Paths in Graphs: Most Direct Route

Michael Levin

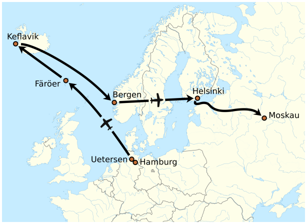
Higher School of Economics

Graph Algorithms
Data Structures and Algorithms

Outline

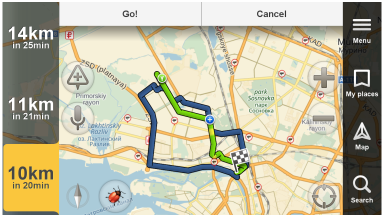
Applications

Applications



Applications

Keflavik
Färöer
Bergen
Helsinki
Moskau
Uetersen
Hamburg



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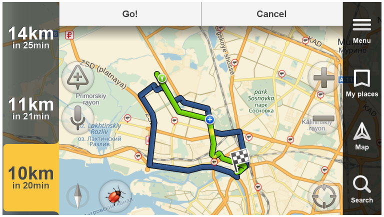

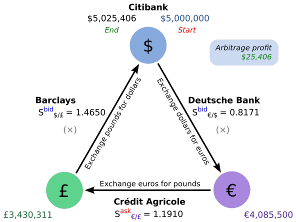

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The most direct route

What is the minimum number of flight segments to get from Hamburg to Moscow?

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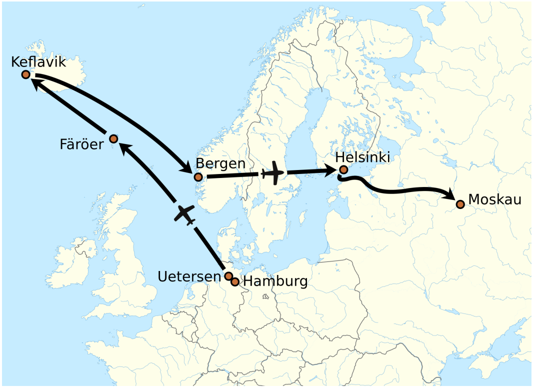
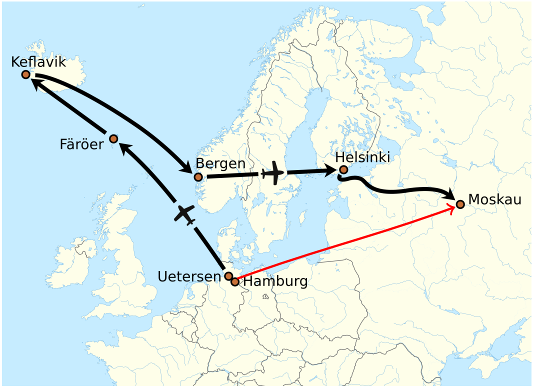
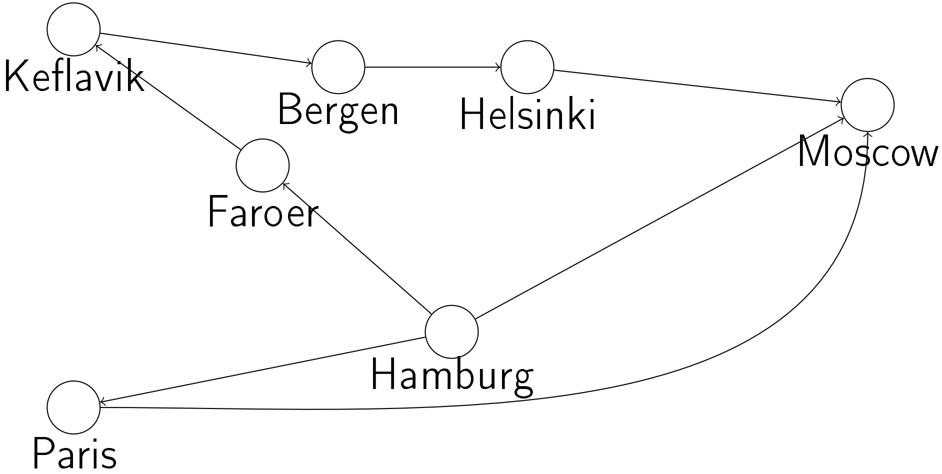


The most direct route

What is the minimum number of flight segments to get from Hamburg to Moscow?





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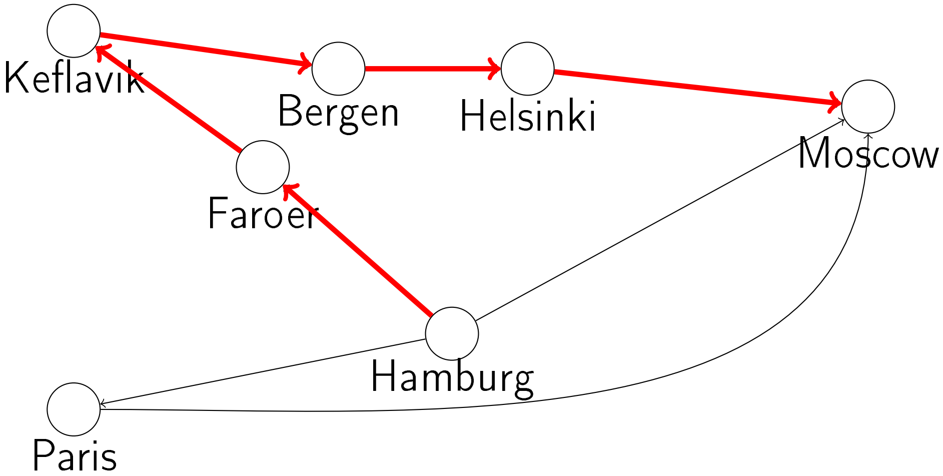
Hamburg
Paris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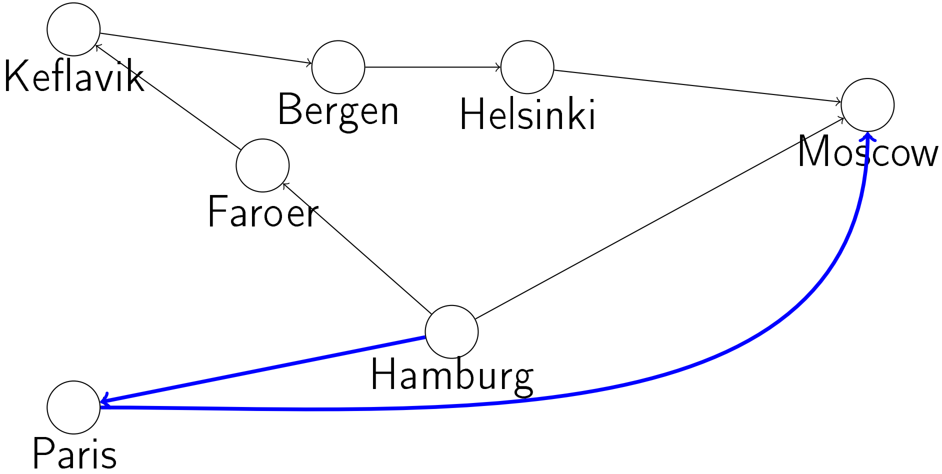
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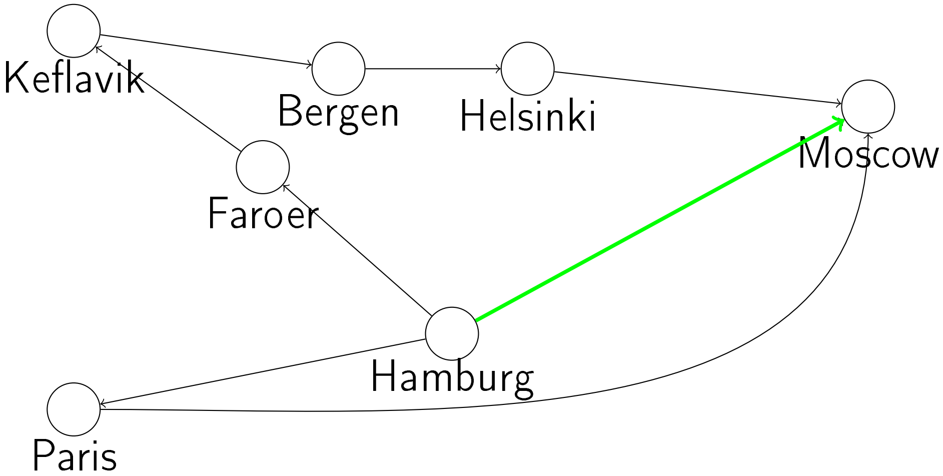
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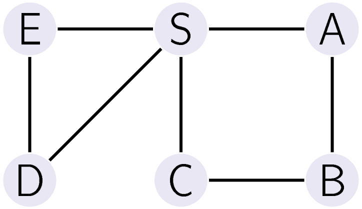
Paths and lengths

Length of the path $L(P)$ is the number of edges in the path.



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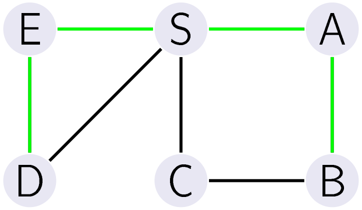

$$L(D - E - S - A - B) = 4$$

Paths and lengths

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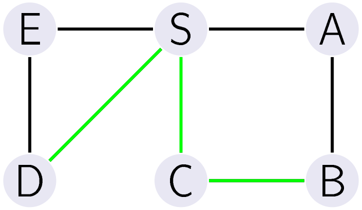
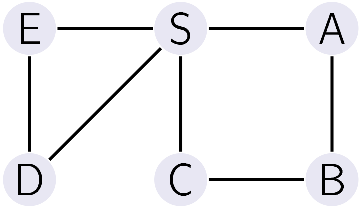


$L(D - E - S - A - B) = 4$

$L(D - S - C - B) = 3$

Distances

The distance between two vertices is the length of the shortest path between them.



Distances

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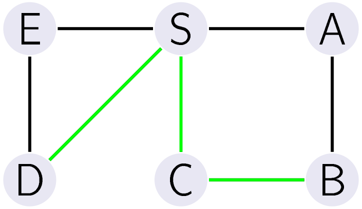
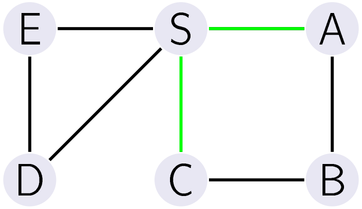


$d(D, B) = 3$

Distances

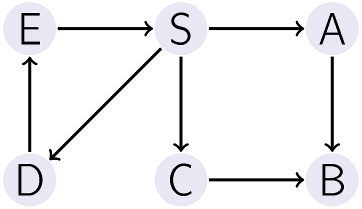
The distance between two vertices is the length of the shortest path between them.



$d(D, B) = 3$
$d(C, A) = 2$

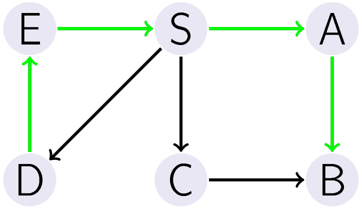
Distances

The distance between two vertices is the length of the shortest path between them.


$$\begin{tikzcd}
E \arrow[r] & S \arrow[r] \arrow[d] \arrow[dl] & A \arrow[d] \\
D \arrow[u] & C \arrow[r] & B
\end{tikzcd}$$

Distances

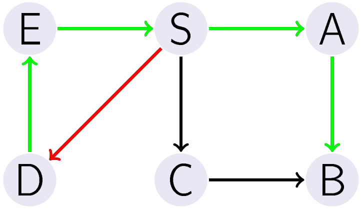
The distance between two vertices is the length of the shortest path between them.



$d(D, B) = 4$

Distances

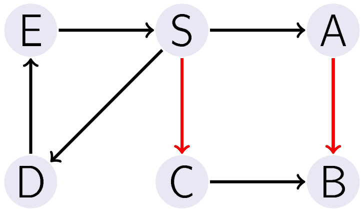
The distance between two vertices is the length of the shortest path between them.



$d(D, B) = 4$

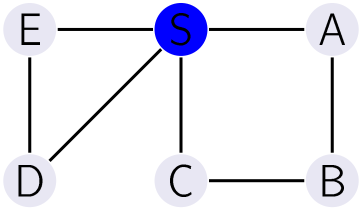
Distances

The distance between two vertices is the length of the shortest path between them.

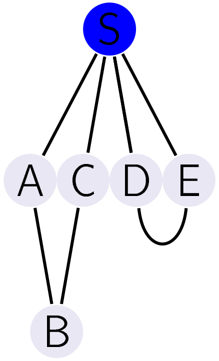


$d(D, B) = 4$
$d(C, A) = \infty$

Distances



Distance layers



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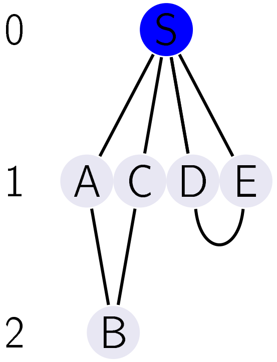

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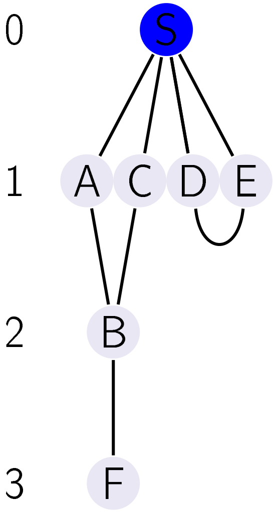

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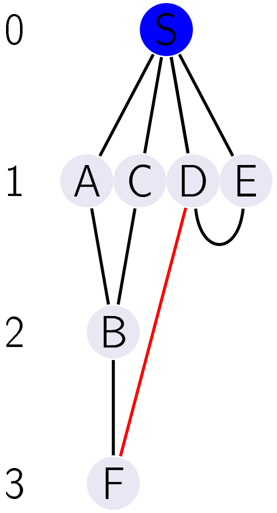

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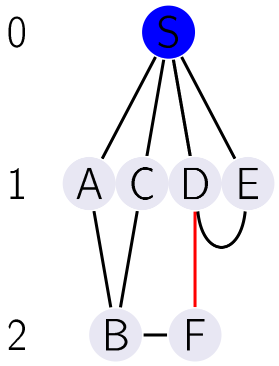

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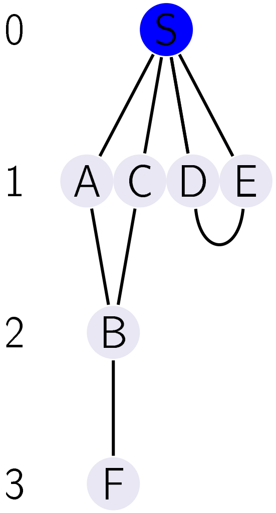

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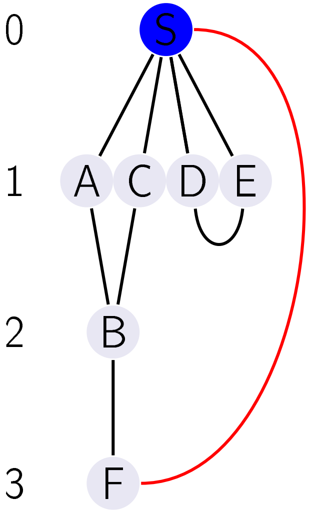

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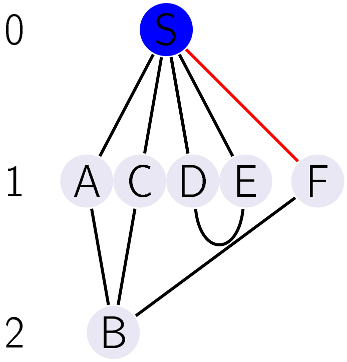

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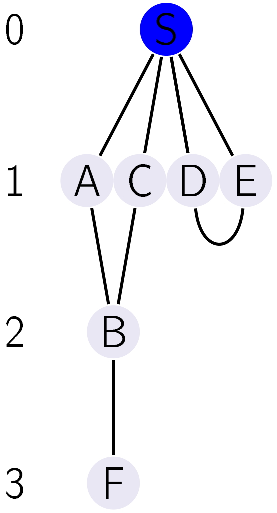

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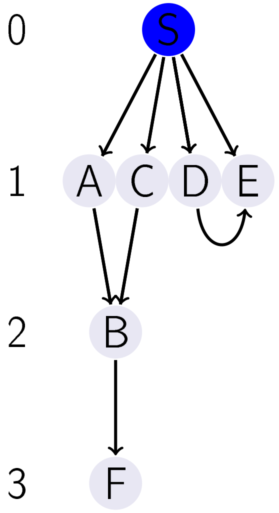

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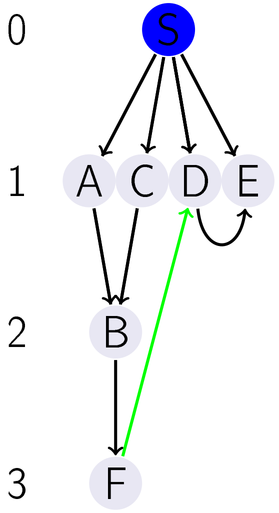

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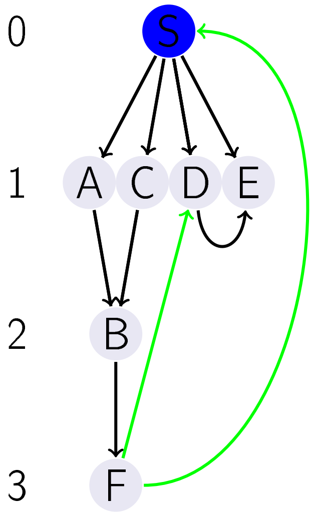

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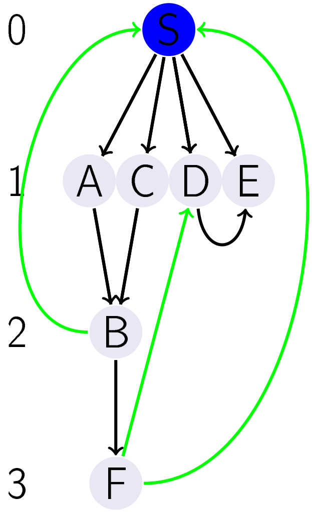

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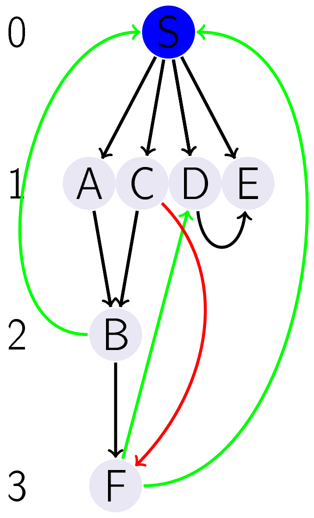
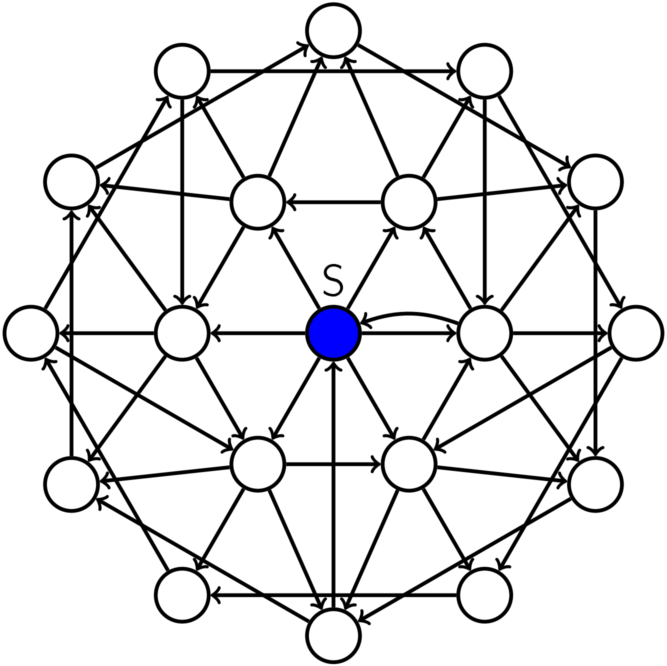


Outline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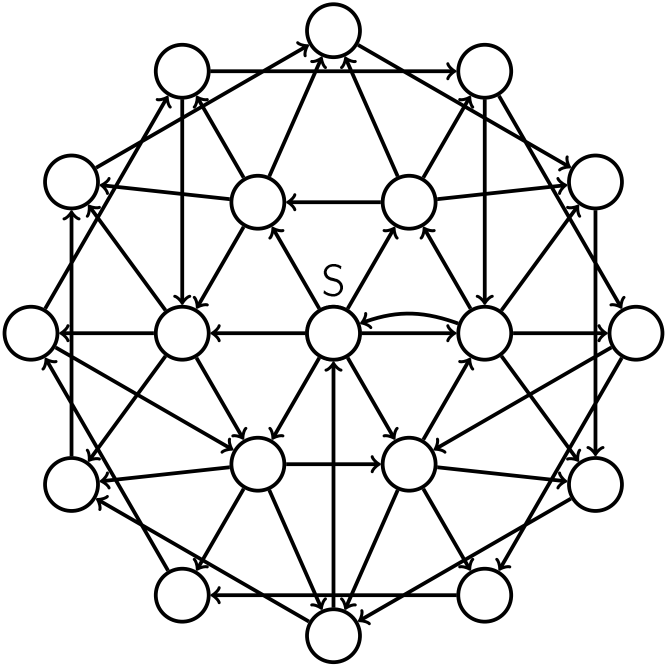
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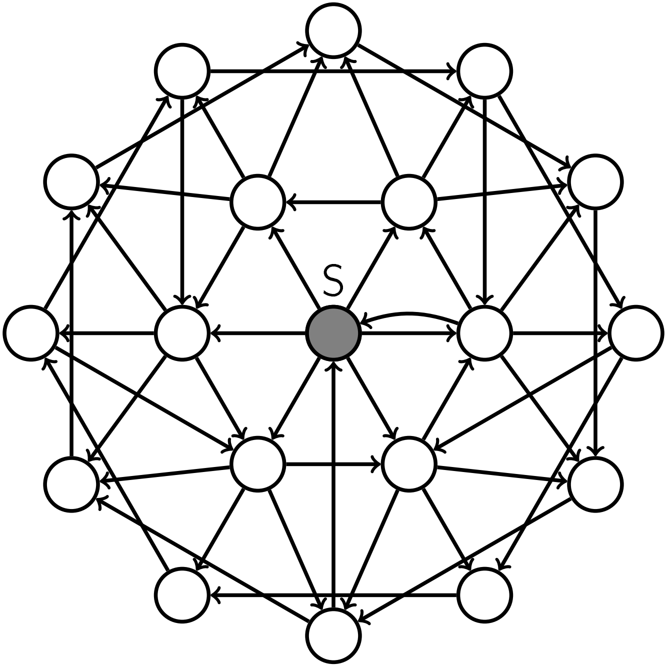
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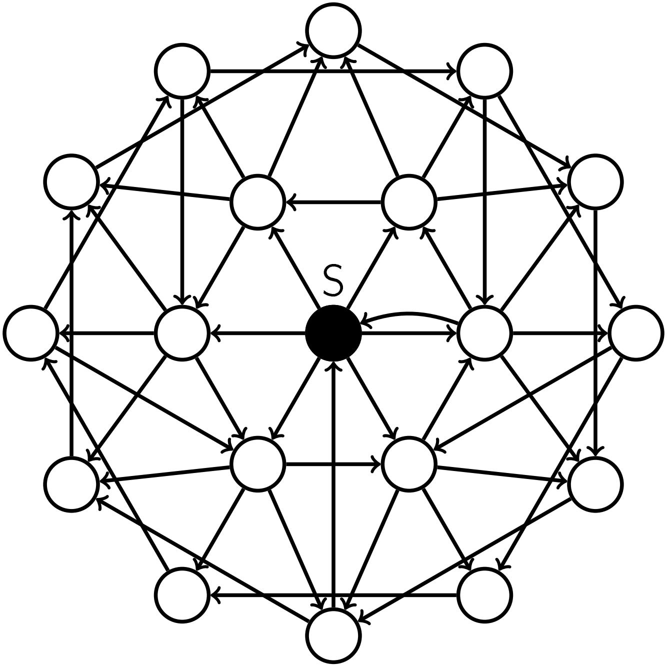
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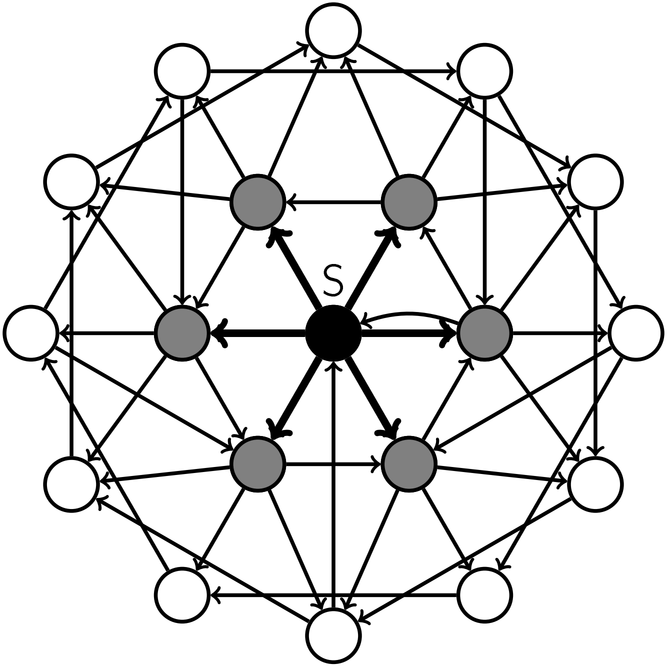
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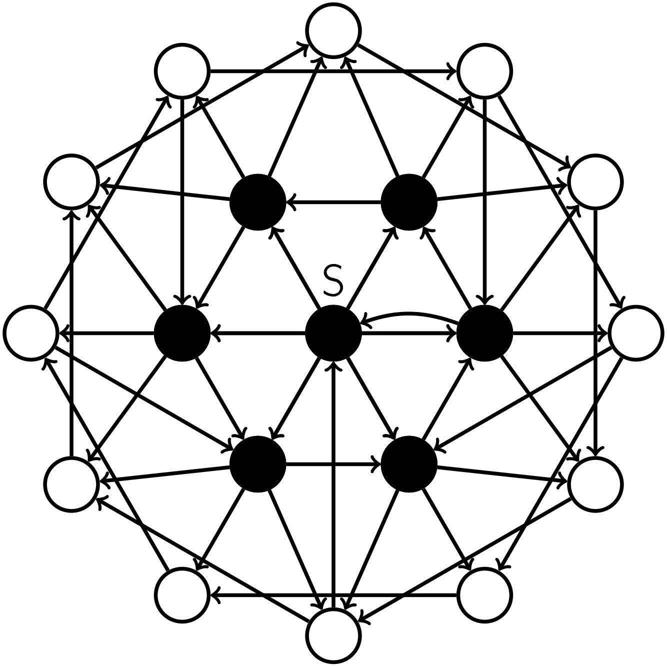
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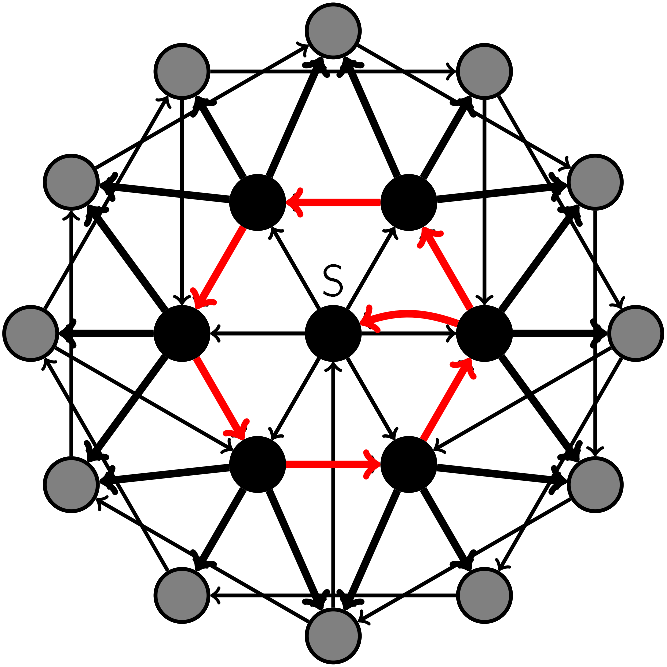
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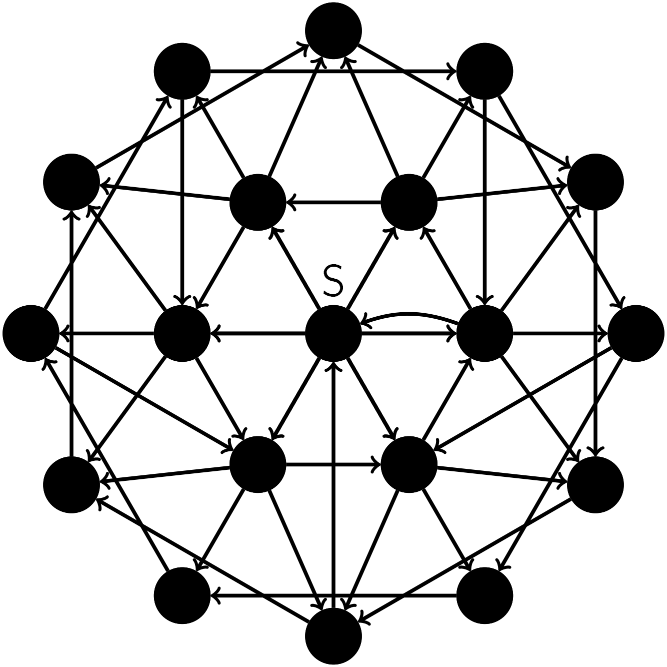
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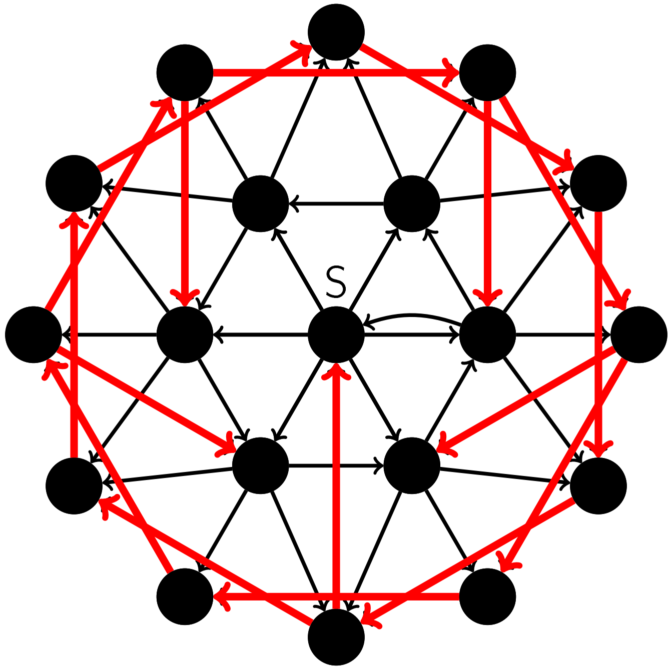
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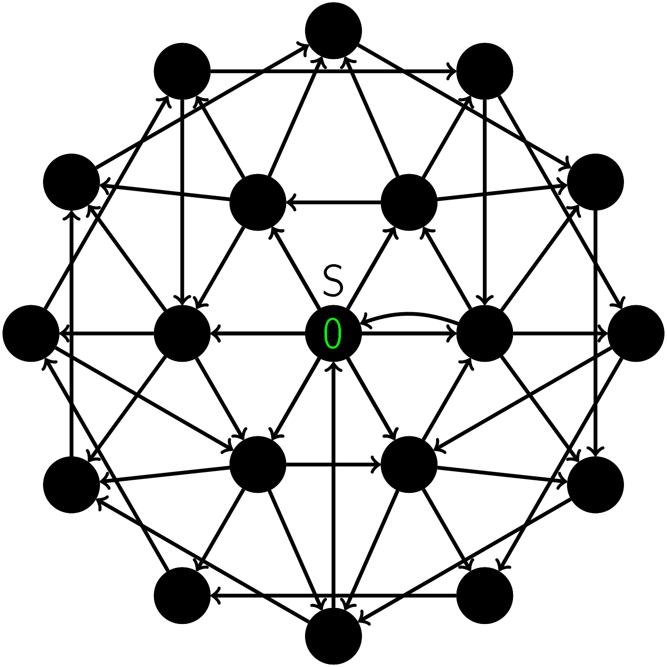
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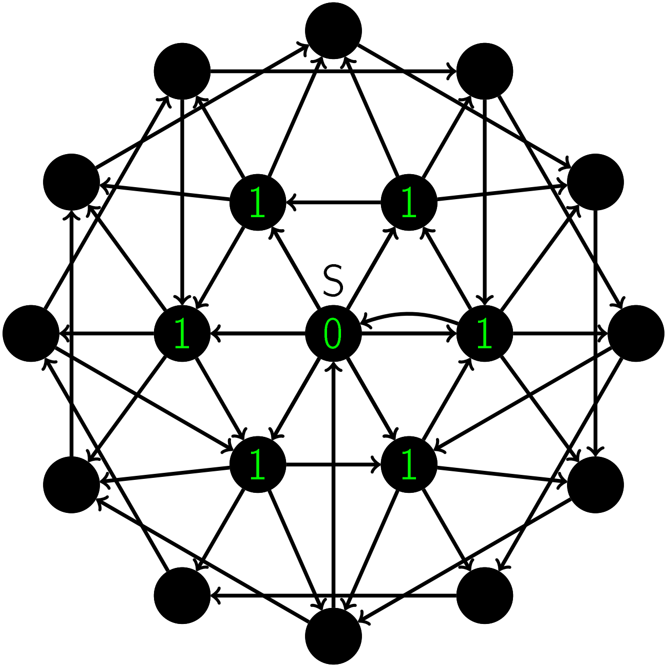

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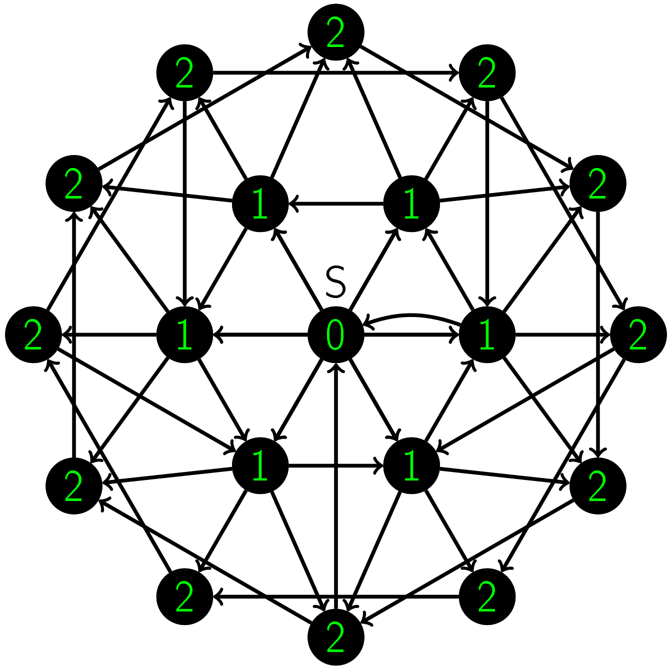
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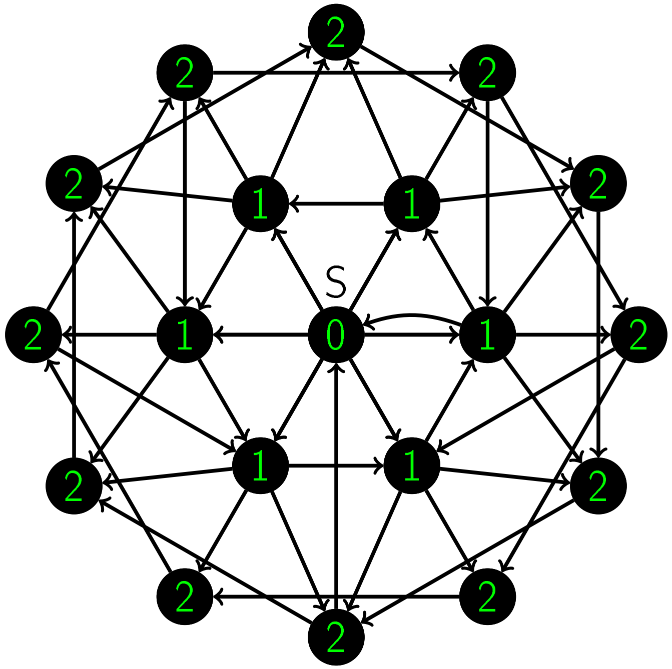

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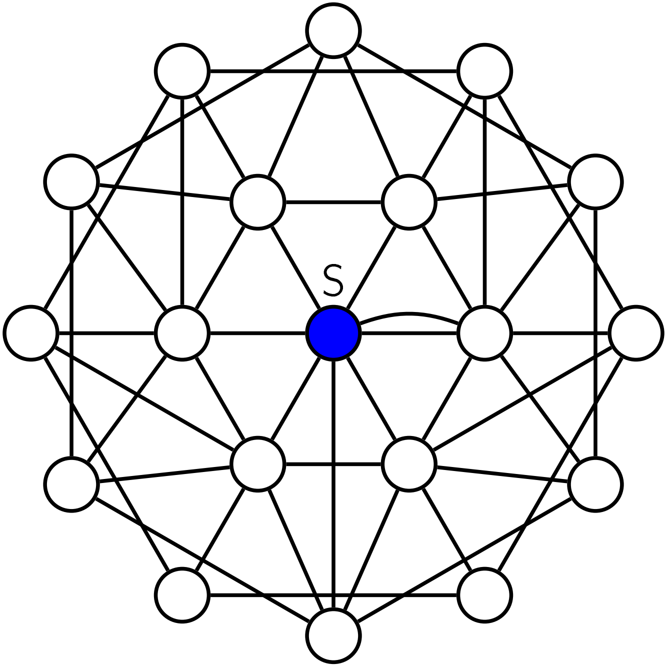
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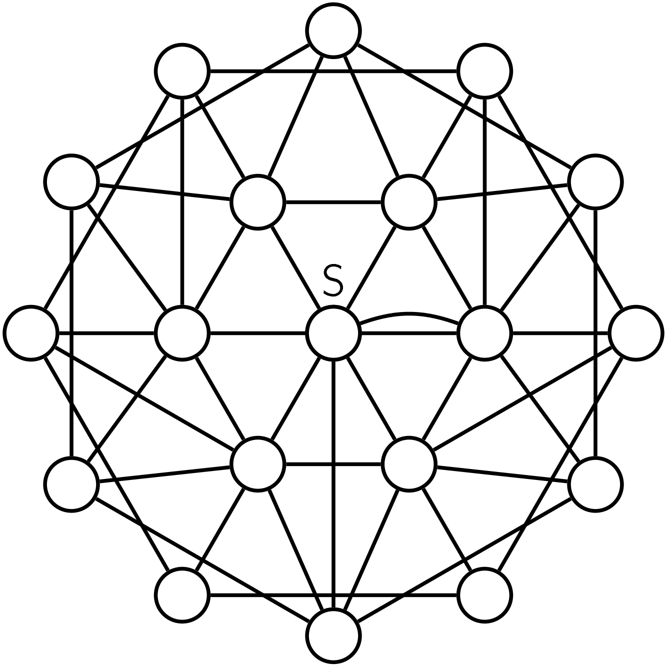
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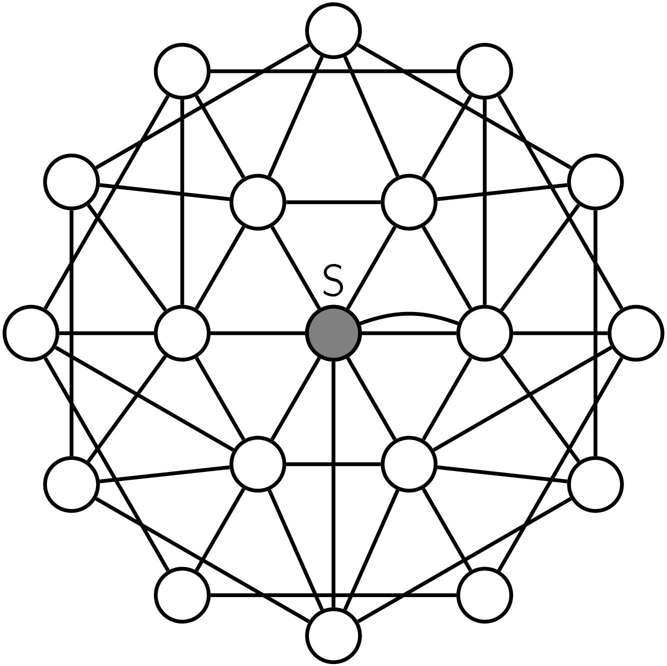
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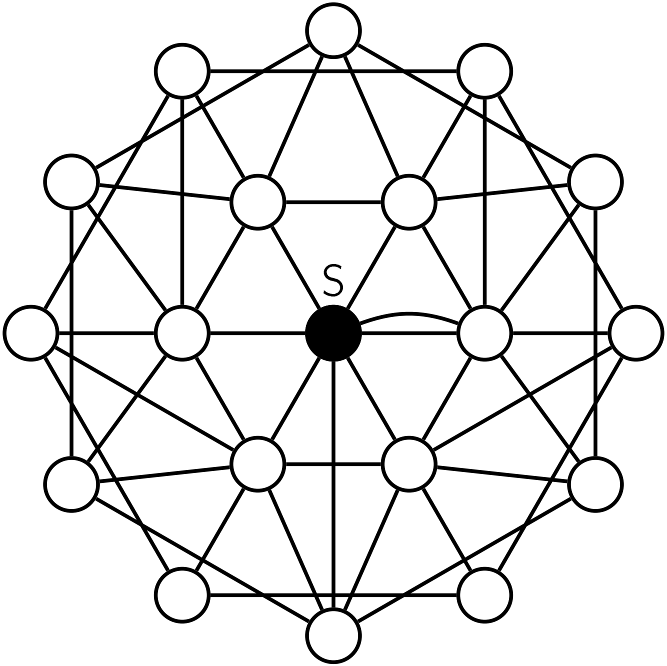
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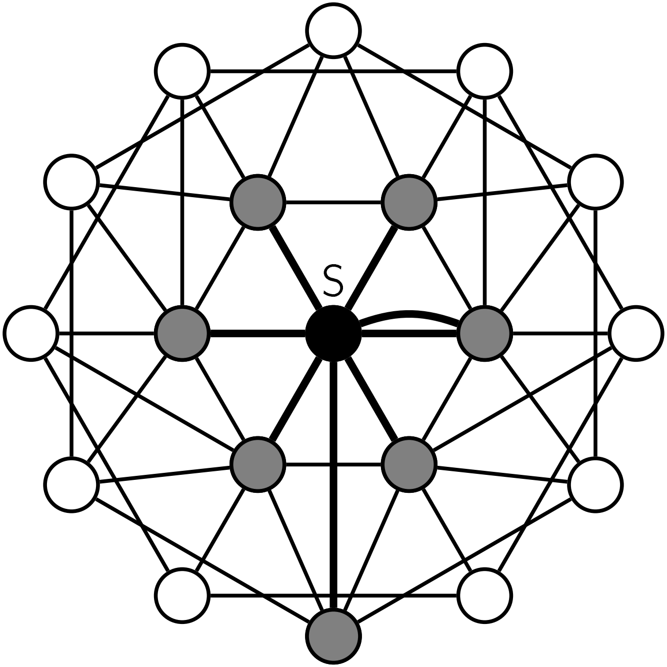
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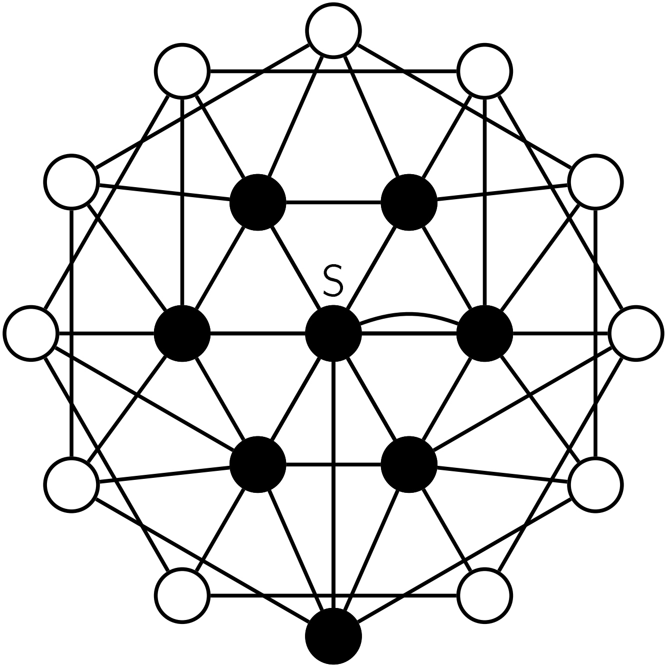
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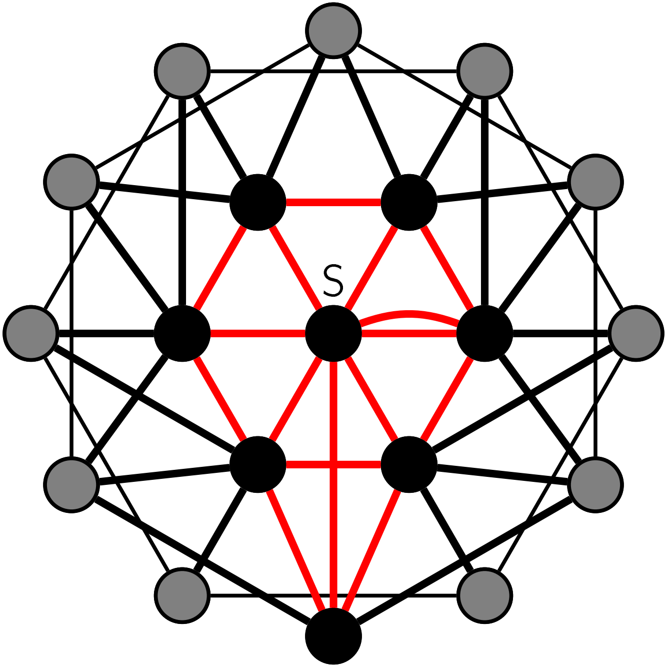
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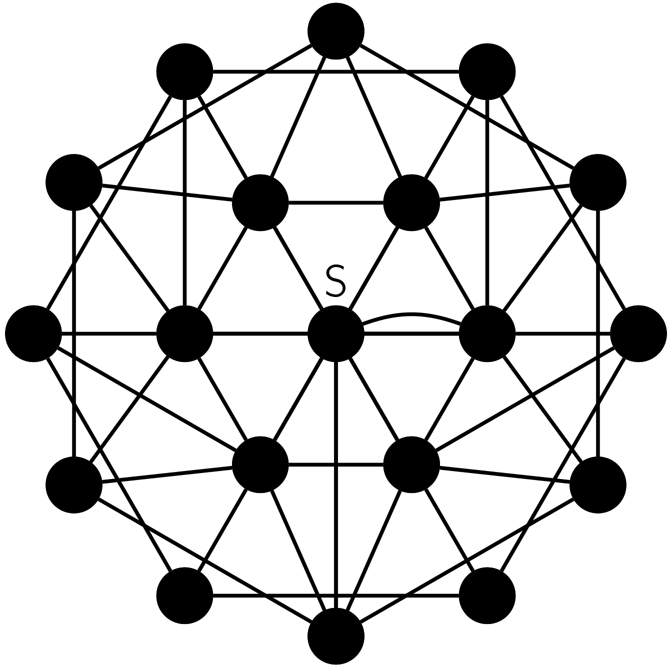
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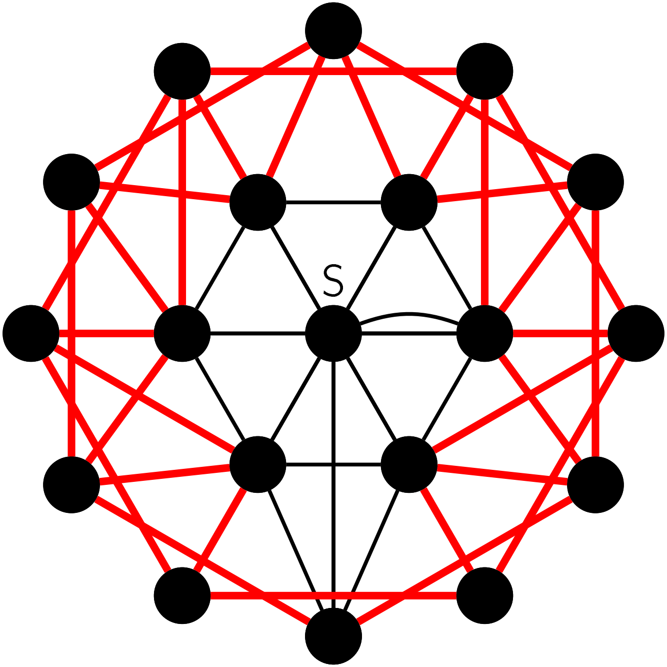
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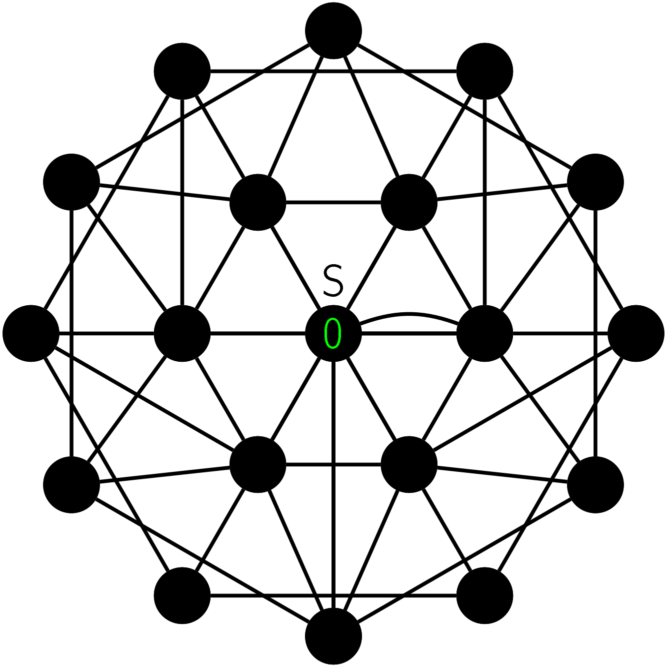
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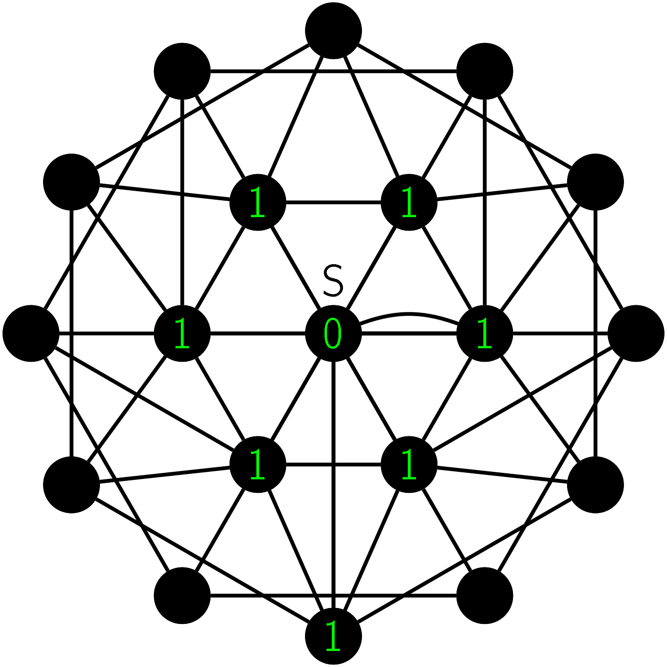
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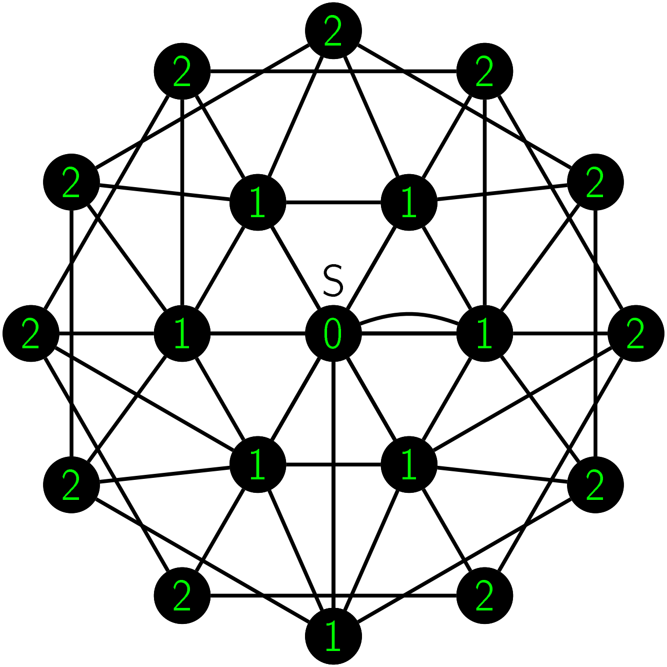
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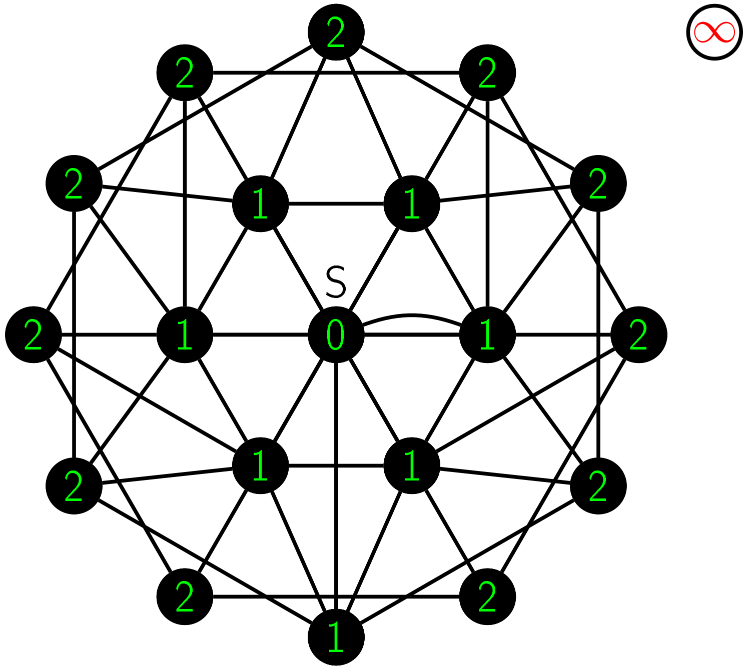
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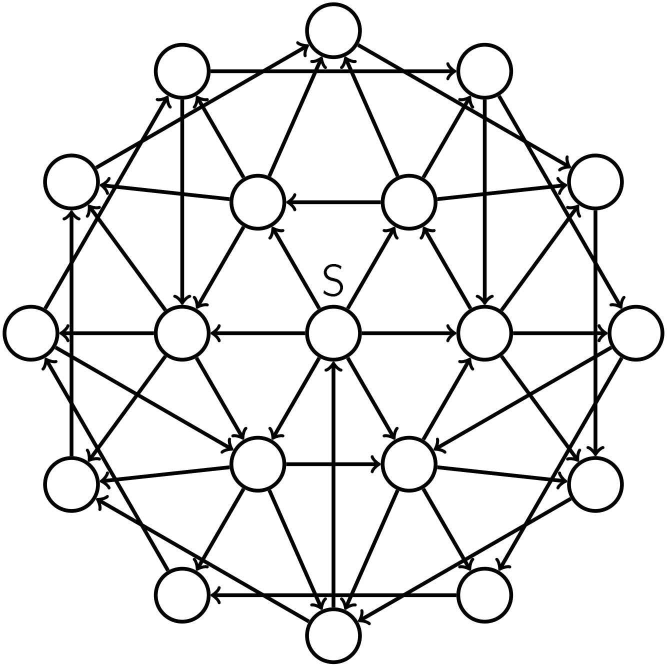
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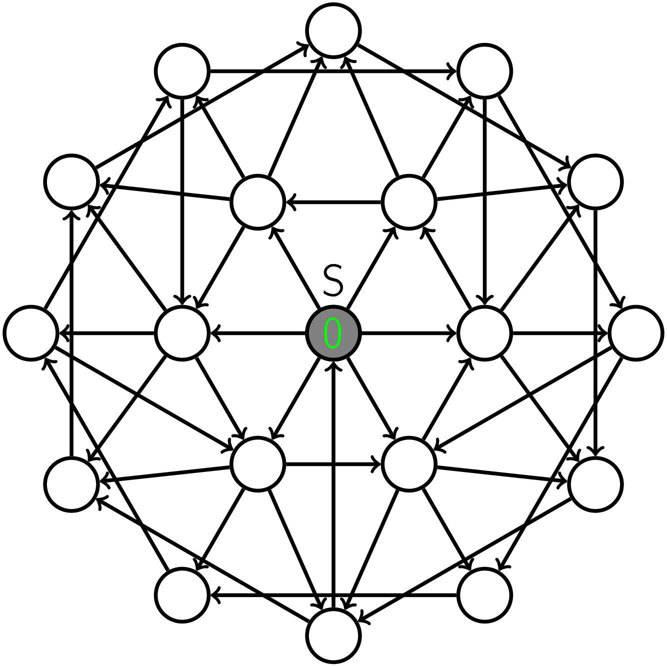
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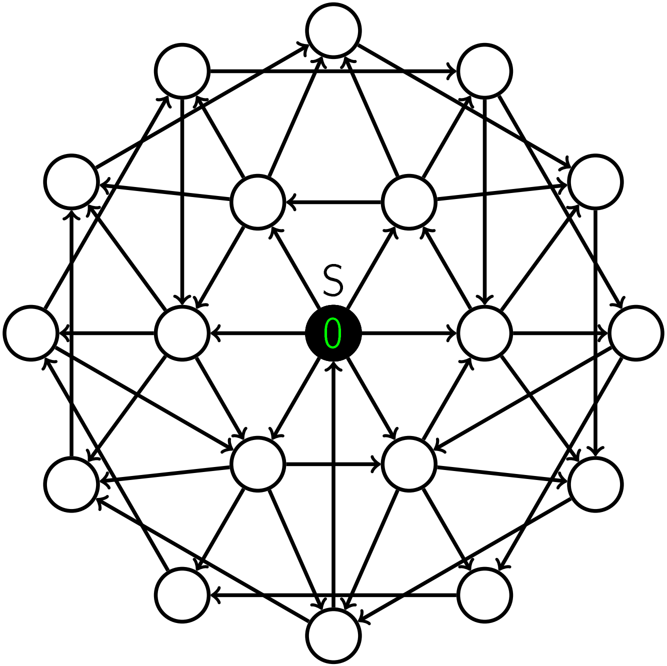
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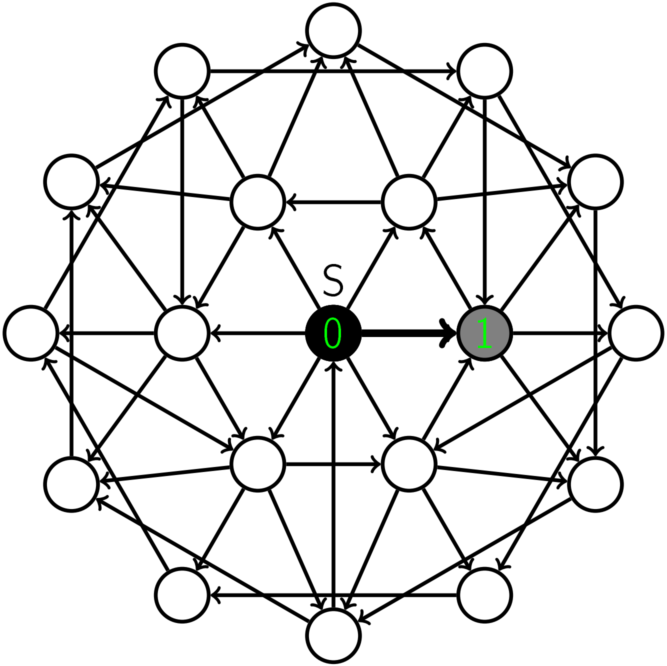
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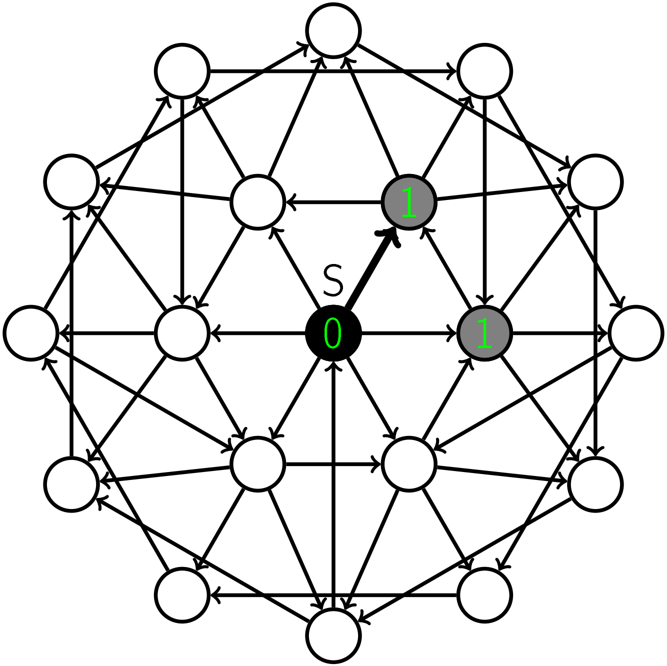
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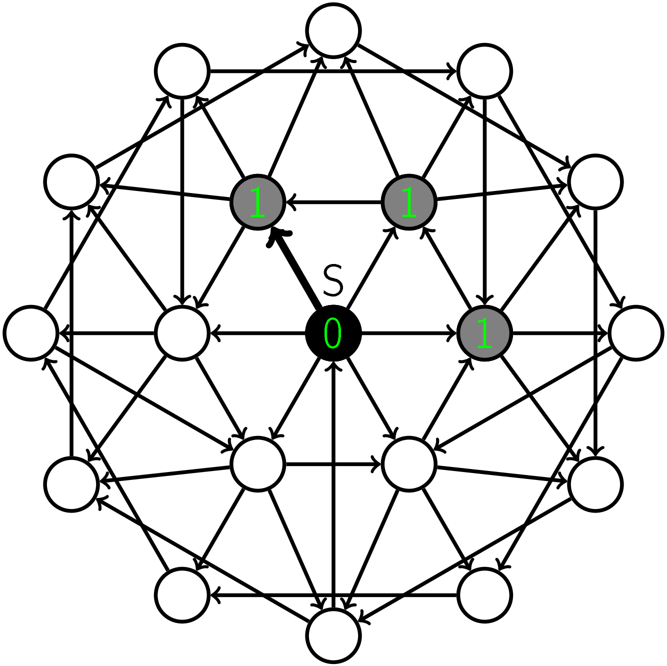
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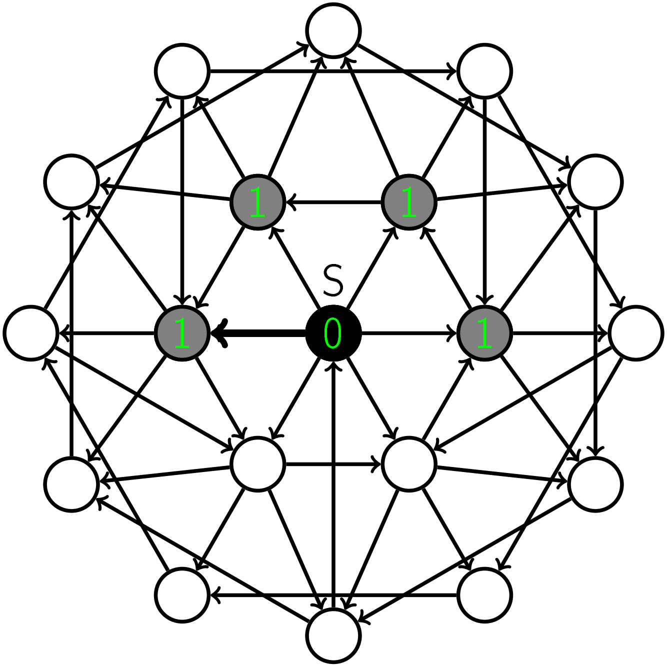
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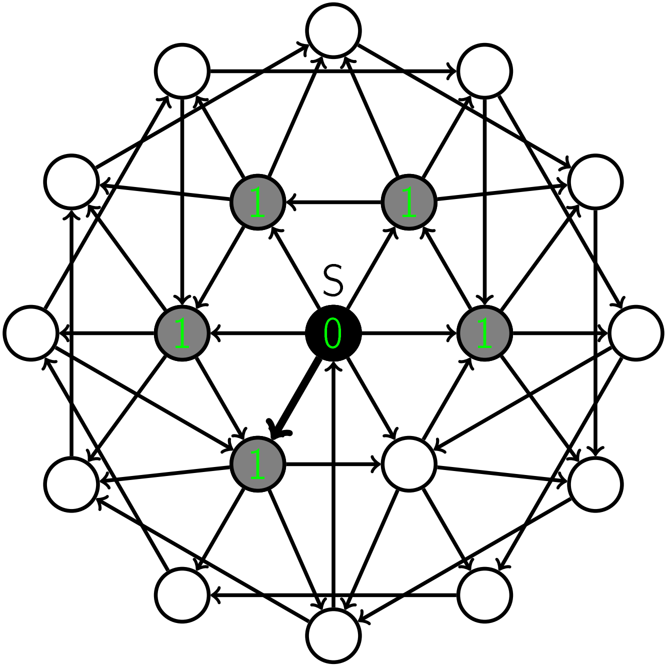
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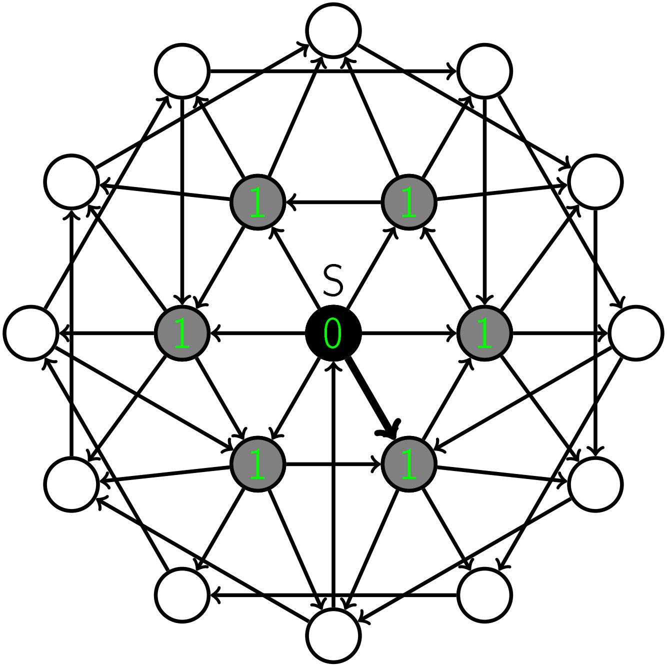
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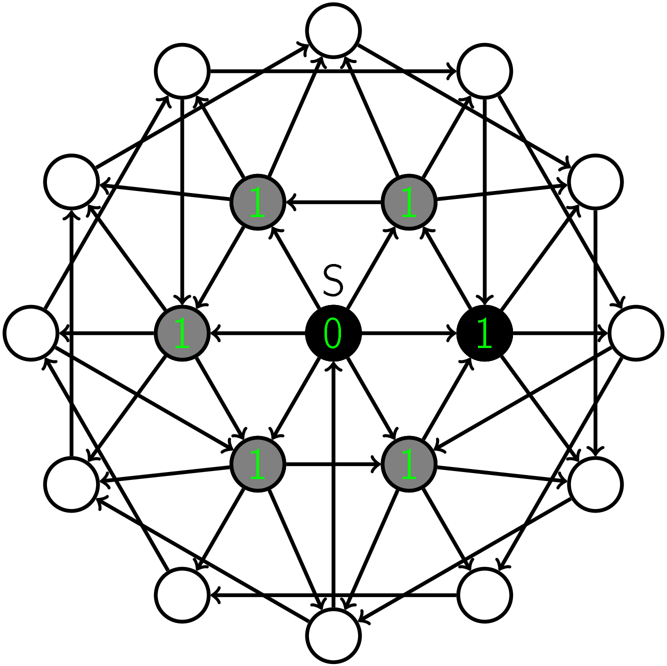

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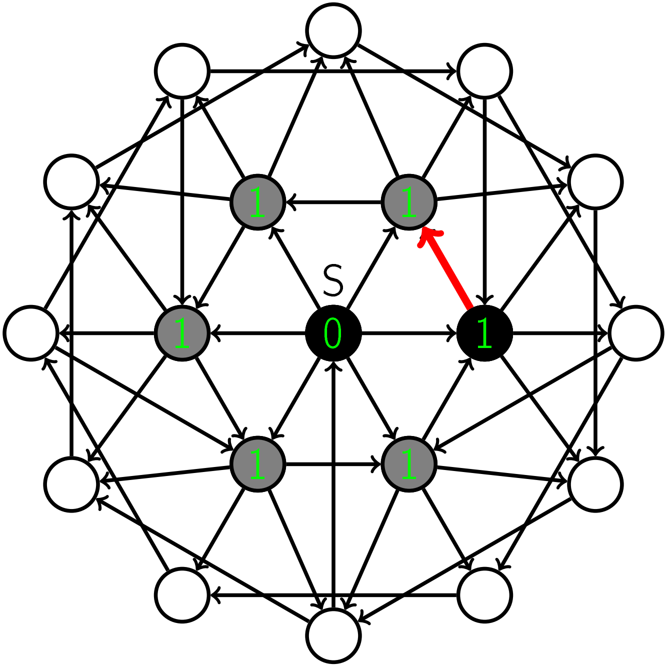
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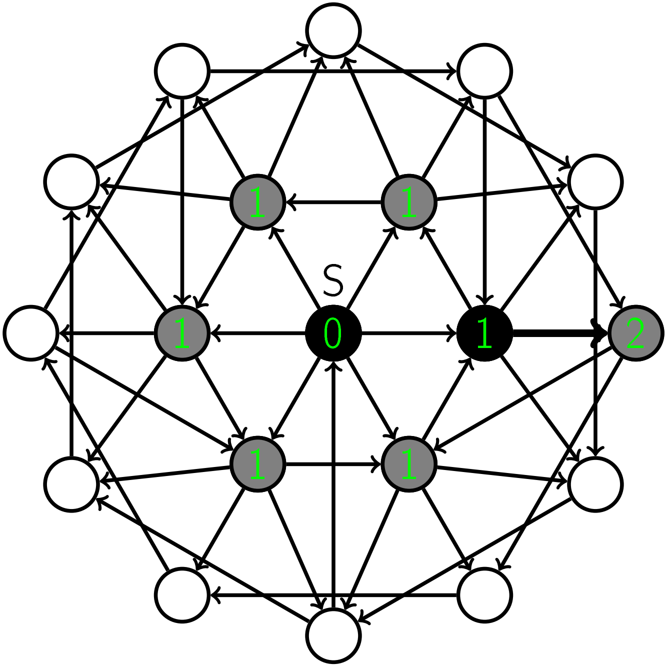
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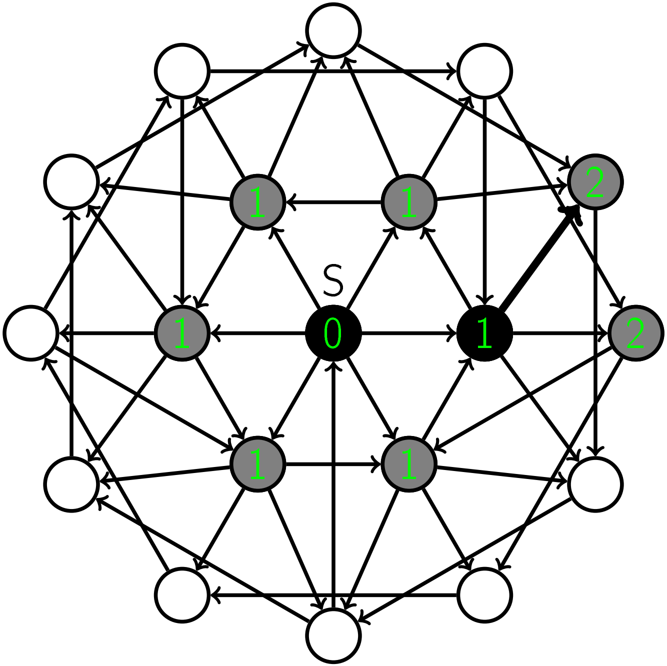
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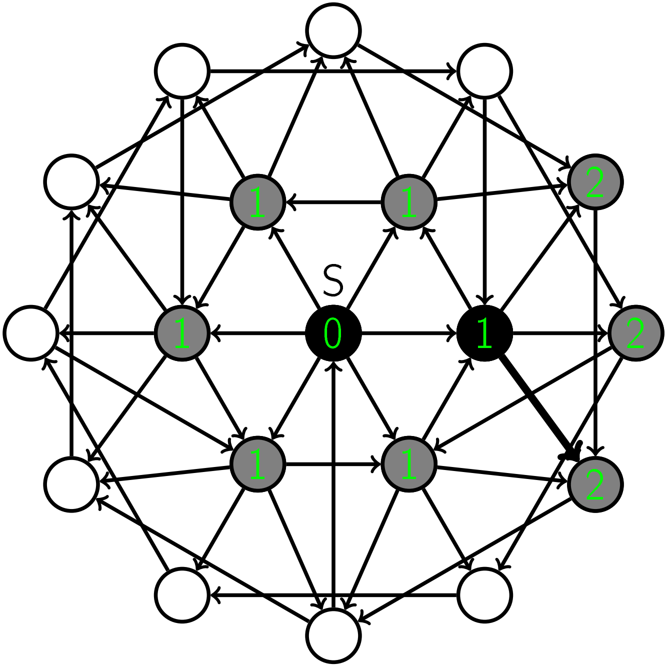
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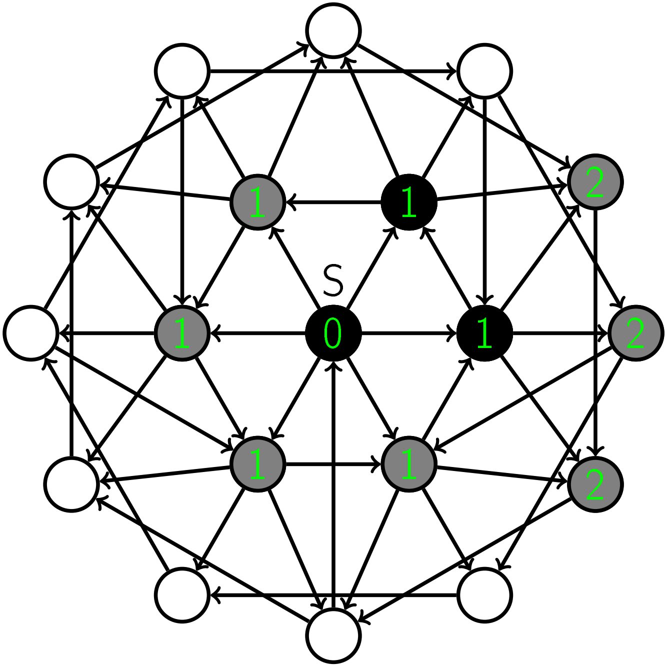

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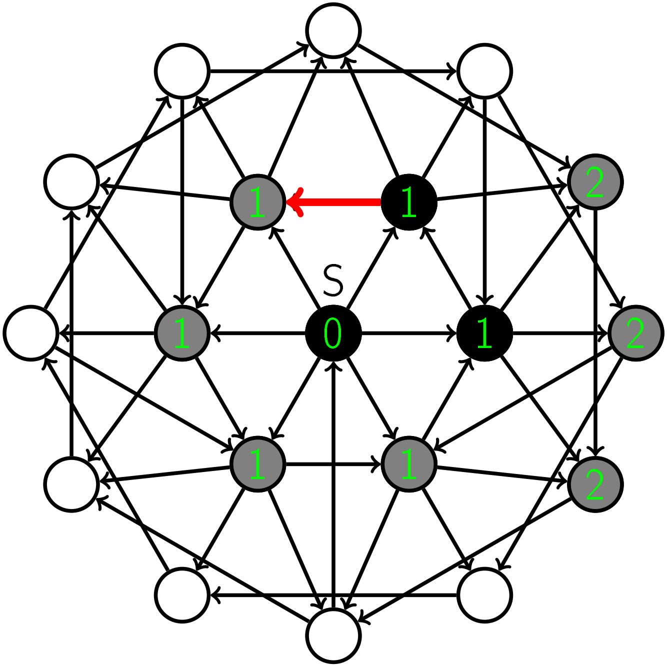
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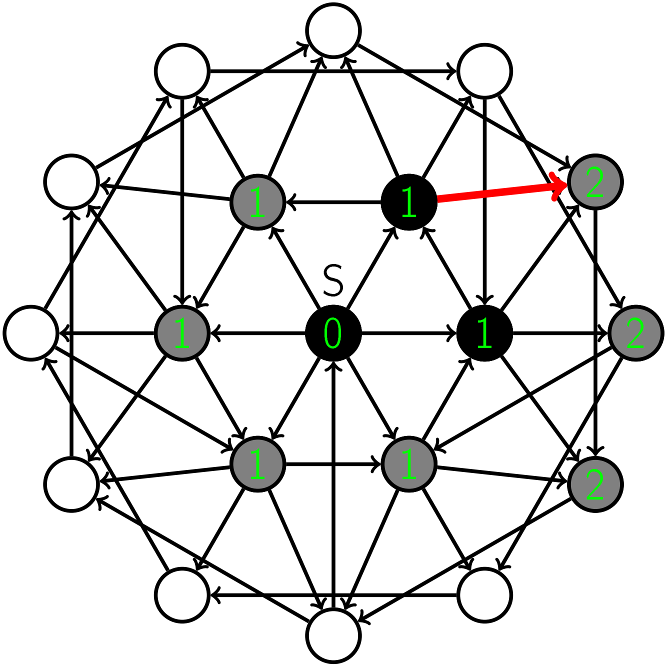

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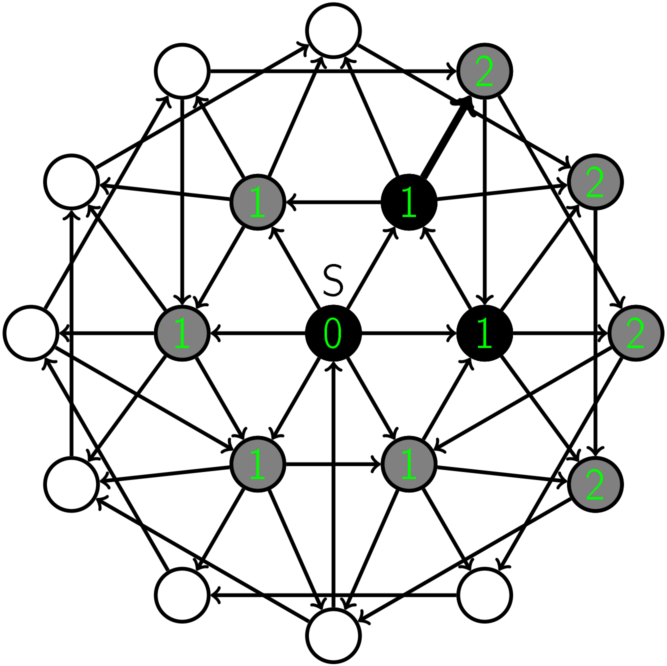
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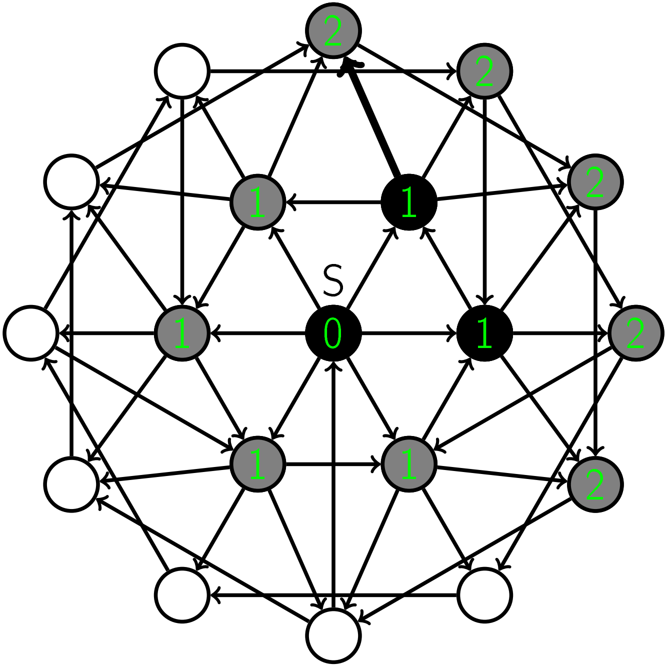

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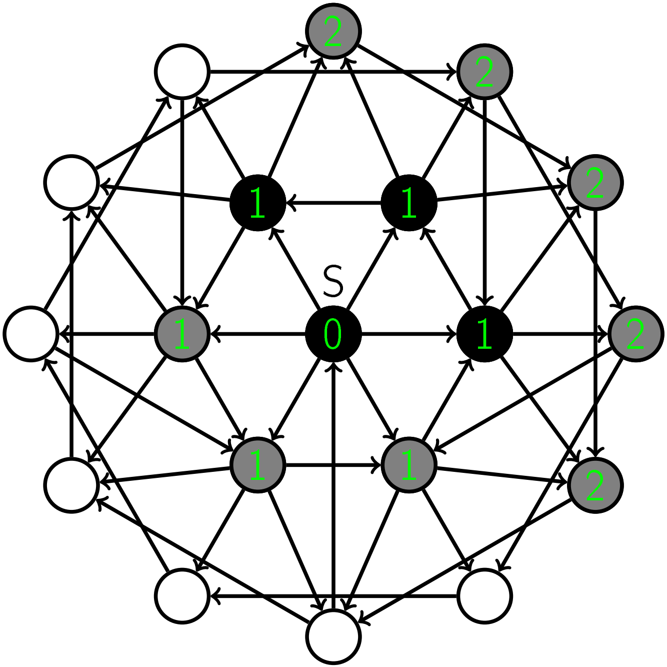
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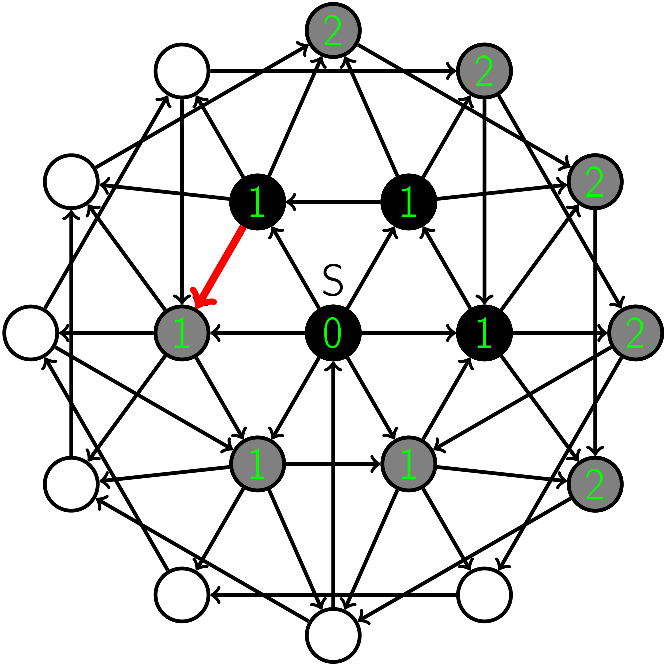

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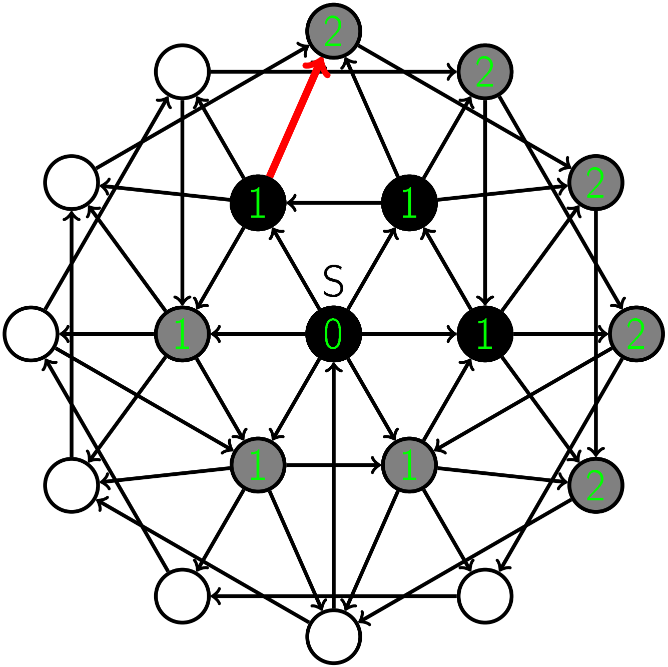
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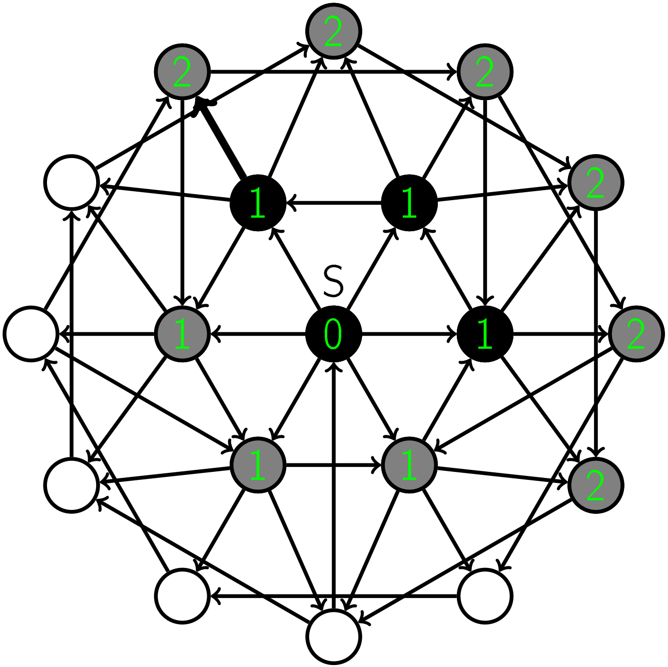

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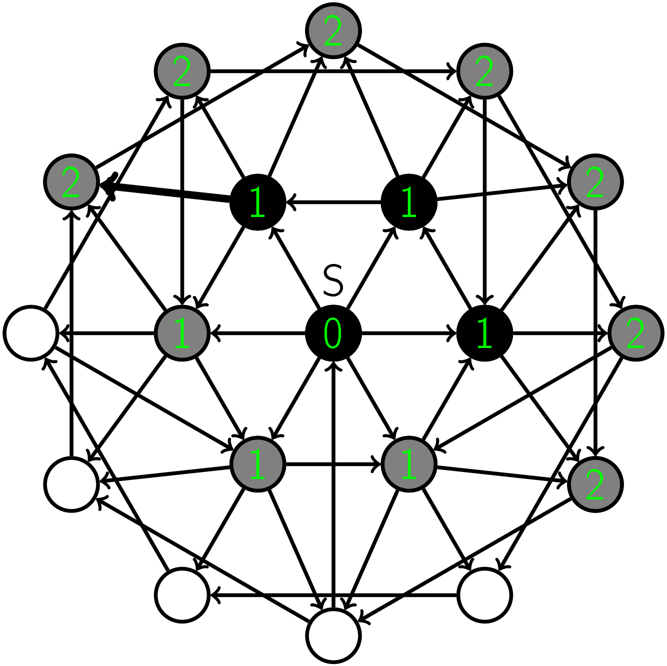
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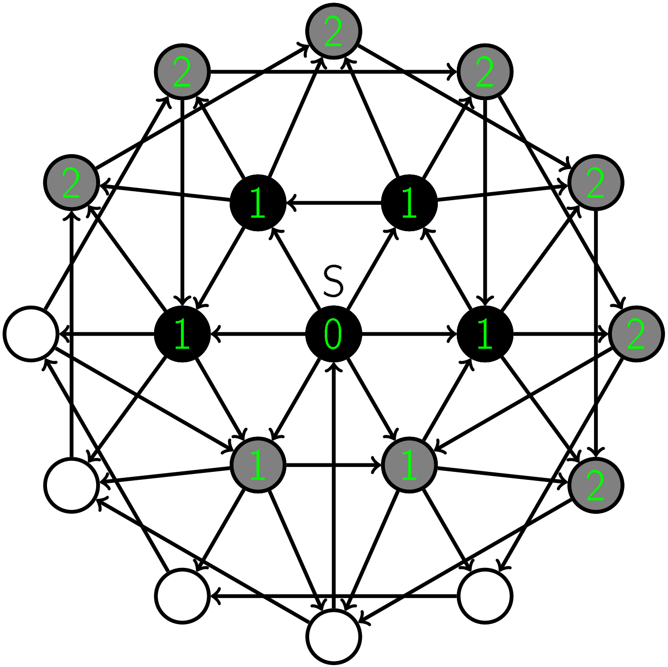

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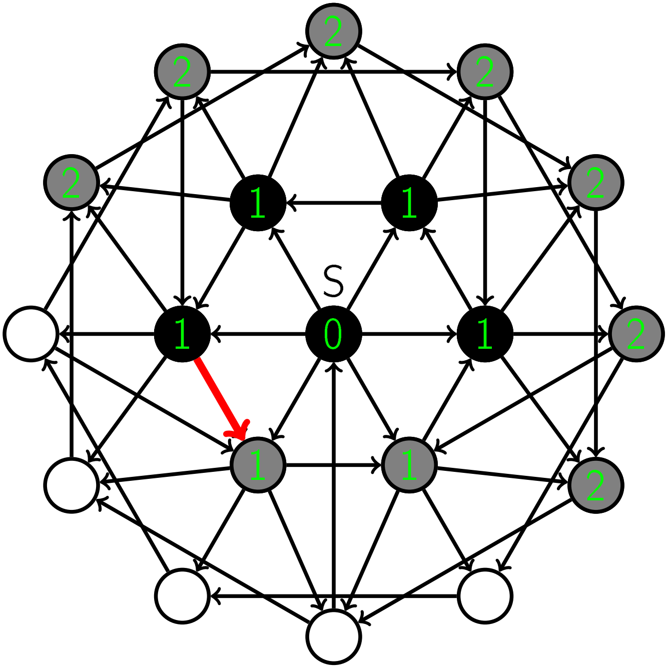

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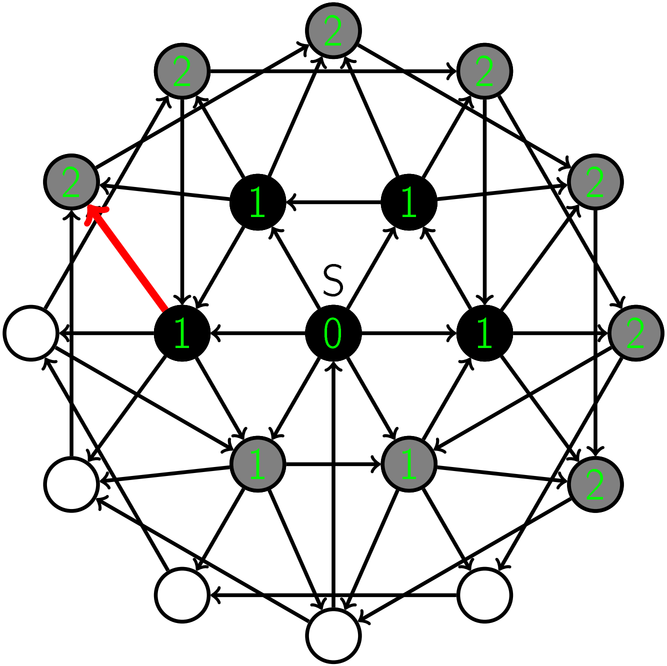
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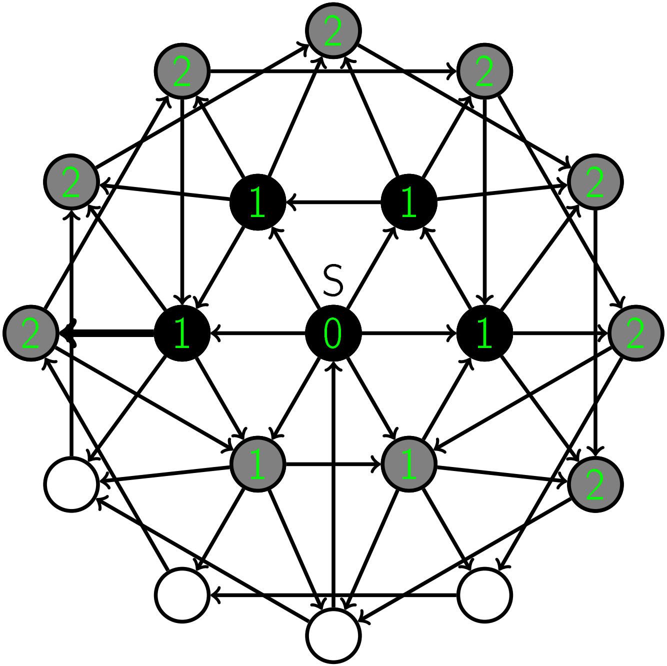
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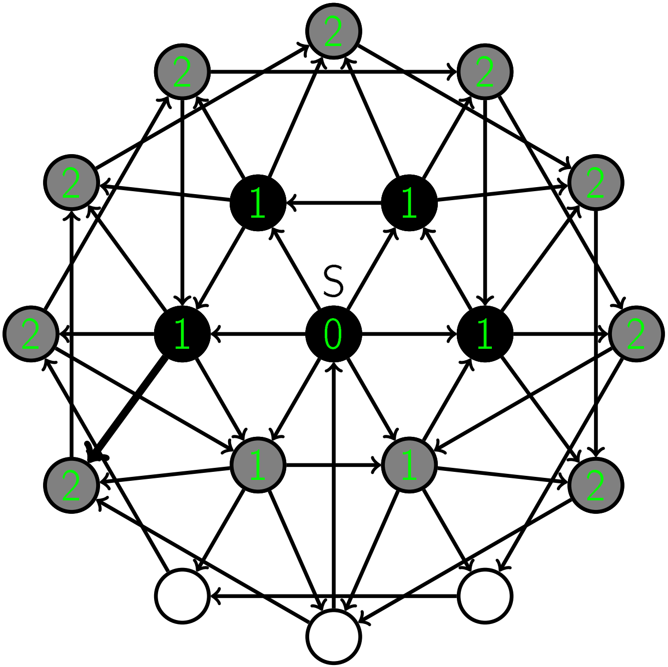
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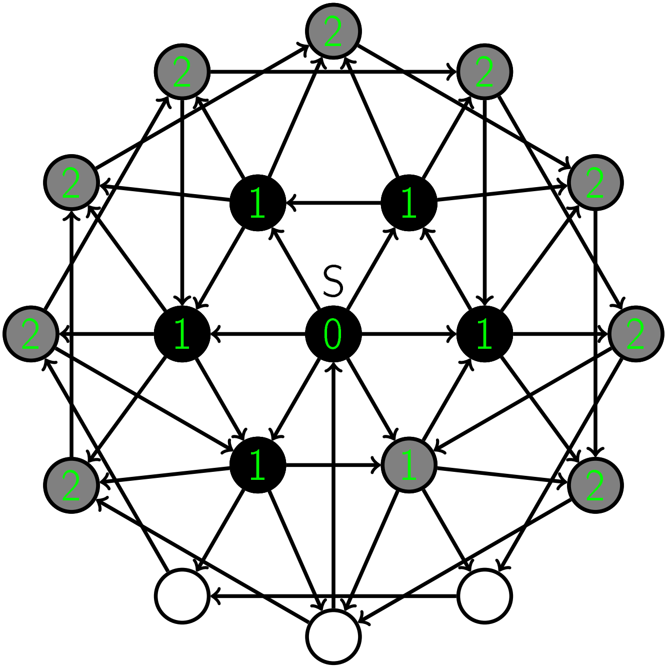

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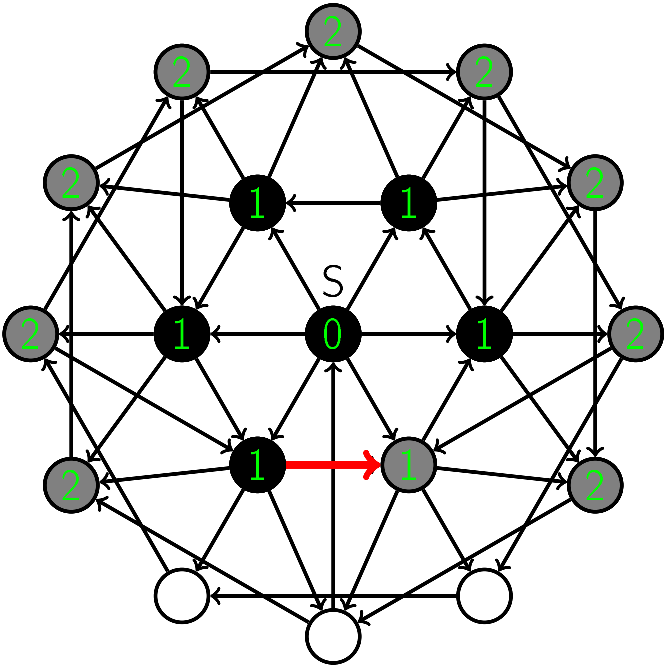
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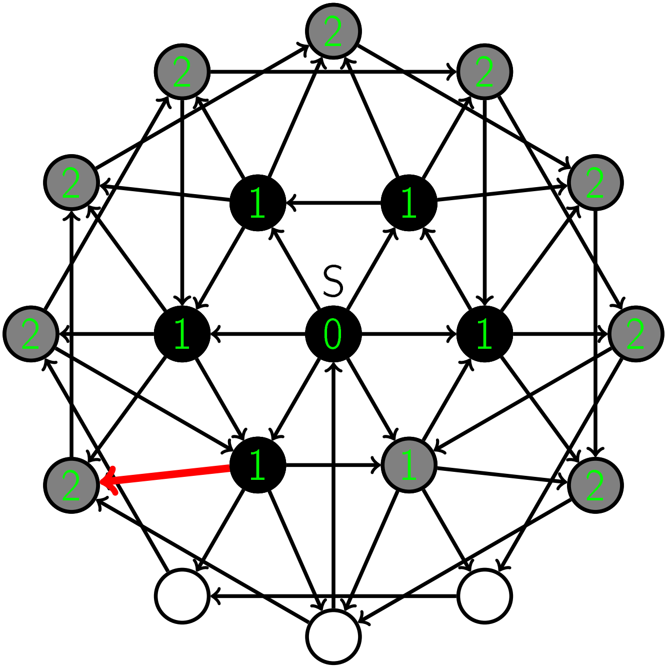
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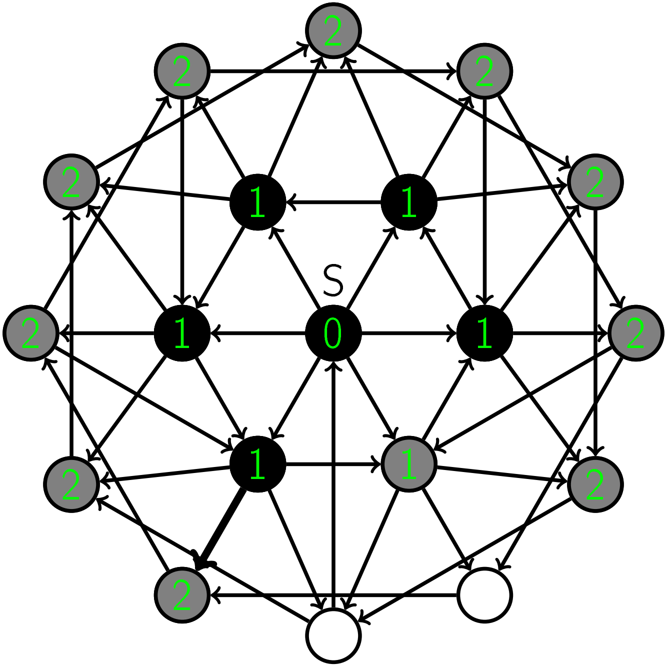

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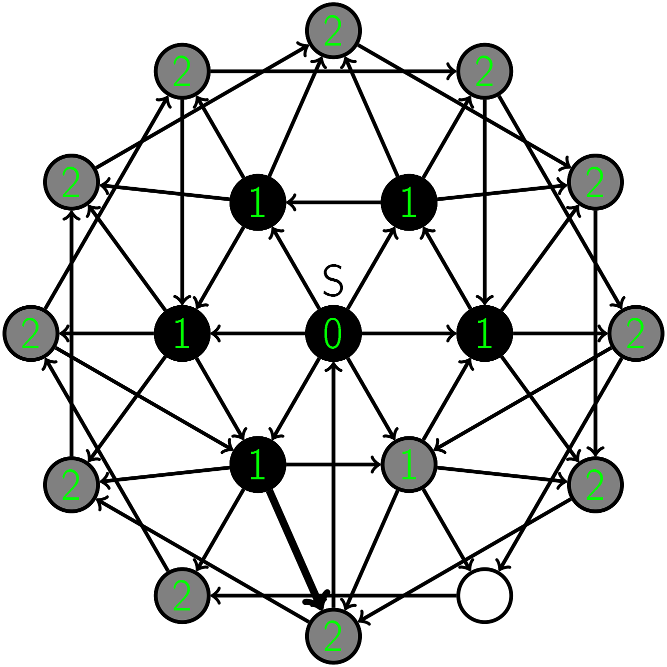

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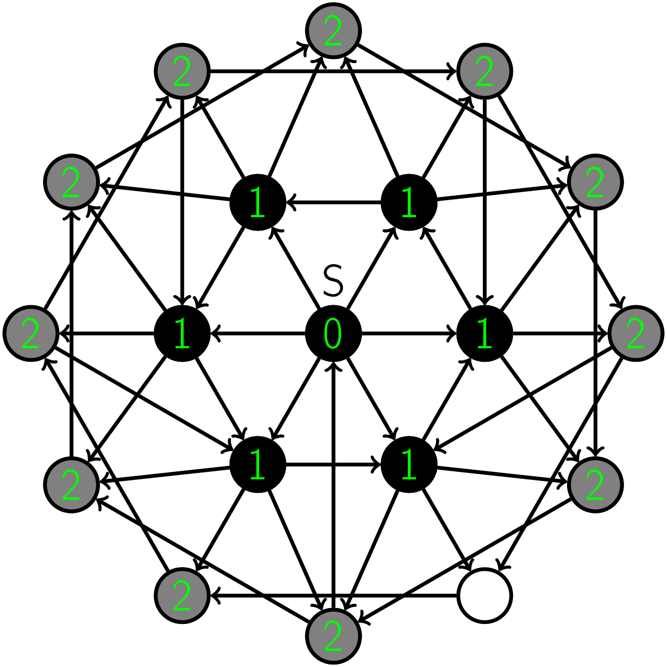
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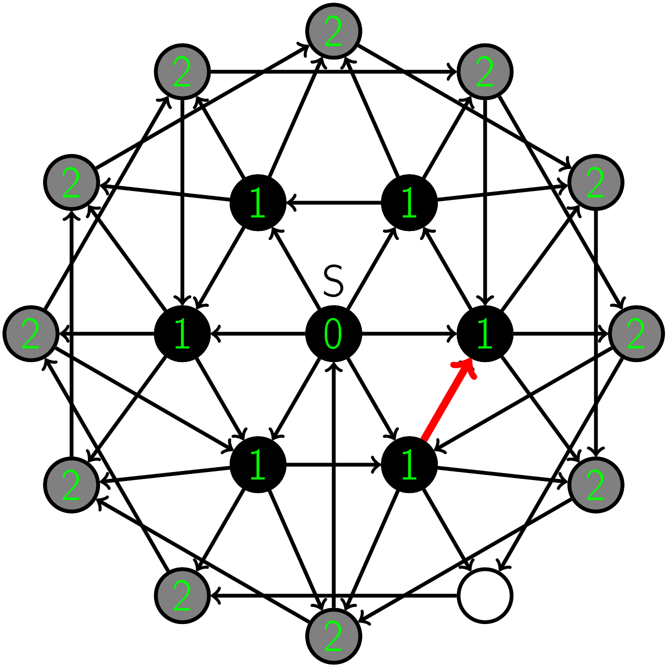
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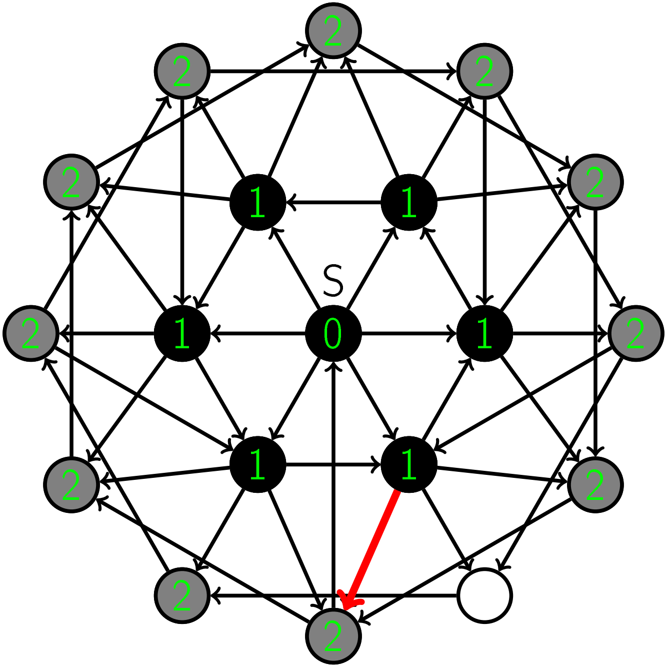
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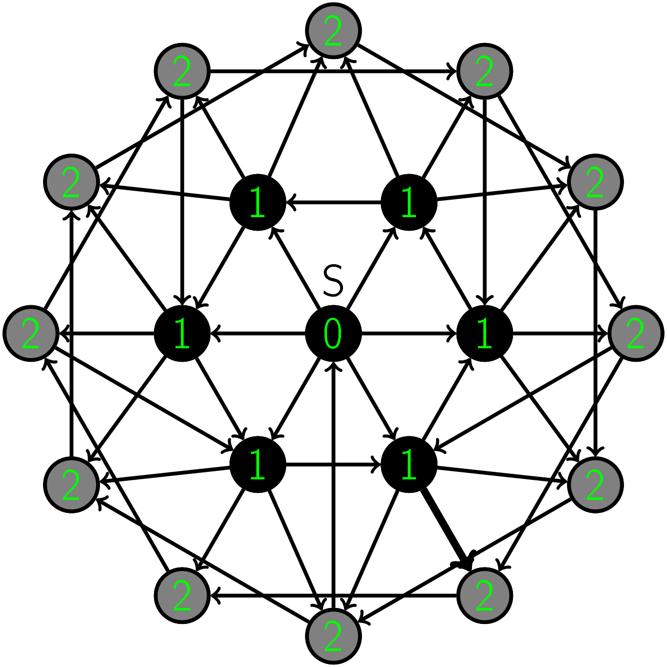
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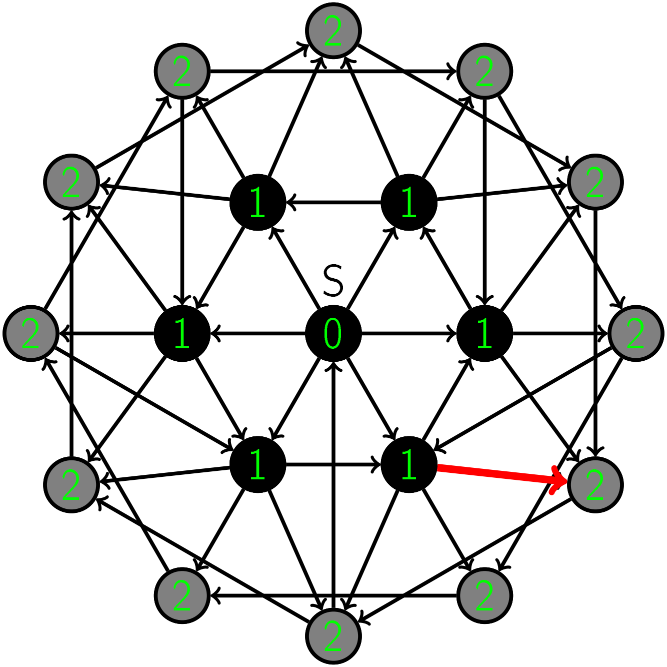
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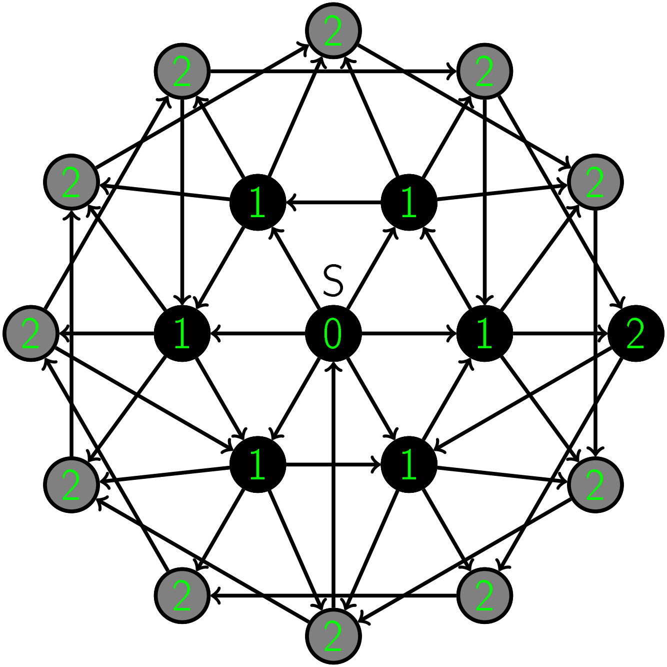
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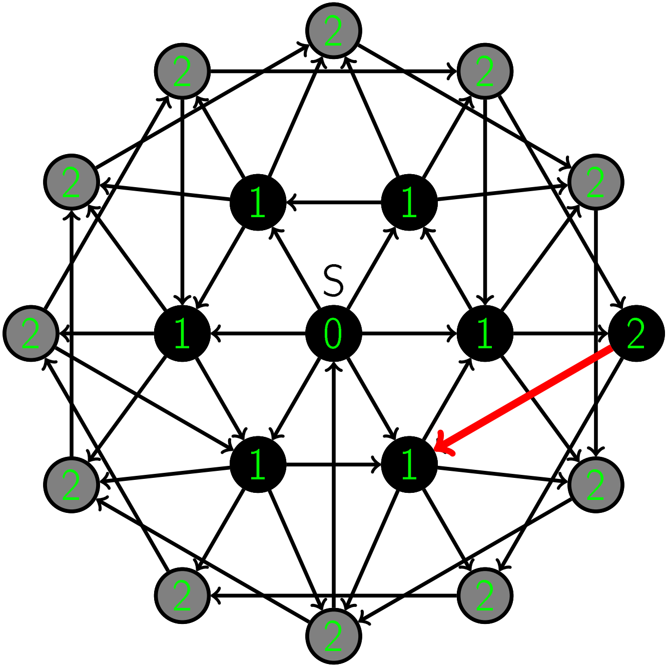
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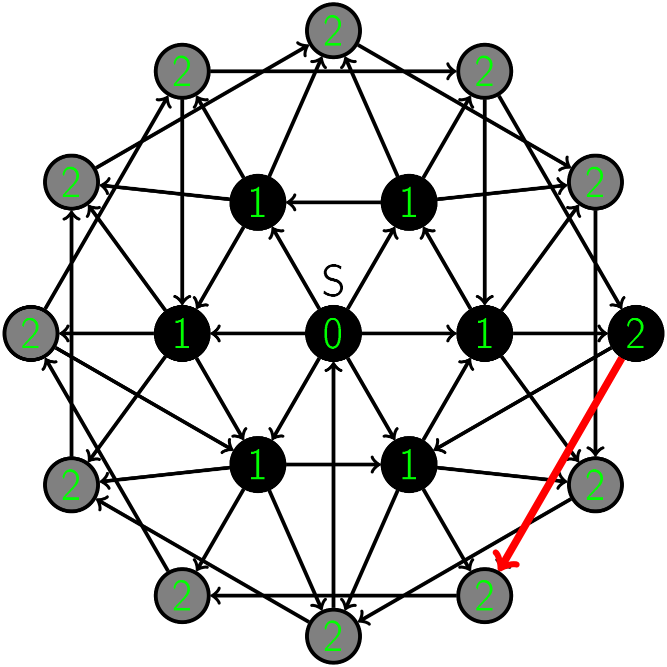
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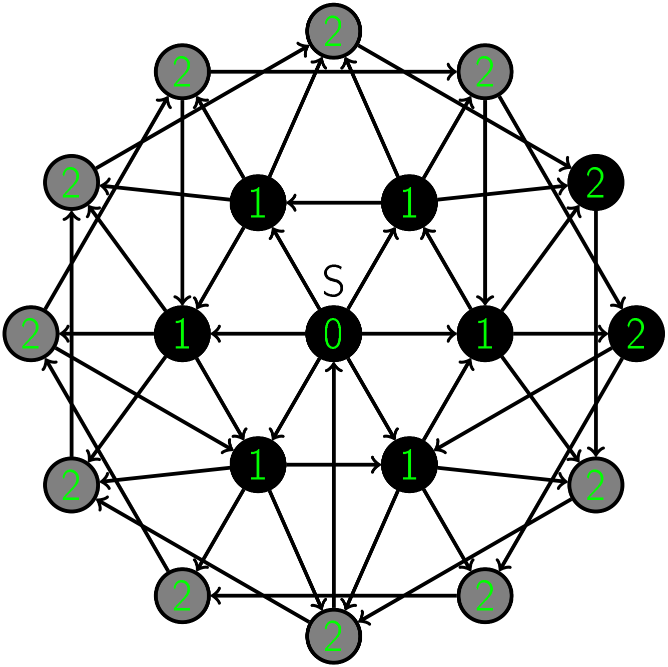
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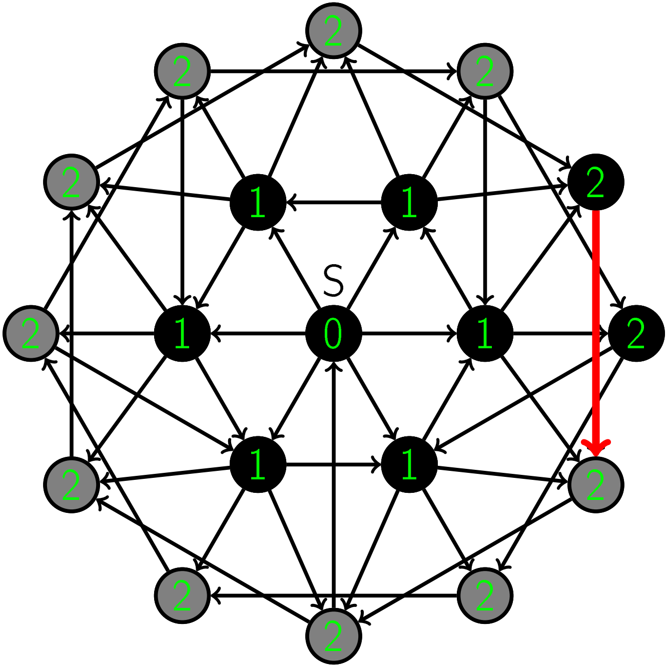
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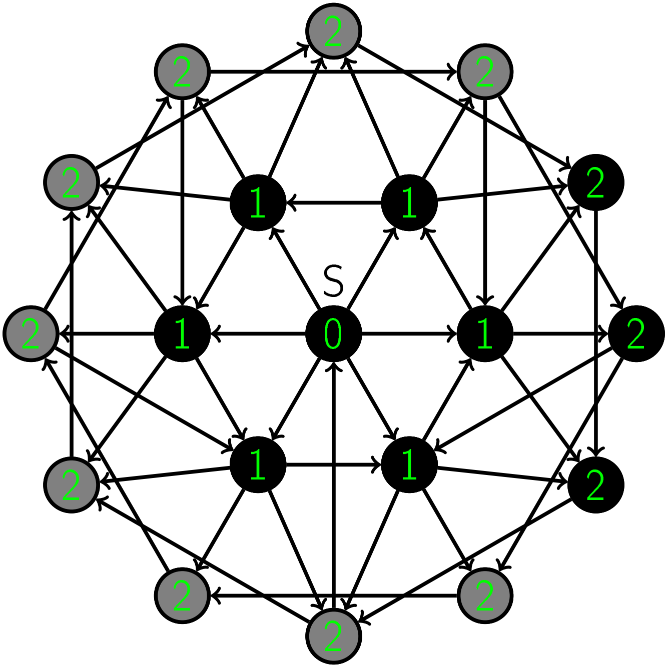
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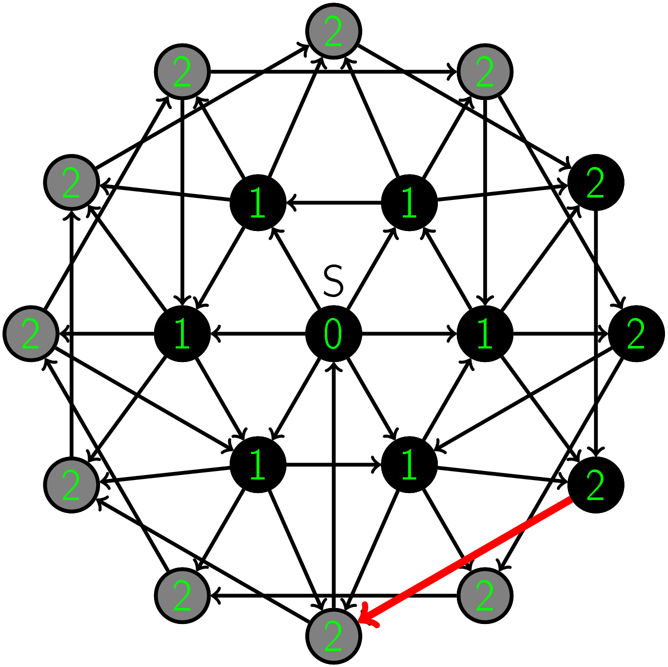
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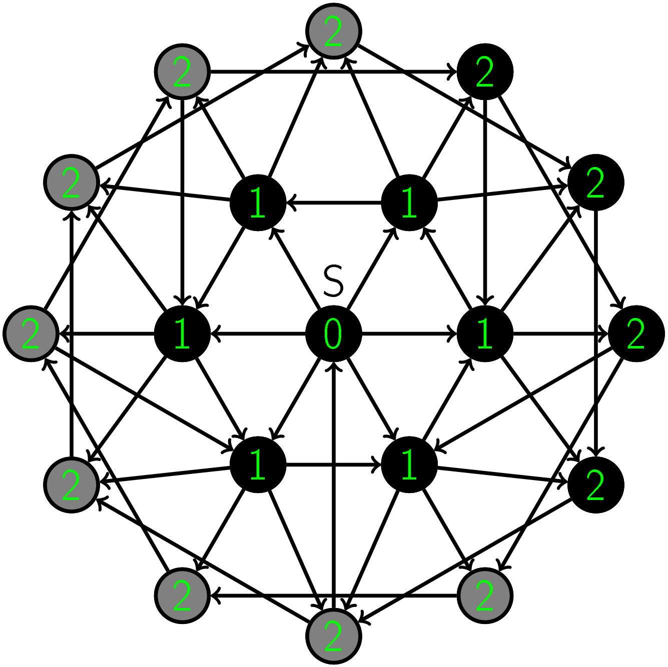

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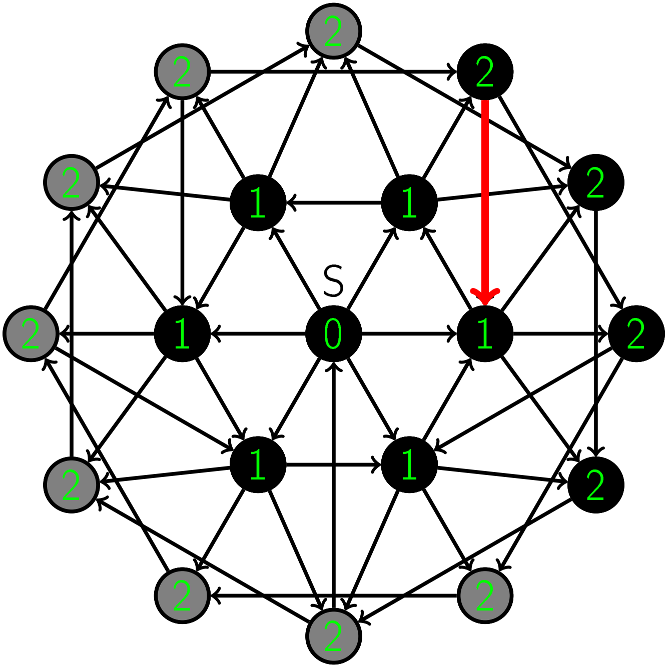

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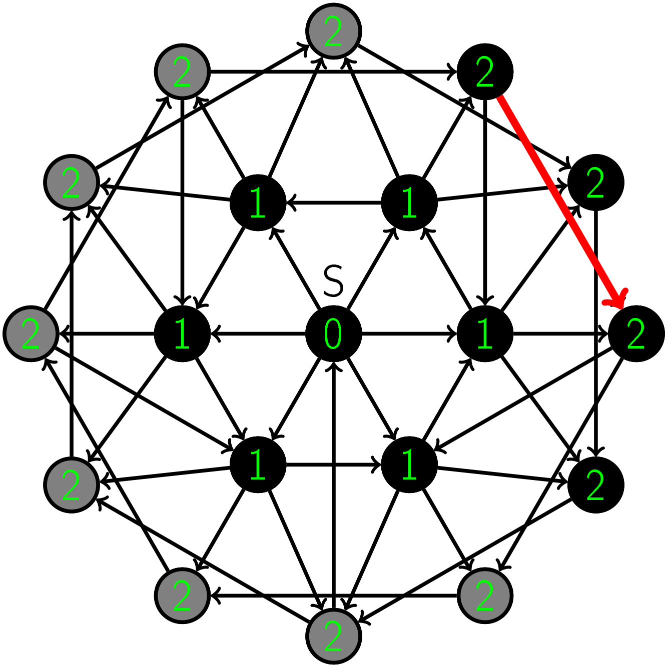
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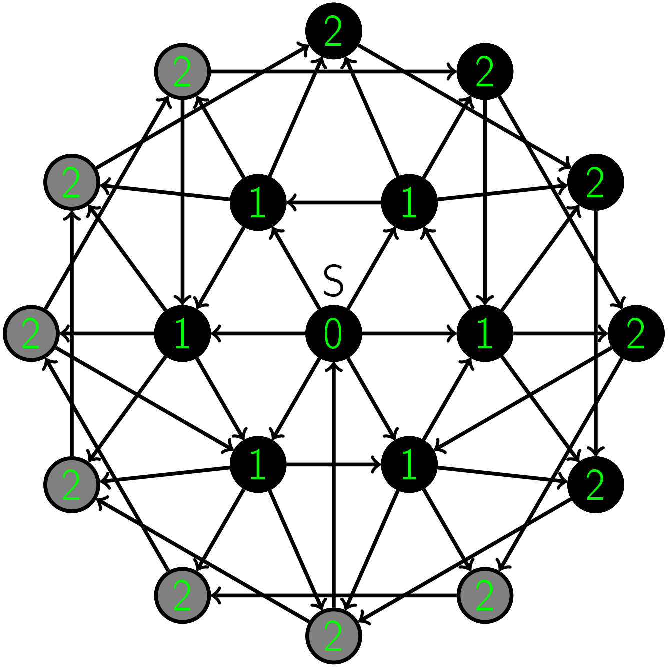
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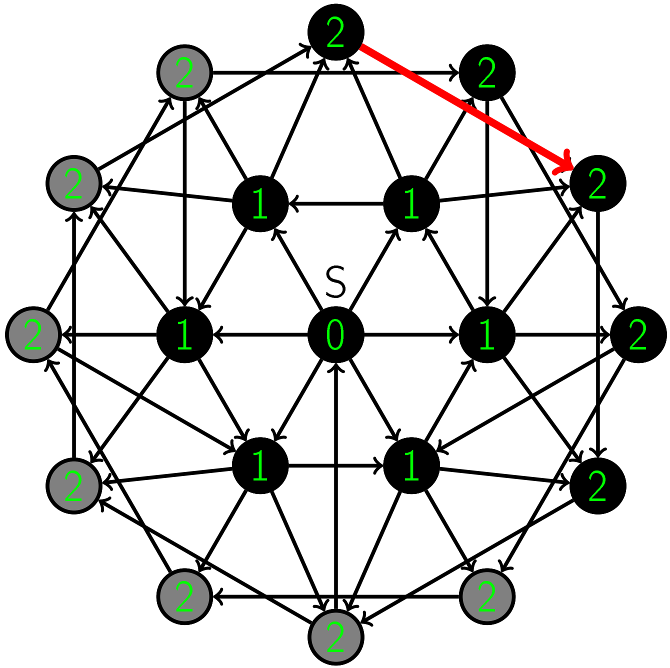

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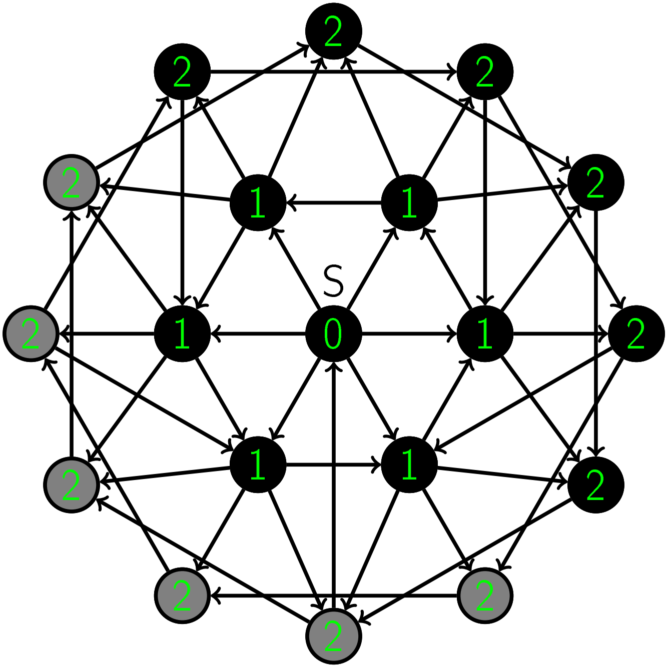
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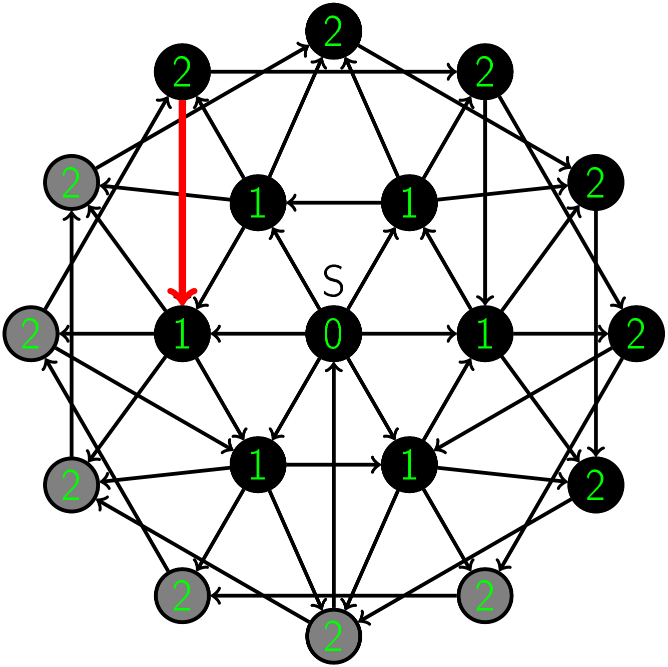
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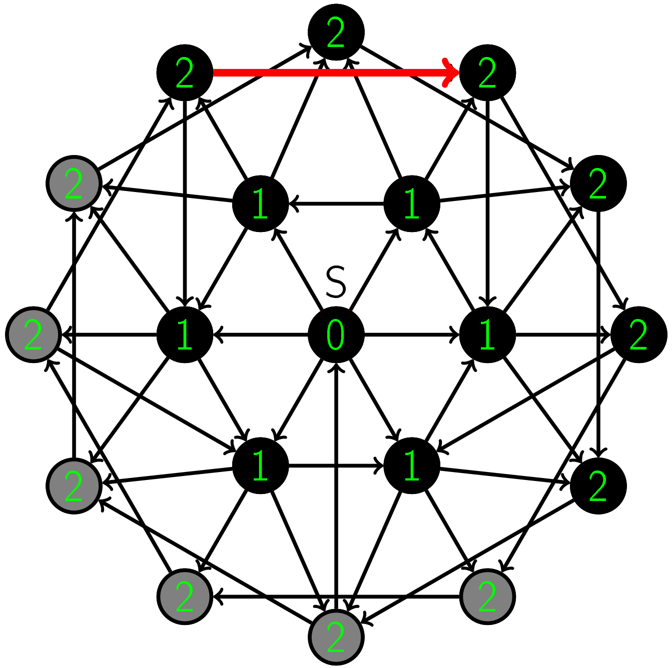

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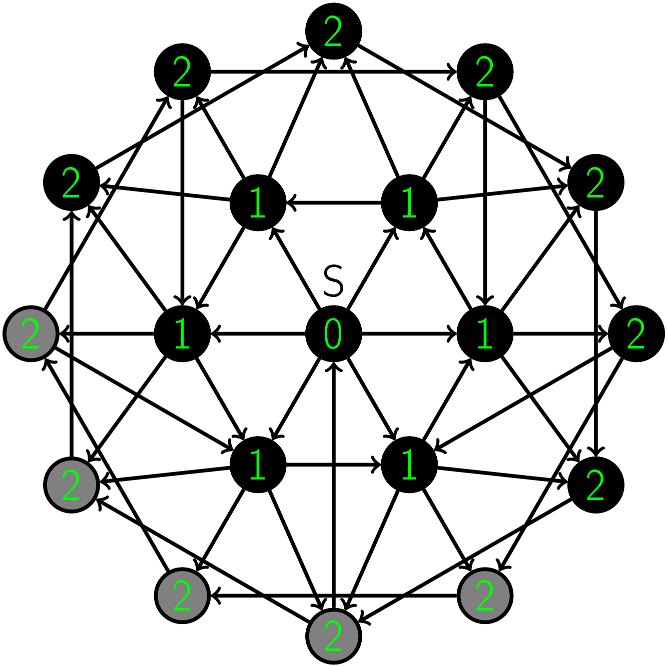
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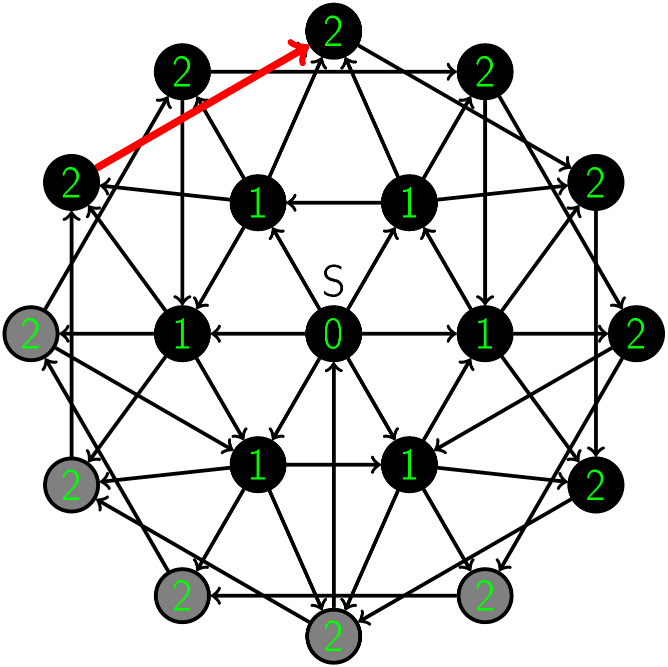

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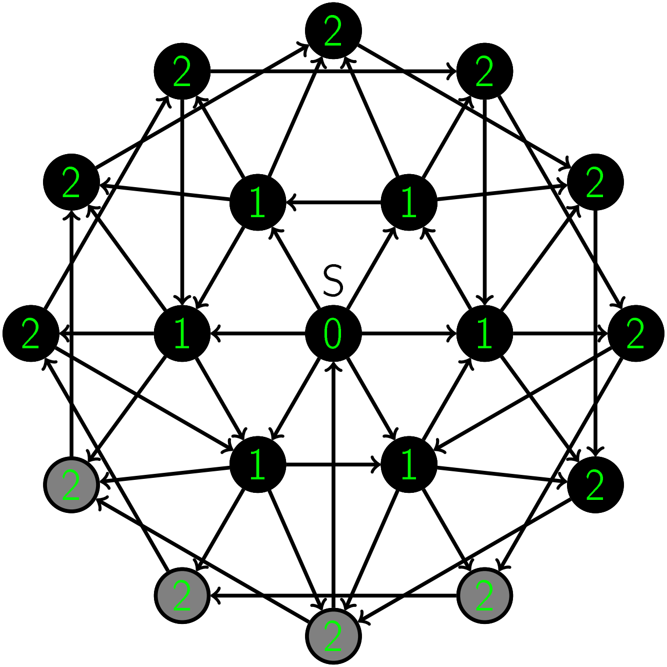
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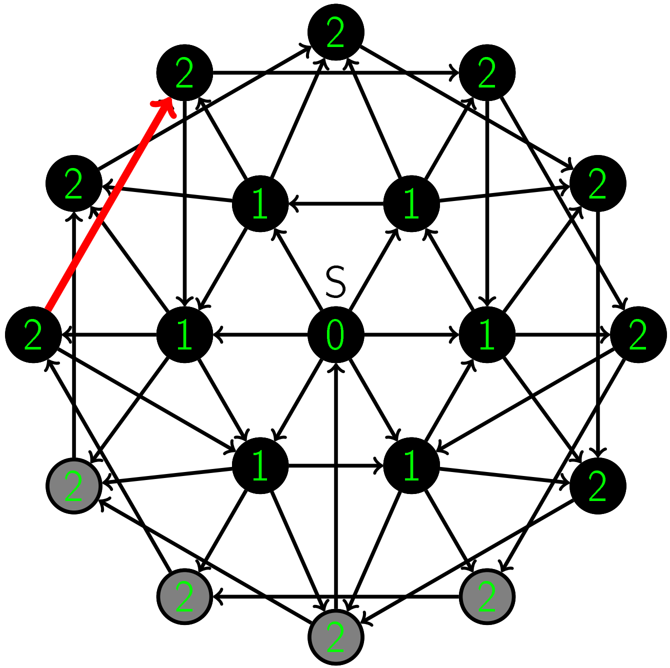

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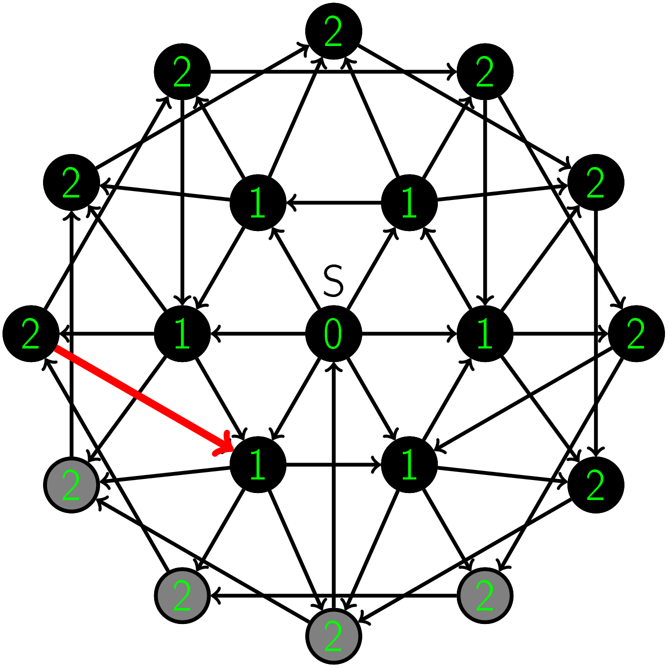
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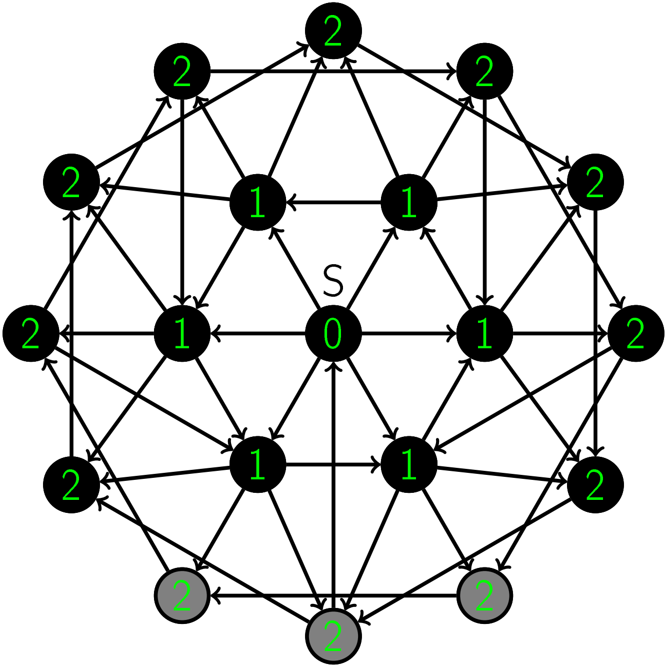
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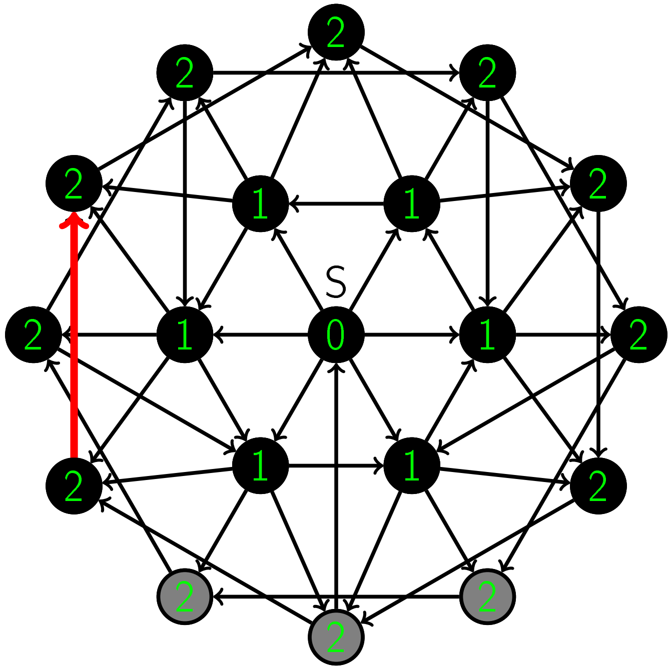
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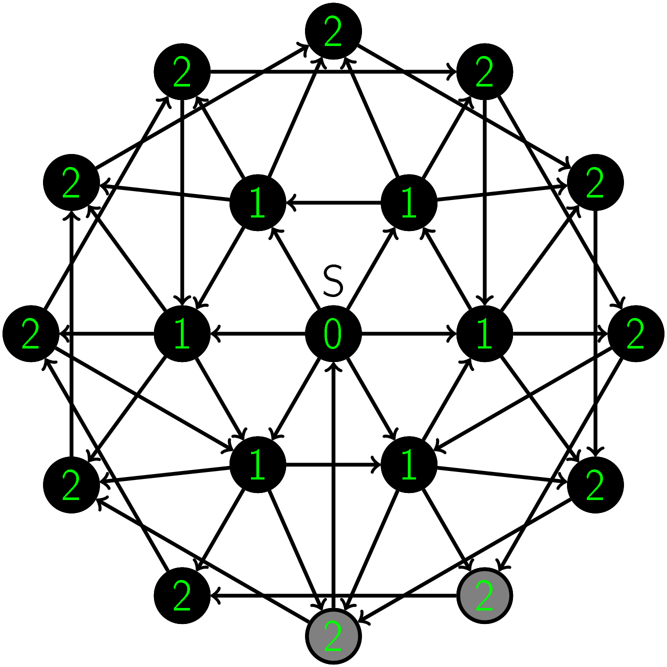
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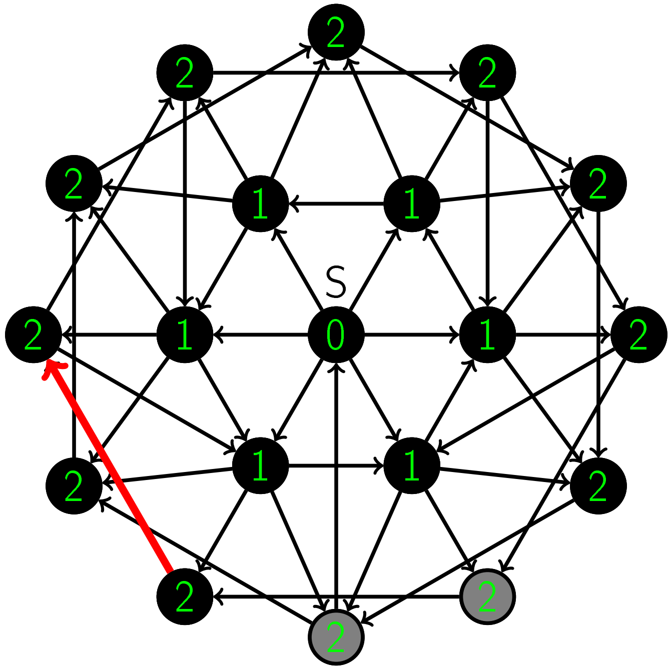
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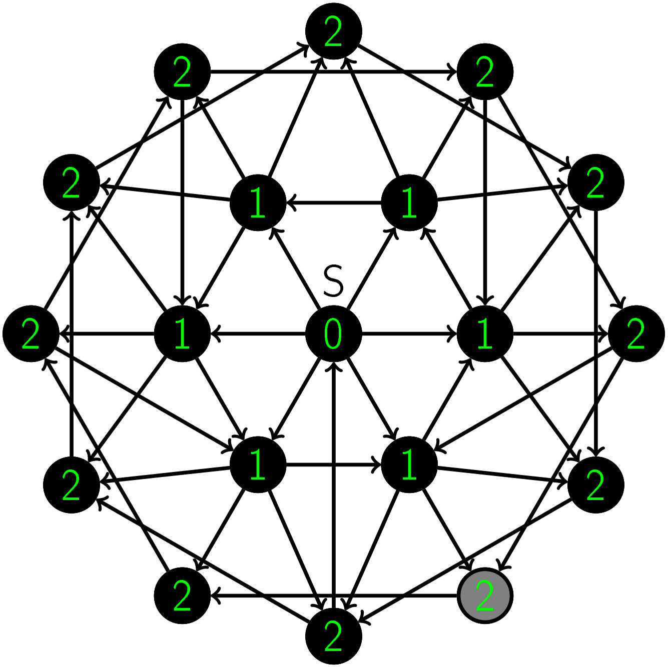
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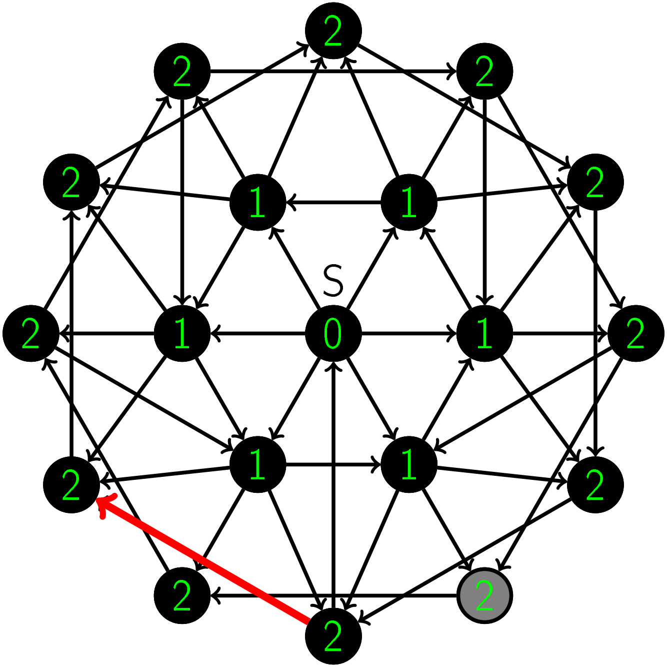
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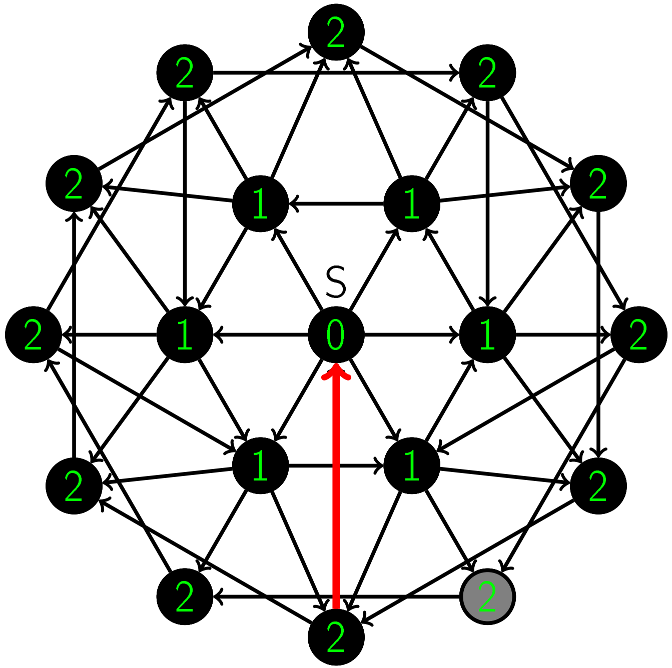
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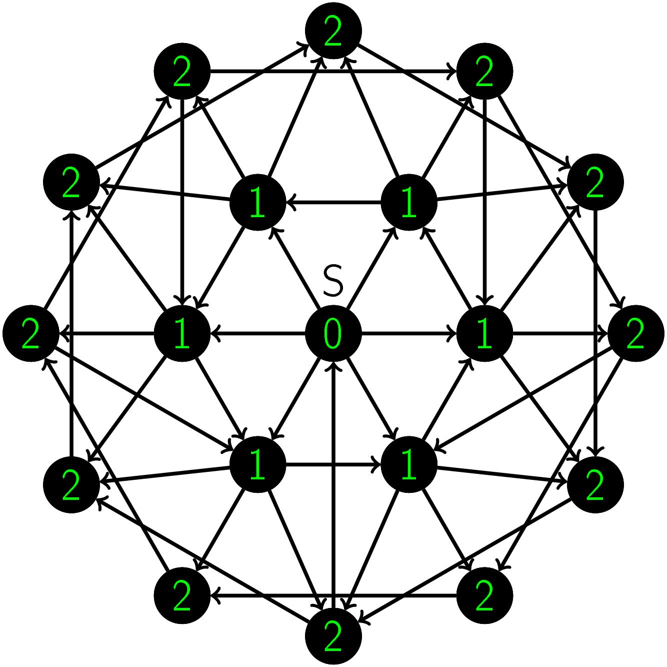
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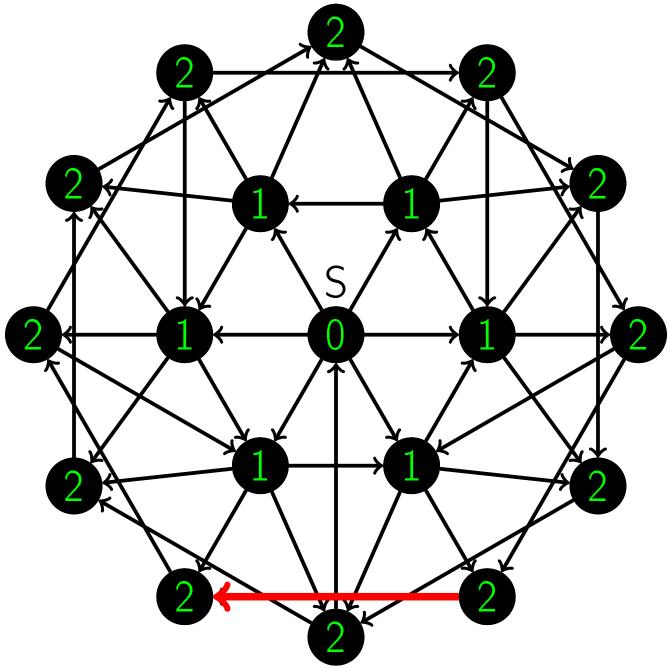
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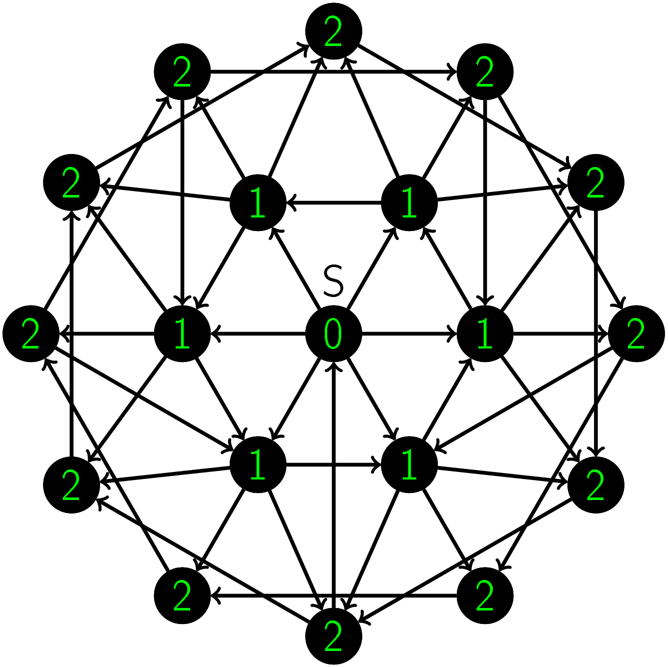
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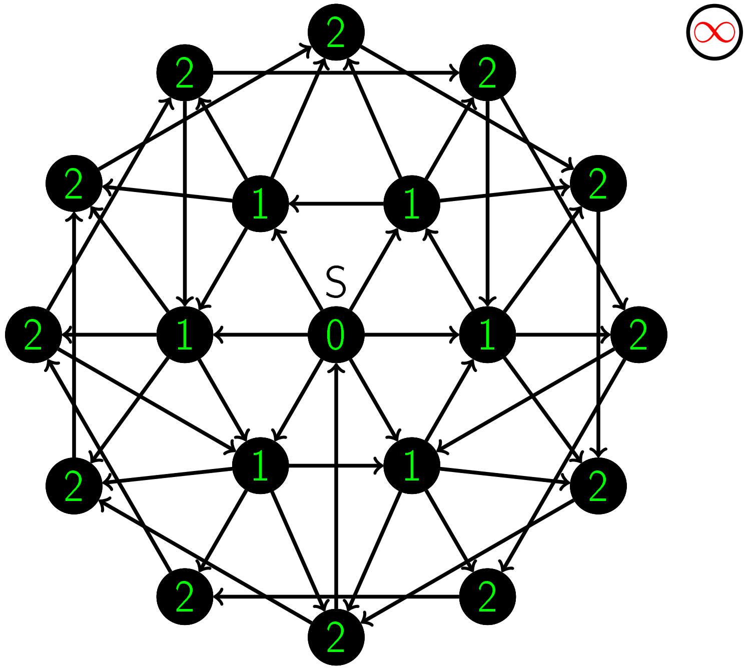
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Outline

Breadth-first search

BFS(G, S)

```
for all $u \in V$:
    dist[$u$] $\leftarrow \infty$
dist[$S$] $\leftarrow 0$
$Q \leftarrow \{S\}$ {queue containing just $S$}
while $Q$ is not empty:
    $u \leftarrow$ Dequeue($Q$)
    for all $(u, v) \in E$:
        if dist[$v$] $= \infty$:
            Enqueue($Q, v$)
            dist[$v$] $\leftarrow$ dist[$u$] $+ 1$
```

Running time

Lemma

The running time of breadth-first search is $O(|E| + |V|)$.

Proof

Running time

Lemma

The running time of breadth-first search is $O(|E| + |V|)$.

Proof

- Each vertex is enqueued at most once

Running time

Lemma

The running time of breadth-first search is $O(|E| + |V|)$.

Proof

- Each vertex is enqueued at most once
- Each edge is examined either once (for directed graphs) or twice (for undirected graphs) $\square$

Outline

Reachability

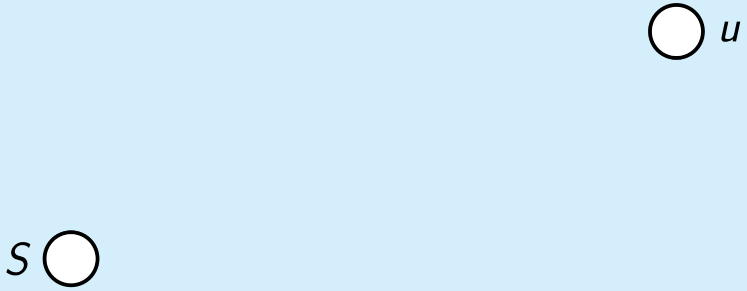
Definition

Node u is reachable from node S if there is a path from S to u

Lemma

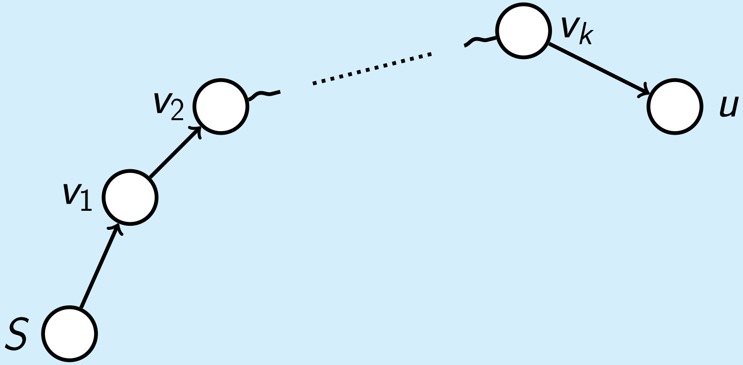
Reachable nodes are discovered at some point, so they get a finite distance estimate from the source. Unreachable nodes are not discovered at any point, and the distance to them stays infinite.

Proof



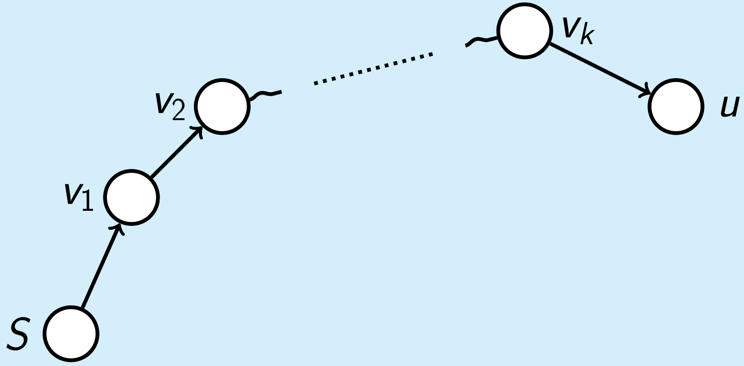
- u — reachable undiscovered closest to S

Proof



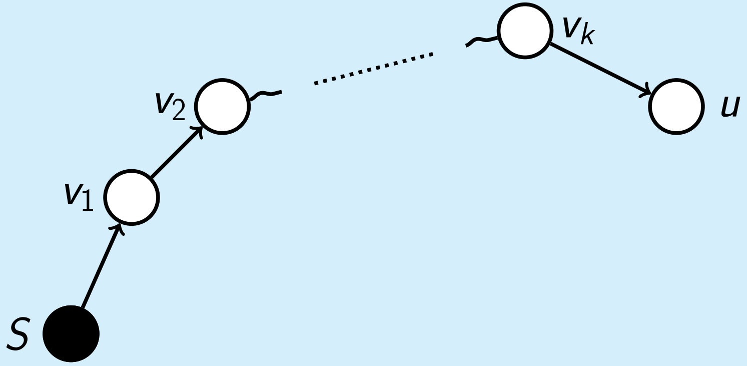
- u — reachable undiscovered closest to S
- $S - v_1 - \cdots - v_k - u$ — shortest path

Proof



- u — reachable undiscovered closest to S
- $S - v_1 - \cdots - v_k - u$ — shortest path
- u is discovered while processing v_k

Proof



- u — reachable undiscovered closest to S
- $S - v_1 - \cdots - v_k - u$ — shortest path
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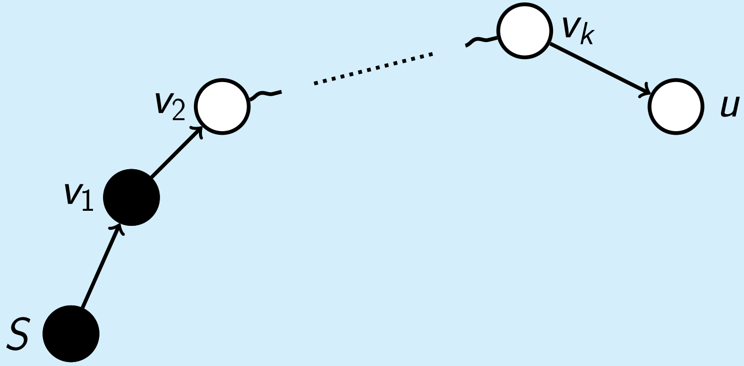
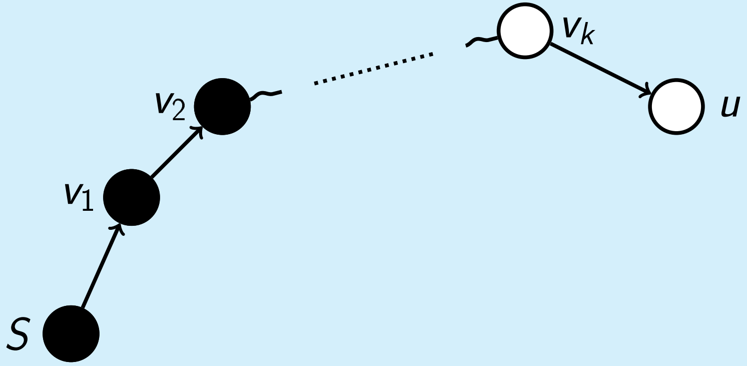
Proof



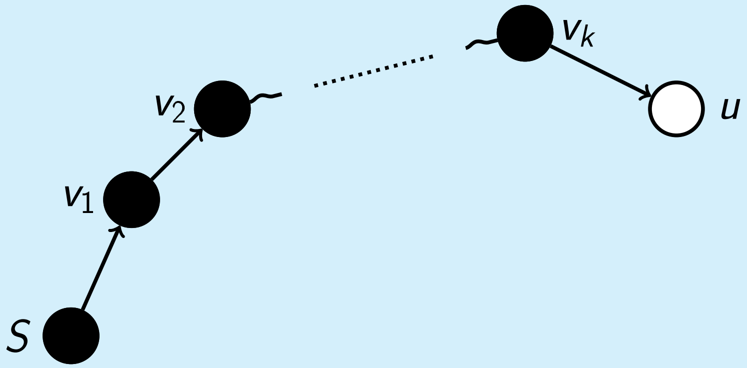
- u — reachable undiscovered closest to S
- $S - v_1 - \cdots - v_k - u$ — shortest path
- u is discovered while processing v_k

Proof



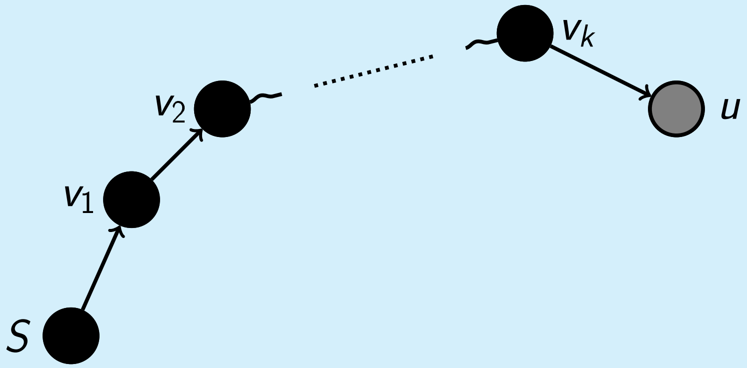
- u — reachable undiscovered closest to S
- $S - v_1 - \cdots - v_k - u$ — shortest path
- u is discovered while processing v_k

Proof



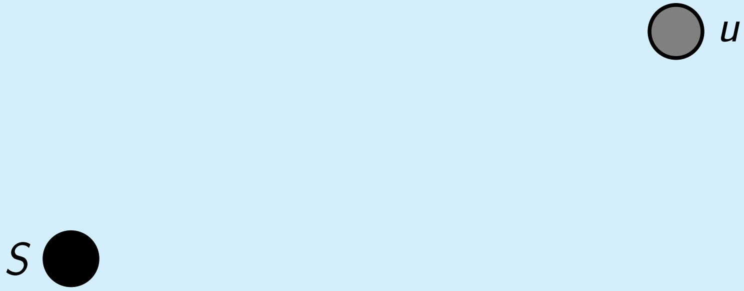
- u — reachable undiscovered closest to S
- $S - v_1 - \cdots - v_k - u$ — shortest path
- u is discovered while processing v_k

Proof



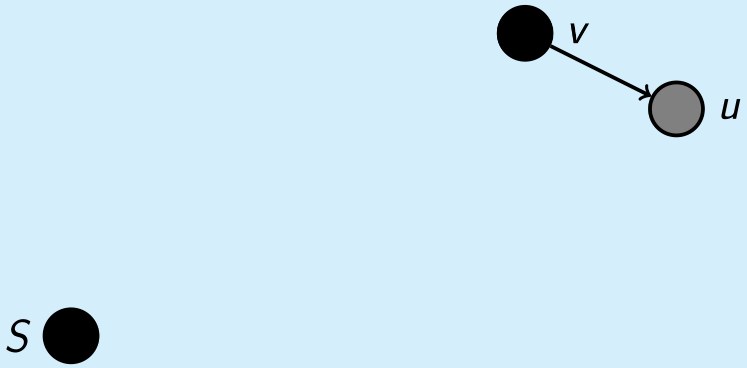
- u — reachable undiscovered closest to S
- $S - v_1 - \cdots - v_k - u$ — shortest path
- u is discovered while processing v_k

Proof



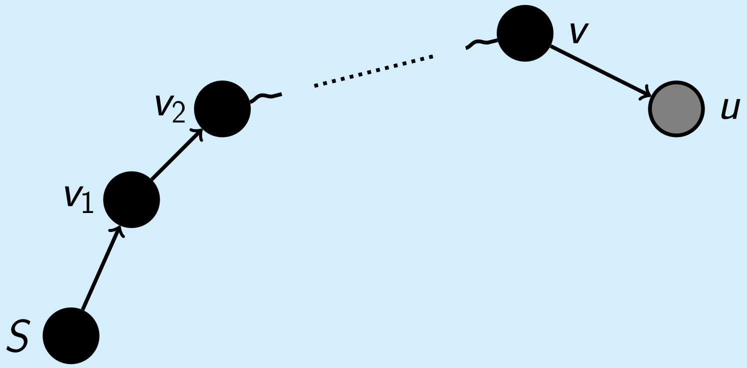
- u — first unreachable discovered

Proof



- u — first unreachable discovered
- u was discovered while processing v

Proof



- u — first unreachable discovered
- u was discovered while processing v
- u is reachable through v □

Order Lemma

Lemma

By the time node u at distance d from S is dequeued, all the nodes at distance at most d have already been discovered (enqueued).

Order Lemma Proof

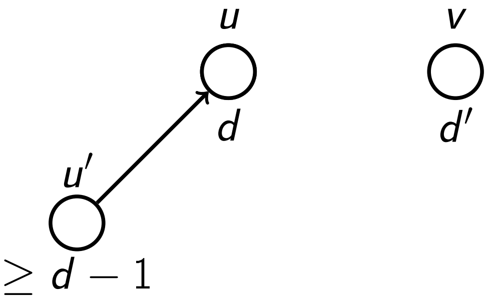
Consider the first time the order was broken

Order Lemma Proof

Consider the first time the order was broken

$d' \leq d$

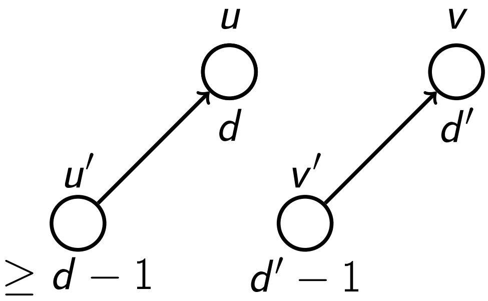
Order Lemma Proof



Consider the first time the order was broken

$d' \leq d$

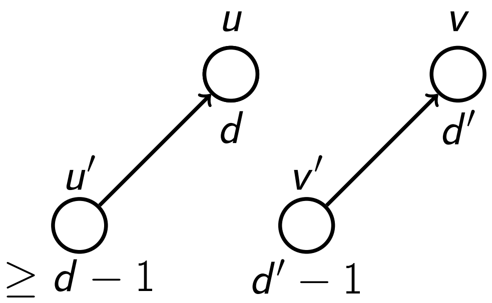
Order Lemma Proof



Consider the first time the order was broken

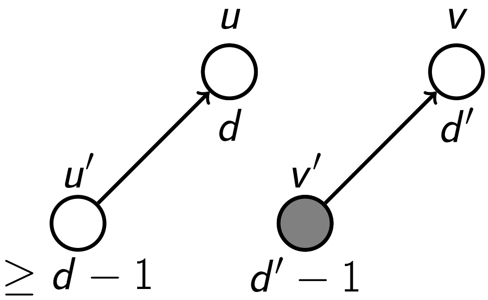
$d' \leq d$

Order Lemma Proof



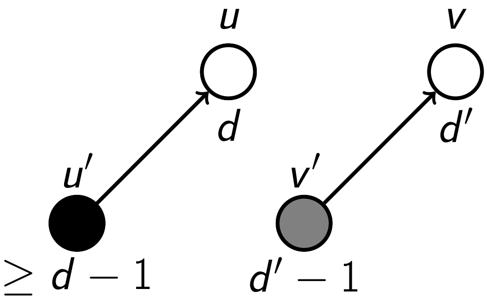
Consider the first time the order was broken
$d' \leq d \quad \Rightarrow d' - 1 \leq d - 1$, so v' was discovered before u' was dequeued

Order Lemma Proof



Consider the first time the order was broken
$d' \leq d \quad \Rightarrow d'-1 \leq d-1$, so v' was discovered before u' was dequeued

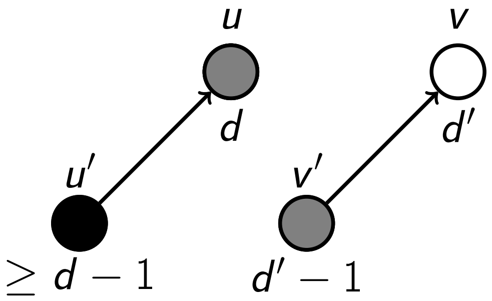
Order Lemma Proof



Consider the first time the order was broken

$d' \leq d \quad \Rightarrow d'-1 \leq d-1$, so v' was discovered before u' was dequeued

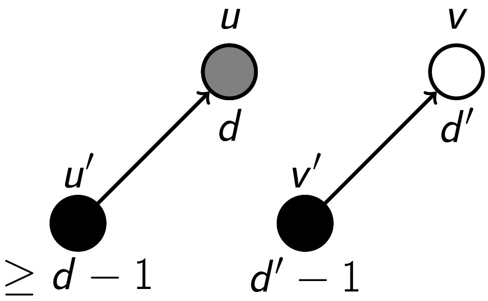
Order Lemma Proof



Consider the first time the order was broken

$d' \leq d \quad \Rightarrow d'-1 \leq d-1$, so v' was discovered before u' was dequeued

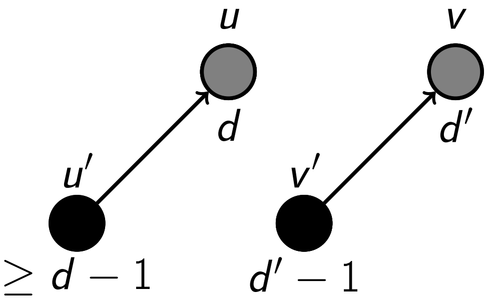
Order Lemma Proof



Consider the first time the order was broken

$d' \leq d \quad \Rightarrow d' - 1 \leq d - 1$, so v' was discovered before u' was dequeued

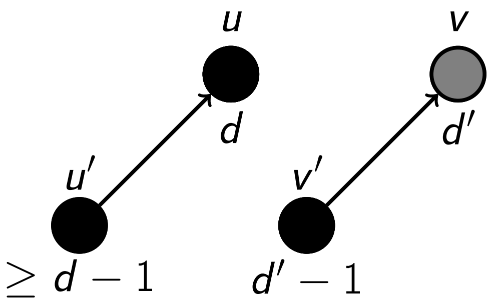
Order Lemma Proof



Consider the first time the order was broken

$d' \leq d \quad \Rightarrow d'-1 \leq d-1$, so v' was discovered before u' was dequeued

Order Lemma Proof



Consider the first time the order was broken
$d' \leq d \quad \Rightarrow d' - 1 \leq d - 1$, so v' was discovered before u' was dequeued

Correct distances

Lemma

When node u is discovered (enqueued), `dist[u]` is assigned exactly $d(S, u)$.

Correct distances

Proof

- Use mathematical induction

Correct distances

Proof

- Use mathematical induction
- Base: when S is discovered, `dist[S]` is assigned $0 = d(S, S)$

Correct distances

Proof

- Use mathematical induction
- Base: when S is discovered, `dist[S]` is assigned $0 = d(S, S)$
- Inductive step: suppose proved for all nodes at distance $\leq k$ from S $\rightarrow$ prove for nodes at distance $k + 1$

Correct distances

Proof

- Take a node v at distance $k + 1$ from S

Correct distances

Proof

- Take a node v at distance $k+1$ from S
- v was discovered while processing u

Correct distances

Proof

- Take a node v at distance $k+1$ from S
- v was discovered while processing u
- $d(S,v) \leq d(S,u)+1 \Rightarrow d(S,u) \geq k$

Correct distances

Proof

- Take a node v at distance $k+1$ from S
- v was discovered while processing u
- $d(S,v) \leq d(S,u)+1 \Rightarrow d(S,u) \geq k$
- v is discovered after u is dequeued, so $d(S,u) < d(S,v) = k+1$

Correct distances

Proof

- Take a node v at distance $k+1$ from S
- v was discovered while processing u
- $d(S,v) \leq d(S,u)+1 \Rightarrow d(S,u) \geq k$
- v is discovered after u is dequeued, so $d(S,u) < d(S,v) = k+1$
- So $d(S,u) = k$, and $\text{dist}[v] \leftarrow \text{dist}[u] + 1 = k+1$ □

Queue property

Queue:

d	d	d	$\dots$	d	d	$d+1$	$d+1$	$\dots$	$d+1$

Lemma

At any moment, if the first node in the queue is at distance d from S, then all the nodes in the queue are either at distance d from S or at distance $d+1$ from S. All the nodes in the queue at distance d go before (if any) all the nodes at distance $d+1$.

Queue property

Proof

- All nodes at distance d were enqueued before first such node is dequeued, so they go before nodes at distance $d + 1$

Queue property

Proof

- All nodes at distance d were enqueued before first such node is dequeued, so they go before nodes at distance $d + 1$
- Nodes at distance $d - 1$ were enqueued before nodes at d, so they are not in the queue anymore

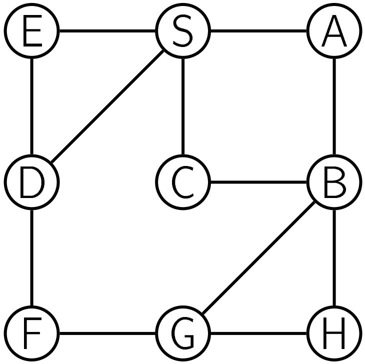
Queue property

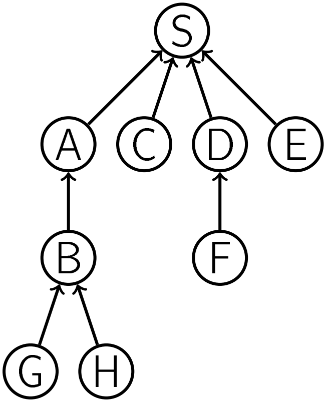
Proof

- All nodes at distance d were enqueued before first such node is dequeued, so they go before nodes at distance $d + 1$
- Nodes at distance $d - 1$ were enqueued before nodes at d, so they are not in the queue anymore
- Nodes at distance $> d + 1$ will be discovered when all d are gone $\square$

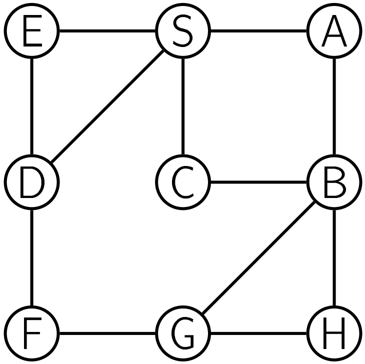
Outline

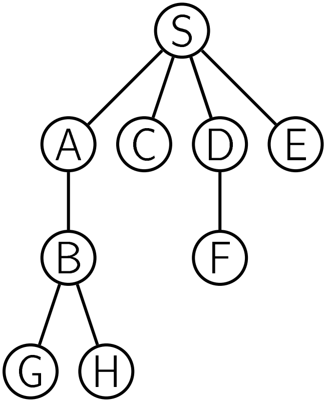
Shortest-path tree





Shortest-path tree

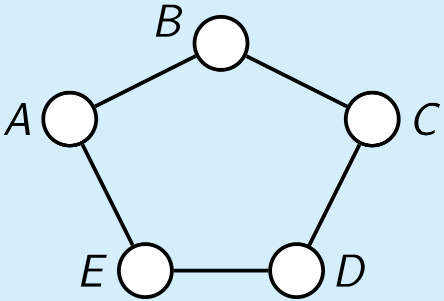




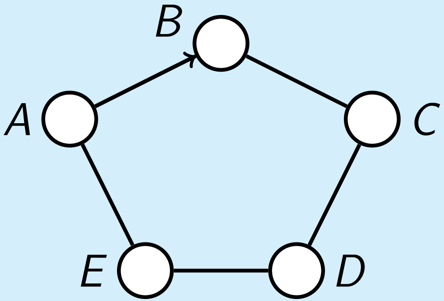
Lemma

Shortest-path tree is indeed a tree, i.e. it doesn't contain cycles (it is a connected component by construction).

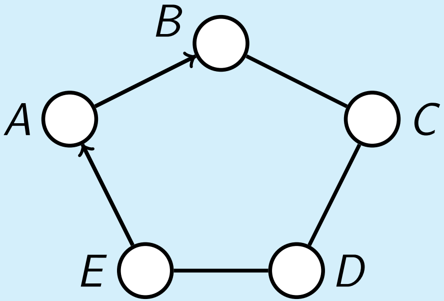
Proof



Proof

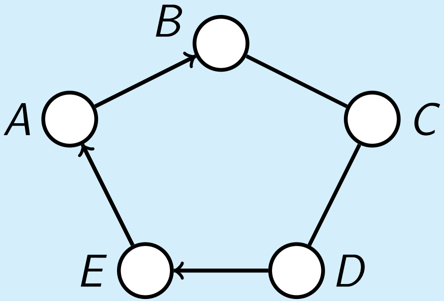


Proof



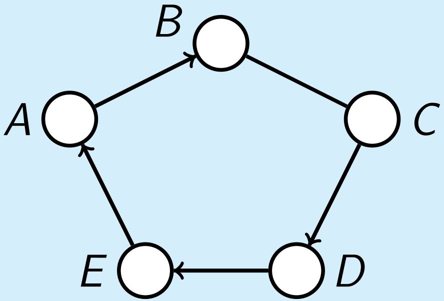
- Only one outgoing edge from each node

Proof



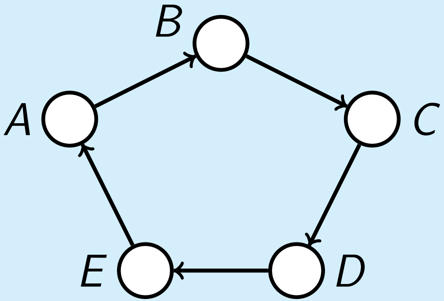
- Only one outgoing edge from each node

Proof



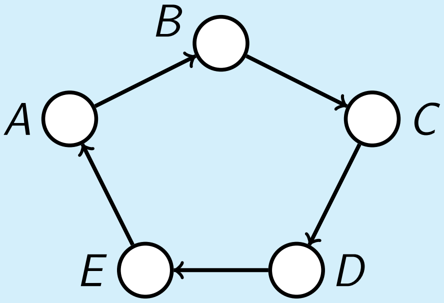
- Only one outgoing edge from each node

Proof



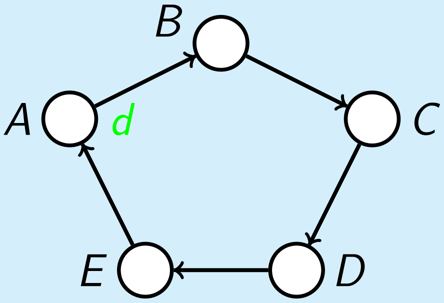
- Only one outgoing edge from each node

Proof



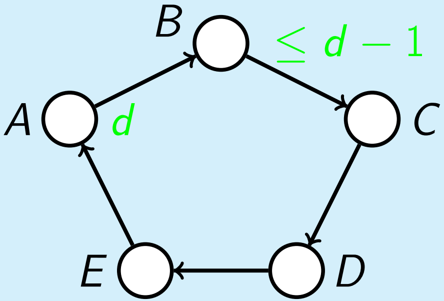
- Only one outgoing edge from each node
- Distance to S decreases after going by edge

Proof



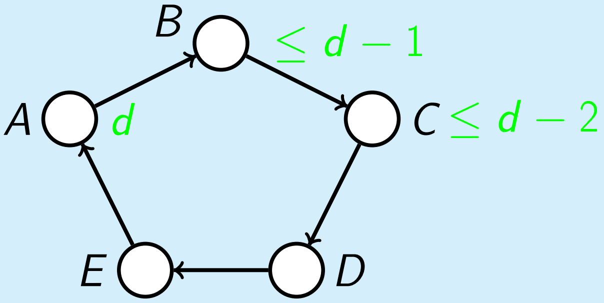
- Only one outgoing edge from each node
- Distance to S decreases after going by edge

Proof



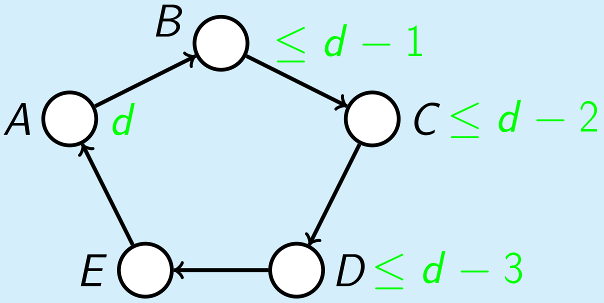
- Only one outgoing edge from each node
- Distance to S decreases after going by edge

Proof



- Only one outgoing edge from each node
- Distance to S decreases after going by edge

Proof



- Only one outgoing edge from each node
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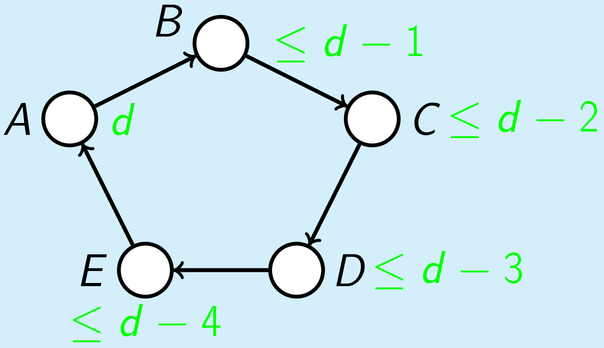
Proof



- Only one outgoing edge from each node
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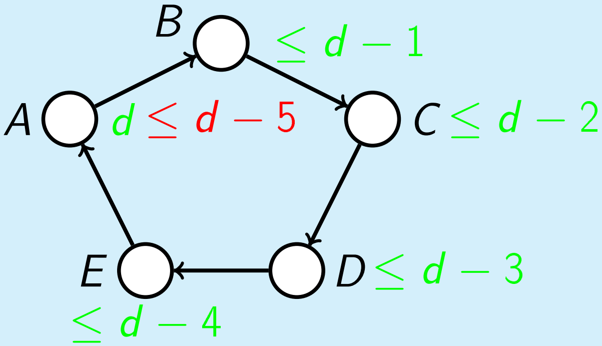
Proof



- Only one outgoing edge from each node
- Distance to S decreases after going by edge

Constructing shortest-path tree

BFS(G, S)

```
for all $u \in V$:
  dist[$u$] $\leftarrow \infty$, prev[$u$] $\leftarrow$ nil
dist[$S$] $\leftarrow 0$
$Q \leftarrow \{S\}$ {queue containing just $S$}
while $Q$ is not empty:
  $u \leftarrow$ Dequeue($Q$)
  for all $(u, v) \in E$:
    if dist[$v$] $= \infty$:
      Enqueue($Q, v$)
      dist[$v$] $\leftarrow$ dist[$u$] $+ 1$, prev[$v$] $\leftarrow u$
```

Reconstructing Shortest Path

```
ReconstructPath($S, u$, prev)

result ← empty
while $u \neq S$:
    result.append($u$)
    $u \leftarrow$ prev[$u$]
return Reverse(result)
```

Conclusion

- Can find the minimum number of flight segments to get from one city to another

Conclusion

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