J. Appl. Prob. **11**, 63–71 (1974)
Printed in Israel

ON THE INVERSES OF SOME PATTERNED MATRICES ARISING IN THE THEORY OF STATIONARY TIME SERIES

R. F. GALBRAITH AND
J. I. GALBRAITH, *University College London*

Abstract

Expressions are obtained for the determinant and inverse of the covariance matrix of a set of n consecutive observations on a mixed autoregressive moving average process. Explicit formulae for the inverse of this matrix are given for the general autoregressive process of order p ($n \geqq p$), and for the first order mixed autoregressive moving average process.

AUTOREGRESSIVE; MOVING AVERAGE; LAURENT MATRIX; INVERSE; DETERMINANT

Introduction

Consider a weakly stationary autoregressive moving average process generated by the relation

$$(1) \quad x_t - \alpha_1 x_{t-1} - \alpha_2 x_{t-2} - \cdots - \alpha_p x_{t-p} = z_t - \beta_1 z_{t-1} - \beta_2 z_{t-2} - \cdots - \beta_q z_{t-q}$$

where the z_t are mutually uncorrelated random variables each with mean zero and variance σ^2. This process is called (Box and Jenkins (1970), Chapter 3) an ARMA(p,q) process and is stationary if all zeros of the polynomial $1 - \alpha_1\zeta - \cdots - \alpha_p\zeta^p$ are greater than one in modulus. The moving average of order q and the autoregression of order p are the special cases MA(q) = ARMA$(0,q)$ and AR(p) = ARMA$(p,0)$ respectively.

A vector $x_n = (x_1, x_2, \cdots, x_n)'$ of n consecutive observations from (1) has covariance matrix $\Gamma_n = E(x_n x_n')$ with (i,j) element $\gamma_{|i-j|}$, where γ_u $(u = 0, 1, 2, \cdots)$, the autocovariance function of the process, can be obtained from the α's, β's and σ^2 by standard methods. The Laurent property of Γ_n, that its (i,j) element is constant for fixed $|i-j|$, is not shared by its inverse although the latter matrix is symmetric about both principal diagonals; i.e., writing $M_n = \sigma^2 \Gamma_n^{-1}$ with (i,j) element m_{ij} not depending on σ^2,

$$(2) \qquad m_{ij} = m_{ji} = m_{n+1-j,n+1-i} = m_{n+1-i,n+1-j}.$$

The second order properties, including Γ_n and M_n, of the process (1) are completely specified and as far as these properties are concerned we may without

Received in revised form 1 February 1973.

loss of generality take the z_t to be independent and normally distributed. The probability density of $\boldsymbol{x}_n$ is then

$$p(\boldsymbol{x}_n \mid \boldsymbol{\alpha}, \boldsymbol{\beta}, \sigma^2) = (2\pi\sigma^2)^{-\frac{1}{2}n} \left| \boldsymbol{M}_n \right|^{\frac{1}{2}} \exp\{-\tfrac{1}{2}\sigma^{-2}\boldsymbol{x}_n'\boldsymbol{M}_n\boldsymbol{x}_n\} \tag{3}$$

which as a function of the α's, β's and σ^2 for fixed $\boldsymbol{x}_n$ is the likelihood function, providing a motive for finding $\boldsymbol{M}_n$ and its determinant. More generally, the α's and β's are usually estimated by those values which minimise $\boldsymbol{x}_n'\boldsymbol{M}_n\boldsymbol{x}_n$. In this context several large sample results have been derived (e.g., Whittle (1951), Chapter 4; Walker (1962)) which bypass the explicit use of $\boldsymbol{M}_n$, and for an invertible process Box and Jenkins ((1970), Chapter 7) give a method for calculating $\boldsymbol{x}_n'\boldsymbol{M}_n\boldsymbol{x}_n$ directly.

In this paper we obtain expressions for $\boldsymbol{M}_n$ and its determinant for the general ARMA (p,q) process and give explicit results in terms of the α's and β's where these are not too complicated. We do not require that the process be invertible. In fact Equations (7) and (8) for the inverse and determinant of the covariance matrix of $\boldsymbol{x}_n$ generated by (1) from $(p+q)$ arbitrary starting values hold for any α's and β's provided that this covariance matrix is non-singular, so that the process from which $\boldsymbol{x}_n$ is a sample need not be stationary. We assume the process to be Gaussian but, as remarked above, this does not restrict the class of covariance matrices $\boldsymbol{\Gamma}_n$ under consideration.

General case

Let $\boldsymbol{v}$ be a $(p+q)$-dimensional normal variable with mean zero and covariance matrix $\sigma^2\boldsymbol{D}^{-1} = E(\boldsymbol{v}\boldsymbol{v}')$. We write

$$\boldsymbol{v} = (x_{1-p}, x_{2-p}, \cdots, x_0, z_{1-q}, z_{2-q}, \cdots, z_0)'$$

and regard $\boldsymbol{v}$ as a vector of p initial x values and q initial z values. Let $\boldsymbol{z}_n = (z_1, \cdots, z_n)'$ be a vector of independent $N(0,\sigma^2)$ variables which is independent of $\boldsymbol{v}$. Then the joint distribution of $\boldsymbol{v}, \boldsymbol{z}_n$ is normal with covariance matrix $\sigma^2\boldsymbol{C}^{-1}$ where

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}.$$

Now define $\boldsymbol{x}_n = (x_1, \cdots, x_n)'$ in terms of $\boldsymbol{v}$, $\boldsymbol{z}_n$ by

$$x_t = \alpha_1 x_{t-1} + \cdots + \alpha_p x_{t-p} + z_t - \beta_1 z_{t-1} - \cdots - \beta_q z_{t-q} \qquad (t = 1, \cdots, n).$$

The transformation from $(\boldsymbol{v}, \boldsymbol{z}_n)$ to $(\boldsymbol{v}, \boldsymbol{x}_n)$ has unit Jacobian and can be written as

$$\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{z}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{A}_n \end{bmatrix} \boldsymbol{x}_n + \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{H} \end{bmatrix} \boldsymbol{v} = \boldsymbol{L}\boldsymbol{x}_n + \boldsymbol{K}\boldsymbol{v}, \text{ say,} \tag{4}$$

where $\mathbf{0}$ is a $(p+q)\times n$ matrix of zeros, A_n is $n\times n$, I is the $(p+q)\times(p+q)$ identity matrix and H is $n\times(p+q)$. The joint density of $(\boldsymbol{v},\boldsymbol{x}_n)$ is therefore obtained from that of $(\boldsymbol{v},\boldsymbol{z}_n)$ by substituting from (4). This is then factorised as

$$p(\boldsymbol{v},\boldsymbol{x}_n)=p(\boldsymbol{v}\,|\,\boldsymbol{x}_n)p(\boldsymbol{x}_n)$$

by decomposing the quadratic form

$$(L\boldsymbol{x}_n+K\boldsymbol{v})'C(L\boldsymbol{x}_n+K\boldsymbol{v})=(\boldsymbol{v}-\hat{\boldsymbol{v}})'K'CK(\boldsymbol{v}-\hat{\boldsymbol{v}})$$
$$+(L\boldsymbol{x}_n+K\hat{\boldsymbol{v}})'C(L\boldsymbol{x}_n+K\hat{\boldsymbol{v}}),$$

where $\hat{\boldsymbol{v}}=E(\boldsymbol{v}\,|\,\boldsymbol{x}_n)$ satisfies the normal equations

$$(K'CK)\hat{\boldsymbol{v}}=-(K'CL)\boldsymbol{x}_n. \tag{5}$$

The density of $\boldsymbol{x}_n$ is then

(6)
$$p(\boldsymbol{x}_n\,|\,\boldsymbol{\alpha},\boldsymbol{\beta},\sigma^2)=(2\pi\sigma^2)^{-\frac{1}{2}n}\,|C|^{\frac{1}{2}}\,|K'CK|^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}\sigma^{-2}(L\boldsymbol{x}_n+K\hat{\boldsymbol{v}})'C(L\boldsymbol{x}_n+K\hat{\boldsymbol{v}})\}$$

where $\hat{\boldsymbol{v}}$ is given by (5). (Box and Jenkins (1970), p. 271 obtain the density of $\boldsymbol{x}_n$ for a MA(q) process in this manner.) Substituting (5) in (6) and comparing with the general expression (3) gives, after some algebra,

$$M_n=A_n'A_n-(H'A_n)'(D+H'H)^{-1}(H'A_n) \tag{7}$$

and

$$|M_n|=|D|\,|D+H'H|^{-1}. \tag{8}$$

To obtain the elements of A_n and H in terms of the α's and β's note that A_n is lower triangular with (r,s) element a_{r-s} where

$$\begin{aligned} a_u&=0 && (u<0),\\ a_u&=\beta_1a_{u-1}+\beta_2a_{u-2}+\cdots+\beta_qa_{u-q}-\alpha_u && (u\geqq 0). \end{aligned} \tag{9}$$

Here and subsequently we use the convention that $\alpha_0=-1$ and $\alpha_u=0$ for $u>p$. It can be shown by straightforward but lengthy algebra that the elements of the columns of $K=[I\,H']'$ also follow this recurrence relation, but with different starting values: writing k_{ij} for the (i,j) element of K,

$$\begin{aligned} k_{ij}&=\delta_{ij} && (1\leqq i\leqq p+q,\ \ 1\leqq j\leqq p+q),\\ &=\beta_1k_{i-1,j}+\beta_2k_{i-2\,j}+\cdots+\beta_qk_{i-q\,j}-\alpha_{i-j-q} && (p+q+1\leqq i\leqq n+p+q,\\ & && 1\leqq j\leqq p),\\ &=\beta_1k_{i-1\,j}+\beta_2k_{i-2\,j}+\cdots+\beta_qk_{i-q\,j} && (p+q+1\leqq i\leqq n+p+q,\\ & && p+1\leqq j\leqq p+q). \end{aligned}$$

The elements of A_n and H are therefore easily computed and the evaluation of M_n requires the inversion of only a $(p+q) \times (p+q)$ matrix. The matrix D could in principle be taken to correspond to any convenient starting distribution. Of course if D were chosen arbitrarily, x_n would not be from a stationary process and M_n would not be the inverse of a Laurent matrix.

In the case where the initial values v come from the stationary ARMA (p, q) process (1),

$$D^{-1} = \begin{bmatrix} M_p^{-1} & B_{p\times q} \\ B'_{p\times q} & I_q \end{bmatrix}$$

where $B_{p\times q}$ is the $p \times q$ matrix with (i,j) element $b_{i-j-p+q}$, the b_u being defined recursively by

$$b_u = 0 \qquad (u < 0),$$

$$b_u = \alpha_1 b_{u-1} + \alpha_2 b_{u-2} + \cdots + \alpha_p b_{u-p} - \beta_u \qquad (u \geqq 0),$$

with the convention that $\beta_0 = -1$ and $\beta_u = 0$ for $u > q$. For a stationary AR(p) process $D = M_p$, while for a MA(q) process $D = I_q$ and $D + H'H = K'K$.

Before giving some particular cases we remark that for an ARMA(p, q) process which is stationary and invertible Wise (1955) gives the expression

$$M = \sigma^2 \Gamma^{-1} = (B^{-1}A)'(B^{-1}A)$$

for the inverse of the semi-infinite covariance matrix $\Gamma = E(xx')$ where $x = (x_{\tau+1}, x_{\tau+2}, \cdots)'$ for arbitrary fixed time τ. The matrices A and B are defined as

$$\begin{aligned} A &= I - \alpha_1 U - \alpha_2 U^2 - \cdots - \alpha_p U^p, \\ B &= I - \beta_1 U - \beta_2 U^2 - \cdots - \beta_q U^q, \end{aligned} \tag{10}$$

where U is the auxiliary identity matrix with (i,j) element $\delta_{i+1\,j}$, all these matrices being infinite. It is easily seen that $B^{-1}A$ is upper triangular with (i,j) element a_{j-i}, the a_u being determined by (9). Hence, for $1 \leqq i, j \leqq n$ the (i,j) element of M is just the $(n+1-j,\, n+1-i)$ element of $A_n' A_n$, and the second term on the right-hand side of (7) can be regarded as a correction for the finite sample. These considerations together with the symmetry conditions (2) suggest that for an invertible process a close approximation to the (r, s) element of M_n is, for large n,

$$\sum_{j=0}^{r-1} a_j a_{j+s-r} - \sum_{j=n+1-s}^{n+r-s} a_j a_{j+s-r} \qquad (1 \leqq r \leqq s \leqq n). \tag{11}$$

Comparison with (15) shows that this is exact for an AR(p) process when $n \geqq p$.

Special cases

1. MA(1)

For the process

$$x_t = z_t - \beta z_{-1}$$

with

$$\sigma^{-2}\Gamma_n = M_n^{-1} =$$

$$\begin{bmatrix} 1+\beta^2 & -\beta & 0 & & \cdots & 0 \\ -\beta & 1+\beta^2 & -\beta & & \cdots & \vdots \\ 0 & -\beta & 1+\beta^2 & & & \vdots \\ \vdots & & & & & 0 \\ \vdots & & & & 1+\beta^2 & -\beta \\ 0 & \cdots & \cdots & & -\beta & 1+\beta^2 \end{bmatrix};$$

the matrices $\boldsymbol{L}$ and $\boldsymbol{K}$ are

$$\boldsymbol{L} = \left[\begin{array}{cccc} 0 & 0 & \cdots & 0 \\ \hline 1 & 0 & \cdots & 0 \\ \beta & 1 & & \\ \beta^2 & \beta & & \vdots \\ \vdots & \vdots & & \\ \beta^{n-1} & \beta^{n-2} & \cdots & 1 \end{array}\right], \quad \boldsymbol{K} = \left[\begin{array}{c} 1 \\ \hline \beta \\ \beta^2 \\ \vdots \\ \\ \beta^n \end{array}\right].$$

Then

$$|\boldsymbol{M}_n|^{-1} = |\boldsymbol{K}'\boldsymbol{K}| = \sum_{j=0}^{n} \beta^{2j}$$
$$= (1-\beta^{2(n+1)})/(1-\beta^2),$$

and the (r,s) elements of $\boldsymbol{A}_n'\boldsymbol{A}_n$ and $(\boldsymbol{H}'\boldsymbol{A}_n)'(\boldsymbol{H}'\boldsymbol{A}_n)$ are respectively

$$\beta^{s-r}\sum_{0}^{n-s}\beta^2 \quad \text{and} \quad \beta^{r+s}\sum_{0}^{n-r}\beta^{2j}\sum_{0}^{n-s}\beta^{2j} \qquad (r \leqq s),$$

so that Formula (7) gives

$$m_{rs} = \beta^{s-r}\frac{(1-\beta^{2r})(1-\beta^{2(n+1-s)})}{(1-\beta^2)(1-\beta^{2(n+1)})} \qquad (r \leqq s)$$

in accordance with the known result for this matrix (see, for example, Shaman (1969)).

2. ARMA(1, 1)

For the simplest mixed process

$$x_t - \alpha x_{t-1} = z_t - \beta z_{t-1}, \qquad |\alpha| < |1|,$$

the autocovariance function is

$$\gamma_0 = \sigma^2 \frac{1 - 2\alpha\beta + \beta^2}{1 - \alpha^2},$$

$$\gamma_u = \sigma^2 \frac{(1 - \alpha\beta)(\alpha - \beta)}{1 - \alpha^2} \alpha^{u-1} \qquad (u = 1, 2, 3, \cdots).$$

Also

$$\sigma^2 D^{-1} = \begin{bmatrix} \gamma_0 & \sigma^2 \\ \sigma^2 & \sigma^2 \end{bmatrix}, \quad D = \frac{1 - \alpha^2}{(\beta - \alpha)^2} \begin{bmatrix} 1 & -1 \\ -1 & \dfrac{1 - 2\alpha\beta + \beta^2}{1 - \alpha^2} \end{bmatrix},$$

and

$$L = \left[\begin{array}{ccccc} 0 & 0 & \cdots & & 0 \\ 0 & 0 & & & 0 \\ \hline 1 & 0 & \cdots & & 0 \\ (\beta - \alpha) & 1 & & & \\ (\beta - \alpha)\beta & (\beta - \alpha) & & & \\ (\beta - \alpha)\beta^2 & (\beta - \alpha)\beta & & & \vdots \\ \vdots & \vdots & & & \\ & & & 1 & 0 \\ (\beta - \alpha)\beta^{n-2} & (\beta - \alpha)\beta^{n-3} & \cdots & (\beta - \alpha) & 1 \end{array}\right], \quad K = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline -\alpha & \beta \\ -\alpha\beta & \beta^2 \\ -\alpha\beta^2 & \\ \vdots & \vdots \\ & \\ -\alpha\beta^{n-1} & \beta^n \end{array}\right].$$

The determinant of M_n is, using (8),

$$|M_n| = \left[\frac{(1 - \alpha\beta)^2 - (\beta - \alpha)^2\beta^{2n}}{(1 - \alpha^2)(1 - \beta^2)}\right]^{-1}.$$

The (r, r) diagonal element of $A_n'A_n$ is

$$[1 - \beta^2 + (\beta - \alpha)^2(1 - \beta^{2(n-r)})]/(1 - \beta^2) \qquad (1 \leqq r \leqq n),$$

while the (r, s) element of $A_n'A_n$ is

$$\beta^{s-r-1}(\beta - \alpha)[(1 - \alpha\beta) - (\beta - \alpha)\beta^{2(n-s)+1}]/(1 - \beta^2) \qquad (1 \leqq r < s \leqq n).$$

The (r, s) element of the correction term $(H'A_n)'(D + H'H)^{-1}(H'A_n)$ is

$$\frac{\beta^{s+r-2}(\beta-\alpha)^2[(1-\alpha\beta)-(\beta-\alpha)\beta^{2(n-r)+1}][(1-\alpha\beta)-(\beta-\alpha)\beta^{2(n-s)+1}]}{(1-\beta^2)[(1-\alpha\beta)^2-(\beta-\alpha)^2\beta^{2n}]}$$
$$(1 \leqq r \leqq s \leqq n).$$

Hence the elements of M_n are

$$m_{rr} = \frac{(1-\alpha\beta)^2(\beta-\alpha)^2[1-\beta^{2(r-1)}][1-\beta^{2(n-r)}]+(1-\beta^2)[(1-\alpha\beta)^2-(\beta-\alpha)^2\beta^{2(n-1)}]}{(1-\beta^2)[(1-\alpha\beta)^2-(\beta-\alpha)^2\beta^{2n}]}$$

(12) $$(1 \leqq r \leqq n),$$

$$m_{rs} = \frac{\beta^{s-r-1}(1-\alpha\beta)(\beta-\alpha)[(1-\alpha\beta)-(\beta-\alpha)\beta^{2r-1}][(1-\alpha\beta)-(\beta-\alpha)\beta^{2(n-s)+1}]}{(1-\beta^2)[(1-\alpha\beta)^2-(\beta-\alpha)^2\beta^{2n}]}$$
$$(1 \leqq r < s \leqq n).$$

This result can be used to obtain the inverse of $\Sigma_n + \lambda I$ for given λ and α ($\alpha^2 \neq 0, 1$) where Σ_n has (i,j) element $\alpha^{|i-j|}$, since

(13) $$(\Sigma_n + \lambda I)^{-1} = \frac{(1-\alpha\beta)(\alpha-\beta)}{\alpha(1-\alpha^2)} M_n,$$

where β satisfies

$$\alpha\lambda\beta^2 - [(1-\alpha^2) + \lambda(1+\alpha^2)]\beta + \alpha\lambda = 0.$$

The roots of this quadratic are β and β^{-1} and it is easily verified from (12) that replacing β by β^{-1} does not alter the value of (13).

3. AR(p)

For an AR(p) process generated by

$$x_t - \alpha_1 x_{t-1} - \alpha_2 x_{t-2} - \cdots - \alpha_p x_{t-p} = z_t$$

$D = M_p$ and the (i,j) elements of A_n and H are $-\alpha_{i-j}$ $(i \geqq j)$ and $-\alpha_{p+i-j}$ respectively.

When $n > p$,

$$A_n = -\begin{bmatrix} \alpha_0 & 0 & & \cdots & & 0 \\ \alpha_1 & \alpha_0 & & \cdots & & 0 \\ \vdots & & & & & \\ \alpha_p & & & & & \vdots \\ 0 & & & & & \vdots \\ \vdots & & & & & 0 \\ 0 & \cdots & 0 & \alpha_p & \cdots \; \alpha_1 & \alpha_0 \end{bmatrix}, \quad H = -\begin{bmatrix} \alpha_p & \alpha_{p-1} & \cdots & \alpha_1 \\ 0 & \alpha_p & \cdots & \alpha_2 \\ \vdots & & & \vdots \\ 0 & & & \alpha_p \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \end{bmatrix},$$

and the last $n - p$ columns of $\boldsymbol{H}'\boldsymbol{A}_n$ consist of zeros so that, whatever the composition of $(\boldsymbol{D} + \boldsymbol{H}'\boldsymbol{H})^{-1}$, the second term on the right-hand side of (7) is a matrix of zeros except for the top left $p \times p$ block, and the elements of $\boldsymbol{M}_n$, apart from the first $p \times p$ block, are the corresponding elements of $\boldsymbol{A}_n'\boldsymbol{A}_n$. It follows immediately that the $m_{rs} = 0$ whenever $|r - s| > p$. When $n \geqq 2p$ the complete matrix $\boldsymbol{M}_n$ can be obtained from $\boldsymbol{A}_n'\boldsymbol{A}_n$ and the symmetry conditions (2). For $p \leqq k < 2p$ the elements of $\boldsymbol{M}_k$ are then easily obtained using the identity

$$(14) \qquad \boldsymbol{x}_n'\boldsymbol{M}_n\boldsymbol{x}_n = \boldsymbol{x}'_k\boldsymbol{M}_k\boldsymbol{x}_k + \sum_{t=k+1}^{n} [x_t - \alpha_1 x_{t-1} - \cdots - \alpha_p x_{t-p}]^2$$

which holds for any k and n such that $p \leqq k < n$ and which follows from the factorisation

$$p(\boldsymbol{x}_n) = p(x_{k+1,k+2}, \cdots, x_n \,|\, \boldsymbol{x}_k)p(\boldsymbol{x}_k),$$

where $\boldsymbol{x}_k$ consists of the first k elements of $\boldsymbol{x}_n$. Taking $p \leqq k < 2p \leqq n$ in (14) and comparing coefficients of $x_r x_s$ gives

$$m_{rs}^{(n)} = m_{rs}^{(k)} + \sum_{j=k+1-s}^{p+r-s} \alpha_j \alpha_{j+s-r}.$$

Hence, combining the above results, the elements of $\boldsymbol{M}_n$ are, for $n \geqq p$,

$$(15) \qquad m_{rs} = \sum_{j=0}^{c} \alpha_j \alpha_{j+s-r} - \sum_{j=d}^{p+r-s} \alpha_j \alpha_{j+s-r} \qquad (1 \leqq r \leqq s \leqq n)$$

where

$$c = \min\{r - 1, p + r - s, n - s\}, \ d = \max\{r - 1, n - s\},$$

and either of the sums is taken to be zero if its upper limit is less than its lower limit. The second sum is zero unless $n - p + 1 \leqq r \leqq s \leqq p$ while both sums are zero if $s - r > p$.

The matrix $\boldsymbol{M}_p$ is of special interest, being proportional to the large sample covariance matrix of the usual estimators of the α's for an AR(p) process (or, with β's in place of α's, of the β's from a MA(p) process). When $n = p$ (15) simplifies to

$$(16) \qquad m_{rs} = \sum_{j=0}^{r-1} \alpha_j \alpha_{j+s-r} - \sum_{j=p+1-s}^{p+r-s} \alpha_j \alpha_{j+s-r} \qquad (1 \leqq r \leqq s \leqq p)$$

which can be written in matrix form as

$$\boldsymbol{M}_p = \boldsymbol{A}_p\boldsymbol{A}_p' - \boldsymbol{H}'\boldsymbol{H}.$$

Equation (8) becomes

$$\begin{aligned} |\boldsymbol{M}_n| &= |\boldsymbol{M}_p| \, |\boldsymbol{A}_p\boldsymbol{A}_p'|^{-1} \qquad (n \geqq p) \\ &= |\boldsymbol{M}_p| \end{aligned}$$

as is well known. Siddiqui (1958) shows how M_n ($n \geqq p$) can be obtained from (14) and (2) but does not give the explicit formula (15). Equation (7) could now be used to obtain M_n for $n < p$.

Acknowledgment

We are grateful to the referee for his constructive comments on an earlier draft of the paper.

References

Box, G. E. P. and Jenkins, G. I. (1970) *Time Series Analysis Forecasting and Control.* Holden Day, San Francisco.

Shaman, P. (1969) On the inverse of the covariance matrix of a first order moving average. *Biometrika* **56**, 595–600.

Siddiqui, M. M. (1958) On the inversion of the sample covariance matrix in a stationary autoregressive process. *Ann. Math. Statist.* **29**, 585–588.

Walker, A. M. (1962) Large sample estimation of parameters for autoregressive processes with moving average residuals. *Biometrika* **49**, 117–131.

Whittle, P. (1951) *Hypothesis Testing in Time-Series Analysis.* Almqvist and Wiksell, Upsala.

Wise, J. (1955) The autocorrelation function and the spectral density function. *Biometrika* **42**, 151–159.