Low Dimensional Physics

Ammar Husain

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Chapter 1

Dimension 1-Part 1

1.1 Free Boson

Insert Ball and spring on a line model. Altland Simons

$$
\begin{aligned}
H &= \sum \frac{p_i^2}{2m} + \frac{K}{2} \sum (x_i - x_{i+1})^2 \\
-m\ddot{x}_l &= K(2x_l - x_{l-1} - x_{l+1}) \\
x_l &\equiv x_0 \cos kal \\
\omega_k^2 &= (2 - 2\cos ka)\frac{K}{m} = \frac{4K}{m} \sin^2 \frac{ka}{2}
\end{aligned}
$$

For periodic boundary conditions with N particles we then get the quantization of k

$$
\begin{aligned}
\cos kaN &= \cos ka0 = 1 \\
kaN &= 2\pi n \\
k &= \frac{2\pi n}{aN} \\
k &\leftrightarrow k + \frac{2\pi}{a}
\end{aligned}
$$

That is an integer worth of solutions k and then mod out by an index N subgroup. We may take a fundamental domain for this with $n \in [0, N)$ integers, but in truth it is a quotient. Replacing the representative by changing $n \to n + N$ will shift $k \to k + \frac{2\pi}{a}$ which will do nothing to the x_l.

$$
\begin{aligned}
H &= \frac{1}{2m}\sum p_k p_{-k} + \frac{m}{2}\sum \omega_k^2 x_k x_{-k} \\
Q_k &= \frac{1}{\sqrt{N}}\sum_l e^{ikal} x_l \\
\Pi_k &= \frac{1}{\sqrt{N}}\sum_l e^{-ikal} p_l \\
H &= \frac{1}{2m}\sum_k \Pi_k \Pi_{-k} + m^2\omega_k^2 Q_k Q_{-k} \\
&= \sum \hbar\omega_k (a_k^\dagger a_k + \frac{1}{2})
\end{aligned}
$$

If we remove the 1/2 and operate at an inverse temperature β in a noninteracting ∞'ly many body theory we get

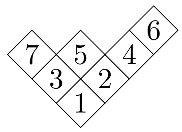
$$
\begin{aligned}
\langle n_k \rangle &= \sum_0^\infty j_k e^{-\beta j_k \hbar\omega_k} / \sum_0^\infty e^{-\beta j_k \hbar\omega_k} \\
&= \sum_0^\infty -\frac{\partial}{\partial(\beta E)} e^{-\beta j_k \hbar\omega_k} / \sum_0^\infty e^{-\beta j_k \hbar\omega_k} \\
&= -\frac{\partial}{\partial \beta E} \sum_0^\infty e^{-\beta\hbar\omega_k j_k} / \sum_0^\infty e^{-\beta j_k \hbar\omega_k} \\
&= -\frac{\partial}{\partial \beta E} \frac{1}{1-e^{-\beta\hbar\omega_k}} / \sum_0^\infty e^{-\beta j_k \hbar\omega_k} \\
&= -\frac{\partial}{\partial \beta E} \frac{1}{1-e^{-\beta\hbar\omega_k}} / \frac{1}{1-e^{-\beta\hbar\omega_k}} \\
&= \frac{e^{\beta\hbar\omega_k}}{(e^{\beta\hbar\omega_k}-1)^2} / \frac{1}{1-e^{-\beta\hbar\omega_k}} \\
&= \frac{1}{e^{\beta\hbar\omega_k}-1}
\end{aligned}
$$

1.2 Free Fermion

We will set up our fermionic Fock space starting with the vacuum vector given by the semi-infinite wedge product

$$
|\,0\rangle = \cdots \wedge e_{-5/2} \wedge e_{-3/2} \wedge e_{-1/2}
$$

The basis vectors are given by Young Tableaux in Russian notation.



The downward sloping lines are the occupied spots and the upward sloping are unoccupied. The empty diagram gives the vacuum vector above.

$$\begin{aligned}
\psi_m \mid \cdots \wedge e_{i_2} \wedge e_{i_1} \rangle &= \mid \cdots \wedge \hat{e}_m \wedge \cdots e_{i_1} \rangle \\
&= 0 \;\; m \notin \{i_n\} \\
\psi_m^\dagger \mid \cdots \wedge e_{i_2} \wedge e_{i_1} \rangle &= \mid \cdots \wedge e_m \wedge \cdots e_{i_1} \rangle \\
&= 0 \;\; m \in \{i_n\}
\end{aligned}$$

This can be regarded as a Hilbert space with a Hamiltonian where each of the are above are eigenvectors with eigenvalues $E_0 \mid \lambda \mid$ For quantum statistical mechanics purposes introduce a parameter q.

$$\begin{aligned}
q &\equiv e^{-\beta E_0} \\
T \to 0 &\implies q \to 0 \\
T \to \infty &\implies q \to 1 \\
Z &= \sum q^{|\lambda|} = (q;q)_\infty = (1-q)(1-q^2)\cdots
\end{aligned}$$

This defines a canonical ensemble.

For an operator diagonal in this basis, we can take it's expectation value in this Gibb's state.

$$\langle f \rangle = (q;q)_\infty \sum_\lambda f(\lambda) q^{|\lambda|}$$

1.2.1 Definition (Schur Measure) *More generally consider a measure using auxiliary variables $x_1 \cdots$ and $y_1 \cdots$. Then define a partition function by*

$$\begin{aligned}
Z &= \sum s_\lambda(x) s_\lambda(y) \\
\mu(\lambda) &= \frac{1}{Z} s_\lambda(x) s_\lambda(y) \\
Z &= \prod (1 - x_i y_j)^{-1} \\
&= \exp(\sum \frac{1}{k} p_k(x) p_k(y))
\end{aligned}$$

by the Cauchy identity. The x and y variables can be permuted without changing the measure so the power sum variables are less stacky.

1.2.2 Corollary *If $p_1(x) = p_1(y) = \sqrt{\xi}$ and the rest are 0 we get the Poissonized Plancherel measure.*

1.3 Luttinger

$$
\begin{aligned}
H_0 &= \sum_k \xi_k c_k^\dagger c_k \\
\xi_k &= \frac{|k|^2}{2m} - \mu \approx v_F(k - k_F)
\end{aligned}
$$

Define $c_{k,R}$ and $c_{k,L}$

$$
\begin{aligned}
c_{k_R} &= \begin{cases} c_k & k \geq 0 \\ 0 & k < 0 \end{cases} \\
c_{k_L} &= \begin{cases} 0 & k \geq 0 \\ c_k & k < 0 \end{cases} \\
c_k &= c_{k,L} + c_{k,R}
\end{aligned}
$$

Plugging this in to the approximate H_0

$$
\begin{aligned}
H_{0,eff} &= v_F \sum_{k>0} k(c_{k,R}^\dagger c_{k,R} - c_{-k,L}^\dagger c_{-k,L}) - k_F v_F (N_R + N_L) \\
H_{0,eff} &= v_F \sum_{k>0} k(c_{k,R}^\dagger c_{k,R} - c_{-k,L}^\dagger c_{-k,L})
\end{aligned}
$$

Suppose there are interactions of the form

$$
H_{int} = \frac{1}{2L} \sum V(q) c_k^\dagger c_{k'}^\dagger c_{k'-q} c_{k+q}
$$

Defining density operators

$$
\begin{aligned}
\rho_R(q) &= \sum_{k>0} c_k^\dagger c_{k+q} \approx_{k>0} c_{k,R}^\dagger c_{k+q,R} \\
\rho_L(q) &= \sum_{k<0} c_k^\dagger c_{k+q} \approx_{k<0} c_{k,L}^\dagger c_{k+q,L} \\
\left[\rho_R(q), \rho_R(-q)\right] &= \sum_{k,k'>0} [c_k^\dagger c_{k+q}, c_{k'}^\dagger c_{k'-q}] \\
&\approx \frac{qL}{2\pi} \\
\left[\rho_L(q), \rho_L(-q)\right] &\approx \frac{-qL}{2\pi}
\end{aligned}
$$

and the rest approximately commute.
In these variables

$$\begin{aligned}
H_{int,1} &= \frac{1}{2L}\sum_q (V(0)-V(2k_F))(\rho_R(q)\rho_L(-q)+\rho_L(q)\rho_R(-q)) \\
H_{int,2} &= \frac{1}{2L}\sum_q V(0)(\rho_R(q)\rho_R(-q)+\rho_L(q)\rho_L(-q)) \\
H_{int} &= \frac{1}{2L}\sum_q (V(0)-V(2k_F))(\rho_L(q)+\rho_R(q))(\rho_L(-q)+\rho_R(-q))
\end{aligned}$$

Changing variables again

$$\begin{aligned}
a_q &= \sqrt{\frac{2\pi}{qL}}\rho_R(q) \\
a_q^\dagger &= \sqrt{\frac{2\pi}{qL}}\rho_R(-q) \\
b_q &= \sqrt{\frac{2\pi}{qL}}\rho_L(-q) \\
b_q^\dagger &= \sqrt{\frac{2\pi}{qL}}\rho_L(q) \\
H_0 &= v_F\sum_{q>0} q(a_q^\dagger a_q + b_q^\dagger b_q) \\
H_{int} &= \frac{1}{2\pi}\sum_{q>0} qV_1(a_q^\dagger a_q + b_q^\dagger b_q + a_q b_q + b_q^\dagger a_q^\dagger) \\
&= \frac{1}{2\pi}\sum qV_1 \begin{pmatrix} a_q^\dagger & b_q \end{pmatrix} \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} a_q \\ b_q^\dagger \end{pmatrix} \\
H &= \sum E_q \gamma_q^\dagger \gamma_q \\
E_q &= qv_F\sqrt{1+\frac{V_1}{\pi v_F}}
\end{aligned}$$

1.4 Charge Density Wave

1.5 Spin Density Wave

1.6 Su-Schrieffer-Heeger

Take polyacetylene where the bond lengths have been distorted by an amount u_n

$$H = -t\sum_{n=1}^{N}(1+u_n)(c_{n\sigma}^\dagger c_{n+1,\sigma} + h.c.) + \sum \frac{k_s}{2}(u_{n+1}-u_n)^2$$

If $u_n = (-1)^n\alpha$ is an alternating patern of single and double bonds

$$H = -t\sum_{n=1}^{N}(1+(-1)^n\alpha)(c^\dagger_{n\sigma}c_{n+1,\sigma}+h.c.)+\frac{Nk_s4\alpha^2}{2}$$

Supposing N is even and using the Bloch operators we get

$$\begin{aligned} H &= \sum_{n=1}^{N=2M}(vc^\dagger_{2n-1\sigma}c_{2n,\sigma}+wc^\dagger_{2n,\sigma}c_{2n-1,\sigma}+h.c.) \\ w &= \mid w \mid e^{i\phi} \end{aligned}$$

$$\begin{aligned} b_{n,\sigma,\sigma'} &\equiv \left(c_{2n-1,\sigma}, c_{2n,\sigma'}\right) \\ b^\dagger_{n,\sigma,\sigma'} &\equiv \left(c^\dagger_{2n-1,\sigma}, c^\dagger_{2n,\sigma'}\right) \\ \psi_{k,\sigma} &\equiv \sum_n e^{ik2an}b_{n,\sigma,\sigma} \\ \psi^\dagger_{k,\sigma} &\equiv \sum_n e^{-ik2an}b^\dagger_{n,\sigma,\sigma} \\ \psi^\dagger_{k,\sigma}\begin{pmatrix} h_{11}(k) & h_{12}(k) \\ h_{21}(k) & h_{22}(k)\end{pmatrix}\psi_{k,\sigma} &= \sum_n e^{ik2an}\psi^\dagger_{k,\sigma}\begin{pmatrix} h_{11}(k)c_{2n-1}+h_{12}(k)c_{2n} \\ h_{21}(k)c_{2n-1}+h_{22}(k)c_{2n}\end{pmatrix} \\ &= \sum_{n,n'} e^{ik2a(n-n')}\left(c^\dagger_{2n'-1}h_{11}(k)c_{2n-1}+c^\dagger_{2n'-1}h_{12}(k)c_{2n}\right. \\ &+ \left. c^\dagger_{2n'}h_{21}(k)c_{2n-1}+c^\dagger_{2n'}h_{22}(k)c_{2n}\right) \\ \sum_k \psi^\dagger_k\begin{pmatrix} h_{11}(k) & h_{12}(k) \\ h_{21}(k) & h_{22}(k)\end{pmatrix}\psi_k &= \sum_k\sum_{n,n'} e^{ik2a(n-n')}\left(c^\dagger_{2n'-1}h_{11}(k)c_{2n-1}+c^\dagger_{2n'-1}h_{12}(k)c_{2n}\right. \\ &+ \left. c^\dagger_{2n'}h_{21}(k)c_{2n-1}+c^\dagger_{2n'}h_{22}(k)c_{2n}\right) \end{aligned}$$

Setting $a = 1$ or alternatively replacing ka by k.

$$\begin{aligned} \begin{pmatrix} h_{11}(k) & h_{12}(k) \\ h_{21}(k) & h_{22}(k)\end{pmatrix} &= h_x(k)\sigma_x + h_y(k)\sigma_y \\ h_x &= Re(v)+\mid w\mid\cos(k+\phi) \\ h_y &= -Im(v)+\mid w\mid\sin(k+\phi) \\ H(k)^2 &= h_x(k)^2+h_y(k)^2+h_xh_y[\sigma_x,\sigma_y]_+ \\ &= (h_x^2+h_y^2)Id_2 = (\mid v\mid^2+\mid w\mid^2+2\mid v\mid\mid w\mid\cos(k+\phi_v+\phi_w))Id_2 \\ E(k) &= \mid v+e^{-ik}w^\dagger\mid = \end{aligned}$$

Returning to our previous case we have $v = 1+\alpha$ and $w = 1-\alpha$. Let us also restore a.

$$\begin{aligned} E(k)^2 &= \mid 1+\alpha+e^{-ika}-e^{-ika}\alpha \mid \\ \Delta_{min} &= E(\frac{\pm\pi}{a}) = 2 \mid \alpha \mid \end{aligned}$$

As k goes around it's S^1, we get a loop in $h_x + ih_y$ space, but it has to stay away from $h=0$ where the energy would go to 0 closing the gap. So all we have to do is count the winding number in $\pi_1(\mathbb{C}^*)$ via $\frac{1}{2\pi i}\int dk \frac{d\log h}{dk}$

Suppose we fill the system halfway for every value of α, then the ground state energy as it varies with α is a double well

1.6.1 Remark how?

$$V(\phi) = -A\phi^2 + B\phi^4$$

1.6.1 Solitons

Interfaces between the two regions mean that there has to have a zero energy state on the defect. Focusing on low energy states given by those with $k = \frac{\pi}{a} + q$ and then expanding in q to give $H = -iv_F\sigma_x\partial_x + m(x)\sigma_y$ which is a Dirac Hamiltonian. Where $m = 2dt$. So for the defect we have changing from +dt to -dt to change which dimerization is favored.

1.6.2 Index

Chiral symmetry $\{\sigma_z, H(k)\} = 0$ so that means that all eigenstates come in pairs with $\pm E$ energies. Except for 0 energy which you count to give a $\mathbb{Z}_2$ invariant.

Chapter 2

Dimension 1 - Spin Chains

2.1 Ising Chain

$$
\begin{aligned}
H &= -J\sum_{i=1}^{L}\sigma_i^z\sigma_{i+1}^z - h\sum\sigma_i^x \\
Z &= \mathrm{Tr}\, e^{-\beta H}
\end{aligned}
$$

In case the transverse field vanishes, we get $\sum_{\{\sigma\}} e^{\beta J\sum\sigma_i^z\sigma_{i+1}^z}$ which is the same result as the classical one-dimensional Ising model.

In case the interaction vanishes, we get $Z = [2cosh(\beta h)]^L$ which is L independent spins as it should be and all is right with the world.

When both nonzero, it maps to the 2D classical Ising model with $N_x = L$, $N_y = N$, $\beta_{cl}J_x = \frac{\beta}{N}J$ and $\beta_{cl}J_y = \gamma$ where $\gamma = \frac{-1}{2}\ln\tanh\frac{\beta h}{N}$.

Now that you have 2d classical Ising without external field, solve it with Onsager.

Corresponding CFT when β_{cl} is just right: $c = 1/2$ Spinless Fermion. The error can be surprisingly small even if you are not even close to the limit.

2.2 Ising+3 Spin XZX

`https://arxiv.org/pdf/1611.10247.pdf`

Rest useless because measuring EE after JW so wrong coproduct

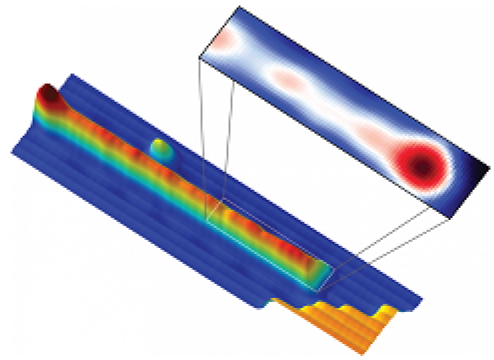


Figure 2.1: Majorana Experiment

$$
\begin{aligned}
H &= \lambda_1 \sum_{i=1}^{L} \sigma_i^z \sigma_{i+1}^z + \lambda_2 \sum \sigma^x \sigma^z \sigma^x + h \sum \sigma_i^x \\
&\to -g \sum (1 - 2c_n^\dagger c_n) - \lambda_1 \sum (c_n^\dagger c_{n+1} + c_n^\dagger c_{n+1}^\dagger + h.c.) \\
&- \lambda_2 \sum (c_{n-1}^\dagger c_{n+1} + c_{n+1} c_{n-1} + h.c.) \\
&= \sum_k \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} H_k \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} \\
H_k &= (h + \lambda_1 \cos k)\sigma^z + (\lambda_1 \sin k - \lambda_2 \sin 2k)\sigma^x \\
H_k(\lambda_2 = 0) &= (h + \lambda_1 \cos k)\sigma^z + (\lambda_1 \sin k)\sigma^x \\
E_k(1, \lambda_1/h \to \lambda_1, \lambda_2/h \to \lambda_2) &= \pm 2\sqrt{1 + \lambda_1^2 + \lambda_2^2 + 2\lambda_1(1 - \lambda_2)\cos k - 2\lambda_2 \cos 2k}
\end{aligned}
$$

h has disappeared. Fix that.

2.3 Majorana Chain

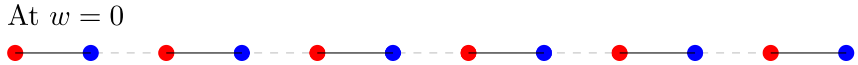
Hamiltonian:

$$
\begin{aligned}
\{c_k, c_l\} &= 2\delta_{kl} \\
H &= \frac{i}{2} v \sum_{j=1}^{N} c_{2j-1} c_{2j} + iw \sum_{j=1}^{N-1} c_{2j} c_{2j+1}
\end{aligned}
$$

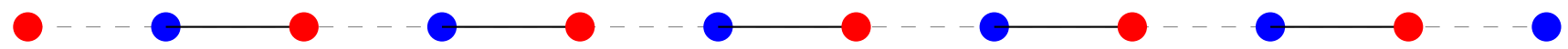
Be careful because if you wrote this in terms of the Dirac fermions rather than the Majorana fermions you would see a number nonconservation term. This means that if you were to actually realize this you would have to put it on the boundary of a superconductor so that the electrons would have somewhere to go.

Ground states:

At $w = 0$



At $v = 0$



and

In the second there are two dangling Majorana's. This leaves a 2-fold almost degenerate system. The two together form a single Dirac fermion. The degeneracy is exact only after passing to the Thermodynamic limit so that the $e^{-\xi/L}$ gap disappears.

2.4 XXX

Hamiltonian:

$$H = \sum \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z + \text{bdry term}$$

Phase Diagram:
Uses $Y(\mathfrak{sl}_2)$

2.4.1 Definition ($Y_\hbar(\mathfrak{sl}_2)$) *Let $i, j = 0, 1$ and $n \in \mathbb{Z}$. Give generators $t_{ij}^{(n)}$.*

$$T(z) \equiv t_{ij}^{(n)} z^n$$

Relations are packaged into the form

$$\begin{aligned} R(u) &= \frac{u}{u+\hbar} I + \frac{\hbar}{u+\hbar} Perm \\ RTT &= TTR \end{aligned}$$

$\hbar$ does not matter too much in this case so sometimes it gets set to 1.

2.4.2 Definition ($Y(\mathfrak{sl}_2)$ in Drinfeld presentation) *This is after applying a Jordan-Wigner transformation, so it is a fermionic perspective rather than a spin perspective.*

2.4.3 Theorem (Bethe Ansatz) *Let $M = 0$ be the zero magnon sector in which all the spins are down. That gives a 1-dimensional subspace of the 2^n total Hilbert space.*

For $M = 1$ magnon sector we make the off shell Bethe vectors by

$$Y(x) = \sum_{i=1}^{n} \left(\prod_{j<i} (x - u_j) \prod_{j>i} (x - u_j + \hbar) e_i \right)$$

where e_i is the vector with a single spin flipped at site i. Then impose the Bethe equations

$$\prod_{j=1}^{n} (x - u_j) = z \prod_{j=1}^{n} (x - u_j + \hbar)$$

2.4.1 Schwinger Boson

Take two harmonic oscillators. The phase space is $\mathbb{R}^4$ thought of as $\mathbb{C}^2$. Fix the total energy so do a symplectic reduction by this $U(1)$ action $z_i \to e^{it} z_i$ This gives $\mathbb{C}^2 // U(1) \simeq \mathbb{CP}^1$.

In fact: Take n harmonic oscillators. The phase space is $\mathbb{R}^{2n}$ thought of as $\mathbb{C}^n$. Fix the total energy so do a symplectic reduction by this $U(1)$ action $z_i \to e^{it} z_i$ This gives $\mathbb{C}^n // U(1) \simeq \mathbb{CP}^n$.

But the case with just two is what we need for Schwinger. We now quantize both sides. We get 2 harmonic oscillators on the boson side and a spin on the other.

This is the phenomenon of quantization commutes with reduction.

2.4.4 Theorem (Borel-Weil-Bott)

The spin j Hilbert space is generated by holomorphic sections of the line bundle $\mathcal{O}(2j)$ which can be expressed as $z^0 \cdots z^{2j}$.

$$
\begin{aligned}
s_x &= -(1-z^2)\partial_z + 2jz \\
s_y &= -i(1+z^2)\partial_z - 2ijz \\
s_z &= -2z\partial_z - 2j
\end{aligned}
$$

2.4.2 Large Spin approximation

$$
\begin{aligned}
s_{xm}s_{xn} + s_{ym}s_{yn} + s_{zm}s_{zn} &= 4j^2 - 4j(z_m - z_n)(\partial_{z_n} - \partial_{z_m}) - 2(z_m - z_n)^2 \partial_{z_m}\partial_{z_n} \\
&\approx 4j^2 - 4j(z_m - z_n)(\partial_{z_n} - \partial_{z_m}) + O(j^0) \\
s_{xm}s_{xn} + s_{ym}s_{yn} - s_{zm}s_{zn} &= -4j^2 - 4j(z_m + z_n)(\partial_{z_n} + \partial_{z_m}) - 2(z_m + z_n)^2 \partial_{z_m}\partial_{z_n} \\
&\approx -4j^2 - 4j(z_m + z_n)(\partial_{z_n} + \partial_{z_m}) + O(j^0)
\end{aligned}
$$

Dropping the order j^0 terms leaves Hamiltonians that are quadratic in the z and ∂_z variables. These are matched with bosonic creation and annihilation so we get a quadratic Hamiltonian that can be diagonalized by Fourier and Bogiliubov.

First the ferromagnetic

$$
\begin{aligned}
J \sum \vec{s} \cdot \vec{s} &\approx 4Jj^2 N - 4jJ \sum (a_m^\dagger - a_{m+1}^\dagger)(a_{m+1} - a_m) \\
&= 4Jj^2 N - 4jJ \sum_k \begin{pmatrix} ? & ? \end{pmatrix} \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} ? \\ ? \end{pmatrix}
\end{aligned}
$$

Now the antiferromagnet

$$
\begin{aligned}
J \sum \vec{s} \cdot \vec{s} - 2s_m^z s_{m+1}^z &\approx -4Jj^2 N - 4jJ \sum (a_m^\dagger + a_{m+1}^\dagger)(a_{m+1} + a_m) \\
&= -4Jj^2 N - 4jJ \sum_k \begin{pmatrix} ? & ? \end{pmatrix} \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} ? \\ ? \end{pmatrix}
\end{aligned}
$$

For the gap between this says that

$$\begin{aligned} \lim_{j\to\infty} \Delta &\propto k_{min} \propto 1/N \\ \lim_{N\to\infty} \lim_{j\to\infty} \Delta &= 0 \end{aligned}$$

This doesn't contradict the Haldane conjecture because of the order of limits. It only says $\lim_{N\to\infty} \Delta$ for fixed integer j is nonzero.

2.4.5 Theorem (Lieb-Schultz-Mattis) *A one dimensional periodic chain of length L with half-integer spin per unit cell has a gap $\leq \frac{C}{L}$ with C some system dependent constant. This contrasts with the Haldane case when a gap develops*

2.5 XXZ Spin 1/2

Hamiltonian:

$$H = \sum \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z + \text{bdry term}$$

Phase Diagram:

Figure 2.2: Phase Diagram as Δ varies

Uses $U_q(\widehat{\mathfrak{sl}_2})$
The R matrix for $V_{1/2}(z_1) \otimes V_{1/2}(z_2)$ where $\frac{z_2}{z_1} = z$ is:
(decide on multiplicative additive etc)

$$R(z) = \begin{pmatrix} ? & 0 & 0 & 0 \\ 0 & ? & ? & 0 \\ 0 & ? & ? & 0 \\ 0 & 0 & 0 & ? \end{pmatrix}$$

In the case of quasi-periodic boundary conditions where sites 1 and N are identified by a matrix

$$H = \begin{pmatrix} & 0 \\ 0 & ? \end{pmatrix}$$

Figure 2.3: Phase Diagram as Δ and h vary

2.6 XXZ with quantum space replaced

2.6.1 $V_1(z_1)$

The R matrix for $V_1(z_1) \otimes V_{1/2}(z_2)$ where $\frac{z_2}{z_1} = z$ is:

2.6.2 Borel rep only

The R matrix makes sense if we put a $U_q(\mathfrak{b})$ representation instead. This is thanks to the factorization property of the universal $\mathcal{R}$. This means that we can use even a prefundamental representation rather than evaluation representations of finite dimensional reps.

2.6.1 Definition (Pre-fundamental) *Hernandez-Jimbo category $\mathcal{O}$*

2.6.2 Theorem (Frenkel-Hernandez)

2.7 XXZ $1/2 \to 1$ for sites

This is for the case of algebraic Bethe ansatz with all quantum and auxiliary spaces replaced by spin 1 rather than 2. For the derivation see `http://arxiv.org/pdf/solv-int/9809001v1.pdf`

Hamiltonian:

$$\begin{aligned}
H &= \sum h_n + H_{bdry} \\
h_n &= J_n \cdot J_{n+1} - (J_n \cdot J_{n+1})^2 \\
&+ \frac{(q-q^{-1})^2}{2}\left(J_n^z J_{n+1}^z + (J_n^z)^2 + (J_{n+1}^z)^2 - (J_n^z J_{n+1}^z)^2\right) \\
&- (q^{1/2}-q^{-1/2})^2\left((J_n^x J_{n+1}^x + J_n^y J_{n+1}^y) J_n^z J_{n+1}^z + J_n^z J_{n+1}^z (J_n^x J_{n+1}^x + J_n^y J_{n+1}^y)\right)
\end{aligned}$$

In particular we can set $q \to 1$ but note that unlike in the $1/2$ case this does not give the Heisenberg point but instead has that extra quartic term still remaining.

$$h_n \to J_n \cdot J_{n+1} - (J_n \cdot J_{n+1})^2$$

Compare to the AKLT below with a -1 instead of $+1/3$.

Phase Diagram:

Uses $U_q(\widehat{\mathfrak{sl}_2})$

$$\Delta E_0 =$$

2.8 XYZ Spin $1/2$

Hamiltonian:

$$H = \sum X\sigma_i^x\sigma_{i+1}^x + Y\sigma_i^y\sigma_{i+1}^y + Z\sigma_i^z\sigma_{i+1}^z + \text{bdry term}$$

Phase Diagram:

Uses Elliptic Quantum Group

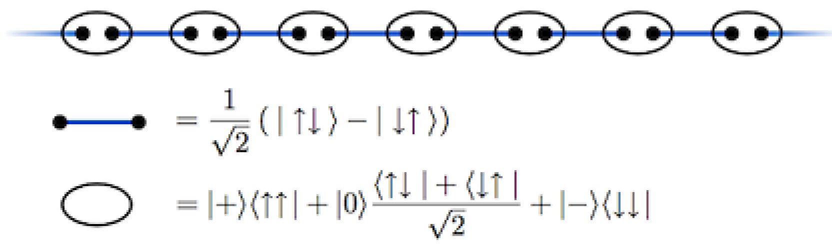


Figure 2.4: AKLT Ground State

2.9 AKLT

Hamiltonian:

$$H = J\sum S_j S_{j+1} + \frac{J}{3}\sum (S_j S_{j+1})^2$$

Ground States:

This was solved by writing the chain in terms of spin 1/2's first and then writing the projector that forbids every pair of spin 1's being in a spin 2 state. If you write the projector to spin 2 you get the AKLT Hamiltonian. Because it comes with a positive coefficient, the low energy states will be in the kernel of these projections. Thankfully you can satisfy them all at once and that is the above ground state. Thinking of it this way only gives the right behavior for energies below the energy associated with the projector because in reality that constraint is strict not only energetic.

At the edges, there are unpaired spin 1/2's. They act independently even though the original chain had only spin 1's. For long chains this gives a four fold degeneracy. It is split into triplet and singlet with gap exponentially small in the length.

2.9.1 Haldane

http://online.kitp.ucsb.edu/online/compqcm10/oshikawa/pdf/Oshikawa_CompQCM.pdf

2.9.1 Conjecture (Haldane Conjecture) *Heisenberg antiferromagnet with half integer is gapless and power law spin correlation decay. But with integer spins, it is gapped and exponential decay for correlations.*

2.10 Fredkin

http://arxiv.org/pdf/1605.03842.pdf

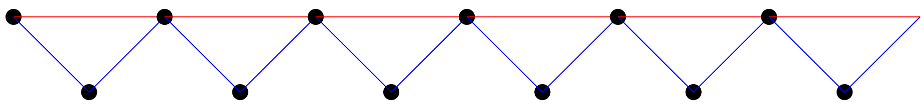
$$\begin{aligned}
H &= H_{left} + H_{right} + \sum_{j=0}^{L-2} Id \otimes H_{j,j+1,j+2} \otimes Id \\
H_{j,j+1,j+2} &= |\uparrow_j\rangle\langle\uparrow_j| \otimes |\, S_{j+1,j+2}\rangle\langle S_{j+1,j+2}\,| + |\, S_{j,j+1}\rangle\langle S_{j,j+1}\,| \otimes |\downarrow_{j+2}\rangle\langle\downarrow_{j+2}| \\
H_{right} &= |\uparrow_L\rangle\langle\uparrow_L| \\
H_{left} &= |\downarrow_0\rangle\langle\downarrow_0|
\end{aligned}$$

The ground state is a equal superpostion over all Dyck paths with up being up spin and down being down spin. In fact this is a 0 eigenvalue eigenstate for every term in the Hamiltonian all of which are positive definite. So it is definitely a ground state and in fact the unique one.

2.10.1 Motzkin

`https://arxiv.org/pdf/1611.03147.pdf`

2.11 Majumdar-Ghosh

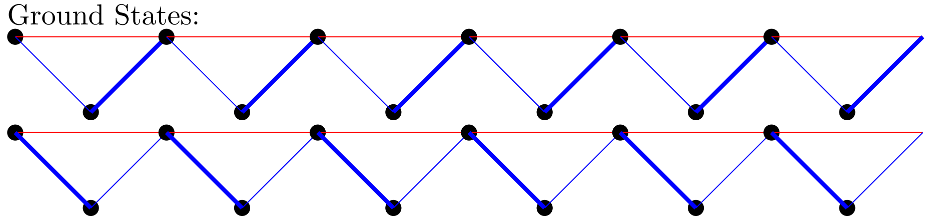


Hamiltonian:

$$H = J\sum S_j S_{j+1} + \frac{J}{2}\sum S_j S_{j+2}$$

The first term comes from the blue bonds in the picture. The second comes from the red bonds. Notice that the red bonds are $\sqrt{2}$ times as long so that we will get a factor of 2 smaller constant if the interactions in the tight binding model fall of with a inverse square law.

Ground States:



2.12 Zig-Zag Ladder

2.13 2-orbital 2-spin

Hund term.

2.14 t-J

$U_q\mathfrak{gl}(1 \mid 1)$

2.15 Perk-Schultz

XXZ and Super

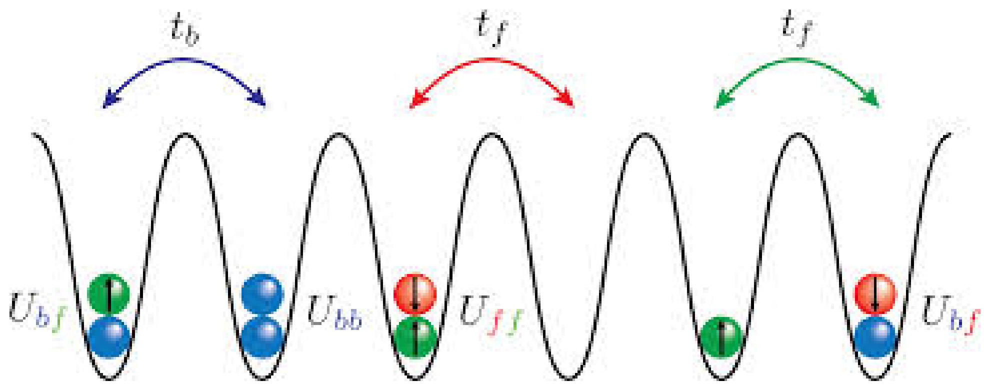


Figure 2.5: Hubbard Model

2.15.1 $GL(2 \mid 1)$

`http://arxiv.org/pdf/1606.03573.pdf`
`http://arxiv.org/pdf/cond-mat/9802204v1.pdf`
`http://arxiv.org/pdf/cond-mat/0309138v1.pdf`

2.16 Hubbard Model

Hamiltonian:

$$H = -\sum_{i,\sigma} t_{i,i+1}(c^\dagger_{i,\sigma}c_{i+1,\sigma} + h.c.) + U\sum_i n_{i,\uparrow}n_{i,\downarrow} + \sum_i \epsilon_i n_i$$

Now assume that the orbitals for all i have the same energy in isolation. So know if we give a filling fraction the last term is a constant which will be dropped. Also this symmetry between all orbitals will turn $t_{i,i+1}$ into a single constant t.

$$H = -t\sum_{i,\sigma}(c^\dagger_{i,\sigma}c_{i+1,\sigma} + h.c.) + U\sum_i n_{i,\uparrow}n_{i,\downarrow}$$

Phase Diagram:

First consider the limit of large $\frac{U}{t}$. In this case the second term dominates and the problem factors into single site problems. Because they don't hop, it gives an insulator.

$$\begin{aligned} Z &= \sum e^{-\beta H + \beta\mu N)} = e^{0+0} + e^{0+\beta\mu} + e^{0+\beta\mu} + e^{-\beta U + 2\beta\mu} \\ \langle n \rangle &= \frac{2(e^{\beta\mu} + e^{-\beta U + 2\beta\mu})}{1 + 2e^{\beta\mu} + e^{-\beta U + 2\beta\mu}} \end{aligned}$$

This looks like a step function which has $\langle n \rangle = 1$ before $\mu = U$ and then when $\mu > U$ it becomes 2 because the chemical potential is high enough to overcome the energetic barrier.

Now consider $U \to 0$. It is a free particle which diagonalizes by passing to momenta. This gives a metal with usual band theory.

At the intermediate we have the Mott transition.

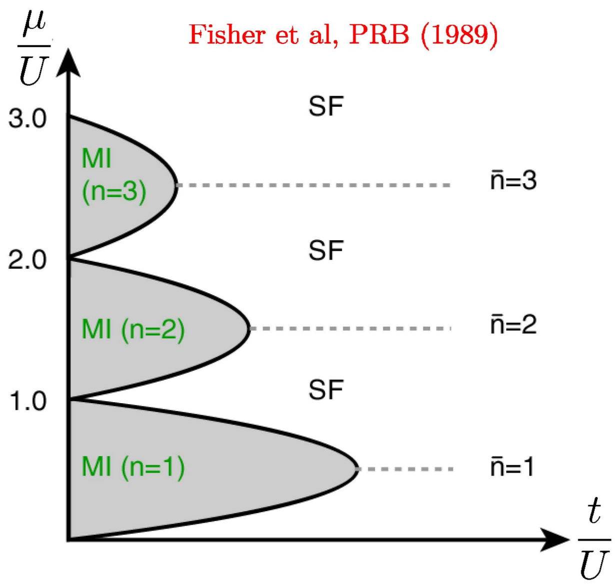


Figure 2.6: Hubbard Model Phase Diagram

Uses $Yangian(sl(2 \mid 2))$ with a central extension.

2.16.1 Definition (Perk-Schultz)

$$R = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}$$

2.17 Fannes-Nachtergale-Werner

`http://projecteuclid.org/euclid.cmp/1104249404`

2.17.1 Theorem (Nachtergale 92) *Construction of Parent Hamiltonians*

Chapter 3

Dimension 1 - Part 2

3.1 Lieb-Liniger,Tonks-Girardeau

3.1.1 Non Linear Schrodinger

$$\mathcal{H} = \frac{\hbar^2}{2m}\int_0^L dx\partial_x\psi^\dagger\partial_x\psi + \frac{g}{2}\int_0^L dx\psi^\dagger\psi^\dagger\psi\psi$$

Define a Fock space with vacuum $\mid 0\rangle$ annihilated by all $\psi(x)$ and normalized. The equation of motion from Heisenberg's equation gives

$$\begin{aligned} i\partial_t\psi(x) &= [\mathcal{H}, \psi(x)] \\ i\partial_t\psi(x) &= -\partial_x^2\psi(x) + 2c\psi^\dagger(x)\psi(x)\psi(x) \end{aligned}$$

If ψ was a classical field instead of a quantum field operator, this would be the non-linear Schrodinger equation. Both

$$\begin{aligned} N &= \int_0^L \psi^\dagger\psi \\ P &= \frac{i}{2}\int_0^L \left[\partial_x, \psi^\dagger\right]\psi \end{aligned}$$

We can restrict them to be eigenvalues.

3.1.2 Lieb-Liniger

In the N particle sector, this reduces to a quantum mechanical problem of

$$H = -\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + 2c\sum_{i<j}^N \delta(x_i - x_j)$$

If we let L be the period of the circle, we find Bethe equations

$$L\lambda_j + \sum 2\text{atan}\frac{\lambda_i - \lambda_j}{c} = 2\pi I_j$$

or in a slightly different parameterization

$$e^{i\lambda_j L} = \prod \frac{\lambda_j - \lambda_l + i\kappa}{\lambda_j - \lambda_l - i\kappa}$$

The $I_1 < I_2 \cdots$ are integers when N is odd and half-integers when N is even. In the ground state, they are integer spaced and $I_1 = -I_N$. That is they arrange in a string like $-N/2 \cdots N/2$. Then solve with these I fixed gives the ground state. Next we switch $I_N \to I_N + n$ or $I_1 \to I_1 - n$. This gives one sort of excitation. Another kind is to choose an n in $- < n \leq N/2$ and replace $I_i \to I_i + 1$ for $i \geq n$. Other integers can be set as well giving yet more kinds of excitations.

In the limit $N, L \to \infty$ with $\frac{N}{L} = \rho$ fixed we get a convergence for $\epsilon_0 = \frac{E_0(N,L)}{N}$. It is given by defining the following:

$$\begin{aligned}
e_B &\equiv \frac{\epsilon_0}{\rho^2} \\
\gamma &\equiv \frac{c}{\rho} \\
L_\kappa f &\equiv \frac{\kappa}{\pi}\int_{-1}^{1} \frac{1}{(x-y)^2+\kappa^2} f(y)dy \\
f_B(x,\kappa) - L_\kappa f_B(x,\kappa) &= 1
\end{aligned}$$

and then fixing which f_B solution and solving for e_B by

$$\begin{aligned}
\frac{\kappa}{\gamma} &= \int_{-1}^{1} f_B(x,\kappa)dx \\
e_B(\gamma) &= \frac{1}{2\pi}\frac{\gamma^3}{\kappa^3}\int_{-1}^{1} x^2 f_B(x,\kappa)dx
\end{aligned}$$

In particular, we can take the weak repulsion limit by $\gamma \to 0^+$ and get

$$e_B(\gamma) = \gamma - \frac{4}{3\pi}\gamma^{3/2} + (\frac{1}{6} - \frac{1}{\pi^2})\gamma^2 + o(\gamma^2)$$

3.1.3 Tonks-Girardeau

Now take the opposite limit, strong repulsion $c \to \infty$

In the $\gamma \to \infty$ limit, the ground state energy per particle goes to $\frac{\pi^2}{3}$.

3.1.4 Super Tonks-Girardeau

3.2 Toda Lattices

3.2.1 Historical Aside

Fermi-Pasta-Ulam is an approximation to the Toda lattice found in numerical experiments with some of the first computers. In the summer of 1953 they saw the lack of ergodicity that is characteristic of having some integrability around.

This was a decade before the integrals of motion for KdV was solved and two decades before Toda. It is a little surprising given that all the discoverers of the equation and Backlund were all 19th century.

3.2.2 Generalized Toda

$$\begin{aligned} V &= (e^{q_i} - e^{q_{i+1}}) \\ H &= \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1}(e^{q_i} - e^{q_{i+1}}) + e^{q_n - q_1} \end{aligned}$$

If we didn't have that last term, the particles would scatter in the fundamental chamber, $q_1 < q_2 < \cdots$ and they would just move further deep into this chamber because the potential is repulsive. The last term is keeping that from happening by penalizing a large distance between the first and last particle.

So how is this Lie algebraic. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ and let σ be the Cartan involution which swaps the E's and F's. What is important here is **real split semisimple**.

Do the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$$

In this case $\mathfrak{k}$ are skew-symmetric matrices, $\mathfrak{a}$ are the diagonals and $\mathfrak{n}$ are the strictly upper triangular ones. Now give the algebra the inner product to identify with $\mathfrak{g}^*$ This will be $\lambda *$Killing. Note that $\mathfrak{k}$ and $\mathfrak{b}$ are not isotropic. In fact the orthogonal to these under the Killing form are $\mathfrak{p}$ and $\mathfrak{n}$ respectively.

3.2.1 Definition (Dressing Transformation)

3.2.2 Lemma (QR Algorithm) *The QR algorithm used to numerically diagonalize A_0 by the following iterative procedure which often converges*

$$\begin{aligned} A_k &= Q_k R_k \\ A_{k+1} &\equiv R_k Q_k = Q_k^T A_k Q_k = Q_{k+1} R_{k+1} \\ A_k &\to A_\infty \end{aligned}$$

which is upper triangular and eigenvalues can be read off. This is to be compared with the action of dressing transformations and the conceptual explanation for stability of the algorithm.

3.2.3 Reduction to get Open Toda

$$\begin{aligned} G_r &= K \times B \subset G \times G \\ T^*G \times O_T // G_r &\simeq O_T \end{aligned}$$

as a symplectic manifold, so instead of treating the orbit as the phase space, we can quantize before reduction.

$$\begin{aligned} H &= (p,p) + 2\sum_\alpha \exp(2\alpha(q)) \\ \{p_i, q_j\} &= 1/2\delta_{ij} \\ \frac{dq}{dt} &= \{H, q\} = p \\ \frac{dp}{dt} &= \{H, p\} = -2\sum_\alpha H_\alpha \exp(2\alpha(q)) \\ \frac{d^2q}{dt^2} &= -2\sum_\alpha H_\alpha \exp(2\alpha(q)) \end{aligned}$$

In the type A_N of $SL(N+1)$ then this reduces to the exponential of difference in coordinates from before. But if we want to use the obvious coordinates on the Cartan as an $N+1$ tuple and then the constraint $\sum q = 0$ we have to change to use the Dirac bracket instead.

$$\begin{aligned} \{p_i, q_j\} &= \frac{1}{2}(\delta_{ij} - \frac{1}{N+1}) \\ H &= (p,p) + 2\sum_\alpha \exp(2(q_i - q_{i+1})) \\ \dot{q}_i &= p_i \\ \dot{p}_1 &= -2e^{2(q_1 - q_2)} \\ \dot{p}_i &= -2e^{2(q_i - q_{i+1})} + 2e^{2(q_{i-1} - q_i)} \\ \dot{p}_{N+1} &= 2e^{2(q_N - q_{N+1})} \end{aligned}$$

This can be written in Lax form

$$\begin{aligned} L &= p + \sum_\alpha n_\alpha e^{\alpha(q)}(E_\alpha + E_{-\alpha}) \\ M &= -\sum n_\alpha e^{\alpha(q)}(E_\alpha - E_{-\alpha}) \end{aligned}$$

Proof Compute $\dot{L}$ and $[M, L]$. □

3.2.3 Definition ($\mathcal{T}(G)$) *Let G be a reductive group with Borel B, Unipotent U and Cartan T. Let ψ be a regular character for $U(\backslash) \to \mathbb{C}$. $\mathcal{D}(G)$ will be the differential operators on G. $U\backslash \otimes U(\backslash)$ maps to $\mathcal{D}(G)$ by taking left and right invariant vector fields due to $\backslash$ and extending to higher order operators. We then do quantum Hamiltonian reduction by $U \times U$ and $\mu = \psi, -\psi$. Call the result $\mathcal{T}(G)$.*

3.2.4 Theorem (Kazhdan-Kostant) *$ZU(\mathfrak{g})$ are the bi-invariant differential operators and they map into this reduction. In fact $U \times U$ acts on the big Bruhat cell freely so doing the quantum Hamiltonian reduction for functions there gives another algebra $\mathcal{D}(C_{w_0})//(U \times U, (\psi, -\psi)) \simeq \mathcal{D}(T)$ that has a map from $\mathcal{T}(G)$. Altogether this gives $ZU(\mathfrak{g}) \to \mathcal{T}(G) \to \mathcal{D}(T)$. The image of this gives the commuting Toda Hamiltonians.*

Proof Semiclassically this is saying we have $T^*T \to T^*G//(U \times U) \to \mathfrak{h}^*/W$ because functions on cotangent turn to D-modules and $SpecZU(\mathfrak{g}) = \mathfrak{h}^*/W$ by polynomials in the fundamental invariants. $\square$

3.2.5 Theorem (Zastava) *The middle step in the classical analog $T^*G//(U \times U)$ is isomorphic to the open Zastava space of degree n based maps from $\mathbb{P}^1$ to itself preserving ∞. This can also be identified with the moduli of $SU(2)$ monopoles of charge n.*

3.2.4 Toda Spectral Curve

3.3 Sine-Gordon and KdV Hierarchy

3.3.1 Sine/Sinh-Gordon

$$\begin{aligned}
\phi_{tt} - \phi_{xx} + \sin\phi &= 0 \\
\mathcal{L} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \cos\phi \\
\frac{d\mathcal{L}}{d\phi} &= -\sin\phi \\
\frac{d\mathcal{L}}{d\partial_\mu\phi} &= \partial^\mu\phi \\
\partial_\mu\partial^\mu\phi + \sin\phi &= 0
\end{aligned}$$

We will see more about it in 2D CFT when looking at affine Toda field theory

3.3.2 KdV

3.3.3 Frenkel-Kontorova

$$\begin{aligned}
H &= \frac{m}{2}\sum \dot{x}_i^2 + \frac{\epsilon_s}{2}\sum(1 - \cos\frac{2\pi x_i}{a}) + \frac{g}{2}\sum(x_{i+1} - x_i - a_0)^2 \\
y_i &\equiv \frac{2\pi}{a}x_i \\
T &= \frac{2\pi}{a}\sqrt{\frac{\epsilon_s}{2m}}t
\end{aligned}$$

In a new set of variables turn this into. This needs to be fixed in order to know how limits like sending $\epsilon_s \to 0$ to go to the chain for a free boson (phonon).

$$\begin{aligned} \frac{d^2 x_n}{dt^2} + \sin x_n - g * (x_{n+1} + x_{n-1} - 2x_n) &= 0 \\ u_n &= x_n - na \end{aligned}$$

If we look at the continuum we may pass to the sine-Gordon model from earlier

3.4 Calegero-Moser Type

3.4.1 Representation Theoretic Aspects

Let $M = T^* End(n, \mathbb{C})$ and $G = PGL(n, \mathbb{C})$. The trace form identifies $\mathfrak{g} = \mathfrak{sl}_n$ with it's dual. Define a moment map by $\mu(X, P) = [X, P]$

3.4.1 Theorem (Gan,Ginzburg) *$Comm(n)/G$ is a reduced scheme isomorphic to $\mathbb{C}^{2n}/S_n$. The Poisson structure is the pushforward of the one from the holomorphic symplectic structure on $\mathbb{C}^{2n}$*

3.4.2 Definition (Hilbert Scheme) *$Hilb^n(\mathbb{C}^2)$ is ideals of length n inside $\mathbb{C}[x, y]$. Geometrically they are n points in $\mathbb{C}^2$ but with collisions when two overlap taken into account, so a resolution of $(\mathbb{C}^2)^n/S_n$. There is the $\epsilon_{1,2}$ action which rotates the base with $\mathbb{C}^*$ actions so we can ask for fixed points under this. These are the torus fixed ideals which can be visualized as Young Tableaux of size n by reading the monomials as $x^n y^m$ if the tableaux contains the point (n,m) when drawn in French notation.*

Let $\mathcal{O}$ be the coadjoint orbit through $diag(-1/n, -1/n, \cdots \frac{n-1}{n})$ for $PGL(n, \mathbb{C})$. Namely it is a special complex coadjoint orbit so hyperKahler. This is the set of traceless matrices such that $T + 1/n$ has rank 1. If this was only the $PSU(n)$ action we would have our density matrices for pure states (as a space not with Fisher). This is not quite the usual definition where we do $[X, Y] - Id$ has rank 1, but they are canonically isomorphic by sending $(X, Y) \to (\tau X, Y)$ from the original definition to the current definition where here $\tau = -1/n$.

3.4.3 Theorem *The action of $PGL(n, \mathbb{C})$ on $\mu^{-1}(\mathcal{O})$ is free and the Hamiltonian reduction μ^{-1}/G is a smooth symplectic variety of dimension 2n. It is connected as well.*

3.4.4 Theorem (Nekrasov-Schwarz) *Moduli space of locally free rank 1 sheaves on a noncommutative projective plane where the graded algebra we are taking (noncommutative) Proj for has x,y,z in degree 1 and $[x, y] = \tau z^2$. They do this by doing a noncommutative version of the twistor transform to pass between instantons on $\mathbb{R}^4$ and sheaves on $\mathbb{P}^2$. The Kronheimer result moves between singular G-instantons on $\mathbb{R}^4$ with prescribed boundary conditions at ∞ and 0 with $\mathfrak{g}^*_{\mathbb{C}}$ coadjoint orbits. How to move from here to Calagero Moser space???*

There are two moduli spaces of coherent sheaves which give this as a Zarisiki open dense subvariety. They provide compactifications. One is the Gieseker compactification and the other is the Uhlenbeck compactificaiton.

In the following $\mathfrak{g}$ is reductive, $\mathfrak{h}$ is a Cartan and W is Weyl.

3.4.5 Theorem (Central Characters) *The center $Z(U(\mathfrak{g}))$ acts on any V_λ for λ highest weight as some scalar. This is by commuting it past until it acts on the v_λ. This gives $xv = \chi_\lambda(x)v$. In addition these characters are equal if and only if $\lambda+\delta$ and $\mu+\delta$ are on the same Weyl orbit. This means that specifying χ_λ is the same data as specifying a Weyl invariant function on the weight space.*

3.4.6 Theorem (Classical Harish-Chandra) *We have the restriction ξ $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \simeq \mathbb{C}[\mathfrak{h}]^W$.*

3.4.7 Theorem (Levasseur-Stafford) *By pulling back on this isomorphism we get an action of $D(\mathfrak{g})^{\mathfrak{g}}$ on $\mathfrak{C}[\mathfrak{h}]^W$. These act by W invariant differential operators but they have some poles. Together this gives a $HC' : \ D(\mathfrak{g})^{\mathfrak{g}} \to D(\mathfrak{h}_{reg})^W$ homomorphism. If you conjugate by $\delta(x) = \prod_{\alpha>0}(\alpha, x)$, you get something that extends without poles. $HC = \delta \circ HC' \circ \delta^{-1}$. In fact it gives a map $D(\mathfrak{g})^{\mathfrak{g}} \to D(\mathfrak{h})^W$ which factors through to give an isomorphism $D(\mathfrak{g})//\mathfrak{g} \simeq D(\mathfrak{h})^W$.*

3.4.8 Example *In particular the Laplacian is an operator we can send through to get something in $D(\mathfrak{h})^W$. If x_i are an orthonormal basis for $\mathfrak{h}$ and e_α and f_α are the paired root elements.*

$$D = \sum_{i=1}^{r} \partial_{x_i}^2 + 2\sum_{\alpha>0} \partial_{f_\alpha}\partial_{e_\alpha}$$

Doing the computation gives that it ends up being the Laplacian on $\mathfrak{h}$, $\Delta_{\mathfrak{h}}$

3.4.9 Proposition *For differential operators with constant coefficients this is the same map as obtained by $(Sym\mathfrak{g})^{\mathfrak{g}} \simeq (Sym\mathfrak{h})^W$*

3.4.10 Theorem (Kirillov Character) *Let $\mathcal{O}$ be a regular coadjoint orbit in the $\mathfrak{g}^*_{cpt}$ compact form and $d\mu(b)$ the measure from KKS symplectic as well as letting $x \in \mathfrak{h}$. Let λ be the point in the dominant chamber of $\mathfrak{h}^*_{cpt}$ it passes through.*

$$\begin{aligned}
\psi_{\mathcal{O}}(x) &\equiv \int_{\mathcal{O}} e^{i(b,x)} d\mu(b) \\
\psi_{\mathcal{O}}(x) &= \delta^{-1}(x) \sum_W (-1)^{\ell(w)} e^{i(w\lambda, x)} \\
Tr_{L_\lambda}(e^x) &= \frac{\delta(x)}{\delta_{tr}(x)} \int_{\mathcal{O}_{\lambda+\rho}} e^{i(b,x)} d\mu(b) \\
\delta_{tr}(x) &\equiv \prod_{\alpha>0} (e^{\alpha(x)/2} - e^{-\alpha(x)/2})
\end{aligned}$$

3.4.11 Theorem (Quantum Hamiltonian Reduction Ideal) *Before we could do Hamiltonian reduction with μ^{-1} of an orbit or any other closed G-invariant subset. This means a Poisson ideal in $Sym\mathfrak{g}$ as far as the polynomial functions and varieties go. So to do the quantum version replace J with $J(I)$ which is now the set $A\mu(I)$ instead of $I = U(\mathfrak{g})^{>0}$ the augmentation ideal which corresponds to the $\mu^{-1}(0)$ case.*

3.4.12 Example $\mathfrak{g} = \mathfrak{gl}_n$, *$A$ is $D(\mathfrak{g})$ which is quantizing $\mathcal{O}(T^*\mathfrak{g})$. Let k be a complex number, W_k be the representation of $\mathfrak{sl}_n$ on functions of the form $(x_1 \cdots x_n)^k f(x_1, \cdots x_n)$ where f is Laurent* *of degree 0. It is a $\mathfrak{gl}_n$ module too by pullback. If I_k is the annihilator of this module in $U(\mathfrak{g})$ we can reduce along it instead. The result is B_k called the spherical rational Cherednik algebra. It gives a family version HC_k parameterized by k that take $D(\mathfrak{g})^{\mathfrak{g}} \to B_k$. This is a flat family of homomorphisms of algebras. If $k = 0$ then it is just $D(\mathfrak{h})^W$ and the ordinary HC map.*

3.4.13 Theorem *B_k acts naturally on $\mathbb{C}[\mathfrak{h}_{reg}]$*

3.4.2 Rational Calagero-Moser

Hamiltonian:

$$H_{cl} = \sum_{i=1}^{N} \frac{p_i^2}{1} - \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}$$

$$H_{qtm} = -\sum_{i=1}^{N} \frac{\partial_i^2}{1} - \sum_{i \neq j} \frac{k(k+1)}{(x_i - x_j)^2}$$

3.4.14 Theorem *Let H_i be the commuting homogenous differential operators with constant coefficients quantizing $Tr(P^i)$ in $A = D(\mathfrak{g})$. They turn into new operators $HC_k(H_i)$ in B_k which subsequently become operators L_i on $\mathbb{C}[\mathfrak{h}_{reg}]$. In particular L_2 is H_{qtm} (slight reparameterization of signs because h vs $-ih$)*

$$\begin{aligned}
H_{qtm} &= \sum_{i=1}^{N} \frac{\partial_i^2}{1} - \sum_{i \neq j} \frac{k(k+1)}{(x_i - x_j)^2} \\
-H_{qtm} &= -\sum_{i=1}^{N} \frac{\partial_i^2}{1} - \sum_{i \neq j} \frac{-k(k+1)}{(x_i - x_j)^2} \\
&= -\sum_{i=1}^{N} \frac{\partial_i^2}{1} - \sum_{i \neq j} \frac{k_2(k_2+1)}{(x_i - x_j)^2} \\
-k(k+1) &= k_2(k_2+1) \\
k_2 &= -1/2(1 \pm \sqrt{1 - 4k(k+1)})
\end{aligned}$$

3.4.3 Trigonometric Calagero-Moser

http://www-math.mit.edu/~etingof/zlecnew.pdf

$$\begin{aligned}
H &= \sum_{i=1}^{N} \frac{(x_i p_i)^2}{1} - \sum_{i \neq j} \frac{x_i x_j}{(x_i - x_j)^2} \\
\tilde{x}_i &= \log x_i \\
\tilde{p}_i &= x_i p_i \\
H &= \sum_{i=1}^{N} \frac{\tilde{p}_i^2}{1} - \sum_{i \neq j} \frac{4}{\sinh^2((\tilde{x}_i - \tilde{x}_j)/2)}
\end{aligned}$$

3.4.15 Theorem *Do the same reduction but with T^*G instead of $T^*\mathfrak{g}$. Again there is a slight mismatch of signs*

$$H_{qtm} = \sum_{i=1}^{N} \frac{\partial_i^2}{1} - \sum_{i \neq j} \frac{k(k+1)}{\sin^2(x_i - x_j)}$$

`http://arxiv.org/pdf/1608.00599.pdf`

$$\begin{aligned}
H^{CS} &= -\sum \frac{\partial}{\partial q_i} + \sum \frac{\beta(\beta-1)}{\sin^2 q_i - q_j} \\
\psi_0 &= |\prod_{i<j} \sin q_i - q_j \,|^{\beta} \\
\psi_0^{-1} H^{CS} \psi_0 &= -\sum (x_i \frac{\partial}{\partial x_i})^2 + \beta \sum \frac{x_i + x_j}{x_i - x_j}(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j}) - 2\beta \sum \frac{x_i x_j}{(x_i - x_j)^2}(1 - P_{ij})
\end{aligned}$$

where q is the $(0, 2\pi)$ coordinate and $x = e^{iq}$

3.4.4 Ruijensaars

$$\begin{aligned}
H &= mc^2 \sum_{i=1}^{N} \cosh \frac{p_i}{mc} \prod_{k \neq i} f(x_i - x_k) \\
f^2 &= \begin{cases} 1 + \frac{g^2}{m^2 c^2 x^2} & \text{rational} \\ 1 + \sin^2 \frac{\nu g}{mc} / \sinh^2 \nu x & \text{trig} \\ 1 + \sinh^2 \frac{\nu g}{mc} / \sin^2 \nu x & \text{trig} \\ a + b\rho(x) & \text{elliptic} \end{cases}
\end{aligned}$$

These are the rational, hyperbolic, trigonometric and elliptic cases respectively in order of more generality.

$$\begin{aligned}
P &= mc \sum_{i=1}^{N} \sinh \frac{p_i}{mc} \prod_{k \neq i} f(x_i - x_k) \\
B &= -m \sum_{j=1}^{N} x_j
\end{aligned}$$

Together they from the Lie algebra of Poincare(1,1) with their Poisson brackets whenever f^2 satisfies the above condition or it's degenerations listed.

Cluster Structure in Ruijensaars

Let $\mathcal{G} = SL(p+1, \mathbb{C})$ and $\mathcal{T}$ be a maximal torus. Equip $\mathcal{G}$ with the standard Poisson-Lie structure.

$$\begin{aligned} W &= S_{p+1} \\ \mathcal{G} &= \bigsqcup_{(u,v)\in W^2} \mathcal{G}_{(u,v)} \end{aligned}$$

The double Bruhat cells are also Poisson

$$\begin{tikzcd} & \mathcal{G}_{(u,v)} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d] \\ (\mathbb{C}^*)^l \arrow[r, Rrightarrow] & \mathcal{G}_{(u,v)}/(Ad\ \mathcal{T}) & \mathcal{G}/(Ad\ \mathcal{T}) \end{tikzcd}$$

where the algebraic torus is equipped with the Poisson structure from the log canonical $\{x_i, x_j\} = B_{ij}x_ix_j$ with the exchange matrix B_{ij}. The quotient by the torus also has an induced Poisson structure because $\mathcal{T}$ is a trivial Poisson subgroup. The many arrows from $(\mathbb{C}^*)^l$ depend on the expression of (u, v) as a reduced word.

Insert Harold citation here

Macdonald Polynomials in Ruijensaars

$$D_r^x \equiv t^{\binom{n}{2}} \sum_{|I|=r} \prod_{j\in I} \frac{tx_i - x_j}{x_i - x_j} \prod T_{q,x_i}$$

3.4.16 Theorem (Macdonald,Ruijsenaars) *The D_i^x for $i = 1 \cdots n$ mutually commute and over the ring of symmetric polynomials, they are diagonalized by Macdonald polynomials (usual type A ones)*

Have x, p and X, P canonical pairs related by canonical transformation symplectomorphism.

$$\begin{aligned} H &= \sum_i \prod_{j\neq i} \frac{1}{1 - e^{x_i - x_j}} e^{p_i} \\ H^* &= \end{aligned}$$

$$
\begin{aligned}
\tau_+(z) &\equiv \prod(1 - ze^{-x_i}) \\
\tau_-(z) &\equiv \prod(1 - e^{x_i}/z) \\
\eta(z) &\equiv \frac{1}{\tau_+\tau_-} \\
\eta_0 &= \oint \frac{1}{2\pi i z}\eta(z) = ?H^* \\
&=?\ H
\end{aligned}
$$

Now look at the dynamics of the x and X

$$
\begin{aligned}
\dot{x} &= \\
\dot{X} &=
\end{aligned}
$$

Could also take $\frac{\dot{\tau_+}}{\tau_+}$

3.4.17 Proposition

$$
\begin{aligned}
\frac{d}{dt}\log\tau_+ &= \frac{-1}{2\pi i}\oint_{C_+}\frac{dw}{w}\frac{z/w}{1-z/w}\frac{1}{\tau_+(w)\tau_-(w)} \\
\frac{d}{dt}\log\tau_- &= \frac{+1}{2\pi i}\oint_{C_-}\frac{dw}{w}\frac{w/z}{1-w/z}\frac{1}{\tau_+(w)\tau_-(w)}
\end{aligned}
$$

where the contours are ???.

Proof $\tau_\pm$ are products so log breaks up as a sum which then .... □

3.4.18 Theorem (Hirota bilinear) *Subtract the two equations to get $\frac{d}{dt}\log\frac{\tau_+}{\tau_-} = \frac{\dot{\tau_+}}{\tau_+} - \frac{\dot{\tau_-}}{\tau_-}$ which on the other side gets $\frac{1}{\tau_+\tau_-} - H$.*

$$
\tau_-\dot{\tau}_+ - \tau_+\dot{\tau}_- = 1 - H\tau_+\tau_-
$$

3.4.19 Definition *Define a recursive system of matrices starting with 2 by 2 identity and related to the next by right multiplication by*

$$
\begin{aligned}
L_0 &= I_2 \\
L_k &= L_{k-1}R_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} \\
R_k &= \begin{pmatrix} 1 & ? \\ ? & 1 \end{pmatrix} \\
\dot{A}_k &= 0 \\
\dot{D}_k &= 0 \\
\dot{B}_k &= \omega B_k \\
\dot{C}_k &= -\omega C_k \\
\dot{\tau}_- &= -\omega C_k \\
\dot{\tau}_+ &= +\omega B_k
\end{aligned}
$$

SUSY and Ruijensaars

Below is taken from Bethe/Gauge in odd dimensions Sciarappa. 6/6/16

N- particle closed	5D $\mathcal{N}=1$ $SU(N)$ on $\mathbb{R}^2_\epsilon \times \mathbb{R}^2 \times S^1$
Spectral curve	SW curve
Bethe Ansatz Eq	SUSY vacuum eq
quantum Hamiltonians	twisted chiral ring
Eigenvalues	Wilson loops
eigenfunctions	full defect
coordinate	FI parameters
boundary monodromy parameter	instanton counting parameter
ω_2	$1/R$
ω_1	$-i\epsilon_1$

Table 3.1: NS Dictionary

3.4.5 Dunkl General

3.4.20 Definition (Dunkl) *For a finite Coxeter group W, $\mathfrak{h}$ the reflection representation with dimension denoted r, $\alpha_s \in \mathfrak{h}^*$ an eigenvector with -1 eigenvalue of the reflection s. $\alpha_s^\vee$ is in the dual and pairs to give 2. Also let c be a conjugation invariant function on the set of reflections S. Then for all $a \in \mathfrak{h}$ define*

$$
\begin{aligned}
D_a(c) &\in \mathbb{C}W \ltimes D(\mathfrak{h}_{reg}) \\
D_a(c) &\equiv \partial_a - \sum \frac{c_s \alpha_s(a)}{\alpha_s}(1-s)
\end{aligned}
$$

3.4.21 Theorem (Dunkl) *Polynomials go back to polynomials and the different D_a and D_b commute.*

3.4.22 Theorem (Olshanetsky-Perelomov/Heckman) *Let P_i be a generator $(Sym\mathfrak{h})^W$ and y_i an orthonormal basis of $\mathfrak{h}$, then define $P(D_{y_i}, \cdots D_{y_r})$. This is an element of $\mathbb{C}W \ltimes D(\mathfrak{h}_{reg})$ and is actually in $(\mathbb{C}W \ltimes D(\mathfrak{h}_{reg}))^W$. We may restrict to an differential operator acting on $\mathbb{C}(\mathfrak{h})^W$ so get a element of $D(\mathfrak{h}_{reg})^W$. Call this result $\bar{L}_i$.*

$$\begin{aligned} \bar{L}_i &= res(P(D_{y_i}, \cdots D_{y_r})) \\ \delta_c &\equiv \prod_s \alpha_s(x)^{c_s} \\ L &\equiv \delta_c^{-1} \circ \bar{L} \circ \delta_c \end{aligned}$$

These become a quantum integrable system and in particular

$$L_2 = \Delta_{\mathfrak{h}} - \sum_S \frac{c_s(c_s+1)(\alpha_s, \alpha_s)}{\alpha_s^2}$$

which is the rational Calagero-Moser quantum Hamiltonian when $W = S_n$ and $c_s = k$ for all s.

3.5 Exclusion Processes

3.5.1 ASEP

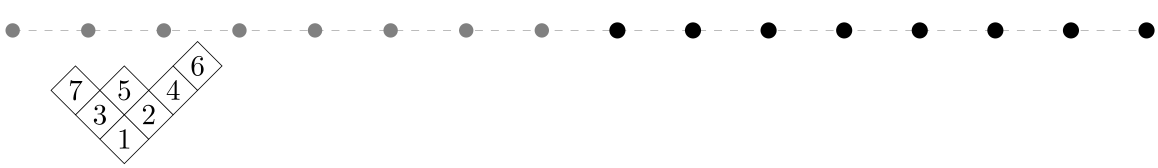
There is a $\mathbb{Z}^d$ of sites. A state is an assignment of occupied or unoccupied to that site. The dynamics are given by the rules followed independently by each particle

- Wait a time given by an exponential distribution with parameter 1
- Choose a $y \in \mathbb{Z}^d$ with probability $p(x, y)$
- If y is vacant, jump there. Else stay put.
- Reset clock

Could replace $\mathbb{Z}^d$ with any countable set S because we did not put any geometric rule on $p(x, y)$ like nearest neighbors only.

3.5.1 Theorem (Spitzer)

3.5.2 1D nearest neighbor ASEP



.

Set $d = 1$ and the function $p(x, y)$ is $p(x, x+1) = p_1$ and $p(x, x-1) = p_2$ with $p_2 > p_1$ to make it asymmetric.

Using the $d = 1$ aspect and if the configuration asymptotes to the step initial condition where all on the left are empty and all on the right filled, we can turn this into a as yet unspecified process on partitions. This is done by putting the partition in Russian notation.

3.5.2 Lemma *For the symmetric case, the generator of theorem 3.5.1 tends to Δ so we approach Brownian motion.*

3.5.3 Theorem (Schutz 95) *ASEP is dual to another ASEP with duality function*

$$D(\eta, A) = \begin{cases} \prod_{x \in A} q^{2N_x(\eta)+2x} & ? \\ 0 & ? \end{cases}$$

3.5.4 Theorem

$$\begin{aligned} L_{SEP}\Delta(u) &= \Delta(u)L_{SEP} \\ &= \Delta(u)L_{SEP}^{\dagger} \end{aligned}$$

So any $\Delta(u)$ can be used as a duality function. Plug in $u \in \mathfrak{sl}_2$ as a particular series in f to recover theorem 3.5.1.

With asymmetry given by q.

$$\begin{aligned} L_{ASEP}\Delta(u)??? &= ???\Delta(u)L_{ASEP} \\ L_{ASEP}^{\dagger} &= V^{-1}L_{ASEP}V \\ L_{ASEP}\Delta(u)V &= \Delta(u)VL_{ASEP}^{\dagger} \end{aligned}$$

So any $\Delta(u)V$ can be used as a duality function. Plug in $u \in U_q\mathfrak{sl}_2$ to recover Schutz.

3.5.3 Plancherel Measures

For a finite group, let $G^{\vee}$ be the set of it's irreps. Define a measure on $G^{\vee}$ by

$$\mu(\pi) = \frac{\dim \pi^2}{\mid G \mid}$$

3.5.5 Example *For $G = S_n$, then we get f^{λ} which is the number of standard Young tableaux of shape λ.*

3.5.6 Example (Poissonized Plancherel) *This is on all partitions instead of just partitions of n. It combines a Poisson distribution for n with the Plancherel measure for S_n as seen in the two factors below.*

$$\mu(\lambda) = \frac{e^{-\theta}\theta^{|\lambda|}}{\mid \lambda \mid!} \frac{(f^{\lambda})^2}{\mid \lambda \mid!}$$

3.5.7 Example (Plancherel Growth) *A random sequence of Young diagrams starting with* (1) *where you add a box according to the transition probability*

$$P(\nu \to \lambda) \quad = \quad P(\lambda^n = \lambda \mid \lambda^{n-1} = \nu) = \frac{f^\lambda}{n f^\nu}$$

It is giving a random walk down Young's lattice. Young's lattice is the Bratelli diagram for $A_i = kS_i$.

3.5.4 1D nearest neighbor TASEP

Use the step initial condition, $p_2 = 1$ and $p_1 = 0$. Let $x_m(t)$ be the position of the m'th particle counting from the left.

3.5.8 Theorem (Johansson) *There is convergence in distribution*

$$\frac{x_m(t) - c_1 t}{c_2 t^{1/3}} \quad \to \quad F_2$$

This is 1/3, KPZ universality class

3.5.9 Definition (Tracy-Widom GUE)

$$\lambda_{max} \quad \approx \quad 2N + N^{1/3} F_2$$

3.5.10 Definition (KPZ Equation)

$$\partial_t h(x,t) \quad = \quad \nu \partial_x^2 h(x,t) + \frac{\lambda}{2}(\partial_x h(x,t))^2 + \sqrt{2D} W_t$$

This is related to the stochastic Burger's PDE.

3.5.11 Definition (Ornstein-Uhlenbeck Process) *A continuum version of* $AR(1)$.

3.5.12 Lemma (Cole-Hopf Transformation) *Let* $= \log$

3.5.5 1D nearest neighbor ASEP again

3.5.6 All the other Macdonald Processes

3.5.13 Remark This has been using ASEP, but there is another way Macdonald polynomials show up. That is in the Quantum Many Problem of the previous section. These can be passed through the Cherednik-Matsuo correspondence to get a solution of Quantum Affine KZ equations. This means it is a correlation function of the XXZ chain with periodic boundary conditions. ◊

3.5.7 XXZ Redux

3.6 Spin Glasses

3.6.1 Edwards Anderson

Hamiltonian:

$$\begin{aligned} H &= -\sum_{<ij>} J_{ij} S_i S_j \\ J_{ij} & \quad N(J_0, \sigma_J^2) \end{aligned}$$

This can be solved with the replica method. It has a vanishing magnetization $< m >= 0$ and a non-vanishing two point correlation function for $\sum_{i=1}^{N} S_i^{\alpha} S_i^{\beta}$ where α β are replica indices.

$$\beta f = -\frac{\beta^2 \sigma_J^2}{4}(1-q)^2 + \frac{\beta J_0 m^r}{2} - \int \exp(\frac{-z^2}{2}) log(2\cosh(\beta \sigma_J z + \beta J_0 m)) dz$$

Chapter 4

2D Topological Field Theory

4.0.1 Definition *A fully extended framed 2 dimensional topological field theory is a symmetric monoidal 2-functor from $Bord_2^{fr}$ to some target 2-category $\mathcal{C}$ (like the Morita 2-category)*

4.0.2 Theorem *They are parameterized by their value on the standard framed pt called $+$.*

`https://www.ma.utexas.edu/users/psafronov/cob/teleman.pdf`

4.0.3 Theorem (4.29 of Teleman 5 Lectures) *In the case of the target being the $(\infty, 2)$ category given by $D^bQCoh(X)$, integral kernels $\Phi \in D^bQCoh(X \times Y)$ and*
Orlov's theorem says that every fully faithful triangulated functor between D^bCoh's of smooth projective varieties is representable by some integral kernel.
The case of $D^bCoh(X)$ with X projective are fully dualizable and to make an $SO(2)$ fixed point means that you have trivialized the Serre functor $\otimes\omega$ so said X was Calabi-Yau by trivializing the canonical bundle.

4.0.4 Theorem (Schommer-Pries) *The functors from $Bord^2 \to AlgBim_k$ are given up to equivalence by the trace-Morita classes of separable symmetric Frobenius algebras that get assigned to the point.*

Proof `https://golem.ph.utexas.edu/category/2008/06/schommerpries.html` □

4.0.5 Theorem (Eilenberg Nakayama) *In vector spaces over a field k, an algebra A can be equipped with a symmetric Frobenius algebra structure if it is separable.*

Proof `https://ncatlab.org/nlab/show/Frobenius+algebra#symmetric_frobenius_algebras` □

4.0.6 Theorem () *The separable algebras over a field k are precisely finite products of matrix algebras over finite-dimensional division algebras D over k (whose centers are separable extensions of k.) The paranthesis is necessary when the field isn't perfect, but characteristic 0 and finite fields are. Worry about $F_p(t)$*

Proof `https://qchu.wordpress.com/2016/03/27/separable-algebras/` □

4.0.7 Lemma *One way to construct finite dimensional associative division algebras over arbitrary fields is to produce quaternion algebras.*

4.0.8 Theorem *To any local field there are just two quaternion algebras.*

4.0.9 Theorem *The quaternion algebras B over $\mathbb{Q}$ are given by completing at all places ν to get B_ν. Each of those is one of the only two possibilities for that local field. So we get a list of places where it is split B_ν is the 2 by 2 matrices over $\mathbb{Q}_\nu$ and the ramified ones where it is not. The places where B ramifies is always finite and even and it detemines B as an algebra.*

4.0.10 Example *Rational Hamilton quaternions ramify at 2 and ∞ and split everywhere else. The rational 2 by 2 matrices split at all places.*

Proof `https://en.wikipedia.org/wiki/Quaternion_algebra` □

4.0.11 Theorem *The Picard-Brauer group (Picard 3-group) of Alg_R algebras over R, bimodules and intertwiners is the same as giving line 2-bundles over $SpecR$ which has homotopy groups as $Br(R)$ $Pic(R)$ and R^*. These match with $H^2_{et}(R, G_m)_{tor}$, $H^1_{et}(R, G_m)$ and $H^0_{et}(R, G_m)$ respectively.*

4.0.12 Theorem *For a global field K like a number field.*

$$0 \longrightarrow Br(K) \longrightarrow \bigoplus_{v \in S} Br(K_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

For local fields that are complete under a discrete valuation like $\mathbb{Q}_p$, we get $Br(K_v) \simeq \mathbb{Q}/\mathbb{Z}$. For the reals the place at ∞ get $\mathbb{Z}_2$ where we regard it as $(1/2)\mathbb{Z}/\mathbb{Z}$.

4.0.13 Corollary *Looking at the exactness for the image of the quaternion algebra $(a, b, -ab)$ over $\mathbb{Q}$ gives quadratic reciprocity by showing the sum over all v giving 0. This is the 2-torsion part of $Br(K)$ given as an even number of $1/2$'s in the $Br(K_\nu)$ and the rest being 0.*

4.1 2D Dijkraaf Witten

Let $A = \mathbb{C}[G]$

4.1.1 Theorem (Class Formula)

Proof

Chapter 5

Dimension 2: Lattices

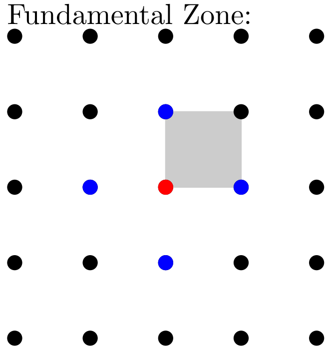
5.0.1 Definition (Wallpaper Group) *A discrete subgroup of the Lie group $O(2) \ltimes \mathbb{R}^2$ that contains two linearly independent translations upon identification of the subgroup $1 \ltimes \mathbb{R}^2$ with it's Lie algebra so that saying linearly independent makes sense. The choice of identification does not affect whether or not they are linearly independent.*

5.1 The Lattices

5.1.1 A1 x A1 : The square

Symmetry Group: $D_4 \ltimes \mathbb{Z}^2$, with the action of $r(x, y) = (y, -x)$ and $f(x, y) = (x, -y)$

Fundamental Zone:



Brillouin Zone:

Materials it appears:

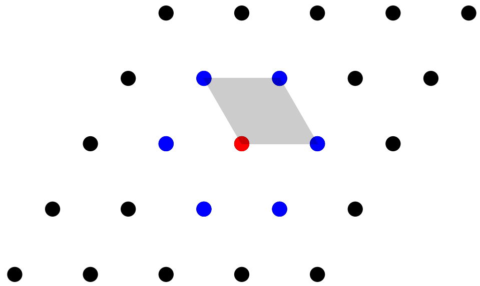
Ising on the square

It is dual to Z_2 gauge theory by the following transformation. Insert here.

5.1.2 A2

Symmetry Group: $\mathbb{Z}_2^2 \ltimes \mathbb{Z}^2$ with the action of $e(x, y) = (y, x)$ and $f(x, y) = (-x, -y)$. where e, f are elements of Klein Four.

Fundamental Zone:



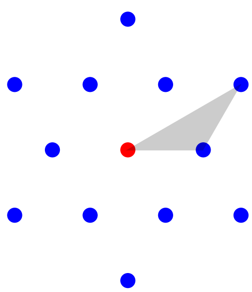
Brilloun Zone:
Places it appears:

5.1.3 B2/C2

http://www.math.ubc.ca/ cass/coxeter/crm1.html

5.1.4 G2

Fundamental Zone:



Brillouin Zone:
Places it appears:

5.1.5 Kagome Lattice

5.1.1 Remark (Pearls)

Spin Liquid

5.2 2D pseudo Lattices

Chapter 6

Dimension 2: Part 1

6.1 Gauge theories

6.1.1 Z2 Gauge theories

6.1.2 Lattice QED in 2+1

6.2 Ising Model

$$
\begin{aligned}
H &= \sum \\
Z &= \sum e^{-\beta H} \\
T &= \\
Z &= TrT^N
\end{aligned}
$$

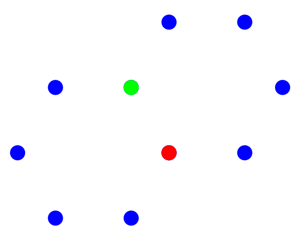
6.2.1 Kramers-Wannier Duality

See `https://arxiv.org/pdf/cond-mat/0404051v2.pdf` and the CFT section where we push to the conformal limit.

$$
\begin{aligned}
e^{\beta J \sigma_i \sigma_j} &= \cosh \beta J + \sinh \beta J \sigma_i \sigma_j \\
&= \cosh \beta J + \sigma_i \sigma_j \sinh \beta J \\
Z &= (\cosh \beta J)^{2N_V} \sum \prod (1 + \sigma_i \sigma_j \tanh \beta J) \\
T >> T_0 &\implies \beta J \to 0 \implies \tanh \beta J \to 0
\end{aligned}
$$

$$\begin{aligned}
K_1 &= \beta_1 J_1 \\
Z(K_2)/e^{NK_2} &= Z(K_1)/2^N \cosh^{2N} 2K_1 \\
\tanh K_1 &= e^{-2K_2} \\
K_1 = K_2 = K_C &\implies \sinh 2K_C = 1 \\
J_1 = J_2 = J &\implies T_C = 2.269J
\end{aligned}$$

6.3 Graphene



6.3.1 Dirac Cone

Assuming a tight binding model and no spin orbit coupling, for every pseudomomentum $k \in \mathbb{T}^2$

$$\begin{aligned}
H &= \begin{pmatrix} 0 & \Delta_k \\ \Delta_k^* & 0 \end{pmatrix} \\
\Delta_k &\equiv -t\sum_{l=1}^{3} e^{ik\cdot\delta_l} \\
\delta_1 &\equiv a/2(1, \sqrt{3}) \\
\delta_2 &\equiv a/2(1, -\sqrt{3}) \\
\delta_3 &\equiv -a(1, 0) \\
E_k &= \pm \mid \Delta_k \mid
\end{aligned}$$

There are 2 special momenta K and $K^{'}$ for which E_k vanishes. That is the gap closes at this point in the Brillouin torus. Expanding in the local coordinate q around K to first order gives a approximate Hamiltonian.

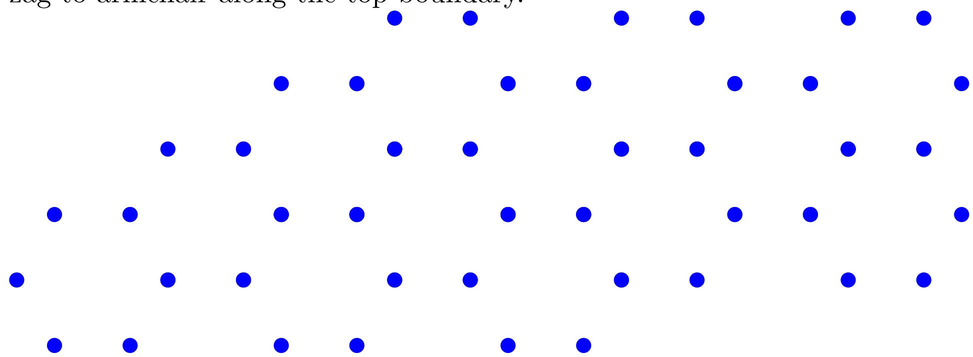
$$H = \hbar v_F(q_x\sigma_x + q_y\sigma_y)$$

with error bounded by $C \mid q \mid^2$.

6.3.1 Remark There is a type error here. We are not using any Lie algebra structure. We are just writing down an operator in a basis. This means we aren't writing σ in it's Lie incarnation or it's Cliff incarnation but just in it's 2 by 2 matrix incarnation. $\Diamond$

6.3.2 Nanoribbons

In this picture you can see both the zig-zag and armchair configurations as well as the change of zig-zag to armchair along the top boundary.



6.4 Charge Density

6.5 Spin Density

6.6 Hubbard

http://quest.ucdavis.edu/tutorial/hubbard7.pdf

Chapter 7

Dimension 2: Integrable Hierarchies

7.1 Dimer Model

http://arxiv.org/pdf/math/0310326.pdf http://arxiv.org/pdf/0910.3129v1.pdf

Let Γ be a bipartite graph drawn on a surface C. Let A be a weight function on its edges. To make a mix of the physical parameters and nice polynomial behavior let $A_e = e^{\beta_0 E_e}$ for the energy E_e and a reference temperature.

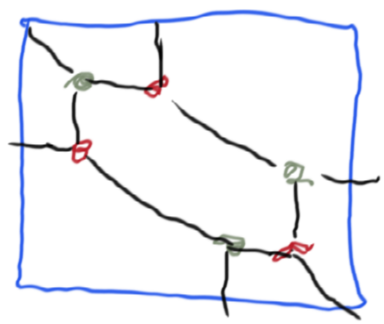


Figure 7.1: Here black and white are colored red and green for contrast. There are 6 choices for D_Γ here.

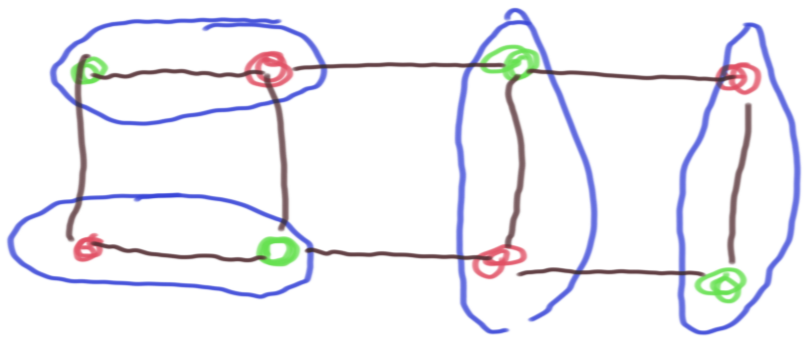
$$Z(\Gamma, A, \beta) = \sum_{D \in D_\Gamma} \prod_{e \in D} A_e^{\beta/\beta_0}$$

where D_Γ is the set of dimer coverings.

If we let $\beta = \beta_0$, then we get a polynomial in the edge variables with unit coefficients.

If we let the graph have incoming and outgoing edges that go to the boundary of C let T_Γ be the set of these edges.

$$Z(\Gamma, A, \beta, \xi) = \sum_{T \subset T_\Gamma} \Big(\sum_{D \in D_\Gamma(T)} \prod_{e \in D} A_e^{\beta/\beta_0} \Big) \prod_{t \in D \bigcap T} \xi_t$$



where $D_\Gamma(T)$ is the set of dimer coverings which have $T \subset T_\Gamma$ occupied and the rest of T_Γ free. ξ are odd variables indexed by T_Γ. The order is given by the order on the boundary circle.

7.1.1 Theorem (Kasteleyn)

$$Z(\Gamma, T^2, A, \beta = \beta_0) \;=\; \frac{1}{2}\big(-Pf(A_1) + Pf(A_2) + Pf(A_3) + Pf(A_4)\big)$$

Proof A_i are labelling the periodic or antiperiodic boundary conditions. These are labelled PP, PA, AP and AA respectively. □

7.1.2 Example *Just counting dimer coverings. so set* $A_e^{\frac{\beta}{\beta_0}} = 1$*. This then gives ...*

7.1.3 Example (Aztec Diamond)

7.1.4 Theorem (Genus g) *More generally a combination of* 2^{2g} *Pfaffians replacing the 4 spin structures in the genus 1 case.*

7.1.5 Theorem *Correlation functions that condition on a dimer being occupied at a particular bond.*

$$\langle I_{e_1} I_{e_2} \cdots I_{e_n} \rangle \;=\; ?$$

Figure 7.2: Partial Gluing of two disks

7.1.1 Cutting and Gluing

$$Z_{gl} = \sum_{gl} Z_1(\Gamma_1, A_1, \beta, \xi_0, \xi_{gl}) Z_2(\Gamma_2, A_2, \beta, \xi_{gl}, \xi_2)$$

7.1.6 Definition (Height function) *Choose height somewhere and everytime you cross a occupied dimer change by ...*
Superimpose two dimer configurations to get a bunch of closed contour cuves. Let the first one be fixed as an implicit reference. In particular this gives a class in $H^1(\Sigma, \mathbb{Z}_2)$

If boundary bond is occupied, that is equivalent to removing that internal vertex as a puncture. This is because that vertex is already on an occupied bond.

7.1.7 Theorem (Limit Shapes) *Specify height function on the boundary as a piecewise continuous function on the boundary which then specializes to the lattice function for every* ϵ *and then compute* Z*, correlation functions, string operators. In fact the probability distribution for possible height functions. Get frozen and liquid regions. Frozen means... and Liquid means ...*

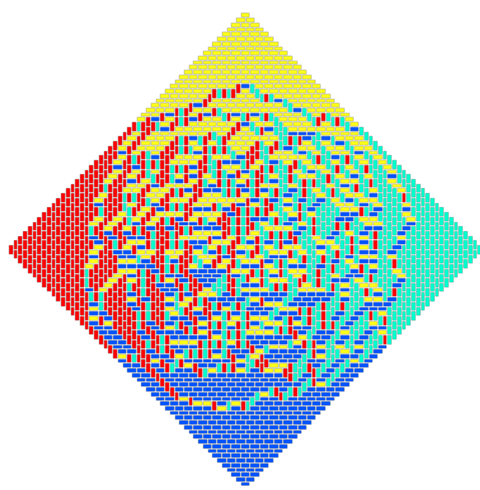


Figure 7.3: Artic Circle

Let e be the edge aligned with a domino in the solid yellow part. It is occupied with high probability so the expectation value stays close to 1.

$$\langle I_e \rangle \approx 1$$

7.1.2 Quantum Dimer Model

7.1.8 Theorem (Rokhsar-Kivelson point)

7.1.9 Conjecture (Moessner-Sondhi)

7.2 6/19/etc vertex models

See XXZ section instead

7.2.1 Theorem (Van Hove) *Mesh size $\to 0$ limit.*

$$\lim_{C\to\infty} \frac{1}{\mid C \mid} ???$$

Limit free energy density exists and does not depend on boundary condition.

Notice that we can avoid this theorem by ???.

7.2.1 Stochastic Six-Vertex Model

Suppose the weights are

- $a_1 =$
- $a_2 =$
- $b_1 =$
- $b_2 =$
- $c_1 =$
- $c_2 =$

Insert picture in quadrant with two linear parts with the liquid in between.

7.2.2 Lemma (Limit $b \to 1$)

7.3 Hitchin System

7.3.1 Moduli stack of bundles

Let C be a curve of genus $g \geq 2$. The moduli space of semistable algebraic vector bundles on C of rank n and degree d is denoted by $U_C(n, d)$. When n and d are relatively prime this is automatically stable and this is a nonsingular projective variety of dimension $n^2(g-1)+1$. Denote $U_C(n)$ as the result of unioning over all degrees, giving all stable vector bundles.

7.3.1 Definition (Stable Vector Bundle)

If you take D modules on the algebraic stack $Bun_G C$, this is the automorphic side of the geometric Langlands conjecture. Above was the GIT quotient that gave an honest space.

The F points of Bun_G

$$Bun_G(C)(F) \quad \simeq \quad G(F)\backslash G(A_F)/G(O_F)$$

Find analogues of eigenfunctions, eigen D-modules. These will be indexed by local systems L on the curve for the dual group. That is the Galois side. In GL case rank N vector bundle with flat connection. On curve automically flat. The D-module is Aut L

For $n = 1$, get Line bundles for Bun which is the Jacobian. We need to produce a D-module on this torus $(\mathbb{C}^*)^{2g}$ based on the data of a local system L on the genus g curve. Because it is only acting as a 1D rep, it factors through homology. So we can say all the monodromies by giving 2g monodromies. So the corresponding D-module is the rank 1 local system on Jac with the connection having prescribed monodromies around each of those $\mathbb{C}^*$ factors.

7.3.2 Hitchin System

We now talk about the Galois side where looking at O-modules on $Loc_{^LG}$ moduli stack of local systems. How do you write a local system? You need to give a holomorphic algebraic bundle E with a holomorphic algebraic connection ∇ on said E.

$$\nabla = d+$$

On this side there are skyscraper sheaves at any particular local system. In the above preview of GL, this was the L we talked about. The other side

Hecke eigenproperty means apply a functor and your object comes back tensored with a vector space. Skyscraper must be such because the support isn't going to change.

On Loc_G can ask about the bad points or torsion sheaves.

7.3.2 Definition (Higgs Bundle) *A Higgs bundle is a pair (E, ϕ) of an algebraic vector bundle on the curve C and a global section $\phi \in \Gamma(C, End(E) \otimes K_C)$. A Higgs bundle is stable if all ϕ invariant proper subbundles to have strictly smaller $\frac{deg}{rank}$. Semistablity allows some to have equal stability condition.*

Like before we will denote the moduli spaces of given ranks and degrees. $\mathcal{H}_C(n, d)$ and without the d if we want to take disjoint union over all d.

Applying Serre duality gives a new Higgs bundle by sending (E, ϕ) to $(E^* \otimes K_C^1, -\phi^\dagger \otimes id)$

where we have seen that $\phi^\dagger\ E^* \otimes K_C^{-1} \to E^*$ induces a map $\phi^\dagger \otimes id\ E^* \otimes K_C^{+1} \to E^* \otimes K_C^2$

If the vector bundle was already stable, then we can give it any Higgs field. Also if the field is 0, then Higgs stability condition is the stability of the bundle. That means that $T^*U_C(n, d) \subset \mathcal{H}_C(n, d)$ where the Higgs field is the cotangent direction. But there is more because we can stabilize an unstable vector bundle with the right Higgs field.

We can use the isomorphism $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[I_1 \cdots I_r]$ of invariant polynomials of degrees d_j. A particular choice of homogenous G-invariant polynomials is tr ϕ^n, the power sums in the eigenvalues. Then we can send the Higgs bundle through this map.

$$(E, \phi) \quad \to \quad (I_1(\phi), \cdots I_r(\phi)) \in \oplus_s H^0(C, K^{\otimes d_s})$$

7.3.3 Definition (Spectral Curve)

7.3.3 Hitchin Hyperkahler

Take a compact Lie group G and a compact Riemann Surface C, get a space $M_G(C)$ which is a fibration over a base B. The complex structures we can put on it are parameterized by $\xi \in \mathbb{CP}^1$. This is the fact that it is hyperkahler. This is chosen so that $\xi = 0$ corresponds to J_3. We already saw one symplectic structure in seeing T^*Bun (if we did the stable stuff the Higgs was better but as stacks we have $Higgs = T^*Bun$)

$$\begin{aligned} \omega_\xi &= -\frac{i}{2\xi}\omega_+ + \omega_3 - \frac{i}{2}\xi\omega_- \\ \omega_+ &= \omega_1 + i\omega_2 \end{aligned}$$

$$\begin{array}{c} M_G(C) = \text{Moduli(Higgs Bundles } (\mathcal{E}\ \Phi)) \\ \downarrow \\ B = \text{Moduli}(S \subset T^*C) \end{array}$$

http://arxiv.org/pdf/0801.0015v3.pdf

7.3.4 Elliptic Spin Calogero-Moser system

7.4 Gaudin Model

https://arxiv.org/pdf/math/0407524v2.pdf

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$ with basis J_a and dual basis J^a. Let z_i be a collection of N positions in the complex plane. Then define the Gaudin Hamiltonians as an element of $U(\mathfrak{g})^{\otimes N}$ where (i) indicates which particle we are to act on.

$$H_i = \sum_{j\neq i}\sum_{a=1}^{d} \frac{J_a^{(i)} J^{a(j)}}{z_i - z_j}$$

This defines a commuting system of operators when we stick modules M_i for all of the particles. In addition to these there is also the $Z(U(\mathfrak{g}))^{\otimes N}$. There are also higher Gaudin Hamiltoinians that do not appear in the 2nd filtration level when the rank of $\mathfrak{g}$ is higher than 1. The situation of literal spins just gives $\mathfrak{sl}_2$ so we don't need them.

http://arxiv.org/pdf/1409.6937v1.pdf

In fact we can insert an automorphism σ to make it look like this model on a wedge geometry instead.

$$\begin{aligned} H_i &= \sum_{p=0}^{T}\sum_{j\neq i}\sum_{a=1}^{d} \frac{J_a^{(i)}\sigma^p J^{a(j)}}{z_i - \omega^{-p} z_j} \\ &+ \sum_{p=0}^{T}\sum_{a=1}^{d} \frac{J_a^{(i)}\sigma^p J^{a(j)}}{z_i - \omega^{-p} z_i} \end{aligned}$$

Here we see the interaction with all of the "image charges" including the images of oneself.

7.4.1 Definition (Oper) *A ^LG oper on a curve X is a triple of a ^LG bundle on X, a connection on this bundle and a reduction to Borel ^LB satisfying a condition with ∇. A Miura ^LG oper has another reduction which is now preserved by the connection.*

7.4.2 Lemma *The space of Miura opers whose underlying oper has regular singularities and trivial monodromy is isomorphic to the complete flag manifold $^LG/^LB$*

7.4.3 Lemma *For opers on SpecR like $R = \mathbb{C}[[t]]$ or $\mathbb{C}((t))$, the action of the gauge group $N(R)$ is free so we can take a unique representative*

$$\nabla = \partial_t + p_{-1} + \sum_{j=1}^{\ell} v_j(t) p_j$$

where p_j are defined by gradation using an $\mathfrak{sl}_2$ triple. $p_{-1}, 2\rho, p_1$

7.4.4 Theorem (PGL_2 and projective connections) *From the canonical form of the lemma we can look at a single function $v(t)$ under change of coordinate $t = \phi(s)$, $v \to v(\phi(s))\phi'(s)^2 - \frac{1}{2}\{\phi, s\}$ like a projective connection.*

In fact in more generality $Op_G(X) \simeq Proj(X) \times \bigoplus_{i=2}^{\ell} \Gamma(X, \Omega^{d_i+1})$

7.4.5 Theorem (Miura)

7.4.6 Theorem (Wakimoto)

7.4.7 Theorem (Feigin-Frenkel) *There is a canonical isomorphism $Z(\hat{g}) \simeq FunOp_{^LG}(D)$ of algebras compatible with $Der\mathcal{O}$ and $Aut\mathcal{O}$*

7.4.8 Definition (Drinfeld-Kohno Operad) *The rth Drinfeld-Kohno Lie algebra is generaded by symbols t_{ij} with $1 \leq i \neq j \leq r$ satisfying*

$$\begin{aligned} [t_{ij}, t_{kl}] &= 0 \ \forall i \neq j \neq k \neq l \\ [t_{ij}, t_{ik} + t_{kj}] &= 0 \ \forall i \neq j \neq k \end{aligned}$$

This is thought of as r strands and each t_{ij} stands for drawing a chord to connect the i and j strands.

This also then makes the operad in the category of Lie algebras structure apparent as a bundling of strands.

7.4.9 Theorem (Arnold-Kohno)

$$\begin{aligned} C^{\bullet}_{CE}(kd(r)) &\simeq H^{\bullet}(E_2(r)) \\ t_{ij}^{\vee} &\to [\frac{d(z_i - z_j)}{z_i - z_j}] \end{aligned}$$

is a dg algebra quasi-isomorphism.

Combining all the r gives $C^\bullet_{CE}(kd)$ the structure of a Hopf dg-cooperad using the cooperad structure from the KD operad and this dg-algebra quasi-isomorphism becomes a quasi-isomorphism of Hopf dg-cooperads $C^\bullet_{CE}(kd) \simeq H^\bullet(E_2)$

7.4.10 Theorem (BV Operad) *The operad BV is the homology of framed little 2-disks. So contrast with $H_\bullet(E_2)$*

7.5 KP Hierarchy

Chapter 8

Dimension 2: CFT

8.1 General CFT Things

8.1.1 Lemma

$$
\begin{aligned}
T_\mu^\nu &= \begin{pmatrix} T_z^z & T_z^{\bar{z}} \\ T_{\bar{z}}^z & T_{\bar{z}}^{\bar{z}} \end{pmatrix} \\
T_{z\bar{z}} + T_{\bar{z}z} &= 0 \\
T_{z\bar{z}} = T_{\bar{z}z} &= 0 \\
T(z) &\equiv T_{zz} \\
\bar{T}(\bar{z}) &\equiv T_{\bar{z}\bar{z}}
\end{aligned}
$$

8.1.2 Definition (Schwarzian Derivative)

$$
\begin{aligned}
\{f(z), z\} &\equiv \frac{f^{(3)}(z)}{f'(z)} - \frac{3}{2}\left(\frac{f^{(2)}(z)}{f'(z)}\right)^2 \\
g(z) = \frac{az+b}{cz+d} &\implies \{g(z), z\} = 0
\end{aligned}
$$

8.1.3 Lemma

$$
\begin{aligned}
T(z_1)T(z_2) &\approx \frac{c/2}{(z_1-z_2)^4} + \frac{2}{(z_1-z_2)^2}T(z_2) + \frac{1}{z_1-z_2}\partial_{z_2}T(z_2) + \cdots \\
z \to \alpha(z) &\implies T(z) \to Jac(\alpha)^{-2}\left(T(z) - \frac{c}{12}\{\alpha(z), z\}\right)
\end{aligned}
$$

8.1.4 Definition (Virasoro Algebra) *Generated by L_n and c with nontrivial brackets given by*

$$
\left[L_m, L_n\right] = (m-n)L_{m+n} + \frac{c}{12}(m(m-1)(m+1))\delta_{m+n,0}
$$

8.1.5 Lemma

$$\begin{aligned}T(z) &= \sum_n L_n z^{-n-2} \\ \bar{T}(\bar{z}) &= \sum_n \bar{L}_n \bar{z}^{-n-2} \\ L_n &= \frac{1}{2\pi i}\oint T(z) z^{n+1}\end{aligned}$$

Proof

$$\begin{aligned}[L_n, L_m] &= \frac{-1}{4\pi^2}\oint\oint z^{n+1}w^{m+1} - z^{m+1}w^{n+1}T(z)T(w) \\ &= \end{aligned}$$

8.1.6 Definition (Primary/Singular) *In a lowest weight irreducible representation c acts by some number also called c. L_0 also central acts by some h. All positive L_n act by 0. Then do induction to get rest. That is there is a primary state and rest are descendants. If just given c, h and not told irreducible have to check for singular vectors.*

$$\begin{aligned}L_{n\geq 1} \mid h\rangle &= 0 \\ L_0 \mid h\rangle &= h \mid h\rangle\end{aligned}$$

Singular Vectors are when this is reducible.

8.1.7 Lemma *If $h = h_{r,s}(c)$ for some r, s then have singular vector at level rs*

$$\begin{aligned}c &= 1 + 6(b + b^{-1})^2 \\ h_{r,s}(c) &= \frac{1}{4}((b + b^{-1})^2 - (br + b^{-1}s)^2)\end{aligned}$$

For example, $2, 1$ gives the singular vector $(L_{-1}^2 + b^2 L_{-2})v$ in $V_{c,-1/2-3/4b^2}$

8.1.1 Order Fields

8.1.8 Definition (OPE)

$$\begin{aligned}\langle \phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)\rangle &= \sum_k \langle C_{12}^k(z_1, z_2)\phi_k(z_{12})\cdots\phi_n(z_n)\rangle \\ C_{12}^k(z_1, z_2) &= \frac{C_{12}^k}{\mid z_1 - z_2 \mid^{2h_1+2h_2-2h_k}}\end{aligned}$$

8.1.2 Twist Fields

8.1.9 Definition (Twist Field) *The picture has an operator with an invisible string attached.*

8.1.3 Verlinde Formula

8.1.10 Theorem

$$N_{\lambda\mu}^{\nu} = \sum_{\sigma} \frac{S_{\lambda\sigma} S_{\mu\sigma} \bar{S_{\nu\sigma}}}{S_{0\sigma}}$$

Proof

8.2 Torus Partition Function

8.3 Fusion Category Things

8.3.1 Definition (Monoidal) *Has tensor product functor, unit object, natural isomorphism giving the associator, natural unitors. These natural isomorphisms satisfy pentagon and triangles respectively.*

8.3.2 Lemma (Associators and Tetrahedra) *Let* $(V_i \otimes V_j) \otimes V_k \simeq \bigoplus F_{ij}^l \otimes V_l \otimes V_k \simeq \bigoplus F_{ij}^l \otimes F_{lk}^m V_m$ *Contrast with* $V_i \otimes (V_j \otimes V_k) \simeq \bigoplus F_{jk}^l \otimes V_i \otimes V_l \simeq \bigoplus F_{jk}^l F_{il}^m V_m$ *This means for the associator we must give isomorphism between the multiplicity spaces* $\bigoplus_l F_{jk}^l \otimes F_{il}^m$ *and* $\bigoplus_l F_{ij}^l \otimes F_{lk}^m$

Insert Tetrahedron Picture

8.3.3 Definition (Rigid) *Objects have duals*

8.3.4 Definition (Fusion) *A rigid semisimple linear (over k) monoidal category with finitely many isomorphism classes of simples such that the endomorphisms of the unit object give k the ground field.*

8.3.5 Definition (Pivotal) *Natural transformation between* Id *and* **

8.3.6 Definition (Traces) *There are right and left traces of* $f\ \ a \to a$

$$1 \xrightarrow{coev} a^* \otimes a^{**} \xrightarrow{1 \otimes dd^{-1}} a^* \otimes a \xrightarrow{1 \otimes f} a^* \otimes a \longrightarrow 1$$

$$1 \xrightarrow{coev} a \otimes a^* \xrightarrow{f} a \otimes a^* \xrightarrow{dd \otimes 1} a^{**} \otimes a^* \longrightarrow 1$$

8.3.7 Definition (Spherical) *Left and right traces coincide.*

8.3.8 Definition (Modular) *For the braiding between simple objects* i *and* j*, construct:*

$$S_{i,j} = \operatorname{tr} R_{i,j} R_{j,i}$$

The matrix built from this should be nondegenerate.

8.4 Segal Axioms

8.4.1 Definition *A functor from $Bord_2^{cx}$ to $Hilb$. For the source, the objects are disjoint unions of the standard circle. The morphisms are Riemann surface with parameterized boundary. These are taken up to biholomorphisms taking the parameterizations to each other. There are no identity morphisms because of the lack of 0 length cylinders/thin annuli. But reparameterization just gives projective rep of $Diff_+$. So insert these limit as morphisms as well. Now give dagger structure to say this is a unitary field theory. The target is Hilbert spaces and linear maps. To get correlation functions we may glue in discs $D(0,1)$ with the desired operators at 0.*

We may pass to the limits of the moduli space of Riemann surfaces.

8.4.2 Definition (Planar Algebra) *Especially consider the case of many inputs, 1 output and genus 0. This forms an operad. If it does depend on the holomorphic structure get a Vertex Operator Algebra. Insert the associativity and locality pictures.*

8.4.3 Example (Free Chiral Charged Fermion) *$H = L^2(S^1)$ with projection to Hardy space pH of the disk for the polarization. Take the Fock space associated. The vacuum is $\Omega = 1 \in \wedge^0 = \mathbb{C}$. Define bounded operators for creation and annihilation.*

8.5 Conformal Nets

8.5.1 Definition (Conformal Net) *H_0 for the vacuum sector Hilbert space. Give $\mathcal{A}(I)$ Von Neumann algebras acting on this for every interval I on that circle.*

8.5.2 Theorem (Tener) *Partially thin annuli converging to relate Segal type to Conformal Net*

Figure 8.1: Partially Thin Annulus

Proof

8.5.3 Theorem (Kawahigashi-Longo-M?)

$$\mu_{\mathcal{A}} = \sum_{irrep} d_\pi^2$$

Because all $d_\pi \geq 1$, that implies that finite $\mu_{\mathcal{A}} < \infty$ immediately implies finitely many irreducibles. Then turns $Rep\mathcal{A}$ into a unitary modular tensor category.

8.5.4 Definition (Orbifold) *Let G be a finite subgroup of the Lie group K, then build invariants for the conformal net $\mathcal{A}$ coming from LK_k loop group.*

8.5.5 Corollary $\mu_{\mathcal{A}^G} = \mid G \mid^2 \mu_{\mathcal{A}}$

Proof In conformal nets, but not VOA. □

8.5.6 Definition (Permutation orbifolds)

8.5.7 Example *Cycle of order n in S_n*

8.5.8 Example *Flip $1 \to n \cdots n \to 1$. A corollory of this is dual subfactor*

8.5.9 Definition (Simple Current Extension)

8.5.10 Example *$LSU(2)_{4k}$ and get $LSO(3)_?$*

8.5.11 Example (Coset) *$SU(N)_{k+l} \to SU(N)_k \times SU(N)_l$, this then gives a subnet $\mathcal{B} \subset \mathcal{A}$, define the coset as the commutant of $\mathcal{B}$ inside $\mathcal{A}$.*

8.5.12 Theorem (GKO) *$N = 2$, and $l = 1$, GKO construction of Virasoro minimal models*

8.5.13 Theorem *... .And a corollary all irreps of $\mathcal{B}$ appear in $H^{i\alpha}$ multiplicity spaces.*

8.5.14 Conjecture *Vague conjecture that all RCFT's are given by these procedures. Add mirror extensions to escape the examples below.*

8.5.15 Example (Mirror Extension) *$SU(2)_{10} \subset Spin(5)_1$*
$SU(2)_{10} \times SU(10)_2 \subset SU(20)_1$
$SU(10)_2 \subset \widetilde{SU(10)_2}$, where $\tilde{?}$ is another rational theory not from conformal inclusion. Not generated by spin 1 currents alone.

8.6 VOA

8.6.1 Definition (Borcherds)

8.6.2 Definition (C_2 algebra) *Given a VOA V, it is canonically filtered so we may form the associated graded grV which is a Poisson vertex algebra. Then take generating ... subring R_V which is now a Poisson algebra. This is called the C_2 algebra of V.*
$\tilde{X}_V = SpecR_V$ is the associated scheme with $X_V = SpecMaxR_V$ the associated variety.
The associated jet space for $\tilde{X}_V$ called J also has structure of a Poisson vertex algebra and there is a surjection $\mathbb{C}[J] \to grV$ which is often an isomorphism as well.

8.6.3 Theorem (C_2 Cofinite) *The dimension of $SpecgrV$ is 0 if and only if $\dim X_V = 0$. This case is called C_2 cofinite.*

8.7 CFT Tables

8.7.1 Virasoro Minimal Models

These are parameterized by two integers p and q.

They have central charge $1 - 6\frac{(p-q)^2}{pq}$
They have primaries with weights

In order to be unitary $q = p + 1$, so they are parameterized by a single integer p.

$$\begin{aligned}
c &= 1-\frac{6}{p(p+1)}\\
&= 1-6(\sqrt{\frac{p}{p+1}}-\sqrt{\frac{p+1}{p}})^2\\
&= 1-6(\frac{p}{p+1}+\frac{p+1}{p}-2)\\
&= 1-6(\frac{p^2+p^2+2p+1}{p(p+1)}-2)\\
&= 1-6(\frac{p^2+p^2+2p+1-2p^2-2p}{p(p+1)})\\
h_{r,s} &= \frac{((p+1)r-ps)^2-1}{4p(p+1)}
\end{aligned}$$

where r runs from $1\cdots p-1$ and s runs from 1 to r. Alternatively let s run all the way to $p-1$ but just remember that the table double counts.

Modular structure on torus partition functions.

$$\begin{aligned}
S_{(r,s),(a,b)} &= 2\sqrt{\frac{2}{pq}}(-1)^{1+sa+rb}\sin(\pi\frac{p}{q}ra)\sin(\pi\frac{p}{q}sb)\\
T_{(r,s),(r,s)} &= e^{2\pi i(h_{r,s}-c/24)}
\end{aligned}$$

8.7.2 $p=2$

Trivial.

8.7.3 $p=3$

$$c = 1-6\frac{1}{12}$$

s\r	1	2
1	$\mathbf{1}$	ϵ
2	σ	σ
3	ϵ	$\mathbf{1}$

	h	$\bar{h}$	meaning
$\mathbf{1}$	0	0	identity
ϵ	1/2	1/2	thermal
σ	1/16	1/16	spin

$$\begin{aligned}
\sigma\times\sigma &= \mathbf{1}+\epsilon\\
\sigma\times\epsilon &= \sigma\\
\epsilon\times\epsilon &= \mathbf{1}
\end{aligned}$$

$$S = \begin{pmatrix} 1/2 & 1/2 & \sqrt{2}/2 \\ 1/2 & 1/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 & 0 \end{pmatrix}$$

The free Majorana fermion also has this CFT.

$$\epsilon = \bar{\psi}\psi$$

but the Ising spin field σ is given by a Jordan-Wigner transformation of the ψ field.

8.7.4 $p = 4$

This is the tricritical Ising model.

$$c = 1 - 6\frac{1}{20} = \frac{7}{10}$$

s\r	1	2	3
1	**1**	$\sigma^{'}$	$\epsilon^{''}$
2	-	σ	$\epsilon^{'}$
3	-	-	ϵ
4	-	-	-

	h	$\bar{h}$	meaning
1	0	0	identity
ϵ	1/10	1/10	thermal
$\epsilon^{'}$	3/5	3/5	thermal
$\epsilon^{''}$	3/2	3/2	thermal
σ	3/80	3/80	spin
$\sigma^{'}$	7/16	7/16	spin

The 5+20 nontrivial fusion rules that don't have an **1**, but commutativity turns this into 15 rules.

$$\begin{aligned}
\epsilon \times \epsilon &= \mathbf{1} + \epsilon^{'} \\
\epsilon \times \epsilon^{'} &= \epsilon + \epsilon^{''} \\
\epsilon \times \epsilon^{''} &= \epsilon^{'} \\
\epsilon^{'} \times \epsilon^{'} &= \mathbf{1} + \epsilon^{'} \\
\epsilon^{'} \times \epsilon^{''} &= \epsilon \\
\epsilon^{''} \times \epsilon^{''} &= \mathbf{1} \\
9 more
\end{aligned}$$

$$
\begin{aligned}
S &= \begin{pmatrix}
s_2 & s_1 & s_1 & s_2 & \sqrt{2}s_1 & \sqrt{2}s_2 \\
s_1 & -s_2 & -s_2 & s_1 & \sqrt{2}s_2 & -\sqrt{2}s_1 \\
s_1 & -s_2 & -s_2 & s_1 & -\sqrt{2}s_2 & \sqrt{2}s_1 \\
s_2 & s_1 & s_1 & s_2 & -\sqrt{2}s_1 & -\sqrt{2}s_2 \\
\sqrt{2}s_1 & \sqrt{2}s_2 & -\sqrt{2}s_2 & -\sqrt{2}s_1 & 0 & 0 \\
\sqrt{2}s_2 & -\sqrt{2}s_1 & \sqrt{2}s_1 & -\sqrt{2}s_2 & 0 & 0
\end{pmatrix} \\
s_1 &= \sin\frac{2\pi}{5} \\
s_2 &= \sin\frac{4\pi}{5}
\end{aligned}
$$

It has supersymmetry that means it is a representation of the super-Virasoro algebra

$$
\begin{aligned}
\left[L_m, L_n\right] &= (m-n)L_{m+n} + \frac{1}{12}cm(m-1)(m+1)\delta_{m+n,0} \\
\{G_m, G_n\} &= 2L_{m+n} + \frac{1}{3}c(m-1/2)(m+1/2)\delta_{m+n,0} \\
\left[L_m, G_n\right] &= (1/2m-n)G_{m+n}
\end{aligned}
$$

For actually using this coincidence, see Section 8.17

8.7.5 $p = 5$

This is the three states Potts model.

$$
c = 1 - 6\frac{1}{30} = \frac{4}{5}
$$

s\r	1	2	3	4
1	-	-	-	-
2	-	-	-	-
3	-	-	-	-
4	-	-	-	-
5	-	-	-	-

8.7.6 Large p

The central charge is very close to 1. The weights are

$$
\frac{r^2p^2 + s^2p^2 + O(p)}{4p^2 + 4p} \to \frac{r^2 + s^2}{4}
$$

8.8 WZW generalities

First we say the nonlinear sigma model with target G. a simple Lie group.

$$\begin{aligned} S^{WZW}_{S^2} &= \tilde{S}_{\partial^{-1}S^2}(g^{-1}dg) \\ \tilde{S} &= S_{CS}(A) + \frac{1}{2}\int_\Sigma \mathrm{tr}\, \bar{A} \wedge A \\ \bar{\partial}_z(g^{-1}\partial_z g) &= 0 \\ \bar{\partial_w}(g^{-1}\partial_w g) &= 0 \\ S &= \int_\Sigma \end{aligned}$$

8.8.1 Theorem (Polyakov-Wiegmann)

$$\begin{aligned} S[gM] &= S[g] + S[M] - \frac{1}{\pi}\int_\Sigma \mathrm{tr}(M^{-1}\bar{\partial}M\partial h h^{-1}) \\ \partial S(M \to gM \;\; g \approx 1) &= -\frac{1}{\pi}\int_\Sigma \mathrm{tr}(\bar{\partial}A_z \delta g g^{-1}) \end{aligned}$$

8.8.2 Definition (Vertex Algebra)

8.8.3 Lemma (Types of Kac-Moody) *These are those that have ....*

- *Finite Type*
- *Affine*
- *Indefinite: including the nicest which are hyperbolic*

8.8.4 Definition (Affine Kac-Moody) *Normalize inner product to 2 as usual.*

$$\begin{aligned} \hat{\mathfrak{g}} &\equiv \mathfrak{g} \otimes \mathbb{C}[t^{\pm}] \oplus \mathbb{C}K \\ \big[J_i \otimes t^n, J_s \otimes t^m\big] &= \big[J_i, J_s\big] \otimes t^{n+m} + \langle J_i \mid\mid J_j \rangle Res\; t^n d(t^m)\; K \end{aligned}$$

In Kac-Moody formalism $e_i f_i h_i$ for $i \in [0, n]$ with 0 being the affine node and others finite nodes.

8.9 $SU(2)_k$

$$c = \frac{k(2^2-1)}{k+2} = \frac{3k}{k+2}$$

Partitions that fit within a 1 by k box, so they will have labels 0 through k.
In the decategorified fusion ring, the fusion rules are:

$$[j_1] \times [j_2] =$$

8.9.1 $SU(2)_1$

$$c = 1$$

See later section ?? for table. Coincidence with easier model.

8.9.2 $SU(2)_2$

$$c = 3/2$$

8.9.3 $SU(2)_3$

8.9.4 $SU(2)_4$

8.9.5 $SU(2)_5$

8.9.6 $SU(2)_6$

8.9.7 $SU(2)_7$

8.9.8 $SU(2)_8$

8.10 $SU(N)$

8.11 Type B

8.12 Type C

8.13 Type D

8.14 E_8

8.14.1 E_8 level 1

8.14.1 Definition (Holomorphic VOA) *$\mathcal{W}$ is called holomorphic if Rep$\mathcal{W}$*

8.15 Kazhdan-Lusztig

8.15.1 Theorem *The categories*

$PE - Rep\ LG \rightleftarrows Rep^{ss} U_q\mathfrak{g}$

are equivalent

Proof (Finkelberg) `https://link.springer.com/article/10.1007%2FBF02247887?LI=true`□

So the above sections can be replaced by considering $U_q(\mathfrak{sl}(2))$ at $q = e^{2\pi i/(k+2)}$ etc.

8.16 Lattice VOA

A lattice can have the following useful properties

- Even If the norms are all even
- Odd otherwise
- Positive Definite if the norm $(a, a) > 0$
- Unimodular if the determinant is ± 1, this also makes the $L \subset L^*$ by the inner product into an isomorphism
- Extremal for a even unimodular lattice is when the minimal norms of the vectors saturate the upper bound $2(\left[n/24\right] + 1)$

8.16.1 Theorem *Any two indefinite unimodular lattices with the same type dimension and signature are isomorphic. Therefore can parameterize them as $I_{m,n}$ or $II_{m,n}$ for the odd and even cases respectively. Here $m, n \geq 1$.*

Proof `https://math.berkeley.edu/~reb/papers/al/al.pdf` □

8.16.2 Theorem (Vinberg) *For unimodular Lorentzian lattices $I_{n,1}$ (this one is odd), have infinite reflection group when $n \geq 2$. The quotient of $Aut(I_{n,1})$ by this reflection group is finite if and only if $n \leq 19$. In fact those cases are subgroups of Co_0 by Conway and Sloane.*

Proof `https://math.berkeley.edu/~reb/papers/al/al.pdf` □

8.16.3 Theorem *In contrast when looking at lattices in inner product spaces so positive definite, we get finite automorphism groups.*

8.16.4 Theorem *The theta function of a unimodular positive definite lattice is a modular form of weight half the rank. If even it has level 1. If odd, it has $\Gamma_0(4)$ structure so level 4.*

$$\begin{aligned}
\Theta_L &= \sum_{x \in L} e^{\pi i \tau |x|^2} = \sum_x q^{|x|^2/2} \\
q &\equiv e^{2\pi i \tau}
\end{aligned}$$

so counting (extremal) minimal vectors means reading that coefficient of q^n for length squared to be $2n$. In particular if $n < 24$ then minimal length squared is ≤ 2, That means looking at coefficient of q picks out roots with length $\sqrt{2}$. For $n = 24$, we get minimal length squared is ≤ 4. If this minimimum is 2, there are roots and it is one of the 23 other Niemier lattice. The extremal one is the Leech lattice. Looking at the coefficent of q gives 0, so need to look at q^2 for the true minimal vectors.

8.16.5 Definition (Lattice VOA) *From the lattice L build a V_L by*

8.16.6 Example ($\boldsymbol{L = \mathbb{Z}\sqrt{n}}$)

8.16.7 Example *Start with a root lattice by*

8.16.8 Theorem *For an even unimodular lattice without roots.* $Aut(V_L^+) \simeq O(\hat{L})/\langle \theta_{V_L} \rangle \simeq Hom(L, \mathbb{Z}_2).(O(L)/(-$

Proof 4.1 of `http://ac.els-cdn.com/S0021869304002960/1-s2.0-S0021869304002960-main.pdf?_tid=1441ef86-ebf6-11e6-9bb0-00000aacb360&acdnat=1486335461_97f578648a00911bc9f32d94e00501`

8.17 N=1 Minimal

[]

There is again a discrete and then a continuous range for central charge. The continuous ones start at $c = 3/2$.

8.18 N=2

There is yet again a discrete and then a continuous range for central charge. The continuous ones start at $c = 3$. The N=2 minimal models have central charges $c = 3 - \frac{6}{n}$ with $n \geq 3$.

A conformal $N = 2$ with target a k dimensional Calabi-Yau has central charge $3k$. This is shown by first looking at $\mathbb{C}^k$ (where it is clear that there are k chiral superfields each contributing 3) and arguing from there.

8.19 Coset Construction

8.19.1 Definition (Conformal Embedding) $\hat{\mathfrak{g}}_1 \subset \hat{\mathfrak{g}}_2$. *Do Segal-Sugawara construction to write down* Vir_{c_2} *from* $\mathfrak{g}_2$ *and* 1 *respectively. Can ask for* $c_2 = c_1$.

8.20 Free Boson

Move this earlier

Begin with the free boson CFT. It's action is

$$S = \frac{g}{2} \int \partial_\mu \phi \partial^\mu \phi$$

Now let's assumed our metric is flat. This means for examples of hyperbolic surfaces we have already applied a Weyl transformation of multiplication by y^2 from the upper half plane with it's usual metric to it's realization as included into the plane. In this $g \geq 2$ case, of course this vanishes at the boundary so we need to make sure we stay in the interior. Alternatively we applied $1+ \mid z \mid^2$ to get to $\mathbb{CP}^1$

$$\langle \phi(z, \bar{z}) \phi(0) \rangle = -\frac{1}{4\pi g} \ln(\mid z \mid^2)$$

This is because of the Green's function for the Laplacian in 2D.

Now set $g = 1/4\pi$ so that we can drop that coefficient. Equivalently we are working with $\sqrt{4\pi g}$ off from what we had before.

From this action you form the stress energy tensor

$$T(z) \quad = \quad \frac{-1}{2} : \partial\phi_2 \partial\phi_2 :$$

You define a primary field $e^{i\sqrt{2}\alpha\phi}$ which has dimension α^2

$$\langle e^{i\sqrt{2}\alpha\phi(z)} e^{i\beta\sqrt{2}\phi(0)} \rangle = e^{-2\alpha\beta(-lnz)} = z^{-2\alpha^2}\delta_{\alpha,-\beta}$$

A way to see this neutrality condition is $\phi \to \phi + a$ is a symmetry so it will act on the correlator namely by making it pick up a factor of $e^{i\sqrt{2}a\sum\alpha_i}$. For this to vanish for all a means that the Coulomb gas has to be neutral.

The trick to keep using is

$$\langle e^{A_1} \cdots e^{A_n} \rangle \quad = \quad exp(\sum_{i<j} \langle A_i A_j \rangle)$$

$$\begin{aligned}
\langle \prod_{i=1}^{N} e^{\frac{\alpha_i}{\sqrt{2}}\phi(z_i)} \rangle &= \exp(\sum \frac{\alpha_i\alpha_j}{2} \langle \phi(z_i)\phi(z_j) \rangle) \\
&= \exp(\sum \frac{\alpha_i\alpha_j}{2}(-\ln \mid z_i - z_j \mid^2)) \\
&= \exp(\sum(\ln \mid z_i - z_j \mid^{-\alpha_i\alpha_j}) \\
&= \prod \mid z_i - z_j \mid^{-\alpha_i\alpha_j}
\end{aligned}$$

which is the N point function of N vertex operator insertions.

In particular let $N = 4$ so get 1 cross ratio left

In mode decomposition

$$\phi(x,t) \quad = \quad \phi_0 + \pi_0 t + \sum_{k>0} + \sum_{k<0}$$

8.20.1 Definition (1+1D Boson algebra (Heisenberg Algebra)) *Let $\mathfrak{h}$ be a finite dimensional vector space with nondegenerate bilinear form $(,)$. This is the flavors of the boson.*

$$
\begin{aligned}
H &= \mathfrak{h} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \\
\hat{H} &= H \bigoplus \mathbb{C}[c] \\
\left[f, g\right] &= (\oint (g, df))c \\
\left[c, -\right] &= 0 \\
\left[x \otimes t^n, y \otimes t^m\right] &= n(x, y)\delta_{n,-m} c
\end{aligned}
$$

To simplify can take $\mathfrak{h} = \mathbb{C}$ with just multplication as bilinear form. Or alternatively that we are working in a orthonormal basis for the bilinear form.

$$
\begin{aligned}
a_n &\equiv 1 \otimes t^n \\
\mid t \mid &= 1 \\
\mid c \mid &= 0 \\
\left[a_n, a_m\right] &= n\delta_{n,-m} c \\
a(z) &\equiv \sum a_n z^{-n-1} \in \hat{H}[[z, z^{-1}]] \\
\left[a(z), a(w)\right] &= c \sum n z^{-n-1} w^{n-1} = \partial_w \sum (\frac{w}{z})^n
\end{aligned}
$$

8.20.2 Lemma *Only trivial finite dimensional representations.*

Proof

$$
Tr(\left[a_n, a_{-n}\right]) = 0 = nTr1
$$

For infinite dimensional representations, let them be graded $V = \bigoplus V_\lambda$ with $\exists \lambda_m \forall \lambda > \lambda_m \ \ V_\lambda = 0$

8.20.3 Lemma

$$
\begin{aligned}
H &\equiv \frac{1}{2} \sum a_n a_{-n} + ? \quad \notin U(\hat{H}) \\
\left[H, a_n\right] &= -n a_n \\
: H : &\equiv \frac{1}{2} \sum : a_n a_{-n} : \ \notin U(\hat{H}) \\
\left[: H :, a_n\right] &= -n a_n
\end{aligned}
$$

This is not in the universal enveloping algebra because tensor product of graded vector spaces uses the definition

$$
\begin{aligned}
T(V)^2_k &= \bigoplus_{i+j=k} (V_i \otimes V_j) \\
&\hookrightarrow \prod_{i+j=k} (V_i \otimes V_j)
\end{aligned}
$$

But even still by our restriction about $\exists \lambda_m$ means that $: H :$ can act on any objects we are considering, the positive energy representations. The eigenvalues of $: H :$ on these V are bounded from below.

8.20.4 Example *Let*

$$
\begin{aligned}
F(\lambda) &\equiv \bigoplus_n \bigoplus_{|\mu|=n} \mathbb{C} a_{-\mu} v_\lambda \\
&= \bigoplus_n F(\lambda)_n \\
\mu &\in Part(n) \\
\lambda &\in \mathbb{C} \\
a_{-\mu} &= a_{-\mu_1} a_{-\mu_2} \cdots a_{-\mu_l} \\
a_0 v_\lambda &= \lambda v_\lambda \\
a_{n>0} v_\lambda &= 0
\end{aligned}
$$

8.20.5 Theorem *Each $F(\lambda)$ irreducible and if a_0 acts diagonalizably in some positive energy rep V, then $V \simeq \bigoplus_\lambda n_\lambda F(\lambda)$*

8.20.6 Lemma (Partition Function)

$$
\begin{aligned}
tr_{F(\lambda)} q^{:H:} &= \sum q^k dim F(\lambda)_k \\
&= q^{1/2\lambda^2} \prod \frac{1}{1-q^j} \\
&= q^{1/2\lambda^2} \sum q^n p(n) \\
q &\equiv e^{-\beta} \\
T \to \epsilon \to 0 &\implies q \to e^{-1/\epsilon} \to 0
\end{aligned}
$$

This is the finite and zero temperature cases.

8.20.7 Theorem (Power Sum Realization) *This Fock space is isomorphic as a vector space to a polynomial algebra in countably many generators $\mathbb{C}[p_1 \cdots]$. This means you can identify the action of a_i $i > 0$ with the action of multiplication by p_i and a_{-n} with $-n\frac{d}{dp_n}$.*

8.21 Free Majorana Fermion

8.22 Liouville

You can couple the boson to the scalar curvature

$$S = \frac{1}{8\pi} \int \sqrt{g}(D_g\phi D_g\phi + 2\gamma\phi R)$$

Now the translation of the field by a is not a symmetry, but by the Gauss-Bonet theorem it just changes the action by $\frac{\gamma a}{4\pi} 8\pi(1-g) = 2\gamma a(1-g)$

We are on the sphere now so $g = 0$ with curvature entirely concentrated at ∞. This gives a new neutrality condition $\sum \alpha_k = -i\sqrt{2}\gamma$ which is equivalent to putting a charge $i\gamma\sqrt{2}$ at ∞. This would make sense only if γ was imaginary which makes the action complex. Parameterizing $\gamma = i\sqrt{2}\alpha_0$

The vertex operators are still primary but now of dimension $\alpha(\alpha - 2\alpha_0)$

Because we put α_0 real notice that for $\alpha = \alpha_0 + ia$ the dimension is $(\alpha_0 + ia)(-\alpha_0 + ia) = -\mid \alpha_0 + ia \mid^2$ which is negative real. This means that either this state $\mid h\rangle$ or $L_1 \mid h\rangle$ have negative norm. We could also make sure we are only looking at real α in which case we get negative dimension in the range between 0 and $2\alpha_0$ when the two factors have opposite signs. Otherwise they have the same sign giving positive dimension.

In particular notice if we write $2\alpha_0 = b - b^{-1} = i\beta + i\beta^{-1} = i(\beta + \beta^{-1})$ we get two dimension 1 operators with $\alpha = b$ and $\alpha = -b^{-1}$ which means that $\oint dz V_\pm$ are dimension zero screening operators.

$$\gamma = \frac{i}{\sqrt{2}}(b - b^{-1}) = \frac{-1}{\sqrt{2}}(\beta + \beta^{-1})$$

Because they are dimension 0 we can insert them into conformal blocks and under change $z \to w$, they won't transform any differently because $\left(\frac{\partial w}{\partial z}\right)^0 = 1$ so we can insert enough of them to satisfy the neutrality condition.

For the physical vertex operators, we should be allowed to have two of them meaning we want $\langle V_\alpha(z) V_\alpha(w)\rangle$ to not be zero, but actually we can't stop insertion of screening currents so we should include them.

$$\begin{aligned} \sum \alpha_i + m\alpha_+ + n\alpha_- &= 2\alpha_0 \\ \sum \alpha_i &= (1-m)\alpha_+ + (1-n)\alpha_- \\ \alpha_{r,s} &= \frac{1}{2}(1-r)\alpha_+ + \frac{1}{2}\alpha_- \end{aligned}$$

This gives the Kac formula for conformal dimensions, but r and s are just unconstrained integers and $\alpha_\pm$ haven't been fixed yet.

In order to cut it down the number of primary fields we can identify the results of $\alpha_{r,s}$ with $\alpha_{r+k*q,s+k*p}$ for some p and q so that the distinct fields are parameterized by $\mathbb{Z}_q \times \mathbb{Z}$. Find the unique representative of a given lattice point in the fundamental strip domain for this action.

For this to make sense for their dimensions we see that $q\alpha_+ + p\alpha_- = 0$.
So you get $\alpha_+ = \sqrt{p/q}$ and $\alpha_- = -\sqrt{q/p}$. This successfully gives us the central charge and conformal dimensions for the (p,q) minimal models. (Unitary if $p = q \pm 1$)

But we still have the entire strip in one direction (the s direction by how we chose the fundamental domain). But then you check the three point functions to see the fusion rules and find that you can consistently look at the subalgebra of the Verlinde algebra with $1 \leq s < p$

The way you check the three point functions is by using the four point functions and knowing that after sending the points to $0, 1, z$ and ∞ they are solutions to a hypergeometric equation in the remaining z variable because every second order linear ODE with three regular singular points can be transformed into the Gauss Hypergeometric equation.

8.22.1 Theorem *A second order Fuchsian equation with n singular points has a D_n acting on it's solutions so in this case we have 24 nice solutions sitting inside this 2 dimensional space. This gives lots of connection formula which tell you how expanding a solution around one point would be expressed in the given basis at another point.*

This is part of a general formalism of the monodromies of KZ equations. Take an N point function of vertex operators where the insertion points are radially ordered as $\mid z_{\sigma(1)} \mid << \mid z_{\sigma(2)} \mid \cdots$ for some permutation σ. Changing the permutation by crossing a wall gives you a monodromy matrix from which you can understand the quantum group at the correct root of unity. We took the extremely deep into a chamber limit because that way you could right down the conformal block easily but you could follow the KZ connection as long as you don't cross a wall where you have to cross a branch cut. Then you are in a different chamber and have to apply a Stokes matrix.

8.22.1 Felder Complex

Let $V_{r,s}^{i,j}$ be the screened vertex operator which has $V_{r,s}$, i α_+ screenings and α_- screenings.

In particular knowing $\frac{1}{s} \oint V_{1,-1}^{0,s-1}$ and the same for $p-s$ gives BRST operator on the the complex which has charged Fock spaces

$$\cdots F_{r,2p-s} \overset{Q_{p-s}}{\to} F_{r,s} \overset{Q_s}{\to} F_{r,-s} \cdots$$

8.22.2 2D Quantum Gravity

There is another perspective that gives the same form of the action. This is given by assuming that ϕ is the Weyl transformation that takes you from metric g to a reference metric $\hat{g}$. Then you plug into the Einstein-Hilbert action with cosmological constant.

$$\begin{aligned} g &= e^{\alpha\phi}\hat{g} \\ S &= \frac{1}{8\pi}\int\sqrt{\hat{g}}(\hat{\nabla}\phi\hat{\nabla}\phi + 2\gamma\phi\hat{R} + \frac{\mu}{\alpha^2}e^{\alpha\phi}) \end{aligned}$$

If $2\gamma = i\sqrt{2}(b - b^{-1})$ and $\alpha = b$ or $-b^{-1}$, then the cosmological constant term would be the screening current. If μ was formal, expanding in μ would give the correct number of screening currents to pull down in order to evaluate a correlation function with Fateev integral methods. Note we would be only pulling down one type of screening current.

But note that in this perspective, ϕ is not a scalar field that lives on the surface; it is a piece of the metric. Under Weyl transformation. This is visible by transformation laws.

$$\begin{aligned} \hat{g} &\rightarrow e^{2\rho}\hat{g} \\ \alpha\phi &\rightarrow \alpha\phi - 2\rho \end{aligned}$$

Under diffeomorphism

$$\alpha\phi \rightarrow \alpha\phi + 2log \mid \frac{\partial w}{\partial z} \mid$$

It is not an ordinary scalar field which would just transform by pullback. It has even more problem than the usual logarithmic OPE.

8.22.3 Higher Rank Toda Field Theory

First for the $\mathfrak{h}$ diagonalize the inner product to break down to $\ell = \dim \mathfrak{h}$ decoupled bosons. Can treat as ℓ tuples of partitions for the Fock space and $p_{n,i}$ and $\frac{d}{dp_{m,i}}$ for $i = 1 \cdots \ell$

8.22.4 Drinfeld Sokolov Reduction

8.23 Boundaries

8.23.1 Definition (Cardy Condition)

8.23.2 Definition (Ishibashi Condition)

8.23.3 Definition (Boundary Changing Operators) *Work on $\mathbb{H}$ with boundary conditions L and R on the left and right part of the real axis.*

8.23.4 Lemma (Annulus) *Work on an annulus viewed as a rectangle with width* 1 *and height* δ*. Periodic boundary conditions identifying the 1 sides. Boundary conditions* a *and* b *on the two sides. The Hamiltonian is the vertical translation which is the rotation of the annulus.*

$$\begin{aligned} q &= e^{-\delta} \\ Z_{ab}(1,\delta) &= \mathrm{tr}_{H_{ab}}\, q^{L_0} \\ &= \sum_{\Delta} n_{ab}^{\Delta}\chi_{\Delta}(q) \end{aligned}$$

Can also look the other way and think of this as a matrix element between a *and* b *instead.*

$$\begin{aligned} \mid a\rangle &= \sum c_{a\Delta} \mid \Delta\rangle \\ \mid b\rangle &= \sum c_{b\Delta} \mid \Delta\rangle \\ \langle a \mid e^{-H*1} \mid b\rangle &= \sum_{\Delta,\Delta'} c_{\bar{a}\Delta} c_{b\Delta'}\langle \Delta \mid q^{-H*1} \mid \Delta'\rangle \\ &= \sum_{\Delta'} c_{a\Delta'} c_{b\Delta'}\chi_{\Delta'}(\tilde{q}) \\ &= \sum_{\Delta'} c_{a\Delta'} c_{b\Delta'} \sum S_{\Delta}^{\Delta'}\chi_{\Delta}(q) \end{aligned}$$

8.23.5 Lemma *This implies*

$$\begin{aligned} c_{a\Delta'}c_{b\Delta'} &= \sum_{\Delta} S_{\Delta}^{\Delta'} n_{ab}^{\Delta} \\ n_{ab}^{\Delta} &= \sum_{\Delta'} S_{\Delta}^{\Delta'} c_{a\Delta'} c_{b\Delta'} \end{aligned}$$

8.23.6 Example (Minimal Models)

8.23.7 Theorem (Fuchs-Runkel-Schweigert) *A full RCFT is given by the chiral part already given and a Frobenius algebra object.*

8.23.8 Example

8.24 Entanglement Entropy

Take the CFT in 1+1 perspective. Suppose there is the ground state ψ, you can compute the induced state for any subsystem. Being a subsystem of a connected 1 dimensional manifold can be described as a disjointed union of intervals. We would like to compute it's Von Neumann Entropy.

$$\begin{aligned}
S_{A\subset X} &= -\operatorname{tr}\rho_{A\subset X}\log\rho_{A\subset X}\\
S_{[x,x+\ell]\subset\mathbb{R}} &= \frac{c}{3}\log\frac{\ell}{a}+\cdots\\
\operatorname{tr}\rho^n_{[0,\ell]\subset\mathbb{R}} &= c_n\frac{\ell}{a}^{-c/6(n-1/n)}\\
S_A &= -\lim_{n\to 1}\frac{\partial}{\partial n}Zn\\
Zn &= \operatorname{tr}((\rho_{[0,\ell]\subset\mathbb{R}})^n)
\end{aligned}$$

Zn is the partition function on a degree n connected cover of the spacetime ramified at the endpoints of the intervals at some instant of time. That is a codimension 2 phenomenon. This is equivalent to taking the correlation function of $2*N$ twist fields which implement this gluing. We think of each of them as having ramification $e_p = d-1$ modelled on z^d singularities.

8.24.1 Theorem (Hurwitz)

$$\begin{aligned}
2g-2 &= d(2h-2)+\sum e_p + d - length(\lambda)\\
e_p &= d-1\\
\mid P\mid &= 2N\\
2g-2 &= (d-1)*(2*N)+d-length(\lambda)+d(2h-2)\\
g &= (d-1)*N+\frac{d-length(\lambda)}{2}+1+d(h-1)\\
g &= (d-1)*(N-1)+(d-1)+\frac{d-length(\lambda)}{2}+1+d(h-1)\\
g &= (d-1)*(N-1)+\frac{d-length(\lambda)}{2}+d*h
\end{aligned}$$

8.24.2 Theorem

$$\begin{aligned}
\lambda &\in Partition(d)\\
r &= (1-2h)d+length(\lambda)+2g-2\\
2g-2 &= r-length(\lambda)+d(2h-2)+d\\
2g-2 &= d(2h-2)+r+d-length(\lambda)
\end{aligned}$$

Set $d - length(\lambda) = e_\infty$ for a marked point so we can call that a point where there may be additional ramification. Usually we will set $\lambda = 1^d$ so that there is no ramification there.

8.24.3 Theorem (Dijkgraaf) *Let $N_{g,d}$ count the number of degree d covers of an elliptic curve $h=1$ ramified simply ($e_p = 1$) at a given set of $r = \sum e_p = 2g-2$ distinct points with $g \geq 2$ so there is some ramification somewhere. This is the way it has to be if $\lambda = 1^d$ the marked point has no ramification. This gives a genus g curve with map to our original E. This defines a groupoid whose objects are covers and morphisms are automorphisms of covers. Take the groupoid cardinality of this and call it $N_{g,d}$. $\sum_{n\geq 1} N_{g,d}q^d$ is a quasimodular form of weight $6g-6$.*

If we were taking 1 instead of $1/\operatorname{Aut}$, this would be the partition function for the trivial theory on all those covering surfaces. Note this was asking for points of ramification 1 rather than $d-1$. This corresponds to elementary transpositions vs the long cycle.

8.25 Orbifolding, Extension, Building Bigger

8.25.1 Theorem (ADE classification of modular invariants) *A_l, D_{2l}, E_6 and E_8. which have Coxeter numbers ... This means that for $SU(2)_k$ can ...*

8.25.2 Definition (Q-System)

8.25.3 Example (D_4)

8.25.4 Definition ($\alpha^{\pm}$ induction)

8.25.5 Example (D_4 Redux)

8.26 Integrable Perturbations

8.26.1 Sin(h)-Gordon

Chapter 9

Fractional Quantum Hall

9.0.1 Remark Gaussian units here. ◊

9.1 Integer

A 2D electron gas in the xy plane in a high magnetic field $B\hat{z} + B_x\hat{x}$. We have rotated the sample so that the magnetic field is in this form, so this is a WLOG.

We can gauge fix the vector potential for this by

$$A = (0, Bx, 0) + (0, 0, B_x y)$$

$$H = \frac{1}{2m}(p - \frac{e}{c}A)^2$$

We now assume that the $B_x = 0$.

$$\begin{aligned} H &= \frac{1}{2m}(p_x^2 + (p_y - \frac{eB}{c}x)^2) \\ &= \frac{1}{2m}p_x^2 + \frac{1}{2m}\frac{e^2B^2}{c^2}(\frac{cp_y}{eB} - x)^2 \\ &= \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2(\frac{\hbar k_y}{m\omega} - x)^2 \end{aligned}$$

The eigenfunctions are of the form

$$\Psi = e^{ik_y y}\psi_{SHO}(x - \frac{p_y}{m\omega})$$

Put this in an L_x by L_y box with Dirichlet boundary conditions. So now the eigenvalues are labelled by two integers

$$\begin{aligned}
k_y &= 2\pi n/L_y \\
E_{n,N} &= \frac{\hbar 4\pi^2 n^2}{L_y^2} + \hbar\omega(N+\frac{1}{2})
\end{aligned}$$

but the center of the oscillator wavefunction is at $\frac{\hbar 2\pi n}{L_y m\omega} = \frac{\hbar 2\pi n c}{L_y eB}$. Constraining this to be within L_x restricts n to be in a finite range in particular

$$\begin{aligned}
0 \leq n &\leq \frac{m\omega L_x L_y}{2\pi\hbar} \\
&= \frac{B L_x L_y}{hc/e} = \frac{\Phi}{\Phi_0} \\
\Phi_0 &= \frac{hc}{e}
\end{aligned}$$

If the particles had charge Ze instead and spin S not locked to the magnetic field, this degeneracy would get multiplied by $Z(2S+1)$.

This is a bit hockey because it counted states of the Harmonic Oscillator if the center of the wavefunction was inside the box. But the Harmonic oscillator states are always infinite in the x direction, so the finite size effects are more subtle even though the area scaling heuristic still holds.

9.1.1 Symmetric Gauge

In this gauge the wave functions are

$$\psi(z) = f(z)e^{-eBz\bar{z}/4\hbar c}$$

where $f(z)$ is analytic. Give the monomials as a basis (that will be completed to a Hilbert Space)

Now the eigenstate with z^n prefactor has expectation value of radius

$$\langle r^2 \rangle = (n+1)\frac{2\hbar c}{eB}$$

So in order to stay within the overall radius R

$$n \leq \frac{\Phi}{hc/e}$$

again this is a rough argument that neglects the actual wavefunction when there is an edge.

9.2 Fractional/Laughlin Wavefunctions

Let there be many noninteracting particles. Now the antisymmeterized wavefunction is

$$\Psi = \prod (z_i - z_j) e^{\sum_i -eBz_i\bar{z}_i/4\hbar c}$$

You can see that the power of each z_i is $N-1$ so this needs to be less than or equal to $\frac{\Phi}{\Phi_0}$. In order to get fractional filling $1/k$ change the power of the Vandermonde part.

$$\Psi = \prod (z_i - z_j)^k e^{\sum_i -eBz_i\bar{z}_i/4\hbar c}$$

Now the highest power is $k(N-1)$ so the the filling is given by $k(N-1) \leq \frac{\Phi}{\Phi_0}$ or the density is $\frac{N-1}{A} \leq \frac{B}{k\Phi_0}$

There are defects you can insert by

$$\Psi = \prod (z_i - w) \prod (z_i - z_j)^k e^{\sum_i -eBz_i\bar{z}_i/4\hbar c}$$

The insertion at w is $1/k$th of an electron in that it takes k of them to make a single electron. It raises the power of the z_i exponents making the number N have to be lower. In particular it lowers the number of particles by $1/k$ to fit within the same radius. Or inserting k of the holes lowers the number of z_i by 1.

9.3 Anomalous and Semiclassical

9.3.1 Remark Switched to SI ◊

9.3.2 Lemma *If we have a symplectic manifold M with global polarization partitioning by Lagrangian N. Say fixing the values of F_i integrable conserved quantities. Then we can identify after taking a cover if necessary with an open subset of T^*N but with $\omega = \omega_{std} + eB_{ij}dx^i dx^j$.*

$$\begin{aligned}
\pi &= \frac{d}{dp_i} \wedge \frac{d}{dq_i} + eB\frac{d}{dp_1} \wedge \frac{d}{dp_2} + \Omega\frac{d}{dq_1} \wedge \frac{d}{dq_2} \\
\dot{q}_i &= \{H, q_i\} = \dot{q}_i^{B=0,\Omega=0} + \Omega\frac{dH}{dq_j} \\
\dot{p}_i &= \{H, p_i\} = \dot{p}_i^{B=0,\Omega=0} + eB\frac{dH}{dp_j}
\end{aligned}$$

Is this bivector symplectic in the cases of flux quantizations? What about integral periods against H_2, 2π factor on Ω

From Sniatycki Let a particle with charge e be in a external electromagnetic field F. The phase space is $T^*\Sigma$ with $\omega_e = d\lambda_Y + e\pi^* F$. We want this to be quantizable, so we get the integrality condition

$$\begin{aligned} -h^{-1}\omega_e &\in H^2(T^*\Sigma, \mathbb{Z}) \\ \int_\Sigma h^{-1} eF &\in \mathbb{Z} \\ \frac{e\Phi}{h} &\in \mathbb{Z} \end{aligned}$$

Fix the scaling and signs above above to get the below.

Take a semiclassical wavepacket

$$\begin{aligned} \dot{r} &= \frac{1}{\hbar}\nabla_k \epsilon_k - \dot{k} \times \Omega \\ \hbar\dot{k} &= -eE - e\dot{r} \times B \\ \Omega &\equiv -Im[\langle \nabla_k u_k \mid\mid \nabla_k u_k \rangle] \\ \epsilon_k &\equiv \epsilon_k^{B=0} - m_k B \\ m_k &\equiv -\frac{e}{2\hbar} Im[\langle \nabla_k u_k \mid H_k - \epsilon_k^{B=0} \mid \nabla_k u_k \rangle] \end{aligned}$$

9.3.3 Definition (Berry connection/curvature)

$$\begin{aligned} \tilde{A} &\equiv -Im[\langle u_k \mid\mid \nabla_k u_k \rangle] \\ \Omega &\equiv -Im[\langle \nabla_k u_k \mid\mid \nabla_k u_k \rangle] \end{aligned}$$

9.4 Spin Hall

Because we can first consider the noninteracting electrons case, the single body Hamiltonian is

$$\begin{aligned} H &= \frac{p_x^2 + p_y^2}{2m_e} + \alpha_{SO}(p_x \times \sigma) \cdot \nabla V + V \\ V &= \frac{y^2}{2m_e \alpha_{SO}^2} \\ H &= \frac{p_y^2}{2m_e} + \frac{1}{2m_e}(p_x - \frac{\sigma_z}{\alpha_{SO}} y)^2 \end{aligned}$$

This looks like quantum Hall for spin up with magnetic field α^{-1} and for down with $-\alpha^{-1}$. So it is reduced to the previously solved case in order to get BF theory from two Chern Simons theories of opposite levels.

9.5 Abelian Chern Simons

$$\begin{aligned} \sigma_{xy} &= \frac{k}{2\pi} \\ k &= \frac{e^2}{\hbar}\nu \end{aligned}$$

where $J_i = (\sigma_{diag}\delta_{ij} + \sigma_{off}\epsilon_{ij})E_j$
`http://physics.stackexchange.com/questions/92809/topological-ground-state-degeneracy-of-sun`
For compact semisimple Lie Group G, WZW at level k gives a Hilbert Space of dimension

$$dimV_{g,k} = (C(k+h)^r)^{g-1} \sum_{\Lambda_k} \prod_{\alpha\in\Delta} (1 - e^{\frac{i\alpha\cdot(\lambda+\rho)}{k+h}})^{1-g}$$

where C is the order of the center of the group. h is the dual Coxeter number ρ is the half sum of positive roots, r is the rank and Λ_k is the set of integrable highest weights of the Kac-Moody algebra.

If $g = 1$ then this just counts the number of elements in Λ_k for $SU(N)$ this is the counting only partitions that fit inside a N by k box.

See the main chapter for more.

9.6 Haldane-Shastry Spin System

Don't call it a spin chain. You don't think of it as a 1 dimensional quantum system. The particles are on the circle inside the complex plane. Because the interactions are not local in the circle, you can't think of it as a 1+1 dimensional system, it is a specific part of a 2+1 dimensional system.

Take a $SU(2)_1$ theory. Fix $z_1 \cdots z_N$. Compute all the chiral correlators with insertions at those points and those points only. Take those as spanning your Hilbert space. Because this is $SU(2)_1$ the fields are identity and $\phi_{1/2}$, the correlators

$$\begin{aligned} J^0(z) &= \frac{i}{\sqrt{2}}\partial\varphi \\ J^{\pm}(z) &= e^{\pm i\sqrt{2}\varphi} \\ \phi_{1/2,s_i}(z_i) &= \rho_{1/2i} : e^{is_i\varphi/\sqrt{2}} : \\ \rho_{1/2,2i} = 1 & \qquad \rho_{1/2,2i+1} = e^{i\pi(s_i-1)} \\ \psi(s_1, z_1 \cdots s_N z_N) &= \langle \phi_{1/2,s_1}(z_1) \cdots \rangle \\ &= \rho_{1/2} \prod (z_i - z_j)^{s_i s_j/2} \end{aligned}$$

That is the amplitude for that particular configuration of ups and downs as $s_i = \pm$ at that site. The Hamiltonian for this is then cooked up

For this it is useful to impose and define.

$$\begin{aligned} \mid z_i \mid &= 1 \\ w_{ik} &\equiv \frac{z_i + z_k}{z_i - z_k} \\ c_i &\equiv \sum_{k \neq i} w_{ki} \end{aligned}$$

Then the Hamiltonian you get is

$$\begin{aligned} \mathcal{H} &= -\frac{1}{2}\sum_{i\neq j} \frac{z_i z_j}{z_{ij}^2} + \frac{N}{4} - \frac{N+1}{6} T^a T^a - 2\sum_{i \neq j} (\frac{z_i z_j}{z_{ij}^2} + \frac{w_{ij}(c_i - c_j)}{12}) t_i^a t_j^a \\ z_j &= e^{2\pi i j/N} \\ \mathcal{H} &= -\sum_{i\neq j} \frac{z_i z_j}{z_{ij}^2} (2t_i^a t_j^a - 1/2) - E_0 - \frac{N+1}{6} T^a T^a \\ (2t_i^a t_j^a - 1/2) &= P_{ij} - 1 \end{aligned}$$

If the z_i are placed around the circle but not at regular intervals you get a generalization of the Haldane-Shastry Model.

9.6.1 Calagero-Moser Relation

Polychronakos's freezing trick

$$H_{spin} = H_{scalar} \pm 4aH_{HS}(x)$$

The first is the spin trigonometric Calagero-Moser-Sutherland, the second is the scalar one. I'm not sure about the sign. But as you send $a \to \infty$, the scalar has a potential that forces everything into evenly spaced positions, but because it's a $\sin^{-2}$ those are angular positions. This recovers the previous z_i positions imposed above.

http://arxiv.org/pdf/1109.5470v2.pdf

9.6.2 Compare Gaudin Model

$$\begin{aligned} Cas_2 L(\lambda) &= \langle \lambda \mid \lambda + 2\rho \rangle \\ \tilde{\lambda} &= \lambda + \rho \\ Cas_2 L(\lambda) &= \langle \tilde{\lambda} - \rho \mid \tilde{\lambda} + \rho \rangle \end{aligned}$$

9.7 To Insert

9.7.1 Definition (Spin Connection)

9.7.2 Definition (Wen-Zee Term)

$$N = \nu N_\phi + \nu \bar{s} \chi$$

where the second term is much smaller because taking a limit of large N_ϕ

9.7.3 Definition (Viscous Response)

9.7.4 Definition (Thermal Hall Conductance)

$$\begin{aligned} J_i^Q &= \kappa_{ij} \frac{\partial_j T}{T} \\ \kappa &= \kappa ? + \kappa_H ? \\ \kappa_H &= c \frac{\pi^2}{3} T \end{aligned}$$

9.7.1 Hofstadter Butterfly

Chapter 10

Chern-Simons

$$\begin{array}{rcl} \Theta & \in & \Omega^1(P,\mathfrak{g})^G \\ \forall g \in G \;\; Ad_g R_g^* \Theta & = & \Theta \\ \theta & = & i_p^* \Theta \quad \in \Omega^1(X_3,\mathfrak{g})^G \\ \Omega & = & F_\Theta = d\Theta - \big[\Theta,\Theta\big] \\ \pi^*\omega & = & \Omega \end{array}$$

10.0.1 Theorem *$Sym^2(\mathfrak{g})^G \simeq H^4(BG,\mathbb{R})$. In the case of connected, simply connected we get only a one parameter choice $\mathbb{R}\cdot Killing$*

10.1 Abelian Chern Simons

10.1.1 Factorization Algebra

$$\Omega^0 \otimes \mathbb{R}^n \longrightarrow \Omega^1 \otimes \mathbb{R}^n \longrightarrow \Omega^2 \otimes \mathbb{R}^n \longrightarrow \Omega^3 \otimes \mathbb{R}^n$$

is the sheaf of fields.

10.1.2 Belov-Moore

`https://sbseminar.wordpress.com/2010/05/14/lattices-and-their-invariants/`
`https://arxiv.org/pdf/hep-th/0505235v1.pdf`
`https://arxiv.org/pdf/0807.2857.pdf`

10.1.1 Theorem (Rokhlin) *For all compact oriented 3-manifolds without boundary there exists a 4-manifold Z such that $\partial Z = X$. This allows us to extend the principle bundle and a connection extending θ.*

10.1.2 Theorem *Let Λ be a lattice with integral bilinear form B, $\Lambda^* = Hom(\Lambda,\mathbb{Z}) \supseteq \Lambda$.*

10.1.3 Definition (Signature) *Extend scalars of the lattice to $\mathbb{R}$. Then use Sylvester's law of inertia to diagonalize the symmetric bilinear form to give some number of $+1$, -1 and 0. We may then call the signature of a definite form as $\sigma = r_+ - r_-$.*

10.1.4 Definition (Discriminant Group) $D = \Lambda^*/\Lambda$. *It inherits a* $\mathbb{Q}/\mathbb{Z}$ *valued bilinear form. In case of even lattice, this can be refined to a quadratic form as well.*

10.1.5 Theorem *A representation of the modular group of a spin Chern-Simons thoery defined by lattice* Λ *is encoded in the invariants* $\sigma mod 24$*, the discriminant group with it's bilinear form and a quadratic refinement thereof, subject to equivalences when* $q_1(?) = q_2(? - \Delta)$ *and when* q *is shifted by constants. The constant constraint can be implemented by a Gauss-Milgram constraint to normalize.*

10.1.6 Corollary *Two quantum spin Chern-Simons theories with lattices* Λ_1 *and* Λ_2 *define the same invariants on all closed 3 manifolds if and only if the invariants are all matched.*

`http://www.maths.ed.ac.uk/~aar/papers/conslo.pdf`

10.1.7 Definition (Witt Group) *The Witt group for a field* k *is given by taking vector spaces over* k *equipped with symmetric bilinear forms and then imposing equivalence by allowing adding metabolic quadratic spaces like* $k \bigoplus k, q = x^2 - y^2$ *(assume char* $\neq 2$ *then get only sums of these). These equivalence classes are the elements of the Witt group and the multiplication comes from orthogonal direct sum. All elements of finite order have order* $2^?$.

10.1.8 Definition (Witt Ring) *Allow tensor product as well. This is totally different from what is also called the ring of Witt vectors.*

10.1.9 Example $W(k^{alg}) = \mathbb{Z}_2$ *for all algebraically closed fields or just quadratically closed.*
$W(\mathbb{R}) = \mathbb{Z}$
$W(F_{q\equiv 3(4)}) = \mathbb{Z}_4$
$W(F_{q\equiv 1(4)}) = \mathbb{Z}_2[F^*/(F^*)^2]$

10.1.10 Theorem (Nikulin 1.13.3) *Suppose we are told that* D *is the discrimanant group of an even, indefinite lattice of rank* $rkL > \ell(D) + 2$ *where* ℓ *is the minimal number of generators as an abelian group.* L *is determined up to isometry by rank,signature and discriminant form.*

10.1.11 Theorem (Nikulin 1.4.1) *Let* L *be even lattice. There exists a bijection between isotropic subgroups of* D_L *and even overlattices* L_G *of* L*. The discriminant form* D_{L_G} *is given by* q_L *restricted to* $G^\perp/G$*. Unimodular lattices* L_G *correspond to isotropic subgroups* H *with* $|H|^2 = |D_L|$

`http://www2.warwick.ac.uk/fac/sci/maths/people/staff/fbouyer/talks/lattices_and_the_picard_group_presentation.pdf`

10.1.12 Lemma *The indefinite lattice* $II_{p,q}$ *admits a symmetry of order* d *whenever* $p + q = \phi(d)$ *and* d *does not have the form* p^e *or* $2p^e$*. This also means* $p - q \equiv 0$
There exists a definite even unimodular lattice of rank $\phi(d)$ *with a symmetry of order* d *whenever* $\phi(d) \equiv 0$ *modulo* 8 *and* $d \neq p^e$ *or* $2p^e$.

`https://dash.harvard.edu/bitstream/handle/1/3446009/McMullen_AutomorphismUnimodular.pdf?sequence=5`

10.1.13 Proposition (3.1) *Take two nondegenerate symmetric bilinear lattices (M_i, f_i, c_i) where f is the bilinear form and $c \in M^*$ is such that $f(x,x) - c(x) \in 2\mathbb{Z}$. c's are in bijection with fractional Wu classes for f. These get sent to discriminants by $G_{M,f} = M^\sharp/M$ with $\phi_{M,f,c}([x]) = \frac{1}{2}(f_{\mathbb{Q}}(x,x) - c_{\mathbb{Q}}(x))$ as a map to $\mathbb{Q}/\mathbb{Z}$.*
These lattices are stably equivalent if and only if the associated discriminant+quadratic functions $G_{M,f}, \phi_{M,f,c}$ are isomorphic. Also any isomorphism downstairs on the $G_{M,f}, \phi_{M,f,c}$ lifts to a stable equivalence. With degeneracy this is no longer true.

```
https://arxiv.org/pdf/math/0301040.pdf
http://www.cornell.edu/video/jacob-lurie-the-siegel-mass-formula
```

10.1.14 Theorem (Siegel Mass Formula) *Let q be a positive definite quadratic form over $\mathbb{Z}$*

$$mass(q) = \sum_{q' \in g(q)} \frac{1}{\mid O_{q'}(\mathbb{Z}) \mid}$$

If it is unimodular meaning nondegenerate modulo p for all p. This requires $8 \mid n$. In this case

$$\begin{aligned} mass(q) &= \frac{\Gamma(1/2)\Gamma(1)\cdots\Gamma(n/2)}{2^{n-1}\pi^{(n+1)n/4}}\zeta(2)\zeta(4)\cdots\zeta(n-2)\zeta(n/2) \\ &= \frac{\Gamma(1/2)\Gamma(1)\cdots\Gamma(n/2)}{2^{n-1}\pi^{(n+1)n/4}}\prod_p \frac{p^{n*(n-1)/2}}{\mid SO_q(\mathbb{Z}_p) \mid} \end{aligned}$$

10.1.15 Example *Let $n = 8$ and q come from the E_8 lattice.*

$$\frac{1}{2^{14}3^5 5^2 7} = \frac{\Gamma(1/2)\Gamma(1)\cdots\Gamma(4)}{2^{8-1}\pi^{(8+1)*8/4}}\zeta(2)\zeta(4)\cdots\zeta(8-2)\zeta(8/2)$$

10.2 Geometric Quantization of CS

10.2.1 Definition (Atiyah-Bott Moduli space)

$$\omega = \int_\Sigma [\delta A \wedge \delta A]$$

10.2.2 Definition (Prequantum Vector Space) *Modulo gauge transformations to get the moduli space $\mathcal{M}^s(G,\Sigma)$ where have imposed stability to avoid stackiness. Now that we have a finite dimensional symplectic manifold with a line bundle over it, we may take the sections.*

10.2.3 Definition (Segal-Bargmann Space) *Holomorphic functions on $\mathbb{C}^n$ such that $\phi(z)e^{-1/(2\hbar)|z|^2}$ is in L^2. Maybe $\hbar = 1$ in some definitions, but that just rescales the $\mathbb{C}^n$. There are annihalation and creation $a_i = \partial_{z_i}$ and $a_j^\dagger = z_j\cdot$ which by Stone-Von-Neumann have a unitary Segal-Bargmann transform to the usual CCR on $L^2(\mathbb{R}^n)$.*

10.2.4 Definition (Complex Polarization) *Pick a point in the Teichmuller space $J \in T$. We may use this complex structure to polarize.*

10.2.5 Theorem (Hitchin-Witten Connection) *This follows for all τ. In fact, we may globalize to build a bundle over T. This bundle has connection given by Hitchin-Witten. In fact it is given explicitly by:*

Proof `AndersenandGamelgaard` □

10.2.6 Lemma (Mapping Class Group Action)

10.2.7 Definition (Gelfand-Zak Transform)

10.3 Perturbative SU(N) - Chern Simons

Let k be large and do expansion in $\frac{1}{k}$. In geometric quantization terms, this can be done by replacing the line bundle for k with the line bundle L^n to go to level nk. Have for large n so have for large level.

Chapter 11

3d Topological

11.1 Fusion Category Things 2

11.1.1 Definition (Braided)

11.1.2 Definition (Symmetric)

11.1.3 Definition (Modular) *For the braiding between simple objects i and j, construct:*

$$S_{i,j} = \operatorname{tr} R_{i,j} R_{j,i}$$

The matrix built from this should be nondegenerate.

11.1.4 Definition (Center)

11.1.5 Theorem *Consider the theory on $\Sigma \times S^1$. Now $S^1 \times S^1$ gets a assigned a vector space. The pair of pants crossed with the S^1 gives a product on this.*

Now that not in a CFT setting, also talk about the more general fusion case without the more physical bells and whistles.

11.1.6 Example (Haagerup) *I don't know of an excuse for this to show up physically. Much like the zoo of finite groups, it goes in a different direction than nature does.*

11.2 3D manifold facts

11.2.1 Definition (Heegard Splitting) *Cut into solid handlebodies of some genus such that they share the boundary Σ_g glued to itself by some $MCG(\Sigma_g)$ twist. The minimal such g is called the Heegard genus of the 3-manifold.*

11.2.2 Theorem (Half-lives and half-dies)

11.2.3 Definition (Skein algebra/module) *The skein algebra is $\Sigma \times I$ with stuff inside. The multiplication operation is stacking along the interval direction.*
The skein module is given by the solid handlebody for Σ. The skein algebra acts by stacking.

11.3 Dijkgraaf Witten

Let G be a finite group, $\alpha \in Z^3(G, U(1))$

$$\begin{aligned}
\mathbb{C}[G] &\simeq \bigoplus V_i^* \otimes V_i \\
\chi_i &\equiv \chi(V_i) = (g \to \operatorname{tr} \rho_{V_i} g) \\
\delta_e(-) &\in \mathbb{C}[G]^* = \sum_i \frac{\dim V_i}{\mid G \mid} \chi_i(-)
\end{aligned}$$

$Hom(\pi_1(X), V)$ is in bijection with G labelings of X. That is each triangle goes to a (g_1, g_2, g_3) subject to $g_1 g_2 g_3 = e$ following the orientation. $G^{|V|}$ acts by multiplying all the labels incident on each vertex $v \in V$.

11.3.1 Closed 3-manifold

The space of fields on X a closed connected oriented 3-manifold, F_X is the finite set of G-bundles over X up to equivalence. It can be described as $\frac{\pi_1(X) \to G}{G}$

$$\begin{aligned}
e^{2\pi i S_x(\phi)} &= e^{2\pi i \langle \bar{F}^* \alpha || [X] \rangle} \\
Z(X) &= \sum \frac{1}{Aut(\phi)} e^{2\pi i S_X(\phi)}
\end{aligned}$$

Can rewrite the space of fields as the framed at x version and then quotienting out this framing. Makes a groupoid. $(\phi, \psi) \to \phi$. Or with multiple points with framings.

$$\begin{aligned}
Z(X) &= \frac{1}{\mid G \mid} \int_{F_{X,x}} e^{2\pi i S_X(\phi, \psi)} \\
Z(X) &= \frac{1}{\mid G \mid^n} \int_{F_{X, x_1 \cdots x_n}} e^{2\pi i S_X(\phi, \psi_1 \cdots \psi_n)}
\end{aligned}$$

11.3.1 Example (S^3) $Z(X) = \frac{1}{|G|}$

11.3.2 Example ($S^2 \times S^1$) $Z(X) = 1$

11.3.2 With Boundary

Replace $[X]$ with relative cohomology.

- Make a choice via triangulations - Dijkgraaf-Witten, ?

- All choices - Freed

Fix triangulation T for X. Then pick a total ordering of the 0-skeleton X^0. So make F_{X,X^0} which is equivalent to $Hom(\pi_{\leq 1}(X, X^0), G)$

11.3.3 Definition ($Co(X,T)$) *A color Q of X, T is a map sending oriented edges to G. By doing the open path holonomy which is OK because we trivialized each of the X^0. This satisfies reversing orientation gives inverse, and 2-simplices satisfy the product.*

Fix $\tau \in Co(\partial X, T_\partial)$, then make $Co(X, T, \tau)$ so that the bundle agrees with the trivialization at the boundary.

Tetrahedron

Insert image of tetrahedron and backbone

11.3.4 Definition (Backbone)

This gives map $\sigma \to BG$ giving the 3-cell in BG given by $\phi = [g \mid h \mid k]$

$$\begin{aligned} w(\sigma,\phi) &= e^{2\pi i \tilde{\alpha}\cdot\phi} \\ Z_X(\tau) &= \frac{1}{\mid G \mid^{X^0-(\partial X)^0}} \sum_{\phi \in Co(X,T,\tau)} \prod_{\sigma} w(\sigma,\phi)^{\pm 1} \end{aligned}$$

11.3.5 Theorem (DW,Walker) *Independent of triangulation, orderings if we fix the boundary triangulation*

11.3.6 Lemma *The Verlinde algebra for this gives*

11.3.3 TQFT

11.3.7 Lemma *The linear map for cobordism*

$$\begin{aligned} \Phi_X(\tau) &= \sum \Phi_X(\tau' \bigsqcup \tau)\tau' \\ \Phi_{X_{12}} &= \Phi_{X_2}\Phi_{X_1} \\ \Phi_{Y\times I} &= \Phi_{Y\times I}\Phi_{Y\times I} \end{aligned}$$

11.4 Turaev-Viro

11.4.1 Kitaev Double

11.4.1 Theorem (Kitaev Kong)

11.4.2 Levin-Wen

11.4.3 Turaev-Viro

11.4.4 Schommer-Pries-Douglas-Snyder

11.5 Reshetikhin Turaev

11.5.1 Definition *(k,l) ribbon graph is an embedding of a disjoint union of $[0,1]^2$ ribbons into $\mathbb{R}^2 \times [0,1]$ such that the intersections with the top and bottom meed at the intervals specified k and l respectively.*

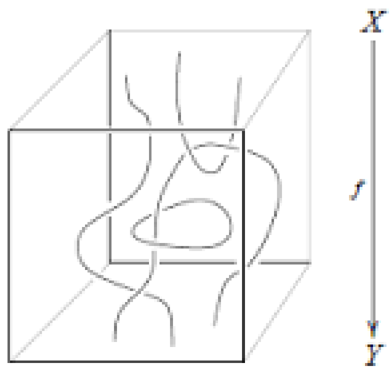


Figure 11.1: A morphism in Tangle, find one with thickened lines

11.5.2 Definition (Homogenous) *It can be isotoped so that consistent forward orientation. Namely no half twists but full twists are allowed.*

11.5.3 Definition (Colored) *A labeling on the ribbon graph by objects of Rep(A) for all the connected components.*

11.5.4 Definition (Target Category) *Objects are $(V_1, \pm_1) \times \cdots (V_k, \pm_k)$. These provide the labels on the $\mathbb{R}^2 \times \{0\}$. The morphisms are then given by (k,l) homogenous colored ribbon graphs such that the sources and target colors and orientations match. These are given up to isotopy. Composition is given by vertical concatenation.*

11.5.5 Definition (Coupon) *Can also stick in coupons decorated by morphisms.*

Figure 11.2: A coupon with ? inputs and ? outputs

11.5.6 Theorem *This is a rigid braided monoidal category.*

Proof Show the pictures for braided and rigid.

11.5.7 Definition (Ribbon Hopf Algebra) *So that RepA can be used requires the following:*

Figure 11.3: Braiding

Figure 11.4: Rigidity

- Δ *giving the monoidal structure.*
- ϵ *giving* $A \to k$
- S *provides* $M^\vee$ *from an object* M*.*
- $R \in A \otimes A$ *that provides quasi-triangularity.*
- θ *with (insert picture of ribbon and braiding interaction)*

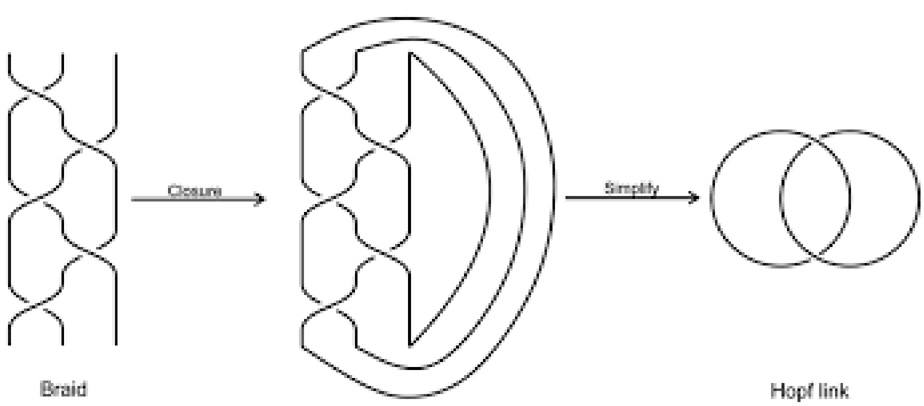
Figure 11.5: Ribbon and Braiding interaction

11.5.8 Theorem *Functor from tangle category to* $\mathcal{C} \equiv (RepA)^{ss}$ *satisfying colors go correctly, preserves tensor products exactly, caps cups go to evaluations and coevaluations. Essentially all the data you built in to build the tangle category matches up.*

Proof Insert more pictures here. □

11.5.9 Theorem (Surgery)

Proof The Kirby moves are . Therefore check the following ...



[scale=.25]

Figure 11.6: Hopf Link: S-matrix

Chapter 12

Poisson-Lie

12.1 Lax Matrices

Let $\mathcal{M}$ be a Poisson manifold.

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "L"] & \mathcal{M} \arrow[d, "L"] \\ \mathfrak{g} \arrow[r, "F'"] & \mathfrak{g} \end{tikzcd}$$

where the top arrow is the flow for time t given by the Hamiltonian and the lower arrow is a local flow on $\mathfrak{g}$ which is tangent on adjoint orbits. Because $L(x) \in \mathfrak{g}$, we can regard L as a matrix with coefficients in $C^\infty(\mathcal{M})$ an element of $\mathfrak{g} \otimes C^\infty(\mathcal{M})$. The isospectrality means that there is another matrix $M \in \mathfrak{g} \otimes C^\infty(\mathcal{M})$ such that the flow downstairs is given by ad_M

The isospectrality property ensures that the following gives more integrals of motion.

$$\begin{aligned} L_V = (\rho \otimes id)L &\in End\ V \otimes C^\infty(\mathcal{M}) \\ det(L_V - \lambda) &= \sum H_i \lambda^i \end{aligned}$$

The H_i are integrals of motion.

12.1.1 Example (KdV)

$$\begin{aligned} L(u, u_x) &= -6\frac{d^2}{dx^2} - u \\ M(u, u_x) &= -4\frac{d^3}{dx^3} - u\frac{d}{dx} - \frac{1}{2}u_x \end{aligned}$$

The Poisson manifold is $T^*Maps_{Schwarz}(\mathbb{R}, \mathbb{R})$ where the base coordinate is u and the fiber is proportional to u_x.

The Poisson bracket is

$$\{f, g\} =$$

Hamiltonian:

$$H_2[u] = \int_{\mathbb{R}} dx(\frac{1}{2}u_x^2 + u^3)$$

12.1.2 Theorem (Kostant bracket) *Let $\mathfrak{g}^*$ be the dual to a Lie algebra. Polynomial functions on this are given by the symmetric algebra of $\mathfrak{g}$. This has a Poisson bracket given by the Lie bracket on linear functions and extending all the way up.*

The center of $S(\mathfrak{g})$ as a Poisson algebra coincides with the subalgebra of ad-invariants also known as the Casimirs.

If you try moving with these Hamiltonian flows, you find no motion whatsoever. $D_C\phi = \{C, \phi\} = 0$. We need a different Poisson structures so these become interesting conserved quantities instead of just constants along each coadjoint orbit.

12.1.3 Definition (Classical r-matrix) *Let r be an endomorphism of the Lie-algebra, we can define a new Lie bracket by*

$$[X, Y]_r = \frac{1}{2}([rX, Y] + [X, rY])$$

If r satisfies the modified classical Yang Baxter Equation

$$[rX, rY] - r[rX, Y] - r[X, rY] + [X, Y] = 0$$

then the bracket defined will satisfy Jacobi.

$$r_\pm = \frac{1}{2}(r \pm id)$$

are Lie algebra homomorphisms $\mathfrak{g}_\mathfrak{r} \to \mathfrak{g}$. They can be extended to maps of Poisson algebras on the associated Symmetric algebras $Sym(\mathfrak{g}_\mathfrak{r}) \to Sym(\mathfrak{g})$ so a map of Poisson manifolds $\mathfrak{g}^* \to \mathfrak{g}_r^*$

$$\begin{aligned} S(\mathfrak{g}_r) \otimes S(\mathfrak{g}) &\to S(\mathfrak{g}) \\ x \cdot y &= \sum r_+(x_i^1) y r_-(S(x_i^2)) \end{aligned}$$

12.1.4 Theorem *Let i_r be the isomorphism of graded linear spaces $S(\mathfrak{g}_r) \to S(\mathfrak{g})$ given by sending $P \in S(\mathfrak{g}_r)$ to $P \cdot 1$. i_r^{-1} restricted to the Poisson center before (the Casimirs) is a Poisson map so we get a new Poisson commuting algebra.*

12.1.5 Remark It can be much smaller for example if $r = \pm id$ then the new Poisson commuting algebra you get is 0. ♢

A common source of classical r-matrices is decompositions of Lie algebras $\mathfrak{g} = \mathfrak{g}_+ \bigoplus \mathfrak{g}_-$ as a **linear space**. Then $r = P_+ - P_-$ will be a solution. In this case the bracket is

$$[X, Y]_r = [X_+, Y_+] - [X_-, Y_-]$$

Then $\mathfrak{g}_r$ splits as a Lie algebra $\mathfrak{g}_r = \mathfrak{g}_+ \bigoplus \mathfrak{g}_-$

$$\begin{aligned}
\mathfrak{g} &= g_+^\perp \oplus g_-^\perp \\
g_+^* &\simeq g_-^\perp \\
g_-^* &\simeq g_+^\perp \\
L &= \sum e_i \otimes e^i \in g_+^* \otimes g_+ \\
L &\in g_-^\perp \otimes g_+ \subset g_-^\perp \otimes S(g_+)
\end{aligned}$$

Wait it says projecting to $S(g_-)$ but before said $S(g_+)$ were supposed to be the observables.

12.1.6 Definition (Manin triple) *Let $\mathfrak{g}$ be a Lie algebra and let the subspaces in the decomposition $\mathfrak{g}_\pm$ be isotropic with respect to the invariant inner product so that $\mathfrak{g}_\pm^\perp = \mathfrak{g}_\pm$.*

12.1.7 Definition (Lie bialgebra) *Let $\mathfrak{a}$ be a Lie algebra with dual $\mathfrak{a}^*$. Give $\mathfrak{a}^*$ a Lie bracket as well. You can also view this as a map $\mathfrak{a} \to \mathfrak{a} \wedge \mathfrak{a}$. Compatibility will require this to be a 1-cocycle with values in $\wedge^2 Adjoint$.*

12.1.8 Proposition (Double of a Lie bialgebra) *Given a Manin triple, identify $\mathfrak{g}$ and $\mathfrak{g}^*$ with an inner product. Give the dual the r-bracket $P_+ - P_-$. This is a Lie bialgebra. $(\mathfrak{g}_+, \mathfrak{g}_-)$ is a Lie sub-bialgebra. In addition from a Lie bialgebra, you can construct a unique Lie algebra called the double which is $\mathfrak{g} \oplus \mathfrak{g}^*$ as a vector space, has the evaluation pairing and the bracket is given by the brackets on the subalgebras.*

$$\begin{aligned}
\left[e^i, e^j\right] &= f_k^{ij} e^k \\
\left[e_i, e_j\right] &= C_{ij}^k e_k \\
\left[e^i, e_j\right] &= C_{jk}^i e^k - f_j^{ik} e_k
\end{aligned}$$

The bracket within $\mathfrak{g}$ and within $\mathfrak{g}^*$ are left as they were.

$$\begin{aligned} \delta e^i &= C^i_{jk} e^j \wedge e^k \\ \delta e_i &= f^{jk}_i e_j \wedge e_k \end{aligned}$$

12.1.9 Definition (Factorizable Lie Bialgebra) *Let r be a classical r-matrix on $\mathfrak{g}$ which has fixed inner product. Also assume that r is skew and satisfies MCYBE. Then $(\mathfrak{g}, \mathfrak{g}^*)$ is a factorizable Lie bialgebra.*

12.1.10 Lemma *We view $T^*\mathfrak{t}$ as the Lie algebra given by the double construction for the Lie bialgebra where the structure constants C^i_{jk} give the bracket as is and the dual bracket $f^{ik}_j = 0$.*

Red for algebra arrows, Yellow for coalgebra arrows

$$\begin{tikzcd} T^*\mathfrak{k} \arrow[r] & \mathfrak{k} \ltimes \mathfrak{k}^* \\ T\mathfrak{k}^* \arrow[r] & \mathfrak{k} \rtimes \mathfrak{k}^* \\ \mathfrak{k} \otimes \mathbb{C} \arrow[r] & \mathfrak{sl}(N, \mathbb{C}) \\ \mathfrak{d}(\mathfrak{k}) \arrow[r] \arrow[dr] & \mathfrak{k} \bowtie \mathfrak{an} \\ & \mathfrak{k} \oplus \mathfrak{an}^{op} \end{tikzcd}$$

12.1.11 Lemma

$$\begin{aligned} \mathfrak{d}(\mathfrak{g}) &\simeq \mathfrak{g} \bowtie \mathfrak{g}^* \\ &\simeq \mathfrak{g} \oplus \mathfrak{g}^{*,op} \end{aligned}$$

where the first line is as algebras and the second line is a Lie coalgebra isomorphism. This gives a natural Lie bialgebra structure on the double for which $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie bialgebra embeddings.

We are free to scale the Lie brackets and cobrackets. Let t scale r and s scale the Lie bracket. Because the cobracket uses the bracket as well as r it will scale with st.

$$\begin{aligned}
[X+\mu, Y+\nu] &= (s[X,Y] + str^\sharp(ad^*_X\nu) - str^\sharp(ad^*_Y\mu) - stad_X r^\sharp\nu + stad_Y r^\sharp\mu, \\
&\quad st[\mu,\nu]_* + s * ad^*_X\nu - s * ad^*_Y\mu) \\
\mathfrak{d}_{t=0}(\mathfrak{g}) &= \mathfrak{g} \ltimes \mathfrak{g}^* \\
\mathfrak{d}_{s=0}(\mathfrak{g}) &= \mathfrak{g}_{ab} \times \mathfrak{g}^* \\
\mathfrak{d}_t(\mathfrak{g}) &= \mathfrak{g} \bowtie \mathfrak{g}^* \\
D_{t=0}(\mathfrak{g}) &= G \ltimes \mathfrak{g}^*
\end{aligned}$$

$\mathfrak{g}^*_t$ is part of the Manin triple $(\mathfrak{d}_t, \mathfrak{g}, \mathfrak{g}^*_t)$

$$\begin{aligned}
s_t\ \mathfrak{g}^* &\to \mathfrak{g}^*_t : \quad A \to t * A \\
\mathfrak{d} &\to \mathfrak{d}_t : \quad X + \mu \to X + t * \mu \\
j_t\ \mathfrak{g}^* &\to \mathfrak{d} : \ \mu \to tr^\sharp(\mu) + \mu \\
s_t j_t\ \mu &\to tr^\sharp(\mu) + t\mu = j_1 s_t(\mu) \\
\sigma_t &= t^{-1} s^*_t \sigma
\end{aligned}$$

$$\begin{aligned}
\mathfrak{g}^* \times G &\to D_{t=0} \\
\mu, g &\to g, Ad^*_g\mu
\end{aligned}$$

$$\begin{tikzcd}
T\mathfrak{g}^* \arrow[rr, "\sigma^\flat"] \arrow[dr] & & T^*\mathfrak{g}^* \arrow[ll, bend right, "\pi^\sharp"'] \arrow[dl] \\
& \mathfrak{g}^* &
\end{tikzcd}$$

$$\begin{aligned}
(\pi^\sigma)^\sharp &= \pi^\sharp(Id + \sigma^\flat\pi^\sharp)^{-1} \\
&= \pi^\sharp(Id - \sigma^\flat\pi^\sharp + (\sigma^\flat\pi^\sharp)^2 + \cdots)
\end{aligned}$$

12.1.12 Proposition (Moser's trick) *$d\sigma_t/dt = -da_t$ for a 1-form, we can make a one parameter family of vector fields $-\pi^\sharp(a_t)$. They are all tangent to symplectic leaves. So we can let them flow and because the leaves are compact there is no problem of incompleteness. So now we have a new Poisson structure for each t given by pushforward along the flow's diffeomorphism. It also has the same leaves.*

The standard r-matrix on a simple Lie algebra

$$
\begin{aligned}
r &= \frac{1}{2}H_i \otimes H_j (A^{-1})_{ij} + \sum_{\Delta^+} X_\alpha \otimes Y_\alpha \\
2s = r + \sigma(r) &= \frac{1}{2}H_i \otimes H_j (A^{-1})_{ij} + \sum_{\Delta^+} X_\alpha \otimes Y_\alpha + Y_\alpha \otimes X_\alpha \\
s &= \frac{1}{2}H_i \otimes H_j (A^{-1})_{ij} + \frac{1}{2}X_\alpha \otimes Y_\alpha + \frac{1}{2}Y_\alpha \otimes X_\alpha \\
\mathfrak{su}(N) &= iH_\alpha, X_\alpha - Y_\alpha, i(X_\alpha + Y_\alpha) \\
\mathfrak{su}(N)^* &= -iH_\alpha^\vee, \frac{1}{2}(X_\alpha^\vee - Y_\alpha^\vee), \frac{-i}{2}(X_\alpha^\vee + Y_\alpha^\vee)
\end{aligned}
$$

Let H_α be the basis of $\mathfrak{h}$ which is an eigenbasis for A^{-1}. Because the eigenvalues are all positive (not a general Kac-Moody algebra) you can rescale with real parameters $\sqrt{\lambda}$ such that the first term of s was $\frac{1}{4}H_\alpha \otimes H_\alpha$

So $\mathfrak{sl}(N)$ is quasitriangular by above, but the real forms are not because can't write $X \otimes Y$ with $X + Y$ and $i(X - Y)$ and only real coefficients. But we can do the trick with the complex version.

Back to complex version momentarily. Make it a braided version. Corollary 3.2 Majid

$$
\begin{aligned}
\delta^{new}(x) &= \frac{1}{2}H_\alpha \otimes [x, H_\alpha] + X_\alpha \otimes [x, Y_\alpha] + Y_\alpha \otimes [x, X_\alpha] \\
\delta^{new}(H_\beta) &= X_\beta \otimes ?Y_\beta + Y_\beta \otimes ?X_\beta \\
\delta^{new}(X_\beta) &= \frac{1}{2}H_\beta \otimes ?X_\beta + X_\beta \otimes ?H_\beta \\
\delta^{new}(Y_\beta) &= \frac{1}{2}H_\beta \otimes ?Y_\beta + Y_\beta \otimes ?H_\beta
\end{aligned}
$$

This is the same cobracket, so we haven't changed at all in making it a braided version. Oh this is example 3.3. The factorizable ones always give KKS back.

What about transmuting along

We have the arrow of lie bialgebras over the reals

$$
\mathfrak{su}(N), [], i\delta \quad \to \quad \mathfrak{sl}(N), [], i\delta_{\mathbb{R}}
$$

by inclusion so we have the dual map

$$
\mathfrak{sl}(N)^* \quad \to \quad \mathfrak{su}(N)^*
$$

of Lie bialgebras so we are almost in the setting of transmutation after using a self duality.

$$
\begin{aligned}
p : \mathfrak{sl}(N), [], i\delta = d_{CE}(i * r) &\rightarrow \mathfrak{su}(N)^* \\
ir &= \frac{-i}{4} iH_\alpha \otimes_{\mathbb{C}} iH_\alpha + iX_\alpha \otimes_{\mathbb{C}} Y_\alpha \\
ir_{\mathbb{R}} &= \\
p(X_\alpha) &= \\
p(iX_\alpha) &= \\
p(Y_\alpha) &= \\
p(iY_\alpha) &= \\
p(H_\alpha) &= \\
p(H_\alpha) &=
\end{aligned}
$$

The lie algebra $\mathfrak{g}^*$ is unaffected but the cobracket changes. $\mathfrak{g}^* \rightarrow \mathfrak{g}^* \otimes \mathfrak{g}^*$ which changes the Lie algebra $\mathfrak{g}$ which changes what you mean by KKS Poisson structure. It will still be linear and will have different symplectic leaves.

$$
\begin{aligned}
\delta^{new}(x) &= \delta^{old}(x) + [p(r^1), x] \otimes p(r^2) - p(r^2) \otimes [p(r^1), x] \\
\delta^{new}(-iH_\alpha^\vee) &= \\
\delta^{new}(\frac{1}{2}(X_\alpha^\vee - Y_\alpha^\vee)) &= \\
\delta^{new}(\frac{-i}{2}(X_\alpha^\vee + Y_\alpha^\vee)) &=
\end{aligned}
$$

12.2 Hamiltonian reduction

Choose a trivialization of $T^*G \simeq G \times \mathfrak{g}^*$ by left translations. Let B be a Lie subgroup B acting by right translations (by b^{-1} so it is a left action) extends to the cotangent bundle. In the trivialization this is given by

$$
\begin{aligned}
R_{b^{-1}}\ (g, \xi) &\rightarrow (gb^{-1}, (Ad^* b)^{-1}\xi) \\
L_b\ (g, \xi) &\rightarrow (bg, \xi)
\end{aligned}
$$

The moment map for this is $(g, \xi) \rightarrow \xi \mid_{\mathfrak{b}}$. So given a point F in $\mathfrak{b}^*$ we can do the symplectic reduction for it.

12.2.1 Theorem (Kazhdan, Kostant, Sternberg) *There is a symplectic induction functor $S_B \rightarrow S_G$ from the category of Hamiltonian B-spaces to Hamiltonian G-spaces. Up to coverings these are coadjoint orbits. The functor is given by $M \rightarrow (T^*G \times M)//B$ where the difference symplectic structure is used on the two factors and B acts diagonally in the middle of the factors so that the moment map is the difference of moment maps. There is a leftover left action by G from the action of G on T^*G by left multiplication and extended to cotangent.*

12.3

Do the example where $B = U(N)$ and $G = U(N+M)$. You start with the coadjoint orbit of ρ.

For example, do $B = G$. Then we get

$$\begin{tikzcd}
m^{-1}(0) \arrow[r] \arrow[d] \arrow[dr, hook] & m^{-1}(0)/G \arrow[dr, hook] & \\
0 \arrow[dr, hook] & T^*G \arrow[r] \arrow[d] & (T^*G)/G \\
& \mathfrak{g}^* &
\end{tikzcd}$$

Apply symplectic induction $S_G \to S_G$ on a point.

12.4 Standard r-matrix

For a **complex simple** Lie algebra, we can write a standard

$$\begin{aligned}
\mathfrak{d} &= \mathfrak{g} \oplus \mathfrak{g} \\
\mathfrak{g} &= \Delta(\mathfrak{g}) \subset \mathfrak{g} \oplus \mathfrak{g} \\
\mathfrak{g}^* &= \{(X_+, X_-) \in \mathfrak{b}_+ \oplus \mathfrak{b}_- \mid \pi(X_+) = -\pi(X_-)\} \\
r &= \sum_{\alpha \in \Delta_+} e_\alpha \wedge e_{-\alpha}
\end{aligned}$$

This extends to loop algebras as well and gives the Trigonometric r-matrix. See page 16 of Semenov-Tian-Shansky for loops in a complex semisimple algebra.

After the dust settles you get an r-matrix, that you can write in terms of the r-matrix for the corresponding finite dimensional Lie algebra.

$$\begin{aligned}
r_{\text{trig}}(x) &= \text{Casimir}\frac{x^{1/2} + x^{-1/2}}{x^{1/2} - x^{-1/2}} + r_{\text{finite}} \\
x &\neq 1 \\
x &= \exp y \quad y \neq 0
\end{aligned}$$

12.5 Functions on the group instead of on a coadjoint orbit

This is changing from linear to quadratic case.

12.5.1 Definition (Poisson Lie Group) *A Lie group with multiplicative Poisson bivector P.*

12.5.2 Definition (Multiplicative Tensor) *For each point in G we have a Q_g which is an element of $\otimes^k T_g G$. That is they are sections of the tensor bundle of the tangent bundle. $\lambda(g)$ left multiplication by G induces a map $\otimes^k T_h G \to \otimes^k T_{gh} G$ and same with right multiplication.*

$$Q_{gh} = \lambda(g)Q_h + Q_g\rho(g)$$

12.5.3 Theorem *If you apply a right trivialization by taking any tensor on G to an element of $\otimes^k \mathfrak{g}$ by bringing it back to the identity via right multiplication with g^{-1}, you get*

$$\begin{aligned}\rho(Q)(gh) &= Q_{gh}h^{-1}g^{-1} = (gQ_h + Q_g h)h^{-1}g^{-1} \\ &= g\rho(Q)(h)g^{-1} + \rho(Q)(g)\end{aligned}$$

That is it is a 1-cocycle on G with values in the kth tensor power of $\mathfrak{g}$ as a power of the adjoint representation of G.

Also do a left trivialization $\lambda(Q)$

$$\begin{aligned}\lambda(Q)(gh) &= h^{-1}g^{-1}Q_{gh} = h^{-1}g^{-1}(gQ_h + Q_g h) \\ &= h^{-1}Q_h + h^{-1}g^{-1}Q_g h = \lambda(Q)(h) + h^{-1}\lambda(Q)(g)h\end{aligned}$$

It is a 1-cocycle on G^{op} with values in the kth tensor power of $\mathfrak{g}$ as a power of adjoint representation of G^{op} so G acts on the right.

12.5.4 Example *Let $q \in \otimes^k \mathfrak{g}$. We can build a multiplicative tensor by $Q_g = g.q - q.g$*

$$\begin{aligned}Q_{gh} &= (gh).q - q.(gh) = g.h.q + g.q.h - g.q.h - q.gh = g.(Q_h) + (Q_g).h \\ \rho(Q)(g) &= Q_g g^{-1} = (g.q - q.g).g^{-1} = g.q.g^{-1} - q\end{aligned}$$

Take differential of $\rho(Q)$ to get a Lie algebra co-cycle instead $\mathfrak{g} \to \otimes^k \mathfrak{g}$ It is $DQ(x) = ad_x q$ which is δq the coboundary of just q a 0-cocycle $\mathbb{R} \to \otimes^k \mathfrak{g}$

Take an $r \in \Lambda^2\mathfrak{g} = Sym^2(\mathfrak{g}[1])$, It turns into a multiplicative tensor $g.r - r.g$, it is a multiplicative bivector but for it to define a Poisson bivector we check Jacobi which means that $T_e(\rho(P))$ as a 1-cocycle $\mathfrak{g} \to \Lambda^2\mathfrak{g}$ needs to be a Lie bracket on $\mathfrak{g}^*$ which amounts to the equation $[[r,r]]$ is Ad invariant.

If r is quasi-triangular with ad-invariant symmetric part satisfying these first conditions listed.

$$\begin{aligned}s &\in \Lambda^2(\mathfrak{g})^{\mathfrak{g}} \\ g.s - s.g &= 0 \\ \langle a,a\rangle + \langle s,s\rangle &= 0 \\ P(g) &= g.r - r.g = g.s - s.g + g.a - a.g = g.a - a.g \\ -\frac{1}{2}[[a,a]] &= \langle a,a\rangle = -\langle s,s\rangle \in (\Lambda^3\mathfrak{g})^{\mathfrak{g}}\end{aligned}$$

12.5.5 Definition (Quasi-Triangular and Factorizable) *Quasi-Traingular Comes from this above. Factorizable if in addition the map induced by s is invertible.*

12.5.6 Definition (Sklyanin bracket)

This ensures that we see the

$$\begin{tikzcd} G^N \arrow[r, "F"] \arrow[d, "M"] & G^N \arrow[d, "M"] \\ G \arrow[r, "F'"] & G \end{tikzcd}$$

as an upgrade of the Lax formalism in the Lie algebra case.

Our subalgebra of integrals of motion are

12.5.1 Hamiltonian reduction

Non-abelian moment map Reyman 96 and Semenov-Tian-Shansky 94

12.6 Quasi-Hamiltonian

12.6.1 Definition (q-Hamiltonian G-Space) *Let G be a Lie group and K be an isomorphism $\mathfrak{g} \to \mathfrak{g}^*$. Use the same letter for the inner product $K(-_1, -_2) = K(-_1) \circ (-_2)$. We can find a group valued moment map $\mu M \to G$ by*

$$\begin{aligned} \xi &\in \mathfrak{g} \\ \omega &\in \Omega^2(M) \\ \theta_{L,R} &\in \Omega^1(G, \mathfrak{g}) \\ i(v_\xi)\omega &= \frac{1}{2}\mu^*(K(\theta_L + \theta_R)(\xi)) \end{aligned}$$

If you want ω to be G-invariant, you have to give up the closed-ness. You are lead to $d_{dR}\omega = -\mu^* \frac{1}{12} K(\theta_{L,R}, [\theta_{L,R}, \theta_{L,R}])$ which is familiar from Chern-Simons. So it is not non-degenerate it's kernel is

$$\ker \omega_x = \{v_\xi \mid \xi \in \ker(Ad_{\mu(x)} + 1)\}$$

Let u be a parameter of homological degree 1. Now instead give the map $uK\ \mathfrak{g} \to \mathfrak{g}^*[1]$

12.6.2 Definition (Cayley Map) *Take a skew Hermitian A and send it to $(I - A)(I + A)^{-1}$ to produce a unitary matrix. Converseley a unitary without a eigenvalue -1 can be transformed back to a skew Hermitian matrix.*

12.6.3 Example • *Conjugacy class in G.*

- *$D(G_2) = G_2 \times G_2$ as a $G = G_2 \times G_2$ space.*
- *q-Hamiltonian reduction for $G_1 \times G_2$ spaces to get a G_2 space*
- *Fusion product to build a $G \times H_1 \times H_2$ space from a $G \times H_1$ and a $G \times H_2$ space*

12.6.4 Theorem *Let us denote a quasi-Hamiltonian G-space M equipped with G action A, symplectic form ω and moment map μ*

Start with a usual Hamiltonian G-space (M, A, σ, Φ)

$$\begin{aligned} M^{new} &= M \\ A^{new} &= A \\ exp_s(-) &= exp(s * -) \\ \omega &= \sigma + \Phi^* Kill^*(\frac{1}{2}\int_0^1 (exp_s^*\bar{\theta}, \frac{d}{ds} exp_s^*\bar{\theta}) ds) \\ \mu &= exp(\Phi) \end{aligned}$$

If $d_{Kill(\xi)}exp$ is a bijection for $\xi \in \Phi(M)$, then this is q-Hamiltonian G-space.

12.6.5 Example *Let M be a coadjoint orbit.*

12.6.6 Definition (Poisson-Lie G space) *M manifold. A is an action of G. ω is a symplectic form. It is not invariant under the action. μ is a moment map but to the dual group G^*.*

$$i(v_\xi^\sharp)\omega = 2\mu^* \langle \bar{\theta}_G \mid \xi \rangle$$

12.6.7 Definition (q-Hamiltonian G space with P valued moment map) *Manifold, action 2 form ω_P and moment map but now P valued.*

$$\begin{aligned} d\omega_P &= -\mu_P^* \chi_P \\ i(v_\xi)\omega_P &= \frac{1}{2i}\mu_P^*(p^*\theta + p^*\bar{\theta}, \xi) \end{aligned}$$

12.6.8 Theorem *Let $j(b) = bb^\dagger$ be the map that intertwines the left multiplication by G to the adjoint action on P, dressing action on G^* . κ is the G^* embedding. I $G_{\mathbb{C}} \to G_{\mathbb{C}}$ by Hermitian conjugation.*

$$
\begin{aligned}
\mu_P &= j\mu \ \ M \to G^* \to P \\
\omega_P &= \omega + \frac{1}{2}\mu^* \kappa^* Im(I^* \bar{\theta}, \theta)
\end{aligned}
$$

This turns a Poisson-Lie G-space to a q-Hamiltonian G-space with P valued moment map.

12.6.9 Theorem (q Hamiltonian P moment map to usual G space) *There exists a canonical 2-form on P τ such that*

$$
\begin{aligned}
\mu &= \log \mu_P \ \ M \to G^* \to \mathfrak{g}^* = \sqrt{-1}\mathfrak{g} \\
\omega &= \omega_P + \mu_P^* \tau
\end{aligned}
$$

12.6.10 Theorem *q-Hamiltonian G space to Hamiltonian LG space with proper moment map*

$$
\begin{tikzcd}
q-Hamiltonian \arrow[r, "12.6.10"] \arrow[d, shift left, "12.6.4"] & Hamiltonian\ LG\ space \\
Hamiltonian \arrow[u, shift left, "12.6.4"] \\
q-HamltonianwithP \arrow[u, "12.6.9"] \\
Poisson-LieG-space \arrow[u, "12.6.8"]
\end{tikzcd}
$$

@articlealekseev1997lie, title=Lie group valued moment maps, author=Alekseev, Anton and Malkin, Anton and Meinrenken, Eckhard, journal=arXiv preprint dg-ga/9707021, year=1997

12.7 Irregular Singularities

Boalch: Stokes Matrices and Poisson Lie Groups

Let $G = GL(n, \mathbb{C})$. We identify G^* with a moduli space of meromorphic connections over the unit disc with irregular singularity at the origin. From this we get: for each $A_0 \in t_{reg}$, there exists a holomorphic map $\mathfrak{g}^* \to G^*$. This map factors through the moduli space and the Riemann-Hilbert correspondence. This is a Poisson map for each A_0 where the left has the Kirillov-Kostant Poisson structure and the right has the standard structure multiplied by $2\pi i$.

If we restrict A_0 to be imaginary diagonal rather than just diagonal and restrict to the real locus we get the Ginzburg-Weinstein diffeomorphism $\mathfrak{k}^* \simeq K^*$ with usual Poisson structure and π rescaled structure.

$$\begin{tikzcd}
\mathfrak{k}^* & K^* \\
\mathfrak{g}^* \arrow[d] & G^* \arrow[d] \\
\mathcal{M}(A_0) \arrow[r, "RH"] & M(A_0)
\end{tikzcd}$$

$\mathcal{M}(A_0)$ is given by isomorphism classes of triples of a meromorphic connection on a rank n vector bundle on the disc with just one second order pole at 0, a framing g_0 at 0 where ∇ is of the form $d - (\frac{A_0}{z^2} + \frac{B}{z})dz + \theta$ where θ is a holomorphic one form. Some are of the form $\nabla = d - (\frac{A_0}{z^2} + \frac{B}{z})dz$. $\mathfrak{g}^*$ is included in by sending $\mathfrak{g}^* \to \mathfrak{g} \to \mathcal{M}(A_0)$ where the last map is setting B.

12.7.1 Lemma *There is a unique formal gauge transformation $\hat{F}$ taking Δ to one of the form $\nabla = d - (\frac{A_0}{z^2} + \frac{B}{z})dz$ with A_0 and B diagonal.*

12.7.2 Theorem *On each sector there is a $\Sigma_i \hat{F}$ that is asymptotic to $\hat{F}$ in each supersector.*

12.8 Dynamical Yang Baxter Equation

In this section we are working with $\mathrm{Rep}\, U_q\mathfrak{g}_{\mathbb{C}}$

12.8.1 Definition (Shapalov Form) *For Verma module M_λ, we can define a pairing $M_\lambda \otimes M_\lambda^* \to \mathbb{C}$ by*

$$\begin{aligned} \langle 1_\lambda \mid 1_\lambda^* \rangle &= 1 \\ \langle e_i 1_\lambda \mid 1_\lambda^* \rangle &= \langle 1_\lambda \mid S(e_i) 1_\lambda^* \rangle \end{aligned}$$

Take the dual of this arrow to give

$$\mathbb{C} \rightarrow (M_\lambda \otimes M_\lambda^*)^*$$

Copy from lectures of Dynamical Yang Baxter Equation by Etingof Schiffmann

Copy from Enriquez Etingof

12.8.2 Definition (Intertwining Operator) *Let $v \in V[\lambda - \mu]$ the specified weight space of V. Then construct $\Phi_\lambda^v\ M_\lambda \to M_\mu \otimes V$ by*

$$\Phi_\lambda^v(x_\lambda) \in x_\mu \otimes v + \sum_{\nu < \mu} M_\mu[\nu] \otimes V$$

Making this an intertwiner is enough to specify it uniquely with that leading term.

12.8.3 Definition (Fusion Matrix)

$$\begin{aligned} J_{WV}(\lambda)\ W \otimes V &\rightarrow W \otimes V \\ w \otimes v &\rightarrow \langle \phi_\lambda^{w,v} \rangle \end{aligned}$$

where we first take v and use it to construct the vertex operator Φ_λ^v and then use w to construct $\Phi_{\lambda - wt(v)}^w$. The composition $M_\lambda \to M_{\lambda - wt(v) - wt(w)} \otimes W \otimes V$ can be done with a unique element of $W \otimes V$ if it was done in one step.

Figure 12.1: Insert a picture of Verma branching of many copies of $\mathbb{C}_{x_i}^2$ evaluation representations.

12.8.4 Proposition *The fusion matrix is rational in λ and strictly lower triangular with 1's on the diagonal. It satisfies a dynamical 2-cocycle condition for triples of modules.*

12.8.5 Definition (Exchange Matrix)

$$R_{VW}(\lambda) \quad = \quad J_{VW}(\lambda)^{-1} J_{WV}^{21}(\lambda)$$

This R matrix satisfies the dynamical Yang Baxter Equation without spectral parameter and step $\gamma = 1$

This also works for quantum groups, but change rational dependence on λ to trigonometric dependence/rational in $q^{\lambda \dot{\alpha}}$

In order to take classical limits is better to calculate $J(\frac{\lambda}{\gamma})$ and $R(\frac{\lambda}{\gamma})$ which now satisfy the equations with step γ. Now take the series as the step goes to 0.

The rational case using $U\mathfrak{g}_{\mathbb{C}}$ goes to the **basic rational dynamical r-matrix** and the trigonometric case using $U_q\mathfrak{g}_{\mathbb{C}}$ as well as setting $q = e^{-\epsilon\gamma/2}$ (so that it goes to 1 in this limit as well) goes to the **basic trigonometric dynamical r-matrix with coupling** ϵ

$U \subset \mathfrak{h}^*$ is the space for the dynamical partameter. Let $X = U \times G \times U$ viewed as a groupoid. H^2 acts on this by

$$\begin{aligned}
(h_1, h_2)(u_1, g, u_2) &= (Ad^*(h_1)u_1, h_1 g h_2^{-1}, Ad^*(h_2)u_2) \\
&= (h_1 u_1 h_1^{-1}, h_1 g h_2^{-1}, h_2 u_2 h_2^{-1}) \\
g^{-1} u_1 g &= u_2 \\
(h_1 g h_2^{-1})^{-1} (h_1 u_1 h_1^{-1}) (h_1 g h_2^{-1}) &= h_2 g^{-1} u_1 g h_2^{-1} \\
&= h_2 u_2 h_2^{-1}
\end{aligned}$$

So that's what the arrows do, they take u_1 to $g^{-1}u_1 g$ so it is the right action. of g on u_1 as witnessed by the fact that g was written to the right side. So this preserves $Y \subset X$ the ones where g actually connects you from u_1 to u_2. Y is modelled on $T^*G \rightrightarrows \mathfrak{g}^*$ actually $T^*G \rightrightarrows U$

So there is a Poisson structure on X by EV1 and similar conditions define coboundary, quasi-triangular and triangular dynamical Poisson-Lie groupoids.

Let $\mathfrak{h}^* = \mathfrak{su}(N)^*$ and U be a subset near the Hermitian matrix 0.

You can use the $H = SU(N)$ action to diagonalize u_1. In X the fiber of the source map is all of $SU(N) \times \mathfrak{su}(N)^*$ because they don't have to connect. In Y the fiber of this is $SU(N)$ and there is a $S(U(1))^N$ action which stabilize the source and target. But the target $S(U(1)^N)$ is the same one after conjugation by g so we can write the space of arrows when one end is diagonalized as $SU(N)/(S(U(1)^N)$ the coadjoint orbit through said reference point. That isn't playing well with the moment map which is $(-s, t)$ so we need to give a coisotropic in $\mathfrak{h}^* \oplus \mathfrak{h}^*$ which would be entire orbits not points.

In terms of shifed Poisson structures `https://arxiv.org/pdf/1706.02623.pdf` by Safranov

[?]

12.9 Nahm's Equations

12.9.1 Definition (Nahm) *Give 3 k by k matrices which depend smoothly on $s \in (-\infty, 0]$*

$$\begin{aligned} \frac{dT_1}{ds} &= [T_2, T_3] \\ \frac{dT_2}{ds} &= [T_3, T_1] \\ \frac{dT_3}{ds} &= [T_1, T_2] \end{aligned}$$

12.9.2 Definition (Kronheimer Space $M(0, \xi, 0)$) *$\xi \in \mathfrak{t}$ is the orbit we are trying to pass through.*

The space of solutions to

Give 3 elements of $\mathfrak{g}$ which depend smoothly on $s \in (-\infty, 0]$

$$\begin{aligned} \frac{dB_1}{ds} &= -[B_2, B_3] \\ \frac{dB_2}{ds} &= -[B_3, B_1] \\ \frac{dB_3}{ds} &= -[B_1, B_2] \end{aligned}$$

satisfying boundary condition $\exists g_0 \in G$

$$\begin{aligned} B_1 &\to 0 \\ B_2 &\to Ad(g_0)(\xi) \\ B_3 &\to 0 \end{aligned}$$

12.9.3 Theorem (Kronheimer90) *Thiis maps diffeomorphically to the adjoint orbit $\mathcal{O}_\xi$ by*

$$\begin{aligned} (B_1, B_2, B_3) &\to B_2(0) + iB_3(0) \\ T_1 &= -B_3 \\ T_2 &= -B_1 \\ T_3 &= -B_2 \\ (T_1, T_2, T_3) &\to -T_3(0) - iT_1(0) \end{aligned}$$

where T is the solution to the original form of Nahm equations which will be more convenient.

12.9.4 Theorem (Nahm equation Lax Form)

$$\begin{aligned}
A(s,\xi) &= T_1 + iT_2 + \xi(-2iT_3) + \xi^2(T_1 - iT_2) \\
A_+ &= -iT_3 + \xi(T_1 - iT_2) \\
\frac{dA}{ds} &= [A, A_+] \\
A &= A_0 + A_1\xi + A_2\xi^2 \\
A_+ &= \frac{1}{2}A_1 + A_2\xi \\
T_i^\dagger = -T_i &\iff A_0^\dagger = -A_2 \;\; A_1^\dagger = A_1
\end{aligned}$$

Because we have a Lax presentation, the characteristic polynomial is a constant of motion independent of s. Say the variable for the characteristic polynomial is η. In TP^1 with coordinates (η, ξ) the characteristic polynomial determines a curve by compactification.

Because only the eigenvalues matter, all solutions of Nahm's equations determine the same spectral curve.

So if the eigenvalues are λ_i are distinct then the spectral curve

$$(\eta - \lambda_1\xi)\cdots(\eta - \lambda_k\xi) = 0$$

decomposes as the union of

$$(\eta - \lambda_i\xi) = 0$$

which are almost disjoint except for $\eta = 0$ and $\xi = 0$ ∞ where they all come together.

- Solution of Bogomolny equations on $\mathbb{R}^3$ with prescribed boundary conditions at ∞
- Spectral curve $S(\xi)$ satisfying a reality condition in $T\mathbb{P}^1$ and a certain line bundle
- Antihermitian solution to Nahm's equations
- Complex Coadjoint orbit

12.10 Springer

Copied from `http://arxiv.org/pdf/math/9802004v3.pdf`

12.10.1 Flag Variety

The set of Borels in $\mathfrak{g}$ is a closed subvariety of the Grassmannian of subspaces in $\mathfrak{g}$ of dimension $dim\mathfrak{b}$ but it is more restrictive by saying that subspace is actually a Borel subalgebra. All Borels are conjugate under the action of G and the isotropy group is the exponentiated version B. So you can start with a reference $\mathfrak{b}$ and then ask what element in G/B is needed to conjugate to your desired Borel. This is a bijection $G/B \simeq \mathcal{B}$. This is a G-equivariant isomorphism of algebraic varieties.

$G/B \longrightarrow K/T \hookrightarrow \mathfrak{k}^*$

12.10.2 Springer

Say we want to consider all nilpotents in $\mathfrak{g}$. The ones where the operator $ad\ x$ is nilpotent. Call this set $\mathcal{N}$. It is closed, Ad G stable subvariety and it is also stable under dilatation. This says $\mathcal{N}$ is a conical variety.

Let $\tilde{\mathcal{N}}$ be the set of pairs of a Borel $\mathfrak{b}$ and a nilpotent $x \in \mathfrak{b}$. The fiber over a given Borel is it's nilpotent elements which are $[\mathfrak{b}, \mathfrak{b}]$. This means we have a vector bundle over the flag variety G/B parameterizing Borels.

Use the Killing form to identify $\mathfrak{g} \simeq \mathfrak{g}^*$. Then there is a natural G equivariant vector bundle isomorphism $\tilde{\mathcal{N}} \simeq T^*\mathcal{B}$

Define the map $\tilde{\mathcal{N}} \to \mathcal{N}$ by projection to the first factor. This is proper and surjective. Every nilpotent lives in some Borel. It is irreducible and a resolution of singularities for $\mathcal{N}$.

In addition to this resolution of singularity perspective, you can also think of it as a moment map for the canonical Hamiltonian G action on $T^*\mathcal{B}$.

The Steinberg variety is the pullback $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$

12.10.1 Theorem *The Steinberg has many components all of the same dimension. One of them is the diagonal.*

12.10.2 Theorem (Gan Ginzburg) *There is an algebra isomorphism from $H^{BM}_{middle}(Z) \otimes \mathbb{C} \simeq \mathbb{C}[W]$ where the left hand side has the convolution product.*

This can be extended to a statement over the rationals which keeps more information like the difference between $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(e)$

12.10.3 Definition *M^+ is the union of the conormal bundles to the B orbits. For the $T^*\mathbb{CP}^1$ example, there is $\mathbb{C}$ and ∞ which upon taking conormal gives the candy wrapper. There is the $\mathbb{P}^1$ and the fiber at ∞ touching at one point.*

12.10.4 Theorem (Gan Ginzburg) *$H^{BM}_d(M^+)$ is a module over $H^{BM}_{2d}(Z)$. In this case we get $\mathbb{C}[W]$ as a regular representation and as an algebra respectively.*

12.10.3 Groethendieck-Springer

Instead of considering x nilpotent only, let it be any element of $\mathfrak{g}$. This still gives a vector bundle over the flag variety but now of rank $dim\mathfrak{b}$ instead of $dim\mathfrak{n}$. It also becomes a resolution $\tilde{\mathfrak{g}} \to \mathfrak{g}$ instead of just the nilpotent cone. You can also do the group version where $\tilde{G}$ is the set of pairs of a Borel subgroups and arbitrary elements in them. This gives $\tilde{G} \to G$

12.10.5 Example *For $G = SL(2)$, the flag variety is $\mathbb{P}^1$, so we can try to explicitly say which vector bundle we are getting. It is $\mathcal{O}(-1) \bigoplus \mathcal{O}(-1)$ for the Groethendieck-Springer resolution and $\mathcal{O}(-2)$ for the Springer resolution. http://arxiv.org/pdf/math/0604445v1.pdf*

12.11 Symplectic Resolutions

[?]

12.11.1 Definition (Symplectic Singularity) *A normal algebraic variety X such that the regular locus for any resolution of singularities carries an algebraic closed nondegenerate 2-form. A symplectic resolution is one where it extends from the regular locus to the entire $\tilde{X}$. It is necessarily Calabi-Yau.*

12.11.2 Definition (Conical symplectic resolution) *An affine symplectic singularity carrying a conical $\mathbb{G}_m$ action that acts on the symplectic form with weight $n > 0$.*

12.11.3 Example *Springer Resolution.*

Consider $T^*\mathcal{B}$ Give two $\mathbb{C}^*$ actions by scaling the fibers call S, and T which is given by a cocharacter extend to cotangent bundle so that it preserves ω. S is what makes it conical.

See category $\mathcal{O}$ discussion above in Beilinson-Bernstein section.

Now just do generally

Let $Z = M \times_{M_0} M$ and let M^+ be the sub of M such that $lim_{t \to 0} tp$ exists. Again $H_{2d}^{BM}(Z)$ is an algebra acting on $H_d^{BM}(M^+)$ as before.

$K(\mathcal{O}) \otimes \mathbb{C} \simeq H_d^{BM}(M^+)$ this was our regular representation so to categorify the $\mathbb{C}[W]$ action, turn this into a bimodule.

Again categorify to get bimodules acting on a module category.

We don't have an X such that $M = T^*X$, instead want a quantization of M.

12.11.4 Definition (Quantization of M Conical symplectic resolution) *a T-equivariant sheaf of filtered algebras on M and a $T \times S$ graded isomorphism $gr\mathcal{A} \simeq Fun_M$*

For the case we did before we get $U(\mathfrak{g})$ some other central character. Twisted D-modules.

If $M = Hilb(\mathbb{C}^2/\Gamma)$ then get a quotient of spherical rational Cherednik algebra.

Again try to take singular support to get cycles on M^+ to get what you call category $\mathcal{O}$. A^+ acts locally finitely. In the springer case A^+ is not $U(\mathfrak{b})$ on the nose, but the local finiteness-ness is the same.

Give a version of Harish-Chandra bimodules

Chapter 13

MSRI LiveTeX

13.0.1 Definition (Noncommutative polynomials)

$$D_1(\tau_{t_1}\tau_{t_2}\tau_{t_1}\cdots) = 1\otimes\tau_{t_2}\cdots + 0 + \tau_{t_1}\tau_{t_2}\otimes 1$$

13.0.2 Definition (Hilbert Space construction) *Complete with respect to the L^2 norm Left multiplication gives an operator on this. It is bounded by the Catalan number calculation*

13.0.3 Lemma (Is a factor) *Assume $n > 1$. This algebra generated by the above operators is a factor.*

Proof Define a densely defined operator by

$$A \rightarrow$$

Extend to all of M by

The graph and then try closing. If closure of graph is still a graph, then that operator is closable and we can form the operator $\bar{T}$. By Hahn-Banach, closability = adjoint is also densely defined.

Know that $\langle x_i p\rangle =$ so by our Schwinger-Dyson can make $D_i^\dagger$

$$D_i^\dagger =$$

Now use the regularization below.

$$\begin{aligned}
\delta_2^{reg} x_1 &= 0\\
\delta_2^{reg}(x_1 z - z x_1) &= 0\\
x_1\delta_2^{reg} z - \delta_2^{reg}(z) x_1 &= 0\\
T x_1 &= x_1 T
\end{aligned}$$

Figure 13.1: Diagram of the derivation

Compact operator that commutes with an operator with continuous spectrum (whose eigenspaces are 0 or infinite dimensional). That then implies that T is

$$\begin{aligned} \delta_2^{reg} x_2 &= 1 \otimes 1 \\ \delta_2^{reg}(zx_2 - x_2 z) &= \end{aligned}$$

13.0.4 Definition (Regularization)

$$\begin{aligned} \eta_\alpha &= \frac{\alpha}{\alpha + \Delta} \\ \zeta_\alpha x &\in Dom() \\ \mid \zeta_\alpha - \alpha \mid &\to 0 \\ \delta_\alpha^{reg} &\equiv \delta_\alpha \zeta_\alpha \end{aligned}$$

So as $\alpha \to 0$ resembles the unregularized on all x

13.0.5 Definition (Spectral Triple)

13.0.6 Theorem (Voiceulescu)

13.0.1 For a general planar algebra

Let P be a (subfactor) planar algebra.

13.0.7 Definition ($Gr_k(P_n)$) *Rectangle with arbitrary number of strings on top, k on sides. They attatch side to side. Trace by $\frac{1}{k}$ then connecting the two sides and putting τ on the top. Where τ is the diagram that looks like doubled Temperley-Lieb capping as shown below:*

13.0.8 Definition (Symmetric Enveloping Algebra) $Gr_k \boxtimes Gr_k^{op}$ *Box decorated on all sides as ...*

13.0.9 Definition (Master Derivation) *On Gr_1 leave the first l on top alone, bring the next two off to the right, the next ? down and then the one that used to go to the right goes all the way around to the left. Now is in $Gr_2 \boxtimes Gr_2^{op}$*

13.0.10 Example

13.0.11 Theorem (Hartglass) *Hilbert bimodule to produce the Hilbert space associated to this more general situation*

13.0.12 Theorem *Is a factor. Need to find an analog of δ_2^{reg} from before.*

Figure 13.2: Diagram of the derivation

Proof To get δ_2 modify by a Jones-Wenzl idempotent

$$\begin{aligned} \delta_2 x_1 &= 0 \\ \delta_2 x_2 &= JW \end{aligned}$$

Now do the same from the polynomial example before

13.0.13 Theorem *The Von Neumann algebra generated by $Gr_k(P)$ is a frator and attatching a disjoint string at the bottom gives $VN(Gr_k) \to VN(Gr_{k+1})$. The Jones projections follow as ..*

13.0.14 Example $Gr_0 \subset Gr_1$

13.1 Noah 20170622

13.1.1 Lemma *If left and right von Neumann dimension for a von Neumann A are finite. Then those A-A bimodules form a tensor category.*

Proof How fusion product of bimodules is defined ...

$$\begin{gathered} {}_A M_A \\ {}_A \bar{M}_A \end{gathered}$$

Suppose M is projective over A
For A a Von Neumann algebra, ∞ dimension then SOL. If finite, then

- $L^2(A) \bigoplus \cdots L^2(A)$
- ?

Write as a sum of numbers that are all < 1. The ones that are from < 1 already have.
Use this to show the existence of duals including evaluation and coevaluation □

13.1.2 Lemma (Pivotal) *The condition for pivotal to make $Id \to \star\star$*
$\star_L \to \star_R$

13.1.3 Theorem *$\mathcal{C}$ planar pivotal and a x gives a planar algebra.*
As this favorite object x varies what changes in the planar algebra.

13.1.4 Definition (From a subfactor) *$A \subset B$, then form ${}_AL^2(B)_A$, $L^2(B)$ is an algebra object in $Bim(A)$. It serves as a favorite object.*

$$\begin{aligned} {}_AA_A &\to {}_AL^2(B)_A \\ {}_AL^2(B)_A \otimes_A L^2(B)_A &\to {}_AL^2(B)_A \end{aligned}$$

[Shaded planar algebra]

13.1.5 Example ($\boldsymbol{Vec_G}$) *$\mathbb{C}[G]$ with g in grading g is an algebra object. gh is in degree gh, no problem*

13.1.6 Definition (Smash Product) *$A \subset A\#\mathbb{C}G$*

13.1.7 Example *For a subgroup $H \subset G$, Vec_H Vec_G*

13.1.8 Example (Functions on $\boldsymbol{G}$) *$\mathcal{O}(G) \subset Rep(G)$ as regular representations $A^G \subset A$ subfactor*

13.1.9 Example *For any object X form $X \otimes X^*$ as a "matrix algebra" (the case of Vec)*

13.1.10 Lemma *Let the favorite object x be an algebra object. Then why is the planar algebra you get shaded? Assume has trace so A is self dual (make unoriented)*
Thicken the algebra diagram arrows into a ribbon diagram.
To go the other way, from a shaded diagram, turn the colored portion into a ribbon planar diagram and put that as the string A.
.
Alternatively color by $\mathcal{C}$ or A module in $\mathcal{C}$. Can forget the color and just remember the underlying object.
If there is the color in between or not $\otimes$ or $\otimes_A$ where $\otimes_A = X \otimes A \otimes Y \to X \otimes Y$ put the second arrow to make this an equalizer diagram

13.1.11 Example (Index 4 planar algebra) *One example is X with dimension 2 and then take matrix algebra construction. In particular suppose X is simple too.*
$RepG$ and a 2 dimensional irrep, take principal graph that defines how $\otimes X$ acts.

13.1.12 Exercise (Dihedral Group) *D_5 (group of order 10) graph for tensoring then gives affine D_5.*

13.1.13 Exercise (Platonic Groups) *Subgroups of $SL(2)$, get the exceptional*

13.1.14 Example *All of $SU(2)$, but now that's infinite line $TL_{\delta=2}$*
For the U_qsl_2 ... change δ