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Estimation of ARMA(p, q) parameters[1]

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Abstract. The need for estimating the parameters of an ARMA(p, q) process arises in many applications both in signal processing and in automatic control. Recently, we proposed an estimation procedure to get the ARMA parameters. The method is based on a 2-step approach: first the AR parameters are estimated using a transient Kalman gain, then the MA parameters are estimated by a fast filtering algorithm. This short paper shows the results of simulations conducted to evaluate this method. We study the consistency and the efficiency of the proposed estimator and we give some examples for ARMA(2, 2), ARMA(4, 2) and ARMA(4, 4) processes.

Zusammenfassung. Eine Schätzung der Parameter eines ARMA(p, q)-Prozesses braucht man für zahlreiche Anwendungen der Signalverarbeitung wie der selbsttätigen Regelung. Kürzlich haben wir eine Schätzprozedur hierfür vorgeschlagen. Sie beruht auf einem zweistufigen Ansatz: Zunächst werden die AR-Parameter mit Hilfe einer transienten Kalman-Verstärkung, danach die MA-Parameter mit einem schnellen Filter-Algorithmus geschätzt. Die Ergebnisse von Simulationen zur Bewertung des Verfahrens werden gezeigt. Wir untersuchen die Konsistenz und die Wirksamkeit des vorgeschlagenen Schätzers, und wir geben einige Beispiele an für ARMA(2, 2)-, ARMA(4, 2)- und ARMA(4, 4)-Prozesse.

Résumé. L'estimation des paramètres d'un modèle ARMA(p, q) est un problème qui se pose à la fois en automatique et en traitement du signal. Nous avons présenté une méthode d'estimation des paramètres d'un modèle ARMA. La procédure d'estimation s'effectue en 2 temps: tout d'abord, l'estimation de la partie AR est faite en utilisant un gain de Kalman transitoire. Puis, la partie MA est estimée grace à un algorithme de filtrage rapide. Cet article présente les résultats de tests effectués pour évaluer cette méthode. Nous étudions la consistance et l'efficacité de l'estimateur proposé et présentons plusieurs exemples pour des processus ARMA(2, 2), ARMA(4, 2) et ARMA(4, 4).

Keywords. ARMA model, parameter-estimation, 1-step predictor, Kalman filter.

1. Introduction

The need for estimating the parameters of an ARMA(p, q) model arises in many applications both in automatic control and in signal processing (see [3, 4, 6, 10–15, 17]).

In this short paper, we evaluate a method (briefly described in Section 2) based on a 2-step approach:

– first, the AR parameters are estimated using a transient Kalman gain [1]
– then, the MA parameters are obtained by a fast filtering algorithm [2, 7].

In Section 3, we are presenting the simulation results obtained first on two AR(p) + noise models (narrow and large bands). We compare the estimation results with the Cramer–Rao Bound for a signal-to-noise ratio varying from 0 to 30 dB.

We have also studied the influence of the window length on the quality of the estimation.

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Second, we consider the ARMA(p, q) case and we present the estimation of poles and zeros with 10 sequences of data for an ARMA(4, 2) model.

Then we conclude in Section 4.

2. ARMA(p, q) model and parameter estimation

2.1. The ARMA model

In this paper, we work on an ARMA(p, q) model based on the following representation [2]:

$$X_{n+1} = JX_n + Ay_n + Be_{n+1}, \tag{1}$$

$$y_n = HX_n, \tag{2}$$

where e_n is a white noise process with a variance $\mathrm{E}[e_n^2]$ equal to σ^2, y_n is a stationary scalar process and if $p > q$:

$$J = \begin{bmatrix} 0 & . & . & 0 \\ 1 & . & . & . \\ . & 1 & . & . \\ 0 & . & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -a_p \\ . \\ . \\ -a_1 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ b_q \\ . \\ 1 \end{bmatrix}, \quad H = [0, \dots, 0, \quad 1\}. \tag{3}$$

The extension of the model to the case $p = q$ [1] or $p < q$ is straightforward.

In the following, the AR and MA polynomials are assumed to be Hürwitz polynomials.

2.2. The 1-step predictor

The filtering equations for the previous model are given by

$$\hat{X}_{n+1/n} = J\hat{X}_{n/n-1} + Ay_n + \tilde{K}_n v_n, \tag{4}$$

with

$$\tilde{K}_n^{\mathrm{T}} = (0, \dots, 0, \tilde{k}_{q,n}, \dots, \tilde{k}_{1,n}),$$

being the transient Kalman gain defined by

$$\tilde{K}_n = (JP_nH^{\mathrm{T}})(HP_nH^{\mathrm{T}})^{-1} \tag{5}$$

where P_n is the error-covariance matrix, $\hat{y}_{n/n-1}$ is the 1-step predictor given by

$$\hat{y}_{n/n-1} = -\sum_{i=1}^{n} a_i^n y_{n-i}. \tag{6}$$

2.3. AR parameter estimation

We have recently shown (see [2, Proof of Theorem 1]) that the coefficients of the AR part of the ARMA model are given, together with the Kalman gain components, by the following system:

$$\sum_{j=1}^{q} a_{p+i-j}^{p+q-j} \tilde{k}_{j,p+q-j} = -a_{p+i}^{p+q}, \quad 1 \leqslant i \leqslant q, \tag{7}$$

$$\begin{cases} a_1 = a_1^{p+q} + \tilde{k}_{1,p+q-1}, \\ a_2 = a_2^{p+q} + \tilde{k}_{1,p+q-1} a_1^{p+q-1} + \tilde{k}_{2,p+q-2}, \\ \vdots \\ a_p = a_p^{p+q} + \sum_{j=1}^{q} \tilde{k}_{j,p+q-j} a_{p-j}^{p+q-j}. \end{cases} \tag{8}$$

Therefore, the algorithm to estimate the AR coefficients is straightforward:

(1) Estimate the 1-step predictor coefficients of increasing orders by using for instance Burg algorithm [5] to get the (a_i^{p+q}) in transient-state.
(2) Use (7) to get the Kalman gain components.
(3) Use (8) to get the AR coefficients $(\hat{a}_i)$.

Tsay and Tiao in [16] have shown that

$$\hat{a}_i = a_i + \mathrm{O}(N^{-1/2}), \quad 1 \leqslant i \leqslant p, \tag{9}$$

where the coefficients $(\hat{a}_i)$ are obtained by solving (8) and N is the length of the observation window. Therefore, the AR estimator is consistent.

As for the study of its efficiency, Tsay and Tiao [16, Lemma 5.5] have proven that the solution of (8) is asymptotically equivalent to the solution of the Modified Yule–Walker equations if the process is stationary (see also [10, Appendix]).

If we now consider the computational cost of the AR estimation method, it depends on the algorithm used to estimate the 1-step predictor coefficients (a_i^j) until order $p+q$. We have used Burg algorithm with Levinson recursion, but other techniques such as exact least-squares, lattice algorithm or gradient lattice algorithm can be used alternatively (see [10, 16]). Then, we have to solve a system of q linear equations (refer to (7)) to get the transient Kalman gain. At last, we have to compute the AR estimated coefficients by means of (8) which is a system of p linear equations. As is obvious, this method is interesting if $q \leqslant p$.

Otherwise it is better to use a dual algorithm [3, 13] which requires the inversion of a lower triangular matrix of dimension $(q \times q)$ before the inversion of a $(p \times p)$ matrix (herein, we are not in the case of an overdetermined system), see [14].

2.4. MA parameter estimation

It can be shown easily [2] that the components of the Kalman filter gain converge towards the MA parameters (provided that the MA polynomial of the ARMA model is an Hürwitz polynomial).

Here, we use a fast filtering algorithm [7] to get the MA parameters introducing the transient Kalman gain for the initialization [2].

The procedure results in a set of $2p$ scalar equations ($l_{i,n}$ is the i-th component of the vector L_n):

$$l_{i,n} = \frac{(1-\delta_{i-1})l_{i-1,n-1} - l_{p,n-1}\tilde{k}_{i,n-1}}{1-(l_{p,n-1})^2} \qquad (10)$$

$$\forall i \in [1, p].$$

$$\tilde{k}_{i,n} = \tilde{k}_{i,n-1} - l_{p,n-1} l_{i,n} \qquad (11)$$

To initialize this algorithm, we can again use the properties of the 1-step predictor (see [2]).

The initial condition for the vector L_n is given by

$$L_0 = A + \tilde{K}_0. \qquad (12)$$

$\tilde{K}_0$ has p components different from zero (see [2]), so has L_0.

It is easy to obtain the first p components of $\tilde{K}_0$ by using only 1-step predictor coefficients and the parameters (a_i):

$$\sum_{j=1}^{n} a_{i-j}^{n-j} \tilde{k}_{j,n-j} = a_i - a_i^n$$

$$\forall n \in [1, p] \quad \forall i \in [1, n], \qquad (13)$$

with the following convention:

$$a_m^l = \begin{cases} 0 & \text{if } m < 0 \\ 1 & \text{if } m = 0 \end{cases} \quad \forall l.$$

We can now summarize the procedure for estimating the MA parameters of an ARMA model:

(1) Compute the first p components of $\tilde{K}_0$ by solving (13) for $n = 1, 2, \ldots, p$.
(2) Then, initialize the first p components of L_0 using the estimated AR parameters and (12).
(3) Solve the fast filtering equations (11) and (12) up to the convergence of the algorithm. Two convergence tests are available: test either if L_n is equal to a null-vector or if $\tilde{K}_n$ is a constant vector.

It is worthwhile noticing that the quality of the MA estimation is strongly dependent upon the quality of the AR estimation through the initialization. But if we suppose that we are able to know the true AR parameters, then the MA estimator converges towards the true MA components. The proof is the following: at the beginning of this paper, we have assumed that the MA polynomial is an Hürwitz polynomial which implies a minimum phase. We know that a Kalman filter always converges to the optimal solution (minimum phase). As the minimum phase solution is unique, the transient Kalman gain asymptotically converges towards the MA parameters.

As for the computational cost, Favier and Alengrin have shown [7] that solving the Riccati equation results in solving a non-linear system of $p(p+1)/2$ equations whereas this method involves only $2p$ equations (without taking account of the initialization in both cases). With the proposed initialization, we need to solve p systems of 1 to p equations to get the first p components of $\tilde{K}_0$.

3. Simulation results

3.1. AR(p)+ noise model

It is well-known that an AR process corrupted by an additive gaussian white noise is equivalent to a special ARMA(p, p) model.

Herein, we study the asymptotic error variance of the ARMA parameters estimated with the previous algorithm. We compare these results to the Cramer–Rao bound (CRB) in order to appreciate the efficiency of the ARMA estimator.

First, we report on results obtained for AR(2) plus noise processes previously used by Friedlander and Sharman [8], AR coefficients of which are, respectively,

$a_1=-1.40$, $a_2=0.95$ for the narrow-band process,

$a_1=-0.45$, $a_2=0.55$ for the broad-band process.

The influence of the SNR on the estimator performance is given in Figs. 1-4. The CRB has been computed for the asymptotic case ($N\to\infty$) of an ARMA model. The tests have been conducted using an SNR varying from 0 to 30 dB, an observation window of 512 points with 100 Monte-Carlo runs in simple precision (the results presented in this paper are somewhat better than those obtained in [18] due to a change of the noise generator).

It is interesting to notice the good performance obtained for the broad-band process. As for the narrow-band case, the variance of the parameters is significantly larger than the CRB at low SNR. This phenomenon has been encountered with other estimation procedures (see [8] for instance). It is always more difficult to estimate narrow-band processes than broad-band ones.

Then, we have studied the influence of the window length N on the estimator performance using an AR(4)+noise model [8]. The AR parameters of this process are the following:

$$a_1=0,\quad a_2=1.4603,\quad a_3=0 \text{ and } a_4=0.8145.$$

The results presented in Figs. 5 and 6 are obtained at 15 dB with N equal to 300 and 3000 points (10 Monte-Carlo runs), respectively. The quality of the estimation is obviously improved with the increase of N.

3.2. ARMA(p, q) model

We have generated 10 sequences of data from the following ARMA(4, 2) model [11]:

$$x_n-2.595x_{n-1}+3.339x_{n-2}-2.20x_{n-3}+0.731x_{n-4} = e_n+0.433e_{n-1}+0.490e_{n-2},$$

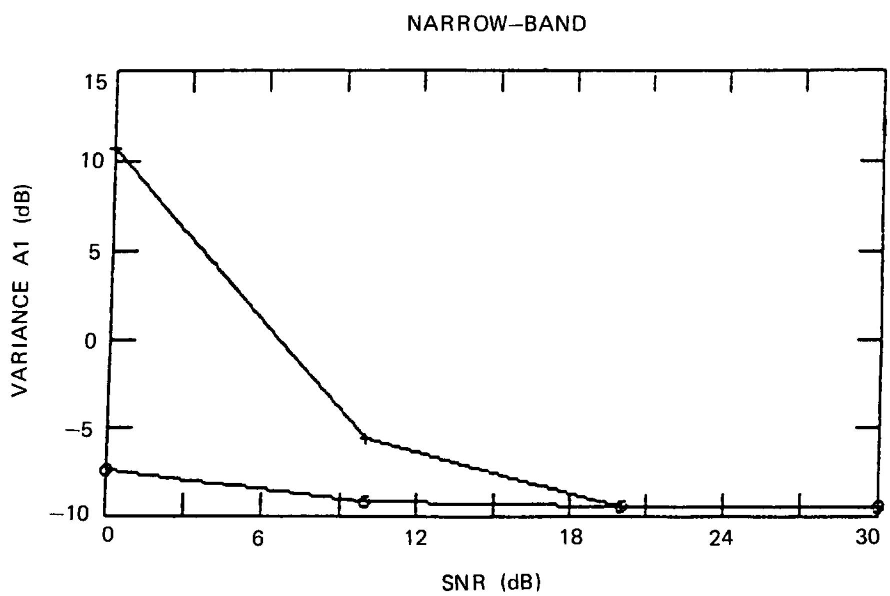

Fig. 1. Comparison to the CRB for the narrow-band process (AR).

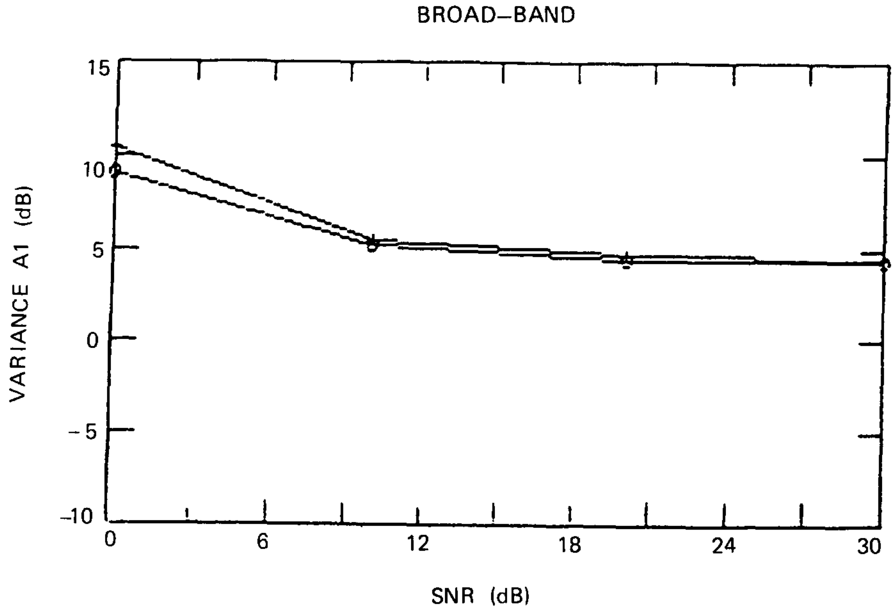

Fig. 2. Comparison to the CRB for the broad-band process (AR).

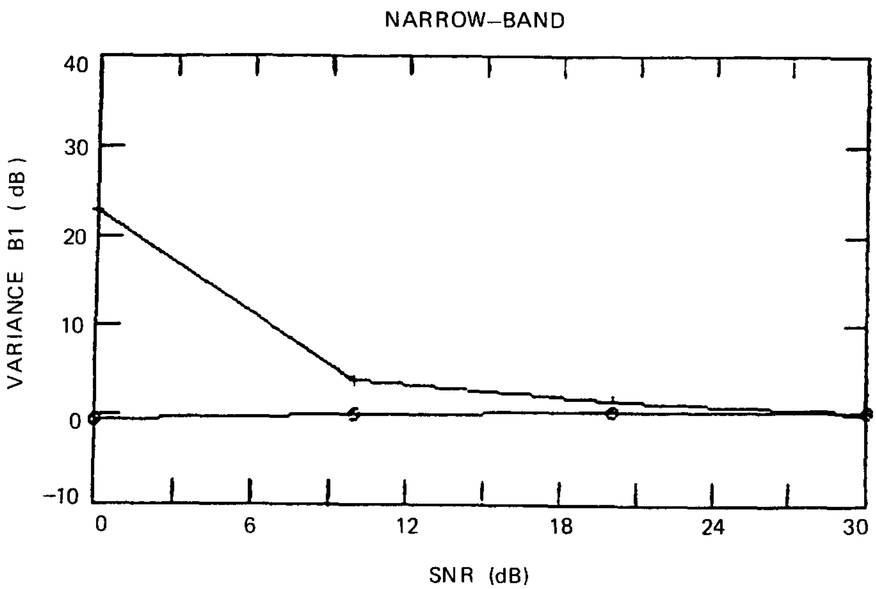

Fig. 3. Comparison to the CRB for the narrow-band process (MA).

which corresponds to two pairs of poles at 0.95 $\exp(\pm j0.2\pi)$ and 0.90 $\exp(\pm j0.3\pi)$ and one pair of zeros at 0.70 $\exp(\pm j0.6\pi)$. We have worked with a window of 512 points and have plotted the corresponding estimated poles and zeros on the unit disk as shown in Fig. 7 ($\triangle$ = poles, $\bigcirc$ = zeros).

For the AR part, a comparison with the MYW estimator can be done using the results obtained in [11] (Fig. 2). The use of Burg algorithm avoids to compute the autocorrelation and allows to work directly on the data which improves the quality of the estimation.

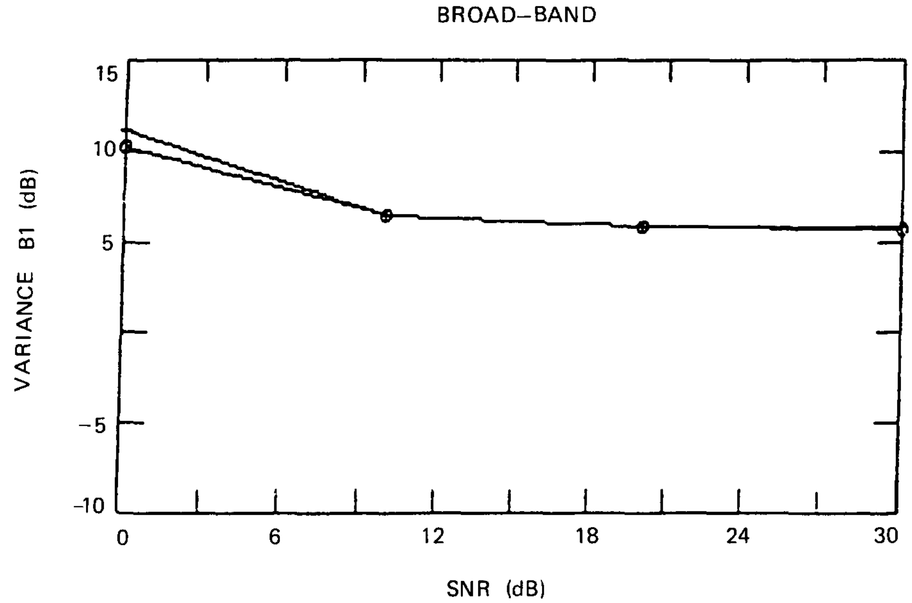

Fig. 4. Comparison to the CRB for the broad-band process (MA).

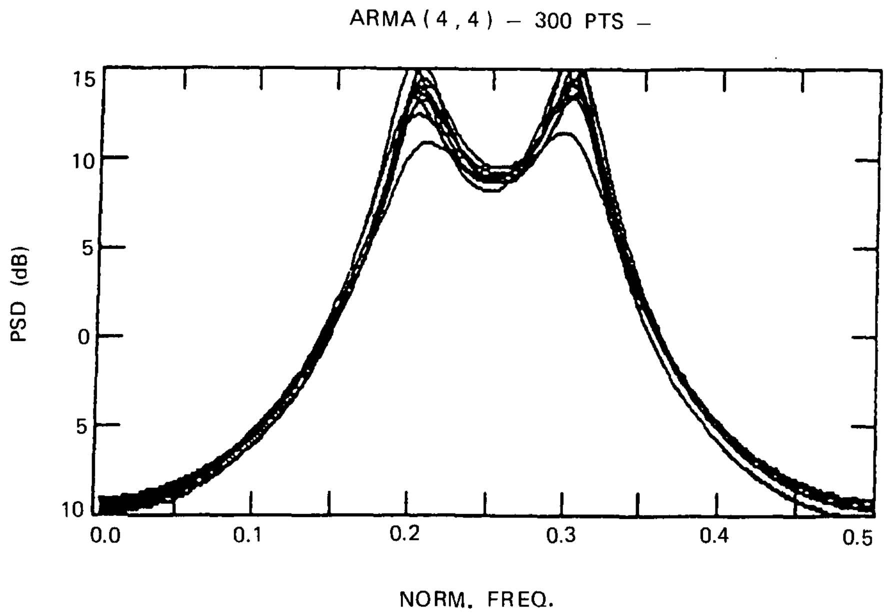

Fig. 5. AR(4)+NOISE, PSD (dB) with 300 PTS

As for the MA part, the Kalman gain asymptotically converges to the MA parameters. The dispersion is greater for the zeros than for the poles, partially due to the initialization (the MA part depends on the AR estimated parameters).

4. Conclusion

In this short paper, we have presented the simulation results of a method to estimate the parameters of an ARMA(p, q) model. Our method is

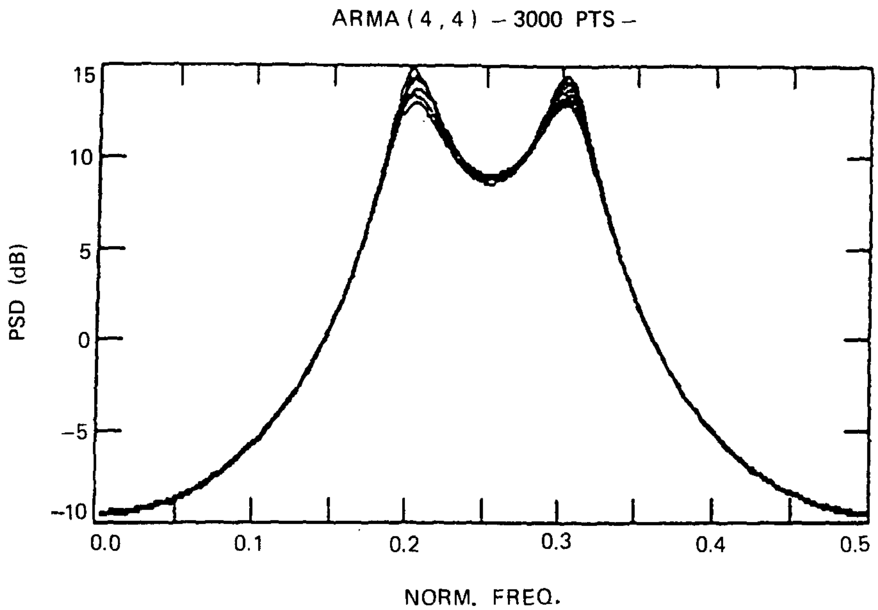

Fig. 6. AR(4)+NOISE, PSD (dB) with 3000 PTS.

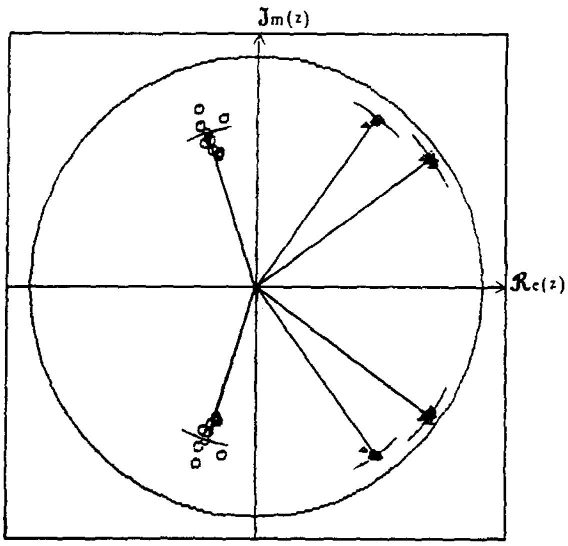

Fig. 7. Poles and zeros for the ARMA(4, 2) process with 10 sequences of data using a window of 512 points.

based on Kalman filtering in transient state. The AR estimation is consistent and numerical results have indicated that the performance of the estimator is interesting. As for the MA estimator, the results are reasonable taking into account they are dependent on the AR estimation. Then, we have given a few examples of both AR(p)+noise and ARMA(p, q) processes.

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