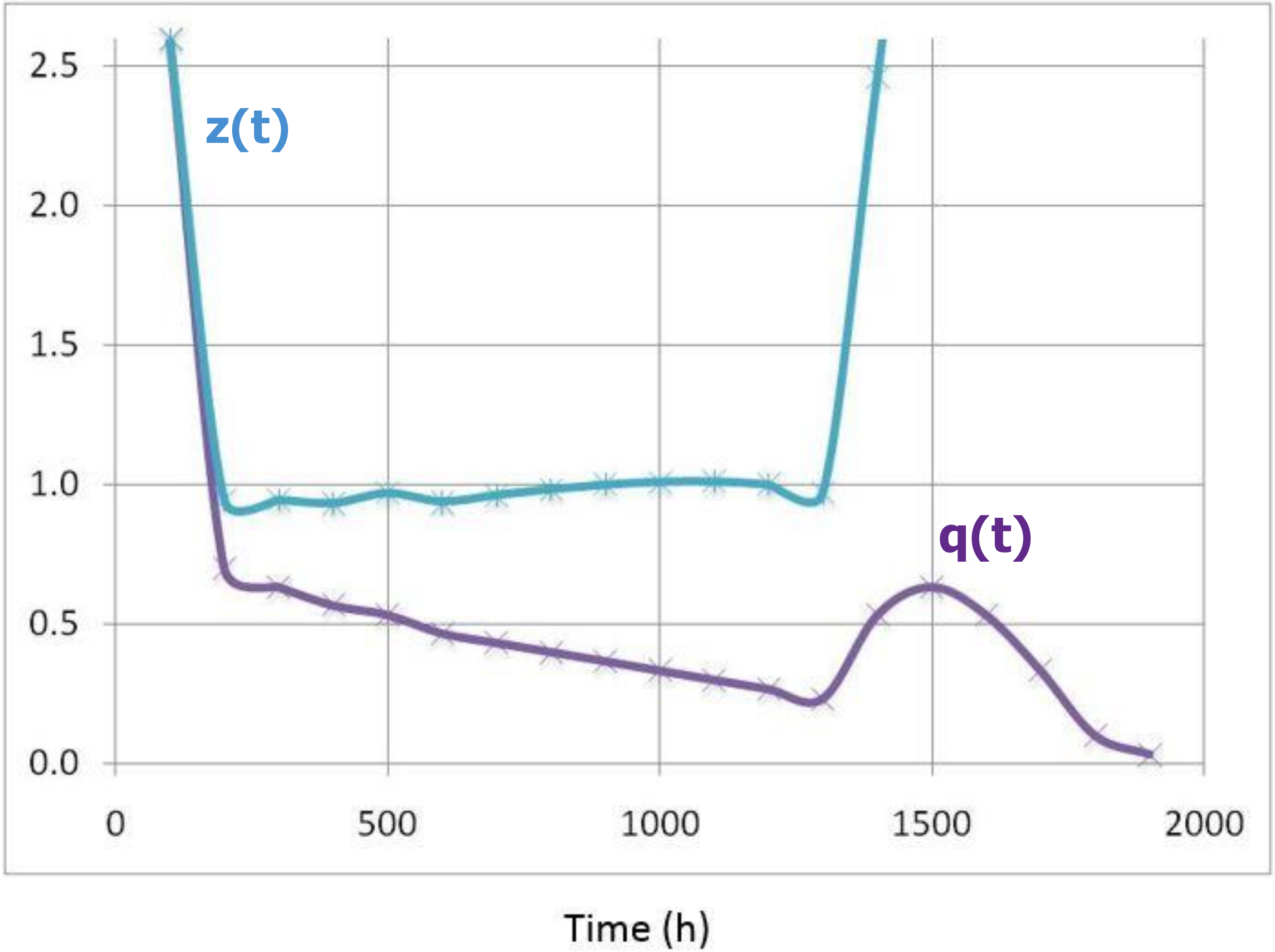
Example – components testing

Time interval (h)	Number of failures in the interval	R(t)	Q(t)	$q(t) \cdot 10^{-3}$	$z(t) \cdot 10^{-3}$
0 - 100	78	0.740	0.260	2.600	2.600
101- 200	21	0.670	0.330	0.700	0.946
201- 300	19	0.607	0.393	0.633	0.945
301- 400	17	0.550	0.450	0.567	0.934
401 - 500	16	0.497	0.503	0.533	0.970
501- 600	14	0.450	0.550	0.467	0.940
601- 700	13	0.407	0.593	0.433	0.963
701- 800	12	0.367	0.633	0.400	0.984
801- 900	11	0.330	0.670	0.367	1.000
901-1000	10	0.297	0.703	0.333	1.010
1001-1100	9	0.267	0.733	0.300	1.011
1101-1200	8	0.240	0.760	0.267	1.000
1201-1300	7	0.217	0.783	0.233	0.972
1301-1400	16	0.163	0.837	0.533	2.462
1401-1500	19	0.100	0.900	0.633	3.878
1501-1600	16	0.047	0.953	0.533	5.333
1601-1700	10	0.013	0.987	0.333	7.143
1701-1800	3	0.003	0.997	0.100	7.500
1801-1900	1	0.000	1.000	0.033	10.000
	Σ=300			Σ=1.00	

Example – component testing



Example – component testing



Exponential law of reliability of a component

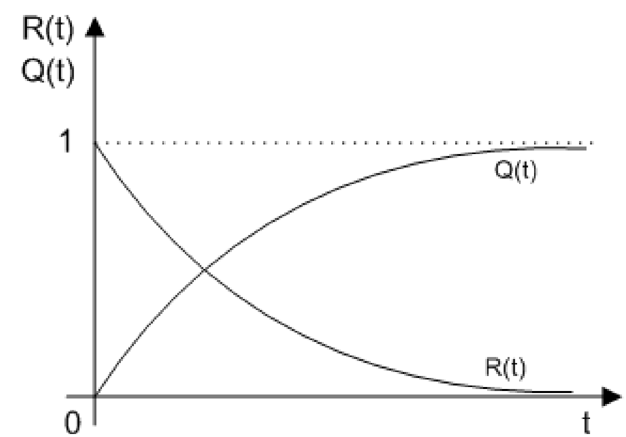
- **Single parameter, time independent, constant failure model**
- Failure rate $z(t) = \lambda = \text{const.}$
- **We observe only the period of random failures!**

- Reliability

$$R(t) = e^{-\int_0^t z(u)du} = e^{-\lambda t}$$

- Unreliability (failure probability)

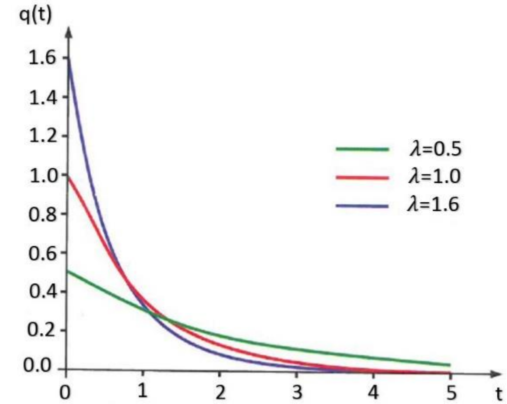
$$Q(t) = 1 - R(t) = 1 - e^{-\lambda t}$$



Exponential law of component reliability

- Failure probability density function:

$$q(t) = \frac{dQ(t)}{dt} = \lambda e^{-\lambda t}$$



- Mean time to failure:

$$\text{MTTF} = T_0 = \int_0^\infty R(t)dt = \int_0^\infty e^{-\lambda t} dt = \frac{1}{\lambda}$$

- Reliability can be expressed as:

$$R(t) = e^{-\lambda t} = e^{-\frac{t}{T_0}}$$

Exponential law of component reliability and reliability "philosophy"

- In the case when exponential law is used, probability for proper operation for a given interval of time $[t,t+\Delta t]$ is independent from the events occurring before time t and depends only on time interval Δt

$$R(t, t+\Delta t) = \frac{R(t+\Delta t)}{R(t)} = \frac{e^{-\lambda(t+\Delta t)}}{e^{-\lambda t}} = e^{-\lambda \Delta t}$$

Other reliabilty laws – other failure rate functions

Normal distribution

- Failure probability density function is given by the equation:

$$q(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-m}{\sigma}\right)^2}, t \geq 0, m > 0, \sigma > 0$$

- Commonly used equation in a mathematical statistics (known as Gaussian distribution)
  - It is a two-parameter distribution with parameters m and σ which represents a good model in cases when there is a gradual aging of the system (components, materials) during its usage, ie. when deterioration occurs

Normal distribution

- In general, the following considerations for the reliability and unreliability functions are valid

$$Q(t) = P(\tau \le \mathrm{t}) = \int_{-\infty}^{t} q(\xi)d\xi$$

- Where τ is time to failure

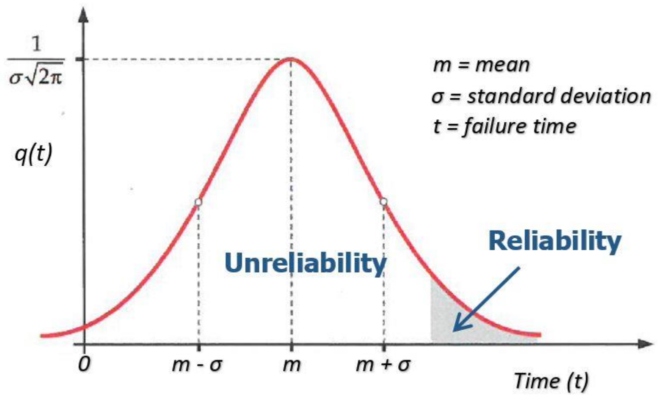
$$R(t) = 1 - Q(t) = P(\tau > \mathrm{t}) = \int_{t}^{+\infty} q(\xi)d\xi$$

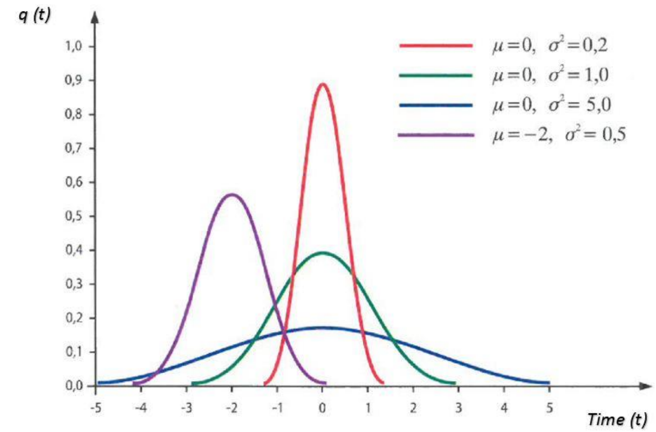
- This means that the integral $\int_{-\infty}^{+\infty} q(t)dt = 1$, ie. for *t ≥ 0* $\int_{0}^{+\infty} q(t)dt = 1$

  – This integral presents the area under the graph of function *q(t)*

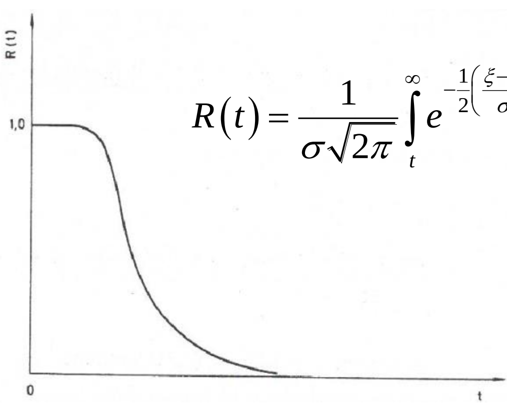
Normal distribution

- Failure probability density function

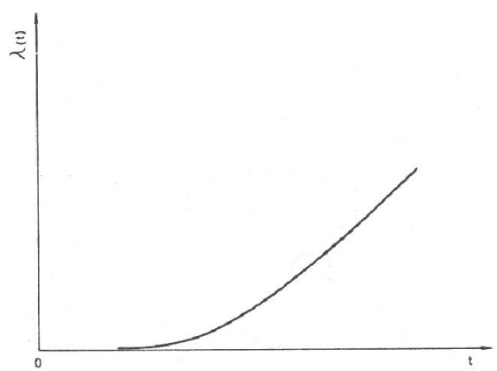




Failure rate function



$$R(t)=\frac{1}{\sigma\sqrt{2\pi}}\int_t^{\infty} e^{-\frac{1}{2}\left(\frac{\xi-m}{\sigma}\right)^2} d\xi$$



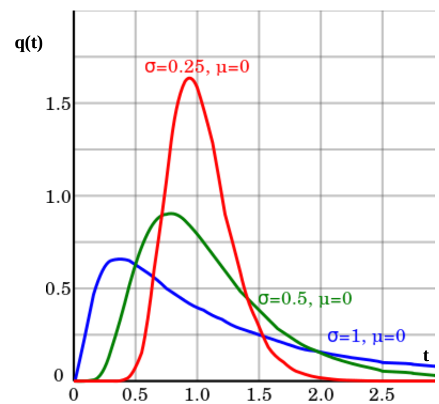
Mean time to failure: $T_0 = m$

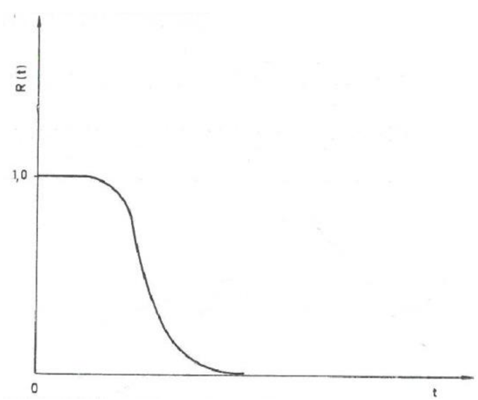
Log-normal distribution

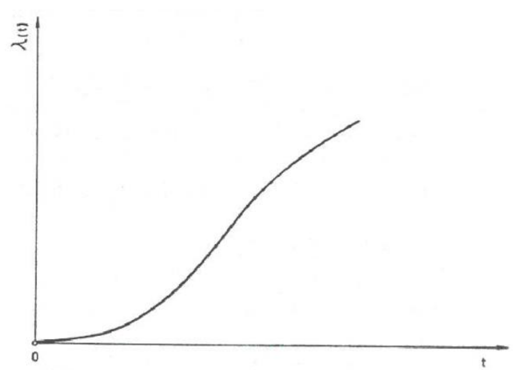
- The distribution of a random variable whose logarithm is described by the normal distribution. It is widely used in the field of system maintenance. It is a good model for studying failures caused by the material fatigue. It is also useful for representing a random variable that results from a combination of many independent random variables.

Failure probability density function is given by the equation:

$$q(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln t - \mu}{\sigma}\right)^2}, t > 0, \mu > 0, \sigma > 0$$







Mean time to failure:

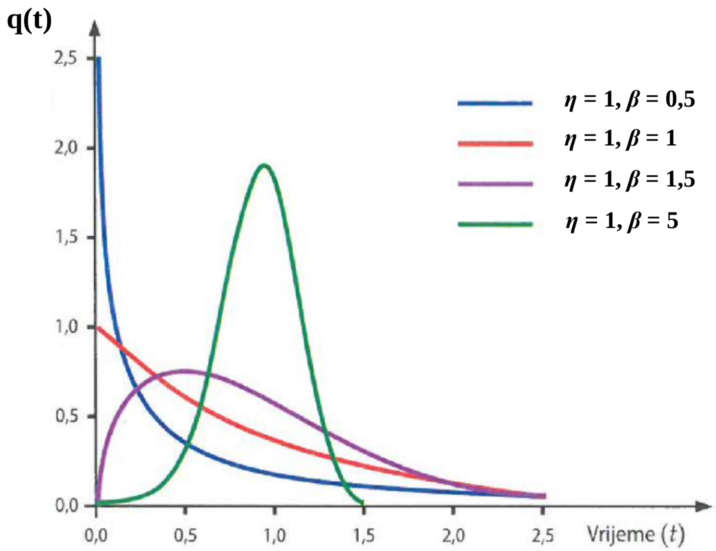
$$T_0 = e^{\mu + \frac{1}{2}\sigma^2}$$

Weibull distribution

- Failure probability density function is given by the equation:

$$q(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} e^{-\left(\frac{t}{\eta}\right)^{\beta}}, t \geq 0, \beta > 0, \eta > 0$$

β – shape parameter
η – scale parameter



Complex function:

– eg. for β = 1 it is the exponential distribution for which *λ = 1/η*

– for greater β values, it is approaching to the normal distribution

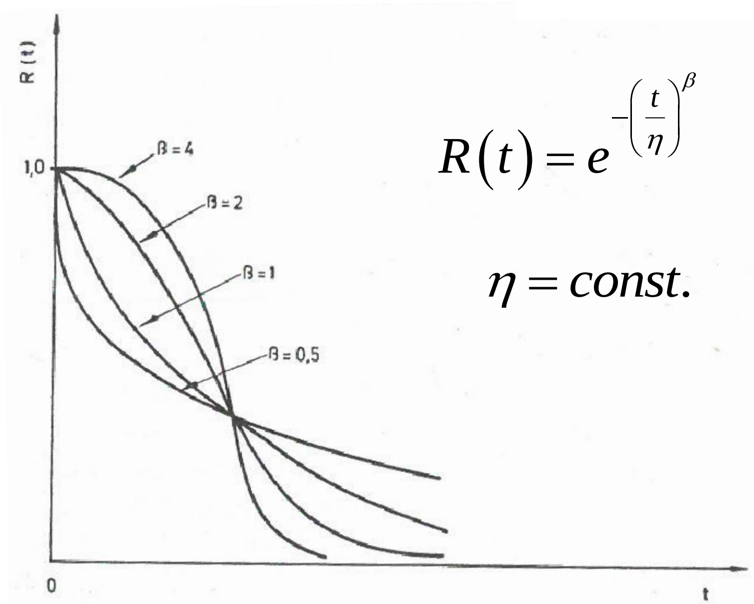
Weibull distribution

- It is often used in practical applications, in analyses of life-time, fatigue, ie. durability of elements, and especially in the field of reliability analyses for the purpose of maintenance processes management
  - many failure modes can be approximated by Weibull distribution because of the parametric character of the failure probability density function
- while the application of the exponential distribution is limited due to the assumption of a constant failure rate, the Weibull distribution covers the decreasing, constant, and increasing failure rate functions
  - For $\beta < 1$ the number of failures decreases over time
  - For $\beta = 1$ the failure rate is constant
  - For $\beta > 1$ the number of failures increases over time, which represents the deterioration of the components

Weibull distribution

- Reliability and failure rate:

$$\lambda(t) = \frac{q(t)}{R(t)} = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}$$



$$R(t) = e^{-\left(\frac{t}{\eta}\right)^{\beta}}$$

$$\eta = const.$$

$$\eta = const.$$