- The complex numbers $\mathbb{C}$ form a plane.
- Their operations are very related to two-dimensional geometry.
- In particular, multiplication by a unit complex number:

$$|z|^2 = 1$$

which can all be written:

$$z = e^{i\theta}$$

gives a *rotation*:

$$R_z(w) = zw$$

by angle θ.

How does this work?

- $\mathbb{C} = \left\{ a + bi : a, b \in \mathbb{R}, \quad i^2 = -1 \right\}$
- Any complex number has a length, given by the Pythagorean formula:

$$|a + bi| = \sqrt{a^2 + b^2}.$$

- We can add and subtract in $\mathbb{C}$. For example:

$$a + bi + c + di = (a + c) + (b + d)i.$$

- We can also multiply, which is much messier:

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

What does this last formula mean?

Fortunately, there is a better way to multiply complex numbers, thanks to Leonhard Euler:

Figure: Handman's portrait of Euler. *Wikimedia Commons.*

who proved:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

Geometrically, this formula says $e^{i\varphi}$ lies on the unit circle in $\mathbb{C}$:

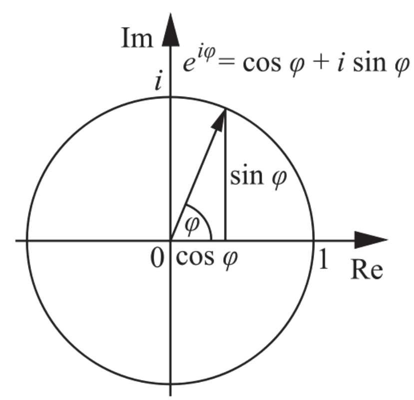


Figure: Euler's formula. *Wikimedia Commons.*

- $e^{i\varphi}$ has unit length.
- If we multiply by a positive number, r, we get a complex number of length r:
$$re^{i\varphi}.$$
- By adjusting the length r and angle φ, we can write any complex number in this way!
- In a calculus class, this trick goes by the name **polar coordinates**.

And this gives a great way to multiply complex numbers:

- Remember our formula was:

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

- Instead, we can write each factor in polar coordinates:

$$a+bi = re^{i\varphi}, \quad c+di = se^{i\theta}$$

- And now:

$$(a+bi)(c+di) = re^{i\varphi}se^{i\theta} = rse^{i(\varphi+\theta)}.$$

- In words: to multiply two complex numbers, *multiply their lengths and add their angles!*

In particular, if we multiply a given complex number z by

$$e^{i\varphi}$$

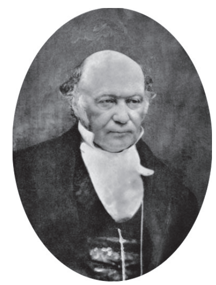
which has unit length 1, the result:

$$e^{i\varphi} z$$

has the same length as z.

It is rotated by φ degrees.

So, we can use *complex arithmetic* (multiplication) to do a *geometric operation* (rotation).



The 19th century Irish mathematician and physicist William Rowan Hamilton was fascinated by the role of $\mathbb{C}$ in two-dimensional geometry.

For years, he tried to invent an algebra of "triplets" to play the same role in three dimenions:

$$a + bi + cj \in \mathbb{R}^3.$$

Alas, we now know this quest was in vain.

Theorem

The only normed division algebras, which are number systems where we can add, subtract, multiply and divide, and which have a norm satisfying

$$|zw| = |z||w|$$

have dimension 1, 2, 4, or 8.

Hamilton's search continued into October, 1843:

Every morning in the early part of the above-cited month, on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me: "Well, Papa, can you multiply triplets?" Whereto I was always obliged to reply, with a sad shake of the head: "No, I can only add and subtract them."

On October 16th, 1843, while walking with his wife to a meeting of the Royal Society of Dublin, Hamilton discovered a 4-dimensional division algebra called the **quaternions**:

That is to say, I then and there felt the galvanic circuit of thought close; and the sparks which fell from it were the fundamental equations between i, j, k; exactly such as I have used them ever since:

$$i^2 = j^2 = k^2 = ijk = -1.$$

Hamilton carved these equations onto Brougham Bridge. A plaque commemorates this vandalism today:

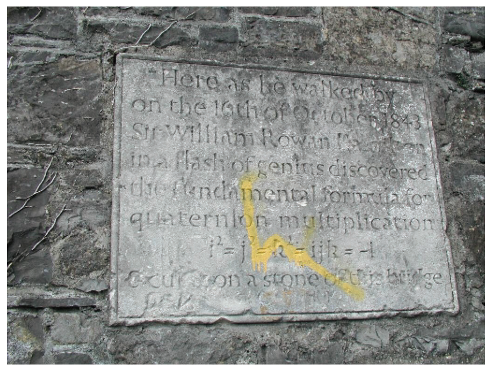


Figure: Brougham Bridge plaque. *Photo by Tevian Dray.*

The quaternions are

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}.$$

- i, j and k are all square roots of -1.
- $ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik$.
- As we shall see, we can use quaternions to do rotations in 3d.

Puzzle

Check that these relations ($ij = k = -ji$, etc) all follow from Hamilton's definition:

$$i^2 = j^2 = k^2 = ijk = -1.$$

- The quaternions don't commute!
- A useful mnemonic for multiplication is this picture:

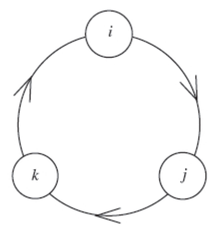


Figure: Multiplying quaternions. *Figure by John Baez.*

- If you have studied vectors, you may also recognize i, j and k as unit vectors.
- The quaternion product is the same as the **cross product** of vectors:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

- Except, for the cross product:

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

  while for quaternions, this is -1.
- In fact, we can think of a quaternion as having a scalar (number) part and a vector part:

$$v_0 + v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} = (v_0, \mathbf{v}).$$

We can use the cross product, and the **dot product**:

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

to define the product of quaternions in yet another way:

$$(v_0, \mathbf{v})(w_0, \mathbf{w}) = (v_0 w_0 - \mathbf{v} \cdot \mathbf{w},\ \ v_0 \mathbf{w} + w_0 \mathbf{v} + \mathbf{v} \times \mathbf{w}).$$

Puzzle

Check that this formula gives the same result for quaternion multiplication as the explicit rules for multiplying i, j, and k.

I promised we could use quaternions to do 3d rotations, so here's how:

- Think of three-dimensional space as being purely imaginary quaternions:

$$\mathbb{R}^3 = \{xi + yj + zk : x, y, z \in R\}.$$

- Just like for complex numbers, the rotations are done using **unit quaternions**, like:

$$\cos\varphi + i\sin\varphi, \quad \cos\varphi + j\sin\varphi, \quad \cos\varphi + k\sin\varphi.$$

- By analogy with Euler's formula, we will write these as:

$$e^{i\varphi}, \quad e^{j\varphi} \quad e^{k\varphi}.$$

But there are many more unit quaternions than these!

- i, j, and k are just three special unit imaginary quaternions.
- Take any unit imaginary quaternion, $\mathbf{u} = u_1 i + u_2 j + u_3 k$. That is, any **unit vector**.
- Then
$$\cos\varphi + \mathbf{u}\sin\varphi$$
is a unit quaternion.
- By analogy with Euler's formula, we write this as:
$$e^{\mathbf{u}\varphi}.$$

Theorem

If $\mathbf{u}$ *is a unit vector, and* $\mathbf{v}$ *is any vector, the expression*

$$e^{\mathbf{u}\varphi}\mathbf{v}e^{-\mathbf{u}\varphi},$$

gives the result of rotating $\mathbf{v}$ *about the axis in the* $\mathbf{u}$ *direction.*

Proof.

I will prove this for $\mathbf{u} = i$, since there is nothing special about the i direction.

$$\begin{aligned}
& e^{i\varphi}(v_1 i + v_2 j + v_3 k)e^{-i\varphi} & \\
= \;& e^{i\varphi}(v_1 i + (v_2 + v_3 i)j)e^{-i\varphi} & \text{Puzzle!} \\
= \;& e^{i\varphi}(v_1 i)e^{-i\varphi} + e^{i\varphi}(v_2 + v_3 i)je^{-i\varphi} & \\
= \;& e^{i\varphi}(v_1 i)e^{-i\varphi} + e^{i\varphi}(v_2 + v_3 i)e^{+i\varphi}j & \text{Puzzle!} \\
= \;& v_1 i + e^{i2\varphi}(v_2 + v_3 i)j & \text{Puzzle!}
\end{aligned}$$

$\square$

Note the 2φ!

Theorem (Improved)

If $\mathbf{u}$ is a unit vector, and $\mathbf{v}$ is any vector, the expression

$$e^{\mathbf{u}\varphi}\mathbf{v}e^{-\mathbf{u}\varphi},$$

gives the result of rotating $\mathbf{v}$ about the axis in the $\mathbf{u}$ direction by 2φ degrees.

Amazingly, this 2φ is important when describing electrons!

Let's write the rotation we get from the unit quaternion $e^{\mathbf{u}\varphi}$ as:

$$R_{e^{\mathbf{u}\varphi}}(\mathbf{v}) = e^{\mathbf{u}\varphi}\mathbf{v}e^{-\mathbf{u}\varphi}$$

This is a rotation by 2φ. To rotate by φ, we need:

$$R_{e^{\mathbf{u}\varphi/2}}(\mathbf{v}) = e^{\mathbf{u}\varphi/2}\mathbf{v}e^{-\mathbf{u}\varphi/2}$$

And to say how this relates to electrons, we need to talk about quantum mechanics.

Quantum mechanics says that particles are represented by waves:

- The simplest kind of wave is a function:

$$\psi\colon \mathbb{R} \to \mathbb{R}$$

- But since we live in three dimensions:

$$\psi\colon \mathbb{R}^3 \to \mathbb{R}$$

- And because it's quantum, it's complex-valued:

$$\psi\colon \mathbb{R}^3 \to \mathbb{C}$$

- Yet, in 1924, Wolfgang Pauli (secretly) discovered that for electrons, it's *quaternion-valued*:

$$\psi\colon \mathbb{R}^3 \to \mathbb{H}$$

If we rotate most particles, we rotate its wave:

$$R_{e^{\mathbf{u}\varphi/2}}\psi(\mathbf{v}) := \psi(R_{e^{-\mathbf{u}\varphi/2}}(\mathbf{v})).$$

But to rotate an electron, Pauli found:

$$R_{e^{\mathbf{u}\varphi/2}}\psi(\mathbf{v}) := e^{\mathbf{u}\varphi/2}\psi(R_{e^{-\mathbf{u}\varphi/2}}(\mathbf{v})).$$

In particular, for a $\varphi = 360°$ rotation:

$$R_{e^{\mathbf{u}\,360°/2}}\psi(\mathbf{v}) = e^{\mathbf{u}\,180°}\psi(\mathbf{v}) = -\psi(\mathbf{v}).$$

Electrons can tell if they have been rotated 360 degrees!