AN OVERVIEW OF PERIODIC TIME SERIES WITH EXAMPLES

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Abstract: Two environmental examples are used to demonstrate modeling with periodic autoregressive moving average (PARMA) processes. These two examples – average monthly temperatures in Big Timber, Montana, USA, and carbon dioxide exchange rates of flowering plants in a growth chamber – are clearly periodic in their first and second moments, and highlight the necessity as well as the pitfalls of PARMA modeling. *Copyright © 2001 IFAC*

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1. INTRODUCTION

Periodicity occurs naturally in many environmental time series – hourly tide levels, daily stream flow, monthly average temperature, and carbon dioxide (CO_2) exchange of growing plants are but a few examples. This paper will focus on the last two examples in a demonstration of methods for analyzing periodically correlated (PC) time series. Specifically, a PC time series will tend to exhibit periodicity not only in its first moment, but also in its second moments. For modeling such phenomena, periodic autoregressive moving average (PARMA) models are discussed in some detail in Section 2.

The first example, shown in Figure 1, is of monthly average temperatures in degrees Celsius, measured over 86 years (1911 through 1996), in Big Timber, Montana, USA (Lund and Seymour, 2001). Beyond the obvious periodicity in the first moment, there is no other significant feature (*e.g.*, linear trend in the first moment of these data. There is also an obvious

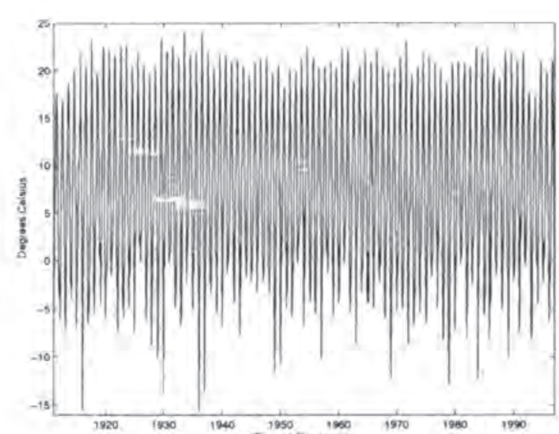

Figure 1. Monthly Average Temperatures (in °C) for Big Timber, Montana, USA, 1911-1996.

periodicity in the second moment of this data – the winter temperatures are more variable than the summer temperatures, which is true in general.

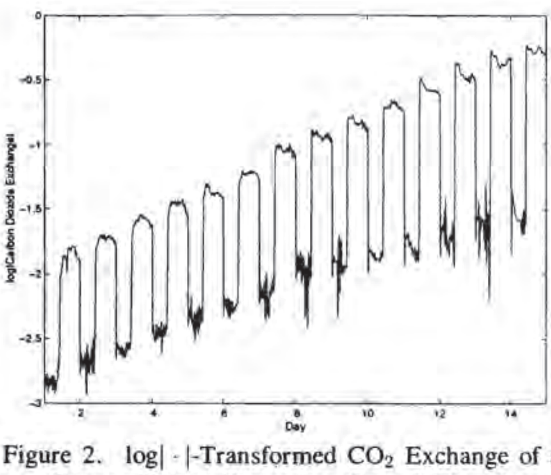

Figure 2. log| · |-Transformed CO_2 Exchange of a Group of Vinca Grown in a Growth Chamber.

Figure 2 shows the second example – the log| · |-transformed amount of CO_2 (originally in micro-mol/second) taken up by a group of thirty-two vinca [*Catharanthus roseus* (L.) G. Don.] plants growing inside a single growth chamber in which temperature and light intensity were controlled, and in which the only changes in CO_2 concentration would be due to the plants within the chamber (Seymour, 2001). The log| · | transform was used since the daylight data are all strictly positive while the darkness data are all strictly negative. The data were recorded every 20 minutes over 14 days, with 14 hours of daylight and 10 hours of darkness. There are 1008 observations: during any given 24-hour period, photosynthesis is occurring during the 42 daylight observations, but not during the 30 darkness observations. Like the temperature series, this series clearly exhibits periodic structure in both its first and second moments.

This paper continues in Section 2 with a description of methods for detecting and modeling periodic time series, as well as practical issues of estimating the covariance function of a periodic time series. A more detailed overview of this methodology may be found in Lund and Basawa (1999). Section 3.1 gives results of applying these methods to the data in Figure 1, while Section 3.2 gives the results of applying these methods to the data in Figure 2. More details on these results may be found in Lund *et al.* (2001) and Seymour (2001), respectively.

2. ESSENTIALS OF PERIODICALLY CORRELATED TIME SERIES

A process $\{X_t\}$ is said to be *periodically correlated with period* T (PC-T) if

$$E(X_{t+T}) = E(X_t) \qquad (1)$$
$$\mathrm{Cov}(X_{s+T}, X_{t+T}) = \mathrm{Cov}(X_s, X_t)$$

for all integers t and s. This property is also called *periodically stationary* or *cyclostationary*. A PC-T process is not (weakly) stationary; however, if a PC-T process is written as a T-variate process by grouping the data into "periods", then it is T-variate stationary (*cf.* Basawa and Lund, 2001). Such a process conveniently allows for a seasonal expression of the mean and covariance. Assume for simplicity that the series length N is evenly divisible by the period T, so that $d = N/T$ is an integer. Then for $\nu = 1, \ldots, T$ and $n = 0, \ldots, d-1$, the mean is a function of only season ν, and the covariance is a function of only season ν and lag h:

$$\mu_\nu = E(X_{nT+\nu}) \qquad (2)$$
$$\gamma_\nu(h) = \mathrm{Cov}(X_{nT+\nu}, X_{nT+\nu+h}), h \geq 0$$

The covariance function $\gamma_\nu(h)$ is not symmetric in h, but $\gamma_\nu(-h) = \gamma_{\nu+h}(h)$. Note also that seasonal notation is used when emphasis upon the seasonality is necessary.

There are various tools for detecting periodicity, all of which are primarily visual. For detecting periodicity in the mean of a given time series $\{X_t\}$, besides visual inspection of $\{X_t\}$ itself, there is $\widehat{\rho}(h)$, the sample autocorrelation function (ACF) of $\{X_t\}$ at lag h (Brockwell and Davis 1996, or any introductory time series text). If $\{X_t\}$ has a substantial periodic component in its first moment, then $\widehat{\rho}(h)$ will have the same periodicity. However, a bit of caution is required in using the sample ACF to infer periodic behavior: a process for which $\widehat{\rho}(h)$ exhibits periodic behavior need not be periodic in its mean (Brockwell and Davis 1991).

Detecting periodicity in the second moments, or covariance structure, of a time series is more challenging. Perhaps the best tool to use is the average squared coherence (ASC) statistic of Goodman (1965), which involves the spectral representation of $\{X_t\}$ (see Bloomfield *et al.* (1994) or Lund *et al.* (1995) for the details). In essence, for a series of length N *from which the first moments have been removed*, the ASC statistic is given by

$$\overline{\kappa}_h = \frac{1}{N}\sum_{j=1}^{N-1}\left(\frac{\left|\sum\limits_{m=0}^{M-1} I_{j+m}\overline{I_{j+h+m}}\right|^2}{\sum\limits_{m=0}^{M-1}|I_{j+m}|^2 \sum\limits_{m=0}^{M-1}|I_{j+h+m}|^2}\right) \qquad (3)$$

at spectral lags h and for a smoothing control M, where $I_j = \frac{1}{\sqrt{2\pi N}}\sum_{t=0}^{N-1} X_{t+1}\exp(-it\lambda_j)$, $i = \sqrt{-1}$ is the discrete Fourier transform of $\{X_t\}$ at the Fourier frequency $\lambda_j = 2\pi j/N$. A plot of $\overline{\kappa}_h$ versus h for all $1 \leq h \leq N/2$ then reveals any significant second-order periodicities: Small values of $\overline{\kappa}_h$ are statistical evidence that $\{X_t\}$ is stationary; large values of $\overline{\kappa}_h$ at some values of h that are integer multiples of $d = N/T$ is statistical evidence that

$\{X_t\}$ is PC-T; and if $\overline{\kappa}_h$ is large at values of h that are not all multiples of some common integer larger than one, this is statistical evidence that the series is neither stationary nor PC.

A well-developed class of models for mean zero stationary time series is the autoregressive moving average model of orders p and q, denoted ARMA(p,q) (Brockwell and Davis, 1991). An ARMA(p,q) series $\{Z_t\}$ is a solution to the linear difference equation

$$Z_t - \sum_{k=1}^{p}\phi_k^{\mathrm{AR}} Z_{t-k} = \varepsilon_t + \sum_{k=1}^{q}\theta_k^{\mathrm{MA}}\varepsilon_{t-k}, \qquad (4)$$

where $\{\varepsilon_t\}$ is white noise with mean zero and variance σ_ε^2, denoted WN$(0,\sigma_\varepsilon^2)$. (Note: for the sake of parsimonious notation, $\{\varepsilon_t\}$ is used henceforth to denote a generic white noise process, while $\{Z_t\}$ is used to denote a generic ARMA(p,q) process.)

The most commonly-used model for seasonal time series based on the ARMA model in (4) is the seasonal autoregressive integrated moving average (SARIMA) model (Brockwell and Davis, 1996), in which an appropriately differenced series behaves as an ARMA process. SARIMA processes are not necessarily stationary; however, it is a simple exercise using (1) to verify that differencing an arbitrary PC-T process at lag T yields a PC-T process with mean zero. Hence the SARIMA models are inappropriate for PC time series.

The appropriate ARMA-type model for a PC-T series is the periodic autoregressive moving average (PARMA) model. The process $\{X_t\}$ is said to be a PARMA series if it is a solution to the periodic linear difference equation

$$(X_{nT+\nu} - \mu_\nu) - \sum_{k=1}^{p_\nu}\phi_{\nu,k}(X_{nT+\nu-k} - \mu_{\nu-k}) \qquad (5)$$
$$= \sum_{k=0}^{q_\nu}\theta_{\nu,k}\,\varepsilon_{nT+\nu-k}$$

in which $n = 0,\ldots,d-1$, $d = N/T$, $\nu = 1,\ldots,T$, $\{\varepsilon_t\} \sim$ WN$(0,1)$, μ_ν are the periodic means, and $\theta_{\nu,0}$ in particular are the periodic white noise variances. Recursive prediction and Gaussian likelihood evaluation for PARMA models is dealt with in Lund and Basawa (2000), but the main difficulty with these models is lack of parsimony. Fitting the PARMA model in (5) requires the estimation of $T + \sum_{\nu=1}^{T}(p_\nu + q_\nu + 1)$ parameters – a potentially formidable task. (For this reason, PARMA notation seldom includes parametric notation except in very simple cases.) A simple solution that has worked in many cases is to *layer*; that is, to fit first a very simple periodic model, and then to fit an ARMA model to the residuals from the first fit. Thus, $\{\varepsilon_t\}$ in (5) would be replaced by the ARMA(p,q) process $\{Z_t\}$ as given in (4).

One layering scheme which has been successfully used in climatology (Amato *et al.*, 1989; Lund and Seymour, 1999; Lund *et al.*, 2001) and in economics (Parzen and Pagano, 1979), and which is used here to model the temperature data in Figure 1, is to *seasonally standardize* the series by subtracting the periodic mean μ_ν, dividing by the periodic standard deviation σ_ν, and then fitting an ARMA(p,q) model to the series that remains – referred to henceforth as SS+ARMA (Bloomfield *et al.*, 1994; Lund *et al.*, 1995). Least-squares estimates of μ_ν and σ_ν are straightforward to calculate: simply average the observations occurring during a fixed season ν to obtain $\widehat{\mu}_\nu$, and compute a sample standard deviation of those observations to obtain $\widehat{\sigma}_\nu$. The seasonally standardized series is then given by $Z_{nT+\nu} = (X_{nT+\nu} - \widehat{\mu}_\nu)/\widehat{\sigma}_\nu$, on which ARMA modeling and diagnostics (Brockwell and Davis 1991) are then employed. The SS+ARMA layering scheme does in fact yield a parsimonious PARMA model with $p_\nu \equiv p$, $q_\nu \equiv q$, and

$$\phi_{\nu,k} = \frac{\sigma_\nu\phi_k^{\mathrm{AR}}}{\sigma_{\nu-k}}, 1 \le k \le p \qquad (6)$$
$$\theta_{\nu,k} = \theta_k^{\mathrm{MA}}\sigma_\nu, 0 \le k < q, \theta_0^{\mathrm{MA}} = 1$$

for $\nu = 1,\ldots,T$. Then the covariance matrix of $\{X_1,\ldots,X_N\}$ is then very easy to calculate. Let $\gamma_{\mathrm{ARMA}}(h)$ be the autocovariance generating function of the model fitted in the ARMA layer. Then the covariance matrix $\Gamma_{\mathrm{SS+ARMA}}$ of the layered SS+ ARMA model is given by

$$\Gamma_{\mathrm{SS+ARMA}} = \qquad (7)$$
$$[\sigma_\nu\sigma_\lambda\gamma_{\mathrm{ARMA}}((n-m)T + (\nu-\lambda))]_{nT+\nu,mT+\nu=1}^{N}$$

where σ_ν^2 is the seasonal variance for season $\nu = 1,\ldots,T$.

Another layering scheme used to model the data in Figure 2 (which has also worked well in climatology) is to fit first a periodic autoregressive process of order 1, denoted by PAR(1), followed by an ARMA(p,q) model – referred to hereafter as PAR+ARMA (Bloomfield *et al.*, 1994; Lund *et al.*, 1995). The process $\{X_t\}$ is said to be PAR(1) if it is a solution to

$$(X_{nT+\nu} - \mu_\nu) - \phi_\nu^{\mathrm{PAR}}(X_{nT+\nu-1} - \mu_{\nu-1}), \qquad (8)$$
$$= \delta_\nu\varepsilon_{nT+\nu}$$

in which $n = 0,\ldots,d-1$, $d = N/T$, $\nu = 1,\ldots,T$, $\{\varepsilon_t\} \sim$ WN$(0,1)$, μ_ν are the periodic means, δ_ν are the periodic white noise variances, and ϕ_ν^{PAR} are the periodic autoregressive parameters. Such a process is PC-T if $|\prod_{\nu=1}^{T}\phi_\nu^{\mathrm{PAR}}| < 1$ (Vecchia, 1985). Note that in this case, $p_\nu \equiv 1$ and $q_\nu \equiv 0$.

Approximate maximum likelihood estimates for the parameters in (8) are

$$\widehat{\phi}_\nu^{\text{PAR}} = \frac{\widehat{\gamma}_\nu(1)}{\widehat{\gamma}_\nu(0)}, \quad \widehat{\delta}_\nu = \widehat{\gamma}_\nu(0) - \widehat{\phi}_\nu^{\text{PAR}} \widehat{\gamma}_\nu(1) \quad (9)$$

where the sample covariance function of the series during season $\nu = 1, \dots, T$ at lag $h \geq 0$ is given by

$$\widehat{\gamma}_\nu(h) = \frac{1}{d} \sum_{n=0}^{d-1} (X_{nT+\nu} - \widehat{\mu}_\nu)(X_{nT+\nu-h} - \widehat{\mu}_{\nu-h}), \quad (10)$$

where $\widehat{\mu}_\nu = \sum_{n=0}^{d-1} X_{nT+\nu}/d$ (Bloomfield *et al.*, 1994). If the PAR(1) model described in (8) is a good fit, then the residuals

$$\frac{(X_{nT+\nu} - \widehat{\mu}_\nu) - \widehat{\phi}_\nu^{\text{PAR}} (X_{nT+\nu-1} - \widehat{\mu}_{\nu-1})}{\widehat{\delta}_\nu} \quad (11)$$

should behave as a WN(0, 1) process. In the case of PAR+ARMA layering, (11) is an ARMA(p, q) process. This layering scheme results in a parsimonious PARMA model which will be used to successfully model the data in Figure 2.

Asymptotic results for PARMA parameter estimates have been recently studied by Basawa and Lund (2001). Taking an estimating equations approach, they establish the asymptotic normality of least squares estimates of the PARMA parameters, which also establishes the asymptotic normality of maximum likelihood estimates of the PARMA parameters under Gaussian assumptions.

The covariance matrix of a PAR+ARMA is complicated; indeed, the covariance function of a PARMA model is very complicated in general. There is a recursive method for calculating the covariance function $\gamma_\nu(h)$ for a PAR(1) (Lund and Basawa, 1999), but it is of no assistance in the PAR+ARMA case. However, a model-based approximation to the covariance matrix is possible based on the results in Lund and Basawa (1999). When the PARMA model is causal, solutions to (5) can be expressed uniquely (in mean square) in the form

$$X_{nT+\nu} = \sum_{k=0}^{\infty} \psi_{\nu,k} \varepsilon_{nT+\nu-k} \quad (12)$$

where $\sum_k |\psi_{\nu,k}| < \infty$ for $\nu = 1, \dots, T$. In the case of PAR+ARMA, the $\psi_{\nu,k}$ are given by

$$\psi_{\nu,k} = \sum_{l=0}^{\infty} \left(\psi_{k-l}^{\text{ARMA}} \delta_{\nu-l} \prod_{i=0}^{l-1} \phi_{\nu-i}^{\text{PAR}} \right), \quad (13)$$

where ψ_k^{ARMA} are the coefficients on the white noise terms in the causal representation of the ARMA process in (11) that is completely analogous to (12). Thus the covariance function of the PAR+ARMA process may be approximated by a finitely summed version of the general PARMA covariance

$$\gamma_\nu(h) = \sum_{k=0}^{\infty} \psi_{\nu,k+h} \psi_{\nu-h,k}. \quad (14)$$

One only needs to check that the convergence of the summands to zero is rapid enough that the tail sum becomes negligible after a point, which is reasonably easy via computer. The covariance matrix for $\{X_1, \dots, X_N\}$ is then given by

$$\Gamma_{\text{PAR+ARMA}} = \left[\gamma_{\max(i,j) - T\left\lfloor \frac{\max(i,j)}{T} \right\rfloor} (|i - j|) \right]_{i,j=1}^{N} \quad (15)$$

where $\lfloor \cdot \rfloor$ is the floor function (*i.e.*, the greatest integer function). Computing the covariance matrix in this way is laborious, even on a computer. Currently, though, it is the only option.

Note that both of the layered estimation procedures given here are a combination of ordinary least squares (OLS) and maximum likelihood (ML): the regression parameters for the overall trend and the periodic mean fluctuations are estimated via OLS, while the PARMA parameters governing the covariance structure are estimated via ML.

To summarize, the PARMA modeling steps are:

1. Transform the data, if necessary.
2. Estimate and the trend function, including the periodic component.
3. Subtract the estimated trend from the data; that is, calculate the residuals.
4. Fit a PARMA model to the residuals obtained in step 3:
   a. Either seasonally standardize the residuals (simply by dividing by the seasonal standard deviation) or fit a PAR(1) to the residuals.
   b. Fit an ARMA model to the residuals from the fit obtained in step 4a.
5. Use the parameter estimates obtained in step 4 to calculate either $\widehat{\Gamma}_{\text{SS+ARMA}}$ or $\widehat{\Gamma}_{\text{PAR+ARMA}}$, based on the fit in step 4 using (13), (14), and (15).

The covariance matrix may then be used, for example, to compute standard errors for the trend estimates found in step 2.

3. RESULTS

In the following sections, the SS+ARMA layering scheme is applied to the temperature data, while the PAR+ARMA is applied to the CO_2 exchange data. Each model is the most parsimonious selection for

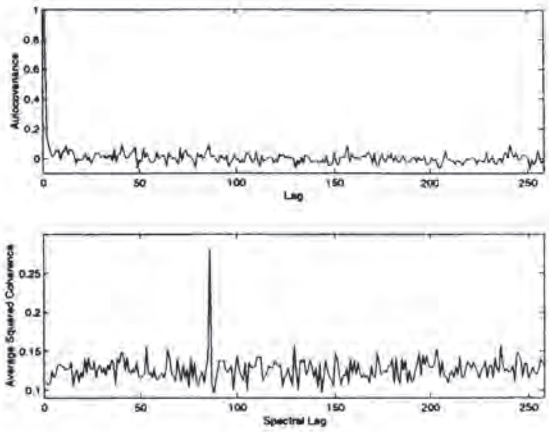

Figure 3. For the Temperature Data: a) ACF After Removal of All First Moments; b) ASC.

the data: in the first example, the PAR+ARMA model overparametrizes the temperature data, while in the second example, SS+ARMA does not account for all of the second-moment periodicity in the CO_2 exchange data.

3.1 SS+ARMA Applied to the Temperature Data.

Figure 3a shows the sample ACF of the temperature data depicted in Figure 1 *after* the removal of all first moments. The sample ACF, which is plotted along with a 99% confidence band for uncorrelated white noise, indicates that there is no further periodicity in the first moment. Figure 3b shows the ASC statistic (3) of the temperature series (recall that the ASC statistic must be calculated on zero-mean data). Note the one very large spike at lag $d = N/T = 1032/12 = 86$. This indicates a strong cycle of period $T = 12$ in the second moments of the series.

Observe the "confidence" line plotted in Figure 3b. The mean and variance of (3) are $E(\overline{\kappa}_h) = 1/M$ and $\mathrm{Var}(\overline{\kappa}_h) = c_M/N$, respectively, where M is a spectral smoothing parameter. The asymptotic normality of (3) is suspected and has been established in unpublished notes only for the case in which $\{X_t\}$ is a moving average of finite order (Bloomfield *et al.*, 1994; Lund *et al.*, 1995). Though (3) is suspected to be Gaussian (particularly in cases in which the data are approximately Gaussian), the constant c_M for spectral smoothing parameter M is still unknown. Calculations of the ASC used $M = 8$; thus the value $c_8 = .1277$, as given in a table of values for c_M simulated from Gaussian white noise (Lund *et al.*, 1995), was required. A "99% confidence" band is therefore $1/M + 2.33\sqrt{c_M/N}$. This value may not be appropriate in the present context. Nevertheless, it does provide a guideline against which spikes in the ASC plot may be evaluated.

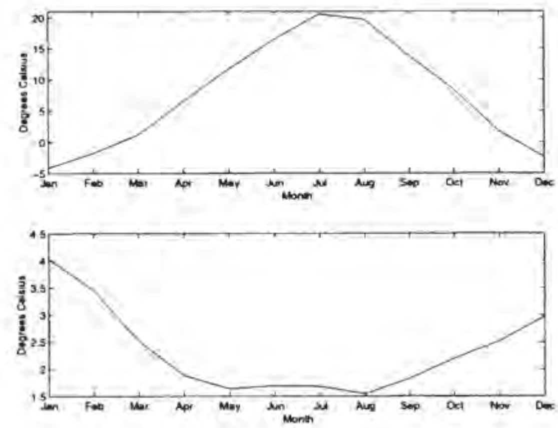

Figure 4. For the Temperature Data: a) Periodic Means; b) Periodic Standard Deviations.

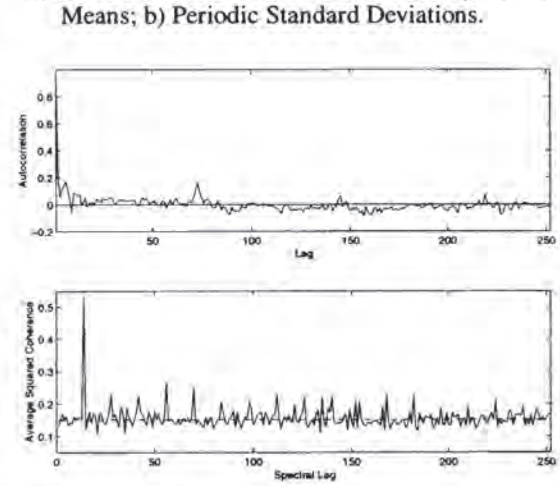

Figure 5. For the CO_2 Exchange Data: a) ACF After Removal of the First Moments; b) ASC.

The SS+ARMA layering approach gave an excellent fit to the data. Figure 4a depicts the seasonal mean temperature for each month, while Figure 4b shows the seasonal standard deviations. These values were used to seasonally standardize the temperature data. The seasonally standardized series was successfully fit with an ARMA(1,3). Hence a PAR+ARMA is a clear overparametrization, even though it may give an acceptable fit

3.2 PAR+ARMA Applied to the CO_2 Exchange Data.

Figure 5a shows the ACF of the CO_2 exchange data depicted in Figure 2. This plot indicates a possibility of seasonal autocorrelation of period $T = 72$. Figure 5b shows the ASC, with one very large spike at lag $d = N/T = 1008/72 = 14$ indicating a strong cycle of period $T = 72$ in the second moments of the series. Note that the "99% confidence" bound isn't

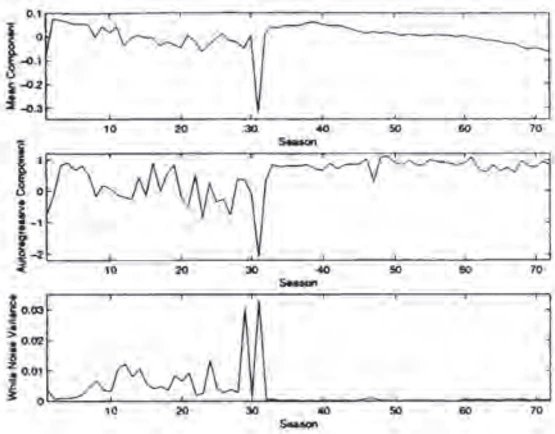

Figure 6. For the CO_2 Exchange Data: a) Periodic Means; b) Periodic Autoregressive Coefficients; c) Periodic White Noise Variances.

as clear as the one in Figure 3b.

Initially, the SS+ARMA model was tried on the CO_2 exchange series after removing the piecewise quadratic trend. In spite of seasonal standardization, the result still contained seasonality in its second moments, according to the ASC of the seasonally standardized series. Thus the SS+ARMA model was inadequate.

However, the PAR+ARMA approach did give a satisfactory model of the PC structure. A mean zero PAR(1) as in (8) was fitted to the detrended data. The fit is summarized in Figure 6: Figure 6a shows the periodic mean fluctuations $\widehat{\mu}_\nu$ about the trend, Figure 6b depicts the periodic autoregressive coefficients $\widehat{\phi}_\nu^{\text{PAR}}$, and Figure 6c gives the white noise variances $\widehat{\delta}_\nu$, each for $\nu = 1, \ldots, T = 72$.

The residuals from the PAR(1) fit were well-modeled by an ARMA(3,1). Thus the PAR+ARMA layering scheme gives a good fit to the CO_2 exchange data.

4. CONCLUSION

It is clear that PARMA models are handy tools for modeling periodic time series. Though parsimony remains a significant problem which is currently being addressed, the layering approaches outlined herein are practical solutions that will model many real time series.

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