Assignment 2

(1) (A) Generate an O(h2) (second-order) approximation to the first derivative of some function f(x) using 3 points, X_0, X_1 and X_2 using the polynomial interpolation approach. Give the finite difference formulas for each of the points X_0, X_1, and X_2. Assume they are equally spaced, with spacing h.

N = 2

$$p(X) = \sum_{k=0}^{N} Y_k L_k(X)$$

$$L_k(X) = \frac{\prod_{i=1, j\neq k}^{N} (X - X_j)}{\prod_{i=1, j\neq k}^{N} (X_k - X_j)}$$

k = 0

$$\frac{d}{dx}\frac{(X-X_1)(X-X_2)}{(X_0-X_1)(X_0-X_2)} = \frac{d}{dx}\frac{X^2 - XX_1 - XX_2 + X_1X_2}{(X_0-X_1)(X_0-X_2)} = \frac{2X - X_1 - X_2}{(X_0-X_1)(X_0-X_2)} = \frac{2X-X_1-X_2}{2h^2}$$

$$\frac{(Y_0)(2X-X_1-X_2)}{2h^2}$$

k = 1

$$\frac{d}{dx}\frac{(X-X_0)(X-X_2)}{(X_1-X_0)(X_1-X_2)} = \frac{d}{dx}\frac{X^2 - XX_0 - XX_2 + X_0X_2}{(X_1-X_0)(X_1-X_2)} = \frac{2X - X_0 - X_2}{(X_1-X_0)(X_1-X_2)} = \frac{2X-X_0-X_2}{-h^2}$$

$$\frac{(Y1)(2X-X_0-X_2)}{-h^2}$$

k = 2

$$\frac{d}{dx}\frac{(X-X_0)(X-X_1)}{(X_2-X_0)(X_2-X_1)} = \frac{d}{dx}\frac{X^2 - XX_0 - XX_1 + X_0X_1}{(X_2-X_0)(X_2-X_1)} = \frac{2X - X_0 - X_1}{(X_2-X_0)(X_2-X_1)} = \frac{2X-X_0-X_1}{2h^2}$$

$$\frac{(Y2)(2X-X_0-X_1)}{2h^2}$$

Now Combine

$$\frac{(Y_0)(2X-X_1-X_2)}{2h^2} + \frac{(Y1)(2X-X_0-X_2)}{-h^2} + \frac{(Y2)(2X-X_0-X_1)}{2h^2}$$

X = X_0

$$\frac{Y_0}{(X_0-X_1)} + \frac{Y_0}{(X_0-X_2)} - \frac{(Y_1*(X_0-X_2))}{((X_0-X_1)(X_1-X_2))} + \frac{(Y_2*(X_0-X_1))}{((X_0-X_2)(X_1-X_2))}$$

$$\frac{Y_0}{(-h)} + \frac{Y_0}{(-2h)} - \frac{(Y_1*(-2h))}{((-h)(-h))} + \frac{(Y_2*(-h))}{((-2h)(-h))}$$

$$\frac{2Y_1}{h} - \frac{3Y_0}{2h} - \frac{Y_2}{2h}$$

X = X_1

$$\frac{Y_1}{(X_1-X_2)} - \frac{Y_1}{(X_0-X_1)} + \frac{(Y_0*(X_1-X_2))}{((X_0-X_1)*(X_0-X_2))} - \frac{(Y_2*(X_0-X_1))}{((X_0-X_2)*(X_1-X_2))}$$

$$\frac{Y_1}{(-h)} - \frac{Y_1}{(-h)} + \frac{(Y_0*(-h))}{((-h)*(-2h))} - \frac{(Y_2*(-h))}{((-2h)*(-h))}$$

$$\frac{y2}{(2*h)} - \frac{Y_0}{(2*h)}$$

X = X_2

$$\frac{(Y_1*(X_0-X_2))}{((X_0-X_1)*(X_1-X_2))} - \frac{Y_2}{(X_1-X_2)} - \frac{(Y_0*(X_1-X_2))}{((X_0-X_1)*(X_0-X_2))} - \frac{Y_2}{(X_0-X_2)}$$

$$\frac{(Y_1*(-2h))}{((-h)*(-h))} - \frac{Y_2}{(-h)} - \frac{(Y_0*(-h))}{((-h)*(-2h))} - \frac{Y_2}{(-2h)}$$

$$\frac{Y_0}{(2*h)} - \frac{(2*Y_1)}{h} + \frac{(3*Y_2)}{(2*h)}$$

(1) (B) Using the error term for polynomial interpolation, show that the error above is O(h^2)

$$f(X) - p(X) = \frac{f^{N+1}(\varepsilon)}{(N+1)!}\prod_{k=0}^{N}(X - X_k)$$

The left hand side is a constant so it can be ignored

$$= \frac{d}{dx}((X - X_0)(X - X_1)(X - X_2))$$

$$= \frac{d}{dx}((X^2 - XX_1 - XX_0 + X_0X_1)(X - X_2))$$

$$= \frac{d}{dx}(X^3 - X^2X_2 - X^2X_1 + XX_1X_2 - X^2X_0 + XX_0X_2 + XX_0X_1 - X_0X_1X_2)$$

$$= 3X^2 - 2XX_2 - 2XX_1 + X_1X_2 - 2XX_0 + X_0X_2 + X_0X_1$$

X = X_0
$3X_0^2 - 2X_0X_2 - 2X_0X_1 + X_1X_2 - 2X_0X_0 + X_0X_2 + X_0X_1$
$3X_0^2 - 2X_0X_1 - 2X_0X_1 + X_1X_2 - 2X_0^2 + X_0X_2 + X_0X_1$
$X_0^2 - 3X_0X_1 + X_0X_2 + X_1X_2$
$(X_0 - X_1) * (X_0 - X_2)$
$(-h) * (-2h)$
$2h^2 = O(h^2)$

X = X_1
$3X_1^2 - 2X_1X_2 - 2X_1X_1 - X_1X_2 - X_1X_0 + X_0X_2 + X_0X_1$
$3X_1^2 - 2X_1X_2 - 2X_1^2 - X_1X_2 - X_1X_0 + X_0X_2 + X_0X_1$
$2X_1^2 + 3X_1X_2 + X_0X_2$
$-(X_0 - X_1) * (X_1 - X_2)$
$-(-h) * (-h)$
$-h^2 = O(h^2)$

X = X_2

$3X_2^2 - 2X_2X_2 - 2X_2X_1 + X_1X_2 - X_2X_0 + X_0X_2 + X_0X_1$
$X_2^2 - 2X_2X_1 + X_1X_2 - X_2X_0 + X_0X_2 + X_0X_1$
$X_2^2 - X_1X_2 + X_0X_1$
$(x0 - x2) * (x1 - x2)$
$(-2h) * (-h)$
$2h^2 = O(h^2)$

(1) (C) Write Matlab code that uses your approximation to compute the derivative of any function f. The code should be a function called FirstDer(). FirstDer takes in two inputs: the first is the set of all points (X) at which we want the derivative, and the second is the set of data/function values at those points (Y = $f|_X$ = f(xk), k = 0, . . . , N). So, you have FirstDer(X,Y). The output is the list of values of the first derivative at each of the points computed by your formula from part a. The function f should be defined as an anonymous function in a file called DriverDer, which calls FirstDer() with the appropriate inputs, collects the outputs, computes errors, and plots them. Assume the points in X are sorted in ascending order. HINT: Loop over x, grab 3 points at a time, apply your formula from part a. Pay careful attention to the first and last points in your list.

Check the Matlab files

(1) (D) Plot the relative error of the approximation to the second derivative as a function of increasing number of points. Does this match your derivation from part b?

Check the Matlab files
Yes it does, as you can see from the graph, it goes down as you would expect in the loglog plot

(2) (A) Generate an approximation to $\int_a^b$f(x)dx using a quadratic polynomial interpolant to f. Assume a you have 3 equispaced points, including the end points a and b; that is, let x_0 = a, x_1 = (a + b)/2 and x_2 = b be the 3 points

$$\int_a^b \frac{Y_0(X-X_1)(X-X_2)}{(X_0-X_1)(X_0-X_2)}\mathrm{dx} + \int_a^b \frac{Y_1(X-X_0)(X-X_2)}{(X_1-X_0)(X_1-X_2)}\mathrm{dx} + \int_a^b \frac{Y_2(X-X_0)(X-X_1)}{(X_2-X_0)(X_2-X_1)}\mathrm{dx}$$

$$\left.\frac{\frac{x^3}{3}+\frac{x^2X_1}{2}+\frac{X^2X_2}{2}+XX_1X_2}{(X_0-X_1)(X_0-X_2)} + \frac{\frac{x^3}{3}+\frac{x^2X_0}{2}+\frac{X^2X_2}{2}+XX_0X_2}{(X_1-X_0)(X_1-X_2)} + \frac{\frac{x^3}{3}+\frac{x^2X_0}{2}+\frac{X^2X_1}{2}+XX_0X_1}{(X_2-X_0)(X_2-X_1)}\right|_{a=X_0}^{b=X_2}$$

$$\frac{(X*(3XX_0^2Y_1-3XX_1^2Y_0-2X^2X_0Y_1+2X^2X_1y_0-3XX_0^2Y_2+3XX_2^2Y_0+2X^2X_0Y_2-2X^2X_2Y_0+3XX_1^2Y_2-3XX_2^2Y_1))}{(6(X_0-X_1)(X_0-X_2)(X_1-X_2))}+$$
$$\left.\frac{(X*(-6X_0X_1^2Y_2+6X_0X_2^2Y_1-6X_1X_2^2Y_0-2X^2X_1Y_2+2X^2X_2Y_1+6X_0^2X_1Y_2-6X_0^2X_2Y_1+6X_1^2X_2Y_0))}{(6(X_0-X_1)(X_0-X_2)(X_1-X_2))}\right|_{a=X_0}^{b=X_2}$$

Answer: Had to break it into two fraction to make it fix in the paper
Same as above

$$\frac{((X_0-X_2)(3X_1^2Y_0-X_0^2Y_1+X_0^2Y_2+X_2^2Y_0+3X_1^2Y_2-X_2^2Y_1-2X_0X_1Y_0+2X_0X_2Y_0-4X_0X_1Y_2))}{(6*(X_0-X_1)(X_1-X_2))}+ \frac{(X_0-X_2)(2X_0X_2Y_1-4X_1X_2Y_0+2X_0X_2Y_2-2*X_1X_2Y_2)}{(6*(X_0-X_1)(X_1-X_2))}$$

$$\frac{((a-b)(3((a+b)/2)^2Y_0-a^2Y_1+a^2Y_2+b^2Y_0+3((a+b)/2)^2Y_2-b^2Y_1-2a((a+b)/2)Y_0+2abY_0-4a((a+b)/2)Y_2))}{(6*(a-((a+b)/2))(((a+b)/2)-b))}+ \frac{(a-b)(2abY_1-4((a+b)/2)bY_0+2abY_2-2*((a+b)/2)bY_2)}{(6*(a-((a+b)/2))(((a+b)/2)-b))}$$

Answer in a, b, and y form

$$-\frac{((a-b)*(Y_0+4*Y_1+Y_2))}{6}$$

Answer in h and y form

$$-\frac{((X_0-X_2)*(Y_0+4*Y_1+Y_2))}{6} = -\frac{((-2h)*(Y_0+4*Y_1+Y_2))}{6}$$

(2) (B) Derive the corresponding error term for this quadrature rule

$C = \frac{M_2}{6}$

$C * \int_a^b (X - X_0)(X - X_1)(X - X_2)$

$C * \int_a^b (X^2 - XX_1 - XX_0 + X_0X_1)(X - X_2)$

$C * \int_a^b X^3 - X^2X_1 - X^2X_0 + XX_0X_1 - X^2X_2 + XX_1X_2 + XX_0X_2 - X_0X_1X_2$

$C * \frac{X^4}{4} - \frac{X^3X_1}{3} - \frac{X^3X_0}{3} + \frac{X^2X_0X_1}{2} - \frac{X^3X_2}{3} + \frac{X^2X_1X_2}{2} + \frac{X^2X_0X_2}{2} - XX_0X_1X_2 \Big|_{a=X_0}^{b=X_2}$

$C * \frac{X^4}{4} + ((-\frac{X_0}{3} - \frac{X_1}{3} - \frac{X_2}{3}) * X^3) + ((\frac{(X_0X_1)}{2} \frac{(X_0X_2)}{2} + \frac{(X_1X_2))}{2}) * X^2) - XX_0X_1X_2 \Big|_{a=X_0}^{b=X_2}$

$C * ((X_0 - X_2)^3 * (X_0 - 2X_1 + X_2))/12$

$C * ((-2 * h)^3 * (X_0 - 2 * (X_0 + h) + (X_1 + h)))/12$

$C * (2 * h^3 * (h + X_0 - X_1))/3$

$C * (2 * h^3 * (h + -h))/3$

$C * (2 * h^3 * (0h))/3$

(2) (C) What is the problem with using equally-spaced points to generate quadrature rules as the number of points increases? How do you alleviate this problem?

The Runge Phenomenon is a problem of oscillation at the edges of an interval that occurs when using polynomial interpolation with polynomials of high degree over a set of equispaced interpolation points.

Chebyshev zeroes and extrema non-uniform points which help against Runge Phenomenon. (And they turn out to be positions of equally spaced points on a semi-circle projected onto the diameter)