

A Journey Through Types

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What is this Talk about?

Types! This presentation hopes to address the following:

- How types can help you write correct software.
 - ▶ This is important when *EVERYTHING* runs software.
 - ▶ Good type systems can make this less horrifying!
- How types make things easier to write in general.
 - ▶ Compiler can automate a lot more.
 - ▶ Compiler can catch many simple issues.

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Somewhat of a whirlwind introduction. Let me know if you're lost, because this talk is all over the place!

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You think that this is normal...

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Traceback (most recent call last):  
  File "<stdin>", line 1, in <module>  
TypeError: 'NoneType' object is not subscriptable
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Traceback (most recent call last):  
  File "<stdin>", line 1, in <module>  
TypeError: 'NoneType' object is not subscriptable
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... It's not!

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- Tell us how to use values.
 - ▶ Tells us what operations are defined on the types.
 - ▶ Can you add things of this type?
 - ▶ Can a function take a value of this type as an argument?
 - ▶ What kind of stuff does this function return?

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- Rejection of general nonsense: 357^{circles}

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▶ NO MORE NULL REFERENCE
EXCEPTIONS!

The types you may have seen: Python

```
def my_sort(xs):
    if xs == []:
        return xs
    else:
        first_elem = xs[0]
        rest = xs[1:]

        smaller = my_sort([x for x in rest if x <= first_elem])
        larger = my_sort([x for x in rest if x > first_elem])

        return smaller + [first_elem] + larger

def my_factorial(n):
    if n == 0:
        return 1
    else:
        return n * my_factorial(n-1)
```

- No types to help document functions.
- No types to catch errors at compile time.
 - ▶ Tests can help...
 - ▶ But it's nice to not have to worry about certain errors at all.

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- What if we could force functions to be compartmentalized?
 - ▶ No sneaky IO
 - ▶ No hidden global states
- Wouldn't it be nice to have a description of what a function can and can't do in a concise format?
- Could the compiler tell us when our function deviates from these descriptions?
 - ▶ Why wait until runtime to find your mistakes?

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```
Integer factorial(Integer n) {  
    if (n == 0) {  
        return 1;  
    }  
    else {  
        return n * factorial(n - 1);  
    }  
}  
  
ArrayList<Integer> my_sort(ArrayList<Integer> xs) {  
    if (xs.size() == 0) {  
        return new ArrayList<Integer>();  
    }  
    else {  
        // ...  
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Types aren't bad... Java is bad.

Greener pastures and better ideas

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 - ▶ “Function does not alter global state”
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 - ▶ If something is “wrong”, then why wait for runtime to tell us?
- Ease reading and writing of programs.
 - ▶ Act as a kind of documentation.
 - ▶ Guide us when writing programs.
 - ▶ Stop us from making mistakes.
- Allow us to make better guarantees.
 - ▶ “Function does not alter global state”
 - ▶ “Function does not read from disk”
- Not too much verbosity.
 - ▶ Nice, clean syntax!

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- Type inference: compiler can figure out the types of things.
- Nice, relatively specific types.

```
-- Causes a type error, because it doesn't make sense.
mySort [factorial, (*2)]
```

Something similar in Python would only be caught at runtime

We need Ord!

You might think you could do this:

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-- Instead of: Ord a => [a] -> [a]
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- *a* could be *any type*
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 - ▶ Like a function, or a picture
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 - ▶ Like a function, or a picture
- Need the constraint so we know we can perform comparisons!

Haskell makes sure we can only perform operations that are defined on values of a given type, but allows us to be general about it. This function works with any orderable element still, and not just a fixed type.

More care in a type

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- This keeps it so that, for the most part, elements of a type act the same way.
- Operations on elements of a type work on all values, so no runtime exceptions are raised!

This helps to keep everything sane!

Maybe maybe!

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- Better than null!
- Type checker can tell us when we need to handle null.
- Compile time errors if we don't handle null!

Maybe examples

```
-- Find out where a value is in a list.
whichIndex :: Eq a => a -> [a] -> Maybe Integer
whichIndex = whichIndexAcc 0

-- Helper function that remembers our position in the list.
whichIndexAcc :: Eq a => Integer -> a -> [a] -> Maybe Integer
whichIndexAcc pos value [] = Nothing
whichIndexAcc pos value (x::xs) = if x == value
                                   then Just pos
                                   else whichIndexAcc (pos+1) xs

-- A dictionary of all the important words.
dictionary :: [String]
dictionary = ["cats", "sandwiches", "hot chocolate"]

main :: IO ()
main = do entry <- getLine
         case whichIndex entry dictionary of
           (Just pos) => putStrLn "Your entry is at position " ++ show
                         pos ++ " in the dictionary."
           Nothing => putStrLn "Your entry does not appear in the
                              dictionary."
```

- You know whichIndex can yield a “null” value (Nothing). Just from type.
- Could also be a Just <Integer>, such as Just 3.
- You have to explicitly unwrap these values (see main) to get at the possible value!

Maybe more!

Seems tedious? It's not! Good syntax makes this easy!

```
-- Look up a word in the same position in a different dictionary.
dictionary :: [String]
dictionary = ["cats", "sandwiches", "hot chocolate"]

synonyms :: [String]
synonyms = ["meows", "bread oreos", "sweet nectar"]

moreSynonyms :: [String]
moreSynonyms = ["floofs", "subs", "hot coco"]

-- Get value at index, Nothing if out of range.
getIndex :: Integer -> [a] -> Maybe a
getIndex _ [] = Nothing
getIndex 0 (x:xs) = Just x
getIndex n (_:xs) = getIndex (n-1) xs
```

More on next slide...

“Do” notation frees us from tedium!

```
lookupSynonyms :: String -> Maybe (String, String)
lookupSynonyms word = do index <- getIndex word dictionary

    -- Lookup my synonyms, if anything fails return Nothing
    .
    firstSynonym <- getIndex index synonyms
    secondSynonym <- getIndex index moreSynonyms

    -- Success! Return Just the synonyms.
    Just (firstSynonym, secondSynonym)

-- lookupSynonyms essentially desugars to this.
-- The compiler can help avoid this tedium!
painfulLookupSynonyms :: String -> Maybe (String, String)
painfulLookupSynonyms word = case getIndex word dictionary of
    Nothing -> Nothing
    (Just index) ->
        case getIndex index synonyms of
            Nothing -> Nothing
            (Just first) ->
                case getIndex index moreSynonyms of
                    Nothing -> Nothing
                    (Just second) -> Just (first,
                                            second)

main :: IO ()
main = do word <- getLine
    case lookupSynonym word of
        Nothing -> putStrLn ("Hmmm, I don't know a synonym for " ++ word)
        (Just synonym) -> putStrLn ("I think " ++ word ++ "'s are a lot like
                                     " ++ synonym ++ "'s!")
```

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- The type checker tell you when you forgot to “unwrap” a promise.
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Having a good type system in an expressive language, like Haskell, can really help ease a lot of the pain you currently suffer.

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Having a good type system in an expressive language, like Haskell, can really help ease a lot of the pain you currently suffer.

Programming can be good?

IO types

```
-- putStrLn :: IO ()  
-- getLine  :: IO String  
  
main :: IO ()  
main = do putStrLn "What is your name?"  
        name <- getLine  
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- IO means a function performs input / output.

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 - ▶ etc...
- () is “void” — no return value.
- No escaping IO. Taints anything using it, so you know if something does input / output.
- Can help avoid unexpected behaviour, similar to global state changing a functions behaviour.

Haskell in summary: what does it buy us?

- We can catch errors at compile time!
 - ▶ Type system lets us describe values in a fair amount of detail, which removes a lot of obviously incorrect programs from the set of programs that compile.
 - ▶ Types don't contain nulls. Very few values which cause explosions at runtime.
- Easier to read and write programs. Types of functions are very descriptive!
 - ▶ Types help in much the same way as test driven development (but they're always there, unlike tests!)
 - ▶ Makes you think about arguments a function takes, and what it returns
 - ▶ Types point out errors when developing, such as forgetting to unwrap a Maybe value.

What more does it buy us?

- Types can be very general, allowing us to reuse functions with any type that makes sense.
 - ▶ `mySort` works with any list of orderable elements!
- It allows us to specify properties and guarantees within our programs.
 - ▶ “This function does not alter global state, or read from a file”.
 - ▶ Functions are “pure”.
 - ▶ Special actions, like IO, are clearly labeled.

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-- Want the integer argument to always be in range so we don't need Maybe!
index :: Integer -> [a] -> a
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```

- Need to encode length of the list into the type.
- Can't do this in Haskell because a type can not depend upon a value.
 - ▶ Length in the type must depend upon the length of the list value.

More motivation...

Also not possible to encode specific properties which depend upon values in types.

```
mySort :: Ord a => [a] -> [a]
mySort [] = []
mySort (first_elem::rest) = smaller ++ [first_elem] ++ larger
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Would be nice to encode into the type of `mySort` that...

- Output list must be in ascending order.
- Output list must contain the same values as the input list.

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Would be nice to encode into the type of `mySort` that...

- Output list must be in ascending order.
- Output list must contain the same values as the input list.

This would prove that the program works! `mySort` would be guaranteed to sort a list in ascending order if the program type checks!

More motivation...

Also not possible to encode specific properties which depend upon values in types.

```
mySort :: Ord a => [a] -> [a]
mySort [] = []
mySort (first_elem::rest) = smaller ++ [first_elem] ++ larger
  where smaller = mySort [x | x <- rest, x <= first_elem]
        larger  = mySort [x | x <- rest, x > first_elem]
```

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We can do this kind of thing with dependent types. We'll look at some basic examples in `Idris`, a programming language like Haskell, but with dependent types.

Dependent types in Idris

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Vectors are a classic example of dependent types!

- Like lists, but...
- They include the length of the list in the type.

```
two_little_piggies : Vect 2 String
two_little_piggies = ["Oinkers", "Snorkins"]

-- This would be a type error, caught at compilation:
three_little_piggies : Vect 3 String
three_little_piggies = two_little_piggies
```


Append and the type level computations

Computations at the type level allow us to make some more complicated, generalized functions.

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- The full `Vect` type constructed from natural number value for length, and a type for the elements.

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```

- The full `Vect` type constructed from natural number value for length, and a type for the elements.
- Two constructors define the type recursively (called an inductive type – we'll see why later).
 - ▶ One for the empty vector.
 - ▶ Single value concatenated to another vector to make a vector with 1 more element. `S` is successor of natural numbers, `+1`.

Types and automation

Idris can help us generate programs based on the types. (All the steps you will see are done automatically by Idris).

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?append_rhs is a hole. It's a stand in for a value we need to provide. Idris can tell us the type of a hole, and potentially fill it in for us. It also tells us types of what's in scope for the hole. This looks like this:

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- + Main.append_rhs [P]
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      m : Nat
      ys : Vect m elem
      n : Nat
      xs : Vect n elem
-----
Main.append_rhs : Vect (n + m) elem
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We can get Idris to do case split on the first argument...

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Which leads to a pattern match on constructors, and two holes:

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```
- + Main.append_rhs_1 [P]
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      elem : Type
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      ys : Vect m elem
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Main.append_rhs_1 : Vect (0 + m) elem

- + Main.append_rhs_2 [P]
'--
      elem : Type
      x : elem
      m : Nat
      ys : Vect m elem
      len : Nat
      xs : Vect len elem
-----
Main.append_rhs_2 : Vect ((S len) + m) elem
```

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Only `ys` satisfies this type. Remember `m` could be any natural.

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So, Idris can tell that it can use `ys` for this case, giving us:

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The second hole is a bit more interesting:

```
- + Main.append_rhs_2 [P]
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    elem : Type
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    m : Nat
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This seems bonkers, so let's look at how Idris did this.

Idris the smart and brave

To get `append (x :: xs) ys = x :: append xs ys` Idris realized a couple things:

```
data Nat : Type where
  0 : Nat -- Zero
  S : Nat -> Nat -- Successor (+1)

(+) : Nat -> Nat -> Nat
(+) 0 m = m
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By definition $(S\ k) + m = S\ (k + m)$, so we have:

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Main.append_rhs_2 : Vect (S (k + m)) elem
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To construct a `Vect (S blah) elem` we need `::!`

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append : Vect n elem -> Vect m elem -> Vect (n + m) elem
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These are the holes:

```
Main.elem_to_concat : elem

Main.rest_of_vect : Vect (k + m) elem
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Idris the smart and brave

The first hole is easy for Idris.

```
- + Main.elem_to_concat [P]
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Second hole involves a bit more work...

Daily dose of recursion

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Idris knows it can call `append` recursively with a `Vect k elem` and a `Vect m elem` to get a `Vect (k + m) elem`.

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Looking at our goal we have such vectors, `ys` and `xs`.

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Idris wrote this function automatically based on a small spec!
It's cool that Idris even gets the order correct because of the types!

Less precise types

Doesn't work well with the less precise list type!

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```
append : List elem -> List elem -> List elem
append [] ys = ?append_rhs1
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```

What do you think will happen when we try the same thing?

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```

What do you think will happen when we try the same thing?

```
append : List elem -> List elem -> List elem
append [] ys = []
append (x :: xs) ys = []
```

Huh... I guess it doesn't realize that append should make a list as long as the inputs combined!

Idris just finds the first possible function. Length isn't encoded in the type to enforce length of output.

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```
index : Fin len -> Vect len elem -> elem
index FZ (x :: xs) = x
index (FS n) (_ :: xs) = myIndex n xs
```

`Fin len` is a type for natural numbers strictly less than `len`

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```

`Fin len` is a type for natural numbers strictly less than `len`

```
data Fin : Nat -> Type where
  FZ : Fin (S k)
  FS : Fin k -> Fin (S k)
```


A tale of cautious indexing

```
cats : Vect 2 String
cats = ["The Panther", "Smoke Smoke"]

-- "The Panther" : String
index 0 cats -- This type checks.

-- (input):1:9:When checking argument prf to function Data.Fin.fromInteger:
--       When using 2 as a literal for a Fin 2
--       2 is not strictly less than 2
index 2 cats -- This is out of bounds, so the program won't even compile!
```

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```

Lots of cool guarantees that we can make with dependent types!

Logic primer

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- Not necessarily true: “ $2 + 2$ is 27”

Logical proofs are used to determine the validity of a proposition.

We could show that “ $2 + 2$ is 27” is false with a logical proof.

Variables

Propositions are often represented by variables, for instance:

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p is a proposition. It could be anything, really...

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And I might have another proposition:

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We can build up more complicated propositions with logical connectives. In this case we might have:

$$p \rightarrow \neg q$$

Which means that if p is true, then q is not true. We'll see more of this shortly.

More logic

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Propositions...

- Are built up from a set of axioms
 - ▶ Just rules which describe your mathematical objects
- Can be combined with logical connectives.

Logic is a sort of metalanguage which describes how you can make judgements about your mathematical objects.

Logical connectives

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- Negation:

- ▶ $\neg p$, meaning “not p ”, “ p is false.”

Quantification

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- ▶ $\forall x \in S, p(x)$, meaning “for every x (in S), $p(x)$ is true.”

- Existential quantification:

- ▶ $\exists x \in S, p(x)$, meaning “there's at least one x (in S), which makes $p(x)$ true.”

Proof and inference rules

You also have some rules for how you can combine these things to form proofs. E.g., Modus ponens

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Or conjunction introduction

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```
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Similarly...

$$p \wedge q$$

Is kind of similar to:

```
hmmm2 : (p, q)
```


Curry-Howard Isomorphism: Propositions as Types!

Conjunction elimination corresponds to destructing a product...

```
-- P /\ Q -> P
fst  : (p, q) -> p
fst  (a, b) = a

-- P /\ Q -> Q
snd  : (p, q) -> q
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Dependent types are needed for quantifiers.

Valuable proofs

If types are propositions, then what are the values?

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Which oddly enough makes a lot of sense as proof!

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Type checker can prevent bogus proofs! Stops us from proving false propositions!

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Can't find a value of type q , since we only have a value of type p !

Proofs in practice

Idris has a type for equality between two things.

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```
equality_good : 2+3 = 5 -- Equality as a proposition in the type!  
equality_good = Refl  
  
-- This fails to type check  
equality_bad : 2+3 = 7  
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Looks a little obtuse... But if we need a something == blah type, we use Refl.

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data (=) : a -> b -> Type where  
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Idris will try to determine if they are equal from the definitions it knows about.

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E.g., $2+3 = 5$, since Idris can evaluate $2+3$ to 5, and see that they are identical.

Congruency

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cong : (f : a -> b) -> x = y -> f x = f y  
cong f prf = ?cong_rhs
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Remember that this just means we need to construct a type using `Ref1`. Idris just needs to show that the left and right hand side are equal.

Congruency

This is our goal:

```
- + Main.cong_rhs [P]
'--
      b : Type
      a : Type
      x : a
      f : a -> b
      y : a
      prf : x = y
-----
Main.cong_rhs : f x = f y
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Looks unimpressive, but it changed our goal:

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Congruency

Refl lets us construct equalities.

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Refl : x = x
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-- So, if we just replace the general "x" above with our "f x" we  
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This concludes the proof of `cong`!

Slightly more complicated proofs

Let's prove the associativity of addition on natural numbers!

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Here's our nice unary representation of natural numbers:

```
data Nat : Type where
  0 : Nat
  S : Nat -> Nat -- Successor, +1

-- 0 = 0
-- S 0 = 1
-- S (S 0) = 2
-- etc...
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-- etc...
```

Addition looks like this:

```
(+) : Nat -> Nat -> Nat
(+) 0 y = y
(+) (S x) y = S (x + y)
```

Associativity

```
plus_assoc : (x, y, z : Nat) -> x + (y + z) = (x + y) + z  
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Case split on x...

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plus_assoc : (x, y, z : Nat) -> x + (y + z) = (x + y) + z  
plus_assoc Z y z = ?plus_assoc_rhs_1  
plus_assoc (S k) y z = ?plus_assoc_rhs_2
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```

Gives us some interesting holes...

```
- + Main.plus_assoc_rhs_1 [P]
' --
      y : Nat
      z : Nat
-----
Main.plus_assoc_rhs_1 : 0 + (y + z) = (0 + y) + z

- + Main.plus_assoc_rhs_2 [P]
' --
      k : Nat
      y : Nat
      z : Nat
--
-----
Main.plus_assoc_rhs_2 : (S k) + (y + z) = ((S k) + y) + z
```


First case...

We have:

```
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Idris will evaluate expressions in an equality type when we use `Ref1`, so this hole is really more like:

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Idris will evaluate expressions in an equality type when we use `Refl`, so this hole is really more like:

```
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  '--
                                     y : Nat
                                     z : Nat
-----
Main.plus_assoc_rhs_1 : y + z = y + z
```

Which is just satisfied with reflexivity...

```
plus_assoc : (x, y, z : Nat) -> x + (y + z) = (x + y) + z
plus_assoc Z y z = Refl
plus_assoc (S k) y z = ?plus_assoc_rhs_2
```

Second case...

Here's our goal:

```
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      k : Nat
      y : Nat
      z : Nat
```

```
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Idris can evaluate this some to simplify as well.

```
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It looks like we need to prove associativity again...

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Recursion is induction!

Idris knows about recursion, so we can actually call `plus_assoc` on `k`, `y`, and `z` to get something with type...

```
attempt : (k y z : Nat) -> k + (y + z) = (k + y) + z
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```

So, now we just need to add `S` to both sides of this... Hmm...

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      y : Nat
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Main.plus_assoc_rhs_2 : S (k + (y + z)) = S ((k + y) + z)
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Enter cong

This is what we use `cong` for, if you remember...

```
cong : (f : a -> b) -> x = y -> f x = f y
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If we give `cong` a function, and an equality type, it will apply the function to both sides!

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This completes the proof!

Enter cong

This is what we use `cong` for, if you remember...

```
cong : (f : a -> b) -> x = y -> f x = f y  
cong f (Refl {x}) = Refl {x = f x}
```

If we give `cong` a function, and an equality type, it will apply the function to both sides!

```
plus_assoc : (x, y, z : Nat) -> x + (y + z) = (x + y) + z  
plus_assoc Z y z = Refl  
plus_assoc (S k) y z = cong S (plus_assoc k y z)
```

This completes the proof!

Neat how applying a theorem is just applying a function. Also neat how recursion and induction are really just the same thing.

Tactics

Can use a different meta-language like Coq's tactics to aid proofs...

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Inductive nat : Type :=
| 0 : nat
| S : nat -> nat.

Fixpoint plus (n m : nat) : nat :=
  match n with
  | 0 => m
  | S n' => S (plus n' m)
  end.

Theorem plus_assoc : forall (x y z : nat), plus x (plus y z) = plus (plus x
  y) z.
Proof.
  intros x y z. induction x as [| k].
  - reflexivity.
  - simpl. (* Simplify with evaluation *)
    rewrite IHk. (* Use induction hypothesis to rewrite terms *)
    reflexivity.
Qed.
```

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Qed.
```

Can allow for very succinct and easy proof development, since meta-language can perform large automated steps!

Conclusion! Questions?

Whirlwind introduction, so you probably have many!

References

-  Edwin Brady. *Type-Driven Development with Idris*. Manning, Mar. 2017. ISBN: 9781617293023. URL: <https://www.manning.com/books/type-driven-development-with-idris>.
-  Benjamin C. Pierce et al. *Software Foundations*. Version 4.2. <http://www.cis.upenn.edu/~bcpierce/sf>. Electronic textbook, 2017.
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These are all good resources! You should look at them!