Low Dimensional Physics

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Contents

Chapter 1

Dimension 0

1.1 Rabi Oscillations

$$\begin{aligned} H &= \begin{pmatrix} E_0+\Delta & W_1-iW_2 \\ W_1+iW_2 & E_0-\Delta \end{pmatrix} \\ E_\pm &= E_0 \pm \sqrt{\Delta^2+\mid W \mid^2} \\ \mid W \mid^2 &= W_1^2+W_2^2 \end{aligned}$$

Define the following auxiliary variables.

$$\begin{aligned} \sin\theta &\equiv \frac{\mid W \mid}{\sqrt{\Delta^2+\mid W \mid^2}} \\ \cos\theta &= \frac{\Delta}{\sqrt{\Delta^2+\mid W \mid^2}} \\ \tan\theta &= \frac{\mid W \mid}{\Delta} \\ W_1+iW_2 &\equiv \mid W \mid e^{-i\phi} \end{aligned}$$

$$\begin{aligned} \mid E_+\rangle &= \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix} \\ \mid E_-\rangle &= \begin{pmatrix} -e^{i\phi}\sin\frac{\theta}{2} \\ cos\frac{\theta}{2} \end{pmatrix} \end{aligned}$$

If we start in $|\uparrow\rangle$ then after time t

$$
\begin{aligned}
\mid \psi(0)\rangle &= |\uparrow\rangle = \cos\frac{\theta}{2} \mid E_+\rangle - \sin\frac{\theta}{2} e^{-i\phi} \mid E_-\rangle \\
\mid \psi(t)\rangle &= \cos\frac{\theta}{2} e^{-iE_+t/\hbar} \mid E_+\rangle - e^{-iE_-t/\hbar} \sin\frac{\theta}{2} e^{-i\phi} \mid E_-\rangle \\
\langle\downarrow|| \psi(t)\rangle &= e^{-i\phi} \sin\frac{\theta}{2} \cos\frac{\theta}{2} (e^{-iE_+t/\hbar} - e^{-iE_-t/\hbar}) \\
\langle\downarrow|| \psi(t)\rangle^2 &= \frac{\mid W \mid^2}{\Delta^2 + \mid W \mid^2} \sin^2 \frac{E_+ - E_-}{2\hbar} t
\end{aligned}
$$

1.2 Oscillator

The Lagrangian and Euler Lagrange equations

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2} L I^2 - \frac{1}{2} \frac{Q^2}{C} \\
&= \frac{L}{2} \dot{Q}^2 - \frac{1}{2C} Q^2 \\
EL &= \{\ddot{Q} + \Omega^2 Q = 0\} \\
\Omega &= \frac{1}{\sqrt{LC}}
\end{aligned}
$$

The Hamiltonian and Hamilton's equations (fix the signs)

$$
\begin{aligned}
\Phi = \frac{d\mathcal{L}}{d\dot{Q}} &= L\dot{Q} = LI \\
H &= \frac{\Phi^2}{2L} + \frac{1}{2C} Q^2 \\
\dot{Q} &= \pm\{H, Q\} = \frac{\Phi}{L} \\
\dot{\Phi} &= \pm\{H, \Phi\} = \frac{-Q}{C}
\end{aligned}
$$

Canonical quantization

$$
\begin{aligned}
\left[\Phi, Q\right] &= -i\hbar \\
H &= \hbar\Omega(a^\dagger a + \frac{1}{2}) \\
\left[a, a^\dagger\right] &= 1
\end{aligned}
$$

The Lagrangian and Euler Lagrange equations

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}C\dot{\Phi}^2 - \frac{1}{2L}\Phi^2 \\ EL &= C\dot{\dot{\Phi}} + \frac{1}{L}\Phi \\ \Omega &= \frac{1}{\sqrt{LC}}\end{aligned}$$

The Hamiltonian and Hamilton's equations

$$\begin{aligned}Q = \frac{d\mathcal{L}}{\dot{\Phi}} &= C\dot{\Phi} \\ H &= \frac{\Phi^2}{2L} + \frac{1}{2C}Q^2 \\ \dot{Q} &= -\frac{\Phi}{L} \\ \dot{\Phi} &= \frac{Q}{C}\end{aligned}$$

Canonical quantization

$$\begin{aligned}\left[Q, \Phi\right] &= -i\hbar \\ H &= \hbar\Omega(a^\dagger a + \frac{1}{2}) \\ \left[a, a^\dagger\right] &= 1\end{aligned}$$

$$\begin{aligned}
\hat{Q} &= -iQ_{ZPF}(a-a^\dagger) \\
\hat{\Phi} &= \Phi_{ZPF}(a+a^\dagger) \\
Q_{ZPF} &= \sqrt{\frac{\hbar}{2Z}} \\
\Phi_{ZPF} &= \sqrt{\frac{\hbar Z}{2}} \\
Z &= \sqrt{\frac{L}{C}} \\
R_Q &= \frac{h}{4e^2} \approx 6453.20 Ohms \\
z &= Z/R_Q \\
Q_{ZPF} &= 2e\sqrt{\frac{1}{4\pi z}} \\
\Phi_{ZPF} &= \Phi_0\sqrt{\frac{z}{4\pi}} \\
\Phi_0 &= \frac{h}{2e}
\end{aligned}$$

Voltage

$$\begin{aligned}
\hat{V} &= \frac{d\hat{\Phi}}{dt} = -iV_{ZPF}(a-a^\dagger) \\
V_{ZPF} &= \Omega\Phi_0\sqrt{\frac{z}{4\pi}}
\end{aligned}$$

Since $a - a^\dagger$ will have expectation order 1, V will have order V_{ZPF}

1.3 Josephson Oscillator

Consider two electrodes with a tunnel junction. Consider the ground states of each electrode only. Now for fixed number of pairs $N = N_L + N_R$ we have a basis for the Hilbert space by

$$\mid m\rangle = \mid N_L - m\rangle \otimes \mid N_R + m\rangle$$

where $\mid N_L - m\rangle$ is the state with that many pairs on the left electrode. There is a tunneling part of the Hamiltonian.

$$\begin{aligned} H_T &= \frac{-1}{2}E_J \sum_m \mid m\rangle\langle m+1 \mid + \mid m+1\rangle\langle m \mid \\ \mid \phi\rangle &= \sum e^{im\phi} \mid m\rangle \\ H_T \mid \phi\rangle &= -E_J \cos\phi \mid \phi\rangle \\ v_{group}(\phi) &= \frac{1}{\hbar}\frac{\partial}{\partial\phi}(-E_J\cos\phi) \\ I(\phi) &= \frac{2e}{\hbar}E_J \sin\phi \end{aligned}$$

This gives a maximal coherent (dissipationless) current $\frac{2e}{\hbar}E_J$. Any more current will cause the voltage to rise above the 2Δ gap and break the approximation that we only need the ground states determined by the number of pairs only.

Now with a Coulomb term as well for this little capacitor

$$H = 4E_C(n-n_g)^2 - E_J\cos\phi$$

1.3.1 Definition (Mathieu Equation)

$$\big[\frac{d^2}{dx^2} + (a - 2q\cos(2x))\big]y = 0$$

For fixed a and q, the solution can be expressed as $F(a,q,x) = e^{i\mu x}P(a,q,x)$ with P π-periodic.

1.3.2 Definition (Mathieu cosine and sine) *The solution with value 1 at $x = 0$ and derivative 0 or vice versa.*

These are real valued solutions.

$$\begin{aligned} C(a,q,x) &= \frac{F(a,q,x) + F(a,q,-x)}{2F(a,q,0)} \\ S(a,q,x) &= \frac{F(a,q,x) - F(a,q,-x)}{2F'(a,q,0)} \end{aligned}$$

If $q = 0$ then these are $\cos\sqrt{a}x$ and $\frac{\sin\sqrt{a}x}{\sqrt{a}}$ respectively. In general they are aperiodic.

For a given value of q there are countably many a that give periodic solutions. For example, if $q = 0$, then $a = (n)^2$ with $n \in \mathbb{Z}$ in order to achieve 2π periodicity.

$$\begin{aligned} H &= -4E_C\frac{\partial^2}{\partial\phi^2} + 8E_C n_g i\frac{\partial}{\partial\phi} + 4E_C n_g^2 - E_J\cos\phi \\ H_2 &= \frac{-H}{4E_C} + n_g^2 = \frac{\partial^2}{\partial\phi^2} - 2n_g i\frac{\partial}{\partial\phi} + \frac{E_J}{4E_C}\cos\phi \\ H_2\psi = \lambda_{H_2}\psi &\implies -4E_C(\lambda_{H_2} - n_g^2) = \lambda_H \end{aligned}$$

Let $\phi = 2x$

$$\begin{aligned} \frac{1}{4}\frac{\partial^2\psi}{\partial x^2} - n_g i\frac{\partial\psi}{\partial x} + \frac{E_J}{4E_C}\cos(2x)\psi &= \lambda\psi \\ \frac{\partial^2\psi}{\partial x^2} - 4n_g i\frac{\partial\psi}{\partial x} + (-4\lambda + \frac{E_J}{E_C}\cos(2x))\psi &= 0 \end{aligned}$$

If $n_g = 0$ then we get $\psi = a_S C(-4\lambda, \frac{-E_J}{2E_C}, \frac{\phi}{2}) + a_A S(-4\lambda, \frac{-E_J}{2E_C}, \frac{\phi}{2})$
Now keep n_g, but define w

$$\begin{aligned} w &= \psi e^{-2in_g x} \\ w &= C_1 C(-4\lambda + 4n_g^2, -\frac{E_J}{2E_C}, x) + C_2 S(-4\lambda + 4n_g^2, -\frac{E_J}{2E_C}, x) \\ \psi &= e^{2in_g x} w \end{aligned}$$

Proof

$$\begin{aligned} \frac{\partial\psi}{\partial x} &= \frac{\partial w}{\partial x}e^{2in_g x} + (2in_g)we^{2in_g x} \\ \frac{\partial^2\psi}{\partial x^2} &= \frac{\partial^2 w}{\partial x^2}e^{2in_g x} + 2in_g\frac{\partial w}{\partial x}e^{2in_g x} + 2in_g\frac{\partial w}{\partial x}e^{2in_g x} - 4n_g^2 we^{2in_g x} \\ \frac{\partial^2\psi}{\partial x^2} - 4n_g i\frac{\partial\psi}{\partial x} + (-4\lambda + \frac{E_J}{E_C}\cos 2x)\psi &= \frac{\partial^2 w}{\partial x^2}e^{2in_g x} + 4n_g^2 we^{2in_g x} - (4\lambda + \frac{E_J}{E_C}\cos 2x)we^{2in_g x} \\ \frac{\partial^2 w}{\partial x^2} + (-4\lambda + 4n_g^2 - 2\frac{E_J}{2E_C}\cos 2x)w &= 0 \\ w &= C_1 C(-4\lambda + 4n_g^2, \frac{E_J}{2E_C}, x) + C_2 S(-4\lambda + 4n_g^2, \frac{E_J}{2E_C}, x) \\ \psi &= C_1 e^{in_g\phi} C(-4\lambda + 4n_g^2, \frac{E_J}{2E_C}, \frac{\phi}{2}) + C_2 e^{in_g\phi} S(-4\lambda + 4n_g^2, \frac{E_J}{2E_C}, \frac{\phi}{2}) \end{aligned}$$

We want ψ to be periodic under $\phi \to \phi + 2\pi$ so $x \to x + \pi$. So w must go to $e^{-2in_g\pi}w$ upon the same shift. This gives countably many allowed values for $-4\lambda + 4n_g^2$ so countably many values for the energy λ_H. That is the spectrum as a function of $E_{C,J}$ and n_g.

Let's say $C_1 = 1$ and $C_2 = 0$. In Mathematica this is given by MathieuCharacteristicA[$-2n_g + k$,$-E_J/2E_C$]. Call it MCA for short. That is under $x \to x + 2\pi$ pick up a $e^{ir2\pi}$.

$$\lambda = n_g^2 - \frac{1}{4}MCA(k - 2n_g, -\frac{E_J}{2E_C})$$

Similarly for the odd functions we have

$$\lambda = n_g^2 - \frac{1}{4}MCB(k - 2n_g, -\frac{E_J}{2E_C})$$

Together the spectrum is the collection of all

$$\begin{aligned}\lambda_H &= E_C MCA(k-2n_g, -\frac{E_J}{2E_C}) \\ \lambda_H &= E_C MCB(k-2n_g, -\frac{E_J}{2E_C})\end{aligned}$$

1.3.3 Remark See the $N=2$ case of the Toda lattice in Section 11.2. There we see the $B2$ periodic Toda and (quasi)periodic solutions to Mathieu equation. $A2$ giving L^2 solutions of Mathieu's modified equation. ◊

1.3.1 Hill Equation

1.3.4 Definition (Hill Operator) *Let $q(x)$ be a periodic real function.*

$$\begin{aligned}Hy &= -y'' + q(x)y \\ Hy &= \lambda y\end{aligned}$$

1.3.5 Theorem (Floquet-Bloch) *Solve the spectrum with periodic boundary conditions to get a countable spectrum $\lambda_0 \leq \lambda_1 \cdots$. Do the same with antiperiodic boundary conditions to give a spectrum $\mu_0 \leq \mu_1 \cdots$. These interlace as $\lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 \cdots$. Then the spectrum for the full operator on $L^2(\mathbb{R})$ is absolutely continuous*

$$\sigma_{ac} = \bigsqcup_{n\in\mathbb{N}} ([\lambda_{2n}, \mu_{2n}] \sqcup [\mu_{2n+1}, \lambda_{2n+1}])$$

The compliment of this is known as the instability set.

Now let the period be π

1.3.6 Theorem (Titschmarsh) *Let $\theta(\lambda, x)$ and $\phi(\lambda, x)$ be solutions with $\theta(\lambda, 0) = \phi'(\lambda, 0) = 1$ and $\theta'(\lambda, 0) = \phi(\lambda, 0) = 0$*

Define the following auxiliary quantities

$$\begin{aligned}\Delta_\pm(\lambda) &\equiv \frac{\theta(\lambda,\pi) \pm \phi'(\lambda,\pi)}{2} \\ F_1(\lambda) &\equiv \int dx f(x)\theta(\lambda, x) \\ F_2(\lambda) &\equiv \int dx f(x)\phi(\lambda, x) \\ \Phi(\lambda, x, f) &\equiv \phi(\lambda,\pi)\theta(\lambda, x)F_1(\lambda) - \theta'(\lambda,\pi)\phi(\lambda, x)F_2(\lambda) \\ &- \Delta_-(\lambda)\theta(\lambda, x)F_2(\lambda) - \Delta_-(\lambda)\phi(\lambda, x)F_1(\lambda)\end{aligned}$$

Then the projector to eigenspaces for a closed Borel subset $\sigma \subset \sigma_{ac}(H)$ of the spectrum gives

$$(P(\sigma)\psi)(x) = \frac{1}{2\pi}\int_\sigma \frac{d\lambda}{\sqrt{1+\Delta_+(\lambda)^2}}\Phi(\lambda,x,\psi)$$

In particular:

$$\psi(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{2\pi}\int_{\lambda_k}^{\mu_{k+1}}\frac{d\lambda}{\sqrt{1+\Delta_+(\lambda)^2}}\Phi(\lambda,x,\psi)$$

By letting $\sigma = \sigma_{ac}(H)$

Proof `https://arxiv.org/pdf/math/0511370.pdf` □

1.3.7 Example (Mathieu) *Let* $q(x) = 2q_0\cos 2x$

$$\begin{aligned}
-y'' + 2q_0\cos(2x)y &= \lambda y \\
y'' + (-\lambda - 2q_0\cos(2x))y &= 0 \\
\theta(\lambda,x) &= C(-\lambda,q_0,x) \\
\phi(\lambda,x) &= S(-\lambda,q_0,x) \\
\Delta_\pm &= \frac{C(-\lambda,q_0,\pi)\pm\partial_x S(-\lambda,q_0,\pi)}{2} \\
F_1(\lambda) &\equiv \int dx f(x)C(-\lambda,q_0,x) \\
F_2(\lambda) &\equiv \int dx f(x)S(-\lambda,q_0,x) \\
\Phi(\lambda,x,f) &\equiv S(-\lambda,q_0,x=\pi)C(-\lambda,q_0,x)F_1(\lambda) - \partial_x C(-\lambda,q_0,x=\pi)S(-\lambda,q_0,x)F_2(\lambda) \\
&- \Delta_-(\lambda)C(-\lambda,q_0,x)F_2(\lambda) - \Delta_-(\lambda)S(-\lambda,q_0,x)F_1(\lambda)
\end{aligned}$$

1.4 Rabi

First suppose the electromagnetic field is purely classical and that the wavelength of the ambient plane wave light is much larger than the scale of the atom so we can treat the field as constant over that relevant scale. We also require that the field strength not be too large.

$$\begin{aligned}
E(r_0,t) &\equiv \hat{e}E(r_0)\cos(\omega t + \phi(r_0)) \\
d &\equiv \langle 1 \mid q(\hat{r}-r_0)\cdot\hat{e} \mid 0\rangle \\
\Omega_1 &\equiv \frac{-dE(r_0)}{\hbar} \\
H &\equiv \hbar\begin{pmatrix} 0 & \Omega_1\cos(\omega t+\phi) \\ \Omega_1\cos(\omega t+\phi) & \omega_0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned} \mid \psi\rangle &\equiv \gamma_a(t) \mid 0\rangle + e^{-i\omega_0 t}\gamma_b(t) \mid 1\rangle \\ i\frac{d}{dt}\gamma_a &= \frac{\Omega_1 e^{i\phi}}{2} e^{i(\omega-\omega_0)t}\gamma_b + \frac{\Omega_1 e^{-i\phi}}{2} e^{-i(\omega+\omega_0)t}\gamma_b \\ i\frac{d}{dt}\gamma_b &= \frac{\Omega_1 e^{-i\phi}}{2} e^{-i(\omega-\omega_0)t}\gamma_a + \frac{\Omega_1 e^{i\phi}}{2} e^{i(\omega+\omega_0)t}\gamma_a \end{aligned}$$

Dropping the terms that oscillate with frequency $\omega + \omega_0$ gives us easier solution and denoting $\delta \equiv \omega - \omega_0$

1.4.1 Exercise *Restore the rapidly oscillating term and show that it does not contribute significantly to the quantities of interest.*

$$\begin{pmatrix}\gamma_a \\ \gamma_b\end{pmatrix} \propto \begin{pmatrix}\Omega_1 e^{i\phi} \\ \delta \pm \sqrt{\Omega_1^2 + \delta^2}\end{pmatrix} e^{-i\frac{\pm\sqrt{\Omega_1^2+\delta^2}}{2}t}$$

If consider the corresponding transition probabilities we see oscillations and the maximum probability is determined by $\frac{\Omega_1^2}{\Omega_1^2+\delta^2}$ which can be 1 when you have the light frequency on resonance with the level spacing and falling off as a Lorentzian function away from that. And as the electric field goes to 0, $\Omega_1 \to 0$ so the oscillations go away. The atom stays in either $\mid 0\rangle$ or $\mid 1\rangle$ with no spontaneous emission or absorptions. We didn't consider any back reaction of the atom on the field let alone quantize the field so we should not be surprised at the lack of capturing this phenomenon.

Now treat the electromagnetic field as quantized. Let a and $a^\dagger$ be annihilation and creation operators for a single mode of te photon field.

$$H - \frac{1}{2}\hbar\omega I = \hbar\omega \begin{pmatrix} a^\dagger a + g(a^\dagger + a) & \Delta \\ \Delta & a^\dagger a - g(a^\dagger + a)\end{pmatrix}$$

P acts as reflection which takes $a^\dagger \to -a^\dagger$, $a \to -a$, and $\mid 1\rangle$ and $\mid 2\rangle$ switch.

1.5 Lennard-Jones/Morse

1.5.1 Lennard-Jones

$$\begin{aligned} V &= \epsilon\big((\frac{r_m}{r})^{12} - 2(\frac{r_m}{r})^6\big) \\ V_{min} &= V(r_m) = -\epsilon \end{aligned}$$

1.5.2 Morse

$$\begin{aligned} V &= D_e(1 - e^{-a(r-r_e)})^2 - D_e \\ &= D_e(e^{-2a(r-r_e)} - 2e^{-a(r-r_e)}) \\ \lim_{r\to+\infty} V &= 0 \end{aligned}$$

Introduce new variables to solve for the v'th eigenstate with energy $E(v)$

$$\begin{aligned} x &= ar \\ x_e &= ar_e \\ \lambda &= \frac{\sqrt{2mD_e}}{a\hbar} \\ \epsilon_v &= \frac{2m}{a^2\hbar^2}E(v) \end{aligned}$$

$$\begin{aligned} \epsilon_n &= 1 - \frac{1}{\lambda^2}(\lambda - n - \frac{1}{2})^2 \\ \psi_n(z) &= N_n z^{\lambda-n-1} e^{-1/2z} L_n^{(2\lambda-2n-1)}(z) \end{aligned}$$

with $L_n^{(\alpha)}$ a generalized Laguerre polynomial in terms of ${}_1F_1$ hypergeometric function.

But because of the limit 0, there are actually only finitely many bound states where this proceduce of listing off the states from lowest to highest works. It stops applying when:

$$\begin{aligned} \nu_0 &= \frac{a}{2\pi}\sqrt{2D_e/m} \\ E(v) &= h\nu_0(v+1/2) - \frac{h^2\nu_0^2(v+1/2)^2}{4D_e} \\ v_{max} &= \frac{2D_e - h\nu_0}{h\nu_0} \end{aligned}$$

Above v_{max} is the continuum of free states.

1.6 Jaynes-Cummings

$$\begin{aligned} H &= H_{em} + H_{atom} + H_{int} \\ H_{em} &= \hbar\omega_c a^\dagger a \\ H_{atom} &= \hbar\omega_a \frac{\sigma_z}{2} \\ H_{int} &= \frac{\hbar\Omega}{2} ES \\ E &= a + a^\dagger \\ S &= \sigma_+ + \sigma_- \end{aligned}$$

In the interaction picture this gives

$$H_{int}(t) = \frac{\hbar\Omega}{2}(a\sigma_- e^{-i(\omega_c+\omega_a)t} + a^\dagger\sigma_+ e^{i(\omega_c+\omega_a)t} + a\sigma_+ e^{i(-\omega_c+\omega_a)t} + a^\dagger\sigma_- e^{-i(-\omega_c+\omega_a)t})$$

Now assume $\mid \omega_c - \omega_a \mid << \omega_c + \omega_a$ to throw away the rapidly oscillating summands and transforming back to Schroedinger picture

$$\begin{aligned} H &= \hbar\omega_c a^\dagger a + \hbar\omega_a \frac{\sigma_z}{2} + \frac{\hbar\Omega}{2}(a\sigma_+ + a^\dagger\sigma_-) \\ &= \hbar\omega_c(a^\dagger a + \frac{\sigma_z}{2}) + H_{II} \\ H_{II} &= \hbar(\omega_a - \omega_c)\frac{\sigma_z}{2} + \frac{\hbar\Omega}{2}(a\sigma_+ + a^\dagger\sigma_-) \\ &= \hbar\delta\frac{\sigma_z}{2} + \frac{\hbar\Omega}{2}(a\sigma_+ + a^\dagger\sigma_-) \end{aligned}$$

This is block form with only 2 by 2 blocks by $\langle n, e \mid$ and $\langle n+1, g \mid$ both with H_I eigenstates of eigenvalue $\hbar\omega_c(n+\frac{1}{2})$ and H_{II} being the off diagonal terms.

$$\begin{aligned} H_{block} &= \hbar \begin{pmatrix} n\omega_c + \frac{\omega_a}{2} & \frac{\Omega}{2}\sqrt{n+1} \\ \frac{\Omega}{2}\sqrt{n+1} & n\omega_c + \frac{\omega_a}{2} - \delta \end{pmatrix} \\ &= \hbar \begin{pmatrix} n\omega_c + \frac{\omega_a}{2} & \frac{\Omega}{2}\sqrt{n+1} \\ \frac{\Omega}{2}\sqrt{n+1} & (n+1)\omega_c - \frac{\omega_a}{2} \end{pmatrix} \\ \lambda_\pm &= \hbar\omega_c(n+\frac{1}{2}) \pm \hbar\sqrt{\delta^2 + \Omega^2(n+1)} \\ \langle n, + \mid &= \cos\frac{\alpha_n}{2}\langle n, e \mid + \sin\frac{\alpha_n}{2}\langle n+1, g \mid \\ \langle n, - \mid &= -\sin\frac{\alpha_n}{2}\langle n, e \mid + \cos\frac{\alpha_n}{2}\langle n+1, g \mid \\ \alpha_n &= \tan^{-1}\frac{\Omega\sqrt{n+1}}{\delta} \end{aligned}$$

1.6.1 Remark Fermi gedanken experiment. There are two atoms a and b separated by distance R. At time $t = 0$, a is in ground state while b is excited. The algebra of observables is $B(H_a) \otimes B(H_b) \otimes B(H_c)$. The initial state is $\omega_a \otimes \omega_b \otimes \omega_c$ where $\omega_a =\mid g\rangle\langle g \mid$ and $\omega_b =\mid e\rangle\langle e \mid$ and ω_c is the vacuum state of the radiation field.

$$\begin{aligned} \omega_t &= \omega_0(e^{itH} \cdot e^{-itH}) \\ P(t) &= \omega_t(E) \\ E &= (1_a - \mid g\rangle\langle g \mid) \otimes 1_b \otimes 1_c \\ P(t) &= 0 \ \ \forall t < R/c \end{aligned}$$

but $\omega_t(E)$ being 0 for all t in some interval then it is identically zero.

1.7 Acceptor/Donor

https://arxiv.org/pdf/1709.04064.pdf

There are 2 2-level systems that are exchanging energy. For example, consider a photosynthesis process. One pigment molecule gets excited and then that oscillates back and forth. That is the case with $\Delta = 0$ and no environment. Then insert a detuning Δ so that the there is a cost to hopping back and forth. Then put in an environment in the form of a single harmonic oscillator with annihalation and creation given by a and $a^\dagger$. The freqency for this oscillator is some ν_{eff}.

$$H = \frac{J}{2}\sigma_a^x\sigma_d^x + \frac{\Delta}{2}\sigma_d^z + \frac{\kappa}{2}\sigma_d^z(a + a^\dagger) + \nu_{eff}a^\dagger a$$

Without the environment

$$\begin{aligned} H &= \begin{pmatrix} \Delta/2 & 0 & 0 & J/2 \\ 0 & \Delta/2 & J/2 & 0 \\ 0 & J/2 & -\Delta/2 & 0 \\ J/2 & 0 & 0 & -\Delta/2 \end{pmatrix} \\ \lambda_\pm &= \pm\sqrt{\Delta^2 + J^2} \end{aligned}$$

Defining $A_\pm = -\frac{-\Delta\pm\sqrt{J^2+\Delta^2}}{J}$ then gives the eigenvectors as

$$\begin{aligned} &A_+, 0, 0, 1 \\ &0, A_+, 1, 0 \\ &A_-, 0, 0, 1 \\ &0, A_-, 1, 0 \end{aligned}$$

with the first 2 for λ_+ and the second 2 for λ_-.

1.8 Wells

1.8.1 Rectangular d-dimensions Infinite

1.8.2 Rectangular d-dimensions Finite

$$V(x_1, \cdots x_d) = -V_0\chi(x_1 \in [-L_1, L_1]) \cdots \chi(x_d \in [-L_d, L_d])$$

The potential is a sum rather than a product so no separation of variables.

1.8.3 Spherical Finite Well

$$\begin{aligned}
\psi_{in} &= Aj_\ell(k'r)Y_{\ell m}(\theta,\phi) \\
k' &= \sqrt{\frac{2\mu(E+V_0)}{\hbar^2}} \\
\psi_{out} &= Bj_\ell(kr) + Cn_\ell(kr) \\
k &= \sqrt{\frac{2\mu(E)}{\hbar^2}}
\end{aligned}$$

The matching conditions at $r = a$ fix B and C both of which are proportional to A.

Define the phase shift δ_ℓ through C and B which is seen by taking the large r limit of ψ_{out} to see the decomposition as incoming and outgoing spherical waves. The $\ell = 0$ has simple expressions for the spherical Bessel functions so in that case it can be written easily.

$$\begin{aligned}
\tan\delta_\ell &= -\frac{C}{B} \\
\tan\delta_{\ell=0} &= \frac{k\cos ka\sin k'a - k'\cos k'a\sin ka}{k\sin ka\sin k'a + k'\cos k'a\cos ka}
\end{aligned}$$

1.8.4 Polygon 2-d Infinite

1.8.1 Definition (Billiard)

1.8.5 Polygon 2-d Finite

1.8.6 Quantum Dot

1.8.2 Definition (Quantum Dot) *A semiconductor nanocrystal consisting of a few thousand atoms on the order of 2-10 nanometers. The electrons and holes thereof behave like they are confined to a three dimensional potential well.*

1.8.3 Definition (Exciton) *An electron hole pair interacting with electrostatic potential.*

$$\begin{aligned}
H_{exciton} &= \frac{-\hbar^2}{2m_e^*}\nabla_e^2 + \frac{-\hbar^2}{2m_h^*}\nabla_h^2 - \frac{e^2}{\epsilon \mid r_e - r_h \mid} \\
&+ \frac{1}{2}m_e^*\omega^2 r_e^2 + \frac{1}{2}m_h^*\omega^2 r_h^2
\end{aligned}$$

After removing the CM motion, this gives

$$\begin{aligned}
m &= \frac{m_e^* m_h^*}{m_e^* + m_h^*} \\
H_{rel} &= \frac{-\hbar^2}{2m}\nabla_r^2 - \frac{e^2}{\epsilon r} + \frac{m\omega^2 r^2}{2}
\end{aligned}$$

1.9 Poisson Geometry

1.9.1 Theorem (Moyal/Kontsevich) *Explicit formula for deformation quantization for Poisson structures on $\mathbb{R}^n$ and less explicit form for arbitrary Poisson manifolds. The set of deformation quantizations up to equivalence $f \to f + \hbar g$ taking one quantization to the other.*

Proof

$$\begin{aligned} f \star_\hbar g &= fg + \sum_{n=1}^{\infty} \hbar^n C_n(f,g) \\ &= fg + \frac{i\hbar}{2}\pi^{ij}\partial_i f \partial_j g - \frac{\hbar^2}{8}\pi^{ij}\pi^{km}\partial_i\partial_k f \partial_j\partial_m g + \cdots \\ &= \mu(e^{\frac{i\hbar}{2}\pi}(f \otimes g)) \end{aligned}$$

where the last line uses BCH to expand out that exponential order by order in $\hbar$

1.9.2 Definition (Quantum Moment Map) *Let A be an algebra with $\mathfrak{g}$ action ϕ $\mathfrak{g} \to Der(A)$. A quantum moment map is $U(\mathfrak{g}) \to A$ associative algebra morphism such that $[\mu(a), b] = \phi(a)b$ for all $a \in \mathfrak{g}$ and $b \in A$.*

1.9.3 Example *If A is a deformation quantization of a Poisson algebra A_0. Also suppose we have a classical $\mathfrak{g}$ action ϕ_0 $\mathfrak{g} \to Der(A_0)$ and a classical moment map μ_0 $U(\mathfrak{g}) \to A_0$. A quantization of this μ is such that μ $U(\mathfrak{g}) \to A[\hbar^{-1}]$ satisfies $\mu(a) = \hbar^{-1}\mu_0(a) + O(1)$*

1.9.4 Definition (Quantum Hamiltonian Reduction) *Let A $\mathfrak{g}$ and μ be given. Then take the $\mathfrak{g}$ invariants $A^{\mathfrak{g}}$ and the ideal J generated by $\mu(a)$ to form the new algebra $A^{\mathfrak{g}}/(J \bigcap A^{\mathfrak{g}})$.*

1.9.5 Remark Other definitions should be used if $\mathfrak{g}$ is not reductive or not acting locally finitely.$\Diamond$

1.9.6 Example *If A is a deformation quantization of $C^\infty(M)$ for a Poisson manifold. Then you can form $\mu_0^{-1}(\mathcal{O})/G = R(M,G,\mathcal{O})$ classically and the quantum Hamiltonian reduction is a deformation of functions on that.*

1.10 Hamilton-Jacobi

1.10.1 Theorem (Hamilton Jacobi) *Let P α be an exact symplectic manifold not necessarily a cotangent bundle with Liouville form. Get a function on parameterized paths γ by:*

$$S(\gamma) = \int_{[\gamma]} \alpha - \int_{t_0}^{t_1} H(\gamma(t))$$

where $[\gamma]$ is the unparameterized path. When not exact, α is now a connection on a line bundle so S is now valued as a section instead.

1.10.2 Theorem (Hamilton-Jacobi-Bellman)

$$\begin{aligned} V(x(0),0) &= min_u L \\ L &= \int_0^T C(x(t),u(t))dt + D(x(T)) \\ \dot{x}(t) &= F(x(t),u(t)) \\ \dot{V}(x,t) &+ min_u \nabla V(x,t)\cdot F(x,u) + C(x,u) = 0 \\ V(x,T) &= D(x) \end{aligned}$$

We then solve this PDE with that terminal condition for $V(x,t)$

Proof

$$\begin{aligned} V(x(t),t) &= min_u\big(V(x(t+dt),t+dt) + \int_t^{t+dt} C(x(s),u(s))ds\big) \\ &\approx min_u\big(V(x(t),t) + \dot{V}(x(t),t)dt + \nabla V(x(t),t)\cdot\dot{x}(t)dt + \int_t^{t+dt} C(x(s),u(s))ds\big) \\ 0 &\approx min_u\big(\dot{V}(x(t),t)dt + \nabla V(x(t),t)\cdot\dot{x}(t)dt + C(x(t),u(t))dt\big) \end{aligned}$$

where the error is $o(dt)$. Taking this limit then provides the required PDE.

1.10.3 Lemma (Principle of Optimality) *An optimal control policy $u(t)$ has the property that if we drop the first step, the remaining policy is optimal for the new state. In other words, an optimal policy is like a shark it doesn't need to look back. It doesn't have a neck.* `http://www.imdb.com/title/tt0584440/quotes`.

1.10.1 Quantized Version

$$\begin{aligned} \psi &= e^{iS/\hbar}(\psi_0 + \hbar\psi_1 + \cdots) \\ \frac{\partial S}{\partial t} &= \frac{1}{2m}(\frac{\partial S}{\partial x})^2 + V(x) \end{aligned}$$

Semiclassical Limit

Suppose we have a state $\psi = e^{if(q)/\hbar}\phi(q)$

$$\begin{aligned} \langle\psi \mid p \mid \psi\rangle &= \int \bar{\phi} e^{i-f/\hbar}(-i\hbar)\frac{\partial}{\partial q}e^{if/\hbar}\phi \\ &= \int \bar{\phi}\phi\frac{\partial f}{\partial q} + O(\hbar) \\ \langle\psi \mid P(p,q) \mid \psi\rangle &= \int \bar{\phi}\phi P(\frac{\partial f}{\partial q}, q) + O(\hbar) \end{aligned}$$

That is the pure state $\langle\psi \mid ? \mid \psi\rangle$ converges as a distribution to the probability measure on the phase space given by with density $\mid \phi \mid^2$ on the Lagrangian subspace $p = \frac{\partial f}{\partial q}$

1.10.4 Example *We may consider the plane wave state where $f = \hbar k * q$ and then we get the support is on $p = \hbar k$. Multiplying by these plane wave factors does what it should and shifts the momenta by $\hbar k$*

1.10.5 Example (Gaussian Wavepackets) *Let $\psi = Ae^{-x^2/2a}$*

$$\langle \psi \mid ? \mid \psi \rangle \quad \rightarrow$$

If we evolve for a time t using the propogator $U(q,t;q^{'},0)$, the state evolves as

$$\begin{aligned} \psi_t(x) &= \int \psi_0(y) U(x,t;y,0) \\ &= \int_{\mathbb{R}^n} \exp() \frac{d^2S}{dxdy}^{1/2} \phi d\mu \\ &\approx \\ \langle \psi_t \mid ? \mid \psi_t \rangle &\rightarrow \end{aligned}$$

WKB version 2

$$\begin{aligned} \hbar^2 \partial_x^2 \psi + (2(m*E - m*V))\psi &= 0 \\ \psi &\equiv \exp \frac{i}{\hbar} \int^x Q(x')dx' \end{aligned}$$

The lower bound is arbitrarily chosen since rephasing doesn't matter. Set $m = 1$. Expand in power series to give a recursive system

$$\begin{aligned} Q &= \sum Q_n \hbar^n \\ Q_0 &= \sqrt{2(E-V)} \\ Q_{n+1} &= \frac{1}{2Q_0}(i\partial_x Q_n - \sum_{k=1}^{n} Q_k Q_{n+1-k} \\ Q &= Q_{odd} + Q_{even} \\ \psi &= \frac{1}{\sqrt{Q_{even}}} \exp \frac{i}{\hbar} \int^x Q_{even}(x')dx' \end{aligned}$$

Suppose V is polynomial. Then $p^2 = 2m(E-V)$ defines an algebraic curve of genus $\mid \frac{d-1}{2} \mid$ depending on degree of potential. This defines a point in the moduli space of hyperelliptic curves which has complex dimension $2g-1$. As E varies this determines a subset thereof.

1.11 Supersymmetric Quantum Mechanics

http://arxiv.org/pdf/hep-th/9405029v2.pdf

Suppose you can factor your hamiltonian $H_1 = Q^\dagger Q$.

$$\begin{aligned} Q &= \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x) \\ Q^\dagger &= -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x) \\ H_1 &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{\hbar}{\sqrt{2m}}\frac{dW}{dx} + W^2(x) \\ H_2 = QQ^\dagger &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\hbar}{\sqrt{2m}}\frac{dW}{dx} + W^2(x) \end{aligned}$$

$$\begin{aligned} H_1\psi_n^{(1)} &= Q^\dagger Q\psi_n^{(1)} = E_n\psi_n^{(1)} \\ H_2 Q\psi_n^{(1)} &= QQ^\dagger Q\psi_n^{(1)} = E_n Q\psi_n^{(1)} \\ H_2\psi_n^{(2)} &= E_n^{(2)}\psi_n^{(2)} \\ H_1 Q^\dagger\psi_n^{(2)} &= E_n^{(2)} Q^\dagger\psi_n^{(2)} \end{aligned}$$

so they have the same spectra, with the possible exception of zero modes $Q\psi_0^{(1)} = 0$ or $Q^\dagger\psi = 0$.

$$\begin{aligned} E_0^{(1)} &= 0 \\ E_n^{(2)} &= E_{n+1}^{(1)} \\ \psi_n^{(2)} &= (E_{n+1}^{(1)})^{-1/2} Q\psi_{n+1}^{(1)} \\ \psi_{n+1}^{(1)} &= (E_n^{(2)})^{-1/2} Q^\dagger\psi_n^{(2)} \end{aligned}$$

In fact you can put these together

$$\begin{aligned} H &= \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \\ Q_2 &= \begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix} \\ Q_2^\dagger &= \begin{pmatrix} 0 & Q^\dagger \\ 0 & 0 \end{pmatrix} \end{aligned}$$

These form the Lie Superalgebra $\mathfrak{sl}(1 \mid 1)$, with Q and $Q^\dagger$ being odd and $H = \left[Q, Q^\dagger\right]_+$ even.

1.11.1 Infinite Square Well

The ground state energy is $E_0 = \frac{\hbar^2\pi^2}{2mL^2}$ so we need to subtract that off first.

$$\begin{aligned} H_1 &= H - E_0 \\ E_n^{(1)} &= \frac{n(n+2)\hbar^2\pi^2}{2mL^2} \\ \psi_n^{(1)} &= \sqrt{\frac{2}{L}}\sin\frac{(n+1)\pi x}{L} \end{aligned}$$

This can be written as SQM with superpotential by solving the Ricati equation.

$$\begin{aligned} W &= -\frac{\hbar}{\sqrt{2m}}\frac{\pi}{L}\cot\frac{\pi x}{L} \\ V_2 &= \frac{\hbar^2\pi^2}{2mL^2}(2\csc^2\frac{\pi x}{L} - 1) \end{aligned}$$

Applying Q gives the eigenvalues for this seemingly difficult potential.

$$\psi_1^{(2)} = \sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}$$

1.11.2 Scattering

Define

$$W_\pm \equiv \lim_{x\to\pm\infty} W = W_\pm$$

Now we have

$$\begin{aligned} \lim_{x\to\pm\infty} V_{1,2} &= W_\pm^2 \\ \psi^{(1,2)}(k, x\to-\infty) &\approx e^{ikx} + R^{(1,2)}(k)e^{-ikx} \\ \psi^{(1,2)}(k_2, x\to+\infty) &\approx T^{(1,2)}(k)e^{ik_2x} \\ k &= (E - W_-^2)^{1/2} \\ k_2 &= (E - W_+^2)^{1/2} \\ R^{(1)}(k) &= \frac{W_- + ik}{W_- - ik}R^{(2)}(k) \\ T^{(1)}(k) &= \frac{W_+ + ik_2}{W_- - ik}T^{(2)}(k) \end{aligned}$$

We have chosen no incoming waves from the positive side.

1.11.1 Remark The right is only the left with lights off.- +44 (paraphrased)

The absolute values for the partners are the same so they have the same reflection probabilities. The same fact for transmission probabilities isn't as obvious but $\mid T \mid^2 = 1 - \mid R \mid^2$ so it is in fact true.

This means you can figure out the scattering coefficients for one potential from it's partner.

Proof `https://arxiv.org/pdf/hep-th/9405029.pdf` Page 33 □

1.11.3 Harmonic Oscillator

1.11.4 Hydrogen Atom

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} &+ (-\frac{e^2}{4\pi\epsilon_0 r} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} - E_0)u = 0 \\
W &= C - \frac{D}{r} \\
C &\equiv \frac{\sqrt{2m}}{\hbar}\frac{e^2}{2 * 4\pi\epsilon_0(l+1)} \\
D &\equiv \frac{\hbar}{\sqrt{2m}}(l+1) \\
V_2 &= -\frac{e^2}{4\pi\epsilon_0 r} + \frac{\hbar^2}{2m}\frac{(l+1)(l+2)}{r^2} + \frac{e^4 m}{32\pi^2\hbar^2\epsilon_0^2(l+1)^2}
\end{aligned}$$

1.11.5 Witten Index

Define an index

$$\begin{aligned}
\Delta(\beta) &= \mathrm{tr}(-1)^F e^{-\beta H} \\
\Delta &= \lim_{\beta\to 0}\Delta(\beta)
\end{aligned}$$

1.11.6 Hierarchy

We can also repeat this process repeatedly

$$\begin{aligned}
H_1 &= Q^\dagger Q \\
H_2 &= QQ^\dagger = Q_2^\dagger Q_2 + E_1^{(1)} \\
H_3 &= Q_2 Q_2^\dagger + E_1^{(1)} = Q_3^\dagger Q_3 + E_2^{(1)} \\
H_m &= Q_m^\dagger Q_m + E_{m-1}^{(1)}
\end{aligned}$$

where Q_n are being inductively defined this way. This can be repeated up to p the number of bound states of H_1 where H_s has the first $s-1$ levels removed.

Shape Invariance

1.11.2 Definition *Let $V_{1,2}$ be partner potentials with parameters a_i. If*

$$\begin{aligned} V_2(x,a_1) &= V_1(x,a_2)+R(a_1) \\ a_2 &= f(a_1) \end{aligned}$$

Repeating with the hierarchy gives

$$\begin{aligned} V_s(x,a_1) &= V_1(x,a_s)+\sum_{k=1}^{s-1} R(a_k) \\ f^{(s-1)}(a_1)=f(f\cdots(a_1)) &= a_s \\ E_0^{(s)} &= \sum_{k=1}^{s-1} R(a_k) \\ E_n^{(1)} &= \sum_{k=1}^{n} R(a_k) \\ E_0^{(1)} &= 0 \\ \psi_n^{(1)}(x,a_1) &= Q^\dagger(x,a_1)Q^\dagger(x,a_2)\cdots Q^\dagger(x,a_n)\psi_0^{(1)}(x,a_{n+1}) \end{aligned}$$

1.11.3 Example

1.11.4 Theorem (Titschmarch's Theorem) *Let $\psi \in L^2(\mathbb{R},\mathbb{C})$, then tfae*

- *It is the boundary value for a $\psi(z)$ that is holomorphic in the upper half plane and square integrable there*
- *The real and imaginary parts are Hilbert transforms of each other*
- *$(\mathcal{F}\psi)(p)$ vanishes for $p<0$*

1.12 Sachdev-Ye-Kitaev

$$\begin{aligned} H &= \frac{1}{4!}\sum j_{ijkl}\psi_i\psi_j\psi_k\psi_l \\ j_{ijkl} &\sim N(\mu=0,\sigma=\frac{J\sqrt{6}}{N^{3/2}}) \\ &\sim \frac{1}{\sigma\sqrt{2\pi}} exp(-N^3 j_{ijkl}/2*6*J^2) \end{aligned}$$

This has a $SL(2,\mathbb{R})$ symmetry given by...

1.13 Quartic Oscillator and Instantons

See Coleman aspects of symmetry book

$$\begin{aligned} H &= \frac{p^2}{2m} - ax^2 + bx^4 \\ U(x_0) = 0 &\implies x_0 = 0 \ \pm \sqrt{\frac{a}{b}} \end{aligned}$$

$$\begin{aligned} E &= \cdots + O(\lambda) \\ E &= \cdots + O(\lambda^2) \\ E &= \end{aligned}$$

1.14 Exponential Potentials

1.14.1 1D

$$\begin{aligned}
\frac{-\hbar^2}{2m}\nabla^2\psi + Ae^{bx}\psi &= E\psi \\
-\nabla^2\psi(x) + \frac{2mA}{\hbar^2}e^{bx} &= \frac{2mE}{\hbar^2}\psi \\
y &\equiv \sqrt{\frac{2mA}{\hbar^2}}\frac{2}{b}e^{bx/2} \\
\psi(x) &= c_{1\pm}I_{\pm i2/b\sqrt{2mE}/\hbar}(y) + c_{2\pm}K_{\pm i2/b\sqrt{2mE}/\hbar}(y) \\
K_{-i2/b\sqrt{2mE}/\hbar}(y^*) &= K_{i2/b\sqrt{2mE}/\hbar}(y)^* \\
\psi(-\infty) &\approx c_{1+}\frac{1}{\Gamma(i2/b\sqrt{2mE}/\hbar+1)}(\frac{y}{2})^{i2/b\sqrt{2mE}/\hbar} + c_{1-}() \\
&= c_{1+}\frac{1}{\Gamma(i2/b\sqrt{2mE}/\hbar+1)}(\frac{\sqrt{2mA}}{\hbar b})^{i2/b\sqrt{2mE}/\hbar}e^{xi\sqrt{2mE}/\hbar} \\
&+ c_{1-}\frac{1}{\Gamma(-i2/b\sqrt{2mE}/\hbar+1)}(\frac{\sqrt{2mA}}{\hbar b})^{-i2/b\sqrt{2mE}/\hbar}e^{-xi\sqrt{2mE}/\hbar} \\
&\propto e^{ix\sqrt{2mE}/\hbar} + \frac{c_{1-}}{c_{1+}}\frac{\Gamma(i2/b\sqrt{2mE}/\hbar+1)}{\Gamma(-i2/b\sqrt{2mE}/\hbar+1)}(\frac{\sqrt{2mA}}{\hbar b})^{-i4/b\sqrt{2mE}/\hbar}e^{-xi\sqrt{2mE}/\hbar} \\
R(E) &= \frac{c_{1-}}{c_{1+}}\frac{\Gamma(i2/b\sqrt{2mE}/\hbar+1)}{\Gamma(-i2/b\sqrt{2mE}/\hbar+1)}(\frac{\sqrt{2mA}}{\hbar b})^{-i4/b\sqrt{2mE}/\hbar} \\
&= \frac{c_{1-}}{c_{1+}}\frac{\Gamma(1+i2/bp/\hbar)}{\Gamma(1-i2/bp/\hbar)}(2mA)^{-2i/bp/\hbar}(\hbar b)^{4i/bp/\hbar} \\
A &\equiv 2\pi\mu \\
b &\equiv 2\tilde{b} \\
R(E) &= \frac{c_{1-}}{c_{1+}}\frac{\Gamma(1+i\tilde{b}^{-1}p/\hbar)}{\Gamma(1-i\tilde{b}^{-1}p/\hbar)}(m\pi\mu/(\hbar^2\tilde{b}^2))^{-i\tilde{b}^{-1}p/\hbar}
\end{aligned}$$

Proof https://physics.stackexchange.com/questions/349207/1d-quantum-scattering-from-exponentia
comment780608_349207 https://physics.stackexchange.com/questions/47128/eigenvalues-and-eigenfun
Good matches with Liouville lectures upon required parameter changes. Is pure phase as it should be. Everything comes back out. □

1.14.2 Spherical

$$\begin{aligned}
\psi &\equiv R(r)Y_l^m \\
u(r) &\equiv rR(r) \\
\frac{-\hbar^2}{2m}\frac{d^2u}{dr^2} + (V(r) + \frac{\hbar^2 l(l+1)}{2mr^2})u(r) &= Eu(r)
\end{aligned}$$

Let $l = 0$ then we get 1d problem but with condition $u(0) = 0$

$$\psi(0) \quad = \quad c_{1+} I_{i2/b\sqrt{2mE}/\hbar}(\sqrt{\frac{2mA}{\hbar^2}}\frac{2}{b}) + c_{1-} I_{-i2/b\sqrt{2mE}/\hbar}(\sqrt{\frac{2mA}{\hbar^2}}\frac{2}{b}) = 0$$

That quantizes E by the condition on zeroes of these Bessel functions. This solves the spectrum for $V(r) = Ae^{br}$ with $A, b > 0$ which has all bound states.

1.15 Localization - Insert Elsewhere

1.15.1 Definition (Landscape Function) *For $V \in L^\infty(\Omega)$ a bounded non-negative potential that is positive on a set of positive measure on the domain $\Omega \subset \mathbb{R}^d$. The solution to*

$$(-\Delta + V)u \quad = \quad 1$$

is a strictly positive C^1 function. It belongs to the Sobelev spaces W_p^2 for all $p < \infty$.

1.15.2 Example *If V is constant $V_0 > 0$, then the assumptions are met and $u = \frac{1}{V_0}$ is the landscape function.*

1.15.3 Example *For a periodic potential $V = a + b\sin x$ with $a \geq b \geq 0$, we get ????*

1.15.4 Definition (Effective Potential) *$W = \frac{1}{u}$ where u is the landscape function.*

1.15.5 Theorem

$$\begin{aligned} L\phi \quad &\equiv \quad \frac{-1}{u^2}\nabla \cdot (u^2 \nabla \phi) \\ (-\Delta + V)(u\phi) \quad &= \quad u(L+W)\phi \\ (-\Delta + V)\psi = \lambda\psi \quad &\iff \quad (L+W)\frac{\psi}{u} = \lambda\frac{\psi}{u} \end{aligned}$$

The hope is that solving the RHS of the correspondence for $\phi \equiv \frac{\psi}{u}$ is easier than the other side. This is thanks to the fact that usually W is much more regular than V. That is it typically has the structure of wells and walls so you can use something localized around a minima of W as the eigenfunctions.

1.15.6 Example *In the constant example as before*

$$\begin{aligned} L\phi \quad &= \quad -V_0^2 \nabla \cdot \frac{1}{V_0^2}\nabla\phi \\ &= \quad -\Delta\phi \\ (L+W) \quad &= \quad -\nabla + V_0 \end{aligned}$$

so in this case both sides of the problem are the same.

1.15.7 Example *Periodic potential example gives the Mathieu example of before.*

Chapter 2

Atoms/Molecule Spectra

2.1 Hydrogen Atom

2.1.1 Bound States

$$\begin{aligned}
\psi_{nlm}(r,\theta,\phi) &= \langle r,\theta,\phi \,||\, nlm\rangle = Ne^{-\rho/2}\rho^l L^{2l+1}_{n-l-1}(\rho)Y^m_l(\theta,\phi) \\
\rho &= \frac{2r}{na_0} \\
N &= \sqrt{\frac{2^3}{(na_0)^3}\frac{(n-l-1)!}{2n(n+l)!}} \\
a_0 &= \\
E_n &= -\frac{m_e e^4}{2(4\pi\epsilon_0)^2\hbar^2}\frac{1}{n^2}
\end{aligned}$$

2.1.2 Scattering

Instead of a particle confined by the Coulomb potential, can do the problem with initial condition of an incoming particle wave.

$$\psi = e^{ikz} + \sum_l$$

2.1.3 Reduction from 4 dimensions.

The classical phase space as stated is $T^*\mathbb{R}^3$ with a potential. But we may also consider it as coming from a symplectic reduction without a potential. This makes the $SO(4)$ of the Coulomb problem including the Runge-Lenz vector manifest. It also has the added advantage of getting rid of the potential.

`http://physics.gmu.edu/~isatija/Phys701/HatomSym.pdf`

2.1.1 Theorem (n-Spherical Harmonics) *The spectrum of the standard Laplacian on the standard S^n is $\lambda_{k,n} = k(k+n-1)$ with multiplicities $m_{k,n} = \binom{k+n}{n} - \binom{k+n-2}{n}$ for $k = 0, 1, 2, \cdots$.*

$$\begin{aligned}
E_{n,k\neq 0} &\equiv B + A\log\lambda_{n,k} \\
Z(s) &= \sum_k m_{n,k} e^{-sE_{n,k}} \\
&= 1 * e^{-sE_{n,0}} + e^{-sB}\sum_{k=1}^{\infty} m_{k,n} e^{-sA\log\lambda_{n,k}} \\
&= e^{-sE_{n,0}} + e^{-sB}\zeta_{S^n}(sA) \\
\sum_{k=1}^{\infty} m_{k,n} e^{-sA\log\lambda_{n,k}} &= \sum_k \frac{m_{n,k}}{\lambda_{n,k}^{sA}} \equiv \zeta_{S^n}(sA)
\end{aligned}$$

Explicitly Zonal spherical harmonics

$$\begin{aligned}
\frac{1}{Area(S^{n-1})}\frac{1-r^2}{\mid x-ry\mid^n} &= \sum_{k=0}^{\infty} r^k Z_x^k(y) \\
\frac{1}{\mid x-y\mid^{n-2}} &= \sum \frac{1}{Area(S^{n-1})}\frac{2k+n-2}{n-2}\frac{\mid x\mid^k}{\mid y\mid^{n+k-2}} Z_{\hat{x}}^k(\hat{y}) \\
Z_x^l(y) &= \frac{Area(S^{n-1})(n-2)}{2l+n-2} C_l^{(n-2)/2}(x\cdot y) \\
\frac{1}{\mid x-y\mid^{n-2}} &= \sum \frac{\mid x\mid^k}{\mid y\mid^{n+k-2}} C_l^{(n-2)/2}(x\cdot y) \\
n=3 &\Longrightarrow \frac{1}{\mid x-y\mid} = \sum \frac{\mid x\mid^k}{\mid y\mid^{k+1}} C_l^{1/2}(\cos\gamma) = \sum \frac{\mid x\mid^k}{\mid y\mid^{k+1}} P_l(\cos\gamma) \\
n=4 &\Longrightarrow \frac{1}{\mid x-y\mid^2} = \sum \frac{\mid x\mid^k}{\mid y\mid^{k+2}} C_l^1(\cos\gamma)
\end{aligned}$$

For a basis of the zonal spherical harmonics of degree l, $Y_{l,m}$

$$\begin{aligned}
Z_x^l(y) &= \sum_m Y_{l,m}(x)\bar{Y_{l,m}}(y) \\
Z_y^l(y) &= \sum_m Y_{l,m}(y)\bar{Y_{l,m}}(y) = Area(S^{n-1})^{-1}\dim H_l
\end{aligned}$$

2.1.2 Definition (Ultraspherical/Gegenbaruer) *The C_l^α are orthogonal polynomials on $[-1,1]$ with respect to $dw = (1-x^2)^{\alpha-1/2}d\mu$. So $n=3$ gave $dw = d\mu$. Since $x = \cos\gamma$ this is $dw = \sin^{2\alpha-1}(\gamma)d\mu_{\cos\gamma}$*

2.1.3 Lemma (Scattering States)

$$
\begin{aligned}
(\frac{-\nabla^2}{2}+\frac{Z}{r})\psi_{\vec{k}}(\vec{r}) &= \frac{k^2}{2}\psi_{\vec{k}}(\vec{r}) \\
\xi &= r+\vec{r}\cdot\hat{k} \\
\zeta &= r-\vec{r}\cdot\hat{k} \\
\psi_{\vec{k}}(\vec{r}) &= \frac{1}{(2\pi)^{3/2}}\Gamma(1\pm i\eta)e^{-\pi\eta/2}e^{i\vec{k}\cdot\vec{r}}\ {}_1F_1(\pm i\eta,1,\pm ikr-i\vec{k}\cdot\vec{r}) \\
\eta &= \frac{Z}{k} \\
\rho &= kr \\
\psi_{\vec{k}}(\vec{r}) &= \frac{1}{(2\pi)^{3/2}}\frac{1}{r}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}4\pi(-i)^l w_l(\eta,\rho)Y_l^m(\hat{r})Y_l^{m*}(\hat{k}) \\
\psi_{klm}(\vec{r}) &= \int\psi_{\vec{k}}(\vec{r})Y_l^m(\hat{k})d\hat{k} \\
&= R_{kl}(r)Y_l^m(\hat{r}) \\
R_{kl}(r) &\equiv \sqrt{\frac{2}{\pi}}(-i)^l\frac{1}{r}w_l(\eta,\rho) \\
\frac{d^2w_l}{d\rho^2}+(1-\frac{2\eta}{\rho}-\frac{l(l+1)}{\rho^2})w_l &= 0 \\
x &= 2i\rho \\
-4\frac{d^2w_l}{dx^2}+(1-\frac{4i\eta}{x}-\frac{-4l(l+1)}{x^2})w_l &= 0 \\
\frac{d^2w_l}{dx^2}+(\frac{-1}{4}+\frac{i\eta}{x}+\frac{-l(l+1)}{x^2})w_l &= 0 \\
-l(l+1) &= 1/4-\mu^2 \\
\mu &= l+1/2 \\
w_l &= M_{i\eta,l+1/2}(x) \\
&= e^{-x/2}x^{l+1/2+1/2}M(l+1/2-i\eta+1/2,1+2(l+1/2),x) \\
&= e^{-x/2}x^{l+1}M(l+1-i\eta,2+2l,x) \\
w_l &= W_{i\eta,l+1/2}(x) \\
&= e^{-x/2}x^{l+1}U(l+1-i\eta,2+2l,x)
\end{aligned}
$$

where we have used the Whittaker differential equation for w_l to write M and W Whittaker functions which are also written in terms of confluent hypergeometric M and U.

2.2 $n+1$ dimensional Radial potentials

$$
\begin{aligned}
H &= \frac{-\hbar^2}{2m}\Delta_{\mathbb{R}^{n+1}}+V(\mid x\mid) \\
\psi(x) &= \phi_{nl}(r)Y_l^m(\hat{r})
\end{aligned}
$$

2.2.1 Hyperbolic Kepler

$$\begin{aligned} H_{hyp} &= -\frac{d^2}{dr^2} + \frac{j(j-1)}{\sinh^2 r} - 2g\coth r \\ H_{euc} &= -\frac{d^2}{dr^2} + \frac{j(j-1)}{r^2} - \frac{2g}{r} \end{aligned}$$

`https://arxiv.org/pdf/1711.02422.pdf`

Is this for $1/r^2$ with ball model or with the actual Green's function for Laplacian in H^3. j $j+1$ issue as well.

2.3 AMO

2.3.1 Rydberg

2.3.1 Definition (Rydberg Atom) *An atom such that one or more electrons have been excited into energy levels of hydrogenic potential with high value of n. This definition requires use of the approximation where can ignore the inter-electron Coulomb potential in order to treat eigenstates as just providing a collection of occupied hydrogenic wavefunctions and creating Slater determinants thereof.*

2.3.2 Lemma (Trilobite State)

2.3.3 Lemma (Butterfly State)

2.3.4 Theorem (Rydberg Blockade) *Suppose have two atoms a distance R apart. Consider two levels for each atom, one being the ground state and the other being a Rydberg state with high value of n. That is there is a 4 dimensional space of states available by taking the span of $\mid 00\rangle$, $\mid 0n\rangle$, $\mid n0\rangle$ and $\mid nn\rangle$.*

The interatomic potential can be such that $\mid 00\rangle$ is low energy and $\frac{1}{\sqrt{2}}(\mid 0n\rangle + \mid n0\rangle)$ is also low lying. But $\mid nn\rangle$ is much higher.

Proof

$$H_{eff} = \begin{pmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix}$$

2.3.2 Optical Pumping

`https://www.youtube.com/watch?v=v4StxGAhm8Y`

Have Rb gas. Split by spin orbit coupling into ${}^2P_{3/2}$ ${}^2P_{1/2}$ and ${}^2S_{1/2}$. This further gets split by nuclear spin. Using Rb^{87} splits into f values 321 21 and 21 respectively. The magnetic field then splits into $2f+1$ states. Those are labeled by m_F. The magnetic field splitting is a much smaller scale.

Now take 2 F levels both with $F=1$. Incoming laser to excite to the higher of the two. The spread is enough to not care about the m_F splitting. Polarize it such that $\Delta m_F = +1$. So the $+1$ on the lower level builds up over time. That is because it can't absorb any of these photons. Eventually all pumped up into this state.

2.4 Molecules

2.4.1 H'uckel

Let $\phi_{i,j}$ be a finite set of vectors in $L^2(\mathbb{R}^3)$ with i indicating an atom in the molecule and j indexing an atomic orbital for that atom. This gives a finite dimensional vector space in which to do the variational method.

$$\begin{aligned}
\psi &= \sum c_\alpha \phi_\alpha \\
E(c_\alpha) &= N^2(\sum c_\alpha^2 H_{\alpha\alpha} + \sum_{\alpha\neq\beta} c_\alpha c_\beta H_{\alpha\beta}) \\
N &= (\sum c_\alpha^2 S_{\alpha\alpha} + \sum_{\alpha\neq\beta} c_\alpha c_\beta S_{\alpha\beta})^{-1/2} \\
S_{\alpha\beta} &= \int \phi_\alpha^\dagger \phi_\beta \\
H_{\alpha\beta} &= \int \phi_\alpha^\dagger H \phi_\beta
\end{aligned}$$

Because the ϕ_α are orthogonal for the same atom those matrix elements of S are all 0. In addition if the atomic orbitals are sufficiently localized, the S matrix elements for those should also be small only due to exponential tail contributions. In total $S_{\alpha\beta}$ may just be replaced by $\delta_{\alpha\beta}$ often. Similarly for $H_{\alpha\beta}$, it should be sparse for a pair of atomic orbitals on vastly seperated atoms because the H operator can not do that much to the exponential tails.

Minimizing $E(c_\alpha)$ gives

$$\det(H_{\alpha\beta} - ES_{\alpha\beta}) = 0$$

The 0 eigenvector is the required c_α, but to just get the minimizing energy which is a good upper bound for the ground state energy is just solving when this determinant polynomial in E has roots.

Ethylene

Let $\phi_{1,2}$ be the $2p_z$ orbitals around the 2 Carbon atoms.

$$\begin{aligned}
H &= \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \\
S &= \begin{pmatrix} 1 & S \\ S & 1 \end{pmatrix} \\
E_\pm &= \frac{\alpha \pm \beta}{1 \pm S} \\
\psi_\pm &= \sqrt{\frac{1}{2 \pm 2S}}\phi_1 \pm \sqrt{\frac{1}{2 \pm 2S}}\phi_2
\end{aligned}$$

The + gives bonding ($\beta < 0$ so this is lower energy). The − gives antibonding. When $S = 0$, they both move β from the middle α. But when $S \neq 0$, the splitting is asymmetric. Bonding is always stablized less than antibonding is destabilized relative to the α diagonal.

π-Stacking

Important because biomolecules have lots of aromatic rings that interact with each other. So this affects their conformational ground states. Protein and DNA folding.

Two aromatic rings in general position. Parameterize by putting one at the origin oriented so that one of the Carbons is on the positive x axis. There is still a residual D_{12} symmetry. Put the other one centered at some $\vec{r}$, oriented in a plane specified by $\vec{n}$ and rotated by an angle θ in that plane. θ is defined up to $\frac{\pi}{3}$ instead of 2π.

- Sandwich, the two rings are right on top of each other with centers vertically separated by R
- T-shape. The second one has vertex pointing down and is above the center of the xy ring.
- Parallel-sandwich. Like a sandwich but the top bread is sliding off.

If there are substituent groups, parameterization loses many of these symmetries.

`https://smartech.gatech.edu/bitstream/handle/1853/5019/Sinnokrot_Mutasem_O_200407_phd.pdf`

2.5 Balslev Combes

2.5.1 Definition (Dilation Analytic) *Let $d, \gamma > 1$ and $\alpha > 0$, then a potential is called dilation analytic if*

The multiplication operator $V(\vec{x})$ is complex valued measurable function $\in L^{\gamma+d/2}(\mathbb{R}^d, \mathbb{C})$. It is $H_0 = \frac{p^2}{2m}$ compact so $Dom(H_0) = H^2(\mathbb{R}^d) \subset Dom(V)$ and $V(H_0+1)^{-1}$ is compact. The function $V_\theta(\vec{x}) \equiv V(e^\theta x)$ extends from $\theta \in \mathbb{R}$ to θ in a strip with $\mid \Im\theta \mid \leq \alpha$ within $L^{\gamma+d/2}(\mathbb{R}^d, \mathbb{C})$. $V_\theta(H_0 - z)^{-1}$ extends as a analytic function from the strip to bounded operators on L^2.

2.5.2 Definition (Unitary Dilation)

$$\begin{aligned} U(\theta)f(x) &= e^{d\theta/2} f(e^\theta x) \\ U(\theta)H_0U(\theta)^{-1} &= e^{-2\theta}H_0 \\ U(\theta)(H_0+V)U(\theta)^{-1} &= e^{-2\theta}(H_0 + e^{2\theta}V_\theta) \end{aligned}$$

2.5.3 Proposition *If V is dilation analytic then the spectrum for $H(\theta)$ is given as $z + e^{-2\theta}[0,\infty)$ where z is a threshold for $H(\theta)$. When θ was 0 all those rays were overlapping making it difficult to extract the points from the rays. After rotating the rays with complex θ, you can easily see the distinct z. So you see a bunch of dots and rays instead of dots on rays.*

2.5.4 Example (Two Body Coulomb problem) *In dimensionless variables we have a spectrum $\frac{-1}{n^2} + \frac{-1}{m^2}$ for two noninteracting electrons in a single Coulomb potential. There is also continuous spectrum with $[-1,\infty)$ because we can put $\frac{-1}{1^2}$ and the other electron with free with arbitrary positive energy. All those bound states eigenvalues are embedded in this ray and we have to extract them to distinguish bound from free.*

Proof `http://www.math.caltech.edu/SimonPapers/20.pdf` □

2.5.5 Example (Molecule) *The hard uniform sphere*

$$V_C(r) = \begin{cases} \frac{1}{2R_c}(3-\frac{r}{R_c})^2 & r \le R_c \\ \frac{1}{r} & r > R_c \end{cases}$$

but if we did a dilation, this would be discontinuous.
has no strip to be dilation analytic. This is fixed by

$$V_C(r) = Ze^2\frac{erf(r/\alpha)}{r}$$

The erf climbs up to 1 as r increases with a characteristic length scale set by α. This is the result of smearing with a Gaussian with variance σ satisfying $\alpha = \sqrt{2\sigma^2}$. So do this for each atom in the molecule N to get a potential for $d = 3N$. Pick an R_0 bigger than all of them and that modifies the Hamiltonian a bit too. Get parameterization $V^{\epsilon}_{R_0}(\theta)$ of the regularized+dilated potentials.

`http://www.math.caltech.edu/SimonPapers/121.pdf http://citeseerx.ist.psu.edu/viewdoc/download?rep=rep1&type=pdf&doi=10.1.1.211.826`

This also allows constant electric fields.

2.5.6 Definition (Autoionization) *Pierre Auger/Lise Meitner effect for when a second electron gets kicked out too. An electron in an inner shell has been removed possibly by a collision, then Auger process is when an outer electron is kicked out as a result of a decay. Usually the decay just emits a photon, but this is when the extra energy is carried out by freeing an electron instead.*

2.6 QuasiProbability distributions

2.6.1 Theorem (Optical Equivalence) *Let Ω be an ordering of the creation and annihilation operators such as normal ordering dictating that annihilation operators should act before creation operators. Let $g_\Omega(a, a^\dagger)$ be an operator which is a power series such that each monomial appearing satisfies the ordering Ω.*

$$\begin{aligned} \rho &\equiv \frac{1}{\pi}\int f_{\overline{Omega}} \mid \alpha\rangle\langle\alpha \mid d^2\alpha \\ \mathrm{tr}\left(\rho g_\Omega(a,a^\dagger)\right) &= \int f_{\overline{Omega}} g_\Omega(\alpha,\alpha^*)d^2\alpha \end{aligned}$$

2.6.2 Definition (Wigner) *Let $\psi \in L^2(\mathbb{R})$ be a wavefunction and give the standard x and p operators though any canonically conjugate pair.*

$$
\begin{aligned}
W_\psi(x,p) &\equiv \frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\psi^*(x+y)\psi(x-y)e^{2ipy/\hbar}dy \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi^*(x+\hbar s/2)\psi(x-\hbar s/2)e^{ips}ds \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi^*(p+\hbar s/2)\phi(p-\hbar s/2)e^{-ixs}ds \\
\langle x \mid\mid \psi\rangle &= \psi(x) \\
W_\rho(x,p) &\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle x-\hbar s/2 \mid \rho \mid x+\hbar s/2\rangle e^{ips}ds \\
W_\psi(x,p) &\in \big[-\frac{2}{\hbar},\frac{2}{\hbar}\big] \\
\int_{-\infty}^{\infty}dpW(x,p) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}ds\int_{-\infty}^{\infty}dpe^{ips}\psi^*(x+\hbar s/2)\psi(x-\hbar s/2) \\
&= \int_{-\infty}^{\infty}ds\frac{1}{2\pi}2\pi\delta(s)\psi^*(x+\hbar s/2)\psi(x-\hbar s/2) \\
&= \psi^*(x)\psi(x) = \mid \psi(x)\mid^2 \\
\int_{-\infty}^{\infty}dxW(x,p) &= \mid \phi(p)\mid^2 \\
W_{\psi^*}(x,p) &= W_\psi(x,-p) \\
W_{\psi(-?)}(x,p) &= W_\psi(-x,-p) \\
W_{\psi(?+a)}(x,p) &= W_\psi(x+a,p) \\
\mid \langle\psi_1 \mid\mid \psi_2\rangle\mid^2 &= 2\pi\hbar\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dpW_{\psi_1}(x,p)W_{\psi_2}(x,p)
\end{aligned}
$$

2.6.3 Definition (Wigner-Weyl Transform) *Let f be a function on the symplectic space $T^*\mathbb{R}$. Let P and Q be operators on a Hilbert space satisfying canonical commutation relations. Suppose the axioms of Stone-Von-Neumann so we can use standard p and x manipulations.*

The Weyl transform of f is then given by doing the Fourier transform of F as a function on $\mathbb{R}^2$. Then doing a version of the inverse Fourier transform but with $p \to P$ and $q \to Q$ in the inverse Fourier integral kernel.

The inverse map takes operators $\Phi\big[f\big]$ back to f and applied to $\langle\psi \mid\mid \psi\rangle$ which is not of the form $\Phi(f)$ gives W_ψ up to a factor of $2\pi\hbar$.

Polynomials $p^m q^n$ go to the sum over all possible words in m P's and n Q's. Using the canonical commutation relations reduces this sum but at the expense of dropping the full S_{n+m} symmetry.

If f is a polynomial of degree at most 2 and g is an arbitrary polynomial $\Phi(\{f,g\}) = \frac{1}{i\hbar}\big[\Phi(f),\Phi(g)\big]$. If we shift the constraint from at most 2 to at most 3, then Grenewold's theorem kicks in and prevents such a canonical quantization exactly preserving the two Lie algebra structures.

Real functions go to self adjoint operators. If f is Schwartz, then $\Phi(f)$ is trace class and the mapping is injective. But for general f, it is just a densely defined potentially unbounded operator.

2.6.4 Corollary (Moyal Quantization) *Specify $f_1 \star_\hbar f_2$ by*

$$\begin{aligned}
\Phi(f_1 \star_\hbar f_2) &= \Phi(f_1)\Phi(f_2) \\
f_1 \star_\hbar f_2 = f_1 f_2 + \frac{i\hbar}{2}\{f_1, f_2\} + O(\hbar^2) & \\
f_1 \star_\hbar f_2 - f_2 \star_\hbar f_1 &= i\hbar\{f_1, f_2\} + O(\hbar^2) \\
\Phi(f_1)\Phi(f_2) - \Phi(f_2)\Phi(f_1) &= i\hbar\Phi(\{f_1, f_2\}) + O(\hbar^2) \\
f_1 \star_\hbar f_2 &= \sum_{n=0}^{\infty} \frac{1}{n!}\big(\frac{i\hbar}{2}\big)^n \Pi^n(f_1, f_2)
\end{aligned}$$

where the $O(\hbar^2)$ is why it is okay as a deformation quantization and escapes no go theorems. Π^n combines degree n differential operators on f_1 and f_2 in an explicit way that specializes to multiplication for $n = 0$ and the Poisson bracket for $n = 1$.

$$\begin{aligned}
\Phi(e^{-a(p^2+q^2)} \star_\hbar e^{-b(p^2+q^2)}) &= \Phi(e^{-a(p^2+q^2)})\Phi(e^{-b(p^2+q^2)}) \\
&= \Phi(\frac{1}{1+\hbar^2 ab} e^{-\frac{a+b}{1+\hbar^2 ab}(q^2+p^2)})
\end{aligned}$$

So on the span of such Gaussians the star product gives a commutative subalgebra.

2.6.5 Corollary *Let O be an observable in terms of x and p.*

$$\begin{aligned}
\Delta^2_\psi O &\equiv \langle O^2\rangle_\psi - \langle O\rangle^2_\psi \\
&= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp W_\psi(x,p)\big(O^2(x,p) - O(x,p)^2\big) \\
&= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp W_\psi(x,p)\big(O(x,p) \star_\hbar O(x,p) - O(x,p)O(x,p)\big) \\
&= 0 + O(\hbar^2)
\end{aligned}$$

If $O(x,p)$ is a sum of Gaussians.

$$\begin{aligned}
O(x,p) &\equiv \sum a_i e^{-b_i(x^2+p^2)} \\
O(x,p) \star_\hbar O(x,p) &= \sum_{i,j} a_i a_j \frac{1}{1+\hbar^2 b_i b_j} e^{-\frac{b_i+b_j}{1+\hbar^2 b_i b_j}(x^2+p^2)} \\
O(x,p)^2 &= \sum_{i,j} a_i a_j e^{-(b_i+b_j)(x^2+p^2)}
\end{aligned}$$

2.6.6 Definition (Husimi Q representation)

2.6.7 Definition (Glauber-Sudarshan P representation)

2.6.8 Definition (Cahill one parameter family)

2.6.1 Harmonic Oscillator Examples

2.6.9 Example (Harmonic Oscillator Eigenstate)

$$\begin{aligned} W_{|m\rangle}(x,p) &= \frac{(-1)^m}{\pi\hbar} e^{\frac{-2E_{cl}(x,p)}{\hbar\omega}} L_m(\frac{4E_{cl}(x,p)}{\hbar\omega}) \\ Q_{|m\rangle}(x,p) &= \frac{2^m (E_{cl}(x,p))^m}{m!(\hbar\omega)^m} e^{-\frac{2E_{cl}(x,p)}{\hbar\omega}} \end{aligned}$$

where L_m are Laguerre polynomials (orthogonal with respect to e^{-x} on the positive reals).

2.6.10 Example (Harmonic Oscillator Coherent State) *Let $\mid \alpha\rangle$ be a coherent state satisfying*

$$\begin{aligned} a \mid \alpha\rangle = \alpha \mid \alpha\rangle & \\ D(\alpha) \mid 0\rangle = \mid \alpha\rangle & \\ D(\alpha) &= e^{\alpha a^\dagger - \alpha^* a} \\ D(\alpha)^\dagger &= D(-\alpha) \\ D(\alpha) a D(-\alpha) = a - \alpha & \\ D(\alpha) a D(-\alpha) D(\alpha) \mid \alpha_1\rangle &= D(\alpha)\alpha_1 \mid \alpha_1\rangle \\ &= \alpha_1 D(\alpha) \mid \alpha_1\rangle \\ (a-\alpha) D(\alpha) \mid \alpha_1\rangle &= \alpha_1 D(\alpha) \mid \alpha_1\rangle \\ a D(\alpha) \mid \alpha_1\rangle &= (\alpha + \alpha_1) D(\alpha) \mid \alpha_1\rangle \\ D(\alpha) \mid \alpha_1\rangle &= e^{i\theta} \mid \alpha + \alpha_1\rangle \end{aligned}$$

$$
\begin{aligned}
\langle\alpha \mid \frac{a+a^\dagger}{2} \mid \alpha\rangle &= \frac{\alpha+\alpha^*}{2}\langle\alpha \mid\mid \alpha\rangle \\
&= Re(\alpha) \\
\frac{a+a^\dagger}{2} &= \sqrt{\frac{m\omega}{2\hbar}}x \\
Re(\alpha) &= \sqrt{\frac{m\omega}{2\hbar}}\langle\alpha \mid x \mid \alpha\rangle \\
\langle\alpha \mid \frac{a-a^\dagger}{2i} \mid \alpha\rangle &= \frac{\alpha-\alpha^*}{2i}\langle\alpha \mid\mid \alpha\rangle \\
&= Im(\alpha) \\
\frac{a-a^\dagger}{2i} &= \frac{1}{\sqrt{2m\omega\hbar}}p \\
Im(\alpha) &= \frac{1}{\sqrt{2m\omega\hbar}}\langle\alpha \mid p \mid \alpha\rangle \\
\langle x \mid\mid \alpha\rangle &= \sqrt[4]{\frac{m\omega}{\pi\hbar}}\exp\big(\frac{-m\omega}{2\hbar}(x-\langle x\rangle_\alpha)^2+\frac{i}{\hbar}\langle p\rangle_\alpha x-\frac{i}{2\hbar}\langle p\rangle_\alpha\langle x\rangle_\alpha\big) \\
&\propto \sqrt[4]{\frac{m\omega}{\pi\hbar}}\exp\big(\frac{-m\omega}{2\hbar}(x-\langle x\rangle_\alpha)^2+\frac{i}{\hbar}\langle p\rangle_\alpha x\big) \\
W_{|\alpha\rangle}(x,p) &= \frac{1}{2\pi}\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}ds\exp\big(\frac{-m\omega}{2\hbar}(x-\langle x\rangle_\alpha+\frac{\hbar s}{2})^2+\frac{-i}{\hbar}\langle p\rangle_\alpha(x+\frac{\hbar s}{2})\big) \\
&\quad \exp\big(\frac{-m\omega}{2\hbar}(x-\langle x\rangle_\alpha-\frac{\hbar s}{2})^2+\frac{i}{\hbar}\langle p\rangle_\alpha(x-\frac{\hbar s}{2})\big)e^{ips}
\end{aligned}
$$

This is a Fourier transform of a Gaussian in the s variable so we just have to simplify the exponent.

$$
\begin{aligned}
E &\equiv \big(\frac{-m\omega}{2\hbar}(x-\langle x\rangle_\alpha+\frac{\hbar s}{2})^2+\frac{-i}{\hbar}\langle p\rangle_\alpha(x+\frac{\hbar s}{2})\big) \\
&+ \big(\frac{-m\omega}{2\hbar}(x-\langle x\rangle_\alpha-\frac{\hbar s}{2})^2+\frac{i}{\hbar}\langle p\rangle_\alpha(x-\frac{\hbar s}{2})\big) \\
&= \frac{-m\omega}{2\hbar}\big((x-\langle x\rangle_\alpha+\frac{\hbar s}{2})^2+(x-\langle x\rangle_\alpha-\frac{\hbar s}{2})^2\big)+\frac{-i}{\hbar}\langle p\rangle_\alpha\hbar s \\
&= \frac{-m\omega}{2\hbar}\big(2(x-\langle x\rangle_\alpha)^2+2(\frac{\hbar s}{2})^2\big)+\frac{-i}{\hbar}\langle p\rangle_\alpha\hbar s \\
W_{|\alpha\rangle}(x,p) &= \frac{1}{2\pi}\sqrt{\frac{m\omega}{\pi\hbar}}\exp(\frac{-m\omega}{\hbar}(x-\langle x\rangle_\alpha)^2)\int ds e^{-\frac{m\omega\hbar}{4}s^2-i\langle p\rangle_\alpha s+ips} \\
&= \frac{1}{2\pi}\sqrt{\frac{m\omega}{\pi\hbar}}\exp(\frac{-m\omega}{\hbar}(x-\langle x\rangle_\alpha)^2)\sqrt{\frac{4\pi}{m\omega\hbar}}e^{(p-\langle p\rangle_\alpha)^2/(m\omega\hbar)} \\
&= \frac{1}{\hbar}\exp(\frac{-m\omega}{\hbar}(x-\langle x\rangle_\alpha)^2)\exp(\frac{1}{m\omega\hbar}(p-\langle p\rangle_\alpha)^2)
\end{aligned}
$$

So a Gaussian ellipse in phase space whose variance in the x direction is $\frac{\hbar}{2m\omega}$ and whose variance in the p direction is $\frac{m\omega\hbar}{2}$.

2.6.2 Quantum Optics Applications

A Concise Treatise on Quantum Mechanics in Phase Space

2.6.11 Definition (Squeezed Light)

2.6.12 Example (Quasiprobability of squeezed light states)

$$\mid \psi\rangle \equiv D(\alpha)S(re^{i\theta}) \mid 0\rangle = S(re^{i\theta})D(\alpha\cosh r + \alpha^* e^{i\theta}\sinh r) \mid 0\rangle$$

Chapter 3

Open Systems

3.1 Lindblad

$$\dot{\rho}(t) \quad = \quad -i[H,\rho(t)] + \sum_{k=1}^{M} L_k \rho(t) L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho(t)\}$$

The action of H on $V \otimes V^\dagger$ is given by coproduct and a sign flip for the $V^\dagger$ (More generally the antipode, but in this case just -1).

$$\Delta H \quad = \quad H \otimes 1 - 1 \otimes H$$

$$\begin{aligned}
\dot{\rho} &\in V \otimes V^\dagger \\
\dot{\rho} &= (-i\Delta H + \mathcal{G})\rho \\
\mathcal{G} &= \sum_{m=0}^{M} \bar{L}_m \otimes L_m - \frac{1}{2} Id \otimes L_m^\dagger L_m - \frac{1}{2} \bar{L}_m^\dagger \bar{L}_m \otimes Id
\end{aligned}$$

3.1.1 Effective Hamiltonian

3.1.1 Definition (H_{eff})

$$H_{eff} \quad = \quad H - i \sum_k \frac{\gamma_k}{2} L_k^\dagger L_k$$

The antiHermitian part is negative for $\gamma_k \geq 0$ in the sense of $L_k^\dagger L_k$ being a positive operator. In Lie algebraic terms $iH_{eff} \in \mathfrak{u}(n) \bigoplus \mathfrak{u}(n)^*_{\geq 0}$

3.1.2 Definition (Iwasawa Poisson-Lie Group) *The one coming from KAN decomposition. So for $GL(n,\mathbb{C})$ regarded as a real Lie group, we get Hermitian and antiHermitian split. TODO: Write down this parameterization.*

$$\frac{d\rho}{dt} \quad = \quad -i(H_{eff}\rho - \rho H_{eff}^\dagger) + \sum_k \gamma_k L_k \rho L_k^\dagger$$

3.2 Linear Open Quantum System

Let the Hilbert space be $L^2(\mathbb{R}^n)$ with operators x_i and p_i combined into a single list as z_m. Let Σ be the matrix encoding the commutation relations.

$$\Sigma = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

3.2.1 Definition (Linear Open Quantum System) *Suppose the Hamiltonian is quadratic in these variables and the L_k are all linear in these variables.*

$$\begin{aligned} H &= \frac{1}{2} z^\dagger M z \\ L_k &= \sum C_{k,m} z_m \end{aligned}$$

M is $2n$ by $2n$ and C is M by $2n$

3.2.2 Theorem *If we start with a Gaussian initial state, then the evolution under the Lindblad equation remains Gaussian. The mean and variance evolve as*

$$\begin{aligned} \frac{d\langle z_m \rangle}{dt} &= \mathcal{A}_{mn} \langle z_n \rangle \\ \frac{dV_{mn}}{dt} &= \mathcal{A}_{mp} V_{pn} + V_{mp} \mathcal{A}^T_{pn} + \mathcal{D}_{mn} \\ \mathcal{A} &= \Sigma(M + Im(C^\dagger C)) \\ \mathcal{D} &= \Sigma(Re(C^\dagger C))\Sigma^T \end{aligned}$$

Proof `https://arxiv.org/pdf/1801.04149.pdf` □

Chapter 4

CS Perspective

4.1 States

4.1.1 Theorem (Gleason's) *Suppose H is a separable Hilbert space of complex dimension at least 3. Then for any quantum probability measure on the lattice Q of self-adjoint projection operators on H there exists a unique trace class operator ρ such that $P(E) = Tr(\rho E)$ for any self-adjoint projection E in Q.*

4.1.2 Theorem (Bell's Inequality)

4.1.3 Definition (Purification) *Let ρ be a state on $B(H_1)$. Would like this to be a marginal for a pure state on $B(H_1 \otimes H_{env})$ where have to say the environment.*

4.1.4 Definition (State Space) *Let us be in $\mathbb{C}^d$. The pure states are parameterized by $\mathbb{CP}^{d-1}$. The space of rank k mixed states is an intersection of the positive cone in the orbit $U(d)/U(n-k) \subset \mathfrak{u}(d)^*$ with unit trace condition. It has dimension $2nk - k^2 - 1$. For example, $d = k = 3$ gives 8 dimensional. This is in fact the big cell for states on qutrits.*

4.1.5 Definition (Segre Embedding) *$P^m \times P^n \to P^{(n+1)(m+1)-1}$ from the $\mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1} \simeq \mathbb{C}^{(n+1)(m+1)}$. This is defined over any field. It gives an image called the Segre variety which is a determinantal variety so cut out by minors of a matrix. In particular 2 by 2 minors. It is the categorical product in schemes. Restricting further to symmetric under exchange is called the Veronese. This characterizes the pure product states among all pure states.*

4.1.6 Example *In the case of 2 qubits, $n = m = 1$ and this gives a quadric in $\mathbb{P}^3$. In the case of a qubit and qutrit, this gives what is called the Segre threefold which intersects a 3-plane $\mathbb{P}^4$ in a twisted cubic.*
For Veronese, similarly do the example from $P(V) \to P(Sym^d V)$ for other d instead of 2. This gives $d = 3$ qubit $V = \mathbb{C}^2$ replicas sitting as a twisted cubic.

4.1.7 Theorem (Hirschfield,Thas 1991) *The automorphisms of the Segre variety are either $PGL(n+1, F) \times PGL(m+1, F)$ or an extra $\mathbb{Z}_2$ when $n = m$.*

4.1.8 Lemma (Schmidt Decomposition) *Let $\mid \psi\rangle \in H_L \otimes H_R$. One may decompose it as $\sum \alpha_i \mid \psi_L\rangle \otimes \mid \psi_R\rangle$. This is done by doing SVD on $\mid \psi_L\rangle\langle\psi_R \mid$. If there is only one nonzero α_i singular value, the state is called separable because there is no entanglement between H_L and H_R.*

4.1.9 Lemma (Peres-Horodecki) *If we look at general density matrices as states on $B(H_L \otimes H_R)$ instead, we may again try to know how they can be written as products of states on $B(H_L)$ and $B(H_R)$ as $\sum p_i \rho_{L,i} \otimes \rho_{R,i}$. A necessary but not sufficient condition for separability is when ρ_B^T has non-negative eigenvalues. When there is a negative eigenvalue, the system is entangled, but if there are only non-negative eigenvalues the test is inconclusive except in $d \leq 6$ cases.*

4.1.10 Example (Werner states)

$$\rho = \frac{1}{4}\begin{pmatrix} 1-p & 0 & 0 & 0 \\ 0 & p+1 & -2p & 0 \\ 0 & -2p & p+1 & 0 \\ 0 & 0 & 0 & 1-p \end{pmatrix}$$

The smallest eigenvalue for ρ^{T_B} is $(1-3p)/4$ so the system is entangled for all $p > 1/3$.

4.1.11 Definition (Entanglement Witness) *https://en.wikipedia.org/wiki/Entanglement_witness*

4.2 Information Extraction

4.2.1 Definition (Relative Entropy)

$$S(\omega, \phi) = \begin{cases} -\langle \xi \mid\mid \log \Delta(\phi/\omega'_\xi)\xi \rangle & \sup \omega \leq \sup \phi \\ +\infty \end{cases}$$

where $\sup \omega$ is the smallest projection such that $\omega(P) = \omega(I)$.
For finite dimensional systems, this reduces to the usual

$$\begin{aligned} \omega(?) &= \mathrm{tr}(\rho ?) \\ \phi(?) &= \mathrm{tr}(\sigma ?) \\ S(\rho, \sigma) &= \mathrm{tr}\, \rho \log \rho - \rho \log \sigma \end{aligned}$$

4.2.2 Lemma *For normal states $S(\omega, \phi) \geq 0$ with equality only for $\omega = \phi$.*

4.2.3 Definition (Ensemble) *Let $\phi = \sum p_x \phi_x$ be an ensemble built from a given set of ϕ_x.*

4.2.4 Lemma (Holevo χ) *If the supports of the ϕ_x are orthogonal projections, then sufficient access to ϕ will be able to recover all the p_x arbitrarily well. If not, there is a bound*

$$\chi(\{p_x\}, \{\phi_x\}) = \sum_x p_x S(\phi_x, \phi)$$

Under extremization over $\{p_x\}, \{\phi_x\}$, this returns to classical capacity of the channel $x \to \phi_x$

4.2.5 Definition (Holevo χ) *Let $x \to \phi_x$ with probabilities p_x be an ensemble $\mathcal{E}$.*

$$\chi(\mathcal{E}) \quad = \quad S(\phi) - \sum_x p(x) S(\phi_x)$$

If each of the ϕ_x are pure then the only contribution is from the first term so it is just the entropy of the composite density operator.

4.2.6 Definition (Subfactor χ) *Let $\mathcal{N} \subset \mathcal{M}$ be the accessible observables inside all observables. $\omega = \sum p_x \omega_x$ is a state on $\mathcal{M}$ and gets restricted to $\mathcal{N}$. We ask for the difference of information extraction of the p_x by looking at all of $\mathcal{M}$ vs only $\mathcal{N}$.*

$$S_{\mathcal{M}|\mathcal{N}}(\{p_x\}, \{\phi_x\}) \quad = \quad \chi(\{p_x\}, \{\phi_x\}) - \chi(\{p_x\}, \{\phi_x \mid \mathcal{N}\})$$

4.2.7 Theorem (Pimsner-Popa)

$$\log[\mathcal{M} : \mathcal{N}] \quad = \quad \sup \sup S_{\mathcal{M}|\mathcal{N}}(\{p_x\}, \{\phi_x\})$$

`https://arxiv.org/pdf/1704.05562.pdf`

4.3 Contextuality

4.3.1 Definition (Quantale) *A closed monoidal suplattice. Closed gives the internal Hom. To say it is a suplattice means it is a poset with all joins. By adjoint functor theorem for posets it has all meets as well so is in fact a complete lattice (all small joins and meets). But this isn't just complete lattices because we don't have to preserve meets, only joins under morphisms of quantales vs we do when we look at their morphisms as complete lattices.*

4.3.2 Example

- $[0, \infty]$ *with order $\geq$, $+$ being the tensor product.*
- *Powerset of strings over an alphabet Σ. Order is by inclusion. Meet is intersection and join is union. The unit is the singleton of the empty string. The monoidal structure is by concatenating every pair of strings in I and J. K/J is removing stuff in J from the right of the strings in K. Similarly for $I \backslash K$*

4.3.3 Definition (Birkhoff-Von Neumann) *Have the Lattice of closed subspaces of a Hilbert space. This has intersection and the linear span as it's operations. These do not distribute creating the difference with classical boolean logic. This also has orthocomplements making it an orthomodular lattice.*

4.3.4 Definition (Enrichment over quantales)

4.4 Hadamard

4.4.1 Definition (Hadamard Gate) *A Hadamard gate of order n and type q is $\frac{1}{\sqrt{n}}$ times a Hadamard matrix H_n of order n and type q.*

$$\begin{aligned} H_n H_n^\dagger &= nI_n \\ (H_n)_{ij}^q &= 1 \end{aligned}$$

So when $q = 2$, this gives real Hadamard matrices. When $q = 4$ this gives the usual definition of complex Hadamard matrix.

4.4.2 Lemma *For $q = 2$, $H_{2^k} = H_2 \otimes H_{2^{k-1}}$ constructs Hadamard matrices for all 2^l. These can be written as $F_n^T F_n$ where F_n is the $n \times 2^n$ matrix listing all the n-bit numbers in counting order.*

$$\begin{aligned} F_1 &= \begin{pmatrix} 0 & 1 \end{pmatrix} \\ F_n &= \begin{pmatrix} 0 & 1 \\ F_{n-1} & F_{n-1} \end{pmatrix} \end{aligned}$$

so F_n is a generating matrix for a length 2^n linear error correcting code of rank n and minimum distance 2^{n-1}. This is called the Walsh code. It is confusingly not the Hadamard code.

4.4.3 Lemma *Given H_n of type q_1 and H_m of type q_2 then $H_n \otimes H_m$ is Hadamard of order nm and type $lcm(q_1, q_2)$*

4.4.4 Lemma *The discrete Fourier transform F_N is in $H(N, q = N)$. Also when F_N and F_M with $(M, N) = 1$ get Hadamard equivalence with F_{NM}*

4.4.5 Theorem *For prime p and $n > 1$, $H(n, p)$ can only exist when $n = mp$. It is open if this is sufficient when $p \geq 3$ as well. $H(6, 3)$??*

4.4.6 Lemma *Complex Hadamard matrices $H(N, \mathbb{C})$ give a basis of maximally entangled pairs in the sense that H can be turned into a state on $\mathbb{C}^N \otimes \mathbb{C}^N$ and this state should be maximally entangled as in $\mathrm{tr}_1 \mid \psi\rangle\langle\psi \mid = \frac{1}{N}$ and the set for all such H then gives an orthonormal basis of this subspace.*

4.5 Solvay-Kitaev

4.5.1 Theorem (Solvay-Kitaev) *Let G be a finitely generated dense subgroup of $SU(2)$. There is a constant c such that for any G and any $\epsilon > 0$ one can choose $l \in O(\log^c \frac{1}{\epsilon})$ such that the length $\leq l$ words in G form a ϵ net for $SU(2)$.*

There is also an algorithm presented for this and polylog reductions upon changing the group with generators.

If we change $SU(2) \to SU(N)$, we get a change in the constant which is exponential in N^2 as well. But N is 2^d so doubly exponential $2^{\#2^{2d}}$. But we have fixed $N = 2^d$ and are only worried about the asymptotics of the number of gates/the required accuracy.

Proof Define an recursive algorithm that takes U and n. For the base case, if $n = 0$, return the closest generator as a length 1 word. Otherwise compute the $n-1$ approximation U_{n-1} of U, Do a decomposition of $UU_{n-1}^\dagger$ into a group commutator of V W. Then do the SK decomposition of those. Then put those all together as $V_{n-1}W_{n-1}V_{n-1}^\dagger W_{n-1}^\dagger U_{n-1}$.

This is correct because $UU_{n-1}^\dagger$ is within ϵ_{n-1} by induction and we must show that the correction with V and W pushes the error down to ϵ_n. But then $V_{n-1}W_{n-1}V_{n-1}^\dagger W_{n-1}^\dagger$ is a $c\epsilon_{n-1}^{3/2}$ approximation of $UU_{n-1}^\dagger$. This means that an ϵ_n which is $c\epsilon_{n-1}^{3/2}$ approximation as the output U_n. This works as long as $\epsilon_{n-1} < 1/c^2$. So we just need to get the ϵ_0 to be small enough since it will just get smaller from there.

Putting this together, we get that for a desired accuracy ϵ, we need a depth of

$$\begin{aligned} n &= \frac{\log \frac{\log \epsilon c^2}{\log \epsilon_0 c^2}}{\log 3/2} \\ \ell_n &\in O(\log^{\log 5/\log \frac{3}{2}} \frac{1}{\epsilon}) \end{aligned}$$

and also polylog for the running time.

4.6 Clifford

4.6.1 Definition (Clifford) *The Clifford group on n qubits is those $U \in PU(2^n)$ such that $\forall \sigma \in \pm P_n^*$ $U\sigma U^\dagger \in \pm P_n^*$ where P_n^* are those strings of $\sigma_{1,a} \otimes \sigma_{2,b} \cdots$ where a, b labels 0,x,y,z. But not all of them 0. That is it is the normalizer of the group consisting of strings of n of the $\sigma_{i,a}$ tensored together.*

Also

$$\begin{aligned} H_i &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ P_i &= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \\ CNOT_{ij} &= \begin{pmatrix} I_2 & 0 \\ 0 & \sigma_x \end{pmatrix} \end{aligned}$$

generate C_n.

4.6.2 Theorem *C_n together with any other new gate forms a universal set of quantum gates.*

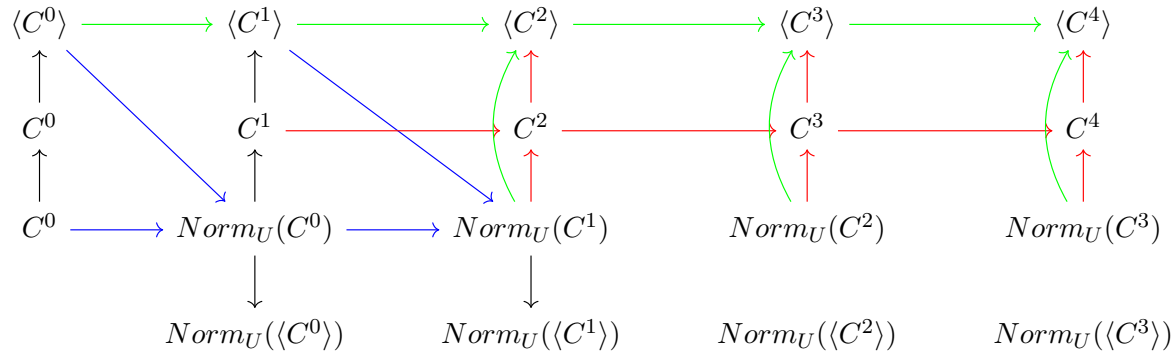
Proof `https://arxiv.org/abs/math/0001038v2` □

4.6.3 Theorem (Kliuchinikov-Maslov-Mosca) *The group $U(2) \bigcap M(2, 2, \mathbb{Z}[\frac{1}{\sqrt{2}}, i]$ is equivalent to the group generated by the Clifford+T.*

`http://www.mathstat.dal.ca/~selinger/newsynth/`
This is related to Langlands by
`https://webusers.imj-prg.fr/~michael.harris/SatoTate/notes/unitarygroups.pdf`

4.6.4 Definition (Clifford Hierarchy) *In addition to generalizing the level of hierarchy, also generalize from 2 to ℓ prime. Let X be the clock operator on $\mathbb{C}^\ell$ in the standard basis which permutes them cyclically. Let Z be multiplication by ω^j where ω is a ℓ'th root of unity. The zeroth level of the Clifford hierarchy is then the group generated by $P \equiv U(1) \times \langle \alpha I, X, Z \rangle$ where $\alpha = \omega$ when $\ell > 2$ and $\alpha = i$ when $\ell = 2$ on any of the n qudits.*
The first level is then defined as $C^1 \equiv \{U \mid \forall p \in P\ UpU^\dagger \in P\}$. That is $N_{U(\ell^n)}(P)$ the normalizer. This generalizes to the next level as $C^{n+1} \equiv \{U \mid \forall p \in P\ UpU^\dagger \in C^n\}$. But this is not a group because conjugating by U_1 lands you in C^n so conjugating by the next U_2 is not guaranteed to be in C^n anymore because that was only quantified over P. So U_1U_2 is not necessarily in C^{n+1}.

4.6.5 Lemma



Red stands for inclusions of sets. Green for inclusions of groups. Blue for normal subgroup inclusions. Black for isomorphisms.

$N_G(S)$ is a subgroup of G. Suppose it contains S, then $\langle S \rangle$ is contained therein. $\langle S \rangle$ is defined to be the intersection of all subgroups containing S. If it doesn't contain S, then there is no inclusion from $\langle S \rangle$ to $N_G(S)$. So whether or not C^n is a subset of $Norm_U(C^n)$ will conclude if there are diagonal green arrows.

$$\begin{aligned} U \in C^1 &\implies U^\dagger \in C^1 \\ U \in C^n &\iff \forall p \in P \exists g \in C^{n-1}\ UpU^\dagger = g \end{aligned}$$

But that is not clear.

S is a subset of $\langle S \rangle$ but x might have the effect of $xsx^{-1} = \prod_{i=1}^n s_i^{\pm} \in \langle S \rangle$. So x is not necessarily in $N_G(S)$. However if S is closed under inverses

$$\begin{aligned} & & x \in N_G(S) &\implies xs_ix^{-1} \in S \\ x(\prod_{i=1}^n s_i^{\pm_i})x^{-1} &= \prod_{i=1}^n xs_i^{\pm_i}x^{-1} & & \\ & \forall i\ xs_i^{\pm_i}x^{-1} &\in S & \\ & & \implies & \prod_{i=1}^n xs_i^{\pm_i}x^{-1} \in \langle S \rangle \\ & & \implies & x \in N_G(\langle S \rangle) \end{aligned}$$

There is also the possibility of taking normalizers repeatedly.

$$\begin{aligned} G_0 &\equiv C^0 \\ G_1 &\equiv Norm_U(C^0) = C^1 \\ G_2 &\equiv Norm_U(C^1) = Norm_U(Norm_U(C^0)) \end{aligned}$$

This means that there is a chain $G_0 \subseteq G_1 \cdots$ which are all normal inclusions.

$$\begin{aligned} g &\in G_{n+1} \\ p \in C^n &\Longrightarrow gpg^\dagger \in C^n \\ p \in P &\Longrightarrow p \in C^n \\ &\Longrightarrow p \in C^{n+1} \\ G_k &\subseteq C^k \\ G_k &\subseteq \langle C^k \rangle \end{aligned}$$

with these being inclusions of sets and groups respectively.

4.6.6 Lemma *Let k be the base field throughout. Let H be an algebraic subgroup of G, then there is a unique algebraic subgroup $N_G(H)$ of G such that for all k-algebras R, the R points of $N_G(H)$ are given by*

$$N_G(H)(R) \equiv \{g \in G(R) \mid gH_Rg^{-1} = H_R\}$$

The key fact is showing that N actually is represented by a closed subscheme of G.

Proof Milne page 33 □

4.6.7 Lemma *Consider $\ell \neq 2$ and the subgroup generated by X and Z and ω.*

$$
\begin{aligned}
XZ \mid j\rangle = X\omega^j \mid j\rangle &= \omega^j \mid j+1\rangle \\
ZX \mid j\rangle = Z \mid j+1\rangle &= \omega^{j+1} \mid j+1\rangle \\
X^{-1}Z^{-1}XZ \mid j\rangle &= X^{-1}Z^{-1}\omega^j \mid j+1\rangle \\
&= \omega^{-1} \mid j\rangle \\
X^{-1}Z^{-1}XZ &= \omega^{-1} \\
XZ &= \omega^{-1}ZX \\
X^bZ^cX^bZ^c &= \omega^{bc}X^{2b}Z^{2c} \\
X^bZ^cX^bZ^cX^bY^c &= \omega^{bc}X^{2b}Z^{2c}X^bY^c \\
&= \omega^{bc+2bc}X^{3b}Z^{3c} \\
(X^bZ^c)^r &= \omega^{bc+2bc+3bc+\cdots(r-1)bc} \\
&= \omega^{bc\frac{r(r-1)}{2}}X^{rb}Z^{rb} \\
g &= \omega^a X^b Z^c \\
g^\ell &= \omega^{\ell a}(X^bZ^c)^\ell \\
&= \omega^{\ell a}\omega^{bc\frac{\ell(\ell-1)}{2}}X^{\ell b}Z^{\ell c}
\end{aligned}
$$

$\ell - 1$ is divisible by 2 because ℓ is an odd prime, so $g^\ell = 1$. Therefore the order of any nontrivial element g is ℓ. If you take the n'th tensor power of this group, then that is still true because the order is given by the g.c.d. of the orders of the factors. So it is still ℓ. One can write this case as $Heis(\mathbb{Z}/p\mathbb{Z})^n$.

When $\ell = 2$, $g^\ell = (-1)^{bc}$ which can be nontrivial. So there are elements of order 4. You could also see this by the fact that it isn't abelian so not every element can square to the identity.

4.7 Complexity Growth

`https://www2.yukawa.kyoto-u.ac.jp/~qist2019/slides/5th/HunterJones.pdf`

4.7.1 Definition (δ-unitary complexity) *A unitary $U \in U(d = q^n)$ has δ-unitary complexity of at most r if there exists $V \in Im(G^r \to U(d))$ such that*

$$\frac{1}{2} \mid \mathcal{U} - \mathcal{V} \mid_\diamond \quad \leq \quad \delta$$

$Im(G^r \to U(d))$ is the set of unitaries that are given as products of r elements from the generating set G. $\mathcal{U}$ and $\mathcal{V}$ are the channels corresponding to the unitaries.

4.7.2 Definition (δ-state complexity) *A state $\mid \psi\rangle$ has δ-state complexity of at most r if there exists $V \in Im(G^r \to U(d))$ such that*

$$\frac{1}{2} \mid\mid \psi\rangle\langle\psi \mid -V \mid 0\rangle\langle 0 \mid V^\dagger \mid_1 \quad \leq \quad \delta$$

4.8 Deutsch

$$\begin{aligned} D(\theta) \mid 1,1,c\rangle &= i\cos\theta \mid 1,1,c\rangle + \sin\theta \mid 1,1,\neg c\rangle \\ D(\theta) \mid a,b,c\rangle &= \mid a,b,c\rangle \\ \lambda_1 &= e^{i*(\frac{\pi}{2}+\theta)} \\ \lambda_2 &= e^{i*(\frac{\pi}{2}-\theta)} \\ \lambda_{3-8} &= 1 \end{aligned}$$

- $\theta = \frac{\pi}{2} \implies D(\theta)^2 = I$
- $\theta = \frac{\pi}{3} \implies D(\theta)^{12} = I$
- $\theta = \frac{\pi}{4} \implies D(\theta)^8 = I$
- $\theta = \frac{\pi}{5} \implies D(\theta)^{20} = I$
- $\theta = \frac{\pi}{6} \implies D(\theta)^6 = I$
- $\theta = \frac{\pi}{7} \implies D(\theta)^{28} = I$
- $\theta = \frac{\pi}{8} \implies D(\theta)^{16} = I$

4.9 Grover Algorithm

Say we have N database entries and a function f which is 1 on one of them and 0 on the rest. We wish to find that value. Let $\mid x\rangle \in (\mathbb{C}^2)^n$ so that $\log_2 N = n$

$$U_f \mid x\rangle \mid y\rangle \equiv \mid x\rangle \mid y \bigoplus f(x)\rangle$$

Initialize by putting in the state

$$\mid Init\rangle = H^{\otimes n} \otimes H \mid 0^{\otimes n}\rangle \mid 1\rangle$$

Then apply the operator

$$\begin{aligned} U_G &= (U_s U_f)^{r(N)} \\ U_s &\equiv (H^{\otimes n}\big(2 \mid 0^{\otimes n}\rangle\langle 0^{\otimes n} \mid -I_n\big)H^{\otimes n}) \otimes I_2 \end{aligned}$$

where $r(N) = \frac{\pi\sqrt{N}}{4}$.

Finally observing the first n qubits in the computational basis gives the desired value ω with high probability

$$\begin{aligned} p &= \sin^2((r+1/2)\theta) \\ \theta &= 2\sin^{-1}\frac{1}{\sqrt{N}} \\ p &= \sin^2((\frac{\pi\sqrt{N}}{2}+1)\sin^{-1}\frac{1}{\sqrt{N}}) \approx \sin^2(\frac{\pi}{2}+\frac{1}{\sqrt{N}}) \approx 1 \end{aligned}$$

If there are k such ω with $f(\omega) = 1$ instead. Run it $\frac{\pi}{4}(\frac{N}{k})^{1/2}$ times instead. When k is an unknown power of 2, do it for all k and still get $O(\sqrt{N})$ algorithm.

$$\sum_{k=2^l=1}^{N} \frac{\pi}{4}(\frac{N}{k})^{1/2} \leq \frac{\pi}{4}N^{1/2}(2+\sqrt{2})$$

`https://en.wikipedia.org/wiki/Grover%27s_algorithm`

4.9.1 Quantum Counting

4.10 Group Problems

4.10.1 Kuperberg's Algorithm

Let G be a group acting on a set transitively. Given x_0 and x_1 find the γ which accomplishes the task $\gamma x_1 = x_0$

$$\begin{aligned} \mid \psi\rangle &= \frac{1}{\sqrt{2 \mid G \mid}} \sum_{g\in G} \mid g,0,gx_0\rangle + \mid g,1,gx_1\rangle \\ &= \frac{1}{\sqrt{2\mid G \mid}} \sum_{g\in G} \mid g,0,gx_0\rangle + \mid g,1,gx_1\rangle \\ g_x x_0 &\equiv x \\ g\gamma x_1 &= gx_0 \\ g_x \gamma x_1 &= x \\ M_x \mid \psi\rangle &= \frac{1}{\sqrt{2}} \mid g_x,0,x\rangle + \mid g_x\gamma,1,x\rangle \end{aligned}$$

where we measure the third register and get some x. This x is irrelevant so throw it away. It is no longer entangled with the rest anyway.

$$\mid \psi_2\rangle = \frac{1}{\sqrt{2}} \mid g_x,0\rangle + \mid g_x\gamma,1\rangle$$

If $G = \mathbb{Z}_N$ and switch to writing additively. Apply QFT to the first register

$$QFT \mid \psi_2\rangle = \frac{1}{\sqrt{2N}} \sum_{k \in G} \zeta_N^{kg_x} \mid k, 0\rangle + \zeta_N^{k(g_x+\gamma)} \mid k, 1\rangle$$

Measure the first register to get k.

$$\begin{aligned} \mid \psi_3\rangle &= \frac{1}{\sqrt{2}} \zeta_N^{kg_x} \mid k, 0\rangle + \zeta_N^{k(g_x+\gamma)} \mid k, 1\rangle \\ &= \frac{1}{\sqrt{2}} \zeta_N^{kg_x} \mid k\rangle \otimes (\mid 0\rangle + \zeta_n^{k\gamma} \mid 1\rangle) \\ \mid \psi_{k,x}\rangle &= \frac{1}{\sqrt{2}} \zeta_N^{kg_x} (\mid 0\rangle + \zeta_n^{k\gamma} \mid 1\rangle) \end{aligned}$$

where dropping the first unentangled register.

$$\begin{aligned} \mid \psi_{k_1,x_1}\rangle \otimes \mid \psi_{k_2,x_2}\rangle &= \frac{1}{2} \zeta_N^{k_1 g_{x_1}} \zeta_N^{k_2 g_{x_2}} (\mid 0\rangle + \zeta_N^{k_1\gamma} \mid 1\rangle) \otimes (\mid 0\rangle + \zeta_N^{k_2\gamma} \mid 1\rangle) \\ &= \frac{1}{2} \zeta_N^{k_1 g_{x_1} + k_2 g_{x_2}} (\mid 00\rangle + \zeta_N^{k_1\gamma} \mid 10\rangle + \zeta_N^{k_2\gamma} \mid 01\rangle + \zeta_N^{(k_1+k_2)\gamma} \mid 11\rangle \\ CNOT(\mid \psi_{k_1,x_1}\rangle \otimes \mid \psi_{k_2,x_2}\rangle) &= \frac{1}{2} \zeta_N^{k_1 g_{x_1} + k_2 g_{x_2}} (\mid 00\rangle + \zeta_N^{k_1\gamma} \mid 11\rangle + \zeta_N^{k_2\gamma} \mid 01\rangle + \zeta_N^{(k_1+k_2)\gamma} \mid 10\rangle \end{aligned}$$

This can be expressed in $\mid \psi_{k_1+k_2,x_3}\rangle$ and $\mid \psi_{k_1-k_2,x_3}\rangle$. Getting $\mid \psi_{N/2,x_3}\rangle$ gives $\mid 0\rangle + (-1)^\gamma \mid 1\rangle$ which is used to evaluate the least significant bit of γ by measurement in the Hadamard basis.

Once you know the least significant bit, you can do this sort of process again to get

$$\mid \psi_{N/4,-}\rangle \propto \mid 0\rangle + \zeta_N^{N/4\gamma}$$

If the least significant bit was 1, apply an $R_z(\zeta_n)$ so that in both cases get

$$\mid 0\rangle + \zeta_N^{N/2 floor(\gamma/2)} \mid 1\rangle = \mid 0\rangle + (-1)^{floor(\gamma/2)} \mid 1\rangle$$

allowing the next least significant bit.

4.10.2 Hidden Subgroup Problem

4.11 Quantum Walks

Let G be an undirected graph with adjacency matrix A and Laplacian L.

4.11.1 Definition (Quantum Walk) $U(t) = e^{-iAt}$ *or* e^{-iLt}

4.11.2 Definition (Oriented Graph Quantum Walk) *On an oriented graph, the adjacency matrix will be antisymmetric so iA can be used as the Hamiltonian in $U(t)$. In this case $U(t)$ is real and orthogonal.*
This is still under the restriction that for any unordered pair there is at most one edge. So graphs with two edges $a \to a$ and $a \to b$ are still not allowed.

4.11.3 Lemma (Products of Directed Multigraphs) *$A_C = A_1 \otimes A_2$ so $U_C(t) = U_1(t) \otimes U_2(t)$ for the adjacency matrix version of quantum walk.*
Here $G_C = G_1 \times G_2$ is the product of graphs which has $V(G_C) = V(G_1) \times V(G_2)$ and $E(G_C) = E(G_1) \times E(G_2)$.
This is the product when you think of individual graphs as objects of $[G^{op}, Set] \simeq PSh(G)$ where G is the category with two objects v and e and two non-identity arrows $s, t\ v \to e$ so that in G^{op} they become usual source and target.
Also works for the product of graphs which has $A_C = A_1 \otimes I + I \otimes A_2$. Then $U_C(t) = U_1(t) \otimes U_2(t)$ because the two terms commute and give $U_1(t) \otimes I$ and $I \otimes U_2(t)$ respectively. (Called Cartesian often, but that is only due to the visual similarity when drawing not any universal property)

4.11.4 Definition (Mixing Matrix) *$M(t) = U(t) \cdot_H U(t)$ which gives the probability of starting out in a state $\mid v\rangle$ for a vertex v and measuring a state $\mid w\rangle$ for a vertex w. The phase factor is gone, just the probability is kept.*
This is a doubly stochastic matrix.

4.11.5 Definition (Uniform Mixing) *The evolution semi-group $U(t)$ is said exhibit uniform mixing at $t_0 > 0$ if $U(t_0) \mid v\rangle = \frac{1}{\sqrt{V}} \sum \gamma_i \mid v_i\rangle$ where each of the $\gamma_i \in U(1)$.*

4.11.6 Definition (Perfect State Transfer) *The evolution semi-group $U(t)$ is said exhibit perfect state transfer if there exists a $t_0 > 0$ such that $U(t_0) \mid v\rangle = \gamma \mid w\rangle$ between two vertices $v \neq w$, $\gamma \in U(1)$. That says that if I start in v, wait for time t_0 and then observe in the vertex basis, I will surely get w.*

4.11.7 Lemma *Perfect State Transfer implies periodicity at $2t_0$.*

Proof

$$\begin{aligned} U(t_0) \mid v\rangle &= \gamma \mid w\rangle \\ U(-t_0) \mid v\rangle &= \overline{\gamma} \mid w\rangle \\ \mid v\rangle &= \gamma^{-1} U(t_0) \mid w\rangle \\ U(t_0) \mid v\rangle &= \gamma^{-1} U(2t_0) \mid w\rangle \\ \gamma \mid w\rangle &= \gamma^{-1} U(2t_0) \mid w\rangle \\ U(2t_0) \mid w\rangle &= \gamma^2 \mid w\rangle \end{aligned}$$

4.11.8 Remark If γ^2 is not a root of unity then γ is not either. So if we find a graph such that $U(2t_0)$ has eigenvalue not a root of unity with eigenvector being a basis vector, there is a chance for perfect state transfer with non-root of unity. But then we would have to make sure that it gave perfect state transfer at t_0. For example, eigenvectors of A or L itself that are basis vectors would give eigenvalues all around the unit circle, but at t_0 they will be still full-periodicity not perfect state transfer.
Let $\lambda_1 \cdots \lambda_n$ be the eigenvalues of A or L as appropriate and $v_1 \cdots v_n$ their eigenvectors. To look for $2t_0$ means looking for those times such a subset of the $e^{-i2t_0\lambda_j}$ are all equal and the span of their associated eigenvectors contains a basis vector $\mid w\rangle = \sum \alpha_i v_i$. Then that basis vector is going to be an eigenvector of $U(2t_0)$. We can then evaluate $U(t_0)$ on this vector. This will give the condition $\forall j, k\ \alpha_{j,k} \neq 0 \implies \exists N \in \mathbb{Z}\ \ t_0\lambda_j = t_0\lambda_k + \pi N$. You do this for each $\mid w\rangle$. $\Diamond$

4.11.9 Definition (Full Periodicity) *The evolution semi-group $U(t)$ is said to exhibit periodicity at vertex $\mid w\rangle$ at time t when as above $t = 2t_0$, $U(2t_0) \mid w\rangle \propto \mid w\rangle$. When $U(t)$ is diagonal itself, then this is much stronger. If we call t_w the time required for each vertex, then this demands that the evolution exhibits periodicity at all vertices with their own times t_w and there is some T that is an natural number multiple of each of them with potentially different natural numbers.*
A graph is fully periodic with adjacency matrix evolution if and only if all the nonzero eigenvalues of the graph (of the adjacency matrix) are rational mutliples of each other.

4.11.10 Example *On the hypercube Q_n whose vertices are bitstrings, there is perfect state transfer at $t_0 = \frac{\pi}{2}$.*

This is a corollary of the lemma for products with adjacency matrix. The graph on 2-vertices exhibits perfect state transfer between $\mid 0\rangle$ and $\mid 1\rangle$ at time $\frac{\pi}{2}$ So the one for Q_n does so as well between any bitstring $b_1 \cdots b_n$ and it's bitwise negation.
Similarly periodicity at time π for any starting bitstring vertex of Q_n.
As well, there is perfect mixing at $\frac{\pi}{4}$ as a corollary.

4.11.11 Theorem (Coutinho-Lu) *For a tree with at least 3 vertices, the quantum walk with Laplacian does not admit perfect state transfer.*

Proof 1408.2935 short proof □

4.11.12 Definition ((Strong) Cospectrality) *Vertices a and b are cospectral if there is an orthogonal matrix Q such that Q commutes with A, it takes $\mid a\rangle$ to $\mid b\rangle$ and vice versa and $Q^2 = I$.*
In particular $Q = \gamma^{-1}U(t_0)$ for perfect transfer does the trick.
Strong cospectrality is when Q is given as a polynomial in A. Here $U(t_0)$ is a power series in A, but then you realize A has an explicit characteristic polynomial so you know how to rewrite that as a polynomial in A, with the coefficients being convergent series in t that you can plug in t_0.

4.11.13 Definition (Exponentials of Graphs) *Directed Multigraphs form a presheaf category $Psh(C)$ so there are exponential objects H^G satisfying $Hom(F, H^G) \simeq Hom(F \times G, H)$.*
Let wv and we stand for walking vertex and walking edge
The vertices of H^G are found by plugging in F_{wv} to be the graph with a single vertex and no edges. So the vertices of H^G are $Hom(F_{wv}, H^G) = Hom(F_{wv} \times G, H)$. But $F_{wv} \times G$ is just the underlying set G as a discrete graph. So the vertices of H^G are all functions $Hom_{Set}(V(G), V(H))$
The edges of H^G are found by plugging in F_{we} to be the graph with two vertices $\pm$ and a single edge $- \to +$. $F_{we} \times G$ is now the graph whose vertices are $(+, v)$ or $(-, v)$ and edges connecting $(-, v)$ to $(+, w)$ whenever there is an edge vw. There are two morphisms $Hom(F_{we} \times G, H) \to Hom(F_{wv} \times G, H)$ given by precomposition with the two morphisms $F_{wv} \to F_{we}$. Those tell you the endpoints of the given edge.
So the edges in H^G are graph homomorphisms from this two layer version of G to H and the endpoints are restricting to each layer.

4.11.14 Remark Much more detail, `https://ncatlab.org/nlab/show/category+of+simple+graphs`
In particular staying within undirected ordinary graphs, by giving C an arrow on the edge object that swaps source and target maps into it (Will get flipped to usual source and target maps when doing Psh(C)). ◊

4.11.15 Lemma *If H is an ordinary digraph rather than a multigraph, then H^G is as well. So restricting to ordinary digraphs mean you can build whatever using exponentials, products (the categorical one not the $A \otimes I + I \otimes B$ one which has a problem with self loops), disjoint unions (colimits exist in topos)*
If H is a multigraph, then either G has any edges in which case you get a multigraph or G has no edges in which case you get a complete digraph with self loops on $v(H)^{v(G)}$ many vertices.

Proof The edges of H^G are graph homomorphisms from a two layer version of G to H. Where the vertices go is specified by the source and target of the edge. The number of edges compatible with the given source and target is counting directed multigraph homomorphisms to H but with all the vertices fixed.
But H is an ordinary digraph so if there is an edge in the source $F_{we} \times G$ going between $(-, v)$ and $(+, w)$ then it must get sent to an edge between $s(v)$ and $t(w)$ where s is the map from vertices of G to vertices of H specifying the source of the edge in H^G and t is the one specifying the target. But there is only at most one such edge. So all such edges in $F_{we} \times G$ must go there if it exists, there are no choices. If it does not exist, there are no graph homomorphisms $F_{we} \times G \to H$ with those specified source and target.
Therefore there are at most one edges between any pair of vertices of H^G.
On the other hand if H has a pair of vertices s_H, t_H with at least 2 edges between them, then for any chosen edge E in G give the function that sends everything to s_H. Make the other function send everything to t_H. In order for $F_{we} \times G \to H$ be a graph homomorphism with these endpoints, every edge goes to one of the many choices of edge from $s_H \to t_H$ in H. So as long as there are any edges in G, this is a multigraph. If G has no edges, then complete digraph with self loops on $v(H)^{v(G)}$ many vertices.

4.11.16 Example (Boring Case 1, G is undirected and H^G stays that way) *$G = F_{wv}$, then the vertices of H^G are vertices of H and edges are any pair of vertices of H. So H^G is the complete directed graph on $\mid V(H) \mid$ with self loops.*

4.11.17 Example (Boring Case 2, G is undirected and H^G stays that way) *$H = F_{wv}$, then the vertices of H^G is just one vertex. The edges are graph homomorphisms from the two layer version of G to H. So a single self loop, this is the case even if G is empty.*

4.11.18 Example (Mildly interesting but nonhermitian, G is directed causing problem) *$G = F_{we}$, then the vertices of H^G are (h_1, h_2) for any pair of vertices of H. The edges are tuples (e, v, w) of two vertices in H and an edge in H. This goes from $(s(e), v)$ to $(w, t(e))$. For each edge there is a self loop at $(s(e), t(e))$. There is an asymmetry because there doesn't have to be an edge $f \in w \to v$ in H that would be needed for a backwards edge $(w, t(e)) \to (s(e), v)$ as the edge corresponding to the tuple $(f, t(e), s(e))$. So as long as there are any pairs of vertices that do not have an edge from one to the other, this adjacency matrix will be asymmetric. This preserves H being a graph rather than a multigraph.*

$$A_{H^G} \quad = \quad Swap(A_H \otimes A_{H^{F_{wv}}})$$

Swap is the flip operator acting on $k^{v(H)} \otimes k^{v(H)}$.

4.11.19 Example (n-fold generalization of above, G directed doesn't cause too much problem) *G is the linear path with n vertices. The vertices of H^G are $(h_1 \cdots h_n)$ for any n-tuple of vertices of H. The edges are tuples $(e_1 \cdots e_{n-1}, v, w)$ of $n-1$ edges and two vertices. This goes between $(s(e_1) \cdots s(e_{n-1}), v)$ to $(w, t(e_1) \cdots t(e_{n-1}))$. For every path in $e_1 \cdots e_{n-1}$ in H that are connected up at endpoints, there is a self loop at $(s(e_1) \cdots s(e_{n-1}), t(e_{n-1}))$ by letting $v = s(e_1)$ and $w = t(e_{n-1})$ and the fact that $e_1 \cdots e_{n-1}$ form a path in H.*
w is not necessarily the source of any edges so that gives asymmetry. Also v being target of any edges. But if this problem doesn't show up (like if H had no sources or sinks even vacuous ones with no edges), then there is a backwards edge. That's because unlike the previous case v and w are now on separate edges and those can be chosen independently.

$$A_{H^G} = Cycle(A_H \otimes \cdots A_H \otimes A_{H^{F_{wv}}})$$

with cycle acting as $v_1 \otimes \cdots v_n \to v_n \otimes v_1 \cdots v_{n-1}$ with each $v_i \in k^{v(H)}$
If H is a graph rather than a multigraph, so is H^G. So despite appearances

$$\begin{aligned}
A_H v_i &= \lambda_i v_i \\
ONE &\equiv (1 \cdots 1) \\
A_{H^{F_{wv}}})ONE &= nONE \\
ONE \cdot w = 0 &\implies A_{H^{F_{wv}}})w = 0w \\
A_{H^G}(v_{\alpha_1} \cdots v_{\alpha_{n-1}} \otimes w) &= \prod_{i=1}^{n-1} \lambda_{\alpha_i} Cycle(v_{\alpha_1} \cdots v_{\alpha_{n-1}} \otimes w) \\
&= \prod_{i=1}^{n-1} \lambda_{\alpha_i} (w \otimes v_{\alpha_1} \cdots v_{\alpha_{n-1}})
\end{aligned}$$

4.11.20 Example (Boring Case 3, H is directed and causes boring for undirected G) *$H = F_{we}$, then the vertices of H^G are subsets of the vertices of G. The edges are graph homomorphisms from the two layer version of G to H. So $S_1 \to S_2$ edge means that S_1 in the top layer goes to $-$, S_1^c in the top layer goes to $+$. S_2 in the bottom layer goes to $-$ and S_2^c in the bottom layer goes to $+$. Any edge from S_1 to S_2 would become a self loop on $-$ which does not exist. Any edge from S_1^c to S_2 would become an edge from $+ \to -$ which does not exist. $S_1 \to S_2^c$ would become $- \to +$ which is fine. $S_1^c \to S_2^c$ would become a self loop on $+$ which does not exist.*
So vertices of H^G are subsets of the vertices of G and there is an edge between $S_1 \to S_2$ if and only if

- *No edges from $v \in S_1 \to w \in S_2$*
- *No edges from $v \in S_1^c \to w \in S_2$*
- *No edges from $v \in S_1^c \to w \in S_2^c$*

The only edges allowed can be from S_1 to S_2^c. Suppose G is such that every edge has a backwards edge and there are no isolated vertices, then every vertex has an edge starting from there so every

vertex must be in S_1 if there are to be any edges in H^G starting from there. Also every vertex is a target ending there so S_2 must be the empty set if there are to be any edges in H^G ending there. So in this case H^G is a graph with $2^{v(G)} - 2$ isolated vertices and the remaining 2 connected with a single directed edge from the full set to the empty set.

4.11.21 Example (Interesting PseudoHermitian case, H is directed causes mild trouble for undirec
Change the above so that H has self loops at both $-$ and $+$. Then the only edges $S_1 \to S_2$ in H^G that are not allowed are when there is an edge S_1^c to S_2 in G.

If $S_1 = \varnothing$, then S_2 must only contain isolated vertices (so probably empty) if there is to be an edge $S_1 \to S_2$. If S_1 is all of G, then there is no constraint on S_2 so that forms a source vertex of H^G. If $S_2 = \varnothing$, then no constraint on S_1 so that forms a sink vertex of H^G. In particular both source and sink have self loops.

Suppose there is symmetry in G (undirected), then S_1^c and S_2 must not have any edges going either way between them. So if G is an ordinary graph such that there is an edge from $S_1 \to S_2$ in H^G, there are no arrows between S_1^c and S_2 in G, then it is also the case there is an arrow from $S_2^c \to S_1^c$ in H^G. So instead of the adjacency matrix being symmetric, it is persymmetric. This not PT symmetry where there would be some operator PT commuting with A. But if $PT \mid S_1\rangle = \mid S_1^c\rangle$, then under $PT(A)PT$, the entry for $S_2 \to S_2^c \to S_1^c \to S_1$ would be the number of edges from $S_2^c \to S_1^c$ which equals the number of edges from $S_1 \to S_2$ not $S_2 \to S_1$ as it would be if $PT(A)PT = A$. The examples, show restricting G does not help, to get an A that satisfies this.
If G had a self loop and there was an arrow in H^G from $S_1 \to S_2$, then the self loop in G must be on a vertex in $S_1 \bigcup S_2^c$. In SimpGraph everything has a self loop so $S_1 \bigcup S_2^c$ needs to be everything which amounts to $S_2 \subseteq S_1$. This means A_{H^G} is triangular with respect to any total order compatible with the partial order on power set.
Closely related to poset structure on power set, but the additional condition that S_1^c and $S_2 \subseteq S_1$ should not only be disjoint as they already are but no edges connecting them. Still a subposet though with the identity on vertices.
In this case self loops in H^G correspond to when S forms a division of G into connected components, so only two self loops when G is connected at the empty set and all of G, which were the self loops we got in general. Otherwise there are 2^n self loops when n is the number of connected components. Trace of adjacency matrix. In the SimpGraph case this says the eigenvalues of A_{H^G} are 1 with multiplicity 2^n and 0 with the remaining multiplicity.

$G \in SimpGraph \to H^G$ which is digraph. PT is permutation of vertices inducing another graph on the same vertices $PT(H^G)$, then take edge union $H^G \bigcup PT(H^G)$. It is not disjoint because 2 self loops on the same vertex become one. The adjacency matrix of this is symmetric.

TODO: What is the adjacency matrix for this when written as an opertor on $\wedge^\bullet k^{v(G)}$? It is not just evaluating $\wedge A_G$ because that breaks down by grading and S_1 and S_2 can have independent sizes.

Assume undirected and all self loops on G until end of example: On $\wedge^0$, then only amplitude is 1 for $S_1 = S_2 = \varnothing$
On $\wedge^1$, then either S_1 is an isolated vertex in which case there is a 1 amplitude to go to itself and a 1 to go $S_2 = \varnothing$. If S_1 is not isolated, then only go to nothing.
On $\wedge^2$, $S_1 = \{a, b\}$, then S_2 can be empty in which case amplitude 1, S_2 can be a or b in which case asking whether there are any edges connecting to a or b respectively. This says that a or b is isolated alone or a tail whose only connection to the rest of the graph is through the other (tails being a case of 1-edge connected but not 2-edge connected in sense of min-cut). The other possibility

of $S_2 = S_1$ means these two vertices should be isolated together or as an edge if the amplitude is to be 1 and not 0.

A permutation of a graph isomorphism of the original G acts on this levelwise by $\wedge P$ such that $(\wedge P)M(\wedge P^{-1}) = M$ where M is the operator from the adjacency matrix. So we can even restrict to sets of cardinality $\leq k$ and permutations will stay within this subspace.

4.11.22 Definition (Persymmetric) *Symmetric across the antidiagonal. For example,*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{12} \\ a_{31} & a_{21} & a_{11} \end{pmatrix}$$

4.11.23 Lemma *Let A be a matrix on k^n and let $\bar{i}$ be an involution on n. If the involution has at most one fixed point (no fixed points in our case), this means that after a permutation the matrix is persymmetric. Suppose A satisfies the following identity*

$$A_{ij} = A_{\bar{j}\bar{i}}$$

Then A^T also satisfies the same property. Then define $B_{ij} \equiv A_{\bar{i}j}$. B is symmetric. Or define C to be the same as B but with A^T taking the place of A. C is symmetric by the same argument.

Proof

$$\begin{aligned} (A^T)_{ij} &= A_{ji} \\ &= A_{\bar{i}\bar{j}} \\ &= (A^T)_{\bar{j}\bar{i}} \end{aligned}$$

$$\begin{aligned} B_{ij} &= A_{\bar{i}j} \\ B_{ji} &= A_{\bar{j}i} \\ A_{\bar{i}j} &= A_{\bar{j}i} \\ B_{ij} &= B_{ji} \\ B_{\bar{i}\bar{j}} &= A_{i\bar{j}} \end{aligned}$$

If we symmeterize a persymmetric A, then it will be centrosymmetric.

$$\begin{aligned} SA &\equiv A + A^T \\ SA_{ij} &= SA_{ji} \\ SA_{ij} &= A_{ij} + (A^T)_{ij} \\ &= A_{\overline{j}\overline{i}} + (A^T)_{\overline{j}\overline{i}} \\ &= SA_{\overline{j}\overline{i}} \\ SA_{ij} &= SA_{\overline{i}\overline{j}} \\ SA &= \begin{pmatrix} SA_{11} & SA_{12} \\ SA_{21} & SA_{22} \end{pmatrix} \end{aligned}$$

where SA_{11} is a symmetric square matrix with dimensions the ceiling of $\frac{n}{2}$. $SA_{12} = SA_{21}^T$ are persymmetric matrices. $SA_{22} = SA_{11}$ or maybe missing the last row and column if n is odd.

4.11.24 Lemma (PT Usual) *Let H be a linear operator commuting with an antilinear involution PT such that a simulataneous eigenstate is given by ϕ, then the eigenvalue of H on this eigenstate is real. Linear commuting operators can be simulataneously diagonalized, but PT being antilinear provides the rub.*
If it anti-commutes with PT, then the eigenvalues on these eigenstates are pure imaginary.
If it q-commutes with PT ($(PT)H = qH(PT)$), then $qE = \bar{E}$ which says that $\mid q \mid = 1$ and the argument of q says which line E can be on interpolating between the real axis and imaginary axis.

Proof

$$\begin{aligned} H \mid v\rangle &= E \mid v\rangle \\ PT \mid v\rangle &= \lambda \mid v\rangle \\ (PT)^2 \mid v\rangle &= PT\lambda \mid v\rangle \\ \mid v\rangle &= \bar{\lambda} PT \mid v\rangle \\ &= \mid \lambda \mid^2 \mid v\rangle \\ (PT)H \mid v\rangle &= (PT)E \mid v\rangle \\ H(PT) \mid v\rangle &= \bar{E}(PT) \mid v\rangle \\ H\lambda \mid v\rangle &= \bar{E}\lambda \mid v\rangle \\ H \mid v\rangle &= \bar{E} \mid v\rangle \end{aligned}$$

Even if all eigenstates of H were simultaneous eigenstates with PT, the spectrum would be purely real but that wouldn't be enough to get Hermitian because the spectral theorem is for normal operators but nothing said that H and $H^\dagger$ commuted. In fact we can know H is not normal assuming it satisfied above but was not already Hermitian. That is because the spectral theorem would then give $H = UDU^\dagger$ with D all real. But that is Hermitian. So it has to have been not normal if this is to provide any use.

If we restrict to the span of these eigenstates, we get a diagonalizable matrix H, but not unitarily diagonalizable. So $H = PDP^{-1}$. Then $e^{-iHt} = Pe^{-iDt}P^{-1}$ has $U(1)$ eigenvalues with the same eigenstates given as Pe_i from the standard basis.

4.12 Quantum Amplification

A coherent state is prepared in a state $\rho =\mid z\rangle\langle z \mid$ for coherent states of $L^2(R)$. The task is to send this to $\mid ga\rangle\langle ga \mid$.

4.12.1 Definition (Two Mode Squeezing)

`https://en.wikipedia.org/wiki/Quantum_amplifier`

We may turn this into a game

Referee sends ρ_x with probability p_x

Player gets ρ does a completely positive trace preserving map $\tilde{\rho} = C(\rho)$. Then sends this to the referee for the test T_x which is 1 for success and 0 for failure.

$$\begin{aligned} F &= sup_C \sum p_x \langle T_x C(\rho_x)\rangle \\ &= min_{\sigma>0\ Tr\sigma=1} \mid I_{out} \otimes \sigma^{-1/2} \Omega I_{out} \otimes \sigma^{-1/2} \mid_\infty \\ \Omega &= \sum_x p_x T_x \otimes \rho_x^T \end{aligned}$$

Can also do the game with the right to pass and compute the score F conditioned on playing the game.

$$\begin{aligned} F &= sup_C \sum p_x \langle T_x C(\rho_x)\rangle \\ &= \mid I_{out} \otimes \rho^{-1/2} \Omega I_{out} \otimes \rho^{-1/2} \mid_\infty \\ \Omega &= \sum_x p_x T_x \otimes \rho_x^T \\ \rho &= \sum p_x \rho_x \end{aligned}$$

So let $x \in \mathbb{C}$ with $p(x) = \lambda e^{-\lambda |x|^2}$ and $\rho_x =\mid x\rangle\langle x \mid$. The test is $T_x =\mid gx\rangle\langle gx \mid$

Suppose the player makes a measurement on ρ get a y and then produces $\tilde{\rho}$ output based on that y. This is a classical strategy that provides a test to see if the player made use of a full quantum strategy.

$$\begin{aligned} F &= sup_{Y,P_y,\tilde{\rho}_y} \sum_{x,y} p_x Tr(T_x \tilde{\rho}_y) Tr(P_y \rho_x) \\ &= min_{\sigma>0\ Tr\sigma=1} \mid I_{out} \otimes \sigma^{-1/2} \Omega I_{out} \otimes \sigma^{-1/2} \mid_x \\ \Omega &= \sum_x p_x T_x \otimes \rho_x^T \end{aligned}$$

where $||_x$ is maximize over product states rather than true operator norm. Called the cross norm.

Again allowing not to play replaces σ by ρ and no minimization.

Calculated maximum expected payoff if using classical and quantum strategies and then see that they are different.

4.12.2 Example *Given N copies of a pure state in $\mathbb{C}^d$ and desire to make M. Send $\mid \alpha\rangle \cdots \mid \alpha\rangle\langle\alpha \mid \cdots \langle\alpha \mid$ to another of the same form but with M instead.*

$$F(N \to M) = \prod_{k=1}^{d-1} \frac{N+k}{N+M+k}$$

https://www.youtube.com/watch?v=ZhLMQT6r5gQ&t=0s

4.12.1 Fidelity

$$\begin{aligned}
D(\alpha) &= e^{\alpha a^\dagger - \bar{\alpha} a} \\
S(z) &= e^{\frac{1}{2}(\bar{z}a^2 - z a^{\dagger 2})} \\
\mid \alpha, z\rangle &= D(\alpha)S(z) \mid 0\rangle \\
\langle \alpha, z \mid a^\dagger a \mid \alpha, z\rangle = E &\implies \mid \alpha \mid^2 + \sinh^2 \mid z \mid = E \\
\mid \langle r_1, \frac{1}{2}\ln d_1 \mid\mid r_2, \frac{1}{2}\ln d_2\rangle \mid^2 &= \frac{2\sqrt{d_1 d_2}}{d_1+d_2} e^{\frac{-2d_1 d_2}{d_1+d_2}(r_2-r_1)^2} \\
F(r_1, r_2, d, d) &= e^{-d(r_2-r_1)^2} \\
F(-\sqrt{E - \frac{1}{4}(\frac{d-1}{\sqrt{d}})^2}, -r_1, d)) &= e^{d^2-(4E+2)d+1} \\
\inf F(-\sqrt{E - \frac{1}{4}(\frac{d-1}{\sqrt{d}})^2}, -r_1, d)) &= e^{-4E^2-4E} \\
d_c(E) &= 2E+1
\end{aligned}$$

Proof https://arxiv.org/pdf/1706.05807.pdf □

4.13 Synchronizing Oscillators

Two oscillators are prepared in coherent states and a part of a two level system.

$$\begin{aligned}
\mid \psi\rangle &= \mid z_1\rangle \mid z_2\rangle \mid\downarrow\rangle \\
H_{sys} &= \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + \frac{\omega_0}{2}\sigma_z + \sum g_k(e^{i\theta_k} a_k e^{-i\theta_k})\sigma_x \\
H_{bath} &= \sum_i \nu_i b_i^\dagger b_i \\
H_{inter} &= \sum_i (\beta_i b_i + \beta_i^\dagger b_i^\dagger)\sigma_x
\end{aligned}$$

https://arxiv.org/pdf/1705.04667.pdf

4.14 Coherent States

4.14.1 Definition (Coherent State) *Let h be a complex separable Hilbert space and let X be a locally compact space with a measure $d\mu$. Then we give a map $X \to h$ denoted by $x \to \mid x\rangle$ that is weakly continuous and provides a resolution of the identity.*

$$\int \mid x\rangle\langle x \mid d\mu \quad = \quad I_h$$

4.14.2 Example (Standard Coherent States) *$X = \mathbb{C}$ and $d\mu = \frac{1}{\pi}dxdy$. This produces the harmonic oscillator example.*

4.14.3 Definition (Group Theoretical Coherent) *Let G be the locally compact group with Haar $d\mu$ acting on h as a continuous irreducible unitary representation.*

$$\begin{aligned} c(\psi) &= \int_G \mid \langle \psi \mid U(g) \mid \psi \rangle \mid^2 d\mu \\ \mid g\rangle &= \frac{1}{c(\psi)} U(g) \mid \psi\rangle \end{aligned}$$

This then provides a cohernet state in the usual sense by $X = G$. In fact we get an linear isometry between h and $L^2(G, \mu)$. As a G representation we get a subrepresentation of the left regular representation on $L^2(G, \mu)$

4.14.4 Definition (Duflo-Moore) *It is a positive self adjoint unbounded invertible operator.*

$$I \quad = \quad \frac{1}{c(\psi)} \int_G d\mu \mid g\rangle\langle g \mid$$

4.14.5 Example (1D Wavelet) *Let G be the affine group of the line that is $x \to ax + b$. It has left Haar measure by $a^{-2}dadb$. This group is acting on $L^2(\mathbb{R})$ manifestly so take that to be h. By the definition we now get states $\mid b, a\rangle$ once we provide ψ called the mother wavelet which needs to have the admissibility condition that $c(\psi) < \infty$.*

$$\begin{aligned}
\psi_{Morlet}(x) &= c_\sigma \pi^{-1/4} e^{-x^2/2}(e^{I\sigma x} - \kappa_\sigma) \\
U(a,b)\psi(x) &= \mid a \mid^{-1/2} \psi(\frac{x-b}{a}) \\
d\mu_L &= \frac{1}{a^2} dadb \\
c(\psi) &= \int_G \frac{dadb}{a^2} \mid \langle \psi \mid U(b,a) \mid \psi \rangle \mid^2 \\
&= \int_{\mathbb{R}^2} \int_G d\xi d\zeta \frac{dadb}{a^2} \mid a \mid e^{ib(\xi-\zeta)} \bar{\hat{\psi}}(\hat{a\xi}) \hat{\psi}(\hat{a\zeta}) \hat{\psi}(\hat{\xi}) \bar{\hat{\psi}}(\hat{\zeta}) \\
&= 2\pi \mid \psi \mid^2 \int \frac{d\xi}{\mid \xi \mid} \mid \hat{\psi(\xi)} \mid^2 \\
c(\psi_{Morlet}) &= 2\pi \int \frac{d\xi}{\mid \xi \mid} \mid \hat{\psi_{Morlet}}(\xi) \mid^2
\end{aligned}$$

The Duflo-Moore operator for this is

$$\begin{aligned}
\hat{C}\hat{\psi}(\xi) &= (\frac{2\pi}{\mid \xi \mid})^{1/2} \hat{\psi}(\xi) \\
C &= F\hat{C}F^{-1} \\
c(\psi) &= \mid C\psi \mid^2
\end{aligned}$$

4.14.6 Example (Dyadic Wavelets)

$$\begin{aligned}
\psi_{j,k} &= 2^{j/2}\psi(2^j x - k) \\
c(\psi) &=
\end{aligned}$$

4.14.7 Definition (Husimi Q Function) *For a possibly mixed state ρ, form the quasi-probability density function $Q_\rho(x) = \langle x \mid \rho \mid x \rangle$. In particular for $\rho =\mid \psi\rangle\langle\psi \mid$ this is $\mid \langle x \mid\mid \psi \rangle \mid^2$. It is not probability because the coherent states pure at z_1 and z_2 are not orthogonal so not mutually exclusive.*

4.14.8 Definition (Monge distance) *$D_M(\rho_1, \rho_2)$ is defined to be the Monge distance for the associated Q-functions.*

4.14.9 Example (Harmonic Oscillator) *The distance between $\rho_i =\mid z_i\rangle\langle z_i \mid$ is*

4.14.10 Lemma *This can be approximated by taking the analog for when Q_1 and Q_2 are supported at N,M points respectively rather than being everywhere. This then turns into the well studying linear programming problem of optimal transport say from N factories with specified supply to M stores with specified demands.* `https://arxiv.org/pdf/quant-ph/9711011.pdf`

Chapter 5

Critical Exponents

5.1 Correlation Length

$$G(x) = \langle s(x)s(0)\rangle$$

in the magnetized phase where $\langle s(x)\rangle \neq 0$

Away from the critical point, $G(x)$ will have scaling behavior

$$G(x) \approx \exp(- \mid x \mid /\xi)$$

5.2 Diverging Correlation Length

This correlation length ξ diverges at the critical point.

$$\begin{aligned} t &= \frac{T - T_c}{T_c} \\ \xi &\approx \mid t \mid^{-\nu} \end{aligned}$$

5.3 Anomalous Dimension

Exactly at $t = 0$, the exponential turns into a power law which goes like

$$G(x) \approx \frac{1}{\mid x \mid^{d-2+\eta}}$$

In $d = 2$ then $h, \bar{h}$ weights of this primary field

5.4 α

The specific heat at fixed $H = 0$ diverges like

$$C_{H=0} \quad = \quad | \, t \, |^{-\alpha}$$

5.5 β

The magnetization tends to zero as

$$M \quad \approx \quad | \, t \, |^{\beta}$$

to calculate this one, put an infinitesimal magnetic field so you can put the system into one of the two selection sectors and then track that sector's behavior as it approaches the critical temperature.

5.6 δ

Keeping $t = 0$, fixed and instead varying the magnetic field gives.

$$M \quad \approx \quad H^{1/\delta}$$

5.7 γ

The final critical exponent is the divergence of the magnetic susceptibility

$$\chi \quad \approx \quad | \, t \, |^{-\gamma}$$

Chapter 6

Quantum Chaos

6.0.1 Definition ($K(q,q',t)$) *In $L^2(\mathbb{R}^n)$ with Hamiltonian $\hat{H}$, eigenstates ϕ_k with eigenvalues E_k*

$$\begin{aligned} i\hbar\partial_t K(q,q',t) &= \hat{H}(q,-i\hbar\nabla)K(q,q',t) \\ K(q,q',0) &= \delta(q-q') \\ K(q,q',t) &= \sum \phi_k^*(q')\phi_k(q)e^{-iE_kt/\hbar} \end{aligned}$$

6.0.2 Definition ($G(q,q',E)$)

$$\begin{aligned} G(q,q',E+i\epsilon) &= \frac{1}{i\hbar}\int_0^\infty dt e^{i(E+i\epsilon)t/\hbar}K(q,q',t) \\ &= \sum \frac{\phi_k^*(q')\phi_k(q)}{E-E_k+i\epsilon} \\ \operatorname{tr} G(q,q',E+i\epsilon) &\equiv \int d^nq G(q,q,E+i\epsilon) = \sum_k \frac{1}{E-E_k+i\epsilon} \end{aligned}$$

6.0.3 Lemma (Density of states)

$$\rho(E) = -\lim_{\epsilon\to 0}\frac{1}{\pi}\Im \operatorname{tr} G(q,q',E+i\epsilon)$$

6.0.4 Theorem (Gutzwiller Trace Formula) *The semiclassical formula for the propogator K (Van Vleck propagator) gives*

$$K_{sc}(q,q',t) = \sum_\gamma \frac{1}{(2\pi i\hbar)^{n/2}} \mid \det(-\partial_q\partial_{q'}R_\gamma) \mid^{1/2} e^{iR_\gamma/\hbar - i\pi i i_\gamma/2}$$

as a sum of γ paths that go from q' to q in time t. $R_\gamma(q,q',t)$ is Hamilton's principal function and i_γ is Maslov index. Plug these in to above and get formula for density of states as a sum over orbits.

Chapter 7

Quantum Gases

7.1 Fermi Gas Generalities

Let there be a countable number of energy levels E_n for the single particle problem, put together as $\vec{E}$.

$$\begin{aligned} Z(\lambda,\beta,\hbar) &\equiv \sum_{\alpha\in 0,1^{\mathbb{N}}} \lambda^{|\alpha|} e^{-\beta\alpha\cdot\vec{E}} \\ &= \prod_{n=0}^{\infty}(1+\lambda e^{-\beta E_n}) \\ E_n &= \hbar\omega(n+1/2) \\ Z(\lambda,\beta,\hbar) &= (-e^{-\hbar\omega\beta/2}\lambda; e^{-\hbar\omega\beta})_\infty \\ &= Li_{2,e^{-\hbar\omega\beta}}(-e^{-\hbar\omega\beta/2}\lambda) \end{aligned}$$

in terms of the quantum dilogarithm.
If we wish to use the pentagon relation

7.1.1 Lemma *If $uv = qvu$, then $\phi(x) = (x,q)_\infty$ satisfies Faddeev-Volkov identity*

$$\phi(v)\phi(u) = \phi(u)\phi(-vu)\phi(v)$$

$$\begin{aligned} u &= -e^{-\hbar\omega\beta/2}\lambda_1 \\ v &= -e^{-\hbar\omega\beta/2}\lambda_2 \\ uv = q\lambda_1\lambda_2 = qvu &\implies q\lambda_1\lambda_2 = q^2\lambda_2\lambda_1 \end{aligned}$$

which would say the chemical potentials $\mu_{1,2}$ form a canonical noncommuting pair.

7.2 Euler-Maclaurin

To justify some of the changes of sums to integrals we need the following theorem.

7.2.1 Theorem

$$\begin{aligned}\sum_{i=m}^{n} f(i) &= \int_m^n f(x)dx + \frac{f(n)+f(m)}{2} \\ &+ \sum_{k=1}^{p/2} \frac{B_{2k}}{(2k)!}(f^{(2k-1)}(n) - f^{(2k-1)}(m)) + R_p\end{aligned}$$

where p is some chosen approximation cutoff so sum is taken up to the floor of $p/2$. B_{2k} are Bernoulli numbers and R_p is the small remaining error.

$$\mid R \mid \leq \frac{2\zeta(p)}{(2\pi)^p}\int_m^n \mid f^{(p)}(x) \mid dx$$

7.3 Fermi Gas Nonrelativistic

7.3.1 $\mathbb{R}^3$

Cube of side length L then

$$\begin{aligned}N &\approx (2s+1)\times\frac{1}{8}\frac{4\pi}{3}n_F^3 = \frac{2s+1}{2}\frac{\pi}{3}n_F^3 \\ s &= 1/2 \\ n_F &= (3N/\pi)^{1/3} \\ \epsilon_F &= \frac{\hbar^2}{2m}\frac{\pi^2 n_F^2}{L^2} \\ U &= 2\sum_{n\leq n_F}\epsilon_n \approx 2\frac{1}{8}4\pi\int_0^{n_F} dn n^2\epsilon_n \\ &= \frac{\pi^3}{2m}\frac{\hbar^2}{L^2}\int_0^{n_F} n^4 \\ &= \frac{3}{5}N\epsilon_F\end{aligned}$$

$$\begin{aligned}N(\leq\epsilon) &= \frac{V}{3\pi^2}(\frac{2m}{\hbar^2})^{3/2}\epsilon^{3/2} \\ dN(\leq\epsilon)/d\epsilon &= D(\epsilon) = \frac{3N(\leq\epsilon)}{2\epsilon} \\ &= \frac{V}{2\pi^2}(\frac{2m}{\hbar^2})^{3/2}\epsilon^{1/2}\end{aligned}$$

To heat a gas from 0 to τ we get a change in U

$$\begin{aligned} U(\tau) &= \int_0^\infty \epsilon D(\epsilon) f(\epsilon,\tau) \\ U(0) &= \int_0^{\epsilon_F} \epsilon D(\epsilon) \\ \frac{d}{d\tau}\Delta U &= \int_0^\infty (\epsilon-\epsilon_F)\frac{df}{d\tau} D(\epsilon) \end{aligned}$$

For τ/ϵ_F small enough $df/d\tau$ is highly peaked (dipped actually) near ϵ_F.

$$\begin{aligned} C &\approx D(\epsilon_F)\int_0^\infty (\epsilon-\epsilon_F) df/d\tau \\ &= D(\epsilon_F)\int_0^\infty (\epsilon-\epsilon_F)\frac{d}{d\tau}(\frac{1}{e^{\beta(\epsilon-\mu)}+1}) \\ &= D(\epsilon_F)\int_0^\infty (\epsilon-\epsilon_F)(\frac{e^{\beta(\epsilon-\mu)}(\epsilon-\mu)(-1/\tau^2)}{(e^{\beta(\epsilon-\mu)}+1)^2}) \\ &\approx \tau D(\epsilon_F)\int_{-\epsilon_F/\tau}^\infty \frac{x^2 e^x}{(e^x+1)^2} \\ &\approx \frac{\pi^2}{3} D(\epsilon_F)\tau \\ &= \frac{\pi^2}{2} N\frac{\tau}{\epsilon_F} \end{aligned}$$

where we have done the $\mu \approx \epsilon_F$ approximation and a $-\epsilon_F/\tau \approx -\infty$. In fact to be more accurate we could say that $\mu = \epsilon_F(1 - \frac{\pi^2}{12}\frac{\tau^2}{\epsilon_F^2} + O(\frac{\tau^3}{\epsilon_F^3}))$

Sommerfeld Theory of Conduction

First let us review the Lorentz oscillator model. In this one there is a electron bound to a infinitely massive nucleus like a spring.

$$\begin{aligned} m\frac{d^2z}{dt^2} &= -k_{spr}z + F_{EM} - m\gamma\frac{dz}{dt} \\ F_{EM} &= eE_{max}\cos(kx-\omega t+\phi) \end{aligned}$$

Complexifying gives

$$
\begin{aligned}
m\ddot{z} + m\gamma\dot{z} + m\omega_0^2 z &= eE_{max}e^{i(kx+\phi)}e^{-i\omega t} \\
z &= z_0(x)e^{-i\omega t} \\
-m\omega^2 z_0(x) - i\omega m\gamma z_0(x) + m\omega_0^2 z_0(x) &= eE_{max}e^{i(kx+\phi)} \\
z_0(x) &= \frac{eE_{max}e^{i(kx+\phi)}}{m\omega_0^2 - m\omega^2 - im\omega\gamma} \\
P_{\mathbb{C}} &= ez(x,t) = \frac{e^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}E_{max}e^{i(kx-\omega t+\phi)} \\
\alpha &= \frac{e^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega} \\
\chi &= n_0\alpha = \frac{ne^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega} \\
P &= Re[ez(x,t)] = \frac{e^2}{m}\frac{E_0(x)}{\sqrt{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}}\cos(\omega t - \varphi) \\
\tan\varphi &= \frac{\gamma\omega}{\omega_0^2 - \omega^2} \\
J_{\mathbb{C}} &= ne\dot{z} = \frac{ne^2/m}{\omega_0^2 - \omega^2 - i\omega\gamma}E_{max}e^{i(kx+\phi)}(-i\omega)e^{-i\omega t} \\
&= \frac{ne^2/m}{-i\frac{\omega_0^2-\omega^2}{\omega} + \gamma}E(x,t) \\
&= \frac{ne^2/m\gamma}{-i\frac{\omega_0^2-\omega^2}{\omega\gamma} + 1}E(x,t)
\end{aligned}
$$

Now let us review the Drude model.

Take an ensemble of particles which collide on average every τ in time. Then $(1 - \frac{dt}{\tau})$ will be accelerating by the electric field and the rest will contribute 0 in expectation to the momentum.

$$
\begin{aligned}
\langle p(t+dt)\rangle &= (1 - \frac{dt}{\tau})(\langle p(t)\rangle + qEdt) \\
\frac{d}{dt}\langle p(t)\rangle &= qE - \frac{\langle p(t)\rangle}{\tau} \\
\langle p(t)\rangle &= q\tau E(1 - e^{-t/\tau}) + \langle p(0)\rangle e^{-t/\tau}
\end{aligned}
$$

Taking the steady state time independent solution gives a current density

$$
\begin{aligned}
\langle p\rangle &= q\tau E \\
J &= nq\langle v\rangle = \frac{nq^2\tau}{m}E
\end{aligned}
$$

You can also take time dependent solutions with $E(t) = Re[E_0(x)e^{-i\omega t}]$ in which case we are back to the previous differential equation but with $\omega_0 = 0$ and $\frac{1}{\tau} = \gamma$

This turns the current density into

$$\begin{aligned} J_{\mathbb{C}} &= \frac{ne^2\tau/m}{-i\frac{0^2-\omega^2}{\omega\gamma}+1}E(x,t) \\ &= \frac{ne^2\tau/m}{i\omega\tau+1}E(x,t) \\ &= \frac{\sigma_0}{1+i\omega\tau}E(x,t) \end{aligned}$$

7.3.2 Fermi Liquid

Suppose there is a dictionary between all operators $c_i(p)$ and $c_i^\dagger(p)$ with new operators that destroy and create corresponding quasiparticles.

$$G(\omega,p) \approx \frac{Z}{\omega+\mu-\epsilon(p)}$$

7.3.1 Lemma (Fermi Golden Rule) *To first order in the perturbation H'*

$$\Gamma = \frac{2\pi}{\hbar} \mid \langle f \mid \epsilon H \mid i \rangle \mid^2 \rho$$

The order of limits in derivation here is that ϵ is taken to first order primarily $\epsilon^2 = 0$ and then a $t \to \infty$ limit.

7.3.3 Constraints on Dynamical Response

`https://arxiv.org/pdf/2003.10390.pdf`

7.3.2 Definition (f-sum) *Suppose a spatially uniform electric current and spatially uniform electric field*

$$j_\mu(\omega) = \sigma_\mu^\nu E_\nu(\omega)$$

Then assuming a Hamiltonian of the form $K + I$ where K is bilinear in particle creation and annihilation operators with usual nonrelativistic form and I is a density-density operator

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\sigma_\mu^\nu(\omega) = \delta_\mu^\nu \frac{\rho}{2m} C(e,\hbar)$$

where ρ is the electron density and m is the electron mass as used in K. $C(e,\hbar)$ is some dimensionality factor that disappears in $e = \hbar = 1$ units.

More generally for $H = K + I$ with other forms of K and I, the right hand side is modified but there is still an f-sum constraint.

7.3.3 Definition (Kohn) *The Drude weight D_μ^ν is the limit as $\omega to 0$ of $\omega Imag(\sigma_\mu^\nu)$. This implies an expansion*

$$\sigma_\mu^\nu(\omega) = \frac{i}{\omega + i\epsilon} D_\mu^\nu + \cdots$$

where the rest of the terms are regular at $\omega = 0$.

The Kohn formula then gives D_μ^ν at $\tau = k_B T = 0$ in terms of the curvature of $E_0(bc)$ as one twists the boundary condition.

7.3.4 General $\mathbb{R}^d$

Now change to d dimensions. Now we must first say what the analog for s is too. This is because we must replace $2s+1$ by dimension of the representation of $Spin(d)$ that we are using. Or we can take the perspective that $d < d_2$ and we are just confining the particles to a hyperplane slice so use $Spin(d_2)$. This is the usual perspective with $d_2 = 3$.

$$\begin{aligned}
N &\approx \dim V(\lambda_{d_2}) \times \frac{1}{2^d} \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} n_F^d \\
n_F^d &= \frac{2^d \Gamma(\frac{d}{2}+1) N}{\pi^{d/2} \dim V(\lambda_{d_2})} \\
n_F &= \frac{2}{\pi^{1/2}} *^d \sqrt{\frac{\Gamma(\frac{d}{2}+1) N}{\dim V(\lambda_{d_2})}} \\
\epsilon_F &= \frac{\hbar^2}{2m} \frac{\pi^2 n_F^2}{L^2} \\
U &= \dim V(\lambda_{d_2}) \sum_{n \le n_F} \epsilon_n \\
&\approx \dim V(\lambda_{d_2}) \frac{1}{2^d} \int_0^{n_F} dn \frac{d\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} n^{d-1} \epsilon_n \\
&= \dim V(\lambda_{d_2}) \frac{1}{2^d} \frac{d\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} \frac{\hbar^2 \pi^2}{2mL^2} \int_0^{n_F} dn n^{d+1} \\
&= \dim V(\lambda_{d_2}) \frac{1}{2^d} \frac{d\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} \frac{\hbar^2 \pi^2}{2mL^2} \frac{n_F^{d+2}}{d+2} \\
&= \frac{N\hbar^2\pi^2}{2mL^2} \frac{d}{d+2} n_F^2 \\
&= \frac{Nd}{d+2} \epsilon_F \\
&= \frac{2\hbar^2\pi}{mL^2} \frac{d}{d+2} (\frac{\Gamma(\frac{d}{2}+1)}{\dim V(\lambda_{d_2})})^{2/d} N^{(d+2)/d}
\end{aligned}$$

$$\begin{aligned}
N(\leq \epsilon) &= \dim V(\lambda_{d_2}) \times \frac{1}{2^d}\frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}(\frac{2mL^2\epsilon}{\hbar^2\pi^2})^{d/2} \\
\frac{dN(\leq \epsilon)}{d\epsilon} &= D(\epsilon) = \dim V(\lambda_{d_2}) \times \frac{1}{2^d}\frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}(\frac{2mL^2}{\hbar^2\pi^2})^{d/2}\frac{d}{2}\epsilon^{d/2-1} \\
&= \dim V(\lambda_{d_2}) \times \frac{m^{d/2}}{2^{d/2}\pi^{d/2}\hbar^d}\frac{1}{\Gamma(\frac{d}{2}+1)}\frac{d}{2}\epsilon^{d/2-1}V \\
&= \frac{1}{\epsilon}N(\leq \epsilon)\frac{d}{2}
\end{aligned}$$

$$\begin{aligned}
C &\approx D(\epsilon_F)\int_0^\infty (\epsilon-\epsilon_F)df/d\tau \\
&= D(\epsilon_F)\int_0^\infty (\epsilon-\epsilon_F)\frac{d}{d\tau}(\frac{1}{e^{\beta(\epsilon-\mu)}+1}) \\
&= D(\epsilon_F)\int_0^\infty (\epsilon-\epsilon_F)(\frac{e^{\beta(\epsilon-\mu)}(\epsilon-\mu)(-1/\tau^2)}{(e^{\beta(\epsilon-\mu)}+1)^2}) \\
&\approx \tau D(\epsilon_F)\int_{-\epsilon_F/\tau}^\infty \frac{x^2e^x}{(e^x+1)^2} \\
&\approx \frac{\pi^2}{3}D(\epsilon_F)\tau \\
&= \frac{\pi^2 d}{6}\frac{N(\epsilon_F)\tau}{\epsilon_F}
\end{aligned}$$

7.3.5 $Spin(d_2 = d)$

7.3.6 Hyperbolic Surface

7.3.4 Theorem

$$\begin{aligned}
g_2 &= e^{2\phi}g_1 \\
\Delta_{g_2} &= e^{-2\phi}\Delta_{g_1} + (n-2)e^{-2\phi}g_1^{ij}\frac{\partial\phi}{\partial x_j}\frac{\partial}{\partial x_i} \\
&= e^{-2\phi}\Delta_{g_1} + 0
\end{aligned}$$

Proof

$$\begin{aligned}
\Delta_{g_1} &= \star_{g_1} d \star_{g_1} d \\
&=
\end{aligned}$$

7.3.5 Theorem (Selberg-Trace Formula) *Let S be a compact hyperbolic surface and h a even test function satisfying some restrictions. Let $\{\lambda_n\}$ be the eigenvalues of the Laplacian and $\mu_n^2 = \lambda_n - \frac{1}{4}$ with μ_n having positive real or imaginary part. Let $\mathcal{G}$ be the set of closed geodesics and $\ell(\gamma)$ their lengths and $m(\gamma)$ their multiplicities.*

$$\begin{aligned}
\sum_{n=0}^{\infty} h(\mu_n) &= \frac{-\chi(S)}{2}\int_{-\infty}^{\infty} h(s)s\tanh\pi s ds + \sum_{\gamma\in\mathcal{G}} R_\gamma \hat{h}(\ell(\gamma)) \\
R_\gamma &= \frac{\ell(\gamma)}{m(\gamma)2\sinh\ell(\gamma)/2} \\
\hat{h}(s) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)e^{-ius}du
\end{aligned}$$

7.3.6 Example (Original) *If no jumps in $m(\gamma)$, keeping β and μ fixed as moving around in Teichmuller space. So no reason for N or U not to change since we are in grand canonical not microcanonical.*

$$\begin{aligned}
\frac{d}{d\tau}R_\gamma &= \frac{1}{m(\gamma)2\sinh\ell(\gamma)/2}\frac{d\ell(\gamma)}{d\tau} - \ell(\gamma)\frac{\cosh\ell(\gamma)/2}{m(\gamma)4\sinh^2\ell(\gamma)/2}\frac{d\ell(\gamma)}{d\tau} \\
&= \frac{1}{\ell(\gamma)}R_\gamma\frac{d\ell(\gamma)}{d\tau} - \frac{\coth\ell(\gamma)/2}{2}R_\gamma\frac{d\ell(\gamma)}{d\tau} \\
&= (\frac{1}{\ell(\gamma)} - \frac{\coth\ell(\gamma)/2}{2})R_\gamma\frac{d\ell(\gamma)}{d\tau}
\end{aligned}$$

$$
\begin{aligned}
i\frac{d}{dt}\psi &= \nabla^2\psi \\
E_n &= \lambda_n = \mu_n^2 + \frac{1}{4} \\
h_N(\mu_n) &= \frac{1}{e^{\beta(E(\mu_n)-\mu)}+1} \\
\sum_{n=0}^{\infty} h_N(\mu_n) &= N \\
\widehat{h_N}(s) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ius}}{e^{\beta(E(u)-\mu)}+1} \\
\frac{\partial N}{\partial \tau} &= \sum_{\gamma} \frac{\partial}{\partial \tau}\big(R_\gamma \hat{h}_N(\ell(\gamma))\big) \\
&= \sum_{\gamma} \frac{\partial}{\partial \tau}(R_\gamma)\hat{h}_N(\ell(\gamma)) + R_\gamma \hat{h}'_N(\ell(\gamma))\frac{\partial \ell(\gamma)}{\partial \tau} \\
&\stackrel{?}{=} \sum_{\gamma}(\frac{1}{\ell(\gamma)} - \frac{\coth \ell(\gamma)/2}{2} + \frac{h'_N}{h_N})R_\gamma \hat{h}_N(\ell(\gamma))\frac{d\ell(\gamma)}{d\tau} \\
h_U(\mu_n) &= \frac{E(\mu_n)}{e^{\beta(E(\mu_n)-\mu)}+1} \\
\sum_{n=0}^{\infty} h_U(\mu_n) &= U \\
\frac{\partial U}{\partial \tau} &= \sum_{\gamma} \frac{\partial}{\partial \tau}\big(R_\gamma \hat{h}_U(\ell(\gamma))\big) \\
&= \sum_{\gamma} \frac{\partial}{\partial \tau}(R_\gamma)\hat{h}_U(\ell(\gamma)) + R_\gamma \hat{h}'_U(\ell(\gamma))\frac{\partial \ell(\gamma)}{\partial \tau} \\
&\stackrel{?}{=} \sum_{\gamma}(\frac{1}{\ell(\gamma)} - \frac{\coth \ell(\gamma)/2}{2} + \frac{h'_U}{h_U})R_\gamma \hat{h}_U(\ell(\gamma))\frac{d\ell(\gamma)}{d\tau}
\end{aligned}
$$

In particular we may move around in Teichmuller space τ and watch these nontrivial function of the lengths vary.

When

$$
\begin{aligned}
u &= \pm\frac{i}{\sqrt{\beta}}\sqrt{\beta - 4\beta\mu + 4(\pi i + 2\pi i C)} \\
C &\in \mathbb{Z}
\end{aligned}
$$

the denominator of $\hat{h}_N(s)$ integral vanishes. So use that do to that integral via contour methods.

7.4 Fermi Gas Relativistic

7.4.1 $\mathbb{R}^3$

Cube of side length L then

$$\begin{aligned}
N &\approx (2s+1) \times \frac{1}{8}\frac{4\pi}{3} n_F^3 = \frac{2s+1}{2}\frac{\pi}{3} n_F^3 \\
s &= 1/2 \\
n_F &= (3N/\pi)^{1/3} \\
\epsilon_F &= \hbar \frac{\pi n_F}{L} c \\
U &= 2 \sum_{n \le n_F} \epsilon_n \approx 2\frac{1}{8} 4\pi \int_0^{n_F} dn n^2 \hbar c \frac{\pi n}{L} \\
&= \frac{\hbar c \pi^2}{L} \int_0^{n_F} dn n^3 \\
&= \frac{\hbar c \pi^2}{L} \frac{n_F^4}{4} \\
&= \frac{3}{4} N \epsilon_F
\end{aligned}$$

$$\begin{aligned}
N(\le \epsilon) &= \\
dN(\le \epsilon)/d\epsilon &=
\end{aligned}$$

To heat a gas from 0 to τ we get a change in U

$$\begin{aligned}
U(\tau) &= \\
U(0) &= \\
\frac{d}{d\tau}\Delta U &=
\end{aligned}$$

For τ/ϵ_F small enough $df/d\tau$ is highly peaked (dipped actually) near ϵ_F.

$$C \approx$$

If we take the exact energy expression instead of the massless limit.

$$
\begin{aligned}
E(k) &= \sqrt{k^2c^2 + m^2c^4} \\
\epsilon(\beta = \infty) &= \frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_F} E(k)k^2dk \\
&= \frac{\epsilon_0}{8}((2x^3 + x)(1 + x^2)^{1/2} - \sinh^{-1}(x)) \\
\epsilon_0 &\equiv \frac{m^4c^5}{\pi^2\hbar^3} \\
x &\equiv \frac{k_F}{mc} \\
p &= \frac{\epsilon_0}{24}((2x^3 - 3x)(1 + x^2)^{1/2} + 3\sinh^{-1}(x))
\end{aligned}
$$

So in the nonrelativistic limit of $x << 1$ we get p is proportional to $\epsilon^{5/3}$. But 4/3 in the relativistic limit.

7.4.2 $\mathbb{R}^d$

7.5 Fermi Gas General Metric

7.6 Bose Gas Nonrelativistic

7.6.1 $\mathbb{R}^3$

$$
\begin{aligned}
f(\epsilon, \mu, \tau) &= \frac{1}{e^{\beta(\epsilon-\mu)} - 1} \\
f(0, \mu, \tau) &= \frac{1}{e^{-\beta\mu} - 1} \\
N &= \lim_{\tau\to 0} f(0, \mu, \tau) \approx \frac{-\tau}{\mu} \\
\mu &\approx \frac{-\tau}{N} \\
\lambda &= e^{\mu\beta}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}(\epsilon) &= \frac{V}{4\pi^2}(\frac{2M}{\hbar^2})^{3/2}\epsilon^{1/2} \\
N &= N_0(\tau) + \int_0^\infty \mathcal{D}(\epsilon)f(\epsilon, \mu, \tau) \\
N_0 &= \frac{1}{\lambda^{-1} - 1} \\
N_e &= \frac{V}{4\pi^2}(\frac{2M}{\hbar^2})^{3/2}\int_0^\infty \frac{x^{1/2}}{\lambda^{-1}e^x - 1}
\end{aligned}
$$

Start at $\mu = 0$.

$$\begin{aligned}
\int_0^\infty \frac{x^{1/2}}{e^x-1} = \frac{\zeta(3/2)}{2}\pi^{1/2} &\approx 1.306\pi^{1/2} = C_{\mu=0}\pi^{1/2} \\
N_e/N &= 2C_{\mu=0}\frac{n_Q}{n} \\
n_Q &\equiv (\frac{M\tau}{2\pi h^2})^{3/2} \\
N_e/N \approx 1 &\implies \tau_E = \frac{2\pi\hbar^2}{M}(\frac{N}{2C_{\mu=0}V})^{2/3} \\
\frac{N_e}{N} &\approx (\frac{\tau}{\tau_E})^{3/2}
\end{aligned}$$

7.6.2 $\mathbb{R}^d$

7.7 Bose Gas Relativistic

7.7.1 $\mathbb{R}^3$

7.7.2 $\mathbb{R}^d$

7.8 Bose Gas General Metric

7.9 Billiard

7.9.1 Definition (Quantum Unique Ergodic) *For Ω a regular subset of $\mathbb{R}^{d\geq 2}$. Bounded sequences of functions of $L^2(\mathbb{R}^n)$ give defect measures on $S^*\mathbb{R}^n$ as limit points called defect measures. Give a linear operator H on $L^2(\bar{\Omega})$, then take the defect measures of $1_\Omega h_\lambda$ for the eigenstates. It is QUE if this is a singleton with the uniform measure on $S^*\Omega$. S^*M is cosphere bundle.*

https://arxiv.org/pdf/1603.00597.pdf

7.9.2 Theorem (Gutzwiller Trace Formula)

$$\begin{aligned}
D &= (-1)^{d+1}\det\begin{pmatrix} \frac{dS}{dr_2,r} & \frac{dS}{dr_2,E} \\ \frac{dS}{dE,r} & \frac{dS}{dE,E} \end{pmatrix} \\
G_{scl}(r,r_2,E) &= \frac{-i}{\hbar}(2\pi i\hbar)^{-(d+1)/2}\sum \mid D \mid^{1/2}\exp(\frac{i}{\hbar}S(r,r_2,E) - i\mu\frac{\pi}{2}) \\
\partial g_{scl}(E) &= \frac{1}{\hbar\pi}\sum_{po}\frac{T_{po}}{\sqrt{\mid \det M - I \mid}}\cos(\frac{S_{po}}{\hbar} - \sigma_{po}\frac{\pi}{2}) \\
\partial N_{scl}(E) &= \int_0^E \partial g_{scl} = \frac{1}{\pi}\sum_{po}\frac{1}{n\sqrt{\mid \det M - I \mid}}\cos(\frac{S_{po}}{\hbar} - \sigma_{po}\frac{\pi}{2}) \\
\partial g(E) &= \sum_{po} e^{-\gamma T_{po}/2\hbar^2}\Im A_{po}(E)\exp(\frac{i}{\hbar}S_{po}(E) + i\phi)
\end{aligned}$$

Proof

$$\begin{aligned}
\langle r \mid\mid n(t)\rangle &= \langle r \mid U(t,t_2) \mid n(t_2)\rangle \\
&= \int dr_2 \langle r \mid U(t,t_2) \mid r_2\rangle\langle r_2 \mid\mid n(t_2)\rangle \\
&= \int dr_2 K(r,r_2,t,t_2)\langle r_2 \mid\mid n(t_2)\rangle \\
K(r,r_2,t-t_2) &= \sum_n \psi_n^\dagger(r_2)\psi_n(r)e^{-i/\hbar E_n(t-t_2)} \\
(-i\hbar\frac{\partial}{\partial t}+H)K &= -i\hbar\delta(r-r_2) \\
G(r,r_2,E) &= -\frac{i}{\hbar}\lim\int d(t-t_2)K(r,r_2,t-t_2)e^{i/\hbar(E+i\epsilon)(t-t_2)} \\
&= \sum\sum_n \psi_n^\dagger(r_2)\psi_n(r)\frac{1}{E-E_n} \\
\operatorname{tr} G(r,r_2,E+i\epsilon) &= \int d^3r G(r,r,E+i\epsilon) = \sum\frac{1}{E-E_n+i\epsilon} \\
g(E) &= \lim\frac{-1}{\pi}\Im\operatorname{tr} G(r,r_2,E+i\epsilon)
\end{aligned}$$

7.9.3 Theorem *Let X be a infinite area hyperbolic surface of finite genus with ends being flares or cusps. The Δ_X has absolutely continuous spectrum $[\frac{1}{4},\infty)$ and finitely many discrete eigenvalues in $(0,\frac{1}{4})$. Define the resonance set as the poles of the analytic continuation of $R_X(s)=(\Delta-s(1-s))^{-1}$ from $Res>1/2$*

7.9.4 Definition (Selberg Zeta Function)

$$Z_X(s) \equiv \prod_{\ell\in L_X}\prod_{k=0}^{\infty}(1-e^{(s+k)\ell}\ )$$

which converges uniformly on compact sets for $Res \geq 1$. Then continue to $\mathbb{C}$. It defines a divisor on $\mathbb{C}$ which has $\sum_{R_X}+m+\sum_{top}+\sum_{poles}-?$. When X has no cusps, then Z_X is entire.

7.9.1 20170419 Topology Talk

$QD \simeq T^*\mathcal{M}_g$. g_t acts by $e^t\ \ e^{-t}$ on QD. Or more generally act on X,q by $GL(2,\mathbb{R})$. Then take orbit closure. Can project to $\mathcal{M}_g$ this gives "complexified" geodesics as an isometrically holomorphic immersion of $\mathbb{H}$. When closed orbit, get a closed Riemann surface, but typically not closed orbit and get dense in $\mathcal{M}_g$ instead.

7.9.5 Theorem *Do billiards on a polygon with rational multiples of π. Unfolds to a polygon with edges identified by translation.*

7.9.2 20171116 Semeyon

Let M be a domain in $\mathbb{R}^d$, free particle in this domain as $E\to\infty$. How does $\mid\psi\mid^2$ concentrate? Does it converge to uniform measure on M?

Instead study on compact hyperbolic to avoid the boundary condition hardness. New Laplacian for the quantum problem and geodesic flow replaces straight lines. Ask same questions about concentration and equidistribution.

7.9.6 Definition *For $a \in C^\infty(T^*M)$ make a pseudodifferential operator. In particular if a is a bump around specific point, localizing in both position and momentum. If polynomial dependence in fiber would have differential operator, but then couldn't make it a bump function.*

$$\begin{aligned} (-\Delta - E_j) \mid \psi_j\rangle &= 0 \\ h_j &= \frac{1}{\sqrt{E_j}} \\ h_j &\to 0 \\ (-h_j^2\Delta - 1) \mid \psi_j\rangle &= 0 \end{aligned}$$

Chapter 8

Band Theory

8.1 Generalities

8.1.1 Definition (Lattice) *A mathematical lattice in a locally compact topological group is a discrete subgroup such that the quotient has finite invariant measure. In particular you take the locally compact topological group to be $\mathbb{R}^n$ for the translations and then you get some $\mathbb{Z}^n \to \mathbb{R}^n$*

8.1.2 Definition (Fundamental Domain) *A lift from the quotient $\mathbb{R}^n/\Gamma$ back to $\mathbb{R}^n$ is a homeomorphism on some restriction of both sides.*

$$\begin{array}{ccc} V & \longrightarrow & \mathbb{R}^n \\ \simeq\uparrow & & \downarrow \\ U & \longrightarrow & \mathbb{R}^n/\Gamma \end{array}$$

That is U is the open cell in the torus that you get after you remove the branch cuts. Like removing -1 for the circle or a figure 8 on T^2.

8.1.3 Definition (Briillouin Zone) *Take the dual lattice inside the Pontryagin dual locally compact abelian topogical group. Then form it's fundamental domain.*

8.1.4 Definition (Bloch Variety) *https://www-fourier.ujf-grenoble.fr/~peters/Articles/pubBourbaki.pdf*

8.2 Dimension 1

8.2.1 Dirac Comb

Consider a periodic identification of a 1d space by Na

$$V(x) \quad = \quad -\alpha \sum_{j=0}^{N-1} \delta(x - ja)$$

Looking in the fundamental domain of $[0, a)$ and $[-a, 0)$ called I and II respectively

$$\begin{aligned}
\psi_I(x) &= A\sin(kx) + B\cos(kx) \\
k &\equiv \frac{\sqrt{2mE}}{\hbar} \\
\psi_{II}(x) &= e^{-iKa}(A\sin(k(x+a)) + B\cos(k(x+a))) \\
K &= \frac{2\pi n}{Na} \ \forall n \in \mathbb{Z}_N \\
\psi_I(0) = B &= \psi_{II}(0) = e^{-iKa}(A\sin ka + B\cos ka) \\
B(e^{iKa} - \cos ka) &= A\sin ka \\
\psi_I^{'}(0) &= Ak \\
\psi_{II}^{'}(0) &= e^{-iKa}(Ak\cos ka - Bk\sin ka) \\
Ak - e^{-iKa}(Ak\cos ka - Bk\sin ka) &= \frac{-2m\alpha}{\hbar^2}B \\
Ak(1 - e^{-iKa}\cos ka) &= (-\frac{2m\alpha}{\hbar^2} - e^{-iKa}k\sin ka)B \\
k(1 - e^{-iKa}\cos ka) &= (-\frac{2m\alpha}{\hbar^2} - e^{-iKa}k\sin ka)\frac{\sin ka}{e^{iKa} - \cos ka} \\
ke^{iKa}(1 - e^{-iKa}\cos ka)^2 &= -\frac{2m\alpha\sin ka}{\hbar^2} - e^{-iKa}k\sin^2 ka \\
ke^{iKa} - 2k\cos ka + ke^{-iKa}\cos^2 ka &= -\frac{2m\alpha\sin ka}{\hbar^2} - e^{-iKa}k\sin^2 ka \\
ke^{iKa} - 2k\cos ka &= -\frac{2m\alpha\sin ka}{\hbar^2} - e^{-iKa}k \\
2\cos Ka &= 2\cos ka - \frac{2m\alpha}{\hbar^2 k}\sin ka
\end{aligned}$$

Now that you have the relation between K and k to provide a consistent patching at 0, you can build up the wavefunction everywhere

$$\psi_{Mth}(x) = e^{-iK(M-1)a}(A\sin(k(x+(M-1)a)) + B\cos(k(x+(M-1)a)))$$

where we've contiued the ordering to define the Mth fundamental zone as before. We had made the entire system periodic with Na as well so that puts the periodicity on M and remembering that x is the coordinate on the universal cover.

So for every K in this discrete set, we have a fiber which is the Hilbert space with basis given by the k which satisfy the constraint. This also gives a spectrum of energy eigenvalues for every K. If you collapse that K information, you get the picture of just the energies without the quasimomentum. You can tell things like insulator, conductor and semiconductor just from this.

8.2.2 Coloumb Comb

8.2.3 Toeplitz Theory

8.2.1 Definition (Toeplitz) *A Toeplitz matrix is a square matrix with constant down each (super and sub)diagonals. A Toeplitz operator is a infinite analog of such. It has symbol* $A(\theta) =$

$\sum_{n=-\infty}^{\infty} a_n u^n = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$. *The spectral distribution is the distribution of* $A(\Theta)$ *with* Θ *taken to be unifomly distributed on the interval* $[-\Pi, \Pi]$. *That is* u *is uniformly distributed on the unit circle if we don't want to pick a coordinate and branch cut for the circle.*

8.2.2 Theorem (Szego) *If only finitely many of the* a_n *are nonzero in a Toeplitz operator, then for all bounded continuous test functions* $f\ \mathbb{C} \to \mathbb{R}$

$$\lim_{N\to\infty} \int f(t) F_N(dt) \quad = \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} f(A(\theta)) d\theta$$

In fact, you can weaken this to the sequence a_i *being in* $\ell^1(\mathbb{Z})$ *instead.*

`http://galton.uchicago.edu/~lalley/Courses/386/Wigner.pdf`

8.3 Dimension 2

8.4 Dimension 3

8.5 Generalities Again

8.5.1 Definition (Plancherel Measure) *A measure on the set* G^* *of irreducible unitary representations of a locally compact group* G. *It is based on how the regular representation breaks up. In the case of LCA then this is proportional to the Haar measure of the Pontryagin dual like the* 2π*'s that show up in the Fourier transform. In the case of semisimple Lie groups it is supported on the tempered representations. So some unitary reps might not show up.* ***https://en.wikipedia.org/wiki/Plancherel_theorem_for_spherical_functions***

8.5.2 Theorem (`http://math.ucr.edu/home/baez/week274.html` Does for general *G*) *Let* G *be a locally compact, Hausdorff, second countable topological group. In this case we have* G^* *the unitary dual of* G *which in the abelian case is the Pontryagin dual. If we have a measure* μ *on* G^* *and a measurable field* H_x, *then we get*

$$H \quad = \quad \int^{\oplus} H_x \otimes R_x d\mu$$

where R_x *is the irrep of* G *given at* x. *This is the direct integral of Hilbert spaces. If* G^* *is a standard Borel space, then we can get all representations of* G *this way.*

8.5.3 Corollary (Original) *Let* $G = \mathbb{Z}^d$ *or* $\times_{i=1}^{d} \mathbb{Z}_{N_i}$ *be the lattice of translations. The full Hilbert space* $L^2(E)$ *comes as the* L^2 *sections of this Hilbert bundle over* G^*. *That is they can then be combined in the direct integral/sum.*

But it comes as an object of a 2-Hilbert space $Hilb^{G^*} \in 2 - Hilb$ *rather than as an object of the default 2-Hilbert space* $Hilb$.

This is what is assigned to the fundamental unit. This is because you first have to give one of those N_i *possibilities for the consistent boundary conditions in each direction. It is only after that, then you can define the Hilbert space. You have to pick the quasimomentum before you get a Hilbert space using only the fundamental cell. If you took the direct sum as a Hilbert space instead, that would be what was assigned to the full* E *rather than this finite piece.*

8.5.1 Integrations over BZ

`https://bandgap.io/blog/numerical_integration_bz/`

For simplicity, suppose no crossings so have labelled each state by specifying the quasimomentum k and a band index n with energy $E_n(k)$. Fix temperature and chemical potential through β and μ throughout.

Energy per unit volume of the crystal is:

$$\begin{aligned} \rho_E &= \frac{2}{(2\pi)^3} \sum_n \int_{BZ} E_n(k) f_{Fermi}(E_n(k)) d^3k \\ f_{Fermi}(E) &= \frac{1}{e^{\beta(E-\mu)}+1} \end{aligned}$$

Current density:

$$j_i = \frac{-2e}{(2\pi)^3} \sum_n \int_{BZ} \nabla_{k_i} E_n(k) f_{Fermi}(E_n(k)) d^3k$$

Energy current density:

$$j_{E,i} = \frac{2}{(2\pi)^3} \sum_n \int_{BZ} E_n(k) \nabla_{k_i} E_n(k) f_{Fermi}(E_n(k)) d^3k$$

Semiclassical conductivity in relaxation time approximation:

$$\sigma_{ij} = \frac{-2e^2}{(2\pi)^3 \hbar} \sum_n \tau_n(E_F) \int_{BZ} \nabla_{k_i} \nabla_{k_j} E_n(k) f_{Fermi}(E_n(k)) d^3k$$

where τ_n is the scattering time in the nth band at the Fermi level.

8.5.2 Bloch Variety

8.5.4 Definition (Bloch Variety) *The subset B of $(\mathbb{C}^*)^d \times \mathbb{C}$ such that all the points $(e^{ik_x}, e^{ik_y} e^{ik_z}, E(k_x, k_y, k_z))$ are on it. It is a complex analytic continuation of the physical locus $U(1)^d \times \mathbb{R}$.*

This assumes that this is algebraically defined rather than just smooth.

8.5.5 Definition (Fermi surface) *Let π be the projection to the complexified energy coordinate. $\pi^{-1}(E) \bigcap B$ defines the complexified Fermi surfaces. Really expected of dimension $d-1$ so surface comes from the usual $d = 3$.*

8.6 Semiconductor Physics

8.6.1 Definition (Semiconductor) *Has a band gap but $\frac{\Delta}{kT}$ is not large. This allows a thermal state to have an appreciable proportion in what would be unfilled at 0 temperature.*

8.6.2 Definition (Direct vs Indirect Bandgap) *The direct bandgap is when $\Delta = min(E_{con}(k) - E_{val}(k))$ while the indirect bandgap is the general case when $min(E_{con}(k)) - max(E_{val}(k'))$ so that momenta don't have to be equal. That is it is allowed for an electron from the maximal valence band state to go to the minimal valence band state even if it is not at the same quasimomentum by being kicked by an appropriate phonon if need be.*

8.6.3 Lemma (Varshni's Empirical) *As the temperature changes, the equilibrium crystal also changes. This effects the spacings and hence the bandgap.*

$$\Delta(T) = \Delta(0) - \frac{\alpha T^2}{T+\beta}$$

This combines a low temperature quadratic behavior with a high temperature linear behavior.

8.6.4 Example *For GaAs, according to empirical fits: $\Delta(0) \approx 1.519eV$, $\alpha \approx 5.405e-4eV/K$ and $\beta \approx 204K$. So*

$$\begin{aligned} \Delta(300K) &= 1.412eV \\ \frac{d\Delta(T)}{dT} &= \frac{-2\alpha T(T+\beta)+\alpha T^2}{(T+\beta)^2} = \frac{-\alpha T(T+2\beta)}{(\beta+T)^2} = \frac{(\Delta(T)-\Delta(0))}{T}\frac{T+2\beta}{T+\beta} \end{aligned}$$

8.6.5 Lemma (Bose-Einstein)

$$\begin{aligned} \Delta(T) - \Delta(0) &= \frac{-2a_B}{\exp \Theta_E/T - 1} \\ \Theta_D &= 4/3\Theta_E = C_B \frac{h}{k_B}(\frac{BV^{1/3}}{M})^{1/2} \end{aligned}$$

where Θ_E is the temperature of phonons interacting with the electrons. Θ_D is Debye temperature, C_B is dimensionless parameter, B is bulk modulus, V is mean atomic volume, M is mean atomic weight.

8.6.6 Definition (Solar Cell/External quantum efficiency) *Various wavelengths come in. If the energy is large enough to excite into the bandgap, get absorption. In more generality have absorption function of frequency which gets put together with the thickness of the device and input light frequency distribution to get what is called external quantum efficiency.*

Get nonabsorption losses if too low frequency incoming light and thermalization if too high energy. Excited into too high an energy level and decays rather than being carried out of the device. Expect a < 30 percent efficiency for a single device. So instead stack them so stuff not absorbed in one layer gets absorbed in the next. But problem with this is that this a circuit in series so current in each part is equal limiting to the worst one in the series.

8.6.1 Doping

8.6.7 Definition (Doping) *Extra or missing electron atom impurities replacing some of the lattice sites. For example, a Phosphorus replacing a Silicon providing an extra electron or Aluminum causing some holes.*

8.6.8 Lemma (Hume Rothery) *When does it substitute rather than intermetallic.*

8.6.2 Diodes

8.6.9 Definition (Diode) *PN junction. Some diffusion to create a P, a middle and N region. Enough to build a electric field to balance diffusion. Equilibrium governs this depletion width.*

$$\begin{aligned} I &= I_S * (e^{qV/kT} - 1) \\ V_T &\equiv kT/q \\ I &= I_S * (e^{V/V_T} - 1) \\ I(V \to -\infty) &= -I_S \\ I(V \to +\infty) &\propto I_S e^{qV/kT} \end{aligned}$$

Nonideal fixes include breakdown at some $V = -V_{brk}$

8.6.3 Transistors

8.7 Numerical Many Body QM

$$\begin{aligned} H &= T + V + U \\ T &= \sum_{i=1}^{N_e} \frac{-\hbar^2}{2m_i} \nabla_i^2 \\ V &= \sum_{i=1}^{N_e} V(r_i) \\ V(r_i) &= \sum_{j=1}^{N_n} \frac{-e * Z_j e}{4\pi\epsilon_0 \mid r_i - R_j \mid} \\ U &= \sum_{i \neq j} U(r_i, r_j) \\ U(r_i, r_j) &= \frac{e^2}{4\pi\epsilon_0 \mid r_i - r_j \mid} \end{aligned}$$

8.7.1 Hartree-Fock

$$
\begin{aligned}
\epsilon_i\psi_i(r) &= \frac{-\hbar^2}{2m}\nabla^2\psi_i(r) + V_{Ne}(r)\psi_i(r) + \sum_j \int dr_2 \frac{\mid \psi_j(r_2) \mid^2}{\mid r - r_2 \mid}\psi_i(r) \\
&+ \sum_j \delta_{\sigma_i\sigma_j}\int dr_2 \frac{\psi_j^*(r_2)\psi_i(r_2)}{\mid r - r_2 \mid}\psi_j(r) \\
\psi_i(r) &\equiv \sum C_{i\nu}\chi_\nu(r) \\
\sum_\nu \epsilon_i C_{i\nu}\chi_\nu &= \sum_\nu \frac{-\hbar^2}{2m}C_{i\nu}\nabla^2\chi_\nu(r) + \sum_\nu V_{Ne}(r)C_{i\nu}\chi_\nu(r) + \sum_{j\mu\nu}\int dr_2 \frac{\mid C_{j\mu}\chi_\mu(r_2) \mid^2}{\mid r - r_2 \mid}C_{i\nu}\chi_\nu(r) \\
&+ \sum_{j\mu\nu\rho}\delta_{\sigma_i\sigma_j}\int dr_2 \frac{C_{j\nu}^*\chi_\nu^*(r_2)C_{i\mu}\chi_\mu(r_2)}{\mid r - r_2 \mid}C_{j\rho}\chi_\rho(r) \\
\langle \xi^\mu \mid\mid \chi_\nu \rangle &\equiv \delta_\nu^\mu \\
\sum_\nu \epsilon_i C_{i\nu}\delta_\nu^\alpha &= \sum_\nu C_{i\nu}\left(\frac{-\hbar^2}{2m}\langle \xi^\alpha \mid\mid \nabla^2\chi_\nu\rangle + \langle \xi^\alpha \mid\mid V_{Ne}\chi_\nu\rangle\right) \\
&+ \sum_{j\mu\nu} \mid C_{j\mu} \mid^2 C_{i\nu}\langle \xi^\alpha \mid\mid \int dr_2 \frac{\mid \chi_\mu(r_2) \mid^2}{\mid r - r_2 \mid}\chi_\nu\rangle \\
&+ \sum_{j\mu\nu\rho} C_{j\nu}^* C_{i\mu} C_{j\rho}\delta_{\sigma_i\sigma_j}\langle \xi^\alpha \mid\mid \int \frac{\chi_\nu^*(r_2)\chi_\mu(r_2)}{\mid r - r_2 \mid}\chi_\rho(r)\rangle
\end{aligned}
$$

Define matrices

$$
\begin{aligned}
\mathbf{C}_i^\nu &\equiv C_{i\nu} \\
(\mathbf{C}^\dagger)_\mu^j &= C_{j\mu}^* \\
\epsilon_i^j &\equiv \epsilon_i\delta_i^j \\
\mathbf{D}_\nu^\alpha &\equiv \left(\frac{-\hbar^2}{2m}\langle \xi^\alpha \mid\mid \nabla^2\chi_\nu\rangle + \langle \xi^\alpha \mid\mid V_{Ne}\chi_\nu\rangle\right) \\
\mathbf{E}_{\beta\nu}^{\mu\alpha} &\equiv \delta_\beta^\mu\langle \xi^\alpha \mid\mid \int dr_2 \frac{\mid \chi_\mu(r_2) \mid^2}{\mid r - r_2 \mid}\chi_\nu\rangle = \delta_\beta^\mu\langle \xi^\alpha \otimes \chi_\mu \mid\mid \frac{1}{\mid r - r_2 \mid}\chi_\nu \otimes \chi_\mu\rangle \\
\mathbf{F}_{\rho\mu}^{\nu\alpha} &\equiv \langle \xi^\alpha \mid\mid \int \frac{\chi_\nu^*(r_2)\chi_\mu(r_2)}{\mid r - r_2 \mid}\chi_\rho(r)\rangle = \langle \xi^\alpha \otimes \chi_\nu \mid\mid \frac{1}{\mid r - r_2 \mid}\chi_\rho \otimes \chi_\mu\rangle \\
(\epsilon\mathbf{C}) &= (\mathbf{CD}) + (\mathrm{tr}_1(\mathbf{C}\otimes\mathbf{C})\mathbf{E}(\mathbf{C}^\dagger\otimes 1)) + \mathrm{tr}_1((\mathbf{C}\otimes\mathbf{C})\mathbf{F}(\mathbf{C}^\dagger\otimes 1)?_{\sigma\sigma}) \\
\sum_{\mu\nu} C_{i\nu}^* C_{i\mu}\int \chi_\nu^*\chi_\mu &= 1
\end{aligned}
$$

Lets say I is the number of i so the number of electrons. Say $\mathcal{V}$ is the number of χ that are decomposed into. This gives $2(I\times\mathcal{V}) + I$ equations from the Hartree-Fock and from the normalizations choice for $2(I\times\mathcal{V}) + 2 * I$ unknowns $\mathbf{C}$ and ϵ variables. Rephasing $C_{i\mu} \to e^{i\theta_i}C_{i\mu}$ takes care of this.

In computing E and F, we may use the Plancherel theorem to view this as an inner product in $L^2(\mathbb{R}^6_{k,k_2})$ instead.

8.7.1 Definition (Roothaan Equations)

$$\begin{aligned}
f(r_1) &= (-\frac{\hbar^2}{2m} + V_{Ne}) + \sum_{occ} 2J_j(r_1) - K_j(r_1) \\
J_j(r_1)\psi_i(r_1) &= (\int \psi_j^\dagger(r_2)\frac{1}{\mid r - r_2 \mid}\psi_j(r_2))\psi_i(r_1) \\
K_j(r_1)\psi_i(r_1) &= (\int \psi_j^\dagger(r_2)\frac{1}{\mid r - r_2 \mid}\psi_i(r_2))\psi_j(r_1) \\
F_{\mu\nu} &= \langle \phi_\mu \mid f(r_1) \mid \phi_\nu \rangle \\
S_{\mu\alpha} &= \langle \phi_\mu \mid\mid \phi_\alpha \rangle \\
\psi_l &= C_{\nu l}\phi_\mu \\
\epsilon_{lm} &= \epsilon_l \delta_{lm} \\
FC &= SC\epsilon
\end{aligned}$$

On a Quantum Computer

`https://arxiv.org/pdf/0905.0887.pdf`

Pick some of the vectors $\mid i\rangle$ to span the relevant subspace of the Hilbert space. Let $c_i^\dagger$ be the associated fermionic creation operators in the second quantized formalism.

Use Jordan-Wigner to turn fermionic operators into spin operators.

Now have the Hamiltonian/propogator in terms of spin operators that one knows how to implement on qubits.

8.7.2 Density Functional

Let $n(r) = N \int d^3r_2 \cdots d^3r_N \psi^*(r_1 \cdots r_N)\psi(r_1 \cdots r_N)$

Because of symmetry it didn't matter which one we kept as long as we integrated out all but one.

8.7.2 Theorem (Hohenberg-Kohn) *This relation $\psi \to n$ can be reversed for the ground state density. That is given $n_0(r)$, we may solve for ψ up to an overall phase. This also means that any observable expectation is a functional of n by $O(n_0) = \langle \psi(n_0) \mid O \mid \psi(n_0) \rangle$.*

The assumptions are requiring each ψ in concern must be a unique ground state of some $H = T + U + V$ for some V defined up to overall constant. Obvious problems like $\psi \to e^{i\theta(x)}\psi$ are forbidden because that takes you outside the domain of consideration. That is a ground state of $e^{i\theta}He^{-i\theta}$ but that is no longer of the required form when θ has complicated position dependence.

This is false if there is a magnetic field.

Proof

$$\begin{aligned} H &= T+U+V \\ H^a &= T+U+V^a \\ \langle\psi \mid H \mid \psi\rangle = E_g &< \langle\psi^a \mid H \mid \psi^a\rangle = \langle\psi^a \mid H^a + V - V^a \mid \psi^a\rangle \\ &= E_g^a + \int n(V-V^a) \\ \langle\psi^a \mid H^a \mid \psi^a\rangle = E_g^a &< \langle\psi \mid H^a \mid \psi\rangle = \langle\psi \mid H + V^a - V \mid \psi\rangle \\ &= E_g + \int n(V^a - V) \end{aligned}$$

So we seek to minimize

$$E(n(r)) = T(n(r)) + U(n(r)) + \int V(r)n(r)dvol$$

$T(n(r))$ and $U(n(r))$ don't depend on the locations of the nuclei so are universal for all materials.

Suppose the problem breaks up as N_e independent problems of a single electron moving in an effective potential with many nuclei and another term taking care of U.

$$\begin{aligned} (\frac{-\hbar^2}{2m_i}\nabla_i^2 + V_{eff}(r))\phi_i &= \epsilon_i\phi_i \\ n(r) &= \sum_i \mid \phi_i \mid^2 \\ V_{eff} &= V_{Ne} + \int \frac{e^2 n}{\mid r-r' \mid} d^3r' + V_{XC}[n_s] \end{aligned}$$

This can be solved iteratively first for the ϕ_i with a given V_{eff} then update V_{eff} to take care of the new $n(r)$.

8.7.3 GW approximation

$$\begin{aligned} G &= G_0 + G_0\Sigma G \\ G_0^{-1}GG^{-1} &= G_0^{-1}G_0G^{-1} + G_0^{-1}G_0\Sigma GG^{-1} \\ G_0^{-1} - G^{-1} &= \Sigma \end{aligned}$$

Chapter 9

Dimension 1-Part 1

9.1 Free Boson

Insert Ball and spring on a line model. Altland Simons

$$\begin{aligned} H &= \sum \frac{p_i^2}{2m} + \frac{K}{2}\sum (x_i - x_{i+1})^2 \\ -m\dot{\dot{x}}_l &= K(2x_l - x_{l-1} - x_{l+1}) \\ x_l &\equiv x_0 \cos kal \\ \omega_k^2 &= (2 - 2\cos ka)\frac{K}{m} = \frac{4K}{m}\sin^2\frac{ka}{2} \end{aligned}$$

For periodic boundary conditions with N particles we then get the quantization of k

$$\begin{aligned} \cos kaN &= \cos ka0 = 1 \\ kaN &= 2\pi n \\ k &= \frac{2\pi n}{aN} \\ k &\leftrightarrow k + \frac{2\pi}{a} \end{aligned}$$

That is an integer worth of solutions k and then mod out by an index N subgroup. We may take a fundamental domain for this with $n \in [0, N)$ integers, but in truth it is a quotient. Replacing the representative by changing $n \to n + N$ will shift $k \to k + \frac{2\pi}{a}$ which will do nothing to the x_l.

$$\begin{aligned} H &= \frac{1}{2m}\sum p_k p_{-k} + \frac{m}{2}\sum \omega_k^2 x_k x_{-k} \\ Q_k &= \frac{1}{\sqrt{N}}\sum_l e^{ikal} x_l \\ \Pi_k &= \frac{1}{\sqrt{N}}\sum_l e^{-ikal} p_l \\ H &= \frac{1}{2m}\sum_k \Pi_k \Pi_{-k} + m^2\omega_k^2 Q_k Q_{-k} \\ &= \sum \hbar\omega_k (a_k^\dagger a_k + \frac{1}{2}) \end{aligned}$$

If we remove the 1/2 and operate at an inverse temperature β in a noninteracting ∞'ly many body theory we get

$$\begin{aligned} \langle n_k \rangle &= \sum_0^\infty j_k e^{-\beta j_k \hbar\omega_k} / \sum_0^\infty e^{-\beta j_k \hbar\omega_k} \\ &= \sum_0^\infty -\frac{\partial}{\partial(\beta E)} e^{-\beta j_k \hbar\omega_k} / \sum_0^\infty e^{-\beta j_k \hbar\omega_k} \\ &= -\frac{\partial}{\partial\beta E}\sum_0^\infty e^{-\beta\hbar\omega_k j_k} / \sum_0^\infty e^{-\beta j_k \hbar\omega_k} \\ &= -\frac{\partial}{\partial\beta E}\frac{1}{1-e^{-\beta\hbar\omega_k}} / \sum_0^\infty e^{-\beta j_k \hbar\omega_k} \\ &= -\frac{\partial}{\partial\beta E}\frac{1}{1-e^{-\beta\hbar\omega_k}} / \frac{1}{1-e^{-\beta\hbar\omega_k}} \\ &= \frac{e^{\beta\hbar\omega_k}}{(e^{\beta\hbar\omega_k}-1)^2} / \frac{1}{1-e^{-\beta\hbar\omega_k}} \\ &= \frac{1}{e^{\beta\hbar\omega_k}-1} \end{aligned}$$

9.1.1 Continuum Limit

Send $N \to +\infty$, $aN \to L$ length and $al \to q$ position along the interval/circle.

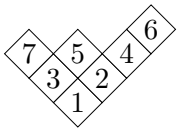
$$H = \int$$

9.2 Free Fermion

We will set up our fermionic Fock space starting with the vacuum vector given by the semi-infinite wedge product

$$| 0\rangle \quad = \quad \cdots \wedge e_{-5/2} \wedge e_{-3/2} \wedge e_{-1/2}$$

The basis vectors are given by Young Tableaux in Russian notation.



The downward sloping lines are the occupied spots and the upward sloping are unoccupied. The empty diagram gives the vacuum vector above.

$$\begin{aligned}
\psi_m \mid \cdots \wedge e_{i_2} \wedge e_{i_1}\rangle &= \mid \cdots \wedge \hat{e}_m \wedge \cdots e_{i_1}\rangle \\
&= 0 \;\; m \notin \{i_n\} \\
\psi_m^\dagger \mid \cdots \wedge e_{i_2} \wedge e_{i_1}\rangle &= \mid \cdots \wedge e_m \wedge \cdots e_{i_1}\rangle \\
&= 0 \;\; m \in \{i_n\}
\end{aligned}$$

This can be regarded as a Hilbert space with a Hamiltonian where each of the are above are eigenvectors with eigenvalues $E_0 \mid \lambda \mid$ For quantum statistical mechanics purposes introduce a parameter q.

$$\begin{aligned}
q &\equiv e^{-\beta E_0} \\
T \to 0 &\implies q \to 0 \\
T \to \infty &\implies q \to 1 \\
Z &= \sum q^{|\lambda|} = (q;q)_\infty = (1-q)(1-q^2)\cdots
\end{aligned}$$

This defines a canonical ensemble.

For an operator diagonal in this basis, we can take it's expectation value in this Gibb's state.

$$\langle f \rangle \quad = \quad (q;q)_\infty \sum_\lambda f(\lambda) q^{|\lambda|}$$

9.2.1 Definition (Schur Measure) *More generally consider a measure using auxiliary variables $x_1 \cdots$ and $y_1 \cdots$. Then define a partition function by*

$$\begin{aligned}
Z &= \sum s_\lambda(x) s_\lambda(y) \\
\mu(\lambda) &= \frac{1}{Z} s_\lambda(x) s_\lambda(y) \\
Z &= \prod (1 - x_i y_j)^{-1} \\
&= \exp(\sum \frac{1}{k} p_k(x) p_k(y))
\end{aligned}$$

by the Cauchy identity. The x and y variables can be permuted without changing the measure so the power sum variables are less stacky.

9.2.2 Corollary *If $p_1(x) = p_1(y) = \sqrt{\xi}$ and the rest are 0 we get the Poissonized Plancherel measure.*

9.3 Luttinger

$$\begin{aligned} H_0 &= \sum_k \xi_k c_k^\dagger c_k \\ \xi_k &= \frac{\mid k \mid^2}{2m} - \mu \approx v_F(k - k_F) \end{aligned}$$

Define $c_{k,R}$ and $c_{k,L}$

$$\begin{aligned} c_{k_R} &= \begin{cases} c_k & k \geq 0 \\ 0 & k < 0 \end{cases} \\ c_{k_L} &= \begin{cases} 0 & k \geq 0 \\ c_k & k < 0 \end{cases} \\ c_k &= c_{k,L} + c_{k,R} \end{aligned}$$

Plugging this in to the approximate H_0

$$\begin{aligned} H_{0,eff} &= v_F \sum_{k>0} k(c_{k,R}^\dagger c_{k,R} - c_{-k,L}^\dagger c_{-k,L}) - k_F v_F(N_R + N_L) \\ H_{0,eff} &= v_F \sum_{k>0} k(c_{k,R}^\dagger c_{k,R} - c_{-k,L}^\dagger c_{-k,L}) \end{aligned}$$

Suppose there are interactions of the form

$$H_{int} = \frac{1}{2L} \sum V(q) c_k^\dagger c_{k'}^\dagger c_{k'-q} c_{k+q}$$

Defining density operators

$$\begin{aligned}
\rho_R(q) &= \sum_{k>0} c_k^\dagger c_{k+q} \approx_{k>0} c_{k,R}^\dagger c_{k+q,R} \\
\rho_L(q) &= \sum_{k<0} c_k^\dagger c_{k+q} \approx_{k<0} c_{k,L}^\dagger c_{k+q,L} \\
\big[\rho_R(q), \rho_R(-q)\big] &= \sum_{k,k'>0} [c_k^\dagger c_{k+q}, c_{k'}^\dagger c_{k'-q}] \\
&\approx \frac{qL}{2\pi} \\
\big[\rho_L(q), \rho_L(-q)\big] &\approx \frac{-qL}{2\pi}
\end{aligned}$$

and the rest approximately commute.
In these variables

$$\begin{aligned}
H_{int,1} &= \frac{1}{2L}\sum_q (V(0) - V(2k_F))(\rho_R(q)\rho_L(-q) + \rho_L(q)\rho_R(-q)) \\
H_{int,2} &= \frac{1}{2L}\sum_q V(0)(\rho_R(q)\rho_R(-q) + \rho_L(q)\rho_L(-q)) \\
H_{int} &= \frac{1}{2L}\sum_q (V(0) - V(2k_F))(\rho_L(q) + \rho_R(q))(\rho_L(-q) + \rho_R(-q))
\end{aligned}$$

Changing variables again

$$\begin{aligned}
a_q &= \sqrt{\frac{2\pi}{qL}}\rho_R(q) \\
a_q^\dagger &= \sqrt{\frac{2\pi}{qL}}\rho_R(-q) \\
b_q &= \sqrt{\frac{2\pi}{qL}}\rho_L(-q) \\
b_q^\dagger &= \sqrt{\frac{2\pi}{qL}}\rho_L(q) \\
H_0 &= v_F \sum_{q>0} q(a_q^\dagger a_q + b_q^\dagger b_q) \\
H_{int} &= \frac{1}{2\pi}\sum_{q>0} qV_1(a_q^\dagger a_q + b_q^\dagger b_q + a_q b_q + b_q^\dagger a_q^\dagger) \\
&= \frac{1}{2\pi}\sum qV_1 \begin{pmatrix} a_q^\dagger & b_q \end{pmatrix} \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} a_q \\ b_q^\dagger \end{pmatrix} \\
H &= \sum E_q \gamma_q^\dagger \gamma_q \\
E_q &= q v_F \sqrt{1 + \frac{V_1}{\pi v_F}}
\end{aligned}$$

9.4 Charge Density Wave

9.5 Spin Density Wave

9.6 Su-Schrieffer-Heeger

Take polyacetylene where the bond lengths have been distorted by an amount u_n

$$H = -t\sum_{n=1}^{N}(1+u_n)(c_{n\sigma}^{\dagger}c_{n+1,\sigma}+h.c.)+\sum\frac{k_s}{2}(u_{n+1}-u_n)^2$$

If $u_n=(-1)^n\alpha$ is an alternating patern of single and double bonds

$$H = -t\sum_{n=1}^{N}(1+(-1)^n\alpha)(c_{n\sigma}^{\dagger}c_{n+1,\sigma}+h.c.)+\frac{Nk_s4\alpha^2}{2}$$

Supposing N is even and using the Bloch operators we get

$$\begin{aligned} H &= \sum_{n=1}^{N=2M}(vc_{2n-1\sigma}^{\dagger}c_{2n,\sigma}+wc_{2n,\sigma}^{\dagger}c_{2n-1,\sigma}+h.c.) \\ w &= \mid w \mid e^{i\phi} \end{aligned}$$

$$\begin{aligned} b_{n,\sigma,\sigma'} &\equiv \left(c_{2n-1,\sigma},c_{2n,\sigma'}\right) \\ b_{n,\sigma,\sigma'}^{\dagger} &\equiv \left(c_{2n-1,\sigma}^{\dagger},c_{2n,\sigma'}^{\dagger}\right) \\ \psi_{k,\sigma} &\equiv \sum_n e^{ik2an}b_{n,\sigma,\sigma} \\ \psi_{k,\sigma}^{\dagger} &\equiv \sum_n e^{-ik2an}b_{n,\sigma,\sigma}^{\dagger} \\ \psi_{k,\sigma}^{\dagger}\begin{pmatrix}h_{11}(k) & h_{12}(k)\\ h_{21}(k) & h_{22}(k)\end{pmatrix}\psi_{k,\sigma} &= \sum_n e^{ik2an}\psi_{k,\sigma}^{\dagger}\begin{pmatrix}h_{11}(k)c_{2n-1}+h_{12}(k)c_{2n}\\ h_{21}(k)c_{2n-1}+h_{22}(k)c_{2n}\end{pmatrix} \\ &= \sum_{n,n'}e^{ik2a(n-n')}\big(c_{2n'-1}^{\dagger}h_{11}(k)c_{2n-1}+c_{2n'-1}^{\dagger}h_{12}(k)c_{2n} \\ &+ c_{2n'}^{\dagger}h_{21}(k)c_{2n-1}+c_{2n'}^{\dagger}h_{22}(k)c_{2n}\big) \\ \sum_k\psi_k^{\dagger}\begin{pmatrix}h_{11}(k) & h_{12}(k)\\ h_{21}(k) & h_{22}(k)\end{pmatrix}\psi_k &= \sum_k\sum_{n,n'}e^{ik2a(n-n')}\big(c_{2n'-1}^{\dagger}h_{11}(k)c_{2n-1}+c_{2n'-1}^{\dagger}h_{12}(k)c_{2n} \\ &+ c_{2n'}^{\dagger}h_{21}(k)c_{2n-1}+c_{2n'}^{\dagger}h_{22}(k)c_{2n}\big) \end{aligned}$$

Setting $a = 1$ or alternatively replacing ka by k.

$$\begin{aligned}
\begin{pmatrix} h_{11}(k) & h_{12}(k) \\ h_{21}(k) & h_{22}(k) \end{pmatrix} &= h_x(k)\sigma_x + h_y(k)\sigma_y \\
h_x &= Re(v) + \mid w \mid \cos(k+\phi) \\
h_y &= -Im(v) + \mid w \mid \sin(k+\phi) \\
H(k)^2 &= h_x(k)^2 + h_y(k)^2 + h_x h_y[\sigma_x, \sigma_y]_+ \\
&= (h_x^2 + h_y^2) Id_2 = (\mid v \mid^2 + \mid w \mid^2 + 2 \mid v \mid\mid w \mid \cos(k + \phi_v + \phi_w)) Id_2 \\
E(k) &= \mid v + e^{-ik} w^\dagger \mid =
\end{aligned}$$

Returning to our previous case we have $v = 1 + \alpha$ and $w = 1 - \alpha$. Let us also restore a.

$$\begin{aligned}
E(k)^2 &= \mid 1 + \alpha + e^{-ika} - e^{-ika}\alpha \mid \\
\Delta_{min} &= E(\frac{\pm\pi}{a}) = 2 \mid \alpha \mid
\end{aligned}$$

As k goes around it's S^1, we get a loop in $h_x + ih_y$ space, but it has to stay away from $h = 0$ where the energy would go to 0 closing the gap. So all we have to do is count the winding number in $\pi_1(\mathbb{C}^*)$ via $\frac{1}{2\pi i}\int dk \frac{d \log h}{dk}$

Suppose we fill the system halfway for every value of α, then the ground state energy as it varies with α is a double well

9.6.1 Remark how?

$$V(\phi) = -A\phi^2 + B\phi^4$$

9.6.1 Solitons

Interfaces between the two regions mean that there has to have a zero energy state on the defect. Focusing on low energy states given by those with $k = \frac{\pi}{a} + q$ and then expanding in q to give $H = -iv_F\sigma_x\partial_x + m(x)\sigma_y$ which is a Dirac Hamiltonian. Where $m = 2dt$. So for the defect we have changing from +dt to -dt to change which dimerization is favored.

9.6.2 Index

Chiral symmetry $\{\sigma_z, H(k)\} = 0$ so that means that all eigenstates come in pairs with $\pm E$ energies. Except for 0 energy which you count to give a $\mathbb{Z}_2$ invariant.

Chapter 10

Dimension 1 - Spin Chains

10.1 Ising Chain

$$\begin{aligned} H &= -J\sum_{i=1}^{L}\sigma_i^z\sigma_{i+1}^z - h\sum\sigma_i^x \\ Z &= \mathrm{Tr}\, e^{-\beta H} \end{aligned}$$

In case the transverse field vanishes, we get $\sum_{\{\sigma\}} e^{\beta J\sum\sigma_i^z\sigma_{i+1}^z}$ which is the same result as the classical one-dimensional Ising model.

In case the interaction vanishes, we get $Z = [2cosh(\beta h)]^L$ which is L independent spins as it should be and all is right with the world.

When both nonzero, it maps to the 2D classical Ising model with $N_x = L$, $N_y = N$, $\beta_{cl}J_x = \frac{\beta}{N}J$ and $\beta_{cl}J_y = \gamma$ where $\gamma = \frac{-1}{2}\ln\tanh\frac{\beta h}{N}$.

Now that you have 2d classical Ising without external field, solve it with Onsager.

Corresponding CFT when β_{cl} is just right: $c = 1/2$ Spinless Fermion. The error can be surprisingly small even if you are not even close to the limit.

10.2 Ising+3 Spin XZX

`https://arxiv.org/pdf/1611.10247.pdf`

Rest useless because measuring EE after JW so wrong coproduct

$$
\begin{aligned}
H &= \lambda_1 \sum_{i=1}^{L} \sigma_i^z \sigma_{i+1}^z + \lambda_2 \sum \sigma^x \sigma^z \sigma^x + h \sum \sigma_i^x \\
&\to -g \sum (1 - 2c_n^\dagger c_n) - \lambda_1 \sum (c_n^\dagger c_{n+1} + c_n^\dagger c_{n+1}^\dagger + h.c.) \\
&- \lambda_2 \sum (c_{n-1}^\dagger c_{n+1} + c_{n+1} c_{n-1} + h.c.) \\
&= \sum_k \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} H_k \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} \\
H_k &= (h + \lambda_1 \cos k)\sigma^z + (\lambda_1 \sin k - \lambda_2 \sin 2k)\sigma^x \\
H_k(\lambda_2 = 0) &= (h + \lambda_1 \cos k)\sigma^z + (\lambda_1 \sin k)\sigma^x \\
E_k(1, \lambda_1/h \to \lambda_1, \lambda_2/h \to \lambda_2) &= \pm 2\sqrt{1 + \lambda_1^2 + \lambda_2^2 + 2\lambda_1(1 - \lambda_2)\cos k - 2\lambda_2 \cos 2k}
\end{aligned}
$$

h has disappeared. Fix that.

10.3 Majorana Chain

Hamiltonian:

$$
\begin{aligned}
\{c_k, c_l\} &= 2\delta_{kl} \\
H &= \frac{i}{2} v \sum_{j=1}^{N} c_{2j-1} c_{2j} + iw \sum_{j=1}^{N-1} c_{2j} c_{2j+1}
\end{aligned}
$$

Be careful because if you wrote this in terms of the Dirac fermions rather than the Majorana fermions you would see a number nonconservation term. This means that if you were to actually realize this you would have to put it on the boundary of a superconductor so that the electrons would have somewhere to go.

Ground states:

At $w = 0$

At $v = 0$

and

In the second there are two dangling Majorana's. This leaves a 2-fold almost degenerate system. The two together form a single Dirac fermion. The degeneracy is exact only after passing to the Thermodynamic limit so that the $e^{-\xi/L}$ gap disappears.

10.4 XXX

Hamiltonian:

$$H = \sum \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z + \text{bdry term}$$

Phase Diagram:
Uses $Y(\mathfrak{sl}_2)$

10.4.1 Schwinger Boson

Take two harmonic oscillators. The phase space is $\mathbb{R}^4$ thought of as $\mathbb{C}^2$. Fix the total energy so do a symplectic reduction by this $U(1)$ action $z_i \to e^{it} z_i$ This gives $\mathbb{C}^2//U(1) \simeq \mathbb{CP}^1$.

In fact: Take n harmonic oscillators. The phase space is $\mathbb{R}^{2n}$ thought of as $\mathbb{C}^n$. Fix the total energy so do a symplectic reduction by this $U(1)$ action $z_i \to e^{it} z_i$ This gives $\mathbb{C}^n//U(1) \simeq \mathbb{CP}^n$.

But the case with just two is what we need for Schwinger. We now quantize both sides. We get 2 harmonic oscillators on the boson side and a spin on the other.

This is the phenomenon of geometric quantization commutes with reduction.

10.4.1 Theorem (Borel-Weil-Bott)

The spin j Hilbert space is generated by holomorphic sections of the line bundle $\mathcal{O}(2j)$ which can be expressed as $z^0 \cdots z^{2j}$.

$$\begin{aligned} s_x &= -(1-z^2)\partial_z + 2jz \\ s_y &= -i(1+z^2)\partial_z - 2ijz \\ s_z &= -2z\partial_z - 2j \end{aligned}$$

10.4.2 Large Spin approximation-Original

$$\begin{aligned} s_{xm}s_{xn} + s_{ym}s_{yn} + s_{zm}s_{zn} &= 4j^2 - 4j(z_m - z_n)(\partial_{z_n} - \partial_{z_m}) - 2(z_m - z_n)^2 \partial_{z_m}\partial_{z_n} \\ &\approx 4j^2 - 4j(z_m - z_n)(\partial_{z_n} - \partial_{z_m}) + O(j^0) \\ s_{xm}s_{xn} + s_{ym}s_{yn} - s_{zm}s_{zn} &= -4j^2 - 4j(z_m + z_n)(\partial_{z_n} + \partial_{z_m}) - 2(z_m + z_n)^2 \partial_{z_m}\partial_{z_n} \\ &\approx -4j^2 - 4j(z_m + z_n)(\partial_{z_n} + \partial_{z_m}) + O(j^0) \end{aligned}$$

Dropping the order j^0 terms leaves Hamiltonians that are quadratic in the z and ∂_z variables. These are matched with bosonic creation and annihilation so we get a quadratic Hamiltonian that can be diagonalized by Fourier and Bogiliubov.

First the ferromagnetic

$$\begin{aligned} J\sum \vec{s}\cdot\vec{s} &\approx 4Jj^2N - 4jJ\sum (a_m^\dagger - a_{m+1}^\dagger)(a_{m+1} - a_m) \\ &= 4Jj^2N - 4jJ\sum_k \begin{pmatrix} ? & ? \end{pmatrix} \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} ? \\ ? \end{pmatrix} \end{aligned}$$

Now the antiferromagnet

$$\begin{aligned} J\sum \vec{s}\cdot\vec{s} - 2s_m^z s_{m+1}^z &\approx -4Jj^2N - 4jJ\sum (a_m^\dagger + a_{m+1}^\dagger)(a_{m+1} + a_m) \\ &= -4Jj^2N - 4jJ\sum_k \begin{pmatrix} ? & ? \end{pmatrix} \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} ? \\ ? \end{pmatrix} \end{aligned}$$

For the gap between this says that

$$\begin{aligned} \lim_{j\to\infty} \Delta &\propto k_{min} \propto 1/N \\ \lim_{N\to\infty} \lim_{j\to\infty} \Delta &= 0 \end{aligned}$$

This doesn't contradict the Haldane conjecture because of the order of limits. It only says $\lim_{N\to\infty} \Delta$ for fixed integer j is nonzero.

10.5 XXZ Spin 1/2

Hamiltonian:

$$H = \sum \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z + \text{bdry term}$$

Phase Diagram:
Uses $U_q(\widehat{\mathfrak{sl}_2})$

$$\Delta E_0 =$$

10.5.1 $\Delta = 0$

10.6 XXZ $1/2 \to 1$ for quantum space

10.7 XXZ $1/2 \to 1$

This is for the case of algebraic Bethe ansatz with all quantum and auxiliary spaces replaced by spin 1 rather than 2. For the derivation see `http://arxiv.org/pdf/solv-int/9809001v1.pdf`
Hamiltonian:

$$\begin{aligned} H &= \sum h_n + H_{bdry} \\ h_n &= J_n \cdot J_{n+1} - (J_n \cdot J_{n+1})^2 \\ &+ \frac{(q-q^{-1})^2}{2}\left(J_n^z J_{n+1}^z + (J_n^z)^2 + (J_{n+1}^z)^2 - (J_n^z J_{n+1}^z)^2\right) \\ &- (q^{1/2} - q^{-1/2})^2\left((J_n^x J_{n+1}^x + J_n^y J_{n+1}^y) J_n^z J_{n+1}^z + J_n^z J_{n+1}^z (J_n^x J_{n+1}^x + J_n^y J_{n+1}^y)\right) \end{aligned}$$

In particular we can set $q \to 1$ but note that unlike in the 1/2 case this does not give the Heisenberg point but instead has that extra quartic term still remaining.

$$h_n \quad \to \quad J_n \cdot J_{n+1} - (J_n \cdot J_{n+1})^2$$

Compare to the AKLT below with a -1 instead of $+1/3$.
Phase Diagram:
Uses $U_q(\widehat{\mathfrak{sl}_2})$

10.8 XYZ Spin 1/2

Hamiltonian:

$$H \quad = \quad \sum X\sigma_i^x\sigma_{i+1}^x + Y\sigma_i^y\sigma_{i+1}^y + Z\sigma_i^z\sigma_{i+1}^z + \text{bdry term}$$

Phase Diagram:
Uses Elliptic Quantum Group

10.9 AKLT

Hamiltonian:

$$H \quad = \quad J\sum S_jS_{j+1} + \frac{J}{3}\sum (S_jS_{j+1})^2$$

Ground States:

This was solved by writing the chain in terms of spin 1/2's first and then writing the projector that forbids every pair of spin 1's being in a spin 2 state. If you write the projector to spin 2 you get the AKLT Hamiltonian. Because it comes with a positive coefficient, the low energy states will be in the kernel of these projections. Thankfully you can satisfy them all at once and that is the above ground state. Thinking of it this way only gives the right behavior for energies below the energy associated with the projector because in reality that constraint is strict not only energetic.

At the edges, there are unpaired spin 1/2's. They act independently even though the original chain had only spin 1's. For long chains this gives a four fold degeneracy. It is split into triplet and singlet with gap exponentially small in the length.

10.9.1 Haldane

`http://online.kitp.ucsb.edu/online/compqcm10/oshikawa/pdf/Oshikawa_CompQCM.pdf`

10.9.1 Conjecture (Haldane Conjecture) *Heisenberg antiferromagnet with half integer is gapless and power law spin correlation decay. But with integer spins, it is gapped and exponential decay for correlations.*

10.10 Fredkin

http://arxiv.org/pdf/1605.03842.pdf

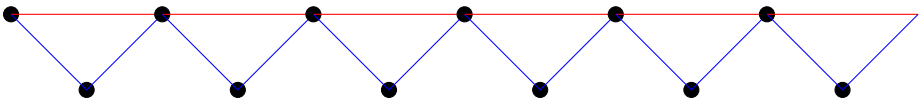
$$\begin{aligned} H &= H_{left} + H_{right} + \sum_{j=0}^{L-2} Id \otimes H_{j,j+1,j+2} \otimes Id \\ H_{j,j+1,j+2} &= |\uparrow_j\rangle\langle\uparrow_j| \otimes |\, S_{j+1,j+2}\rangle\langle S_{j+1,j+2}\,| + |\, S_{j,j+1}\rangle\langle S_{j,j+1}\,| \otimes |\downarrow_{j+2}\rangle\langle\downarrow_{j+2}| \\ H_{right} &= |\uparrow_L\rangle\langle\uparrow_L| \\ H_{left} &= |\downarrow_0\rangle\langle\downarrow_0| \end{aligned}$$

The ground state is a equal superpostion over all Dyck paths with up being up spin and down being down spin. In fact this is a 0 eigenvalue eigenstate for every term in the Hamiltonian all of which are positive definite. So it is definitely a ground state and in fact the unique one.

10.10.1 Motzkin

https://arxiv.org/pdf/1611.03147.pdf

10.11 Majumdar-Ghosh

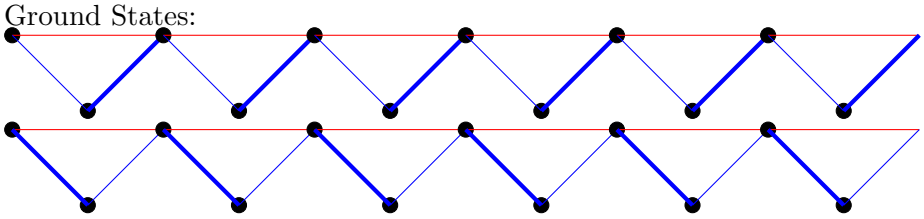


Hamiltonian:

$$H = J \sum S_j S_{j+1} + \frac{J}{2} \sum S_j S_{j+2}$$

The first term comes from the blue bonds in the picture. The second comes from the red bonds. Notice that the red bonds are $\sqrt{2}$ times as long so that we will get a factor of 2 smaller constant if the interactions in the tight binding model fall of with a inverse square law.

Ground States:



10.12 Zig-Zag Ladder

10.13 2-orbital 2-spin

Hund term.

10.14 Perk-Schultz

XXZ and Super

10.14.1 $GL(2 \mid 1)$

http://arxiv.org/pdf/1606.03573.pdf
http://arxiv.org/pdf/cond-mat/9802204v1.pdf
http://arxiv.org/pdf/cond-mat/0309138v1.pdf

10.15 Hubbard Model

Hamiltonian:

$$H = -\sum_{i,\sigma} t_{i,i+1}(c^\dagger_{i,\sigma} c_{i+1,\sigma} + h.c.) + U\sum_i n_{i,\uparrow} n_{i,\downarrow} + \sum_i \epsilon_i n_i$$

Now assume that the orbitals for all i have the same energy in isolation. So know if we give a filling fraction the last term is a constant which will be dropped. Also this symmetry between all orbitals will turn $t_{i,i+1}$ into a single constant t.

$$H = -t\sum_{i,\sigma}(c^\dagger_{i,\sigma} c_{i+1,\sigma} + h.c.) + U\sum_i n_{i,\uparrow} n_{i,\downarrow}$$

Phase Diagram:

First consider the limit of large $\frac{U}{t}$. In this case the second term dominates and the problem factors into single site problems. Because they don't hop, it gives an insulator.

$$\begin{aligned} Z &= \sum e^{-\beta H + \beta\mu N)} = e^{0+0} + e^{0+\beta\mu} + e^{0+\beta\mu} + e^{-\beta U + 2\beta\mu} \\ \langle n\rangle &= \frac{2(e^{\beta\mu} + e^{-\beta U+2\beta\mu})}{1 + 2e^{\beta\mu} + e^{-\beta U+2\beta\mu}} \end{aligned}$$

This looks like a step function which has $\langle n\rangle = 1$ before $\mu = U$ and then when $\mu > U$ it becomes 2 because the chemical potential is high enough to overcome the energetic barrier.

Now consider $U \to 0$. It is a free particle which diagonalizes by passing to momenta. This gives a metal with usual band theory.

At the intermediate we have the Mott transition.

Uses $Yangian(sl(2 \mid 2))$ central extension.

10.16 Fannes-Nachtergale-Werner

10.16.1 Theorem (Fannes-Nachtergale-Werner 92) *Construction of Parent Hamiltonians* `http://link.springer.com/article/10.1007/BF02099178`

Chapter 11

Dimension 1 - Part 2

11.1 Lieb-Liniger,Tonks-Girardeau

11.1.1 Non Linear Schrodinger

$$\mathcal{H} = \frac{\hbar^2}{2m}\int_0^L dx \partial_x\psi^\dagger\partial_x\psi + \frac{g}{2}\int_0^L dx\psi^\dagger\psi^\dagger\psi\psi$$

Define a Fock space with vacuum $\mid 0\rangle$ annihilated by all $\psi(x)$ and normalized. The equation of motion from Heisenberg's equation gives

$$\begin{aligned} i\partial_t\psi(x) &= [\mathcal{H},\psi(x)] \\ i\partial_t\psi(x) &= -\partial_x^2\psi(x) + 2c\psi^\dagger(x)\psi(x)\psi(x) \end{aligned}$$

If ψ was a classical field instead of a quantum field operator, this would be the non-linear Schrodinger equation. Both

$$\begin{aligned} N &= \int_0^L \psi^\dagger\psi \\ P &= \frac{i}{2}\int_0^L \left[\partial_x,\psi^\dagger\right]\psi \end{aligned}$$

We can restrict them to be eigenvalues.

11.1.2 Lieb-Liniger

In the N particle sector, this reduces to a quantum mechanical problem of

$$H = -\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + 2c\sum_{i<j}^N \delta(x_i - x_j)$$

If we let L be the period of the circle, we find Bethe equations

$$L\lambda_j + \sum 2\text{atan}\frac{\lambda_i - \lambda_j}{c} = 2\pi I_j$$

or in a slightly different parameterization

$$e^{i\lambda_j L} = \prod \frac{\lambda_j - \lambda_l + i\kappa}{\lambda_j - \lambda_l - i\kappa}$$

The $I_1 < I_2 \cdots$ are integers when N is odd and half-integers when N is even. In the ground state, they are integer spaced and $I_1 = -I_N$. That is they arrange in a string like $-N/2 \cdots N/2$. Then solve with these I fixed gives the ground state. Next we switch $I_N \to I_N + n$ or $I_1 \to I_1 - n$. This gives one sort of excitation. Another kind is to choose an n in $- < n \leq N/2$ and replace $I_i \to I_i + 1$ for $i \geq n$. Other integers can be set as well giving yet more kinds of excitations.

In the limit $N, L \to \infty$ with $\frac{N}{L} = \rho$ fixed we get a convergence for $\epsilon_0 = \frac{E_0(N,L)}{N}$. It is given by defining the following:

$$\begin{aligned} e_B &\equiv \frac{\epsilon_0}{\rho^2} \\ \gamma &\equiv \frac{c}{\rho} \\ L_\kappa f &\equiv \frac{\kappa}{\pi}\int_{-1}^{1} \frac{1}{(x-y)^2 + \kappa^2} f(y)dy \\ f_B(x,\kappa) - L_\kappa f_B(x,\kappa) &= 1 \end{aligned}$$

and then fixing which f_B solution and solving for e_B by

$$\begin{aligned} \frac{\kappa}{\gamma} &= \int_{-1}^{1} f_B(x,\kappa)dx \\ e_B(\gamma) &= \frac{1}{2\pi}\frac{\gamma^3}{\kappa^3}\int_{-1}^{1} x^2 f_B(x,\kappa)dx \end{aligned}$$

In particular, we can take the weak repulsion limit by $\gamma \to 0^+$ and get

$$e_B(\gamma) = \gamma - \frac{4}{3\pi}\gamma^{3/2} + (\frac{1}{6} - \frac{1}{\pi^2})\gamma^2 + o(\gamma^2)$$

11.1.3 Tonks-Girardeau

Now take the opposite limit, strong repulsion $c \to \infty$

In the $\gamma \to \infty$ limit, the ground state energy per particle goes to $\frac{\pi^2}{3}$.

11.1.4 Super Tonks-Girardeau

11.2 Toda Lattices

11.2.1 Historical Aside

Fermi-Pasta-Ulam is an approximation to the Toda lattice found in numerical experiments with some of the first computers. In the summer of 1953 they saw the lack of ergodicity that is characteristic of having some integrability around.

This was a decade before the integrals of motion for KdV was solved and two decades before Toda. It is a little surprising given that all the discoverers of the equation and Backlund were all 19th century.

11.2.2 Generalized Toda

$$\begin{aligned} V &= (e^{q_i} - e^{q_{i+1}}) \\ H &= \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1}(e^{q_i} - e^{q_{i+1}}) + e^{q_n - q_1} \end{aligned}$$

If we didn't have that last term, the particles would scatter in the fundamental chamber, $q_1 < q_2 < \cdots$ and they would just move further deep into this chamber because the potential is repulsive. The last term is keeping that from happening by penalizing a large distance between the first and last particle.

So how is this Lie algebraic. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ and let σ be the Cartan involution which swaps the E's and F's. What is important here is **real split semisimple**.

Do the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{t} + \mathfrak{a} + \mathfrak{n}$$

In this case $\mathfrak{t}$ are skew-symmetric matrices, $\mathfrak{a}$ are the diagonals and $\mathfrak{n}$ are the strictly upper triangular ones. Now give the algebra the inner product to identify with $\mathfrak{g}^*$ This will be $\lambda * \text{Killing}$. Note that $\mathfrak{t}$ and $\mathfrak{b}$ are not isotropic. In fact the orthogonal to these under the Killing form are $\mathfrak{p}$ and $\mathfrak{n}$ respectively.

11.2.3 Reduction to get Open Toda

$$\begin{aligned} G_r &= K \times B \subset G \times G \\ T^*G \times O_T //G_r &\simeq O_T \end{aligned}$$

as a symplectic manifold, so instead of treating the orbit as the phase space, we can quantize before reduction.

$$\begin{aligned} H &= (p,p) + 2\sum_\alpha \exp(2\alpha(q)) \\ \{p_i, q_j\} &= 1/2\delta_{ij} \\ \frac{dq}{dt} &= \{H, q\} = p \\ \frac{dp}{dt} &= \{H, p\} = -2\sum_\alpha H_\alpha \exp(2\alpha(q)) \\ \frac{d^2q}{dt^2} &= -2\sum_\alpha H_\alpha \exp(2\alpha(q)) \end{aligned}$$

In the type A_N of $SL(N+1)$ then this reduces to the exponential of difference in coordinates from before. But if we want to use the obvious coordinates on the Cartan as an $N+1$ tuple and then the constraint $\sum q = 0$ we have to change to use the Dirac bracket instead.

$$\begin{aligned} \{p_i, q_j\} &= \frac{1}{2}(\delta_{ij} - \frac{1}{N+1}) \\ H &= (p,p) + 2\sum_\alpha \exp(2(q_i - q_{i+1})) \\ \dot{q}_i &= p_i \\ \dot{p}_1 &= -2e^{2(q_1-q_2)} \\ \dot{p}_i &= -2e^{2(q_i-q_{i+1})} + 2e^{2(q_{i-1}-q_i)} \\ \dot{p}_{N+1} &= 2e^{2(q_N-q_{N+1})} \end{aligned}$$

This can be written in Lax form

$$\begin{aligned} L &= p + \sum_\alpha n_\alpha e^{\alpha(q)}(E_\alpha + E_{-\alpha}) \\ M &= -\sum n_\alpha e^{\alpha(q)}(E_\alpha - E_{-\alpha}) \end{aligned}$$

Proof Compute $\dot{L}$ and $[M, L]$. □

11.2.1 Definition ($\mathcal{T}(G)$) *Let G be a reductive group with Borel B, Unipotent U and Cartan T. Let ψ be a regular character for $U(\backslash) \to \mathbb{C}$. $\mathcal{D}(G)$ will be the differential operators on G. $U\backslash \otimes U(\backslash)$ maps to $\mathcal{D}(G)$ by taking left and right invariant vector fields due to $\backslash$ and extending to higher order operators. We then do quantum Hamiltonian reduction by $U \times U$ and $\mu = \psi, -\psi$. Call the result $\mathcal{T}(G)$.*

11.2.2 Theorem (Kazhdan-Kostant) *$ZU(\mathfrak{g})$ are the bi-invariant differential operators and they map into this reduction. In fact $U \times U$ acts on the big Bruhat cell freely so doing the quantum Hamiltonian reduction for functions there gives another algebra $\mathcal{D}(C_{w_0})//(U \times U, (\psi, -\psi)) \simeq \mathcal{D}(T)$ that has a map from $\mathcal{T}(G)$. Altogether this gives $ZU(\mathfrak{g}) \to \mathcal{T}(G) \to \mathcal{D}(T)$. The image of this gives the commuting Toda Hamiltonians.*

Proof Semiclassically this is saying we have $T^*T \to T^*G//(U \times U) \to \mathfrak{h}^*/W$ because functions on cotangent turn to D-modules and $SpecZU(\mathfrak{g}) = \mathfrak{h}^*/W$ by polynomials in the fundamental invariants. □

11.2.3 Theorem (Zastava) *The middle step in the classical analog $T^*G//(U \times U)$ is isomorphic to the open Zastava space of degree n based maps from $\mathbb{P}^1$ to itself preserving ∞. This can also be identified with the moduli of $SU(2)$ monopoles of charge n.*

11.2.4 Toda Spectral Curve

11.3 Sine-Gordon and KdV Hierarchy

11.3.1 Sine/Sinh-Gordon

$$\begin{aligned}
\phi_{tt} - \phi_{xx} + \sin\phi &= 0 \\
\mathcal{L} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \cos\phi \\
\frac{d\mathcal{L}}{d\phi} &= -\sin\phi \\
\frac{d\mathcal{L}}{d\partial_\mu\phi} &= \partial^\mu\phi \\
\partial_\mu\partial^\mu\phi + \sin\phi &= 0
\end{aligned}$$

We will see more about it in 2D CFT when looking at affine Toda field theory

11.3.2 KdV

11.3.3 Frenkel-Kontorova

$$\begin{aligned}
H &= \frac{m}{2}\sum \dot{x}_i^2 + \frac{\epsilon_s}{2}\sum(1 - \cos\frac{2\pi x_i}{a}) + \frac{g}{2}\sum(x_{i+1} - x_i - a_0)^2 \\
y_i &\equiv \frac{2\pi}{a}x_i \\
T &= \frac{2\pi}{a}\sqrt{\frac{\epsilon_s}{2m}}t
\end{aligned}$$

In a new set of variables turn this into. This needs to be fixed in order to know how limits like sending $\epsilon_s \to 0$ to go to the chain for a free boson (phonon).

$$\begin{aligned}
\frac{d^2x_n}{dt^2} + \sin x_n - g * (x_{n+1} + x_{n-1} - 2x_n) &= 0 \\
u_n &= x_n - na
\end{aligned}$$

If we look at the continuum we may pass to the sine-Gordon model from earlier

11.4 Calegero-Moser Type

11.4.1 Representation Theoretic Aspects

Let $M = T^*End(n, \mathbb{C})$ and $G = PGL(n, \mathbb{C})$. The trace form identifies $\mathfrak{g} = \mathfrak{sl}_n$ with it's dual. Define a moment map by $\mu(X, P) = [X, P]$

11.4.1 Theorem (Gan,Ginzburg) *$Comm(n)/G$ is a reduced scheme isomorphic to $\mathbb{C}^{2n}/S_n$. The Poisson structure is the pushforward of the one from the holomorphic symplectic structure on $\mathbb{C}^{2n}$*

11.4.2 Definition (Hilbert Scheme) *$Hilb^n(\mathbb{C}^2)$ is ideals of length n inside $\mathbb{C}[x, y]$. Geometrically they are n points in $\mathbb{C}^2$ but with collisions when two overlap taken into account, so a resolution of $(\mathbb{C}^2)^n/S_n$. There is the $\epsilon_{1,2}$ action which rotates the base with $\mathbb{C}^*$ actions so we can ask for fixed points under this. These are the torus fixed ideals which can be visualized as Young Tableaux of size n by reading the monomials as $x^n y^m$ if the tableaux contains the point (n,m) when drawn in French notation.*

Let $\mathcal{O}$ be the coadjoint orbit through $diag(-1/n, -1/n, \cdots \frac{n-1}{n})$ for $PGL(n, \mathbb{C})$. Namely it is a special complex coadjoint orbit so hyperKahler. This is the set of traceless matrices such that $T+1/n$ has rank 1. If this was only the $PSU(n)$ action we would have our density matrices for pure states (as a space not with Fisher). This is not quite the usual definition where we do $[X, Y] - Id$ has rank 1, but they are canonically isomorphic by sending $(X, Y) \to (\tau X, Y)$ from the original definition to the current definition where here $\tau = -1/n$.

11.4.3 Theorem *The action of $PGL(n, \mathbb{C})$ on $\mu^{-1}(\mathcal{O})$ is free and the Hamiltonian reduction μ^{-1}/G is a smooth symplectic variety of dimension 2n. It is connected as well.*

11.4.4 Theorem (Nekrasov-Schwarz) *Moduli space of locally free rank 1 sheaves on a noncommutative projective plane where the graded algebra we are taking (noncommutative) Proj for has x,y,z in degree 1 and $[x, y] = \tau z^2$. They do this by doing a noncommutative version of the twistor transform to pass between instantons on $\mathbb{R}^4$ and sheaves on $\mathbb{P}^2$. The Kronheimer result moves between singular G-instantons on $\mathbb{R}^4$ with prescribed boundary conditions at ∞ and 0 with $\mathfrak{g}^*_{\mathbb{C}}$ coadjoint orbits. How to move from here to Calagero Moser space???*

There are two moduli spaces of coherent sheaves which give this as a Zarisiki open dense subvariety. They provide compactifications. One is the Gieseker compactification and the other is the Uhlenbeck compactificaiton.

In the following $\mathfrak{g}$ is reductive, $\mathfrak{h}$ is a Cartan and W is Weyl.

11.4.5 Theorem (Central Characters) *The center $Z(U(\mathfrak{g}))$ acts on any V_λ for λ highest weight as some scalar. This is by commuting it past until it acts on the v_λ. This gives $xv = \chi_\lambda(x)v$. In addition these characters are equal if and only if $\lambda + \delta$ and $\mu + \delta$ are on the same Weyl orbit. This means that specifying χ_λ is the same data as specifying a Weyl invariant function on the weight space.*

11.4.6 Theorem (Classical Harish-Chandra) *We have the restriction ξ $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \simeq \mathbb{C}[\mathfrak{h}]^W$.*

11.4.7 Theorem (Levasseur-Stafford) *By pulling back on this isomorphism we get an action of $D(\mathfrak{g})^{\mathfrak{g}}$ on $\mathfrak{C}[\mathfrak{h}]^W$. These act by W invariant differential operators but they have some poles. Together this gives a HC' : $D(\mathfrak{g})^{\mathfrak{g}} \to D(\mathfrak{h}_{reg})^W$ homomorphism. If you conjugate by $\delta(x) = \prod_{\alpha>0}(\alpha, x)$, you get something that extends without poles. $HC = \delta \circ HC' \circ \delta^{-1}$. In fact it gives a map $D(\mathfrak{g})^{\mathfrak{g}} \to D(\mathfrak{h})^W$ which factors through to give an isomorphism $D(\mathfrak{g})//\mathfrak{g} \simeq D(\mathfrak{h})^W$.*

11.4.8 Example *In particular the Laplacian is an operator we can send through to get something in $D(\mathfrak{h})^W$. If x_i are an orthonormal basis for $\mathfrak{h}$ and e_α and f_α are the paired root elements.*

$$D = \sum_{i=1}^{r} \partial_{x_i}^2 + 2\sum_{\alpha>0} \partial_{f_\alpha}\partial_{e_\alpha}$$

Doing the computation gives that it ends up being the Laplacian on $\mathfrak{h}$, $\Delta_{\mathfrak{h}}$

11.4.9 Proposition *For differential operators with constant coefficients this is the same map as obtained by $(Sym\mathfrak{g})^{\mathfrak{g}} \simeq (Sym\mathfrak{h})^W$*

11.4.10 Theorem (Kirillov Character) *Let $\mathcal{O}$ be a regular coadjoint orbit in the $\mathfrak{g}^*_{cpt}$ compact form and $d\mu(b)$ the measure from KKS symplectic as well as letting $x \in \mathfrak{h}$. Let λ be the point in the dominant chamber of $\mathfrak{h}^*_{cpt}$ it passes through.*

$$\begin{aligned}
\psi_{\mathcal{O}}(x) &\equiv \int_{\mathcal{O}} e^{i(b,x)} d\mu(b) \\
\psi_{\mathcal{O}}(x) &= \delta^{-1}(x) \sum_{W} (-1)^{\ell(w)} e^{i(w\lambda, x)} \\
Tr_{L_\lambda}(e^x) &= \frac{\delta(x)}{\delta_{tr}(x)} \int_{\mathcal{O}_{\lambda+\rho}} e^{i(b,x)} d\mu(b) \\
\delta_{tr}(x) &\equiv \prod_{\alpha>0} (e^{\alpha(x)/2} - e^{-\alpha(x)/2})
\end{aligned}$$

11.4.11 Theorem (Quantum Hamiltonian Reduction Ideal) *Before we could do Hamiltonian reduction with μ^{-1} of an orbit or any other closed G-invariant subset. This means a Poisson ideal in $Sym\mathfrak{g}$ as far as the polynomial functions and varieties go. So to do the quantum version replace J with $J(I)$ which is now the set $A\mu(I)$ instead of $I = U(\mathfrak{g})^{>0}$ the augmentation ideal which corresponds to the $\mu^{-1}(0)$ case.*

11.4.12 Example *$\mathfrak{g} = \mathfrak{gl}_n$, A is $D(\mathfrak{g})$ which is quantizing $\mathcal{O}(T^*\mathfrak{g})$. Let k be a complex number, W_k be the representation of $\mathfrak{sl}_n$ on functions of the form $(x_1 \cdots x_n)^k f(x_1, \cdots x_n)$ where f is Laurent of degree 0. It is a $\mathfrak{gl}_n$ module too by pullback. If I_k is the annihilator of this module in $U(\mathfrak{g})$ we can reduce along it instead. The result is B_k called the spherical rational Cherednik algebra. It gives a family version HC_k parameterized by k that take $D(\mathfrak{g})^{\mathfrak{g}} \to B_k$. This is a flat family of homomorphisms of algebras. If $k = 0$ then it is just $D(\mathfrak{h})^W$ and the ordinary HC map.*

11.4.13 Theorem *B_k acts naturally on $\mathbb{C}[\mathfrak{h}_{reg}]$*

11.4.2 Rational Calagero-Moser

Hamiltonian:

$$H_{cl} = \sum_{i=1}^{N} \frac{p_i^2}{1} - \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}$$

$$H_{qtm} = -\sum_{i=1}^{N} \frac{\partial_i^2}{1} - \sum_{i \neq j} \frac{k(k+1)}{(x_i - x_j)^2}$$

11.4.14 Theorem *Let H_i be the commuting homogenous differential operators with constant coefficients quantizing $Tr(P^i)$ in $A = D(\mathfrak{g})$. They turn into new operators $HC_k(H_i)$ in B_k which subsequently become operators L_i on $\mathbb{C}[\mathfrak{h}_{reg}]$. In particular L_2 is H_{qtm} (slight reparameterization of signs because h vs $-ih$)*

$$\begin{aligned}
H_{qtm} &= \sum_{i=1}^{N} \frac{\partial_i^2}{1} - \sum_{i \neq j} \frac{k(k+1)}{(x_i - x_j)^2} \\
-H_{qtm} &= -\sum_{i=1}^{N} \frac{\partial_i^2}{1} - \sum_{i \neq j} \frac{-k(k+1)}{(x_i - x_j)^2} \\
&= -\sum_{i=1}^{N} \frac{\partial_i^2}{1} - \sum_{i \neq j} \frac{k_2(k_2+1)}{(x_i - x_j)^2} \\
-k(k+1) &= k_2(k_2+1) \\
k_2 &= -1/2(1 \pm \sqrt{1 - 4k(k+1)})
\end{aligned}$$

11.4.3 Trigonometric Calagero-Moser

`http://www-math.mit.edu/~etingof/zlecnew.pdf`

$$\begin{aligned}
H &= \sum_{i=1}^{N} \frac{(x_i p_i)^2}{1} - \sum_{i \neq j} \frac{x_i x_j}{(x_i - x_j)^2} \\
\tilde{x}_i &= \log x_i \\
\tilde{p}_i &= x_i p_i \\
H &= \sum_{i=1}^{N} \frac{\tilde{p}_i^2}{1} - \sum_{i \neq j} \frac{4}{\sinh^2((\tilde{x}_i - \tilde{x}_j)/2)}
\end{aligned}$$

11.4.15 Theorem *Do the same reduction but with T^*G instead of $T^*\mathfrak{g}$. Again there is a slight mismatch of signs*

$$H_{qtm} = \sum_{i=1}^{N} \frac{\partial_i^2}{1} - \sum_{i \neq j} \frac{k(k+1)}{\sin^2(x_i - x_j)}$$

http://arxiv.org/pdf/1608.00599.pdf

$$\begin{aligned} H^{CS} &= -\sum \frac{\partial}{\partial q_i} + \sum \frac{\beta(\beta-1)}{\sin^2 q_i - q_j} \\ \psi_0 &= |\prod_{i<j} \sin q_i - q_j\ |^{\beta} \\ \psi_0^{-1} H^{CS} \psi_0 &= -\sum (x_i \frac{\partial}{\partial x_i})^2 + \beta \sum \frac{x_i + x_j}{x_i - x_j} (x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j}) - 2\beta \sum \frac{x_i x_j}{(x_i - x_j)^2} (1 - P_{ij}) \end{aligned}$$

where q is the $(0, 2\pi)$ coordinate and $x = e^{iq}$

11.4.4 Spin Calagero-Moser

Kolya 20200521 talk in Poisson Zoom seminar

Classical Periodic

Classical Open

Quantum Periodic

Quantum Open

11.4.5 Ruijensaars

$$\begin{aligned} H &= mc^2 \sum_{i=1}^{N} \cosh \frac{p_i}{mc} \prod_{k \neq i} f(x_i - x_k) \\ f^2 &= \begin{cases} 1 + \frac{g^2}{m^2 c^2 x^2} & \text{rational} \\ 1 + \sin^2 \frac{\nu g}{mc} / \sinh^2 \nu x & \text{trig} \\ 1 + \sinh^2 \frac{\nu g}{mc} / \sin^2 \nu x & \text{trig} \\ a + b\rho(x) & \text{elliptic} \end{cases} \end{aligned}$$

These are the rational, hyperbolic, trigonometric and elliptic cases respectively in order of more generality.

$$\begin{aligned} P &= mc \sum_{i=1}^{N} \sinh \frac{p_i}{mc} \prod_{k \neq i} f(x_i - x_k) \\ B &= -m \sum_{j=1}^{N} x_j \end{aligned}$$

Together they from the Lie algebra of Poincare(1,1) with their Poisson brackets whenever f^2 satisfies the above condition or it's degenerations listed.

Cluster Structure in Ruijensaars

Let $\mathcal{G} = SL(p+1, \mathbb{C})$ and $\mathcal{T}$ be a maximal torus. Equip $\mathcal{G}$ with the standard Poisson-Lie structure.

$$\begin{aligned} W &= S_{p+1} \\ \mathcal{G} &= \bigsqcup_{(u,v)\in W^2} \mathcal{G}_{(u,v)} \end{aligned}$$

The double Bruhat cells are also Poisson

$$\begin{array}{ccc} \mathcal{G}_{(u,v)} & \longrightarrow & \mathcal{G} \\ \downarrow & & \downarrow \\ (\mathbb{C}^*)^l \Longrightarrow \mathcal{G}_{(u,v)}/(Ad\,\mathcal{T}) & & \mathcal{G}/(Ad\,\mathcal{T}) \end{array}$$

where the algebraic torus is equipped with the Poisson structure from the log canonical $\{x_i, x_j\} = B_{ij}x_ix_j$ with the exchange matrix B_{ij}. The quotient by the torus also has an induced Poisson structure because $\mathcal{T}$ is a trivial Poisson subgroup. The many arrows from $(\mathbb{C}^*)^l$ depend on the expression of (u, v) as a reduced word.

Macdonald Polynomials in Ruijensaars

$$D_r^x \equiv t^{\binom{n}{2}} \sum_{|I|=r} \prod_{j\in I} \frac{tx_i - x_j}{x_i - x_j} \prod T_{q,x_i}$$

11.4.16 Theorem (Macdonald,Ruijsenaars) *The D_i^x for $i = 1 \cdots n$ mutually commute and over the ring of symmetric polynomials, they are diagonalized by Macdonald polynomials (usual type A ones)*

Have x, p and X, P canonical pairs related by canonical transformation symplectomorphism.

$$\begin{aligned} H &= \sum_i \prod_{j\neq i} \frac{1}{1 - e^{x_i - x_j}} e^{p_i} \\ H^* &= \end{aligned}$$

$$\begin{aligned} \tau_+(z) &\equiv \prod (1 - ze^{-x_i}) \\ \tau_-(z) &\equiv \prod (1 - e^{x_i}/z) \\ \eta(z) &\equiv \frac{1}{\tau_+\tau_-} \\ \eta_0 &= \oint \frac{1}{2\pi i z} \eta(z) = ? H^* \\ &= ? \quad H \end{aligned}$$

Now look at the dynamics of the x and X

$$\begin{aligned} \dot{x} &= \\ \dot{X} &= \end{aligned}$$

Could also take $\frac{\dot{\tau_\pm}}{\tau_+}$

11.4.17 Proposition

$$\begin{aligned} \frac{d}{dt}\log\tau_+ &= \frac{-1}{2\pi i}\oint_{C_+}\frac{dw}{w}\frac{z/w}{1-z/w}\frac{1}{\tau_+(w)\tau_-(w)} \\ \frac{d}{dt}\log\tau_- &= \frac{+1}{2\pi i}\oint_{C_-}\frac{dw}{w}\frac{w/z}{1-w/z}\frac{1}{\tau_+(w)\tau_-(w)} \end{aligned}$$

where the contours are ???.

Proof $\tau_\pm$ are products so log breaks up as a sum which then .... □

11.4.18 Theorem (Hirota bilinear) *Subtract the two equations to get* $\frac{d}{dt}\log\frac{\tau_+}{\tau_-} = \frac{\dot{\tau_+}}{\tau_+} - \frac{\dot{\tau_-}}{\tau_-}$ *which on the other side gets* $\frac{1}{\tau_+\tau_-} - H$.

$$\tau_-\dot{\tau}_+ - \tau_+\dot{\tau}_- = 1 - H\tau_+\tau_-$$

11.4.19 Definition *Define a recursive system of matrices starting with 2 by 2 identity and related to the next by right multiplication by*

$$\begin{aligned} L_0 &= I_2 \\ L_k &= L_{k-1}R_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} \\ R_k &= \begin{pmatrix} 1 & ? \\ ? & 1 \end{pmatrix} \\ \dot{A}_k &= 0 \\ \dot{D}_k &= 0 \\ \dot{B}_k &= \omega B_k \\ \dot{C}_k &= -\omega C_k \\ \dot{\tau}_- &= -\omega C_k \\ \dot{\tau}_+ &= +\omega B_k \end{aligned}$$

SUSY and Ruijensaars

Below is taken from Bethe/Gauge in odd dimensions Sciarappa. 6/6/16

N- particle closed	5D $\mathcal{N}=1$ $SU(N)$ on $\mathbb{R}^2_\epsilon \times \mathbb{R}^2 \times S^1$
Spectral curve	SW curve
Bethe Ansatz Eq	SUSY vacuum eq
quantum Hamiltonians	twisted chiral ring
Eigenvalues	Wilson loops
eigenfunctions	full defect
coordinate	FI parameters
boundary monodromy parameter	instanton counting parameter
ω_2	$1/R$
ω_1	$-i\epsilon_1$

Table 11.1: NS Dictionary

11.4.6 Dunkl General

11.4.20 Definition (Dunkl) *For a finite Coxeter group W, $\mathfrak{h}$ the reflection representation with dimension denoted r, $\alpha_s \in \mathfrak{h}^*$ an eigenvector with -1 eigenvalue of the reflection s. $\alpha_s^\vee$ is in the dual and pairs to give 2. Also let c be a conjugation invariant function on the set of reflections S. Then for all $a \in \mathfrak{h}$ define*

$$\begin{aligned} D_a(c) &\in \mathbb{C}W \ltimes D(\mathfrak{h}_{reg}) \\ D_a(c) &\equiv \partial_a - \sum \frac{c_s \alpha_s(a)}{\alpha_s}(1-s) \end{aligned}$$

11.4.21 Theorem (Dunkl) *Polynomials go back to polynomials and the different D_a and D_b commute.*

11.4.22 Theorem (Olshanetsky-Perelomov/Heckman) *Let P_i be a generator $(Sym\mathfrak{h})^W$ and y_i an orthonormal basis of $\mathfrak{h}$, then define $P(D_{y_i}, \cdots D_{y_r})$. This is an element of $\mathbb{C}W \ltimes D(\mathfrak{h}_{reg})$ and is actually in $(\mathbb{C}W \ltimes D(\mathfrak{h}_{reg}))^W$. We may restrict to an differential operator acting on $\mathbb{C}(\mathfrak{h})^W$ so get a element of $D(\mathfrak{h}_{reg})^W$. Call this result $\bar{L}_i$.*

$$\begin{aligned} \bar{L}_i &= res(P(D_{y_i}, \cdots D_{y_r})) \\ \delta_c &\equiv \prod_s \alpha_s(x)^{c_s} \\ L &\equiv \delta_c^{-1} \circ \bar{L} \circ \delta_c \end{aligned}$$

These become a quantum integrable system and in particular

$$L_2 = \Delta_{\mathfrak{h}} - \sum_S \frac{c_s(c_s+1)(\alpha_s, \alpha_s)}{\alpha_s^2}$$

which is the rational Calagero-Moser quantum Hamiltonian when $W = S_n$ and $c_s = k$ for all s.

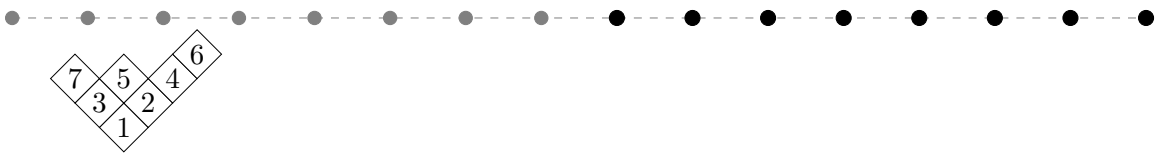
11.5 Exclusion Processes

11.5.1 ASEP

There is a $\mathbb{Z}^d$ of sites. A state is an assignment of occupied or unoccupied to that site. The dynamics are given by the rules followed independently by each particle

- Wait a time given by an exponential distribution with parameter 1
- Choose a $y \in \mathbb{Z}^d$ with probability $p(x, y)$
- If y is vacant, jump there. Else stay put.
- Reset clock

11.5.2 1D nearest neighbor ASEP



.

Set $d = 1$ and the function $p(x, y)$ is $p(x, x + 1) = p_1$ and $p(x, x - 1) = p_2$ with $p_2 > p_1$ to make it asymmetric.

Using the $d = 1$ aspect and if the configuration asymptotes to the step initial condition where all on the left are empty and all on the right filled, we can turn this into a as yet unspecified process on partitions. This is done by putting the partition in Russian notation.

11.5.1 Definition (Current) *Using the step initial condition Let $J_t(x)$ be the number of occupied sites to the left minus the unoccupied on the right. At first $J_0(n) = \pm n$. Can do more generally with other initial conditions.*

11.5.3 Plancherel Measures

For a finite group, let $G^\vee$ be the set of it's irreps. Define a measure on $G^\vee$ by

$$\mu(\pi) \quad = \quad \frac{\dim \pi^2}{\mid G \mid}$$

11.5.2 Example *For $G = S_n$, then we get f^λ which is the number of standard Young tableaux of shape λ.*

11.5.3 Example (Poissonized Plancherel) *This is on all partitions instead of just partitions of n. It combines a Poisson distribution for n with the Plancherel measure for S_n as seen in the two factors below.*

$$\mu(\lambda) \quad = \quad \frac{e^{-\theta}\theta^{|\lambda|}}{|\ \lambda\ |!}\frac{(f^\lambda)^2}{|\ \lambda\ |!}$$

11.5.4 Example (Plancherel Growth) *A random sequence of Young diagrams starting with* (1) *where you add a box according to the transition probability*

$$P(\nu \to \lambda) \quad = \quad P(\lambda^n = \lambda \mid \lambda^{n-1} = \nu) = \frac{f^\lambda}{nf^\nu}$$

It is giving a random walk down Young's lattice. Young's lattice is the Bratelli diagram for $A_i = kS_i$.

11.5.4 1D nearest neighbor TASEP

Use the step initial condition, $p_2 = 1$ and $p_1 = 0$. Let $x_m(t)$ be the position of the m'th particle counting from the left.

11.5.5 Theorem (Johansson) *There is convergence in distribution*

$$\frac{x_m(t) - c_1 t}{c_2 t^{1/3}} \quad \to \quad F_2$$

This is 1/3, KPZ universality class

11.5.6 Definition (Tracy-Widom GUE)

$$\lambda_{max} \quad \approx \quad 2N + N^{1/3}F_2$$

11.5.7 Definition (KPZ Equation)

$$\partial_t h(x,t) \quad = \quad \nu\partial_x^2 h(x,t) + \frac{\lambda}{2}(\partial_x h(x,t))^2 + \sqrt{2D}W_t$$

This is related to the stochastic Burger's PDE.

11.5.8 Definition (Ornstein-Uhlenbeck Process) *A continuum version of* $AR(1)$.

11.5.9 Lemma (Cole-Hopf Transformation) *Let* $= \log$

11.5.5 1D nearest neighbor ASEP again

11.5.6 All the other Macdonald Processes

11.5.10 Remark This has been using ASEP, but there is another way Macdonald polynomials show up. That is in the Quantum Many Problem of the previous section. These can be passed through the Cherednik-Matsuo correspondence to get a solution of Quantum Affine KZ equations. This means it is a correlation function of the XXZ chain with periodic boundary conditions. ◊

11.5.7 XXZ Redux

11.6 Spin Glasses

11.6.1 Edwards Anderson

Hamiltonian:

$$\begin{aligned} H &= -\sum_{<ij>} J_{ij} S_i S_j \\ J_{ij} & \quad N(J_0, \sigma_J^2) \end{aligned}$$

This can be solved with the replica method. It has a vanishing magnetization $< m >= 0$ and a non-vanishing two point correlation function for $\sum_{i=1}^{N} S_i^\alpha S_i^\beta$ where α β are replica indices.

$$\beta f = -\frac{\beta^2 \sigma_J^2}{4}(1-q)^2 + \frac{\beta J_0 m^r}{2} - \int \exp(\frac{-z^2}{2}) log(2\cosh(\beta \sigma_J z + \beta J_0 m)) dz$$

Chapter 12

2D Topological Field Theory

12.0.1 Definition *A fully extended framed 2 dimensional topological field theory is a symmetric monoidal 2-functor from $Bord_2^{fr}$ to some target 2-category $\mathcal{C}$ (like the Morita 2-category)*

12.0.2 Theorem *They are parameterized by their value on the standard framed pt called $+$.*

`https://www.ma.utexas.edu/users/psafronov/cob/teleman.pdf`

12.0.3 Theorem (4.29 of Teleman 5 Lectures) *In the case of the target being the $(\infty, 2)$ category given by $D^bQCoh(X)$, integral kernels $\Phi \in D^bQCoh(X \times Y)$ and*
Orlov's theorem says that every fully faithful triangulated functor between D^bCoh's of smooth projective varieties is representable by some integral kernel.
The case of $D^bCoh(X)$ with X projective are fully dualizable and to make an $SO(2)$ fixed point means that you have trivialized the Serre functor $\otimes\omega$ so said X was Calabi-Yau by trivializing the canonical bundle.

12.0.4 Theorem (Schommer-Pries) *The functors from $Bord^2 \to AlgBim_k$ are given up to equivalence by the trace-Morita classes of separable symmetric Frobenius algebras that get assigned to the point.*

Proof `https://golem.ph.utexas.edu/category/2008/06/schommerpries.html` □

12.0.5 Theorem (Eilenberg Nakayama) *In vector spaces over a field k, an algebra A can be equipped with a symmetric Frobenius algebra structure if it is separable.*

Proof `https://ncatlab.org/nlab/show/Frobenius+algebra#symmetric_frobenius_algebras`□

12.0.6 Theorem () *The separable algebras over a field k are precisely finite products of matrix algebras over finite-dimensional division algebras D over k (whose centers are separable extensions of k.) The paranthesis is necessary when the field isn't perfect, but characteristic 0 and finite fields are. Worry about $F_p(t)$*

Proof `https://qchu.wordpress.com/2016/03/27/separable-algebras/` □

12.0.7 Lemma *One way to construct finite dimensional associative division algebras over arbitrary fields is to produce quaternion algebras.*

12.0.8 Theorem *To any local field there are just two quaternion algebras.*

12.0.9 Theorem *The quaternion algebras B over $\mathbb{Q}$ are given by completing at all places ν to get B_ν. Each of those is one of the only two possibilities for that local field. So we get a list of places where it is split B_ν is the 2 by 2 matrices over $\mathbb{Q}_\nu$ and the ramified ones where it is not. The places where B ramifies is always finite and even and it detemines B as an algebra.*

12.0.10 Example *Rational Hamilton quaternions ramify at 2 and ∞ and split everywhere else. The rational 2 by 2 matrices split at all places.*

Proof `https://en.wikipedia.org/wiki/Quaternion_algebra` □

12.0.11 Theorem *The Picard-Brauer group (Picard 3-group) of Alg_R algebras over R, bimodules and intertwiners is the same as giving line 2-bundles over $SpecR$ which has homotopy groups as $Br(R)$ $Pic(R)$ and R^*. These match with $H^2_{et}(R, G_m)_{tor}$, $H^1_{et}(R, G_m)$ and $H^0_{et}(R, G_m)$ respectively.*

12.0.12 Theorem *For a global field K like a number field.*

$$0 \longrightarrow Br(K) \longrightarrow \bigoplus_{v \in S} Br(K_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

For local fields that are complete under a discrete valuation like $\mathbb{Q}_p$, we get $Br(K_v) \simeq \mathbb{Q}/\mathbb{Z}$. For the reals the place at ∞ get $\mathbb{Z}_2$ where we regard it as $(1/2)\mathbb{Z}/\mathbb{Z}$.

12.0.13 Corollary *Looking at the exactness for the image of the quaternion algebra $(a, b, -ab)$ over $\mathbb{Q}$ gives quadratic reciprocity by showing the sum over all v giving 0. This is the 2-torsion part of $Br(K)$ given as an even number of $1/2$'s in the $Br(K_\nu)$ and the rest being 0.*

12.1 2D Dijkraaf Witten

Let $A = \mathbb{C}[G]$

12.1.1 Theorem (Class Formula)

Proof

Chapter 13

Dimension 2: Lattices

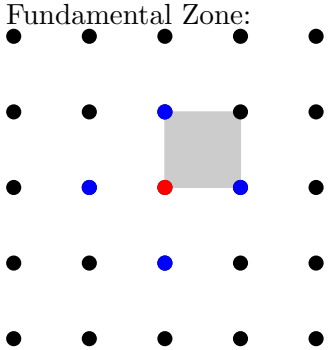
13.0.1 Definition (Wallpaper Group) *A discrete subgroup of the Lie group $O(2) \ltimes \mathbb{R}^2$ that contains two linearly independent translations upon identification of the subgroup $1 \ltimes \mathbb{R}^2$ with it's Lie algebra so that saying linearly independent makes sense. The choice of identification does not affect whether or not they are linearly independent.*

13.1 The Lattices

13.1.1 A1 x A1 : The square

Symmetry Group: $D_4 \ltimes \mathbb{Z}^2$, with the action of $r(x, y) = (y, -x)$ and $f(x, y) = (x, -y)$

Fundamental Zone:



Brillouin Zone:
Places it appears:

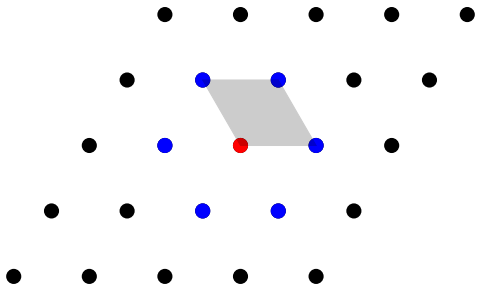
Ising on the square

It is dual to Z_2 gauge theory by the following transformation. Insert here.

13.1.2 A2

Symmetry Group: $\mathbb{Z}_2^2 \ltimes \mathbb{Z}^2$ with the action of $e(x, y) = (y, x)$ and $f(x, y) = (-x, -y)$. where e, f are elements of Klein Four.

Fundamental Zone:



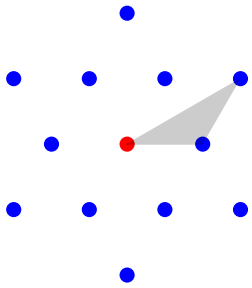
Brilloun Zone:
Places it appears:

13.1.3 B2/C2

http://www.math.ubc.ca/ cass/coxeter/crm1.html

13.1.4 G2

Fundamental Zone:



Brillouin Zone:
Places it appears:

13.1.5 Kagome Lattice

13.1.1 Remark (Pearls)

Spin Liquid

13.2 2D pseudo Lattices

Chapter 14

Dimension 2: Standard

14.1 Gauge theories

14.1.1 Z2 Gauge theories

14.1.2 Lattice QED in 2+1

14.2 Ising Model

$$\begin{aligned} H &= \sum \\ Z &= \sum e^{-\beta H} \\ T &= \\ Z &= TrT^N \end{aligned}$$

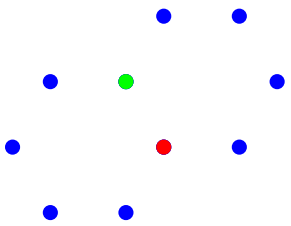
14.2.1 Kramers-Wannier Duality

See `https://arxiv.org/pdf/cond-mat/0404051v2.pdf` and the CFT section where we push to the conformal limit.

$$\begin{aligned} e^{\beta J \sigma_i \sigma_j} &= \cosh \beta J + \sinh \beta J \sigma_i \sigma_j \\ &= \cosh \beta J + \sigma_i \sigma_j \sinh \beta J \\ Z &= (\cosh \beta J)^{2N_V} \sum \prod (1 + \sigma_i \sigma_j \tanh \beta J) \\ T >> T_0 &\implies \beta J \to 0 \implies \tanh \beta J \to 0 \end{aligned}$$

$$\begin{aligned} K_1 &= \beta_1 J_1 \\ Z(K_2)/e^{NK_2} &= Z(K_1)/2^N \cosh^{2N} 2K_1 \\ \tanh K_1 &= e^{-2K_2} \\ K_1 = K_2 = K_C &\implies \sinh 2K_C = 1 \\ J_1 = J_2 = J &\implies T_C = 2.269J \end{aligned}$$

14.3 Graphene



14.3.1 Dirac Cone

Assuming a tight binding model and no spin orbit coupling, for every pseudomomentum $k \in \mathbb{T}^2$

$$\begin{aligned} H &= \begin{pmatrix} 0 & \Delta_k \\ \Delta_k^* & 0 \end{pmatrix} \\ \Delta_k &\equiv -t \sum_{l=1}^{3} e^{ik\cdot\delta_l} \\ \delta_1 &\equiv a/2(1, \sqrt{3}) \\ \delta_2 &\equiv a/2(1, -\sqrt{3}) \\ \delta_3 &\equiv -a(1, 0) \\ E_k &= \pm \mid \Delta_k \mid \end{aligned}$$

There are 2 special momenta K and $K^{'}$ for which E_k vanishes. That is the gap closes at this point in the Brillouin torus. Expanding in the local coordinate q around K to first order gives a approximate Hamiltonian.

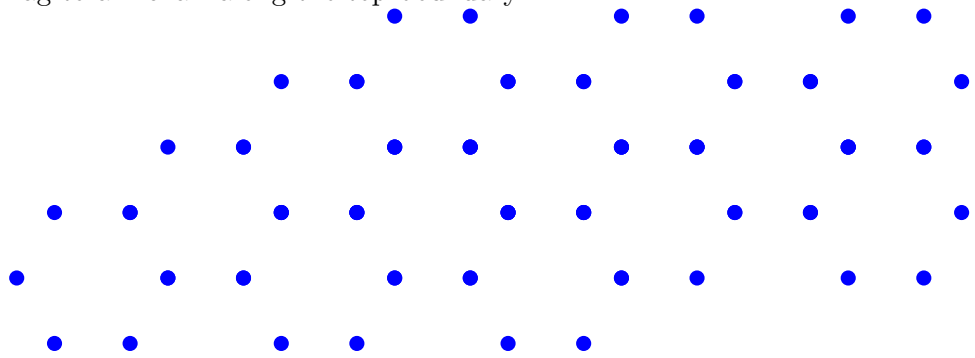
$$H = \hbar v_F (q_x \sigma_x + q_y \sigma_y)$$

with error bounded by $C \mid q \mid^2$.

14.3.1 Remark There is a type error here. We are not using any Lie algebra structure. We are just writing down an operator in a basis. This means we aren't writing σ in it's Lie incarnation or it's Cliff incarnation but just in it's 2 by 2 matrix incarnation. $\Diamond$

14.3.2 Nanoribbons

In this picture you can see both the zig-zag and armchair configurations as well as the change of zig-zag to armchair along the top boundary.



14.4 Charge Density

14.5 Spin Density

14.6 Hubbard

`http://quest.ucdavis.edu/tutorial/hubbard7.pdf`

Chapter 15

Dimension 2: Integrable Hierarchies

15.1 Dimer Model

http://arxiv.org/pdf/math/0310326.pdf http://arxiv.org/pdf/0910.3129v1.pdf

Let Γ be a bipartite graph drawn on a surface C. Let A be a weight function on its edges. To make a mix of the physical parameters and nice polynomial behavior let $A_e = e^{\beta_0 E_e}$ for the energy E_e and a reference temperature.

$$Z(\Gamma, A, \beta) \;=\; \sum_{D \in D_\Gamma} \prod_{e \in D} A_e^{\beta/\beta_0}$$

where D_Γ is the set of dimer coverings.

If we let $\beta = \beta_0$, then we get a polynomial in the edge variables with unit coefficients.

If we let the graph have incoming and outgoing edges that go to the boundary of C let T_Γ be the set of these edges.

$$Z(\Gamma, A, \beta, \xi) \;=\; \sum_{T \subset T_\Gamma} \Big(\sum_{D \in D_\Gamma(T)} \prod_{e \in D} A_e^{\beta/\beta_0} \Big) \prod_{t \in D \bigcap T} \xi_t$$

where $D_\Gamma(T)$ is the set of dimer coverings which have $T \subset T_\Gamma$ occupied and the rest of T_Γ free. ξ are odd variables indexed by T_Γ. The order is given by the order on the boundary circle.

15.1.1 Theorem (Kasteleyn)

$$Z(\Gamma, T^2, A, \beta = \beta_0) \;=\; \frac{1}{2}\big(- Pf(A_1) + Pf(A_2) + Pf(A_3) + Pf(A_4)\big)$$

Proof Insert proof □

15.1.2 Example *Just counting dimer coverings. so set* $A_e^{\frac{\beta}{\beta_0}} = 1$. *This then gives on an* N *by* M *square lattice,*

$$\mid DimerCovers \mid \;=$$

15.1.3 Example (Aztec Diamond)

15.1.4 Theorem *Correlation functions that condition on a dimer being occupied at a particular bond.*

$$\langle I_{e_1} I_{e_2} \cdots I_{e_n} \rangle \quad = \quad ?$$

15.1.1 Cutting and Gluing

Figure 15.1: Partial Gluing of two disks

$$Z_{gl} \quad = \quad \sum_{gl} Z_1(\Gamma_1, A_1, \beta, \xi_0, \xi_{gl}) Z_2(\Gamma_2, A_2, \beta, \xi_{gl}, \xi_2)$$

15.1.5 Definition (Height function) *Choose height somewhere and everytime you cross a occupied dimer change by ± 1 depending on if you cross it with the black vertex is on the left or right as you cross. Either convention works. This defines a well defined function for bipartite graphs on oriented connected simply connected surfaces. Dropping simply connected gets you a section of line bundle with the monodromy providing some $\pi_1 \to \mathbb{Z}$ ambiguity. Call this extended to the domain D in piecewise manner as θ.*
Rescale the height function by for example $h = \epsilon\theta(\frac{x}{\epsilon}, \frac{y}{\epsilon})$ so can take limits under refining the graph. Make it so that we get convergence in a weak sense to a Lipschitz function ($\frac{f(x)-f(y)}{x-y} \leq C$) h_0 on the surface to $\mathbb{R}$. Again has monodromy but now $\pi_1 \to \mathbb{R}$.

If boundary bond is occupied, that is equivalent to removing that internal vertex as a puncture. This is because that vertex is already on an occupied bond.

15.1.6 Theorem (Limit Shapes) *Specify height function on the boundary as a piecewise continuous function on the boundary which then specializes to the lattice function for every ϵ and then compute Z, correlation functions, string operators. In fact the probability distribution for possible height functions. Get frozen and liquid regions. Frozen means... and Liquid means ...*

15.1.7 Theorem *Has an action principle by*

$$\begin{aligned} S[h] \quad &= \quad \int_D \sigma(\partial_x h, \partial_y h) \\ \sigma \quad &= \quad \\ EL \quad &= \quad Burgers = 0 \end{aligned}$$

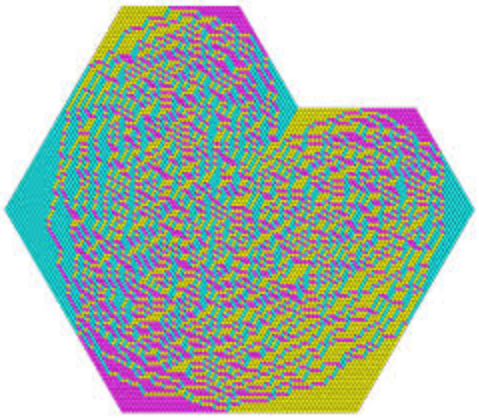


Figure 15.2: Artic Circle

15.1.2 Quantum Dimer Model

15.1.8 Theorem (Rokhsar-Kivelson point)

15.1.9 Conjecture (Moessner-Sondhi)

15.2 6/19/etc vertex models

See XXZ section instead

15.2.1 Theorem (Van Hove) *Mesh size* $\to 0$ *limit.*

$$\lim_{C\to\infty} \frac{1}{\mid C \mid} ???$$

Limit free energy density exists and does not depend on boundary condition.

Notice that we can avoid this theorem by ???.

15.2.1 Stochastic Six-Vertex Model

Suppose the weights are

- $a_1 =$
- $a_2 =$
- $b_1 =$
- $b_2 =$
- $c_1 =$
- $c_2 =$

Insert picture in quadrant with two linear parts with the liquid in between.

15.3 Hitchin System

15.3.1 Moduli stack of bundles

Let C be a curve of genus $g \geq 2$. The moduli space of semistable algebraic vector bundles on C of rank n and degree d is denoted by $U_C(n, d)$. When n and d are relatively prime this is automatically stable and this is a nonsingular projective variety of dimension $n^2(g-1)+1$. Denote $U_C(n)$ as the result of unioning over all degrees, giving all stable vector bundles.

15.3.1 Definition (Stable Vector Bundle)

If you take D modules on the algebraic stack $Bun_G C$, this is the automorphic side of the geometric Langlands conjecture. Above was the GIT quotient that gave an honest space.

The F points of Bun_G

$$Bun_G(C)(F) \simeq G(F)\backslash G(A_F)/G(O_F)$$

Find analogues of eigenfunctions, eigen D-modules. These will be indexed by local systems L on the curve for the dual group. That is the Galois side. In GL case rank N vector bundle with flat connection. On curve automically flat. The D-module is Aut L

For $n = 1$, get Line bundles for Bun which is the Jacobian. We need to produce a D-module on this torus $(\mathbb{C}^*)^{2g}$ based on the data of a local system L on the genus g curve. Because it is only acting as a 1D rep, it factors through homology. So we can say all the monodromies by giving 2g monodromies. So the corresponding D-module is the rank 1 local system on Jac with the connection having prescribed monodromies around each of those $\mathbb{C}^*$ factors.

15.3.2 Hitchin System

We now talk about the Galois side where looking at O-modules on Loc_{L_G} moduli stack of local systems. How do you write a local system? You need to give a holomorphic algebraic bundle E with a holomorphic algebraic connection ∇ on said E.

$$\nabla = d+$$

On this side there are skyscraper sheaves at any particular local system. In the above preview of GL, this was the L we talked about. The other side

Hecke eigenproperty means apply a functor and your object comes back tensored with a vector space. Skyscraper must be such because the support isn't going to change.

On Loc_G can ask about the bad points or torsion sheaves.

15.3.2 Definition (Higgs Bundle) *A Higgs bundle is a pair (E, ϕ) of an algebraic vector bundle on the curve C and a global section $\phi \in \Gamma(C, End(E) \otimes K_C)$. A Higgs bundle is stable if all ϕ invariant proper subbundles to have strictly smaller $\frac{deg}{rank}$. Semistablity allows some to have equal stability condition.*

Like before we will denote the moduli spaces of given ranks and degrees. $\mathcal{H}_C(n,d)$ and without the d if we want to take disjoint union over all d.

Applying Serre duality gives a new Higgs bundle by sending (E,ϕ) to $(E^* \otimes K_C^1, -\phi^\dagger \otimes id)$

where we have seen that $\phi^\dagger\ E^* \otimes K_C^{-1} \to E^*$ induces a map $\phi^\dagger \otimes id\ E^* \otimes K_C^{+1} \to E^* \otimes K_C^2$

If the vector bundle was already stable, then we can give it any Higgs field. Also if the field is 0, then Higgs stability condition is the stability of the bundle. That means that $T^*U_C(n,d) \subset \mathcal{H}_C(n,d)$ where the Higgs field is the cotangent direction. But there is more because we can stabilize an unstable vector bundle with the right Higgs field.

We can use the isomorphism $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[I_1 \cdots I_r]$ of invariant polynomials of degrees d_j. A particular choice of homogenous G-invariant polynomials is tr ϕ^n, the power sums in the eigenvalues. Then we can send the Higgs bundle through this map.

$$(E,\phi) \quad \to \quad (I_1(\phi), \cdots I_r(\phi)) \in \oplus_s H^0(C, K^{\otimes d_s})$$

15.3.3 Definition (Spectral Curve)

15.3.3 Hitchin Hyperkahler

Take a compact Lie group G and a compact Riemann Surface C, get a space $M_G(C)$ which is a fibration over a base B. The complex structures we can put on it are parameterized by $\xi \in \mathbb{CP}^1$. This is the fact that it is hyperkahler. This is chosen so that $\xi = 0$ corresponds to J_3. We already saw one symplectic structure in seeing T^*Bun (if we did the stable stuff the Higgs was better but as stacks we have $Higgs = T^*Bun$)

$$\begin{aligned} \omega_\xi &= -\frac{i}{2\xi}\omega_+ + \omega_3 - \frac{i}{2}\xi\omega_- \\ \omega_+ &= \omega_1 + i\omega_2 \end{aligned}$$

$$\begin{array}{c} M_G(C) = \text{Moduli(Higgs Bundles } (\mathcal{E}\ \Phi)) \\ \Big\downarrow \\ B = \text{Moduli}(S \subset T^*C) \end{array}$$

`http://arxiv.org/pdf/0801.0015v3.pdf`

15.3.4 Elliptic Spin Calogero-Moser system

15.4 Gaudin Model

`https://arxiv.org/pdf/math/0407524v2.pdf`

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$ with basis J_a and dual basis J^a. Let z_i be a collection of N positions in the complex plane. Then define the Gaudin Hamiltonians as an element of $U(\mathfrak{g})^{\otimes N}$ where (i) indicates which particle we are to act on.

$$H_i = \sum_{j\neq i}\sum_{a=1}^{d}\frac{J_a^{(i)}J^{a(j)}}{z_i-z_j}$$

This defines a commuting system of operators when we stick modules M_i for all of the particles. In addition to these there is also the $Z(U(\mathfrak{g}))^{\otimes N}$. There are also higher Gaudin Hamiltoinians that do not appear in the 2nd filtration level when the rank of $\mathfrak{g}$ is higher than 1. The situation of literal spins just gives $\mathfrak{sl}_2$ so we don't need them.

`http://arxiv.org/pdf/1409.6937v1.pdf`

In fact we can insert an automorphism σ to make it look like this model on a wedge geometry instead.

$$\begin{aligned} H_i &= \sum_{p=0}^{T}\sum_{j\neq i}\sum_{a=1}^{d}\frac{J_a^{(i)}\sigma^p J^{a(j)}}{z_i-\omega^{-p}z_j} \\ &+ \sum_{p=0}^{T}\sum_{a=1}^{d}\frac{J_a^{(i)}\sigma^p J^{a(j)}}{z_i-\omega^{-p}z_i} \end{aligned}$$

Here we see the interaction with all of the "image charges" including the images of oneself.

15.4.1 Definition (Oper) *A ${}^LG$ oper on a curve X is a triple of a ${}^LG$ bundle on X, a connection on this bundle and a reduction to Borel ${}^LB$ satisfying a condition with ∇. A Miura ${}^LG$ oper has another reduction which is now preserved by the connection.*

15.4.2 Lemma *The space of Miura opers whose underlying oper has regular singularities and trivial monodromy is isomorphic to the complete flag manifold ${}^LG/{}^LB$*

15.4.3 Lemma *For opers on SpecR like $R=\mathbb{C}[[t]]$ or $\mathbb{C}((t))$, the action of the gauge group $N(R)$ is free so we can take a unique representative*

$$\nabla = \partial_t + p_{-1} + \sum_{j=1}^{\ell} v_j(t)p_j$$

where p_j are defined by gradation using an $\mathfrak{sl}_2$ triple. $p_{-1}, 2\rho, p_1$

15.4.4 Theorem (PGL_2 and projective connections) *From the canonical form of the lemma we can look at a single function $v(t)$ under change of coordinate $t=\phi(s)$, $v\to v(\phi(s))\phi'(s)^2 - \frac{1}{2}\{\phi,s\}$ like a projective connection.*

In fact in more generality $Op_G(X)\simeq Proj(X)\times\bigoplus_{i=2}^{\ell}\Gamma(X,\Omega^{d_i+1})$

15.4.5 Theorem (Miura)

15.4.6 Theorem (Wakimoto)

15.4.7 Theorem (Feigin-Frenkel) *There is a canonical isomorphism $Z(\hat{g}) \simeq FunOp_{^LG}(D)$ of algebras compatible with $Der\mathcal{O}$ and $Aut\mathcal{O}$*

15.4.8 Definition (Drinfeld-Kohno Operad) *The rth Drinfeld-Kohno Lie algebra is generaded by symbols t_{ij} with $1 \leq i \neq j \leq r$ satisfying*

$$\begin{aligned} \left[t_{ij}, t_{kl}\right] &= 0 \ \forall i \neq j \neq k \neq l \\ \left[t_{ij}, t_{ik} + t_{kj}\right] &= 0 \ \forall i \neq j \neq k \end{aligned}$$

This is thought of as r strands and each t_{ij} stands for drawing a chord to connect the i and j strands.

This also then makes the operad in the category of Lie algebras structure apparent as a bundling of strands.

15.4.9 Theorem (Arnold-Kohno)

$$\begin{aligned} C^{\bullet}_{CE}(kd(r)) &\simeq H^{\bullet}(E_2(r)) \\ t^{\vee}_{ij} &\to [\frac{d(z_i - z_j)}{z_i - z_j}] \end{aligned}$$

is a dg algebra quasi-isomorphism.

Combining all the r gives $C^{\bullet}_{CE}(kd(r))$ the structure of a Hopf dg-cooperad using the cooperad structure from the KD operad and this dg-algebra quasi-isomorphism becomes a quasi-isomorphism of Hopf dg-cooperads $C^{\bullet}_{CE}(kd) \simeq H^{\bullet}(E_2)$

15.5 KP Hierarchy

Chapter 16

Dimension 2: CFT

16.1 General CFT Things

16.1.1 Lemma

$$\begin{aligned} T_\mu^\nu &= \begin{pmatrix} T_z^z & T_z^{\bar{z}} \\ T_{\bar{z}}^z & T_{\bar{z}}^{\bar{z}} \end{pmatrix} \\ T_{z\bar{z}} + T_{\bar{z}z} &= 0 \\ T_{z\bar{z}} = T_{\bar{z}z} &= 0 \\ T(z) &\equiv T_{zz} \\ \bar{T}(\bar{z}) &\equiv T_{\bar{z}\bar{z}} \end{aligned}$$

16.1.2 Definition (Schwarzian Derivative)

$$\begin{aligned} \{f(z), z\} &\equiv \frac{f^{(3)}(z)}{f'(z)} - \frac{3}{2}(\frac{f^{(2)}(z)}{f'(z)})^2 \\ g(z) = \frac{az+b}{cz+d} &\implies \{g(z), z\} = 0 \end{aligned}$$

16.1.3 Lemma (Riccati)

$$\begin{aligned} \{w(z), z\} &= f \\ y &\equiv w^{(2)}/w' \\ y' &= f + 0y + \frac{1}{2}y^2 \\ y' &= q_0(z) + q_1(z)y + q_2(z)y^2 \\ y &= -u'/(q_2 u) = -2u'/u \\ u'' + \frac{1}{2}fu &= 0 \\ w^{(2)}/w' = -2u_1'/u_1 &\implies w' = C/u_1^2 \\ w' &= C/u_1^2 = \frac{W(u_1, u_2)}{u_1^2} = (u_2/u_1)' \end{aligned}$$

u_1 and u_2 are independent solutions with Wronskian C. Then can trace back and get $w = u_2/u_1$ as a solution as well as all it's $SL(2,\mathbb{C})$ fractional linear transforms including $w \to w + b$

16.1.4 Lemma

$$\begin{aligned} T(z_1)T(z_2) &\approx \frac{c/2}{(z_1-z_2)^4} + \frac{2}{(z_1-z_2)^2}T(z_2) + \frac{1}{z_1-z_2}\partial_{z_2}T(z_2) + \cdots \\ z \to \alpha(z) &\implies T(z) \to Jac(\alpha)^{-2}\left(T(z) - \frac{c}{12}\{\alpha(z), z\}\right) \end{aligned}$$

16.1.5 Definition (Virasoro Algebra) *Generated by L_n and c with nontrivial brackets given by*

$$\left[L_m, L_n\right] = (m-n)L_{m+n} + \frac{c}{12}(m(m-1)(m+1))\delta_{m+n,0}$$

16.1.6 Lemma

$$\begin{aligned} T(z) &= \sum_n L_n z^{-n-2} \\ \bar{T}(\bar{z}) &= \sum_n \bar{L}_n \bar{z}^{-n-2} \\ L_n &= \frac{1}{2\pi i}\oint T(z) z^{n+1} \end{aligned}$$

Proof

$$\begin{aligned} \left[L_n, L_m\right] &= \frac{-1}{4\pi^2}\oint\oint z^{n+1}w^{m+1} - z^{m+1}w^{n+1}T(z)T(w) \\ &= \end{aligned}$$

16.1.7 Definition (Primary/Singular) *In a lowest weight irreducible representation c acts by some number also called c. L_0 also central acts by some h. All positive L_n act by 0. Then do induction to get rest. That is there is a primary state and rest are descendants. If just given c, h and not told irreducible have to check for singular vectors.*

$$\begin{aligned} L_{n\geq 1} \mid h\rangle &= 0 \\ L_0 \mid h\rangle &= h \mid h\rangle \end{aligned}$$

Singular Vectors are when this is reducible.

16.1.8 Lemma *If $h = h_{r,s}(c)$ for some r, s then have singular vector at level rs*

$$\begin{aligned} c &= 1 + 6(b + b^{-1})^2 \\ h_{r,s}(c) &= \frac{1}{4}((b+b^{-1})^2 - (br + b^{-1}s)^2) \end{aligned}$$

For example, $2,1$ gives the singular vector $(L_{-1}^2 + b^2 L_{-2})v$ in $V_{c,-1/2-3/4b^2}$

16.1.1 Order Fields

16.1.9 Definition (OPE)

$$\begin{aligned}\langle\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)\rangle &= \sum_k \langle C_{12}^k(z_1,z_2)\phi_k(z_{12})\cdots\phi_n(z_n)\rangle \\ C_{12}^k(z_1,z_2) &= \frac{C_{12}^k}{\mid z_1 - z_2 \mid^{2h_1+2h_2-2h_k}}\end{aligned}$$

16.1.2 Twist Fields

16.1.3 Verlinde Formula

$$N_{\mu\nu}^{\lambda}$$

16.2 Torus Partition Function

16.2.1 Fusion Category Things

16.2.1 Definition (Monoidal) *Has tensor product functor, unit object, natural isomorphism giving the associator, natural unitors. These natural isomorphisms satisfy pentagon and triangles respectively.*

16.2.2 Lemma (Associators and Tetrahedra) *Let* $(V_i \otimes V_j)\otimes V_k \simeq \bigoplus F_{ij}^l \otimes V_l \otimes V_k \simeq \bigoplus F_{ij}^l \otimes F_{lk}^m V_m$ *Contrast with* $V_i \otimes (V_j \otimes V_k) \simeq \bigoplus F_{jk}^l \otimes V_i \otimes V_l \simeq \bigoplus F_{jk}^l F_{il}^m V_m$ *This means for the associator we must give isomorphism between the multiplicity spaces* $\bigoplus_l F_{jk}^l \otimes F_{il}^m$ *and* $\bigoplus_l F_{ij}^l \otimes F_{lk}^m$
Insert Tetrahedron Picture

16.2.3 Definition (Rigid) *Objects have duals*

16.2.4 Definition (Fusion) *A rigid semisimple linear (over k) monoidal category with finitely many isomorphism classes of simples such that the endomorphisms of the unit object give k the ground field.*

16.2.5 Definition (Pivotal) *Natural transformation between Id and* **

16.2.6 Definition (Traces) *There are right and left traces of* $fa \to a$

$$1 \xrightarrow{coev} a^* \otimes a^{**} \xrightarrow{1\otimes dd^{-1}} a^* \otimes a \xrightarrow{1\otimes f} a^* \otimes a \longrightarrow 1$$

$$1 \xrightarrow{coev} a \otimes a^* \xrightarrow{f} a \otimes a^* \xrightarrow{dd\otimes 1} a^{**} \otimes a^* \longrightarrow 1$$

16.2.7 Definition (Spherical) *Left and right traces coincide.*

16.3 Segal Axioms

16.3.1 Definition *A functor from $Bord_2^{cx}$ to $Hilb$. For the source, the objects are disjoint unions of the standard circle. The morphisms are Riemann surface with parameterized boundary. These are taken up to biholomorphisms taking the parameterizations to each other. There are no identity morphisms because of the lack of 0 length cylinders/thin annuli. But reparameterization just gives projective rep of $Diff_+$. So insert these limit as morphisms as well. Now give dagger structure to say this is a unitary field theory. The target is Hilbert spaces and linear maps. To get correlation functions we may glue in discs $D(0,1)$ with the desired operators at 0.*

We may pass to the limits of the moduli space of Riemann surfaces.

16.3.2 Definition (Planar Algebra) *Especially consider the case of many inputs, 1 output and genus 0. This forms an operad. If it does depend on the holomorphic structure get a Vertex Operator Algebra. Insert the associativity and locality pictures.*

16.3.3 Example (Free Chiral Charged Fermion) *$H = L^2(S^1)$ with projection to Hardy space pH of the disk for the polarization. Take the Fock space associated. The vacuum is $\Omega = 1 \in \wedge^0 = \mathbb{C}$. Define bounded operators for creation and annihilation.*

16.4 Conformal Nets

16.4.1 Definition (Conformal Net) *H_0 for the vacuum sector Hilbert space. Give $\mathcal{A}(I)$ Von Neumann algebras acting on this for every interval I on that circle.*

16.4.2 Theorem (Tener) *Partially thin annuli converging to relate Segal type to Conformal Net*

Figure 16.1: Partially Thin Annulus

Proof

16.4.3 Theorem (Kawahigashi-Longo-M?)

$$\mu_{\mathcal{A}} = \sum_{irrep} d_\pi^2$$

Because all $d_\pi \geq 1$, that implies that finite $\mu_{\mathcal{A}} < \infty$ immediately implies finitely many irreducibles. Then turns $Rep\mathcal{A}$ into a unitary modular tensor category.

16.4.4 Definition (Orbifold) *Let G be a finite subgroup of the Lie group K, then build invariants for the conformal net $\mathcal{A}$ coming from LK_k loop group.*

16.4.5 Corollary $\mu_{\mathcal{A}^G} =\mid G \mid^2 \mu_{\mathcal{A}}$

Proof In conformal nets, but not VOA. □

16.4.6 Definition (Permutation orbifolds)

16.4.7 Example *Cycle of order n in S_n*

16.4.8 Example *Flip $1 \to n \cdots n \to 1$. A corollory of this is dual subfactor*

16.4.9 Definition (Simple Current Extension)

16.4.10 Example *$LSU(2)_{4k}$ and get $LSO(3)_?$*

16.4.11 Example (Coset) *$SU(N)_{k+l} \to SU(N)_k \times SU(N)_l$, this then gives a subnet $\mathcal{B} \subset \mathcal{A}$, define the coset as the commutant of $\mathcal{B}$ inside $\mathcal{A}$.*

16.4.12 Theorem (GKO) *$N = 2$, and $l = 1$, GKO construction of Virasoro minimal models*

16.4.13 Theorem *... .And a corollary all irreps of $\mathcal{B}$ appear in $H^{i\alpha}$ multiplicity spaces.*

16.4.14 Conjecture *Vague conjecture that all RCFT's are given by these procedures. Add mirror extensions to escape the examples below.*

16.4.15 Example (Mirror Extension) $SU(2)_{10} \subset Spin(5)_1$
$SU(2)_{10} \times \widetilde{SU(10)_2} \subset SU(20)_1$
$SU(10)_2 \subset \tilde{SU}(10)_2$, *where $\tilde{?}$ is another rational theory not from conformal inclusion. Not generated by spin 1 currents alone.*

16.5 CFT Tables

16.5.1 Virasoro Minimal Models

These are parameterized by two integers p and q.

They have central charge $1 - 6\frac{(p-q)^2}{pq}$
They have primaries with weights

In order to be unitary $q = p + 1$, so they are parameterized by a single integer p.

$$\begin{aligned} c &= 1 - \frac{6}{p(p+1)} \\ &= 1 - 6(\sqrt{\frac{p}{p+1}} - \sqrt{\frac{p+1}{p}})^2 \\ &= 1 - 6(\frac{p}{p+1} + \frac{p+1}{p} - 2) \\ &= 1 - 6(\frac{p^2 + p^2 + 2p + 1}{p(p+1)} - 2) \\ &= 1 - 6(\frac{p^2 + p^2 + 2p + 1 - 2p^2 - 2p}{p(p+1)}) \\ h_{r,s} &= \frac{((p+1)r - ps)^2 - 1}{4p(p+1)} \end{aligned}$$

where r runs from $1 \cdots p-1$ and s runs from 1 to r. Alternatively let s run all the way to $p-1$ but just remember that the table double counts.

Modular structure on torus partition functions.

$$\begin{aligned} S_{(r,s),(a,b)} &= 2\sqrt{\frac{2}{pq}}(-1)^{1+sa+rb}\sin(\pi\frac{p}{q}ra)\sin(\pi\frac{p}{q}sb) \\ T_{(r,s),(r,s)} &= e^{2\pi i(h_{r,s}-c/24)} \end{aligned}$$

16.5.2 $p = 2$

Trivial.

16.5.3 $p = 3$

$$c = 1 - 6\frac{1}{12}$$

s\r	1	2
1	**1**	ϵ
2	σ	σ
3	ϵ	**1**

	h	$\bar{h}$	meaning
1	0	0	identity
ϵ	1/2	1/2	thermal
σ	1/16	1/16	spin

$$\begin{aligned} \sigma \times \sigma &= \mathbf{1} + \epsilon \\ \sigma \times \epsilon &= \sigma \\ \epsilon \times \epsilon &= \mathbf{1} \end{aligned}$$

$$S = \begin{pmatrix} 1/2 & 1/2 & \sqrt{2}/2 \\ 1/2 & 1/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 & 0 \end{pmatrix}$$

The free Majorana fermion also has this CFT.

$$\epsilon = \bar{\psi}\psi$$

but the Ising spin field σ is given by a Jordan-Wigner transformation of the ψ field.

16.5.4 $p = 4$

This is the tricritical Ising model.

$$c = 1 - 6\frac{1}{20} = \frac{7}{10}$$

s\r	1	2	3
1	**1**	$\sigma^{'}$	$\epsilon^{''}$
2	-	σ	$\epsilon^{'}$
3	-	-	ϵ
4	-	-	-

	h	$\bar{h}$	meaning
1	0	0	identity
ϵ	1/10	1/10	thermal
$\epsilon^{'}$	3/5	3/5	thermal
$\epsilon^{''}$	3/2	3/2	thermal
σ	3/80	3/80	spin
$\sigma^{'}$	7/16	7/16	spin

The 5+20 nontrivial fusion rules that don't have an **1**, but commutativity turns this into 15 rules.

$$\begin{aligned}
\epsilon \times \epsilon &= \mathbf{1} + \epsilon^{'} \\
\epsilon \times \epsilon^{'} &= \epsilon + \epsilon^{''} \\
\epsilon \times \epsilon^{''} &= \epsilon^{'} \\
\epsilon^{'} \times \epsilon^{'} &= \mathbf{1} + \epsilon^{'} \\
\epsilon^{'} \times \epsilon^{''} &= \epsilon \\
\epsilon^{''} \times \epsilon^{''} &= \mathbf{1} \\
9more
\end{aligned}$$

$$\begin{aligned}
S &= \begin{pmatrix}
s_2 & s_1 & s_1 & s_2 & \sqrt{2}s_1 & \sqrt{2}s_2 \\
s_1 & -s_2 & -s_2 & s_1 & \sqrt{2}s_2 & -\sqrt{2}s_1 \\
s_1 & -s_2 & -s_2 & s_1 & -\sqrt{2}s_2 & \sqrt{2}s_1 \\
s_2 & s_1 & s_1 & s_2 & -\sqrt{2}s_1 & -\sqrt{2}s_2 \\
\sqrt{2}s_1 & \sqrt{2}s_2 & -\sqrt{2}s_2 & -\sqrt{2}s_1 & 0 & 0 \\
\sqrt{2}s_2 & -\sqrt{2}s_1 & \sqrt{2}s_1 & -\sqrt{2}s_2 & 0 & 0
\end{pmatrix} \\
s_1 &= \sin\frac{2\pi}{5} \\
s_2 &= \sin\frac{4\pi}{5}
\end{aligned}$$

It has supersymmetry that means it is a representation of the super-Virasoro algebra

$$\begin{aligned}
\left[L_m, L_n\right] &= (m-n)L_{m+n} + \frac{1}{12}cm(m-1)(m+1)\delta_{m+n,0} \\
\{G_m, G_n\} &= 2L_{m+n} + \frac{1}{3}c(m-1/2)(m+1/2)\delta_{m+n,0} \\
\left[L_m, G_n\right] &= (1/2m-n)G_{m+n}
\end{aligned}$$

16.5.5 $p = 5$

This is the three states Potts model.

$$c = 1 - 6\frac{1}{30} = \frac{4}{5}$$

s\r	1	2	3	4
1	-	-	-	-
2	-	-	-	-
3	-	-	-	-
4	-	-	-	-
5	-	-	-	-

16.5.6 Large p

The central charge is very close to 1. The weights are

$$\frac{r^2p^2 + s^2p^2 + O(p)}{4p^2 + 4p} \to \frac{r^2 + s^2}{4}$$

16.6 WZW generalities

First we say the nonlinear sigma model with target G. a simple Lie group.

$$\begin{aligned}
S^{WZW}_{S^2} &= \tilde{S}_{\partial^{-1}S^2}(g^{-1}dg) \\
\tilde{S} &= S_{CS}(A) + \frac{1}{2}\int_\Sigma \operatorname{tr} \bar{A} \wedge A \\
\bar{\partial_z}(g^{-1}\partial_z g) &= 0 \\
\bar{\partial_w}(g^{-1}\partial_w g) &= 0 \\
S &= \int_\Sigma
\end{aligned}$$

16.6.1 Theorem (Polyakov-Wiegmann)

$$\begin{aligned} S[gM] &= S[g] + S[M] - \frac{1}{\pi}\int_\Sigma \mathrm{tr}(M^{-1}\bar\partial M \partial h h^{-1}) \\ \partial S(M \to gM \;\; g \approx 1) &= -\frac{1}{\pi}\int_\Sigma \mathrm{tr}(\bar\partial A_z \delta g g^{-1}) \end{aligned}$$

16.7 $SU(2)_k$

$$c = \frac{k(2^2-1)}{k+2}$$

Partitions that fit within a 1 by k box, so they will have labels 0 through k.
In the decategorified fusion ring, the fusion rules are:

$$[j_1] \times [j_2] =$$

16.7.1 $SU(2)_1$

$$c = 1$$

Also see later section.

16.7.2 $SU(2)_2$

$$c = 3/2$$

16.13.1 Theorem (Kazhdan-Lusztig-Finkelberg [?]) *There is an equivalence of fusion categories between the semisimplified representation category of $U_q\mathfrak{g}$ at a root of unity and the positive energy representations of $\hat{\mathfrak{g}}_k$ with k determined by the root of unity.*

$$q \quad =$$

So the above can be replaced by considering $U_q(\mathfrak{sl}(2))$ at $q = e^{2\pi i/(k+2)}$

16.14 Lattice VOA

A lattice can have the following useful properties

- Even If the norms are all even
- Odd otherwise
- Positive Definite if the norm $(a, a) > 0$
- Unimodular if the determinant is ± 1, this also makes the $L \subset L^*$ by the inner product into an isomorphism
- Extremal for a even unimodular lattice is when the minimal norms of the vectors saturate the upper bound $2(\lfloor n/24 \rfloor + 1)$

16.14.1 Theorem *Any two indefinite unimodular lattices with the same type dimension and signature are isomorphic. Therefore can parameterize them as $I_{m,n}$ or $II_{m,n}$ for the odd and even cases respectively. Here $m, n \geq 1$.*

Proof `https://math.berkeley.edu/~reb/papers/al/al.pdf` □

16.14.2 Theorem (Vinberg) *For unimodular Lorentzian lattices $I_{n,1}$ (this one is odd), have infinite reflection group when $n \geq 2$. The quotient of $Aut(I_{n,1})$ by this reflection group is finite if and only if $n \leq 19$. In fact those cases are subgroups of Co_0 by Conway and Sloane.*

Proof `https://math.berkeley.edu/~reb/papers/al/al.pdf` □

16.14.3 Theorem *In contrast when looking at lattices in inner product spaces so positive definite, we get finite automorphism groups.*

16.14.4 Theorem *The theta function of a unimodular positive definite lattice is a modular form of weight half the rank. If even it has level 1. If odd, it has $\Gamma_0(4)$ structure so level 4.*

$$\begin{aligned} \Theta_L &= \sum_{x \in L} e^{\pi i \tau |x|^2} = \sum_x q^{|x|^2/2} \\ q &\equiv e^{2\pi i \tau} \end{aligned}$$

so counting (extremal) minimal vectors means reading that coefficient of q^n for length squared to be $2n$. In particular if $n < 24$ then minimal length squared is ≤ 2, That means looking at coefficient of q picks out roots with length $\sqrt{2}$. For $n = 24$, we get minimal length squared is ≤ 4. If this minimimum is 2, there are roots and it is one of the 23 other Niemier lattice. The extremal one is the Leech lattice. Looking at the coefficent of q gives 0, so need to look at q^2 for the true minimal vectors.

16.14.5 Theorem *For L,M ranks less than 43, the minimal vectors of $L \otimes M$ are pure. In particular their lengths are bound by $4(\lfloor l/24 \rfloor + 1)(\lfloor m/24 \rfloor + 1)$ rather than the a priori $2(\lfloor lm/24 \rfloor + 1)$*

16.14.6 Corollary *For the Leech lattice, $Aut(\Lambda \otimes_{\mathbb{Z}} \Lambda) \simeq (Co_0^2 \rtimes \mathbb{Z}_2)/(\pm 1)$*

16.14.7 Theorem *For an even unimodular lattice without roots. $Aut(V_L^+) \simeq O(\hat{L})/\langle \theta_{V_L} \rangle \simeq Hom(L, \mathbb{Z}_2).(O(L)/(-1))$*

Proof 4.1 of `http://ac.els-cdn.com/S0021869304002960/1-s2.0-S0021869304002960-main.pdf?_tid=1441ef86-ebf6-11e6-9bb0-00000aacb360&acdnat=1486335461_97f578648a00911bc9f32d94e00501`

16.15 N=1 Minimal

There is again a discrete and then a continuous range for central charge. The continuous ones start at $c = 3/2$.

16.16 N=2

There is yet again a discrete and then a continuous range for central charge. The continuous ones start at $c = 3$. The N=2 minimal models have central charges $c = 3 - \frac{6}{n}$ with $n \geq 3$.

A conformal $N = 2$ with target a k dimensional Calabi-Yau has central charge $3k$. This is shown by first looking at $\mathbb{C}^k$ (where it is clear that there are k chiral superfields each contributing 3) and arguing from there.

16.17 Coset Construction

16.18 Free Boson

Move this earlier

Begin with the free boson CFT. It's action is

$$S = \frac{g}{2} \int \partial_\mu \phi \partial^\mu \phi$$

Now let's assumed our metric is flat. This means for examples of hyperbolic surfaces we have already applied a Weyl transformation of multiplication by y^2 from the upper half plane with it's usual metric to it's realization as included into the plane. In this $g \geq 2$ case, of course this vanishes at the boundary so we need to make sure we stay in the interior. Alternatively we applied $1+ \mid z \mid^2$ to get to $\mathbb{CP}^1$

$$\langle \phi(z, \bar{z}) \phi(0) \rangle = -\frac{1}{4\pi g} \ln(\mid z \mid^2)$$

This is because of the Green's function for the Laplacian in 2D.

Now set $g = 1/4\pi$ so that we can drop that coefficient. Equivalently we are working with $\sqrt{4\pi g}$ off from what we had before.

From this action you form the stress energy tensor

$$T(z) = \frac{-1}{2} : \partial\phi_2 \partial\phi_2 :$$

You define a primary field $e^{i\sqrt{2}\alpha\phi}$ which has dimension α^2

$$\langle e^{i\sqrt{2}\alpha\phi(z)} e^{i\beta\sqrt{2}\phi(0)} \rangle = e^{-2\alpha\beta(-lnz)} = z^{-2\alpha^2} \delta_{\alpha,-\beta}$$

A way to see this neutrality condition is $\phi \to \phi + a$ is a symmetry so it will act on the correlator namely by making it pick up a factor of $e^{i\sqrt{2}a\sum \alpha_i}$. For this to vanish for all a means that the

Coulomb gas has to be neutral.

The trick to keep using is

$$\langle e^{A_1} \cdots e^{A_n} \rangle = exp(\sum_{i<j} \langle A_i A_j \rangle)$$

$$\begin{aligned} \langle \prod_{i=1}^{N} e^{\frac{\alpha_i}{\sqrt{2}}\phi(z_i)} \rangle &= \exp(\sum \frac{\alpha_i \alpha_j}{2} \langle \phi(z_i)\phi(z_j) \rangle) \\ &= \exp(\sum \frac{\alpha_i \alpha_j}{2} (-\ln \mid z_i - z_j \mid^2)) \\ &= \exp(\sum (\ln \mid z_i - z_j \mid^{-\alpha_i \alpha_j}) \\ &= \prod \mid z_i - z_j \mid^{-\alpha_i \alpha_j} \end{aligned}$$

which is the N point function of N vertex operator insertions.

In particular let $N = 4$ so get 1 cross ratio left

16.18.1 Definition (1+1D Boson algebra (Heisenberg Algebra)) *Let $\mathfrak{h}$ be a finite dimensional vector space with nondegenerate bilinear form $(,)$. This is the flavors of the boson.*

$$\begin{aligned} H &= \mathfrak{h} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \\ \hat{H} &= H \bigoplus \mathbb{C}[c] \\ \big[f, g\big] &= (\oint (g, df))c \\ \big[c, -\big] &= 0 \\ \big[x \otimes t^n, y \otimes t^m\big] &= n(x, y)\delta_{n,-m}c \end{aligned}$$

To simplify can take $\mathfrak{h} = \mathbb{C}$ with just multplication as bilinear form. Or alternatively that we are working in a orthonormal basis for the bilinear form.

$$\begin{aligned} a_n &\equiv 1 \otimes t^n \\ |\, t \,| &= 1 \\ |\, c \,| &= 0 \\ \big[a_n, a_m\big] &= n\delta_{n,-m}c \\ a(z) &\equiv \sum a_n z^{-n-1} \in \hat{H}[[z, z^{-1}]] \\ \big[a(z), a(w)\big] &= c\sum n z^{-n-1} w^{n-1} = \partial_w \sum (\frac{w}{z})^n \end{aligned}$$

16.18.2 Lemma *Only trivial finite dimensional representations.*

Proof

$$Tr(\big[a_n, a_{-n}\big]) = 0 \quad = \quad nTr1$$

For infinite dimensional representations, let them be graded $V = \bigoplus V_\lambda$ with $\exists \lambda_m \forall \lambda > \lambda_m \;\; V_\lambda = 0$

16.18.3 Lemma

$$\begin{aligned} H &\equiv \frac{1}{2}\sum a_n a_{-n} + ? \quad \notin U(\hat{H}) \\ \big[H, a_n\big] &= -n a_n \\ :H: &\equiv \frac{1}{2}\sum :a_n a_{-n}: \; \notin U(\hat{H}) \\ \big[\,:H:, a_n\big] &= -n a_n \end{aligned}$$

This is not in the universal enveloping algebra because tensor product of graded vector spaces uses the definition

$$\begin{aligned} T(V)_k^2 &= \bigoplus_{i+j=k} (V_i \otimes V_j) \\ &\hookrightarrow \prod_{i+j=k} (V_i \otimes V_j) \end{aligned}$$

But even still by our restriction about $\exists \lambda_m$ means that $:H:$ can act on any objects we are considering, the positive energy representations. The eigenvalues of $:H:$ on these V are bounded from below.

16.18.4 Example *Let*

$$\begin{aligned} F(\lambda) &\equiv \bigoplus_n \bigoplus_{|\mu|=n} \mathbb{C} a_{-\mu} v_\lambda \\ &= \bigoplus_n F(\lambda)_n \\ \mu &\in Part(n) \\ \lambda &\in \mathbb{C} \\ a_{-\mu} &= a_{-\mu_1} a_{-\mu_2} \cdots a_{-\mu_l} \\ a_0 v_\lambda &= \lambda v_\lambda \\ a_{n>0} v_\lambda &= 0 \end{aligned}$$

16.18.5 Theorem *Each $F(\lambda)$ irreducible and if a_0 acts diagonalizably in some positive energy rep V, then $V \simeq \bigoplus_\lambda n_\lambda F(\lambda)$*

16.18.6 Lemma (Partition Function)

$$\begin{aligned} tr_{F(\lambda)} q^{:H:} &= \sum q^k dim F(\lambda)_k \\ &= q^{1/2\lambda^2} \prod \frac{1}{1-q^j} \\ &= q^{1/2\lambda^2} \sum q^n p(n) \\ q &\equiv e^{-\beta} \\ T \to \epsilon \to 0 &\implies q \to e^{-1/\epsilon} \to 0 \end{aligned}$$

This is the finite and zero temperature cases.

16.18.7 Theorem (Power Sum Realization) *This Fock space is isomorphic as a vector space to a polynomial algebra in countably many generators $\mathbb{C}[p_1 \cdots]$. This means you can identify the action of a_i $i > 0$ with the action of multiplication by p_i and a_{-n} with $-n\frac{d}{dp_n}$.*

16.19 Free Majorana Fermion

16.20 Liouville

You can couple the boson to the scalar curvature

$$S = \frac{1}{8\pi} \int \sqrt{g} (D_g \phi D_g \phi + 2\gamma \phi R)$$

Now the translation of the field by a is not a symmetry, but by the Gauss-Bonet theorem it just changes the action by $\frac{\gamma a}{4\pi} 8\pi(1-g) = 2\gamma a(1-g)$

We are on the sphere now so $g = 0$ with curvature entirely concentrated at ∞. This gives a new neutrality condition $\sum \alpha_k = -i\sqrt{2}\gamma$ which is equivalent to putting a charge $i\gamma\sqrt{2}$ at ∞. This would make sense only if γ was imaginary which makes the action complex. Parameterizing $\gamma = i\sqrt{2}\alpha_0$

The vertex operators are still primary but now of dimension $\alpha(\alpha - 2\alpha_0)$

Because we put α_0 real notice that for $\alpha = \alpha_0 + ia$ the dimension is $(\alpha_0 + ia)(-\alpha_0 + ia) = - \mid \alpha_0 + ia \mid^2$ which is negative real. This means that either this state $\mid h\rangle$ or $L_1 \mid h\rangle$ have negative norm. We could also make sure we are only looking at real α in which case we get negative dimension in the range between 0 and $2\alpha_0$ when the two factors have opposite signs. Otherwise they have the same sign giving positive dimension.

In particular notice if we write $2\alpha_0 = b - b^{-1} = i\beta + i\beta^{-1} = i(\beta + \beta^{-1})$ we get two dimension 1 operators with $\alpha = b$ and $\alpha = -b^{-1}$ which means that $\oint dz V_\pm$ are dimension zero screening operators.

$$\gamma = \frac{i}{\sqrt{2}}(b - b^{-1}) = \frac{-1}{\sqrt{2}}(\beta + \beta^{-1})$$

Because they are dimension 0 we can insert them into conformal blocks and under change $z \to w$, they won't transform any differently because $\left(\frac{\partial w}{\partial z}\right)^0 = 1$ so we can insert enough of them to satisfy the neutrality condition.

For the physical vertex operators, we should be allowed to have two of them meaning we want $\langle V_\alpha(z) V_\alpha(w)\rangle$ to not be zero, but actually we can't stop insertion of screening currents so we should include them.

$$\begin{aligned} \sum \alpha_i + m\alpha_+ + n\alpha_- &= 2\alpha_0 \\ \sum \alpha_i &= (1-m)\alpha_+ + (1-n)\alpha_- \\ \alpha_{r,s} &= \frac{1}{2}(1-r)\alpha_+ + \frac{1}{2}\alpha_- \end{aligned}$$

This gives the Kac formula for conformal dimensions, but r and s are just unconstrained integers and $\alpha_\pm$ haven't been fixed yet.

In order to cut it down the number of primary fields we can identify the results of $\alpha_{r,s}$ with $\alpha_{r+k*q,s+k*p}$ for some p and q so that the distinct fields are parameterized by $\mathbb{Z}_q \times \mathbb{Z}$. Find the unique representative of a given lattice point in the fundamental strip domain for this action.

For this to make sense for their dimensions we see that $q\alpha_+ + p\alpha_- = 0$.

So you get $\alpha_+ = \sqrt{p/q}$ and $\alpha_- = -\sqrt{q/p}$. This successfully gives us the central charge and conformal dimensions for the (p, q) minimal models. (Unitary if $p = q \pm 1$)

But we still have the entire strip in one direction (the s direction by how we chose the fundamental domain). But then you check the three point functions to see the fusion rules and find that you can consistently look at the subalgebra of the Verlinde algebra with $1 \leq s < p$

The way you check the three point functions is by using the four point functions and knowing that after sending the points to $0, 1, z$ and ∞ they are solutions to a hypergeometric equation in the remaining z variable because every second order linear ODE with three regular singular points can be transformed into the Gauss Hypergeometric equation.

16.20.1 Theorem *A second order Fuchsian equation with n singular points has a D_n acting on it's solutions so in this case we have 24 nice solutions sitting inside this 2 dimensional space. This gives lots of connection formula which tell you how expanding a solution around one point would be expressed in the given basis at another point.*

This is part of a general formalism of the monodromies of KZ equations. Take an N point function of vertex operators where the insertion points are radially ordered as $\mid z_{\sigma(1)} \mid << \mid z_{\sigma(2)} \mid \cdots$ for some permutation σ. Changing the permutation by crossing a wall gives you a monodromy matrix from which you can understand the quantum group at the correct root of unity. We took the extremely deep into a chamber limit because that way you could right down the conformal block easily but you could follow the KZ connection as long as you don't cross a wall where you have to cross a branch cut. Then you are in a different chamber and have to apply a Stokes matrix.

16.20.1 Felder Complex

Let $V_{r,s}^{i,j}$ be the screened vertex operator which has $V_{r,s}$, i α_+ screenings and α_- screenings.

In particular knowing $\frac{1}{s}\oint V_{1,-1}^{0,s-1}$ and the same for $p-s$ gives BRST operator on the the complex which has charged Fock spaces

$$\cdots F_{r,2p-s} \stackrel{Q_{p-s}}{\to} F_{r,s} \stackrel{Q_s}{\to} F_{r,-s} \cdots$$

16.20.2 2D Quantum Gravity

There is another perspective that gives the same form of the action. This is given by assuming that ϕ is the Weyl transformation that takes you from metric g to a reference metric $\hat{g}$. Then you plug into the Einstein-Hilbert action with cosmological constant.

$$\begin{aligned} g &= e^{\alpha\phi}\hat{g} \\ S &= \frac{1}{8\pi}\int \sqrt{\hat{g}}(\hat{\nabla}\phi\hat{\nabla}\phi + 2\gamma\phi\hat{R} + \frac{\mu}{\alpha^2}e^{\alpha\phi}) \end{aligned}$$

If $2\gamma = i\sqrt{2}(b - b^{-1})$ and $\alpha = b$ or $-b^{-1}$, then the cosmological constant term would be the screening current. If μ was formal, expanding in μ would give the correct number of screening currents to pull down in order to evaluate a correlation function with Fateev integral methods. Note we would be only pulling down one type of screening current.

But note that in this perspective, ϕ is not a scalar field that lives on the surface; it is a piece of the metric. Under Weyl transformation. This is visible by transformation laws.

$$\begin{aligned} \hat{g} &\to e^{2\rho}\hat{g} \\ \alpha\phi &\to \alpha\phi - 2\rho \end{aligned}$$

Under diffeomorphism

$$\alpha\phi \to \alpha\phi + 2log \mid \frac{\partial w}{\partial z} \mid$$

It is not an ordinary scalar field which would just transform by pullback. It has even more problem than the usual logarithmic OPE.

16.20.3 Higher Rank Toda Field Theory

First for the $\mathfrak{h}$ diagonalize the inner product to break down to $\ell = \dim \mathfrak{h}$ decoupled bosons. Can treat as ℓ tuples of partitions for the Fock space and $p_{n,i}$ and $\frac{d}{dp_{m,i}}$ for $i = 1 \cdots \ell$

16.20.4 Drinfeld Sokolov Reduction

16.21 Boundaries

16.21.1 Definition (Cardy Condition)

16.21.2 Definition (Ishibashi Condition)

16.21.3 Definition (Boundary Changing Operators) *Work on $\mathbb{H}$ with boundary conditions L and R on the left and right part of the real axis.*

16.21.4 Lemma (Annulus) *Work on an annulus viewed as a rectangle with width 1 and height δ. Periodic boundary conditions identifying the 1 sides. Boundary conditions a and b on the two sides. The Hamiltonian is the vertical translation which is the rotation of the annulus.*

$$\begin{aligned} q &= e^{-\delta} \\ Z_{ab}(1,\delta) &= \mathrm{tr}_{H_{ab}}\, q^{L_0} \\ &= \sum_{\Delta} n_{ab}^{\Delta} \chi_{\Delta}(q) \end{aligned}$$

Can also look the other way and think of this as a matrix element between a and b instead.

$$\begin{aligned} \mid a\rangle &= \sum c_{a\Delta} \mid \Delta\rangle \\ \mid b\rangle &= \sum c_{b\Delta} \mid \Delta\rangle \\ \langle a \mid e^{-H*1} \mid b\rangle &= \sum_{\Delta,\Delta'} \bar{c_{a\Delta}} c_{b\Delta'} \langle \Delta \mid q^{-H*1} \mid \Delta'\rangle \\ &= \sum_{\Delta'} c_{a\Delta'} c_{b\Delta'} \chi_{\Delta'}(\tilde{q}) \\ &= \sum_{\Delta'} c_{a\Delta'} c_{b\Delta'} \sum S_{\Delta}^{\Delta'} \chi_{\Delta}(q) \end{aligned}$$

16.21.5 Lemma *This implies*

$$\begin{aligned} c_{a\Delta'} c_{b\Delta'} &= \sum_{\Delta} S_{\Delta}^{\Delta'} n_{ab}^{\Delta} \\ n_{ab}^{\Delta} &= \sum_{\Delta'} S_{\Delta}^{\Delta'} c_{a\Delta'} c_{b\Delta'} \end{aligned}$$

16.21.6 Example (Minimal Models)

16.21.7 Theorem (Fuchs-Runkl-Schweigert)

16.22 Entanglement Entropy

Take the CFT in 1+1 perspective. Suppose there is the ground state ψ, you can compute the induced state for any subsystem. Being a subsystem of a connected 1 dimensional manifold can be described as a disjointed union of intervals. We would like to compute it's Von Neumann Entropy.

$$\begin{aligned} S_{A\subset X} &= -\operatorname{tr} \rho_{A\subset X} \log \rho_{A\subset X} \\ S_{[x,x+\ell]\subset\mathbb{R}} &= \frac{c}{3} \log \frac{\ell}{a} + \cdots \\ \operatorname{tr} \rho_{[0,\ell]\subset\mathbb{R}}^{n} &= c_n \frac{\ell}{a}^{-c/6(n-1/n)} \\ S_A &= -\lim_{n\to 1} \frac{\partial}{\partial n} Zn \\ Zn &= \operatorname{tr}((\rho_{[0,\ell]\subset\mathbb{R}})^n) \end{aligned}$$

Zn is the partition function on a degree n connected cover of the spacetime ramified at the endpoints of the intervals at some instant of time. That is a codimension 2 phenomenon. This is equivalent to taking the correlation function of $2 * N$ twist fields which implement this gluing. We think of each of them as having ramification $e_p = d - 1$ modelled on z^d singularities.

16.22.1 Theorem (Hurwitz)

$$\begin{aligned}
2g-2 &= d(2h-2)+\sum e_p + d - length(\lambda) \\
e_p &= d-1 \\
\mid P \mid &= 2N \\
2g-2 &= (d-1)*(2*N)+d-length(\lambda)+d(2h-2) \\
g &= (d-1)*N+\frac{d-length(\lambda)}{2}+1+d(h-1) \\
g &= (d-1)*(N-1)+(d-1)+\frac{d-length(\lambda)}{2}+1+d(h-1) \\
g &= (d-1)*(N-1)+\frac{d-length(\lambda)}{2}+d*h
\end{aligned}$$

16.22.2 Theorem

$$\begin{aligned}
\lambda &\in Partition(d) \\
r &= (1-2h)d+length(\lambda)+2g-2 \\
2g-2 &= r-length(\lambda)+d(2h-2)+d \\
2g-2 &= d(2h-2)+r+d-length(\lambda)
\end{aligned}$$

Set $d-length(\lambda)=e_\infty$ for a marked point so we can call that a point where there may be additional ramification. Usually we will set $\lambda=1^d$ so that there is no ramification there.

16.22.3 Theorem (Dijkgraaf) *Let $N_{g,d}$ count the number of degree d covers of an elliptic curve $h=1$ ramified simply ($e_p=1$) at a given set of $r=\sum e_p=2g-2$ distinct points with $g\geq 2$ so there is some ramification somewhere. This is the way it has to be if $\lambda=1^d$ the marked point has no ramification. This gives a genus g curve with map to our original E. This defines a groupoid whose objects are covers and morphisms are automorphisms of covers. Take the groupoid cardinality of this and call it $N_{g,d}$. $\sum_{n\geq 1} N_{g,d}q^d$ is a quasimodular form of weight $6g-6$.*

If we were taking 1 instead of $1/\operatorname{Aut}$, this would be the partition function for the trivial theory on all those covering surfaces. Note this was asking for points of ramification 1 rather than $d-1$. This corresponds to elementary transpositions vs the long cycle.

16.23 Orbifolding, Extension, Building Bigger

16.23.1 Theorem (ADE classification of modular invariants) *A_l, D_{2l}, E_6 and E_8. which have Coxeter numbers ... This means that for $SU(2)_k$ can ...*

16.23.2 Definition (Q-System)

16.23.3 Example (D_4)

16.23.4 Definition ($\alpha^\pm$ induction)

16.23.5 Example (D_4 Redux)

Chapter 17

Fractional Quantum Hall

17.0.1 Remark Gaussian units here. ◊

17.1 Integer

A 2D electron gas in the xy plane in a high magnetic field $B\hat{z} + B_x\hat{x}$. We have rotated the sample so that the magnetic field is in this form, so this is a WLOG.

We can gauge fix the vector potential for this by

$$\begin{aligned} A &= (0, Bx, 0) + (0, 0, B_x y) \end{aligned}$$

$$\begin{aligned} H &= \frac{1}{2m}(p - \frac{e}{c}A)^2 \end{aligned}$$

We now assume that the $B_x = 0$.

$$\begin{aligned} H &= \frac{1}{2m}(p_x^2 + (p_y - \frac{eB}{c}x)^2) \\ &= \frac{1}{2m}p_x^2 + \frac{1}{2m}\frac{e^2B^2}{c^2}(\frac{cp_y}{eB} - x)^2 \\ &= \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2(\frac{\hbar k_y}{m\omega} - x)^2 \end{aligned}$$

The eigenfunctions are of the form

$$\begin{aligned} \Psi &= e^{ik_y y}\psi_{SHO}(x - \frac{p_y}{m\omega}) \end{aligned}$$

Put this in an L_x by L_y box with Dirichlet boundary conditions. So now the eigenvalues are labelled by two integers

$$\begin{aligned} k_y &= 2\pi n/L_y \\ E_{n,N} &= \frac{\hbar 4\pi^2 n^2}{L_y^2} + \hbar\omega(N+\frac{1}{2}) \end{aligned}$$

but the center of the oscillator wavefunction is at $\frac{\hbar 2\pi n}{L_y m\omega} = \frac{\hbar 2\pi nc}{L_y eB}$. Constraining this to be within L_x restricts n to be in a finite range in particular

$$\begin{aligned} 0 \le n &\le \frac{m\omega L_x L_y}{2\pi\hbar} \\ &= \frac{BL_xL_y}{hc/e} = \frac{\Phi}{\Phi_0} \\ \Phi_0 &= \frac{hc}{e} \end{aligned}$$

If the particles had charge Ze instead and spin S not locked to the magnetic field, this degeneracy would get multiplied by $Z(2S+1)$.

This is a bit hockey because it counted states of the Harmonic Oscillator if the center of the wavefunction was inside the box. But the Harmonic oscillator states are always infinite in the x direction, so the finite size effects are more subtle even though the area scaling heuristic still holds.

17.1.1 Symmetric Gauge

In this gauge the wave functions are

$$\psi(z) = f(z)e^{-eBz\bar{z}/4\hbar c}$$

where $f(z)$ is analytic. Give the monomials as a basis (that will be completed to a Hilbert Space)

Now the eigenstate with z^n prefactor has expectation value of radius

$$\langle r^2\rangle = (n+1)\frac{2\hbar c}{eB}$$

So in order to stay within the overall radius R

$$n \le \frac{\Phi}{hc/e}$$

again this is a rough argument that neglects the actual wavefunction when there is an edge.

17.2 Fractional/Laughlin Wavefunctions

Let there be many noninteracting particles. Now the antisymmeterized wavefunction is

$$\Psi = \prod (z_i - z_j) e^{\sum_i -eBz_i\bar{z}_i/4\hbar c}$$

You can see that the power of each z_i is $N-1$ so this needs to be less than or equal to $\frac{\Phi}{\Phi_0}$. In order to get fractional filling $1/k$ change the power of the Vandermonde part.

$$\Psi = \prod (z_i - z_j)^k e^{\sum_i -eBz_i\bar{z}_i/4\hbar c}$$

Now the highest power is $k(N-1)$ so the the filling is given by $k(N-1) \leq \frac{\Phi}{\Phi_0}$ or the density is $\frac{N-1}{A} \leq \frac{B}{k\Phi_0}$

There are defects you can insert by

$$\Psi = \prod (z_i - w) \prod (z_i - z_j)^k e^{\sum_i -eBz_i\bar{z}_i/4\hbar c}$$

The insertion at w is $1/k$th of an electron in that it takes k of them to make a single electron. It raises the power of the z_i exponents making the number N have to be lower. In particular it lowers the number of particles by $1/k$ to fit within the same radius. Or inserting k of the holes lowers the number of z_i by 1.

17.3 Anomalous and Semiclassical

17.3.1 Remark Switched to SI ◊

17.3.2 Lemma *If we have a symplectic manifold M with global polarization partitioning by Lagrangian N. Say fixing the values of F_i integrable conserved quantities. Then we can identify after taking a cover if necessary with an open subset of T^*N but with $\omega = \omega_{std} + eB_{ij}dx^i dx^j$.*

$$\begin{aligned}
\pi &= \frac{d}{dp_i} \wedge \frac{d}{dq_i} + eB \frac{d}{dp_1} \wedge \frac{d}{dp_2} + \Omega \frac{d}{dq_1} \wedge \frac{d}{dq_2} \\
\dot{q_i} &= \{H, q_i\} = \dot{q_i}^{B=0,\Omega=0} + \Omega \frac{dH}{dq_j} \\
\dot{p_i} &= \{H, p_i\} = \dot{p_i}^{B=0,\Omega=0} + eB \frac{dH}{dp_j}
\end{aligned}$$

Is this bivector symplectic in the cases of flux quantizations? What about integral periods against H_2, 2π factor on Ω

From Sniatycki Let a particle with charge e be in a external electromagnetic field F. The phase space is $T^*\Sigma$ with $\omega_e = d\lambda_Y + e\pi^*F$. We want this to be quantizable, so we get the integrality condition

$$\begin{aligned} -h^{-1}\omega_e &\in H^2(T^*\Sigma, \mathbb{Z}) \\ \int_\Sigma h^{-1}eF &\in \mathbb{Z} \\ \frac{e\Phi}{h} &\in \mathbb{Z} \end{aligned}$$

Fix the scaling and signs above above to get the below which is how it appears in `Joel` or `https://pentagono.uniandes.edu.co/~acardona/GQMMQHE.pdf` -Original

Take a semiclassical wavepacket

$$\begin{aligned} \dot{r} &= \frac{1}{\hbar}\nabla_k \epsilon_k - \dot{k} \times \Omega \\ \hbar\dot{k} &= -eE - e\dot{r} \times B \\ \Omega &\equiv -Im[\langle \nabla_k u_k \mid\mid \nabla_k u_k \rangle] \\ \epsilon_k &\equiv \epsilon_k^{B=0} - m_k B \\ m_k &\equiv -\frac{e}{2\hbar} Im[\langle \nabla_k u_k \mid H_k - \epsilon_k^{B=0} \mid \nabla_k u_k \rangle] \end{aligned}$$

17.4 Spin Hall

Because we can first consider the noninteracting electrons case, the single body Hamiltonian is

$$\begin{aligned} H &= \frac{p_x^2 + p_y^2}{2m_e} + \alpha_{SO}(p_x \times \sigma) \cdot \nabla V + V \\ V &= \frac{y^2}{2m_e \alpha_{SO}^2} \\ H &= \frac{p_y^2}{2m_e} + \frac{1}{2m_e}(p_x - \frac{\sigma_z}{\alpha_{SO}} y)^2 \end{aligned}$$

This looks like quantum Hall for spin up with magnetic field α^{-1} and for down with $-\alpha^{-1}$. So it is reduced to the previously solved case in order to get BF theory from two Chern Simons theories of opposite levels.

17.5 Abelian Chern Simons

$$\begin{aligned} \sigma_{xy} &= \frac{k}{2\pi} \\ k &= \frac{e^2}{\hbar}\nu \end{aligned}$$

where $J_i = (\sigma_{diag}\delta_{ij} + \sigma_{off}\epsilon_{ij})E_j$
http://physics.stackexchange.com/questions/92809/topological-ground-state-degeneracy-of-sun
For compact semisimple Lie Group G, WZW at level k gives a Hilbert Space of dimension

$$dimV_{g,k} = (C(k+h)^r)^{g-1}\sum_{\Lambda_k}\prod_{\alpha\in\Delta}(1-e^{\frac{i\alpha\cdot(\lambda+\rho)}{k+h}})^{1-g}$$

where C is the order of the center of the group. h is the dual Coxeter number ρ is the half sum of positive roots, r is the rank and Λ_k is the set of integrable highest weights of the Kac-Moody algebra.

If $g = 1$ then this just counts the number of elements in Λ_k for $SU(N)$ this is the counting only partitions that fit inside a N by k box.

See the main chapter for more.

17.6 Haldane-Shastry Spin System

Don't call it a spin chain. You don't think of it as a 1 dimensional quantum system. The particles are on the circle inside the complex plane. Because the interactions are not local in the circle, you can't think of it as a 1+1 dimensional system, it is a specific part of a 2+1 dimensional system.

Take a $SU(2)_1$ theory. Fix $z_1 \cdots z_N$. Compute all the chiral correlators with insertions at those points and those points only. Take those as spanning your Hilbert space. Because this is $SU(2)_1$ the fields are identity and $\phi_{1/2}$, the correlators

$$\begin{aligned} J^0(z) &= \frac{i}{\sqrt{2}}\partial\varphi \\ J^{\pm}(z) &= e^{\pm i\sqrt{2}\varphi} \\ \phi_{1/2,s_i}(z_i) &= \rho_{1/2i} : e^{is_i\varphi/\sqrt{2}} : \\ \rho_{1/2,2i} = 1 & \qquad \rho_{1/2,2i+1} = e^{i\pi(s_i-1)} \\ \psi(s_1, z_1 \cdots s_N z_N) &= \langle \phi_{1/2,s_1}(z_1)\cdots\rangle \\ &= \rho_{1/2}\prod(z_i - z_j)^{s_i s_j/2} \end{aligned}$$

That is the amplitude for that particular configuration of ups and downs as $s_i = \pm$ at that site. The Hamiltonian for this is then cooked up

For this it is useful to impose and define.

$$\begin{aligned} \mid z_i \mid &= 1 \\ w_{ik} &\equiv \frac{z_i + z_k}{z_i - z_k} \\ c_i &\equiv \sum_{k \neq i} w_{ki} \end{aligned}$$

Then the Hamiltonian you get is

$$\begin{aligned} \mathcal{H} &= -\frac{1}{2}\sum_{i \neq j} \frac{z_i z_j}{z_{ij}^2} + \frac{N}{4} - \frac{N+1}{6} T^a T^a - 2\sum_{i \neq j} (\frac{z_i z_j}{z_{ij}^2} + \frac{w_{ij}(c_i - c_j)}{12}) t_i^a t_j^a \\ z_j &= e^{2\pi i j/N} \\ \mathcal{H} &= -\sum_{i \neq j} \frac{z_i z_j}{z_{ij}^2} (2 t_i^a t_j^a - 1/2) - E_0 - \frac{N+1}{6} T^a T^a \\ (2 t_i^a t_j^a - 1/2) &= P_{ij} - 1 \end{aligned}$$

If the z_i are placed around the circle but not at regular intervals you get a generalization of the Haldane-Shastry Model.

17.6.1 Calagero-Moser Relation

Polychronakos's freezing trick

$$H_{spin} = H_{scalar} \pm 4a H_{HS}(x)$$

The first is the spin trigonometric Calagero-Moser-Sutherland, the second is the scalar one. I'm not sure about the sign. But as you send $a \to \infty$, the scalar has a potential that forces everything into evenly spaced positions, but because it's a $\sin^{-2}$ those are angular positions. This recovers the previous z_i positions imposed above.

http://arxiv.org/pdf/1109.5470v2.pdf

17.6.2 Compare Gaudin Model

$$\begin{aligned} Cas_2 L(\lambda) &= \langle \lambda \mid \lambda + 2\rho \rangle \\ \tilde{\lambda} &= \lambda + \rho \\ Cas_2 L(\lambda) &= \langle \tilde{\lambda} - \rho \mid \tilde{\lambda} + \rho \rangle \end{aligned}$$

Figure 17.1: Picture of butterfly

17.7 To Insert

17.7.1 Definition (Spin Connection)

17.7.2 Definition (Wen-Zee Term)

$$N = \nu N_\phi + \nu \bar{s} \chi$$

where the second term is much smaller because taking a limit of large N_ϕ

17.7.3 Definition (Viscous Response)

17.7.4 Definition (Thermal Hall Conductance)

$$\begin{aligned} J_i^Q &= \kappa_{ij} \frac{\partial_j T}{T} \\ \kappa &= \kappa? + \kappa_H? \\ \kappa_H &= c \frac{\pi^2}{3} T \end{aligned}$$

17.7.1 Hofstaeder Butterfly

17.7.5 Remark Simpson's "It was the butterfly" ◊

Consider a two dimensional electron system on a square lattice with lattice spacing a. Fix the factors of a and $\hbar$

$$\begin{aligned} H &= T_x + T_x^\dagger + \lambda(T_y + T_y^\dagger) \\ T_x T_x^\dagger &= T_y T_y^\dagger = 1 \\ T_x T_y &= e^{i\phi/\phi_0} T_y T_x \\ \phi = 0 &\implies E = 2\cos k_x a + 2\lambda \cos k_y a \end{aligned}$$

If we write $T_x = e^{i\Pi_x a/\hbar}$ we get the effective Piererls-Onsager Hamiltonian

$$\begin{aligned} H &= 2\cos \Pi_x \frac{a}{\hbar} + 2\lambda \cos \Pi_y \frac{a}{\hbar} \\ \left[\Pi_x, \Pi_y\right] &= i\phi \frac{\hbar^2}{a^2 \phi_0} \end{aligned}$$

Chapter 18

Chern-Simons

$$\begin{aligned} \Theta &\in \Omega^1(P,\mathfrak{g})^G \\ \forall g \in G \;\; Ad_g R_g^* \Theta &= \Theta \\ \theta &= i_p^* \Theta \;\; \in \Omega^1(X_3,\mathfrak{g})^G \\ \Omega &= F_\Theta = d\Theta - \big[\Theta,\Theta\big] \\ \pi^*\omega &= \Omega \end{aligned}$$

18.0.1 Theorem *$Sym^2(\mathfrak{g})^G \simeq H^4(BG,\mathbb{R})$. In the case of connected, simply connected we get only a one parameter choice $\mathbb{R}\cdot Killing$*

18.1 Abelian Chern Simons

18.1.1 Factorization Algebra

$$\Omega^0 \otimes \mathbb{R}^n \longrightarrow \Omega^1 \otimes \mathbb{R}^n \longrightarrow \Omega^2 \otimes \mathbb{R}^n \longrightarrow \Omega^3 \otimes \mathbb{R}^n$$

is the sheaf of fields.

18.1.2 Belov-Moore

`https://sbseminar.wordpress.com/2010/05/14/lattices-and-their-invariants/`
`https://arxiv.org/pdf/hep-th/0505235v1.pdf`
`https://arxiv.org/pdf/0807.2857.pdf`

18.1.1 Theorem (Rokhlin) *For all compact oriented 3-manifolds without boundary there exists a 4-manifold Z such that $\partial Z = X$. This allows us to extend the principle bundle and a connection extending θ.*

18.1.2 Theorem *Let Λ be a lattice with integral bilinear form B, $\Lambda^* = Hom(\Lambda,\mathbb{Z}) \supseteq \Lambda$.*

18.1.3 Definition (Signature) *Extend scalars of the lattice to $\mathbb{R}$. Then use Sylvester's law of inertia to diagonalize the symmetric bilinear form to give some number of $+1$, -1 and 0. We may then call the signature of a definite form as $\sigma = r_+ - r_-$.*

18.1.4 Definition (Discriminant Group) *$D = \Lambda^*/\Lambda$. It inherits a $\mathbb{Q}/\mathbb{Z}$ valued bilinear form. In case of even lattice, this can be refined to a quadratic form as well.*

18.1.5 Theorem *A representation of the modular group of a spin Chern-Simons thoery defined by lattice Λ is encoded in the invariants $\sigma mod 24$, the discriminant group with it's bilinear form and a quadratic refinement thereof, subject to equivalences when $q_1(?) = q_2(? - \Delta)$ and when q is shifted by constants. The constant constraint can be implemented by a Gauss-Milgram constraint to normalize.*

18.1.6 Corollary *Two quantum spin Chern-Simons theories with lattices Λ_1 and Λ_2 define the same invariants on all closed 3 manifolds if and only if the invariants are all matched.*

`http://www.maths.ed.ac.uk/~aar/papers/conslo.pdf`

18.1.7 Definition (Witt Group) *The Witt group for a field k is given by taking vector spaces over k equipped with symmetric bilinear forms and then imposing equivalence by allowing adding metabolic quadratic spaces like $k \bigoplus k, q = x^2 - y^2$ (assume char $\neq 2$ then get only sums of these). These equivalence classes are the elements of the Witt group and the multiplication comes from orthogonal direct sum. All elements of finite order have order $2^?$.*

18.1.8 Definition (Witt Ring) *Allow tensor product as well. This is totally different from what is also called the ring of Witt vectors.*

18.1.9 Example *$W(k^{alg}) = \mathbb{Z}_2$ for all algebraically closed fields or just quadratically closed.*
$W(\mathbb{R}) = \mathbb{Z}$
$W(F_{q\equiv 3(4)}) = \mathbb{Z}_4$
$W(F_{q\equiv 1(4)}) = \mathbb{Z}_2[F^*/(F^*)^2]$

18.1.10 Theorem (Nikulin 1.13.3) *Suppose we are told that D is the discrimanant group of an even, indefinite lattice of rank $rkL > \ell(D) + 2$ where ℓ is the minimal number of generators as an abelian group. L is determined up to isometry by rank,signature and discriminant form.*

18.1.11 Theorem (Nikulin 1.4.1) *Let L be even lattice. There exists a bijection between isotropic subgroups of D_L and even overlattices L_G of L. The discriminant form D_{L_G} is given by q_L restricted to $G^\perp/G$. Unimodular lattices L_G correspond to isotropic subgroups H with $\mid H \mid^2 = \mid D_L \mid$*

`http://www2.warwick.ac.uk/fac/sci/maths/people/staff/fbouyer/talks/lattices_and_the_picard_group_presentation.pdf`

18.1.12 Lemma *The indefinite lattice $II_{p,q}$ admits a symmetry of order d whenever $p + q = \phi(d)$ and d does not have the form p^e or $2p^e$. This also means $p - q \equiv 0$*
There exists a definite even unimodular lattice of rank $\phi(d)$ with a symmetry of order d whenever $\phi(d) \equiv 0$ modulo 8 and $d \neq p^e$ or $2p^e$.

`https://dash.harvard.edu/bitstream/handle/1/3446009/McMullen_AutomorphismUnimodular.pdf?sequence=5`

18.1.13 Proposition (3.1) *Take two nondegenerate symmetric bilinear lattices (M_i, f_i, c_i) where f is the bilinear form and $c \in M^*$ is such that $f(x,x) - c(x) \in 2\mathbb{Z}$. c's are in bijection with fractional Wu classes for f. These get sent to discriminants by $G_{M,f} = M^\sharp/M$ with $\phi_{M,f,c}([x]) = \frac{1}{2}(f_{\mathbb{Q}}(x,x) - c_{\mathbb{Q}}(x))$ as a map to $\mathbb{Q}/\mathbb{Z}$.*
These lattices are stably equivalent if and only if the associated discriminant+quadratic functions $G_{M,f}, \phi_{M,f,c}$ are isomorphic. Also any isomorphism downstairs on the $G_{M,f}, \phi_{M,f,c}$ lifts to a stable equivalence. With degeneracy this is no longer true.

`https://arxiv.org/pdf/math/0301040.pdf`
`http://www.cornell.edu/video/jacob-lurie-the-siegel-mass-formula`

18.1.14 Theorem (Siegel Mass Formula) *Let q be a positive definite quadratic form over $\mathbb{Z}$*

$$mass(q) \quad = \quad \sum_{q' \in g(q)} \frac{1}{\mid O_{q'}(\mathbb{Z}) \mid}$$

If it is unimodular meaning nondegenerate modulo p for all p. This requires $8 \mid n$. In this case

$$\begin{aligned} mass(q) &= \frac{\Gamma(1/2)\Gamma(1)\cdots\Gamma(n/2)}{2^{n-1}\pi^{(n+1)n/4}}\zeta(2)\zeta(4)\cdots\zeta(n-2)\zeta(n/2) \\ &= \frac{\Gamma(1/2)\Gamma(1)\cdots\Gamma(n/2)}{2^{n-1}\pi^{(n+1)n/4}}\prod_p \frac{p^{n*(n-1)/2}}{\mid SO_q(\mathbb{Z}_p) \mid} \end{aligned}$$

18.1.15 Example *Let $n = 8$ and q come from the E_8 lattice.*

$$\frac{1}{2^{14}3^5 5^2 7} \quad = \quad \frac{\Gamma(1/2)\Gamma(1)\cdots\Gamma(4)}{2^{8-1}\pi^{(8+1)*8/4}}\zeta(2)\zeta(4)\cdots\zeta(8-2)\zeta(8/2)$$

18.2 Geometric Quantization of CS

18.2.1 Definition (Atiyah-Bott Moduli space)

$$\omega \quad = \quad \int_\Sigma \left[\delta A \wedge \delta A\right]$$

18.2.2 Definition (Prequantum Vector Space) *Modulo gauge transformations to get the moduli space $\mathcal{M}^s(G,\Sigma)$ where have imposed stability to avoid stackiness. Now that we have a finite dimensional symplectic manifold with a line bundle over it, we may take the sections.*

18.2.3 Definition (Segal-Bargmann Space) *Holomorphic functions on $\mathbb{C}^n$ such that $\phi(z)e^{-1/(2\hbar)|z|^2}$ is in L^2. Maybe $\hbar = 1$ in some definitions, but that just rescales the $\mathbb{C}^n$. There are annihalation and creation $a_i = \partial_{z_i}$ and $a_j^\dagger = z_j \cdot$ which by Stone-Von-Neumann have a unitary Segal-Bargmann transform to the usual CCR on $L^2(\mathbb{R}^n)$.*

$$\begin{aligned}
(Bf)(z) &= \int_{\mathbb{R}^n} \exp(-(z\cdot z - 2\sqrt{2}z\cdot x + x\cdot x)/2) f(x)dx \\
\rho &= \pi^{-n} \mid (Bf)(z) \mid^2 \exp(- \mid z \mid^2) \\
f(x) &= \int_{\mathbb{C}^n} \exp(-(\bar{z}\cdot\bar{z} - 2\sqrt{2}\bar{z}\cdot x + x\cdot x)/2)(Bf)(z)e^{-|z|^2}dz \\
f(x) &= \pi^{-n/4}(2\pi)^{-n/2}\exp(- \mid x \mid^2 /2)\int_{\mathbb{R}^n}(Bf)(x+iy)\exp(- \mid y \mid^2 /2)dy \\
A_j &= (a_j + a_j^\dagger)/2 \\
B_j &= (a_j - a_j^\dagger)/2
\end{aligned}$$

18.2.4 Definition (Complex Polarization) *Pick a point in the Teichmuller space $J \in T$. We may use this complex structure to polarize.*

18.2.5 Theorem (Hitchin-Witten Connection) *This follows for all τ. In fact, we may globalize to build a bundle over T. This bundle has connection given by Hitchin-Witten. In fact it is given explicitly by:*

Proof `AndersenandGamelgaard` □

18.2.6 Lemma (Mapping Class Group Action)

18.2.7 Definition (Gelfand-Zak Transform)

Chapter 19

3d Topological

19.1 Fusion Category Things 2

19.1.1 Definition (Braided)

19.1.2 Definition (Symmetric)

19.1.3 Definition (Modular)

19.1.4 Definition (Center)

19.1.1 Pointed

19.1.5 Definition (Pointed) *A tensor category is called pointed if every simple object is invertible.*

19.1.6 Definition ((Pre)Metric Group) *A pre-metric group is a finite abelian group A equipped with a quadratic form. A quadratic form on a group satisfies $q(g) = q(g^{-1})$ such that the symmetric function $b(g,h) \equiv \frac{q(gh)}{q(g)q(h)}$ is a symmetric bicharacter (bilinearity if written additively). If that symmetric form is nondegenerate, then (A,q) is a metric group not just a pre-metric group. An orthogonal homomorphism $(A,q) \to (A',q')$ is a homomorphism of groups $f \in Hom_{Ab}(A,A')$ satisfying $q' \odot f = q$.*

19.1.7 Theorem (Eilenberg-Mac Lane) *Let k be an algebraically closed field of characteristic 0 and $H^3_{ab}(A,k^*)$ denotes a quotient of the group of pairs (ω,c) which are functions on A^3 and A^2 respectively landing in k^* such that using ω for the associator and c for the braiding on a skeletal pointed braided fusion category whose group of simple objects is A. Explicitly this means the following equations.*

$$\begin{aligned}
\omega(g_1g_2,g_3,g_4)\omega(g_1,g_2,g_3g_4) &= \omega(g_1,g_2,g_3)\omega(g_1,g_2g_3,g_4)\omega(g_2,g_3,g_4) \\
\omega(g_2,g_3,g_1)c(g_1,g_2g_3)\omega(g_1,g_2,g_3) &= c(g_1,g_3)\omega(g_2,g_1,g_3)c(g_1,g_2) \\
\omega(g_3,g_1,g_2)^{-1}c(g_1g_2,g_3)\omega(g_1,g_2,g_3)^{-1} &= c(g_1,g_3)\omega(g_1,g_3,g_2)^{-1}c(g_2,g_3)
\end{aligned}$$

The quotient is where (ω,c) are of the form

$$\begin{aligned}\omega(g_1,g_2,g_3) &= k(g_2,g_3)k(g_1g_2,g_3)^{-1}k(g_1,g_2g_3)k(g_1,g_2)^{-1}\\ c(g_1,g_2) &= k(g_1,g_2)k(g_2,g_1)^{-1}\end{aligned}$$

The operation is pointwise multiplication. On this quotient, we can send the equivalence class of (ω, c) to $q \equiv c(-,-)$ which is a quadratic form on A with values in k^*.

This map is in fact an isomorphism.

Proof Section 8.4 of EGNO Tensor Categories book □

19.1.8 Theorem *There is an equivalence of categories between the category of pointed braided fusion categories and isomorphism classes of braided tensor functors and the category of pre-metric groups and orthogonal homomorphisms between them.*

19.2 Dijkgraaf Witten

Let G be a finite group, $\alpha \in Z^3(G, U(1))$

$$\begin{aligned}\mathbb{C}[G] &\simeq \bigoplus V_i^* \otimes V_i\\ \chi_i &\equiv \chi(V_i) = (g \to \operatorname{tr} \rho_{V_i} g)\\ \delta_e(-) &\in \mathbb{C}[G]^* = \sum_i \frac{\dim V_i}{\mid G \mid}\chi_i(-)\end{aligned}$$

$Hom(\pi_1(X), V)$ is in bijection with G labelings of X. That is each triangle goes to a (g_1, g_2, g_3) subject to $g_1g_2g_3 = e$ following the orientation. $G^{|V|}$ acts by multiplying all the labels incident on each vertex $v \in V$.

19.2.1 Closed 3-manifold

The space of fields on X a closed connected oriented 3-manifold, F_X is the finite set of G-bundles over X up to equivalence. It can be described as $\frac{\pi_1(X)\to G}{G}$

$$\begin{aligned}e^{2\pi i S_x(\phi)} &= e^{2\pi i \langle \bar{F}^*\alpha || [X]\rangle}\\ Z(X) &= \sum \frac{1}{Aut(\phi)} e^{2\pi i S_X(\phi)}\end{aligned}$$

Can rewrite the space of fields as the framed at x version and then quotienting out this framing. Makes a groupoid. $(\phi, \psi) \to \phi$. Or with multiple points with framings.

$$\begin{aligned}Z(X) &= \frac{1}{\mid G \mid}\int_{F_{X,x}} e^{2\pi i S_X(\phi,\psi)}\\ Z(X) &= \frac{1}{\mid G \mid^n}\int_{F_{X,x_1\cdots x_n}} e^{2\pi i S_X(\phi,\psi_1\cdots\psi_n)}\end{aligned}$$

19.2.1 Example (S^3) $Z(X) = \frac{1}{|G|}$

19.2.2 Example ($S^2 \times S^1$) $Z(X) = 1$

19.2.2 With Boundary

Replace $[X]$ with relative cohomology.

- Make a choice via triangulations - Dijkgraaf-Witten, ?
- All choices - Freed

Fix triangulation T for X. Then pick a total ordering of the 0-skeleton X^0. So make F_{X,X^0} which is equivalent to $Hom(\pi_{\leq 1}(X, X^0), G)$

19.2.3 Definition ($Co(X,T)$) *A color Q of X, T is a map sending oriented edges to G. By doing the open path holonomy which is OK because we trivialized each of the X^0. This satisfies reversing orientation gives inverse, and 2-simplices satisfy the product.*

Fix $\tau \in Co(\partial X, T_\partial)$, then make $Co(X, T, \tau)$ so that the bundle agrees with the trivialization at the boundary.

Tetrahedron

Insert image of tetrahedron and backbone

19.2.4 Definition (Backbone)

This gives map $\sigma \to BG$ giving the 3-cell in BG given by $\phi = [g \mid h \mid k]$

$$\begin{aligned} w(\sigma, \phi) &= e^{2\pi i \tilde{\alpha} \cdot \phi} \\ Z_X(\tau) &= \frac{1}{\mid G \mid^{X^0 - (\partial X)^0}} \sum_{\phi \in Co(X,T,\tau)} \prod_\sigma w(\sigma, \phi)^{\pm 1} \end{aligned}$$

19.2.5 Theorem (DW,Walker) *Independent of triangulation, orderings if we fix the boundary triangulation*

19.2.3 TQFT

19.2.6 Lemma *The linear map for cobordism*

$$\begin{aligned} \Phi_X(\tau) &= \sum \Phi_X(\tau' \bigsqcup \tau)\tau' \\ \Phi_{X_{12}} &= \Phi_{X_2}\Phi_{X_1} \\ \Phi_{Y\times I} &= \Phi_{Y\times I}\Phi_{Y\times I} \end{aligned}$$

19.2.7 Theorem (Kuperberg) *If you have some formal expression in the structure constants for a finite dimensional semisimple Hopf algebra. It gives you a 3-manifold. You can prove an identity for the Hopf algebra from the axioms of Hopf algebras if and only if the 3-manifolds agree.*

Proof Heegard splitting presentation and each elementary move of Heegard diagrams gives a basic Hopf algebra identity. □

19.2.8 Example *Associativity of multiplication.*

19.2.9 Theorem (Easier Version) *Semisimple algebras with a trace. You get a triangulated surface. Changing triangulations corresponds to identities in semisimple algebras with trace.*

19.3 Kitaev Double

19.3.1 Definition (Smash Product) *A smash product of A and H with H acting on A by $\triangle$. As a vector space it is the tensor product of the underlying vector spaces, but the product structure is*

$$(a \otimes h)(a' \otimes h') \quad = \quad \sum a(h_{(1)} \triangle a') \otimes h_{(2)} h'$$

H sits inside as $1 \otimes H$ and you can check that with the product above this inclusion map intertwines with the product in H.

See Susan Montgomery actions of finite dimensional Hopf algebras and smash products

19.4 Levin-Wen

19.4.1 Definition (Dualizing Tensor Autoequivalence) *For a fusion category $\mathcal{C}$. A tensor autoequivalence $F\, \mathcal{C} \to \mathcal{C}$ is called dualizing if $F(X) \simeq X^*$ for all objects X.*

19.4.2 Proposition ([?] Cor 2.2) *A dualizing tensor autoequivalence F is unique up to the composition with a soft tensor autoequivalence (one isomorphic as a functor to the identity functor).*

19.4.3 Theorem ([?] Thm 2.9) *The Drinfeld center $\mathcal{Z}(G)$ of a finite group has a dualizing braided autoequivalence if and only if the group G has a double class-inverting automorphism.*

19.4.4 Example (Abelian) *An abelian group has $\phi(g) = g^{-1}$ which inverts classes.*

19.4.5 Example (Alternating Groups , [?] 2.22) *The alternating groups A_n has a class-inverting automorphism if and only if $n = 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14$*

19.5 Turaev-Viro

19.6 Schommer-Pries-Douglas-Snyder

19.7 Reshetikhin Turaev

19.7.1 Definition *(k, l) ribbon graph is an embedding of a disjoint union of $[0, 1]^2$ ribbons into $\mathbb{R}^2 \times [0, 1]$ such that the intersections with the top and bottom meed at the intervals specified k and l respectively.*

Figure 19.1: Insert figure here

19.7.2 Definition (Homogenous) *It can be isotoped so that consistent forward orientation. Namely no half twists but full twists are allowed.*

19.7.3 Definition (Colored) *A labeling on the ribbon graph by objects of Rep(A) for all the connected components.*

19.7.4 Definition (Target Category) *Objects are $(V_1, \pm_1) \times \cdots (V_k, \pm_k)$. These provide the labels on the $\mathbb{R}^2 \times \{0\}$. The morphisms are then given by (k, l) homogenous colored ribbon graphs such that the sources and target colors and orientations match. These are given up to isotopy.*

19.7.5 Definition (Coupon) *Can also stick in coupons decorated by morphisms.*

19.7.6 Theorem *This is a rigid braided monoidal category.*

Proof Show the pictures for braided and rigid.

Figure 19.2: Braiding

Figure 19.3: Rigidity

19.7.7 Definition (Ribbon Hopf Algebra)

19.7.8 Theorem *Functor from tangle category to $(RepA)^{ss}$ satisfying colors go correctly, preserves tensor products exactly, caps cups go to evaluations and coevaluations. Essentially all the data you built in to build the tangle category matches up.*

Chapter 20

K Theory Classification

20.1 Wigner Classification

20.1.1 Theorem (Wigner's Theorem) *We will call this classifier $\phi\, G \to \mathbb{Z}_2$.*

Proof

20.1.2 Definition (Time Reversal) *t which says whether it reverses time.*

20.1.3 Lemma *Unless the spectrum is symmetric, $\phi = t$.*

Proof

$$\begin{aligned}
\rho(g)U(\tau)\rho(g)^{-1} &= U(t(g)\tau) \\
\rho(g)H\rho(g)^{-1} &= \phi(g)t(g)H \\
\rho(g)iH\rho(g)^{-1} &= \phi(g)i\rho(g)H\rho(g)^{-1} = \phi(g)i\phi(g)t(g)H = t(g)iH \\
\rho(g)iH &= t(g)iH\rho(g) \\
t(g) = -1 &\implies \{\rho(g), iH\} = 0 \\
\phi(g)t(g) = -1 &\implies \{\rho(g), H\} = 0 \\
\rho(g)U_2(\tau)\rho(g)^{-1} &= U_2(t_2(g)\tau) \\
\rho(g)H_2\rho(g)^{-1} &= \phi(g)t_2(g)H_2
\end{aligned}$$

The first one is time translation and it's generator Hamiltonian. The second pair is another continuous symmetry and it's generator. For example it could be translation then τ would be the distance x of translation. I haven't said whether U and U_2 commute. In this example, there could be Ahranov Bohm phases when doing the loop in x-t space.

20.1.4 Example *Consider a free fermion on a circle, the Hilbert space has time translation and translation along the circle. We then ask for symmetries g that send time translation to itself possibly flipped.*

τ doesn't have to be a single number for U_2, $\frac{i}{\hbar}\tau_a H^a$ like if this was symmetry under arbitrary rotations. Then $t_2(g)$ has to be an automorphism of the Lie algebra which sends $\frac{i}{\hbar}\tau_a H^a$ to $\frac{i}{\hbar}\tau'_a H^a$

Why does $t(g) = \pm 1$? Why not a symmetry that rescales the clock by α. Then repeated applications of $\rho(g)$ would keep scaling τ up further and further which conflicts with the fact that $\rho(g)^n = \rho(g^n) = Id$. Therefore we require that $t_2(g)$ the automorphism of the Lie algebra needs to be finite order. $t_2\ G_{fin} \to Aut(\mathfrak{g})$

20.1.5 Theorem (Kramers) *If $c = \phi t = 1$ time-reversal nature of a symmetry and antiunitarity are the same thing and there is a $T \in G$ such that $T^2 = -1$, then the Hilbert space has a quaternionic structure given by the i we already have as a complex Hilbert space. T as a new complex structure which anticommutes with the original complex structure and their product $k = iT$.*

20.1.6 Corollary (Kramer's Degeneracy) *Let's say the Hilbert space splits as positive and negative energies then it can be regarded as a $\mathbb{Z}_2$ graded vector space. g such that $\phi(g)t(g) = -1$ switch positive and negative eigenvectors so they are the odd part of the supervector space $End(\mathcal{H})$.*

$$\begin{aligned} \phi(g)t(g)H \mid \lambda\rangle &= \phi(g)t(g)\lambda \mid \lambda\rangle = \rho(g)H\rho(g)^{-1} \mid \lambda\rangle \\ H(\rho(g)^{-1} \mid \lambda\rangle) &= \phi(g)t(g)\lambda(\rho(g)^{-1} \mid \lambda\rangle) \end{aligned}$$

This is except for zero modes in which case $\rho(g)^{-1} \mid \lambda\rangle$ just becomes a zero mode and there is no reason for it not to be the same state.

20.1.7 Definition (Witten Index) *$Tr(-1)^F e^{-\beta H}$ In a supersymmetric theory, every nonzero energy eigenvalue contains an equal number of bosonic and fermionic states. Therefore the trace just focuses on the zero eigenvalues and hence doesn't depend on temperature β. If SUSY is spontaneously broken, then there are no zero energy ground states so the Witten index is 0.* [1]

So if the system is gapped and has such a $\phi(g)t(g) = -1$, then take $\rho(g)^{-1}$ as an odd operator that takes the negative energy states which are odd to the positive which are graded even and vice versa or you may also use the opposite gradings. From this you can construct

$$\begin{aligned} Q &= 0_{-\to+} \bigoplus \pm i\sqrt{\mid \lambda \mid}\rho(g)^{-1}_{+\to-} \\ Q^\dagger &= 0_{+\to-} \bigoplus \mp i\sqrt{\mid \lambda \mid}\phi(g)\rho(g)_{-\to+} \\ Q^\dagger &= 0_{+\to-} \bigoplus \rho(g)\sqrt{\mid \lambda \mid}(\mp i) \\ QQ^\dagger &= 0_{+\to+} \bigoplus \mid \lambda \mid \phi(g)^2_{-\to-} \\ Q^\dagger Q &= 0_{-\to-} \bigoplus \mid \lambda \mid \phi(g)^2_{+\to+} \\ Q \to e^{i\theta}Q \qquad & \qquad Q^\dagger \to Q^\dagger e^{-i\theta} \end{aligned}$$

20.2 Clifford Algebra and the 10 fold way

Say we have two symmetries implemented antilinearly. $\hat{T}$ and $\hat{C}$. As a group, before considering their representation we have a Klein-four group.

Pick a subgroup III that is supposed to be protected, then we get a ten-fold way by choosing the subgroup and signs for T^2 and C^2 of the lifts.

[1] If there is a single complex parameter z and we're on a $4k$ dimensional field theory, you could get the Witten genus which is a modular form of weight $2k$ when the first Pontryagin class vanishes.

20.2.1 Clifford algebra essentials

20.2.1 Theorem *$Cliff$ is a functor with objects vector spaces with quadratic form going to superalgebras. For morphisms .... It is monoidal for $\bigoplus$ on the source and $\hat{\otimes}$ for the target.*

Proof As superalgebras for $U \bigoplus V$

Send $u \bigoplus v \to u \otimes 1 + 1 \otimes v$ in $Cl(U)\hat{\otimes}Cl(V)$ where doing the tensor product as Z_2 graded associative algebras. Then after a quick calculation, by universality there is homomorphism $Cliff(U \bigoplus V) \to Cliff(U)\hat{\otimes}Cliff(V)$. It is an isomorphism as superalgebras.

A caution for the usual notation which labels a positive or negative definite V as $\pm n$ depending on signature. So $Cl_{+n} \equiv Cl(\mathbb{R}^{n,0})$ and $Cl_{-n} \equiv Cl(\mathbb{R}^{0,n})$. Then $Cl_n \otimes Cl_m \simeq Cl_{n+m}$ if and only if they have the same sign. Otherwise you are getting $R^{n,m}$ but the thing you labelled was R^{n+m} when $n \geq 0$ and $m \leq 0$. But even if you don't have the same sign behavior, you still get a Morita equivalence.

20.2.2 Proposition $Cliff(V_{\mathbb{C}}, Q_{\mathbb{C}}) \simeq Cliff(V, Q) \otimes \mathbb{C}$

20.2.3 Theorem *The Clifford algebra for $(U, Q_U) \bigoplus (V, Q_V)$ with U even dimensional is isomorphic to the tensor product $Cliff(U, Q_U) \otimes Cliff(V, (-1)^{dim(U)/2} d(U) Q_V)$. This is using the ungraded tensor product so it is not useful anymore.*

20.2.2 10 Fold Way

$$\begin{array}{rcl} CiC & = & -iCC \\ C\ CT & \neq & -CT\ C \\ C\ iCT & = & -iCT\ C \\ iC\ iCT = CCT & & iCT\ iC = -CTC = -CCT \\ iC\ CT & \neq & -CT\ iC \end{array}$$

This means that $C\ iC$ and iCT form a Clifford algebra where you need to specify the those 3 signs but the first two need to be the same. They give C^2 and the last gives $iCTiCT = -C^2T^2$

The signs could be

- $++-$ This means $C^2 = +1$ and $T^2 = +1$ so the time reversal turns it into a real vector space and the C becomes a $Cliff(+1)$ structure on the real vector space given by the fixed points of T.

- $+++$ This means $C^2 = +1$ and $T^2 = -1$ so that time reversal turns it into a quaternionic vector space. But all three together give a $Cliff(+3)$ structure on the underlying real space.

- $--+$ This means that $C^2 = -1$ and $T^2 = +1$ so that time reversal turns it into a real vector space and the C becomes a $Cliff(-1)$ structure on the fixed subspace of T.

- $---$ This means that $C^2 = -1$ and $T^2 = -1$ so that time reversal turns it into a quaternionic vector space. But all three give a $Cliff(-3)$ structure on the real space.

5. If the group is trivial, we get nothing more just the $Cliff(0) = \mathbb{C}$ action already present as a vector space.

6. If the group is the diagonal there is an odd linear operator S which squares to 1. It generates the $Cliff^{\mathbb{C}}(+1)$ Using iS would give -1 instead, but they are Morita equivalent.

Next 2 choices: The group is just T.

- $T^2 = +1$ That means we take the fixed points of the Hilbert space and that's it so we are doing $Cliff(0) = \mathbb{R}$

- $T^2 = -1$ That means we just have a quaternionic vector space. So look at $V \bigoplus V[1] = V \otimes \mathbb{R}^{1|1}$ as a real super vector space and the quaternion structure on V gives a Cliff(+4) structure on this.

Last two choices. The group is just C.

- $C^2 = -1$ so on the underlying real space we have iC, C giving a $Cliff(-2)$

- $C^2 = +1$ so on the underlying space we have iC, C giving a $Cliff(+2)$

These are all 10 classes.

20.2.4 Theorem (Kane-Mele invariants) *Strong invariants are*
The weak invariants are

- $BiSe_2$

20.3 Twisted K Theory

Give a $\mathbb{Z}_2$ graded vector bundle over every object. That we already have. It is the bands given by $H(k)$. Now for every arrow instead of just giving a map between those vector spaces at each fiber, we screwed up by a line.

$$\begin{aligned} L^\tau &\downarrow \mathcal{G}_1 \\ L^\tau &= (G^\tau \times \mathbb{C})/U(1) \\ (\tilde{g}, z)\lambda &= (\tilde{g}\lambda, \lambda^{-1}z) \\ \lambda_{\gamma_2\,\gamma_1}{}^{\phi}(\gamma_1) L^\tau_{\gamma_2} \otimes L^\tau_{\gamma_1} &\to L^\tau_{\gamma_2\gamma_1} \\ \rho^\tau(\gamma)^\phi(\gamma)(L^\tau_\gamma[1 \text{ if } c(\gamma) == -1] \otimes W_{x_0}) &\to W_{x_1} \end{aligned}$$

where ϕ indicates whether or not to conjugate the vector space.

So you know you have an action of one of these Clifford algebras on your Hilbert space, what do you do. But my Hilbert space came to me as a direct sum over characters of the translation group. $Hom(\mathbb{Z}_L^d, U(N))$. That is $\alpha^{1/L} e^{i2\pi m/L}$ where m is any integer But only $m(modL)$ matters. α is the phase you pick up if you do a full loop in configuration space around the given cycle. Because of this magnetic field it is not possible to ask for the trivial representation of translations. We only have a torsor for Hom That parameterizes the real space connection. Doing a loop in the Brillouin Zone means doing a loop $m \to m + L$. This takes you back to the same point in the quasimomentum torus, but it does give nontrivial action on the fiber through Berry's phase.

20.3.1 Example *Picture this as a 2d dimensional lattice. The first d are the spatial points and the last d and are given by $(n_i a_i)$ where the a_i are primitive translation vectors. $n_i \to n_i + L_i$ are identified. The last d are the quasimomenta, they are parameterized by $\alpha_{1/L_i} e^{i2\pi m_i/L_i}$ with $m_i \to m_i + L_i$.*

Above every point (n_i, m_i), look at the fundamental unit cell around site n_i and with patching conditions to the neighboring cells given by m_i. So if I give you a section u of a bundle above the fundamental cell with periodic boundary conditions, I can give you a new section with the correct boundary conditions specified by m_i by multiplying $u(\hat{x_i})$ where $\hat{x_i}$ is the representative in the fundamental cell $e^{(x_i - n_i)(2i\pi m_i/L_i + \log\alpha/L_i)}$. This is a section over all real space which by construction has all the right monodromies for the magnetic field. But this depends on a choice for the m_i as integers rather than as elements of $\mathbb{Z}_{L_i}$. That is the choice of a fundamental Brilloiun zone.

20.3.2 Definition (Vectorial Bundle)

20.3.3 Lemma (D.15 of Freed-Moore) *Let X and Y be locally compact Hausdorff topological spaces with Borel measures and assume Y is compact. Let $\mathcal{L} \to X \times Y$ be a Hermitian line bundle. The Hilbert spaces $\mathcal{E}_x = L^2(Y, \mathcal{L}_{x\times Y})$ combine as fibers of a Hilbert bundle $\mathcal{E} \to X$ and there exists a isomorphism of Hilbert spaces $L^2(X \times Y, \mathcal{L}) \simeq L^2(X, \mathcal{E})$*

20.3.4 Corollary *Let $X = X_\Pi$ and $Y = E/\Pi$ be the Brillouin Zone Torus and Fundamental Unit Cell Torus respectively. On this product there is the Poincare Line bundle defined by the condition*

$$f(\lambda, x + \xi) \quad = \quad \lambda(\xi) f(\lambda, x)$$

on functions on $X_\Pi \times E$ which can now be viewed as a line bundle over $X_\Pi \times E/\Pi$.
Then we get isomorphisms of Hilbert spaces $L^2(E, \mathbb{C}) \simeq L^2(X_\Pi \times E/\Pi, \mathcal{L}) \simeq L^2(X_\Pi, \mathcal{E})$. In fact we may apply the functors $\otimes W$ and the Fock functor to these isomorphisms.

20.3.5 Corollary *If $F \xleftarrow[p]{i} \mathcal{E}$ is an inclusion and orthogonal projection of a finite rank sub-bundle which is cut out by a spectral projection of an elliptic Hamiltonian, we may induce a new connection $p\nabla^{\mathcal{E}} i$ and in fact in this case F is a smooth vector bundle because of elliptic regularity.*

20.3.1 Examples for some Wallpaper Groups

`https://arxiv.org/pdf/1701.08725.pdf` computes ${}^{\phi}K_P^{\tau}(T^2)$ for P a wallpaper point group.

20.3.2 3D groups

Computations for all the space groups are tabulated by giving ...

20.4 $K(BG)$

20.4.1 Localization Theorem

Had ${}^{\phi}K_P^{\tau}(X_\Pi)$. Let $k \in X_\Pi$ be a point with increased stabilizer $e \subset P_k \subset P$.
Relation with $\bigoplus_k K(BP_k)$?

20.4.1 Theorem $K_G(X) \to K_G^{Bor}(X)$
Fixed loci in X...

20.4.2 p-adic

20.4.2 Theorem ([?] Lück) *For finite groups* G[2]

$$\begin{aligned} K^0(BG) &= \mathbb{Z} \times \prod_p (\mathbb{Z}_{(p)})^{r(p,G)} \\ K^1(BG) &= 0 \end{aligned}$$

where $r(p,G)$ is the number of conjugacy classes C such that $g \in C$ will have order p^d for some $d \geq 1$. Similarly define $\tilde{r}(p,G) = r(p,G) + 1$ with $d = 0$ also allowed.

Similar formulas for other discrete groups.

20.4.3 Definition (Residue Field) *Take the ring of integers $\mathcal{O}$ in a nonArchimedian field given by $\mid a \mid \leq 1$. This has a unique non-zero prime ideal $\mid a \mid < 1$. The quotient $\mathcal{O}/m$ is called the residue field. In particular $\mathbb{Z}_{(p)}$ gives $\mathbb{Z}_p$.*

20.5 1D Electric Polarization

`http://www.physics.rutgers.edu/~dhv/talks/rahman.pdf`

For band theory you have to do periodic boundary conditions, so it is a little tricky. You don't have excess charge at the ends that you can look at. Instead you calculate the holonomy of the Berry connection around the entire circle. Both are defined modulo integers. For polarization, this is because on an interval you can add an electron at the end which is the same as moving everything over together. The Berry's phase has a similar ambiguity because you are looking at a $U(1)$ bundle on a circle so there are large gauge transformations where the phase changes by a multiple of 2π as you make a full loop. The change in polarization is well defined however, because now we have something to compare it to.

Construct Wannier orbitals which are Fourier transforms of Bloch states. They are associated to each lattice point and they are nice and localized. These are ambiguous by changing $\mid u(k)\rangle \to f(k) \mid u(k)\rangle$. Now Fourier transforming this will totally change the state.

[2]Parenthesis are used to distinguish p-adic vs cyclic groups.

20.6 TKNN

2-Torus Brillouin zone, take the x-axis loop at 0, and move it up to the x-axis loop at 1. Those are really the same line so they must give the same Berry's phase holonomy up to an integer. You can ask what that integer is. This is the first Chern number.

More invariantly this is given as $\int_{Sq} dA = \int_{0\times x} A - \int_{1\times x} A \in \mathbb{Z}$ where Sq is the rectangle bounded by the sides of the torus that are being identified. This is also just the first Chern class of the bundle that the Berry is a connection on. In fact you could just take the full K-theory class of this bundle without having to do any Chern character to map to cohomology and then subsequently integration to get to numbers.

Laughlin argument, change the integer quantum hall effect picture into a cylinder at thread a magnetic flux from 0 to one flux quantum along the axis. The 1D polarization changes by a number n*e. In the one dimensional picture the variables are k_x and the flux threaded through, because now you don't have k_y

For integer quantum hall with 0 periodic potential, you can artificially define a lattice by having lattice translations that commute even though their infinitisimals do not because magnetic flux.

20.7 Graphene

Similarly work out $H(k_x, k_y) = d(k_x, k_y) \cdot \sigma$, d has vortices so it must go to 0 at the center of that vortex. Expand around those points and see a Dirac cone. Berry's phase of π upon going around the cone. This was without spin.

If inversion or time reversal are broken, the two Dirac cones open up. Broken P example Boron Nitride, the masses at the $\pm k_0$ are the same. So $d(k)$ goes around the equator and goes up to one of the poles at the conical points. So this is the trivial element of $\pi_2(S^2)$ This gives 0 Chern number. If you break time reversal, then the masses are opposite on the two sides, so now you wrap the sphere once.

20.7.1 Edge States

20.7.1 Remark ?

On one side, the two masses have the same sign and on the other they have opposite signs. So write plane waves in the y-direction and now it is reduced to Jackiw-Rebbi problem. What is the dispersion as a function of k_y it is linear. This gives a chiral Dirac fermion. They only have one side of the cone.

Electrical transmission in these states is perfect even in the presence of disorder. In addition they are states that can't exist in purely 1 dimensional materials. If it was purely 1D the dispersion would have to be a periodic function on the Brillouin zone so it has to have downward sloping dispersion relation somewhere.

Index of left moving - right moving is the topological invariant that can't change without closing the bulk energy gap.

20.8 Haldane Honeycomb

`http://www.int.washington.edu/talks/WorkShops/int_15_1/People/Esslinger_T/Esslinger.pdf`

20.9 p+ip superconductor

20.9.1 Example *Consider a spinless toy example*

$$\begin{aligned} H &= \sum_k c_k^\dagger (\frac{k^2}{2m} - \mu) c_k + \Delta e^{i\phi} c_k^\dagger (k_x + i k_y) c_{-k}^\dagger + h.c. \\ &= \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} \begin{pmatrix} \frac{k^2}{2m} - \mu & \Delta e^{i\phi} k_x + i\Delta e^{i\phi} k_y \\ \Delta e^{-i\phi} k_x - i\Delta e^{-i\phi} k_y & \frac{-k^2}{2m} + \mu \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} \\ H(k) &= \begin{pmatrix} \frac{k^2}{2m} - \mu & \Delta e^{i\phi} k_x + i\Delta e^{i\phi} k_y \\ \Delta e^{-i\phi} k_x - i\Delta e^{-i\phi} k_y & \frac{-k^2}{2m} + \mu \end{pmatrix} \end{aligned}$$

From this we have $H(k) \in Maps(\mathbb{R}^2, i\mathfrak{su}(2))$ but if we ask that it extend to S^2 and never go to 0 we get a map from $S^2 \to \mathbb{R}^3 \setminus \{0\}$. Homotopy classes thereof are classified by $\pi_2(S^2) \simeq \mathbb{Z}$.

20.10 Weyl Semimetal

`https://arxiv.org/pdf/1301.0330v1.pdf`

There are linear dispersion relations at 2-fold degeneracy points. Near these degeneracy points in the Brillouin zone, the states are described by a Weyl equation

$$\begin{aligned} H &= -\sum \left[2t_x(\cos k_x - \cos k_0) + m(2 - \cos k_x - \cos k_y) \right] \sigma_x \\ &+ 2t_y \sin k_y \sigma_y + 2t_z \sin k_z \sigma_z \\ p_c &= (\mp k_0, 0, 0) \\ p_{\pm x} &= \pm k_x \mp k_0 \\ p_{y,z} &= k_{y,z} \\ H &= v_\mu p_{\pm\mu} \sigma^\mu \\ v_x &= 2t_x \sin k_0 \\ v_{y,z} &= -2t_{y,z} \end{aligned}$$

where σ^μ form a realization of $Cliff(?)$ (put the consistent convention)

These come in pairs so that the total $\int_\Sigma F = 0$ for all Σ. But if they are separated in this torus and the perturbations which mix these sectors H_{p_1} and H_{p_2} remains weak.

Figure 20.1: Insert Brillouin Zone picture here

Project to one of the planes to see a Fermi arc

20.10.1 Hopf Semimetal

$$\begin{aligned}
H &= S(k)\cdot\sigma \\
S^i(k) &= z^\dagger \sigma^i z \\
z &= \frac{1}{\sqrt{\mid \eta_\uparrow \mid^{2p} + \mid \eta_\downarrow \mid^{2q}}} \begin{pmatrix} \eta_\uparrow^p \\ \eta_\downarrow^q \end{pmatrix} \\
z &\in T^3 \to S^3 \\
S &\in T^3 \to S^2 \\
\eta_\uparrow(k) &= \sin k_x + i \sin k_y \\
\eta_\downarrow(k) &= \sin k_z + i(\cos k_x + \cos k_y + \cos k_z - m) \\
S_x + iS_y &= \frac{2\eta_\uparrow^p \bar{\eta_\downarrow}^q}{\mid \eta_\uparrow \mid^{2p} + \mid \eta_\downarrow \mid^{2q}} \\
S_z &= \frac{\mid \eta_\uparrow \mid^{2p} - \mid \eta_\downarrow \mid^{2q}}{\mid \eta_\uparrow \mid^{2p} + \mid \eta_\downarrow \mid^{2q}} \\
\Gamma(p,q) &= \begin{cases} 0, \mid m \mid > 3 \\ pq \; 1 < \mid m \mid < 3 \\ -2pq \;\; \mid m \mid < 1 \end{cases}
\end{aligned}$$

20.10.2 Dirac Semimetal

Figure 20.2: Insert Brillouin Zone picture here

Consider in particular a Weyl semimetal and it's orientation reversal. This produces a 4-fold band crossing given as 2+2 from the Weyl points of both.

20.10.3 Transport

Large Negative Magnetoresistance

20.11 Bordism Classification

20.11.1 Theorem (Spin Bordism)

- $\Omega_0^{Spin} = \mathbb{Z}$ *by counting + points*
- $\Omega_1^{Spin} = \mathbb{Z}_2$ *by R-NS conditions*
- $\Omega_2^{Spin} = \mathbb{Z}_2$ *by square of the antiperiodic above*
- $\Omega_3^{Spin} = 0$
- $\Omega_4^{Spin} = \mathbb{Z}$ *by Kummer surface*
- $\Omega_5^{Spin} = 0$
- $\Omega_6^{Spin} = 0$
- $\Omega_7^{Spin} = 0$

- $\Omega_8^{Spin} = \mathbb{Z}^2$ *by* $\mathbb{HP}^2$ *and* $\frac{1}{4}[K3]^2$

20.11.2 Theorem (Pontryagin dual of Ω_3^{Spin}) *Evaluate on various BG.* `https://arxiv.org/pdf/1612.02860.pdf` *Provides a computation of* $Hom(\Omega_3^{Spin}(B\Gamma), \mathbb{R}/\mathbb{Z})$ *as a contravariant functor from the homotopy category to compact abelian groups category. In particular can input* $B\Gamma$ *for various groups as objects of the homotopy category. Can also look at the group maps that provide* $B\Gamma_1 \to B\Gamma_2$ *morphisms in the homotopy categrory.*

Think of this as being representable functor as $[B\Gamma, X]$ where X is the representing object in the homotopy category. Right now this looks like a contravariant functor from finite groups to sets. Need a H-space kind of structure to say that this mapping space has an abelian group structure.

20.11.3 Theorem (Smith Isomorphism) $\Omega_{d+1}^{Sp}(B\mathbb{Z}_2)' \simeq \Omega_d^{Pin^-}(*)$ *where the* $'$ *indicates the direct summand* $\Omega_{d+1}^{Sp}(B\mathbb{Z}_2)' \bigoplus \Omega_{d+1}^{Sp}(*) \simeq \Omega_{d+1}^{Sp}(B\mathbb{Z}_2)$

20.12 20170503 Talk - SchaffidiParkerVasseur

$$H = \sum -\sigma_{i-1}^z \sigma_i^x \sigma_{i+1}^z$$

1 is trivial, 3 is twisted. 2 and 4 are perturbed of 1 and 3 respectively on A sites with criticality.

$$\begin{aligned} H_{1A} &= \\ H_{1B} &= \\ H_{2A} &= \\ H_{2B} &= \\ H_{3A} &= \\ H_{3B} &= \\ H_{4A} &= \\ H_{4B} &= \end{aligned}$$

Chapter 21

Superconductivity and Superfluidity

21.0.1 Definition (Superconductor)

21.1 Meissner Effect

21.2 BCS Mechanism

21.2.1 Cooper Pairing

$$\begin{aligned}\psi_0 &= \sum_k (g_k e^{ik\cdot r_1} e^{-ik\cdot r_2}) \mid r_1, r_2\rangle \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \\ (E - 2\epsilon_k) g_k &= \sum_{k>k_F} V_{kk'} g_{k'}\end{aligned}$$

For example, we can take the Fourier transform of the pairing potential to be the following:

$$V_{kk'} = \begin{cases} -V & E_F < \epsilon_k < E_F + \hbar\omega_c \\ 0 & \end{cases}$$

21.2.2 BCS Hamiltonian

$$\begin{aligned}H &= \sum \epsilon_k n_k + \sum V_{kl} c_k^\dagger c_{-k}^\dagger c_l c_{-l} \\ H - \mu N &= \sum \xi_k n_{k\sigma} + \sum V_{kl} c_{k,\uparrow}^\dagger c_{-k,\downarrow}^\dagger c_{l,\downarrow} c_{-l,\uparrow} \\ b_k &\equiv \langle c_{-k\downarrow} c_{k\uparrow}\rangle \\ c_{-k\downarrow} c_{k\uparrow} &= b_k + (c_{-k\downarrow} c_{k\uparrow} - b_k) \\ H - \mu N &\approx \sum \xi_k c_{k,\sigma}^\dagger c_{k,\sigma} + \sum V_{kl} c_{k\uparrow}^\dagger c_{-k,\downarrow}^\dagger b_l + b_k^\dagger c_{-l,\downarrow} c_{l,\uparrow} - b_k^\dagger b_l \\ \Delta_k &\equiv \sum V_{kl} b_l \\ H - \mu N &= \sum \xi_k c_{k\sigma}^\dagger c_{k\sigma} + \sum \Delta_k c_{k\uparrow}^\dagger c_{-k,\downarrow}^\dagger + \Delta_k^* c_{-k,\downarrow} c_{k,\uparrow} - \Delta_k b_k^*\end{aligned}$$

21.2.1 Definition (Bogoliubov Transformation)

$$\begin{aligned} H-\mu N &= \sum \xi_k c_{k\sigma}^\dagger c_{k\sigma} + \sum \Delta_k c_{k\uparrow}^\dagger c_{-k,\downarrow}^\dagger + \Delta_k^* c_{-k,\downarrow} c_{k,\uparrow} - \Delta_k b_k^* \\ H-\mu N &= \sum \begin{pmatrix} c_{k\uparrow} & c_{-k\downarrow}^\dagger \end{pmatrix} \begin{pmatrix} -\xi_k & -\Delta_k^\star \\ -\Delta_k & \xi_{-k} \end{pmatrix} \begin{pmatrix} c_{k\uparrow}^\dagger \\ c_{-k\downarrow} \end{pmatrix} - \Delta_k b_k^* \end{aligned}$$

After diagonalizing this mean field Hamiltonian we get

$$\begin{aligned} \mid \Psi\rangle &= \prod (u_k + v_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger) \mid 0\rangle \\ \langle E\rangle_{super} - \langle E\rangle_{normal} &= \frac{-1}{2} N(0)\Delta^2 \end{aligned}$$

21.3 Critical Temperature

Returning to the definition of Δ_k

$$\begin{aligned} \Delta_k &= -\sum V_{kl} \langle c_{-l\downarrow} c_{l\uparrow}\rangle \\ &= -\sum V_{kl} \langle 1 - \gamma_{l\uparrow}^\dagger \gamma_{l\uparrow} - \gamma_{-l\downarrow}^\dagger \gamma_{-l\downarrow}\rangle \\ &= -\sum V_{kl} u_l^* v_l (1 - 2 f_{FD}(E_l)) \\ &= -\sum V_{kl} \frac{\Delta_l}{2E_l} \tanh \frac{\beta E_l}{2} \end{aligned}$$

Plugging in a particular choice for the pairing potential like the one before gives a formula to solve for β_c. This is when the $\Delta_k \to 0$. In the simple pairing from before

$$kT_c \approx 1.13 \hbar \omega_c e^{-1/N(0)V}$$

21.4 Andreev Reflection

Put a normal metal and a superconductor next to each other.

`http://gsfp.physi.uni-heidelberg.de/graddays_oktober_2009/content/en/zubehoer/anhaenge/dolcini/Supercond-Meso-Lecture-4.pdf`

$$\begin{aligned}
\begin{pmatrix} H_e & \Delta \\ \Delta^* & -H_e^\dagger \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} &= E \begin{pmatrix} u \\ v \end{pmatrix} \\
\Delta &= \theta(x)\Delta_0 e^{i\phi} \\
E &\equiv E_x + E_{transverse} \\
\epsilon_{Fn} &\equiv \epsilon_F - E_n \\
\begin{pmatrix} H_{free1d} - \epsilon_{Fn} + \Lambda\delta(x) & \Delta \\ \Delta^* & -H_{free1d} + \epsilon_{Fn} - \Lambda\delta(x) \end{pmatrix} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} &= E \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \\
H_{free1d} &\equiv \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}
\end{aligned}$$

Plugging in $\Delta(x)$, you can solve on both sides and then glue with the boundary condition given by Λ

21.5 High Tc

21.5.1 Definition (Pseudogap)

21.5.2 Example (Cuprates)

21.5.3 Example (YBCO - $YBa_2Cu_3O_7$)

21.5.4 Example (Pnictides) *P,N for Phosphorus and Nitrogen*

21.5.5 Example (Iron Pnictides)

21.5.6 Definition (Second Harmonic Generation) *Send in polarized laser at ω frequncy. Reflected light comes out at ω' with other polarization.*

Mostly $\omega' = \omega$ linear theory. But also have some $\omega' = 2\omega$.

Rotate the sample to also see the dependence on crystal axes.

21.6 Superfluid

21.6.1 Example (Helium)

Chapter 22

To Sort

22.1 Tight Binding

Supposing the wave function is a linear combination of atomic orbitals at each site R_n of the lattice gives

$$\begin{aligned}
V &= \sum_{R_n} V_{Coulomb} + \Delta U \\
\psi &= \sum b_m(R_n)\phi_m(r - R_n) \\
\psi_m(r + R) &= e^{ik\dot{R}}\psi_m(r) \\
\epsilon_m(k) &= E_m - \frac{\beta_m + \sum_{R\neq 0}\sum_l e^{ik\dot{R}}\gamma_{m,l}(R)}{1 + \sum_{R\neq 0}\sum_l e^{ik\dot{R}}\alpha_{m,l}(R)} \\
\beta_m &= -\int \phi_m^* \Delta U \phi_m \\
\gamma(R) &= -\int \phi_m^*(r)\Delta U(r)\phi_m(r - R) \\
\alpha(R) &= \int \phi_m^*(r)\phi_m(r - R)
\end{aligned}$$

At every k this is giving a number that depends on the lattice. If you scale the lattice up by a factor ξ, you will see $\alpha(\xi R_n)$ instead which is like $\alpha(R_n)^\xi$ if the overlaps exponentially decay with distance. If the overlaps scale as power laws with distance like $1/R_n^p$ then rescaling will give factors of ξ^p. Want to redefine k such that $e^{ik\dot{\xi}R_n} = f(\xi)\xi^{-p}e^{ik\dot{R}_n}$ Impossible to do for all n. Can't cook up a modular form. How about if you supposed only the 2 nearest neighbors and α and γ fall to 0 after that? That seems bad because the next furthest could be around the same distance, for example the $\tau = 1.0001i$ torus.

22.2 Rashba

Put because `https://arxiv.org/pdf/0907.2021v1.pdf`

Consider the Hilbert space given by $\mathbb{C}^4$ as $\mathbb{C}\bigoplus\mathbb{C}^3$ of s and p orbitals tensored with the spin Hilbert space $\mathbb{C}^2$. This is done over a 2d lattice in the xy plane.

$$\begin{aligned} H_{SO} &= \Delta_{SO} \sum_j L_j \otimes \sigma_j \\ H_E &= E_0 z \end{aligned}$$

we then ask for the hopping term for $\mid p_z\rangle \otimes \mid i\rangle \otimes \mid \sigma\rangle$ and $\mid p_{x,y}\rangle \otimes \mid j\rangle \otimes \mid \sigma^{'}\rangle$ with the H_E Hamiltonian. The spin orbit summand then gives an amplitude for changing to $\mid p_z\rangle \otimes \mid j\rangle \otimes \mid \sigma^{''}\rangle$ so switch between $p_{x,y}$ and $\uparrow$ and p_z with $\downarrow$.

$$\begin{aligned} \alpha &\approx \frac{at_0\Delta_{SO}}{\Delta_{BG}} \\ H_R &= \alpha(\sigma \times p) \cdot \hat{z} \end{aligned}$$

22.3 Van Hove Singularities

22.4 Fermion Generalities

A finite dimensional fermionic system is the following data.

- A real mode space $\mathcal{M}$ endowed with a positive symmetric bilinear form. These are the Majorana modes for single excitations
- An extension of $Cliff(\mathcal{M}, Q) \otimes \mathbb{C}$ to a $\star$ algebra by $(e \otimes z)^* = e \otimes \bar{z}$ and extend with $(ab)^* = (-1)^{|a||b|} b^* a^*$
- A Hilbert space H_F Fock space. Together with a $\star$ homomorphism that realizes the above as linear operators.

The Clifford algebra acts on the exterior algebra with$\rho(v) = i(v) + m(v)$ where i is removing a copy of v by moving s to the front and replacing it with $Q(v, v_s)$ for all s. m is inserting v at the front. This satisfies the Clifford relation so extend to the entire algebra from the underlying vectors.

This is $\Lambda^\bullet M$ which is too big, in order to pick out the right $2^{N/2}$ dimensional Fock space, do the following.

$$\begin{aligned} \mathcal{M} \otimes \mathbb{C} &\simeq W \bigoplus \bar{W} \\ \Lambda^\bullet(\mathcal{M} \otimes \mathbb{C}) &= \Lambda^\bullet W \otimes \Lambda^\bullet \bar{W} \\ H_F &= \Lambda^\bullet W \end{aligned}$$

This is a representation for $Cliff(V,Q)$ by $\rho(w) = m(w)$ the creation operator and $\rho(\bar{w}) = i(w)$ the annihilation.

This Fock space is graded by fermion parity and in fact forms a graded representation where odd in the clifford algebra means that the parity is reversed.

$$Spin(2n) \subset Cliff(M,Q) \subset Cliff(M,Q) \otimes \mathbb{C} = Cliff(V,Q)$$

Decomposing as irreps for Spin(2n) then gives S^+ and S^- by the bosonic and fermionic sectors.

$H_{F,W}$ becomes a unitary representation of $Spin(2n)$ via the inner product $det(Q(\bar{w}_i, w'_j))$ between $w_1 \cdots w_k$ and $w'_1 \cdots w'_k$ and zero if they don't have the same fermion number.

Now we worry about changing complex structure. These are parameterized by $O(2n,\mathbb{R})/U(n)$ which has two components.

$O(q,\mathbb{C})$ is the transformations which preserve CCR's. The Bogoulibov transformations.

If you look at those that come from changing the complex structure on M then you get the intersection with $U(2n) \bigcap O(q,\mathbb{C})$

The ones that are missing are the ones that change the $\star$ structure so $f_i^\star \neq f_i$. They are the ones that change what you mean by Hermitian.

22.4.1 Definition (Clifford Group) *If V is a finite dimensional real vector space with positive or negative definite quadratic form then the Clifford group maps onto the orthogonal group of V. This is the Cartan-Dieudonne theorem. The kernel is the nonzero elements of the underlying field. If you take the degree 0 parts then it only hits SO.*

22.4.2 Definition (Spinor Norm) *Spinor norm is take the Clifford group and send x to $x^t x$*

The pin group is the subgroup of Clifford group of spinor norm ± 1. Spin being those of determinant 1.

$$Cliff(\pm 8) \simeq End(\mathbb{R}^{8|8})$$

It is the first time that the Clifford algebra is a matrix superalgebra.

22.4.3 Theorem (Atiyah Skew) *The map α $\mathcal{F}^k_\bullet(H_R) \to \Omega(\mathcal{F}^{k-1}(H_R))$ given by $c_{k-1}\cos \pi t + A \sin \pi t$ gives a homotopy equivalence. That means that $\mathcal{F}^k_\bullet(H_R)$ is a classifying space for KR^{-k}. Be careful that in his indexing $\mathcal{F}^k$ anticommuted with $k-1$ of the k clifford algebra generators leaving the Hamiltonian for the k'th. Also he has everything square to -1 and c_i being anti-Hermitian.*

A triality is a map $V_1 \times V_2 \times V_3 \to \mathbb{R}$ such that specifying any two as nonzero vectors gives a nondegenerate linear form on the third. We can use this to get a multiplication $V \times V \to V$ with unit. This depends on choosing an e_1 and e_2 and e_3 such that $e_1 \times e_2 \times e_3 \to 1$

This means that it is a division algebra and all division algebras give trialities.

22.4.4 Proposition *Let X be a compact, connected oriented 4-dimensional manifold. Write $TX \bigoplus TX$. This induces a $SO(8)$ principal bundle via the diagonal.*

$$w_2(2TX) = 2w_0w_2 + w_1w_1$$

This means that there is a spin structure a $Spin(8)$ principal bundle.

A normed triality is one where $t(v_1, v_2, v_3) \leq \mid v_1 \mid\mid v_2 \mid\mid v_3 \mid$ when we have three inner product spaces. That bound also needs to be saturable given the first two you should be able to find a saturating third.

In particular there is a normed triality $V_8 \times S_8^+ \times S_8^- \to \mathbb{R}$. It gives the octonions.

There is an isomorphism $Cliff(n-1) \to Cliff_0(n)$ by $\phi(e_i) = e_ie_n$ where the subscript is the even sector of $Cliff_n$. The spinors in n dimensions are then pinors in $n-1$ dimensions.

$$V_n \times P_n^{\pm} \to P_n^{\pm} \; n = 3, 7 mod 8$$
$$V_n \times S_n^{\pm} \to S_n^{\pm} \; n = 0, 4 mod 8$$

Dualizing this gives the triality $V_n \times S_n^+ \times S_n^- \to \mathbb{R}$ in either 4 or 8 dimensions. Let's focus on the 8 dimensional case.

The triality has automorphisms by specifying $f_iV_i \to V_i$ such that the product is preserved. If this was done after turning them all into the same vector space, and specializing to 8 this would be $Spin(8)$ as the automorphisms of the triality. The automorphisms of the octonions G_2 sits inside.

But there are also the ones that permute the V_i

$$\begin{array}{rcl} \forall g \in Spin(8) \; \exists! \quad g_{\pm} & \quad s.t. t(g(v_1), g_+(v_2), g_-(v_3)) = t(v_1, v_2, v_3) \\ \alpha_+ g & \to & g_+ \\ \alpha_- g & \to & g_- \end{array}$$

Those transformations are outer automorphsims of $Spin(8)$. In fact you can do any permutation in addition to the ones just described which switched vector with $\pm$ spinor.

22.5 Boltzmann Equation

We have a probability measure on the full N body phase space $T^*\mathbb{R}^{dN}$. We may push this down to a single body measure as well. In the case of identical classical particles, we may as well assume that this is for the first particle.

$$\begin{aligned}
\mu &= f^0(x_1 \cdots x_N, p_1 \cdots p_N)\frac{1}{h^d}\nu \\
\nu &= d^d x_1 d^d p_1 \cdots d^d x_N d^d p_N \\
\pi_* \mu &= f(r,p,t) d^d r d^d p \\
f(r,p,t) &= N \int \prod_{i=2}^{N} d^d x_i d^d p_i f(r \cdots x_N, p \cdots p_N)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial (fvol)}{\partial t} &= -L_{X_H}(fvol) \\
&= -d i_{X_H}(fvol) \\
\frac{\partial f}{\partial t} &= -\{f, H\} \\
\frac{\partial f}{\partial t} + \{f, H\} &= (\frac{\partial f}{\partial t})_{coll}
\end{aligned}$$

22.6 XY Model

22.6.1 XY

$$\begin{aligned}
\Phi &\in \Sigma \times \mathbb{R} \to \mathbb{C} \\
\mathcal{L} &= \mid \partial_\mu \Phi \mid^2 + r \mid \Phi \mid^2 + \\
\mathcal{L} &= \mid \partial_\mu - i A_\mu \Phi \mid^2 + r \mid \Phi \mid^2 +
\end{aligned}$$

Number of bosons conserved $ndxdy$ number density The current J_μ should really be $J^{\mu\nu}$ because currents are really $n-1$ forms so they can be integrated on the time slice. External field coupled $A_\mu J_\mu$ Boson creation operator Apply B_ext Gapless superfluid goes to gapped insulator as make r increase

22.6.2 Abelian Higgs

$$\begin{aligned}
\mathcal{L} &= \mid (\partial_\mu - ia)\psi \mid^2 + r_2 \mid \psi \mid^2 + \cdots + \frac{1}{2g^2} f_{\mu\nu} f^{\mu\nu} \\
\mathcal{L} &= \mid (\partial_\mu - ia)\psi \mid^2 + r_2 \mid \psi \mid^2 + \cdots + \frac{1}{2g^2} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2\pi} A da
\end{aligned}$$

Gauge flux $n = \frac{f_{xy}}{2\pi}$ $\frac{1}{2\pi} \star da$ This becomes $\frac{1}{2\pi} \star da A_\mu$ Adds 1 to flux so makes a basic monopole Introduce $n_v = \frac{B_{ext}}{2\pi}$ Higgs phase gapped goes to coulomb phase as r_2 goes up

22.7 KT Transition

$$\begin{aligned} F &= E - TS = \#log\frac{r}{a} - T\#log\frac{r}{a} \\ sign(F) &= sign(\# - T\#) \end{aligned}$$

So above a critical T_c it becomes more favorable to produce a pair of vortices and let them go free to produce a vortex gas instead of confined few (if any) vortices.

22.8 Haldane's path integral

A d dimensional antiferromagnet of spin S. The field is given by a map into $\mathbb{CP}^{2S+1-1}$. So from Space cross with a circle for periodic Euclidean time β. In $d = 1$ with periodic boundary conditions on space this is a T^2 mapping in.

$$[T^2, X] = (a, b) \in \pi_1 \; ab = ba \quad \times \pi_2(X)/(t - t^a, t - t^b)$$

This is because first you specify the parallel and meridean axes of the torus and then you get a null homotopy of $aba^{-1}b^{-1}$ from the 2-cell. But such null-homotopies form a torsor for $\pi_2(X)$ you start with your favorite null-homotopy and you smash with $[S^2, X]$ to get the next one. If you change a or b by some homotopy t, again identified with π_2 changes that previous null homotopy by $t - t^b$. So that π_2 is meant as a torsor for π_2 not π_2 itself and changing the origin won't affect $t - t^a$ so we're still good with the quotient because under shifts of the original definition they both shift together and if you shift t you get a $s - s^a$ which is still supposed to be quotiented by.

Here we are going into a $\mathbb{CP}$ so we don't need the above because for mapping in such small things, it might as well be a $\mathbb{CP}^\infty$ or $\mathbb{CP}^1$. Either way we just get $\mathbb{Z}$. At least for the infinity version it is because $K(Z, 2)$. So what are the characters of $\mathbb{Z}$ that we can use as weights?

But $[T^2, \mathbb{RP}^2] = \{0, 0\} \times \mathbb{Z} \bigcup (0, 1), (1, 0), (1, 1) \times \mathbb{Z}_2$

Notice the 6 elements in there. What phases do they come with in a 1D nematic system. The π_1 part tells you the phase of starting aligned vertically and doing half twists or full integer twists 1 or 0.

So what are the phases we can assign to these different classes. On π_1 we get the 4 different characters of the Klein four group. π_1 acts on π_2 here by shifts by 2 so that last $\mathbb{Z}_2$ is a torsor for the even or oddness of the integer. Whatever character we pick on this factor needs to be invariant under this action. The choices are $\mp$ and $\pm$ on even/odd as well as $++$ coming from the trivial character on $\mathbb{Z}$ which gives only $+$ on the torsor. Note that because it is only a torsor for $\mathbb{Z}$ we don't have to give 0 the phase 1.

So 12 choices, 4 from the π_1 and 3 from the π_2

http://math.stackexchange.com/questions/36488/how-to-compute-homotopy-classes-of-maps-on-the-2-torus

22.9 Renormalization-Physicist Version

22.9.1 Lemma (Callan-Symanzik Equation)

$$\big(M\frac{\partial}{\partial M}+\beta\frac{\partial}{\partial\lambda}+n\gamma\big)G^{(n)}(x_1\cdots x_n,M,\lambda) \;=\; 0$$

Proof

$$\begin{aligned}
M &\to M+\delta M\\
\lambda &\to \lambda+\delta\lambda\\
\phi &\to (1+\delta\eta)\phi\\
G^{(n)}(x_1\cdots x_n) &= \langle 0\mid T\phi(x_1)\cdots\phi(x_n)\mid 0\rangle\\
G^{(n)} &\to (1+n\delta\eta)G^{(n)}\\
dG^{(n)} &= \frac{\partial G^{(n)}}{\partial M}\delta M+\frac{\partial G^{(n)}}{\partial\lambda}\delta\lambda\\
&= n\delta\eta G^{(n)}\\
\beta &\equiv \frac{M}{\delta M}\delta\lambda\\
\gamma &\equiv \frac{M}{\delta M}\delta\eta\\
\big(M\frac{\partial}{\partial M}+\beta\frac{\partial}{\partial\lambda}+n\gamma\big)G^{(n)}(x_1\cdots x_n,M,\lambda) &= 0
\end{aligned}$$

22.10 Wilson-Fisher Fixed Point

22.11 Jordan-Wigner

$$\begin{aligned}
\sigma_j^+ &= e^{-i\pi\sum_L a_k^\dagger a_k}a_j^\dagger\\
\sigma_j^- &= e^{i\pi\sum_L a_k^\dagger a_k}a_j\\
\sigma_j^z &= 2a_j^\dagger a_j-1
\end{aligned}$$

a_j and $a_j^\dagger$ form a free fermion canonical anticommutation relations CAR algebra.

22.12 Anderson Localization

Let

$$H_\Lambda = \gamma\Delta + V$$

for some subset $\Lambda \subset \mathbb{Z}^d$. This Hamiltonian acts on $\ell^2(\Lambda)$. V is a random potential which since the underlying space is discrete is just IID for each site where the potential for each site is drawn from probability density ρ.

For sufficiently small $\mid \gamma \mid$, the eigenstates are localized. It is only pure point spectrum. See the extreme case of $\gamma = 0$ when $\psi_q(p) = \delta_{p,q}$ will be eigenstates. In this case the eigenvalues are $V(q_1)$ and $V(q_2)$ etc. So those are all independent. This fact continues approximately in the sense that if we take eigenfunctions that are localized around far apart places have nearly independent eigenvalues.

22.12.1 Theorem (RAGE Ruelle, Amrein, Georcescu, and Enss) *The eigenvectors of pure point spectrum has*

$$\lim_{R\to\infty} \sup_t \mid \chi(\mid x \mid > R)e^{-itH}\psi \mid = 0$$

Meanwhile the continuous spectrum escapes the compact balls.

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} \mid \chi(\mid x \mid > R)e^{-itH}\psi \mid = 0$$

22.12.2 Example

$$\begin{aligned} H &= \sum_{i=1}^{L} h_i S_i^z + J S_i \cdot S_{i+1} \\ h_i &\in U([-h,h]) \end{aligned}$$

If $J = 0$, then the Hamiltonian is sum of commuting one site contributions. Therefore get a basis of product states for eigenstates. That is their energies are $\sum \pm h_i$ where the $\pm$ are picking which product state we are using.
If $h = 0$, then $h_i = 0$ for all i so we are in pure Heisenberg chain which will delocalized momentum basis eigenstates.

$$\begin{aligned} m_{i\alpha}^n &\equiv \langle n \mid S_i^z \mid n\rangle_\alpha \\ C(h,L,\Delta) &= \langle\langle m_{i\alpha}^n - m_{i\alpha}^{n+\Delta}\rangle\rangle \end{aligned}$$

If the eigenstate thermalization hypothesis holds then the difference between the local (to the observables on site i) restrictions of $\mid n\rangle_\alpha\langle n \mid_\alpha$ and $\mid n+1\rangle_\alpha\langle n+1 \mid_\alpha$ is a slight difference in temperature. That implies that $C(n,L,1)$ has exponential decay with L.

22.12.1 MBL-Many Body Localization

Given a 1D system and a highly excited state. Take a region look at the entanglement entropy for the restriction of that state. Then look at the scaling under the size of the region.

22.12.3 Definition (l-bits) *A quasi-local Pauli operator τ_i^z around site i. Quasi-locality means*

$$\begin{aligned} \tau_i^z &= \sum_{j,\alpha} K_{ij}^{\alpha}\sigma_j^{\alpha} + K_{ijk}^{\alpha\beta}\sigma_j^{\alpha}\sigma_k^{\beta} + \cdots \\ K_{i,j_1\cdots j_m}^{\alpha_1\cdots\alpha_m} &\approx \exp(-\frac{d_m}{\xi} - \frac{m}{\theta}) \\ d_m &\equiv diam(i, j_1 \cdots j_m) \end{aligned}$$

We may also ask for it to approximately commute with the Hamiltonian.

$$| \, [H, \tau_i^z] \, | \quad \in \quad O(Ke^{-L/\xi})$$

22.13 Optical Traps

22.13.1 Two Cold Atoms

`https://www.researchgate.net/profile/Thomas_Busch3/publication/257822286_Two_Cold_Atoms_in_a_Harmonic_Trap/links/581e99f608aea429b295d4a0.pdf`

$$H = \frac{-\hbar^2}{2m}\nabla_1^2 + \frac{-\hbar^2}{2m}\nabla_2^2 + \frac{1}{2}m_1\omega^2 r_1^2 + \frac{1}{2}m_2\omega^2 r_2^2 + 4\pi a_0 \delta_{reg}^{(3)}(r_1 - r_2)$$

Chapter 23

Classical Integrable Appendix

23.1 Lax Matrices

Let $\mathcal{M}$ be a Poisson manifold.

where the top arrow is the flow for time t given by the Hamiltonian and the lower arrow is a local flow on $\mathfrak{g}$ which is tangent on adjoint orbits. Because $L(x) \in \mathfrak{g}$, we can regard L as a matrix with coefficients in $C^\infty(\mathcal{M})$ an element of $\mathfrak{g} \otimes C^\infty(\mathcal{M})$. The isospectrality means that there is another matrix $M \in \mathfrak{g} \otimes C^\infty(\mathcal{M})$ such that the flow downstairs is given by ad_M

The isospectrality property ensures that the following gives more integrals of motion.

$$\begin{aligned} L_V = (\rho \otimes id)L &\in End\, V \otimes C^\infty(\mathcal{M}) \\ det(L_V - \lambda) &= \sum H_i \lambda^i \end{aligned}$$

The H_i are integrals of motion.

23.1.1 Example (KdV)

$$\begin{aligned} L(u, u_x) &= -6\frac{d^2}{dx^2} - u \\ M(u, u_x) &= -4\frac{d^3}{dx^3} - u\frac{d}{dx} - \frac{1}{2}u_x \end{aligned}$$

The Poisson manifold is $T^*Maps_{Schwarz}(\mathbb{R}, \mathbb{R})$ where the base coordinate is u and the fiber is proportional to u_x.

The Poisson bracket is

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{M} \\ \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle L} \\ \mathfrak{g} & \xrightarrow{F'} & \mathfrak{g} \end{array}$$

$$\{f, g\} \quad =$$

Hamiltonian:

$$H_2[u] \quad = \quad \int_{\mathbb{R}} dx(\frac{1}{2}u_x^2 + u^3)$$

23.1.2 Theorem (Kostant bracket) *Let $\mathfrak{g}^*$ be the dual to a Lie algebra. Polynomial functions on this are given by the symmetric algebra of $\mathfrak{g}$. This has a Poisson bracket given by the Lie bracket on linear functions and extending all the way up.*

The center of $S(\mathfrak{g})$ as a Poisson algebra coincides with the subalgebra of ad-invariants also known as the Casimirs.

If you try moving with these Hamiltonian flows, you find no motion whatsoever. $D_C\phi = \{C, \phi\} = 0$. We need a different Poisson structures so these become interesting conserved quantities instead of just constants along each coadjoint orbit.

23.1.3 Definition (Classical r-matrix) *Let r be an endomorphism of the Lie-algebra, we can define a new Lie bracket by*

$$[X, Y]_r \quad = \quad \frac{1}{2}([rX, Y] + [X, rY])$$

If r satisfies the modified classical Yang Baxter Equation

$$[rX, rY] - r[rX, Y] - r[X, rY] + [X, Y] \quad = \quad 0$$

then the bracket defined will satisfy Jacobi.

$$r_\pm \quad = \quad \frac{1}{2}(r \pm id)$$

are Lie algebra homomorphisms $\mathfrak{g}_\mathfrak{r} \to \mathfrak{g}$. They can be extended to maps of Poisson algebras on the associated Symmetric algebras $Sym(\mathfrak{g}_\mathfrak{r}) \to Sym(\mathfrak{g})$ so a map of Poisson manifolds $\mathfrak{g}^* \to \mathfrak{g}_r^*$

$$\begin{aligned} S(\mathfrak{g}_r) \otimes S(\mathfrak{g}) \quad &\to \quad S(\mathfrak{g}) \\ x \cdot y \quad &= \quad \sum r_+(x_i^1) y r_-(S(x_i^2)) \end{aligned}$$

23.1.4 Theorem *Let i_r be the isomorphism of graded linear spaces $S(\mathfrak{g}_r) \to S(\mathfrak{g})$ given by sending $P \in S(\mathfrak{g}_r)$ to $P \cdot 1$. i_r^{-1} restricted to the Poisson center before (the Casimirs) is a Poisson map so we get a new Poisson commuting algebra.*

23.1.5 Remark It can be much smaller for example if $r = \pm id$ then the new Poisson commuting algebra you get is 0. $\Diamond$

A common source of classical r-matrices is decompositions of Lie algebras $\mathfrak{g} = \mathfrak{g}_+ \bigoplus \mathfrak{g}_-$ as a **linear space**. Then $r = P_+ - P_-$ will be a solution. In this case the bracket is

$$[X, Y]_r = [X_+, Y_+] - [X_-, Y_-]$$

Then $\mathfrak{g}_r$ splits as a Lie algebra $\mathfrak{g}_r = \mathfrak{g}_+ \bigoplus \mathfrak{g}_-$

$$\begin{aligned} \mathfrak{g} &= g_+^\perp \oplus g_-^\perp \\ g_+^* &\simeq g_-^\perp \\ g_-^* &\simeq g_+^\perp \\ L &= \sum e_i \otimes e^i \in g_+^* \otimes g_+ \\ L &\in g_-^\perp \otimes g_+ \subset g_-^\perp \otimes S(g_+) \end{aligned}$$

Wait it says projecting to $S(g_-)$ but before said $S(g_+)$ were supposed to be the observables.

23.1.6 Definition (Manin triple) *Let $\mathfrak{g}$ be a Lie algebra and let the subspaces in the decomposition $\mathfrak{g}_\pm$ be isotropic with respect to the invariant inner product so that $\mathfrak{g}_\pm^\perp = \mathfrak{g}_\pm$.*

23.1.7 Definition (Lie bialgebra) *Let $\mathfrak{a}$ be a Lie algebra with dual $\mathfrak{a}^*$. Give $\mathfrak{a}^*$ a Lie bracket as well. You can also view this as a map $\mathfrak{a} \to \mathfrak{a} \wedge \mathfrak{a}$. Compatibility will require this to be a 1-cocycle with values in $\wedge^2 Adjoint$.*

23.1.8 Proposition (Double of a Lie bialgebra) *Given a Manin triple, identify $\mathfrak{g}$ and $\mathfrak{g}^*$ with an inner product. Give the dual the r-bracket $P_+ - P_-$. This is a Lie bialgebra. $(\mathfrak{g}_+, \mathfrak{g}_-)$ is a Lie sub-bialgebra. In addition from a Lie bialgebra, you can construct a unique Lie algebra called the double which is $\mathfrak{g} \oplus \mathfrak{g}^*$ as a vector space, has the evaluation pairing and the bracket is given by the brackets on the subalgebras.*

$$\begin{aligned} \left[e^i, e^j\right] &= f_k^{ij} e^k \\ \left[e_i, e_j\right] &= C_{ij}^k e_k \\ \left[e^i, e_j\right] &= C_{jk}^i e^k - f_j^{ik} e_k \end{aligned}$$

The bracket within $\mathfrak{g}$ and within $\mathfrak{g}^*$ are left as they were.

$$\begin{aligned} \delta e^i &= C^i_{jk} e^j \wedge e^k \\ \delta e_i &= f_i^{jk} e_j \wedge e_k \end{aligned}$$

23.1.9 Definition (Factorizable Lie Bialgebra) *Let r be a classical r-matrix on $\mathfrak{g}$ which has fixed inner product. Also assume that r is skew and satisfies MCYBE. Then $(\mathfrak{g}, \mathfrak{g}^*)$ is a factorizable Lie bialgebra.*

23.1.10 Lemma *We view $T^*\mathfrak{t}$ as the Lie algebra given by the double construction for the Lie bialgebra where the structure constants C^i_{jk} give the bracket as is and the dual bracket $f_j^{ik} = 0$.*

Red for algebra arrows, Yellow for coalgebra arrows

$T^*\mathfrak{k} \longrightarrow \mathfrak{k} \ltimes \mathfrak{k}^*$

$T\mathfrak{k}^* \longrightarrow \mathfrak{k} \rtimes \mathfrak{k}^*$

$\mathfrak{k} \otimes \mathbb{C} \longrightarrow \mathfrak{sl}(N, \mathbb{C})$

$\mathfrak{d}(\mathfrak{k}) \longrightarrow \mathfrak{k} \bowtie \mathfrak{an}$

$\mathfrak{k} \oplus \mathfrak{an}^{op}$

23.1.11 Lemma

$$\begin{aligned} \mathfrak{d}(\mathfrak{g}) &\simeq \mathfrak{g} \bowtie \mathfrak{g}^* \\ &\simeq \mathfrak{g} \oplus \mathfrak{g}^{*,op} \end{aligned}$$

where the first line is as algebras and the second line is a Lie coalgebra isomorphism. This gives a natural Lie bialgebra structure on the double for which $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie bialgebra embeddings.

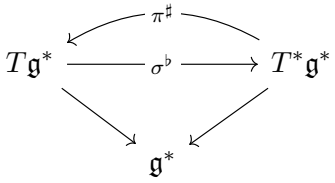
We are free to scale the Lie brackets and cobrackets. Let t scale r and s scale the Lie bracket. Because the cobracket uses the bracket as well as r it will scale with st.

$$\begin{aligned}
[X+\mu, Y+\nu] &= (s[X,Y] + str^\sharp(ad^*_X\nu) - str^\sharp(ad^*_Y\mu) - stad_X r^\sharp\nu + stad_Y r^\sharp\mu, \\
&\quad st[\mu,\nu]_* + s*ad^*_X\nu - s*ad^*_Y\mu) \\
\mathfrak{d}_{t=0}(\mathfrak{g}) &= \mathfrak{g} \ltimes \mathfrak{g}^* \\
\mathfrak{d}_{s=0}(\mathfrak{g}) &= \mathfrak{g}_{ab} \times \mathfrak{g}^* \\
\mathfrak{d}_t(\mathfrak{g}) &= \mathfrak{g} \bowtie \mathfrak{g}^* \\
D_{t=0}(\mathfrak{g}) &= G \ltimes \mathfrak{g}^*
\end{aligned}$$

$\mathfrak{g}_t^*$ is part of the Manin triple $(\mathfrak{d}_t, \mathfrak{g}, \mathfrak{g}_t^*)$

$$\begin{aligned}
s_t\ \mathfrak{g}^* &\to \mathfrak{g}_t^* : \quad A \to t*A \\
\mathfrak{d} &\to \mathfrak{d}_t : \quad X+\mu \to X + t*\mu \\
j_t\ \mathfrak{g}^* &\to \mathfrak{d} : \quad \mu \to tr^\sharp(\mu) + \mu \\
s_t j_t\ \mu &\to tr^\sharp(\mu) + t\mu = j_1 s_t(\mu) \\
\sigma_t &= t^{-1} s_t^* \sigma
\end{aligned}$$

$$\begin{aligned}
\mathfrak{g}^* \times G &\to D_{t=0} \\
\mu, g &\to g, Ad^*_g\mu
\end{aligned}$$



$$\begin{aligned}
(\pi^\sigma)^\sharp &= \pi^\sharp(Id + \sigma^\flat\pi^\sharp)^{-1} \\
&= \pi^\sharp(Id - \sigma^\flat\pi^\sharp + (\sigma^\flat\pi^\sharp)^2 + \cdots)
\end{aligned}$$

23.1.12 Proposition (Moser's trick) *$d\sigma_t/dt = -da_t$ for a 1-form, we can make a one parameter family of vector fields $-\pi^\sharp(a_t)$. They are all tangent to symplectic leaves. So we can let them flow and because the leaves are compact there is no problem of incompleteness. So now we have a new Poisson structure for each t given by pushforward along the flow's diffeomorphism. It also has the same leaves.*

The standard r-matrix on a simple Lie algebra

$$
\begin{aligned}
r &= \frac{1}{2}H_i \otimes H_j (A^{-1})_{ij} + \sum_{\Delta^+} X_\alpha \otimes Y_\alpha \\
2s = r + \sigma(r) &= \frac{1}{2}H_i \otimes H_j (A^{-1})_{ij} + \sum_{\Delta^+} X_\alpha \otimes Y_\alpha + Y_\alpha \otimes X_\alpha \\
s &= \frac{1}{2}H_i \otimes H_j (A^{-1})_{ij} + \frac{1}{2} X_\alpha \otimes Y_\alpha + \frac{1}{2} Y_\alpha \otimes X_\alpha \\
\mathfrak{su}(N) &= iH_\alpha, X_\alpha - Y_\alpha, i(X_\alpha + Y_\alpha) \\
\mathfrak{su}(N)^* &= -iH_\alpha^\vee, \frac{1}{2}(X_\alpha^\vee - Y_\alpha^\vee), \frac{-i}{2}(X_\alpha^\vee + Y_\alpha^\vee)
\end{aligned}
$$

Let H_α be the basis of $\mathfrak{h}$ which is an eigenbasis for A^{-1}. Because the eigenvalues are all positive (not a general Kac-Moody algebra) you can rescale with real parameters $\sqrt{\lambda}$ such that the first term of s was $\frac{1}{4}H_\alpha \otimes H_\alpha$

So $\mathfrak{sl}(N)$ is quasitriangular by above, but the real forms are not because can't write $X \otimes Y$ with $X + Y$ and $i(X - Y)$ and only real coefficients. But we can do the trick with the complex version.

Below is wrong: didn't want to use the standard r-matrix. That is the one that comes from double the borel not double the real form. It's r_{Gauss} not $r_{Iwasawa}$.

The trick to picking out the real version is

$$A^\sigma \hookrightarrow A$$

$$A^\sigma \otimes_{\mathbb{R}} A^\sigma \longrightarrow A \otimes_{\mathbb{R}} A \longrightarrow A \otimes_{\mathbb{C}} A$$

$$
\begin{array}{ccccc}
A & \xrightarrow{\delta} & A \otimes_{\mathbb{C}} A & & \\
\downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma \otimes_{\mathbb{C}} \sigma} & & \\
A & \xrightarrow{\delta} & A \otimes_{\mathbb{C}} A & & \\
\uparrow & & \uparrow & & \\
(A^\sigma) & \xrightarrow{\delta} & (A \otimes_{\mathbb{C}} A)^\sigma & \longleftarrow & A^\sigma \otimes_{\mathbb{R}} A^\sigma \\
 & & & \nwarrow & \\
 & & & & A^{-\sigma} \otimes_{\mathbb{R}} A^{-\sigma}
\end{array}
$$

There is a $U(1)$ worth of choices for every single term in how to lift $a \otimes b$ to any of $az \otimes z^{-1}b$ which are now different.

$$\begin{aligned}
r(z_\alpha, w_\alpha) &= \frac{1}{4} z_\alpha H_\alpha \otimes z_\alpha^{-1} H_\alpha + w_\alpha X_\alpha \otimes w_\alpha^{-1} Y_\alpha \\
\delta(H_\beta) &= [r, v_1 H_\beta \otimes v_1^{-1} + v_2^{-1} \otimes v_2 H_\beta] \\
&= w_\alpha v_1 [X_\alpha, H_\beta] \otimes v_1^{-1} w_\alpha^{-1} Y_\alpha + w_\alpha v_2 X_\alpha \otimes w_\alpha^{-1} v_2^{-1} [Y_\alpha, H_\beta] \\
v_1 &= v_2 \\
\delta(H_\alpha) &= 0 \\
\delta(X_\beta) &= \frac{1}{4} z_\alpha v_1 [H_\alpha, X_\beta] \otimes z_\alpha^{-1} v_1^{-1} H_\alpha + \frac{1}{4} z_\alpha v_1 H_\alpha \otimes z_\alpha^{-1} v_1^{-1} [H_\alpha, X_\beta] \\
&+ w_\alpha v_1 [X_\alpha, X_\beta] \otimes w_\alpha^{-1} v_1^{-1} Y_\alpha + w_\alpha v_1 X_\alpha \otimes w_\alpha^{-1} v_1^{-1} [Y_\alpha, X_\beta] \\
w_\alpha &= z_\alpha \\
w_\alpha &= z \\
r(z) &= \frac{1}{4} z H_\alpha \otimes z^{-1} H_\alpha + z X_\alpha \otimes z^{-1} Y_\alpha \\
\delta(X_\beta) &= \frac{1}{4} z v a_{\alpha\beta} X_\beta \otimes (zv)^{-1} H_\alpha - (zv) X_\alpha \otimes (zv)^{-1} ? \delta_{\alpha\beta} H_\alpha + \frac{1}{4} (zv) H_\alpha \otimes (zv)^{-1} a_{\alpha\beta} X_\beta \\
&+ zv [X_\alpha, X_\beta] \otimes (zv) Y_\alpha
\end{aligned}$$

End of below is useless.

Back to complex version momentarily. Make it a braided version. Corollary 3.2 Majid

$$\begin{aligned}
\delta^{new}(x) &= \frac{1}{2} H_\alpha \otimes [x, H_\alpha] + X_\alpha \otimes [x, Y_\alpha] + Y_\alpha \otimes [x, X_\alpha] \\
\delta^{new}(H_\beta) &= X_\beta \otimes ? Y_\beta + Y_\beta \otimes ? X_\beta \\
\delta^{new}(X_\beta) &= \frac{1}{2} H_\beta \otimes ? X_\beta + X_\beta \otimes ? H_\beta \\
\delta^{new}(Y_\beta) &= \frac{1}{2} H_\beta \otimes ? Y_\beta + Y_\beta \otimes ? H_\beta
\end{aligned}$$

This is the same cobracket, so we haven't changed at all in making it a braided version. Oh this is example 3.3. The factorizable ones always give KKS back.

What about transmuting along

We have the arrow of lie bialgebras over the reals

$$\mathfrak{su}(N), [], i\delta \quad \to \quad \mathfrak{sl}(N), [], i\delta_{\mathbb{R}}$$

by inclusion so we have the dual map

$$\mathfrak{sl}(N)^* \quad \to \quad \mathfrak{su}(N)^*$$

of Lie bialgebras so we are almost in the setting of transmutation after using a self duality.

$$\begin{aligned}
p : \mathfrak{sl}(N), [], i\delta = d_{CE}(i * r) &\rightarrow \mathfrak{su}(N)^* \\
ir &= \frac{-i}{4} iH_\alpha \otimes_{\mathbb{C}} iH_\alpha + iX_\alpha \otimes_{\mathbb{C}} Y_\alpha \\
ir_{\mathbb{R}} &= \\
p(X_\alpha) &= \\
p(iX_\alpha) &= \\
p(Y_\alpha) &= \\
p(iY_\alpha) &= \\
p(H_\alpha) &= \\
p(H_\alpha) &=
\end{aligned}$$

The lie algebra $\mathfrak{g}^*$ is unaffected but the cobracket changes. $\mathfrak{g}^* \rightarrow \mathfrak{g}^* \otimes \mathfrak{g}^*$ which changes the Lie algebra $\mathfrak{g}$ which changes what you mean by KKS Poisson structure. It will still be linear and will have different symplectic leaves.

$$\begin{aligned}
\delta^{new}(x) &= \delta^{old}(x) + [p(r^1), x] \otimes p(r^2) - p(r^2) \otimes [p(r^1), x] \\
\delta^{new}(-iH_\alpha^\vee) &= \\
\delta^{new}(\frac{1}{2}(X_\alpha^\vee - Y_\alpha^\vee)) &= \\
\delta^{new}(\frac{-i}{2}(X_\alpha^\vee + Y_\alpha^\vee)) &=
\end{aligned}$$

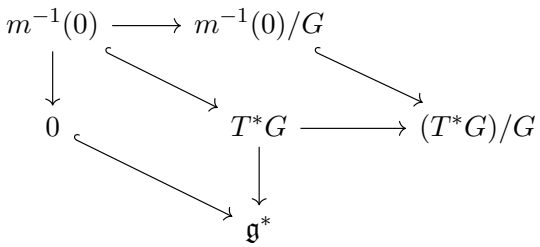
23.2 Hamiltonian reduction

Choose a trivialization of $T^*G \simeq G \times \mathfrak{g}^*$ by left translations. Let B be a Lie subgroup B acting by right translations (by b^{-1} so it is a left action) extends to the cotangent bundle. In the trivialization this is given by

$$\begin{aligned}
R_{b^{-1}}\ (g, \xi) &\rightarrow (gb^{-1}, (Ad^* b)^{-1}\xi) \\
L_b\ (g, \xi) &\rightarrow (bg, \xi)
\end{aligned}$$

The moment map for this is $(g, \xi) \rightarrow \xi\ |_{\mathfrak{b}}$. So given a point F in $\mathfrak{b}^*$ we can do the symplectic reduction for it.

23.2.1 Theorem (Kazhdan, Kostant, Sternberg) *There is a symplectic induction functor $S_B \rightarrow S_G$ from the category of Hamiltonian B-spaces to Hamiltonian G-spaces. Up to coverings these are coadjoint orbits. The functor is given by $M \rightarrow (T^*G \times M)//B$ where the difference symplectic structure is used on the two factors and B acts diagonally in the middle of the factors so that the moment map is the difference of moment maps. There is a leftover left action by G from the action of G on T^*G by left multiplication and extended to cotangent.*



23.3

Do the example where $B = U(N)$ and $G = U(N + M)$. You start with the coadjoint orbit of ρ.
For example, do $B = G$. Then we get
Apply symplectic induction $S_G \to S_G$ on a point.

23.4 Standard r-matrix

For a **complex simple** Lie algebra, we can write a standard

$$\begin{aligned}
\mathfrak{d} &= \mathfrak{g} \oplus \mathfrak{g} \\
\mathfrak{g} &= \Delta(\mathfrak{g}) \subset \mathfrak{g} \oplus \mathfrak{g} \\
\mathfrak{g}^* &= \{(X_+, X_-) \in \mathfrak{b}_+ \oplus \mathfrak{b}_- \mid \pi(X_+) = -\pi(X_-)\} \\
r &= \sum_{\alpha \in \Delta_+} e_\alpha \wedge e_{-\alpha}
\end{aligned}$$

This extends to loop algebras as well and gives the Trigonometric r-matrix. See page 16 of Semenov-Tian-Shansky for loops in a complex semisimple algebra.

After the dust settles you get an r-matrix, that you can write in terms of the r-matrix for the corresponding finite dimensional Lie algebra.

$$\begin{aligned}
r_{\text{trig}}(x) &= \text{Casimir} \frac{x^{1/2} + x^{-1/2}}{x^{1/2} - x^{-1/2}} + r_{\text{finite}} \\
x &\neq 1 \\
x &= \exp y \quad y \neq 0
\end{aligned}$$

23.5 Functions on the group instead of on a coadjoint orbit

This is changing from linear to quadratic case.

23.5.1 Definition (Poisson Lie Group) *A Lie group with multiplicative Poisson bivector P.*

23.5.2 Definition (Multiplicative Tensor) *For each point in G we have a Q_g which is an element of $\otimes^k T_g G$. That is they are sections of the tensor bundle of the tangent bundle. $\lambda(g)$ left multiplication by G induces a map $\otimes^k T_h G \to \otimes^k T_{gh} G$ and same with right multiplication.*

$$Q_{gh} = \lambda(g)Q_h + Q_g\rho(g)$$

23.5.3 Theorem *If you apply a right trivialization by taking any tensor on G to an element of $\otimes^k \mathfrak{g}$ by bringing it back to the identity via right multiplication with g^{-1}, you get*

$$\begin{aligned}\rho(Q)(gh) &= Q_{gh}h^{-1}g^{-1} = (gQ_h + Q_g h)h^{-1}g^{-1} \\ &= g\rho(Q)(h)g^{-1} + \rho(Q)(g)\end{aligned}$$

That is it is a 1-cocycle on G with values in the kth tensor power of $\mathfrak{g}$ as a power of the adjoint representation of G.

Also do a left trivialization $\lambda(Q)$

$$\begin{aligned}\lambda(Q)(gh) &= h^{-1}g^{-1}Q_{gh} = h^{-1}g^{-1}(gQ_h + Q_g h) \\ &= h^{-1}Q_h + h^{-1}g^{-1}Q_g h = \lambda(Q)(h) + h^{-1}\lambda(Q)(g)h\end{aligned}$$

It is a 1-cocycle on G^{op} with values in the kth tensor power of $\mathfrak{g}$ as a power of adjoint representation of G^{op} so G acts on the right.

23.5.4 Example *Let $q \in \otimes^k \mathfrak{g}$. We can build a multiplicative tensor by $Q_g = g.q - q.g$*

$$\begin{aligned}Q_{gh} &= (gh).q - q.(gh) = g.h.q + g.q.h - g.q.h - q.gh = g.(Q_h) + (Q_g).h \\ \rho(Q)(g) &= Q_g g^{-1} = (g.q - q.g).g^{-1} = g.q.g^{-1} - q\end{aligned}$$

Take differential of $\rho(Q)$ to get a Lie algebra co-cycle instead $\mathfrak{g} \to \otimes^k \mathfrak{g}$ It is $DQ(x) = ad_x q$ which is δq the coboundary of just q a 0-cocycle $\mathbb{R} \to \otimes^k \mathfrak{g}$

Take an $r \in \Lambda^2\mathfrak{g} = Sym^2(\mathfrak{g}[1])$, It turns into a multiplicative tensor $g.r - r.g$, it is a multiplicative bivector but for it to define a Poisson bivector we check Jacobi which means that $T_e(\rho(P))$ as a 1-cocycle $\mathfrak{g} \to \Lambda^2\mathfrak{g}$ needs to be a Lie bracket on $\mathfrak{g}^*$ which amounts to the equation $[[r, r]]$ is Ad invariant.

If r is quasi-triangular with ad-invariant symmetric part satisfying these first conditions listed.

$$\begin{aligned}s &\in \Lambda^2(\mathfrak{g})^{\mathfrak{g}} \\ g.s - s.g &= 0 \\ \langle a, a\rangle + \langle s, s\rangle &= 0 \\ P(g) &= g.r - r.g = g.s - s.g + g.a - a.g = g.a - a.g \\ -\frac{1}{2}[[a, a]] &= \langle a, a\rangle = -\langle s, s\rangle \in (\Lambda^3\mathfrak{g})^{\mathfrak{g}}\end{aligned}$$

$$\begin{array}{ccc} G^N & \xrightarrow{F} & G^N \\ \downarrow M & & \downarrow M \\ G & \xrightarrow{F'} & G \end{array}$$

23.5.5 Definition (Quasi-Triangular and Factorizable) *Quasi-Traingular Comes from this above. Factorizable if in addition the map induced by s is invertible.*

23.5.6 Definition (Sklyanin bracket)

This ensures that we see the
as an upgrade of the Lax formalism in the Lie algebra case.

Our subalgebra of integrals of motion are

23.5.1 Hamiltonian reduction

Non-abelian moment map Reyman 96 and Semenov-Tian-Shansky 94

23.6 Twistings

23.7 L_∞ matching

Now let's repeat everything above in a way that's suitable to L_∞ algebras. In particular weak Lie 2-bialgebras regarded as concentrated in degrees 0 and ± 1. The graded dual of a L_∞ algebra is

23.7.1 Definition (q-Hamiltonian G-Space) *Let G be a Lie group and K be an isomorphism $\mathfrak{g} \to \mathfrak{g}^*$. Use the same letter for the inner product $K(-_1, -_2) = K(-_1) \circ (-_2)$. We can find a group valued moment map $\mu M \to G$ by*

$$\begin{aligned} \xi &\in \mathfrak{g} \\ \omega &\in \Omega^2(M) \\ \theta_{L,R} &\in \Omega^1(G, \mathfrak{g}) \\ i(v_\xi)\omega &= \frac{1}{2}\mu^*(K(\theta_L + \theta_R)(\xi)) \end{aligned}$$

If you want ω to be G-invariant, you have to give up the closed-ness. You are lead to $d_{dR}\omega = -\mu^* \frac{1}{12} K(\theta_{L,R}, [\theta_{L,R}, \theta_{L,R}])$ which is familiar from Chern-Simons. So it is not non-degenerate it's kernel is

$$\ker \omega_x = \{v_\xi \mid \xi \in \ker(Ad_{\mu(x)} + 1)\}$$

Let u be a parameter of homological degree 1. Now instead give the map $uK\ \mathfrak{g} \to \mathfrak{g}^*[1]$

23.7.2 Definition (Cayley Map) *Take a skew Hermitian A and send it to $(I-A)(I+A)^{-1}$ to produce a unitary matrix. Converseley a unitary without a eigenvalue -1 can be transformed back to a skew Hermitian matrix.*

23.7.3 Example

- *Conjugacy class in G.*
- *$D(G_2) = G_2 \times G_2$ as a $G = G_2 \times G_2$ space.*
- *q-Hamiltonian reduction for $G_1 \times G_2$ spaces to get a G_2 space*
- *Fusion product to build a $G \times H_1 \times H_2$ space from a $G \times H_1$ and a $G \times H_2$ space*

23.7.4 Theorem *Let us denote a quasi-Hamiltonian G-space M equipped with G action A, symplectic form ω and moment map μ*

Start with a usual Hamiltonian G-space (M, A, σ, Φ)

$$\begin{aligned} M^{new} &= M \\ A^{new} &= A \\ exp_s(-) &= exp(s * -) \\ \omega &= \sigma + \Phi^* Kill^*(\frac{1}{2}\int_0^1 (exp_s^*\bar{\theta}, \frac{d}{ds} exp_s^*\bar{\theta})ds) \\ \mu &= exp(\Phi) \end{aligned}$$

If $d_{Kill(\xi)}exp$ is a bijection for $\xi \in \Phi(M)$, then this is q-Hamiltonian G-space.

23.7.5 Example *Let M be a coadjoint orbit.*

23.7.6 Definition (Poisson-Lie G space) *M manifold. A is an action of G. ω is a symplectic form. It is not invariant under the action. μ is a moment map but to the dual group G^*.*

$$i(v_\xi^\sharp)\omega = 2\mu^*\langle \bar{\theta}_G \mid \xi \rangle$$

23.7.7 Definition (q-Hamiltonian G space with P valued moment map) *Manifold, action 2 form ω_P and moment map but now P valued.*

$$\begin{aligned} d\omega_P &= -\mu_P^*\chi_P \\ i(v_\xi)\omega_P &= \frac{1}{2i}\mu_P^*(p^*\theta + p^*\bar{\theta}, \xi) \end{aligned}$$

23.7.8 Theorem *Let $j(b) = bb^\dagger$ be the map that intertwines the left multiplication by G to the adjoint action on P, dressing action on G^* . κ is the G^* embedding. I $G_{\mathbb{C}} \to G_{\mathbb{C}}$ by Hermitian conjugation.*

$$\begin{aligned} \mu_P &= j\mu \;\; M \to G^* \to P \\ \omega_P &= \omega + \frac{1}{2}\mu^*\kappa^* Im(I^*\bar{\theta}, \theta) \end{aligned}$$

This turns a Poisson-Lie G-space to a q-Hamiltonian G-space with P valued moment map.

23.7.9 Theorem (q Hamiltonian P moment map to usual G space) *There exists a canonical 2-form on P τ such that*

$$\begin{aligned} \mu &= \log \mu_P \;\; M \to G^* \to \mathfrak{g}^* = \sqrt{-1}\mathfrak{g} \\ \omega &= \omega_P + \mu_P^* \tau \end{aligned}$$

23.7.10 Theorem *q-Hamiltonian G space to Hamiltonian LG space with proper moment map*

$$\begin{array}{ccc} q-Hamiltonian & \xrightarrow{23.7.10} & Hamiltonian\ LG\ space \\ {\scriptstyle 23.7.4}\uparrow\downarrow{\scriptstyle 23.7.4} & & \\ Hamiltonian & & \\ {\scriptstyle 23.7.9}\uparrow & & \\ q-HamltonianwithP & & \\ {\scriptstyle 23.7.8}\uparrow & & \\ Poisson-LieG-space & & \end{array}$$

@articlealekseev1997lie, title=Lie group valued moment maps, author=Alekseev, Anton and Malkin, Anton and Meinrenken, Eckhard, journal=arXiv preprint dg-ga/9707021, year=1997

`https://arxiv.org/pdf/1706.07493.pdf`

LG is Banach Lie group of G valued loops for some Sobelev class $s > 1/2$.

23.7.11 Theorem *The tangent bundle of any proper Hamiltonian loop group space $\mathcal{M}$ has a canonically defined LG equivariant completion $\bar{T}\mathcal{M}$ such that $\bar{T}\mathcal{M} \to T^*\mathcal{M}$ becomes a strong symplectic form isomorphism.*

23.7.12 Theorem *$\bar{T}\mathcal{M}$ has a distinguished LG invariant polarization. It admits a invariant compatible J within this polarization class unique up to homotopy.*
Defines a bundle of clifford algebras $\mathbb{C}l(\bar{T}\mathcal{M})$ and that acts on $S_{\bar{T}\mathcal{M}}$

23.8 Irregular Singularities

Boalch: Stokes Matrices and Poisson Lie Groups

Let $G = GL(n, \mathbb{C})$. We identify G^* with a moduli space of meromorphic connections over the unit disc with irregular singularity at the origin. From this we get: for each $A_0 \in t_{reg}$, there exists a holomorphic map $\mathfrak{g}^* \to G^*$. This map factors through the moduli space and the Riemann-Hilbert correspondence. This is a Poisson map for each A_0 where the left has the Kirillov-Kostant Poisson structure and the right has the standard structure multiplied by $2\pi i$.

If we restrict A_0 to be imaginary diagonal rather than just diagonal and restrict to the real locus we get the Ginzburg-Weinstein diffeomorphism $\mathfrak{k}^* \simeq K^*$ with usual Poisson structure and π rescaled structure.

$$\begin{array}{ccc} \mathfrak{k}^* & & K^* \\ \\ \mathfrak{g}^* & & G^* \\ \downarrow & & \downarrow \\ \mathcal{M}(A_0) & \xrightarrow{RH} & M(A_0) \end{array}$$

$\mathcal{M}(A_0)$ is given by isomorphism classes of triples of a meromorphic connection on a rank n vector bundle on the disc with just one second order pole at 0, a framing g_0 at 0 where ∇ is of the form $d-(\frac{A_0}{z^2}+\frac{B}{z})dz+\theta$ where θ is a holomorphic one form. Some are of the form $\nabla = d-(\frac{A_0}{z^2}+\frac{B}{z})dz$. $\mathfrak{g}^*$ is included in by sending $\mathfrak{g}^* \to \mathfrak{g} \to \mathcal{M}(A_0)$ where the last map is setting B.

23.8.1 Lemma *There is a unique formal gauge transformation $\hat{F}$ taking Δ to one of the form $\nabla = d - (\frac{A_0}{z^2} + \frac{B}{z})dz$ with A_0 and B diagonal.*

23.8.2 Theorem *On each sector there is a $\Sigma_i \hat{F}$ that is asymptotic to $\hat{F}$ in each supersector.*

23.9 Dynamical Yang Baxter Equation

In this section we are working with $\mathrm{Rep}\, U_q\mathfrak{g}_{\mathbb{C}}$

23.9.1 Definition (Shapalov Form) *For Verma module M_λ, we can define a pairing $M_\lambda \otimes M_\lambda^* \to \mathbb{C}$ by*

$$\begin{aligned} \langle 1_\lambda \mid 1_\lambda^* \rangle &= 1 \\ \langle e_i 1_\lambda \mid 1_\lambda^* \rangle &= \langle 1_\lambda \mid S(e_i) 1_\lambda^* \rangle \end{aligned}$$

Take the dual of this arrow to give

$$\mathbb{C} \to (M_\lambda \otimes M_\lambda^*)^*$$

Copy from lectures of Dynamical Yang Baxter Equation by Etingof Schiffmann

Copy from Enriquez Etingof

23.9.2 Definition (Intertwining Operator) *Let $v \in V[\lambda - \mu]$ the specified weight space of V. Then construct $\Phi_\lambda^v\ M_\lambda \to M_\mu \otimes V$ by*

$$\Phi_\lambda^v(x_\lambda) \in x_\mu \otimes v + \sum_{\nu < \mu} M_\mu[\nu] \otimes V$$

Making this an intertwiner is enough to specify it uniquely with that leading term.

23.9.3 Definition (Fusion Matrix)

$$\begin{aligned} J_{WV}(\lambda)\ W \otimes V &\to W \otimes V \\ w \otimes v &\to \langle \phi_\lambda^{w,v} \rangle \end{aligned}$$

where we first take v and use it to construct the vertex operator Φ_λ^v and then use w to construct $\Phi^w_{\lambda - wt(v)}$. The composition $M_\lambda \to M_{\lambda - wt(v) - wt(w)} \otimes W \otimes V$ can be done with a unique element of $W \otimes V$ if it was done in one step.

Figure 23.1: Insert a picture of Verma branching of many copies of $\mathbb{C}^2_{x_i}$ evaluation representations.

23.9.4 Proposition *The fusion matrix is rational in λ and strictly lower triangular with 1's on the diagonal. It satisfies a dynamical 2-cocycle condition for triples of modules.*

23.9.5 Definition (Exchange Matrix)

$$R_{VW}(\lambda) \quad = \quad J_{VW}(\lambda)^{-1} J^{21}_{WV}(\lambda)$$

This R matrix satisfies the dynamical Yang Baxter Equation without spectral parameter and step $\gamma = 1$

This also works for quantum groups, but change rational dependence on λ to trigonometric dependence/rational in $q^{\lambda\dot{\alpha}}$

In order to take classical limits is better to calculate $J(\frac{\lambda}{\gamma})$ and $R(\frac{\lambda}{\gamma})$ which now satisfy the equations with step γ. Now take the series as the step goes to 0.

The rational case using $U\mathfrak{g}_{\mathbb{C}}$ goes to the **basic rational dynamical r-matrix** and the trigonometric case using $U_q\mathfrak{g}_{\mathbb{C}}$ as well as setting $q = e^{-\epsilon\gamma/2}$ (so that it goes to 1 in this limit as well) goes to the **basic trigonometric dynamical r-matrix with coupling** ϵ

$U \subset \mathfrak{h}^*$ is the space for the dynamical partameter. Let $X = U \times G \times U$ viewed as a groupoid. H^2 acts on this by

$$\begin{aligned}
(h_1, h_2)(u_1, g, u_2) &= (Ad^*(h_1)u_1, h_1 g h_2^{-1}, Ad^*(h_2)u_2) \\
&= (h_1 u_1 h_1^{-1}, h_1 g h_2^{-1}, h_2 u_2 h_2^{-1}) \\
g^{-1}u_1 g &= u_2 \\
(h_1 g h_2^{-1})^{-1}(h_1 u_1 h_1^{-1})(h_1 g h_2^{-1}) &= h_2 g^{-1} u_1 g h_2^{-1} \\
&= h_2 u_2 h_2^{-1}
\end{aligned}$$

So that's what the arrows do, they take u_1 to $g^{-1}u_1 g$ so it is the right action. of g on u_1 as witnessed by the fact that g was written to the right side. So this preserves $Y \subset X$ the ones where g actually connects you from u_1 to u_2. Y is modelled on $T^*G \rightrightarrows \mathfrak{g}^*$ actually $T^*G \rightrightarrows U$

So there is a Poisson structure on X by EV1 and similar conditions define coboundary, quasi-triangular and triangular dynamical Poisson-Lie groupoids.

Let $\mathfrak{h}^* = \mathfrak{su}(N)^*$ and U be a subset near the Hermitian matrix 0.

You can use the $H = SU(N)$ action to diagonalize u_1. In X the fiber of the source map is all of $SU(N) \times \mathfrak{su}(N)^*$ because they don't have to connect. In Y the fiber of this is $SU(N)$ and there is a $S(U(1))^N$ action which stabilize the source and target. But the target $S(U(1)^N)$ is the same one after conjugation by g so we can write the space of arrows when one end is diagonalized as $SU(N)/(S(U(1)^N)$ the coadjoint orbit through said reference point. That isn't playing well with the moment map which is $(-s, t)$ so we need to give a coisotropic in $\mathfrak{h}^* \oplus \mathfrak{h}^*$ which would be entire orbits not points.

In terms of shifed Poisson structures `https://arxiv.org/pdf/1706.02623.pdf` by Safranov

[**?**]

23.10 Nahm's Equations

23.10.1 Definition (Nahm) *Give 3 k by k matrices which depend smoothly on* $s \in (-\infty, 0]$

$$\begin{aligned} \frac{dT_1}{ds} &= \big[T_2, T_3\big] \\ \frac{dT_2}{ds} &= \big[T_3, T_1\big] \\ \frac{dT_3}{ds} &= \big[T_1, T_2\big] \end{aligned}$$

23.10.2 Definition (Kronheimer Space $M(0, \xi, 0)$) $\xi \in \mathfrak{t}$ *is the orbit we are trying to pass through.*

The space of solutions to

Give 3 elements of $\mathfrak{g}$ *which depend smoothly on* $s \in (-\infty, 0]$

$$\begin{aligned} \frac{dB_1}{ds} &= -\big[B_2, B_3\big] \\ \frac{dB_2}{ds} &= -\big[B_3, B_1\big] \\ \frac{dB_3}{ds} &= -\big[B_1, B_2\big] \end{aligned}$$

satisfying boundary condition $\exists g_0 \in G$

$$\begin{aligned} B_1 &\to 0 \\ B_2 &\to Ad(g_0)(\xi) \\ B_3 &\to 0 \end{aligned}$$

23.10.3 Theorem (Kronheimer90) *Thiis maps diffeomorphically to the adjoint orbit* $\mathcal{O}_\xi$ *by*

$$\begin{aligned} (B_1, B_2, B_3) &\to B_2(0) + iB_3(0) \\ T_1 &= -B_3 \\ T_2 &= -B_1 \\ T_3 &= -B_2 \\ (T_1, T_2, T_3) &\to -T_3(0) - iT_1(0) \end{aligned}$$

where T *is the solution to the original form of Nahm equations which will be more convenient.*

23.10.4 Theorem (Nahm equation Lax Form)

$$\begin{aligned} A(s,\xi) &= T_1 + iT_2 + \xi(-2iT_3) + \xi^2(T_1 - iT_2) \\ A_+ &= -iT_3 + \xi(T_1 - iT_2) \\ \frac{dA}{ds} &= \left[A, A_+\right] \\ A &= A_0 + A_1\xi + A_2\xi^2 \\ A_+ &= \frac{1}{2}A_1 + A_2\xi \\ T_i^\dagger = -T_i \quad &\Longleftrightarrow \quad A_0^\dagger = -A_2 \;\; A_1^\dagger = A_1 \end{aligned}$$

Because we have a Lax presentation, the characteristic polynomial is a constant of motion independent of s. Say the variable for the characteristic polynomial is η. In TP^1 with coordinates (η, ξ) the characteristic polynomial determines a curve by compactification.

Because only the eigenvalues matter, all solutions of Nahm's equations determine the same spectral curve.

So if the eigenvalues are λ_i are distinct then the spectral curve

$$(\eta - \lambda_1\xi)\cdots(\eta - \lambda_k\xi) \;\; = \;\; 0$$

decomposes as the union of

$$(\eta - \lambda_i\xi) \;\; = \;\; 0$$

which are almost disjoint except for $\eta = 0$ and $\xi = 0\,\infty$ where they all come together.

- Solution of Bogomolny equations on $\mathbb{R}^3$ with prescribed boundary conditions at ∞
- Spectral curve $S(\xi)$ satisfying a reality condition in $T\mathbb{P}^1$ and a certain line bundle
- Antihermitian solution to Nahm's equations
- Complex Coadjoint orbit

23.11 Springer

Copied from http://arxiv.org/pdf/math/9802004v3.pdf

23.11.1 Flag Variety

The set of Borels in $\mathfrak{g}$ is a closed subvariety of the Grassmannian of subspaces in $\mathfrak{g}$ of dimension $dim\mathfrak{b}$ but it is more restrictive by saying that subspace is actually a Borel subalgebra. All Borels are conjugate under the action of G and the isotropy group is the exponentiated version B. So you can start with a reference $\mathfrak{b}$ and then ask what element in G/B is needed to conjugate to your desired Borel. This is a bijection $G/B \simeq \mathcal{B}$. This is a G-equivariant isomorphism of algebraic varieties.

$G/B \longrightarrow K/T \hookrightarrow \mathfrak{t}^*$

23.11.2 Springer

Say we want to consider all nilpotents in $\mathfrak{g}$. The ones where the operator $ad\ x$ is nilpotent. Call this set $\mathcal{N}$. It is closed, Ad G stable subvariety and it is also stable under dilatation. This says $\mathcal{N}$ is a conical variety.

Let $\tilde{\mathcal{N}}$ be the set of pairs of a Borel $\mathfrak{b}$ and a nilpotent $x \in \mathfrak{b}$. The fiber over a given Borel is it's nilpotent elements which are $[\mathfrak{b}, \mathfrak{b}]$. This means we have a vector bundle over the flag variety G/B parameterizing Borels.

Use the Killing form to identify $\mathfrak{g} \simeq \mathfrak{g}^*$. Then there is a natural G equivariant vector bundle isomorphism $\tilde{\mathcal{N}} \simeq T^*\mathcal{B}$

Define the map $\tilde{\mathcal{N}} \to \mathcal{N}$ by projection to the first factor. This is proper and surjective. Every nilpotent lives in some Borel. It is irreducible and a resolution of singularities for $\mathcal{N}$.

In addition to this resolution of singularity perspective, you can also think of it as a moment map for the canonical Hamiltonian G action on $T^*\mathcal{B}$.

The Steinberg variety is the pullback $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$

23.11.1 Theorem *The Steinberg has many components all of the same dimension. One of them is the diagonal.*

23.11.2 Theorem (Gan Ginzburg) *There is an algebra isomorphism from* $H^{BM}_{middle}(Z) \otimes \mathbb{C} \simeq \mathbb{C}[W]$ *where the left hand side has the convolution product.*

This can be extended to a statement over the rationals which keeps more information like the difference between $\mathbb{Q}(\sqrt{5})$ *and* $\mathbb{Q}(e)$

23.11.3 Definition M^+ *is the union of the conormal bundles to the B orbits. For the* $T^*\mathbb{CP}^1$ *example, there is* $\mathbb{C}$ *and* ∞ *which upon taking conormal gives the candy wrapper. There is the* $\mathbb{P}^1$ *and the fiber at* ∞ *touching at one point.*

23.11.4 Theorem (Gan Ginzburg) $H^{BM}_d(M^+)$ *is a module over* $H^{BM}_{2d}(Z)$*. In this case we get* $\mathbb{C}[W]$ *as a regular representation and as an algebra respectively.*

23.11.3 Groethendieck-Springer

Instead of considering x nilpotent only, let it be any element of $\mathfrak{g}$. This still gives a vector bundle over the flag variety but now of rank $dim\mathfrak{b}$ instead of $dim\mathfrak{n}$. It also becomes a resolution $\tilde{\mathfrak{g}} \to \mathfrak{g}$ instead of just the nilpotent cone. You can also do the group version where $\tilde{G}$ is the set of pairs of a Borel subgroups and arbitrary elements in them. This gives $\tilde{G} \to G$

23.11.5 Example *For $G = SL(2)$, the flag variety is $\mathbb{P}^1$, so we can try to explicitly say which vector bundle we are getting. It is $\mathcal{O}(-1) \bigoplus \mathcal{O}(-1)$ for the Groethendieck-Springer resolution and $\mathcal{O}(-2)$ for the Springer resolution. http://arxiv.org/pdf/math/0604445v1.pdf*

23.12 Symplectic Resolutions

[?]

23.12.1 Definition (Symplectic Singularity) *A normal algebraic variety X such that the regular locus for any resolution of singularities carries an algebraic closed nondegenerate 2-form. A symplectic resolution is one where it extends from the regular locus to the entire $\tilde{X}$. It is necessarily Calabi-Yau.*

23.12.2 Definition (Conical symplectic resolution) *An affine symplectic singularity carrying a conical $\mathbb{G}_m$ action that acts on the symplectic form with weight $n > 0$.*

23.12.3 Example *Springer Resolution.*

Consider $T^*\mathcal{B}$ Give two $\mathbb{C}^*$ actions by scaling the fibers call S, and T which is given by a cocharacter extend to cotangent bundle so that it preserves ω. S is what makes it conical.

See category $\mathcal{O}$ discussion above in Beilinson-Bernstein section.

Now just do generally

Let $Z = M \times_{M_0} M$ and let M^+ be the sub of M such that $lim_{t\to 0} tp$ exists. Again $H^{BM}_{2d}(Z)$ is an algebra acting on $H^{BM}_d(M^+)$ as before.

$K(\mathcal{O}) \otimes \mathbb{C} \simeq H^{BM}_d(M^+)$ this was our regular representation so to categorify the $\mathbb{C}[W]$ action, turn this into a bimodule.

Again categorify to get bimodules acting on a module category.

We don't have an X such that $M = T^*X$, instead want a quantization of M.

23.12.4 Definition (Quantization of M Conical symplectic resolution) *a T-equivariant sheaf of filtered algebras on M and a $T \times S$ graded isomorphism $gr\mathcal{A} \simeq Fun_M$*

For the case we did before we get $U(\mathfrak{g})$ some other central character. Twisted D-modules.

If $M = Hilb(\mathbb{C}^2/\Gamma)$ then get a quotient of spherical rational Cherednik algebra.

Again try to take singular support to get cycles on M^+ to get what you call category $\mathcal{O}$. A^+ acts locally finitely. In the springer case A^+ is not $U(\mathfrak{b})$ on the nose, but the local finiteness-ness is the same.

Give a version of Harish-Chandra bimodules

Chapter 24

To Sort 2

24.1 Cactus Group

Define the cactus group as the orbifold fundamental group of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})/S_n$ where the S_n permutes the first n marked points. If we gave it as generators and relations they would be

$$\begin{aligned} 1 \leq p &< q \leq n \\ s_{p,q}^2 &= e \\ s_{p_1,q_1}s_{p_2,q_2} &= s_{p_2,q_2}s_{p_1,q_1} \quad [p_1,q_1]\bigcap[p_2,q_2] = \varnothing \\ s_{p_1,q_1}s_{p_2,q_2} &= s_{p_1+q_1-q_2,p_1+q_1-p_2}s_{p_1,q_1} \quad [p_2,q_2] \subseteq [p_1,q_1] \end{aligned}$$

One can interpret $s_{p,q}$ as taking the circle with $n+1$ marked points, taking the part between p and q and pinching it off and putting it back backwards. The pinching off part is the only part when you leave $\mathcal{M}_{0,n+1}(\mathbb{R})$ to a nodal curve before coming back.

With the exact sequence of groups

$1 \longrightarrow PJ_n \longrightarrow J_n \longrightarrow S_n \longrightarrow 1$

$s_{p,q}$ goes to the permutation $(1 \cdots n) \to (1 \cdots (p-1), q \cdots p, (q+1) \cdots n)$. The kernel PJ_n is the fundamental group of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$.

24.1.1 Remark (Coxeter overgroup TODO)

$$s_{p_1,q_1}s_{p_2,q_2}s_{p_1,q_1}s_{p_2,q_2} = s_{p_1+q_1-q_2,p_1+q_1-p_2}s_{p_2,q_2}$$

If $[p_1+q_1-q_2, p_1+q_1-p_2] = [p_2,q_2]$, then they we have $(s_{p_1,q_1}s_{p_2,q_2})^2 = e$
If $[p_1+q_1-q_2, p_1+q_1-p_2]\bigcap[p_2,q_2] = \varnothing$, then they commute and we have $(s_{p_1,q_1}s_{p_2,q_2})^4 = e$
Otherwise we have to keep using the rules to figure out the minimal n such that $(s_{p_1,q_1}s_{p_2,q_2})^n = e$

Have results that embed J_n into right angled Coxeter groups but this goes the other way. Also have results saying it is residually nilpotent. (G embeds into it's pro-nilpotent completion, take inverse limit (limit of the diagram in categorical parlance) over $G \to H$ where H is a nilpotent group)

24.1.2 Definition (Coboundary Category) *A monoidal category with a functorial involutive isomorphism $s_{X,Y}$ $X \otimes Y \to Y \otimes X$ called the commutor. This does not have to come from a braided monoidal category. PJ_n acts by endomorphisms of $X_1 \cdots X_n$ for any collection of n objects from a coboundary category.*

24.1.3 Example (The category of $\mathfrak{g}$ crystals) *For any Kac-Moody $\mathfrak{g}$, the category of $\mathfrak{g}$ crystals is a coboundary category. The tensor product is not symmetric unlike $U(\mathfrak{g})$ modules.*

24.1.4 Definition ($J_{\mathfrak{g}}$ for finite dimensional complex semisimple Lie algebra) *Generators are s_I for any subdiagram I of the Dynkin diagram of $\mathfrak{g}$.*

$$\begin{aligned} s_I^2 &= 1 \\ s_I s_{I'} &= s_{I'} s_I \quad I \bigcap I' = \varnothing \\ s_I s_{I'} &= s_{\theta_I(I')} s_I \quad I' \subseteq I \end{aligned}$$

$$1 \longrightarrow PJ_{\mathfrak{g}} \longrightarrow J_{\mathfrak{g}} \longrightarrow W_{\mathfrak{g}} \longrightarrow 1$$

θ_I is the diagram automorphism defined using the longest element of the Weyl group of $\mathfrak{g}_I$. So for all $i \in I$

$$\alpha_{\theta_I(i)} = -w_0^I \cdot \alpha_i$$

This gives another simple root so we see $\theta_I(i)$ corresponds to a node of the diagram as well. Extending to subsets this becomes $\theta_I(I')$ becomes another subdiagram.

Like with the definition as a fundamental group $PJ_{\mathfrak{g}} \simeq \pi_1(\overline{\mathbb{P}(\mathfrak{h}_{\mathbb{R},reg})})$. In the case of $\mathfrak{sl}_N$, this is back to $\overline{\mathcal{M}}_{0,N+1}(\mathbb{R})$ which is compactification of specifying $N+1$ points of RP^1 then subtracting off the thick diagonal that makes sure the points are pairwise distinct, then modding out by the overall action of $PGL_2(\mathbb{R})$ which acts as usual on each of the RP^1 factors.

24.1.5 Example *The above case is the one corresponding to $\mathfrak{g} = \mathfrak{sl}_N$ and $s_{p,q}$ corresponds to the inclusion of the subdiagram between nodes p and $q-1$. When I is the full Dynkin diagram, $\theta_I(i) = N - i$ and otherwise it is flipping I and seeing what happens to the I' inside.*

24.1.6 Theorem *For any $\mathfrak{g}$ crystal $B(\lambda)$ we have an action of $J_{\mathfrak{g}}$ defined by $s_I(b) = \xi_{B_I}(b)$ where ξ are Schutzenberger involutions that swap highest and lowest weight elements and the corresponding e_i and f_i Kashiwara operators. This happens on each connected component of the crystal which each are of the form $B(\mu)$ for some μ. ξ_{B_I} means restrict the crystal to the subdiagram I first.*

24.1.7 Conjecture *Let A_μ be the shift of argument algebra which is a Poisson commutative subalgebra of $Sym(\mathfrak{g})$ with KKS bracket. It is generated by some F_i which are elements of the Poisson center of $Sym(\mathfrak{g})$ and $\partial_\mu^n F_i$ for any $\mu \in \mathbb{P}(\mathfrak{h}_{\mathbb{R},reg})$.*
This family of algebras admits a compactification of the base to $\overline{\mathbb{P}(\mathfrak{h}_{\mathbb{R},reg})}$ and quantization to $U(\mathfrak{g})$. If you specify the representation $V(\lambda)$ A_μ acts with simple spectrum. So you can specify a covering

E which above any point μ is the set of one dimensional eigenspaces. That then implies a monodromy action by the fundamental group $PJ_{\mathfrak{g}}$.
As general abstract nonsense goes, Fib gives a functor from the category of covering spaces of X to $\Pi_1(X) \to Set$ functors where Π_1 is fundamental groupoid. That is send E to the function $x \to E_x$ and $\gamma \in x \to y$ goes to $E_x \to E_y$ defined by path lifting. In fact reconstruction functor backwards shows this as an equivalence of categories.
$\Pi_1(E) \simeq \int_{\Pi_1(X)} Fib_E$ *doing a Grothendieck construction on* $Fib_E\, \Pi_1(X) \to Set$.

24.1.8 Theorem *If $\mathfrak{g}$ is $\mathfrak{sl}_N$, then this holds. In fact this action is the same as the action on $B(\lambda)$*

Proof `http://www.math.toronto.edu/ivahal/A%20cactus%20group%20action%20on%20crystals.pdf` □

24.1.9 Remark (TODO) Other times you have families parameterized over $\overline{\mathbb{P}(\mathfrak{h}_{\mathbb{R},reg})}$ look for such an action of PJ. Any time you have a covering space $\phi\ \ E \to \overline{\mathbb{P}(\mathfrak{h}_{\mathbb{R},reg})})$ you get a monodromy action $PJ_{\mathfrak{g}}$ acting on $\phi^{-1}(\mu)$ where μ is a basepoint.

Fisher information ???, above each point in $\mathbb{P}(\mathfrak{h}_{\mathbb{R},reg})$, if all the signs are the same we can normalize it to a ρ_{diag} by an overall scalar to make sure the sum is 1. Then above that we have a K/T with Fisher metric which is the set of mixed states which have the same eigenvalues for ρ and generic. These are the ones that have all the same entropies for every kind of entropy (all of those only depend on eigenvalues). Does the converse hold?

Any other time seen covering spaces of $\overline{\mathbb{P}(\mathfrak{h}_{\mathbb{R},reg})}$??

24.2 Modular Operads

`https://arxiv.org/pdf/dg-ga/9408003.pdf` `https://arxiv.org/pdf/2103.01383.pdf`

24.2.1 Definition (Stable S-module) *A collection of chain complexes $P(g,n)$ for $g,n \geq 0$ and $2g+n-2<0$. Each $P(g,n)$ is equipped with an S_n action. For any finite set I of cardinality n, form the direct sum*

$$P(g,I) \equiv \Big(\bigoplus_{f \in Bij([n],I)} P(g,n)\Big)_{S_n}$$

Here the $_{S_n}$ indicates taking coinvariants.

For a multigraph (with loops) G with a function g $V_G \to \mathbb{N}$ and satisfying the stability condition on each vertex when interpreted as $g(v)$ being genus and each half edge being a puncture. This graph is encoding a thick-thin decomposition of the surface with the edges being thin cylinders that are being cut, this is known as the dual graph of a stable curve when all the thin cyclinders have become nodes and the curve has broken into components with n marked points and glued together at node points. The overall genus is $\sum_v g(v) + rank H_1(G)$

$$\begin{aligned} P(G) &\equiv \bigotimes_{v \in V_G} P(g(v), Leg(v)) \\ &\equiv \bigotimes_{v \in V_G} \Big(\bigoplus_{f \in Bij([n],Leg(v))} P(g(v),n)\Big)_{S_n} \end{aligned}$$

24.2.2 Definition (As a modular operad) *Extend P to be a functor on the category of stable graphs. So have maps $P(h)$ induced on h $G \to G'$ of stable graphs satisfying $P(h_1 h_0) = P(h_1)P(h_0)$ on composable morphisms.*

24.2.3 Definition ($\mathbb{M}$ monad) *Let $\mathbb{M}$ be the endofunctor on the category of stable $\mathbb{S}$-modules by*

$$(\mathbb{M}\mathcal{V})(g,n) \equiv \bigoplus_{G \in |\Gamma(g,n)|} \mathcal{V}(G)_{Aut(G)}$$

where $|\Gamma(g,n)|$ is the finite set of isomorphism classes of objects in the category whose objects are stable graphs of genus g and bijections ρ between legs of that graph and $[n]$ and morphisms are morphisms of stable graphs that preserve these labelling data ρ and ρ'. The terminal object in this category is the graph with no edges but one vertex of genus g and n half edges eminating from it.

24.2.4 Definition (Modular Operad) *An algebra for the monad $\mathbb{M}$ in stable $\mathbb{S}$-modules. Equivalently it can be a stable $\mathbb{S}$-module $\mathcal{A}$ equipped with a μ $\mathbb{M}\mathcal{A} \to \mathcal{A}$ that extends to a functor on the category of stable graphs. The first condition on μ gives a candidate for the structure map but not that it satisfies the axioms for giving an algebra structure with the associative $\mathbb{M}^2 \to \mathbb{M}$ and $1 \to \mathbb{M}$ rules.*

24.2.5 Example (Deligne-Mumford) *First define the general modular operad but with orbifolds instead of chain complexes. In this example, use the orbifold of stable curves whose dual graph is G and then takes it's closure within moduli of stable curves of genus g and n marked points.*

Taking homology gives a modular operad in the category of graded vector spaces instead of chain complexes. Then an algebra over this operad is a CohFT in the sense of Kontsevich-Manin.

24.2.6 Definition (Feynman Transform)

24.3 Free Probability

24.3.1 Definition (Non-Commutative Probabilty Space) *A von Neumann algebra $\mathcal{A}$ with a tracial state τ $\mathcal{A} \to \mathbb{C}$*

$$\begin{aligned} \tau(XY) &= \tau(YX) \\ \tau(X^\dagger X) &\geq 0 \\ \tau(1) &= 1 \end{aligned}$$

24.3.2 Definition (Freely Independent) *Subalgebras $\mathcal{A}_1 \cdots \mathcal{A}_k$ are freely independent if*

$$\tau(X_1 \cdots X_m) = 0$$

whenever all the X_j are from some $\mathcal{A}_i$ with none of the adjacent ones in the same subalgebra and $\tau(X_j) = 0$ for all j.

Variables $X_1 \cdots X_m$ are called independent when $\mathcal{A}_i = \langle X_i \rangle$ are independent as subalgebras.

24.3.3 Example *Let $L(F_2)$ be the von-Neumann algebra generated by left translation operators by $g \in F_2 = \langle a, b \rangle$ acting on $\ell^2(F_2)$. Let the trace be $\tau(X) = \langle \delta_e \mid X\delta_e \rangle_{\ell^2(F_2)}$.*
Then a and b are freely independent. $a + a^{-1}$ and $b + b^{-1}$ are also freely independent.

This generalizes to free product of any two groups, not just $\langle a \rangle \star \langle b \rangle = \mathbb{Z} \star \mathbb{Z}$

24.3.4 Theorem (Cauchy Transform)

24.3.5 Theorem (Central Limit)

`https://www.math.ucla.edu/~davidjekel/Bozeman_Talk.pdf`

24.3.1 Application

$\rho = e^{-\beta H}$. We don't have tracial. But we could restrict to stuff that commutes with H first.

24.4 Modular Representation

`https://arxiv.org/pdf/2002.01604.pdf`

Throughout this section, work in $T^*\mathbb{R}^d$ as the phase space being quantized.

24.4.1 Definition (Weyl Operator)

$$\begin{aligned} W_{a,b} &\equiv e^{i(b\cdot q - a\cdot p)/\hbar} \\ W_{a,b}W_{a',b'} &= e^{\frac{1}{2}i(b\cdot a' - a\cdot b')/\hbar} W_{a+a',b+b'} \end{aligned}$$

24.4.2 Theorem (Gelfand-Naimark) *Once a commutative C^* subalgebra A of $\mathcal{W}$ is chosen, we get an associated topological space X such that $A \simeq C^*(X, \mathbb{C})$. If we choose A to be generated by all the $W_{0,b}$, then we get the usual $L^2(\mathbb{R}^d)$*

24.4.3 Definition (Modular Representation) *Notice that $W_{a,b}$ and $W_{a',b'}$ commute if and only if $\frac{1}{2\pi\hbar}(a' \cdot b - a \cdot b') \in \mathbb{Z}$. So the subset $(a, b) \in \mathbb{R}^{2d}$ satisfying this condition forms a lattice Λ.*
We can always do a symplectic coordinate transformation of $\mathbb{R}^{2d}$ to get $\Lambda = \{(\lambda n, \overline{\lambda}\overline{n})\}$ where λ and $\overline{\lambda}$ are diagonal matrices such that $\lambda\overline{\lambda} = 2\pi\hbar I$ and $n, \overline{n} \in \mathbb{Z}^d$
The common eigenvectors for this subalgebra are $\mid X\rangle$ for any X in $\mathbb{R}^{2d}$ where the action is

$$\begin{aligned} W_{k,\tilde{k}} \mid X\rangle &= e^{\frac{1}{2}ik\cdot\tilde{k}/\hbar} e^{i\omega(K,X)/\hbar} \mid X\rangle \\ \mid X + K\rangle &= e^{\frac{1}{2}ik\cdot\tilde{k}/\hbar} e^{\frac{1}{2}i\omega(K,X)/\hbar} \mid X\rangle \end{aligned}$$

The rest of $\mathcal{W}$ act as

$$W_Y \mid X\rangle = e^{\frac{1}{2}i\omega(Y,X)/\hbar} \mid X + Y\rangle$$

Because of the quasi-periodicity $\mid X+K\rangle$ *for any* $K \in \Lambda$*, we can get a complete and orthonormal basis by restricting* X *to a fundamental domain of* $T_\Lambda \equiv \mathbb{R}^{2d}/\Lambda$*.* $\langle Y \mid\mid X\rangle = \delta(X-Y)$

General states are of the form

$$\mid \Phi\rangle \quad = \quad \int_{T_\Lambda} d^{2d}X \Phi(X) \mid X\rangle$$

Where to be well defined $\Phi(X)$ *must have the opposite quasi-periodicity as* $\mid X\rangle$ *so that* $\Phi(X) \mid X\rangle$ *is well defined on* T_Λ*.*

More fundamentally they are sections of a $U(1)$ *bundle over* T_Λ *with the specified quasi-periodicity giving the particular bundle. There is the connection* $A_\mu dX^\mu$ *with* $A_\mu = (\frac{1}{2}\tilde{x}_\nu, \frac{-1}{2}x^\gamma)$ *where* $\mu = (\nu, \gamma)$

24.4.4 Definition (Gelfand-Zak Transform)

$$\mid X\rangle \quad = \quad (\det \tilde{\lambda})^{-1/2} e^{\frac{1}{2}ix\cdot\tilde{x}/\hbar} \sum_n e^{i\tilde{x}\cdot\lambda n/\hbar} \mid x+\lambda n\rangle$$

where the left hand side has $X \in \mathbb{R}^{2d}$ *and the modular representation. The right hand side has the usual Schrodinger representation.*

24.4.5 Definition (Gauge Transformation) *Send* $\mid X\rangle \to e^{i\alpha(X)} \mid X\rangle$ *for any smooth function* $\alpha \in C^\infty(\mathbb{R}^{2d})$*. The line bundles and the action of* $\mathcal{W}$ *change accordingly as well as the modular connection.*

24.4.6 Corollary *Take* $\lambda = \ell I_d$ *and* $\tilde{\lambda} = \frac{2\pi\hbar}{\ell} I_d$*. Send* $\ell \to \infty$ *and choose* $\alpha = \frac{-1}{2\hbar} x \cdot \tilde{x} + C$