Contents

Chapter 1

Point-Set Topology

1.1 What is a topology?

Specifying a topology means specifying what you mean by the word open.

1.1.1 Definition (Topological Space) *Subset of the power set that says what you call open. The empty set and the whole space must be open. Arbitrary unions and finite intersections of opens are open. If we say the set is denoted X and we call the above subset of the power set of X by τ, then we get a space (X, τ). A common abuse of notation is to describe the topological space with just X, but the data of which topology needs to be given too. It is just that their is usually a default topology for that set, so you don't need to write it.*

1.1.2 Definition (Basis) *Give a collection of subsets where every point is in at least one and if x is in the intersection of two B_1 and B_2 then there is another $B_3 \subset B_1 \bigcap B_2$ which also has x. Take the topology generated by this by taking arbitrary unions and finite intersections. In fact you can just give any open as unions from the basis.*

1.1.3 Example (Discrete) *Suppose we declare that every subset is open. That works as a topology on any set X, so we get a space $(X, \tau_{discrete})$*

1.1.4 Example (Indiscrete) *Just the empty set and the entire thing are open.*

1.1.5 Example (Real Line) *The usual topology on $\mathbb{R}$ is given by a basis consisting of subsets of the form (a, b). That is we use the order by $<$ to define opens.*

1.1.6 Example (Long Ray) $\omega_1 \times [0, 1)$ *equipped with the order topology. The order is lexicographic (the ω_1 part is the first letter so matters more).*

1.1.7 Exercise (Product of Long Lines) *Give a basis for the topology of this product.*

1.2 Continuity

1.2.1 Definition (Continuous) *Inverse images of opens are open. You must give the function and the topologies on the spaces omiting notation only if you know what you're doing.*

1.2.2 Exercise (Epsilon-Delta) *Show that this matches the δ-ϵ definition you learned before.*

1.2.3 Exercise *When are functions from a space with a discrete topology to something else continuous?*

1.2.4 Definition (Homeomorphism) *A continuous map such that it is bijective and the inverse map is also continuous. This is the notion of isomorphism in the category of topological spaces and continuous maps.*

1.3 Separation Axioms

1.3.1 Definition (Hausdorff) *For every pair of distinct points, p and q, there exist nbhds of each U and V respectively such that U and V are disjoint. That is the points can be "housed off"*

1.3.2 Exercise (Line with Two Origins) *Is this Hausdorff? Why or why not?*

1.3.3 Exercise (Diagonal Map) *Prove that a space X is Hausdorff if and only if the diagonal $\Delta = \{(x, y) \in X \times X \mid x = y\}$ is closed.*

1.3.4 Definition (T3/Regular) *A point x and a closed set F can be separated by disjoint open sets.*

1.3.5 Definition (T4/Normal Regular) *Disjoint closed sets can be separated.*

1.3.6 Lemma (Urysohn) *A topological space is normal if and only if any two disjoint closed subsets can be separated by a function. That is there is a real valued function that takes value 0 on A and 1 on B.*

1.3.7 Remark (Perfectly Normal) If someone calls something weird, you can say it is perfectly normal by stating why it satisfies that assumption. $\Diamond$

1.3.8 Exercise *Give an example of a space that is Hausdorff but not Normal Regular.*

1.4 Connectedness

1.4.1 Definition (Connected)

1.4.2 Definition (Path Connected)

1.4.3 Theorem (Intermediate Value Theorem)

1.4.4 Exercise `https://math.stackexchange.com/questions/4784531/for-all-x-y-in-a-connected-m`

Suppose M is a m dimensional topological manifold. Suppose for all pairs x, y there is an open neighborhood $U_{x,y}$ of both which is homeomorphic (via some h) to $\mathbb{R}^m$. Then we get that the space is connected because we can immediately say it is path connected using the inverse image along h of the line connecting $h(x)$ and $h(y)$ to get a path in $U_{x,y} \subseteq M$ which connects x and y. Path connected being even stronger than connected.

Prove the converse is also true. That is to say, prove that once you have this statement about existence of such $U_{x,y}$'s, prove that M must be connected.

1.5 Compactness

1.5.1 Definition (Compact) *Every cover has a finite subcover that still does the job.*

1.5.2 Theorem (Other Compactness) *Sequential compactness and limit point compactness. Depending on other properties of the space we have implications among these notions. Won't put that here but see Munkres.*

1.5.3 Definition (Lindelöf) *Every open cover has a countable subcover.*

1.5.4 Definition (Locally Compact) *For every point $x \in X$, we have a compact neighborhood around it. That means we have an open neighborhood U contained in a compact set K so that $x \in U \subseteq K \subseteq X$.*

1.5.5 Exercise (One Point Compactification) *Give a topology on a space Y that has an extra point ∞ as an addition to a locally compact but not compact Hausdorff space X such that Y is compact. The opens of X should not be messed with.*

1.5.6 Exercise *What is the one point compactification of $\mathbb{R}^n$ (given it's usual topology)? Hint: start with $n = 1$ and $n = 2$ so you can draw pictures.*

Chapter 2

Manifolds

2.1 So you think you can $\mathbb{R}^n$?

2.1.1 Definition (Derivative)

2.1.2 Exercise (Directional Derivatives can lead you the wrong way)

$$f(x,y) = \frac{xy^2}{x^2+y^4}$$

Find the directional derivatives. Is this differentiable?

2.1.1 Function Spaces

For more detailed descriptions, click the link to the appropriate section in function spaces chapter.

- C^k Section 21.6
- C^∞ Section 21.7
- C^k_{cs} Section 21.8
- C^ω Section 21.9
- Smooth Distributions Section 21.10

2.2 What is a manifold?

The configuration space of a system (possibly with constraints) is a differentiable manifold. (We will not be concerned with the distinction if it is just topological because otherwise any calculus or physics would be impossible.)

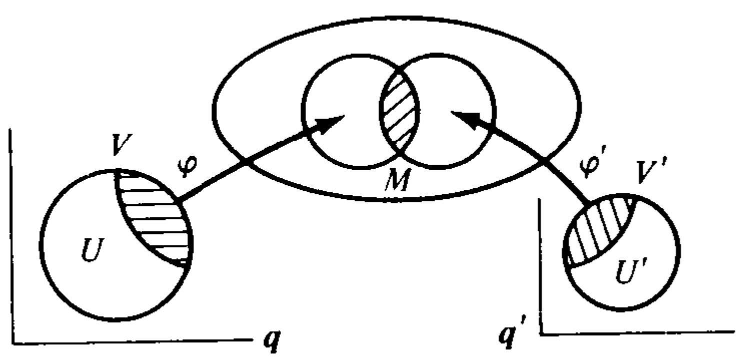

Figure 2.1: 2 Compatible Charts: (Figure Copied from Master Arnold)

2.2.1 Definition (Chart) *A chart on an n-dimensional manifold is a diffeomorphism between* $\mathbb{R}^n$ *and some open subset of the manifold.*

2.2.2 Definition (Atlas) *Most manifolds can't be covered with just one chart. This is one of those things where your physics classes have lied to you or at least mislead you. An atlas is a collection of charts. If we take all possible consistent charts, that is a maximal atlas. It is one of the things that exists but it is a terribly uncountable collection so not terribly useful.*

2.2.3 Definition (Groethendieck Sites) *A category together with a Groethendieck topology. Rather than state the definition fully, just consider the basic example $Op(X)$. That is the the category whose objects are open subsets of a topological space and morphisms are inclusions. Then the Groethendieck topology amounts to covers in the usual sense. The axioms of Grothendieck topology turn this into topology in the usual sense*

$$\begin{tikzcd}
(1,2) \arrow[r] & (1,4) \arrow[r] & (-10,10) \\
(2,3) \arrow[urr] & & \\
\cdots \arrow[uurr] & &
\end{tikzcd}$$

2.2.4 Example (Euclidean Space) $\mathbb{R}^n$ *with the entire space as a single chart. This is* $\mathbb{R}^n$ *regarded as a manifold not as a vector space. You cannot add points or multiply by scalars.*

2.2.5 Example (Circle or to be more accurate just a loop) *Let us give it in the standard presentation as the unit circle and measuring angles from the positive x-axis. This is more structure than is necessary, but it is so familiar. For example, we don't have any sense of whether this is supposed to be round or flat.* $U_1 = (-\pi, \pi)$ *and* $U_2 = (0, 2\pi)$ *give 2 charts. The first one does not include the point* $(0, -1)$ *while the second fails to include* $(0, 1)$. *We have also done the equivalence between an open interval and the entire real line. For example, you could do this with an arctangent function.*

2.2.6 Example (Surfaces) *Take the surface of a pretzel. Again we are regarding this only as a manifold right now. We may give it more structure later. Rather than give equations, here is a picture.*

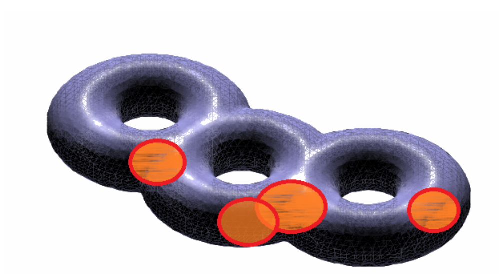

Figure 2.2: Some charts on a genus 3 surface

2.2.7 Exercise (Figure 8) *Figure 8. Is it a manifold? Prove why or why not.*

2.2.8 Exercise (Long Line) *Is the long line a smooth manifold?*

2.2.9 Exercise (Products of Manifolds) *Product topology of manifolds M and N. Show this product of topological spaces as a manifold.*

2.2.10 Exercise (n-Sphere) *Give coordinate charts for this. Give the gluing between all pairs. Are ϕ and θ coordinates on the whole sphere? Why or why not?*

2.2.11 Exercise (Grassmannian) *Do the same for the space of 2-planes in $\mathbb{R}^4$.*

2.2.12 Exercise (Manifold with Boundary) *Is the half plane (such as given by $y \geq 0$) a manifold? Prove why or why not. Is a manifold with boundary a manifold?*

2.3 Partitions of Unity

2.3.1 Definition (Bump Function) *A smooth function that is compactly supported.*

2.3.2 Exercise *Show that $f(x) = e^{-1/(1-x^2)}$ for $|\, x \,| < 1$ and 0 outside is a smooth bump function.*

2.3.3 Definition (Partition of Unity) *A partition of unity is a set of continuous functions from M to $[0,1]$ such that for each $x \in M$ there exists a neighborhood U where all but finitely many f_j (for $j \in J$ the indexing set not necessarily countable) have $f_j(U) = 0$. Also they partition unity meaning that they sum to 1.*

2.3.4 Theorem (Existence of Riemannian Metric) *We will get to this in more detail in Section 10.1*

Proof Let $\{(U_\alpha, \phi_\alpha)\}$ be a locally finite atlas. Each chart is diffeomorphic to $\mathbb{R}^n$ so pull the usual Euclidean metric back along the map defining that chart. Now take a partition of unity and use these as coefficients of a convex combination to add all these pieces together. You can then show that this is a Riemannian metric. □

2.4 Immersions

2.4.1 Definition (Immersion) *The differential gives an injective map $T_pM \to T_{f(p)}N$. This also means the differential has constant rank of dimension of M everywhere.*

2.4.2 Example *The standard picture of the Klein bottle in $\mathbb{R}^3$ gives such an immersion. It intersects itself which shows that even though f is not injective, it's differential is and that is all that matters.*

2.5 Submersions

2.5.1 Definition (Submersion) *The differential gives an surjective map $T_pM \to T_{f(p)}N$. This also means the differential has constant rank of dimension of N everywhere.*

2.5.2 Definition (Regular/Critical Point/Value) *If the differential gives a surjective map $T_pM \to T_{f(p)}N$ for a point p then p is called a regular point. If not, it is a critical point. For $q \in N$, if all of $f^{-1}(q)$ are regular points, then q is a regular value. Critical values are then the images of the critical points.*

2.6 Sard's Theorem

2.6.1 Theorem (Sard) *For a smooth function from a manifold M to some $\mathbb{R}^n$ the images of the critical points has Lebesgue measure 0. This image is called critical values. In fact all you need is C^k with $k \geq max(m-n+1, 1)$ rather than C^∞.*

2.7 Whitney Embedding

2.7.1 Theorem (Whitney Strong Embedding) *Any smooth real m-manifold can be smoothly embedded in $\mathbb{R}^{2m}$. Real projective spaces $\mathbb{RP}^m$ with $m = 2^n$ show that this $2m$ can't be made smaller.*

2.7.2 Remark (Nash Embeddings) There is also Nash embedding. Every Riemannian manifold can be isometrically embedded into some Euclidean space. There are C^1 and C^k $3 \leq k \leq \infty$ versions. Something something Beautiful Mind. ♢

2.8 The Categories

2.8.1 Definition (DiffMan) *Objects C^∞ manifolds. Smooth maps are the morphisms. Composition of smooth stays smooth.*

2.8.2 Definition (TopMan) *The category of topological manifolds and continuous maps. Composition of continuous stays continuous. Be wary that this is often called Top which leads to confusion with the category of topological spaces and continuous maps.*

2.8.3 Definition (PLMan) *A piecewise linear manifold is a topological manifold together with an atlas such that the transition functions are piecewise linear functions rather than merely continuous transition functions.*

2.8.4 Remark (Generalized Poincaré Conjecture)

2.9 Sheaves

2.9.1 Definition (Locally Ringed Space)

Using the concept of definition 2.2.3, it is easier to understand sheaves.

2.9.2 Definition (Pre-Sheaf) *A functor from $Opens(X)^{op}$ to Set. On each object open, we get the set $\mathcal{F}(U)$. Morphisms are inclusions flipped around, so they turn into restriction maps on these sets.*

2.9.3 Definition (Sheaf) *Now we use the covering sieve to say about the sheaf condition.*

2.9.4 Definition (Stalk)

2.9.5 Example (Functions) *This is the main example you should think that you are trying to distill the key properties of. It assigns to every open U the smooth functions on U. The sheaf condition is the concept of putting a function together based on its restrictions. This is customarily called $\mathcal{O}$.*

When we get to bundles in chapter 7, we will see generalizations called sections.

2.9.6 Definition (Cosheaf)

2.9.1 Physical

2.9.7 Example (Fields)

2.9.8 Example (Observables) *Observables in QFT. The cosheaf property has an obvious interpretation as the observations you can do in this piece of spacetime or in this other piece of spacetime. There is some more subtlety but we will get to those more when we see Section 8.1*

2.9.9 Remark This is the philosophy that is advocated in Factorization Algebras such as the book by Costello and Gwilliam. ♢

2.10 Graded and Super

Even without assuming physical supersymmetry (See the notion of superspace for how to use this to make supersymmetric QFTs.) these are very useful to rephrase more conventional notions like differential forms and spinors. Philosophically it is getting you to learn to change your ambient topos, but we can wait until later to learn philosophy.

2.10.1 Definition (Graded Manifold)

2.10.2 Definition (Super Manifold) *Instead of locally modelling the functions on the space as* $C^\infty(\mathbb{R}^m)$ *we instead have* $C^\infty(\mathbb{R}^m) \otimes \wedge^\bullet \mathbb{R}^n$. *This is using the terminology of locally ringed spaces.*

The body is the underlying space which is an ordinary manifold. The soul is the nilpotent directions, you can't get it as a set or a space but it is relevant to characterizing the entire supermanifold as a whole.

2.10.3 Example (It's a Bird. It's a $\mathbb{R}^2$. No, it's SuperPoint!) *We want the body to just be a point. This is then* $\mathbb{R}^{0|q}$ *so that it's algebra of functions is just the exterior algebra part only tensoring with* $C^\infty(\mathbb{R}^0) \simeq \mathbb{R}$ *constants.*

2.10.4 Theorem (Batchelor's Theorem) *Every supermanifold is noncanonically of the form* ΠE *where* Π *is parity reversal on the fibers of the vector bundle* E. *The base is still* M *and the sheaf of superalgebras is the associated sheaf of* $\wedge^\bullet E^*$. *Locally we can see the coordinates in the fiber directions of the bundle given by* E^* *are now odd, instead of being even as they would be on the total space of* E.

Chapter 3

Vector Fields and Differential Operators

3.1 The Tangent Bundle as a manifold

$$
\begin{array}{c}
TM \\
\downarrow \\
M
\end{array}
$$

3.2 Vector Fields

The dynamics defines a vector field Δ on TQ. If you flow along that vector field for time t, that is evolving the system for time t.

$$\Delta = \dot{q}\frac{\partial}{\partial q} + \ddot{q}(q, \dot{q})\frac{\partial}{\partial \dot{q}}$$

Writing $\ddot{q}$ as a function on TQ is where the details of the physics goes.

This is a differential operator. It maps functions to functions and obeys a Leibniz rule. It is a vector field.

3.3 One Forms

3.3.1 Definition (One Form)

Dual to vector fields are one forms. This means they eat a vector field and give a function.

In the coordinates we have the dual basis of the coordinate vector fields

$$\begin{aligned} dq(\frac{\partial}{\partial q}) &= 1 \\ d\dot{q}(\frac{\partial}{\partial \dot{q}}) &= 1 \end{aligned}$$

The other one forms will be linear combinations of these

$$\omega = \eta dq + \beta d\dot{q}$$

where η and β are functions on TQ functions of q and $\dot{q}$.

3.4 k-Forms

3.4.1 Definition (k Form)

3.4.2 Definition (Wedge Product)

3.4.3 Theorem $C^\infty(T[1]M) = \Omega^\bullet(M)$. *This is why we introduced graded manifolds by calling the fiber coordinates degree 1 instead in definition 2.10.1. If you forget from 1 to just odd, you get a supermanifold definition 2.10.2 (That would forget the* $\mathbb{Z}$ *grading down to a* $\mathbb{Z}_2$*).*

3.5 deRham Operator

3.5.1 Definition (deRham d)

3.5.2 Theorem ($d^2 = 0$) *This means that* $Imd \subset Kerd$*. So form the quotient as vector spaces. This is called the deRham cohomology and it detects some of the topology.*

3.5.3 Example (Pendulum)

If we have a function L, its differential defines a one-form

$$dL = \frac{\partial L}{\partial q}dq + \frac{\partial L}{\partial \dot{q}}d\dot{q}$$

If we pair this with the vector field Δ from before we get

$$\begin{aligned} dL(\Delta) &= (\frac{\partial L}{\partial q}dq + \frac{\partial L}{\partial \dot{q}}d\dot{q})(\Delta) \\ &= \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} \\ &= \dot{L} \end{aligned}$$

3.6 Integration and Stokes

3.6.1 Theorem (Generalized Stokes)

3.7 Hodge Star

This is how to go back to the div, grad, curl that is an arcane waste of time. Why is it strictly worse? Because what if I had to use generalized coordinates because of something like being on subject to a constraint.

3.7.1 Definition (Hodge Star)

3.7.2 Exercise *Write the analog of curl of a vector field in this formalism where you also use the map g from 1-forms to vector fields. Check this if the metric g is the Euclidean metric on $\mathbb{R}^3$ as well as if it is the same space in spherical coordinates. Check these with* `https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates`.

3.7.3 Exercise *Write what the Cross Product means with metric, Hodge star and wedge.*

3.7.4 Exercise *Do Gauss' Law.*

3.7.5 Definition *Let $d^\dagger = (-1)^? \star d \star$ and $\Delta = d^\dagger d + dd^\dagger$*

3.7.6 Exercise *Do the Laplacian on the functions on the standard metric sphere. Check your answer with* `https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates`

3.8 Lie Derivative

3.8.1 Theorem (Cartan's Magic Formula)

$$L_V = di_V + i_V d$$

3.8.2 Theorem (Lie Bracket)

$$\begin{aligned} U &= L_X L_Y - L_Y L_X \\ U &= L_Z \\ Z &\equiv [X, Y] \end{aligned}$$

Now think of this as a vast generalization of matrices and the commutator.

3.8.3 Theorem (Frobenius)

3.8.4 Example (Contact Geometry) *Why this example is important will come back in Section 5.2.1*

3.9 Integral Curves

Remember existence and uniqueness from DiffEq?

3.10 Einstein Notation

More invariantly this tells you which sections of $TX^{\otimes r} \otimes T^*X^{\otimes s}$ your equation is being defined in.

3.10.1 Remark Where does it live? $\Diamond$

3.10.2 Definition (Upsie Index)

3.10.3 Definition (Downsie Index)

3.11 Gerstenhaber Algebra

For when we get to Section 8.2

Chapter 4

Lagrangian Mechanics

4.1 Euler Lagrange

$$\begin{aligned}
\theta_L &= \frac{\partial L}{\partial \dot q} dq \\
L_\Delta \theta_L &= \frac{d}{dt}\frac{\partial L}{\partial \dot q} dq + \frac{\partial L}{\partial \dot q} d\dot q \\
dL &= \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot q} d\dot q \\
L_\Delta \theta_L - dL &= 0 = a dq + b d\dot q \\
a = 0 &\implies \frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = 0 \\
b = 0 &\implies \frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial \dot q} = 0
\end{aligned}$$

4.2 Noether Theorem

4.2.1 Definition (Killing Vector Field) *A symmetry or at more properly an infinitesimal version thereof.*

Let X be a Killing vector field.

$$\begin{aligned}
L_X L &= 0 \\
i_X dL &= i_X L_\Delta \theta_L = i_X i_\Delta d\theta_L + i_X d i_\Delta \theta_L \\
&= i_X i_\Delta d\theta_L + i_X d i_\Delta \theta_L + i_X i_\Delta d\theta_L - i_X i_\Delta d\theta_L
\end{aligned}$$

$$L_X L = \langle \theta_L \mid [X, \Delta] \rangle + L_\Delta \langle \theta_L \mid X \rangle$$

4.3 Jet Bundle

Recall Section 2.9

4.3.1 Definition (Jet Bundle)

4.4 Helmholtz Conditions

4.4.1 Theorem (Helmholtz) *Suppose $F_i(t, x_j, \dot x_j, \ddot x_j) = 0$ is a system of n second order differential equations. In order for these to be Euler-Lagrange equations of a Lagrangian $L(t, x_j, \dot x_j)$ it is equivalent to ask that*

$$\begin{aligned}
\frac{\partial F_i}{\partial \ddot{x}_j} &= \frac{\partial F_j}{\partial \ddot{x}_i} \\
\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i} &= \frac{1}{2}\frac{d}{dt}\left[\frac{\partial F_i}{\partial \dot{x}_j} - \frac{\partial F_j}{\partial \dot{x}_i}\right] \\
\frac{\partial F_i}{\partial \dot{x}_j} + \frac{\partial F_j}{\partial \dot{x}_i} &= 2\frac{d}{dt}\left[\frac{\partial F_j}{\partial \ddot{x}_i}\right]
\end{aligned}$$

One consequence of these is that F_i must be linear in $\ddot{x}_j$ for all j so can be written as $P_i(t, x_j, \dot{x}_j) + \sum_\ell Q_{i,\ell}(t, x_j, \dot{x}_j)\ddot{x}_\ell$ where j as arguments to the function stands for all j just like it did in F_i.

If F_i are not Euler-Lagrange equations of any such L, the dynamics might still be. Because you can replace $F_i \to \sum_\ell \Lambda_{i\ell} F_\ell$ with some invertible n by n matrix function Λ which depends on $t, x_j, \dot{x}_j, \ddot{x}_j$.

4.4.2 Example (1 dimension) *If there is only one index for i and j, then the first two conditions are vacuously satisfied. So we are reduced to only the third.*

$$\begin{aligned}
2\frac{\partial F}{\partial \dot{x}} &= 2\frac{d}{dt}\left[\frac{\partial F}{\partial \ddot{x}}\right] \\
\frac{\partial P}{\partial \dot{x}} + \frac{\partial Q}{\partial \dot{x}}\ddot{x} &= \frac{dQ}{dt}
\end{aligned}$$

Let $F = m(t)\ddot{x} + b(t)\dot{x} + k(t)x + g(t)$ which is the general form of a second order linear differential equation. $Q = m(t)$ and $P = b\dot{x} + kx + g$.

$$b(t) + 0 = \dot{m}$$

which amounts to saying $\frac{d}{dt}(m\dot{x}) = -kx - g = -\frac{d}{dx}(\frac{1}{2}kx^2 + gx)$. So this describes a changing mass on a spring and a constant force.

If we allow Λ and say Λ depends on $\ddot{x}$ in a polynomial manner, then in order for the product ΛF to have a chance at satisfying Helmholtz conditions Λ cannot depend on $\ddot{x}$ at all because F was already linear in $\ddot{x}$ and so the only way for the product to be linear in $\ddot{x}$ is for Λ to not have such dependence.

$$
\begin{aligned}
P' &= \Lambda(t,x,\dot{x})b(t)\dot{x} + \Lambda kx + \Lambda g \\
Q' &= \Lambda m \\
\frac{\partial P'}{\partial \dot{x}} &= \Lambda b(t) + \frac{\partial \Lambda}{\partial \dot{x}}(b\dot{x} + kx + g) \\
\frac{\partial Q'}{\partial \dot{x}}\ddot{x} &= \frac{\partial \Lambda}{\partial \dot{x}} m\ddot{x} \\
\Lambda b(t) + \frac{\partial \Lambda}{\partial \dot{x}} F &= \frac{d\Lambda}{dt} m + \Lambda \frac{dm}{dt} \\
\Lambda b(t) &= \frac{d\Lambda}{dt} m + \Lambda \frac{dm}{dt} \\
\frac{d\Lambda}{dt} &= \Lambda\left(\frac{-1}{m}\frac{dm}{dt} + \frac{b}{m}\right) \\
\Lambda &= \exp\left(\int \frac{-1}{m}\frac{dm}{dt} + \frac{b}{m} dt\right)
\end{aligned}
$$

where the ambiguity of the integral only affects Λ by a nonzero multiplicative constant so only affects ΛF and then therefore L by that multiplicative constant.

4.4.3 Exercise *Let $m = m_0 + m_1 t + m_2 t^2$ and $b = b_0 + b_1 t + m(t)\tilde{b}(t)$ with $\tilde{b}$ another polynomial. Also assume that $m > 0$ for all t. Come up with a Lagrangian with the same dynamics.*

$$
\begin{aligned}
I &= \int \frac{(b_0 - m_1) + (b_1 - 2m_2)t}{m_0 + m_1 t + m_2 t^2} + \tilde{b}(t) dt \\
\Lambda &= e^I
\end{aligned}
$$

This integral can be done with trig substitution.

Chapter 5

Hamiltonian Classical Mechanics

5.1 Symplectic Linear Algebra

5.2 Symplectic Manifolds

5.2.1 Definition (Symplectic Form) *A closed nondegenerate 2-form. This means we can pair vectors in T_pM in the same manner as symplectic linear algebra by $i_v i_w \omega$.*

5.2.2 Example (Cotangent bundle) $\begin{array}{c} T^*S \\ \downarrow \\ S \end{array}$

Using coordinates q_i on the basis and p_i for the corresponding fiber coordinate, the total space T^*S has a symplectic form $\sum dp \wedge dq$.

5.2.3 Theorem (Weinstein Neighborhood Theorem)

5.2.1 Contact Boundary

5.3 Hamiltonian Vector Fields

5.4 Conserved Quantities

Here we state probably the most important theorem.

5.4.1 Theorem (KAM)

We will return to it later.

5.5 Canonical Transformations

5.5.1 Definition (Symplectomorphism) *A symplectomorphism of M is a diffeomorphism f such that $f^*\omega = \omega$.*

5.5.2 Exercise (Hamiltonian Symplectomorphisms) *Give something in Symp but not in Ham for the symplectic manifold (pick something). Is Ham a normal subgroup of Symp?*

5.5.3 Theorem (Banyaga Theorem) *For (M, ω) a compact symplectic manifold.*

$$0 \longrightarrow \mathrm{Ham}(M,\omega) \longrightarrow \mathrm{Symp}_0(M,\omega) \longrightarrow H^1(M,\mathbb{R})/\Gamma \longrightarrow 0$$

The subscript 0 denotes identity component and Γ is a countable subgroup.

Proof Define the flux homomorphism Flux from $\tilde{\mathrm{Symp}}_0(M,\omega)$ to $H^1(M,\mathbb{R})$. Here the source is smooth paths in $\mathrm{Symp}_0(M)$ whose value at the start of the path $t = 0$ is the identity and whose endpoint is arbitrary. The paths are taken up to smooth homotopies that fix their endpoints. The endpoint map sending $\psi_t \to \psi_1$ shows this as a universal cover of $\mathrm{Symp}_0(M)$. We can take the product structure to be either pointwise multiplications on the target symplectomorphism group (one can also use shifting, squeezing and concatenating paths but that will be equivalent).

$$\begin{aligned} \mathrm{Flux}(\psi_t) &= \int_0^1 [i_{X_t}\omega] \\ \frac{d\psi_t}{dt} &= X_t \circ \psi_t \end{aligned}$$

This is invariant under the homotopies so the morphism is well defined. If the path is homotopic to a path that stays within $\mathrm{Ham}(M,\omega)$ then $i_{X_t}\omega = 0$ for all t so the flux is 0. This is in fact an if and only if. Let Γ be the image of this map on those loops that also end at the identity. By usual universal cover arguments, this flux homorphism descends to $\mathrm{Symp}_0(M,\omega) \to H^1(M,\mathbb{R})/\Gamma$. Changing the preimage of ψ_1 from ψ_t to ϕ_t is equivalent to multiplication by $\phi_t\psi_t^{-1}$ in the source and that maps to something in Γ. Therefore by quotienting Γ in the target, the map is well defined. The statements about the flux of isotopies that stay with $\mathrm{Ham}(M,\omega)$ allow us to show that the kernel of this descended map is precisely $\mathrm{Ham}(M,\omega)$.

5.5.4 Corollary *The inclusion* $Ham(M,\omega) \hookrightarrow Symp_0(M,\omega)$ *induces an isomorphism on* π_k *for* $k \geq 2$.

Proof Long exact sequence of homotopy groups induced by the above as a fibration. □

5.5.5 Theorem (Banyaga Theorem) *The commutator group* $[G,G]$ *of* $G = Symp_0(M,\omega)$ *is the* $Ham(M,\omega)$ *subgroup. By general results of Epstein that* $[G,G]$ *is simple for certain classes of groups that act as homeomorphisms, this shows that* $Ham(M,\omega)$ *is simple.*

5.5.6 Definition (Generating Functions)

5.6 Liouville's Theorem

5.7 Hamilton-Jacobi

5.8 Action Angle Variables

5.8.1 Theorem (KAM) *Now we can be more accurate about Theorem 5.4.1*

Proof

5.9 Chaos

5.9.1 Remark You should always feel it impending on you from all sides. Do you feel it? - Chaos Chaos. Cherish integrability like the beautiful gem that it is. ◊

5.9.2 Theorem (Poincare Recurrence)

5.10 Contact Geometry

5.10.1 Thermodynamics

Consider the example of $\mathbb{R}^{2n+1}$ and the contact form of the form

$$\alpha \equiv du - p_i dq^i$$

u is called the thermodynamic potential, q^i are called fundamental variables and p_i their conjugates. We can change coordinates to change these special functions as well. This is possible thanks to a Darboux theorem like in the symplectic case. Some special coordinate systems encode the first law of thermodynamics

$$\begin{aligned}
\alpha &= dU + PdV - TdS \\
&= d[H \equiv U + PV] - VdP - TdS \\
&= d[F \equiv U - TS] + PdV + SdT \\
&= d[G \equiv H - TS] - VdP + SdT \\
&= dS - \beta dU + \frac{P}{T} dV
\end{aligned}$$

The particular transformations of replacing $u \to u + p_i * q^i$ and doing the appropriate swap of p_i and q^i up to a sign is a Legendre transform and is a particular sort of contactomorphism. It can be done as a flow of a one parameter family of transformations determined by $F = \frac{1}{2}(p_i^2 + (q^i)^2)$ for time $\frac{\pi}{2}$. This is a spiral motion which rotates the p_i, q^i plane by a quarter turn as the vertical direction u gradually changes by a net $p_i * q^i$. Because in this case F is not dependent on any of the other fundamental variables or their conjugates, they remain unchanged.

$$\begin{aligned}
\dot{q}^i &= \frac{\partial F}{\partial p_i} \\
\dot{p}_i &= -\frac{\partial F}{\partial q^i} - p_i \frac{\partial F}{\partial u} \\
\dot{u} &= p_i \frac{\partial F}{\partial p_i} - F
\end{aligned}$$

If we call this vector field X_F we can calculate that

$$\begin{aligned}
i_{X_F} \alpha &= -F \\
L_{X_F} \alpha &= \frac{-\partial F}{\partial u} \alpha
\end{aligned}$$

which shows X_F preserves contact structure. We don't have to demand $L_V \alpha = 0$ because we are allowed to change $\alpha \to f\alpha$ while still keeping the maximally nonintegrable distribution of hyperplanes in the tangent spaces the same.

Chapter 6

Lie Groups

6.0.1 Definition (Lie Group) *A group object in the category of smooth manifolds. That is first and foremost an object of SmMan and then all the axioms for groups need to be Homs in SmMan namely smooth maps. So multiplication is smooth and inversion is smooth and all those compositions you build from them are too.*

6.0.2 Remark Caution: It is not a group which is also a smooth manifold. They need to play well together. ♢

6.0.3 Example ($SU(2)$) *We can parametrize a coordinate patch by*

$$
\begin{aligned}
g &= \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \\
a &= r_1 e^{i\theta_1} \\
b &= \sqrt{1-r_1^2} e^{i\theta_2}
\end{aligned}
$$

with coordinates r_1, $\theta_{1,2}$ in a patch so that we have gotten rid of the usual ambiguity in angles.

6.0.4 Exercise *Show that multiplication in $SU(2)$ is smooth.*

6.1 Invariant Vect

We have left and right trivializations of the bundle TG.

6.1.1 Lemma *The relation between brackets of vector fields defined before and Lie bracket on $\mathfrak{g} = T_e G$.*

6.2 Cartan Dynkin and Killing

6.2.1 Definition (Cartan Subalgebra)

6.2.2 Definition (Killing Form) *It is a symmetric bilinear form that is invariant in the sense that $B([x,y],z) = B(x,[y,z])$. It is given by $Tr(adxady)$ in the adjoint representation.*

6.2.3 Lemma *In a simple Lie algebra, all symmetric invariant bilinear forms must be multiples of this Killing form.*

6.2.4 Theorem (Cartan Criterion) *A lie algebra is semisimple if and only if the Killing form is nondegenerate.*

6.2.5 Definition (Roots, coroots and weights)

6.2.1 Type A

$A_n \equiv \mathfrak{sl}(n+1,\mathbb{C})$

6.2.2 Type B

$B_n \equiv \mathfrak{so}(2n+1,\mathbb{C})$

6.2.3 Type C

$C_n \equiv \mathfrak{sp}(2n, \mathbb{C})$

6.2.4 Type D

$D_n \equiv \mathfrak{so}(2n, \mathbb{C})$

6.2.5 Type E,F and G

They are the exception. Cue Mongols scene of Crash Course World History.

6.2.6 Definition (Maurer-Cartan Form)

6.3 Universal Enveloping Algebra

6.3.1 Definition (Chevalley-Eilenberg)

6.3.2 Theorem (Duflo Isomorphism)

6.3.3 Theorem (Hopf Structure)

6.3.4 Remark If we give $\mathfrak{g}^*$ a Lie bracket too in the right way, we can get Lie bi-algebras and Poisson-Lie Groups which is the beginning of the story for how to deform this entire subject to quantum groups. $\Diamond$

Chapter 7

Principal Bundles

7.1 Abelian

$$\begin{array}{ccc} U(1) & \hookrightarrow & L \\ & & \downarrow \\ & & S \end{array}$$

7.2 Frame Bundle

$$\begin{array}{ccc} GL(n) & \hookrightarrow & L \\ & & \downarrow \\ & & S \end{array}$$

7.3 Other Non-abelian

$$\begin{array}{ccc} SU(3) & \hookrightarrow & L \\ & & \downarrow \\ & & S \end{array}$$

7.4 Classifying Spaces

$$\begin{array}{ccc} P & \longrightarrow & EG \\ \downarrow & & \downarrow \\ X & \longrightarrow & BG \end{array}$$

For this we need some Section 9.2

7.5 Connections

Reluctantly, I'm going to pick a basis for the Lie algebra called t^a. These might be the standard Pauli matrices.

7.6 Curvature

7.6.1 Theorem (Bianchi)

7.7 Chern-Weil

$H^\bullet(BG,\mathbb{C}) \qquad \mathbb{C}[\mathfrak{g}]^G \qquad H^\bullet(M,\mathbb{C})$

7.8 Chern-Simons

Remember definition 6.2.6. It's back to haunt us.

$$S \;=\;$$

7.9 Classical Yang-Mills

$$S \;=\;$$

7.9.1 Instantons

One simpler way to satisfy the Yang-Mills equations above is to have $\star F = \pm F$. That way the nontrivial $d_A \star F = 0$ would be automatically solved by the trivial one which is just the Bianchi identity.

$$\int \mathrm{tr}\, F \wedge \star F \;\;\geq\;\; |\int \mathrm{tr}\, F \wedge F\,|$$

We also ask that F vanishes at ∞ so that we can talk about it on S^3 at ∞ instead of $\mathbb{R}^4$.

7.9.1 Theorem (Atiyah-Donaldson-Hitchin-Manin) *Let V and W be complex vector spaces of dimensions k and N. Define $B_{1,2} \in Hom(V,V)$, $I \in Hom(W,V)$ and $J \in Hom(V,W)$.*

Consider the locus satisfying

$$\begin{aligned} \mu_r &\equiv \left[B_1, B_1^\dagger\right] + \left[B_2, B_2^\dagger\right] + II^\dagger - J^\dagger J = 0 \in Hom(V,V) \\ \mu_c &\equiv \left[B_1, B_2\right] + IJ = 0 \in Hom(V,V) \end{aligned}$$

From this we can construct an anti-self-dual instanton in $SU(N)$ gauge theory with instanton number k. All the anti-self-dual instantons can be obtained this way. Two different solutions of the above that differ by $U(k)$ action induced by acting on V give the same instanton. We can give the moduli space of instantons a metric and that metric is the one inherited from the flat metric of the B, I and J since that is just an affine space of complex dimension $2k^2 + 2kN$.

Proof Let the point of spacetime x be written in quaternionic form

$$x_{ij} = \begin{pmatrix} z_2 & z_1 \\ -\bar{z}_1 & \bar{z}_2 \end{pmatrix}$$

Arrange I, J, B_1 and B_2 in the following matrix.

$$\Delta =$$

The moment map equations $\mu_r = \mu_c = 0$ imply that

$$\Delta\Delta^\dagger = \begin{pmatrix} f^{-1} & 0 \\ 0 & f^{-1} \end{pmatrix}$$

The null space of Δ can be arranged into a unitary matrix U

$$A_m = U^\dagger \partial_m U$$

Everything depends on the spacetime position through $z_{1,2}$ and $\bar{z}_{1,2}$.

7.9.2 Remark If we let the spacetime be noncommutative, then we can set μ to something else that depends on this noncommutativity. In this we get instantons even in the $U(1)$ case. These are due to Nekrasov-Schwarz. $\diamondsuit$

Chapter 8

Algebraic Topology 1

8.1 Homotopy Theory 1 - Fundamental Groups

8.1.1 Theorem (Seifert-Van Kampen) *Let X be a topological space which is the union of U_1 and U_2 which are open and path connected. Suppose $U_1 \bigcap U_2$ is path connected and not empty. Let $x_0 \in U_1 \bigcap U_2$ be the basepoint.*

$$\begin{tikzcd}
 & \pi_1(U_1) \arrow[dr, dotted] & \\
\pi_1(U_1 \bigcap U_2, x_0) \arrow[ur] \arrow[dr] & & \pi_1(X) \\
 & \pi_2(U_2) \arrow[ur, dotted] &
\end{tikzcd}$$

The arrows are from the inclusion maps. The dotted arrows mean that $\pi_1(X)$ is the pushout. That translates to the fact that $\pi_1(X)$ is isomorphic to $\pi_1(U_1) \star_{\pi_1(U_1 \bigcap U_2)} \pi_2(U_2)$. This is an amalgamation that generalizes the free product of groups.

If each of the ingredients are finitely presented. Then you can immediately write down a finite presentation of the amalgamation.

You can be more general if you use groupoids. This avoids the issue with x_0 and path connectedness.

8.1.2 Exercise *What is the fundamental group of a n-torus ($S^1 \times \cdots S^1$ n times)*

8.1.3 Exercise *What is the fundamental group of a figure 8? Of a flower shape with k petals (where the figure 8 has 2 petals)?*

8.2 Derived Category of R-modules

8.2.1 Abelian Category Crash Course

8.2.1 Definition (Additive Category)

8.2.2 Definition (Abelian Category)

8.2.2 (Co)Chain Complexes

$$C^i \longrightarrow C^{i-1} \longrightarrow C^{i-2} \longrightarrow \cdots \longrightarrow C^j$$

There is a reason we said Theorem 3.5.2 was the most important formula of all.

8.2.3 Definition (Ext)

8.2.4 Exercise *Compute $Ext^1(\mathbb{Z}_2, \mathbb{Z}_3)$.*

8.2.5 Definition (Tor)

Figure 8.1: Insert a Simplicial Complex Picture

8.3 Homology

8.3.1 Exercise *Compute the simplicial homology of fig. 8.1*

8.3.2 Theorem (Universal Coefficient)

8.4 Cohomology

8.4.1 Theorem (Universal Coefficient)

8.4.1 deRham Cohomology

Remember Section 3.5?

8.4.2 Integration As Pushforward

Back when we did Section 3.6

8.5 Hodge Theory

8.6 Morse Theory

8.6.1 Definition (Morse Function) *A function f in the set J of all smooth functions $M \to \mathbb{R}$ (equipped with C^∞ topology) is called Morse if critical points are isolated and nondegenerate and all critical values are distinct.*

8.6.2 Lemma *Morse functions are open and dense in J.*

Proof For dense, do the perturbation in local models. □

8.6.3 Theorem (Witten's Perspective) *Supersymmetric Quantum Mechanics*

8.6.1 Cerf Theory

8.6.4 Definition (Cerf) *There is a stratification of J by strata of various codimensions. Then a one parameter family f_s that hits the codimension 1 walls transversely at isolated points.*

As you surf along the parameter, you will encounter some s_i where you see births and deaths of critical points.

8.6.5 Exercise *What are the critical values of $f(x) = x^3 + (1-s)x$ as a function of the parameter s?*

8.6.6 Lemma (Codim 2, two parameters s and t) *Some of the possible codimension 2 strata describe the blinking eye where nothing to a birth death process back to nothing, can also get a swallowtail. There is also the death, birth pair turning into disjoint.*

Figure 8.2: Pictures of hitting this strata

Chapter 9

2 Algebraic 2 Topology

9.1 K - Theory

9.1.1 Theorem (Atiyah-Singer Index)

Proof Pushforward proof of Family Index Theorem □

9.1.2 Remark We will see this really shine in it's really useful example Theorem 14.4.3 ◊

9.1.1 Real K-Theory

9.1.3 Exercise *Sing Twinkle Twinkle Little Star. Now sing the following:* $\mathbb{Z}_2$ $\mathbb{Z}_2$ 0 $\mathbb{Z}$ 0 0 0 $\mathbb{Z}$ *to the same tune.*

9.2 Homotopy Theory 2 - General

9.2.1 Definition (Suspension)

9.3 Spectra

9.3.1 Definition (Spectrum)

9.3.2 Example (Suspension)

9.3.3 Example (Eilenberg-MacLane)

9.4 Extraordinary Cohomology

9.5 Ring Spectra

9.5.1 Remark Make a Wagner Ring Cycle pun here. ◊

9.6 Cobordism

9.6.1 Definition (Cobordism Ring) *A cobordism ring is the graded ring whose elements in degree n are classes of n-dimensional smooth manifolds modulo cobordism. The product operation is given by Cartesian product. The sum operation is given by disjoint union. What is meant by cobordism can vary.*

9.6.2 Example (Oriented Cobordism Ring) $\Omega_\bullet^{SO}$ *is the cobordism ring where cobordisms between manifolds means oriented cobordism.*
So for example there is a cobordism from $+\bigsqcup-$ *to the empty set where* $+$ *and* $-$ *denote different orientations for points. However there is not a cobordism from* $+\bigsqcup+$ *to the empty set because the cobordism connecting the dots has to have opposite orientations at it's two endpoints. This in particular shows that* $\Omega_0^{SO} \simeq \mathbb{Z}$ *by counting the difference between number of positive and negative points.*

9.6.3 Example (Unoriented Cobordism Ring) *$\Omega^O_\bullet$ is the cobordism ring where cobordisms between manifolds means unoriented cobordism.*
Unlike before, cobordisms can now connect any pair of dots to the empty set. So instead we get $\Omega^O_0 \simeq \mathbb{Z}_2$

9.6.4 Definition (Relative Cobordism Ring) *Elements are classes modulo cobordism over X equipped with continuous functions to a fixed manifold X.*
Multiplication is given by taking $f_1\ \Sigma_1 \to X$ and $f_2\ \Sigma_2 \to X$ to the pullback

$$\begin{array}{ccc} \Sigma_1 \times_{f_3,f_2} \Sigma_2 & \longrightarrow & \Sigma_1 \\ \downarrow & & \downarrow \\ \Sigma_2 & \longrightarrow & X \end{array}$$

where f_3 is a perturbation of f_1 so that the maps become transversal.

9.6.5 Definition (Relative Cobordism Ring with Boundary) *Now in addition to the condition that Σ map to X, there is also the condition that $\partial\Sigma$ must land in A.*
In fact this fits to a functor which on objects acts as $(X, A) \to \Omega^G_\bullet(X, A)$ where G is as SO and O before controlling what sorts of cobordisms are allowed.
This is actually a representable functor and so by Brown representability there is a universal Thom spectra MG that represents it. It is a ring spectrum under Whitney sum of universal vector bundles.

9.6.6 Corollary (Evaluate on a point)

$$\Omega^G_\bullet(pt) \quad = \quad MG_\bullet(pt) = \pi_\bullet(MG)$$

9.6.7 Example *If the structure is framing structure rather than orientation or spin etc, then $MFr \simeq S$ the sphere spectrum. So the framed cobordism ring is the ring structure for stable homotopy groups of spheres.*

9.6.8 Definition (Genus) *A ring homomorphism out of some $\Omega_\bullet$ to any other ring R.*
From the realization that $\Omega_\bullet$ is the ring of coefficents for some MG, it is natural to ask for a map or ring spectra $MG \to E$ such that on coefficients it becomes $\Omega_\bullet \to R$.
From a formal power series Q in variable z, we can define a genus by

$$\begin{aligned} K(p_1 \cdots) &\equiv Q(z_1)Q(z_2)\cdots \\ \phi(X) &= K(p_1, \cdots) \end{aligned}$$

where in the first line p_i are elementary symmetric functions in the z variables. The RHS of that definition is symmetric so we know that such an expression can be written.
Then to evaluate ϕ on the class of the manifold X, plug in the certain classes of X for the p_i into K. You only need to worry about the terms of the dimension of X, so the computation is not impossible.

9.6.9 Example (Signature)

$$Q(z) = \frac{\sqrt{z}}{\tanh(\sqrt{z})} = \sum_{k\geq 0} \frac{2^{2k} B_{2k} z^k}{(2k)!}$$

The signature σ is the signature of the intersection form on the $2n$'th cohomology group of a $4n$ dimensional manifold. In the particular case of $n=1$, this is very useful chapter 17.

$$\sigma(M) = \langle L_n(p_1(M)\cdots p_n(M))[M]\rangle$$

where L_n is the term with the degree $4n$ where each of the p_i are in degree $4i$ when expanding K.

9.6.10 Example (Todd Genus)

$$Q(z) = \frac{z}{1-e^{-z}}$$

The Todd class of all $\mathbb{C}P^n$ are 1 so it agrees with arithmetic genus for all algebraic varieties. This is part of Riemann-Roch ideas.

Plug in c_i for p_i and interpret them as Chern classes.

9.6.11 Example ($\hat{A}$)

$$Q(z) = \frac{\sqrt{z}/2}{\sinh(\sqrt{z}/2)}$$

If the manifold is spin, $\hat{A}(M)$ is an integer. If the manifold is even dimensional and spin, $\hat{A}(M)$ is an even integer. In fact for spin manifolds this value is the index of the Dirac operator as we will see in Theorem 9.1.1.

9.6.12 Definition (Elliptic Genus) *If $f(z)$ satisfies $(f')^2 = 1 - 2\delta f^2 + \epsilon f^4$ for some constants δ and ϵ, $Q(z) = \frac{z}{f(z)}$ defines a genus.*

- *$f(z) = \tanh(z)$, satisfies the equation with $\delta = \epsilon = 1$ giving back example 9.6.9.*
- *$f(z) = \tanh(\sqrt{\delta}z)/\sqrt{\delta}$, satisfies the equation with $\delta^2 = \epsilon$ generalizing the example 9.6.9 in a one parameter family.*
- *$f(z) = 2\sinh(z/2)$, satisfies the equation with $\delta = \frac{-1}{8}$ and $\epsilon = 0$ giving back example 9.6.11.*

9.6.13 Definition (Witten Genus)

$$\begin{aligned} Q(z) &= \frac{z}{\sigma_L(z)} = \exp(\sum_{k\geq 2} \frac{2G_{2k}(\tau) z^{2k}}{(2k)!}) \\ &= \frac{z/2}{\sinh(z/2)} \prod_{n\geq 1} \frac{(1-q^n)^2}{(1-q^n e^z)(1-q^n e^{-z})} \end{aligned}$$

where σ_L is the Weierstrauss σ function for the lattice L. $G_{2k}(\tau)$ are multiples of Eisenstein series. So we can see that this defines a genus $\Omega_\bullet^{SO} \to \mathbb{Q}[[q]]$ by looking at the formula in terms of q rather than τ or L.

The Witten genus of a $4k$ dimensional compact oriented smooth spin manifold with vanishing p_1 is a modular form with weight $2k$ and integral Fourier coefficents.

On manfiolds with rational string structure, it is a q-expansion of a modular form. If the manifold has string structure rather than just rational string structure, then we get more specifically topological modular forms which is the coefficent ring of the ring spectrum tmf. Ordinary modular forms are seen by taking sections of ω^k where ω is the standard line bundle on $\overline{\mathcal{M}_{ell}}$ as compared to tmf which are global sections of the derived structure sheaf of $\overline{\mathcal{M}_{ell}}$. In particular there is a map $tmf_{2\bullet} \to MF_\bullet$ which realizes this forgetting from actual to rational string structure.

Chapter 10

Riemannian Geometry

10.1 Metrics

Recall we did existence of Riemannian metrics in Section 2.3

Maybe Section 3.7 needs to go here.

10.2 Christ-Awful Symbols

Remember Section 7.5 and Section 7.2

10.2.1 Theorem (Levi-Cevita)

10.2.2 Exercise *Compute the Christoffel symbols for the standard sphere of radius R on the usual ϕ θ coordinate patch.*

10.3 Riemann and Ricci

10.4 Laplacians

10.4.1 Definition (Integration Pairing) *Let V be vector bundle over M and $\langle -, - \rangle_V$ be an inner product on it's fibers. By an abuse of notation let it also be the pairing that takes two sections to a function on M.*

$$\langle \alpha \mid\mid \beta \rangle \quad \equiv \quad \int_M \langle \alpha \mid \beta \rangle_V dVol_M$$

This requires integrating a smooth function against the volume density $dVol_M$, so the $dVol_M$ has to exist and the integral must converge. This is why we impose that M is a compact Riemannian manifold without boundary.

If we don't need this much generality, one can treat V as something of the form $TX^{\otimes r} \otimes T^*X^{\otimes s}$ as in Section 3.10 so only considering vector bundles that come from M intrinsically.

Here we are only talking about real vector bundles and the inner product is on real vector spaces.

10.4.2 Definition (Laplace-Beltrami)

10.4.3 Definition (Hodge Laplacian) *We can use Hodge star operator definition 3.7.1 to define a codifferential*

$$\delta \quad \equiv \quad (-1)^{\cdots} \star d \star$$

Then from that the Hodge Laplacian is

$$\Delta \quad \equiv \quad \left[d, \delta \right]_+ = d\delta + \delta d$$

Note that because d and δ cohomological degree ± 1 which is odd, the anti-commutator is the god-given sign rule for a bracket between them.

10.4.4 Exercise *Do the cases of $\mathbb{R}^n$ in Euclidean and spherical coordinates and $\mathbb{S}^{n-1}$*

10.4.1 Ray-Singer Torsion

10.4.5 Definition *Let E be a vector bundle and consider the Laplacians Δ_p acting on sections of $\Omega^p \otimes E$. Then we can construct the alternating product*

$$T(M,E) \equiv \prod_p \left(\det_{reg} \Delta_{p,E}\right)^{(-1)^{p+1}\frac{p}{2}}$$

For this we need a quick aside on ζ-regularization because otherwise we would be taking a product of infinitely many terms in a nonsense way.

10.4.6 Definition (Zeta Regularize) *For s with large enough real part that this converges, define*

$$\zeta(s) = \sum_{\lambda>0} \lambda^{-s}$$

where λ are eigenvalues of the operator in question ($\Delta_{p,E}$ above).
Then

$$\begin{aligned} e^{-\frac{d}{ds}\zeta(0)} & \quad \sum e^{\lambda^{-s}\log\lambda|_{s=0}} \\ &= \sum e^{\log\lambda} = \prod \lambda \end{aligned}$$

where the   is for the intuition as to why this works, but not strictly true because $s=0$ is not in the domain of $\zeta(s)$ before analytic continuation. That explicit sum formula does not apply in a neighborhood of $s=0$ of course unless the operator was just a finite dimensional matrix and the constriction on the real part of s was vacuous and analytic continuation was unnecessary.

10.4.7 Theorem (Cheeger-Muller)

10.5 Information Geometry

10.5.1 Definition (Conjugate Connections)

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z)$$

Conjugation for a fixed metric g gives an involution on the space of connections.

10.5.2 Lemma *Dual parallel transport preserves the metric. That is to say if you have two vectors u and v in T_pM and take their inner product with g_p, you can transport one along a path with ∇ and the other along that path with ∇^* and that inner product with g_q will still give you the same real number.*

10.5.3 Corollary *Levi-Cevita is self conjugate. In fact if you take any dual pair and average $\frac{\nabla+\nabla^*}{2}$ you get a self-conjugate connection which coincides with Levi-Cevita by theorem 10.2.1 and the definition of conjugate connections in the self-conjugate case.*

10.5.4 Lemma (α family) *For any (M, g, ∇, ∇^*) we can form a one parameter family ∇^α such that $(\nabla^\alpha)^* = \nabla^{-\alpha}$.*

Proof

$$\begin{aligned} C(\partial_i, \partial_j, \partial_k) = C_{ijk} &= \Gamma^k_{ij} - (\Gamma^*)^k_{ij} \\ &\iff C(X, Y, Z) \equiv g(\nabla_X Y - \nabla^*_X Y, Z) \end{aligned}$$

This defines a symmetric cubic tensor.
From this define ∇^α

$$g(\nabla^\alpha_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{\alpha}{2} C(X, Y, Z)$$

or by it's Christoffel symbols

$$(\Gamma^\alpha)_{ij;k} = \frac{1+\alpha}{2}\Gamma_{ij;k} + \frac{1-\alpha}{2}\Gamma^*_{ij;k}$$

10.5.5 Theorem (Constant Curvature) *If ∇ is a torsion free affine connection with curvature κ, then ∇^* does too.*
This is of particular interest when $\kappa = 0$.

10.5.6 Definition (Divergence) *Let S be a space of probability distributions on a common support. Then $D(- \mid -) \in S \times S \to \mathbb{R}$ is called a divergence if it satisfies*

$$\begin{aligned} D(p \mid q) &\geq 0 \\ D(p \mid q) = 0 &\iff p = q \end{aligned}$$

It does not have to be symmetric or satisfy the triangle inequality, so is weaker than giving S the structure of a metric space.

If f is a convex function on $\mathbb{R}_+$ with $f(1) = 0$ and the space of common support has an integration measure $d\mu_X$, an f divergence is the divergence

$$D_f(p \mid q) \equiv \int_X p(x) f(\frac{q(x)}{p(x)}) d\mu_X$$

10.5.7 Example (Kullback-Leibler)

$$\begin{aligned} D_{KL}(p \mid q) &\equiv \int_X p(x) \ln(\frac{p(x)}{q(x)}) d\mu_X \\ &= \int_X p(x)(-\ln(\frac{q(x)}{p(x)})) d\mu_X \end{aligned}$$

So the f divergence where $f(u) = -\ln u$

10.5.8 Example (Reverse Kullback-Leibler)

$$
\begin{aligned}
D_{RKL}(p \mid q) &\equiv D_{KL}(q \mid p) \\
&= \int_X q(x) \ln(\frac{q(x)}{p(x)}) d\mu_X \\
&= \int_X p(x) \big(\frac{q(x)}{p(x)} \ln(\frac{q(x)}{p(x)}) \big) d\mu_X
\end{aligned}
$$

So the f divergence where $f(u) = u \ln u$

10.5.9 Example (Squared Hellinger)

$$
\begin{aligned}
D_{Hellinger}(p \mid q) &\equiv \int_X (\sqrt{p(x)} - \sqrt{q(x)})^2 d\mu_X \\
&= \int_X p(x) - 2\sqrt{p(x)q(x)} + q(x) d\mu_X \\
&= \int_X p(x)(1 - 2\sqrt{\frac{p(x)}{q(x)}} + \frac{q(x)}{p(x)}) d\mu_X \\
&= \int_X p(x)(1 - \sqrt{\frac{p(x)}{q(x)}})^2 d\mu_X
\end{aligned}
$$

$f(u) = (1 - \sqrt{u})^2$

10.5.10 Example (α-divergence) *The f-divergences where the function*

$$
\begin{aligned}
D_\alpha(p \mid q) &\equiv \frac{4}{1-\alpha^2}(1 - \int_X p(x)^{(1-\alpha)/2} q(x)^{(1+\alpha)/2} d\mu_X) \\
&= \frac{4}{1-\alpha^2}(1 - \int_X p(x) p(x)^{-(1+\alpha)/2} q(x)^{(1+\alpha)/2} d\mu_X) \\
&= \frac{4}{1-\alpha^2} \int_X p(x)(1 - \frac{q(x)}{p(x)}^{(1+\alpha)/2}) d\mu_X \\
&= \int_X p(x) \big(\frac{4}{1-\alpha^2}(1 - \frac{q(x)}{p(x)}^{(1+\alpha)/2}) \big) d\mu_X
\end{aligned}
$$

$f(u) = \big(\frac{4}{1-\alpha^2}(1 - u^{(1+\alpha)/2}) \big)$.
If you plug in $\alpha = 0$,

$$
\begin{aligned}
D_{\alpha=0}(p \mid q) &= 4 - 2\int_X 2\sqrt{p(x)q(x)}d\mu_X \\
&= 2\int_X p(x)d\mu_X + 2\int_X q(x)d\mu_X - 2\int_X 2\sqrt{p(x)q(x)}d\mu_X \\
&= 2(\int_X p(x) + q(x) - 2\sqrt{p(x)q(x)})d\mu_X \\
&= 2\int_X (\sqrt{p(x)} - \sqrt{q(x)})^2 d\mu_X \\
&= 2D_{Hellinger}(p \mid q)
\end{aligned}
$$

Similarly for $\alpha \to \pm 1$ to give quantities related to D_{KL} and D_{RKL} respectively.

10.5.11 Theorem (Dual Connections from Divergence)

10.5.12 Theorem (α paramaterization) *All invariant standard f-divergences give the Fisher information metric usually done from Kullback-Leibler. The α in the definition of α-divergences lines up with ∇^α*

Chapter 11

Pseudo-Riemannian Geometry aka GR

11.1 Einstein Equations

An equation for sections of S^2T^*X recalling Section 3.10

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^4}T_{\mu\nu}$$

where $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar as in Section 10.3. The same formulas apply but now for this indefinite signature.

11.2 Black Holes

11.2.1 Schwarschild

In this chosen coordinate patch, the metric is diagonal and gives the line element:

$$\begin{aligned} c^2 d\tau^2 &= (1 - \frac{r_s}{r})c^2dt^2 - (1 - \frac{r_s}{r})^{-1}dr^2 - r^2 d\Omega^2 \\ r_s &= \frac{2GM}{c^2} \end{aligned}$$

11.2.2 Kerr

Yes to rotation. No to charge.

11.2.1 Definition (Penrose Process)

11.2.3 Reissner-Nordstr'́om

No to rotation. Yes to charge.

11.2.4 Kerr-Newman

Yes to rotation. Yes to charge.

11.3 Cosmological

11.3.1 Definition (FLRW)

$$d^2 \equiv -dt^2 + a^2(t)(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2))$$

Suppose there is a perfect fluid for the stress-energy tensor.

$$T_{\mu\nu} = (p+\rho)U_\mu U_\nu + pg_{\mu\nu}$$

The isotropy of the fluid and the isotropy of the metric that we are taking as the ansatz imply that $U^\mu = \partial_t = (1,0,0,0)$.

$$\begin{aligned} T^\mu_\nu &= \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \\ T &= -\rho + 3p \end{aligned}$$

Einstein equations become the Friedmann equations.

$$\begin{aligned} \frac{\dot{a}^2 + kc^2}{a^2} &= \frac{8\pi G_N \rho + \Lambda c^2}{3} \\ \frac{\ddot{a}}{a} &= \frac{-4\pi G_N}{3}(\rho + \frac{3p}{c^2}) + \frac{\Lambda c^2}{3} \end{aligned}$$

One can absorb Λ *into the fluid itself, but let's leave it explicit.*

11.4 Conformal Perspective

11.4.1 Definition (Conformal Transformation)

11.4.2 Lemma (Causality)

We didn't say it in the Riemannian geometry chapter, but by the way for surfaces we have

11.4.3 Theorem (Uniformization Theorem)

11.4.4 Definition (Penrose Diagram)

11.4.5 Definition (Causal Features)

- i^+
- i^-
- i^0
- $\mathcal{I}^+$
- $\mathcal{I}^-$

11.4.6 Example (Minkowski) *Write the metric in spherical coordinates as* $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$.

Change coordinates by

$$\begin{aligned}
u &\equiv t - r \\
v &\equiv t + r \\
ds^2 &= -du dv + r^2 d\Omega^2 \\
\tilde{u} &\equiv \tanh u \\
\tilde{v} &\equiv \tanh v \\
ds^2 &= \frac{-1}{(1-\tilde{u}^2)(1-\tilde{v}^2)} d\tilde{u} d\tilde{v} + r^2 d\Omega^2
\end{aligned}$$

$-1 < \tilde{u} \leq \tilde{v} < 1$. *Suppressing the* $d\Omega^2$*, we get a diagram of all of Minkowski space in a triangle. The causal structure is still manifest so that the* 45^o *lines are the projections of the light geodesics (dropping the angular information).*

11.5 Energy Conditions

11.5.1 Definition (Perfect Fluid) $T^{ab} = \rho u^a u^b + p(g^{ab} + u^a u^b)$ *where* u *is the normalized d-velocity (*$d = 4$ *physically) of the matter in the fluid. Normalization means that* $u_a u^a = -1$.

11.5.2 Example (Cosmological Constant) *A cosmological constant is the same as putting* $\rho = \Lambda$ *and* $p = -\Lambda$.

11.5.3 Definition (Null Energy condition) *Let* k *be a future pointing null vector field. The null energy condition states that* $T_{ab}k^a k^b \geq 0$
The averaged null energy condition weakens this to $\int_C d\lambda T_{ab} k^a k^b \geq 0$ *for every integral curve* C *of* k*. This is weaker than the pointwise condition above which will fail immediately upon quantum effects because of the Casimir effect.*
In the perfect fluid case, null energy condition becomes $\rho + p \geq 0$*. Cosmological constant satisfies this.*

11.5.4 Definition (Weak Energy condition) *Let* k *be a timelike vector field. The weak energy condition states that* $T_{ab}k^a k^b \geq 0$
In the perfect fluid case, weak energy condition becomes $\rho \geq 0 \wedge \rho + p \geq 0$*. Nonnegative cosmological constant satisfies this, but negative does not.*

11.5.5 Definition (Dominant Energy condition) *Let* k *be a future pointing causal (timelike or null) vector field. The dominant energy condition states that* $-T^\mu_\nu k^\nu$ *is a future pointing causal vector field too.*
In the perfect fluid case, dominant energy condition becomes $\rho \geq \mid p \mid \geq 0$*. Nonnegative cosmological constant satisfies this, but negative does not.*

11.5.6 Definition (Strong Energy condition) *Let* k *be a timelike vector field. The strong energy condition states that* $(T_{ab} - \frac{1}{n-2} T g_{ab}) k^a k^b \geq 0$.
In the perfect fluid case, strong energy condition becomes $\rho + p \geq 0 \wedge (n-3)\rho + (n-1)p \geq 0$*. Nonpositive cosmological constant satisfies this, but positive does not.*

11.6 Singularity Theorems

11.6.1 Theorem (Raychaudhuri) *Consider a smooth congruence of timelike geodesics.* [1] *Let ξ^a be the associated smooth vector field to this congruence, by parameterizing all the curves with proper time, we can assume that $\xi^a \xi_a = -1$.*

From this vector field define the quantities

$$\begin{aligned} B_{ab} &\equiv \nabla_b \xi_a \\ h_{ab} &\equiv g_{ab} - \xi_a \xi_b \\ \theta &\equiv B^{ab} h_{ab} \\ \sigma_{ab} &\equiv B_{(ab)} - \frac{1}{3}\theta h_{ab} \\ \omega_{ab} &\equiv B_{[a,b]} \end{aligned}$$

These quantities decompose B_{ab} into trace, symmetric and antisymmetric parts and obey the following system of first order differential equations.

$$\begin{aligned} \xi^c \nabla_c \theta = \frac{d\theta}{d\tau} &= \frac{-1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{cd}\xi^c\xi^d \\ \xi^c \nabla_c \sigma_{ab} &= \frac{-2}{3}\theta \sigma_{ab} - \sigma_{ac}\sigma^c_b - \omega_{ac}\omega^c_b \\ &+ \frac{1}{3}h_{ab}(\sigma_{cd}\sigma^{cd} - \omega_{cd}\omega^{cd}) + C_{cbad}\xi^c\xi^d + \frac{1}{2}(h_{ac}h_{bd}R^{cd} - \frac{1}{3}h_{ab}h_{cd}R^{cd}) \\ \xi^c \nabla_c \omega_{ab} &= \frac{-2}{3}\theta\omega_{ab} - 2\sigma^c_{[b}\omega_{a]c} \end{aligned}$$

In particular for a congruence that is hypersurface orthogonal we have $\omega_{ab} = 0$ so if $R_{cd}\xi^c\xi^d \geq 0$ by some energy/curvature condition, we get

$$\begin{aligned} \frac{d\theta}{d\tau} &\leq \frac{-1}{3}\theta^2 \\ \frac{-1}{\theta^2}\frac{d\theta}{d\tau} &\geq \frac{1}{3} \\ \frac{d}{d\tau}(\frac{1}{\theta}) &\geq \frac{1}{3} \\ \frac{1}{\theta} - \frac{1}{\theta_0} &\geq \frac{1}{3}\tau \end{aligned}$$

which if θ_0 is negative means that $\theta \to -\infty$ within proper time $\frac{3}{-\theta_0}$. This holds along each geodesic in the congruence.

[1] A congruence in O for an open subset of spacetime means that there is a family of curves and that for each point $p \in O$, there is exactly one member of the family passing through. If you take the tangents of these curves you get a vector field.

11.6.2 Theorem (Penrose) *Suppose the spacetime M is connected globally hyperbolic, contains a non-compact Cauchy hypersurface Σ, a closed future-trapped surface $S \subset \Sigma$ and $R_{\mu\nu} u^\mu u^\nu \geq 0$ holds for all null vector fields u. Then there are future-directed incomplete null geodesics.*

Future-trapped means that a spacelike submanifold S has the condition that the mean curvature vector field H^μ is timelike and future pointing everywhere on S.

Proof Because M is globally hyperbolic, there exists a time function $t\ M \to \mathbb{R}$. We can arrange that time function such that $\Sigma = t^{-1}(0)$. Let v be the gradient vector field of t. Each (fully extended) integral curve of v intersects Σ exactly once. This follows from v being timelike and Σ being a Cauchy hypersurface. They intersect $\partial I^+ S$ (the boundary of the chronological future of S) at most once by lemma 11.6.3. Use this to define a continuous injective map $\pi\ \ \partial I^+ S \to \Sigma$ with open image. Do this by taking the integral curve for v passing through $x \in \partial I^+ S$ and finding it's intersection with Σ called z. This is injective because if we had two integral curves one connecting $x_1 \to z$ and another connecting $x_2 \to z$ then the fact they both pass through z means they must be the same curve by uniqueness of ODE given initial conditions. But then this would be an integral curve intersecting $\partial I^+ S$ twice which cannot happen. Continuity is lemma 11.6.4

Let γ be a future-directed null geodesic orthogonal to S. It must have a conjugate point within an affine parameter distance $\frac{-2}{\theta_0}$ to the future of S lemma 11.6.5. After that it must be in $I^+ S$ by a later lemma lemma 11.6.6. By lemma 11.6.7 this would mean $\partial I^+ S$ would be compact. The image under π would also be compact being the continuous image of a compact set. Compact and Hausdorff then implies closed. Therefore this image must in Σ must be a component of Σ. Σ is connected because $M \simeq \mathbb{R} \times \Sigma$ is. Either this image is the empty set, but that is excluded because $\partial I^+ S$ is not empty. So it must be all of Σ. A continuous bijective map from a compact space to a Hausdorff space is a homeomorphism. This then contradicts the assumption that Σ was noncompact.

11.6.3 Lemma ($\partial I^+ S$ intersection) *Suppose γ is a timelike curve. It can intersect $\partial I^+ S$ at most once.*

Proof Suppose it intersects twice first at x, then y. Then $y \in I^+(x)$. But if $x \in \partial I^+(S)$ and $y \in I^+(x)$ then $y \in I^+ S$ not the boundary.

To see this note that $x \in I^-(y)$. Take an open neighborhood of x inside this open set. With x on the boundary of $I^+(S)$ we must have that this neighborhood intersected with $I^+ S$ is nonempty. So $I^-(y) \bigcap I^+ S$ is nonempty. Let z be a point in this intersection. We have a chronological curve $z \to y$ and a chronological curve $s \to z$ for some s. Connect these and patch up the neighborhood of z for any necessary differentiability. This gives $y \in I^+ S$. □

11.6.4 Lemma (Continuity of π)

11.6.5 Lemma ()

11.6.6 Lemma ()

11.6.7 Lemma ()

11.6.8 Theorem (Hawking)

Chapter 12

Complex Geometry

12.1 Almost Complex

12.1.1 Definition (Nijenhuis)

12.2 Hodge Theory

12.3 Kähler

12.3.1 Example (Complex Projective Space)

12.3.2 Theorem (Hard Lefschetz) *On a compact Kähler manifold (X,ω) of complex dimension d, we can define an operator $L = \wedge\omega$. In this case L^k is an isomorphism on $H^{d-k} \simeq H^{d+k}$. This further means that we can decompose further*

$$\begin{aligned} P^{d-k} &\equiv ker(L^{k+1} \in H^{d-k} \to H^{d+k+2}) \\ P^{d-2k} &\equiv ker(L^{2k+1} \in H^{d-2k} \to H^{d+2k+2}) \\ H^d &\simeq \bigoplus L^k P^{d-2k} \end{aligned}$$

The idea is L^k is an isomorphism, but if you do it just one more time you get a little bit of kernel. That is why this notion of primitive cohomology classes P^{d-k} are defined.

In fact knowing that ω is $(1,1)$ in the Dolbeault sense we can see that $L^{d-(p+q)}$ gives the isomorphism $H^{p,q} \simeq H^{d-q,d-p}$.

$$\begin{aligned} p + (d-(p+q)) &= d-q \\ q + (d-(p+q)) &= d-p \end{aligned}$$

So it is not just the H^{p+q} are isomorphic to $H^{2d-(p+q)}$ via $L^{d-(p+q)}$ but also the individual summands in the Dolbeaut decomposition.

12.4 Kodaira

12.4.1 Remark If we had an excuse to put Kummer, then we could have climbed the K3 surface.$\Diamond$

12.5 Holomorphic Symplectic

12.5.1 Remark The main reference for this section has been `https://math.unice.fr/~beauvill/conf/Lisbon.pdf` $\Diamond$

In Section 5.2, we discussed the real case. But we can replace the 2-form ω with ω_{hol} which is holomorphic. So instead of something like $dp \wedge dq$ with p and q being real valued coordinates.

$$\omega_{hol} = \sum_{i=1}^{n} dz_i \wedge dw_i$$

with z_i and w_i are complex valued. This is no longer a true symplectic form because of vector fields like $\frac{\partial}{\partial \bar{z}_i}$. So ω_{hol} is not quite non-degenerate for all vector fields, but only holomorphic ones.

12.5.2 Definition (Holomorphic Symplectic Manifold) *X is a compact, Kähler, simply-connected manifold such that it admits a ω_{hol} which is unique up to $\mathbb{C}^*$.*
A consequence of this is that the complex dimension of X is even $2r$ for some r, so it is of real dimension $4r$. The canonical bundle $K_X \equiv \Omega_X^{2r}$ is trivial and generated by ω_{hol}^r.

12.5.3 Theorem (Decomposition Theorem) *If X is compact, Kähler with $K_X \simeq \mathcal{O}_X$, then there exists $\tilde{X} \to X$ étale finite cover such that*

$$\tilde{X} = T \times \prod_i Y_i \times \prod_j Z_j$$

with $T = \mathbb{C}^g/L$ for some lattice, Y_i being holomorphic symplectic and Z_j being simply-connected projective Calabi-Yau manifolds of real dimension $2n \geq 3$ with $h^{0,0} = 1 = h^{0,n}$ and otherwise $h^{0,j} = 0$.

So Calabi-Yau and holomorphic symplectic manifolds form the harder building blocks of these sort of K_X trivial compact Kähler manifolds.

12.5.4 Example *If S is a K3 surface as in Section 12.4, then $Hilb^n S$ are holomorphic symplectic of complex dimension $2n$.*

12.5.5 Remark There are also generalized Kummer manifolds which are constructed similarly and 2 isolated examples in complex dimension 6 and 10. Every known example fits somewhere in this classification. ♢

12.5.6 Proposition (Period Map) *There exists a q quadratic form on $H^2(X, \mathbb{Z})$ with values in $\mathbb{Z}$ such that there exists an $f \in \mathbb{Z}$ satisfying*

$$\int_X \alpha^{2r} = f q(\alpha)^r$$

for all $\alpha \in H^2(X, \mathbb{Z})$. So for any other lattice, we can make a space $\mathcal{M}_L$ parameterizing pairs (X, λ) where λ is an isomorphism $(H^2(X, \mathbb{Z}), q) \simeq L$.

In particular, we can complexify λ to get an isomorphism $H^2(X, \mathbb{C}) \simeq L_{\mathbb{C}}$ and then plug in $\mathbb{C}\omega_{hol}$ to get a point in $Proj(L_{\mathbb{C}})$. This is called the period map.

12.5.7 Theorem *The period map lands in the locus Ω where $q(x) = 0$ and $q(x, \bar{x}) > 0$. In fact by Beauville there is a local isomorphism $\mathcal{M}_L \to \Omega$. By Huybrechts, the period map is surjective and by Verbitsky the restriction to any connected component of $\mathcal{M}_L$ is generically injective.*

12.6 Quantum Fisher Information

This is the quantum analog of Section 10.5 section before.

Consider a finite dimensional Hilbert space of dimension N. The states are parameterized by $\mathbb{CP}^N$.

12.6.1 Definition (Symmetric Logarithmic Derivative) *Let $\rho(\lambda)$ be a family of states The symmetric logarithmic derivative $L(\rho)$ is the operator that satisfies*

$$\dot{\rho} = \frac{1}{2}(L\rho + \rho L)$$

12.6.2 Lemma *Let $\rho = e^{D(\lambda)}$ like some Gibbs ensemble with varying temperatures, chemical potentials etc.*
Then L can be calculated as

$$\begin{aligned} f &\equiv \frac{\tanh t/2}{t/2} \\ L &= f(\mathcal{C})(\dot{D}) \end{aligned}$$

Proof `https://arxiv.org/pdf/1912.12313.pdf`

$$\begin{aligned} \mathcal{C}(L) &\equiv [D, L] \\ r(t) &= \frac{e^t + 1}{2} \\ \dot{\rho}\rho^{-1} &= r(\mathcal{C})(L) \end{aligned}$$

where $r(\mathcal{C})(L)$ is interpreted as expanding $r(t)$ as a power series in t, then replacing t^n by the commutator with $[D, -]$ applied n times on L.

Suppose $L = f(\mathcal{C})(\dot{D})$ for some power series f. Do some more algebra

12.6.3 Definition (Quantum Fisher Metric)

12.6.4 Exercise *How is the Fisher Information metric on this related to the Fubini-Study metric?*

12.6.5 Remark It would be better if you just did the exercise yourself, but here are places with the solution.
John Baez Link
Contreras Ercolessi and Schiavina KKS ◊

Chapter 13

Spin Geometry

13.1 Clifford Algebra

13.1.1 Definition (Cliff) *For a quadratic vector space (V, Q), take the tensor algebra of V modulo the relation $v \otimes v + Q(v)1 = 0$. It is a quantization of the exterior algebra which is the case when $Q = 0$. They are $Cliff(V, q) \simeq Cliff(V, 0)$ as vector spaces. This is natural if char $k \neq 2$.*

This has the universal property for all j satisfying $j(v)^2 = -Q(v)1_A$

$$
\begin{tikzcd}
V \arrow[r, hook] \arrow[dr, "j"'] & Cliff(V,Q) \arrow[d, dashed, "!"] \\
& A
\end{tikzcd}
$$

Figure 13.1: Universal property: Cyan are vector space maps, blue are associative algebra maps

It also has the nice functorial properties. Under base change (V_k, Q) to $(V_k \otimes_k l, Q \otimes 1)$ gives $Cliff(V_l, Q_l) = l \otimes_k Cliff(V_k, Q)$. $Cliff(V, -Q) \simeq Cliff(V, Q)^{opp}$. (Careful that this is opposite as superalgebras) It takes direct sums of quadratic vector spaces (with $Q''(x + x') = Q(x) + Q(x')$) to the tensor product as superalgebras.

13.1.2 Exercise *Show that after realizing the superalgebra structure what looks like trying to identify $\mathbb{R}^2$ and $\mathbb{C}$ for $Cliff(1, -)$ and $Cliff(1, +)$ becomes something actuatlly true. This is the lesson that algebras were a mistake.*

$$
\begin{tikzcd}
& & Cliff(\mathbb{R}^{n-1},+) \arrow[d, "\wr\, \phi"] \\
& & Cliff_0(\mathbb{R}^n,+) \arrow[d] \\
Spin(\mathbb{R}^n,+) \arrow[r, hook] \arrow[urr, hook] \arrow[d] & Pin(\mathbb{R}^n,+) \arrow[r, hook] \arrow[d] & Cliff(\mathbb{R}^n,+) \\
SO(n) \arrow[r, hook] & O(n)
\end{tikzcd}
$$

Figure 13.2: Green are arrows for the group/multiplicative structure. Red is a algebra arrow. Blue is a superalgebra arrow where the source happens to have nothing odd.

The Pin is included as the group generated by the unit vectors $v \in V$. That is generated by all reflections. The Spin then just keeps the part in even grading. More detail about the red arrow is given by: If $V = k\langle a \rangle \bigoplus U$ as an orthogonal direct sum, then $Cliff_0(V, Q) \simeq Cliff(U, Q(a)Q \mid_U)$ as associative algebras only not superalgebras.

The only reason for $Pin_\pm$ is the drop in notation of the quadratic form. Pin_- is just included into $Cliff(\mathbb{R}^n, -)$ instead. These $Pin_\pm$'s are not isomorphic as groups even though they fit in the diagram below.

$$\begin{tikzcd}
Spin(n,+) \arrow[r, hook] \arrow[d, "\wr"] & Pin_+(n) \arrow[r, hook] & Cliff(n,+) \arrow[d, "\wr"] \\
Spin(n,-) \arrow[r, hook] & Pin_-(n) \arrow[r, hook] & Cliff(n,-)
\end{tikzcd}$$

Figure 13.3: Green:Grp and Cyan:SVect

13.1.3 Remark Under the functor J described in `http://mathoverflow.net/questions/185645/what-are-the-correct-conventions-for-defining-clifford-algebras` flips the two rows of this diagram. $\diamondsuit$

13.1.4 Theorem (Bott Periodicity) *In the superMorita bicategory of superalgebras, bimodules and bimodule homomorphisms, the superBrauer group is the abelian group of objects up to equivalence in this bicategory. The operation is from tensoring superalgebras. The class of* $Cliff(1,+) = R + Ri[1]$ *with* $i[1]^2 = -1$ *generates a* $\mathbb{Z}_8$ *here.*

Proof $Cliff(1,-) = R + Re[1]$ $(e[1]^2 = 1)$ is it's inverse. The map $Cliff(1,-) \otimes Cliff(1,+) \to Hom(\mathbb{R}^{1|1}, \mathbb{R}^{1|1})$ is

$$\begin{aligned}
e \otimes i &\to \\
e \otimes 1 &\to \\
1 \otimes i &\to
\end{aligned}$$

So $[Cliff(1,+)]$ is invertible. Now to find it's order. It's third power $[Cliff(3,+)]$ is equivalent to $H \otimes Cliff(1,-)$ so the fourth power will be equivalent to a pure bosonic H after cancelling the $Cliff(1,\pm)$ factors. And H is it's own inverse by conjugation. Together this gives order 8.

13.2 Representation Theory of *Spin*

13.2.1 $Spin(2,1)$

Isomorphic as a Lie group to $SL(2,\mathbb{R})$ accidentally.

Real Representation Category

Complex Representation Category

13.2.2 $Spin(3)$

Isomorphic as a Lie group to $SU(2)$ accidentally.

Real Representation Category

Complex Representation Category

13.2.3 $Spin(3,1)$

Isomorphic as a Lie group to $SL(2,\mathbb{C})$ accidentally. (remember this is not as an algebraic group over $\mathbb{C}$ so you do have to complexify again for complex representations.)

Real Representation Category

Complex Representation Category

Irreducible objects are labelled by pairs of half-integers. This is by the description as pairs of $SL(2,\mathbb{C})$ representations.

- $(0,0)$ Complex scalar
- $(1/2,0)$ Left handed Weyl
- $(0,1/2)$ Right handed Weyl
- $(1/2,0)\bigoplus(0,1/2)$ Dirac Spinor
- $(1/2,1/2)$ Complex Vector
- $(1,0)$
- $(0,1)$
- $(1,0)\bigoplus(0,1)$
- $(1,1/2)$
- $(1/2,1)$
- $(1,1/2)\bigoplus(1/2,1)$
- $(1,1)$

13.2.1 Remark Who else has to do the hand trick with your left hand making an L every time to tell which one is which? We haven't broken parity yet so we shouldn't really say which one is left or right. That's what we can tell ourselves is our excuse for not being able to master this basic thing. $\Diamond$

13.2.4 $Spin(4)$

Isomorphic as a Lie group to $SU(2)\times SU(2)$ accidentally.

Real Representation Category

Complex Representation Category

Irreps are pairs of integers again.

13.3 Spinor Bundles

13.3.1 Definition (Fermion Field) *A fermion field is a section of a spinor bundle. The adjectives Majorana, Dirac and Weyl say which representation is used in the spinor bundle. Adjectives like R or NS are used to indicate which cohomology class is being used for the topological classification. If you want to make it charged give a vector bundle E for the representation of the gauge group you want.*

13.3.2 Example *For example, E is the associated bundle for the fundamental $SU(3)$ representation gives color. Physics usually assumes a contractible spacetime implying a fortiori that fermions can exist and there are no periodic/antiperiodic choices to worry about. All you need to worry about is what bundles E you want to couple to.*

13.3.1 Stieffel-Whitney

13.3.3 Theorem (Spin Structure) *For a spin structure to exist on a Riemannian manifold M, we must have an orientation and $w_2(TM) \in H^2(M, \mathbb{Z}_2)$ needs to vanish. As with taxes, $W2$ is a pain.*

13.3.4 Theorem (Pin Structure) *A $Pin^{\pm}$ bundle is a principal bundle with that $Pin^{\pm}$ structure group and an isomorphism from the ± 1 quotient to the orthonormal frame bundle (an isomorphism of $O(n)$ principal bundless).*

A $Pin(\mathbb{R}^n, -)$ structure exists when $w_2(TM) = 0$ and $Pin(\mathbb{R}^n, +)$ structure needs $w_2 + w_1^2$ to vanish. $\overline{w}_2 = w_2 + w_1^2$ under the group structure by the total Stieffel-Whitney class w which takes direct sum of bundles to cup product. This means that $\overline{w}_1$ is the w_1 for a different bundle $E^{\perp}$.

13.3.5 Remark Getting Mathematica code to show the inverse

$\left(\frac{1}{\sum_{i=1}^{10} x_i + 1} /.\, \text{Thread}\left[x_{\text{i\_}} \to s^i x_i\right]\right) + O(s)^5$

Then you can easily do the reduction to characteristic 2. ◊

Proof For example, `http://arxiv.org/pdf/1604.06527.pdf`. Our conventions for Clifford algebras are switched from Freed's so don't forget to switch. □

Chapter 14

Dirac Operators

14.1 Atiyah-Singer Dirac

14.2 Hodge-Dirac

14.3 Kahler-Dirac

14.4 Arrange as appropriate

14.4.1 Lemma (Fermions on bulk and boundary) *Because of the red arrow in fig. 13.2 which is given by sending $\tilde{\gamma}_i \to \gamma_0 \tilde{\gamma}_i$ we may write the following.*

$$\begin{aligned} i\gamma^0\left(\frac{\partial}{\partial x_0} - \sum \phi(\tilde{\gamma}_i)\frac{\partial}{\partial x_i}\right) &= i\gamma^0\frac{\partial}{\partial x_0} - \sum i\gamma_0^2\tilde{\gamma}_i\frac{\partial}{\partial x_i} \\ &= i\sum \gamma_\mu \frac{\partial}{\partial x_\mu} \end{aligned}$$

Notice we are being agnostic as to which representation of the clifford algebra we are using. We haven't yet specified which spinor bundle the fermions use which is encoded in a representation of the γ_i.

14.4.2 Theorem (Fierz Identity)

Proof Take the image of the identity in $V \otimes V^*$ in $Cliff(V,Q) \otimes Cliff(V^*,Q^*)$. That is in physics notation $(\gamma^\mu)^\beta_\alpha (\gamma_\mu)^\eta_\delta$ where the different subscripts and superscripts are used to say that you really mean $\sum \gamma^\mu \otimes \gamma_\mu$ so they are acting in different spaces. So what is the result of that? Well we said $Cliff$ was a functor taking direct sum to tensor product in superalgebras. So we have given an element of $Cliff(V \bigoplus V^*, Q \bigoplus Q^*)$ $\square$

14.4.3 Theorem (Atiyah-Singer Index Theorem)

14.4.4 Theorem (Atiyah-Patodi-Singer) *What about when there is boundary?*

14.4.1 Rarita-Schwinger

Suppore (M, g) is an n-dimensional Riemannian manifold with complexified tangent bundle $T^{\mathbb{C}}M$. Then we can twist the Dirac operator with $T^{\mathbb{C}}M$ to give

$$\begin{aligned} \phi &\in \Gamma(S_{1/2} \otimes T^{\mathbb{C}}M) \\ D_{TM}\phi &\equiv \sum_{k=1}^{n} (e_k \cdot - \otimes id_{T^{\mathbb{C}}M}) \nabla_{e_k} \phi \\ \phi &= \psi \otimes X \\ D_{TM}\phi &= D\psi \otimes X + \sum e_i \cdot \phi \otimes \nabla_{e_i} X \end{aligned}$$

where e_k is local frame and $e_k \cdot$ is Clifford multiplication by the specified e_k.

There is a $Spin(n)$ equivariant decomposition $S_{1/2} \otimes T^{\mathbb{C}}M \simeq S_{1/2} \bigoplus S_{3/2}$ so we can write D_{TM} in block diagonal form

$$D_{TM} \equiv \begin{pmatrix} \frac{2-n}{n} D & 2P^* \\ \frac{2}{n} P & \text{Rarita} \end{pmatrix}$$

with $P\ \Gamma(S_{1/2}) \to \Gamma(S_{3/2})$ being the so called Penrose twistor operator which applies ∇ to go to $S_{1/2} \otimes T^{\mathbb{C}}M$ and then projects to $S_{3/2}$. It's adjoint map P^* can be written

$$P^*\psi = -\sum_i (\nabla_{e_i} \psi) e_i$$

with considering $\psi \in \Gamma(S_{3/2})$ as a spinor $S_{1/2}$ valued one-form.

14.4.5 Remark D, D_{TM} and Rarita are all formally self-adjoint. The only reason it doesn't appear so in the block diagonal form is because of the way it is written using an inclusion ι of $S_{1/2}$ in the splitting of $S_{1/2} \otimes T^{\mathbb{C}}M$ which does not take the inner product into account.

$$\begin{aligned} \iota(\phi) &= -\frac{1}{n} \sum_i e_i \cdot \phi \otimes e_i \\ \tilde{\iota} &\equiv \alpha\iota \\ \tilde{\iota}^{-1} &= \frac{1}{\alpha}\iota^{-1} \end{aligned}$$

so in order to take that into account $2P^*$ is really replaced by $2\iota P^* = \frac{2}{\alpha}\tilde{\iota}P^*$ and $\frac{2}{n}P$ by $\frac{2}{n}P\iota^{-1} = \frac{2}{n}P\alpha\tilde{\iota}^{-1}$. If $\alpha = \sqrt{n}$ we get $\frac{2}{\sqrt{n}}\tilde{\iota}P^*$ and $\frac{2}{\sqrt{n}}P\tilde{\iota}^{-1}$ more symmetrically. But $\tilde{\iota}$ does not split the map $S_{1/2} \otimes T^{\mathbb{C}}M \to S_{1/2}$ defined by $\phi \otimes X \to X \cdot \phi$, ι does.

14.4.6 Lemma (Weitzenbock Formulas)

$$\begin{aligned} Rarita^2 \mid_{ImP} &= \frac{(n-2)^2}{n^2}(\Delta_{S_{3/2}} + \frac{1}{8}R) \\ Rarita^2 \mid_{KerP^*} &= \Delta_{S_{3/2}} + \frac{n-8}{8n}R \end{aligned}$$

This is similar to the Weitzenbock formulas for $S_{1/2}$ which read

$$\begin{aligned} \Delta_{S_{1/2}} &= \nabla^*\nabla + \frac{R}{8} \\ &= D^2 - \frac{R}{8} \\ D^2 &= \nabla^*\nabla + \frac{R}{4} \end{aligned}$$

More generally if there is a nontrivial connection A on the determinant line bundle but more specifically if $n = 4$, there is an extra term $\frac{1}{2}\langle F_A^+, ?\rangle$ where self-dual 2-forms act by Clifford multiplication.

14.4.7 Lemma (Rarita-Schwinger Commutation Relations)

$$\begin{aligned} RaritaP &= \frac{n-2}{n}PD \\ P^*Rarita &= \frac{n-2}{n}DP^* \\ \Delta_{S_{1/2}}D &= D\Delta_{S_{3/2}} \\ \Delta_{S_{1/2}}P^* &= P^*\Delta_{S_{3/2}} \\ \Delta_{S_{3/2}}P &= P\Delta_{S_{1/2}} \\ \Delta_{S_{3/2}}Rarita &= Rarita\Delta_{S_{3/2}} \end{aligned}$$

Proof `https://arxiv.org/pdf/2001.06167.pdf` `https://arxiv.org/pdf/1804.10602.pdf` □

14.4.8 Theorem (Atiyah Singer) *If $n = 2m$ is even, then $S_{3/2} \simeq S_{3/2}^+ \bigoplus S_{3/2}^-$ so Rarita decomposes as two pieces that interchange $+$ and $-$. Call these Q^+ and Q^- where the superscript indicates the domain.*

$$\begin{aligned} indexRarita &\equiv \dim\ker Q^+ - \dim\ker Q^- \\ &= (-1)^m\big(\frac{ch(S_{3/2}^+) - ch(S_{3/2}^+)}{e(TM)}\hat{A}(TM)^2\big)[M] \\ &= indexD_{TM} + indexD \end{aligned}$$

Proof `https://arxiv.org/pdf/1804.10602.pdf` □

Chapter 15

Hyperbolic Geometry

15.1 Fuchsian Groups

15.2 Role In AdS

Consider the Riemannian manifold $H^3 \times \mathbb{R}$.

15.3 Geometerization Theorem

Remember the Uniformization Theorem Theorem 11.4.3

- Spherical
- Euclidean
- Hyperbolic
- Spherical Times $\mathbb{R}$
- Hyperbolic Times $\mathbb{R}$
- $SL(2,\mathbb{R})$
- Nil
- Sol

15.4 Topology vs Geometry

15.4.1 Theorem (Mostow Rigidity) *Suppose M and N are complete finite volume hyperbolic manifolds of dimension ≥ 3. If there exists an isomorphism of $\pi_1(M) \simeq \pi_1(N)$, then it is induced by a unique isometry $M \to N$. This means that you can reduce to information about the fundamental group and it's outer automorphisms.*

15.4.2 Definition (Knot Complement) *A knot complement is S^3/K. It may be that this can be given complete hyperbolic metric so we get a hyperbolic 3-manifold. If so this is unique up to isometry by Theorem 15.4.1 since we already have π_1 fixed. This means any geometric invariants of this hyperbolic metric are in fact invariants of the topological space.*

15.4.3 Remark This is unlike usual topological invariants. This is an object by object invariant. There is no sense of Mayer-Vietoris or gluing in this definition. It does not deserve the same treatment as true topological invariants. $\lozenge$

15.4.4 Conjecture (Volume Conjecture)

$$\begin{aligned}
\langle K \rangle &\equiv \lim_{q \to e^{2\pi i/N}} \frac{J_{K,N}(q)}{J_{O,N}(q)} \\
Vol(S^3/K, g_{Hyp}) &= \lim_{N \to \infty} \frac{2\pi \log |\langle K \rangle|}{N} \\
Vol(S^3/K, g_{Hyp}) + CS(S^3/K) &= \lim_{N \to \infty} \frac{2\pi \log \langle K \rangle}{N}
\end{aligned}$$

15.4.5 Corollary (Murakami-Murakami) *If the volume conjecture holds, every knot that is different from the trivial knot has at least one different Vasiliev invariant.*

15.5 Ideal Triangulations

15.5.1 Definition (Ideal Triangles) *An ideal triangle in H^2 is a hyperbolic triangle such that all three vertices are at the boundary. All ideal triangles are congruent. The boundary vertices are all at cusps at infinity so have angle 0. They have area π which is maximal among all hyperbolic triangles.*

15.5.2 Definition (Triangle Group) *Fix a hyperbolic triangle. Let a b and c be the isometries of H^2 that are reflections across the three different edges. Together they give a subgroup of $PSL(2,\mathbb{R})$. Let the angles be $\frac{\pi}{l}$,$\frac{\pi}{m}$ and $\frac{\pi}{n}$. Then this is called the (l,m,n) triangle group.*
If you take the index two subgroup of words of even length in the generators a,b,c then you get a discrete subgroup of orientation-presrving isometries which is Section 15.1.
If the triangle was ideal then this would be the (∞,∞,∞) triangle group whiich is isomorphic to $\mathbb{Z}_2 \star \mathbb{Z}_2 \star \mathbb{Z}_2$. That is to say there are no relations between the above a,b and c besides the obvious fact that they each square to the identity.

15.5.3 Definition (Bloch-Wigner Dilogarithm)

$$
\begin{aligned}
Li_2(z) &= \sum_{k=1}^{\infty} \frac{z^k}{k^2} \\
&= -\int_0^z \log(1-t)\frac{dt}{t} \\
D_2(z) &= Im(Li_2(z)) + arg(1-z)\log \mid z \mid \quad \forall z \in \mathbb{C} \setminus \{0,1\}
\end{aligned}
$$

where the integration contour connecting 0 and z avoids the cut 1 to ∞ along the real axis. D_2 is real analytic on it's domain and satisfies several useful identities

15.5.4 Definition (Ideal Tetrahedra) *An ideal triangle in H^3 is a hyperbolic tetrahedron such that all four vertices are at the boundary. Every ideal tetrahedron is isomorphic to one with vertices at $0,1,z,\infty$ for some complex number z with $Imz > 0$. This is when regarding the boundary of hyperbolic spaces as $\mathbb{CP}^1$ with $\mathbb{C}$ and ∞.*
The volume of this tetrahedron is $D_2(z)$ where D_2 is the Bloch-Wigner dilogarithm. In general the volume is D_2 of the cross ratio of the 4 vertices $D_2(\frac{(p_0-p_2)(p_1-p_3)}{(p_0-p_1)(p_2-p_3)})$ where again the p_i are regarded in $\mathbb{C}$ or ∞
For a general hyperbolic 3 manifold formed by gluing n tetrahedra, add up the volumes of the individual tetrahedra giving a sum of dilogarithms
The 6 edges of a tetrahedron come in opposite pairs. The edge connecting p_0p_1 is paired with the one for p_2p_3 and so on

15.5.5 Lemma *If we don't order the points of the tetrahedra, we get different cross ratios z, z' and z''. They satisfy*

$$
\begin{aligned}
zz'z'' &= -1 \\
z + z' - 1 &= 0
\end{aligned}
$$

It doesn't matter which we plug into D_2 thanks to those many useful identities.

15.5.6 Lemma *Let M be a 3-manifold that is the interior of a manifold with boundary $\bar{M}$ such that all its boundary components are tori.*
Triangulate M by ideal tetrahedra in a more general sense allowing vertices in the boundary components and edges being geodesics in M instead of just the H^3 case above.
Then the number of edges is the same as the number of tetrahedra

Proof Euler characteristic of the boundary is 0 and the boundary gets an induced triangulation with $2E$ vertices, $3F$ edges and $4n$ triangles and $2F = 4n$ (each tetrahedron has 4 faces and each face is on 2 tetrahedra) so $E = n$
More generally $2E - 2n$ is the Euler characteristic of the boundary.

15.5.7 Lemma (Neumann-Zagier Matrix) *`https://arxiv.org/pdf/2002.10356.pdf` 2.26*

Chapter 16

Riemann Surfaces

16.1 Goldman Lie bialgebra

16.1.1 Definition (Goldman Lie bialgebra) *Consider the module generated by free homotopy classes of loops in an oriented surface*
This module has a Lie bracket defined by

$$[\gamma_1, \gamma_2] \equiv \sum_p sgn(p) \gamma_1 \star_p \gamma_2$$

where the right hand side is a sum over intersection points. $sgn(p)$ *indicates how* $T_p\gamma_{1,2}$ *are oriented with respect to the orientation of the surface. That is do they form an oriented basis or not.* $\gamma_1 \star_p \gamma_2$ *means the path that starts at* p*, follows* γ_1 *until hitting* p *again and then using* γ_2*. In order to calculate this summand you may have to perturb the representatives* γ_1 *and* γ_2 *first so that intersections are transverse in order to define both* $sgn(p)$ *and* $\gamma_1 \star_p \gamma_2$*.*

It works over the integers so to get a Lie algebra, you need to make a vector space by tensoring $\otimes_{\mathbb{Z}} k$ *for some field.*

16.1.2 Definition (String Topology Operations)

16.2 Riemann Rocks

16.2.1 Theorem (Riemann-Roch) *Define* $\mathcal{L}(D)$ *to be the line bundle associated to the Cartier divisor* D*. That means that* $D = \sum d_i z_i$ *tells you that if* $d_i < 0$*, then there must be a zero at* z_i *of multiplicity at least* d_i *and if* $d_i > 0$*, then you can have a pole of at most order* d_i *there.*

Define K *as the canonical divisor.*

$$\begin{aligned} \ell(D) &\equiv \dim_{\mathbb{C}} H^0(\Sigma, \mathcal{L}(D)) \\ \ell(K-D) &\equiv \dim_{\mathbb{C}} H^0(\Sigma, \mathcal{L}(K-D)) \\ \ell(D) - \ell(K-D) &= deg(D) - g + 1 \end{aligned}$$

16.2.2 Corollary (Plane Curves) *An irreducible plane algebraic curve of degree* d *has degree* $\frac{(d-1)(d-2)}{2} - g$ *singularities when properly counted. This is the difference between the arithmetic genus which is the first term and the geometric genus* g*. Reversing the formula and letting* δ_P *be the delta invariant for the singularity at* P*, so* $\sum \delta_P$ *is the singularities properly counted.*

$$g = \frac{(d-1)(d-2)}{2} - \sum \delta_P$$

16.2.3 Corollary (Ellliptic Functions) *If* $D = nP$ *for some point* P *and* $g = 1$*, then* $\ell(D) = n - 1 + 1 + \ell(K-D)$*. When* $n > 0$*,* $K - D$ *is strictly negative so* $\ell(K-D) = 0$*. That means we* *get* $1, 1, 2, 3, 4, 5 \cdots$ *for the sequence of* $\ell(D)$ *as* $n = 0, 1, \cdots$.

A basis for $H^0(\Sigma, \mathcal{L}(D))$ *can be given with the theory of elliptic functions.*

16.2.4 Definition (Weierstrass points) *If we look at the sequence* $\ell(D = nP)$ *is always of the form* $1, ? \cdots ?, g, g+1, g+2 \cdots$ *where there are* $2g-2$ *intermediate* $?$. $g-1$ *of them are where there is no step from the previous value of* n *and* $g-1$ *of them are where the step is* 1.
A non-Weierstrass point is when the sequence is $1, 1 \cdots 1, 2, 3, 4, \cdots g-1, g, g+1, g+2, \cdots$. *A Weierstrass point is when the seqeuence of no steps and steps is different.*

16.3 Jacobians

16.3.1 Theorem (Abel-Jacobi) *If you pick* $p_0 \in \Sigma$ *and define the map* $p \in \Sigma \to \int_{p_0}^{p} \omega_i$ *where* $\omega_1 \cdots \omega_g$ *are* g *linearly independent holomorphic differential forms. If you change the integration path* $p_0 \to p$, *this map only changes by something in the lattice* Λ *defined by the vectors* $(\int_{\gamma_j} \omega_1 \cdots \int_{\gamma_j} \omega_g)$ *where* γ_j *are* $2g$ *loops generating* $H_1(\Sigma, \mathbb{Z})$. *This gives a lattice in* $\mathbb{C}^g \simeq \mathbb{R}^{2g}$.
In total we get a well defined map $\Sigma \to Jac(\Sigma) \equiv \mathbb{C}^g/\Lambda$ *for every point* p_0 *and if you change the point* p_0, *you change the map by post-composition with a translation of the resulting* $Jac(\Sigma)$ *which has a translation action on itself.*

16.3.2 Theorem (Torelli) *A nonsingular projective algebraic curve (compact Riemann surface)* Σ *is determined by* $Jac(\Sigma)$ *as a principally polarized abelian variety. In fact this works over all algebraically closed fields.*

16.4 Moduli Space

16.4.1 Definition ($\mathcal{M}_{g,n}$) *This is the compactified moduli space of stable curves of genus* g *and* n *marked points.*

16.4.2 Lemma (Zagier-Harer) *One can calculate the orbifold Euler characteristic using the Riemann zeta function as follows*

$$\begin{aligned}
\chi(\mathcal{M}_{g,1}) &= \zeta(1-2g) \\
&= \frac{-B_{2g}}{2g} \\
\chi(\mathcal{M}_{1,1}) &= \zeta(-1) = \frac{-1}{12}
\end{aligned}$$

Chapter 17

Four Manifolds

17.0.1 Definition (Four Manifold) *A four manifold is either a topological or smooth manifold of dimension 4.*[1]*. That is either locally homeomorphic/diffeomorphic to* $\mathbb{R}^4$ *with this distinction being critical.*

17.0.2 Remark If forgotten, connectedness is also assumed. Disjoint union of two components may induce obvious counterexamples to statements that are improperly stated. ♢

17.1 Topological

17.1.1 Definition (Intersection Form) *The middle dimensional (co)homology on a oriented compact 4-manifold has an intersection form with values in the integers. Call this* Q*.*
In particular it will be unimodular, so $\det Q = \pm 1$ *when thinking about it as a matrix rather than a bilinear form.*
By Sylvester's law of Inertia, you can diagonalize over $\mathbb{R}$ *to get* $b_2^{\pm}$*. There are no zero eigenvalues because* Q *was unimodular.*

17.1.2 Definition (Particular Lattices)

- $I_{m,n}$ *- the lattices formed by the inclusion* $\mathbb{Z}^{m+n} \to \mathbb{R}^{m,n}$ *of signature* $(+, \cdots +, - \cdots -)$
- $II_{m,n}$ *when* $m - n$ *is divisible by 8. The lattice specified by the above along with those* $(a_1 \cdots a_{m+n})$ *where they are all half-integers with even sum. In particular* $II_{8,0}$ *is* E_8

17.1.3 Theorem (Signature) *The 4-manifold bounds a 5-manifold if and only if the signature is zero.*
A spin 4-manifold has signature that is a multiple of 8.
Rokhlin's theorem states the a smooth compact 4-manifold's signature is a multiple of 16.

17.1.4 Theorem (Freedman) *The homeomorphism type of a simply connected compact 4-manifold is determined by this intersection form and a* $\mathbb{Z}_2$ *Kirby-Siebenmann invariant.*
Given any unimodoular symmetric bilinear form over the integers Q*, you can cook up a simply-connected closed 4-manifold with that intersection form. If* Q *is even, then there is only one such manifold. If not, then there are two. At least one of them has no smooth structure. These two are distinguished by different Kirby-Siebenmann invariants.*
In particular this implies that if two smooth simply-connected 4-manifolds have the same intersection form, they are homeomorphic, but does not give a diffeomorphism between them.

17.1.5 Example *This last consequence can be specialized to the case of a smooth simply-connected 4-manifold that has the same intersection as* S^4*. It is homeomorphic to* S^4*, but it still might be an exotic sphere.*
This is the open case of smooth Poincaré conjecture in dimension 4 remark 2.8.4.

[1] Daughter: I study 4 manifolds. Dad: Which 4?

17.2 Smooth

17.2.1 Theorem (Donaldson) *A definite intersection form of a compact, oriented, simply-connected smooth 4-manifold is diagonalizable.*
Positive/Negative definite implies it diagonalizes to $\pm I$ over the integers.

17.2.2 Example (E_8 manifold) *Let Q be the intersection form of the E_8 lattice. It defines a simply connected closed topological manifold by theorem 17.1.4. However because it is not the diagonal intersection form, theorem 17.2.1 says that it is not from a smooth simply connected manifold. Therefore it is a topological manifold that is not smoothable. In fact, it is not even triangulable as a simplicial complex.*

17.3 Algebraic

We can also consider complex manifolds with complex dimension 2. More strictly we can consider complex algebraic varieties of complex dimension 2. Confusingly these are called surfaces even though they are 4 real dimensional. These give smooth 4-manifolds because the transition maps being holomorphic imply smooth.

17.3.1 Definition (Minimal Surface) *Given any point p on a surface S, we may blow it up to get Bl_pS. If S was nonsingular, then p was nonsingular and the blowup was unnecessary from the point of view of using blow ups to resolve singularities. So if X is a nonsingular surface that cannot be expressed as Bl_pS for some other non-singular surface S, we call X minimal. That is to say it involves no unnecessary blow ups.*

17.3.2 Theorem (Castelnuovo Contraction Theorem) *A nonsingular surface X is minimal if and only if there are no smooth rational curves with self intersection number -1. The more modern definition of minimal is to say K_X is nef (nonnegative degree on every closed irreducible curve where regarding as being associated to a divisor).*

So if there is a smooth rational curve with self intersection -1, we may write $X = Bl_pY$ and an accompanying blowdown map $X \to Y$ with $f^{-1}p$ being the aforementioned curve.

Proof The easy direction : If X is not minimal, then it has been a presentation as $X = Bl_pY \to Y$. The exceptional $f^{-1}(p)$ has self intersection -1. Contrapositively, this means that no such smooth rational curves of self-intersection -1 implies minimal. □

17.3.3 Theorem (Enriques-Kodaira) *There are 10 classes of compact complex surfaces (not necesssarily algebraic). Each one has an associated moduli space.*

- *Rational - these are the ones with the same function field as $\mathbb{CP}^2$ (birational). For example $\mathbb{CP}^2$ itself, $\mathbb{CP}^1 \times \mathbb{CP}^1$, Hirzebruch surfaces, quadrics, cubic surfaces, del Pezzo, Veronese.*
- *Ruled with genus $g > 0$ - A smooth morphism to Σ_g and the fibers are all $\mathbb{CP}^1$. For example, $\Sigma_g \times \mathbb{CP}^1$. Any such ruled surface is birational with $\mathbb{CP}^1 \times \Sigma_g$*
- *Type VII - These are neither algebraic nor Kahler.*

- *K3 - These are Kahler and have trivial canonical bundle. All are diffeomorphic to each other and only distinguished from each other with complex geometry. By theorem 17.1.4 we know that giving the homeomorphism type means providing an intersection form and Kirby invariant. In this case it is* $II_{3,19}$ *which is even so no Kirby needed.*
- *Enriques - Close to* $K3$ *but instead of the canonical line bundle being trivial, only it's square is. They are quotients of* $K3$ *by a* $\mathbb{Z}_2$
- *Kodaira - Never algebraic but they do have non-constant meromorphic functions. For example take a non-trivial line bundle over an elliptic curve, remove the zero section then quotient out by the action of* q^n *for some* $q \in \mathbb{C}^*$ *acting fiberwise*
- *Toric*
- *Hyperelliptic - Take the product of two elliptic curves and quotient by a finite group of automorphisms (7 choices of finite group available)*
- *Properly quasi-elliptic*
- *General type*

The first 3 *have Kodaira dimension* $-\infty$ *which means that* $P_d = \dim H^0(X, K_X^d)$ *all vanish (these are called the plurigenera).*

Rational surfaces are birational with $P^1 \times P^1$ *so they are ruled with genus* 0*. That is why we had to single out* $g > 0$*, in the second class.*

The K3 class has Kodaira dimension 0 *so the* $P_d \in O(d^0)$ *which means they are bounded. This is similar to the case of elliptic curves in dimension* 1 *where* K_X^d *is always trivial and so* $P_d = 1$ *always. Here again the canonical line bundle is trivial.*

Enriques also have Kodaira dimension 0 *as do Kodaira.*

17.4 Gauge Theoretic

17.4.1 Donaldson

17.4.2 Seiberg-Witten

Let (M, g) be a compact oriented smooth 4-manifold. Every such manifold admits a $Spin^c$ structure by a theorem of Hirzebruch and Hopf. This in particular says $w_2(M) \in H^2(M, \mathbb{Z}_2)$ actually comes from $K \in H^2(M, \mathbb{Z})$. With this we can use the notions from chapter 13 to give $W^+ \bigoplus W^-$ as the positive and negative spinor bundles acted on by $Spin(4)$ and the $U(1)$ acting by multiplication to give the full $Spin^c$. There is a Clifford bundle $Cliff(M, g)$ and it acts on $W^+ \bigoplus W^-$. Restricting to the 1-forms inside the Clifford bundle means that to every 1-form we get $\gamma(a)$ defined as bundle morphisms $W^\pm \to W^\mp$.

17.4.1 Definition (Sieberg-Witten equations) *Let* ϕ *be a section of* W^+ *and let* ∇^A *be the unique connection satisfying*

$$\begin{aligned} \nabla^A_X(\gamma(a)) &= \left[\nabla^A_X, \gamma(a)\right] \\ &= \gamma(\nabla^g_X a) \end{aligned}$$

where $\nabla_0 + A$ *is a connection on* $\det W^+ \simeq \det W^-$, X *is a vector field and* a *is a 1-form.*

$$\begin{aligned} D^A\phi &= 0 \\ F_A^+ &= (\phi h(\phi,-) - \frac{1}{2}h(\phi,\phi)1_{W^+}) + i\omega \end{aligned}$$

Counting solutions to the above equations (with signs) modulo gauge equivalence gives the Seiberg-Witten invariant of the 4-manifold with specified $Spin^c$ *structure.*

Chapter 18

Algebraic Geometry

18.1 Generalities

18.1.1 Definition (Affine) *An affine variety over an algebraically closed $\overline{k}$ is a zero locus of some finite family of polynomials in n variables with coefficients in $\overline{k}$ so that it defines a subset of $\overline{k}\mathbb{A}^n$. Those $f_1 \cdots f_n$ generate some ideal I in $\overline{k}[x_1 \cdots x_n]$ and we want that to be a prime ideal. The quotient describes the space of regular functions on the variety. They are just the polynomial functions on the ambient space, but we don't care about anything in I because those are the functions that vanish on the variety so they can all be treated as 0.*

18.1.2 Definition (Projective) *A projective variety over algebraically closed $\overline{k}$ is a zero locus of some finite family of homogeneous polynomials in $n+1$ variables with coefficients in $\overline{k}$ so that it defines a subset of $\overline{k}\mathbb{P}^n$. Those $f_1 \cdots f_n$ generate some ideal I in $\overline{k}[x_0 \cdots x_n]$ and we want that to be a prime ideal.*

18.1.3 Lemma (Projective Closure) *Let I give a prime ideal defining an affine variety. Homogenize everything in I by inserting powers of x_0 as necessary so that on the $A_n \subset P^n$ identified as rescaling nonzero x_0 to 1 we get the original f_i. This might introduce singularities, but it is now projective.*

18.1.4 Definition (Hilbert Polynomial) *Because I is a homogeneous prime ideal, $\overline{k}[x_0 \cdots x_n]/I$ is a $\mathbb{N}$-graded ring. It is called the homogeneous coordinate ring. Taking the dimensions of each piece (denoted R_n) gives a sequence of natural numbers. For sufficiently large n, there is a polynomial P which interpolates all of these values. That is to say $\exists N \; \forall n > N \; P(n) = \dim R_n$. The degree of P as a polynomial is the dimension of the variety.*

18.1.5 Definition (Quasi-projective)

18.1.6 Definition (Intersection Pairing) *For algebraic curves in algebraic surfaces, we want to define an intersection number similar to how one would do so in differential topology for complementary dimensional submanifolds. However, we have much less flexibility now.*

If the field is of characteristic 0, so we can think of the surface analytically we can use the map from classes of curves to $H^2(X, \mathbb{R})$ and use the intersection pairing there.

Without characteristic 0, we can use

$$\begin{aligned}
(C, D) &\equiv \chi(\mathcal{O}) - \chi(\mathcal{O}(-C)) - \chi(\mathcal{O}(-D)) + \chi(\mathcal{O}(-C) \otimes \mathcal{O}(-D)) \\
\chi(L) &\equiv \dim H^0(X, L) - \dim H^1(X, L) + \dim H^2(X, L)
\end{aligned}$$

18.1.7 Definition (Birational) *A rational map from one variety V to another W gives a field homomorphism $\overline{k}(W) \to \overline{k}(V)$ where $\overline{k}(V)$ means the field of functions on that variety. In the affine case, this is the field of fractions for the integral domain given by the ring of regular functions on that variety. It is not necessarily well defined everywhere when thinking about points in V as a space.*

A birational equivalence is then a pair of rational maps $V \to W$ and $W \to V$ that are mutually inverse wherever the composition makes sense. This is when we give an isomorphism of fields between the corresponding field of functions.

A birational morphism is then when the forward direction of the birational map is well defined when thinking about V as a space. The inverse only has to be a birational map. The prototypical example is a blowup $Bl_p S \to S$ is a birational map and we have well definedness on everywhere in the source, but in the other direction p can come from an entire P^1 of choices. In that direction, it is only a map not a morphism.

18.2 Curves

18.2.1 Definition (Genus) *We are no longer exclusively thinking with complex coefficients, so we need the more algebraic definition of* $\dim H^0(\Sigma, \Omega^1) = \dim H^1(\Sigma, \Omega^0)$ *where the equality is from Serre duality. This quantity is the genus.*

18.2.2 Theorem (Riemann-Roch theorem 16.2.1)

18.2.3 Lemma () *Let* $P_d \equiv \dim H^0(\Sigma, K_\Sigma^d)$*. If* $g = 0$*, then* P_d *gives the global sections of* $\mathcal{O}(-2d)$ *which remain* 0*. If* $g = 1$*, then* K *is trivial and so* $P_d = 1$*. If* $g \geq 2$*, then* $P_d = (2d-1)(g-1)$ *once* $d \geq 2$*.*

Proof If d is such that the divisor for K^d has degree at least $2g-1$, then

$$\begin{aligned}
\ell(K^d) &= \deg(K^d) - g + 1 \\
P_d &= \deg(K^d) - g + 1 \\
\deg(K^d) &= (2g-2)d \\
P_d &= (2g-2)d - g + 1 = (g-1)(2d-1)
\end{aligned}$$

If $g \geq 2$, then the $2g-1$ condition for the correction to vanish becomes $(2g-2)d \geq 2g-1$ which implies $d \geq 2$. When $g = 0$ or $g = 1$, the condition is impossible to satisfy with any natural number d.

18.2.4 Corollary (Plane Curves corollary 16.2.2)

18.3 Surfaces

18.3.1 Generalities

Expanding on Section 17.3. But now we are restricting ourselves to algebraic world so we only get the cases as being rational, ruled with genus $g > 0$ for Kodaira dimension $-\infty$. Enriques surfaces, Hyperelliptic surfaces, K3 surfaces and Abelian surfaces with Kodaira dimension 0, elliptic surfaces with Kodaira dimension 1 (excluding cases that fell into earlier classes) and then general type for the remaining. This is 8 classes and is a different, but overlapping classification than we did before.

18.3.1 Theorem *Every algebraic surface is birational with one that is projective, nonsingular and minimal.*

Proof Take an open affine subset S_0 of your surface, That is birational to the whole surface. Nothing is changed in the field of functions by this throwing out. Then S_0 can be included into A^n for some n by definition of it being affine. Then extend this to P^n.

Now we have a projective, possibly singular and possibly non-minimal algebraic surface. For the codimension 1 singularities, normalization works. For codimension 2 singularities blow up points. Each time new singularities, may be introduced, but the process does eventually terminate.

Now we have a projective, nonsingular, but possibly non-minimal algebraic surface. This is where we keep blowing down any rational curves of self-intersection -1 we can. This is using Castelnuovo's contraction theorem. Blowup increases the dimension of H^2 by 1 and blowing down decreases that. So because we know that H^2 has bounded rank, we know this process will terminate.

18.3.2 Invariants

18.3.2 Definition (Kodaira Dimension) *For any d, the vector space $H^0(X, K_X^d)$ is a birational invariant. That is we get canonical vector space isomorphisms between $H^0(X, K_X^d)$ and $H^0(Y, K_Y^d)$ when X and Y are birationally equivalent smooth projective varieties. In particular the dimensions P_d are invariant. The case for curves was described earlier with lemma 18.2.3. Then the growth rate as the minimal κ such that $P_d \in O(d^\kappa)$ is also an invariant. This gives either $-\infty$ if $P_d = 0$ for all $d > 0$ or a natural number between 0 and the dimension of the variety inclusive. Note in particular how the case for curves of genus ≥ 2 had P_d grow linearly with d.*

18.3.3 Lemma (Nakamura-Ueno) *Let $\pi\ V \to W$ be an analytic fiber bundle of compact complex manifolds. In particular, the fibers are all isomorphic as complex manifolds. Suppose that fiber F satisfies the technical Moishezon property (Compact complex algebraic varieties meet this criterion). Then the Kodaira dimension of the total space is the sum of the Kodaira dimensions of the base and fiber.*

18.3.4 Definition (Geometric Genus) $P_g \equiv h^2(\Omega^0) = h^0(\Omega^2)$

18.3.5 Definition (Holomorphic Euler Characteristic) $\chi_{hol} \equiv h^0(\mathcal{O}) - h^1(\mathcal{O}) + h^2(\mathcal{O})$

18.3.6 Definition (Hodge numbers) $h^{p,q} \equiv \dim H^q(\Omega^p)$ *We assume the duality $h^{p,q} = h^{n-p,n-q}$, but not $h^{p,q} = h^{q,p}$ which is the case for complex projective algebraic surfaces.*

If we can assume complex coefficients then we have the relation to the topological Betti numbers.

$$\begin{aligned}
P_g &= h^{2,0} = h^{0,2} \\
q &\equiv P_g - P_a = h^{0,1} = h^{1,0} \\
P_a &= h^{2,0} - h^{1,0} \\
\chi_{hol} &= h^{0,0} - h^{0,1} + h^{0,2} \\
&= h^{0,0} + P_a = 1 + P_a
\end{aligned}$$

18.3.7 Definition (Chern Numbers) *Chern classes evaluated on the class of the space to give a number rather than a cohomology class.*

The only ones of concern are c_2 and c_1^2. c_2 give the topological Euler characteristic. c_1^2 gives the self intersection of the canonical class K.

18.3.8 Theorem (Max Noether)

$$12\chi_{hol} = c_1^2 + c_2$$

18.3.3 Kodaira Dimension $-\infty$

Rational

Birational with P^2.

Consider smooth hypersurfaces in P^3. If the degree is 1, then this is just isomorphic to P^2. If the degree is 2, then we can put this in the form $wx = yz$ and that is isomorphic to $P^1 \times P^1$ as the image of Segre embedding. If the degree is 3, we get a cubic surface. Degree 4 and above are not rational.

18.3.9 Example (Cubic Surface) 18.3.10 Theorem (27 lines)

How many curves through P^2 blown up at n general position points?

First, there are n from the inverse images of those blown up points. Then for each pair of points we get a line with self intersection $1-1-1=-1$. Then we can take 5 and consider a conic through them and cook up such an exceptional rational curve. Then a cubic through 7 points where one of the points is a double point. Then at $n=8$, we see a new phenomenon of quartics with 3 being double points, a degree 5 curves with 5 being double points and a degree 6 curve with one triple point and the rest double points. With degree $n=9$, all hell breaks loose and we get infinitely many exceptional curves.

This gives the sequence $0, 1, 3, 6, 10, 16, 27, 56, 240$.

Each exceptional curve is an element of the Picard group which is $\mathbb{Z}^{1,n}$ where the signature describes the self-intersection pairing. The canonical K corresponds to $(3, 1, ...1)$ which has norm $9-n$. An exceptional curve is then an element r with norm -1 and $(r, K) = 1$ using the fact that it is rational. With K having positive norm, there are only finitely many solutions to these integer equations. This explains the trichotomy of $n < 9$, $n = 9$ and $n > 9$.

For $n \leq 8$, the orthogonal complement to K gives a negative definite lattice and that is the root lattices of various Lie algebras (with sign flip). The exceptional curves then correspond to certain weight vectors to the corresponding Lie algebra. Not directly because $K^{\perp}$ would be pairing with K to give 0 rather than 1.

18.3.11 Example (Hirzebruch Surfaces) *Make a projective P^1 bundle over P^1. By classification of vector bundles over P^1, we can assume that the vector bundle is $\mathcal{O}(n) \bigoplus \mathcal{O}(m)$. Tensoring by $\mathcal{O}(-min(n, m))$ and switching summands does not change the result so we might as well assume that we have $\mathcal{O}(0) \bigoplus \mathcal{O}(n)$ with $n \geq 0$. These give Hirzebruch surfaces H_n. H_0 is using a trivial bundle so that is just $P^1 \times P^1$. The case $n = 1$ ends up giving P^2 blown up at one point. So H_1 is not minimal. All the rest are minimal rational surfaces.*

Ruled Surfaces with $g > 1$

Something birational with $\Sigma_g \times P^1$. More generally P^1 bundle over Σ_g.

18.3.4 Kodaira Dimension 0

18.3.12 Lemma *Assuming connected complex projective of Kodaira dimension 0, the possibilities for the Hodge numbers are*

- $h^{0,2} = 0$, $h^{0,1} = 0$ $b_2 = 10$, $h^{1,1} = 10$
- $h^{0,2} = 0$, $h^{0,1} = 1$ $b_2 = 2$, $h^{1,1} = 2$
- $h^{0,2} = 1$, $h^{0,1} = 0$ $b_2 = 22$, $h^{1,1} = 20$
- $h^{0,2} = 1$, $h^{0,1} = 1$ $b_2 = 14$, $h^{1,1} = 12$
- $h^{0,2} = 1$, $h^{0,1} = 2$ $b_2 = 6$, $h^{1,1} = 4$

Proof If the Kodaira dimension is 0, the possibilities for $h^{0,2}$ are either 0 or 1. If the canonical class is trivial, then $h^{0,2}$ will be 1. Otherwise it will be 0. In addition c_1^2 vanishes in this case.

Now write Max Noether's theorem for this case

$$12 - 12h^{0,1} + 12h^{0,2} \quad = \quad 2 - 4h^{0,1} + b_2$$

where we have used that the surface is complex projective and connected to condense the information about rows 0, 1, 3 and 4 of the Hodge diamond into just $2 - 4h^{0,1}$.

$$10 + 12h^{0,2} \quad = \quad 8h^{0,1} + b_2$$

$h^{0,2}$ is either 0 or 1 and $h^{0,1}$ and b_2 are natural numbers so we get only a few possibilities.

18.3.13 Corollary *All but the 4th are actually realized and they correspond to Enriques surfaces, Hyperelliptic surfaces, K3 surfaces and Abelian surfaces respectively*

18.3.14 Example (Abelian Surface) *These are of the form $\mathbb{C}^2/\Lambda$ where Λ is a lattice. As a space it is $(S^1)^4$. This is fine as an abelian group without any algebraic structure, but when we want it to be algebraic we get restriction on Λ. This is not seen for $\mathbb{C}/\Lambda$ elliptic curves when the restriction gets satisfied for all choices.*

There must be a Hermitian form such that the imaginary part is integral on the lattice. This is known as the Riemann bilinear relations.

Examples include products of elliptic curves and Jacobians of genus 2 curves.

18.3.15 Example (Hyperelliptic Surface) *Quotient out an abelian surface by a group acting without fixed points. Sometimes you get another abelian surface, but when you don't these are hyperelliptic and fit in the second class of the above. There are 7 possibilities for the group that you quotient by that gives such a non-abelian-surface result. This means that there are 7 families of such surfaces, each with their own moduli (and more discrete parameters, and possible coincidences among them).*

18.3.16 Example ($K3$) *Caution, we are only assuming the algebraic case so we only are within the 19 dimensional moduli not the full 20. There are various ways to construct them. One is $x^4 + y^4 + z^4 + w^4 = 0$ or any other such smooth degree 4 hypersurface in P^3.*

18.3.17 Example (Enriques Surface) *Take a $K3$ surface and quotient by a $\mathbb{Z}_2$ acting fixed point freely.*

18.3.5 Kodaira Dimension 1

18.3.18 Lemma *Assuming connected complex projective of Kodaira dimension 1, the result is an elliptic surface. This means it has a map to some base C and the fibers are mostly elliptic curves. There can be some exceptional singular fibers. Obviously the converse is not true, because $E \times P^1$ is manifestly ruled with Kodaira dimension $-\infty$ and $E_1 \times E_2$ is manifestly an Abelian surface with Kodaira dimension 0.*

Proof We can always map $S \to Proj \bigoplus \Gamma(K^n)$. When the Kodaira dimension is 1, this Proj construction gives a curve and the fibers of this map are mostly elliptic curves.

The singular fibers are classified by affine ADE Dynkin diagrams where each node is a P^1 and the linking describes how they intersect.

18.3.6 Kodaira Dimension 2

All hell breaks loose. There are only a few equalities and inequalities.

- $c_2 > 0$

- $c_1^2 > 0$

- $12\chi_{\text{hol}} = c_1^2 + c_2 \implies c_1^2 + c^2 \text{mod} 12 = 0$

- $5c_1^2 - c_2 + 36 \geq 0$

- BMY inequality $c_1^2 \leq 3c_2$

If we plot all possible values of c_1^2 and c_2 we get that they must be on integer points on certain lines and within a region bounded by the inequalities.

Chapter 19

Microlocal Analysis

19.1 Differential Operators

19.2 Pseudo-differential Operators

19.2.1 Smoothing Operators

Chapter 20

Reference for Differential Equations

20.1 Sturm-Liouville

20.1.1 Definition (Sturm-Liouville Operator)

$$
\begin{aligned}
Lu &\equiv \frac{-1}{w(x)}\Big(\frac{d}{dx}(p(x)\frac{du}{dx}) + q(x)u\Big) \\
Lu &= \lambda u
\end{aligned}
$$

Regularity demands that $p(x)$ and $w(x)$ are positive and that $p(x)$, $p'(x)$, $q(x)$ and $w(x)$ are continuous on the interval $\big[a, b\big]$.

If the boundary conditions can be written with α_i and β_i as

$$
\begin{aligned}
\alpha_1 u(a) + \alpha_2 u'(a) &= 0 \\
\beta_1 u(b) + \beta_2 u'(b) &= 0
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_1^2 + \alpha_2^2 &> 0 \\
\beta_1^2 + \beta_2^2 &> 0
\end{aligned}
$$

20.1.2 Theorem *With the Hermitian product on $L^2([a, b], w(x))$ and boundary conditions, this defines a self adjoint operator. Therefore the eigenvalues are real and eigenfunctions of different eigenvalues are orthogonal.*
When we break the continuity on the closed interval assumption or let the interval become unbounded, there can be continuous spectrum.

20.1.3 Theorem (Bochner) *Suppose you have a sequence of polynomials $\phi_n(x)$ of degree n for each $n \in \mathbb{N}$. This solve a Sturm-Liouville operator*

$$
a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = \lambda_n y(x)
$$

if and only if the sequence (up to complex change of variable and constant factors) is Jacobi polynomials $P_n^{\alpha,\beta}$*, Laguerre* L_n^{α}*, Hermite* H_n*,* x^n*, or Bessel* $B_n^{\alpha,\beta}$*.*

Proof First plug in ϕ_0 ϕ_1 and ϕ_2.

$$\begin{aligned}
\phi_0 &\equiv A \\
\phi_1 &\equiv Bx + C \\
\phi_2 &\equiv Dx^2 + Ex + F \\
a_0(x)A &= \lambda_0 A \\
a_1(x)(B) + a_0(x)(Bx + C) &= \lambda_1(Bx + C) \\
a_2(x)2D + a_1(x)(2Dx + E) + a_0(x)(Dx^2 + Ex + F) &= \lambda_2(Dx^2 + Ex + F)
\end{aligned}$$

This implies that $a_0(x)$ must be a constant. Then that $a_1(x)$ must be a linear polynomial. Then $a_2(x)$ must be a quadratic polynomial.

20.2 Method of Characteristics

20.3 Elliptic , Parabolic , Hyperbolic

20.3.1 Remark (Elliptic Complex) When the C^i in Section 8.2.2 are $\Gamma(E_i)$ given as global sections of vector bundles E_i on M and the differential operators P_i forming the complex have symbols $\sigma(P_i)$ such that

$$0 \longrightarrow \pi^* E_0 \longrightarrow \pi^* P_1 \qquad \cdots \qquad \pi^* E_k \longrightarrow 0$$

is exact outside the zero section. π^* pulls back vector bundles on M to T^*M.

This is called an elliptic complex. When there are only two terms in the sequence, this gives back the previously defined elliptic operator.

Chapter 21

Reference for Various Function Spaces

21.1 Banach

21.1.1 Definition (Banach Space) *A normed vector space V in which all Cauchy sequences converge, aka completeness.*

21.1.2 Example ()

21.2 Frechet

21.2.1 Definition (Frechet) *A topological vector space which must be Hausdorff and have it's topology induced by a countable family of seminorms. It must also be complete with respect to this family.*

21.2.2 Definition (Frechet-Alternative) *Locally convex with topology induced by a translation invariant metric and a complete metric space.*

21.2.3 Example (Banach) *Every Banach space. Use the alternative definition and the metric given by $\mid x - y \mid$ with the Banach norm.*

21.3 Bornology

21.3.1 Definition (Bornological)

21.4 Convenient

21.4.1 Definition (Convenient Vector Space)

21.5 Nuclear

21.5.1 Definition (Nuclear Space)

In summary

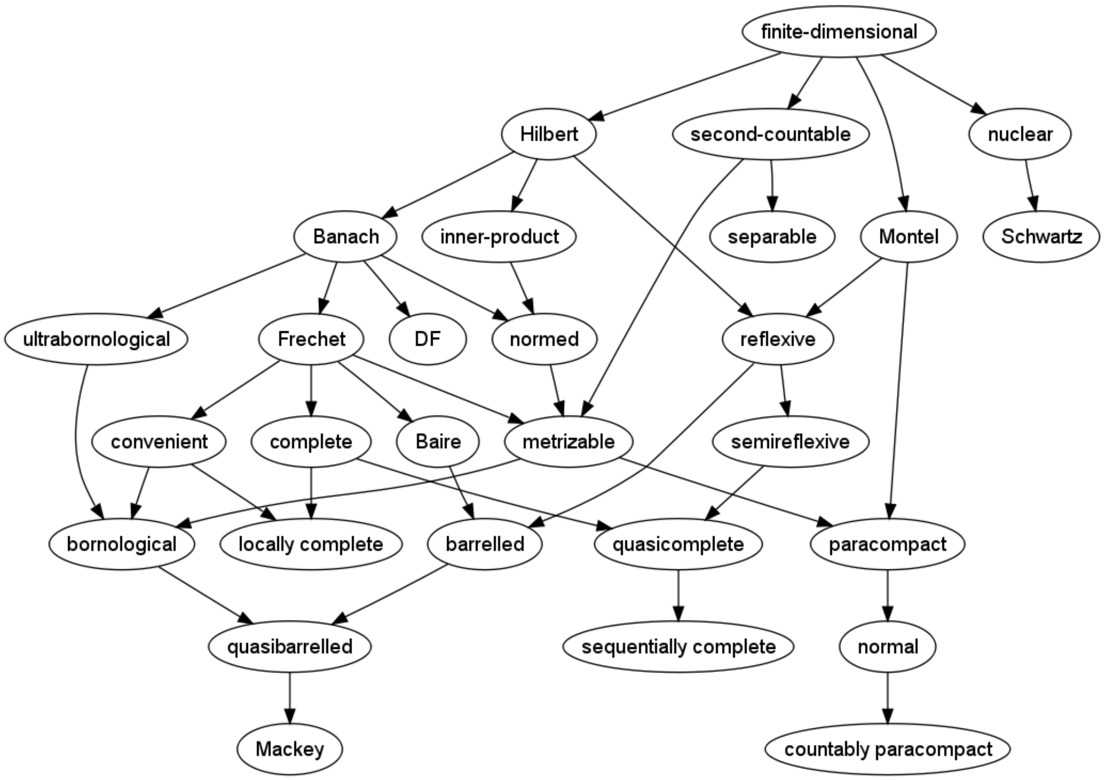

Figure 21.1: Click for MO Source

21.6 $C^k(U \to \mathbb{R})$

21.7 $C^\infty(U \to \mathbb{R})$

21.8 $C^k_{cs}(U \to \mathbb{R})$

21.9 $C^\omega(U \to \mathbb{R})$

21.10 Distributions for $C^\infty(U \to \mathbb{R})$

21.11 $C^k(K \to \mathbb{R})$

This is Banach with norm

$$\begin{aligned} \| f \|_{C^k} &= \sum_{i=0}^{k} | f^{(i)} |_\infty \\ | f^{(i)} |_\infty &= \sup_{x \in K} | f^{(i)}(x) | \end{aligned}$$

21.12 $C^\infty(K \to \mathbb{R})$

21.13 Distributions for $C^\infty(K \to \mathbb{R})$

21.14 $L^p(U \to \mathbb{C}, \mu)$

21.15 $L^2(U \to \mathbb{C}, \mu)$

21.16 $C^{k,\alpha}(U \to \mathbb{R})$

21.17 $W^{k,p}(U \to \mathbb{C}, \mu)$

21.18 Regularity Structures

`http://www.hairer.org/notes/RoughPaths.pdf`

21.18.1 Definition (Regularity Structure) *Let $A \subsetneq \mathbb{R}$ be an index set containing 0 and bounded from below and locally finite.*
Let $T = \bigoplus_A T_\alpha$ be an A-graded vector space such that $T_0 \simeq \mathbb{R}$. Make each T_α a Banach space with norm $||_\alpha$. By abuse of notation let $| x |_\alpha$ mean the α norm of the appropriate component.
Let G be a group of continuous linear operators acting on T such that for all $\Gamma \in G$

$$\Gamma \tau_\alpha - \tau_\alpha \in \bigoplus_{\beta < \alpha} T_\beta$$

21.18.2 Example (Taylor Series) *Let A be the natural numbers. Let T_n be the span of monomials in d variables of total degree n. Let $G \simeq \mathbb{R}^d$ acting on T by $\Gamma_h P(X) = P(X+h)$ for any polynomial in d variables P.*
This is lower triangular because shifting arguments only introduces lower order terms by binomial formulas.

21.18.3 Definition (Model for a regularity structure) *A model for a regularity structure on $\mathbb{R}^d$ is given by a pair of maps Π and Γ*

$$\begin{aligned} \Pi &\in \mathbb{R}^d \to \mathcal{L}(T, \mathcal{D}'(\mathbb{R}^d)) \\ \Gamma &\in \mathbb{R}^d \times \mathbb{R}^d \to G \\ \Gamma_{xy}\Gamma_{yz} &= \Gamma_{xz} \\ \Pi_x \Gamma_{xy} &= \Pi_y \end{aligned}$$

Π_x *tells how to realize elements of T as distributions with a basepoint x. Γ_{xy} says how to change those basepoints from y to x.*

21.18.4 Definition (Modeled Distributions)

21.18.5 Theorem (Reconstruction Theorem)

Chapter 22

Reference for Other Analysis Facts

22.1 Kramers-Kronig

22.1.1 Theorem (Sokhotski-Plemelj) *Let C be a smooth closed simple curve in the complex plane and φ an analytic function on C.*

We can't use the Cauchy formula

$$\phi(z) = \frac{1}{2\pi i} \oint_C \frac{\varphi(\zeta)}{\zeta - z} d\zeta$$

for $z \in C$, but we can do it inside and outside and then take the limit.

$$\lim_{w \to z} \phi_{in}(w) = \frac{1}{2\pi i} P \oint_C \frac{\varphi(\zeta)}{\zeta - z} d\zeta + \frac{1}{2}\varphi(z)$$
$$\lim_{w \to z} \phi_{out}(w) = \frac{1}{2\pi i} P \oint_C \frac{\varphi(\zeta)}{\zeta - z} d\zeta - \frac{1}{2}\varphi(z)$$

If the curve is no longer closed for the complex plane, we can still do a version of this theorem. In particular we can be in the situation of the real line which still divides the plane into two pieces (though now they are lower and upper instead of outside and inside).

22.1.2 Theorem (Kramers-Kronig)

$$\lim_{\epsilon \to 0} \int_{-a}^{b} \frac{f(s)}{s - x \pm i\epsilon} ds \quad = \quad P \int_{-a}^{b} \frac{f(s)}{s - x} ds \mp \pi i f(x)$$

22.1.3 Corollary (Kramers-Kronig Relations) *Let $f = f_R + f_I$ and suppose f extends analytically in the upper half plane and vanishes faster than $1/\mid z \mid$ so we can extend $(-a, b)$ to the entire real line and close off the integral on the left hand side into a closed contour that encloses no poles.*

$$\begin{aligned}
\pi i f(x) &= P\int_{-\infty}^{\infty}\frac{f(s)}{s-x}ds \\
if(x) &= \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{f(s)}{s-x}ds \\
Re(if(x)) = -f_I &= \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{f_R(s)}{s-x}ds \\
Im(if(x)) = f_R &= \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{f_I(s)}{s-x}ds
\end{aligned}$$

Chapter 23

Surgery (Place appropriately later)

23.1 Heegard Splittings

Let M be a closed connected orientable 3-manifold.

23.1.1 Definition (Heegard) *A pair of genus g handle-bodies $H_{1,2}$ in M where the union is M. $H_1 \bigcap H_2 = \partial H_i = \Sigma$*

23.1.2 Lemma (Alexander) *All that admit genus 0 splittings are S^3.*

23.1.3 Example (Torus decomposition of S^3)

23.1.4 Example (Identity Gluing) *Glue to itself with Id_Σ. This gives $\#^g S^1 \times S^2$*

23.1.5 Example (T^3) *The fundamental group of T^3 has 3 generators so the Heegard splitting must have at least $g = 3$. Here is a such genus 3 splitting.*

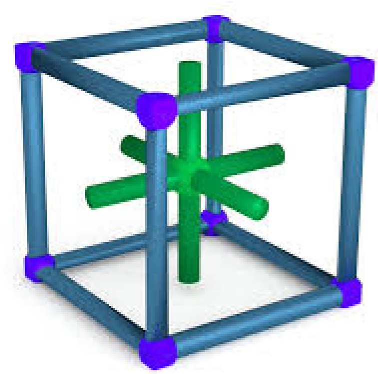

Figure 23.1: A Heegard splitting of T^3 presented with a cube with sides identified.

23.1.6 Theorem (Edwards) *Double suspension homology 3-sphere*

23.1.7 Theorem () *All 3-manifolds admit Heegard splittings.*

Proof Morse functions proof. □

Proof Triangulate M by $\mid K \mid \simeq M$. $\mid K^i \mid$ is notation for number of i cells. Thicken up the 1-skeleton to a handlebody. This is a graph thickened up so genus $E - E_T = E - V + 1$ (T is the maximal spanning tree). $\mid K^1 \mid$ edges and $\mid K^0 \mid$ vertices so genus $\mid K^1 \mid - \mid K^0 \mid +1$. Let Γ^1 be the dual 1-skeleton. Thicken this up and get another handlebody. This graph has $2* \mid K^2 \mid + \mid K^3 \mid$ vertices and $2* \mid K^2 \mid$ edges. So this handlbody has genus $\mid K^2 \mid - \mid K^3 \mid +1$. These two equal by Poincare duality. □

23.1.8 Lemma () *Given a splitting (Σ^g, H_1, H_2) can get a new splitting by stabilization. Take an arc in Σ and finish it off with another arc in H_i so together bounds disc D. Add the handle to Σ.*

Figure 23.2: Picture of handle attachment

23.1.9 Theorem (Waldhausen) *All Heegard splittings of S^3 are stabilizations of the genus 0 splitting.*

23.1.10 Conjecture *Have g_1 and g_2 Heegard splittings and want a common stabilization has genus $max(g_1, g_2) + 1$. By Reidemeister Thm know common stabilization appears eventually.*

23.1.11 Definition (Heegard Diagram)

23.1.1 Lens Spaces

We know S^3 is the only genus 0 3-manifold. What if we have a genus 1 splitting?

To do the gluing along tori, specify $SL_2(\mathbb{Z})$ which is the associated map on homology. This is enough information even though a priori you would need to specify a homeomorphism of the torus, but we are only concerned about building the 3 manifold up to homeomorphism allowing you to reduce the data for isotopy classes.

Chapter 24

Catastrophy Theory (Place appropriately later)

24.1 Classical

24.1.1 Theorem (Arnold ADE classification)

- A_1 *which corresponds to* $\pm x^2 + ax$
- A_2 *the fold singularity* $x^3 + ax$ *which has stable and unstable equilibria for negative* a *and then none for positive* a.
- A_3 *the cusp singularity* $x^4 + ax^2 + bx$
- A_4 *the swallowtail* $x^5 + ax^3 + bx^2 + cx$
- A_5 *the butterfly*
- A_k
- D_4^-
- D_4^+
- D_5
- D_k
- E_6
- E_7
- E_8

24.2 Quantum

https://arxiv.org/pdf/1912.06223.pdf

Chapter 25

Modular Stuff

25.1 Trigonometry Revisited

25.1.1 Trigonometry

Explanation due to Jan Vonk's lecture introducing Complex Multiplication

Start out with the real numbers. Now pick a discrete subgroup thereof. WLOG take it to be $\mathbb{Z}$ instead of the (not really) more general $r\mathbb{Z}$.

We would like functions on $\mathbb{R}$ that are invariant under the shifts by $\mathbb{Z}$. The functions on the circle $\mathbb{R}/\mathbb{Z}$. Forgetting the existence of trigonometry, let us do the naive thing of summing terms that are all related by a shift. Because $\mathbb{Z}$ is infinite, we can't average so we just sum.

$$f_g(x) \equiv \sum_{\lambda \in \mathbb{Z}} g(x-\lambda)$$

Choose a nice function g by $\frac{1}{x^k}$ with $k \geq 2$. In this way

$$\alpha_k(x) \equiv \sum_{\lambda \in \mathbb{Z}} \frac{1}{(x-\lambda)^k}$$

is well defined for all points not in $\mathbb{Z}$ and is invariant under shifts. There are poles but they are only on the lattice points $\mathbb{Z}$.

We can't do with $k=1$ because that would be divergent everywhere in the same manner as a harmonic series. But we can fix that

$$\begin{aligned} \alpha_1(x) &= \frac{1}{x} + \sum_{\lambda \in \mathbb{Z} \setminus 0} \frac{1}{x-\lambda} + \frac{1}{\lambda} \\ \alpha_1(x+n) &= \alpha_1(x) \end{aligned}$$

Now find a solution $s(x)$ to

$$\begin{aligned}
\frac{d}{dx}\log s(x) &= \alpha_1(x) \\
s(z) &\equiv Cz\prod_{\lambda\in\mathbb{Z}\setminus 0}(1-z/\lambda)e^{z/\lambda} \\
\frac{d}{dz}\log s(z) &= \frac{d\log}{dz}C+\frac{d\log}{dz}z+\sum_{\lambda\in\mathbb{Z}\setminus 0}\frac{d\log}{dz}(1-z/\lambda)e^{z/\lambda} \\
&= 0+\frac{1}{z}+\sum\frac{1}{z-\lambda}+\frac{1}{\lambda}=\alpha_1(z)
\end{aligned}$$

$$\begin{aligned}
s(z+1) &\neq s(z) \\
s(z+n) &= (-1)^n s(z)
\end{aligned}$$

So what we see is that
If we fix the constant to be π, we get the infinite product expansion of $\sin(\pi z)$.

$$\begin{aligned}
\alpha_1(z) &= \frac{1}{z}+\sum_{\lambda}\sum_{n\geq 1}(-1)\frac{x^n}{\lambda^{n+1}} \\
&= \frac{1}{z}-\sum_{n\geq 1}x^n\sum_{\lambda}\lambda^{n+1} \\
\Omega_n &\equiv \sum_{\lambda\in\mathbb{Z}\setminus 0}\lambda^n \\
\alpha_1(z) &= \frac{1}{z}+\sum_{k\geq 1}\Omega_{k+1}(k)z^k \\
\alpha_1(z) &= \frac{d\log}{dz}\sin(\pi z)=\frac{\pi\cos(\pi z)}{\sin(\pi z)} \\
&= \pi\cot(\pi z)
\end{aligned}$$

Now match terms in Taylor expansions to get the relationship between values of the Riemann zeta function at nonpositive integers an Bernoulli numbers.

25.1.2 Elliptic

Now we move from $\mathbb{R}\to\mathbb{C}$ and $r\mathbb{Z}\to\Lambda\equiv\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$.

$$\alpha_k(z) \equiv \sum_{\lambda\in\Lambda}\frac{1}{(z-\lambda)^k}$$

Now for convergence $k\geq 3$ and we need to slight fix ups for α_2 and α_1 instead of just α_1.

$$\begin{aligned}
\alpha_2(z) &= \frac{1}{z^2} + \sum_{\lambda\neq 0\in\Lambda} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} = \wp(z) \\
\alpha_1(z) &= \frac{1}{z} + \sum_{\lambda\neq 0\in\Lambda} \frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \\
\alpha_1(z+\omega_i) &\neq \alpha_1(z) \\
\alpha_1(z+\omega_i) &= \eta_i\alpha_1(z)
\end{aligned}$$

Again solve the differential equation of the $d\log$.

$$\begin{aligned}
\frac{d\log}{dz}\sigma(z) &= \alpha_1(z) \\
\sigma(z) &= z\prod_{\lambda\neq 0\in\Lambda}(1-z/\lambda)e^{z/\lambda+z^2/2\lambda^2} \\
\sigma(z+\omega_i) &= (-1)e^{\eta_i(z+\frac{\omega_i}{2})}\sigma(z)
\end{aligned}$$

25.1.1 Definition (Weirstrauss Model) *Send z in $\mathbb{C}$ to $\alpha_2(z), \alpha_3(z)$. If we call those X and Y respectively, they satisfy a cubic equation. If you specify X and Y as two complex numbers satisfying this equation, then there exists a z mapping to it and it is well defined up to all shifts by Λ. So you get the non-origin points of $\mathbb{C}/\Lambda$. Then you put in a point at infinity which represents the coset of Λ which is the origin point of $\mathbb{C}/\Lambda$.*

You could do this in the real case and map $\mathbb{R}$ to a parabola (and point at infinity) such that it descends to a map from $\mathbb{R}/\mathbb{Z}$ which sends the $\mathbb{Z}$ to the point at infinity.

25.1.2 Definition (Jacobi Model) *Send z in $\mathbb{C}$ to $\sigma(z), \sigma(z+\frac{\omega_1}{2}), \sigma(z+\frac{\omega_2}{2}), \sigma(z+\frac{\omega_1+\omega_2}{2})$. But that isn't invariant under shifts of z by Λ, but we can make all 4 have the same factors of automorphy under the shifts by putting in some prefactors on each of those 4 maps. Together this gives a map $\mathbb{C}\to\mathbb{C}^4\setminus 0$ which descends to a well defined map $\mathbb{C}/\Lambda$ to $\mathbb{CP}^3$. The image is characterized as the intersection of 2 quadrics, $X^2+Y^2=W^2$ and $Z^2+k^2X^2=W^2$ where k encodes the elliptic modulus (which depends on the lattice). Note the only dependce on the lattice is through k^2.*

You could do this in the real case and map $\mathbb{R}$ to a circle by using $s(z)$ and $s(z+\frac{1}{2})$ which are basically $\sin$ *and* $\cos$ *respectively. The invariance is only for $2\mathbb{Z}$ (when there are an even number of sign errors) which is the distinction between $\mathbb{RP}^1$ and the double cover S^1.*

25.2 j-Invariant

25.2.1 Definition *Functions invariant under $\tau\to\tau+1$ can be written as functions of $q=e^{2\pi i\tau}$. If want invariance under $SL_2(\mathbb{Z})$ acting by fractional linear transformations on the upper half plane, then you need j.*

$$j(\tau) = q^{-1} + 744 + 196884q + \cdots$$

25.2.2 Theorem *Let a, b and c be integers $j(\frac{a+\sqrt{-b}}{c})$ has an exact expression. In fact when $a=-1$, $-b$ is the discriminant of one of Euler's quadratic polynomials that produces primes and $c=2$, then the result is an integer.*

Euler's quadratics that produce primes are

$$
\begin{aligned}
&x^2-x+41\\
&x^2-x+17\\
&x^2-x+11
\end{aligned}
$$

in which you plug in the numbers 1 through 40, 16 and 10 respectively. They have discriminants -163, -67 and -43 respectively.

25.2.3 Corollary (Almost Integers) *Suppose q is very small in magnitude and τ is of the above form so that $j(\tau)$ is an integer, then*

$$
q^{-1} \quad = \quad j(\tau) - 744 + (-196884q + \cdots)
$$

where the term in parenthesis is small because q is so small. In particular, use $\tau = \frac{-1+\sqrt{-163}}{2}$ in which case $q = e^{\pi i(-1+i\sqrt{163})} = -e^{-\pi\sqrt{163}}$. This is a very small number and of the form for the above theorem to hold. Therefore $q^{-1} = -e^{\pi\sqrt{163}}$ is almost an integer. $e^{\pi\sqrt{163}}$ is close to $-j(\tau)+744$. Explicitly it is close to $262537412680768000 + 744$.

You could also use $\tau = \frac{-1+\sqrt{-67}}{2}$ and $\tau = \frac{-1+\sqrt{-43}}{2}$ but they won't work as well because q is not as small in these cases.