CHAPTER 12

# VARIABLE-FREQUENCY NETWORK PERFORMANCE

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## THE LEARNING GOALS FOR THIS CHAPTER ARE:

- **Understand the variable-frequency performance of the basic circuit elements: *R, L,* and *C***
- **Learn the different types of network functions and the definition of poles and zeros**
- **Be able to sketch a Bode plot for a network function**
- **Know how to create a Bode plot**
- **Know how to analyze series and parallel resonant circuits**
- **Be introduced to the concepts of magnitude and frequency scaling**
- **Learn the characteristics of basic filters such low-pass, high-pass, band-pass, and band rejection**
- **Know how to analyze basic passive and active filters**

**Music and Frequency** Time and frequency are two sides of the same coin, describing signals in two ways. We hear music, for example, as a time signal but the sampling rate for its digital recording, mastering, and manufacturing depends on signal bandwidth—a frequency description. We utilize both "languages"—time and frequency—to design recording and playback equipment. For iPods and MP3 players on laptops, the bandwidth has to be large enough to reproduce music in frequency ranges from bass to violin and from tuba to flute. For reference, concert flutes have frequencies from 262 Hz to over 2 kHz (middle C to three octaves above).

Your hearing must have a bandwidth that extends to frequencies at least as high as the highest frequency present in the musical signal. Voice or other signals can be modulated to a band of much higher frequencies, so high in some cases that they can be heard only by dogs or Superman. High-frequency hearing tends to deteriorate as you get older. This can be an advantage when you're young. For example, you can download high-frequency ringtones that are inaudible to the average professor. On the other hand, some convenience stores discourage loitering by broadcasting high-frequency signals that most adults can't hear, but that are very irritating to teenagers.

Variable frequency is as important in electric circuits as it is in describing musical signals. Frequency response plots of voltage transfer functions show magnitude in decibels and phase in degrees on linear scales versus radian frequency on a logarithmic scale. Filter design using variable frequency techniques results in series or parallel circuits that yield desired frequency spectra—low-pass, high-pass, or band-pass—for recording, communication, and radar systems. Your cell phone, television, and iPod or MP3 player use these principles. The music beat delivered from high quality recordings and the clear tones of live performances both uplift the listener—provided the bandwidth is adequate.

## 12.1 Variable Frequency-Response Analysis

In previous chapters we investigated the response of *RLC* networks to sinusoidal inputs. In particular, we considered 60-Hz sinusoidal inputs. In this chapter we allow the frequency of excitation to become a variable and evaluate network performance as a function of frequency. To begin, let us consider the effect of varying frequency on elements with which we are already quite familiar—the resistor, inductor, and capacitor. The frequency-domain impedance of the resistor shown in Fig. 12.1a is

$$\mathbf{Z}_R = R = R\underline{/0^\circ}$$

The magnitude and phase are constant and independent of frequency. Sketches of the magnitude and phase of $\mathbf{Z}_R$ are shown in Figs. 12.1b and c. Obviously, this is a very simple situation.

For the inductor in Fig. 12.2a, the frequency-domain impedance $\mathbf{Z}_L$ is

$$\mathbf{Z}_L = j\omega L = \omega L\underline{/90^\circ}$$

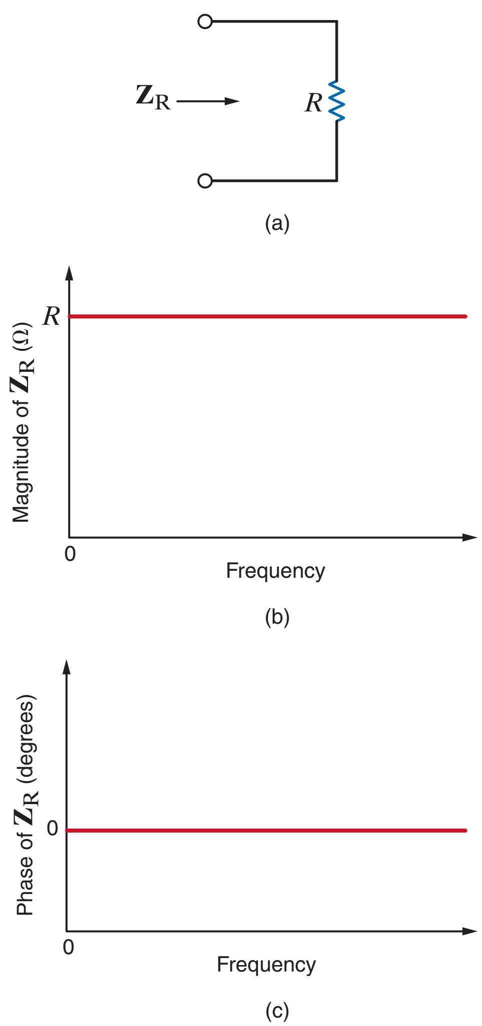



**Figure 12.1**

Frequency-independent impedance of a resistor.

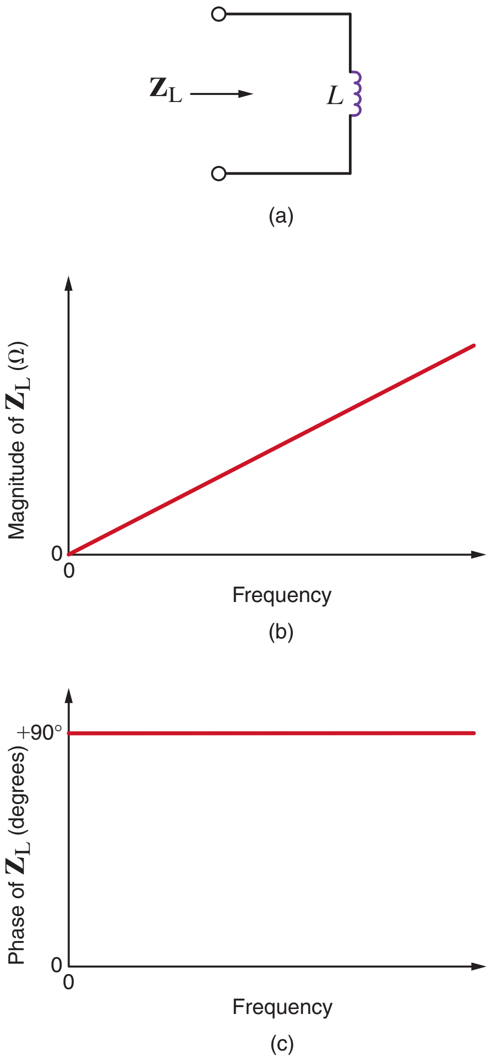



**Figure 12.2**
Frequency-dependent impedance of an inductor.

The phase is constant at 90°, but the magnitude of $\mathbf{Z}_L$ is directly proportional to frequency. Figs. 12.2b and c show sketches of the magnitude and phase of $\mathbf{Z}_L$ versus frequency. Note that at low frequencies the inductor's impedance is quite small. In fact, at dc, $\mathbf{Z}_L$ is zero, and the inductor appears as a short circuit. Conversely, as frequency increases, the impedance also increases.

Next consider the capacitor of Fig. 12.3a. The impedance is

$$\mathbf{Z}_C = \frac{1}{j\omega C} = \frac{1}{\omega C} \underline{/-90^\circ}$$

Once again the phase of the impedance is constant, but now the magnitude is inversely proportional to frequency, as shown in Figs. 12.3b and c. Note that the impedance approaches infinity, or an open circuit, as $\omega$ approaches zero and $\mathbf{Z}_C$ approaches zero as $\omega$ approaches infinity.

Now let us investigate a more complex circuit: the *RLC* series network in Fig. 12.4a. The equivalent impedance is

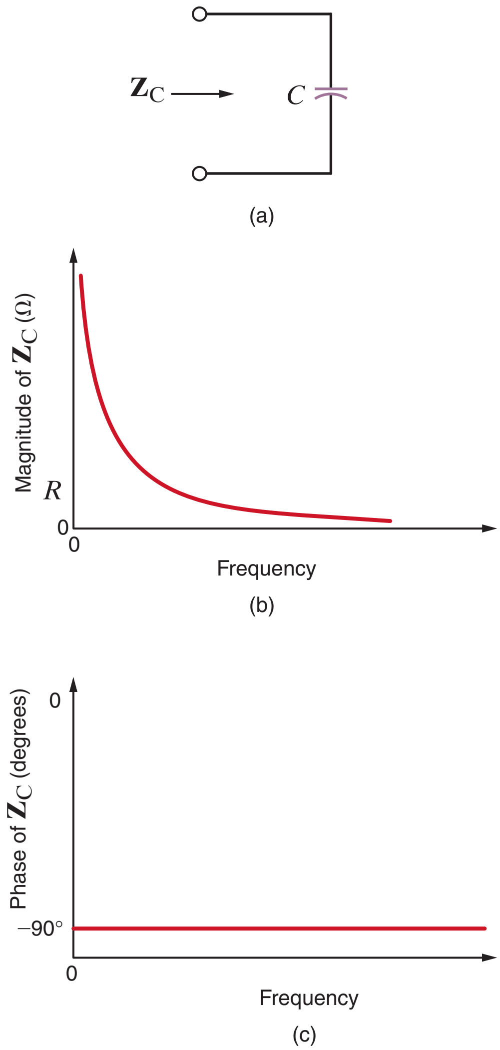
$$\mathbf{Z}_{eq} = R + j\omega L + \frac{1}{j\omega C}$$

or

$$\mathbf{Z}_{eq} = \frac{(j\omega)^2 LC + j\omega RC + 1}{j\omega C}$$

**Figure 12.3**

Frequency-dependent impedance of a capacitor.




Sketches of the magnitude and phase of this function are shown in Figs. 12.4b and c.

Note that at very low frequencies, the capacitor appears as an open circuit, and, therefore, the impedance is very large in this range. At high frequencies, the capacitor has very little effect and the impedance is dominated by the inductor, whose impedance keeps rising with frequency.

As the circuits become more complicated, the equations become more cumbersome. In an attempt to simplify them, let us make the substitution $j\omega = s$. (This substitution has a more important meaning, which we will describe in later chapters.) With this substitution, the expression for $\mathbf{Z}_{\text{eq}}$ becomes

$$\mathbf{Z}_{\text{eq}} = \frac{s^2LC + sRC + 1}{sC}$$

If we review the four circuits we investigated thus far, we will find that in every case the impedance is the ratio of two polynomials in $s$ and is of the general form

$$\mathbf{Z}(s) = \frac{N(s)}{D(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0} \qquad \textbf{12.1}$$

where $N(s)$ and $D(s)$ are polynomials of order $m$ and $n$, respectively. An extremely important aspect of Eq. (12.1) is that it holds not only for impedances but also for all voltages, currents, admittances, and gains in the network. The only restriction is that the values of all circuit elements (resistors, capacitors, inductors, and dependent sources) must be real numbers.

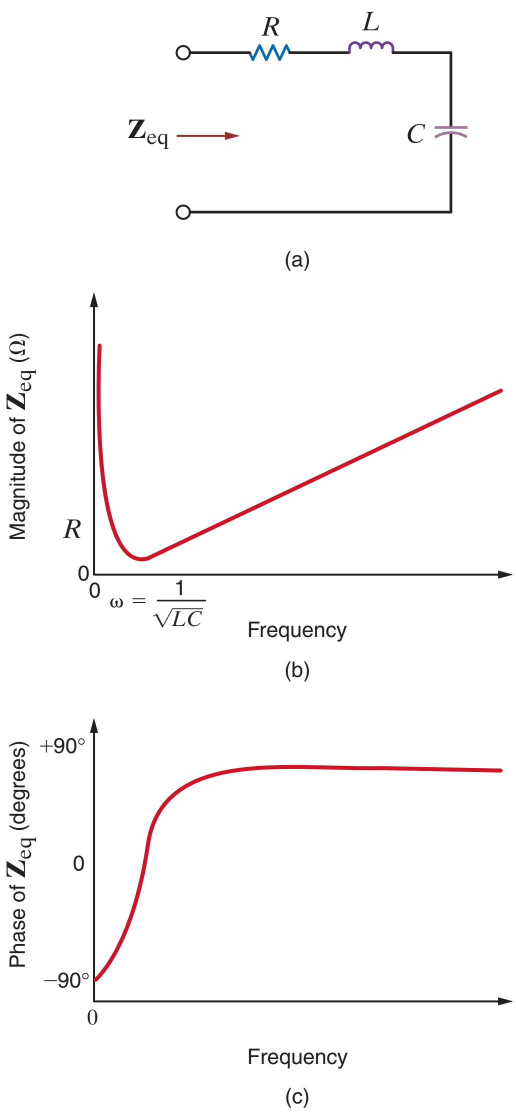



**Figure 12.4**
Frequency-dependent impedance of an *RLC* series network.

Let us now demonstrate the manner in which the voltage across an element in a series *RLC* network varies with frequency.

**EXAMPLE 12.1**

Consider the network in Fig. 12.5a. We wish to determine the variation of the output voltage as a function of frequency over the range from 0 to 1 kHz.

**SOLUTION**

Using voltage division, we can express the output as

$$\mathbf{V}_o = \left( \frac{R}{R + j\omega L + \dfrac{1}{j\omega C}} \right) \mathbf{V}_S$$

or, equivalently,

$$\mathbf{V}_o = \left( \frac{j\omega CR}{(j\omega)^2 LC + j\omega CR + 1} \right) \mathbf{V}_S$$

Using the element values, we find that the equation becomes

$$\mathbf{V}_o = \left( \frac{(j\omega)(37.95 \times 10^{-3})}{(j\omega)^2(2.53 \times 10^{-4}) + j\omega(37.95 \times 10^{-3}) + 1} \right) 10\underline{/0^\circ}$$

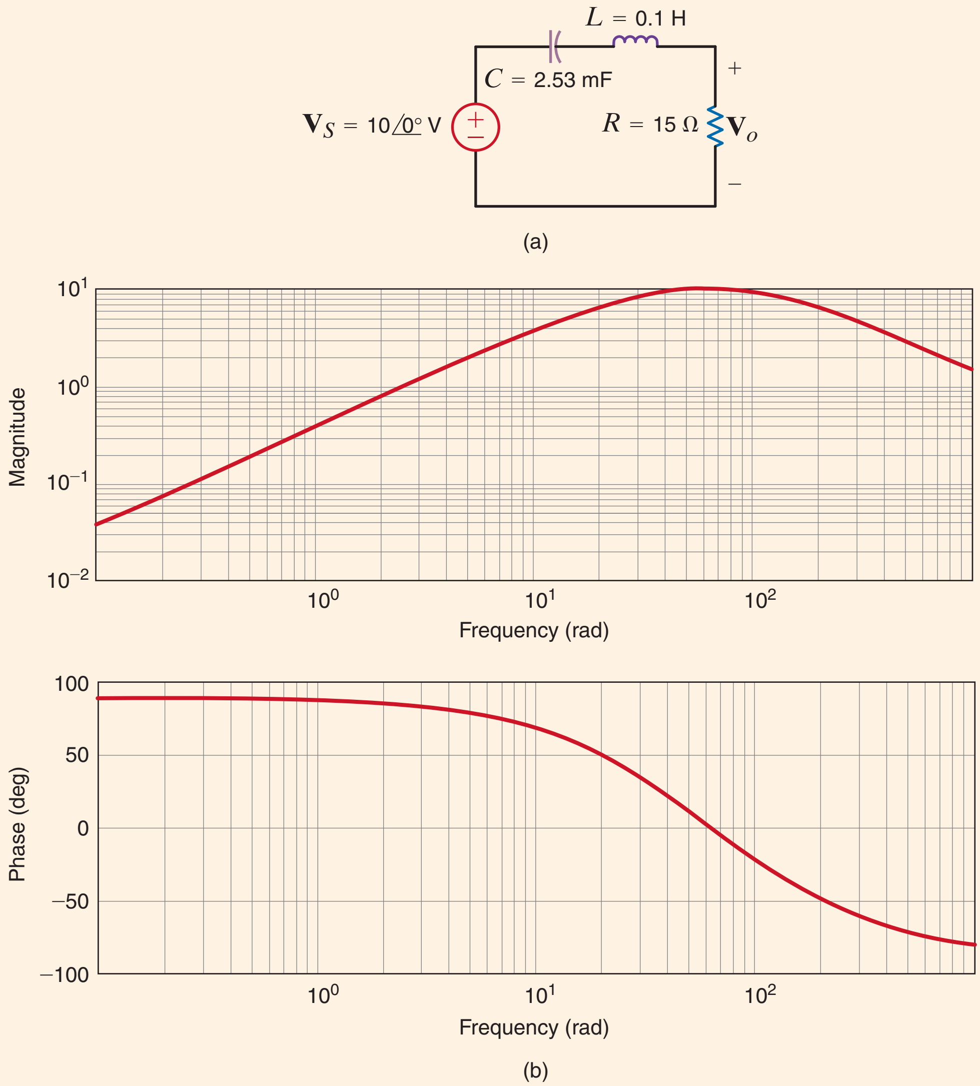



**Figure 12.5**

(a) Network and (b) its frequency-response simulation.

The resultant magnitude and phase characteristics are semilog plots in which the frequency is displayed on the log axis. The plots for the function $\mathbf{V}_o$ are shown in Fig. 12.5b.

In subsequent sections we will illustrate that the use of a semilog plot is a very useful tool in deriving frequency-response information.

As an introductory application of variable frequency-response analysis and characterization, let us consider a stereo amplifier. In particular, we should consider first the frequency range over which the amplifier must perform and then exactly what kind of performance we desire. The frequency range of the amplifier must exceed that of the human ear, which is roughly 50 Hz to 15,000 Hz. Accordingly, typical stereo amplifiers are designed to operate in the frequency range from 50 Hz to 20,000 Hz. Furthermore, we want to preserve the fidelity of the signal as it passes through the amplifier. Thus, the output signal should be an exact duplicate of the input signal times a gain factor. This requires that the gain be independent of frequency over the specified frequency range of 50 Hz to 20,000 Hz. An ideal sketch of this requirement for a gain of 1000 is shown in Fig. 12.6, where the midband region is defined as

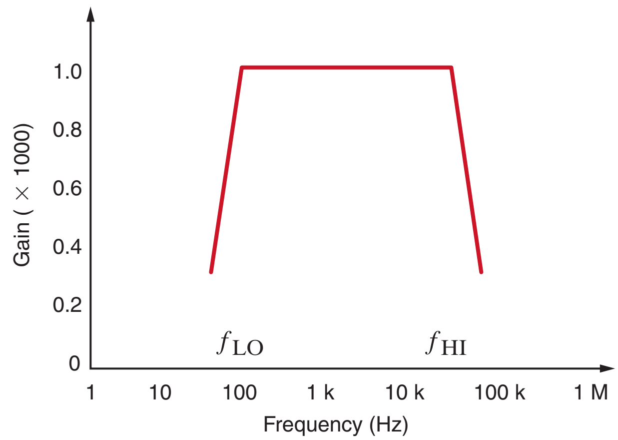



**Figure 12.6**
Amplifier frequency-response requirements.

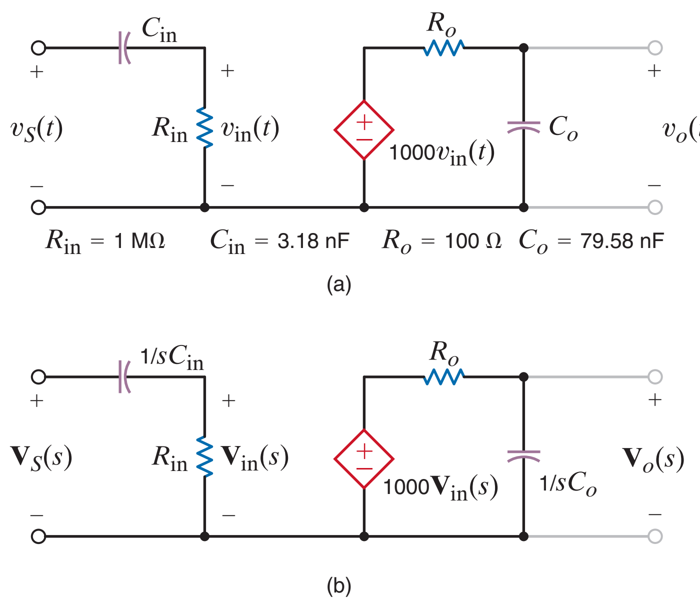



**Figure 12.7**
Amplifier equivalent network.

that portion of the plot where the gain is constant and is bounded by two points, which we will refer to as $f_{LO}$ and $f_{HI}$. Notice once again that the frequency axis is a log axis and, thus, the frequency response is displayed on a semilog plot.

A model for the amplifier described graphically in Fig. 12.6 is shown in Fig. 12.7a, with the frequency-domain equivalent circuit in Fig. 12.7b.

If the input is a steady-state sinusoid, we can use frequency-domain analysis to find the gain

$$\mathbf{G}_v(j\omega) = \frac{\mathbf{V}_o(j\omega)}{\mathbf{V}_S(j\omega)}$$

which with the substitution $s = j\omega$ can be expressed as

$$\mathbf{G}_v(s) = \frac{\mathbf{V}_o(s)}{\mathbf{V}_S(s)}$$

Using voltage division, we find that the gain is

$$\mathbf{G}_v(s) = \frac{\mathbf{V}_o(s)}{\mathbf{V}_S(s)} = \frac{\mathbf{V}_{in}(s)}{\mathbf{V}_S(s)}\frac{\mathbf{V}_o(s)}{\mathbf{V}_{in}(s)} = \left[\frac{R_{in}}{R_{in} + 1/sC_{in}}\right](1000)\left[\frac{1/sC_o}{R_o + 1/sC_o}\right]$$

or

$$\mathbf{G}_v(s) = \left[\frac{sC_{in}R_{in}}{1 + sC_{in}R_{in}}\right](1000)\left[\frac{1}{1 + sC_oR_o}\right]$$

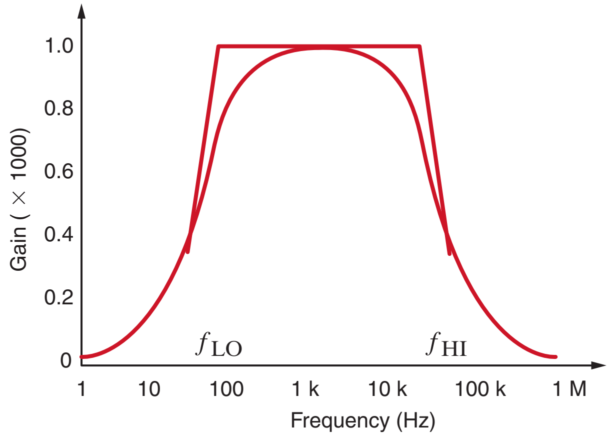



**Figure 12.8**
Exact and approximate amplifier gain versus frequency plots.

Using the element values in Fig. 12.7a,

$$\mathbf{G}_v(s) = \left[\frac{s}{s + 100\pi}\right](1000)\left[\frac{40{,}000\pi}{s + 40{,}000\pi}\right]$$

where $100\pi$ and $40{,}000\pi$ are the radian equivalents of 50 Hz and 20,000 Hz, respectively. Since $s = j\omega$, the network function is indeed complex. An exact plot of $\mathbf{G}_v(s)$ is shown in Fig. 12.8 superimposed over the sketch of Fig. 12.6. The exact plot exhibits smooth transitions at $f_{\text{LO}}$ and $f_{\text{HI}}$; otherwise the plots match fairly well.

Let us examine our expression for $\mathbf{G}_v(s)$ more closely with respect to the plot in Fig. 12.8. Assume that $f$ is well within the midband frequency range; that is,

$$f_{\text{LO}} \ll f \ll f_{\text{HI}}$$

or

$$100\pi \ll |s| \ll 40{,}000\pi$$

Under these conditions, the network function becomes

$$\mathbf{G}_v(s) \approx \left[\frac{s}{s}\right](1000)\left[\frac{1}{1 + 0}\right]$$

or

$$\mathbf{G}_v(s) = 1000$$

Thus, well within midband, the gain is constant. However, if the frequency of excitation decreases toward $f_{\text{LO}}$, then $|s|$ is comparable to $100\pi$ and

$$\mathbf{G}_v(s) \approx \left[\frac{s}{s + 100\pi}\right](1000)$$

Since $R_{\text{in}}C_{\text{in}} = 1/100\pi$, we see that $C_{\text{in}}$ causes the rolloff in gain at low frequencies. Similarly, when the frequency approaches $f_{\text{HI}}$, the gain rolloff is due to $C_o$.

Through this amplifier example, we have introduced the concept of frequency-dependent networks and have demonstrated that frequency-dependent network performance is caused by the reactive elements in a network.

**NETWORK FUNCTIONS** In the previous section, we introduced the term *voltage gain*, $\mathbf{G}_v(s)$. This term is actually only one of several network functions, designated generally as $\mathbf{H}(s)$, which define the ratio of response to input. Since the function describes a reaction due to an excitation at some other point in the circuit, network functions are also called *transfer functions*. Furthermore, transfer functions are not limited to voltage ratios. Since in electrical networks inputs and outputs can be either voltages or currents, there are four possible network functions, as listed in Table 12.1.

There are also *driving point functions*, which are impedances or admittances defined at a single pair of terminals. For example, the input impedance of a network is a driving point function.

**TABLE 12.1** Network transfer functions

| INPUT | OUTPUT | TRANSFER FUNCTION | SYMBOL |
|---|---|---|---|
| Voltage | Voltage | Voltage gain | $\mathbf{G}_v(s)$ |
| Current | Voltage | Transimpedance | $\mathbf{Z}(s)$ |
| Current | Current | Current gain | $\mathbf{G}_i(s)$ |
| Voltage | Current | Transadmittance | $\mathbf{Y}(s)$ |

## EXAMPLE 12.2

We wish to determine the transfer admittance $[\mathbf{I}_2(s)/\mathbf{V}_1(s)]$ and the voltage gain of the network shown in Fig. 12.9.

**SOLUTION**

The mesh equations for the network are

$$(R_1 + sL)\mathbf{I}_1(s) - sL\mathbf{I}_2(s) = \mathbf{V}_1(s)$$

$$-sL\mathbf{I}_1(s) + \left(R_2 + sL + \frac{1}{sC}\right)\mathbf{I}_2(s) = 0$$

$$\mathbf{V}_2(s) = \mathbf{I}_2(s)R_2$$

Solving the equations for $\mathbf{I}_2(s)$ yields

$$\mathbf{I}_2(s) = \frac{sL\mathbf{V}_1(s)}{(R_1 + sL)(R_2 + sL + 1/sC) - s^2L^2}$$

Therefore, the transfer admittance $[\mathbf{I}_2(s)/\mathbf{V}_1(s)]$ is

$$\mathbf{Y}_T(s) = \frac{\mathbf{I}_2(s)}{\mathbf{V}_1(s)} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (L + R_1R_2C)s + R_1}$$

and the voltage gain is

$$\mathbf{G}_v(s) = \frac{\mathbf{V}_2(s)}{\mathbf{V}_1(s)} = \frac{LCR_2s^2}{(R_1 + R_2)LCs^2 + (L + R_1R_2C)s + R_1}$$

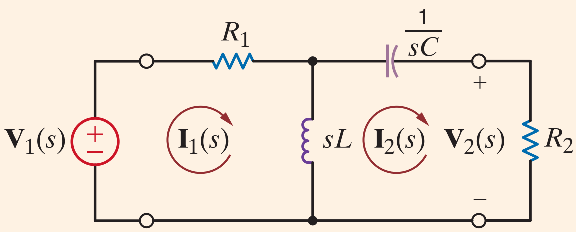



**Figure 12.9** Circuit employed in Example 12.2.

**POLES AND ZEROS** As we have indicated, the network function can be expressed as the ratio of the two polynomials in $s$. In addition, we note that since the values of our circuit elements, or controlled sources, are real numbers, the coefficients of the two polynomials will be real. Therefore, we will express a network function in the form

$$\mathbf{H}(s) = \frac{N(s)}{D(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0} \qquad \textbf{12.2}$$

where $N(s)$ is the numerator polynomial of degree $m$ and $D(s)$ is the denominator polynomial of degree $n$. Equation (12.2) can also be written in the form

$$\mathbf{H}(s) = \frac{K_0(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)} \qquad \textbf{12.3}$$

where $K_0$ is a constant, $z_1, \ldots, z_m$ are the roots of $N(s)$, and $p_1, \ldots, p_n$ are the roots of $D(s)$. Note that if $s = z_1$, or $z_2, \ldots, z_m$, then $\mathbf{H}(s)$ becomes zero and hence $z_1, \ldots, z_m$ are called zeros of the transfer function. Similarly, if $s = p_1$, or $p_2, \ldots, p_n$, then $\mathbf{H}(s)$ becomes infinite and, therefore, $p_1, \ldots, p_n$ are called poles of the function. The zeros or poles may actually be complex. However, if they are complex, they must occur in conjugate pairs since the coefficients of the polynomial are real. The representation of the network function specified in Eq. (12.3) is extremely important and is generally employed to represent any linear time-invariant system. The importance of this form lies in the fact that the dynamic properties of a system can be gleaned from an examination of the system poles.

# Learning Assessments

**E12.1** Find the driving-point impedance at $\mathbf{V}_S(s)$ in the amplifier shown in Fig. 12.7b.

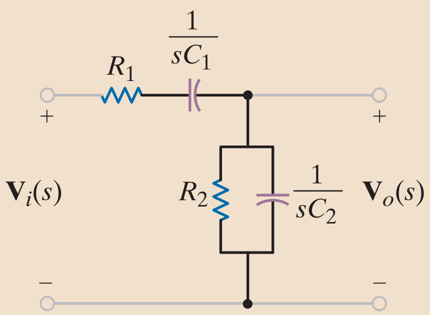
**ANSWER:**

$$\mathbf{Z}(s) = R_{\text{in}} + \frac{1}{sC_{\text{in}}} = \left[1 + \left(\frac{100\pi}{s}\right)\right] \text{M}\Omega.$$

**E12.2** Find the pole and zero locations in hertz and the value of $K_0$ for the amplifier network in Fig. 12.7.

**ANSWER:** $z_1 = 0$ Hz (dc); $p_1 = -50$ Hz; $p_2 = -20{,}000$ Hz; $K_0 = (4 \times 10^7)\,\pi$.

**E12.3** Determine the voltage transfer function $\mathbf{V}_o(s)/\mathbf{V}_i(s)$ as a function of $s$ in Fig. PE12.3.




**ANSWER:**

$$\frac{s}{R_1C_2\left[s^2 + \dfrac{C_1R_2 + C_2R_2 + C_1R_1}{R_1R_2C_1C_2}s + \dfrac{1}{R_1R_2C_1C_2}\right]}$$

Figure E12.3

## 12.2 Sinusoidal Frequency Analysis

Although in specific cases a network operates at only one frequency (e.g., a power system network), in general we are interested in the behavior of a network as a function of frequency. In a sinusoidal steady-state analysis, the network function can be expressed as

$$\mathbf{H}(j\omega) = M(\omega)e^{j\phi(\omega)} \tag{12.4}$$

where $M(\omega) = |\mathbf{H}(j\omega)|$ and $\phi(\omega)$ is the phase. A plot of these two functions, which are commonly called the *magnitude* and *phase characteristics,* displays the manner in which the response varies with the input frequency $\omega$. We will now illustrate the manner in which to perform a frequency-domain analysis by simply evaluating the function at various frequencies within the range of interest.

**FREQUENCY RESPONSE USING A BODE PLOT** If the network characteristics are plotted on a semilog scale (that is, a linear scale for the ordinate and a logarithmic scale for the abscissa), they are known as *Bode plots* (named after Hendrik W. Bode). This graph is a