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Higher order energy decay for damped wave equations with variable coefficients ☆

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ABSTRACT

Under appropriate assumptions the higher order energy decay rates for the damped wave equations with variable coefficients $c(x)u_{tt} - \operatorname{div}(A(x)\nabla u) + a(x)u_t = 0$ in $\mathbb{R}^n$ are established. The results concern weighted (in time) and pointwise (in time) energy decay estimates. We also obtain weighted L^2 estimates for spatial derivatives.

1. Introduction

Consider the following dissipative hyperbolic equation:

$$c(x)u_{tt} - \operatorname{div}\bigl(A(x)\nabla u\bigr) + a(x)u_t = 0, \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

where $A(x) = (a_{ij}(x))$ are symmetric and positively definite matrices for all $x \in \mathbb{R}^n$ and $a_{ij}(x)$ are smooth functions in $\mathbb{R}^n$. The coefficients a, c are C^1 functions satisfying the assumptions listed below, and the initial data

$$u|_{t=0} = u_0, \qquad u_t|_{t=0} = u_1$$

have compact support

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$$u_0(x) = 0 \quad \text{and} \quad u_1(x) = 0 \quad \text{for } |x| > R.$$

Such a system appears in models for traveling waves in a heterogeneous medium with damping that changes with the position (given by $a(x)u_t$). The coefficient $c(x)$ accounts for variable mass density, while $A(x)$ is responsible for material.

If $A(x) = I$ is the unit matrix, and $c(x) = 1$, this problem has been studied intensively for the homogeneous medium, see [9,10] and many others. In [13], a strengthened multiplier method has been developed and used to derive the weighted energy of solutions for Eq. (1.1) with space dependent damping $a(x)u_t$. The results on this problem are scarce when $A(x) = b(x)I$ is the diagonal matrix. We first mention the work of [11] in which the authors established weighted L^2-estimates for dissipative wave equations by using the multiplier method. Higher order energy decay rates were studied in [12] for a damped wave equations with variable coefficients $b(x)$ and $a(x)$. If $A(x)$ is a general matrix, such a problem is called a wave equation with variable coefficients in principle. In the case of variable coefficients in principle, this problem has received considerable attention in the literature. See [1–8,14–16]. More prominent among them is [14], where the geometric method is first used to cope with variable coefficients in principle. It is well known that the Riemannian geometry method is a powerful tool to cope with variable coefficients in principle, subsequently this method was extended in [1–8,15–17] and many others.

Behaviors of solutions to problem (1.1) are closely related to a Riemannian metric, given by (1.2) below. This is because wave propagate along rays which are the geodesics in the metric (1.2). In the case of constant coefficients, the metric is the dot metric of the Euclidean space and rays are straight lines. However, if the system has a variable coefficient principle part like (1.1), structures of geodesics are very complicated, while Riemannian geometry has shown us how to understand structures of geodesics by the curvature theory, for example, see [15]. Motivated by [12,17], we are interested in higher order energy decay rates of solutions to (1.1) with variable coefficients in principle.

We next introduce some notations. Let

$$g = A^{-1}(x) \quad \text{for } x \in \mathbb{R}^n \tag{1.2}$$

be a Riemannian metric on $\mathbb{R}^n$ and consider the couple $(\mathbb{R}^n, g)$ as a Riemannian manifold. For each $x \in \mathbb{R}^n$, the Riemannian metric g induces the inner product and the norm on the tangent space $\mathbb{R}^n_x = \mathbb{R}^n$ by

$$\langle X, Y\rangle_g = \langle A^{-1}(x)X, Y\rangle, \qquad |X|_g^2 = \langle X, X\rangle_g, \quad X, Y \in \mathbb{R}^n, \tag{1.3}$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product of the Euclidean space $\mathbb{R}^n$. For any $w \in H^1(R^n)$, we define

$$|\nabla_g w|_g^2 = \sum_{i,j=1}^n a_{ij}(x) w_{x_i}(x) w_{x_j}(x) \quad \text{for } x \in \mathbb{R}^n, \tag{1.4}$$

where ∇_g is the gradient of the Riemannian metric g. Let $\rho(x) = d(x, 0)$ be the distance function from $x \in \mathbb{R}^n$ to the origin 0 in the Riemannian metric g, given by (1.2). Let

$$G(x) = A^{-1}(x) = \big(g_{ij}(x)\big) \quad \text{for } x \in \mathbb{R}^n. \tag{1.5}$$

Then the metric can be rewritten as

$$g = \sum_{i,j=1}^n g_{ij}(x)\, dx_i\, dx_j \quad \text{for } x \in \mathbb{R}^n. \tag{1.6}$$

2. Main results

Let a, c be C^1 functions satisfying the conditions:

$$a_0\big(1+\rho(x)\big)^{-\alpha} \leqslant a(x) \leqslant a_1\big(1+\rho(x)\big)^{-\alpha}, \tag{2.1}$$

$$c_0\big(1+\rho(x)\big)^{-\gamma} \leqslant c(x) \leqslant c_1\big(1+\rho(x)\big)^{-\gamma}, \tag{2.2}$$

where a_i, c_i, $i = 0, 1$ are positive constants and $\rho(x) = d(x, 0)$ is the distance function from $x \in \mathbb{R}^n$ to the origin 0 in the Riemannian metric g, given by (1.2). The exponents α, γ satisfy:

$$0 < \gamma < 2, \qquad 0 < 2\alpha - \gamma < 2.$$

We consider initial data

$$u(0, x) = u_0 \in H^{k+1}\big(\mathbb{R}^n\big), \qquad u_t(0, x) = u_1 \in H^k\big(\mathbb{R}^n\big), \tag{2.3}$$

where k is a nonnegative integer. These two functions u_0, u_1 have compact support:

$$u_0(x) = 0, \qquad u_1(x) = 0 \quad \text{for } |x| > R. \tag{2.4}$$

Before announcing our main results we state the finite speed of propagation result which will be used throughout the paper. This property reflects the impact of the variable coefficients $A(x)$ and $c(x)$ on the size of the support of the solution.

We assume the following:

$$\text{There exists } q \in C^1, \quad 0 < q'\big(\rho(x)\big) \leqslant \sqrt{c(x)}, \quad x \in \mathbb{R}^n, \tag{2.5}$$

where $\rho(x) = d(x, 0)$ is the distance function from $x \in \mathbb{R}^n$ to the origin 0 in the Riemannian metric g, given by (1.2).

Proposition 2.1 *(Finite speed of propagation). Let u be a solution of (1.1) with initial data $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. If (2.5) holds, then $u_0(x) = u_1(x) = 0$ for $|x| > R$ implies*

$$u(x, t) = 0 \quad \text{for } q(\rho) > q(R_0) + t, \tag{2.6}$$

where $R_0 = \sup_{|x| \leqslant R} \rho(x)$.

From the above result, we can find more explicit support of the solution u.

Corollary 2.1. *Let u be a solution of (1.1) with initial data (u_0, u_1), such that $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Then $u_0(x) = u_1(x) = 0$ for $|x| > R$ implies*

$$u(x, t) = 0 \quad \text{for } \rho(x) > R_0 + q_0 t^{\frac{2}{2-\gamma}},$$

where $q_0 = (\frac{2-\gamma}{2\sqrt{c_0}})^{\frac{2}{2-\gamma}}$.

Remark 1. As a consequence of the above result, the support of the solution u at time t satisfies

$$\operatorname{supp} u(\cdot, t) \subseteq \big\{x \in \mathbb{R}^n\colon\ \rho(x) \leqslant R_0 + q_0 t^{\frac{2}{2-\gamma}}\big\}. \tag{2.7}$$

The following cases give the relationship between the distance function $\rho(x) = d(x, 0)$ of the metric g and $|x|$ of the Euclidean metric, and by using (2.7) we can get the support of the solution u in the Euclidean metric.

Case 1. Consider a metric on $\mathbb{R}^n$, given by

$$g = b^{-1}(x)\langle\cdot,\cdot\rangle \quad \text{for } x \in \mathbb{R}^n$$

where $b(x)$ satisfies

$$b_0\bigl(1 + |x|\bigr)^{\beta} \leqslant b(x) \leqslant b_1\bigl(1 + |x|\bigr)^{\beta} \quad \text{for } x \in \mathbb{R}^n,$$

where $\beta < 2$. Then we have $|x| = O(\rho^{2/(2-\beta)})$. And the support of the solution u in the Euclidean metric is

$$\operatorname{supp} u(\cdot, t) \subseteq \bigl\{x \in \mathbb{R}^n\colon\ |x| \leqslant R + Ct^{\frac{4}{(2-\gamma)(2-\beta)}}\bigr\},$$

for some $C > 0$, which coincides with the result on the support of the solution in [12].

Case 2. If all sectional curvatures of $(\mathbb{R}^n, g)$ are non-positive, then

$$\rho(x) \geqslant c|x| \quad \text{for } x \in \mathbb{R}^n$$

and for some constant $c > 0$.

Case 3. If there are constants $m, M > 0$ such that

$$m|X|^2 \leqslant \bigl\langle A(x)X, X\bigr\rangle \leqslant M|X|^2 \quad \text{for } X \in \mathbb{R}^n_x,\ x \in \mathbb{R}^n,$$

then we have

$$d_0\rho(x) \leqslant |x| \leqslant d_1\rho(x) \quad \text{for } x \in \mathbb{R}^n.$$

In the above two cases, we can easily obtain the support of the solution u in the Euclidean metric.

Let us define the energy of the solution of problem (1.1) as

$$E(t; u) = \frac{1}{2}\int \bigl[cu_t^2 + |\nabla_g u|_g^2\bigr]\, dx,$$

where $|\nabla_g u|_g^2$ is defined by (1.4). And we can easily get that

$$\frac{d}{dt}E(t; u) + \int a u_t^2\, dx = 0. \tag{2.8}$$

Now we are in position to state our main results.

Theorem 2.1. *Let $a, c \in C^1(\mathbb{R}^n)$ be smooth coefficients which satisfy (2.1), (2.2). Let $\theta > 0$, and let k be a nonnegative integer. Then the solution u of problem (1.1) with initial data, which satisfy the condition (2.3), satisfies the following weighted energy estimate:*

$$\int_{T_0}^{T} (1+t)^{\theta+2k} E\bigl(t; \partial_t^k u\bigr)\, dt \lesssim (1+T_0)^{\nu}\left[\sum_{i=0}^{k} E\bigl(T_0; \partial_t^i u\bigr)\right] + \int_{T_0}^{T} (1+t)^{\theta} E(t; u)\, dt, \tag{2.9}$$

where $T_0 > 0$, $\nu = \theta + \omega + 2$, and ω is defined by (3.6).

The above arguments allow us to also obtain pointwise (in time) decay rates for the energy as stated in the following

Theorem 2.2. *Under the same assumptions of Theorem 2.1, with the same notation and parameters, we have*

$$E\big(T;\partial_t^k u\big) \lesssim (1+T)^{-\theta-2k-1}\left\{(1+T_0)^{\nu}\left[\sum_{i=0}^{k} E\big(T_0;\partial_t^i u\big)\right] + \int_{T_0}^{T}(1+t)^{\theta}E(t;u)\,dt\right\}, \tag{2.10}$$

for $T_0 < t < T$.

Theorem 2.3. *Let $a, c \in C^2(\mathbb{R}^n)$ be coefficients which satisfy (2.1) and (2.2). Let $Mu = a^{-1}\operatorname{div}(A\nabla u)$. Then for the solution u of (1.1) with initial conditions (2.3) and (2.4), the following weighted estimates hold:*

(i)

$$\int_{T_0}^{T}\int (1+t)^{\theta+1}a(Mu)^2\,dx\,dt \lesssim (1+T_0)^{\theta+3}\left[\sum_{i=0}^{1} E\big(T_0;\partial_t^i u\big)\right] + \int_{T_0}^{T}(1+t)^{\theta}E(t;u)\,dt, \tag{2.11}$$

(ii)

$$\int_{T_0}^{T}\int (1+t)^{\theta+3-\lambda_2}a\big(M^2u\big)^2\,dx\,dt \lesssim (1+T_0)^{\theta+3}\left[\sum_{i=0}^{3} E\big(T_0;\partial_t^i u\big)\right] + \int_{T_0}^{T}(1+t)^{\theta}E(t;u)\,dt, \tag{2.12}$$

where $\theta > 0$ and λ_1, λ_2 are given by (3.20) and (3.21).

3. Proof of main results

Let u be a solution to problem (1.1) and let $\rho(x) = d(x, 0)$ is the distance function from $x \in \mathbb{R}^n$ to the origin 0 in the Riemannian metric g, given by (1.2). The finite speed of propagation property of the wave with variable coefficients can be proved as follows.

Proof of Proposition 2.1. Define the energy

$$e(t) = \int_{q(\rho)>q(R_0)+t} \big(cu_t^2 + |\nabla_g u|_g^2\big)\,dx,$$

where $|\nabla_g u|_g^2$ is defined by (1.4). By using the Fubini theorem in the Euclidean metric we have

$$e(t) = \int_{q(R_0)+t}^{\infty}\int_{\Gamma(\tau)} \frac{cu_t^2 + |\nabla_g u|_g^2}{q'(\rho)|\nabla\rho|}\,d\Gamma(\tau)\,d\tau,$$

where $d\Gamma(\tau)$ is the area element of the Euclidean metric on the surface

$$\Gamma(\tau) = \big\{x \in \mathbb{R}^n \,\big|\, q(\rho) = \tau\big\}.$$

After a simple computation, we can get

$$e'(t) = -\int\limits_{q(\rho)=q(R_0)+t} \frac{cu_t^2 + |\nabla_g u|_g^2}{q'(\rho)|\nabla\rho|}\, d\Gamma + 2\int\limits_{q(\rho)>q(R_0)+t} \big[cu_t u_{tt} + \langle \nabla_g u, \nabla_g u_t\rangle_g\big]\, dx \triangleq I_1 + I_2. \tag{3.1}$$

Using the divergence theorem of the Euclidean metric and Eq. (1.1) we can have

$$\begin{aligned} I_2 &= 2\int\limits_{q(\rho)>q(R_0)+t} \big[cu_t u_{tt} - u_t \operatorname{div}(A\nabla u) + \operatorname{div}(u_t A\nabla u)\big]\, dx \\ &= \int\limits_{q(\rho)>q(R_0)+t} -2au_t^2\, dx + 2\int\limits_{q(\rho)=q(R_0)+t} \left\langle u_t A\nabla u, \frac{q'(\rho)\nabla\rho}{q'(\rho)|\nabla\rho|}\right\rangle d\Gamma. \end{aligned} \tag{3.2}$$

By Cauchy–Schwarz inequality and Cauchy's inequality, the second term in (3.2) can be estimated as

$$\left|2\left\langle u_t A\nabla u, \frac{\nabla\rho}{|\nabla\rho|}\right\rangle\right| = \frac{2|u_t|}{|\nabla\rho|}\big|\langle \nabla_g u, \nabla_g \rho\rangle_g\big| \leqslant \frac{cu_t^2 + |\nabla_g u|_g^2}{\sqrt{c}|\nabla\rho|}. \tag{3.3}$$

Combining (2.5), (3.2) and (3.3), we can get

$$e'(t) \leqslant -\int\limits_{q(\rho)>q(R_0)+t} 2au_t^2\, dx \leqslant 0.$$

And

$$e(0) = \int\limits_{q(\rho)>q(R_0)} \big(cu_1^2 + |\nabla_g u_0|_g^2\big)\, dx = \int\limits_{\rho>R_0} \big(cu_1^2 + |\nabla_g u_0|_g^2\big)\, dx = \int\limits_{|x|>R} \big(cu_1^2 + |\nabla_g u_0|_g^2\big)\, dx.$$

Taking into account the condition (2.4) we have $e(0) = 0$. Thus (2.6) follows from $e(0) = 0$. □

Proof of Corollary 2.1. We already know that $u(x,t) = 0$ if $q(\rho(x)) > t + q(R_0)$, where $0 < q'(\rho(x)) \leqslant \sqrt{c(x)}$. It is sufficient to find q, such that $q'(\rho) = \sqrt{c_0}(1+\rho)^{-\frac{\gamma}{2}}$, $q(0) = 0$, where the constant c_0 is given in (2.2). Solving for q, we obtain

$$q\big(\rho(x)\big) = \frac{2\sqrt{c_0}}{2-\gamma}\big[\big(1+\rho(x)\big)^{\frac{2-\gamma}{2}} - 1\big]. \tag{3.4}$$

By solving the inequality

$$q\big(\rho(x)\big) > t + q(R_0),$$

where the function q is given by (3.4). Then we have

$$u(x,t) = 0 \quad \text{if } \rho(x) > R_0 + q_0 t^{\frac{2}{2-\gamma}},$$

where $q_0 = (\frac{2-\gamma}{2\sqrt{c_0}})^{\frac{2}{2-\gamma}}$. □

Before proving our main results, we state a simple consequence.

Proposition 3.1. *Assume that* $0 \leqslant T_0 < T$ *and* $\mu > 0$. *Then*

$$\int_{T_0}^{T}\int (1+t)^{\mu} a(x) u_t^2 \, dx \, dt \leqslant (1+T_0)^{\mu} E(T_0; u) + \mu \int_{T_0}^{T} (1+t)^{\mu-1} E(t; u) \, dt. \tag{3.5}$$

Proof. Multiplying (2.8) by $(1+t)^{\mu}$ and then integrating on $[T_0, T]$, we can easily get (3.5). □

We will proceed to estimate higher order norms and energies. To simplify notations, let $v = u_t$. Our next result is

Proposition 3.2. *Assume that the functions* a *and* c *satisfy the assumptions of Theorem 2.1. Then for every* $\theta > 0$ *and* ω *such that*

$$\max\left\{0, \frac{2(\alpha-\gamma)}{2-\gamma}\right\} < \omega < 1, \tag{3.6}$$

we have:

$$\int_{T_0}^{T} (1+t)^{\theta+2} E(t; v) \, dt \leqslant (1+T_0)^{\theta+\omega+2} \big[E(T_0; v) + E(T_0; u)\big] + \int_{T_0}^{T} (1+t)^{\theta} E(t; u) \, dt. \tag{3.7}$$

Proof. Let $v = u_t$. Differentiating (1.1) with respect to t,

$$c(x) v_{tt} - \operatorname{div}\big(A(x)\nabla v\big) + a(x) v_t = 0, \quad x \in \mathbb{R}^n, \ t > 0,$$

and

$$\frac{1}{2}\frac{d}{dt}\int \big(c v_t^2 + |\nabla_g v|_g^2\big) \, dx + \int a v_t^2 \, dx = 0. \tag{3.8}$$

In order to obtain integrals of $E(t; v)$, we multiply the equation for v with $\frac{1}{2}W(t)v$, where W will be determined later. Integration on $\mathbb{R}^n$ yields

$$\frac{1}{2}\frac{d}{dt}\int \left(W c v_t v + \frac{W a - W_t c}{2} v^2\right) dx - \int W c v_t^2 \, dx + \frac{1}{2}\int \left(W c v_t^2 + W|\nabla_g v|_g^2 + \frac{W_{tt} c - W_t a}{2} v^2\right) dx = 0. \tag{3.9}$$

Adding (3.8) and (3.9), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \left(c v_t^2 + |\nabla_g v|_g^2 + W c v_t v + \frac{W a - W_t c}{2} v^2\right) dx + \int (a - W c) v_t^2 \, dx \\ &\quad + \frac{1}{2}\int \left(W c v_t^2 + W|\nabla_g v|_g^2 + \frac{W_{tt} c - W_t a}{2} v^2\right) dx = 0. \end{aligned} \tag{3.10}$$

We will choose W such that

$$\begin{aligned}
&\text{(i)} \quad W(t) \leqslant \inf_{x \in \operatorname{supp} u(\cdot,t)} \frac{a(x)}{c(x)},\\
&\text{(ii)} \quad W_{tt}c - W_t a \geqslant 0,\\
&\text{(iii)} \quad 0 \leqslant cv_t^2 + |\nabla_g v|_g^2 + Wcv_t v + \frac{Wa - W_t c}{2} v^2 \leqslant 2cv_t^2 + |\nabla_g v|_g^2 + C_0 W a v^2,\\
&\text{(iv)} \quad w_1(1+t)^{-\omega} \leqslant W(t) \leqslant w_2(1+t)^{-\omega}, \quad 0 < \omega < 1,
\end{aligned} \tag{3.11}$$

where $0 < w_1 \leqslant w_2$, C_0 is some constant.

Next multiply identity (3.10) by $(1+t)^\nu$. And integrate on $[T_0, T]$, where $\nu > 0$ is arbitrary and $T_0 > 0$. Using (3.11)(i), (ii) we obtain the following inequality:

$$\begin{aligned}
&\frac{1}{2}\int_{T_0}^{T} (1+t)^\nu \frac{d}{dt} \int \left(cv_t^2 + |\nabla_g v|_g^2 + Wcv_t v + \frac{Wa - W_t c}{2} v^2 \right) dx\, dt\\
&\quad + \frac{1}{2}\int_{T_0}^{T} \int (1+t)^\nu \left(Wcv_t^2 + W|\nabla_g v|_g^2 \right) dx\, dt \leqslant 0.
\end{aligned}$$

Integrating by parts (in time) in the first integral, and using the left inequality of (3.11)(iii) it follows that

$$\begin{aligned}
&\frac{1}{2}\int_{T_0}^{T} \int (1+t)^\nu \left(Wcv_t^2 + W|\nabla_g v|_g^2 \right) dx\, dt\\
&\quad \leqslant \frac{1}{2}(1+T_0)^\nu \left[\int \left(cv_t^2 + |\nabla_g v|_g^2 + Wcv_t v + \frac{Wa - W_t c}{2} v^2 \right) dx \right]_{t=T_0}\\
&\qquad + \frac{\nu}{2}\int_{T_0}^{T} \int (1+t)^{\nu-1} \left(cv_t^2 + |\nabla_g v|_g^2 + Wcv_t v + \frac{Wa - W_t c}{2} v^2 \right) dx\, dt.
\end{aligned}$$

Using the right inequality of (3.11)(iii) it follows that

$$\begin{aligned}
&\frac{1}{2}\int_{T_0}^{T} \int (1+t)^\nu \left(Wcv_t^2 + W|\nabla_g v|_g^2 \right) dx\, dt\\
&\quad \leqslant (1+T_0)^\nu \left[\int \left(cv_t^2 + |\nabla_g v|_g^2 + C_0 W a v^2 \right) dx \right]_{t=T_0}\\
&\qquad + \nu \int_{T_0}^{T} \int (1+t)^{\nu-1} \left(cv_t^2 + |\nabla_g v|_g^2 + C_0 W a v^2 \right) dx\, dt.
\end{aligned}$$

Choose T_0 sufficiently large, such that

$$(1+t)^\nu W(t) > 4\nu(1+t)^{\nu-1}, \quad \text{for } t > T_0. \tag{3.12}$$

So by combining similar terms we obtain

$$\int_{T_0}^{T} \int (1+t)^\nu \left(Wcv_t^2 + W|\nabla_g v|_g^2 \right) dx\, dt$$

$$\lesssim (1+T_0)^\nu \left[\int \left(cv_t^2+|\nabla_g v|_g^2+C_0 W a v^2\right) dx\right]_{t=T_0} + \nu C_0 \int_{T_0}^{T} \int (1+t)^{\nu-1} W a v^2 \, dx \, dt. \tag{3.13}$$

Since the weighted function W satisfies (3.11)(iv):

$$w_1(1+t)^{-\omega} \leqslant W(t) \leqslant w_2(1+t)^{-\omega},$$

where $0<\omega<1$ and $w_2 \geqslant w_1 > 0$. Then (3.13) becomes

$$\int_{T_0}^{T} \int (1+t)^{\nu-\omega}\left(cv_t^2+|\nabla_g v|_g^2\right) dx\, dt$$

$$\lesssim (1+T_0)^\nu \left[\int \left(cv_t^2+|\nabla_g v|_g^2+W a v^2\right) dx\right]_{t=T_0} + \int_{T_0}^{T} \int (1+t)^{\nu-\omega-1} a v^2 \, dx\, dt.$$

By using the estimate from Proposition 3.1 with $\mu=\nu-\omega-1$ and we derive the following estimate

$$\int_{T_0}^{T} (1+t)^{\nu-\omega} E(t;v)\, dt \lesssim (1+T_0)^\nu \left[E(T_0;v)+E(T_0;u)\right] + \int_{T_0}^{T} (1+t)^{\nu-\omega-2} E(t;u)\, dt. \tag{3.14}$$

Set $\theta=\nu-\omega-2$. Then the inequality (3.7) follows from (3.14). The proof of Proposition 3.2 is complete provided we show:

Existence of the weight function W. Let

$$W(t)=w_0(1+t)^{-\omega}, \tag{3.15}$$

where:

(i) $w_1 \leqslant w_0 \leqslant w_2$,
(ii) the exponent ω is chosen such that

$$\max\left\{0, \frac{2(\alpha-\gamma)}{2-\gamma}\right\} < \omega < 1.$$

This choice for the weight W satisfies all the constraints of (3.11) as we show below:

By (2.1),(2.2) it suffices to show:

$$w_0(1+t)^{-\omega} \leqslant \frac{a_0}{c_1} \inf_{\rho(x)\leqslant R_0+q_0 t^{\frac{2}{2-\gamma}}} \left(1+\rho(x)\right)^{\gamma-\alpha}. \tag{3.16}$$

For $\alpha \leqslant \gamma$, this is obvious since $(1+t)^{-\omega} \to 0$ as $t\to\infty$, while in the right hand side we have $(1+\rho)^{\gamma-\alpha} > 1$. For $\alpha>\gamma$, we need to show

$$w_0(1+t)^{-\omega} \leqslant \frac{a_0}{c_1}\left(1+R_0+q_0 t^{\frac{2}{2-\gamma}}\right)^{\gamma-\alpha},$$

but for sufficiently large times t this holds since $\omega > \frac{2(\alpha-\gamma)}{2-\gamma}$.

By (3.15), $W_{tt} \geqslant 0$, $W_t < 0$. So $W_{tt}c - W_t a \geqslant 0$.

For (3.11)(iii), since $|Wcv_t v| \leqslant cv_t^2 + \frac{W^2c}{4}v^2$. So we left to show

$$2Wa - 2W_t c - W^2 c \geqslant 0.$$

By using (2.1), (2.2) and (3.15) we only need to show that

$$2w_0 a_0 (1+\rho)^{\gamma-\alpha} + 2\omega w_0 c_0 (1+t)^{-1} - c_1 w_0^2 (1+t)^{-\omega} \geqslant 0. \tag{3.17}$$

For large t, the above inequality is true by the assumption on ω.

We follow a similar approach for the right inequality in (3.11)(iii); we need to show

$$4C_0 Wa - W^2 c - 2Wa + 2W_t c \geqslant 0. \tag{3.18}$$

In order to prove the above inequality, it is enough to show

$$(4C_0 - 2) w_0 a_0 (1+t)^{\gamma-\alpha} - 2\omega w_0 c_1 (1+t)^{-1} - w_0^2 c_1 (1+t)^{-\omega} \geqslant 0,$$

which is done exactly as above when proving (3.17). $\square$

Proof of Theorem 2.1. A simple induction argument applied to (3.14) yields the decay estimates stated in Theorem 2.1. $\square$

Proof of Theorem 2.2. Let k be a nonnegative integer. Since $\partial_t^k u$ is the solution of

$$c(x)\big(\partial_t^k u\big)_{tt} - \operatorname{div}\big(A(x)\nabla \partial_t^k u\big) + a(x)\big(\partial_t^k u\big)_t = 0, \quad x \in \mathbb{R}^n,$$

the energy of the solution $\partial_t^k u$ satisfies

$$\frac{d}{dt}E\big(t; \partial_t^k u\big) + \int a\big(\partial_t^{k+1} u\big)^2\, dx = 0. \tag{3.19}$$

We conclude from the above equality that

$$E\big(T; \partial_t^k u\big) \leqslant E\big(t; \partial_t^k u\big),$$

for $0 < t < T$. Hence

$$\int_{T_0}^{T} (1+t)^{\theta+2k} E\big(t; \partial_t^k u\big)\, dt \geqslant E\big(T; \partial_t^k u\big) \int_{T_0}^{T} (1+t)^{\theta+2k}\, dt.$$

Simplifying the above inequality and using (2.9) we obtain the desired estimate. $\square$

Proposition 3.3. *Under the assumptions of Theorem 2.1, we have*

$$\int a u_t^2\, dx \lesssim \big[E(t;u)E(t;u_t)\big]^{\frac{1}{2}}.$$

Proof. By Eq. (1.1) we have

$$au_t^2 = -cu_t u_{tt} + u_t \operatorname{div}(A\nabla u),$$

then by the divergence formula and the Cauchy inequality,

$$\begin{aligned}\int au_t^2\,dx &= -\int cu_t u_{tt}\,dx - \int \langle \nabla_g u, \nabla_g u_t\rangle_g\,dx \\ &\leqslant \left(\int cu_t^2\,dx\right)^{\frac{1}{2}}\left(\int cu_{tt}^2\,dx\right)^{\frac{1}{2}} + \left(\int |\nabla_g u|_g^2\,dx\right)^{\frac{1}{2}}\left(\int |\nabla_g u_t|_g^2\,dx\right)^{\frac{1}{2}}.\end{aligned}$$

From the above inequality we completes the proof of Proposition 3.3. □

Let $Mu = a^{-1}\operatorname{div}(A\nabla u)$. Given $\lambda_1, \lambda_2 \in [0,1]$, and the coefficients satisfy

$$\sup_{x\in \operatorname{supp} u(\cdot,t)}\left[\frac{c(x)}{a(x)} + \frac{1}{a(x)}\left|\nabla_g \frac{c(x)}{a(x)}\right|_g^2\right] \lesssim (1+t)^{\lambda_1}, \tag{3.20}$$

$$\sup_{x\in \operatorname{supp} u(\cdot,t)}\left[\frac{1}{a(x)}\operatorname{div}\left(A(x)\nabla\frac{c(x)}{a(x)}\right)\right]^2 \lesssim (1+t)^{\lambda_2}. \tag{3.21}$$

We will test the feasibility of the above two conditions in Appendix A.

Proposition 3.4. *Let $a, c \in C^2(\mathbb{R}^n)$ be coefficients which satisfy (2.1) and (2.2). Let $Mu = a^{-1}\operatorname{div}(A\nabla u)$ and assume that u is a solution of (1.1).*

(i) *If (3.20) holds, then*

$$a(Mu)^2 \lesssim (1+t)^{\lambda_1} cu_{tt}^2 + au_t^2.$$

(ii) *If (3.20) and (3.21) hold, then*

$$a\left(M^2u\right)^2 \lesssim (1+t)^{3\lambda_1} c\left(\partial_t^4 u\right)^2 + (1+t)^{\lambda_1}\left[c\left(\partial_t^3 u\right)^2 + |\nabla_g u_{tt}|_g^2\right] + (1+t)^{\lambda_2} au_{tt}^2.$$

Proof. (i) From Eq. (1.1) we have that $Mu = \frac{c}{a}u_{tt} + u_t$, and

$$a(Mu)^2 \leqslant \frac{2c}{a} cu_{tt}^2 + 2au_t^2,$$

then claim (i) follows from (3.20).

(ii) Using the definition of Mu, and

$$M(uv) = uMv + vMu + 2a^{-1}\langle A\nabla u, \nabla v\rangle,$$

we have the chain of identities

$$\begin{aligned}M^2 u &= M\left(\frac{c}{a}u_{tt} + u_t\right) \\ &= u_{tt}M\frac{c}{a} + \frac{c}{a}Mu_{tt} + 2a^{-1}\left\langle A\nabla\frac{c}{a}, \nabla u_{tt}\right\rangle + Mu_t \\ &= \frac{c^2}{a^2}\partial_t^4 u + \frac{2c}{a}\partial_t^3 u + \left(M\frac{c}{a} + 1\right)u_{tt} + 2a^{-1}\left\langle A\nabla\frac{c}{a}, \nabla u_{tt}\right\rangle.\end{aligned}$$

Then

$$a\big(M^2u\big)^2 \leqslant 4a\left[\frac{c^4}{a^4}\big(\partial_t^4 u\big)^2 + \frac{4c^2}{a^2}\big(\partial_t^3 u\big)^2 + \left(M\frac{c}{a}+1\right)^2 u_{tt}^2 + \frac{4}{a^2}\left|\nabla_g \frac{c}{a}\right|_g^2 |\nabla_g u_{tt}|_g^2\right]$$
$$\lesssim \frac{c^3}{a^3}c\big(\partial_t^4 u\big)^2 + \frac{c}{a}c\big(\partial_t^3 u\big)^2 + \left(M\frac{c}{a}+1\right)^2 a u_{tt}^2 + \frac{1}{a}\left|\nabla_g \frac{c}{a}\right|_g^2 |\nabla_g u_{tt}|_g^2.$$

We deduce from (3.20) and (3.21) that

$$a\big(M^2u\big)^2 \lesssim (1+t)^{3\lambda_1} c\big(\partial_t^4 u\big)^2 + (1+t)^{\lambda_1}\big[c\big(\partial_t^3 u\big)^2 + |\nabla_g u_{tt}|_g^2\big] + (1+t)^{\lambda_2} a u_{tt}^2. \qquad \square$$

Proof of Theorem 2.3. (i) Recall the estimate of u in Proposition 3.1

$$\int_{T_0}^{T}\int (1+t)^{\theta+1} a(x) u_t^2\,dx\,dt \lesssim (1+T_0)^{\theta+1} E(T_0;u) + \int_{T_0}^{T} (1+t)^{\theta} E(t;u)\,dt.$$

And using the estimate in Theorem 2.1 with $k=1$:

$$\int_{T_0}^{T} (1+t)^{\theta+2} E(t;u_t)\,dt \lesssim (1+T_0)^{\nu}\left[\sum_{i=0}^{1} E\big(T_0;\partial_t^i u\big)\right] + \int_{T_0}^{T} (1+t)^{\theta} E(t;u)\,dt,$$

where $T_0>0$ and $\nu=\theta+\omega+2$. Since $\lambda_1 \leqslant 1$, the claim follows from Proposition 3.4(i).

(ii) Using the equality (3.19) with $k=1$, we have

$$\frac{d}{dt}E(t;u_t) + \int a u_{tt}^2\,dx = 0. \tag{3.22}$$

Multiplying (3.22) by $(1+t)^{\theta+3}$ and then integrating on $[T_0,T]$, we can easily get

$$\int_{T_0}^{T}\int (1+t)^{\theta+3} a(x) u_{tt}^2\,dx\,dt \lesssim (1+T_0)^{\theta+3} E(T_0;u_t) + \int_{T_0}^{T} (1+t)^{\theta+2} E(t;u_t)\,dt. \tag{3.23}$$

Using the estimate in Theorem 2.1 with $k=2,3$ respectively:

$$\int_{T_0}^{T} (1+t)^{\theta+4} E(t;u_{tt})\,dt \lesssim (1+T_0)^{\nu}\left[\sum_{i=0}^{2} E\big(T_0;\partial_t^i u\big)\right] + \int_{T_0}^{T} (1+t)^{\theta} E(t;u)\,dt, \tag{3.24}$$

$$\int_{T_0}^{T} (1+t)^{\theta+6} E\big(t;\partial_t^3 u\big)dt \lesssim (1+T_0)^{\nu}\left[\sum_{i=0}^{3} E\big(T_0;\partial_t^i u\big)\right] + \int_{T_0}^{T} (1+t)^{\theta} E(t;u)\,dt. \tag{3.25}$$

Adding (3.23), (3.24) and (3.25), and recalling the estimate in Proposition 3.4(ii), we complete the proof of Theorem 2.3. $\square$

4. Application in the case $c = 1$

In this section we will derive explicit decay estimates for (1.1) when $c(x) \equiv 1$, i.e. for

$$u_{tt} - \operatorname{div}\bigl(A(x)\nabla u\bigr) + a(x)u_t = 0, \quad x \in \mathbb{R}^n,\ t > 0. \tag{4.1}$$

To state these results we introduce the following:

Assumption (H). Let a satisfy the growth condition listed in (2.1). Then there exists a subsolution $w(x)$ to the problem

$$\operatorname{div}\bigl(A(x)\nabla w(x)\bigr) \geqslant a(x) \quad \text{for } x \in \mathbb{R}^n, \tag{4.2}$$

which satisfies the following properties:

$$\begin{aligned}
&\text{(p1)} \qquad w(x) \geqslant 0 \quad \text{for all } x \in \mathbb{R}^n;\\
&\text{(p2)} \qquad w(x) \leqslant c\rho^{2-\alpha} \quad \text{for } \rho = \rho(x),\ x \in \mathbb{R}^n, \text{ and some } c > 0;\\
&\text{(p3)} \qquad \mu := \liminf_{x\to\infty} \frac{a(x)w(x)}{|\nabla_g w(x)|_g^2} > 0,
\end{aligned}$$

where the definition of $|\nabla_g w(x)|_g^2$ is given by (1.4).

Theorem 4.1. *(See [17].) Let Assumption (H) holds and let $a(x)$ satisfies (2.1). Then for every $\delta > 0$ the solution of the problem (4.1) satisfy*

$$\int e^{(\mu-\delta)\frac{w(x)}{t}} a(x)u^2\,dx \leqslant C_\delta\bigl(\|\nabla_g u_0\|^2 + \|u_1\|_2^2\bigr)t^{2\delta-\mu},$$

$$\int e^{(\mu-\delta)\frac{w(x)}{t}} \bigl(u_t^2 + |\nabla_g u|_g^2\bigr)\,dx \leqslant C_\delta\bigl(\|\nabla_g u_0\|^2 + \|u_1\|_2^2\bigr)t^{2\delta-\mu-\alpha},$$

for all $t \geqslant T_0$. Here $w(x)$ and μ are given in Assumption (H).

An immediate consequence of the above theorem which follows (p1) is

Corollary 4.1. *Under the assumptions of Theorem 4.1, we have*

$$\int a(x)u^2\,dx \leqslant C_\delta\bigl(\|\nabla_g u_0\|^2 + \|u_1\|_2^2\bigr)t^{2\delta-\mu}, \tag{4.3}$$

$$E(t;u) = \int \bigl(u_t^2 + |\nabla_g u|_g^2\bigr)\,dx \leqslant C_\delta\bigl(\|\nabla_g u_0\|^2 + \|u_1\|_2^2\bigr)t^{2\delta-\mu-\alpha}. \tag{4.4}$$

Theorem 4.2. *Let a satisfies (2.1). Then for every $\delta > 0$ and $T_0 > 0$ sufficiently large, we have the following estimates:*

$$\int_{T_0}^{T} (1+t)^{\theta+2k} E\bigl(t;\partial_t^k u\bigr)\,dt \lesssim (1+T_0)^{\nu}\left[\sum_{i=0}^{k} E\bigl(T_0;\partial_t^i u\bigr)\right] + T^{1+\theta+2\delta-\mu-\alpha}, \tag{4.5}$$

$$E\bigl(T;\partial_t^k u\bigr) \lesssim (1+T)^{-\theta-2k-1}\left\{(1+T_0)^{\nu}\left[\sum_{i=0}^{k} E\bigl(T_0;\partial_t^i u\bigr)\right] + T^{1+\theta+2\delta-\mu-\alpha}\right\}, \tag{4.6}$$

$$\int a u_t^2\,dx \lesssim \left[E(t;u)E(t;u_t)\right]^{\frac{1}{2}} \lesssim (1+T)^{-\theta-2}\left\{(1+T_0)^{\nu}\left[\sum_{i=0}^{k} E\left(T_0;\partial_t^i u\right)\right] + T^{1+\theta+2\delta-\mu-\alpha}\right\}, \tag{4.7}$$

where $\nu = \theta+\omega+2$. *And*

$$\int_{T_0}^{T}\int (1+t)^{\theta+1} a(Mu)^2\,dx\,dt \lesssim (1+T_0)^{\theta+3}\left[\sum_{i=0}^{1} E\left(T_0;\partial_t^i u\right)\right] + T^{1+\theta+2\delta-\mu-\alpha}. \tag{4.8}$$

$$\int_{T_0}^{T}\int (1+t)^{\theta+3-\lambda_2} a\left(M^2 u\right)^2\,dx\,dt \lesssim (1+T_0)^{\theta+3}\left[\sum_{i=0}^{3} E\left(T_0;\partial_t^i u\right)\right] + T^{1+\theta+2\delta-\mu-\alpha}. \tag{4.9}$$

Remark. We deduce from (4.4) and (4.6) that the k-th order energy has a polynomial decay of order $\mu+\alpha+2k-\delta$, and the first order energy has a decay order $\mu+\alpha-\delta$. The estimate (4.3) indicates that the L^2 norm of u has a decay rate $\mu-2\delta$, while the damping term has the decay rate $1+\alpha+\mu-2\delta$.

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Appendix A. Feasibility of conditions (3.20), (3.21)

With two metrics on $\mathbb{R}^n$ in mind, one the Euclidean metric and the other the Riemannian metric g, we have to deal with various notations carefully. Let

$$\begin{aligned} a(x) &= a\left(1+\rho(x)\right)^{-\alpha}, \quad a_0 \leqslant a \leqslant a_1, \\ c(x) &= c\left(1+\rho(x)\right)^{-\gamma}, \quad c_0 \leqslant c \leqslant c_1, \end{aligned} \tag{A.1}$$

where $a_i, c_i, i = 0, 1$ are given in (2.1) and (2.2), and the exponents α, γ satisfy:

$$0 < \gamma < 2, \qquad 0 < 2\alpha - \gamma < 2. \tag{A.2}$$

By a simple computation we have that

$$\frac{c(x)}{a(x)} + \frac{1}{a(x)}\left|\nabla_g \frac{c(x)}{a(x)}\right|_g^2 = \frac{c}{a}\left(1+\rho(x)\right)^{\alpha-\gamma} + \frac{c^2}{a^3}(\alpha-\gamma)^2\left(1+\rho(x)\right)^{3\alpha-2\gamma-2}. \tag{A.3}$$

For the exponents $\alpha-\gamma$ and $3\alpha-2\gamma-2$, we can get that $3\alpha-2\gamma-2 < \alpha-\gamma$ by (A.2).

We next divide into three cases to discuss the feasibility of condition (3.20).

Case i. $3\alpha-2\gamma-2 > 0$. In this case (A.3) can be estimated as follows:

$$\frac{c(x)}{a(x)} + \frac{1}{a(x)}\left|\nabla_g \frac{c(x)}{a(x)}\right|_g^2 \lesssim \left(1+R_0+q_0 t^{\frac{2}{2-\gamma}}\right)^{\alpha-\gamma} \lesssim (1+R_0)^{\alpha-\gamma} + q_0^{\alpha-\gamma} t^{\frac{2(\alpha-\gamma)}{2-\gamma}},$$

for $x \in \operatorname{supp} u(\cdot,t) \subseteq \{x \in \mathbb{R}^n:\ \rho(x) \leqslant R_0 + q_0 t^{\frac{2}{2-\gamma}}\}$. By (A.2), we can easily get

$$0 < \frac{2(\alpha-\gamma)}{2-\gamma} < 1.$$

Set $\lambda_1 = \frac{2(\alpha-\gamma)}{2-\gamma}$, then yields condition (3.20).

Case ii. $\alpha - \gamma < 0$.

In this case, condition (3.20) can be satisfied with any $\lambda_1 \geqslant 0$.

Case iii. $3\alpha - 2\gamma - 2 < 0 < \alpha - \gamma$. Set $\lambda_1 = \frac{2(\alpha-\gamma)}{2-\gamma}$, the condition (3.20) is automatically satisfied.

As for condition (3.21), using the divergence theorem of the Euclidean metric, and after a simple computation, we can have

$$\frac{1}{a(x)} \operatorname{div}\left(A(x)\nabla \frac{c(x)}{a(x)}\right) = \frac{c}{a^2}(\alpha-\gamma)\big(1+\rho(x)\big)^{2\alpha-\gamma-2}\big[\big(1+\rho(x)\big)\operatorname{div}\nabla_g\rho(x) + \alpha - \gamma - 1\big],$$

where ∇_g is the gradient of the Riemannian metric g. In [8] we have listed several examples in which $\operatorname{div}\nabla_g\rho(x)$ has been computed as $\operatorname{div}\nabla_g\rho(x) = \frac{n-1}{\rho(x)}$. Then by (A.2), we can get that

$$\left[\frac{1}{a(x)} \operatorname{div}\left(A(x)\nabla \frac{c(x)}{a(x)}\right)\right]^2 \lesssim \big(1+\rho(x)\big)^{4\alpha-2\gamma-4} \lesssim (1+t)^{\lambda_2},$$

for any $\lambda_2 \geqslant 0$. $\square$

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