Reliability and Availability Assessment Methods

Reliability of the system made of independent, irreparable and irreplaceable components

The system made of independent, irreparable and irreplaceable components

The meaning of terms:

- **System** → a set of components with known reliabilities; we will always observe components with constant failure rates;
- **Independent components** → <u>operation or failure of any component does not influence reliability of other components</u>
- **Irreparable and irreplaceable components**

Reliability models

- The complex system is divided into subsystems
- <u>Knowing the system structure</u> (components connections) and the system operation is crucial for determination if failure of one or more components will also cause the system to fail
- For this purpose **reliability models** are constructed
- Difference between the reliability model and the "physical" model of the system:
  - physical relationship between components
  - functional dependence between components (which component must operate properly for the system to fulfill its purpose)

Reliability model structure

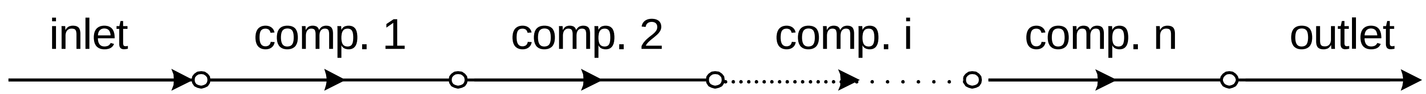
- Reliability models will consist of a set of **oriented lines (blocks)** which represent components
- The minimum number of lines is equal to number of components, but there can be more lines if some component is found also in some other **path**
- **path** → a set of lines or components that transmit energy, matter, information or signal from the system's <u>inlet to the outlet</u>, in the in the indicated direction, enabling correct system operation
- **Nodes** connect components and constitute the fixed model structure

Reliability models and calculation of reliability

Based on the reliability models, we develop **mathematical models** which specify functional relationships between components and, through a failure rate of each component, calculate reliability or probability of correct system operation.

Reliability of series systems

- Equivalent model of reliability



$x_i \quad \bar{x}_i \quad P(x_i) \quad P(\bar{x}_i) \quad R_s(t) \quad Q_s(t)=1-R_s(t)$

Proper system operation = Intersection of events $x_1, x_2, \dots, x_n$

$$R_s(t) = P(x_1 x_2 x_3 \cdot \ldots \cdot x_n)$$

$$R_s(t) = P(x_1)P(x_2 / x_1)P(x_3 / x_1 x_2)\cdot \ldots \cdot P(x_n / x_1 x_2 \ldots x_{n-1})$$

Reliability of series systems – other possible approach

$$R_s(t) = P(x_1)P(x_2)P(x_3)\cdot\ldots\cdot P(x_n)$$

- There is another possible method for reliability calculation by determining probability of failure (unreliability)
- Union of events

$$\overline{x}_1, \ \overline{x}_2, \ \ldots, \ \overline{x}_n$$

$$Q_s(t) = P(\overline{x}_1 + \overline{x}_2 + \overline{x}_3 + \ldots + \overline{x}_n)$$

Reliability of series systems

$$Q_s(t) = \sum_i P(\bar{x}_i) - \sum_i \sum_j P(\bar{x}_i \bar{x}_j) + \sum_i \sum_j \sum_k P(\bar{x}_i \bar{x}_j \bar{x}_k) - \cdots \pm \cdot\cdot P(\bar{x}_1 \bar{x}_2 \cdots \bar{x}_n) =$$

$$= \left[\mathrm{P}(\bar{\mathrm{x}}_1) + \overline{\mathrm{P}(\mathrm{x}_2)} + \ldots + \overline{\mathrm{P}(\mathrm{x}_n)}\right] - \left[\mathrm{P}(\bar{\mathrm{x}}_1 \bar{\mathrm{x}}_2) + \mathrm{P}(\bar{\mathrm{x}}_1 \bar{\mathrm{x}}_3) + \ldots + \mathrm{P}(\bar{\mathrm{x}}_i \bar{\mathrm{x}}_j)\right] +$$

$$+ \ldots - (-1)^n \mathrm{P}(\bar{\mathrm{x}}_1 \bar{\mathrm{x}}_2 \ldots \bar{\mathrm{x}}_n), \; \mathrm{i} < \mathrm{j}$$

Total $2^n - 1$ terms

$$R_s(t) = 1 - P(\bar{x}_1) - P(\bar{x}_2) - \ldots - P(\bar{x}_n) + P(\bar{x}_1) P(\bar{x}_2 / \bar{x}_1) +$$

$$+ P(\bar{x}_1) P(\bar{x}_3 / \bar{x}_1) + \ldots + P(\bar{x}_i) P(\bar{x}_j / \bar{x}_i) + \ldots$$

$$+ (-1)^n P(\bar{x}_1) P(\bar{x}_2 / \bar{x}_1) \ldots P(\bar{x}_n / \bar{x}_1 \bar{x}_2 \ldots \bar{x}_{n-1})$$

$P(\bar{x}_3 / \bar{x}_1 \bar{x}_2)$ *becomes* $P(\bar{x}_3)$ *for independent components*

Reliability of series systems

$$R_s(t) = 1 - P(\overline{x}_1) - P(\overline{x}_2) - \ldots + P(\overline{x}_1)P(\overline{x}_2) +$$

$$+ \ldots + P(\overline{x}_i)P(\overline{x}_j) + \ldots + (-1)^n P(\overline{x}_1)P(\overline{x}_2)\ldots P(\overline{x}_n)$$

$$z(t) = \lambda$$

$$R_s(t) = P(x_1)P(x_2)\ldots P(x_n) =$$

$$= \prod_{i=1}^{n} P(x_i) = \prod_{i=1}^{n} e^{-\lambda_i t} = e^{-t\sum_{i=1}^{n}\lambda_i} = e^{-\lambda_s t}$$

$$\lambda_s = \sum_{i=1}^{n} \lambda_i$$ Failure rate of series system

Conditions of application of the derived equation

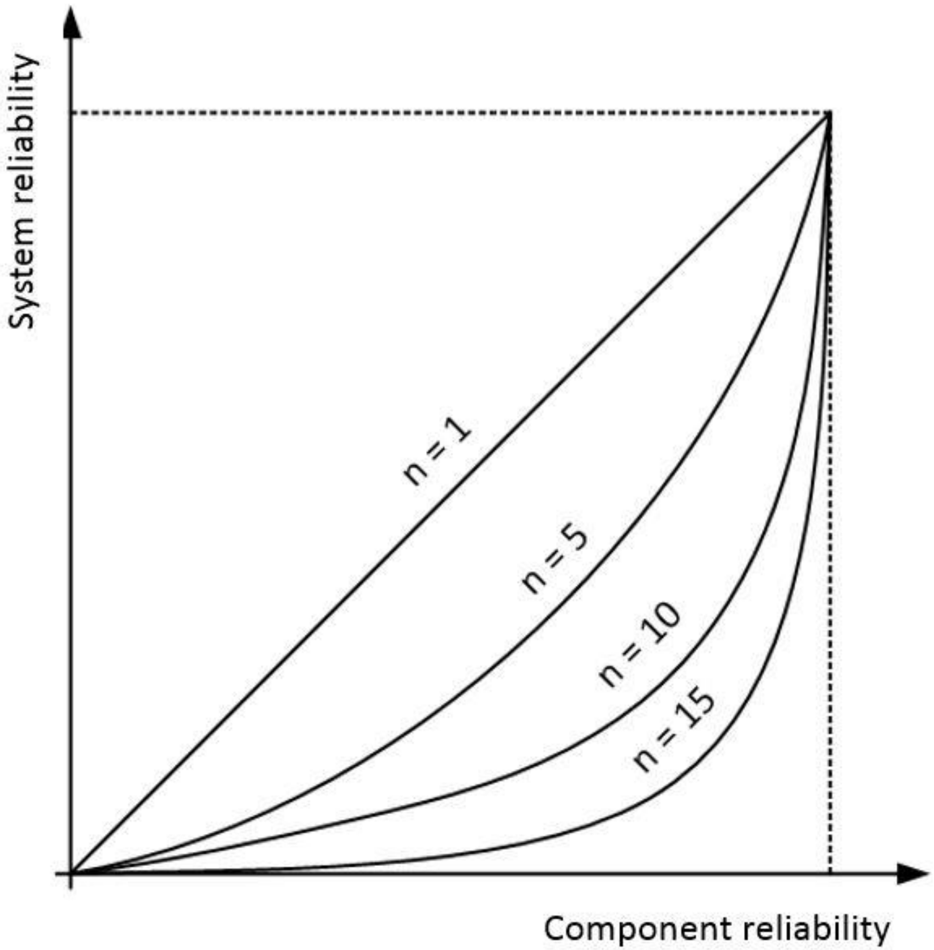
The derived equation is the simplest and the most frequently used formula in reliability theory, but often incorrectly.

Let's observe the conditions which must be met when applying the equation:

1. **Reliability model has to be serial;**
2. **Components must be independent;**
3. **Component failure rates must be constant.**

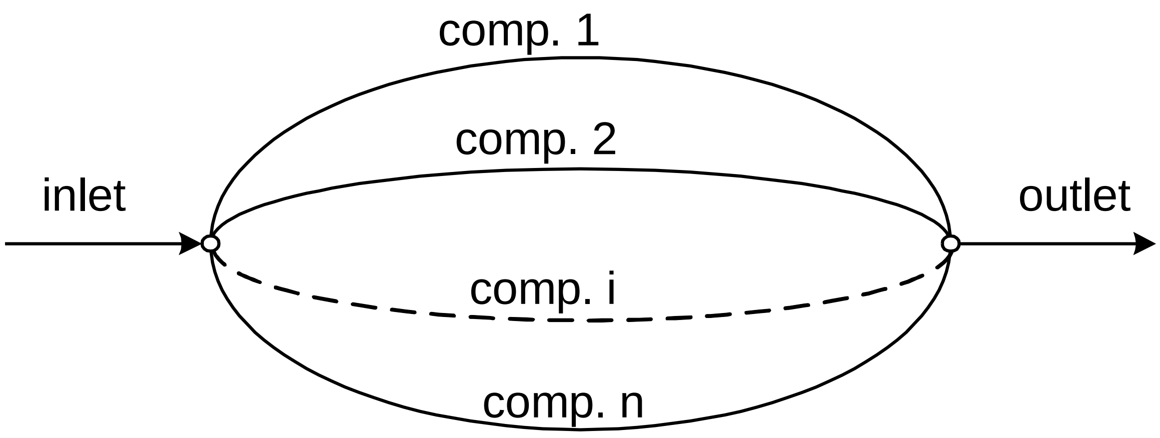
Reliability of series systems

- Reliability dependence on the number of components



Reliability of parallel systems

In the cases when the system made up of *n* components is operational if at least one of the units (components) is operational, the reliability model is represented with the parallel model. (This model is also known as full active redundancy.) Equivalent model of reliability is shown in the figure:



Reliability of parallel systems

$$R_p(t) = P(x_1 + x_2 + \ldots + x_n)$$

$$Q_p(t) = P(\overline{x}_1 \overline{x}_2 \ldots \overline{x}_n)$$

$$R_p(t) = 1 - P(\overline{x}_1 \overline{x}_2 \ldots \overline{x}_n)$$

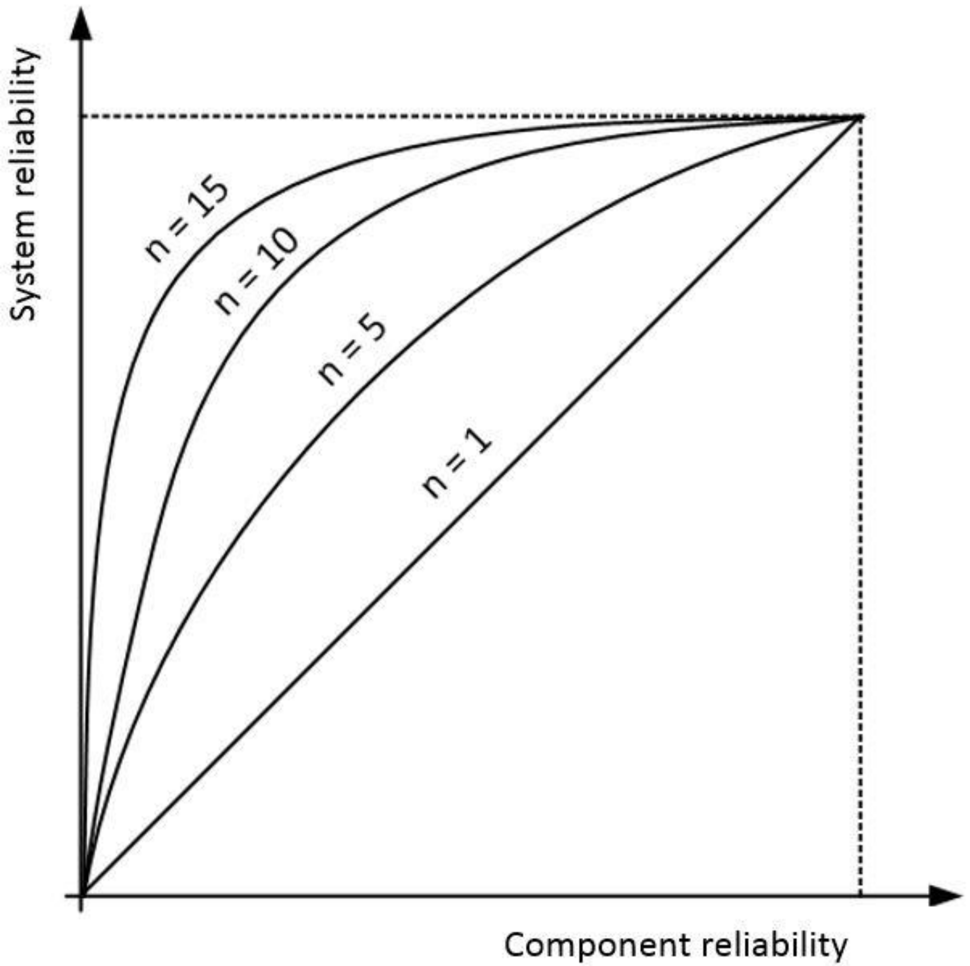
$$R_p(t) = 1 - P(\overline{x}_1) P(\overline{x}_2) \ldots P(\overline{x}_n)$$

$$P(\overline{x}_i) = 1 - e^{-\lambda_i t}$$

$$R_p(t) = 1 - \left[\prod_{i=1}^{n} (1 - e^{-\lambda_i t}) \right]$$

Reliability of parallel systems

- Reliability dependence on the number of components



Redundancy system

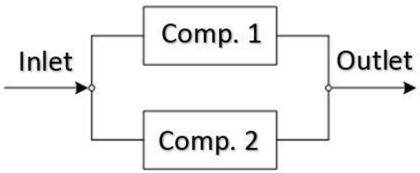
- Reliability of the system increases with the increasing number of components
  - It is used when the required reliability of the system is greater than the reliability of a single component, then the required number of components is added in parallel to meet the system requirements
- Except the increasing number of components, the number of spare parts in the warehouse can be increased
  - Special case is when we have a spare component which is not used until the one that is in usage does not fail
- The drawbacks are: an increase in the mass and volume of the system, difficult maintenance, and increase in cost
- They are important in highly reliable systems (hospitals, aircrafts, nuclear power plants,…) or when the cost of damages (and repairs) in case of a failure is too high

Types of redundancy systems

- **Fully active redundancy**
  - When all the components are operating simultaneously and share the loading, while only one component would be enough for the system to be operable
- **Partially active redundancy**
  - When n components are operating simultaneously, while r out of n components are required for the system to be operable ($1<r<n$)
- **Passive redundancy (backup system)**
  - When only one component is operating, while the other components are on standby, there is a switch that turns on a backup component when the component in operation fails
  - We can observe two cases – the first is that a backup component can fail, and the second that the backup component is maintained regularly, and its failure rate is negligible.

Fully active redundancy

- The reliability of this systems is described with the same equation as in the case of parallel systems
- Let's observe the system with two components
  - The reliability can be calculated in three ways:



1. Directly

$$R(t)=P(x_1+x_2)=P(x_1)+P(x_2)-P(x_1)P(x_2)=R_1(t)+R_2(t)-R_1(t)R_2(t)$$

2. Through unreliability

$$R(t)=1-Q(t)=1-P(\overline{x}_1)P(\overline{x}_2)=1-\left(1-R_1(t)\right)\cdot\left(1-R_2(t)\right)=R_1(t)+R_2(t)-R_1(t)R_2(t)$$

3. Functionally – the system will work if component 1 works, or when component 1 fails, but component 2 is operable

$$R(t)=R_1(t)+Q_1(t)R_2(t)=R_1(t)+\left(1-R_1(t)\right)\cdot R_2(t)=R_1(t)+R_2(t)-R_1(t)R_2(t)$$

Partially active redundancy

- In many cases the system will work properly if r out of the total n components are in operation (n powerlines connect two cities when only r is enough and $r<n$, etc.)
- if r functional components are needed for the proper functioning of the system (assuming, for the time being, that the components are identical), it is clear that the system will work correctly if also $r + 1$, $r + 2$..., $n - 1$ or n components are functional, and the probability of the correct system operation will be equal to the sum of probabilities that r, $r + 1$, ..., n components are operational

Partially active redundancy

If we denote p as the probability that any component is functional, the probability that r out of n components are operational is:

$$\binom{n}{r} p^r (1-p)^{n-r}$$

The reliability of the system is:

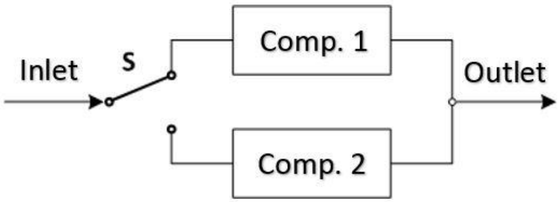
$$R(t) = \sum_{k=r}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

If we assume constant failure rate of the components λ, we get

$$R(t) = \sum_{k=r}^{n} \binom{n}{k} e^{-k\lambda t} (1-e^{-\lambda t})^{n-k}$$

Passive redundancy

- The system with a switch
  - Let's assume first that a backup component can fail with a constant failure rate and that a switch is not an ideal one and can fail in two ways
    - It can be activated too early - Q'_S
    - It does not activate on a demand - Q_S



- The system will work properly if:
  1. Component 1 works properly R_1, the switch is not activated too early R'_S
  2. Component 1 fails $Q_1 = 1 - R_1$, the switch is activated on time R_S, and component 2 works properly R_2
  3. Component 1 works properly R_1, the switch is activated too early $Q'_S = 1 - R'_S$, and component 2 works properly R_2

$$R = R_1 R'_S + (1 - R_1) R_S R_2 + R_1 (1 - R'_S) R_2$$

Passive redundancy

- Since these are mutually exclusive events, the system will work properly in the case that either of these three events occurs

$$R = R_1 R'_S + (1 - R_1) R_S R_2 + R_1 (1 - R'_S) R_2$$

- We must come to the same expression when looking at a complementary event – failure of the system Q, it follows $R = 1 - Q$
- The system will fail if:
  1. Component 1 works properly R_1, the switch is activated too early Q'_S and component 2 fails Q_2
  2. Component 1 fails Q_1, the switch is not activated on time Q_S
  3. Component 1 fails Q_1, the switch is activated on time R'_S, and component 2 fails Q_2
- The probability of the system working properly is:

$$R = 1 - Q = 1 - \left(R_1 Q'_S Q_2 + Q_1 Q_S + Q_1 R_S Q_2 \right) = R_1 R'_S + (1 - R_1) R_S R_2 + R_1 (1 - R'_S) R_2$$

Example

Let’s consider the system consisting of 4 identical components with constant failure rate and assume that the system is operational if at least 2 components are operational. Reliability function is then:

Reliability of the system with 4 identical components out of which 2 or more are operational

$$R(t) = \sum_{k=2}^{4} \binom{4}{k} e^{-\lambda t k} (1 - e^{-\lambda t})^{4-k} =$$

$$= \binom{4}{2} e^{-2\lambda t} (1 - e^{-\lambda t})^2 + \binom{4}{3} e^{-3\lambda t} (1 - e^{-\lambda t}) + \binom{4}{4} e^{-4\lambda t} =$$

$$= 6 e^{-2\lambda t} - 8 e^{-3\lambda t} + 3 e^{-4\lambda t}$$

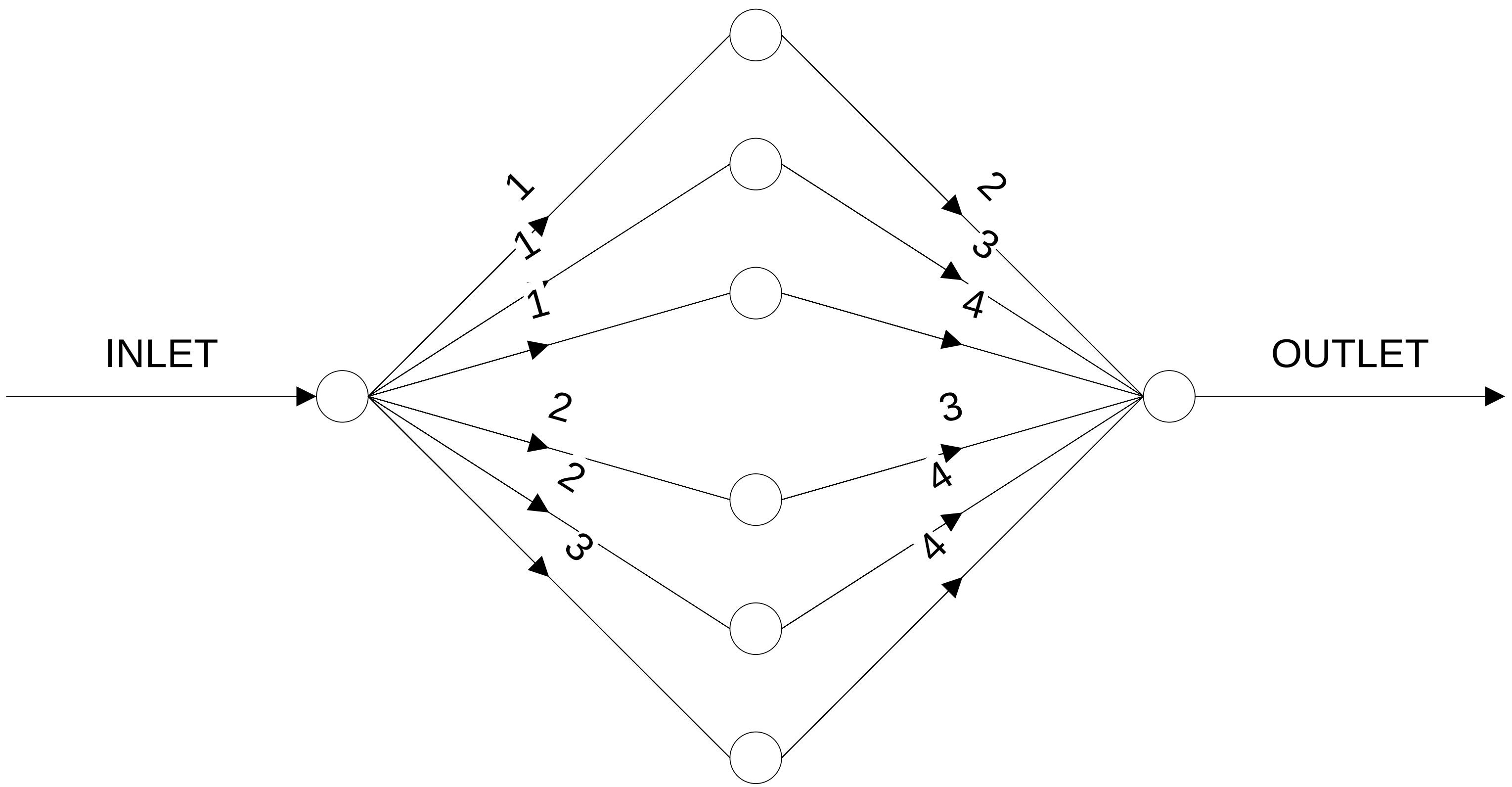
Reliability of the system with 4 different components out of which 2 or more are operational

In the case that the components are different, it is required to count in all possible combinations of successful operation. The reliability models are then very helpful.

In our case with 2-out-of-4 components it is required to list all paths containing two different components. This corresponds to the number of 2 combinations from a set of 4 elements. Therefore, there are:

$$\binom{4}{2} = \frac{4 \cdot 3}{1 \cdot 2} = 6$$ paths in the reliability model

Reliability of the system with 4 different components out of which 2 or more are operational



Reliability of the system with 4 different components out of which 2 or more are operational

The system will work successfully if at least one path is functional. The system reliability is then equal to the union of events of successful operation of individual paths:

$$R(t) = P\left(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4\right)$$

Expanding the above expression we get $2^6 - 1 = 63$ terms, and by its reduction:

$$= P(x_1x_2) + P(x_1x_3) + P(x_1x_4) + P(x_2x_3) + P(x_2x_4) + P(x_3x_4) - 2P(x_1x_2x_3) -$$
$$- 2P(x_1x_2x_4) - 2P(x_1x_3x_4) - 2P(x_2x_3x_4) + 3P(x_1x_2x_3x_4)$$

Reliability of the system with 4 different components out of which 2 or more are operational

(It is easy to check the expression; for independent, equal components with constant failure rates we obtain:

$$R(t) = 6e^{-2\lambda t} - 8e^{-3\lambda t} + 3e^{-4\lambda t}.)$$

If we introduce the equivalence $p_i = e^{-\lambda_i t}$, we get the expression:

$$R(t) = p_1p_2 + p_1p_3 + p_1p_4 + p_2p_3 + p_2p_4 +$$

$$+ p_3p_4 - 2p_1p_2p_3 - 2p_1p_2p_4 - 2p_1p_3p_4 - 2p_2p_3p_4 +$$

$$+ 3p_1p_2p_3p_4$$

(Frequent) mistake in reliability calculation

Note the (frequent) mistake in reliability calculation. Specifically, the system will be working properly for as long as at least one path of the system is correct:

$$R(t) = P\left(s_1 + s_2 + s_3 + s_4 + s_5 + s_6\right)$$

(In the considered case the system consists of 6 paths: sets, assemblies of the components which connect system inlet with the outlet; their proper work allows the system to function – to transfer energy from input to output of the system.)

(Frequent) mistake in reliability calculation

Of course, the system reliability can also be determined based on the probability of the system failure.

Specifically, the system will fail only when all the paths fail, so it is a cross section of events

which mean the failure of a particular path:

$$Q(t) = \overline{R(t)} = P\left(\overline{s_1}\,\overline{s_2}\,\overline{s_3}\,\overline{s_4}\,\overline{s_5}\,\overline{s_5}\,\overline{s_6}\right)$$

Denote $\overline{S}_i$ as the failure of i-th path

The system reliability is:

$$R(t) = 1 - Q(t) = 1 - P\left(\overline{s_1}\,\overline{s_2}\,\overline{s_3}\,\overline{s_4}\,\overline{s_5}\,\overline{s_6}\right)$$

(Frequent) mistake in reliability calculation

The path is operable if both components work correctly, i.e. the path is inoperable if the first or the second component are inoperable

Therefore:

$$P\left(\overline{s_i}\right) = 1 - P\left(s_i\right) = 1 - P(x_i x_j) = 1 - P(x_i)P(x_j) = 1 - p_i p_j \,,$$

So the reliability of the observed system can be expressed by the following formula (component independence):

$$R(t) = 1 - \left(1 - p_1 p_2\right)\left(1 - p_1 p_3\right)\left(1 - p_1 p_4\right)\left(1 - p_2 p_3\right)\left(1 - p_2 p_4\right)\left(1 - p_3 p_4\right)$$

(Frequent) mistake in reliability calculation

Up to this point, the calculation is performed correctly. Now, however, the mistake is made when trying to shorten the calculation by inserting immediately numerical values for component reliabilities (pi). **Namely, the paths are independent, therefore they are not mutually exclusive.**

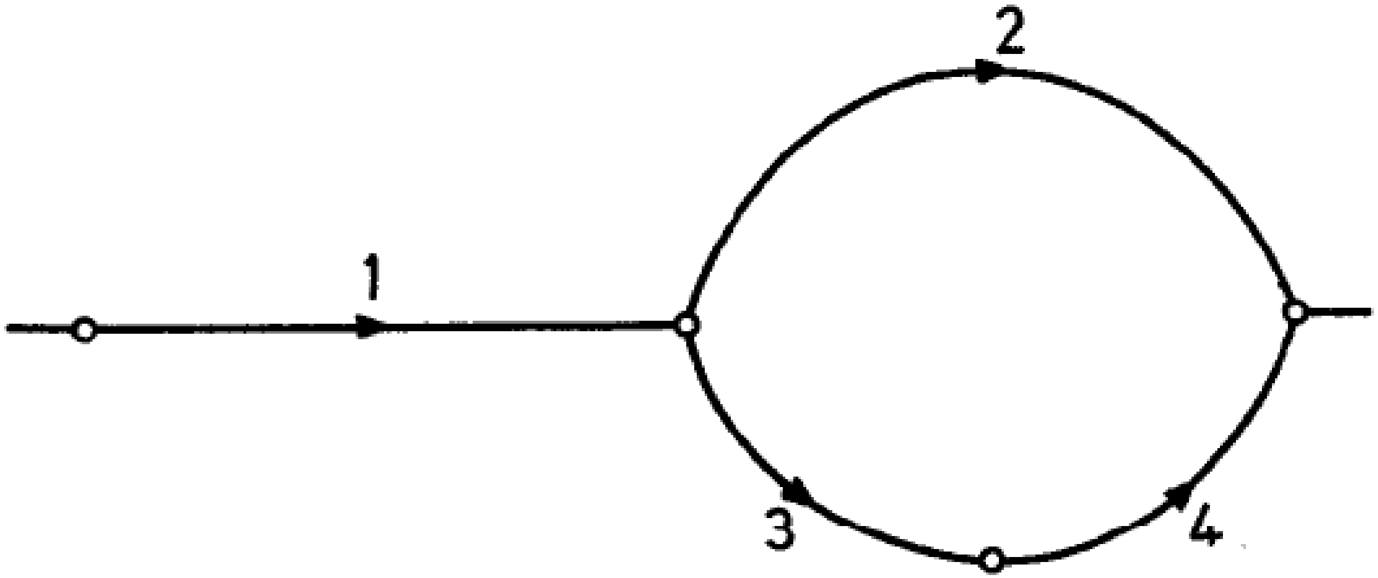
The expressions in parentheses must be multiplied but it needs to be kept in mind that the product $\mathbf{p_i \cdot p_i}$ is equal to $\mathbf{p_i}$ and not $\mathbf{p_i^2}$; it is the probability of intersection of the event with itself: $\mathbf{P(x_i x_i) = P(x_i)}$, and not $\mathbf{P(x_i) \cdot P(x_i) = [P(x_i)]^2}$.

In other words, by multiplying the parts of the expression, we get the same expression, while performing the work of the same extent.

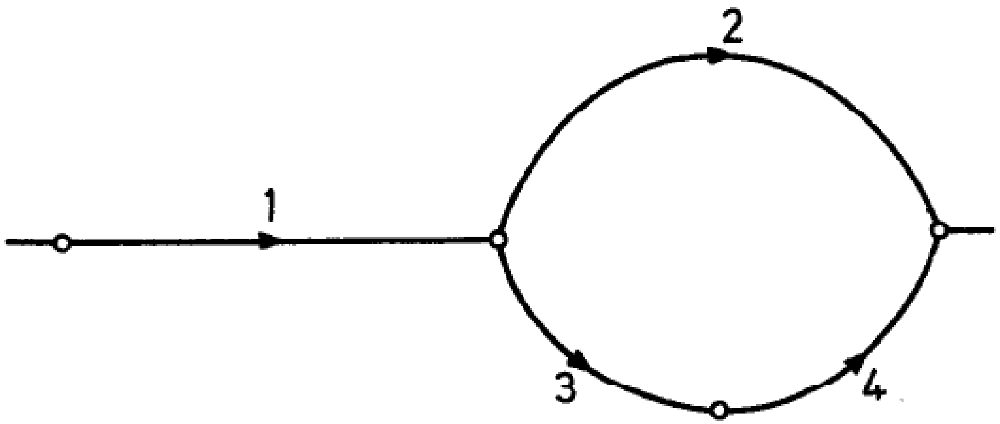
Reliability of a system with serial-parallel reliability model

- If the components of a system form a serial-parallel structure, the reliability model will be a combination of a serial and a parallel model
- consequently, the expression for reliability will be a combination of the previously derived equations
- For example, let's determine the reliability of the systems represented with these models:

Reliability of a system with serial-parallel reliability model



Reliability of a system with serial-parallel reliability model



$$R(t) = P[x_1(x_2+x_3x_4)]$$

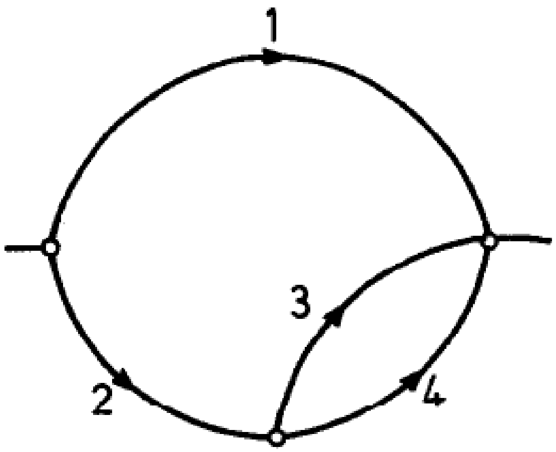
$$R\ (t) = P\ (x_1)\ P\ (x_2 + x_3x_4) = P\ (x_1) / \ P\ (x_2) +$$

$$+ P\ (x_3x_4) - P\ (x_2x_3x_4) / = P\ (x_1)\ P\ (x_2) + P(x_1)$$

$$P\ (x_3)\ P\ (x_4) - P\ (x_1)\ P\ (x_2)\ P\ (x_3)\ P\ (x_4) =$$

$$= e^{-2\lambda t} + e^{-3\lambda t} - e^{-4\lambda t},$$

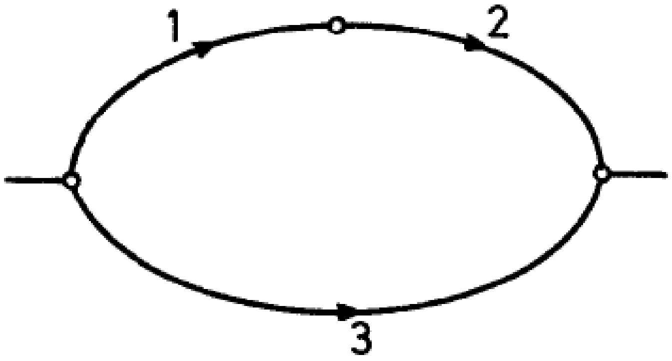
Reliability of a system with serial-parallel reliability model



$$R(t) = P[x_1 + x_2(x_{3+}x_4)]$$

$$R(t) = P(x_1 + x_2x_3 + x_2x_4) = P(x_1) + P(x_2x_3) +$$
$$+ P(x_2x_4) - P(x_1x_2x_3) - P(x_1x_2x_4) - P(x_2x_3x_4) +$$
$$+ P(x_1x_2x_3x_4) = e^{-\lambda t} + 2e^{-2\lambda t} - 3e^{-3\lambda t} + e^{-4\lambda t}$$

Reliability of a system with serial-parallel reliability model



$$R(t) = P(x_1 x_2 + x_3)$$

$$R(t) = P(x_1 x_2) + P(x_3) - P(x_1 x_2 x_3) =$$

$$= e^{-2\lambda t} + e^{-\lambda t} - e^{-3\lambda t}$$

An example of the theoretical background

Let’s assume we need to build a highly reliable transmission system. It will transmit the power of 50 MW, 40 MW, 30 MW, 20 MW and 10 MW. We will analyze three possibilities of transmission:

a) A single power line of 50 MW;

b) Two power lines of 25 MW each;

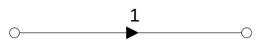
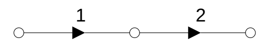
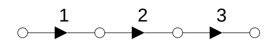
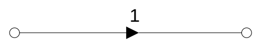
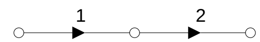
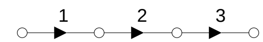
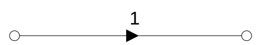
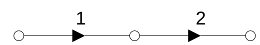
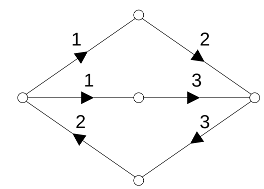
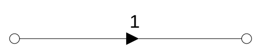
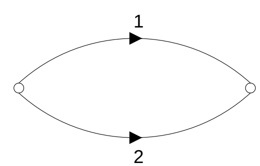
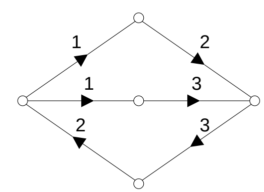
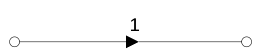
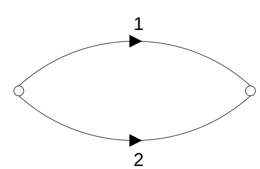
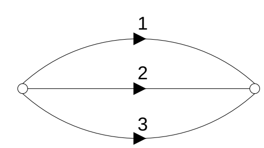
c) Three power lines of 17 MW each.

An example of the theoretical background

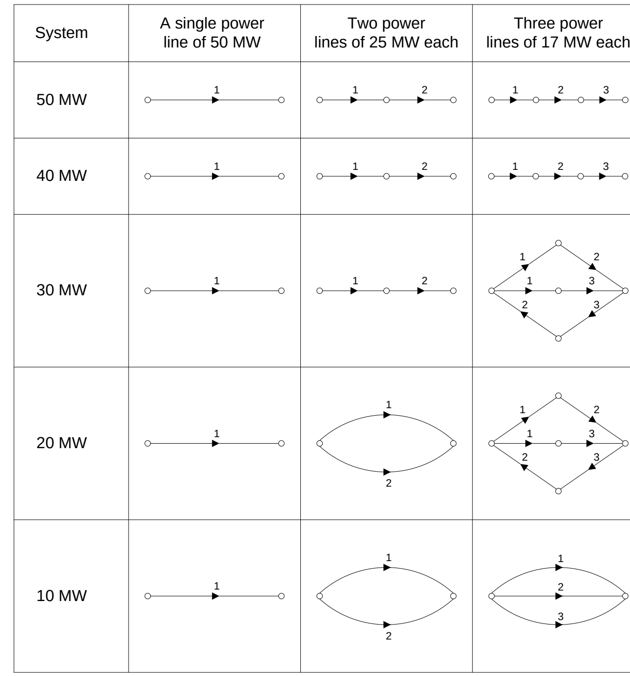
Naturally, each option has its advantages and disadvantages considering the size, price, execution, etc., which we will disregard at the moment. We will opt for the best solution based on the reliability of transmission of each power.

Since there are in fact five different system states depending on the power transmitted, we will need to determine the reliability of each of those states. Firstly, we are going to determine and illustrate the reliability models.

An example of the theoretical background

System	A single power line of 50 MW	Two power lines of 25 MW each	Three power lines of 17 MW each
50 MW			
40 MW			
30 MW			
20 MW			
10 MW			

An example of the theoretical background



System	A single power line of 50 MW	Two power lines of 25 MW each	Three power lines of 17 MW each
50 MW	$R(t)=e^{-\lambda t}$	$R(t)=e^{-2\lambda t}$	$R(t)=e^{-3\lambda t}$
40 MW	$R(t)=e^{-\lambda t}$	$R(t)=e^{-2\lambda t}$	$R(t)=e^{-3\lambda t}$
30 MW	$R(t)=e^{-\lambda t}$	$R(t)=e^{-2\lambda t}$	$R(t)=3e^{-2\lambda t}-2e^{-3\lambda t}$
20 MW	$R(t)=e^{-\lambda t}$	$R(t)=2e^{-\lambda t}-e^{-2\lambda t}$	$R(t)=3e^{-2\lambda t}-2e^{-3\lambda t}$
10 MW	$R(t)=e^{-\lambda t}$	$R(t)=2e^{-\lambda t}-e^{-2\lambda t}$	$R(t)=1-\left(1-e^{-\lambda t}\right)^3$

An example of the theoretical background

The reliability of one power line is

$$R(t) = e^{-\lambda t},$$

regardless the transmitted power.

The reliability of transmitting 40 MW and 50 MW power is, if the two lines are used,

$$R(t) = e^{-2\lambda t},$$

and in the case of the three lines,

$$R(t) = e^{-3\lambda t},$$

because we have the serial model of the component reliability.

An example of the theoretical background

The reliability of the transmission of 30 MW when two power lines are used will be

$$R(t) = e^{-2\lambda t},$$

because both power lines of 25 MW must work properly to transmit the power of 30 MW, and when three power lines are used, the reliability is:

$$R(t)=P(x_1x_2 + x_1x_3 + x_2x_3)=P(x_1x_2)+P(x_1x_3) +$$
$$+P(x_2x_3)-P(x_1x_2x_3)-P(x_1x_2x_3)-P(x_1x_2x_3)+ P(x_1x_2x_3)=$$
$$=P(x_1x_2) + P(x_1x_3) +P(x_2x_3) - 2P(x_1x_2x_3)$$

The same expression is used also for transmission of 20 MW.

An example of the theoretical background

Since we assume that the components are independent, the reliability equals:

$$R(t) = P(x_1)P(x_2) + P(x_1)P(x_3) +$$

$$+P(x_2)P(x_3) - 2P(x_1)P(x_2)P(x_3) =$$

$$= 3e^{-2\lambda t} - 2e^{-3\lambda t}$$

An example of the theoretical background

Finally, the reliability of the transmission of 10 MW will be $e^{-\lambda t}$ in case of a single power line,

$2e^{-\lambda t} - e^{-2\lambda t}$ in case of two power lines (the same expression as for 20 MW), and

$1 - (1 - e^{-\lambda t})^3$ in case of three power lines.

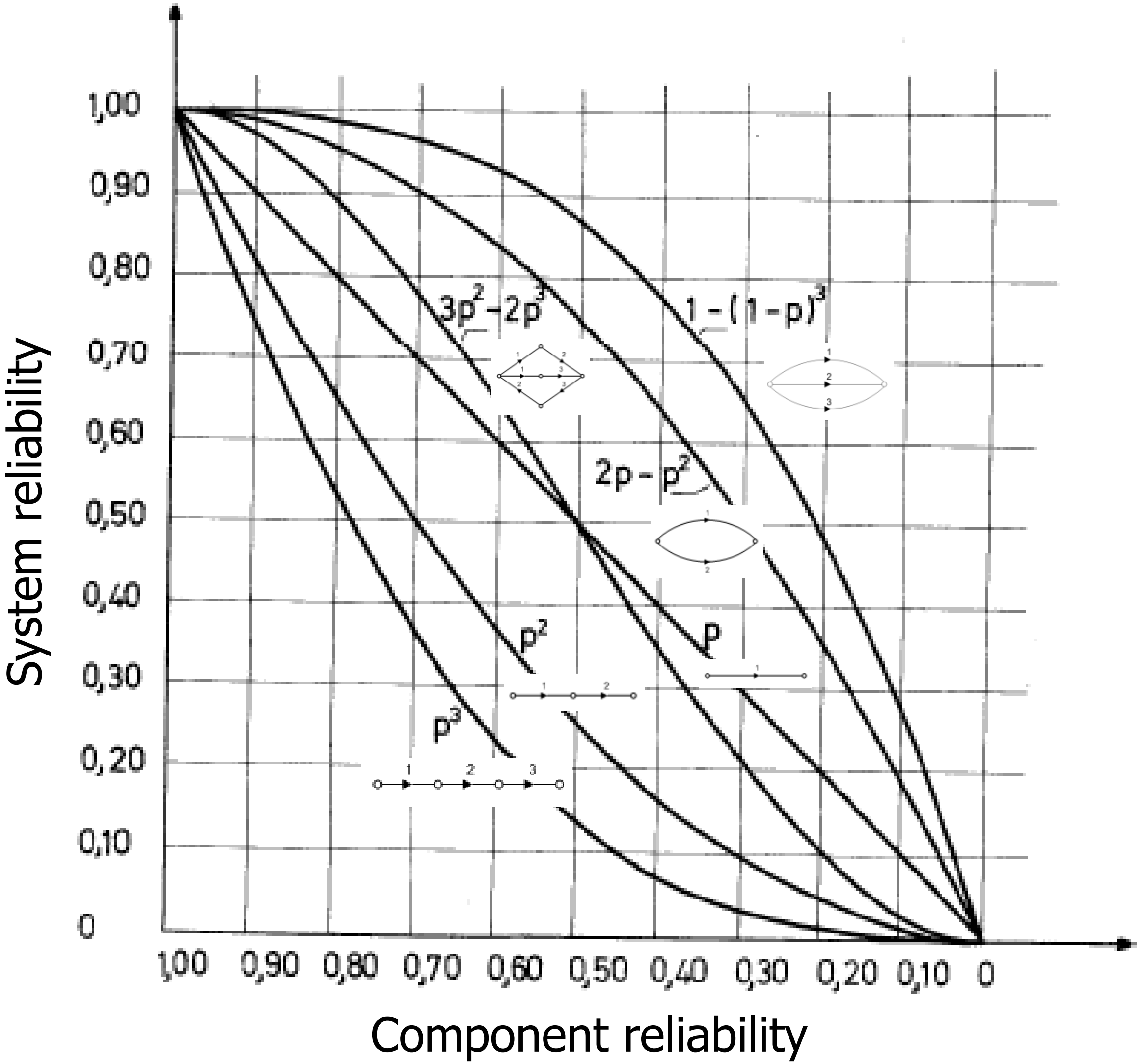
The last expression is derived based on the reliability model; all three paths must be disfunctional to prevent the transmission of 10 MW.

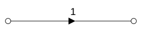
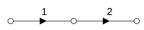
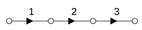
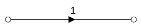
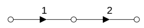
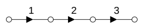
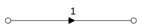
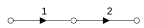
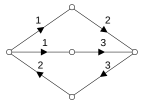
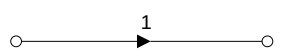
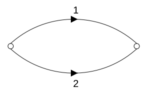
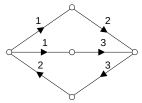
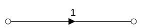
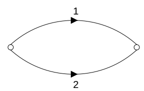
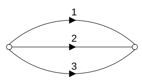
An example of the theoretical background

Now we have six different expressions for reliability. To make their comparison easier we will neglect the influence of time and replace $e^{-\lambda t}$ with p in our equations.

The results are shown in the figure, where p is the parameter.

An example of the theoretical background



System	A single power line of 50 MW	Two power lines of 25 MW each	Three power lines of 17 MW each
50 MW			
40 MW			
30 MW			
20 MW			
10 MW			

An example of the theoretical background

- The transmission reliability for 10 MW is the highest when three power lines are used because in that case the reliability model is parallel.

  (Needless to say, we should take into account that the power lines will be transmitting only 1/5 of the nominal power.)

- When transmitting 20 MW, two power lines are the most reliable option, while three power lines are better than a single power line if the lines are highly reliable, i.e. as long as $0.5 \leq p$, and worse if $0 < p < 0.5$.

An example of the theoretical background

- However, a serial configuration is always worse than a single component option and the transmission of 50 MW/40 MW will be the most reliable when a single power line is used. The final decision will therefore be influenced by the odds of transmission frequency of a particular power.
- For example, if the power of 30 MW is the most frequently transmitted, we will choose, based on the transmission reliability, solution with three power lines of 17 MW each, if the power lines reliabilities are greater than 0.5.