Super Field Theories

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Chapter 1

Organization of Chapters

Insert chart here with links to appropriate chapter titles.

$$\begin{tikzcd}
TheoryX \arrow[d] \arrow[ddr] & \\
ClassS & \\
& RW
\end{tikzcd}$$

Chapter 2

Necessary structures on manifolds

2.1 Spin Review

2.1.1 Super Poincare

In 4 dimensions we have the useful accidental isomorphism $Spin(4) \simeq SU(2) \times SU(2)$. Let $S^{\pm}$ be the fundamental representations of each factor then regarded as representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. These are both 2 complex dimensional representations. Let $T_{\mathbb{C}}$ be the complexified translations. There is an isomorphism as $\mathfrak{spin}(4) \otimes_{\mathbb{R}} \mathbb{C}$ representations $T_{\mathbb{C}} \simeq S^+ \otimes S^-$ The isomorphism can be scaled up if so chosen.

2.1.1 Definition (Super-translation)

$$T_{\mathbb{C}} \bigoplus (S^+ \bigoplus S^-)[1]$$
$$\left[Q^{+,\alpha}, Q^{-,\dot{\alpha}}\right] = Q^+ \otimes Q^- \in S^+ \otimes S^- \simeq T_{\mathbb{C}}$$

2.1.2 Definition (Extended SUSY) *Replace $S^+ \to S^+ \otimes W$ and $S^- \to S^- \otimes W^*$. Use the evaluation pairing in addition to the bracket of two odd elements.*
If $W = \mathbb{C}^k$, call the result $T^{\mathcal{N}=k}$ for the $N = k$ super translation algebra.

2.1.3 Definition (Super Poincare) $Spin(4) \ltimes T^{\mathcal{N}=k}$

2.1.2 Spin structures on 4 manifolds

Let X be a smooth 4 manifold with a Riemannian metric g. The structure group of the tangent bundle is $SO(4)$. A spin structure lifts the principal $SO(4)$ bundle of trivializations to $Spin(4)$. So if a spin structure exists, then we can choose one of them as it is an affine space over $H^1(M, \mathbb{Z}_2)$. In which case we can use the accidental isomorphism to give a pair of rank two vector bundles $S_{\pm}$ (The globalizations of the vector spaces we had before).

The closed orientable 4 manifold may not admit a spin structure ($w_2(M) \neq 0$) in which case we have to settle for a $Spin^c$ structure.

$$
\begin{aligned}
Spin^c(4) &\simeq (Spin(4) \times U(1))/\mathbb{Z}_2 \simeq (SU(2) \times SU(2) \times U(1))/\mathbb{Z}_2 \\
Spin^c(4) &\rightrightarrows (SU(2) \times U(1))/\mathbb{Z}_2 \simeq U(2)
\end{aligned}
$$

This gives the structure of a pair of $U(2)$ bundles. $V_\pm$. If the manifold happened to be spin then you can even write the structure as $S_\pm \otimes \mathcal{L}$ using a complex line bundle.

So locally break up $V_\pm$ into $S_\pm \otimes \mathcal{L}$. This is possible because in a small enough patch, a spin structure does exist and is unique up to isomorphism. On $\mathcal{L}$ you can start writing a unitary connection and on $S_\pm$ you can look at the Levi-Cevita connection. Put them together to define a connection on $S_\pm \otimes \mathcal{L}$ in this patch. This is the connection that can be globalized because the others are on nonexistent bundles.

2.1.4 Definition (*Spin*^{c} connection) *A connection that is locally of this type for one and hence any local decomposition into $S_\pm \otimes \mathcal{L}$ bundles over patches.*

The curvature of a $Spin^c$ connection is a $\mathfrak{u}(2)$ valued 2-form. We can project to $\mathfrak{su}(2)$ and to $\mathfrak{u}(1)$. The first gives the Riemann tensor of X, and the second gives a closed 2-form F.

In addition to the possibly nonexistent line bundle $\mathcal{L}$, we have one that is well defined with the data we have given called det V_+. If $\mathcal{L}$ did exist then this one is it's square $\mathcal{L}^{\otimes 2}$. So the curvature on the det bundle is 2F. It is classified by first Chern class $\frac{2F}{2\pi}$

On a Spin manifold we have $S_+ \otimes S_+ \simeq \Omega^0 \bigoplus \Omega^{2,+}$. So there is the bilinear map σ $S_+ \otimes S_+ \to \Omega^{2+}$. $\bar{S}_+$ is naturally isomorphic to it's complex conjugate because the fundamental representation of SU(2) is equivalent to the complex conjugate. For a $Spin^c$ structure, you give a map $V_+ \otimes \bar{V_+} \to \Omega^{2+}$. If it was spin then this would be $S_+ \otimes \mathcal{L} \otimes \bar{\mathcal{L}} \otimes \bar{S}_+ \simeq S_+ \otimes \bar{S}_+$ and the map would then be the σ from before.

2.1.5 Lemma *Page 10 Seiberg Senthil Wang Witten says*

"A $Spin_c$ connection is locally the same as a $U(1)$ gauge field, but it's Dirac quantization is different. Its fluxes satisfy

$$
\int_C \frac{dA}{2\pi} = \frac{1}{2} \int_C w_2 \; mod \, \mathbb{Z}
$$

where $C \subset X$ is an oriented two-cycle ..."

What that really means is that we are taking the $Spin_c$ connection to be Levi-Cevita tensored with that A on said nonexistent line bundle $\mathcal{L}$. Also w_2 is only a $\mathbb{Z}_2$ cohomology class so that really means evaluate on C and identify the $\mathbb{Z}_2$ with $\{0,1\} \in \mathbb{Z}$.

2.1.6 Remark See how you're using metrics before you need to. Potential for confusion. If you used a different metric, you would change your A which would affect this interpretation.

2.1.7 Theorem (Lichnerowicz formula)

$$D_A^\dagger D_A \psi \quad = \quad \nabla_A^\dagger \nabla_A \psi + \frac{1}{4} R \psi + \frac{1}{2} \langle F_A^+ \psi \rangle$$

where D_A is the Dirac operator $\Gamma(S^+) \to \Gamma(S^-)$
∇_A is the covariant derivative $\Gamma(S^+) \to \Gamma(S^+ \otimes T^* M)$
and $\langle F_A^+ \psi \rangle$ is defined by regarding $\Omega^{2,+}$ as a sub-bundle of the $End(S^+)$ bundle.

2.1.8 Corollary *There are no nontrivial harmonic spinors on a positively curved manifold.*

Proof $\nabla^\dagger \nabla$ is a positive operator and by assumption $R \geq 0$ everywhere. $\square$

2.1.3 Topological Perspective

Let M be an oriented manifold. For it to have a $Spin^c$ structure the third-integral Steiffel-Whitney class must vanish. $\beta w_2 = W_3 \in H^3(M, \mathbb{Z})$. If so the actual structures form a torsor over $H^2(M, \mathbb{Z})$.

2.1.9 Definition (Dixmier-Douady class)

2.2 Symplectic Review

2.2.1 Symplectic

2.2.1 Definition (Symplectic) *A symplectic manifold is a pair (X, ω) where $\omega \in \Omega^2(X)$ is closed and nondegenerate.*

2.2.2 Definition (Exact Symplectic) *A symplectic manifold equipped with a choice of primitive Liouville form λ such that $d\lambda = \omega$*

2.2.3 Definition (LagCor) *A Lagrangian correspondence between M_1 and M_2 is a lagrangian in $M_1 \times \bar{M}_2$. We can use these as "morphisms". Let this be the "category" with symplectic manifolds and Lagrangian correspondences. Composition does not quite work correctly because of smoothness for pullbacks.*

$$\begin{tikzcd} & L_{12} \arrow[dl] \arrow[dr] & & L_{23} \arrow[dl] \arrow[dr] & \\ M_1 & & M_2 & & M_3 \end{tikzcd}$$

2.2.4 Theorem *For nonlinear sigma model from $[t_1, t_2]$ to (M, ω) we get the projection of the Euler Lagrange as a Lagrangian $L \subset \mathcal{F}_\partial \simeq M \times \bar{M}$. We may hope that this is the the graph of a Hamiltonian diffeomorphism. In that case just have to specify the initial values in order to solve the EL equations and then read off the final point in phase space.*

2.2.5 Definition (Moment Map) *If you have a G real Lie group acting on the symplectic manifold by symplectomorphisms, then let ρ be the associated infinitesimal action $\mathfrak{g} \to Vect(X)$. A moment map is a G equivariant map $X \to \mathfrak{g}^*$ map satisfying $d(\mu(Z)) = i_{\rho(Z)} \omega$ for all $Z \in \mathfrak{g}$.*

2.2.6 Definition (Symplectic quotient) *A symplectic quotient is* $X//G = \mu^{-1}(0)/G$

$$\begin{tikzcd} & \mu^{-1}(0) \arrow[dl] \arrow[dr] & \\ X//G & & X \end{tikzcd}$$

is a Lagrangian correspondence. This quantizes to quantum Hamiltonian reduction reviewed in Section 27.1 and a bimodule between the algebras on both sides.

2.2.7 Example T^*S^2. *This is exact symplectic. We can consider rotating the sphere.*

2.2.8 Theorem (Duistermaat-Heckman) *Let* M *be a compact symplectic manifold of dimension* $2n$ *with a* $U(1)$ *action and moment map for that* μ.

$$\int_M \frac{\omega^n}{n!} e^{-\mu} = \sum_i \frac{e^{-\mu(x_i)}}{e(x_i)}$$

2.2.9 Theorem (Berligne-Vergne-Atiyah-Bott) *This is generalized to a general compact manifold* M *with* $U(1)$ *action with vector field* V *and an equivariantly closed form* α.

$$\int_M \alpha = \sum \frac{\pi^n \alpha_0(x_i)}{\sqrt{\det \partial_\mu V^\nu(x_i)}}$$

Proof Integration over $T[1]M$. □

2.2.10 Definition (Weinstein Manifold) $(M^{2n}, \omega, \alpha, f)$ *where* M, ω *is a symplectic manifold with Liouville form* α *and a proper bounded below Morse function. We want* V *around critical points to be given as a gradient flow for* f *with some choice of metric in adapted coordinates around these. That is* $Cr(f) = Z(V) = Z(\alpha)$. *So let* $S_p^{\pm}$ *be the stable and unstable manifolds for* $p \in Crit(f)$

2.2.11 Example $\mathbb{C}^n$ *with* $\omega = \sum dx_j \wedge dy_j$, $V = \frac{1}{2} \sum x_i \partial_{x_i} + y_i \partial_{y_i}$ *and* $f = \sum x_j^2 + y_j^2$

2.2.12 Example *Cotangent bundles of closed manifolds with usual* $V = \sum p_j \partial_{q_j}$ *and* $f = \sum p_j^2$

2.2.13 Example *Products with* $V_1 \times V_2$ *and* $f_1 \bigoplus f_2$. *In particular, the stabilization is the product with* $\mathbb{C}$ *from the first example.*

2.2.14 Definition (Weinstein Cobordism) *If we have manifold with boundary* $\partial M = \partial_- M \bigsqcup \partial_+ M$, *then make the boundaries regular level sets of* f *with values min and max respectively. The completenss of the Liouville vector field is replaced by the condition of pointing inwards at the incoming boundary and out at the outgoing boundary.*
In particular if $\partial_- M = \emptyset$ *then Weinstein domain.*

2.2.15 Definition (Skeleton) *Let* L *be the union over all critical points* $\bigcup_{p \in Crit(f)} S_p^+$.

2.2.16 Lemma *S_p^+ are isotropic. S_p^- are coisotropic.*

For a given manifold, we may look at the space of possible structures to attatch such as the space of Weinstein structures being $Weinstein(M)$, they have maps between each other as follows:

$$\begin{tikzcd} Stein(M) \arrow[r] & Weinstein(M) \arrow[r] \arrow[d] & Morse(M) \\ & Liouville(M) & \end{tikzcd}$$

Proof S_p^+ are invariant under V so ϕ_t^V moves along S_p^+. But this is equivalent to $\alpha \mid_{S_p^+} = 0$. □

2.2.17 Corollary *The skeleton is isotropic and their topology is concentrated in degrees $\leq n$. The picture is a body and then a cylindrical end once we get $f^{-1}(b, \infty)$ for b greater than all the critical values.*

2.2.18 Definition *Notions of equivalences can be symplectomorphisms that approach the identity for the cylindrical ends. Another notion of equivalence is homotopy respecting the end. There are also Lagrangian correspondences but those are not necessarily invertible morphisms.*

2.2.19 Proposition (Moser Rigidity) *Have a homotopy then fix to a symplectomorphism.*

2.2.20 Example *$M = T^*X$ for X compact. A choice of Riemannian metric then gives a function $\frac{p^2}{2m}$. This has critical manifold X. with stable and unstable manifold X and T^*X.*

2.2.21 Theorem *If there is only one critical manifold then $M \simeq_W T^*\mathbb{R}^m$ is Weinstein homotopic.*

2.2.22 Definition (Handle Attachment) *Do Morse handle attachment procedure in the exact symplectic category. So start from the bottom. Next critical point has it's unstable manifold $S_{p_2}^-$ this gets thickened up for attachment. Along isotropic spheres and extra data in contact level sets.*

2.2.23 Lemma () *If $\dim S_p^+ < n$ for all p, then $M \simeq_W M_2 \times \mathbb{C}$. This says that the most important information is for those $\dim S_p^+ = n$*

2.2.24 Example *To study a symplectic 4 manifold we need to know about Legendrian knots in contact 3-manifolds. Legendrian knots are much more than their underlying knot.*

2.2.2 Contact

2.2.25 Example *Let N be a hypersurface in a Liouville symplectic manifold such that transverse to the Liouville vector field. In a Riemannian case can do the unit cotangent bundle $U^*M \subset T^*M$. Can also do the ST^*M for the sphere bundle defined via a quotient rather than a sub.*

The restriction of the Liouville λ gives the α.

2.2.26 Definition (Contactification) *For an exact symplectic manifold T^*M we can contactify to get $T^*M \times \mathbb{R}$ with $\alpha = \lambda - dt$*

2.2.27 Definition (Symplectization) *$M \times \mathbb{R}$ with $\omega = d(e^t\alpha)$ Only the $\xi = \ker \alpha$ matters up to equivalence.*

2.2.28 Lemma *$T^*M \times \mathbb{R}_t \times \mathbb{R}_s$ with form $d(e^s(\lambda - dt)) = e^s\omega + e^s ds \wedge \lambda - e^s ds \wedge dt$*

2.2.29 Definition (Reeb) *A Reeb vector field is given by $\alpha(R) = 1$ and $d\alpha(R, -) = 0$. Taking the time 1 orbits for the flow by the Reeb vector field gives a Reeb orbit.*

2.2.30 Definition (Legendrian)

2.2.31 Example (Legendrian knots)

2.3 Kahler Review

2.3.1 Definition (Hermitian Metric) *Let (X, I) be an almost complex manifold, A Hermitian metric on it is a Riemannian metric obeying $g(IX, IY) = g(X, Y)$. In the complexification this says g is of type $(1,1)$.*

2.3.2 Lemma *$h = g - i\omega$ defines a Hermitian metric on the complex vector bundle TX which is linear in first slot and antilinear in the second.*

2.3.3 Definition (Kahler) *(X, g, I) is Kahler if $\nabla_g I = 0$. ω is called the Kahler form.*

2.3.4 Definition (Plurisubharmonic)

2.3.5 Definition (Stein) *$M \subset \mathbb{C}^N$, $f = \mid \vec{z} - \vec{z_0} \mid^2$ from a generic point provides a plurisubharmonic function that can serve as a Kähler potential.*

2.3.6 Theorem (C-Eliashberg) *Weinstein up to homotopy and Stein up to homotopy.*

In order to motivate the next sections consider the decomposition theorem

2.3.7 Theorem (Beauville) *Let M be a compact Kahler manifold with $c_1(TM) = 0$. There exists a finite ètale cover M' that is a product of three types.*

The first kind are complex tori.
The second kind are Calabi-Yau. In particular they need to be simply-connected projective manifolds of dimension greater than 2 such that $H^0(X, \Omega^\bullet_X) \simeq \mathbb{C} \bigoplus \mathbb{C}\omega_{top}$ where ω_{top} is a generator for sections of the canonical bundle K_X.
The third kind are compact simply connected and satisfying $H^0(X, \Omega^\bullet_X) \simeq \mathbb{C}[\sigma]$ where σ is a section of Ω^2_X which is every non-degenerate. These are irreducible holomorphic symplectic.

Proof For (M, g, J) a complex manifold, the following hold. The condition of being Kahler is equivalent to $H_p \subseteq U(T_pM)$.
Ricci-flatness is equivalent to the statement that $H_p \subseteq SU(T_pM)$.
Existence of a holomorphic symplectic parallel with respect to the Levi-Cevita connection is equivalent to $H_p \subseteq Sp(T_pM)$.

Assume compact and simply connected. We can assume that the holonomy representation is irreducible because if not then the deRham theorem says that M splits as a product.

By a special case of Berger's thesis says that the holonomy group can only be U, SU or Sp for Kahler manifolds that are not symmetric spaces and have irreducible holonomy representation.

Assume M compact Kahler simply-connected with $K_M \simeq \mathcal{O}_M$. Yau's theorem implies that it carries a Kahler metric that is Ricci flat. Apply the deRham theorem to split into factors and apply Berger's theorem to say that the holonomy groups are $SU(n)$ and $Sp(r)$

Bochner's principle says that on a compact Kahler Ricci-flat manifold a holomorphic tensor field is parallel.

So if $H = SU(n)$ then the only invariant holomorphic tensor field that we can make is the one coming from the determinant. The only nontrivial tensor preserved by $SU(n)$. This leaves $\mathbb{C} \bigoplus \mathbb{C}\omega_{top}$

If $H = Sp(r)$, the only invariants are the powers of the symplectic form. This gives $\mathbb{C}[\sigma]$ as the only holomorphic tensor fields.

Cheeger-Gromoll says that when M is compact Kahler and Ricci-flat, the universal cover is isomorphic to $\tilde{M} \simeq \mathbb{C}^k \times X$ where X is compact and simply connected. So $M = \tilde{M}/\Gamma$ $Aut(X)$ is finite, so we can pick a finite index subgroup $\Gamma_2 \subset \Gamma$ that acts trivially on the X factor. Now we pick a further finite index subgroup Γ_2 that acts on the $\mathbb{C}^k$ by translations by Bieberbach's theorem. $\tilde{M}/\Gamma_3$ is then the desired cover. The $\mathbb{C}^k/\Gamma_3$ gives the complex torus and on X we satisfy the compact simply-connected assumptions that make the above holonomy considerations valid to decompose into Calabi-Yau and holomorphic symplectic factors.

2.4 Holomorphic Symplectic Review

2.4.1 Definition (Holomorphic Symplectic) *A complex manifold with $\Omega \in \Omega^{2,0}(X)$ which is closed and nondegenerate in the holomorphic sense. Namely $T^{(1.0)} \to (T^{(1,0)})^*$ is an isomorphism. It is not nondegenerate on all of $TX \otimes \mathbb{C}$. In particular it kills all the antiholomorphic vectors.*

2.4.2 Example $Hilb^n(\mathbb{C}^2)$

2.4.3 Lemma *More generally for other S smooth complex surface (4 real dimensions) with nowhere vanishing holomorphic 2 forms. Can make $Hilb^n(S)$*

Proof Mukai 84. Beauville 83 □

2.4.4 Theorem (Yau) *Beauville, Variétés Kählеriennes dont la première classe de Chern est nulle, J. Diff. Geom 1983*

2.5 HyperKahler Review

2.5.1 Definition (HyperKahler) (X, g, I_1, I_2, I_3) *such that* (X, g, I_i) *are all Kahler and* $I_1 I_2 = I_3$. *These complex structures satisfy the quaternion relations. In fact can use* $u_1 I + u_2 J + u_3 K$ *with* u_i *real but all together normalized to be on a* S^2.

A convenient reparameterization of u_i is

$$
\begin{aligned}
u_1 + iu_2 &\;=\; \frac{2\xi}{1+\mid \xi \mid^2} \\
\xi \to -1/\xi &\;\to\; \frac{-2/\xi}{1+\frac{1}{|\xi|^2}} = \frac{-2\xi^\dagger}{1+\mid \xi \mid^2} \\
\xi \to -1/\xi &\;\to\; u_1 + iu_2 \to -u_1 + iu_2 \\
K \to (0,0,1) &\;\to\; \xi = 0 \\
J \to (0,1,0) &\;\to\; \xi = i \\
I \to (1,0,0) &\;\to\; \xi = 1
\end{aligned}
$$

2.5.2 Proposition *HyperKahler implies holomorphic symplectic. In particular use $\Omega = \omega_2 + i\omega_3$ is holomorphic symplectic with respect to I_1. You can also cyclically permute. More generally*

$$
\begin{aligned}
\Omega(\xi) &= (\omega_2 + i\omega_3) + 2i\omega_1\xi + (\omega_2 - i\omega_3)\xi^2 \\
\Omega(0) &= (\omega_2 + i\omega_3) \\
\Omega(1) &= 2\omega_2 + 2i\omega_1
\end{aligned}
$$

2.5.3 Theorem (Calabi Theorem)

$\frac{\Omega(\xi)}{2\xi}$ and $\xi \in \mathbb{C}^*$ gives a symplectic form. Assume circle action X

$$
\begin{aligned}
\mathcal{L}_X(\omega_2 + i\omega_3) &= di_X(\omega_2 + i\omega_3) \\
[\frac{\Omega(\xi)}{2\xi}] &= [\frac{\Omega(\xi_2)}{2\xi_2}] = [i\omega_1]
\end{aligned}
$$

2.5.4 Definition (Twistor Space) *So now form the twistor space $M \times \mathbb{CP}^1$. It's complex structure at (x, ξ) is given by J_ξ and the usual complex structure of complex projective space.*

2.5.5 Theorem (??) .

- *If M, g is HyperKahler of dimension $4r$ then there exist a holomorphic fibration $Z \to \mathbb{CP}^1$ whose fibers $p^{-1}(\xi)$ are M in complex structure ξ.*
- *$\exists$ a holomorphic section of $\Omega^2_{Z/CP^1} \otimes \mathcal{O}(2)$ such that restrictions to fibers are ω_ξ.*
- *$\xi \to -1/\bar{\xi}$ lifts to an antiholomorphic involution of the total space Z.*
- *For all points x in M, there is a holomorphic section of the fibration $\mathbb{CP}^1 \to Z$ with normal bundle $\mathcal{O}(1)^{2r}$.*

2.5.6 Example ($M = \mathbb{R}^4$) *Use the usual structure as $\mathbb{C}^2$. Z becomes the total space of $\mathcal{O}(-1)^{\bigoplus 2} \to \mathbb{CP}^1$. A particularly useful correspondence in this case is that $H^1(Z, \mathcal{O}(-2)) \simeq \{\phi \in C^\infty(\mathbb{R}^4) \;\; \Delta\phi = 0\}$. This can be promoted from an isomorphism on sets of solutions of PDEs to a correspondence of Lagrangians as well.*

$$
\begin{aligned}
A &\in \Omega^{0,1}(Z, \mathcal{O}(-2)) \simeq \Omega^{0,1}(Z, K^{1/2}) \\
S(A) &= \int_Z A\overline{\partial}A \\
\phi &\in C^\infty(\mathbb{R}^4) \\
S(\phi) &= \int_{\mathbb{R}^4} \phi\Delta\phi
\end{aligned}
$$

This sort of correspondence also gives equivalences as classical field theories

- *Holomorphic BF theory and Self-Dual Yang-Mills*
- *Holomorphic Chern-Simons for $\mathfrak{g}$ (for specific pole conditions) and σ model with target G.*

2.5.7 Lemma (Behavior under quotients) *If M is a HyperKahler manifold with Hamiltonian action by G. If the three moment maps are $\mu_1 \cdots \mu_3$ and we do the HyperKahler quotient $\mu^{-1}(0)/G$ then the corresponding result on the twistor space is $Z \to \nu^{-1}(0)/G^{\mathbb{C}}$ where $G^{\mathbb{C}}$ is the complexified group which now acts on Z.*

2.5.8 Example *If Z is $V \otimes \mathcal{O}(1) \bigoplus V^* \otimes \mathcal{O}(1)$ with V a 2 dimensional vector space and $\mathbb{C}^*$ acting by λ and λ^{-1} on the two summands fiberwise.*
The reduction gives $T^*P(V)$ when we use ν that comes directly from pairing V and V^*. The metric that you get when you see this T^*S^2 differential geometrically is known as the Eguchi-Hanson metric.

2.5.1 NonSUSY BPS/Instanton

$$
\begin{aligned}
F &= \pm \star F \\
S_{YM}(A) &\geq
\end{aligned}
$$

Saturate this bound for that given k.

2.5.9 Theorem (ADHM) *A quiver with two vertices, edges connecting them back and forth and 2 self loops on one of the vertices.*

$$
\begin{aligned}
I &\in \mathbb{C}^N \to \mathbb{C}^k \\
J &\in \mathbb{C}^k \to \mathbb{C}^N \\
B_1 &\in \mathbb{C}^k \to \mathbb{C}^k \\
B_2 &\in \mathbb{C}^k \to \mathbb{C}^k \\
\mu_r &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J \\
\mu_c &= [B_1, B_2] + IJ
\end{aligned}
$$

A HKLR reduction with these moment maps.

2.6 Calabi-Yau Review

2.6.1 Definition (Calabi-Yau) *M is CY of complex dimension n if it is a compact Kahler manifold satisfying one of the equivalent conditions:*

- *The Canonical bundle is trivial.*
- *M has a holomorphic n-form that vanishes nowhere.*
- *The structure group can be reduced from $U(n)$ to $SU(n)$.*
- *M has a Kahler metric with global holonomy $\subset SU(n)$*

In particular $c_1(M) = 0$.

2.6.2 Example (Quintic in $\mathbb{CP}^4$)

$$
z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 + 5\psi z_1 z_2 z_3 z_4 z_5 = 0
$$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Figure 2.1: Hodge Diamond of non-singular quintic

2.6.3 Example (Conifold AKA the Toblerone) *The resolved conifold is the total space of $\mathcal{O}(-1) \bigoplus \mathcal{O}(-1)$ over $\mathbb{CP}^1$.*
T^*S^3.
At infinity look like cones over $S^2 \times S^3$. That means after cutting out a compact region then mapping remaining part at infinity to portion of cone over $S^2 \times S^3$.

2.7 Toric Review

See also Section 27.2

2.7.1 Toric

2.7.1 Definition (Toric Manifold) *A toric manifold is a compact Kahler manifold of complex dimension n with an effective action of $(S^1)^n$ by holomorphic isometries. The ω is the curvature of a line bundle $\mathcal{L}$ and this torus action extends to an action on $\mathcal{L}$. The action of $(S^1)^n$ on the manifold extends to an action of $(\mathbb{C}^*)^n$ on the manifold. One of the orbits of this becomes an open dense subset such that the manifold is some particular way of making a closure.*

2.7.2 Definition (Integral Delzant Polytope) *A convex polytope in $\mathbb{R}^n$ defined by a finite number of faces. The vertices must all be in $\mathbb{Z}^n$. Each face is of the form $\lambda_r \cdot x \geq c_r$ where $\lambda_r \in \mathbb{Z}^n$ specifies the normal of that face and $x \in \mathbb{R}^n$ is in the polytope if it meets all the conditions. Around every vertex, the λ_r for the adjacent faces form a basis of $\mathbb{Z}^n$ so that we can make an $SL(n, \mathbb{Z})$ transformation to get the standard octant (analog of octant if $n \neq 3$). Call these M_α for each vertex α.*

2.7.3 Theorem (Guillemin-Sternberg Convexity / Delzant construction) *Let (M^{2n}, ω) be a compact symplectic manifold with effective $(S^1)^n$ action with moment map μ. Then the image of the moment map inside $\mathbb{R}^n$ is a convex polytope. It is the convex hull of the images of the torus fixed points.*

Conversely given a Delzant polytope $\Delta \subseteq \mathbb{R}^n$, the above construction gives a compact symplectic toric manifold whose moment map image is as prescribed. Translating the polytope gives equivariantly symplectomorphic results.

2.7.2 Hypertoric

2.8 Fano Review

2.9 Supermanifold Review

2.9.1 Definition *A $p \mid q$ supermanifold M has body M_{red} and structure sheaf modeled on $\mathcal{O}_U \otimes \wedge \mathbb{R}^q$ for the subset $U \simeq \mathbb{R}^p \subset M_{red}$*

2.9.2 Theorem (Batchelor's Theorem) *Take a vector bundle on an ordinary manifold, then apply parity reversal to the fibers. This gives a supermanifold. This gives all of them.*

2.9.3 Example *Let $E = S^+ \bigoplus S^-$ on a spin 4-manifold, possibly $\mathbb{R}^4$. Then can form the superspace that we will use for writing down supersymmetric Lagrangians. See later section ???.*

2.9.4 Definition (Odd Tangent Bundle) *Consider the presheaf on supermanifolds defined by $SMan(\mathbb{R}^{0|1}, X)$ for an ordinary manifold X. This is $T[1]X$ so that it's functions are $\Omega^\bullet$ collapsed to $\mathbb{Z}_2$ grading. Also call it $\mathcal{S}X$*

2.9.5 Theorem *$SDiff \subset SMan(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ acts on this by precomposition. This super Lie group is translations and dilations. It's Lie algebra gives an odd Q for infinitesimal translation and an even N for infinitesimal dilation. The relation is $[N,Q]=Q$. On the space of functions $\Omega^\bullet$, N picks out the $\mathbb{Z}$ grading and Q gives the deRham differential. Call this super Lie group $\mathcal{D}$.*

2.9.6 Lemma *Let G be a Lie Group acting on X by ρ, then can form $\mathcal{S}G$ acting on $\mathcal{S}X$ by $f \in S \times \mathbb{R}^{0|1} \to G$ and $g \in S \times \mathbb{R}^{0|1} \to X$ gets sent to $\rho(f \times g)$ to give a new S point of $\mathcal{S}X$. The associated Lie group is then $G \ltimes \mathfrak{g}[1]$ where $\mathfrak{g}$ has addition and adjoint action. This has automorphisms by $\mathcal{D}$. Turns $Lie(\mathcal{S}G)$ into a dgla. The generators get sent to L_v and i_v for the prescribed Killing vector fields.*

2.9.7 Theorem *Take $\mathcal{D} \ltimes \mathcal{S}G$. $[Q, i_v] = L_v$ in the Lie algebra there. This gets sent to the Cartan magic formula $[d, i_X] = L_X$*

2.9.8 Example *$G = SO(4)$ acts on $X = \mathbb{R}^4$ so $\mathcal{D} \ltimes (SO(4) \ltimes \mathfrak{so}(4)[1])$ acts on $T[1]\mathbb{R}^4$.*

2.9.9 Example *$G = PSL(2)$ acts on $X = \mathbb{H}^2$ so $\mathcal{D} \ltimes (PSL(2) \ltimes \mathfrak{psl}(2)[1])$ acts on $T[1]\mathbb{H}^2$.*
Use the accident with $SO(2,1)$ where $SO(2,1) \ltimes \mathfrak{so}(2,1)$ ends up being the corresponding Poincare group.

2.9.10 Definition (Tangent 2-Group) *For G a Lie group TG is defined from the crossed module with $H = \mathfrak{g}$ in the adjoint representation and for h a 2-arrow from $1 \in G$ to $t(h) \in G$ $t(h) = 1$.*

2.9.11 Definition ((Inner) Automorphism 2-Group) *For G a Lie group $Inn(G)$ is defined from the crossed module with $H = G$ with conjugation action and $t(h) = h$.*
Can also make $Aut(H)$ with $G = Aut(H)$ with H any Lie Group. So just keeping the $H \subset G$ inner automorphisms would be the previous.

2.9.1 Berezin Integration

2.9.12 Lemma *There is a unique right supermodule Ber such that changes of local coordinates in even variables transforms with determinant of the Jacobian and change of the odd variables changes by inverse determinant.*

2.9.13 Definition (Berezin) *Twist the sheaf Ber by relative orientations and then when expand a function on the supermanifold read off the coefficient.*
This is sufficient for compactly supported, but not if break that assumption.

2.9.14 Definition (Atiyah-Bott Localization)

2.10 Twisting - Insert Elsewhere

If you take any odd element Q, it is of degree (0,1). Compare that to the BRST operator Q_{BRST} it is of degree (1,0). If we had a parameter of degree (1,1) we could form the combination $Q_{BRST} + tQ$ and act like that was our new BRST operator. This would give a deformation that lived over $\mathbb{C}[[\cdots]][[t]]$ where the $\cdots$ are deformations we've already done like maybe foolishly we quantized and understood super Yang-Mills before trying to make the theory simpler and understand Donaldson theory. I can not emphasize again how dumb of an idea that would be.

2.10.1 Definition (Twisting data) *Twisting data for a supersymmetric field theory consists of an odd square 0 element Q,* $[Q,Q]=0$ *and a group homomorphism* $\rho\ \mathbb{C}^* \to G_R$ *such that* $\rho(\lambda)(Q) = \lambda Q$

This means we can promote the $\mathbb{Z}\times\mathbb{Z}_2$ grading of ghost/super to a $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_2$ of $\mathbb{C}^*$ weight/ghost/super.

This will allow us to get a t that sits in degree $(0,0)$ by rearranging gradings. Call the new grading system $(a,b,c) \to (a,b+a,c+a)$. Q had weight 1 so t had weight -1. This means that t now has ghost/super degrees $(0,0)$ as desired and the weight is still -1. Because the $\mathbb{C}^*$ action can move t around, the theory only cares about $t=0$ or $t \neq 0$.

So tQ has degree $(-1,0,0)+(1,1,2)=(0,1,0)$ same as Q_{BRST}.

Chapter 3

Supersymmetry and Morse Theory

3.1 Supersymmetric Quantum Mechanics

Suppose you can factor your hamiltonian $H_1 = Q^\dagger Q$.

$$\begin{aligned}
Q &= \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x) \\
Q^\dagger &= -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x) \\
H_1 &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{\hbar}{\sqrt{2m}}\frac{dW}{dx} + W^2(x) \\
H_2 = QQ^\dagger &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\hbar}{\sqrt{2m}}\frac{dW}{dx} + W^2(x)
\end{aligned}$$

$$\begin{aligned}
H_1\psi_n^{(1)} &= Q^\dagger Q\psi_n^{(1)} = E_n\psi_n^{(1)} \\
H_2 Q\psi_n^{(1)} &= QQ^\dagger Q\psi_n^{(1)} = E_n Q\psi_n^{(1)} \\
H_2\psi_n^{(2)} &= E_n^{(2)}\psi_n^{(2)} \\
H_1 Q^\dagger\psi_n^{(2)} &= E_n^{(2)} Q^\dagger\psi_n^{(2)}
\end{aligned}$$

so they have the same spectra, with the possible exception of zero modes $Q\psi_0^1 = 0$ or $Q^\dagger\psi = 0$.

$$\begin{aligned}
E_0^{(1)} &= 0 \\
E_n^{(2)} &= E_{n+1}^{(1)} \\
\psi_n^{(2)} &= (E_{n+1}^{(1)})^{-1/2} Q\psi_{n+1}^{(1)} \\
\psi_{n+1}^{(1)} &= (E_n^{(2)})^{-1/2} Q^\dagger\psi_n^{(2)}
\end{aligned}$$

In fact you can put these together

$$\begin{aligned}
H &= \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \\
Q_2 &= \begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix} \\
Q_2^\dagger &= \begin{pmatrix} 0 & Q^\dagger \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

These form the Lie Superalgebra $\mathfrak{sl}(1 \mid 1)$, with Q and $Q^\dagger$ being odd and $H = \left[Q, Q^\dagger\right]_+$ even.

3.1.1 Infinite Square Well

The ground state energy is $E_0 = \frac{\hbar^2\pi^2}{2mL^2}$ so we need to subtract that off first.

$$\begin{aligned}
H_1 &= H - E_0 \\
E_n^{(1)} &= \frac{n(n+2)\hbar^2\pi^2}{2mL^2} \\
\psi_n^{(1)} &= \sqrt{\frac{2}{L}} \sin \frac{(n+1)\pi x}{L}
\end{aligned}$$

This can be written as SQM with superpotential by solving the Ricati equation.

$$\begin{aligned}
W &= -\frac{\hbar}{\sqrt{2m}} \frac{\pi}{L} \cot \frac{\pi x}{L} \\
V_2 &= \frac{\hbar^2\pi^2}{2mL^2} (2 \csc^2 \frac{\pi x}{L} - 1)
\end{aligned}$$

Applying Q gives the eigenvalues for this seemingly difficult potential.

$$\psi_1^{(2)} = \sin \frac{\pi x}{L} \sin \frac{2\pi x}{L}$$

3.1.2 Scattering

Define

$$W_\pm \equiv \lim_{x \to \pm\infty} W = W_\pm$$

Now we have

$$
\begin{aligned}
\lim_{x\to\pm\infty} V_{1,2} &= W_\pm^2 \\
\psi^{(1,2)}(k, x\to-\infty) &\approx e^{ikx} + R^{(1,2)}(k)e^{-ikx} \\
\psi^{(1,2)}(k_2, x\to+\infty) &\approx T^{(1,2)}(k)e^{ik_2x} \\
k &= (E - W_-^2)^{1/2} \\
k_2 &= (E - W_+^2)^{1/2} \\
R^{(1)}(k) &= \frac{W_- + ik}{W_- - ik} R^{(2)}(k) \\
T^{(1)}(k) &= \frac{W_+ + ik_2}{W_- - ik} T^{(2)}(k)
\end{aligned}
$$

The absolute values for the partners are the same so they have the same reflection and transmission probabilities.

3.1.3 Harmonic Oscillator

3.1.4 Hydrogen Atom

3.1.5 Witten Index

3.1.1 Definition (Witten Index)

$$\Delta(\beta) = \mathrm{tr}(-1)^F e^{-\beta H}$$

3.1.2 Lemma

$$\Delta = \lim_{\beta\to 0} \Delta(\beta)$$

Proof In a supersymmetric theory, every nonzero eigenvalue has an equal number of bosonic and fermionic eigenstates so they contribute $\pm e^{-\beta\lambda}$ in cancelling pairs. The only terms that do not cancel out are from zero eigenvalues which contribute ± 1. So $\Delta(\beta)$ is a signed count of the number of zero eigenvalue eigenstates. It does not depend on β so it can be evaluated in the limit of $\beta \to 0$. □

3.1.6 Hierarchy

We can also repeat this process repeatedly

$$
\begin{aligned}
H_1 &= Q^\dagger Q \\
H_2 &= QQ^\dagger = Q_2^\dagger Q_2 + E_1^{(1)} \\
H_3 &= Q_2 Q_2^\dagger + E_1^{(1)} = Q_3^\dagger Q_3 + E_2^{(1)} \\
H_m &= Q_m^\dagger Q_m + E_{m-1}^{(1)}
\end{aligned}
$$

where Q_n are being inductively defined this way. This can be repeated up to p the number of bound states of H_1 where H_s has the first $s-1$ levels removed.

Shape Invariance

3.1.3 Definition *Let $V_{1,2}$ be partner potentials with parameters a_i. If*

$$
\begin{aligned}
V_2(x, a_1) &= V_1(x, a_2) + R(a_1) \\
a_2 &= f(a_1)
\end{aligned}
$$

Repeating with the hierarchy gives

$$
\begin{aligned}
V_s(x, a_1) &= V_1(x, a_s) + \sum_{k=1}^{s-1} R(a_k) \\
f^{(s-1)}(a_1) = f(f \cdots (a_1)) &= a_s \\
E_0^{(s)} &= \sum_{k=1}^{s-1} R(a_k) \\
E_n^{(1)} &= \sum_{k=1}^{n} R(a_k) \\
E_0^{(1)} &= 0 \\
\psi_n^{(1)}(x, a_1) &= Q^\dagger(x, a_1) Q^\dagger(x, a_2) \cdots Q^\dagger(x, a_n) \psi_0^{(1)}(x, a_{n+1})
\end{aligned}
$$

3.1.4 Example

3.2 Morse Theory

This is in 0+1 dimensions for the spacetime and there is a target manifold which is a Riemannian n manifold with metric g_{IJ}.

$$\begin{aligned}\mathcal{L} &= g_{IJ}\dot{q}^I\dot{q}^J - g^{IJ}\partial_I h \partial_J h \\ &- g_{IJ}\tilde{\chi}^I D_t \chi^J - g^{IJ} D_I D_J h \tilde{\chi}^I \chi^J - R_{IJKL}\chi^I\chi^J\chi^K\chi^L\end{aligned}$$

The perturbative vacuaa (in $1/\lambda$ as we crank up the potential via $h \to \lambda h$) give $h^{'}(p) = 0$. That is the critical points of the Morse function. The bosonic part is a Gaussian sharply peaked around the critical point with covariance determined by the Hessian of h. The fermionic portion of the state keeps track of the Morse index to say which linear combination of fermion creation operators were applied as determined by the negative eigenvectors of the Hessian. In particular the fermionic portion keeps track of how many negative eigenvalues there are at the given critical point.

The Betti numbers are the dimension of the Q-cohomology, and this is independent of h because Q is just the twisted deRham differential by the exact dh. The grading is by how many $\overline{\chi}$ creation operators were applied which can be from 0 through the dimension of the target manifold.

Realizing the supersymmetric ground states will be in the span of these perturbative ground states and that the grading is respected gives the Morse inequality $b_p \leq M_p$ where b_p is the Betti number and M_p is the number of Morse critical points of index p.

3.2.1 Definition (Heat Kernel)

$$\begin{aligned} s &\in \Gamma(M, E) \\ K_t &\in \Gamma(M \times M, E \boxtimes E^{\vee}) \\ \int_M K_t(x,y)s(y)dvol_M &= (e^{-t\Delta_{Q,Q^\dagger}} s)(x)\end{aligned}$$

3.2.1 Twisted deRham (one form twists)

Let α be a closed 1-form on a manifold Y, then we can form $\Omega^\bullet(Y, d + \alpha\wedge)$. This new differential d_α is still closed because α is. This defines a twisted cohomology $H^\bullet(Y, d_\alpha)$.

Suppose β is another closed 1-form in the same cohomology class, then there exists a solution to $df = \alpha - \beta$. Then $\gamma \to e^f\gamma$ defines a cochain map between the two. The space of the choices for f is a torsor for Ω^0_{cl}, and if we only care about what the chain map is we only need to specify in the quotient abelian group $\Omega^0_{cl}/(2\pi i\mathbb{Z})$ (assuming our forms are complex valued).

3.2.2 Remark (Odd form twists) If we do not demand that the grading by form degree be kept and only the underlying $\mathbb{Z}_2$ grading, we can twist by closed odd degree forms. Most notable are 3-form twists, but that is not relevant right now. $\Diamond$

3.2.2 S^1 valued functions

The potential might not be well defined as a real/complex number. Instead there may be some $\mathbb{Z}$ ambiguity which turns the well defined values of the function to be in $\mathbb{R}/r\mathbb{Z}$ or $\mathbb{C}/c\mathbb{Z}$ where r and c are some real/complex numbers to describe the change that can happen from two different choices of auxiliary data that we don't have. Consider Chern-Simons/complexified Chern-Simons. But we still get dh even though it is not interpreted as a potential function on the target anymore.

3.2.3 Definition (Morse-Novikov Complex) *Let Y be a closed Riemannian manifold and f an $\mathbb{R}/\mathbb{Z}$ valued function on it. Lift this to a real valued F on $\tilde{Y}$. The critical points of F form a set with an action of $\mathbb{Z}$ and linearizing we get a $k[\mathbb{Z}] \simeq k[t, t^{-1}]$ module structure. Just as with ordinary Morse there is a differential d_{MN} counting gradient flow lines, but now on $\mathcal{N}_\bullet \equiv \mathcal{M}_\bullet \otimes_{\mathbb{C}[t,t^{-1}]} \mathbb{C}((t))$.*

3.2.4 Theorem (Novikov) *$H^\bullet(\mathcal{N}_\bullet, d_{MN})$ is isomorphic to $H^\bullet(Y, \mathbb{C}_{\tau df}) \otimes \mathbb{C}((t))$ where τ is a generic element of $\mathbb{C}/2\pi i\mathbb{Z}$. For the lack of dependence on τ generically we need another Novikov theorem which states that this happens for Y a finite CW-complex and $\alpha \in H^1(Y, \mathbb{Z})$. In fact, it is only finitely many bad values of τ which is even better than just operating within an open dense subset.*

3.3 Morse Theory Revisited

This is now in 1+1 dimensions

Can do this in a family parameterized by $z \in C$. Do a path in C. Can make a 1+1 QFT by

$$h = \int_{\mathbb{R}} \phi^*(\lambda) + \Re(\zeta^{-1} W(\phi, z(x)))dx$$

In general the interfaces $Br(T_1, T_2)$ are morphisms of an A_∞ category. There are 2-morphisms parameterized by boundary changing operators.

Chapter 4

MSSM

4.1 Superfields

4.1.1 Definition (Superspace $\mathbb{R}^{3,1|4}$) *The supermanifold with bosonic coordinates x^μ and fermionic coordinates θ^α and $\theta^\dagger_{\dot\alpha}$. To define the version with lowered indices use the usual pseudo-Riemannian metric for the bosonic variables and ϵ^{ab} for the fermionic coordinates. Here $\epsilon^{12} = -\epsilon^{21} = 1$. So in particular we have θ_α where $\theta_1 = \theta^2$ and vice versa.*

4.1.2 Definition (Superfield) *A function on such a superspace. In particular it is of the form*

$$\begin{aligned}
S(x,\theta,\theta^\dagger) &= a(x) + \theta^\alpha \xi_\alpha(x) + \theta^\dagger_{\dot\alpha}\chi^{\dagger,\dot\alpha}(x) + \theta^\alpha\theta_\alpha b(x) + \theta^\dagger_{\dot\alpha}\theta^{\dagger,\dot\alpha}c(x) \\
&+ \theta^{\dagger,\dot\alpha}\overline{\sigma}^\mu_{\alpha,\dot\alpha}\theta^\alpha v_\mu(x) + \theta^\dagger_{\dot\alpha}\theta^{\dagger,\dot\alpha}\theta^\alpha\eta_\alpha(x) + \theta^\alpha\theta_\alpha\theta^\dagger_{\dot\alpha}\zeta^{\dagger,\dot\alpha}(x) \\
&+ \theta^\alpha\theta_\alpha\theta^\dagger_{\dot\alpha}\theta^{\dagger,\dot\alpha}d(x)
\end{aligned}$$

4.1.3 Definition (Super differential operators)

$$\begin{aligned}
Q_\alpha &\equiv i\frac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha,\dot\alpha}\theta^{\dagger,\dot\alpha}\partial_\mu \\
Q^\alpha &\equiv -i\frac{\partial}{\partial\theta_\alpha} + \theta^{\dagger,\dot\alpha}\overline{\sigma}^{\mu,\alpha,\dot\alpha}\partial_\mu \\
Q^{\dagger,\dot\alpha} &\equiv i\frac{\partial}{\partial\theta^\dagger_{\dot\alpha}} - \overline{\sigma}^{\mu,\alpha,\dot\alpha}\theta_\alpha\partial_\mu \\
Q^\dagger_{\dot\alpha} &\equiv -i\frac{\partial}{\partial\theta^{\dagger,\dot\alpha}} + \theta^\alpha\sigma^\mu_{\alpha,\dot\alpha}\partial_\mu
\end{aligned}$$

These are differentials but the product rule is altered by a $(-1)^{|S|}$ so that $Q_\alpha(ST) = Q_\alpha(S)T + (-1)^{|S|}SQ_\alpha T$ for commuting the Q_α passed the S to apply on the T directly.

For infinitesimal ϵ^α and $\epsilon^\dagger_{\dot\alpha}$, then define the supersymmetry transformation as

$$
\begin{aligned}
\delta_{\epsilon,\epsilon^\dagger} &\equiv \frac{-i}{\sqrt{2}}(\epsilon^\alpha Q_\alpha + \epsilon^\dagger_{\dot\alpha} Q^{\dagger,\dot\alpha} \\
\sqrt{2}\delta_{\epsilon,\epsilon^\dagger} S(x,\theta,\theta^\dagger) &= S(x^\mu + i\epsilon^\alpha \sigma^\mu_{\alpha,\dot\alpha}\theta^{\dagger,\dot\alpha} + i\epsilon^{\dagger,\dot\alpha}\overline{\sigma}^\mu_{\alpha,\dot\alpha}\theta^\alpha, \theta+\epsilon, \theta^\dagger + \epsilon^\dagger)
\end{aligned}
$$

So this can be interpreted as an infinitesimal translation in a complexified sense. The bosonic coordinates are interpreted as $\mathbb{C}^4$ instead of $\mathbb{R}^4$.

$$
\begin{aligned}
\theta^\alpha &\to \theta^\alpha + \epsilon^\alpha \\
\theta^\dagger_{\dot\alpha} &\to \theta^\dagger_{\dot\alpha} + \epsilon^\dagger_{\dot\alpha} \\
x^\mu &\to x^\mu + i\epsilon^\alpha \sigma^\mu_{\alpha,\dot\alpha}\theta^{\dagger,\dot\alpha} + i\epsilon^{\dagger,\dot\alpha}\overline{\sigma}^\mu_{\alpha,\dot\alpha}\theta^\alpha
\end{aligned}
$$

These extend the Lie algebra of translations generated by the P_μ by

$$
\begin{aligned}
\{Q_\alpha, Q^\dagger_{\dot\beta}\} &= -2\sigma^\mu_{\alpha,\dot\beta} P_\mu \\
\{Q_\alpha, Q_\beta\} &= 0 \\
\{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\} &= 0
\end{aligned}
$$

The $\dagger$ is to denote these are adjoints when considering the space of complex superfields with the inner product given by $\langle T \mid\mid S\rangle \equiv \int d^4x \int d^2\theta \int d^2\theta^\dagger T^* S$

4.1.4 Definition (Chiral Covariant Derivatives) *$\delta_{\epsilon,\epsilon^\dagger}\frac{\partial S}{\partial\theta^\alpha}$ does not equal the quantity with the order of δ_ϵ and $\frac{\partial}{\partial\theta^\alpha}$ reversed. Define the quantities*

$$
\begin{aligned}
D_\alpha &\equiv \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha,\dot\alpha}\theta^{\dagger,\dot\alpha}\partial_\mu \\
D^\alpha &\equiv -\frac{\partial}{\partial\theta_\alpha} + i\theta^\dagger_{\dot\alpha}\overline{\sigma}^{\mu,\alpha,\dot\alpha}\partial_\mu \\
\overline{D}_{\dot\alpha} &\equiv -\frac{\partial}{\partial\theta^{\dagger,\dot\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha,\dot\alpha}\partial_\mu \\
\overline{D}^\alpha &\equiv \frac{\partial}{\partial\theta^\dagger_{\dot\alpha}} - i\overline{\sigma}^{\mu,\alpha,\dot\alpha}\theta_\alpha\partial_\mu \\
\overline{D}_{\dot\alpha}S^* &= (D_\alpha S)^*
\end{aligned}
$$

$$
\begin{aligned}
\{Q_\alpha, D_\beta\} &= 0 \\
\{Q^\dagger_{\dot\alpha}, D_\beta\} &= 0 \\
\{Q_\alpha, \overline{D}_{\dot\beta}\} &= 0 \\
\{Q^\dagger_{\dot\alpha}, \overline{D}_{\dot\beta}\} &= 0 \\
\delta_{\epsilon,\epsilon_\dagger} D_\beta S &= \frac{-i}{\sqrt{2}}\left(\epsilon^\alpha Q_\alpha + \epsilon^\dagger_{\dot\alpha} Q^{\dagger,\dot\alpha}\right) D_\beta S \\
&= \frac{-i}{\sqrt{2}}\left(\epsilon^\alpha Q_\alpha D_\beta + \epsilon^\dagger_{\dot\alpha} Q^{\dagger,\dot\alpha} D_\beta\right) S \\
&= \frac{i}{\sqrt{2}}\left(\epsilon^\alpha D_\beta Q_\alpha + \epsilon^\dagger_{\dot\alpha} D_\beta Q^{\dagger,\dot\alpha}\right) S \\
&= \frac{-i}{\sqrt{2}}\left(D_\beta \epsilon^\alpha Q_\alpha + D_\beta \epsilon^\dagger_{\dot\alpha} Q^{\dagger,\dot\alpha}\right) S \\
&= D_\beta \frac{-i}{\sqrt{2}}\left(\epsilon^\alpha Q_\alpha + \epsilon^\dagger_{\dot\alpha} Q^{\dagger,\dot\alpha}\right) S \\
&= D_\beta \delta_{\epsilon,\epsilon_\dagger} S
\end{aligned}
$$

4.1.5 Definition (Chiral Superfield) *A chiral superfield is a superfield such that $\overline{D}_{\dot\alpha} S = 0$. An anti-chiral superfield is one such that $D_\alpha S = 0$. Alternatively a chiral superfield is one where the only $\theta^\dagger$ dependence is through $y^\mu \equiv x^\mu + i\theta^{\dagger,\dot\alpha}\overline{\sigma}^\mu_{\alpha,\dot\alpha}\theta^\alpha$. So we can write it as a function of only the y^μ and θ^α. Similarly anti-chiral superfields can be written with $y^{\mu,\star} \equiv x^\mu - i\theta^{\dagger,\dot\alpha}\overline{\sigma}^\mu_{\alpha,\dot\alpha}\theta^\alpha$.*

If we take a general superfield and apply $\overline{D}_{\dot\alpha}\overline{D}^{\dot\alpha}$, we get a chiral superfield. This is because of the property that applying 3 $\overline{D}$ gives 0 no matter how they are indexed with $\dot\alpha$, $\dot\beta$ and $\dot\gamma$. Similarly for anti-chiral superfields with D replacing $\overline{D}$.

In addition for any holomorphic function of a n complex variables W, plug in n chiral superfields as the arguments. In order to make sense of this expand $W = \sum a_i z_1^{b_{i,1}} \cdots z_n^{b_{i,n}}$ with its power series representation around the origin of $\mathbb{C}^n$.

4.1.6 Definition (Vector Superfield) *Impose the reality condition on a general superfield as*

$$
\begin{aligned}
a(x) &= a^*(x) \\
\chi^\dagger(x) &= \xi^\dagger(x) \\
c(x) &= b^*(x) \\
v_\mu(x) &= v^*_\mu(x) \\
\zeta^\dagger(x) &= \eta^\dagger(x) \\
d(x) &= d^*(x)
\end{aligned}
$$

This is saying $S = S^*$ where θ^* where this conjugation operation interchanges θ and $\theta^\dagger$.

4.1.7 Definition (D-Term) *Let V be a vector superfield. Then $\left[V\right]_D$ is defined to be $\int d^2\theta \int d^2\theta^\dagger V$. This picks out the real valued $d(x)$. It is often written as $\frac{1}{2}D + \frac{1}{4}\partial_\mu\partial^\mu a$ where this gives the definition of the real valued $D(x)$ in terms of the previously defined d and a. This is real valued because V satisfied the reality condition.*

If we take the D term of a chiral superfield instead, we get a $d(x) = \frac{1}{4}\partial_\mu\partial^\mu a(x)$ so this can be dropped because of the lack of boundary and the decay condition at infinity.

4.1.8 Definition (F-Term) *For a chiral superfield Φ, define $\left[\Phi\right]_F$ as $\int d^2\theta\Phi\mid_{\theta^\dagger=0}$. This picks out the $b(x)$ from Φ when written as a general superfield. The name F comes from writing with different letters specifically for chiral superfields specifically only ϕ, ψ and F with all the usual superfields being expressed in terms of these 3. This is complex valued.*

4.2 $\mathcal{N}=1$ SQED

A vector superfield $V(x,\theta,\theta^\dagger)$

$$\begin{aligned}
V &= a + \theta^\alpha\xi_\alpha + \theta^{\alpha\dagger}\xi^\dagger_\alpha + \theta^\alpha\theta^\beta b_{\alpha\beta} \\
&+ \theta^{\dagger\alpha}\theta^{\dagger\beta}b_{\alpha\beta} + \theta^\dagger\bar{\sigma}^\mu\theta A_\mu + \theta^\dagger\theta^\dagger\theta(\lambda - \frac{i}{2}\sigma^\mu\partial_\mu\xi^\dagger) \\
&+ \theta\theta\theta^\dagger(\lambda^\dagger - \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\xi) \\
&+ \theta\theta\theta^\dagger\theta^\dagger(\frac{1}{2}D + \frac{1}{4}\partial_\mu\partial^\mu a)
\end{aligned}$$

We treat this as gauge with gauge transformations given by shifts by $i(\Omega^* - \Omega)$ for chiral superfields Ω. In particular the $A_\mu \to A_\mu + \partial_\mu(\phi + \phi^*)$ so some $\mathfrak{u}(1)$ connection.

A choice of partial gauge fixing is given by Wess-Zumino by

$$V = \theta^\dagger\bar{\sigma}^\mu\theta A_\mu + \theta^\dagger\theta^\dagger\theta\lambda + \theta\theta\theta^\dagger\lambda^\dagger + \frac{1}{2}\theta\theta\theta^\dagger\theta^\dagger D$$

Ordinary gauge transformations are still left unfixed. Also supersymmetry transformations Q will take you out of this choice and you will have to do another gauge transformation to reaffirm this choice.

From this define chiral and anti-chiral superfields by

$$\begin{aligned}
W_\alpha &= \frac{-1}{4}\bar{D}\bar{D}D_\alpha V \\
W^\dagger_{\dot{\alpha}} &= \frac{-1}{4}DD\bar{D}_{\dot{\alpha}}V
\end{aligned}$$

These are gauge invariant so the computation can be done in Wess-Zumino gauge to define and compute

$$\begin{aligned}\int \mathcal{L} &= \int\int d^2\theta \frac{1}{4} W^\alpha W_\alpha \mid_{\theta^\dagger = 0} + c.c. \\ &= \int \frac{1}{4} W^\alpha W_\alpha \mid_{\theta\theta} + c.c. \\ &= \int \frac{1}{2} D^2 + i\lambda^\dagger \bar{\sigma}^\mu \partial_\mu \lambda - \frac{1}{4} F \wedge \star F\end{aligned}$$

Many terms were dropped by integration by parts including a $\frac{i}{4} F \wedge F$ term.

4.2.1 Remark A feature/bug of all of supersymmetry literature is integration by parts always with choice of decay to zero assumptions. $\Diamond$

One can also add $-2\kappa[V]_D$ as well that is the $\theta\theta\theta^\dagger\theta^\dagger$ term.

4.3 $\mathcal{N} = 1$ SQCD

4.4 Minimal SUSY Standard Model

Chapter 5

Linear Sigma Models

5.1 Linear Sigma Model

5.2 Gauged Linear Sigma Models

5.2.1 Example *$U(1)^s$ gauge group with vector superfields $V_1 \cdots V_s$ and n chiral superfields with charges $Q_{i,a}$ for the ith chiral superfield under the ath factor of the gauge group.*

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_{kin} + \mathcal{L}_{gauge} + \mathcal{L}_{D,\theta} \\
U &= \sum_{a=1}^{s} \frac{e_a^2}{2} \big(\sum_{i=1}^{n} Q_{i,a} \mid \phi_i \mid^2 - r_a \big)^2 \\
r_a &\in \mathbb{R}
\end{aligned}
$$

The ground states are given by

$$
\big(\sum_{i=1}^{n} Q_{i,a} \mid \phi_i \mid^2 - r_a \big) = 0
$$

which is (for appropriate parameters) a $n-s$ dimensional normal toric variety whose fan has n edges. That is what is cut out above for $\mid \phi \mid$ with torus fibers above it for the phases.

Chapter 6

Seiberg Witten

6.1 Seiberg Witten Gauge Theory

The acheivement of Seiberg-Witten is control of renormalization in $\mathcal{N} = 2$ SYM.

$$\tau_{eff} = \frac{\theta_{eff}}{\pi} + \frac{8\pi i}{g^2_{eff}} = \frac{8\pi i}{g^2_{bare}} + \frac{2i}{\pi} \log \frac{a^2}{\Lambda^2} - \frac{i}{\pi} \sum_{\ell=0}^{\infty} c_\ell (\frac{\Lambda}{a})^{4\ell}$$

This is the effective renormalized gauge coupling with Λ the scale at which the gauge coupling is strong and a is the scale at which we are considering.

The fields of pure $\mathcal{N} = 2$ is a vector multiplet in the adjoint representation. In $\mathcal{N} = 1$ language this is written as a pair of chiral multiplets.

$$\begin{aligned}
\mathbf{W} &= W^i_\alpha + \lambda^i_\alpha \theta + A^i_\mu \theta\theta \\
\mathbf{\Phi} &= \phi^i + \psi^{\beta i}\theta + F\theta\theta
\end{aligned}$$

Continue to a $\mathcal{N} = 2$ gauge theory and then the Seiberg-Witten theory.

6.2 Marcolli Notation

6.2.1 Remark There is no control of the gauge group in the following. $\Diamond$

$$\begin{aligned}
S &= \int dvol \mid D_A \psi \mid^2 \\
&+ \mid F^+_{\hat{A}} - \frac{1}{4} \langle e_i e_j \psi \mid\mid psi \rangle e^i \wedge e^j \mid^2 \\
&= \int dvol \mid \nabla_A \psi \mid^2 + \mid F^+_{\hat{A}} \mid^2 + \frac{\kappa}{4} \mid \psi \mid^2 + \frac{1}{8} \mid \psi \mid^8
\end{aligned}$$

Distinguishing A for the spin connection and $\hat{A}$ for the $\mathfrak{u}(1)$ connection.

6.2.2 Corollary *If the scalar curvature is nonnegative, then all solutions will have $\psi = 0$. If $b_2^+ > 0$, then for generic metric $\hat{A}$ will also be flat.*

Proof The equation of motion for ψ in the second formulation after applying Weitzenbock. This then makes $F_{\hat{A}}^+ = 0$ but to get all of $F_{\hat{A}} = 0$ we see that

$$\begin{aligned} c_1(L) = \frac{1}{2\pi}[F_{\hat{A}}] &= \frac{1}{2\pi}[F_{\hat{A}}^-] \\ &\in H^{2-}(X, \mathbb{R}) \\ c_1(L) &\in H^2(X, \mathbb{Z})/Torsion \\ \frac{1}{2\pi}[F_{\hat{A}}] &\in H^{2-}(X, \mathbb{R}) \bigcap H^2(X, \mathbb{Z})/Torsion \end{aligned}$$

If $b_2^+ > 0$, this intersection is generically only trivial. Then $[F_{\hat{A}}] = 0$ giving a global 1-form χ so must show $F_{\hat{A}} = 0$ to prove $\hat{A}$ actually flat.

6.3 Moduli Space

$$\begin{aligned} D_A\psi &= 0 \\ F_A^+ &= (\psi \otimes \psi)_+ + i\omega \\ \omega &\in \Omega^{2,+}_{\mathbb{R}} \end{aligned}$$

Again implicitly we have used the projection $\sigma \;\; S_+ \otimes S_+ \to \Omega^{2,+}$ in the beginning section.

Say you have chosen a value $u = \mathrm{tr}(\phi^2)$ of ϕ the adjoint valued scalar field and tr is meant to indicate the suibtably normalized Killing form. The point of this is simply to give a coordinate on the adjoint quotient $\frac{\mathfrak{g}}{\mathfrak{g}}$.

6.3.1 Remark We have to say what space of functions this was in and the topology of that space of functions, this will also allow defining the moduli as a topological space rather than just a set. Someone has to stop a villian from giving it the indiscrete topology and making compactness useless. To be inserted. $\Diamond$

6.3.2 Theorem () *Given a sequence of solutions of the Seiberg-Witten equations in (what topologized function space), there exist a subsequence and gauge transformations for them such that the transformed subsequence converges to a smooth solution of the Seiberg-Witten equations.*

You get a specific Kahler manifold where the Kahler metric is given by

$$
\begin{aligned}
g_{i\bar{j}} & \\
g_{i\bar{j}} &= Im\tau_{ij}(a) \\
\tau_{ij} &= \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}
\end{aligned}
$$

and $\tau_{ij}(u)$ is a period matrix for an an abelian variety $\mathbb{C}^r/(\mathbb{Z}^r \bigoplus \tau\mathbb{Z}^r)$. All together this entire family forms a complex symplectic manifold. Some fibers may degenerate. These abelian varieties are Lagrangian.

Proof

$$
\sum da^i \wedge da_{Di} = 0
$$

Implies locally exists $\mathcal{F}$ such that $\frac{\partial \mathcal{F}}{\partial a^i}$

6.4 Seiberg Witten Curve

Consider the example of $N_f = 2$ and we have rescaled Λ to 1.

So over this parameter space write the elliptic curves

$$
y^2 = (x-1)(x+1)(x-u)
$$

For each u, pick a homology basis of E_u. Make that continuously varying in u. Also make them a dual basis so that their intersection pairing $\gamma_1 \cdot \gamma_2 = 1$. This can be paired with $H^1(E_u, \mathbb{C})$ meromorphic $(1,0)$ forms with vanishing residues modulo exact by

$$
\gamma_i \rightarrow \int_{\gamma_i} \lambda
$$

The vanishing residues ensures γ can even cross a pole no problem. In particular for E_u pick λ as

$$
\begin{aligned}
\lambda_1 &= \frac{dx}{y} \\
\lambda_2 &= \frac{xdx}{y} \\
b_i &= \int_{\gamma_i} \lambda_1 \\
\frac{b_1}{b_2} &= \tau_u
\end{aligned}
$$

Under change of basis of γ_i, this gets the $SL(2,\mathbb{Z})$ action on τ

6.4.1 Definition (a, a_D Lattice)

$$
\begin{aligned}
a_D &= \int_{\gamma_1} a_1(u)\lambda_1 + a_2(u)\lambda_2 \\
a &= \int_{\gamma_2} a_1(u)\lambda_1 + a_2(u)\lambda_2
\end{aligned}
$$

Again changing γ_i amounts to an $SL(2,\mathbb{Z})$ transformation.

Make a choice for $a_{1,2}(u)$ that is better for some reason I don't know.
Now you look at the asymptotics of $a(u)$ and $a_D(u)$ around ∞
This is given by

$$
\begin{aligned}
a_D(u) &\approx \frac{i}{\pi}\sqrt{2u}\log\frac{u}{\Lambda^2} \\
a(u) &= \sqrt{u/2}
\end{aligned}
$$

Therefore as you make a loop around $u = \infty$, you get

$$
\begin{pmatrix} a_D^{new} \\ a^{new} \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_D^{old} \\ a^{old} \end{pmatrix}
$$

Similarly we can do this for any (g, q) dyon

$$
\begin{pmatrix} a_D^{new} \\ a^{new} \end{pmatrix} = \begin{pmatrix} 1+qg & q^2 \\ -g^2 & 1-gq \end{pmatrix} \begin{pmatrix} a_D^{old} \\ a^{old} \end{pmatrix}
$$

6.4.2 Remark Something wrong $Tr = -2$ vs $Tr = 2$ $\diamondsuit$

The two singularities that actually show up are $(1, 0)$ and $(1, -2)$. That is those are the dyons that really do go massless in this setup. These generate the subgroup $\Gamma_0(4)$ of the modular group. Notice in particular that S is not in this subgroup. It is not a duality of this theory unlike $\mathcal{N} = 4$

6.5 Charge Lattice

You have a lattice Γ with an antisymmetric pairing $\Gamma \times \Gamma \to \mathbb{Z}$

6.5.1 Definition (Central Charge) *Z is a homomorphism from $\Gamma \to \mathbb{C}$*

6.5.2 Definition (BPS state,particle and antiparticle) *A state is BPS if $m = \mid Z(\gamma) \mid$. For a convention we can call a BPS particle when $Z(\gamma) \in \mathbb{H}$ and antiparticle if in $-\mathbb{H}$*

6.5.3 Lemma *The γ of all particles form a strict convex cone in $\Gamma \otimes \mathbb{R}$ if $\Gamma = \bigoplus_{i=1}^{r} \mathbb{Z}e_i$*

In the case of this assumption being satisfied, we may construct a quiver with vertices e_i and $\langle e_i, e_j \rangle$ arrows from $e_i \to e_j$. Also get a superpotential W which is a a sum of terms with complex coeffecients multiplied by cycles in this quiver.

6.5.4 Definition (Path algebra) *$PA = \mathbb{C}Q/\partial W$*

6.5.5 Definition (Stable Particle) *A particle is stable if $\forall Y \quad 0 < argZ(Y) < argZ(X) \leq \pi$. This depends on the choice of half plane.*

6.5.6 Remark The dependence on these choices is because all that matters is the $D^b(PA - mod)$ so there are other quivers that can be used. That is the ones related by Seiberg dualities that give the same theory. $\diamondsuit$

`http://scgp.stonybrook.edu/video_portal/video.php?id=1595`

6.5.7 Definition (BPS Jumping)

Chapter 7

Topological String Theory

7.1 $(2, 2)$ SUSY

The local superspace coordinates:

$$\begin{array}{rcl} \mathbb{C}(z, \bar{z}, \theta^{\pm}, \bar{\theta}^{\pm}) & & \\ \theta^{\pm} & \to & e^{\pm i\alpha/2}\theta^{\pm} \\ \bar{\theta}^{\pm} & \to & e^{\pm i\alpha/2}\bar{\theta}^{\pm} \end{array}$$

The action consists of D terms like

$$S_D = \int d^2 z d^4 \theta K(\Phi^i, \bar{\Phi}^i)$$

and

7.2 A Model

$$Q_A$$

The Euler-Lagrange equations are

7.2.1 Definition (Lagrangian Grassmannian) *$LagGr(2n)$ is the space of Lagrangian $\mathbb{R}^n \subset \mathbb{R}^{2n}$.*
A Lagrangian submanifold $L^n \subset M^{2n}$ defines a map to $LagGr(2n)$ by taking the tangent spaces $T_l L \subset T_l M$ for points $l \in L$.
It can be stabilized to $LagGr(\infty)$. It is a delooping of $\mathbb{Z} \times BO$ on the Bott clock.

7.2.2 Remark In particular you can map $L \to LagGr(\infty) \to BPic(S) \to BPic(R)$ where the map to $BPic(S)$ is the delooping of the J-homomorphism $\mathbb{Z} \times BO \to Pic(S)$ and $Pic(S) \to Pic(R)$ is the map you get because S is initial among ring spectra so can use any commutative ring spectrum R.
`https://arxiv.org/pdf/1704.04291.pdf` and `https://arxiv.org/pdf/1707.07663.pdf`
For example (caution the capital letters), $Pic(TMF) = \mathbb{Z}_{576}$ `http://www.math.harvard.edu/~amathew/picard_revision.pdf` $\Diamond$

7.2.1 Poisson Sigma Model

We could also consider an AKSZ sigma model with target $T^*[1]P$ where P is a Poisson manifold.

7.2.3 Theorem (Branes) *Branes of the Poisson sigma model are given by $N^*[1]C$ for C a coisotropic in P. In particular we could look at the case when P happens to be symplectic and the coisotropic C happens to be Lagrangian in there.*

Proof Cattaneo-Felder $\square$

7.3 Gromov-Witten

7.3.1 Version 1

7.3.1 Definition (ψ, λ and ϕ Classes) *ψ classes are those that come from cotangents of the marked points. λ are Chern classes of the vector bundle over the moduli space that encodes the canonical bundle. ϕ classes are those that come from the target space.*

$$\begin{aligned} \mathcal{L}_i \mid_{\Sigma_g} &= T^*_{p_i}\Sigma_g \\ \psi_i &= c_1(\mathcal{L}_i) \\ \mathcal{E} \mid_{\Sigma_g} &= H^0(\Sigma_g, K_g) \\ \lambda_j &= c_j(\mathcal{E}) \\ \phi_i &= ev_i^* \phi_X \\ \phi_X &\in H^\bullet(X) \end{aligned}$$

7.3.2 Definition (Descendents) *The classes $\tau_{k_i}(\phi_i) = \psi_i^{k_i}\phi_i$ are the descendent insertions. The descendent GW invariant is then the corresponding integral*

$$\langle \tau_{k_1}(\phi_1) \cdots \tau_{k_m}(\phi_m) \rangle^{GW}_{g,\beta} \equiv \int_{\left[\bar{M}_{g,m}(X,\beta)\right]^{vir}} \tau_{k_1}(\phi_1) \cdots \tau_{k_m}(\phi_m)$$

We can form a generating function

$$
\begin{aligned}
\tilde{F}_g(t) &= \sum_{\beta \in H_2(X,\mathbb{Z})} e^{t\cdot\beta} \int_{[virt]} 1 = \langle\langle 1 \rangle\rangle \\
t &\in H_2^\vee \\
\tilde{W}_{g,n}(t, \{z_1 \cdots z_n\}) &= \langle\langle \frac{\phi_{1,a_1}}{z_1 - \psi_1} \cdots \frac{\phi_{n,a_n}}{z_n - \psi_n} \rangle\rangle \\
&= z_1^{-1} \cdots z_n^{-1} \langle\langle \frac{\phi_{1,a_1}}{1 - z_1^{-1}\psi_1} \cdots \frac{\phi_{n,a_n}}{1 - z_n^{-1}\psi_n} \rangle\rangle
\end{aligned}
$$

so extracting the coefficient of $z_1^{-1-k_1} \cdots z_n^{-1-k_n}$ *gives*

$$
\langle\langle \psi_1^{k_1} \phi_{1,a_1} \cdots \psi_n^{k_n} \phi_{n,a_n} \rangle\rangle = \sum_\beta e^{t\cdot\beta} \langle \tau_{k_1}(\phi_{1,a_1}) \cdots \tau_{k_m}(\phi_{n,a_n}) \rangle_{g,\beta}^{GW}
$$

7.3.3 Definition (Negative Descendents) *This can be extended to include negative values of* k *by introducing symbols* $\tau_k(\gamma)$ *for all* $k \in \mathbb{Z}$ *with* $\gamma \in H^\bullet(X, \mathbb{C})$*. The relations are*

$$
\left[\tau_k(\alpha), \tau_m(\beta)\right] = (-1)^k \frac{\delta_{k+m+1}}{u^2} \int_X \alpha \cdot \beta
$$

This forms an algebra over $\mathbb{C}(u)$ *of Heisenberg form. Diagonalizing the inner product of* $H^\bullet(X, \mathbb{C})$ *decomposes how many 1d bosonic fields there are. If just kept the* $k \in \mathbb{N}$*, we would have a commutative algebra.*

7.3.2 $g = 0$ in particular

Reference : Kontsevich Quantum Spectrum in Algebraic Geometry I talk given on 20200127

7.3.4 Theorem (Properties in $g = 0$ and with only ϕ classes)

- *Symmetry under* S_n *action.*
- $\langle \phi_1 \cdots \phi_n \rangle_{0,\beta}^{GW}$ *not vanishing implies either* $\beta = 0$ *or* $\int_\beta \omega > 0$
- $\langle (ev_1^* 1_X) \cdots \phi_n \rangle_{0,\beta \neq 0}^{GW} = 0$

7.3.5 Theorem (WDVV equations)

7.3.6 Definition (Alternate $g = 0$ generating function) *Let* Δ_i *be basis of the cohomology of* X

$$
F_0 = \sum_{n \geq 0} \sum_{i_1 \cdots i_n} \sum_\beta \langle ev_1^* \Delta_{i_1} \cdots ev_n^* \Delta_{i_n} \rangle_{0,\beta} t_{i_1} \cdots t_{i_n} q^\beta
$$

Choose a map B from the torsion of $H_2(X) \to \mathbb{Q}/\mathbb{Z}$ and for the free part choose $\omega_1 \cdots \omega_{b_2}$ which are all symplectic forms whose classes form a basis of $H^2(X,\mathbb{Q})$ but are in the image of the integral cohomology. Then using this data we can replace the formal symbol q^β with

$$q^\beta \quad \to \quad e^{2\pi i B(\beta_{tors})} \prod_{k=1}^{b_2(X)} q_i^{\int_\beta \omega_i}$$

So $F_0(X,B)$ is a series in $t_1 \cdots t_{\sum b_i(X)}$, $q_1 \cdots q_{b_2}$ with coefficients in a cyclotomic field k (depending on the torsion and B). For the t_i that correspond to H^2, we have t_i acts like $\log q_i$ if you line up the bases of H^2 consistently. So leave only the q variables for H^2 and keep the t variables for the rest of $H^\bullet$.

From this definition the quantum product is

$$\Delta_i \cdot \Delta_j \quad = \quad \frac{\partial^3 F_0(X,B)}{\partial t_i \partial t_j \partial t_k} \Delta^k$$

7.3.7 Definition (Quantum Connection) *Define a bundle over the product of the formal spectrum of $k[[t_1 \cdots t_{\sum b_i}, q_{b_0+b_1} \cdots q_{b_0+b_1+b_2}]]$ and $Speck[u]$ for a new variable u. Let the fiber be $H^\bullet(X,k)$.*

From this we define a connection

$$\begin{aligned}
\nabla_u &\equiv \partial_u + \frac{1}{u^2}K + \frac{1}{u}G \\
\nabla_{t_i} &\equiv \partial_{t_i} + \frac{1}{u}A_i \\
\nabla_{q_i} &\equiv \partial_{q_i} + \frac{1}{u}A_i \\
K &\equiv \\
A_i? &\equiv \Delta_i \cdot ? \\
G &\equiv \bigoplus_{i=0}^{2N} \frac{i-N}{2} id_{H^i}
\end{aligned}$$

This is a flat connection.

7.4 B Model

Twist the $(2,2)$ Σ model by the following procedure:

$$Q_B$$

Chapter 8

Topological String Theory Part 2

8.1 Kodaira Spencer

8.2 Topological Vertex

8.2.1 Vertex

8.2.1 Definition (Topological Vertex)

8.2.2 Example ($\mathbb{C}^3$) *Has* $(0,1)$*,* $(1,0)$ *and* $(-1,-1)$*. telling you the degenerations for the* $T^2 \times \mathbb{R}$ *fibration.*

8.2.3 Lemma *Has cyclic symmetry of the 3 legs.*

8.2.4 Theorem (Orthant Box Counting) *Equal to the generating function for counting 3d partitions in the 3d positive orthant subject to asymptotic boundary conditions of* $Y_{1,2,3}$ *along the three axes.*

Proof Okounkov,Reshetikhin,Vafa □

8.2.2 Quantum Curve

Schwarz

`http://arxiv.org/pdf/1609.00882.pdf`

8.2.5 Definition (Schwarz) *A pair of ordinary differential operators such that* $[P,Q] = c$ *specify a quantum curve. Correspondingly a pair* $KL = qLK$ *specifies a discrete quantum curve.*

8.2.6 Definition *The Kac-Schwarz operators for string equations for the* (a,b) *minimal models for 2D quantum gravity.*

$$\begin{aligned} A &= \frac{1}{bz^{b-1}}\frac{d}{dz} + z^a + \cdots \\ B &= z^b \\ \left[A,B\right] &= 1 \end{aligned}$$

Points of the Sato Grassmannian are given by subspaces of $V = \mathbb{C}((z^{-1}))$ invariant under A, B

8.2.7 Definition *We can also ask for an admissible basis of W given by $\Phi_j = Gx^{-j}$ where $x = z^{-1}$ where G is a invertible q-difference operator $G(x, q^{xd/dx})$*

$$\begin{aligned} A &\equiv Gq^{-D}G^{-1} \\ B &\equiv Gx^{-1}G^{-1} \\ AB &= qBA \end{aligned}$$

In particular $A\Phi_0 = \Phi_0$

If you replace the q^{-D} by $-D$ we get $\left[A,B\right] = B$ instead.

`https://arxiv.org/pdf/1401.1574v2.pdf`

Marino

`https://arxiv.org/pdf/1410.3382.pdf`

$$\begin{aligned} \hat{X} &\equiv Spec(\mathbb{C}[u,v,\alpha,\alpha^{-1},\beta,\beta^{-1}]/(uv - H(\alpha,\beta))) \\ \Sigma_X &\equiv Spec(\mathbb{C}[\alpha,\alpha^{-1},\beta,\beta^{-1}]/(H(\alpha,\beta))) \subset \mathbb{G}_m^2 \end{aligned}$$

Turn H into a homogenous polynomial in 3 variables so defines a genus $g = \frac{1}{2}(d-1)(d-2)$ curve.

8.2.8 Example (Hofstaeder Hamiltonian) *Let α and β represent translation operators in 2d with some flux that causes them to not commute. Still no position dependent potential even though these two operators look like e^x and e^p for a 1d problem.*

For the algebraic geometry problem in the commuting case, multiply through by $\alpha\beta$ and homogenize the polynomial with a γ

$$H = \alpha + \alpha^{-1} + \lambda\beta + \lambda\beta^{-1} \quad \rightarrow \quad \alpha^2\beta + \beta\gamma^2 + \lambda\beta^2\alpha + \lambda\alpha\gamma^2$$

For this $d = 3$ so for each λ get a genus 1 curve. Then there is the involution $\alpha \rightarrow 1/\bar{\alpha}$ that fixes the real circle locus where they live before complexification.

8.2.9 Definition (Segal-Bargmann Space) *Holomorphic functions on $\mathbb{C}^n$ such that $\phi(z)e^{-1/(2)|z|^2}$ is in L^2. There are annihalation and creation $a_i = \partial_{z_i}$ and $a_j^\dagger = z_j \cdot$ which by Stone-Von-Neumann have a unitary Segal-Bargmann transform to the usual CCR on $L^2(\mathbb{R}^n)$.*

$$
\begin{aligned}
(Bf)(z) &= \int_{\mathbb{R}^n} \exp(-(z \cdot z - 2\sqrt{2} z \cdot x + x \cdot x)/2) f(x) dx \\
\rho &= \pi^{-n} \mid (Bf)(z) \mid^2 \exp(- \mid z \mid^2) \\
f(x) &= \int_{\mathbb{C}^n} \exp(-(\bar{z} \cdot \bar{z} - 2\sqrt{2}\bar{z} \cdot x + x \cdot x)/2)(Bf)(z) e^{-|z|^2} dz \\
f(x) &= \pi^{-n/4}(2\pi)^{-n/2} \exp(- \mid x \mid^2 /2) \int_{\mathbb{R}^n} (Bf)(x+iy) \exp(- \mid y \mid^2 /2) dy \\
A_j &= (a_j + a_j^\dagger)/2 \\
B_j &= (a_j - a_j^\dagger)/2 \\
e^{mA_j + nB_j} &= e^{-m^2/8+n^2/8} e^{(m/2+n/2)a_j} e^{(m/2-n/2)a_j^\dagger} \\
&= e^{m^2/8-n^2/8} e^{(m/2-n/2)a_j^\dagger} e^{(m/2+n/2)a_j} \\
e^{mA_j+nB_j} g(z) &= e^{-m^2/8+n^2/8} e^{(m/2+n/2)a_j} e^{(m/2-n/2)z_j} g(z_1 \cdots z_d) \\
&= e^{-m^2/8+n^2/8} e^{(m/2-n/2)(z_j+m/2+n/2)} g(z_1, z_j + m/2 + n/2, \cdots z_d) \\
&= e^{m^2/8-n^2/8} e^{(m/2-n/2)(z_j)} g(z_1, z_j + m/2 + n/2, \cdots z_d) \\
e^{mA_j+nB_j} g(z) &= e^{m^2/8-n^2/8} e^{(m/2-n/2)z_j} g(z_1, z_j + m/2 + n/2, \cdots z_d)
\end{aligned}
$$

8.2.3 Refined Topological Vertex

8.3 Relation with Chern Simons

8.3.1 Unrefined

Let M be the closed 3-manifold on which we wish to compute $U(N)_k$ on. Then take T^*M with N Lagrangians $M \hookrightarrow T^*M$. This is a setting for which open A-model can be put on.

$$
g_s = \frac{2\pi}{k+N}
$$

`https://arxiv.org/pdf/hep-th/9207094.pdf`

8.3.1 Conjecture *For $M = S^3$, as $N \to \infty$, the N Lagrangian D-branes collapse and creates a transition to the resolved conifold $\mathcal{O}(-1) \bigoplus \mathcal{O}(-1) \to \mathbb{P}^1$ with no more D-branes. That results in evaluating closed topological string on this geometry.*

$$
\begin{aligned}
t &= \int_{\mathbb{P}^1} \omega + iB \\
t &= i g_s N = \frac{iN2\pi}{k+N} = 2\pi i \frac{N}{k+N} \to 2\pi i (1 - \frac{k}{N} + (\frac{k}{N})^2 + \cdots)
\end{aligned}
$$

If there are knots, then we have $N^*K \hookrightarrow T^*M$. In the case of $M = S^3$, they become other objects after the conifold transition.

8.3.2 Refined

8.4 Brane Tiling

Recall Dimer models:

8.4.1 Dimer Model

Let Γ be a bipartite graph drawn on a surface C. Let A be a weight function on its edges. To make a mix of the physical parameters and nice polynomial behavior let $A_e = e^{-\beta_0 E_e}$ for the energy E_e and a reference temperature.

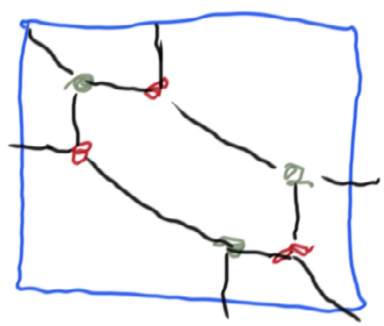


Figure 8.1: Letting x and y be the variables for crossing the identification sides polynomial is $x + y + \frac{1}{xy} + 1 + 1 + 1$ for the 6 different coverings. Here black and white are colored red and green for contrast.

$$Z(\Gamma, A, \beta) = \sum_{D \in D_\Gamma} \prod_{e \in D} A_e^{\beta/\beta_0}$$

where D_Γ is the set of dimer coverings.

If we let $\beta = \beta_0$, then we get a polynomial in the edge variables with unit coefficients.

If we let the graph have incoming and outgoing edges that go to the boundary of C let T_Γ be the set of these edges.

$$Z(\Gamma, A, \beta, \xi) = \sum_{T \subset T_\Gamma} (\sum_{D \in D_\Gamma(T)} \prod_{e \in D} A_e^{\beta/\beta_0}) \prod_{t \in D \bigcap T} \xi_t$$

where $D_\Gamma(T)$ is the set of dimer coverings which have $T \subset T_\Gamma$ occupied and the rest of T_Γ free. ξ are odd variables indexed by T_Γ. The order is given by the order on the boundary circle.

8.4.1 Theorem (Kasteleyn)

$$Z(\Gamma, T^2, A, \beta = \beta_0) \quad = \quad \frac{1}{2}\Big(- Pf(A_1) + Pf(A_2) + Pf(A_3) + Pf(A_4)\Big)$$

Proof Insert proof □

8.4.2 Example *Just counting dimer coverings. so set* $A_e^{\frac{\beta}{\beta_0}} = 1$*. This then gives ...*

8.4.3 Example (Aztec Diamond)

8.4.4 Theorem *Correlation functions that condition on a dimer being occupied at a particular bond.*

$$\langle I_{e_1} I_{e_2} \cdots I_{e_n} \rangle \quad = \quad ?$$

8.4.5 Definition (Height function) *Choose height somewhere and everytime you cross a occupied dimer change by ...*

8.4.2 Brane Tiling

8.4.6 Definition (Brane Tiling)

8.4.7 Definition (Zig Zag path) *For each white node turn right as you come into it and for each black node turn left. A zig zag path is maximal with respect to this property on the graph* Γ*. The surface* C *provides the ambient orientation.*

Figure 8.2: Same example as above.

8.4.8 Definition (Associated Double Suspension) *Take* $uv = P(x, y)$ *for the Laurent polynomial given from the graph on the torus. Call this* Y*. See* $uv = H(\alpha, \beta)$ *in the Hofstaeder case.*

Proof `https://arxiv.org/abs/0907.2063` □

8.4.9 Theorem $D^b FukY$ *is equivalent to* $D^b Coh$ *of the associated toric Calabi-Yau 3-fold.*

Chapter 9

Donaldson-Thomas and GV

9.1 Bridgeland Stability Conditions

See Section 28.4.1

9.2 CY3 categories from quivers

See chapter 35

9.3 Donaldson-Thomas Physical

Consider the analog of the Chern-Simons action as follows

$$S(A) \quad \propto \quad \int_X \mathrm{tr}(\nabla_{A_0}\alpha \wedge \alpha + \frac{2}{3}\alpha \wedge \alpha \wedge \alpha) \wedge \theta$$

where $A = A_0 + \alpha$ is a holomorphic connection on a bundle E over a CY3 X. Note the use of the complex volume form θ.

The critical points of this are holomorphcally flat connections where instead of asking the cuvature to be 0, we ask that the $(0,2)$ part of the curvature vanishes. These critical points define holomorphic structures on the bundle E.

9.3.1 Definition (Casson Invariant) *Suppose M is an oriented integral homology 3-sphere equipped with a Heegard splitting as $M_1 \bigcup_\Sigma M_2$. The Casson invariant makes precise the idea that each handle body defines a Lagrangian in the moduli space of flat connections on Σ by those flat connections that extend into M_1 or M_2 respectively. The end result is the problem of counting flat connections on X.*

9.4 Donaldson-Thomas

9.4.1 Definition (Curve Counting) *Consider the space of embedded curves in a nonsingular projective variety. We actually want them to be in embedded in the algebraic sense so we can't be as floppy as if we were only thinking about the underlying topological surfaces in the topological 6 manifold. In Section 7.3 when we didn't put any cohomology classes of the target space and just did* $\tilde{F}_g(t) = \sum_{\beta \in H_2(X,\mathbb{Z})} e^{t \cdot \beta} \int_{[virt]} 1 = \langle\langle 1 \rangle\rangle$ *for* $t \in H_2^\vee$*, that was acting like such a count, but multiple covers counted differently because we remembered the source in* $\mathcal{M}_{g,n}(X, \beta)$ *rather than just it's result as an embedded cuve in* X*.*

9.4.2 Example (Quintic) *You wake up an algebraic geometer in the middle of the night and say 27 and they will respond straight lines on cubic surface* $X \subset \mathbb{P}^3$*. Straight lines means* $\mathbb{P}^1 \subset \mathbb{P}^3$*. This can also be stated as rational lines with self-intersection* -1*. These are permuted by the* 27 *dimensional fundamental representation of* W_{E_6}*.*

9.4.3 Definition (Ideal Sheaf) *Just like an ideal is a subobject of* R *within* R *modules, an ideal sheaf is a subobject of* $\mathcal{O}_X$ *in* $\mathcal{O}_X$ *modules.*

Given a scheme X *and an ideal sheaf* $\mathcal{I}$*, then we can give the sheaf* $\mathcal{O}_X/\mathcal{I}$ *which will have support that is a closed subset* Z *of* X *in such a way that* $(Z, \mathcal{O}_X/\mathcal{I})$ *will be a closed subscheme.*

Conversely given a closed immersion $\iota \; Z \to X$ *then there is an associated map* $\iota^\sharp$ *which goes from* $\mathcal{O}_X$ *to* $i_* \mathcal{O}_Z$ *as sheaves on* X *which is surjective on stalks. We can take the kernel of this* $\iota^\sharp$ *to get a quasi-coherent ideal sheaf.*

This definition explicitly presents the ideal sheaf as sitting inside the exact sequence $0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$*.*

9.4.4 Definition (Moduli of Ideal Sheaves) $I_n(X, \beta)$ *is defined as the moduli space of ideal sheaves* $\mathcal{I}$ *satisfying* $\chi(\mathcal{O}_Z) = n$ *where* $\mathcal{O}_Z$ *is the closed subscheme that follows from* $\mathcal{J}$ *and also that the class of* Z *in homology is given by a specified* $\beta \in H_2(X, \mathbb{Z})$*. Here* $\chi(\mathcal{O}_Z)$ *is holomorphic Euler characteristic* $\sum (-1)^i \dim H^i(Z, \mathcal{O}_Z)$*.*

9.4.5 Lemma (*I_n $Hilb^n$* relationship)

9.4.6 Example *Let* $E \to M$ *be a vector bundle. Have a section* s*, take* X *scheme theoretic zero section. Take the Chow class that it represents* $[X]$*.*
Want something without the embedding dimension. Scale s *to* λs*. In the limit, is invariant under scaling action of fibers. Get normal none of* $X \subset M$ *as embedded in* E*.*

Proof See details in MSRI Intro Beyond Numbers Talk □

9.4.7 Definition *Let* $\mathcal{F}$ *be the universal sheaf over* $Hilb^n(Y) \times Y$*. Let* π *be the map to* $Hilb^n(Y)$*. Build* E *over* X *by* $R\pi_*$*. Hopefully can massage into something concentrated in degrees* 0 *and* 1*.*

9.4.8 Corollary Y *a CY 3-fold. Obstruction space dual to deformation space.*

Proof

$$\begin{aligned} Ext^1(F,F) &\simeq Ext^{3-1}(F, F \otimes \omega)^\vee \\ &\simeq Ext^2(F,F)^\vee \end{aligned}$$

9.4.9 Definition (Almost Closed 1-form) *$d\omega$ does not have to be 0 but only 0 when restricted back to X so 0 modulo the specified ideal. The special case of $\omega = df$ for usual derived critical locus of a regular function.*

9.4.10 Theorem *Every symmetric obstruction theory is locally given by an almost closed 1-form.*

9.4.11 Definition (Behrend function) *Constructible function χ built by...*

9.4.12 Theorem (Behrend) *If take weighted Euler characteristic weighted by χ above ...*

9.4.13 Theorem (Kashiwara-MacPhereson Microlocal Index Theorem) *If X has a symmetric obstruction theory and is compact then count is euler characteristic weighted by the constructible function χ above.*

9.4.14 Example *$Y = \mathbb{C}^3$ and $X = Hilb^n(Y)$. Let T be the torus with $t_1t_2t_3 = 1$ so that it preserves CY structure.*

$$\chi(Hilb^n\mathbb{C}^3, \nu) = \sum_{monomial} (-1)^?$$

9.4.1 Motivic Donaldson-Thomas

9.4.15 Definition (MacMahon Function) *$M(x)$ is the generating function for plane partitions and $M(a, b, c)$ for the number that fit in an a by b by c box.*

$$\begin{aligned} M(x) &= \sum_{n=0}^{\infty} PL(n)x^n \\ &= \prod \frac{1}{(1-x^k)^k} \\ M(a,b,c) &= \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2} \end{aligned}$$

9.4.16 Theorem (Ben Young)

9.4.17 Definition (Kapranov (Hilbert) Motivic Zeta Function) *Consider $K_0(Var/k)$ for a field k. Then one can form the classes of Sym^nX and $Hilb^nX$ allowing definition of*

$$\begin{aligned} Z_{mot}(X,t) &\equiv \sum_{n\geq 0} [Sym^n X]t^n \\ Z_{hilbmot}(X,t) &\equiv \sum_{n\geq 0} [Hilb^n X]t^n \end{aligned}$$

If we have a motivic measure $K_0(Var/k) \to R$, then we can get elements of $R[[t]]$.

9.4.18 Example

$$Z_{mot}(Speck, t) = \frac{1}{1-t}$$

9.4.19 Theorem *Let C be a reduced curve over an algebraically closed field. $Z_{hilbmot}(C, t)$ is a rational function of t with constant term 1.*

Proof For smooth curve this coincides with Z_{mot} and then rationality is a theorem of Kapranov. But otherwise Z_{mot} is insensitive to singularities. `https://arxiv.org/pdf/1710.04198.pdf` □

9.4.20 Definition (Ganter-Kapranov Zeta function) *Let k be a field of characteristic 0 and form $K_0(dg - cat/k)$*

$$Z_{cat}(\mathcal{C}, t) \equiv \sum_{n \geq 0} [Sym^n \mathcal{C}] t^n$$

There is a motivic measure μ_{dg} $K_0(Var/k) \to K_0(dg - cat/k)$ by sending the class of of X to the dg-enhanced version of $D^b CohX$. It is well defined under the equivalence relations for making motives.

9.4.21 Theorem *Let X be a smooth projective variety with dimension ≤ 2 or that the class of X is a polynomial in $[A^1]$. Then:*

$$Z_{cat}(\mu_{dg}(X), t) = \prod_{k \geq 1} \mu_{dg}(Z_{mot}(X, t^k))$$

Proof `https://arxiv.org/pdf/1506.05831.pdf` □

9.4.22 Example

$$\begin{aligned} X &= Speck \\ Z_{cat}(dg - Vect_k, t) &= \prod_{k \geq 1} \frac{1}{1-t^k} \end{aligned}$$

9.5 MNOP

9.5.1 Conjecture (MNOP)

9.6 Gopukumar-Vafa

`https://arxiv.org/pdf/math/0701568v2.pdf`

Chapter 10

Landau Ginzburg Models

10.1 B-side:Matrix Factorization Categories

For a evil liar (physicist), someone would say there is a supersymmetric quantum mechanics with target $Maps(D, X)$ with D one dimensional and possibly with boundary.

10.1.1 Definition (Matrix Factorization) *A matrix factorization X of $W(x) \in \mathbb{C}[x]$ is a $\mathbb{Z}_2$ graded free $\mathbb{C}[x]$ module with an odd $\mathbb{C}[x]$ linear operator d_X*

$$\begin{aligned} d &= \begin{pmatrix} 0 & d_X\mid_1 \\ d_X\mid_0 & 0 \end{pmatrix} \\ d_X^2 &= W \cdot Id \end{aligned}$$

10.1.2 Example

$$W = x^3$$
$$d = \begin{pmatrix} 0 & x \\ x^2 & 0 \end{pmatrix}$$

10.1.3 Example

$$W = y^5 - x^3$$
$$d = \begin{pmatrix} 0 & 0 & x^2 & -y \\ 0 & 0 & y^4 & -x \\ -x & y & 0 & 0 \\ -y^4 & x^2 & 0 & 0 \end{pmatrix}$$

10.1.4 Definition (Matrix Factorization Category) *The category of Matrix Factorizations of W. It is the $\mathbb{Z}_2$ dg category with objects $V_0 \mid V_1$ equipped with an odd endomorphism $d = (0, d_1; d_0, 0)$ such that $d_0 d_1 = W Id_{V_1}$ and $d_1 d_0 = W Id_{V_0}$. Here V_i are the even and odd components of the $\mathbb{Z}_2$-graded free $k[z_1, \cdots z_n]$ module. n is fixed because in order to say what W was we had to say how many variables it had.*

10.1.5 Theorem *This is equivalent to the derived category of singularities of the scheme* $X = Spec(k[z_1 \cdots z_n]/W)$. $D_{sing}(X) \equiv D^b Coh(X)/Perf(X)$. *Here k is a field of (???).*

Proof On objects send a matrix factorization to the $coker(d_1)$
`https://arxiv.org/pdf/math/0302304.pdf` □

10.1.6 Theorem (Bertin) *Jose Bertin - Clifford Algebras and Matrix Factorizations*

Let W *be homogeneous quadratic in* $k[z_1 \cdots z_r]$. *This can be interpreted as a bilinear form on* k^r *so we can form* $Cliff(k^r, w)$. *Let us also suppose that this bilinear form is nondegenerate.*

We can get an equivalence of categories

Proof Suppose we are given a matrix factorization $(d_0 = P, d_1 = Q)$ with P, Q both being matrices over $k[z_1 \cdots z_r]$.

$$
\begin{aligned}
w &\equiv \sum w_{ij} x_i x_j \\
\gamma_i &\equiv \begin{pmatrix} 0 & \frac{\partial Q}{\partial x_i}(0) \\ \frac{\partial P}{\partial x_i}(0) & 0 \end{pmatrix} \\
\gamma_i \gamma_j + \gamma_j \gamma_i &= 2 w_{ij} Id \\
\gamma_i &\in End(k^{n_1} \bigoplus k^{n_2})
\end{aligned}
$$

This defines a Clifford algebra representation. Here it is essential that W is of degree 2 so that one can relate $\frac{\partial^2 W}{\partial x_i \partial x_j}$ with $\frac{\partial P}{\partial x_i}(0)\frac{\partial Q}{\partial x_j}(0) + \frac{\partial P}{\partial x_j}(0)\frac{\partial Q}{\partial x_i}(0)$.

In the other direction we send a Clifford module $A_0 \bigoplus A_1$ to $V_i = k[V^*] \otimes A_i$ with $d_{0,1}(1 \otimes a) = \sum z_i \otimes \gamma_i a$ and extended $k[z_1, \cdots z_r] = k[V^*]$ linearly. Here γ_i is the corresponding basis for the degree 1 part of the Clifford algebra matching the z_i.

Then see that both of these actually define functors and that they give an equivalence.

10.1.7 Theorem (Abouzaid-Auroux-Efimov-Katzarkov-Orlov) $D^b WrFuk(W^{-1}(0)) \simeq D^b_{sing}(cohZ) \simeq D^b_{sing}(mod - \frac{\mathbb{C}Q}{(\partial\Phi, \Phi)})$ *where* Z *is union of toric divisors in the toric CY 3-fold.*

10.1.8 Theorem (K-Theory) *For a scheme that admits an ample family of line bundles, Thomas-Trobaugh K-theory of* $Perf(X)$ *coincides with Quillen K-theory of* $Vect(X)$. *When smooth,* $Perf(X) \simeq D^b(X)$

10.1.9 Theorem $K_i(D_{sing}(X), \mathbb{Z}/l^\nu) = coker(M)$ *or* $ker(M)$ *or* 0 *depending on* i.
`https://arxiv.org/pdf/1502.05364.pdf` for the Kleinian singularities like $k[u, v, w]/(u^n + vw)$ *which gives* A_{n-1} $\mathbb{C}^2/\Gamma$
`https://arxiv.org/pdf/1512.01205.pdf` for more general affinizations of $\mathbb{A}^d/\mathbb{Z}_n$ *where acting by some* ζ^{a_i} *diagonal subgroup of* $SL(d, \mathbb{C})$. *That is the abelian part of the Shephard group* $G(n, n, d)$ *without the* S_d. *There are* $\leq \binom{n-1}{d-1}$ *choices. Precisely that many when* n *is prime.*
Because this is taking Spec of $k[x_1 \cdots x_d]^G$, *then use Noether theorem to write as* $k[R_G(x_\bullet^\beta)]$ *for* $\mid \beta \mid \leq \mid G \mid$. *But those have relations now (syzygies). How many* β *are there given* $\mid G \mid$ *and* d? *A composition of* $n + d$ *in exactly* d *parts for all* $n \leq \mid G \mid$ *which is* $\binom{n+d-1}{d-1}$ *which is* $\binom{d+|G|}{d-1}\frac{|G|+1}{d}$

10.1.10 Example *Since already know the case for $d = 2$, try $d = 3$. $n = 5$ with $1, 2, 2$ weights This is the first example computed and shows $K_i(D_{sing}(X))$ to be uniquely l divisible for all primes $\neq 2, 13$
Looking at invariants $R_G(x^\beta)$ gives lots of monomials. Counting when $5 \mid a_1 + 2a_2 + 2a_3$ for $R_G(x_1^{a_1} x_2^{a_2} x_3^{a_3})$ not to vanish.*

10.1.11 Example (Pair of Pants) *Define the n-dimensional pair of pants be the complement of $n + 2$ generic hyperplanes in $\mathbb{CP}^{n+2}$. So if $n = 1$ thrice punctured sphere but viewed complex geometrically instead of as real only. The mirror of this should be $\mathbb{C}^{n+2}$ with $W = z_1 \cdots z_{n+2}$*

$$k[x, y, z]/(xyz) \quad =$$

10.1.12 Theorem (Orbifold) *If we take Y/G for a finite group, then $Br(X, W)$ is defined by taking G equivariant $Br(Y, W)$.
$Br(X, W) = Perf(\mathcal{O}_Y \rtimes \mathbb{C}G, W)$ where taking the noncommutative curved algebra.
The closed string space then gives $(\bigoplus_g \Omega^\bullet_{Y^g}, dW_g \wedge)/G$
In particular for $Y = A^2$ and $W = 0$, $Perf(\mathbb{C}[x, y] \rtimes \mathbb{C}G = H_{0,0}(G))$*

10.1.13 Theorem (Fusion and Folding) *For another superpotential $W_1 \pm W_2$ from knowledge of individual W_i. That is given a TCFT from $W_1 \in \mathbb{C}[x_1 \cdots x_n]$ and $W_2 \in \mathbb{C}[y_1 \cdots y_m]$, you can make a defect between them by $W_1 - W_2 \in \mathbb{C}[x_1 \cdots x_n, y_1 \cdots y_m]$.
There is also the procedure to fuse $W_1 - W_2$ with $W_2 - W_3$ to produce $W_1 - W_3$. The matrix factorization starts of infinite rank over the remaining variables, but need to reduce to finite by cancelling trivial pairs.*

Proof `https://arxiv.org/pdf/0707.0922.pdf` □

10.1.14 Theorem (Hochschild) *When W has isolated singularities the Hochschild homology is the Jacobi ring J_W with $\Omega^n_X[-n]/(dW)$ being a J_W torsor after peeling off a volume form. If not $(\Omega^\bullet_X, \wedge dW)$ replaces it.*

Proof `https://arxiv.org/pdf/0904.1339.pdf` □

10.1.15 Example *For $W =$, the Jacobi ring is $J_W \simeq$.*

10.1.16 Definition (General Matrix Factorization) *For a more general category $\mathcal{C}$ and an endomorphism η*

$$A \underset{b}{\overset{a}{\rightleftarrows}} B$$

with $ab = \eta_B$ and $ba = \eta_A$.

10.1.17 Theorem (Polishchuk-Vaintrob 3.14) *Let X be a smooth FCDRP (finite cohomological dimension and resolution property). W a potential not a zero divisor. Then the functor*

$$DMF(X, W) \longrightarrow D^b_{sing}(X_0 = W^{-1}(0))$$

is an equivalence of triangulated categories.

Proof `https://arxiv.org/pdf/1011.4544.pdf` □

10.1.18 Lemma *For U a Noetherian scheme and G reductive algebraic group. Assume U has ample family of G equivariant line bundles. Then U/G is FCDRP.*

10.1.19 Example

10.2 TODO: Merge in correct location.

Let W be a family of polynomial maps $\mathbb{C}^N \to \mathbb{C}$, the parameters for this family are denoted by $\mathcal{M}$.

10.2.1 Example

$$\begin{aligned} \mathcal{M} &= \mathbb{C} \\ a &\in \mathcal{M} \\ W &\in \mathbb{C}^1 \to \mathbb{C} \\ W &= \frac{x^3}{3} - ax \end{aligned}$$

The vacuaa are solutions to $dW = 0$. Assume isolated.

Label the vacuaa i, j et cetera.

A kink approaches some vacuum i at $\sigma \to -\infty$ and j at $\sigma \to +\infty$, then solve this equation for $X(\sigma)$.

$$\frac{dX}{d\sigma} = \alpha \frac{\partial \bar{W}}{\partial X}$$

In the target of W plane, we see a straight line connecting $W(vac_i)$ and $W(vac_j)$.

Vanishing cycles of vac_i and vac_j are Δ_i and Δ_j.

The net number of kinks n_{ij} is the intersection number $\Delta_i \circ \Delta_j$.

As we move around in $\mathcal{M}$, these vanishing cycles change, this also changes the n_{ij}

Let V be $\mathbb{Z}^v$ where v is the number of vacuaa. Define an operator

$$T_{ij} = 1 + n_{ij} e_{ij}$$

Let $W_{ij} = W(vac_j) - W(vac_i)$. Do for all the kinks, before and after moving in $\mathcal{M}$

Monodromy matrices are $\prod_{ij} T_{ij}$ where the order is by the phase of W_{ij}. The eigenvalues of M do not change no matter where you are in $\mathcal{M}$ even though the individual T_{ij} have changed.

10.2.2 Example

$$\begin{aligned}
W &= x^3 \\
M &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \\
Spec(M) &= \{\omega_6, \omega_6^{-1}\} \\
\omega_6^6 &= 1
\end{aligned}$$

Chapter 11

Mirror Symmetry

11.1 SYZ

11.1.1 Conjecture (SYZ Mirror Symmetry) *If X and $X^\vee$ are a mirror pair of Calabi-Yau n-folds, then they are related by*

$$\begin{tikzcd} X \arrow[dr] & & X^\vee \arrow[dl] \\ & B & \end{tikzcd}$$

where the fibers are special Lagrangian and generically dual n-tori.

11.1.2 Example (Action-angle Variables)

11.2 HMS

11.2.1 Definition (Fukaya Category) *For a symplectic manifold, (M, ω), $Fuk(M)$ is defined to be the A_∞ category whose objects are Lagrangian submanifolds (equipped with some extra data). The morphisms $CF^\bullet(L_0, L_1)$ are Floer complexes which use intersection points and counts of holomorphic strips.*
The higher A_∞ structures m_n come from mapping in polygons with boundary conditions on the prescribed Lagrangians.

11.2.2 Example (Torus) *Restrict attention to M being a standard square torus with standard symplectic structure. Also restrict the Lagrangians to be the ones that come from straight lines in the plane with rational slopes.*

Get relation from these to Jacobi theta functions. example 11.2.7

11.2.3 Definition (Fukaya Seidel Category)

11.2.4 Definition (Wrapped Fukaya Category)

11.2.5 Conjecture (Homological Mirror Symmetry)

11.2.6 Conjecture (Cosheaf of Categories) *Let S be a Stein manifold with skeleton X. Then X carries a cosheaf of categories. Taking global sections should give the wrapped Fukaya category of S.* `https://arxiv.org/pdf/1604.06448.pdf`

11.2.7 Example (Elliptic Curve) *Can also be seen as cut out by a cubic in $\mathbb{P}^2$, but let's give the differential viewpoint first.*

If take the basis of $Hom(\mathcal{O},\mathcal{L}) \simeq Hom_{Fuk}(L_1, L_2)$ given by the intersection points of L_1 and L_2 you get Jacobi theta functions.

11.3 Miscellaneous

11.3.1 Example (Quintic) *Take a quintic equation in 5 variables defining $X \subset \mathbb{CP}^4$. Quotient by the subquotient $\mathbb{Z}_5^3$ of $\mathbb{Z}_5^5$ (5th roots of unity on all the coefficients) that preserves the quintic so $\sum a_i \equiv 0$ and modulo the trivial diagonal action. This is singular with 10 $\mathbb{P}^1$ where there is an extra $\mathbb{Z}_5$ stabilizer and 10 points where there is an extra $\mathbb{Z}_5^2$.*
Do a resolution of singularites and build the mirror.

11.3.2 Conjecture (Atiyah-Floer) *It is not a symplectic manifold because it is singular, but we can construct $Loc_G(\Sigma)$ with Atiyah-Bott 'symplectic' structure. There are prefered Lagrangians $L_{1,2}$ for flat connections that extend to handlebodies. That means we should be able to take $\mathcal{F}_{Loc_G(\Sigma)}(L_1, L_2)$. Philosophically we've given a flat connection on Σ that extends to both sides of a Heegard splitting so should be related to something that is generated by flat connections on the entire 3-manifold, the Instanton Floer Homology.*

11.3.3 Definition (Log CY) *Let Y be a smooth projective and $D \subset Y$ normal crossing divisor (think transversality). $U = Y \setminus D$ is called log Calabi Yau if $K_Y + D = 0$ so there exists a holomorphic top form nowhere zero with simple poles along D.*

We can also ask for positivity which if D supports an ample divisor.

Can ask for maximal boundary if D has 0-stratum such as if D has n branches. (cutting out all the axes in $\mathbb{C}^n$)

11.3.4 Example (Projective Space) *Mirror for $\mathbb{CP}^n$ is LG-B $(\mathbb{C}^*)^n$ with superpotential $z_1 + \cdots z_n + \frac{a}{z_1 \cdots z_n}$*

11.3.5 Example (Toric Variety) $U = (\mathbb{C}^*)^n$

11.3.6 Definition (Blowups)

11.4 (Co)Homological Consequences

11.4.1 Definition (Hochschild (Co)Homology of a category) *Bimodules given as $Func(\mathcal{A} \otimes \mathcal{B}, Mod_k)$*
Let $\mathcal{A}_\Delta$ be the $\mathcal{A}$-$\mathcal{A}$ bimodule defined as $F(X \otimes Y) = Hom(X, Y)$, this can be also stated as an $\mathcal{A}^e$ module. Then define $\mathcal{A}_\Delta \otimes^L_{\mathcal{A}^e} \mathcal{A}_\Delta$. This is called $CC(\mathcal{A})$

11.4.2 Definition (Smooth or proper) *Smooth means $\mathcal{A}_\Delta$ is perfect over $\mathcal{A}^e$. That is a finite resolution. $Hom_{\mathcal{A}^e}(\mathcal{A}_\Delta, -)$ preserves limits?*

11.4.3 Example *dg category of $Coh(X)$, then matches with smooth and properness of X.*

11.4.4 Lemma *If $\mathcal{A}$ is smooth and proper, the diagonal and the right dual of the diagonal and left/smooth dual all give functors from $Perf_{\mathcal{A}}$ to itself. The first gives identity functor, the second gives Serre functor and last gives inverse Serre functor.*

11.4.5 Definition (Calabi-Yau Category) *A Calabi-Yau A_∞ category $\mathcal{F}$ of dimension d has morphism of chain complexes*

$$\begin{tikzcd} Hom_{\mathcal{F}}(a,b) \otimes Hom_{\mathcal{F}}(b,a) \arrow[r] \arrow[d, "\sigma_{a,b}"] & k[d] \\ Hom_{\mathcal{F}}(a,b) \otimes Hom_{\mathcal{F}}(b,a) \arrow[ur] & \end{tikzcd}$$

that is nondegenerate and symmetric under $\sigma_{a,b}$. It also should be cyclic invariant.

- *Strong*
- *Weak*
- *Left for smooth case: $k[d] \to CC(\mathcal{A})$ factoring through invariants. Then compose with $CC(?) \to Hom(?, ?)$.*
- *Right for $\mathcal{A}$ proper: $\Theta\; CC_-(\mathcal{A}) \to k[-d]$ with nondegeneracy condition.*

11.4.6 Example *$D^b(CohX)$ for a smooth projective Calabi-Yau variety of dimension d. Serre duality with the trivialized canonical class implements this.*

11.4.7 Theorem (Seidel) *For a closed symplectic manifold, the Hochschild homology of $\mathcal{F}(M,\omega)$ gives a map $HH_\bullet(\mathcal{F}(M,\omega), \mathcal{F}(M,\omega)) \to QH^\bullet(M,\omega)$. This is expected to be an isomorphism as sought by Kontsevich.*

11.4.8 Theorem *The small quantum cohomology of a product $X \times Y$ is given by $QH^\bullet(X) \otimes QH^\bullet(Y)$*

Proof

For noncompact:

$$HH_\bullet(\mathcal{WF}(M)) \longrightarrow SH^\bullet(M,\omega) \longrightarrow HH^\bullet(\mathcal{WF}(M,\omega))$$

11.5 Givental

11.5.1 Definition (I-function)

11.5.2 Definition (J-function)

11.5.3 Definition (K-Theoretic J-Function)

11.5.4 Definition (Mirror Map)

11.5.5 Definition (Lagrangian Cone)

11.6 Table

Table 11.1: Mirror Pairs

A	B
$WrFuk(Pants)$	$xy = 0 \subset \mathbb{A}^2$
$WrFuk(1 + x_1 + \cdots x_n = 0 \subset (\mathbb{C}^*)^n)$	$D^b_{sing}(-z_1 \cdots z_{n+1} = 0 \subset \mathbb{A}^{n+1})$
$(\mathbb{C}^*)^n \setminus H$	$D^b_{coh}(H)$

Chapter 12

Matrix Models

12.1 General

$$
\begin{aligned}
g_{s,3,4,\cdots} &\in \mathbb{R} \\
V &\in Herm(N) \simeq \mathfrak{u}(N)^* \to \mathbb{R} \\
V(M) &= \frac{-1}{g_s}\operatorname{tr}(\frac{1}{2}M^2 + \sum_{p\geq 3}\frac{g_p}{p}M^p) \\
Z(N, g_s, g_{3,4,\cdots}) &= \frac{1}{\mid U(N) \mid}\int dM e^{V(M)}
\end{aligned}
$$

That is we are giving the volume for a certain kind of measure on $\mathfrak{u}(N)^*$ which is Lie-Poisson. $\mid U(N) \mid$ is given by a choice of normalization for the Haar measure. This is also unimodular. By diagonalization of M this gives

12.1.1 Orthogonal Polynomials

12.2 Ginzburg-Weinstein

12.2.1 Theorem (Ginzburg-Weinstein) *For a compact Lie group K with standard Poisson structure, $(\mathfrak{k}^*, \pi_{KS}) \simeq (K^*, \pi_{PLD})$ as Poisson manifolds where they are given Kostant-Souriu and Poisson-Lie dual group structures respectively. This in particular allows us to push forward desired measures.*

$$
\begin{aligned}
Z(N, g_s, g_{3,4,\cdots}) &= \frac{1}{N!}\int\prod\frac{d\lambda_i}{2\pi}\Delta^2(\{\lambda\})e^{V(\{\lambda_i\})} \\
V(\{\lambda_i\}) &\equiv V(\mathrm{diag}(\lambda_1\cdots\lambda_N)) \\
\Delta(\{\lambda\}) &\equiv \prod_{i<j}(\lambda_i-\lambda_j) \\
\Delta^2(\{\lambda\}) &= (-1)^{N(N-1)/2}\prod_{i\neq j}(\lambda_i-\lambda_j) > 0 \\
V_{eff} &\equiv V(\{\lambda\}) + \sum_{i\neq j}\log\mid\lambda_i-\lambda_j\mid \\
V_{eff} &\in (\mathbb{R}^N\to\mathbb{C}/(2\pi i\mathbb{Z},+))^{S_N} \\
V_{eff}(x) &= \frac{-1}{g_s}(\frac{1}{2}\mid x_1\cdots x_N\mid^2 + \sum_p\frac{g_p}{p}\mid x_1\cdots x_N\mid^p \\
&- g_s\sum_{i\neq j}\log\mid x_i-x_j\mid)) \\
\mid x_1\cdots x_N\mid^p &\equiv \sum_i x_i^p
\end{aligned}
$$

This is an integration over the reals (λ_i all come to you as real values) a priori, but then it must be deformed in the complex plane. This gives another ambiguity which becomes another auxiliary choice to pick the contour.

$$
\begin{tikzcd}
 & Meas(\mathfrak{u}(N)^*)^{U(N)} \arrow[d] \arrow[dr] & \\
 & Meas(\mathfrak{u}(N)^*) \arrow[d, leftrightarrow, "\simeq_{GWGT}"] & Meas(\mathfrak{t}(N)^*)^W \\
Im(Meas(\mathfrak{u}(N)^*)^{U(N)}) \arrow[r] & Meas(U(N)^*) &
\end{tikzcd}
$$

12.2.2 Remark You need a pair of these to then try to get a function on $U(N)^*$ which can be thought of as a classical limit for an element of a quantum group now that it is a function on a Poisson-Lie group. $\Diamond$

12.3 1/2/4

Similarly we may find a measure over $\mathfrak{o}(N)^*$ and $\mathfrak{sp}(N)^*$

12.3.1 Theorem (Semicircular Law)

12.3.2 Theorem *The pair correlation of the zeros of Riemann zeta function is that of GUE. This leads to a conjecture that zeroes are given by some linear operator. A conjecture stronger than Riemann Hypothesis.*

12.3.3 Conjecture *The distribution of spacings for zeroes of not only the Riemann zeta function, but also other automorphic L-functions over $\mathbb{Q}$ are all given by GUE measure.* *https://web.math.princeton.edu/~nmk/RMFEM.pdf http://ac.els-cdn.com/S0022314X12001928/1-s2.0-S0022314X12001928-main.pdf?_tid=4dde4f94-e7e3-11e6-8379-00000aab0f01&acdnat=1485887593_7484e11d1645ba04c3f144ce48a2a94c*

12.4 β

The way they are described in https://arxiv.org/pdf/math-ph/0206043v1.pdf is by taking symmetric tridiagonal matrices where the diagonal

12.4.1 Toda Lattice Reminder

$$
\begin{aligned}
L &= \begin{pmatrix} a_0 & b_0 & 0 & 0 \\ b_0 & a_1 & b_1 & 0 \\ 0 & b_1 & a_2 & b_2 \\ 0 & 0 & b_2 & a_3 \end{pmatrix} \\
\frac{dL}{dt} &=
\end{aligned}
$$

That is we have given a probability measure on the phase space of the open Toda lattice. $a_n = \frac{1}{2} p_n$ are distributed normally $\frac{1}{\sqrt{2}} N(0,2)$ and $b_n = \frac{1}{2} e^{(q_{n+1}-q_n)/2}$ are distributed like $\chi_{\beta(N-1-n)}$ so the separations are distributed like a log chi distribution whose moments are given in polygamma functions. In particular, we may fix also the trace to work at a specific total momentum rather than the full phase space. We let $\beta = 1$ because we are in the real case right now. Of course the distribution evolves with the Toda flow.

$$
\begin{aligned}
Q(t)R(t) &\equiv e^{tL(0)} \\
L(t) &= Q(t)^T L(0) Q(t) \\
L(0) &= Q(t) L(t) Q(t)^T
\end{aligned}
$$

In the notation of Theorem 2.12, $L(0)$ is what is drawn from the distribution $H_{\beta=1}$, and then we are considering when $L(t)$ approaches a diagonal matrix, so Q there is $Q(t \to \pm\infty)$ (get the sign if this is repulsive or attractive lattice as given). The first row of this matrix given with nonnegative signs, is distributed with all entries as $\chi_{\beta=1}$ and then pushed down from $\mathbb{R}^N_{\geq 0}$ to the corresponding portion of the S^{N-1} (The row isn't the zero vector so this doesn't run into that ambiguity). The eigenvalues are distributed according to $c_{GOE} \prod_{i<j} \mid \lambda_i - \lambda_j \mid e^{-1/2 \sum \lambda_i^2}$

These are classical mixed states, and they should be limits of semiclassical mixed states. They are mixed as now they are not concentrated on Lagrangians $p = \frac{df}{dq}$ with density $\mid \phi \mid^2$ like the limit states associated with $\psi(q) = e^{if(q)/\hbar} \phi(q)$ pure state wavefunctions.

12.5 Vogel

12.5.1 Definition (Vogel Plane) *$\alpha\beta\gamma$ as homogenous coordinates in $\mathbb{CP}^2$. You then form the quotient by S_3 permuting them.*

Dynkin	Algebra/Parameters	α	β	γ	$t(-2,\beta,\gamma)=h^\vee$
A_{N-1}	$sl(N)$	-2	2	N	N
$B_{(N-1)/2}$	$so(N)$	-2	4	$N-4$	$N-2$
$D_{N/2}$	$so(N)$	-2	4	$N-4$	$N-2$
$C_{N/2}$	$sp(N)$	-2	1	$N/2+2$	$N/2+1$
$Exc(-2/3)=G_2$	G_2	-2	$2n+4=8/3$	$n+4=10/3$	$3n+6=4$
$Exc(0)=D_4$	$SO(8)$	-2	$2n+4=4$	$n+4=4$	$3n+6=6$
$Exc(1)=F_4$	F_4	-2	$2n+4=6$	$n+4=5$	$3n+6=9$
$Exc(2)=E_6$	E_6	-2	$2n+4=8$	$n+4=6$	$3n+6=12$
$Exc(4)=E_7$	E_7	-2	$2n+4=12$	$n+4=8$	$3n+6=18$
$Exc(8)=E_8$	E_8	-2	$2n+4=20$	$n+4=12$	$3n+6=30$

$$
\begin{aligned}
t &= \alpha+\beta+\gamma \\
dim\mathfrak{g} &= \frac{(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma}
\end{aligned}
$$

Upon switching $\alpha \to \beta$ as one of the allowed symmetries, we then rescale to keep $\alpha=-2$. This then switches $sl(N)\to sl(-N)$, $so(N)\to sp(-N)$ and $sp(N)\to so(-N)$. In addition it also has the following effect on the exceptionals.

- $G_2 \to -2, 3/2, -5/2$
- $D_4 \to -2, 1, -2$ which gives $sp(-8)$
- $F_4 \to -2, 2/3, -5/3$
- $E_6 \to 8, -2, 6 \to -2, 1/2, -3/2$
- $E_7 \to 12, -2, 8 \to -2, 1/3, -4/3$
- $E_8 \to 20, -2, 12 \to -2, 1/5, -6/5$

so the exceptional ones (not counting D_4 which is special for other reasons) don't come back to anything in the table.

12.5.2 Definition (Universal functions)

12.5.3 Example

$$
\chi_{ad}(x\rho) = \frac{\sinh(x\frac{\alpha-2t}{4})}{\sinh(x\frac{\alpha}{4})}\frac{\sinh(x\frac{\beta-2t}{4})}{\sinh(x\frac{\beta}{4})}\frac{\sinh(x\frac{\gamma-2t}{4})}{\sinh(x\frac{\gamma}{4})}
$$

12.5.4 Example (Not relevant yet wrong parameters) *For all classical Lie groups, define the following*

$$
\begin{aligned}
C_p &\equiv Tr_{fund}(\hat{X_{\mu_1}} \cdots \hat{X_{\mu_p}}) X^{\mu_1} \cdots X^{\mu_p} \\
C_G(\lambda, z) &\equiv \sum_{p=0}^{\infty} C_p z^p \\
&= z^{-1}(1 + \frac{\beta z}{2 - 2(2\alpha+1)z})(1 - \Pi_G(\lambda, z)) \\
\Pi_G(\lambda, z) &\equiv \prod_i (1 - \frac{z}{1 - m_i z}) \\
m_i &\equiv l_i + \alpha \\
l_i &\equiv \lambda_i + r_i \quad i > 0 \\
l_{-i} &\equiv -l_i \\
l_0 &\equiv 0
\end{aligned}
$$

The parameters $\alpha\beta r_i$ are given as for example $\alpha = (n-1)/2$ and $\beta = 0$ for $SU(n)$.

12.5.5 Theorem *The partition function for $SU(N)$ matrix model is essentially Barnes G function $G(1+N)$. More generally writes volume as a function of the Vogel parameters. Not invariant under the permutations or rescaling. But it does scale as power under λ so defines a section of a line bundle $\mathcal{L}$ over $\mathbb{P}^2$. Put the different S_3 related actions into the fiber to form an associated vector bundle putting them all together. $\mathcal{L}^{\bigoplus 3!}$.*

Proof `https://arxiv.org/pdf/1602.00337v1.pdf` □

12.5.6 Theorem (Kinkelin's relation on Barnes function)

$$
\begin{aligned}
G(z+1) &= \Gamma(z) G(z) \\
G(1) &= 1 \\
\log \frac{G(1+N)}{G(1-N)} &= N \log 2\pi - \int_0^N dx \pi x \cot(\pi x)
\end{aligned}
$$

12.5.7 Theorem (Riemann Zeta Relation)

$$
\begin{aligned}
\log G(1+z) &= \frac{z}{2} \log 2\pi - \frac{z + (1+\gamma) z^2}{2} + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k+1} z^{k+1} \\
z &\in (0,1) \\
\exp(\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k+1} z^{k+1}) &= \prod_{k=1}^{\infty} \left((1 + \frac{z}{k})^k \exp(\frac{z^2}{2k} - z)\right)
\end{aligned}
$$

12.6 Random Supersymmetric Model

Instead of randomly choosing an N dimensional Hermitian H as in the GUE, we randomly choose the operator Q which is the odd operator serving as the square root of both partner Hamiltonians.

$$
\begin{aligned}
Q &\equiv \begin{pmatrix} 0 & P \\ P^\dagger & 0 \end{pmatrix} \\
H &= Q^2 = \begin{pmatrix} PP^\dagger & 0 \\ 0 & P^\dagger P \end{pmatrix}
\end{aligned}
$$

The measure is weighted by $e^{-N \operatorname{tr} f(Q^2)}$ where f is some suitable function. Mimicing the general formalism before gives

$$
\begin{aligned}
f(H = Q^2) &= \frac{-1}{g_s}(\frac{1}{2}M^2 + \sum_{p \geq 3} \frac{g_p}{p} M^p) \\
Z(N, g_s, g_{3,4,\cdots}) &= \int dP e^{-N \operatorname{tr} f(Q^2)}
\end{aligned}
$$

12.7 ABJM Theory

This is the theory that describes the worldvolume of M2 branes in M theory. For contrast, the M5 case gives theory X which is expanded on in a later chapter.

Chapter 13

q deformed 2D Yang-Mills

13.1 2D Yang-Mills

$$S = \frac{k}{e^2}\int \mathrm{tr}\, F \wedge \star F$$

using the $k\,\mathrm{tr}$ to keep track of which multiple of Killing form has been used.

For comparison to `https://arxiv.org/pdf/1305.1580v2.pdf`, $k = -1/2$ and $e^2 = g_s$.

Let Σ_h be an oriented surface equipped with volume form ω, then we may take $F = f\omega$ for some lie algebra valued function f. This then turns the action into

$$S = \frac{k}{2e^2}\int dvol Tr f^2$$

13.1.1 BF-Like

$$\begin{aligned} S_{\phi F}(e^2,\rho) &= \frac{-e^2}{2}\int dvol Tr\phi^2 - i\int Tr\phi F \\ \rho &\equiv \int dvol \\ Z_{\phi F}(e^2,\rho) &= Z_{YM}(e^2,\rho) \end{aligned}$$

Send $e^2 \to 0$ in this action to just get a BF.

$$S_{\phi F}(0,-) = -i\int Tr\phi F$$

13.1.2 Chern Simons

Let $X = S^1 \times \Sigma$ which projects to Σ via w and put Chern Simons on this. If the connection is of the form $w^*\phi dt + w^*A$ we may write this as $\frac{ik}{2\pi}\int_\Sigma Tr\phi F = S_{\phi F}(0,\rho)$ but with ρ irrelevant so replace with $-$ in notation. These specific types of connections are only some of them. But thinking classically, we see only flat connections and irreducible flat connections are pullbacks of flat connections on Σ assuming $Z(G) = \{e\}$. For finite center, take copies.

$$Z_{CS,X}(k) \quad \approx \quad \mid Z(G) \mid e^{\Delta v \chi(\Sigma)} Z_{\phi F}(0,-)$$

13.1.1 Definition (Migdal's formula)

$$Z_{YM}(g_s, \Sigma_h) \quad = \quad \left(\frac{Vol(G)}{(2\pi)^{\dim G}}\right)^{2h-2} \sum_\lambda (\dim V_\lambda)^{2-2h} e^{-g_s/2C_2(\lambda)}$$

summing over isomorphism classes of unitary irreps. Note that this already regularized answer still diverges for example if $g_s = 0$ and $h = 1$.

13.2 q deformation

Let p be a positive integer and $q = e^{-g_s}$

$$\begin{aligned} Z_M^{(p)}(q, \Sigma_h) &= \sum_\lambda (\dim_q \lambda)^{2-2h} e^{-pg_s/2C_2(\lambda)} \\ \dim_q \lambda &= s_\lambda(q^\rho) = q^{|\lambda|/2} s_\lambda(1, q, \cdots q^{N-1}) \end{aligned}$$

When $q = e^{2\pi i/(k+h^\vee)}$, we truncate the sum to an alcove. It is the partition function for a Chern-Simons gauge theory at level k on a circle bundle of degree p over Σ_h.

13.2.1 Lemma ($p = 1$) *For $p = 1$ it is related to (G, k) WZW on Σ_h.*

13.2.2 Lemma ($p = 0$) *q deformed BF*
Verlinde formula for dimension of conformal blocks
Gauged G/G WZW
Chern-Simons on $\Sigma \times S^1$

13.3 β deformation

13.4 q, t deformation

$$\begin{aligned}
Z_h(q,t;p) &= \sum_{\lambda\in\Lambda_+} \frac{(\dim_{q,t} R_\lambda)^{2-2h}}{g_\lambda^{1-h}} q^{p/2\langle\lambda||\lambda\rangle} t^{p\langle\lambda||\lambda\rangle} \\
\dim_{q,t} R_\lambda &= \prod_{m=0}^{\beta-1}\prod_{\alpha} \frac{\left[\langle\lambda+\beta\rho \,||\, \alpha\rangle + m\right]_q}{\left[\langle\beta\rho \,||\, \alpha\rangle + m\right]_q} \\
\left[x\right]_q &= \frac{q^{x/2}-q^{-x/2}}{q^{1/2}-q^{-1/2}}
\end{aligned}$$

where Λ_+ are dominant weights, h is the genus of the closed oriented surface, $\mid q \mid, \mid t \mid < 1$ for convergence. $t = q^\beta$ with $\beta > 0$ a natural number.

13.4.1 Remark We will need some analog of Bohr-Mollerup theorem to pick out the correct analytic continuation from the positive integers to all $\mathbb{C}$. What reason does the free energy have to be convex as a function of β? $\Diamond$

13.4.2 Example ($U(N)$)

$$\dim_{q,t} R_\lambda = t^{1/2(|\lambda^t|_2^2 - N|\lambda|_1^1)} \prod_{\Box\in\lambda} \frac{1-t^{N-r(\Box)+1}q^{c(\Box)-1}}{1-t^{\lambda^T_{c(\Box)}-r(\Box)+1}q^{\lambda_{r(\Box)}-c(\Box)}}$$

Chapter 14

$\mathcal{N} = 4$ SYM

14.1 Untwisted

14.2 GL-Twist

14.2.1 Definition (Hitchin Equations) *On a Riemann surface* Σ *and gauge group* $\mathfrak{g}$

$$\begin{aligned} F_D + [\phi, \phi^\dagger] &= 0 \\ \bar{\partial_D}\phi &= 0 \end{aligned}$$

14.2.2 Definition (Nahm Equations)

14.2.3 Definition (Bogomolony Equations)

14.2.4 Theorem (Kapustin-Witten) *SYZ mirror of* $\mathcal{M}_G()$ *is ...*

14.3 Donaldson Twist

14.4 Untwisted

$$S_{bos} =$$

14.4.1 Planar

$AdS_5 \times S^5$

14.5 Amplitudehedron

https://arxiv.org/pdf/1703.04541.pdf

14.5.1 Positive Grassmannian

14.5.1 Definition (Positive Grassmannian) *Section 33.1*

14.5.2 ϕ^3 version 20171018 Nima

Let p_i be all incoming momenta. They add up to 0 so treat them like edges of a N-gon for the N point scattering amplitude.

$$\mathcal{A}_N^{\phi^3} = \sum_{triangulation} \prod_{\substack{triangle \\ ijk}} \frac{1}{X_{ij}}$$

The X_{ij} form a coordinate system for all the $(p_a \cdot p_b)$ after taking into account the momentum conservation. They are indexed by diagonals of the polygon connecting vertices i and j where vertex i is the starting vertex for the edge labelled p_i. We are indexing in ?clockwise order around the polygon. They indicate $(\sum p_l)^2$ for all the edges on one side of the diagonal cut by connecting vertices ij. It doesn't matter which side you choose because of momentum conservation.

$$-2p_i \cdot p_j = X_{ij} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j}$$

14.5.2 Definition (Polytopal Realization) *Given a fan $\mathcal{F}$ a polytopal realization P is a polytope such that the normal vectors of all the faces, etc gives back $\mathcal{F}$. This is called the normal fan. For example, in A_2 the fan looks like*

???

so we draw $N = 5$ perpendiculars to these lines and connect them all up into a polytope which in this case will be a certain pentagon.

This Nima result can be phrased as talking about the space of all polytopal realizations of the g vector fan in linear type A_n.

14.5.3 Definition (Type Cone [?]) *In order to specify a polytopal realization of a fan given by N rays, it is enough to specify how far along each ray the half spaces to go. So for ray defined by vector p_i, specifying $h_i \geq 0$ to give the half-space $p_i \cdot x \leq h_i$. We then intersect all these half-spaces to give the polytope. You can't specify the h_i arbitrarily because it might not close up to the right polytope, but it is sufficient data to specify the polytope. If there are N rays, this is open cone in $\mathbb{R}^N$. This is called the type cone of $\mathcal{F}$ and is denoted by $TC(\mathcal{F})$.*

14.5.4 Theorem ([?]) *If $TC(\mathcal{F})$ is simplicial and if K is a $(N-n)$ by N matrix sufficiently describing exchange relations. Then the space of all polytopal realizations is given by $\bigsqcup_{\ell \in \mathbb{R}^{N-n}_{>0}} \{z \in \mathbb{R}^N \mid Kz = \ell, z \geq 0\}$.*

Chapter 15

Spectral Networks

15.1 Physics

Fix an $\mathcal{N} = 2$ theory T in $d = 4$ and a point of the Coulomb branch. In particular let $T = S[\mathfrak{sl}_K, C, D]$ where C is a punctured Riemann surface and D are a collection of defects at the punctures. In these theories a point of the Coulomb branch is a tuple $(\phi_2 \cdots \phi_K)$ of ϕ_r being a meromorphic r-differential on C with poles at the defects.

Let the single particle Hilbert space be $\mathcal{H} = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma$. Decompose each of these sectors as representations of the Super-Poincare algebra.

15.1.1 Definition (BPS count $\boldsymbol{\Omega(\gamma)}$) *This is an integer which counts with signs the number of copies of the small irrep in $\mathcal{H}_\gamma$.*

15.1.2 Definition (Class S) *Let C be a compact Riemann surface, $\{z_i\}$ be a collection of punctures and $\mathfrak{g}$ be a Lie algebra of ADE type. For this data, we associate a 4d quantum field theory. In these theories, we can compute $\Omega(\gamma)$ without lying even though the previous definition was lying.*

15.2 Quadratic Differentials

15.2.1 Definition (Meromorphic r-differential) *An r differential on C is a section of $K_C^{\otimes r}$ holomorphic away from the punctures and presribed singularities at the punctures.*

Let ϕ_2 be a quadratic differential and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Away from zeroes and poles find a coordinate w such that $\phi_2 = (dw)^2$. Namely $w = \int_{z_0}^{z} \sqrt{f(z)}dz$. Now draw a foliation such that they are straight lines of inclination θ in w coordinate.

15.2.2 Example $\phi_2 = m\frac{dz^2}{z^2}$ *the trajectories spiral into the pole at $z = 0$. The precise way clcockwise or counterclockwise depends on $me^{-i\theta}$. If it's square has imaginary part less than or greater than 0. In other cases it is a star pattern or a annulus pattern but that is when $me^{-i\theta}$ is purely real or purely imaginary. Think of these as exceptions.*

15.2.3 Lemma *For generic θ every trajectory has at least one end on a pole.*

Proof Strebel ∎

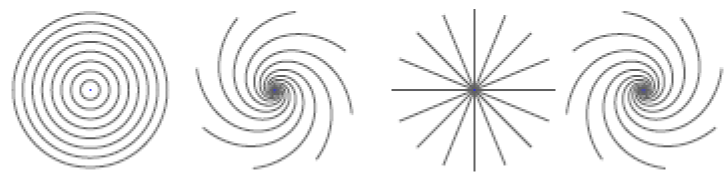


Figure 23: Behavior of WKB curves near a singularity.

At the simple zeroes get a 3-pronged singularity. 3 distinguihed trajectories. Only use these leaves of the foliation.

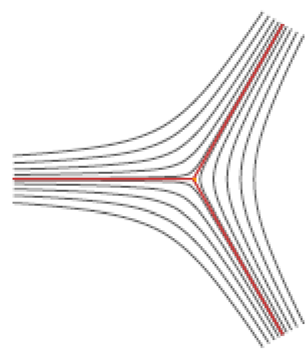


Figure 24: Behavior of the WKB foliation near a turning point. Generic WKB curves are shown as thin black curves, separating ones as thicker red curves.

15.2.4 Definition (Spectral Network/Critical Graph) *Pick a generic phase θ. Take each zeroes and emit these trajectories from each. They go around the surface for a bit and eventually end up at the poles.*

15.2.5 Lemma *For small changes in the phase or quadratic differential from a generic starting point, the network changes by isotopy. That is exists an ϵ neighborhood for this data when you are away from some codimension 1 walls.*

At these critical changes when something changes get two types of special trajectories.

15.2.6 Definition (Special trajectories) *Saddle connections are when goes from one zero to another zero. Closed trajectories are when comes out of a zero and returns to itself. These come in pairs.*

15.2.7 Lemma *These special trajectories can only show up at countably θ.*

Proof Each special trajectory lifts to a 1-cycle on the spectral cover. with a class in $H^1(\Sigma) = \Gamma$.

15.2.8 Definition (Central Charge) *Integrate the tautological Liouville form $i^*\lambda$ restricted to the spectral curve along the cycle $\gamma \in \Gamma$.*

If γ is the charge of a special trajectory, then $Z_\gamma \in e^{i\theta}\mathbb{R}_-$. There are countably many γ so there are countably many θ that this can be solved for.

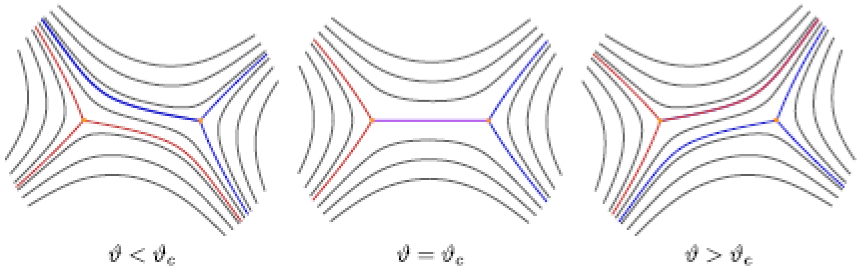


Figure 27: The jump of the WKB foliation as ϑ crosses a critical ϑ_c at which a finite WKB curve appears, corresponding to a BPS hypermultiplet.

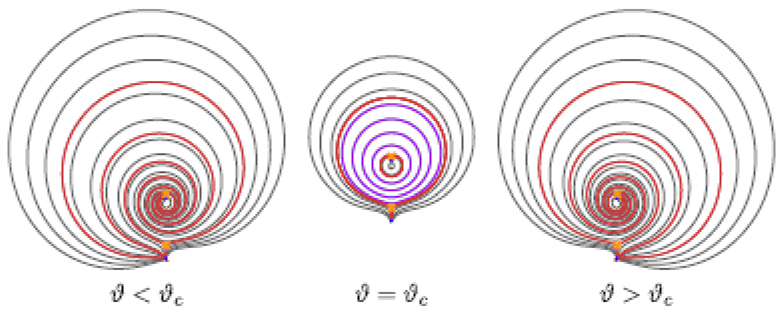


Figure 29: An annular region of the WKB foliation, near a critical $\vartheta = \vartheta_c$ at which a family of closed WKB curves representing a BPS vectormultiplet appears.

15.2.9 Definition (DT invariants)

$$\Omega(\gamma, \phi_2) \quad = \quad SC(\gamma, \phi_2) - 2CL(\gamma, \phi_2)$$

where SC are the number of saddle connections and CL are the number of closed loop pairs. In both cases $\theta = \arg Z_\gamma$.

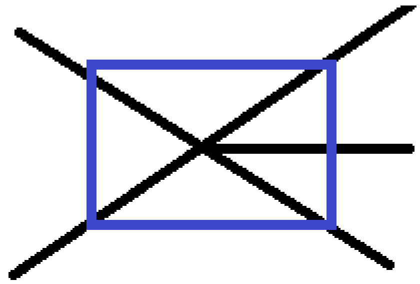
15.2.10 Definition *$T = Hom(\Gamma, \mathbb{C}^*)$ is a $\mathbb{C}^{*n}$ algebraic torus with canonical coordinate functions X_γ given by evaluation. This has birational symplectic automorphisms $X_\gamma \to X_\gamma(1 - \pm X_\mu)^{\langle \mu\gamma \rangle}$ where that sign is $\sigma(\mu)$*

15.2.11 Theorem *For any contractible closed path on $\mathbb{B} \times S^1$, then $\prod_{p \bigcap D} K_\gamma^{\pm\Omega(\gamma)} = 1$ where the product is taken in the order along the path. $\mathcal{B}$ is the parameter space of quadratic differentials.*

15.2.12 Example *In particular take a rectangular path for the 2 to 3 case illustrated below which is when $\langle \gamma_1 \mid\mid \gamma_2 \rangle = 1$*

$$K_{\gamma_1} K_{\gamma_2} \quad = \quad K_{\gamma_2} K_{\gamma_1 + \gamma_2} K_{\gamma_1}$$

Here the left side is the quadratic differential $(z^3 - z)dz^2$ on $\mathbb{CP}^1$. The vertical axis is θ. The two lines indicate the $\gamma_{1,2}$ saddle connections that appear. Those are two classes in $H_1(\Sigma)$. The right side is the quadratic differential $(z^3 - 1)dz^2$ on $\mathbb{CP}^1$. The three lines indicate the $\gamma_{1,2}$ and $\gamma_1 + \gamma_2$ saddle connections that appear. This in the family of quadratic differentials on $\mathbb{CP}^1$ with singularity of order 7 at ∞.



15.2.13 Example *If $\langle \gamma_1 \mid\mid \gamma_2 \rangle = 2$ then get*

$$K_{\gamma_1} K_{\gamma_2} = \prod_{n=0}^{\infty} K_{\gamma_2 + n(\gamma_1+\gamma_2)} K^{-2}_{\gamma_1+\gamma_2} \prod_{n=\infty}^{0} K_{\gamma_1+n(\gamma_1+\gamma_2)}$$

For example consider $\mathbb{CP}^1$ with $\phi = (\frac{1}{z} + z)\frac{dz^2}{z^2}$. This has 2 saddle connections as θ varies. $\Omega(\pm\gamma_1) = \Omega(\pm\gamma_2) = 1$. Then we change the quadratic differential to $\phi = (\frac{1}{z} + 8i + z)\frac{dz^2}{z^2}$. This has $\Omega(\pm\gamma_1 \pm n(\gamma_1+\gamma_2)) = 1$ and $\Omega_{\gamma_1+\Omega_2} = -2$ because of saddle connections and closed loop pairs respectively.

15.2.1 Higgs Perspective

Now replace quadratic differentials with Higgs fields (E, ϕ) Locally

$$\phi(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix} dz$$

The quadratic differential is $Tr\phi^2$. The spectral curve is $\{\det \phi(z) - \lambda = 0\} \subset T^*C$. So even get the line bundle over the spectral curve. In general $\bigoplus_{r=1}^{r-1} H^0(K^{\otimes r})$.

Flat $SL(2,\mathbb{C})$ connection. In a local patch.

$$\nabla s = (A_z s + \partial_z s)dz + (A_{\bar{z}} s + \partial_{\bar{z}} s)d\bar{z}$$

15.2.14 Lemma *Flat connection gives some local bases $(s_1, s_2 \cdots)$ with $\nabla s_i = 0$. Basis for bundle E. We will seek to find these local bases.*

Do analytic continuation in a loop to get $R_\nabla\ \pi_1 \to SL(2,\mathbb{C})$ up to equivalence. Start with a s_1 s_2 and as you come back, you will be off from what you started with by a monodromy. Can also do the reverse process to get the connection up to gauge transformation.

15.2.15 Theorem *Given a (stable) Higgs bundle there is a corresponding family of connections parameterized by $\zeta \in \mathbb{C}^*$. This is a family of flat connections.*

$$\nabla(\zeta) = \frac{\phi}{\zeta} + D + \phi^\dagger \zeta$$

where D is unitary with respect to some metric on E.

Actually ϕ is meromorphic so when simple poles change to $\pi_1(C\backslash\{z_i\})$ and give $\nabla(\zeta)$ regular singularities. If higher order poles replace with Stokes data. In particular, with irregular singularities, can even get nontrivial examples with $\mathbb{CP}^1$ and 1 puncture.

Proof Hitchin, Donaldson, Simpson, Biquard-Boalch

The flatness of this is implied by knowing solutions to Hitchin's equations.

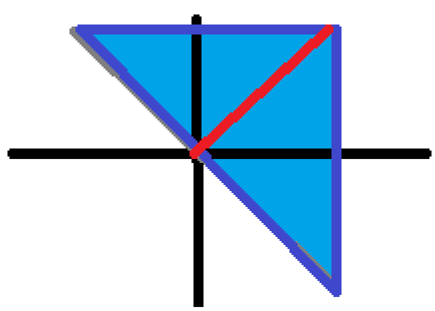
How does this family of representations $\pi_1 \to SL(2,\mathbb{C})$ behave as $\zeta \to 0$. Find those approximate solutions $\nabla(\zeta)s = 0$

15.2.16 Proposition

$$\begin{aligned}
\phi(z)e^{(i)}(z) &= \lambda^{(i)}(z)e^{(i)}(z) \\
(\phi\zeta + \partial)s &= 0 \\
s &\approx exp(-\zeta^{-1}\int_{z_0}^{z} \lambda^{(i)})e^{(i)}(z) \\
s &= exp(-\zeta^{-1}\int_{z_0}^{z} \sum_{n=0}^{\infty} \lambda_n^{(i)}\zeta^n) \sum_{n=0}^{\infty} \zeta^n e_n^{(i)}(z)
\end{aligned}$$

Substitute back in get iterative solutions for λ_n and e_n. But this series doesn't converge. Zero radius of convergence. How is this obviously not convergent? Because know as move around branch point the eigenvalues swap. So the monodromies would exchange and a factor from some integrals of eigenvalues. This contradicts the fact that $\nabla(\zeta)$ is flat even for this small nonzero ζ. So it must be only an asymptotic expansion, cannot plug in $\zeta \neq 0$.

Now instead look in half spaces determined by some θ. Say that this is sector in which those asymptotic expansion holds.



15.2.17 Theorem *For each generic point on the curve and given eigenvalue of Higgs field and generic half space, then there exists an actual solution with the above prescribed asymptotics. These don't patch together, but have walls where they jump. This allows the full $\nabla(\zeta)$ to be flat. The codimension 1 walls in C are the spectral network for this quadratic differential and θ.*

$$\begin{aligned} TrR_{\nabla(\zeta)}(\mathcal{P}) &= \sum_{\gamma\in\Gamma} \bar{\Omega}(\mathcal{P},\gamma)\mathcal{X}_\gamma(\zeta) \\ X_\gamma(\zeta) &\approx exp(\zeta^{-1}Z_\gamma)\ \ \forall\zeta\in H_\theta\to 0 \end{aligned}$$

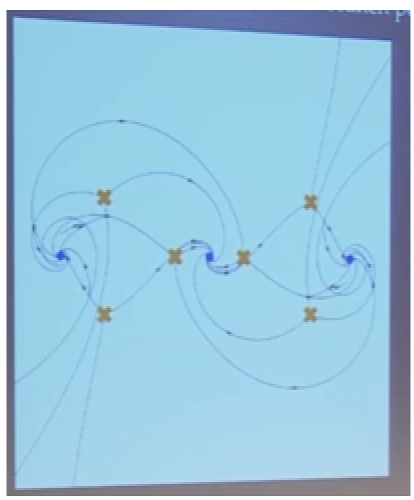
Take the asymptotics by picking the dominant term.

15.2.18 Theorem *$\nabla(\zeta)$ $SL(2,\mathbb{C})$ connection on C and $\pi_*\nabla^{ab}(\zeta)$ the pushforward of a $\mathbb{C}^*$ connection over Σ are related by cutting and gluing along the spectral network via unipotent matrices.*

As spectral network jumps, the intersections governing the integers $\bar{\Omega}$ jump so the $\mathcal{X}_\gamma$ must also jump to keep the sum the same. The jump of the abelianized connection gives the K_γ wall crossing symplectomorphism of the torus. The X_γ are defined with just knowing where you are in $\mathcal{B}\times S^1_\theta$ space so as travels around contractible closed loop have to return to same X_γ. That gives the wall crossing formula $\prod\mathcal{K}=1$

15.2.19 Definition *Each Stokes line carries label ij locally defined on C obeying $(\lambda_i-\lambda_j)\dot{z}=e^{i\theta}$. These kinds of labels mix up globally so no longer leaves of a global foliation.*

Each branch point emits 3 lines again. When ij and jk intersect they give birth to a new ik line.

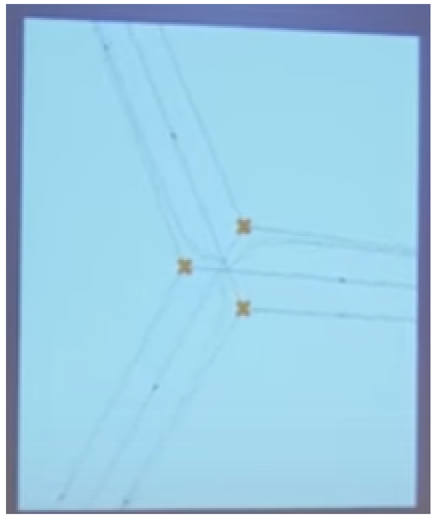


There are new special trajectories besides saddles and closed loop pairs. There are many more possibilities. Can have trees, loops connecting all these branch points.

This $ij\ jk\ ki$ junction also happens to give $\Omega=1$.

15.3 Nietzke Talk at MSRI

- $G=U(K)$ and $G_{\mathbb{C}}=GL(K)$



- Riemann surface C with $n \geq 0$ punctures
- rank K Hermitian vector bundle E over C
- m_i and $m_i^{\mathbb{R}}$ parameters at each puncture.

15.3.1 Definition (Hitchin System) *$\mathcal{M}(G, C)$ moduli space ϕ $E \to E \otimes K_C$*

$$\begin{aligned} F_D + \left[\phi, \phi^{\dagger}\right] &= 0 \\ \bar{\partial_D}\phi &= 0 \end{aligned}$$

15.3.2 Theorem *It is hyperKahler. In fact even better it is CY. It comes with a torus fibration*

$$\begin{array}{c} \mathcal{M} \\ \Big\downarrow \\ B \end{array}$$

to the Hitchin base with generic fibers compact torus. The bad fibers have these degenerate.

15.3.3 Definition $\Sigma_u = \{\det(\phi - \lambda) = 0\} \subset T^*C$

For $\mid \zeta \mid = 1$ on the twistor sphere this is a special Lagrangian fibration. For $\zeta = 0$ complex structure get a holomorphic fibration.

15.3.4 Definition (Scattering Diagram) *Fix $\zeta \in \mathbb{C}^*$. Draw a scattering diagram on B by marking the points for exceptional. They emit walls. When walls meet, they emit even more walls.*

15.3.5 Lemma *Neighborhood of C in T^*C is incomplete hyperKahler.*

In some structures Σ is Lagrangian and look for holomorphic discs in T^*C with boundary Σ_u Look at the set of $u \in B$ such that there exists these discs on Σ_u

15.3.6 Remark Σ_u will not be in that small neighborhood, but can tune parameter of C to make that neighborhood bigger and then get Σ_u to stay in neighborhood away from the punctures where can't hope for it to stay close. $\Diamond$

15.3.7 Conjecture *Given $u \in B \setminus D$ $\exists$ canonical I_ζ holomorphic Darboux coordinate system on U with $\pi^{-1}(u) \subset U$.*

Build a network $W(u) \subset C$ where $W(u)$ is the set of points such that exists an I_ζ holomorphic bigon in T^*C with boundary on Σ_u and $T^*_z C$

15.3.8 Theorem (Kashiwara-Schiapira) *Microsupports are coisotropic. These then give $N^*[1]\mu supp(F)$ are Lagrangians in $T^*[1]T^*M$ that can serve as Poisson Sigma Model boundary conditions.*

15.3.9 Theorem (Nadler Zaslow) *For a compact manifold M, $D_{con}(M)$ derived category of constructible sheaves on M is equivalent to $DFuk(T^*M)$ by sending k_U to the graph Λ_f approximating the microsupport.*

15.3.10 Example *If we have a constructible sheaf on a space with a list of strata, then can just take the dimensions of the stalks in each strata. For $D_{con}(M)$ then need to take h^i to get an actual constructible sheaf so get a table indexed by which stratum we are in and an integer for homological dimension.*
In particular if looking at $f_*\mathbb{C}_Y$ of the constant sheaf, then are listing the dimensions of $H^\bullet(f^{-1}p, k)$ for p in each stratum and $\bullet$ for the homological grading.

15.3.11 Theorem (Tamarkin) *For A, B arbitrary subsets in T^*M (not conic so not microsupports of anything). Let F be on $M \times \mathbb{R}$ with $\rho\mu supp(F) \subset A$ where $\rho\ T^*(M \times \mathbb{R}) \to T^*M$.*

If $\lim Hom(F, t_{c\star}G) \neq 0$, then A, B are nondisplacable.

15.3.12 Definition *Build a coordinate system by sending a patch of Ψ_u $\mathcal{M}(G, C) \to \mathcal{M}(GL(1), \Sigma_u)$*

On the complement of the network identify by $\nabla \simeq \pi_*\nabla_u^{ab}$

Parallel transport for ∇ is parallel transport for $\pi_*\nabla$ but with corrections from I_ζ holomorphic bigons.

For example say the holomorphic bigon connects sheets 2 and 3 then follow the edge of the bigon to get a different lifted path that computing holonomy along. Throw in this unipotent part.

Take $\mathbb{Z}[\pi_{\leq 1}(C)]$ to $\mathbb{Z}[\pi_{\leq 1}(\Sigma_u)]$ by lifts and the bigon corrections.

For any closed curve $\mathbf{Tr}(\mathbf{Hol}(\gamma, \nabla)) = \sum_{\gamma^l} \underline{\bar{\Omega}}(\gamma, \gamma^l)\mathbf{Hol}(\gamma^l \nabla_u^{ab})$

15.3.13 Theorem *If $K = 2$ and $n \geq 1$, these are the Fock-Goncharov $\mathcal{X}$ coordinates using the ideal triangulation $T(u)$ with walls going from branch points to punctures. The walls never meet. As make a change in the base u get a change of triangulation topology. Do diagonal flips for the triangulation.*

Have a holomorphic disc ending on the spectral curve. If the path doesn't hit this disc then do nothing. If it does go all the way around the disc detour or don't. Add both possible terms. This is an automorphism of $\mathbb{Z}[\pi_{\leq 1}\Sigma]$

If pass to the subquotient $\mathbb{Z}[H_1(\Sigma)]$ get the cluster $\mathcal{X}$ move

$$X_\gamma \quad \to \quad X_\gamma(1 + X_\mu)^{\langle \mu | \gamma \rangle}$$

15.4 Motohico Mulase at SCGP and Dumitrescu at String-Math

http://scgp.stonybrook.edu/video_portal/video.php?id=2719

http://video.upmc.fr/differe.php?collec=C_string_math_2016&video=27

Want to quantize a Hitchin Spectral curve.

$$\begin{array}{ccc} \Sigma & & T^*C \\ & & \\ & & C \end{array}$$

want a unique holomorphically depending globally defined Rees differential operator $P(\hbar\frac{d}{dz}, z, \hbar)$ on C with semiclassical limit Σ

Choose C genus ≥ 2 $K_C^{1/2}$.

Higgs moduli spaces over Hitchin base $\pi_H \to B = \bigoplus H^0(C, K_C^{m_i+1})$ want to go to deRham. Take Hitchin section L_H to a holomorphic Lagrangian. Biholomorphic map γ takes to $Op_C(G)$. This is a quantization and semiclassical the other way. Want to find γ.

15.4.1 Definition (Hitchin Section) *Take principal $SL(2)$ in $G = SL(n,\mathbb{C})$ called $H, X_\pm$ for diagonal, upper and lower respectively. L_H is given by bundle E_0 and $\phi(q)$ with $E_0 = K_C^{(n-1)/2} \bigoplus \cdots K_C^{-(n-1)/2}$ so total degree 0. $phi(q) = X_- + \sum q_i X_+^i$. Stable.*

15.4.2 Definition (Oper) *E holomorphic vector bundle (dimension n) over C, holomorphic connection ∇. Filtration $0 = F_n \subset F_{n-1} \cdots E$. Griffiths transversality $\nabla\ F_i \to F_{i-1} \otimes K_C$. $\bar{\nabla}\ F_i/F_{i+1} \to F_{i-1}/F_i \otimes K_C$ is an $\mathcal{O}_C$ iso of line bundles.*

Suppose have (E, ϕ) stable Higgs pair, Skipped some

15.4.1 What is the limiting oper?

Let U_α and U_β have coordinates z_α and z_β glued by Mobius. Chose the half Canonical so know how to fix the signs so in $SL(2,\mathbb{R})$ rather than $SL(2,\mathbb{C})$. So call $\xi_{\alpha\beta} = \pm(c_{\alpha\beta}z_\beta + d_{\alpha\beta})$ for the 1-cocycle defining the bundle $K_C^{1/2}$.

Use this to define a bundleE_0 with $(\xi_{\alpha\beta})^H = e^{H \log \xi_{\alpha\beta}}$ gluing and $E_\hbar = e^{H \log \xi_{\alpha\beta}} e^{X_+ \hbar \frac{d}{dz_\beta} \log \xi_{\alpha\beta}}$. In the $n=2$ this is an extension of half and inverse half canonicals whereas the E_0 is just the direct sum. For higher n filtered extension.

Proof The formula said for $E_\hbar$ was actually a cocycle so this is a vector bundle $\square$

Define $\nabla^\hbar = d + \frac{1}{\hbar}X_-$ on U_α is a globally defined connection in $E_\hbar$, this matches with E_0, X_- on the other side. For $E_0, \phi(q) = X_- + \sum q_i X_+^i$ it goes to $E_\hbar, d + \frac{1}{\hbar}\phi(q)$. This is an oper obtained by the limit. By a gauge transformation can take it to $d + X_- + \frac{1}{\hbar^2}\sum q_i X_+^i$

Chapter 16

Framed BPS in Two and Four Dimensions

Greg Moore's Talk at String-Math 2016

http://video.upmc.fr/differe.php?collec=C_string_math_2016&video=1

16.1 $d = 4$

Work on Coulomb branch $\mathcal{B}$. There is a local system of lattices Γ which are equipped with pairings $\Gamma_u \times \Gamma_u \to \mathbb{Z}$.

16.1.1 Definition (Framed BPS) *Those states in sector $H_{L,\gamma}$ which saturate the bound $E \geq -Re(Z_\gamma/\zeta)$*

We are looking at this subspace of the Hilbert space.

16.1.2 Definition (Framed Index)

16.1.3 Definition (Vanilla)

Vanilla BPS states can bind to framed BPS states with γ_h combining with a core γ_c to give $\gamma_h + \gamma_c$ with approximate binding radius:

$$r = \frac{\langle \gamma_h \mid\mid \gamma_c \rangle}{2\Im(Z_{\gamma_h}(u)/\zeta)}$$

16.1.4 Definition (K-wall) *For each γ_h the subset of $\mathcal{B}$ where $Z_{\gamma_h}(u)$ and ζ are parallel is called the K_{γ_h} wall. That is where the radius above blows up.*

16.2 $d = 2$

See SUSY and Morse theory in chapter 3 first.

16.2.1 Example *Consider the interface from trivial theory to itself. Get the category of chain complexes.*

Proof

Chapter 17

Analytically Continued Chern Simons

17.1 Finite Dimensional Examples

17.1.1 Definition (Lefschetz Thimble Construction) *Take the downward gradient flow of a function $h = \Re I$. So I is a complex valued function on a Riemannian manifold (M, g). The coordinates are given by u^i that are flowing along gradients.*

$$
\begin{aligned}
h &\equiv \Re I \\
\frac{du^i}{dt} &= -g^{ij}\frac{dh}{du^j} \\
\frac{dh}{dt} &= \sum \frac{\partial h}{\partial u^i}\frac{du^i}{dt} = -\sum \mid \frac{dh}{du^i} \mid^2 < 0
\end{aligned}
$$

17.1.2 Lemma *For a Kahler metric g, ω, J, then gradient flow of ReI and metric g is the same as Hamiltonian flow with ImI and symplectic form ω.*

Proof Compare the differential equations for the different flows in u_1^i and u_2^i again with u^i labeling the coordinates.

$$
\begin{aligned}
\frac{du_1^i}{dt} &= -g^{ij}\frac{dR}{du_1^j} \\
\frac{du_2^i}{dt} &= -\omega^{ij}\frac{dI}{du_2^j} \\
&=
\end{aligned}
$$

Proof

$$\begin{aligned}
ds^2 &= \mid dx \mid^2 \\
\frac{dx}{dt} &= -\frac{\partial \bar{I}}{\partial \bar{x}} \\
\frac{d\bar{x}}{dt} &= -\frac{\partial I}{\partial x} \\
\frac{d}{dt}(\frac{x+\bar{x}}{2}) &= \frac{1}{2}(-\frac{\partial I}{\partial x} - \frac{\partial \bar{I}}{\partial \bar{x}}) \\
&= \\
\frac{dImI}{dt} &= \frac{1}{2i}\frac{d}{dt}(I-\bar{I}) = \frac{1}{2i}(\frac{\partial I}{\partial x}\frac{dx}{dt} - \frac{\partial \bar{I}}{\partial \bar{x}}\frac{d\bar{x}}{dt}) = \frac{1}{2i}\big(- \frac{\partial I}{\partial x}\frac{\partial \bar{I}}{\partial \bar{x}} + \frac{\partial I}{\partial x}\frac{\partial \bar{I}}{\partial \bar{x}}\big) \\
&= 0
\end{aligned}$$

17.2 $SL(2)$

- $\Gamma(SU(2))$ is the contour chosen such that for integer k coincides with the compact Chern Simons
- $\tilde{A}(SL(2,\mathbb{C}))$ is the universal cover of $(A(SL(2,\mathbb{C}))$ the $SL(2,\mathbb{C})$ flat connections modulo gauge)
- $M_{\alpha,f}$ is a component of the moduli space of flat $SL(2,\mathbb{C})$ connections which also comes with an integer f labelling
- $\tilde{M}_{\alpha,f}$ is without the quotient for the base point so just $Hom(\pi_1, SL(2,\mathbb{C})) \times \mathbb{Z}$
- $\Gamma_{\alpha,f,\theta}$ all the steepest descent trajectories with the function ? from $\tilde{M}_{\alpha,\theta}$. Middle dimensional in $\tilde{A}$. $k = \mid k \mid e^{i\theta}$

$$\begin{aligned}
\Gamma(SU(2)) &= \sum n_{\alpha\theta}\Gamma_{\alpha\theta} \\
I_{\alpha,\theta} &= \int_{\Gamma_{\alpha,\theta}} DAe^{2\pi ikS(A)} \\
Z &= \int_{\Gamma\subset\tilde{A}(SL(2,\mathbb{C}))} DAe^{2\pi ikS(A)} \\
&= \sum n_{\alpha,f,\theta}I_{\alpha,f,\theta} \sum_{\alpha,f} n_{\alpha,f,\theta}e^{2\pi ikS_\alpha}Z_\alpha^{pert}(k) \\
ns_\alpha &\equiv \sum_f n_{\alpha,f,0}
\end{aligned}$$

If $k \in \mathbb{N}_+$ there is no dependence on f for $I_{\alpha,f,\theta=0}$ so just say $e^{2\pi i k CS_\alpha} Z_\alpha^{pert}(k)$

$$Z(k \in \mathbb{N}_+) \quad = \quad \sum n_{\alpha,f,\theta=0} I_{\alpha,f,\theta=0} = \sum_\alpha n s_\alpha e^{2\pi i k CS_\alpha} Z_\alpha^{pert}(k)$$

For large k integers this tends to give $n_\alpha = 1$ for the $SU(2)$ connections and 0 for the ones with negative imaginary part. Those are the terms that would be $e^{+\#k}$ divergence.

17.2.1 Relation with BV

$\otimes_{\mathbb{R}} \mathbb{C}$ as a real algebra vs $\otimes_{\mathbb{R}} Cliff(\mathbb{R}, +)$ as a real superalgebra.

Chapter 18

Theory X

Has it's origin as the worldvolume theory for N $M5$ branes. But let's just take M theory and string theories as motivational rather than serious merit.

18.0.1 Definition (Theory X) *The worldvolume theory for a stack of $M5$ branes. It is parameterized by a real Lie algebra $\mathfrak{g}$ with invariant inner product b normalized such that coroots have length 2 and a full overlattice Γ of the coroot lattice Γ' such that the inner product is even integral on Γ.*
We don't know much more about it.

18.0.2 Lemma (ADE) *By taking an ADE Lie algebra, Killing form and $\Gamma = \Gamma'$. Some literature saying this is the only possibility (some extra assumption missing?)*

18.0.3 Example $\mathfrak{g} = L \otimes \mathbb{R}$ *an abelian Lie algebra and the lattice is L.*

18.0.4 Theorem (Nikulin 1.4.1) *Let L be even lattice. There exists a bijection between isotropic subgroups of D_L and even overlattices L_G of L. The discriminant form D_{L_G} is given by q_L restricted to $G^{\perp}/G$. Unimodular lattices L_G correspond to isotropic subgroups H with $\mid H \mid^2 = \mid D_L \mid$*

Proof `http://www2.warwick.ac.uk/fac/sci/maths/people/staff/fbouyer/talks/lattices_and_the_picard_group_presentation.pdf` □

18.1 Relation to Twisted Cohomotopy

The Sullivan minimal model for the rationalized S^4 (go all the way to real coefficents instead of just rational) is the following:

$$(S^4)_{\mathbb{R}} \quad \equiv \quad \mathbb{R}[\omega_4, \omega_7]/(d\omega_4 = 0, d\omega_7 = -\omega_4 \wedge \omega_4)$$

so when consider the map $X \to (S^4)_{\mathbb{R}}$ from the 11 dimensional target spacetime, we interpret that as a closed 4-form G_4 on X as well as a 7-form G_7 satisfying $d(2G_7) = -G_4 \wedge G_4$. That was without twist.

So this changes to a twisted version.

That was rationalized, what if it factored through a map to the actual 4-sphere, not $(S^4)_{\mathbb{R}}$? With the proper adjustments for twisted version, that is called hypothesis H.

18.1.1 Theorem *Hypothesis H implies that the action for the anomaly theory on closed 7 manifolds has action that lands in the integers rather than merely the reals. This means that for 6 manifolds, the procedure of pick a bounding 7-manifold and compute the action for that theory gives a well defined element of $\mathbb{R}/\mathbb{Z}$. In this way $e^{2\pi i S}$ will be well defined for any particular field configuration. So there is hope for it to quantize in a well defined manner.*

Proof `https://arxiv.org/pdf/1906.07417.pdf` □

18.2 Little String Theory

This is still part of the string theory assertions that we take as oracles for now.

18.2.1 Mina's 16/09/12

`https://www.youtube.com/watch?v=ZldWJtBUPGE`

Consider $\mathfrak{g}$ little string theory on $\Sigma \times \mathbb{R}^4$ where Σ is a flat Riemann surface.

18.2.1 Remark Do we need Σ to be connected? ◊

This little string theory is realized given by IIB on $Y \times \Sigma \times \mathbb{R}^4$ where Y is the ALE resolution of $\mathbb{C}^2/\Gamma$ for the finite group Γ given by the McKay correspondence. Section 20.1.

18.2.2 Theorem (Nakajima) *Y is hyperKahler, ALE Riemannian, has 2 cycles labelled by nodes of the Dynkin diagram. That is a basis of $H_2(Y, \mathbb{Z})$ This has inner product by intersection form. This is the root lattice. The $H_2(Y, \partial Y, \mathbb{Z})$ gives the weight lattice.*

Put $D5$ branes of IIB on $A \times p \times \mathbb{R}^4$ where A is a 2-cycle in Y. This then becomes a codimension 2 defect for the little string theory.

18.2.3 Remark Is it IIB on $Y \times X$ for other 6-manifolds? Just requires flatness? ◊

The effective field theory on the D5 branes is a quiver gauge theory. Fix the following quiver.

$$\begin{tikzcd}
\boxed{W_1} & \boxed{W_2} & \boxed{W_3} & \boxed{W_4} \\
V_1 \arrow[u] \arrow[r, dash] & V_2 \arrow[u] \arrow[r, dash] & V_3 \arrow[u] \arrow[r, dash] & V_4 \arrow[u]
\end{tikzcd}$$

where $\dim(V_a) = d_a$ and $\dim W_a = m_a$ which are determined by the classes in $H_2(Y, \mathbb{Z})$

18.2.4 Theorem (Claim) *Taking the partition function of this $\mathfrak{g}$ type little string theory on $\Sigma \times \mathbb{R}^4$ gives the equivariant K-theoretic instanton partition function for a $\mathfrak{g}$ type quiver gauge theory on $\mathbb{R}^4$. This quiver depends on the defect choices.*

18.2.5 Theorem (2.4 Nakajima Yoshioka) *http://arxiv.org/pdf/math/0505553v1.pdf*

18.2.6 Example (Single Puncture on C in limit) *Let $w_{0\cdots n}$ be a collection of $n+1$ weights which sum to 0 and are in the Weyl orbit of the fundamentals ω_i.*

18.2.7 Remark Dot or undot action ◊

18.2.8 Definition *Let $a_{i,F}$ be the equivariant parameters of homological degree 2 for G_F. Let $a_{j,G}$ be similar for G. Let $\epsilon_{1,2}$ be the ones for the rotations of $\mathbb{R}^4$. These are all given by choices of maximal tori. We have to give an excuse for why we don't care about conjugacy. There are also parameters τ which correspond to the moduli of Y's symplectic structure.*

18.2.9 Definition (Stable Basis)

18.2.10 Theorem (Claim-Elliptic Stable Basis) *Partition function of which 3d gauge theory on $T^2 \times I$ with the changed boundary conditions on either side. This gives the matrix element for the change of basis.*

18.2.2 Haozi/Schmid 2016/11/14 and 2017/08/30

18.2.11 Definition (Theory on a codimension 2 defect) *On the setup of type IIB on $\mathbb{C}^2/\Gamma \times T^2 \times \mathbb{R}^4$, blowup the ALE and make 2 cycles that correspond to some nonnegative integer combination of the nodes of the Dynkin diagram.*

18.2.12 Definition (Conjugacy classes of Levi subalgebras) *Take the Dynkin diagram for $\mathfrak{g}$ and then remove some subset of the nodes. Use the Cartan and only the e_α and f_α for the kept α nodes.*

18.2.13 Example (E_6)

18.2.14 Theorem *Little string goes to Theory X as take $\ell \to 0$*
Codimension 2 defect goes to 4d $\mathcal{N}=4$
Labelled by Levi vs labelled by nilpotent orbits via Bala-Carter

18.2.15 Theorem (Bala-Carter) *Nilpotent orbits in $\mathfrak{g}$ are labelled by Levi subalgebras up to conjugacy. Generally there is some extra data too.*

18.3 Theory X Proper

https://www.youtube.com/channel/UCFxegb9gYX5eVK3oSNcM_mw/playlists

Chapter 19

String Theories

A small chapter on some parts, that are useful to this more mathematical applications of SUSY field theories.

19.1 M Theory

19.2 F Theory

19.2.1 Definition (Elliptic Fibration) *A bundle of elliptic curves with singular curves along some divisors in the base B.*

19.2.2 Example (Elliptic Surface) *If the base B is a curve, there is no problem of intersecting divisors. Therefore the problem is more tractable. The singular fibers follow an ADE classification for the intersection matrix of the components. That is the intersection matrix of components is a Cartan matrix for an Affine Lie algebra.*

- *I_n for $n \geq 0$ with I_n for $n \geq 2$ corresponding to $\hat{A}_{n-1}$ affine Lie algebra where there are n self intersection points making the curve not smooth.*
- *${}_mI_v$ with $v \geq 0$ and $m \geq 2$*
- *II which has one cusp.*
- *III which has two components meeting in a common cusp. Corresponds to $\hat{A}_1$, degeneration when I_2 come together into a single cusp.*
- *IV in which 3 components all meet in a point corrsponds to $\hat{A}_2$*
- *I_v^* with $v \geq 0$ corresponding to $\hat{D_{4+v}}$*
- *IV^* corresponding to $\hat{E}_6$*
- *III^* corresponding to $\hat{E}_7$*
- *II^* corresponding to $\hat{E}_8$*

If we look at the monodromy of $H^1(E_t, \mathbb{Z})$ as t loops around the singular fiber, that gives specified matrices.

If we look at the smooth locus of the singular fiber, that still carries a group law like the elliptic curve group law on the other fibers. Those are also itemized in terms of above.

Chapter 20

Quiver Gauge Theory

20.0.1 Remark These are also called Moose diagrams because of the resemblance of some of the originally studied examples with the antlers of a moose. $\Diamond$

20.1 D-brane Motivation

Consider type II string theory on $\mathbb{R}^{1,p} \times \mathbb{R}^{9-p}/\Gamma$ with a Dp brane on $\mathbb{R}^{1,p} \times \{0\}$. Depending on the parity of p says IIA or IIB. Consider the low energy effective action for open strings ending on this Dp brane.

Move slightly off $\{0\}$ and get $\mid \Gamma \mid$ preimages on the Γ cover. Now have $\mid \Gamma \mid$ Dp branes coming together and the action of Γ acts on this open strings spaces from $x_i \to x_j$.

20.1.1 Constructing the quiver

The gauge group is $\prod_{i \in Irrep(\Gamma)} U(Nr_i)$ where $r_i = \dim R_i$

Let V be the vector representation of $Spin(9-p)$ and res_Γ it's restriction. Then in the category $\Gamma\ - mod$

$$R_i \otimes res_\Gamma(V) \quad \simeq \quad \bigoplus_j \mathbb{C}^{b_{ij}} \otimes R_i$$

Draw a quiver with vertices irreducibles and b_{ij} arrows between R_i and R_j.

Now to each vertex assign the vector space $\mathbb{C}^{Nr_i}$ as a Hermitian vector space not as a representation of Γ. The scalars will take values in the arrows. For the fermions make a similar quiver but tensor with S instead of with V.

$$\begin{aligned}
\phi &\in \bigoplus_{ij} \mathbb{C}^{b_{ij}} \otimes Hom(E_i, E_j) \\
\phi &\in Hom(\mathbb{C}^N \otimes \mathbb{C}^\Gamma, \mathbb{C}^N \otimes \mathbb{C}^\Gamma \otimes V)^\Gamma \\
\psi &\in Hom(\mathbb{C}^N \otimes \mathbb{C}^\Gamma, \mathbb{C}^N \otimes \mathbb{C}^\Gamma \otimes S)^\Gamma
\end{aligned}$$

If the group Γ is a subgroup of a special holonomy group $\Gamma \subset \{G_2, \text{etc}\} \subset Spin(9-p)$ then can get these two quivers to be the same. This is called supersymmetric orbifolding.

`http://scgp.stonybrook.edu/video_portal/video.php?id=1599`

20.1.1 Definition (Exceptional Collection) *An exceptional object has $Hom(E, E[k]) = 0$ for all $k \neq 0$ and $\mathbb{C}$ for $k = 0$. An ordered collection of these is called an exceptional collection if $RHom(E_j, E_k) = 0 \ \forall j > k$*

20.1.2 Definition (Mutation) *Define $L_E F$ and $R_F E$ by distinguished triangles*

$$L_E F \longrightarrow RHom(E, F) \otimes E \longrightarrow F$$

$$E \longrightarrow RHom(E, F)^* \otimes F \longrightarrow R_F E$$

Then define R_{E_i} of a collection $(E_0 \cdots E_n)$ by $E_0 \cdots E_{i-1}, E_{i+1}, R_{i+1} E_i \cdots E_n$ and similarly $L_{E_{i+1}}$ by $E_0 \cdots E_{i-1}, L_{E_i} E_{i+1}, E_i \cdots E_n$

20.1.3 Proposition *A mutation makes a new exceptional collection. If it generated the category, the mutation still does.*

20.1.4 Theorem (Bondal 90) *Take the direct sum of the objects in the exceptional collection of objects in a derived category. The endomorphism algebra of this is the path algebra of a corresponding quiver. This allows identification of a derived category of coherent sheaves and the of quiver representations.*

`https://books.google.com/books?hl=en&lr=&id=UHrhWUZykOEC&oi=fnd&pg=PA75&dq=bondal+helices+quiver+koszul+algebras&ots=b28WfhBpNl&sig=pH63bsRw5jpI_EJm8iv-h54pGIo#v=onepage&q=bondal%20helices%20quiver%20koszul%20algebras&f=false`

Chapter 21

Nekrasov Nonsense

21.0.1 Remark Credit/blame for this title goes to Vivek Shende in the Theory X conference. $\Diamond$

21.1 Instanton Partition Function

`http://quarks.inr.ac.ru/2008/proceedings/p5_FT/nekrasov.pdf`

21.1.1 Theorem (Donaldson) *Identify $\mathcal{M}(r, n)$ instantons moduli space with the moduli space of rank r holomorphic bundles on $\mathbb{CP}^2$ with given $c_2 = n$ and trivialized at the line $\mathbb{CP}^1$ at $\infty = [a, b, 0]$. Torsion free sheaves is bigger and gives a partial compactification.*

There is a class in $T \equiv (\mathbb{C}^*)^{2+r}$ equivariant cohomology that we are integrating.

21.1.2 Theorem (General Localization) *Letting $\mathcal{X}_n$ be an element of $H_T^\bullet(\mathcal{M}_n)$*

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} q^n \int_{\mathcal{M}_n} \mathcal{X}_n \\ &= \sum_{n=0}^{\infty} q^n \sum_{fix} \frac{\mathcal{X}_n(f)}{\prod w_i(f)} \end{aligned}$$

where $\mathcal{X}_n(f) \in H_T^\bullet(f = pt)$ and the $w_i \in \mathfrak{t}^*$. The denominator is in $Sym(\mathfrak{t}^*)$ a polynomial ring. For K_0 replace with $(1 - e^{-w_i(f)})$

21.1.3 Remark

21.1.4 Theorem *Let $\mathcal{M}_n$ be the Hilbert scheme of n points on $\mathbb{C}^2$ with $T = (\mathbb{C}^*)^2$ acting by action on the base. The fixed points are indexed by partitions which indicate the monomial ideals.*

$$
\begin{aligned}
\chi = \sum e^{w_i} &= \sum_{\square} q_1^{a(\square)+1} q_2^{-l(\square)} + q_1^{-a(\square)} q_2^{l(\square)+1} \\
q_1 q_2 = e^{\epsilon_1} e^{\epsilon_2} = 1 &\implies \chi = \sum_{\square} q_1^{a(\square)+1+l(\square)} + q_1^{-a(\square)-l(\square)-1} \\
&\implies \chi = 2 \sum_{\square} \cosh(\epsilon_1 (a+l+1))
\end{aligned}
$$

Comparing to `https://arxiv.org/pdf/0905.2555v3.pdf` *formula 2.5 gives* $t = q_2^{-1}$ *and* $q = q_1^{-1}$

21.2 SCGP Video

$\mathcal{N} = 2$ 4d SYM in the case of $N_f = 2N_c = 2L$. So $G = U(N_c)$

$$
\begin{aligned}
y + q\frac{AD}{y} &= T(x) \\
At &= y \\
At + qDt^{-1} &= (1+q)T_L
\end{aligned}
$$

where ADT are degree L polynomials in x

If quantize so $t = e^{\epsilon \partial_x}$ and act on $Q(x)$ get a Baxter relation for $SL(L)$ spin chain

$$
A(x)Q(x+\epsilon) + qD(x)Q(x-\epsilon) = (1+q)T_L(x)Q(x)
$$

21.3 4d to 2d

One way is to compactify on a torus and get a theory with infinitely many fields KK modes. Another way is to put on ϵ_1 background with $\epsilon_2 = 0$.

21.3.1 Definition (Twisted Effective Superpotential on Coulomb Branch) *For* $N = 2^*$ *with* $U(N_c)$

$$
\tilde{W}_{eff}(a, q = e^{i\tau} = e^{-\beta}, m, \epsilon) = \lim_{\epsilon_2 \to 0} \epsilon_2 \log Z_{total}
$$

$$
\tilde{W}_{eff} = \frac{\tilde{f}(a, \cdots)}{\epsilon} + \cdots
$$

The SW prepotential $\tilde{f}$ has a classical algebraic integrable system interpretation.

$$a_i^D = \frac{\partial \tilde{f}}{\partial a_i}$$

where the a_i are using A-cycles and a_i^D are using B-cycles.

21.3.2 Example *Pure $\mathcal{N} = 2$ $U(N)$ gauge theory broken low energy to $U(1)^N$ get periodic Toda chain with phase space $X^{2N} \simeq (\mathbb{C} \times \mathbb{C}^*)^N$ given by (p_i, e^{q_i})*

The a and a_D are deemed to be the action variables. They aren't independent so pick half of them.

21.3.3 Example 2^* *with adjoint massive hyper. Let $\tau_{bare} = \frac{4\pi i}{g_{bare}^2} + \frac{\theta_{bare}}{2\pi}$ be the microscopic coupling. Turns into elliptic Calagero-Moser with that τ*

Now quantize the integrable system. This is done by deforming the gauge theory.

$$\begin{aligned} H_2 &= \sum p_i^2 + m(m+\hbar)\sum \rho(q_i - q_j \mid i\tau) \\ H_2 &= \sum -\hbar^2 \frac{\partial^2}{\partial q^2} + m(m+\hbar)\sum \rho(q_i - q_j \mid i\tau) \end{aligned}$$

21.3.4 Definition (Ω deformation) *Write $\mathbb{R}^{3.1} = \mathbb{R}^2 \times \mathbb{R}^{1,1}$ and replace $\mathbb{R}^2$ with $\mathbb{R}^2_{\epsilon_1}$*

21.3.5 Definition (Effective twisted superpotential)

$$\frac{\partial W}{\partial a_i} = 2\pi i n_i$$

This gives the Bethe equations if identify W with the Yang-Yang function.

21.3.1 Full Deformation

Also deform the $\mathbb{R}^{1,1}$ with ϵ_2. Compute 4d partition function as a function of the Coulomb branch. Compute via localization.

$$\begin{aligned} Z(a_i, \epsilon_1, \epsilon_2, m, \tau) &= Z_{tree+1-loop} Z_{instanton} \\ Z^{inst} &= \sum_{\lambda^1 \cdots \lambda^N} f(a, \epsilon_1, \epsilon_2) q^{\sum |\lambda^1|} \\ q &= exp(2\pi i \tau) \end{aligned}$$

where f is a rational expression with integer coefficients. Poles when $a_j = a_i + p\epsilon_1 + q\epsilon_2$

In $\epsilon_2 \to 0$ limit $\epsilon_1 = \hbar$ we get

$$\begin{aligned}
\lim_{\epsilon_2 \to 0} Z &\approx exp(1/\epsilon_2 W + \cdots) \\
\lim_{\epsilon_{1,2} \to 0} W &\approx \frac{1}{\hbar}\mathcal{F}
\end{aligned}$$

21.4 Nekrasov Shatashvilli

21.4.1 Bethe/Gauge Correspondence 2d version

Supersymmetric Vacuaa of the two dimensional $\mathcal{N} = 2$ theories and stationary eigenstates of quantum integrable system.

$$\{Q_\pm, \bar{Q}_\pm\} = 2(H \pm P)$$

21.4.1 Definition (Twisted Chiral Ring) *Operators which anticommute with $\mathcal{Q}_A = Q_+ + \bar{Q}_-$ which has $\{\mathcal{Q}_A, \mathcal{Q}_A^\dagger\} = H$*

21.4.2 Example *In the absence of massless charged matter fields, we may give $\frac{1}{k!(2\pi i)^k} \operatorname{tr} \sigma^k$ where σ is scalar of the vector multiplet.*

21.4.3 Definition (Chiral Ring) *Operators which anticommute with $\mathcal{Q}_B = Q_+ + Q_-$ which has $\{\mathcal{Q}_B, \mathcal{Q}_B^\dagger\} = H$*

The local operators $\mathcal{O}(x)$ in the twisted chiral ring are independent of x up to Q_A commutators.

$$\mathcal{O}(x) = \mathcal{O}(y) + \{Q_A, -\}$$

So ignoring Q_A exactness we get a commutative associative ring, by picking arbitary points for the insertions, doing an OPE and then ignoring the position again.

$$\begin{aligned}
\mathcal{O}_i &\equiv [\mathcal{O}_i(x)] \ \forall x \\
\mathcal{O}_j &\equiv [\mathcal{O}_j(y)] \ \forall y \\
\mathcal{O}_i\mathcal{O}_j &= [\mathcal{O}_i(x)\mathcal{O}_j(y)]
\end{aligned}$$

More properly, we have an $Chains(E_2)$ algebra, but this is only looking at the degree 0 operations and only up to homology for now.

The cohomology of the operator $\mathcal{Q}_{A/B}$ is then identified with a Hilbert space for a quantum integrable system.

21.4.4 Definition (Yang-Yang Function)

$$\frac{1}{2\pi i}\frac{\partial Y}{\partial \lambda_i} = n_i$$

gives the Bethe equations.

21.4.5 Example (Toda)

21.4.6 Theorem (Lebedev) *The relativistic Toda lattice has an associated* $MD(U_q\mathfrak{sl}_2(\mathbb{R}))$

21.4.7 Example ($N = 2$)

$$\begin{aligned} H &= e^x + e^{-x} + R^2(e^p + e^{-p}) \\ \tilde{H} &= e^{\tilde{x}} + e^{-\tilde{x}} + \tilde{R}^2(e^{\tilde{p}} + e^{-\tilde{p}}) \\ \tilde{p} &= \frac{2\pi}{\hbar}p \\ \tilde{x} &= \frac{2\pi}{\hbar}x \\ \tilde{\hbar} &= \frac{4\pi^2}{\hbar} \\ \tilde{R} &= R^{2\pi/\hbar} \end{aligned}$$

21.4.8 Example (XYZ)

21.4.2 Bethe/Gauge Correspondence 4d version

Chapter 22

3d Theories

22.1 N=4 SUSY gauge theory

Consider a compact Lie group G and a complex representation M equipped with an antilinear j with $j^2 = -1$. This gives the vector space M an action of the quaternions.

In particular let $M = N \bigoplus N^*$ with $G \to Symp(N \bigoplus N^*)$

- $A_\mu \in$ G-conn
- $\sigma \in \mathfrak{g}$
- $\phi \in \mathfrak{g} \otimes \mathbb{C}$
- $X \in N$
- $Y \in N^*$

22.1.1 Definition (Moduli space of Vacuaa)

$$\begin{aligned} [\sigma, \phi] &= 0 \\ [\phi, \phi^\star] &= 0 \\ \mu_{\mathbb{R}}(X,Y) = \mu_{\mathbb{C}}(X,Y) &= 0 \\ \phi(X) = \phi(Y) &= 0 \\ \sigma(X) = \sigma(Y) &= 0 \end{aligned}$$

22.1.2 Definition (Coulomb branch) *Set X and Y to 0. The first two equations give $\sigma \in \mathfrak{t}$ $\phi \in \mathfrak{t}_{\mathbb{C}}$. The T is the stabilizer of a generic σ ϕ pair. These leftover abelian gauge fields give this branch it's name. Turn the gauge field to a scalar by dualizing. All together this gives $\mathfrak{t}_{\mathbb{C}} \times T^\vee$ modulo Weyl. Think of this as $T^*(T^\vee) \otimes_{\mathbb{R}} \mathbb{C}$ before the Weyl quotienting where the tensoring is happening in the fiber.*

22.1.3 Definition (Quantum Corrections) *We don't get the classical coulomb branch $T^*T^\vee/W$ but instead something merely birational to it.*

22.1.4 Definition (Braverman-Finkelberg-Nakajima) *When the symplectic representation is $N \bigoplus N^*$ as we have stated all along. $M_{Coulomb,G,N}$ is an affine algebraic variety Spec $A_{G,N}$ of an algebraic Poisson algebra. Generically $M_{C,G,N}$ is symplectic.*

22.1.5 Theorem *The map to fibers has Lagrangian fibers and allows definition of an integrable system.*

Proof $A_{G,N}$ is defined as $H^{BM}_\bullet$ of $\mathcal{S}_{G,N}$ with convolution product which turns out to be commutative. It is Poisson because it has a noncommutative deformation by taking equivariant homology $H^{\mathbb{C}^*}_\bullet$. Taking the equivariant parameter to 0 leaves a Poisson bracket as the remnant of noncommutativity at first order.
The Poisson commuting integrals of motion come from a map $H^{\mathbb{C}^*}_\bullet(???) \to H^{\mathbb{C}^*}_\bullet(S_{G,N})$. This is a commutative algebra in the deformation which becomes a Poisson commuting algebra inside the Poisson algebra. $\square$

22.1.6 Example ($N = 0$) *$\mathcal{S}_{G,0}$ is $G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})$. Or bundle over disc with two origins together with a section. This then gives $A_{G,0} = H^{G(\mathcal{O})}_\bullet(AfGr_G)$ so $M_{C,G,0}$ is the BFM (Bezrukavnikov-Finkelberg-Mirkovic) space for ^LG*
In particular for $G = SU(N)$, this gives the space of $SU(2)$ monopoles of charge N.

22.1.7 Example ($N = 0$ and $G = T$) *In this case get $T^*T^\vee$ which is birational but not equal to the classical Coulomb branch $T^*T^\vee/W$*

22.1.8 Example ($N = Adj_G$) *No quantum corrections just $T^*T^\vee/W$*

22.1.9 Example ($N = \mathbb{C}$ and $G = \mathbb{C}^*$) *Let G act with weights k.*

22.1.10 Example ($N = \mathbb{C}^k$ and $G = \mathbb{C}^*$) *Let G act with weights 1 on all factors.*

22.1.11 Definition (Flavored Coulomb branch) *If $G_F \subset Aut_G(N)$ then deformation space over $\mathfrak{t}_F/W_F$. That is the fibers are deformations of $M_{C,G,N}$ also of the Poisson algebras $A_{C,G,N}$*

In $4d$ version of $M_{C,G,N}$ get $(T \times T^\vee)/W$ instead so Lie algebra to group when you go from 3 to 4 dimensions. Again this deforms over T_F/W_F and gives a deformation quantized algebra over $\mathbb{C}[q, q^{-1}]$. Can attempt to change complex structure over all of $\mathbb{P}^1$. In some points, the complex manifold changes and in fact stops being affine. It should become the Seiberg-Witten integrable system.

22.1.12 Example ($N = 0$) *$N = 0$ gives universal centralizer in $\mathfrak{g}^L_{\mathbb{C}}$ so given by pairs (x, g) where x is regular and $gxg^{-1} = x$ all up to conjugation. In case of $GL(n)$ also interpret as $SU(2)$ framed monopoles on $\mathbb{R}^3$ with charge n by Donaldson or quasimaps $\mathbb{P}^1 \to \mathbb{P}^1$ of degree n.*

22.1.13 Example (Quiver) *Take oriented quiver Q and form $Q^\heartsuit$ the framed quiver. $G_Q = \prod GL(V_i)$ and $N = \bigoplus Hom(V_i, V_j) \bigoplus \bigoplus Hom(V_i, W_i)$. Here G_F can be taken to be all of $\prod GL(W_i)$ from the framing vertices. M_C then becomes singular monopoles for G_Q.*
If W_i are all 0 nonsingular.

22.1.14 Theorem (Monopole Equation)

22.1.15 Definition (Higgs Branch) *Set σ and ϕ to 0. Then the equations become imposing the HKLR moment maps quotiented by G. In other words $R \bigoplus R^* ///G$ or equivalently $R \bigoplus R^* //G_{\mathbb{C}}$.*

22.1.16 Definition (Mixed Branch) *Neither the Coulomb nor Higgs special cases. These are other irreducible components of the full moduli space of vacua. Think of $xyz = 0$ which is reducible. We have looked at 2 of the irreducible components but there is one other left.*

22.2 Rozansky-Witten

22.2.1 Courant Sigma Model

`https://arxiv.org/pdf/0906.3167v2.pdf`

Use AKSZ Sigma model $T[1]X \to T^*[2]T^*[1]M$ for the hyperKahler manifold M. Upon a certain gauge fixing and a degree 2 parameter modification this becomes the Rozansky-Witten model.

`https://arxiv.org/pdf/0911.0993.pdf` Let $L \bigoplus L^*$ be given the structure of a Lie bialgebroid by the given algebroid on L and the zero anchor/bracket one on L^*. For example, we can say that L is TP and L^* is T^*P but we have scaled the Poisson bivector making the anchor/bracket to 0. Now make this into a Courant algebroid on $L \bigoplus L^*$. If you put Courant with this target $T^*[2]L[1]$ on $\Sigma \times I$ and dimensionally reduce with $L[1]$ boundary conditions you get 2d AKSZ model with target $T^*[1]L^*$.
Boundary conditions for this 3d theory are given by $N^*[2]K[1]$ for K a subalgebroid of L. For the 2d theory they are $N^*[1]C$ for C a coisotropic in L^* which might be $N^*[1]K^\perp$ where K is a subalgebroid of L

B-model can be made by using a complex Lie algebroid L which comes from an ordinary complex structure. Similarly for a symplectic structure to make A-model. Either way the target is something modeled on T^*L^* where the Poisson structure on L^* is constructed differently.

22.2.2 With a Holomorphic Symplectic Manifold

Insert Reference for below

22.2.1 Theorem (In quotes) *Let $M = T^*Y$ for Y a complex manifold. Then we dimensionally reduce on a circle and get the 2d TQFT giving the B-model of Y. This gives us $Maps(T[1](\Sigma \times S^1), T^*[2]T[1]T^*Y)$.*

For mirror symmetry, we would want to compare to the Poisson Sigma model with $Y^\vee$ so $Maps(T[1]\Sigma, T^*[1]Y^\vee)$. These are very different but remember we only want a equivalence of derived categories so it's not totally hopeless.

22.2.2 Example *Suppose we look at a nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$. By Kronheimer, we can identify this with a moduli space of monopoles and it is hyperkahler (almost all). For the entire nilpotent cone, we can take the Springer resolution $T^*(G/B) \to \mathcal{N}$.*

22.2.3 Kapranov Perspective

22.2.3 Definition (Weight System)

22.3 3d theories labelled by 3-manifolds

22.3.1 Definition ($T^{DGG}(M)$)

22.3.2 Conjecture *For the S^3_b as $b^2(x^2+y^2)+\frac{1}{b^2}(z^2+w^2)=1$ and a hyperbolic manifold M then*

$$\Re \log Z_{T[M]}(S^3_b) \approx \frac{-1}{2\pi b^2} vol(M) + O(\frac{1}{b})$$

22.4 3D Mirror Symmetry

Chapter 23

Class S Theories

23.0.1 Definition ($S_{\mathfrak{g},\Sigma}$) *4D theories that are obtained from compactifying theory X on a surface Σ.*

23.0.2 Example ($\Sigma = T^2$) *This gives an $\mathcal{N} = 4$ theory with parameter τ.*

23.0.3 Lemma *The mapping class group of Σ acts as "dualities" for these theories. For example, the $SL(2, \mathbb{Z})$ in the case of T^2*

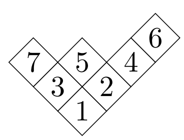
Chapter 24

Symmetric Functions Appendix

http://www.math.umn.edu/~reiner/Classes/HopfComb.pdf

24.1 Combinatorics

24.1.1 Definition ((Semi)Standard Tableaux) *Standard means that the labels $\{1 \cdots n\}$ are strictly increasing in rows and columns. For semi-standard it goes to $\mathbb{N}$ and only has to be weakly increasing along each row. Turn this into the fermionic (Russian) way of drawing the diagram.*

24.1.2 Example (Standard) 

24.1.3 Theorem

$$| \, SYT(\lambda) \, | \quad = \quad \frac{n!}{\prod_{x \in \lambda} h(x)}$$

24.1.4 Definition ((Co)charge) *The cocharge of a tableaux T with content μ is the unique integer invariant under jeu-de-taquin slides, 0 for single row T and under swapping disconnected pieces goes like $cc(T_{old}) = cc(T_{new}) - \mid X \mid$ where X is what used to be above and left in French notation.*

Charge is $n(\mu) - cc(T)$

24.1.5 Example *The following diagram has cocharge 9.*

1	2	4	6
3	5		
7			

```
Tableau([[1,2,4,6],[3,5],[7]]).cocharge()
```

24.1.6 Definition (Kostka $K_{\lambda\mu}$ Polynomials)

$$\begin{aligned} K_{\lambda\mu} &= \sum_T t^{c(T)} \\ \tilde{K}_{\lambda\mu} &= \sum_T t^{cc(T)} \end{aligned}$$

*This can also be interpreted in terms of rigged configurations which are in bijection with semistandard tableaux. It interchanges weight of configuration with charge of tableaux. This enumerates Behte vectors in the $GL_?$ $X??$ model with $V =$ by magnon number.[**?**]*

24.1.7 Example (Forests)

24.1.8 Example (Parking Functions)

24.2 Hurwitz/Dijkgraaf

24.2.1 Theorem (Hurwitz)

$$\begin{aligned} 2g-2 &= d(2h-2) + \sum e_p + d - length(\lambda) \\ e_p &= d-1 \\ \mid P \mid &= 2N \\ 2g-2 &= (d-1)*(2*N) + d - length(\lambda) + d(2h-2) \\ g &= (d-1)*N + \frac{d - length(\lambda)}{2} + 1 + d(h-1) \\ g &= (d-1)*(N-1) + (d-1) + \frac{d - length(\lambda)}{2} + 1 + d(h-1) \\ g &= (d-1)*(N-1) + \frac{d - length(\lambda)}{2} + d*h \end{aligned}$$

24.2.2 Theorem

$$\begin{aligned} \lambda &\in Partition(d) \\ r &= (1-2h)d + length(\lambda) + 2g - 2 \\ 2g-2 &= r - length(\lambda) + d(2h-2) + d \\ 2g-2 &= d(2h-2) + r + d - length(\lambda) \end{aligned}$$

Set $d - length(\lambda) = e_\infty$ for a marked point so we can call that a point where there may be additional ramification. Usually we will set $\lambda = 1^d$ so that there is no ramification there.

24.2.3 Theorem (Dijkgraaf) *Let $N_{g,d}$ count the number of degree d covers of an elliptic curve $h = 1$ ramified simply ($e_p = 1$) at a given set of $r = \sum e_p = 2g-2$ distinct points with $g \geq 2$ so there is some ramification somewhere. This is the way it has to be if $\lambda = 1^d$ the marked point has no ramification. This gives a genus g curve with map to our original E. This defines a groupoid whose* *objects are covers and morphisms are automorphisms of covers. Take the groupoid cardinality of this and call it* $N_{g,d}$. $\sum_{n\geq 1} N_{g,d}q^d$ *is a quasimodular form of weight* $6g-6$.

If we were taking 1 *instead of* $1/\operatorname{Aut}$, *this would be the partition function for the trivial theory on all those covering surfaces. Note this was asking for points of ramification* 1 *rather than* $d-1$. *This corresponds to elementary transpositions vs the long cycle.*

24.2.4 Theorem (Bloch/Okounkov) *http://people.mpim-bonn.mpg.de/zagier/files/doi/10.1007/s11139-015-9730-8/bloch-okounkov.pdf https://arxiv.org/abs/alg-geom/9712009*

24.3 Hopf Algebra of Symmetric Functions

24.3.1 Theorem (Exp Algebra) *Let CRing* $\to$ *Ab be the functor sending* R *to the formal power series* $R[[t]]$ *with constant term 1 and multiplication. Adjoint to this is a functor Ab* $\to$ *CRing. Applying this to* $\mathbb{Z}$ *gives* Λ. *In particular let it be the commutative ring generated by symbols* $\forall g \in G \;\; g_i$ *such that*

$$\begin{aligned}
e^{gt} &= 1 + g_1t + g_2t^2 + \cdots \\
e^{gt}e^{ht} &= e^{(g+h)t} \\
(1 + g_1t + g_2t^2 + \cdots)(1 + h_1t + h_2t^2 + \cdots) &= 1 + (g_1 + h_1)t + (g_2 + g_1h_1 + h_2)t^2 + \cdots \\
(g+h)_1 &= g_1 + h_1 \\
(g+h)_2 &= (g_2 + g_1h_1 + h_2)
\end{aligned}$$

This can be given a Hopf algebra structure by saying

$$\begin{aligned}
\Delta e^{gt} &= e^{gt} \otimes e^{gt} \\
\Delta g_1 &= g_1 \otimes 1 + 1 \otimes g_1 \\
\Delta g_2 &= 1 \otimes g_2 + g_1 \otimes g_1 + g_2 \otimes 1 \\
\Delta g_3 &= 1 \otimes g_3 + g_1 \otimes g_2 + g_2 \otimes g_1 + g_3 \otimes 1 \\
\Delta g_n &= \cdots \\
S(e^{gt}) &= e^{-gt} \\
S(g_1) &= -g_1 \\
S(g_2) &= g_2 \\
S(g_n) &= \cdots
\end{aligned}$$

24.3.2 Example *So we have* g_i *generators.*

$$\begin{aligned}
(ng) = (g + \cdots_n + g)_1 &= g_1 + \cdots + g_1 \\
(ng)_2 &= n * g_2 + \binom{n}{2} * g_1g_1
\end{aligned}$$

24.3.3 Theorem

$$\begin{aligned}
E(t) &= 1 + e_1 t + e_2 t^2 + \cdots \\
H(t) &= 1 + h_1 t + h_2 t^2 + \cdots \\
E(-t)H(t) &= 1 \\
\sum_{i+j=n} (-1)^i e_i h_j &= \delta_{0,n} \\
S(e_n) &= (-1)^n h_n \\
S(h_n) &= (-1)^n e_n
\end{aligned}$$

24.3.4 Theorem

$$\begin{aligned}
\omega(p_\lambda) &= \\
\omega(e_\lambda) &= \\
\omega(h_\lambda) &= \\
\omega(f_\lambda) &= \\
\omega(s_\lambda) &= \\
\omega(s_{\lambda\backslash\mu}) &= s_{\lambda^t\backslash\mu^t} \\
S(s_{\lambda\backslash\mu}) &= (-1)^{|\lambda\backslash\mu|} s_{\lambda^t\backslash\mu^t}
\end{aligned}$$

24.3.5 Definition ($\boldsymbol{\lambda}$ operations)

$$\lambda^n(e_1) = e_n$$

24.3.6 Definition (Hall Inner Product) *There is an inner product on each graded piece given by*

$$\begin{aligned}
\langle s_\lambda \mid\mid s_\mu \rangle &= \delta_{\lambda,\mu} \\
\langle p_\lambda \mid\mid p_\mu \rangle &= z_\lambda \delta_{\lambda,\mu} \\
z_\lambda &= \prod_{i \geq 1} i^{m_i} m_i! \\
\langle h_\lambda \mid\mid m_\mu \rangle &= \delta_{\lambda,\mu}
\end{aligned}$$

24.3.7 Theorem (Dyson Constant Term conjecture) *For the C(O/U/S)E*

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \prod \mid e^{i\theta_j} - e^{i\theta_k} \mid^\beta d\theta_1 \cdots d\theta_n$$

Then for the q,t deformations:

$$\int_0^{2\pi} \cdots =$$

24.3.8 Definition (MacDonald Inner Product)

$$
\begin{aligned}
\langle p_\lambda \mid\mid p_\mu \rangle_{q,t} &= z_\lambda \delta_{\lambda,\mu} \\
z_\lambda &= \prod_{i\geq 1} i^{m_i} m_i! \prod_{i=1}^{l(\mu)} \frac{1-q^{\mu_i}}{1-t^{\mu_i}} \\
\omega_{q,t} p_\mu &= (-1)^{|\mu|+l(\mu)} p_\mu \prod_{i=1}^{l(\mu)} \frac{1-q^{\mu_i}}{1-t^{\mu_i}}
\end{aligned}
$$

So that $\omega_{q,t}$ is symmetric operator (defined on the dense linear span of power sums) for this inner product. It's inverse is manifestly $\omega_{t,q}$ by looking on p_μ basis.

24.3.9 Corollary *The norm for Jack polynomials given by setting $t = q^\alpha$ and sending $q \to 1$.*

$$
\begin{aligned}
\langle J^\alpha_\lambda \mid\mid J^\alpha_\mu \rangle &= \\
\lim_{t=q^\alpha, q\to 1} \langle p_\lambda \mid\mid p_\mu \rangle &= \delta_{\lambda,\mu} \lim_{q\to 1} \prod_{i\geq 1} i^{m_i} m_i! \prod_{i=1}^{l(\mu)} \frac{1-q^{\mu_i}}{1-q^{\alpha\mu_i}} \\
? &=
\end{aligned}
$$

24.3.10 Lemma (Cauchy Identities)

$$
\begin{aligned}
\sum_\lambda t^{|\lambda|} s_\lambda(X) s_\lambda(Y) &= \prod (1 - t x_i y_j)^{-1} \\
\sum_\lambda h_\lambda(X) m_\lambda(Y) &= \prod (1 - x_i y_j)^{-1} \\
\sum_\lambda z_\lambda^{-1} p_\lambda(X) p_\lambda(Y) &= \prod (1 - x_i y_j)^{-1}
\end{aligned}
$$

24.3.11 Theorem (Zelivinsky's Theorem of positive self-dual Hopf algebras) *If a graded connected Hopf algebra A over a field of characteristic 0 has $I = \mathfrak{p} \bigoplus I^2$, then the inclusion $\mathfrak{p} \to A$ extends to a Hopf algebra isomorphisms $Sym\mathfrak{p} \simeq A$. In particular A is both commutative and cocommutative.*

Proof `https://arxiv.org/pdf/1409.8356.pdf` page 70 □

24.4 Character Theory

24.4.1 Theorem *There is a $\mathbb{Q}$ graded algebra isomorphism*

$$
\begin{aligned}
F : \bigoplus K_0(Rep(S_n)) \otimes_{\mathbb{Z}} \mathbb{Q} &\to \Lambda \\
F_A(z) &= \frac{1}{n!} \sum_{w \in S_n} \chi^A(w) p_{\tau(w)}(z)
\end{aligned}
$$

where $\tau(w)$ is the cycle type giving a partition of n. The left hand side has inner product by inner product of characters and multiplication by $[A]\cdot[B] = [Ind_{S_n\times S_m}^{S_{n+m}} A\boxtimes B]$

This extends to taking representations of S_n in $Vect_A$ for a locally compact abelian group A with Pontryagin dual A^*. For example, if $A=\mathbb{Z}^2$.

$$\begin{aligned} F_A(z,q,t) &= \sum F_{A_{r,s}}(z)\otimes q^s t^r \ \in \widehat{\Lambda\otimes_{\mathbb{Q}}\mathbb{C}(q,t)} \\ (q,t) &\in Hom(A,U(1))\simeq (U(1))^2 \\ q^r t^s &\in \Gamma(A^*,\mathcal{O}_{A^*}) \end{aligned}$$

24.5 Categorical/Species

24.5.1 Definition (Plethysm) *To do a plethysm of f by $A\in\Lambda\otimes_{\mathbb{Q}}\mathbb{Q}(q,t)$, do the following:*

$$\begin{aligned} f &= \sum f_\lambda(q,t)p_\lambda \\ f[A] &= \sum f_\lambda(q,t)p_{\lambda_1}[A]\cdots p_{\lambda_n}[A] \\ p_k[A] &= A\mid_{q\to q^k,t\to t^k,z_i\to z_i^k} \\ p_k[z_1+z_2\cdots] &= p_k(z) \\ p_k[-z_1-z_2\cdots] &= -p_k(z) \\ p_k[tz_1+tz_2+\cdots] &= t^k p_k(z) \\ p_k[\frac{1}{1-t}Z] &= p_k(z)\frac{1}{1-t^k} \\ p_k[(1-t)Z] &= (1-t^k)p_k(z) \\ p_k[\frac{1-q}{1-t}Z] &= p_k(z)\frac{1-q^k}{1-t^k} \end{aligned}$$

24.5.2 Definition (Schur Functor) *Let R_n be a representation of S_n, then*

$$S_{R_n}(-) = R_n\otimes_{\mathbb{C}[S_n]}(-)^{\otimes n}$$

Or if we have such for each $n\geq 0$, but only finitely many nonzero then we get

$$S_R(-) = \bigoplus_{n\geq 0} R_n\otimes_{\mathbb{C}[S_n]}(-)^{\otimes n}$$

24.5.3 Example

- $V\to V^{\otimes n}$

- $V \to F(V) \bigoplus G(V)$

- $V \to F(V) \otimes G(V)$

- $V \to F(G(V))$

24.5.4 Definition (Schur) *The category with objects Schur functors and morphisms are natural transformations. R as described is a functor $core(FinSet) \to Vect$ by sending $n \to R_n$ and the S_n worth of morphisms $n \to n$ to make it an S_n representation. That is representations of this groupoid. But the $R_N = 0$ restriction means just polynomial species.*

24.5.5 Definition (Schur) *Let $\mathcal{C}$ be a symmetric monoidal Cauchy complete linear category.*

Define Schur functors in $\mathcal{C} \to \mathcal{C}$ by

$$\begin{array}{rcl} p_\lambda :\ V^{\otimes n} & \to & V^{\otimes n} \\ S_\lambda(V) :\ V & \to & coker(p_\lambda) \end{array}$$

$$\begin{tikzcd} X^{\otimes n} \arrow[r] \arrow[d, "f^{\otimes n}"] & S_\lambda(X) \arrow[d, "S_\lambda(f)"] \\ Y^{\otimes n} \arrow[r] & S_\lambda(Y) \end{tikzcd}$$

For $R = \bigoplus V_{\lambda_i}$ do the direct sum of the above $\bigoplus S_{\lambda_i}(-)$

In particular, consider a morphism $f\ X \to X$ and take $S_\lambda(f)$. It is an element of $Hom_{\mathcal{C}}(S_\lambda(X), S_\lambda(X)) \in Obj(\mathcal{E})$ where $\mathcal{E}$ is the enriching category.

24.5.6 Example *For the category of $SVect$*

24.5.7 Example *For the category of $gr - Vect$*

24.5.8 Example *For the category of $Rep_{fd}(\mathbb{Z}^2)$*

24.5.9 Example *For the category of $Vect_{sm}(T^2)$*

24.5.10 Example *For the category of $Vect_{alg}(\mathbb{G}_m^2)$*

24.5.11 Theorem *Plethysm and composition of Schur Functors*

24.6 Hopf Algebra of Quasi-Symmetric Functions

https://arxiv.org/pdf/1003.2124v1.pdf

24.6.1 Definition (*QSym*) *The quasisymmetric functions over the same alphabet x_i are those such that the coefficient of $x_{i_1}^{a_1} \cdots x_{i_l}^{a_l}$ and $x_{j_1}^{a_1} \cdots x_{j_l}^{a_l}$ are equal whenever both $i_1 < \cdots < i_l$ and $j_1 < \cdots < j_l$.*

24.6.2 Theorem *A basis for QSym is given by the monomial quasisymmetric functions for a composition α*

$$M_\alpha = \sum_{i_1 < \cdots < i_l} x_{i_1}^{\alpha_1} \cdots x_{i_l}^{\alpha_l}$$

If the number of variables is finite only use compositions of length $\leq | I |$. This is also graded by letting $Comp_n$ be compositions of n and then using the span of those M_α to give $QSym_n$. That is the same as giving all of the variables homological degree 2 and then looking at the piece with homological degree $2n$. We should really call it $QSym_{2n}$.

24.6.3 Corollary *An alternative basis is the fundamental basis F_α which can be expressed as $\sum_{\beta \leq \alpha} M_\beta$ where $\beta \leq \alpha$ means that α is obtained from β by adding together adjacent parts.*

24.7 Graded Dual Hopf Algebra of $QSym$

24.7.1 Definition (*NSym*)

24.8 Hall Algebra

24.9 Macdonald Polynomials

24.9.1 Proposition

$$F_A[Z(1-t)] = \sum_{k \geq 0} (-t)^k F_{\wedge^k \mathbb{C}^n \otimes A}(z;t)$$

$$F_A[Z(1-t)] = \sum_{k \geq 0} (t)^k F_{S^k \mathbb{C}^n \otimes A}(z;t)$$

24.9.2 Definition

$$\begin{aligned}
\langle f \mid\mid g \rangle_{0,t} &= \langle f \mid\mid g[Z\frac{1}{1-t}] \rangle \\
\langle f \mid\mid g \rangle_{q,t} &= \langle f \mid\mid g[Z\frac{1-q}{1-t}] \rangle \\
\langle p_k \mid\mid p_l \rangle_{q,t} &= \langle p_k[Z\frac{1-q^{1/2}}{1-t^{1/2}}] \mid\mid p_l[Z\frac{1+q^{1/2}}{1+t^{1/2}}] \rangle_H \\
\langle P_\lambda \mid\mid P_\lambda \rangle_{q,t} &= \frac{h'_\lambda}{h_\lambda} \\
&= \frac{\prod(1-q^{a+1}t^l)}{\prod(1-q^a t^{l+1})}
\end{aligned}$$

24.9.1 Relation to Quantum Symmetric Spaces

24.9.3 Theorem (Quantum Symmetric Spaces) $A_q(G)$ $A_q(T)$ $\mathbb{C}(t_1 \cdots t_n)$

Proof Noumi, Letzter □

24.9.2 Jack Polynomials

24.9.4 Definition (Jacks) *The unique elements of $\Lambda \otimes \mathbb{Q}(\alpha)$ which are orthogonal, triangular with respect to the monomial symmetric functions*

$$J_\lambda = \sum_\mu v_\lambda^\mu(\alpha) m_\mu$$

and normalized with coefficient $n!$ for $\mu = 1^n$.

24.9.5 Theorem *They are the homogenous polynomial eigenfunctions for the operator $\sum x_i^2 \partial_i^2 + \frac{2}{\alpha} \sum \frac{x_i^2}{x_i - x_j} \partial_i$*

24.9.6 Theorem (Fractional Quantum Hall Wavefunctions)

24.9.3 Geometric Understanding

24.9.7 Definition (Taub-Nut Spacetime) *Topology of $\mathbb{R} \times S^3$ given a Lorentzian metric by*

$$\begin{aligned}
ds^2 &= -dt^2 \frac{1}{U(t)} + 4\ell^2 U(t)(d\psi + \cos\theta d\phi)^2 + (t^2+\ell^2)(d\theta^2 + \sin^2\theta d\phi^2) \\
U(t) &= \frac{2mt + \ell^2 - t^2}{t^2 + \ell^2}
\end{aligned}$$

which is a solution of vacuum Einstein equations. This is related to $\mathbb{C}^2/\Gamma$ by

24.9.8 Theorem (Haiman)

24.9.4 Koornwinder Polynomials

Above we were in a type A_n/periodic mindset, but let us move to a BC_n/open mindset.

24.10 $n!$ and diagonal coinvariants

24.10.1 Definition (Diagonal Coinvariants)

$$\begin{aligned} I &= \mathbb{Q}[x_1 \cdots x_n, y_1 \cdots y_n]^{S_n} \prod (x, y) \\ R_n &= \mathbb{Q}[x_1 \cdots x_n, y_1 \cdots y_n]/I \\ R_n &= \bigoplus_{r,s} (R_n)_{r,s} \\ \dim R_n &= (n+1)^{n-1} \\ \mathcal{F}(z, q, t) &= \sum_{r,s} q^r t^s F char(R_{n,r,s}) \end{aligned}$$

where r and s are weights for $x_i \to \lambda x_i$ and $y_j \to \mu y_j$. So just count x's and y's.

24.10.2 Example ($n = 2$) *The classes of 1 x_2 and y_2 form a basis of R_2. They are in degrees $(0,0)$, $(1,0)$ and $(0,1)$.*

24.10.3 Theorem *Let ∇ be diagonal in modified Macdonald basis.*

$$\begin{aligned} \nabla H_\mu &= t^{n(\mu)} q^{n(\mu^T)} H_\mu \\ n(\mu) &= \sum (i-1)\mu_i \\ \mathcal{F} &= \nabla e_n(z) \\ \langle \nabla e_n \mid\mid e_n \rangle &= C_n(q, t) \\ C_n(1,1) &= C_n = \frac{1}{n+1}\binom{2n}{n} \end{aligned}$$

24.10.4 Definition (Procesi bundle) *For a partition μ label the cells by (p_j, q_j) coordinates written in French notation. Let $x_1 \cdots x_n$ and $y_1 \cdots y_n$ be indeterminates.*

$$\begin{aligned} \Delta_\mu &= \det x_i^{p_j} y_i^{q_j} \\ \mathcal{L}_\mu &\equiv \bigoplus \langle \partial_{x_s}^{a_s} \partial_{y_t}^{b_t} \Delta_\mu \rangle \end{aligned}$$

for all a and b. So all partial derivatives in all variables. Then have taken linear span of these. Δ_μ is only defined up to sign because there is no $1 \cdots n$ ordering on the boxes and changing any given ordering permutes the columns of the matrix.

This generalizes the Vandermonde matrix and it's determinant. $\Delta_{\mu=1^n}$ gives the Vandermonde in the x_i variables. $\Delta_{\mu=(n)}$ gives the Vandermonde in the y_i variables.

24.10.5 Theorem *The dimension of this is $n!$*

Proof This is isomorphic as a bigraded module to $R_\mu \equiv \mathbb{C}[x_1 \cdots x_n, y_1 \cdots y_n]/J_\mu$ where $J_\mu = \mathcal{L}_\lambda^\perp$. We need R_μ to be a quotient of the above R_n. $\square$

24.10.1 Odd Version

`https://arxiv.org/pdf/2003.10031.pdf`

Let G be a finite group and V an n dimensional representation such that $\wedge^i V$ are pairwise non-isomorphic irreducible modules. Let θ_i be a basis of V and ξ_i be the dual basis of V^*.

In particular let $G = S_{n+1}$ and V be the reflection representation (the subspace in $\mathbb{C}^{n+1}$ such that the sum is 0). This satisfies the assumption by a general result of Steinberg.

$$\delta_W \equiv \sum \theta_i \xi_i$$

24.10.6 Proposition *δ_W generates $\wedge(V \otimes V^*)^W$.*

Proof $\wedge(V \otimes V^*)_{i,j} = \wedge^i V \otimes \wedge^j V^*$. The W invariant subspace of this is the isotypic component of the identity. But both tensorands are irreducibles and only dual to each other if $i = j$. So the invariant subspace is either 1 dimensional if $i = j$ or trivial. $\delta_W \in \wedge^1 V \otimes \wedge^1 V^*$ so that shows the case that δ_W provides a basis for $\wedge(V \otimes V^*)_{1,1}$. Then δ_W^i provides the same for the i, i bigraded piece as long as it is nonzero. That is proven by a quick computation. That covers all the nonzero pieces of the invariant subring $\wedge(V \otimes V^*)^W$. $\square$

24.10.7 Theorem

$$\begin{aligned}
DR_W &\equiv \wedge(V \otimes V^*)/(\wedge(V \otimes V^*)_+^W) \\
grFrob(DR_W, q, t) &\equiv \sum_{i,j} Frob(DR_{W,i,j}) q^i t^j \\
i + j > n \implies DR_{W,i,j} &= 0 \\
i + j \leq n \implies \left[DR_{W,i,j}\right] &= \left[\wedge^i V\right] \cdot \left[\wedge^j V^*\right] - \left[\wedge^{i-1} V\right] \cdot \left[\wedge^{j-1} V^*\right]
\end{aligned}$$

where the last equation takes place in $K^0(RepW)$.

We can recover the permutation representation for $W = S_n$ (Replacing $n+1 \to n$ for convenience), $U = \mathbb{C}^n$ by noting that it is $V \bigoplus \mathbb{C}$ as W representations. This then gives an isomorphism

$$
\begin{aligned}
\wedge(V\otimes V^*)/\wedge(V\otimes V^*)^W_+ &\simeq \wedge(U\otimes U^*)/\wedge(U\otimes U^*)^W_+\\
\mathrm{grFrob}(DR_{S_n},q,t) &\equiv \sum_{i,j}\left(s_{(n-i,1^i)}\star s_{n-j,1^j}-s_{n-i+1,1^{i-1}}\star s_{n-j+1,1^{j-1}}\right)q^it^j
\end{aligned}
$$

where $s_lambda \star s_\mu = Frob(S^\lambda \otimes S^\mu)$ where S^λ is the corresponding irreducible representation of S_n.

24.10.8 Remark One could do the combination $A = \mathbb{C}[x_1\cdots x_n, y_1\cdots y_n]\wedge(\theta_1\cdots\theta_n\cdots\xi_n)$. This is the algebra of polynomial coefficient holomorphic differential forms on $\mathbb{C}^{2n}$. S_n acts diagonally on all 4 sets of variables and we can again form the coinvariants $A/A_+^{S_n}$. $\Diamond$

24.11 Random Partitions

24.11.1 Plancherel Measure

Irreducible representations of S_n weighted by the squares of their dimensions $p_\lambda = \frac{(\dim\lambda)^2}{n!}$

24.11.2 Poissonized Plancherel Measure

Poisson process that controls n and then the Plancherel measure.

$$
\begin{aligned}
p_\lambda &= e^{-\theta}\frac{\theta^{|\lambda|}}{|\lambda|!}\frac{(\dim\lambda)^2}{|\lambda|!}\\
E(|\lambda|) &= \sum_{n=1}^{\infty}e^{-\theta}\frac{\theta^{n-1}\theta}{(n-1)!}=\theta
\end{aligned}
$$

where the first part is the Poisson and the second is the Plancherel. All together gives a measure on $\bigsqcup_n S_n^\vee$

See the app on GitHub for falling blocks animation.

24.11.3 3D Partitions

24.11.1 Theorem (MacMahon)

$$
\begin{aligned}
Z(q) &\equiv \sum_{n=0}^{\infty}PL(n)q^n\\
&= \prod_{k=0}^{\infty}\frac{1}{(1-q^k)^k}
\end{aligned}
$$

24.11.2 Theorem *Dimers on a domain in the hexagonal lattice. View as viewing this setup along the* $(1,1,1)$ *axis.*

24.12 Shuffle Algebra

https://arxiv.org/pdf/1702.08060.pdf

24.12.1 Definition (Space of Theta Functions) *Let $\Theta_k^-(z, y, \lambda)$ be the space of entire holomorphic functions $f(t_1 \cdots t_k)$ that are symmetric under S_k and*

$$\begin{aligned} g(t_1 \cdots t_k) &\equiv \frac{f(t_1 \cdots t_k)}{\prod_{j=1}^{k} \prod_{a=1}^{n} \theta(t_j - z_a)} \\ g(t_1 \cdots, t_i + r + s\tau, \cdots t_k) &= e^{2\pi i s(\lambda - ky)} g(t_1 \cdots t_k) \end{aligned}$$

That is it gives sections of some line bundle over $Sym^k E(\tau)$. The only parameter that changes the isomorphism class of this line bundle is actually $\sum z_a + \lambda - ky$. The dimension of this vector space is $\binom{}{n+k-1k}$

24.12.2 Definition (Shuffle Product) $\Theta_{k'}^{\pm}(z', y, \lambda + y(n'' - 2k'')) \otimes \Theta_{k''}^{\pm}(z'', y, \lambda) \longrightarrow \Theta_k^{\pm}(z, y, \lambda)$

$$\begin{aligned} \sum_{a=1}^{n'} z'_a + \lambda + y(n'' - 2k'') - k'y &= \\ \sum_{a=1}^{n''} z''_a + \lambda - k''y &= \\ \sum_{a=1}^{n'+n''} z_a + \lambda - (k' + k'')y &= \end{aligned}$$

Chapter 25

Spin Geometry Conventions

25.1 Clifford Algebra

25.1.1 Definition (Cliff) *For a quadratic vector space (V, Q), take the tensor algebra of V modulo the relation $v \otimes v + Q(v)1 = 0$. It is a quantization of the exterior algebra which is the case when $Q = 0$. They are $Cliff(V, q) \simeq Cliff(V, 0)$ as vector spaces. This is natural if char $k \neq 2$.*

This has the universal property for all j satisfying $j(v)^2 = -Q(v)1_A$

$$\begin{tikzcd} V \arrow[r, hook] \arrow[dr, "j"'] & Cliff(V,Q) \arrow[d, dashed, "!"] \\ & A \end{tikzcd}$$

Figure 25.1: Universal property: Cyan are vector space maps, blue are associative algebra maps

It also has the nice functorial properties. Under base change (V_k, Q) to $(V_k \otimes_k l, Q \otimes 1)$ gives $Cliff(V_l, Q_l) = l \otimes_k Cliff(V_k, Q)$. $Cliff(V, -Q) \simeq Cliff(V, Q)^{opp}$. It takes direct sums of quadratic vector spaces (with $Q^{''}(x + x') = Q(x) + Q(x')$) to the tensor product as superalgebras.

$$\begin{tikzcd} & & Cliff(\mathbb{R}^{n-1}, +) \arrow[d, "\wr\, \phi"] \\ & & Cliff_0(\mathbb{R}^n, +) \arrow[d] \\ Spin(\mathbb{R}^n, +) \arrow[r, hook] \arrow[urr, hook] & Pin(\mathbb{R}^n, +) \arrow[r, hook] & Cliff(\mathbb{R}^n, +) \end{tikzcd}$$

Figure 25.2: Green are group arrows for the multiplications. Red is a algebra arrow. Blue is a superalgebra arrow where the source happens to have nothing odd.

The Pin is included as the group generated by the unit vectors $v \in V$. That is generated by all reflections. The Spin then just keeps the part in even grading. More detail about the red arrow is given by: If $V = k\langle a\rangle \bigoplus U$ as an orthogonal direct sum, then $Cliff_0(V,Q) \simeq Cliff(U, Q(a)Q\mid_U)$ as associative algebras only not superalgebras.

The only reason for $Pin_\pm$ is the drop in notation of the quadratic form. Pin_- is just included into $Cliff(\mathbb{R}^n, -)$ instead. They are not isomorphic as groups even though we see an isomorphism of groups on the left and isomorphism of supervector spaces on the right in the diagram below.

$$\begin{tikzcd}
Spin(n,+) \arrow[r, hook] \arrow[d, "\wr"] & Pin_+(n) \arrow[r, hook] & Cliff(n,+) \arrow[d, "\wr"] \\
Spin(n,-) \arrow[r, hook] & Pin_-(n) \arrow[r, hook] & Cliff(n,-)
\end{tikzcd}$$

25.1.2 Remark Under the functor J described in `http://mathoverflow.net/questions/185645/what-are-the-correct-conventions-for-defining-clifford-algebras` flips the two rows of this diagram. ♢

25.2 Representation Theory of *Spin*

25.2.1 $Spin(2,1)$

Isomorphic as a Lie group to $SL(2,\mathbb{R})$ accidentally.

Real Representation Category

Complex Representation Category

25.2.2 $Spin(3)$

Isomorphic as a Lie group to $SU(2)$ accidentally.

Real Representation Category

Complex Representation Category

25.2.3 $Spin(3,1)$

Isomorphic as a Lie group to $SL(2,\mathbb{C})$ accidentally. (remember this is not as an algebraic group over $\mathbb{C}$ so you do have to complexify again for complex representations.)

Real Representation Category

Complex Representation Category

Irreducible objects are labelled by pairs of half-integers. This is by the description as pairs of $SL(2,\mathbb{C})$ representations.

- $(0,0)$ Complex scalar
- $(1/2,0)$ Left handed Weyl
- $(0,1/2)$ Right handed Weyl
- $(1/2,0)\bigoplus(0,1/2)$ Dirac Spinor
- $(1/2,1/2)$ Complex Vector
- $(1,0)$
- $(0,1)$
- $(1,0)\bigoplus(0,1)$
- $(1,1/2)$
- $(1/2,1)$
- $(1,1/2)\bigoplus(1/2,1)$
- $(1,1)$

25.2.1 Remark Who else has to do the hand trick with your left hand making an L every time to tell which one is which? We haven't broken parity yet so we shouldn't really say which one is left or right. That's what we can tell ourselves is our excuse for not being able to master this basic thing. $\Diamond$

25.2.4 *Spin*(4)

Isomorphic as a Lie group to $SU(2)\times SU(2)$ accidentally.

Real Representation Category

Complex Representation Category

Irreps are pairs of integers again.

25.3 Spinor Bundles

25.3.1 Definition (Fermion Field) *A fermion field is a section of a spinor bundle. The adjectives Majorana, Dirac and Weyl say which representation is used in the spinor bundle. Adjectives like R or NS are used to indicate which cohomology class is being used for the topological classification. If you want to make it charged give a vector bundle E for the representation of the gauge group you want.*

25.3.2 Example *For example, E is the associated bundle for the fundamental $SU(3)$ representation gives color. Physics usually assumes a contractible spacetime implying a fortiori that fermions can exist and there are no periodic/antiperiodic choices to worry about. All you need to worry about is what bundles E you want to couple to.*

25.3.3 Example (Differential Forms) *Take the full $\wedge^\bullet T^*M$ exterior bundle. Give this bundle the structure of an associated bundle by*

25.3.1 Stieffel-Whitney

25.3.4 Theorem (Spin Structure) *For a spin structure to exist on a Riemannian manifold M, we must have an orientation and $w_2(TM) \in H^2(M, \mathbb{Z}_2)$ needs to vanish. As with taxes, $W2$ is a pain.*

25.3.5 Theorem (Pin Structure) *A $Pin^{\pm}$ bundle is a principal bundle with that $Pin^{\pm}$ structure group and an isomorphism from the ± 1 quotient to the orthonormal frame bundle (an isomorphism of $O(n)$ principal bundless).*

A $Pin(\mathbb{R}^n, -)$ structure exists when $w_2(TM) = 0$ and $Pin(\mathbb{R}^n, +)$ structure needs $w_2 + w_1^2$ to vanish.

Proof For example, `http://arxiv.org/pdf/1604.06527.pdf`. Our conventions for Clifford algebras are switched from Freed's so don't forget to switch. □

`https://arxiv.org/abs/1606.07894`

25.4 Atiyah-Singer

25.4.1 Theorem (Atiyah-Singer Index Theorem)

25.4.2 Example *Let there be a free Dirac theory on a Riemannian 4-manifold.*

$$\begin{aligned} Z[m, A, g] &= \int D\psi e^{-S[m,A,g]} \\ S[m, A, g] &= \int dVol_g \bar{\psi}(\not{D} + m)\psi \\ Z[m, A, g] &= \det(\not{D} + m) \end{aligned}$$

The chirality operator γ^5 anticommutes with $i\not{D}$. Therefore nonzero modes for the Dirac operator come in pairs λ and $-\lambda$. The zero modes do not necessarily come in pairs. Say there are $N_{\pm}$ for $\gamma^5 = \pm 1$.

$$\begin{aligned} Z[m, A, g] &= (\prod_{\lambda > 0} (i\lambda + m)(-i\lambda + m))(m^{N_+ + N_-}) \\ &= (\prod_{\lambda > 0} (\lambda^2 + m^2))(m^{N_+ + N_-}) \\ \frac{Z[m, A, g]}{Z[-m, A, g]} &= (-1)^{N_+ + N_-} = (-1)^{N_+ - N_- + 2N_-} = (-1)^{N_+ - N_-} \end{aligned}$$

$$\begin{aligned} N_+ - N_- &= \frac{1}{8\pi^2} \int F \wedge F - \frac{\sigma}{8} \\ \sigma &= \frac{-1}{24\pi^2} \int \mathrm{tr}(R_g \wedge R_g) \end{aligned}$$

In more abstract terms, a spin structure gives a KO orientation that means we have a class $[M] \in KO_n(M)$. This can be evaluated against the various $\pi^j(TM) \in KO^0(M)$ and polynomials thereof to get elements of $KO_n(pt)$. The Dirac operators are representatives thereof and the evaluations give us indices on various vector bundles (add flavor toppings to the fermions).

For example, $\pi^0(TM)$ provides a map $\Omega_\bullet^{Sp} \to KO_\bullet$. When $4 \mid n$, this is the $\hat{A}$ genus up to factors of 1/2 in dimensions not divisible by 8.

- $\Omega_0^{Spin} = \mathbb{Z}$ by counting + points
- $\Omega_1^{Spin} = \mathbb{Z}_2$ by R-NS conditions
- $\Omega_2^{Spin} = \mathbb{Z}_2$ by square of the antiperiodic above
- $\Omega_3^{Spin} = 0$
- $\Omega_4^{Spin} = \mathbb{Z}$ by Kummer surface
- $\Omega_5^{Spin} = 0$
- $\Omega_6^{Spin} = 0$
- $\Omega_7^{Spin} = 0$
- $\Omega_8^{Spin} = \mathbb{Z}^2$ by $\mathbb{HP}^2$ and $\frac{1}{4}[K3]^2$

$$\begin{tikzcd}
MString_\bullet \arrow[r] \arrow[d] & tmf_\bullet \arrow[d] \\
\Omega_\bullet^{St,\mathbb{Q}} \arrow[r] \arrow[d] & MF_\bullet \arrow[d] \\
MSpin_\bullet \arrow[r] \arrow[d] & \mathbb{Z}[[q]] \arrow[d] \\
MSO_\bullet \arrow[r] & \mathbb{Q}[[q]]
\end{tikzcd}$$

The picture is to put a string theory on a worldsheet E_τ on the various oriented/spin/rationally string/honestly string manifolds that represent classes on the left hand side. Taking the partition functions then gives functions of τ and therefore functions of the nome $q = e^{2\pi i \tau}$. In good cases the equivalent worldsheets related by modular transformations should give related by partition functions. After all it looks like the same source and target. $\tau \in \frac{1}{2\pi i} \log \mathbb{Q}_{(0,1)}$ would be all along the positive imaginary axis of the upper half plane. This turns into special values to evaluate modular functions.

25.4.3 Theorem (Atiyah-Patodi-Singer) *What about when there is boundary?*

25.4.4 Definition (η ρ ...)

$$\begin{aligned}
\eta &\equiv \\
\rho &\equiv
\end{aligned}$$

Chapter 26

D'oh!-AHA!

26.1 HA! - Hecke Algebra

26.1.1 Definition (Bruhat Order) *A partial order on the Coxeter system by $x \leq w$ by x a prefix for a reduced word for w.*

26.1.2 Definition (Hecke Algebra) (W, S) *Coxeter system with coxeter Matrix M_{st}, R commutative unital ring. $q_s \in R^*$ are units in R. Form the R algebra generated by symbols T_s with relations:*

$$\begin{aligned} T_s T_t T_s \cdots &= T_s T_t T_s \cdots \\ (T_s - q_s^{1/2})(T_s + q_s^{-1/2}) &= 0 \end{aligned}$$

where $\cdots$ depend on M_{st} for how many. In particular, main example let $R = \mathbb{Z}[q, q^{-1}]$

H_w where choose any reduced word and write $H_{s_{i_1}} \cdots H_{s_{i_m}}$. By Matsumoto only need braid relations to change between reduced words so this only depends on w not the choice.

26.1.1 Kazhdan-Lusztig

26.1.3 Theorem (Kazhdan-Lusztig) *First give the involution as a ring map (but not an R module map) by*

$$\begin{aligned} H_s &\to H_s^{-1} \\ q &\to q^{-1} \end{aligned}$$

Also define an anti-involution by same on generators but anti-multiplicative map.

, now the following special elements of the algebra.

$$\underline{H}_s = H_s + qH_e$$

A priori we held the proofs to be self evident that all bases were created equal, but this involution has single out the better basis.

26.1.4 Theorem (PBW like Basis)

26.1.5 Theorem (Intersection Cohomology) *In different convention*

$$P_{y,w}(q) = \sum_i q^i \dim IH^{2i}_{X_y}(\bar{X_w})$$

where X_w means the Schubert cell for w and the subscript means take a stalk at any point in X_y.

26.1.6 Theorem *Inside $D^b_{B \times B}(G, \mathbb{C})$ there are IC sheaves $IC(B\bar{w}B) = IC_w$. The category from direct sums and shifts of these give the Hecke category.*

26.1.7 Definition (Soergel Presentation) *Free $\mathbb{Z}[v, v^{-1}]$ algebra with basis H_w and multiplication*

$$\begin{aligned} H_w H_s &= H_{ws} \\ &= (v^{-1} - v)H_w + H_{ws} \end{aligned}$$

26.1.8 Definition (IC sheaf) *Perverse sheaf with only one nonzero on the diagonal and only nonzeroes in the table in the triangle below.*

26.1.9 Theorem (Decomposition Theorem) *For proper maps, the pushforward preserves shifted semisimples. As a consequence $IC_\lambda \simeq \bigoplus_X f_* \mathbb{C}_X \bigoplus_u IC_u$ for some with point fibers and some of IC sheaves that already computed.*

26.1.10 Theorem (Soergel Categorification) *Let $\mathfrak{h}$ be a reflection faithful representation of W, then take $R = Sym_k(\mathfrak{h}^*[2]) = Spec(\mathfrak{h}[-2])$ ($SpecR^W$ affinization of $\mathfrak{g}[-2]/G$ moduli of vacua) Make $B_s \equiv R \otimes_{R^s} R[1]$ as an $R-R$ graded module. B_w by Bott-Samelson bimodule. Then take the minimal strictly full subcategory of $R - gmod - R$ including these. Call that $SorBim_{BS}(W, \mathfrak{h})$ Then take Karoubi envelope to get something called Soergel bimodules $SorBim(W, \mathfrak{h})$. The split Grothendieck group of this is the Hecke algebra.*

Proof `https://arxiv.org/pdf/1703.01576.pdf` Because $H_{B\times B}(pt) = R \otimes_{\mathbb{C}} R$, taking hypercohomology lands you in $R - R$ bimodules by a fully faithful monoidal functor. In particular $\mathcal{H}(IC_s) = B_s = R \otimes_{R^s} R[1]$ so actually lands in Soergel bimodules. In fact equivalence. We already knew the Hecke category categorified the Hecke algebra so this does too, but now for more general Coxeter systems. □

26.1.11 Definition (Demazure Operator)

$$\begin{aligned} D_s &\in R \to R^s[-2] \subset R[-2] \\ D_s f &\equiv \frac{f - s(f)}{\alpha_s} \\ D_s(\frac{\alpha_s}{2}) &= 1 \\ D_s(fg) &= D_s f \cdot g + s(f) D_s(g) \\ \langle f \mid\mid g \rangle &\equiv D_s(fg) \\ D_s^2 &= 0 \\ D_s D_t &= \end{aligned}$$

D_s is a map of $R^s - R^s$ bimodules.

$D_w = D_{s_{i_1}} \cdots D_{s_{i_n}}$ is well defined because satisfies the braid relation so doesn't depend on the reduced word decomposition.

26.1.12 Theorem (Frobenius Extension) *$R^s \subset R$ is*

26.1.13 Definition (StdBim) *For $x \in W$, R_x is R as a left R module and the right module structure is $m \cdot r \to x(r) \cdot m$. The bimodule monoidal structure gives $R_x \otimes R_y \simeq R_{xy}$. That is we have embedded W graded version of trivial category.*

Rouqier Complexes

26.1.14 Theorem (Hard Lefschetz)

Consider the bounded homotopy category $K^b(SorBim)$.

26.1.15 Definition (Rouqier Complex) *Any object of $K^b(SorBim)$ isomorphic to some ... built from $F_s^{\pm}$ which are the invertible objects given as ...*

By writting each graded piece in indecomposables, can write all the differentials as matrices. When find an isomorphism between two, can remove those subcomplexes.

26.1.16 Example *$m = 3$ case $F_s F_t F_s$ vs $F_t F_s F_t$*

26.1.17 Theorem *$\sigma_s \to F_s$ map from B_W to $K^b(SorBim)$ extends as a strict monoidal functor where treating group as a monoidal category. In finite type, this gives an injection. For affine type and general Coxeter open problem.*

26.1.18 Theorem (Minimal Representative Complex)

Using both the shifts internal to $SorBim$ at the shifts due to taking homotopy category can write them in grid form as .... Because of this define ≥ 0 and ≤ 0 for support above and below diagonal in this grid.

26.1.19 Lemma *The functor of induction $R^I \to R^J$ structure on one side of the bimodule is represented by R^J as an $R^I - R^J$ bimodule. Define Res_{IJ} to be that but with a grading shift by $l(w_{0J}) - l(w_{0I})$ lengths of longest words. Using lots of different parabolics.*
Singular Bott Samuelson bimodule 2-category
Objects are R^I with W_I finite.
1-morphisms are the ones that build from inductions and restrictions bimodules above.

26.1.20 Lemma *$Hom(R, R)$ has all Bott Samuelson by letting the I's that show up always as R^{s_i} one at a time. But can also induce to R^{s_1,s_2} in one step. Could do s, t, u in one step etc.*
But $Hom(R, R)$ is already $SorBim$ because could build those things that look new from the summands, envelope construction that we did to define the category $SorBim$ from the basic construction of Bott-Samuelson bimodules. Nothing new in the singular setting just from $R \to R$ but there are new stuff for other parabolics.

26.1.21 Lemma *Sorgel bimodules $Hom(R, R)$ to Sorgel modules by making right side acting by 0. Get projectives in trivial block of category $\mathcal{O}$.*
Do same for $Hom(R, R^I)$ get singular blocks.

26.1.22 Theorem (Soergel-Williamson) *Singular Sorgel bimodules ... parameterized by set of double cosets $W^J \backslash W / W^I$*
Therefore this 2-category categorifies the Hecke algebroid (Morita equivalent to Schur ....)

26.1.23 Theorem *2-functor from 2 colored (s, t) Temperley Lieb with $\delta = -[2]_q$ to Singular SorBim.*
s,t dot goes to ${}_{R^s}R_{R^t}$ break with empty region colored R in between on the other side as ${}_{R^s}R_R \otimes_R {}_R R_{R^t}$

26.1.24 Theorem *Can change the t-structure in $K^b(Sor\bar{B}im)$ thought of as $D^b(\mathcal{O}_0)$. It is called the perverse t-structure, it mixes the internal and homological gradings. The heart of this called $\mathcal{O}_0^{perv}$ has shifts that combine internal and homological by the same value.*
So can do calculations in $D^b(\mathcal{O}_0^{perv})$ instead.

26.1.25 Example ($\boldsymbol{SL(2)}$) *Index by $W \times \mathbb{Z}$ by $w, n \to B_w[n]$.*
In this example the Vermas go to R. The finite dimensional L_s goes to B_s.
In $D^b(\mathcal{O}_0)$, know that $0 \to R \to ? \to B_s \to 0$ because of how finite dimensional sit inside Vermas with another Verma as quotient. That extension is the cone of the map $B_s \to R(1)$. That fits in a short exact sequence of complexes in the homotopy category. The triangle that gives the name to triangulated category. But these are all perverse so have a short exact in $\mathcal{O}_0^{perv}$

$$\begin{array}{ccccc}
0 & \Delta_s & T_s & \Delta_?(1) & 0 \\
 & R(1) & R(1) & & \\
 & B_s & B_s & & \\
 & & R(-1) & R(-1) &
\end{array}$$

T_s is called tilting object. Can also describe as $K^b(Tilt)$. Indecomposable tiltings are parameterized by $W \times \mathbb{Z}$

26.1.26 Theorem (Ringel Duality) *Use this to write the complicated $Ext(\bigoplus S, \bigoplus S)$ into $End(\bigoplus T)$. This guarantees formality.*

26.1.27 Theorem (Ganter-Ram) *$h_T(G/B) = R \otimes_{R^{W_0}} R$ where $R = h_T(pt)$ and $R^{W_0} = h_G(pt)$. In particular take the example of ordinary cohomology to give $h_T(pt) = Sym(\mathfrak{h}[2]^*) \simeq \mathbb{C}[x_1 \cdots x_n]$. To make SorBim would have to get $s \in W_0$ coinvariants instead and be able to take tensor products of these. It's like just having B_s.*
So $h_T(G/B) \otimes_{h_T(pt)} h_T(G/B)$

26.1.28 Theorem (Generalized Demazure Operators) *Take the quotient map π $G/B \to G/P_J$. Then form $\pi_J^*(\pi_J)_!$ as a map on $h_T(G/B)$ as ?? (as $h_G(pt)$ bimodules? or as $h_T - h_G$ bimodules?). In particular let P_J be the parabolic where only change by allowing a single entry below diagonal, just complete one of the sl_2 triples. They satisfy $D_i^2 = D_i p(x_{\alpha_i}, x_{-\alpha_i})$. So if cohomology nil-Hecke and if K-theory 0-Hecke??*

Proof `https://arxiv.org/pdf/1212.5742.pdf` □

26.1.29 Definition (Nil Affine Hecke)

$$\begin{array}{rcl}
x_{\lambda+\mu} &=& x_\lambda + x_\mu - p(x_\lambda, x_\mu) x_\lambda x_\mu \\
x_{\lambda+\mu} &=& x_\lambda + x_\mu - 0 * x_\lambda x_\mu \\
(1 - e^{\lambda+\mu}) &=& (1 - e^\lambda) + (1 - e^\mu) - 1(1 - e^\lambda)(1 - e^\mu) \\
H &=& (S \otimes_L S) \ltimes L[W_0] \\
x_\mu &=& x_\mu \otimes 1 \ltimes 1 \\
y_\mu &=& 1 \otimes x_\mu \ltimes 1 \\
t_w &=& 1 \otimes 1 \otimes t_w \\
y_{\lambda+\mu} &=& y_\lambda + y_\mu - p(y_\lambda, y_\mu) y_\lambda y_\mu \\
x_\lambda y_\mu &=& y_\mu x_\lambda \\
t_v t_w &=& t_{vw} \\
t_w y_\lambda &=& y_\lambda t_w \\
t_w x_\lambda &=& x_{w\lambda} t_w
\end{array}$$

$$\begin{aligned}
\frac{1}{y_{-\alpha}} + \frac{1}{y_{\alpha}} &= p(y_\alpha, y_{-\alpha}) \\
\frac{1}{x_{-1}x_{-2}} - \frac{1}{x_{-2}x_{-3}} + \frac{1}{x_1 x_{-3}} &= \frac{1}{x_{-1}x_{-2}} - \frac{1}{x_{-2}x_{-3}} - \frac{(1-p()x_{-1})}{x_{-1}x_{-3}} \\
&= \frac{x_{-3}}{x_{-1}x_{-2}x_{-3}} - \frac{x_{-1}}{x_{-1}x_{-2}x_{-3}} - \frac{x_{-2}(1-p()x_{-1})}{x_{-1}x_{-2}x_{-3}} \\
&\propto x_{-3} - x_{-1} - x_{-2} + p()x_{-2}x_{-1} = 0 \\
y_\alpha &= -y_{-\alpha}(1 - p(y_\alpha, y_{-\alpha})y_{-\alpha})^{-1}
\end{aligned}$$

26.1.2 NilHecke

26.2 AHA! - Affine Hecke Algebra

26.2.1 Definition (Affine Weyl Group) *$W_0 \ltimes \mathbb{Z}^n$ where W_0 is the usual Weyl group (S_n in type A). Alternative presentation by*

26.2.2 Definition

$$\begin{aligned}
T_i T_j T_i \cdots &= T_j T_i T_j \cdots \\
\pi T_i \pi^{-1} &= T_{\pi(i)} \\
(T_i - t^{1/2})(T_i + t^{1/2}) &= 0 \quad 0 \le i \le n
\end{aligned}$$

26.2.3 Theorem (PBW like Basis)

26.3 Elliptic Hall Algebra

26.4 D'oh! - Double Affine Hecke Algebra

26.4.1 Definition (Type A)

$$\begin{aligned}
(T_i - t^{1/2})(T_i + t^{-1/2}) &= 0 \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \\
T_i T_j &= T_j T_i \quad \mid i-j \mid \geq 2 \\
T_i X_i T_i &= X_{i+1} \\
T_i X_j &= X_j T_i \;\; j \neq i, i+1 \\
T_i Y_i T_i &= Y_{i+1} \\
T_i Y_j &= Y_j T_i \;\; j \neq i, i+1 \\
X_1^{-1} Y_2^{-1} X_1 Y_2 &= T_1^2 \\
Y_i \prod X_i &= q(\prod X_i) Y_i \\
X_i \prod Y_i &= q^{-1}(\prod Y_i) X_i \\
\left[X_i, X_j\right] &= 0 \\
\left[Y_i, Y_j\right] &= 0
\end{aligned}$$

26.4.2 Theorem (Polynomial Representation) *A representation ρ on $\mathbb{C}[X_i^{\pm}]$ by X_i acts like X_i, T_i acts by*

$$\begin{aligned}
T_i &\to t^{1/2} s_i + \frac{t^{1/2} - t^{-1/2}}{X_i / X_{i+1} - 1}(s_i - 1) \\
Y_i &\to t^{(N-1)/2} \rho(T_i^{-1} \cdots T_{N-1}^{-1}) \omega \rho(T_1 \cdots T_{i-1}) \\
\omega(f) &= f(qX_N, X_1 \cdots X_{N-1})
\end{aligned}$$

26.4.3 Definition (Rectangular Representation) *A rectangular representation for the rectangular partition k^N is given by*

26.4.4 Theorem (Jordan-Vazirani) *Under $\mathcal{F}^d$ $\mathcal{D}_q(\frac{G}{G}) \to DAHA{-}mod$ from [**?**] with $d = kN$, $\mathcal{O}_q(G)$ goes to k^N rectangular representation.*

Proof Use a Peter-Weyl theorem then get $(V^{\otimes d} \otimes \bigoplus_\lambda V_\lambda \otimes V_\lambda^*)^{U_q\mathfrak{g}}$. This is to build the result as a vector space. For individual λ, build an AHA action. Then put all λ together and turn into a DAHA action. Then check the actions of the Y_i and compare with the potential irreducible DAHA modules. Find that it is k^N. d needs to be multiple of N so that get invariants in the first place.

So if $(V^{\otimes d} \otimes V(\lambda)) \otimes V(\lambda)$ invariants are given by looking at the parenthesis and counting to make sure get back to $V(\lambda)$. Each V gives a step of some ϵ_i. In $\mathfrak{sl}_2$ this is the modifying spin by ± 1. So counting d step paths that start and end in λ and they must stay in the dominant chamber. This gives a basis for the invariant space. Translate this into standard tableaus by putting λ and

λ^* in an N by N box. Sliding apart k horizontally. Then putting standard tableau structure on the holes. This counts the same problem.
Have basis and action of Y_i on this. $\square$

26.4.5 Lemma (3.1 `https://arxiv.org/pdf/1611.10216.pdf`) *Let f be a polynomial and $Y_i(f)$ be $Y_i T_{i-1}^{-1} \cdots T_1^{-1} f(X_1^{-1}) T_1 \cdots T_{i-1}$. In particular $Y_i = Y_i(1)$. These are pairwise commuting. So you can define their elementary summetric function to build $M_r(f) = e_r(Y_1(f) \cdots Y_N(f))$ multiplied by the symmeterizer e on the right. In the polynomial representation this acts by Macdonald difference operators.*

Proof We already know that $Y_i(1)$ commute with each other. But we can now specialize to the polynomial representation and provide a conjugation that takes us to the realization of $Y_i(f)$ from the realization of $Y_i(1)$. That is by conjugating by $g(X_1) \cdots g(X_N)$ where $g(qX) = g(X) f(X^{-1})$ is a meromorphic function $g(X) = \prod_{m=1}^{\infty} f(q^{-m} X^{-1})$ $(q > 1)$. $\square$

26.4.6 Lemma *Let $f(X) = (X - Z_1) \cdots (X - Z_l)$. Then let $Z_1 \cdots Z_l = (-1)^l$ by rescaling the Z_i simultaneously. The nonsymmetric Macdonald $F()$ was a joint eigenfunction of $Y_i(1)$, then $g(X_1) \cdots g(X_N) F()$ is a joint eigenfunciton of the $Y_i(f)$ by the conjugation argument above.*

26.4.7 Lemma *To get eigenfunctions of $\phi(Y_i(f))$ apply a difference Fourier transform $\mathcal{F}$*

$$
\begin{aligned}
Y_i(f) \mid g(X_1) \cdots g(X_N) F() \rangle &\propto \mid g(X_1) \cdots g(X_N) F() \rangle \\
\phi(Y_i(f)) &= T_{i-1}^{-1} \cdots T_1^{-1} D_1^{(l)} T_1^{-1} \cdots T_{i-1}^{-1} \\
D_1^{(l)} &= X_1^{-1} (Y_1 - Z_1) \cdots (Y_1 - Z_l) \\
\phi(Y_i(f)) \mathcal{F} \mid g(X_1) \cdots g(X_N) F() \rangle &= \mathcal{F} \mathcal{F}^{-1} \phi(Y_i(f)) \mathcal{F} \mid g(X_1) \cdots g(X_N) F() \rangle \\
&\propto \mathcal{F} \mid g(X_1) \cdots g(X_N) F() \rangle
\end{aligned}
$$

26.4.8 Theorem (PBW like Basis)

26.4.1 $PSL(2, \mathbb{Z})$ action

26.4.2 Rank 1

Consider the problem of 2 particles on a torus. We may reduce the problem to a single particle on a once punctured torus modulo a $\mathbb{Z}_2$ action using the difference structure and the fact that these are identical particles. Now there are 3 generators of π_1 to emphasize. X, Y for going around each of the loops of the torus and a T to half-wrap the puncture which is swapping the 2 particles in the original picture.

26.5 Haiman

26.5.1 Definition (Isospectral Hilbert Scheme)

26.6 Degenerations

26.6.1 Rational

26.6.1 Definition (Dunkl Operator)

$$\begin{aligned} D &= \frac{d}{dx} - \frac{k}{x}(s-1) \\ D^2 \mid_{sym} &= \frac{d^2}{dx^2} + \frac{2k}{x}\frac{d}{dx} \end{aligned}$$

is the radial part of Laplace operator.

More generally

$$\begin{aligned} D_i &= \frac{d}{dx_i} + k\sum_{j\neq i}\frac{s_{ij}-1}{x_i-x_j} \\ \left[D_i, D_j\right] &= 0 \\ sD_is^{-1} &= D_{s(i)} \end{aligned}$$

26.6.2 Theorem (Etingof-Ginzburg) *2001 V a symplectic vector space over $\mathbb{C}$ and $G \subset Sp(V)$ be a finite group. Then deformations of $\mathbb{C}G \ltimes Weyl(V)$ are unobstructed and parameterized by $HH^2(A) \simeq \mathbb{C}[S]^G$ where S is the set of symplectic reflections inside G and that set is acted on by G through conjugation.*

Explicitly

$$\left[x, y\right] = \omega(x,y)1 + \sum_g c_g\omega((g-1)x,(g-1)y)g$$

26.6.3 Example *Let $V \simeq \mathfrak{h}\bigoplus\mathfrak{h}^*$ and G be the Weyl group acting symplectically through it's cotangent lift.*

Continue here

26.6.2 Trigonometric

26.6.4 Definition *The algebra over $\mathbb{C}[\hbar,\kappa]$ generated by $S_N \ltimes \mathbb{Z}^N$ of s_i and X_i respectively as well as y_i.*

$$\begin{aligned} s_iy_i &= y_{i+1}s_i + \kappa \\ \left[y_i, y_j\right] &= 0 \\ \left[y_i, X_j\right] &= \kappa X_js_{ij} \;\; i > j \\ \left[y_i, X_j\right] &= \kappa X_is_{ij} \;\; i < j \\ \left[y_i, X_i\right] &= \hbar X_i - \kappa\sum_{r<i}X_rs_{ir} - \kappa\sum_{r>i}X_is_{ir} \end{aligned}$$

This is bigraded by $\mid X_i \mid= (2,0)$ *and* $\mid s_i \mid= (0,0)$ *and rest* $(0,2)$

26.6.5 Theorem (Cherednik) *There is a representation on* $\mathbb{C}[X_i^{\pm}]$ *Laurent polynomial ring, by sending* s_i *to the swap maps,* X_i *to multiplications and* y_i *to* D_i^{trig}

$$\begin{aligned} D_i^{rat} &= \hbar \frac{\partial}{\partial X} - \sum_{j \neq i} \frac{\kappa}{X_i - X_j}(1 - s_{ij}) \\ D_i^{trig} &= X_i D_i^{rat} - \kappa \sum_{j<i} s_{ij} \end{aligned}$$

26.6.6 Definition (Spherical subalgebra) *Define the projector* $e = \frac{1}{N!} \sum_{S_N} s$*. Then* eHe *is the spherical subalgebra. It has a polynomial representation on the invariant Laurent polynomials. In particular* $\sum y_i^2$ *which is part of the set of* $\sum y_i^p$ *provides a realization of the trigonometric Calogero-Moser Hamiltonian.*
Similarly can define $e_- = \frac{1}{N!} \sum_{S_N} sgn(s)s$ *and* $e_- H e_-$ *for anti-spherical version. In fact* $eH_c e \simeq e_- H_{c+1} e_-$ *where c is .. in terms of* κ *above.*

26.6.3 $N \to \infty$

20160227 String-math seminar

Take $N \to \infty$ DAHA.

26.6.7 Theorem $H_K \subset \bigoplus_k \prod_n K^T(Hilb^{n+k}\mathbb{A}^2 \times Hilb^n \mathbb{A}^2) \otimes_R K$ *with* $R = K^T(pt) \simeq \mathbb{C}(q,t)$ *equipped with convolution product is isomorphic to a one dimensional central extension of spherical DAHA of* GL_∞*.* $\bigoplus_n K^T(Hilb^n)$ *is isomorphic to the standard representation on* $\mathbb{C}(q^{1/2}, t^{1/2}) \otimes \Lambda$ *the symmetric polynomials in countably infinite variables.*

Proof `https://arxiv.org/pdf/0905.2555.pdf` □

26.6.4 Cyclotomic

`https://arxiv.org/pdf/1611.10216.pdf`

26.7 Cherednik 20170623

TODO: Break up this seminar notes into the appropriate sections. Video is locked on slide, but writing on board :(

26.7.1 Definition (Elliptic Braid Group) T_i *are half turns*

26.7.2 Definition (DAHA) *Group algebra of elliptic braid group modulo the relations* $T_i^2 + aT_i + b = 0$

Did the rank 1 DAHA picture. Already have above.

26.7.3 Definition (Macdonald Operators)

$$\begin{aligned}
L_f &= f(Y_1 \cdots Y_N) \\
L_f P_\mu(x) &= f(t^{\rho_N} q^\mu) P_\mu(x) \\
L_{P_\lambda} P_\mu(t^\rho) &= S^\mu_\lambda
\end{aligned}$$

In this case Rogers polynomials

26.7.4 Definition (Torus Knot) $x^r = y^s \bigcap S^3_\epsilon \subset \mathbb{C}^2$

26.7.5 Definition (DAHA invariant) *From r, s make $\gamma \in SL_2(\mathbb{Z})$ then evaluate $\gamma \frac{P_\lambda}{P_\lambda(t^\rho)}$. Take coinvariant by an evaluation.*

$$\begin{aligned}
\gamma &= \tau_+ \tau_-^2 \\
DJ_{3.2} &= \gamma \to Y(X^2) \\
&= t^{-1/2} q^{-1} X^2 - t^{1/2} + t^{-1/2} \\
X^2 \to t^{-1} &\implies 1 - qt^2 + qt
\end{aligned}$$

Similar to extend to higher rank A_n

26.7.6 Definition (Iterated Torus Knot)

26.7.7 Definition (Plane Curve) *Take $f(x, y) = 0 \subset \mathbb{C}^2$ with origin as special point.*

26.7.8 Definition *Euler numbers of $Hilb^n C$ then form generating function*

$$\begin{aligned}
K &\equiv \bigoplus K_T(Hilb^d) \otimes_{\mathbb{C}[q^\pm, t^\pm]} \mathbb{C}(q^\pm, t^\pm) \\
K &\simeq Pol \\
I_\lambda \otimes 1 &\to \tilde{H}_\lambda(x, q, t) = t^{n(\lambda)} J_\lambda(q, t^{-1}) \mid_P \\
P &= p_k \to \frac{p_k}{1 - t^{-k}} \\
n(\lambda) &= \sum l(\Box) \\
\nabla \tilde{H}_\lambda &= q^{n(\lambda^T)} t^{n(\lambda)} \tilde{H}_\lambda \\
\nabla &= \mathcal{O}(1) \times
\end{aligned}$$

26.7.9 Theorem (Oblomkov-Shende)

$n_C(i)$ are positive integers

26.7.10 Conjecture (ORS) *Use nested Hilbert schemes to make*

$$\sum q^? a^? t^? (Hilb^{l,l+m}(C)) \quad \to \quad Kh$$

26.7.11 Conjecture (ChD)

26.7.12 Conjecture (Ch-Philipp) *Let $t = 1/p^l$ size of the finite field look at standard fields over F_{p^l} the DAHA super polynomial is in terms of the variety of standard flags $\mathcal{F}(F_{p^l})$*

26.7.13 Definition (Period of Cusp form) $\int_\gamma z^k \Phi_\chi(z) dz \to$ *p-adic measures*

26.8 Okounkov ∞-dim alg seminar

$\chi^\lambda(g)$ are functions on G that are eigenfunctions under some $\mathcal{D}$ differential operators on G. But invariant under conjugation by G/T so turn into differentail operator acting on functions on T. But now there is the Weyl chamber problem. There are hyperplanes separating regions of T that are redundant under the diagonalization.

These Differential operators on T become q-difference operators.

Define Φ as functions on $(z, a) \in T \times T^\vee$ and q, t deformation parameters. Starts with $\exp(\langle \log z \mid \log a \rangle / \log q)$ using the pairing between $\mathfrak{t}$ and it's dual.

Symmetry under exchanging T and $T^\vee$.

In principal infinite series so far, just solving the difference equation in a triangular manner. Terms keep being added to make fix the errors in the difference equation not being satisfied.

Has been calling t as $\hbar$ in this talk.

Take limit $q \to 1$ but not through real numbers but as $qe^{2\pi i/m}$ and $m \to \infty$ and $a = q^{-c}$ c/m remaining fixed.

$\log a$ stays well defined in the limit. The function that used to be a function of z and a. For a sector on the circle around that limit $a = q^{-c}$ get a function expressed as an integral.

Let X be an algebraic symplectic variety. (comments about smooth or resolution of singularities not being important for today)

Chapter 27

Algebraic Geometry Appendix 1

27.1 Quantum Hamiltonian Reduction

27.1.1 Definition (Quantum Moment Map) *Let A be an algebra with $\mathfrak{g}$ action ϕ $\mathfrak{g} \to Der(A)$. A quantum moment map is $U(\mathfrak{g}) \to A$ associative algebra morphism such that $[\mu(a), b] = \phi(a)b$ for all $a \in \mathfrak{g}$ and $b \in A$.*

27.1.2 Example *If A is a deformation quantization of a Poisson algebra A_0. Also suppose we have a classical $\mathfrak{g}$ action ϕ_0 $\mathfrak{g} \to Der(A_0)$ and a classical moment map μ_0 $U(\mathfrak{g}) \to A_0$. A quantization of this μ is such that μ $U(\mathfrak{g}) \to A[\hbar^{-1}]$ satisfies $\mu(a) = \hbar^{-1}\mu_0(a) + O(1)$*

27.1.3 Definition (Quantum Hamiltonian Reduction) *Let A $\mathfrak{g}$ and μ be given. Then take the $\mathfrak{g}$ invariants $A^{\mathfrak{g}}$ and the ideal J generated by $\mu(a)$ to form the new algebra $A^{\mathfrak{g}}/(J \bigcap A^{\mathfrak{g}})$.*

27.1.4 Definition (Quantum Hamiltonian Reduction-2) *Let G compact Lie group act on superalgebra W with comoment map $\mu^* \mathfrak{g} \to W$.*

$$
\begin{aligned}
W//G &\equiv (W/ < \mu^* \mathfrak{g} >)^G \\
&\simeq W^G/(W^G \bigcap < \mu^* \mathfrak{g} >) \\
&\simeq End_W(W/ < \mu^* \mathfrak{g} >)
\end{aligned}
$$

$W/ < \mu^* \mathfrak{g} >$ provides a bimodule between W and $W//G$.

27.1.5 Example *If A is a deformation quantization of $C^\infty(M)$ for a Poisson manifold. Then you can form $\mu_0^{-1}(\mathcal{O})/G = R(M, G, \mathcal{O})$ classically and the quantum Hamiltonian reduction is a deformation of functions on that.*

27.2 Toric Geometry

27.2.1 Definition *A toric variety is an irreducible variety X such that there is a Zarisiki open $(\mathbb{C}^*)^n$ and the action of the torus on itself extends to all of X.*

27.2.2 Theorem (Toric Ideal) *Let V be an affine variety. Then the following are equivalent*

- *V is a toric variety*
- *V is Spec of the $\mathbb{C}[S]$ for some finitely generated integral semigroup S where integral means that S can be embedded as a subsemigroup inside some lattice $\mathbb{Z}^m$*
- *V is the vanishing set for a toric ideal I where toric ideal in $\mathbb{C}[x_1 \cdots x_n]$ means that it can be generated by binomials.*

27.2.3 Definition *Let $N = \mathbb{Z}^d$. Take a convex cone C with vertex the origin that doesn't contain any full lines. Then construct Spec $\bigoplus \sigma^\vee$ where $\bigoplus \sigma^\vee$ is the semigroup algebra of the dual cone.*

The picture is to put the integral Delzant polytope as $(1, P) \subset \mathbb{R}^{n+1}$. Then form the cone whose integer points are $(k, n \in kP)$ for all $k \in \mathbb{N}$.

27.2.4 Example ($\mathbb{CP}^1$)

27.2.5 Example ($\mathbb{CP}^2$) *$\mathbb{CP}^2$ can be represented as $\mid z_1 \mid^2 + \mid z_2 \mid^2 + \mid z_3 \mid^2 = 1$ modulo phase. That is doing the real rescaling first and leaving the remaining $U(1)$ for later. Defining $x_i = \mid z_i \mid$ gives a triangle $x_1 + x_2 + x_3 = 1$. Above every point inside we have the two torus of the relative phase between z_1 and z_2 and between z_2 and z_3. That is as long as none of them were 0. In those boundary cases, the fiber becomes a circle instead.*

27.2.6 Example () *Take the symplectic quotient $X = \mathbb{C}^{k+3}//G$ where $G = U(1)^k$ each α factor acting by $\phi_i \to e^{iQ_i^\alpha \theta_\alpha}\phi_i$ for some integer vectors Q^α. Take the moment maps to be*

$$\mu = \sum_i Q_i^\alpha \mid X \mid_i^2 - r^\alpha$$

This is just a symplectic manifold right now, but it is Calabi-Yau-able when $\sum_i Q_i^\alpha = 0$

27.2.7 Definition (Hypertoric)

27.3 Categorical Quotients

27.3.1 Definition (Categorical Quotient) *Let $\mathcal{C}$ be the category ... X an object with a G action given by morphisms σ_g $\forall g$. Then the categorical quotient is the universal object Z for all arrows f $X \to Y$ satisfying $f \cdot \sigma_g = f$*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ Z \arrow[ur, dashed] & \end{tikzcd}$$

27.3.2 Example (Trivial Action) *$Z = X$*

Now specialize to $\mathcal{C} = \mathbb{C} - Alg^{op} \underset{\Gamma}{\overset{Spec}{\rightleftarrows}} Aff_{Spec\,\mathbb{C}}$ so $X = SpecA$.

27.3.3 Definition (Affine Quotient) $Spec\Gamma(X, \mathcal{O})^G = SpecA^G$

27.3.4 Definition (Projective Quotient)

27.3.5 Definition (Stacky Quotient)

Let red stand for arrows as schemes. Purple for arrows as affine schemes (Has red in it too). Green is for arrows as stacks.

$$\begin{tikzcd}
Spec(A) \arrow[r] \arrow[rr, bend left] \arrow[d] & Y & Spec(B) \\
Spec(A^G) & &
\end{tikzcd}$$

$Proj \bigoplus_n A^{\chi^n}$

27.4 Equivariant (Co)(BM) Homology

27.4.1 Cohomology

Let the following: torus $T = (\mathbb{C}^*)^r$, $V_N = (\mathbb{C}^{N+1})^r$, $V \setminus \{0\} \to (\mathbb{P}^N)^r$ so that when $N \to \infty$ this gives $ET \to BT$ and M isa algebraic variety with an algebraic T action. Also M has a locally closed T embedding into a smooth projective T variety.

27.4.1 Definition

$$\begin{aligned}
M_V &= (V_N \setminus \{0\}) \times_T M \\
H^i_{T,V_N}(M) &= H^i((V_N \setminus \{0\}) \times_T M) \;\; N \geq i \\
H^i_{T,V_{N_1}}(M) &\simeq H^i_{T,V_{N_2}}(M) \;\; \forall N_1\, N_2 \geq i
\end{aligned}$$

27.4.2 Lemma

$$\begin{aligned}
H^i_{T,V_N}(pt) &= H^i((\mathbb{P}^N)^r) \\
H^\bullet(\mathbb{P}^N) &= \mathbb{C}[a]/(a^{N+1} = 0) \\
H^\bullet_T(pt) &\simeq \mathbb{C}[a_1 \cdots a_r] \\
\mid a_i \mid &= 2
\end{aligned}$$

This is a ring isomorphism under cup product.

27.4.3 Lemma *For trivial action.*

$$\begin{aligned} H_T^\bullet(M) &\simeq H^\bullet(M) \otimes_{\mathbb{C}} H_T^\bullet(pt) \\ M &= pt \\ H_T^\bullet(pt) &\simeq H^\bullet(pt) \otimes_{\mathbb{C}} H_T^\bullet(pt) = \mathbb{C} \otimes_{\mathbb{C}} H_T^\bullet(pt) \end{aligned}$$

For a T-equivariant continuous map $f\ M_1 \to M_2$

$$\begin{aligned} H_T^\bullet(M_2) &\to H_T^\bullet(M_1) \\ i\ M &\to pt \\ H_T^\bullet(pt) &\to H_T^\bullet(M) \end{aligned}$$

For a free action.

$$H_T^\bullet(M) \simeq H^\bullet(M/T)$$

For inclusions of tori $T' \hookrightarrow T$ and restriction of action.

$$H_T^\bullet(M) \to H_{T'}^\bullet(M)$$

27.4.4 Example ($\mathbb{P}^{N-1}$ with $T = (\mathbb{C}^*)^N$) *Taking the obvious action on the $\mathbb{C}^N$ then $H_T^\bullet(\mathbb{P}^{N-1}) \simeq \frac{\mathbb{C}[c_1(V), u_1 \cdots u_N]}{Rel}$ where $c_1(V)$ is the Chern class of the tautological line bundle. The relations Rel are $(c - u_1) \cdots (c - u_N)$ because of the restrictions of the tautological bundle to the fixed points which were the coordinate lines.*

The classes of the fixed points p give $\prod_{k \neq p}(c_1(V) - u_k)$. They are eigenvectors for multiplication by $c_1(V)$ with eigenvalue u_p.

$$\begin{aligned} c_1(V) \prod_{k \neq p}(c_1(V) - u_k) &= (c_1(V) - u_p + u_p) \prod_{k \neq p}(c_1(V) - u_k) \\ &= 0 + u_p \prod_{k \neq p}(c_1(V) - u_k) \end{aligned}$$

27.4.5 Definition (Pairing on $H_T^\bullet(X)$) *Do cup product and integrate X. The matrix realizing this pairing gives*

$$Y(x) = \sum_{p \in X^T} [c_p] \otimes [c_p]$$

Substitute x for $c_1(V) \otimes 1$ etc for everything in the first factor. In the example of $X = \mathbb{P}^{N-1}$

$$Y(x) = \sum_{p \in X^T} \prod_{k \neq p} (x - u_k) \otimes [c_p]$$

27.4.2 Borel-Moore Homology

27.4.6 Definition

$$\begin{aligned} H^{T,BM}_{i,V}(M) &\equiv H^{lf}_{i+2dimV-2dimT}(M_V) \\ H^{T,BM}_{i,V}(M) &\simeq H^{T,BM}_{i,W}(M) \end{aligned}$$

27.4.7 Lemma *It is a module over $H^\bullet_T(M)$ and by $H^\bullet_T(pt) \to H^\bullet_T(M)$*

27.4.3 Ordinary Equivariant Homology

The graded dual of the equivariant cohomology $H^\bullet_T(M)$. It is a comodule for $(H^\bullet_T(pt))^\star$

27.4.4 Equivariant K theory

27.4.8 Definition

$$\begin{aligned} K^G_i(X) &\equiv \pi_i(B^+ Coh^G(X)) \\ K^G_0(X) &\equiv K(Coh^G(X)) \end{aligned}$$

27.4.9 Theorem *Let $Z \to X$ be a closed immersion of equivariant algebraic schemes. and open immersion $Z - U \to X$.*

$$K^G_i(Z) \longrightarrow K^G_i(X) \longrightarrow K^G_i(U) \longrightarrow K^G_{i-1}(Z) \longrightarrow K^G_{i-1}(X) \longrightarrow K^G_{i-1}(U)$$

27.4.10 Example *Let X have isolated fixed points Z.*

27.5 Quasimaps

Let X be the full flag variety $SL(r+1)/B$. We have the Plucker embedding which maps into $\Pi \equiv \prod_{i=1}^r \mathbb{CP}^{n_i+1}$ as $n_i = \binom{r+1}{i}$. For holomorphic degree d maps from $\mathbb{CP}^1 \to \mathbb{CP}^N$ we compactify to a $(N+1)(d+1)-1$ complex projective space. Do this on each factors of Π with $d_1 \cdots d_r$ to get a compactification of the space of maps $\mathbb{CP}^1 \to X$.

27.5.1 Definition (General quasimap) *Put in more general definition for other curves and G/P*

27.6 Wonderful Compactification

Let λ be a regular dominant weight and $G_{\mathbb{C}}$ be a reductive algebraic group over $\mathbb{C}$

$$\begin{tikzcd} G_{\mathbb{C}} \arrow[r] \arrow[d] & Aut(V_\lambda) \arrow[r] & End(V_\lambda) \setminus \{0\} \arrow[d] \\ G_{\mathbb{C}}/Z(G) \arrow[rr, "\psi"] & & P(End(V_\lambda)) \end{tikzcd}$$

$$\overline{G_{adj}} = \overline{\psi(G_{adj})}$$

This is an arrow of complex varieties with $G_{\mathbb{C}}/Z(G) \times G_{\mathbb{C}}/Z(G)$ action by $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$

If you change, λ, you get isomorphic varieties.

27.6.1 Example (PSL2) *We are sending this inside of* $\mathbb{CP}^3$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to [a : b : c : d]$$

But we have the restriction that the rank is 2. That is:

$$ad - bc \neq 0$$

This picks an open locus of $\mathbb{CP}^3$ *because it is a homogenous polynomial and not equaling 0 is a condition that can be imposed projectively. What we are missing is the rank 1 operators which are a* $\mathbb{CP}^1 \times \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^3$, *The compactification puts that locus back in. It makes it all of* $\mathbb{CP}^3$

This is related to the twistor transform in that

27.6.2 Example (PSL3)

27.6.1 Peter Weyl

27.6.3 Theorem (Algebraic Peter-Weyl)

$$Hom_G(V, \mathbb{C}[G]) \simeq V^*$$

$\mathbb{C}[G]$ is a direct sum of irreducibles because G is reductive. Because of the Froebenius reciprocity statement above, you know that the multiplicity space for V will be V^*

$$\mathbb{C}[G] \quad \simeq \quad \bigoplus_{irrep} V \otimes V^*$$

The analogous compact form uses a real polarization instead of holomorphic polarization. As well as requiring Hilbert space completion.

$$L^2(G_{compact}) \quad \simeq \quad \bigoplus \widehat{V \otimes V^*}$$

27.6.2 Vinberg Semigroup

27.7 Bott Samuelson

For a word w write $P_{s_{i_1}} \times_B \cdots P_{s_{i_n}}$ then mod out by B. Multiply everything together and you get something in G/B. If w reduced this is a resolution of singularities of a Schubert cell. If read these products one at a time you see that this is an iterated bundle of $\mathbb{P}^1$'s. So a $\mathbb{P}^1$ bundle over a $\mathbb{P}^1$ bundle over a $\mathbb{P}^1$ bundle over a $\mathbb{P}^1$ bundle over a etc.

27.7.1 Example *Take $\mathbb{C}_Y[3]$ on P_{sts} mapping to $SL(3)/B$. There are 6 orbits corresponding to the S_3. Get 4 fibers as points and 2 as $\mathbb{P}^1$'s. That allows filling the table of dimensions remembering to shift by 3 homologically.*

27.7.1 Matsuki Correspondence

$$\begin{tikzcd}
 & G(\mathbb{C}) & \\
K \arrow[ur] & G_{\mathbb{R}} \arrow[u] & G_{cpt} \arrow[ul] \\
 & K_{cpt} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

where K_{cpt} is the maximal compact of $G_{\mathbb{R}}$. K is complexification of K_{cpt} and G_{cpt} is the maximal compact for $G(\mathbb{C})$.

27.7.2 Example $G_{\mathbb{R}} = GL_n(\mathbb{R})$, $G_{cpt} = SU_n$, $K_{cpt} = SO_n(\mathbb{R})$ *and* $K = SO_n(\mathbb{C})$

27.7.3 Theorem (Matsuki) *There is an anti-isomorphism of orbit posets between $K\backslash\mathcal{B}$ and $G_{\mathbb{R}}\backslash\mathcal{B}$.*

27.7.4 Theorem (Kashiwara) *There is an equivalence between bounded constructible K-equivariant derived categories $D_c(K\backslash\mathcal{B})$ and the $G_{\mathbb{R}}$ equivariant version $D_c(G_{\mathbb{R}}\backslash\mathcal{B})$*

Chapter 28

Algebraic Geometry Appendix 2

28.1 Moduli stack of bundles

Let C be a curve of genus $g \geq 2$. The moduli space of semistable algebraic vector bundles on C of rank n and degree d is denoted by $U_C(n, d)$. When n and d are relatively prime this is automatically stable and this is a nonsingular projective variety of dimension $n^2(g-1)+1$. Denote $U_C(n)$ as the result of unioning over all degrees, giving all stable vector bundles.

28.1.1 Definition (Stable Vector Bundle)

If you take D modules on the algebraic stack $Bun_G C$, this is the automorphic side of the geometric Langlands conjecture. Above was the GIT quotient that gave an honest space.

The F points of Bun_G

$$Bun_G(C)(F) \quad \simeq \quad G(F)\backslash G(A_F)/G(O_F)$$

Find analogues of eigenfunctions, eigen D-modules. These will be indexed by local systems L on the curve for the dual group. That is the Galois side. In GL case rank N vector bundle with flat connection. On curve automically flat. The D-module is Aut L

28.1.1 Preview of GL

Geometric Class Field Theory if n=1.

Get Line bundles for Bun which is the Jacobian. We need to produce a D-module on this torus $(\mathbb{C}^*)^{2g}$ based on the data of a local system L on the genus g curve. Because it is only acting as a 1D rep, it factors through homology. So we can say all the monodromies by giving 2g monodromies. So the corresponding D-module is the rank 1 local system on Jac with the connection having prescribed monodromies around each of those $\mathbb{C}^*$ factors.

28.2 Hitchin System

We now talk about the Galois side where looking at O-modules on $Loc_{^LG}$ moduli stack of local systems. How do you write a local system? You need to give a holomorphic algebraic bundle E with a holomorphic algebraic connection ∇ on said E.

$$\nabla = d+$$

On this side there are skyscraper sheaves at any particular local system. In the above preview of GL, this was the L we talked about. The other side

Hecke eigenproperty means apply a functor and your object comes back tensored with a vector space. Skyscraper must be such because the support isn't going to change.

On Loc_G can ask about the bad points or torsion sheaves.

28.2.1 Definition (Higgs Bundle) *A Higgs bundle is a pair (E, ϕ) of an algebraic vector bundle on the curve C and a global section $\phi \in \Gamma(C, End(E) \otimes K_C)$. A Higgs bundle is stable if all ϕ invariant proper subbundles to have strictly smaller $\frac{deg}{rank}$. Semistablity allows some to have equal stability condition.*

Like before we will denote the moduli spaces of given ranks and degrees. $\mathcal{H}_C(n, d)$ and without the d if we want to take disjoint union over all d.

Applying Serre duality gives a new Higgs bundle by sending (E, ϕ) to $(E^* \otimes K_C^1, -\phi^\dagger \otimes id)$

where we have seen that $\phi^\dagger\ E^* \otimes K_C^{-1} \to E^*$ induces a map $\phi^\dagger \otimes id\ E^* \otimes K_C^{+1} \to E^* \otimes K_C^2$

If the vector bundle was already stable, then we can give it any Higgs field. Also if the field is 0, then Higgs stability condition is the stability of the bundle. That means that $T^*U_C(n, d) \subset \mathcal{H}_C(n, d)$ where the Higgs field is the cotangent direction. But there is more because we can stabilize an unstable vector bundle with the right Higgs field.

We can use the isomorphism $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[I_1 \cdots I_r]$ of invariant polynomials of degrees d_j. A particular choice of homogenous G-invariant polynomials is tr ϕ^n, the power sums in the eigenvalues. Then we can send the Higgs bundle through this map.

$$(E, \phi) \rightarrow (I_1(\phi), \cdots I_r(\phi)) \in \oplus_s H^0(C, K^{\otimes d_s})$$

28.2.2 Definition (z-connection) *A z-connection is a triple (V, ∇, z) such that ∇ is a differential operator on sections of the principal G-bundle V, but not a connection because Leibnitz rule fails by z.*

$$\nabla(fg) =$$

If $z \neq 0$ then we can rescale to get an honest connection. For $z = 0$ we get a Higgs bundle.

$$D^b_{coh}(Loc, \mathcal{O}) \xrightarrow{z \to 0} D^b_{coh}(Higgs, \mathcal{O})$$

$$D^b_{coh}({}^L Bun, \mathcal{D}) \xrightarrow{z \to 0} D^b_{coh}({}^L Bun, gr\mathcal{D}) \xrightarrow{\simeq} D^b_{coh}({}^L Higgs, \mathcal{O})$$

28.2.3 Theorem *As a stack $T^* Bun_G(\Sigma)$ and stack of Higgs bundles.*

28.2.4 Theorem (Gaiotto Lagrangian) *For X a symplectic vector space with Hamiltonian G action as well as a $\mathbb{G}_m$ action where $\mid \omega \mid = 2$. Then get a Lagrangian by ??? ...*
More generally let X be a smooth symplectic algebraic manifold with $G \times \mathbb{G}_m$ action where the weight $\mid \omega \mid = \ell \geq 1$ and $K_\Sigma^{1/\ell}$ is provided. No longer the special case of $X = \mathbb{C}^{2n}$ and $\ell = 2$.

Proof arXiv: 1703.08578 □

28.3 HyperKahler

28.3.1 HKLR

This does not stand for HyperKahLeR. It stands for the authors of the original paper.

28.3.2 Complex Coadjoint Orbits

A quick diversion before we go back to Higgs Bundles.

(moduli spaces of singular G-instantons on $\mathbb{R}^4$) = ($\mathfrak{g}^*_\mathbb{C}$ coadjoint orbits). Have hyperkahler metrics invariant under G. (not $G_\mathbb{C}$ invariant) See Biquard, Kovalev and Kroenheimer.

28.3.3 Hitchin Integrable System

Take a compact Lie group G and a compact Riemann Surface C, get a space $M_G(C)$ which is a fibration over a base B. The complex structures we can put on it are parameterized by $\xi \in \mathbb{CP}^1$. This is the fact that it is hyperkahler. This is chosen so that $\xi = 0$ corresponds to J_3. We already saw one symplectic structure in seeing T^*Bun (if we did the stable stuff the Higgs was better but as stacks we have $Higgs = T^*Bun$)

$$\begin{aligned} \omega_\xi &= -\frac{i}{2\xi}\omega_+ + \omega_3 - \frac{i}{2}\xi\omega_- \\ \omega_+ &= \omega_1 + i\omega_2 \end{aligned}$$

$$
\begin{array}{c}
M_G(C) = \text{Moduli(Higgs Bundles } (\mathcal{E}\ \Phi)) \\
\downarrow \\
B = \text{Moduli}(S \subset T^*C)
\end{array}
$$

28.3.1 Definition *In given complex structure, right the Kahler form approximately from the spectral curve.*

$$
\begin{aligned}
\omega^{semiflat} &\equiv \partial\bar{\partial}K^{sf} + \omega_{fiber} \\
K^{sf} &\equiv \int_{\Sigma} i^*(\lambda \wedge \bar{\lambda})
\end{aligned}
$$

where i is the inclusion of the spectral curve Σ into T^*C and λ is the Liouville 1-form. The base of the Hitchin fibration tells you what $\Sigma = S$ is. ω_{fiber} is a translation invariant 2-form on the fibers using only homology data.

Has a corresponding metric $g^{semiflat}(-,-) = \omega^{semiflat}(-, I-)$

This formula only works away from discriminant locus where the tori of the integrable system pinch.

28.3.2 Conjecture (Gaiotto-Moore-Neitzke) *For a ray going to ∞ avoiding the discriminant, $g - g^{semiflat} \to 0$*
In fact the correction is given through a function $\Omega = DT\ H_1(\Sigma, \mathbb{Z}) \to \mathbb{Z}$. The heuristic for $DT(\gamma)$ is counting special Lagrangian 2-chains whose boundary has class γ on Σ. This doesn't make sense because can't define special Lagrangian in T^*C. Only okay for a neighborhood of C in T^*C.

28.4 Harder-Narasimhan Filtration

28.4.1 Definition (Slope) *The essential feature of a slope is that it gives a function from the set of objects of an abelian category to a preorder such that the below seesaw inequalities hold. However the general definition involves a positive linearly independent system of r additive functions on $K_0(\mathcal{A})$. The target for the slope is $\mathbb{R}^{r-1}$ with lexicographic order. Usually we take $r = 2$ so there is only one number for the slope.*

28.4.2 Lemma (Short Exact Inequalities) *For a short exact sequence, we have inequalities among the slopes of the 3 pieces. These are called see-saw inequalities.*

Suppose the short exact sequence was $0 \to E \to F \to G \to 0$.

$$
\begin{aligned}
\mu(E) < \mu(F) &\leftrightarrow \mu(E) < \mu(F) < \mu(G) \\
\mu(E) = \mu(F) &\leftrightarrow \mu(E) = \mu(F) = \mu(G) \\
\mu(E) > \mu(F) &\leftrightarrow \mu(E) > \mu(F) > \mu(G)
\end{aligned}
$$

This is satisfied when μ is a ratio of functions which are additive with respect to short exact sequences. That is the $r = 2$ case of the definition above, but the same argument shows see-saw behavior with larger r as well.

Proof Suppose $\mu = f(\frac{r}{d})$ and r and d are additive with respect to short exact sequences, and f is a monotonic function (preferrably with codomain $(0, 1]$ via a usual arctangent trick). f does not change the conclusion about squeezing so we can treat it as identity.

Consider the case with $d \neq 0$ on any of the objects in question.

$$
\begin{aligned}
r(F) &= r(E) + r(G) \\
d(F) &= d(E) + d(G) \\
\frac{r(F)}{d(F)} &= \frac{r(E) + r(G)}{d(E) + d(G)}
\end{aligned}
$$

This is the elementary schoolers dream of adding fractions, known technically as the mediant. Fitting with the name $\mu(F)$ is squeezed between $\mu(E)$ and $\mu(G)$ and same with nontrivial f.

28.4.3 Definition (Semistable Object) *All proper subobjects V of the starting W have slope that is less than or equal to that of the original object. If the inequality is strict instead, we call that object stable rather than semistable.*

28.4.4 Definition (Harder-Narasimhan Filtration) *A filtration $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$ of an object X such that X_i/X_{i-1} are all semistable and the slopes are ordered $\mu(X_1/X_0) > \mu(X_2/X_1) \cdots \mu(X_n/X_{n-1})$*

28.4.5 Theorem *Harder-Narasimhan Filtrations exist and are unique.*

Proof Start with a lemma saying that for any object there exists a unique subobject whose slope is maximal among the slopes of all subobjects. A fortiori, X_1 itself is semistable.

Also assume there is some natural number analog of total dimension such that the quantity for X/X_1 is smaller than that of X whenever X_1 is a subobject and we are taking the cokernel of the inclusion.

By the first lemma, we get a HN filtration of X/X_1 and a quotient map π from X to this quotient.

$$
\begin{array}{lll}
0 = Y_0 \subsetneq Y_1 & \cdots & Y_{s-1} = X/X_1 \\
X_1 \subsetneq \pi^{-1} Y_1 & \cdots & X
\end{array}
$$

The successive quotients are all semistable and there is the inequality on slopes because of induction. Putting $X_0 = 0$ at the beginning maintains the semistability of successive quotients by construction that X_1 was semistable. The slope of X_1/X_0 is that of X_1 which is greater than that of X_2/X_1. To see this look at $X_1 \to X_2 \to X_2/X_1$ with $\mu(X_2) < \mu(X_1)$ which gives the inequalities $\mu(X_1) > \mu(X_2) > \mu(X_2/X_1)$ using lemma 28.4.2.

Uniqueness is similarly done with induction and using these lemmas.

28.4.6 Example (Quiver Representations) *Let Θ be a set of integers for each vertex of the finite quiver. Similarly let σ be the same, but with $\mathbb{N}_{>0}$ instead. Now let the slope be $\frac{\Theta \cdot \dim M}{\sigma \cdot \dim M}$. Usually one takes $\sigma_v = 1$ for all vertices, but we don't have to.*

28.4.7 Example (Vector Bundles On a Curve) *For holomorphic vector bundles on algebraic curves, the slope is $\frac{deg}{rank}$. This slope is measure of ampleness so the stability condition means sub-bundles are at most only as ample than the original.*

28.4.1 Bridgeland Stability

28.4.8 Definition (Slicing) *A slicing of a derived category $\mathcal{D}$ such as $\mathcal{D}^b(Coh(X))$ is a collection of subcategories $\mathcal{P}_\phi$ for all $\phi \in \mathbb{R}$ with the following conditions:*

$\mathcal{P}_\phi[1] = \mathcal{P}_{\phi+1}$ which means that ϕ can be thought of as having the important part being valued in $\mathbb{R}/\mathbb{Z}$ and the integer part is encoding a shift.

$\phi_1 > \phi_2$ implies that $Hom(A, B) = 0$ for $A \in \mathcal{P}_{\phi_1}$ and $B \in \mathcal{P}_{\phi_2}$.

For all $E \in \mathcal{D}$, there are real numbers $\phi_1 > \phi_2 \cdots \phi_m$ and objects $E_i \in \mathcal{D}$ and $A_i \in \mathcal{P}_{\phi_i}$ such that one has a Harder-Narasimhan filtration of the form

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{m-1} \arrow[r] \arrow[d] & E_m = E \arrow[d] \\
& A_1 \arrow[ul] & A_2 \arrow[ul] & & A_{m-1} \arrow[ul] & A_m \arrow[ul]
\end{tikzcd}$$

If we choose the closure of $\mathcal{P}_\phi$ for $\phi \in [0, 1)$ then we get a heart of t-structure. This recovers abelian category from derived category.

28.4.9 Definition (Bridgeland Stability) *A Bridgeland stability condition is assigned to the pair of a derived category $\mathcal{D}^b(Coh(X))$ and a surjective homomorphism $v\ K_0(X) \to \Lambda$ to a finite rank lattice with a norm $| \bullet |_\Lambda$. The data of the stability condition is a pair of a slicing of $\mathcal{D}$ and a function $Z\ \Lambda \to \mathbb{C}$ such that*

$$E \in \mathcal{P}_\phi \implies Z(v([E])) \in \mathbb{R}_{>0} e^{i\pi\phi}$$

The set of $\frac{|Z(v([E]))|}{|v([E])|}$ as E ranges over all nonzero objects in all the $\mathcal{P}_\phi$ is a set of nonnegative real numbers. We demand that the infimum of this set be nonzero.

28.4.10 Theorem (Space of Stability Conditions) *The set of stability conditions $Stab(X, \Lambda, | \bullet |_\Lambda, v)$ on a given $\mathcal{D}^b(Coh(X))$ and $v\ K_0(X) \to \Lambda$ can be given a topology and with that topology it is homeomorphic to a complex manifold of dimension the rank of Λ. This is because only remembering $Z \in Hom(\Lambda, \mathbb{C})$ gives a local homeomorphism. This can be used to build the homeomorphism to a complex manifold.*

A priori $Stab(X, \Lambda, |\bullet|_\Lambda, v)$ *is only a topological space so all properties assigned as a complex manifold must use the additional data pulled back from this particular homeomorphism.*

Proof Bridgeland 2007 - Stability Conditions on Triangulated Categories $\square$

28.4.11 Lemma (Shift by 2) *The action of shifts by arbitrary integers acts on the slicing data in the obvious way by reindexing* $\phi \to \phi + k$. *Then in order to act on the central charge* Z *in a compatible manner* $Z \to (-1)^k Z$. *In particular if we want to leave the forgetful map to the central charge alone, we still have the action by even shifts. This means we can quotient by the action of* $[2]$ *on* $Stab(X, \Lambda, |\bullet|_\Lambda, v)$ *to get a new complex manifold of the same dimension. The difference here is akin to the difference between* $\mathbb{C}^*$ *and the infinite sheeted universal cover as in* $\log(z)$.

28.4.12 Example *If* $K_0(X)$ *is finite dimensional already, we might as well take* $\Lambda = K_0(X)$ *and* v *being the identity. We still have to give the additional data of the norm on this lattice to define* $|v([E])|$.

In this case drop the Λ *and* v *from the* $Stab(X, \Lambda, |\bullet|_\Lambda, v)$ *notation to just have* $Stab(X, |\bullet|_{K_0})$

28.4.13 Example (Finite Type, CY_3, v on Euler form) *Suppose further that* $\mathcal{D}$ *is of finite type. This means that for every pair of objects* A *and* B. *the hom spaces are in finitely many degrees and each one is finite dimensional. This can be condensed into just saying* $\dim_{\mathbb{C}} \bigoplus_{i \in \mathbb{Z}} Hom^i_{\mathcal{D}}(A, B) < \infty$ *by saying none of the direct summands are infinite dimensional and there are finitely many that are nontrivial.*

This descends to an Euler form

$$
\begin{aligned}
\chi(A, B) &\equiv \sum_{i \in \mathbb{Z}} (-1)^i \dim Hom^i_{\mathcal{D}}(A, B) \\
\chi([A], [B]) &\equiv \chi(A, B)
\end{aligned}
$$

where we have abused notation to define χ *both on pairs from the set of objects as well as* $K_0(\mathcal{D})$. *The behavior of* Hom^i *under exact sequences is needed to show that* χ *actually is defined on* K_0. *Now if* $Hom^i_{\mathcal{D}}(A, B) \simeq Hom^{3-i}_{\mathcal{D}}(B, A)^*$, *then this Euler form is skew symmetric.*

We then would like to demand that Λ *has a skew-symmetric (possibly also demanding symplectic)* $\langle -, - \rangle$ *form that extends this. This means that on the image of* v, $\langle v(x), v(y) \rangle = Eul(x, y)$. *We also demand that this extension is a non-degenerate skew-symmetric bilinear form on* Λ.

In particular you could always add copies of $\mathbb{Z}^{1,1}$ *to* Λ *and have them be orthogonal to the image of* v. *The satisfaction of the condition would remain the same. This would give a new space of stability conditions with complex dimension* 2 *greater.*

28.4.14 Theorem (Macri) *For* X *a smooth projective curve of genus* $g \geq 1$, $K_0(X)$ *is not finite rank, but we can consider* $\Lambda = H^\bullet(X, \mathbb{Z})$ *and* $v = ch$. *For this choice* $Stab(X, \Lambda, |\bullet|_\Lambda, v)$ *is homeomorphic to* $\mathbb{C} \times \mathbb{H}$

28.4.15 Theorem (Okada) *For* $X = \mathbb{CP}^1$, *the stability conditions with auxiliary data being the implicit one are* $\mathbb{C}^2$.

28.5 P=W

Let $\mathcal{M}_{Dol}$ be the moduli space of Higgs bundles of rank n and degree d in Dolbeaut sense above. Let $\mathcal{M}_{Betti}$ be the Betti moduli space giving $2g$ elements of GL_n representing the holonomies of the generators of $\pi_1(C)$ subject to the constraint that the product of commutators is $e^{2\pi i d/n} I_n$ up to simultaneous conjugation by PGL_n as a GIT quotient.

$$\begin{aligned} X &\equiv \{A_1 B_1 \cdots A_g B_g \mid A_1^{-1} B_1^{-1} A_1 B_1 \cdots A_g B_g = e^{\frac{2\pi i d}{n}} Id\} \\ \mathcal{M}_{Betti} &\equiv X//PGL_n \end{aligned}$$

The non-Abelian Hodge theorem gives the diffeomorphism between them, but it is not algebraic so we can ask what happens to algebraically defined structures on either side under this diffeomorphism. In particular we can ask about the perverse filtration or the weight filtration.

28.5.1 Definition (Deligne Weight Filtration) *For any complex algebraic variety X there is a filtration*

$$W_{0,k} \subset W_{1,k} \subset \cdots W_{2k,k} = H^k(X, \mathbb{Q})$$

$$\begin{aligned} WH(X,q,t) &\equiv \sum \dim \frac{W_{i,k}}{W_{i-1,k}} t^k q^{\frac{i}{2}} \\ WH(X,1,t) &= \sum \dim H^k t^k = P(X,t) \\ q^{d_X} WH(X,1/q,-1) &= E(X,q) \end{aligned}$$

where the last equation also assumes X is smooth to make contact with the E polynomial (as in $K_0(Var_{\mathbb{C}})$).

28.5.2 Definition (Perverse Filtration) *Let f $X \to Y$ be a proper map of relative dimension d, then you get a filtration on $H^k(X)$ using the decomposition theorem on the $Rf_* \mathbb{Q}_X$ so that $P_0 \subset P_1 \cdots P_k = H^k(X)$*

When X is smooth and Y is affine, then we can easily compute $P_{k-i-1} H^k(X)$ by taking the kernel of i^* where i is the inclusion of $f^{-1}(Y_i) \hookrightarrow X$. The Y_i are generic dimension i sub of Y.

28.5.3 Conjecture (P=W) *Use the perverse filtration on the Dolbeaut side where there is the Hitchin map providing f. Use the weight filtration on the Betti side.*

Chapter 29

Algebraic Geometry Appendix 3

29.1 Regular Singularities

Let P be the differential operator

$$P = a_m(z)\frac{d^m}{dz^m} + \cdots + a_0(z)$$

Let z_0 be a zero of a_m. It is regular when the order of the a_k at this point satisfy

$$ord_{z=z_0}a_k \geq ord_{z=z_0}a_m - (m-k)$$

where we take the order of the zero/pole in the sense of divisors.

The local solution is of the form

$$u(z) \approx (z-z_0)^\lambda(c_0 + c_1(z-z_0) + \cdots) + logarithmic$$

there are m solutions of this form.

Taking solutions $\{u \in \mathcal{O} \mid Pu = 0\}$ gives a local system outside the singularities

Regular holonomic D-module

If a_m is a constant, Turn this into a first order system by

$$(\frac{\partial}{\partial z} + \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 \\ a_0(z)/a_m & a_1(z)/a_m & \cdots & a_{m-1}(z)/a_m \end{pmatrix}) \begin{pmatrix} u(z) \\ u'(z) \\ \cdots \\ u^{(m-1)}(z) \end{pmatrix}$$

`https://arxiv.org/pdf/math/0407524v2.pdf`

29.1.1 Definition (Oper) *A ^LG oper on a curve X is a triple of a ^LG bundle on X, a connection on this bundle and a reduction to Borel ^LB satisfying a condition with ∇. A Miura ^LG oper has another reduction which is now preserved by the connection.*

29.1.2 Lemma *The space of Miura opers whose underlying oper has regular singularities and trivial monodromy is isomorphic to the complete flag manifold $^LG/^LB$*

29.1.3 Lemma *For opers on $SpecR$ like $R = \mathbb{C}[[t]]$ or $\mathbb{C}((t))$, the action of the gauge group $N(R)$ is free so we can take a unique representative*

$$\nabla \;\; = \;\; \partial_t + p_{-1} + \sum_{j=1}^{\ell} v_j(t) p_j$$

where p_j are defined by gradation using an $\mathfrak{sl}_2$ triple. $p_{-1}, 2\rho, p_1$

29.1.4 Theorem (PGL_2 and projective connections) *From the canonical form of the lemma we can look at a single function $v(t)$ under change of coordinate $t = \phi(s)$, $v \to v(\phi(s))\phi'(s)^2 - \frac{1}{2}\{\phi, s\}$ like a projective connection.*

In fact in more generality $Op_G(X) \simeq Proj(X) \times \bigoplus_{i=2}^{\ell} \Gamma(X, \Omega^{d_i+1})$

29.1.5 Theorem (Quadratic Differentials) *$Proj(X)$ is an affine space modeled on $H^0(X, \Omega^2)$ the space of quadratic differentials.*

29.1.6 Theorem (Miura)

29.1.7 Theorem (Wakimoto)

29.1.8 Theorem (Feigin-Frenkel) *There is a canonical isomorphism $Z(\hat{\mathfrak{g}}) \simeq FunOp_{^LG}(D^*)$ of algebras compatible with $Der\mathcal{O}$ and $Aut\mathcal{O}$*

29.1.9 Theorem (Feigin-Frenkel) *$W_k(\widehat{\mathfrak{g}})$ and $W_{^Lk}(\widehat{^L\mathfrak{g}})$*

29.1.10 Theorem (Frenkel-Zhu) *Any flat G bundle on D^* admits an oper structure. That is the forgetful map $Op_G(D^*) \to Loc_G(D^*)$ is surjective.*

$$\begin{tikzcd}
 & Op_{^LG}(\Sigma \setminus \{p,q,r\}) \arrow[dl, "r?"'] \arrow[d, "q?"] \arrow[dr, "p?"] & \\
Op_{^LG}(D^*) \arrow[d, two heads] & Op_{^LG}(D^*) \arrow[d, two heads] & Op_{^LG}(D^*) \arrow[d, two heads] \\
Loc_{^LG}(D^*) \arrow[d, "forget"] & Loc_{^LG}(D^*) \arrow[d, "forget"] & Loc_{^LG}(D^*) \arrow[d, "forget"] \\
Bun_{^LG}(D^*) & Bun_{^LG}(D^*) & Bun_{^LG}(D^*)
\end{tikzcd}$$

More generally $Op_G(X)$ is the fiber above $\mathcal{F}_{oper}$ for the forget $Loc \to Bun$. So this a special about D^*

29.1.11 Theorem (Frenkel-Gaitsgory) *Take points in $\sigma \in Loc_G(D^*)$ thought of as skyscraper sheaves objects in $\mathcal{O}(Loc_G(D^*))$. Then pick a preimage χ under this surjective map. Use this to determine a central character and get a representation of $U(\hat{\mathfrak{g}}_{\kappa_c})$.*

$$\begin{array}{rcl} V_{0,\chi} & = & Ind^{\hat{\mathfrak{g}}_{\kappa_c}}_{\mathfrak{g}(\mathcal{O}_x)\bigoplus \mathbb{C}\mathbf{1}}\mathbb{C} \\ \Delta_{\kappa_c,x} \;\; \hat{\mathfrak{g}}_{\kappa_c,x} - mod^{G(\mathcal{O}_x)} & \to & \mathcal{D}_{\kappa_c} - mod = \mathcal{D}_{K^{1/2}} - mod(G_{out}\backslash G()/G(\mathcal{O}_x)) \\ \Delta_{\kappa_c,x}V_{0,\chi} & = & \end{array}$$

(G simply connected). This construction gives the Hecke eigensheaf on the $D-mod$ side with eigenvalue E_χ

29.1.12 Theorem (Riemann-Hilbert) *Let X be a complex manifold.*

$$D^b_{reg\,hol}(\mathcal{D}_X) \underset{\psi_X}{\overset{DR_X}{\rightleftarrows}} D^b_{const}(X)$$

$$\begin{array}{rcl} DR_X(\mathcal{L}) & = & \Omega^{top}_X \otimes^L_{\mathcal{D}} \mathcal{L} \\ \psi_X(L) & = & Tempered\ hom(L^*, \mathcal{O}_X)[dim X] \end{array}$$

29.1.13 Definition (Picard-Fuchs)

29.1.14 Definition (Gauss-Manin)

29.1.1 q-W Algebra

Now take the $U_q(\widehat{\mathfrak{g}})$ analog.

29.1.15 Theorem (Miura)

29.1.16 Theorem (Wakimoto)

29.2 Singular Support

29.2.1 Definition (Singular Support/ Microsupport) *Let X be the ambient manifold and F the sheaf of concern. Pick a ball B_x around the point $x \in X$ and a function $f_x \in B_x \to \mathbb{R}$ such that $f(x) = 0$ and $df(x) = \xi$ where $\xi \in T^*_x X$. Then we get a subset of the ball B_x by taking the subset such that $f(y) < 0$ and calling that N_f. Now look at the cone $H^j(B_x, F) \to H^j(N_f, F)$ and take the limit as shrink B_x around x. If those maps are all isomorphisms, then it is said that F propogates at x in the direction ξ. Otherwise if F does not propogate at (x, ξ) then (x, ξ) is said to be in the singular support of F.*

29.2.2 Proposition

- *$SS(F)$ is conical*

- *If $(x,p) \in SS(F)$, then $x \in Supp(F)$*
- $Supp(F) = SS(F) \bigcap (X \simeq 0_X \subset T^*X)$
- *If $F_{1,2,3}$ form an exact sequence, then $SS(F_i) \subset SS(F_j) \bigcup SS(F_k)$ for any permutation of 123. The symmetric difference of any two is contained in the third. This generalizes to a property of all distinguished triangles when considering objects in the derived category.*

29.3 Quantum Geometric Langlands

$D^b(\mathcal{O} - mod(Loc_{^LG})) \qquad D^b(\mathcal{D} - mod(Bun_G))$

These are equivalent in case of GL_1. See above at the Section 28.1.1

Stable maps $\mathbb{CP}^1 \to X$ with X Nakajima quiver variety.

29.3.1 Quantum q-Geometric Langlands

29.4 Irregular Singularities

If you have a irregular holonomic $\mathcal{D}_X$ module, you can take the regularized version $\psi_X(DR_X(\mathcal{M}))$. On the constructible side these two become the same.

`https://arxiv.org/pdf/1705.07610.pdf`

29.4.1 Example

$$Airy \quad =$$

29.4.2 Theorem (Biquard-Boalch)

29.4.3 Remark Boalch's article from 2001 put online instead of print only `https://arxiv.org/pdf/2002.00052.pdf`

29.4.4 Definition ($\mathfrak{g}$ Quasi-Poisson)

29.4.5 Definition (Fusion of $\mathfrak{g}$ Quasi-Poisson Manifolds)

29.4.6 Definition (Quasi-Hamiltonian G-space)

29.5 Satake

Place somewhere else later

Copied from `http://math.mit.edu/~ptingley/QuantumGroupsSpring2011/lecture11.pdf`

29.5.1 Theorem (Classical Satake) *Let G be a Chevalley group and $G^\vee$ it's Langlands dual corresponding to switching roots and coroots. Let K be a non-Archimedean local field like $\mathbb{Q}_p$ or more concretely $\mathbb{C}((t))$ and $\mathcal{O}$ it's ring of integers like $\mathbb{Z}_p$ or $\mathbb{C}[[t]]$. Then there is an isomorphism from compactly supported functions on the double quotient of $G(K)$ by $G(\mathcal{O})$ to $K_0 RepG^\vee \otimes \mathbb{C}$*

$$\mathbb{C}_c[G(\mathcal{O})\backslash G(K)/G(\mathcal{O})] \quad \simeq \quad K_0 RepG^\vee \otimes \mathbb{C}$$

29.5.2 Example *Let G be a torus T. So get no dominant condition because trivial to take Weyl group invariants just compactly supported functions on the lattice.*

$$1_\lambda \star 1_\mu \quad = \quad 1_{\lambda+\mu}$$

29.5.3 Theorem (Geometric Satake) *This can be upgraded to a categorical level.*

$$\begin{tikzcd}
Perv[G(\mathcal{O})\backslash G(K)/G(\mathcal{O})] \arrow[r, "\simeq"] \arrow[d, "\boldsymbol{H}"] & Rep\ G^\vee \arrow[d, "Forget"] \\
Vect \arrow[r] & Vect
\end{tikzcd}$$

Given just the left down arrow, you know that it has to be a representation category so the work goes into seeing that it is actually G^v

Proof See Lusztig, Drinfeld, Ginzburg, Mirkovic-Vilonen □

Those perverse sheaves can be viewed instead as $G(\mathcal{O})$ equivariant perverse sheaves on the affine grassmannian $G(K)/G(\mathcal{O})$

Fix a maximal torus $T \subset G$ and let W be the Weyl group. Then the coweight lattice is $Hom(G_m, T)$ and given a coweight we can construct a map from the formal punctured disk $SpecK$ to G by composition.

$$SpecK \longrightarrow G_m \xrightarrow{\lambda} T \longrightarrow G$$

Call the composition t^λ. It has a very concrete description in the case of GL. It is the diagonal matrix with entries t^{λ_i}

The T fixed points on the affine Grassmannian are precisely these elements. In addition each $G(\mathcal{O})$ orbit contains the W orbit of a unique dominant coweight t^λ. You can find which one by diagonalizing the matrix over the Taylor series ring.

Therefore you can index $G(\mathcal{O})$ orbits by dominant coweights as Gr^λ. This gives a intersection homology sheaf as an example perverse sheaf given by IC_{Gr^λ}. This gets sent to some representation. In fact it is $V(\lambda)$

Choose a pair of opposite Borels B and B^- such that their intersection is the torus T. Let N and N^- be their unipotent radicals. Each $N(K)$ orbit contains a unique torus fixed point. This follows from Iwasawa KAN decomposition. Call the $N(K)$ orbit running through t^ν by S^ν. Do the same with N^- and call those R^μ

29.5.4 Theorem (Mirkovic-Vilonen [?]) *There is an isomorphism of functors from hypercohomology to the following*

$$\begin{aligned} \boldsymbol{H}^\bullet &\xrightarrow{\simeq} \bigoplus_\nu H_c^{2\rho(\nu)}(S^\nu, -) \\ V(\lambda) = \boldsymbol{H}^\bullet(IC_{\bar{Gr^\lambda}}) &\simeq \bigoplus_\nu H_c^{2\rho(\nu)}(S^\nu \bigcap Gr^\lambda, IC_{\bar{Gr^\lambda}}) \\ &= \bigoplus_\nu H_c^{2\rho(\nu+\lambda)}(S^\nu \bigcap Gr^\lambda, \mathbb{C}) \end{aligned}$$

This means that the irreducible components of $Gr^\lambda \bigcap S^\nu$ index a basis for the weight space $V(\lambda)_{-\nu}$. You might as well use components of the closure. These are called Mirkovec-Vilonen cycles

29.5.5 Definition (Perverse Sheaf)

29.5.6 Example *For smooth X, $\mathbb{C}_X[dimX]$ is self dual because it'll get shifted by $dim_R X = 2dimX$ upon Poincare duality. Checking 0 above the diagonal is now easy.*

29.5.7 Example (Proper pushforwards) *For pushforward along proper maps*

29.6 Affine Springer Fiber

29.6.1 Definition (Affine Springer) *Replace $T^*(G/B) \simeq \tilde{\mathcal{N}} \to \mathcal{N}$ with affine grassmanian case Gr. Let x be a t-adically nilpotent element of the loop group's Lie algebra. This will replace the nilpotent element of before.*

29.6.2 Example *Infinite chain of $\mathbb{P}^1$ in the case of $SL_2(t)$*

Local picture of Hitchin fibrations. Replacing the entire curve with formal discs.

29.6.3 Definition (Compactified Jacobian) *If the curve is smooth, then already compact. Otherwise have to compactify. Want to do so in in a modular way. Use all torsion free degree 0 rank 1 sheaves instead of line bundles. Alternatively quotient by lattice action that changes degrees rather than picking degree 0 representative on each component.*

29.6.4 Lemma (Ideal Sheaves) *Plane curve. Take point from $Hilb^n(C)$ this gives point in Pic by Abel-Jacobi map. From there go to Springer fiber.*

If n large enough compared to $2g-2$, then AJ isomorphism when curve is irreducible.

29.7 Resurgence

29.7.1 Definition (Borel Sum) *Given an element of $A[[g]]$ expressed as $\sum_{n=0}^{\infty} c_n g^n$ with A being an algebra over $\mathbb{Q}$ usually $\mathbb{C}$, the Borel transform is $\sum_{n=0}^{\infty} \frac{c_n}{n!} t^n dt$ as an element of $A[[t]]dt$. This is thought of as a holomorphic one form on the t plane, but we haven't imposed any convergence yet.*

29.7.2 Definition (Resummation Operator)

$$
\begin{aligned}
L_\theta(\hat{\phi}(t)dt) &= \int_0^{e^{i\theta}\infty} e^{-t/a}\hat{\phi}(t)dt \\
S_\theta &\equiv L_\theta(B(-)) \\
&\in A[[g]] \to A[[a]]
\end{aligned}
$$

29.7.3 Definition (Stokes Automorphism) *The two Laplace transforms for angles $\theta + \epsilon$ and $\theta - \epsilon$ are related by a precomposition by an automorphism on the convolutive algebra of the Borel plane. This automorphism is called the Stokes automorphism.*

It still can be thought of as a flow of a vector field of the alien derivatives

$$
\mathfrak{S}_\theta = \exp(\sum_{\omega \in \Gamma_\theta} \Delta_\omega)
$$

the reason it is called alien is it is a derivation for the convolution product rather than the usual product.

29.7.4 Lemma (Relation to Dirichlet series)

$$
\begin{aligned}
\tilde{f}(s) &\equiv \sum_{n=1}^{\infty} f_n n^{-s} \\
F(t) &\equiv Z\left[\tilde{f}\right] = \sum_{n=1}^{\infty} f_n (e^{-t})^n \\
\Gamma(s+1)\tilde{f}(s+1) &= \int_0^\infty t^s F(t) = \int_0^\infty (-\log a)^s F(t)
\end{aligned}
$$

Mellin transform relating ordinary generating series with parameter $a = e^{-t}$ and Dirichlet series.

https://en.wikipedia.org/wiki/Zeta_function_regularization

Chapter 30

Symplectic Resolutions

[?]

30.0.1 Definition (Symplectic Singularity) *A normal algebraic variety X such that the regular locus for any resolution of singularities carries an algebraic closed nondegenerate 2-form. A symplectic resolution is one where it extends from the regular locus to the entire $\tilde{X}$.*

30.0.2 Definition (Conical symplectic resolution) *An affine symplectic singularity carrying a conical $\mathbb{G}_m$ action that acts on the symplectic form with weight $n > 0$.*

Unwrapping this definition, X is affine normal and algebraic and carries a Poisson structure. The resolution of singularities $\tilde{X}$ is smooth and carries an algebraic closed nondegenerate 2-form ω. The $\mathbb{G}_m$ action on both compatible with π $\tilde{X} \to X$ acts with weight n.

30.0.3 Theorem *Let (V, ω, G) be an irreducible symplectic reflection group. Then V/G admits a symplectic resolution if and only if it admits a smooth Poisson deformation. This is also equivalent to $X_c(G) = SpecZ(H_{0,c}(G)) = Spec(eH_{0,c}(G)e)$ being smooth for generic c*

$$
\begin{aligned}
H_{t,c}(G) &= \frac{TV^* \rtimes \mathbb{C}G}{(u \otimes v - v \otimes u = t\omega(u,v) - 2\sum_s c(s)\omega_s(u,v)s)} \\
t &\in \mathbb{C} \\
c &\in S \to \mathbb{C}
\end{aligned}
$$

c is a complex function of the symplectic reflections, the ones with $rk(1-s) = 2$ (complex reflection would be 1).

30.1 Beilinson Bernstein

G acts on X so $\mathfrak{g}$ gives me a vector field. This gives an algebra map $U(\mathfrak{g}) \to \Gamma(X, Diff(X))$ global sections of the sheaf of differential operators namely differential operators.

$$\begin{tikzcd}
ker \arrow[rr] & & U(\mathfrak{g}) \arrow[rr] \arrow[d] & & Diff(X) \\
& & U(\mathfrak{g})_0 = U(\mathfrak{g})/ker \arrow[urr, "iso"] & &
\end{tikzcd}$$

$$U(\mathfrak{g})_0 - mod \underset{\Gamma}{\overset{Loc}{\rightleftarrows}} D_\lambda - mod$$

Have a D module so can take it's microlocal support which is a cycle on T^*X, but wanted cycles on M^+ which is a sub so need to take a subcategory. This is called category O.

30.1.1 Definition (Category O) *Finitely generated modules for $U(\mathfrak{g})_0$ that are locally finite for the action of $U(\mathfrak{b})$. Alternatively don't insist center act with fixed central character but instead Cartan acts semisimply. Soergel says these are equivalent.*

30.1.2 Theorem $U(\mathfrak{g})_0 - bimod \xrightarrow{Loc} D_X \boxtimes D_X^{op} - mod \xrightarrow{SingSupp}$ *cycle on* $M \times M$

but wanted a cycle on Z so pick Harish-Chandra bimodules inside which are G equivariant sub of $D_X \boxtimes D_X^{op} - mod$

30.1.3 Theorem *HC_0 is a tensor category acting on $\mathcal{O}_0$. Support intertwines derived tensor product with the product in $\mathbb{C}[W]$ $\exists H_w$ such that $H_w \otimes^L -$ is an autoequivalence and there is a natural equivalence $\theta_w\theta_{w'} \simeq \theta_{ww'}$ only when the lengths add. This still gives you a categorical action of the braid group. If you took Groethendieck group you would lose down to the W action.*

30.2 Springer Resolution

Say we want to consider all nilpotents in $\mathfrak{g}$. The ones where the operator $ad\ x$ is nilpotent. Call this set $\mathcal{N}$. It is closed, Ad G stable subvariety and it is also stable under dilatation. This says $\mathcal{N}$ is a conical variety.

Let $\tilde{\mathcal{N}}$ be the set of pairs of a Borel $\mathfrak{b}$ and a nilpotent $x \in \mathfrak{b}$. The fiber over a given Borel is it's nilpotent elements which are $[\mathfrak{b}, \mathfrak{b}]$. This means we have a vector bundle over the flag variety G/B parameterizing Borels.

Use the Killing form to identify $\mathfrak{g} \simeq \mathfrak{g}^*$. Then there is a natural G equivariant vector bundle isomorphism $\tilde{\mathcal{N}} \simeq T^*\mathcal{B}$

Define the map $\tilde{\mathcal{N}} \to \mathcal{N}$ by projection to the first factor. This is proper and surjective. Every nilpotent lives in some Borel. It is irreducible and a resolution of singularities for $\mathcal{N}$.

In addition to this resolution of singularity perspective, you can also think of it as a moment map for the canonical Hamiltonian G action on $T^*\mathcal{B}$.

The Steinberg variety is the pullback $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$

30.2.1 Theorem *The Steinberg has many components all of the same dimension. One of them is the diagonal.*

30.2.2 Theorem (Gan Ginzburg) *There is an algebra isomorphism from $H^{BM}_{middle}(Z) \otimes \mathbb{C} \simeq \mathbb{C}[W]$ where the left hand side has the convolution product.*

This can be extended to a statement over the rationals which keeps more information like the difference between $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(e)$

30.2.3 Definition *M^+ is the union of the conormal bundles to the B orbits. For the $T^*\mathbb{CP}^1$ example, there is $\mathbb{C}$ and ∞ which upon taking conormal gives the candy wrapper. There is the $\mathbb{P}^1$ and the fiber at ∞ touching at one point.*

30.2.4 Theorem (Gan Ginzburg) *$H^{BM}_d(M^+)$ is a module over $H^{BM}_{2d}(Z)$. In this case we get $\mathbb{C}[W]$ as a regular representation and as an algebra respectively.*

30.3 Return to generality

Consider $T^*\mathcal{B}$ Give two $\mathbb{C}^*$ actions by scaling the fibers call S, and T which is given by a cocharacter extend to cotangent bundle so that it preserves ω. S is what makes it conical.

Now just do generally

Let $Z = M \times_{M_0} M$ and let M^+ be the sub of M such that $lim_{t \to 0} tp$ exists. Again $H^{BM}_{2d}(Z)$ is an algebra acting on $H^{BM}_d(M^+)$ as before.

$K(\mathcal{O}) \otimes \mathbb{C} \simeq H^{BM}_d(M^+)$ this was our regular representation so to categorify the $\mathbb{C}[W]$ action, turn this into a bimodule.

Again categorify to get bimodules acting on a module category.

We don't have an X such that $M = T^*X$, instead want a quantization of M.

30.3.1 Definition (Quantization of M Conical symplectic resolution) *a T-equivariant sheaf of filtered algebras on M and a $T \times S$ graded isomorphism $gr\mathcal{A} \simeq Fun_M$*

For the case we did before we get $U(\mathfrak{g})$ some other central character. Twisted D-modules.

If $M = Hilb(\mathbb{C}^2/\Gamma)$ then get a quotient of spherical rational Cherednik algebra. Remember this is $Hilb$ of a Slodowy slice.

Again try to take singular support to get cycles on M^+ to get what you call category $\mathcal{O}$. A^+ acts locally finitely. In the springer case A^+ is not $U(\mathfrak{b})$ on the nose, but the local finiteness-ness is the same.

Give a version of Harish-Chandra bimodules

30.4 Stable Envelope

30.4.1 Definition (Attracting Correspondence) *Z^σ is the set of $(x, y) \in X \times X^A$ such that the limit as $t \to 0$ is $\sigma(t) \cdot x \to y$ for σ a map $\mathbb{C}^* \to A$ where A are symplectomorphisms of the conical symplectic resolution and T is $A \times \mathbb{C}^*$ that scales it too.*

30.4.2 Definition (Cohomological Stable Envelope)

30.4.3 Definition (K-theoretic Stable Envelope) *$K_T(X^A) \to K_T(X)$ specified by the following conditions*

30.4.4 Definition (Elliptic Stable Envelope)

30.4.5 Definition (Nearby Cycle Functor)

30.5 Symplectic Duality

30.5.1 Definition (dg-Morita theory) *A is a graded algebra $A_0 \bigoplus_{j>0} A_j$ where A_0 is semisimple and as an A module admits a graded projective resolution where the ith degree component is generated over A in degree i. That makes it a Koszul ring. Then make $Ext_A(A_0, A_0)$.*

$$A - mof \qquad\qquad A^! - mof$$

$$D^b(A - mof) \xrightarrow{\simeq} D^b(A^! - mof)$$

equivalence of triangulated categories.

30.5.2 Example (Koszul Duality of Exterior/Symmetric) *Polynomial ring written as $Sym(V^*)$. Let $\dim V = n$. Get the following Koszul resolution, which is written condensedly as $Sym(V^*[1] \to V^*)$*

$$\cdots \qquad \wedge^2 V^* \otimes Sym(V^*) \longrightarrow V^* \otimes Sym(V^*) \longrightarrow Sym(V^*) \longrightarrow k$$

finite free resolution of k even as a dg algebra. This is taking place in dg $Sym(V^*)$ modules. Use this to compute the desired $Ext(k,k)$ so that have $Hom(\wedge^\bullet V^* \otimes Sym(V^*), k) \simeq Hom(\wedge^\bullet V^*, \underline{Hom}(S(V), k))$

Get $Sym(V[1])$ the exterior algebra, the Spec is $V^*[-1]$.

Alternatively if want to resolve in $Sym(V[1]) = \Lambda^\bullet V$ dg modules. Want to get $SpecExt_\Lambda(k,k) = SpecS = V[2]$ so to resolve k as $Sym(V[2] \to V[1])$

30.5.3 Theorem *Relation with odd cotangent bundle.*

Proof `https://arxiv.org/pdf/1004.0096.pdf` □

30.5.4 Example *For a reductive group G, $A = H^\bullet(BG, \mathbb{C})$ and $A^! = H^\bullet(G, \mathbb{C})$*
Think functions on $T[1]BG$ and $T[1]G$ for a deRham model $T[1]\mathfrak{g}[0/-1] \to \mathfrak{g}[0/-1] \bigoplus \mathfrak{g}[1/0]$. This is polynomial/exterior on some generators like $c_2 \cdots c_{2n}$. So $H^n()$ is made up of terms indexed by partitions of n into those $2+2+4=8$ becomes $c_2^2 c_4 \in H^8$
`https://ncatlab.org/nlab/show/Koszul+duality`

$$\begin{array}{ccc} D^b(H^\bullet(G,\mathbb{C}) - mod) & \xrightarrow{\simeq} & D^b(H^\bullet(BG,\mathbb{C}) - mod) \\ \downarrow{\scriptstyle res} & & \\ D^b(H^0(G,\mathbb{C}) - mod) & & \end{array}$$

Field coefficents makes all the Kunneth theorems if necessary easy.

30.5.5 Example (Quadratic Poisson Deformation) *For the polynomial/exterior Koszul duality, we may consider giving a quadratic Poisson bracket that is attempting to deform these polynomial algebras. So* $\pi = \sum c_{ij}^{kl} x_i x_j \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_l}$. *Then* $\pi^!$ *on* $\wedge^\bullet[\xi_1 \cdots \xi_n]$.
In fact there are isomorphisms

$$\begin{aligned}
\pi^! &= \sum c_{ij}^{kl} \xi_k \xi_l \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \xi_j} \\
HP^\bullet(A) &\simeq HP^\bullet(A^!) \\
HP_\bullet(A) &\simeq HP^{-\bullet}(A^!, Hom(A^!, \mathbb{R}))
\end{aligned}$$

When unimodular quadratic Poisson algebra, then the first is promoted to an isomorphism of BV *algebras.*

https://arxiv.org/pdf/1701.06112.pdf

30.5.6 Example *For the relativistic Toda system we have* $c_0 \dots c_n$ *and* $d_0 \cdots d_n$ *with Poisson structure*

$$\begin{aligned}
\pi &= \sum 2c_k d_k \frac{\partial}{\partial c_k} \frac{\partial}{\partial d_k} \\
&- \sum 2c_k d_{k+1} \frac{\partial}{\partial c_k} \frac{\partial}{\partial d_{k+1}} - \sum 2c_k c_{k+1} \frac{\partial}{\partial c_k} \frac{\partial}{\partial c_{k+1}} \\
A\{t^{-1} \log c_k, t^{-1} \log d_k\} &= At^{-2} 2 \\
A\{t^{-1} \log c_k, t^{-1} \log d_{k+1}\} &= At^{-2}(-2) \\
A\{t^{-1} \log c_k, t^{-1} \log c_{k+1}\} &= At^{-2}(-2)
\end{aligned}$$

Change of variables to $T^*\mathbb{C}^{n+1}$ *explicitly given. Can't build functions like* p_θ *from Laurent polynomials in c's and d's because* log *transcendental. Also can only access* $e^{q_i - q_j}$ *not* q_i *or even* e^{q_i}. *There is a nontrivial relation like* $d_0 c_0^{-1} \cdots d_n c_n^{-1} = ?^{n+1} e^{q_0 - q_0} = ?$. *Say* $\hat{T}$ *is the ring in the true differential world for the correct level of regularity imposed.*

$A = \mathbb{C}[c,d] \subset \mathbb{C}[c^\pm, d^\pm] = T$ *which gives* $(\mathbb{C}^*)^n \subset \mathbb{C}^n$ *Poisson map.*

$$\begin{tikzcd}
D^b(A - grmod) \arrow[d, "\simeq"] & D^b(T - mod) \arrow[l, "res"'] & D^b(\hat{T} - mod) \arrow[l, "res"'] \\
D^b(A^! - grmod)
\end{tikzcd}$$

$$(H - \lambda)\psi = 0 \quad \implies \quad \frac{A}{(H - \lambda)} \in A - mod$$

provides some functions on the semiclassical Lagrangian for the state which is a a variety when regarded in the $\mathbb{C}^{2n+2}$ *before dropping to the real* $\mathbb{R}_+^{2n+2} \subset \mathbb{C}^{2n+2}$ *and then further taking the log to pass back to real canonical coordinates.*

30.5.7 Theorem (Beilinson-Ginzburg-Soergel) *Let $L \in \mathcal{O}$ be the direct sum of all the simple highest weight modules $L(\lambda)$ with trivial infinitesimal character. LetP the direct sum of their projective covers.*

$$\begin{aligned} End_{\mathcal{O}}(P) &\simeq Ext_{\mathcal{O}}(L, L) \\ \mathcal{O}_0 &\simeq End_{\mathcal{O}}(P) - mof \\ Ext_{\mathcal{O}}(L, L) &= Ext_{\mathcal{O}}(L, L)^! \end{aligned}$$

For dominant integral weights λ possibly with $\lambda + \rho$ on the walls of the fundamental chamber. Then do $\mathcal{O}_\lambda$ where now generalized infinitesimal character is that of $L(\lambda)$. Again form $L = \sum_{x \in W^\lambda} L(x \cdot \lambda)$ and $P = \bigoplus P(x \cdot \lambda)$. Also for q parabolic with $W^q = W^\lambda$, form $\mathcal{O}^q$ for q-locally finite instead of just $\mathfrak{b}$ locally finite. Form $L^q = \bigoplus L_x^q$ and $P^q = \bigoplus P_x^q$ analogously.

$$\begin{aligned} End_{\mathcal{O}_\lambda}(P) &\simeq Ext_{\mathcal{O}^q}(L^q, L^q) \equiv A^Q \\ End_{\mathcal{O}^q}(P^q) &\simeq Ext_{\mathcal{O}_\lambda}(L, L) \equiv A_Q \end{aligned}$$

with the two rows being Koszul dual $E(A^Q) = A_Q$ to each other.

$$\begin{aligned} \mathcal{O}^q &\simeq A^Q mof \\ \mathcal{O}_\lambda &\simeq A_Q mof = E(A^Q) - mof \\ A^! &\equiv E(A)^{opp} \end{aligned}$$

This Koszul duality takes indecomposable injectives to simples and simples to indecomposable projectives. It takes (graded) dual Vermas in $\mathcal{O}_\lambda$ to (graded) parabolic Vermas in $\mathcal{O}^q$

30.5.8 Proposition *There is an equivalence $D^b(\mathcal{O}^q) \simeq \mathcal{D}_B(G/Q)$ where if we take heart of obvious t-structure on left we get $\mathcal{O}^q \simeq Perv_B(G/Q)$. This takes the simples L_x^q to the IC complexes L_x^Q of the closures of BxQ/Q*

Proof `http://www.ams.org/journals/jams/1996-9-02/S0894-0347-96-00192-0/S0894-0347-96-00192-0.pdf` □

30.5.9 Conjecture (General Symplectic Resolution) *Let M_0 be a conical symplectic singularity with M a symplectic resolution thereof. Form it's $\mathbb{C}$ Picard group H. It will be acted on by some Coxeter group. Form H/W which is $HP^2(M_0)$, the base of the universal Poisson deformation. Let $H_{\mathbb{Z}}$ be the Picard group of that.*
There is also some torus acting Hamiltonian-ly commuting with the conical structure. That is $T \subset G$ where G is a Levi complement of the group of Hamiltonian automorphisms commuting with conical structure. That means you can form a Weyl group $\mathbb{W}$. This is used to construct a $(M)^!$
It is expected $\mathbb{W} \simeq W^!$ as Coxeter group isomorphisms and $\mathfrak{t} \simeq H^!$ conical equivariant isomorphisms.
This means that $HP^2(M_0) \simeq H/W \simeq \mathfrak{t}^!/\mathbb{W}^!$

30.5.10 Example (BGS) *$M_0 = \mathcal{N}$ nilpotent cone for $\mathfrak{g}$, then get the usual H and W. The symplectic dual for $M = T^*G/B$ should be T^*G^L/B^L*

30.5.11 Example (S3 Variety-Webster) *Let Ξ^μ_ν be the preimage from the map $T^2Fl(\nu) \to Slodowy(O_\mu)$, the algebra you get upon quantization is a primitive quotient of the finite W-algebra W_μ (call that ideal J_ν). The dual is Ξ^ν_μ.*

$D^b(W_\mu/J_\nu - mof) \xrightarrow{\simeq} D^b(W_\nu/J_\mu - mof)$

$$\begin{tikzcd} D^b(W_\mu - mof) \arrow[d, "Skryabin"] & D^b(W_\nu - mof) \arrow[d, "Skryabin"] \\ D^b(Whittaker_{?\mu}) & D^b(Whittaker_{?\nu}) \end{tikzcd}$$

30.5.12 Example *$Hilb^n(\mathbb{C}^2) \to S^n\mathbb{C}^2$, the rational line bundle that goes into the definition of K-theoretic stable envelope is a point in H.*

30.5.13 Definition (Koszul Duality for hypertoric category $\mathcal{O}$)

30.5.14 Definition (Truncated Shited Yangian) *Y^λ_μ is an algebra which quantizes the affine Poisson variety that is the affine Grassmannian slice Gr^λ_μ. In fact there are a family of these parameterized by $R \in \prod \mathbb{C}^{\lambda_i}$ where λ_i is the coefficent of ω_i in λ.*

Proof `https://arxiv.org/pdf/1806.07519.pdf` and sources within □

30.6 Implosions and the Moore-Tachikawa Category

`https://arxiv.org/pdf/2004.09620.pdf`

30.6.1 Symplectic Implosion

30.6.1 Definition (Symplectic Implosion) *Let M be a space with a Hamiltonian action of a compact group K. The symplectic implosion of (M, K) is defined as the space M_{impl} with a Hamiltonian action of the maximal torus T such that $M_{impl}//_\xi T \simeq M//_\xi K$ if $\xi \in \overline{\mathfrak{t}^*_+}$ (implicitly giving a subset of $\mathfrak{k}^*$ as well thanks to Killing).*

Suppose $M = T^*K$ with the $K \times K$ action. We can implode with respect to the right action to get $(T^*K)_{impl}$. The left K action is unchanged and the right action has been reduced to a T action. If we know how to do this for all K, then we know how to do it for any M. What we do is do symplectic reduction of $M \times (T^*K)_{impl}$ by the diagonal K action using the given action on M and the remaining left action.

30.6.2 Moore-Tachikawa

30.6.2 Definition (Moore-Tachikawa) *A category whose objects are either complex semisimple or reductive groups. Morphisms are given by complex-symplectic manifolds with $G_1 \times G_2$ action* *as well as a circle action that gives the symplectic form weight 2. These are the elements of $Hom(G_1, G_2)$.*

Compositon of morphisms is given by taking the complex-symplectic quotient of $X \times Y$ by the G_2 action for the object in the middle. This leaves an element in $Hom(G_1, G_3)$. $T^*K_{\mathbb{C}}$ is the identity in $Hom(K_{\mathbb{C}}, K_{\mathbb{C}})$

30.6.3 Lemma (Forgetting and Trivial Actions) *Suppose M be a complex-symplectic manifold with circle action of weight 2. Let $x \in Hom_{Grp}(G_1 \times G_2, Aut(M))$. For every such x, we have an element of $Hom(G_1, G_2)$. In particular, the trivial action where x sends everything to the identity is always available. That means every Hom set between any two groups has a forgetful map that just gives all complex-symplectic manifolds with circle actions weight 2.*

30.6.4 Corollary *The analog of symplectic implosion is then given by a post composition. Starting with a complex-symplectic manifold with an action of $K_{\mathbb{C}}$, we can think of that in $Hom(1, K_{\mathbb{C}})$. The analog $(T^*K_{\mathbb{C}})_{impl}$ is a complex-symplectic manifold giving an element in $Hom(K_{\mathbb{C}}, T_{\mathbb{C}})$. The process of symplectic implosion is then given by composition.*

Chapter 31

Taming the wild world of algebra

31.1 Tameness and Hereditary

https://mathoverflow.net/questions/5895/what-are-tame-and-wild-hereditary-algebras

31.1.1 Definition *A tame k algebra is one where for all d you can parameterize all isoclasses of indecomposables of dimension d by a finite number of 1 parameter families. A wild algebra is one where mod_A has $mod_{k<x,y>}$ inside as a subcategory.*

31.1.2 Theorem (Drozd Dichotomy) *A finite dimensional algebra is wild or tame.*
In fact if you have A is wild, then mod_A has mod_B inside for all finite dimensional algebras B.

31.1.3 Definition (Finite Global Dimension) *Every $X \in A - mod$ (finitely generated left A modules) admits finite projective resolution. Take the maximal over simple objects of their minimal length resolutions. This is preserved among Morita equivalence so we can give some basic forms for A as something more manageable.*

31.1.4 Theorem *If we further assume hereditary (global dimension 1) over an algebraically closed field, then it is Morita equivalent to a path algebra without oriented cycles. Then the tame or wild dichotomy becomes is it Dynkin/extended Dynkin or not respectively.*
If we just have finite global dimension, but not hereditary it is a quiver with relations $A \simeq kQ/I$ for some admissible two sided ideal.

Proof https://arxiv.org/pdf/1209.2093.pdf □

31.2 Hall Algebra

31.2.1 Definition (Hall Algebra) *The Hall algebra for a small abelian category is defined as taking constructible functions on objects modulo equivalence. It is an algebra by*

$$f \star g(M) \quad = \quad \int f(\lambda)g(m/\lambda)d\lambda$$

In particular let f be the characteristic function for an object M and g characteristic for object N then this gives $f \star g(P) =| \{0 \to M \to P \to N \to 0\} |$ (maybe switched the order, need to check)

The unit for this algebra is the characteristic function for the unit object.

https://arxiv.org/pdf/math/0611617v2.pdf

31.2.2 Theorem (Ringel) *For a quiver Q let $\mathcal{A}$ be finite dimensional kQ modules over $\mathbb{F}_q$. Q defines a generalized Cartan matrix which makes a derived Kac-Moody $\mathfrak{g} = \mathfrak{n}^+ \bigoplus \mathfrak{h} \bigoplus \mathfrak{n}^-$. Then we get $U_{\sqrt{q}}\mathfrak{n}^+ \hookrightarrow H_{tw}(\mathcal{A})$ the twisted Hall algebra of said category as algebras. It is an isomorphism in the simply laced Dynkin example. E_i goes to the simple module sitting at that vertex. This generates.*

31.2.3 Theorem (Green,Xiao) *In the hereditary case, one can give a categorical coproduct and antipode as well. So that the Hopf algebra $U_{\sqrt{q}}\mathfrak{n}^+$ is described in terms of $\mathcal{A}$.*

31.2.4 Theorem (1.1) *Let $\mathcal{A}$ be the $\mathbb{F}_q$ reps of a finite quiver without oriented cycles, then $U_{\sqrt{q}}\mathfrak{g} \hookrightarrow DH_{red}(\mathcal{A})$. It is an isomorphism when simply laced Dynkin.*

31.2.5 Theorem (1.2) *Let $\mathcal{A}$ be artinian or noetherian and satisfy the following. Essentially small with finite morphism spaces, linear over F_q, finite global dimension and enough projectives, actually make that global dimension 1 and nonzero objects go to nonzero classes in K group. Once we have such $\mathcal{A}$ then $DH(\mathcal{A})$ the derived Hall algebra is equivalent to the Drinfeld double of the extended twisted Hall algebra of the category. The reduced then makes it reduced double.*

31.2.6 Example *Look at 1.1. Then we have*

$$\begin{array}{ccccc} U_{\sqrt{q}}\mathfrak{g} & \xrightarrow{\simeq} & DH_{red}(\mathcal{A}) & \xrightarrow{\simeq} & D_{red}(H^e_{tw}(\mathcal{A})) \\ & & & & \\ D(U_{\sqrt{q}}\mathfrak{n}) & & & & D(H_{tw}(\mathcal{A})) \end{array}$$

https://arxiv.org/pdf/1111.0745v1.pdf

31.2.7 Theorem (Feldvoss-Witherspoon) *For a simple Lie algebra, the principal block of the small quantum group $u_\zeta\mathfrak{g}$ is tame if and only if $\mathfrak{g} = \mathfrak{sl}_2$*

31.2.8 Theorem (Kulshammer) *Auslander-Reiten theory for small quantum groups. Any tame block of $u_\zeta\mathfrak{g}$ is Morita equivalent to a block of $u_\zeta\mathfrak{sl}_2$.*

https://arxiv.org/pdf/1601.06687v1.pdf for some more about Auslander/Hopf interaction.

31.3 Finite W Algebras

https://arxiv.org/pdf/0912.0689v2.pdf

31.3.1 Example ($e = 0$) *Then the W-algebra is $U(\mathfrak{g})$.*

31.3.2 Example (The Regular Nilpotent) *Then the W-algebra is $Z(U(\mathfrak{g}))$.*

`http://arxiv.org/pdf/1505.08048v1.pdf`

31.3.3 Lemma (3.7 Losev) *Let G be a semisimple group and O a nilpotent orbit. There is a natural bijection between the set of quantizations of O with Hamiltonian G action and the set of primitive ideals $J \subset U$ with associated variety $\bar{O}$ and multiplicity U/J on O is 1. This is also in bijection with 1 dimensional A stable W modules. For example, central characters in the regular case.*

`https://arxiv.org/pdf/0912.0689v2.pdf`

31.3.4 Definition (3.2) *W_χ is the set $\bar{y} \in U(\mathfrak{g})/I_\chi$ such that $(a - \chi(a))y = 0$ for all $a \in \mathfrak{m}$*

31.3.5 Lemma

$$\begin{aligned}
\Delta(y + i_\chi) &= y \otimes 1 + 1 \otimes y + i_\chi \otimes 1 + 1 \otimes i_\chi \\
\Delta^2(y + i_\chi) &= y \otimes 1 \otimes 1 + 1 \otimes y \otimes 1 + 1 \otimes 1 \otimes y + i_\chi \otimes 1 \otimes 1 + 1 \otimes i_\chi \otimes 1 + 1 \otimes 1 \otimes i_\chi
\end{aligned}$$

31.3.6 Lemma (Lemma 35) *Given a Whittaker $\mathfrak{g}$ module E with action ρ, The subspace of Whittaker vectors $Wh(E)$ is a W_χ module by $\bar{y}.v = \rho(y)v$. Conversely, for V a W_χ module $Q_\chi \otimes_{W_\chi} V$ is a Whittaker $\mathfrak{g}$ module by*

$$\begin{aligned}
y.(q \otimes v) &= (y.q) \otimes v \\
y &\in U(\mathfrak{g}) \\
q &\in Q_\chi = U(\mathfrak{g})/I_\chi
\end{aligned}$$

`http://arxiv.org/pdf/1004.1669v1.pdf`

31.3.7 Theorem (1.1) *Let $\tilde{O}$ be a G equivariant covering of O. Pick a point $x \in \tilde{O}$ over e and set $\Gamma = G_x/(G_x)^o$. Then the set of quantizations of $\tilde{O}$ is in bijection with the set of $(Id^1(W))^\Gamma$. of Γ fixed points in the set of two-sided ideals of codimension 1 in W denoted by Id^1.*

31.3.8 Theorem (Brieskorn and Slodowy) *The Slodowy slice for a subregular nilpotent e in simple Lie algbra $\mathfrak{g}$ is isomorphic as a Poisson variety to $\mathbb{C}^2/\Gamma$ where Γ is the finite subgroup corresponding to that ADE Dynkin diagram of $\mathfrak{g}$.*

Proof `http://arxiv.org/pdf/0905.0686v2.pdf` Page 4 ☐

31.3.9 Definition (Translation) *$V \in Rep^{fd}(U(\mathfrak{g}))$ acts by $- \otimes V$ on Whittaker modules. Transporting through Skryabin equivalence gives an exact endofunctor. $\otimes V$ and $\otimes V^*$ are biadjoint functors. That is a module category for $Rep\, U\mathfrak{g}$ so something that can show up at a boundary.*

31.3.10 Theorem (Ginzburg-Kumar) .

$$\begin{tikzcd}
HH^{2\bullet}(u_\xi) & \\
H^{2\bullet}(u_\xi, u_\xi^{ad}) \arrow[u, "\simeq"] & k(\tilde{N}) \\
H^{2\bullet}(u_\xi, k^{triv}) \arrow[u, "inj"] \arrow[r, "GK"] & k^\bullet(\mathcal{N}) \arrow[u, "inj"'] \\
Ext^{2\bullet}_{u_\xi}(k^\epsilon, k^{triv}) \arrow[u, "\simeq"] &
\end{tikzcd}$$

$$Ext^\bullet_{u_\xi\, bimod}(u_\xi, M) \xrightarrow{\simeq} H^\bullet(u_\xi, M^{ad})$$

31.3.11 Theorem *Cyclic homology of the Taft algebras and of Auslander algebras. Rachel Taillefer. Have a* $\Lambda_n \simeq u_q^+\mathfrak{sl}_2$ *for* q *nth root. This then computes Hochschild homology, and cyclic homology of this because it is of the form* kQ/m^n *for a given quiver and ideal* m^n*.*

$$\begin{aligned}
HH_{2c,cn} &= k^{n-1} \\
HH_{2c-1,cn} &= k^{n-1} \\
HH_{0,0} &= k^n
\end{aligned}$$

and the rest 0*.*

Analogously

$$\begin{aligned}
HC_{2c} &= k^n \\
HC_{2c+1} &= k^{n-1}
\end{aligned}$$

for $c \in \mathbb{N}$

31.3.1 Affine Analog

`https://arxiv.org/pdf/1611.04937.pdf`

31.3.12 Theorem (Affine Skryabin) *There is a canonical equivalence* $Whit(\hat{\mathfrak{g}_\kappa}) - dg - mod$ *to* $\mathcal{W}_\kappa - dg - mod$ *such that the composition from* $\hat{\mathfrak{g}}_\kappa - dg - mod$ *to the above then on to dg-Vect is computed by the Drinfeld Sokolov functor.*

31.3.13 Theorem (Categorical Feigin-Frenkel) *There is an equivalence* $Whit(\hat{\mathfrak{g}_\kappa}) - dg - mod$ *with* $Whit(\mathfrak{g}^{\hat{L}}_{\kappa_L}) - dg - mod$*. In particular, we can take* κ *to be the critical level in which case have an isomorphism* $W_k(\widehat{\mathfrak{g}}) \simeq Z(\widehat{\mathfrak{g}})$

31.3.14 Theorem (Gaiotto State)

Chapter 32

Nakajima Appendix

32.1 Quiver Varieties

`http://arxiv.org/pdf/0905.0686v2.pdf`

32.1.1 Definition (Free/Walking/2-Kronecker quiver) *This is the category with 2 objects called E and V. The 2 identity morphisms $E \to E$ and $V \to V$ and 2 morphisms s, t from $E \to V$. Call this category $\mathbf{2}\mathcal{K}$ for 2-Kronecker.*

32.1.2 Definition *The category of quivers in $\mathcal{C}$ is then the functor category $\mathbf{2}\mathcal{K} \to \mathcal{C}$.*

32.1.3 Definition (Quiver as a functor) *A quiver is then an object of this category for $FinSet$. So a particular functor $\mathbf{2}\mathcal{K} \to FinSet$. That is a finite set of vertices and edges.*

32.1.4 Definition (Quiver as a category itself) *Objects are the vertices and morphisms are the arrows.*

32.1.5 Definition (Path Algebra) *The algebra with basis paths in the quiver and product is given by concatenation and extended linearly.*

32.1.6 Theorem (Bialgebra or Hopf structure)

Proof `http://arxiv.org/pdf/1310.6501.pdf` and `https://www.math.ntnu.no/~oyvinso/Papers/newhopf.pdf` □

32.1.7 Definition ($Q^{\heartsuit}$) *The vertex set is $V \bigsqcup V'$ where V' is another copy of V. There are the original edges E for the V vertices and new edges connecting V with the corresponding vertices in V'.*

32.1.8 Example ($Q = A_3$) .

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet & \bullet & \bullet \end{tikzcd}$$

32.1.9 Definition ($\overline{Q}$) *Same vertex set as Q but now there are backwards edges for every edge in Q.*

32.1.10 Example ($Q = A_3$) .
$\bullet \overset{\longrightarrow}{\longleftarrow} \bullet \overset{\longrightarrow}{\longleftarrow} \bullet$

32.1.11 Definition (Preprojective algebra) $(\mathbb{C}\overline{Q})/I_\lambda$ *where I is the two sided ideal generated by*

$$\sum_{x \in Q} xx^\dagger - x^\dagger x - \sum_{v \in V} \lambda_v 1_v$$

where λ_i are parameters. Let $x, x^\dagger$ have degree 2 and λ_i degree 4. This is for the grading by path length before taking the quotient. Giving λ_i degree 4 is so this ideal is homogeneous and the quotient gets a grading. In fact usually we set these parameters all to 0. This grading does not pay attention to the difference between edges from the original Q and their reversed versions x^* so this is called the unoriented grading. An alternative grading is to say all the x^* have degree 2 and the λ_i have degree 2. This again gives a homogenous ideal. So we have P is an algebra over $\mathbb{C}$ which can be regarded as a graded algebra in these two different ways (among other ways).

32.1.12 Theorem *For $\lambda_v = 0$ the quotient $B = k\bar{Q}/I_0$ is $(h-2, 2)$-Koszul when Q ADE Dynkin and actually Koszul otherwise. The λ_v ruin the purely quadratic nature of the relations in favor of a quadratic linear scalar type, and Koszul computations behave better with the purely quadratic case.*

Proof `http://people.bath.ac.uk/masadk/papers/alkos.pdf` □

32.1.13 Lemma

$$\begin{aligned} Rep(\overline{Q}, v) &\simeq Rep(Q, v) \times Rep(Q^{op}, v) \\ Rep(\overline{Q}, v) &\simeq Rep(Q, v) \times Rep(Q, v)^* \simeq T^*(Rep(Q, v)) \end{aligned}$$

These are isomorphisms as
F

32.1.14 Definition ($\overline{Q^\heartsuit}$)

32.1.15 Definition (Rep Variety) *This is the affine variety specified by the representations of specified quiver and dimension vectors. They come with actions of the group $GL(\vec{v})$ that changes the bases of the vector spaces assigned to the vertices (or possibly a subset of those vertices).*

32.1.16 Definition (Nakajima Variety)

32.1.17 Example *Let Q be the quiver with 1 vertex and 1 self loop. $\bar{Q}$ is 1 vertex and 2 self loops. The dimension vector is a single integer v. This means that the Rep variety before quotienting is $\mathfrak{gl}_v \times \mathfrak{gl}_v$. Then doing*

32.1.18 Theorem *They have no odd cohomology, the cohomology is pure and their homology can be represented by T invariant cycles enabling localization.*

Proof

32.1.19 Theorem (Kronheimer 89) *Take an affine Dynkin diagram of type $\hat{A}\hat{D}\hat{E}$. Remove the special vertex 0 which is labelled 1. Turn it into the framing vector spaces W_i attatched to the finite ADE quiver for wherever 0 is attached. Then give $V_i = \mathbb{C}^{d_i}$ as labelled by the diagram. This produces a quiver $\bar{Q}^{\heartsuit}$*

32.2 Hilbert Scheme

32.2.1 Theorem (Haiman) *Consider ordinary Hilbert scheme $Hilb^n(\mathbb{C}^2)$. Then there is a rank $n!$ bundle P on it called the Processi bundle. This is the bundle that gives the $\tilde{H}_\lambda$ all the Macdonald polynomials calculations (transformed from usual type A Macdonald polynomial by ???). It satisfies*

$$\begin{aligned} Hom(P,P) &\simeq \mathbb{C}[x_1 \cdots x_n, y_1 \cdots y_n] \rtimes \mathbb{C}[S_n] \\ Ext^i(P,P) &\simeq 0 \;\; \forall i > 0 \end{aligned}$$

The direct summands of P generate $D^b(Hilb^n(\mathbb{C}^2))$.

The functor $RHom(P,-)$ goes from $D^b(Hilb^n(\mathbb{C}^2))$ to $D^b(Hom(P,P)-mod)$ and it is an equivalence.

32.2.2 Theorem (McKay) *Take $\mathbb{C}^2/\Gamma$. Resolve by taking $(Hilb^{|\Gamma|}(\mathbb{C}^2))^\Gamma$ such that $\mathbb{C}[x,y]/I$ is isomorphic to the regular representation of Γ. This is a minimal resolution.*

$$\begin{tikzcd}
 & & Hilb^n(X) \times Hilb^{n-i}(X) \times X \arrow[dl] \arrow[dr] & & \\
 & Hilb^n(X) \times X \arrow[dl] \arrow[dr] & & Hilb^{n-i}(X) \times X \arrow[dl] \arrow[dr] & \\
Hilb^n(X) & & X & & Hilb^{n-i}(X)
\end{tikzcd}$$

32.2.3 Theorem *The Slodowy slice for a subregular nilpotent e in simple Lie algbra $\mathfrak{g}$ is isomorphic as a Poisson variety to $\mathbb{C}^2/\Gamma$ where Γ is the finite subgroup corresponding to that ADE Dynkin diagram of $\mathfrak{g}$.*

Proof `http://arxiv.org/pdf/0905.0686v2.pdf` Page 4 □

32.2.4 Definition

$$\begin{array}{rcl}\forall i > 0 \; P[i] & \subset & \bigsqcup_n Hilb^n(X) \times Hilb^{n-i}(X) \times X \\ P[i] & \equiv & \{(I_1, I_2, x) \mid \; I_1 \subset I_2 \; Supp(I_2/I_1) = \{x\}\} \\ q_{1n} : \; P[i] & \to & Hilb^n(X) \\ q_{2n} : \; P[i] & \to & Hilb^{n-i}(X) \times X\end{array}$$

32.2.5 Definition

$$\begin{array}{rcl}H_\bullet^{T,BM}(Hilb^{n-i}(X) \times X) & \simeq & H_\bullet^{T,BM}(Hilb^{n-i}(X)) \otimes_A H_\bullet^{T,BM}(X) \\ Conv_1 : \; H_\bullet^{T,BM}(Hilb^{n-i}(X)) \otimes_A H_\bullet^{T,BM}(X) & \to & H_\bullet^{T,BM}(Hilb^n(X)) \\ Conv_1 : ? & \to & q_{1n\star}(q_{2n}^\star(?) \cap [P[i]]) \\ P_{-i}(\beta) : \; H_\bullet^{T,BM}(Hilb^{n-i}(X)) & \to & H_\bullet^{T,BM}(Hilb^n(X)) \\ P_{-i}(\beta)(?) & = & Conv_1(? \otimes \beta) \\ \beta & \in & H_\bullet^{T,BM}(X)\end{array}$$

$$\begin{array}{rcl}Conv_2 : \; H_\bullet^{T,BM}(Hilb^n(X)) & \to & H_\bullet^{T,BM}(Hilb^{n-i}(X)) \otimes_A H_\bullet^{T,BM}(X) \\ Conv_2 : ? & \to & (-1)^i q_{2n\star}(q_{1n}^\star(?) \cap [P[i]]) \\ P_{+i}(\alpha) : \; H_\bullet^{T,BM}(Hilb^n(X)) & \to & H_\bullet^{T,BM}(Hilb^{n-i}(X)) \\ \alpha_0 & \in & H_\bullet^T(X) \\ \alpha = a_{M\star}(? \cap \alpha_0) : \; H_{|?|}^{T,BM}(M) & \to & H_T^{dimM-|?|-|\alpha_0|}(pt) \\ P_{+i}(\alpha)(?) & = & (id \otimes_A \alpha) Conv_2(?)\end{array}$$

32.3 Stable Envelopes

32.3.1 Definition (Attracting Correspondence) *Z^σ is the set of $(x, y) \in X \times X^A$ such that the limit as $t \to 0$ is $\sigma(t) \cdot x \to y$ for σ a map $\mathbb{C}^* \to A$ where A are symplectomorphisms of the conical symplectic resolution and T is $A \times \mathbb{C}^*$ that scales it too.*

32.3.2 Definition (Cohomological Stable Envelope)

32.3.3 Definition (K-theoretic Stable Envelope) $K_T(X^A) \to K_T(X)$

32.3.4 Definition (Elliptic Stable Envelope)

32.3.5 Theorem

$$R \;\; = \;\; Stab_?^{-1} Stab_?$$

See dynamical R-matrix where factor $R = J_\lambda^{-1} J_?$

Chapter 33

Fock-Goncharov Appendix

33.1 Total Positivity

https://arxiv.org/pdf/math/9912128.pdf

33.1.1 Definition ($GL(n,\mathbb{R})_{\geq 0}/GL(n,\mathbb{R})_{>0}$) *Matrices such that all minors are nonnegative/positive. This is a strengthening of positive (semi)definite where the condition is only on principal minors.*

33.1.2 Lemma *Both of the above are closed under matrix multiplication. Also if one of the two factors is in the positive set rather than merely the nonnegative set, the product is also in the positive set.*

Proof The Cauchy-Binet theorem states that

$$\Delta_{I,J}(AB) = \sum_{K} \Delta_{I,K}(A)\Delta_{K,J}(B)$$

where A is m by p and B is p by n and I, J, K are subsets of $[m]$, $[n]$ and $[p]$ respectively all of the same size so we can take determinant of the square matrix extracted by selecting the relevant rows/columns.

33.1.3 Theorem (Loewner-Whitney) *Any invertible totally nonnegative matrix is a product of elementary Jacobi matrices with nonnegative matrix entries.*

Elementary Jacobi matrices are those matrices that can be written as either $I+tE_{i,i+1}$, $I+tE_{i+1,i}$ or $I+(t-1)E_{i,i}$ with $E_{i,j}$ being matrices with 1 precisely in the specified position and 0 elsewhere. These are typically denoted $x_i(t)$, $x_{\bar{i}}(t)$ and $x_{ⓘ}(t)$ respectively. For the totally nonnegative criterion, we simply restrict $t > 0$ in all these cases.

33.1.4 Definition (Product Map) *Let w be a word of length ℓ in the alphabet $1 \cdots (n-1)$, ①$\cdots$ⓝ, $\overline{1} \cdots \overline{n-1}$. This defines a product map $(\mathbb{C}^*)^\ell \to G$ by taking the product of the relevant elementary Jacobi matrix with the parameter t prescribed by that factor in the input. This can be interpreted as a planar network with n horizontal lines weighted by 1 and depending on the letter type either a diagonal line connecting i and $i \pm 1$ with weight t or one of the horizontal lines weighted by t instead. Then computing entries of the product is given by a sum of paths connecting the relevant nodes where each path is weighted by the product of all it's edge weights.*

33.1.5 Definition (Totally Positive Part of Flag Variety) *Recall that flag varieties over reals are given as $GL(n,\mathbb{R})/P$ where the parabolic P is indicating the steps of the flag. If we take P to be B this is the complete flag variety, but here we can be uniform in both partial and complete flag varieties.*

So instead of taking the quotient from $GL(n,\mathbb{R})$, restrict to $GL(n,\mathbb{R})_{>0}$. This gives a subset of the corresponding flag variety which we call the totally positive part.

The totally nonnegative part is then the closure of this set inside the corresponding flag variety.

33.1.6 Remark The totally nonnegative part is not $GL(n,\mathbb{R})_{\geq 0}/P$. This only gives a subset of the totally nonnegative part of $GL(n,\mathbb{R})/P$. Taking the closure before quotienting by P and taking the closure after are different operations. $\diamondsuit$

33.1.7 Definition (Plucker Positive Flag) *All Plucker coordinates are positive (or more precisely, they are all nonzero and they all have the same sign so that the statement makes sense projectively). Plucker nonnegative is when they are all nonnegative with a similar projectivity caveat.*

33.1.8 Theorem (Bloch-Karp `https://arxiv.org/pdf/2206.05806.pdf`) *The following are equivalent.*

- *The totally positive part of the flag variety coincides with the Plucker positive part of the flag variety*
- *The totally nonnegative part of the flag variety coincides with the Plucker nonnegative part of the flag variety*
- *The middle steps of the flag are consecutive integers (not counting the first inclusion $0 \subset V_1$ or the last $V_{k_l} \subset \mathbb{R}^n$)*

This in particular shows the equality in the case of Grassmannians where there are no middle steps and in complete flags where all the steps are 1 not just the middle ones.

In the other cases, the inclusion that follows immediately from the definitions of totally positive/nonnegative including into the Plucker positive/nonnegative is a proper inclusion. In these cases, one cannot determine positivity/nonnegativity from such a naive criterion on just the Plucker coordinates.

33.2 Cluster Algebra

Copied from `http://arxiv.org/pdf/math.QA/0208033.pdf`

33.2.1 Definition *Let B be an adjacency matrix for a quiver with n vertices. Also start with n variables $\{f_1, \cdots f_n\}$ and say the first m of them will be mutatable and the rest will be frozen.*

33.2.2 Definition (Mutation) *For each $i \in [1,m]$, define the transformation*

$$
\begin{aligned}
T_i(f_i)f_i &= \prod_{B_{ik}>0} f_k^{B_{ik}} + \prod_{B_{ik}<0} f_k^{-B_{ik}} \\
T_i(f_j) &= f_j
\end{aligned}
$$

At the same time the quiver transforms by flipping all vertices and deleting loops.

33.2.3 Example (A2 quiver) *The variables start as $\{x_1, x_2\}$ and the quiver with an arrow from 1 to 2. WLOG we can consider the sequence of mutations 12121 because doing 11 or 22 would be the identity.*

$$
\begin{aligned}
\{x_1, x_2\} &\rightarrow \{\frac{x_2+1}{x_1}, x_2\} \\
&\rightarrow \{\frac{x_2+1}{x_1}, \frac{1}{x_2}\frac{x_2+1+x_1}{x_1}\} \\
&\rightarrow \{\frac{1+x_1}{x_2}, \frac{1}{x_2}\frac{x_2+1+x_1}{x_1}\} \\
&\rightarrow \{\frac{1+x_1}{x_2}, x_1\} \\
&\rightarrow \{x_2, x_1\}
\end{aligned}
$$

This example is 5-periodic. There are a finite number of cluster coordinate charts. This is finite type.

33.2.4 Definition (g-vector)

33.3 Bruhat Cells

33.3.1 Definition (Bruhat Cell) *BwB*

33.3.2 Definition (Schubert Cell) *Take $BwB/B \subset G/B$ instead. Then take the closures.*

33.3.3 Definition *Let P be a parabolic in $GL(N)$ and let it act on $Gr(k, N)$* **??**

33.3.1 Double Bruhat Cells

33.3.4 Definition *A group G has two Bruhat decompositions with respect to B_+ and it's opposite B_-. Each of them are indexed by elements of the Weyl group.*

$$
G^{u,v} = B_+uB_+ \bigcap B_-vB_-
$$

33.3.5 Theorem (FZ1) *This variety is biregularly isomorphic to a Zarisiki open in $\mathbb{C}^{r+l(u)+l(v)}$. The big cell is the given by the longest elements.*

For every pair of reduced words j and k that represent u and v respectively, you can define a new word i of length $m = l(u) + l(v)$ by shuffling $-j$ and k. Then you can write a map

$$x_i(h,t) = h \prod_{\nu=1}^{m} x_{i_\nu}^{sign(i_\nu)}(t_\nu)$$

For example if we are looking at $SU(3)$ and we want the words $u = (12)$ and $v = (12)(23)(12)$ which we shuffle like $1, -1, 2, 1$ then we are looking at the factorization

$$g = hexp(t_1 e_1)exp(t_2 f_1)exp(t_3 e_2)exp(t_4 e_1)$$

For every i, we get a family of log-canonical coordinates on a Zarisiki open subset of $G^{u,v}$

We can change the presentations j and k by Coxeter moves and these will induce mutations on the factorization parameters.

33.3.6 Lemma *The product of two Bruhat cells is given by $B\dot{s}B \times B\dot{w}B$ is $B\dot{s}\dot{w}B$ if the lengths add and the union of $B\dot{w}B$ and $B\dot{s}\dot{w}B$ otherwise. Repeatedly apply this for two words w_1 and w_2 by giving a presentation in simple reflections and applying the above rule.*

33.3.7 Theorem (Richardson-Springer) *If you take the 0-Hecke monoid determined by $s^2 = s$ replacing $s^2 = 1$. The unique dense open $B \times B$ orbit in $BwB \times Bw'B$ is $Bw''B$ for $h(w)h(w') = h(w'')$ in the 0-Hecke monoid and h is the bijection from $W \to 0 - H$*

33.3.2 Log Canonical

A log canonical Poisson structure is one where the coordinate functions satisfy $\{x_j, x_k\} = c_{jk} x_j x_k$ for some constant skew-symmetric matrix C. This is compatible with mutations in the sense that the new coordinate functions satisfy the same form of Poisson bracket but with a new $\tilde{C}$

This is to be compared with canonical commutation relations $\{x, p\} = 1$ which implies that $\{x, e^{bp}\} = b + b^2 p + \cdots \frac{b^n}{n!} n p^{n-1} + \cdots = b e^{bp}$ and $\{e^{ax}, e^{bp}\} = abe^{bp} + \frac{a^2}{2} 2xbe^{bp} + \cdots \frac{a^n}{n!} n x^{n-1} b e^{bp} + \cdots = abe^{ax}e^{bp}$. This shows how if you exponentiate canonical coordinates, you get log-canonical or if you take logs of log-canonical coordinates you get canonical commutation relations

A Poisson bracket of degree d-1 comes from the $Pois_d$ operad. It is $(anti)^d$ symmetric.

$$
\begin{aligned}
0 = \{x_i, x_j x_j^{-1}\} &= x_j\{x_i, x_j^{-1}\} + \{x_i, x_j\}x_j^{-1} \\
x_j\{x_i, x_j^{-1}\} &= -c_{ij}x_i x_j x_j^{-1} \\
\{x_i, x_j^{-1}\} &= -c_{ij}x_i x_j^{-1}
\end{aligned}
$$

33.4 Tropical

Define new coordinates as $\xi_i = t^{-1}\log x_i = T\log x_i$ on some chart where this makes sense. Write the Poisson bracket in these coordinates

$$
\{\xi_i, \xi_j\} = t^{-2}c_{ij} = T^2 c_{ij}
$$

Replace all variables with $x_i = e^{\xi_i/T}$. This intertwines addition $(a,b) \to a+b$ and multiplication of exponentials.

$$
(a,b) \to (e^{a/T}, e^{b/T}) \to exp((Tln(e^{a/T} * e^{b/T}))/T) = exp((a+b)/T) \to a+b
$$

We can also do the addition map on the exponentials and see it become a new operation $a \star_T b$

$$
(a,b) \to (e^{a/T}, e^{b/T}) \to exp((Tln(e^{a/T} + e^{b/T}))/T) \approx exp(max(a,b)/T) \to max(a,b)
$$

where we have taken the $T \to 0$ limit in the approximation. The unapproximated expression is complicated ln $\exp((T\ln(e^{a/T} + e^{b/T})))$

So if we want to use the ring structure on exponentiated functions $e^{a/T}$ for low temperature T, we end up seeing the tropical (max , +) semiring on the unexponentiated functions.

33.4.1 Definition (Amoeba) *Suppose we have a variety $V(I)$ in $(\mathbb{C}^*)^n$. Then take $Re(\xi_i) = t^{-1}\log \mid z_i \mid$ coordinates of those points in the variety. In particular let $V(I) = (f)$ for the hypersurface case.*

$$
\begin{aligned}
f &= \sum a_I x^I \\
T\log \mid f \mid &= T\log \mid \sum a_I x^I \mid
\end{aligned}
$$

As $T \to 0^+$, a triangle inequality and a log *being convex, to give* $max(T \log a_I + I \cdot Re(\xi))$ *where only need to look at those* I *where* $a_I \neq 0$*. We wanted to look for* $f = 0$ *so this expression needs to be very negative say* $-M$ *even with the* T *rescaling. This means we get some half spaces, one for each of the linear equations. This allows you to figure out a region of potential* ξ *where all the terms are small and no cancellation is needed.*

We could also replace $T \to -i\hbar/x$. But notice we used the commutativity of a and b when using the Baker-Campbell-Hausdorff formula in the addition operation otherwise we would see more terms like $\frac{1}{2T}[a,b] + \cdots$ and if we still want to take $T \to 0$ we should say that the brackets all have a factor of T so that we only see exponents that look like c/T for some c that does not depend on T.

Another way they come up is if you apply a valuation

$$\begin{aligned}
\psi_{1,WKB} &\approx A_1(x)e^{iS_1(x)/\hbar} \\
\psi_{2,WKB} &\approx A_2(x)e^{iS_2(x)/\hbar} \\
\psi_{1+2,WKB} &\approx
\end{aligned}$$

33.4.2 Definition (Valuation)

Archimedian and non-Archimedean. The prototypic non-Archimedean one is how you make the p-adics

33.4.3 Example

$$\begin{aligned}
val(\sum_{n=-\nu}^{\infty} a_n p^n) &= ??\nu \\
val(ab) &= \\
val(a+b) &=
\end{aligned}$$

Let's say you have a curve defined over $\mathbb{Q}_p$*, you can turn it into a tropical curve by*

33.4.4 Lemma (Sturmfels) *Say you have n points and all pairwise distances. In order to know whether this is a metric, namely satisfies the triangle inequality you check whether* $D \circ D = D$ *under tropical matrix multiplication with min,+ convention instead.*
It is a metric coming from a metrized tree if and only if $-D$ *gives a point of* $Gr_{2,n}(\mathbb{T})$*. This is also called the 4-point condition in phylogenetics.*

33.4.5 Definition (Hyperfield) *A commutative hyperring takes ring axioms, but now allows multivalued result of addition. Additive inverse now means that* 0 *is in the set* $x + (-x)$ *instead of* $x + -x = 0$*. If demand all nonzero have multiplicative inverse, that is what is called a hyperfield.*

33.4.6 Example (F_1)

$$\begin{aligned} 1+1 &= \{0,1\} \\ 1+0 &= \{1\} \\ 0+0 &= \{0\} \end{aligned}$$

33.4.7 Example ($\mathbb{R} \bigsqcup -\infty$) *Wrap everything in $i()$ to indicate when as T rather than when as their usual operations.*

$$\begin{aligned} i(a)+i(a) &= \{i(c) \mid c \leq a\} \\ i(a)+i(b) &= \{i(max(a,b))\} \\ i(a)*i(b) &= i(a+b) \end{aligned}$$

$i(0)$ is multiplicative identity. and $i(-\infty)$ additive identity. This is the analog of the max-plus tropical semiring as a hyperring. The difference is in allowing all the $c \leq a$ when adding $i(a)+i(a)$.

33.5 Lattice Poisson/Quantized

33.5.1 Definition (Skew Lattice Poisson Algebra) *If we have a lattice $\mathbb{Z}^r$ with skew-symmetric integer-valued bilinear form $\langle -,- \rangle$ we get a log-canonical Poisson algebra structure on $k[[x_1, \cdots x_r]]$ with k being a field of characteristic 0 which we might as well take to be $\mathbb{Q}$. We will be doing lots of divisions by natural numbers so we will need all of them to be invertible. We regard this only as a vector space over k and impose a commutative algebra and Poisson bracket on it momentarily.*

$$\begin{aligned} x^\alpha x^\beta &= (-1)^{\langle \alpha,\beta \rangle} x^{\alpha+\beta} \\ \{x^\alpha, x^\beta\} &= (-1)^{\langle \alpha,\beta \rangle} \langle \alpha,\beta \rangle x^{\alpha+\beta} \\ &= \langle \alpha,\beta \rangle x^\alpha x^\beta \end{aligned}$$

Here the α are in $\mathbb{N}^r$. They are akin to the functions $x_1^{\alpha_1} \cdots x_r^{\alpha_r}$, but not quite. We are only formal in the variables so they are not like the $\mathbb{C}^*$-valued coordinate functions on $(\mathbb{C}^*)^r$. In addition, the data of the reduction of the skew-symmetric bilinear form modulo 2 comes into the product. That means that even without formalness, we are in a twisted group algebra rather than a group algebra of $\mathbb{Z}^r$.

In general, there are Poisson algebra automorphisms, where we use Li_2 as it's formal power series in order to plug in the formal symbol x^α.

$$
\begin{aligned}
T_\alpha(x^\beta) &\equiv x^\beta(1-x^\alpha)^{\langle\alpha,\beta\rangle} \\
T_\alpha &= Ad\exp\big(-\sum_{n\geq 1}\frac{x^{n\alpha}}{n^2}\big) \\
&= Ad\exp\big(-Li_2(x^\alpha)\big)
\end{aligned}
$$

Here Ad means build up from the ad action. The latter is the action by Poisson bracketing. Namely $ad_y(x) = \{y, x\}$.

33.5.2 Remark One can specialize to the case where r is the number of vertices of the quiver $\mid Q_0 \mid$ and the skew-symmetric bilinear form is built from anti-symmeterizing the Euler-Ringel form.

$$
\begin{aligned}
\langle\alpha,\beta\rangle_{ER} &= -\sum_{s\to t\in Q_1}\alpha_s\beta_t + \sum_{s\to t\in Q_1}\alpha_t\beta_s \\
&= \sum_{s\to t\in Q_1}(\alpha_t\beta_s - \alpha_s\beta_t)
\end{aligned}
$$

33.5.3 Remark If we pass to an subgroup of $\mathbb{Z}^r$ where the skew-symmetric bilinear form is even, we get the usual untwisted product. We could always take the subgroup of index 2^r by demanding all the components be even, but depending on the bilinear form, we might be able to use something closer to the full $\mathbb{Z}^r$ as well. $\Diamond$

Proof

$$
\begin{aligned}
\{x^\alpha, x^\beta x^\gamma\} &= \{x^\alpha, (-1)^{\langle\beta,\gamma\rangle}x^{\beta+\gamma}\} \\
&= (-1)^{\langle\beta,\gamma\rangle}\langle\alpha,\beta+\gamma\rangle x^\alpha x^{\beta+\gamma} \\
&= \langle\alpha,\beta+\gamma\rangle x^\alpha x^\beta x^\gamma \\
&= \langle\alpha,\beta\rangle x^\alpha x^\beta x^\gamma + \langle\alpha,\gamma\rangle x^\alpha x^\beta x^\gamma \\
&= \{x^\alpha, x^\beta\}x^\gamma + x^\beta\{x^\alpha, x^\gamma\}
\end{aligned}
$$

For the Poisson automorphism

$$
\begin{aligned}
T_\alpha(x^\beta) &= \operatorname{Ad}\exp\big(-Li_2(x^\alpha)\big)(x^\beta) \\
\operatorname{Ad}\exp\big(\frac{-x^{n\alpha}}{n^2}\big)(x^\beta) &= \sum_{k\geq 0}\frac{1}{k!}\frac{(-1)^k}{n^{2k}}\operatorname{ad}^k_{x^{n\alpha}}(x^\beta) \\
\operatorname{ad}_{x^{n\alpha}}(x^\beta) &= n\langle\alpha,\beta\rangle x^{n\alpha}x^\beta \\
\operatorname{ad}^k_{x^{n\alpha}}(x^\beta) &= (n\langle\alpha,\beta\rangle)^k(x^{n\alpha})^k x^\beta \\
\sum_{k\geq 0}\frac{1}{k!}\frac{(-1)^k}{n^{2k}}\operatorname{ad}^k_{x^{n\alpha}}(x^\beta) &= \sum_{k\geq 0}\frac{1}{k!}\frac{(-1)^k}{n^k}(\langle\alpha,\beta\rangle)^k(x^{n\alpha})^k x^\beta \\
&= \exp\big(\frac{-\langle\alpha,\beta\rangle x^{n\alpha}}{n}\big)x^\beta \\
T_\alpha(x^\beta) &= \big(\exp\sum_{n\geq 1}\frac{-\langle\alpha,\beta\rangle x^{n\alpha}}{n}\big)x^\beta \\
&= \big(\exp\sum_{n\geq 1}\frac{-x^{n\alpha}}{n}\big)^{\langle\alpha,\beta\rangle}x^\beta \\
&= \big(\exp\log(1-x^\alpha)\big)^{\langle\alpha,\beta\rangle}x^\beta \\
&= (1-x^\alpha)^{\langle\alpha,\beta\rangle}x^\beta
\end{aligned}
$$

33.5.4 Lemma (Quantization of Skew Lattice Poisson Algebra) *Quantize the Poisson algebra above into an associative algebra in such a way that $q^{1/2}=1$ is the classical limit.*

$$
X^\alpha \circ X^\beta = (-q^{1/2})^{\langle\alpha,\beta\rangle}X^{\alpha+\beta}
$$

The Poisson algebra automorphisms T_α turn into algebra automorphisms

$$
\begin{aligned}
T_\alpha = Ad\exp\big(-Li_2(x^\alpha)\big) &\rightarrow \tilde{T}_\alpha \equiv Ad\exp\big(-qLi_2(x^\alpha)\big) \\
qLi_2(x) &\equiv \sum_{n\geq 1}\frac{1}{n}\frac{(q^{1/2}x)^n}{q^n-1}
\end{aligned}
$$

Proof The bracket on the quantized associative algebra side is taking the commutator divided by $q-1$. This is what needs to recover the classical Poisson bracket.

$$
\begin{aligned}
\frac{X^\alpha\circ X^\beta - X^\beta\circ X^\alpha}{q-1} &= \frac{(-q^{1/2})^{\langle\alpha,\beta\rangle}-(-q^{1/2})^{-\langle\alpha,\beta\rangle}}{q-1}X^{\alpha+\beta} \\
\lim\frac{X^\alpha\circ X^\beta - X^\beta\circ X^\alpha}{q-1} &= (-1)^{\langle\alpha,\beta\rangle}\langle\alpha,\beta\rangle x^{\alpha+\beta} = \{x^\alpha,x^\beta\}
\end{aligned}
$$

One could also state the limit algebraically by working modulo the ideal generated by $q^{1/2}-1$. To simplify notation one could write as $\frac{z^n - z^{-n}}{z^2-1} = n + O(z-1)$ and then specialize to $z = -q^{1/2}$ and $n = \langle \alpha, \beta \rangle$.

$$
\begin{aligned}
\tilde{T}_\alpha^{-1}(X^\beta) &= \exp\big(qLi_2(X^\alpha)\big) \circ X^\beta \circ \exp\big(-qLi_2(X^\alpha)\big) \\
&= X^\beta \circ \exp\big(qLi_2(q^{\langle\alpha,\beta\rangle}X^\alpha)\big) \circ \exp\big(-qLi_2(X^\alpha)\big) \\
&= X^\beta \circ \exp\left(\sum_{n\geq 1} \frac{1}{n} \frac{q^{n\langle\alpha,\beta\rangle}(q^{1/2}X^\alpha)^n - (q^{1/2}X^\alpha)^n}{q^n - 1}\right) \\
&= X^\beta \circ \exp\left(\sum_{n\geq 1} \frac{1}{n} \frac{1-q^{n\langle\alpha,\beta\rangle}}{1-q^n}(q^{1/2}X^\alpha)^n\right) \\
&= X^\beta \circ \exp\left(\sum_{n\geq 1} \frac{1}{n} \langle\alpha,\beta\rangle (q^{1/2}X^\alpha)^n\right) + O(q^{1/2}-1) \\
&= X^\beta \circ \left(\exp\left(\sum_{n\geq 1} \frac{1}{n}(q^{1/2}X^\alpha)^n\right)\right)^{\langle\alpha,\beta\rangle} + O(q^{1/2}-1) \\
&= X^\beta \circ \Big(\exp\big(-\log(1-X^\alpha)\big)\Big)^{\langle\alpha,\beta\rangle} + O(q^{1/2}-1) \\
&= X^\beta \circ \Big(1-X^\alpha\Big)^{-\langle\alpha,\beta\rangle} + O(q^{1/2}-1)
\end{aligned}
$$

which shows how we lift the special Poisson algebra automorphisms to special associative algebra automorphisms.

33.5.5 Definition (Plethystic Exponential) *Let m be the maximal ideal of $\mathbb{Q}(q^{1/2})[[x_1 \cdots x_r]]$. On this define the Plethystic exponential PLEXP*

$$
\begin{aligned}
PLEXP(f+g) &= PLEXP(f)PLEXP(g) \\
PLEXP(q^{k/2}x^\alpha) &= \frac{1}{1-q^{k/2}x^\alpha}
\end{aligned}
$$

See definition 24.5.1 for more on Plethysm.

Let PLLOG be the inverse map $1+m \to m$. One can combine these to give $1+m \to 1+m$ operations given by $PLEXP(gPLLOG(-))$ for all g in the algebra. These act like raising to the g'th power for arbitrary elements of the algebra.

33.5.6 Example

$$\begin{aligned}
PLEXP(\frac{x}{q^{-1/2}-q^{1/2}}) &= PLEXP(\frac{q^{1/2}x}{1-q}) \\
&= \prod_{n\geq 0} PLEXP(q^{1/2}xq^n) \\
&= \prod_{n\geq 0} PLEXP(q^{(2n+1)/2}x) \\
&= \prod_{n\geq 0} \frac{1}{1-q^n q^{1/2}x} \\
&= \frac{1}{(q^{1/2}x;q)_\infty} \\
&= \sum_{n\geq 0} \frac{(q^{1/2}x)^n}{(q;q)_n} \\
&= \exp\big(\sum_{n\geq 1} \frac{(q^{1/2}x)^n}{n(1-q^n)}\big) \\
&= \exp(-qLi_2(x))
\end{aligned}$$

33.5.7 Example (m-Kronecker Quiver) *Let $r=2$ and the skew-symmetric bilinear form be $m(\alpha_1\beta_2-\alpha_2\beta_1)$. This is the one from the Kronecker quiver with m arrows instead of the usual 2.*

The Poisson automorphisms satisfy numerous identities. In particular, we get identities of the form

$$T_{1,0}T_{0,1} = \prod_{\frac{a}{b}\uparrow} T_{a,b}^{\Omega_{a,b}}$$

where the $\Omega_{a,b}$ are integers depending on $(a,b)\in\mathbb{N}^2\setminus(0,0)$. The arrow indicates which order to take the factors, namely by increasing order of the quotient $\frac{a}{b}$ (when $a>0$ and $b=0$, regard that as large corresponding to $+\infty$)

$m=1$ and $m=2$ are the typical cases. The $m=1$ case gives $T_{0,1}T_{1,1}T_{1,0}$ on the other side so all the other numerical DT invaraints in this case are 0. In the $m=2$, there are 1 for all the $(n,n\pm 1)$ and -2 for $(1,1)$. This gives an infinite product of Poisson automorphisms.

Further one may promote this to an identity among the quantized $\tilde{T}$ automorphisms as well. In order to state the precise form, we realize they are all adjoint actions by some elements of the algebra E_α and those elements of the algebra are all in $1+m$ so we can state the identity as an identity among these and their Plethystic powers.

$$E(x_1)E(x_2) = \prod_{\frac{a}{b}\uparrow} PLPOW(\ E(x^{(a,b)})\ ,\ \Omega_{a,b,ref}\ )$$
$$= \prod_{\frac{a}{b}\uparrow} PLEXP(\frac{\Omega_{a,b,ref}x^{(a,b)}}{q^{-1/2}-q^{1/2}})$$

Here the $\Omega_{a,b,ref}$ are refined DT invariants that are Laurent polynomials in $q^{1/2}$ and plugging in $q^{1/2}=1$ gives the classical unrefined limit.

33.6 Cluster Category

33.6.1 Theorem *Let $\mathcal{D} = D^b(mod - kQ)$. Because kQ is hereditary we have a simple structure. The indecomposables are $M[i]$ where M is an indecomposable kQ module. The morphisms between the indecomposables are*

$$\begin{aligned} Hom_D(M[i], N[j]) = Ext_A^{j-i}(M,N) &= Hom_A(M,N)\ \ i=j \\ &= Ext_A^1(M,N)\ \ j=i+1 \\ &= 0 \end{aligned}$$

33.6.2 Theorem (Auslander-Reiten) *$\mathcal{D}$ has Serre duality τ which*

$$Hom_D(M, N[1]) \simeq Hom_D(N, \tau(M))^*$$

33.6.3 Definition *Let $F = \tau^{-1}[1]$. Then form the orbit category $\mathcal{D}/F$ as $\mathcal{C}$. It is a triangulated category with induced Auslander-Reiten translation from $\mathcal{D}$.*

33.6.4 Theorem (Relation with Matrix Factorizations) *For a Dynkin ADE quiver $D^b(kQ-$mod) is equivalent to a triangulated category of graded matrix factorizations for the corresponding polynomial of type ADE. Those are explicitly given as*

- $x^{l+1}+yz$ *with* $h=l+1$ *at* A_l *and* $l \geq 1$
- $x^2y+y^{l-1}+z^2$ *with* $h=2(l-1)$ *at* D_l *and* $l \geq 4$
- $x^3+y^4+z^2$ *for* $h=12$ *and* E_6
- $x^3+xy^3+z^2$ *for* $h=18$ *and* E_7
- $x^3+y^5+z^2$ *for* $h=30$ *and* E_8

Proof `https://arxiv.org/pdf/math/0511155.pdf` □

33.6.5 Theorem (Relation to Cluster Algebra) *The indecomposable rigid objects become cluster variables. The cluster-tilting objects $T_1 \bigoplus T_2 \cdot T_n$ gets sent to a cluster $x_{T_1} \cdots x_{T_n}$. When $Ext^1_{\mathcal{C}}(M,N) = \mathbb{C}$ x_M and x_N form an exchange pair that can be mutated from a cluster with x_M to a cluster with x_N instead. Note that the sum of monomials is lost. For that see, definition 33.6.13*

33.6.6 Definition (Spherical Object) *An object of a triangulated category $\mathcal{C}$ with Serre functor S such that $SA = A[n]$ and $Hom(A, A[k])$ is $\mathbb{C}$ for $k = 0, n$ only. This says that A has the homology of an n-sphere.*

33.6.7 Definition (Spherical Twist) *The autoequivalence T_A of $\mathcal{C}$ defined on objects $X \to T_A(X)$ by making $Hom^\bullet(A,X) \otimes A \to X \to T_A(X)$ into an exact triangle. The first arrow is the evaluation and the second is the autoequivalence.*

33.6.8 Remark The spherical twists often satisfy a braid relation up to natural transformation. `https://arxiv.org/pdf/math/0001043v2.pdf` ♢

33.6.9 Example *For an elliptic curve:* $0 \longrightarrow 2\mathbb{Z} \times Aut(E) \ltimes Pic(E)^0 \longrightarrow Aut(D^b(CohE)) \longrightarrow Sp(1,\mathbb{Z})$

33.6.10 Example (Seidel-Brav-Thomas) *For X minimal resolution of $\mathbb{C}^2/G$ for G ADE finite group. Then there is*

$$B_\Gamma \longrightarrow AutD^b(Coh(X))$$

sending s_i to generalized twists around the Lagrangian spheres which on B-side are implemented by $\mathcal{O}_{C_i}[1]$ where C_i are irreducible components of $\pi^{-1}(0)$.

33.6.11 Definition (S Duality Group) *First form the auto-equivalences of the triangulated cluster category $\operatorname{Aut}\mathcal{C}$. Cluster tilting objects go to other cluster tilting objects. The physically trivial subgroup Aut^0 are those autoequivalences that whenever you provide a family of objects and apply the automorphism the family gets reparameterized. The S-Duality group is then defined to be the quotient $Aut(\mathcal{C})/Aut^0(\mathcal{C})$*

33.6.12 Lemma *Because it acts on the electromagnetic charges the is a map $S \to G_{\mathbb{Z}} \subset Sp(2g, \mathbb{R})$ which is an arithmetic subgroup. More generally it is simply commensurable with such a discrete arithmetic subgroup.*

33.6.13 Definition (Monoidal Categorification) *An abelian monoidal category $\mathcal{M}$ such that it's Grothendieck ring is the cluster algebra $\mathcal{A}$. The cluster variables x are the classes of real prime simple objects S_x. Two cluster variables belong to the same cluster if and only if their product $S_x \otimes S_y$ is also simple. Other cluster monomials $xy \cdots z$ is the class of $S_x \otimes S_y \cdots S_z$. They are real simple objects. Exchange relations come from exact sequences of the form*

$$0 \longrightarrow M \longrightarrow S_x \otimes S_{x^*} \longrightarrow M' \longrightarrow 0$$

$$0 \longrightarrow M' \longrightarrow S_{x^*} \otimes S_x \longrightarrow M \longrightarrow 0$$

If $S_x^{\otimes 2}$ is simple, the object is called real. Prime means that there is no factorization $S \simeq S_1 \otimes S_2$.

33.6.14 Definition ($\mathcal{C}_l$) *For ADE Dynkin color the vertices with $\xi_i = 0/1$. Then take the subcategory of finite dimensional representations of $U_q\hat{\mathfrak{g}}$ consisting ofobjects such that any simple composition factor and index $i \in I$, the roots of that Drinfeld polynomial are in the set $\{q^{-2k+\xi_i}\}$ for $k \in \mathbb{Z}$.*
Define $\mathcal{C}_l$ by only allowing $k \in [0,l]$ instead.
$K_0(\mathcal{C}_{\mathbb{Z}})$ is generated by $[V_{i,q^{-2k+\xi_i}}]$ and similarly for $K_0(\mathcal{C}_l)$ but in the $\mathcal{C}_l$ case it is a polynomial ring.
(Mixing up some k and $-k$ here. Straighten out which one goes where.)
They all sit inside via χ_q mapping to $\mathbb{Z}[Y_{i,a}]$

33.6.15 Conjecture (Leclerc) *Let Δ be a Dynkin diagram and $l \geq 1$ integer. Then the category of finite dimensional $U_q\hat{\mathfrak{g}}$ for q not root of unity and $\mathfrak{g}$ corresponding to Δ has a monoidal abelian subcategory $\mathcal{M}_{\Delta,l}$ ($\mathcal{C}_l$ up to that mix up k/l problem above) which categorifies the quiver cluster algebra from $Q_{\Delta,l}$.*

33.6.16 Example *https://webusers.imj-prg.fr/~bernhard.keller/publ/ClusterAlgQuantAffAlg.pdf*

33.7 $\mathcal{A}$ and $\mathcal{X}$

33.7.1 Definition ($Pos(k)$) *The category of split algebraic tori over k and morphisms are positive rational maps. A positve rational function is $\frac{p(x)}{q(x)}$ where p and q are polynomials with $a_\alpha X^\alpha$ monomials for $\alpha \in \mathbb{N}^d$ and $a_\alpha \in \mathbb{N}$. A positive rational map preserves the semifields of these positive rational functions.*

33.7.2 Definition (Positive Space) *A functor from a groupoid to $Pos(k)$*

33.7.3 Definition (Cluster Ensemble) *A pair of positive spaces $\mathcal{G} \to Pos(k)$ and a natural transformation between them.*

`https://arxiv.org/pdf/1704.06586.pdf`

33.8 Zeitlin Talk

Let $\{y_i, y_j\} = b_i j y_i y_j$ be a cluster Poisson bracket.

$$Li_2(-y_k), y_i = -b_{ki} \log(1 + y_k) y_i$$

provides an infinetesimal form of the mutation of the X variables.

RTGC: Penner coordinates on super-Teichmuller spaces - Zeitlin

Let $\Sigma_{g,s}$ be our surface with genus g and s punctures.

Figure 33.1: Insert figure

Figure 33.2: Insert figure

33.8.1 Definition (Penner coordinates) *Do an ideal triangulation of* $\Sigma_{g,s}$.

Assign λ *lengths to each edge. Regularize the length of geodesics by horocycles. This gives decorated Teichmuller.*

33.8.2 Theorem (Ptolemy around 100 CE)

$$ef = ac + bd$$

33.8.3 Definition (Trivalent Fat Graph)

33.8.4 Definition (A/X surface) *Bipartite graph, make a surface with or without twisting upon going black to white.*

33.8.5 Definition (Super Upper Half Plane) $(z_i \mid \theta_j)$ *of* $\mathbb{C}^{1|N}$ *with* $Imz_i > 0$. *Similiarly positive orthant of* $\mathbb{R}^{M|N}$ *with all the of the* M $x_i > 0$

33.8.6 Definition (Super-Fuchsian Group) *Subgroup of* $OSp(1 \mid 2)$ *such that projects to Fuchsian group.*

33.8.7 Theorem (Relation with combinatorial spin structure) *Orient the fat graph to tell you which spin structure*

33.9 Harold Talk 20170131

33.9.1 Theorem *The* K_0 *of the category of* $SL_n\mathcal{O} \times \mathbb{C}^*$ *equivariant coherent sheaves on affine Grassmanian is equivalent to the quantum cluester algebra with quiver given by* $n-1$ *Jordan quivers patched together as follows. Insert figure here. And all cluster variables are classes of simple perverse coherent sheaves. This provides an additive categorification of this cluster algebra.*

33.9.2 Definition (BFM Category) *Heart of t-structure for* $D^b(Coh^{G(\mathcal{O})}(Gr_{G(\mathcal{K})})$ *specified by the condition that ...*

Chapter 34

Hodge Structures

34.0.1 Definition (Pure Hodge Structure) *An abelian group $H_{\mathbb{Z}}$ equipped with a decomposition $H_{\mathbb{Z}} \otimes \mathbb{C} \simeq \bigoplus_{p,q} H^{p,q}$ with $\bar{H^{p,q}} = H^{q,p}$. That is it acts like $H^n(X,\mathbb{Z})$ of a compact Kahler manifold.*

34.0.2 Definition (Mixed Hodge Structure)

34.0.3 Definition (Polarized Hodge Structure of weight n) *A Hodge structure of weight n and a nondegenerate integer bilinear form Q on $H_{\mathbb{Z}}$ extended as $Q_{\mathbb{C}}$ satisfying ...*

called the Hodge Riemann bilinear relations.

34.0.4 Example *$\mathbb{Z}(1)$ is the copy of $\mathbb{Z}$ sitting as $2\pi i\mathbb{Z}$. Upon tensoring with $\mathbb{C}$ this gets put as $H^{-1,-1}$. A pure Hodge structure of weight -2. $\mathbb{Z}(n) \equiv \mathbb{Z}(1)^{\otimes_{\mathbb{Z}} n}$ still 1-dimensional, but now of weight $-2n$.*

34.0.5 Definition (Variation of Hodge Structures) *Over a family given by a complex manifold X. In particular variation of Hodge structure of weight n on X is given by a locally constant sheaf S of finitely generated abelian groups together with a decreasing Hodge filtration F on $S \otimes \mathcal{O}_X$ so that each stalk of S gets a Hodge structure by F and the connection on $S \otimes \mathcal{O}_X$ (S locally constant and d deRham) takes $F^n \to F^{n-1} \otimes \Omega^1$.*

34.0.6 Lemma *This connection is flat and is a Gauss-Manin connection and can be described by (generalized) Picard-Fuchs.*

`http://people.math.umass.edu/~cattani/ICTP/cattani_vhs.pdf`

34.0.7 Definition (Classifying Space of Hodge structures) *Provide all the combinatorial data $V_{\mathbb{Z}}$ lattice with Q of parity $(-1)^k$, integer k to indicate weight and all the dimensions $h^{p,q}$. Now ask for all Hodge structures that realize these numbers and polarization. This is called D. Also make one for the compatible filtrations with those dimensions specified instead. Call it $\tilde{D}$.*

34.0.8 Theorem *$\tilde{D} = G_{\mathbb{C}}/B$ and $D = G/V$ where these are defined by ...*

34.0.9 Example ($\boldsymbol{k = 1}$) *Since k is odd, there will be no possible $h^{p,p}$ so $\dim V_{\mathbb{C}}$ will be even. Let $h^{1,0} = h^{0,1} = n$. Then we can take a basis of $\mathbb{V}_{\mathbb{C}}$ to get Q in block diagonal form like i times the standard symplectic pairing. This then turns $D \simeq Sp(n,\mathbb{R})/U(n)$. The Siegel upper half space of symmetric n by n matrices over $\mathbb{C}$ with positive definite imaginary part.*

34.0.10 Example *A genus g Riemann surface. It determines a point in D. The question of finding which are realizable this way is Schottky problem. $3g - 3 \leq \frac{g(g+1)}{2}$. At $g = 2$ $3 = 3$. At $g = 3$ $6 = 6$. At $g = 4$ $9 \leq 10$ and it just get's worse from there.*

34.0.11 Example ($\boldsymbol{k = 2}$) *Let $h^{2,0} = h^{0,2}$ and $h^{1,1}$ be the only nonzero specified. Now get $O(2h^{2,0}, h^{1,1})/(U(2h^{2,0}) \times O(h^{1,1}))$. We may ask which of these are realized as $H^2(X,\mathbb{Z})$ for some compact Kahler X with the prescribed Hodge numbers.*

34.0.12 Lemma (Period Map) *Given an abstract variation of Hodge structure $\mathbb{V}$ ∇_{GM} $\mathcal{Q}$ $\mathbb{F}^p$ over base B with basepoint b_0. So above each point in B we identify those fibers which are polarized Hodge structures on $\mathbb{V}_b$. Those can be identified with polarized Hodge structures on $\mathbb{V}_{b_0}$, but that introduces an ambiguity of Γ from π_1. Altogether:*

$$B \longrightarrow \Gamma\backslash D \simeq \Gamma\backslash G/V$$

More fundamentally we are giving a map from $Map(I, B, 0 \to b_0)$ to D. But then you realize that it only depends on homotopy class in the source.

$$\begin{tikzcd}
HMap(I, B, 0 \to b_0) \arrow[r] \arrow[d, "end"] & D \\
B & \Gamma\backslash D
\end{tikzcd}$$

34.0.13 Example ($\boldsymbol{k = 1}$ and $\boldsymbol{n = 1}$) *The lattice in this case is $\mathbb{Z} \bigoplus \mathbb{Z}$. Now $D = \mathbb{H}$ upper half plane since 1 by 1 matrix. The quotient by a discrete subgroup of $SL(2,\mathbb{Z})$ then gives something like a modular curve. Pull back forms from $\Gamma\backslash\mathbb{H}$ to B if desired. This is the example made by considering a family of elliptic curves. Taking their H^1 and mapping to the Hodge structure on $\pi^{-1}b_0$ they realize.*

Chapter 35

CoHA

35.1 Calabi-Yau Category motivation

35.1.1 Definition (Calabi-Yau Category) *A Calabi-Yau A_∞ category $\mathcal{F}$ of dimension d has morphism of chain complexes*

$$\begin{tikzcd} Hom_{\mathcal{F}}(a,b) \otimes Hom_{\mathcal{F}}(b,a) \arrow[r] \arrow[d, "\sigma_{a,b}"] & k[d] \\ Hom_{\mathcal{F}}(a,b) \otimes Hom_{\mathcal{F}}(b,a) \arrow[ur] & \end{tikzcd}$$

that is nondegenerate and symmetric under $\sigma_{a,b}$. It also should be cyclic invariant in the sense that.

$$\langle m_{n-1}(\alpha_0 \otimes \cdots \otimes \alpha_{n-2}), \alpha_{n-1}\rangle \quad = \quad \pm\langle m_{n-1}(\alpha_1 \otimes \cdots \otimes \alpha_{n-1}), \alpha_0\rangle$$

with sign determined by n and the degrees of the a_i. This is a sign that appears because we are permuting α_0 past all the α_i so we should get lots of $(-1)^{|\alpha_0||\alpha_i|}$ factors. Here α_i are morphisms in $Hom_{\mathcal{F}}(a_i, a_{i+1})$ (with indices modulo n) so that the LHS m_{n-1} is the one for $Hom(a_0, a_1) \cdots Hom(a_{n-2}, a_{n-1}) \to Hom(a_0, a_{n-1})$ and being paired with $Hom(a_{n-1}, a_{n \equiv 0})$ and on the RHS is the one for $Hom(a_1, a_2) \cdots Hom(a_{n-1}, a_0) \to Hom(a_1, a_0)$ and being paired with $Hom(a_0, a_1)$.

We will say why this is relevant to include in the next two sections for how we can get Calabi-Yau categories of dimensions 2 and 3 that come from quivers rather than from sheaves on smooth projective Calabi-Yau varieties. In the smooth projective CY variety case, the nondegenerate symmetric (under $\sigma_{a,b}$ which hides the necessary signs) pairing comes from Serre duality.

35.2 Quiver without Potential

35.2.1 Definition (Kac Polynomial) *Let Q be a locally finite quiver with vertex set Q_0 and edge set Q_1. Let $d \in \mathbb{N}^{Q_0}$ be a dimension vector. Define $A_{Q,d}(q = p^n)$ to be the number of absolutely indecomposable representations of Q over F_q of dimension d.*

35.2.2 Theorem (Kac) *$A_{Q,d}(q)$ is initially defined only for prime powers q, but it is in fact given by the evaluation of a polynomial with integer coefficients that we will also denote $A_{Q,d}$. This polynomial is independent of orientations on Q_1 and is monic of degree $1 - \langle d, d \rangle$ where $\langle d, d' \rangle \equiv d^T (Id - Adj) d'$ where Adj_Q is the adjacency matrix of Q.*

35.2.3 Theorem (Hausel-Letellier-Rodriguez-Villegas) *$A_{Q,d} \in \mathbb{N}[t]$ not just $\mathbb{Z}[t]$.*

35.2.4 Example *Let Q be the Jordan quiver with 1 vertex and 1 arrow. Then the free path algebra over F_q is $F_q[x]$ which is a PID. Wanting an indecomposable representation thereof amounts to saying $M \simeq F_q[x]/\langle f^r(x) \rangle$ where f is irreducible polynomial. But we want absolutely indecomposable so we must have that f is linear and because we only care about the ideal we can take WLOG take the leading coefficient to be 1. This leaves q choices for f. So there are q absolutely indecomposable representations. r is now d in order for the dimension to be fixed. Doing Jordan normal form of the d by d matrix demands there only be a single Jordan block and we have q choices for the eigenvalue.*

Extending this as a polynomial gives $A_{Q,d}(t) = t$.

35.2.5 Definition (Cyclic Path Algebra) *$k[Q]$ is given as $\bigoplus_{d=0}^{\infty} (k^{Q_1}) \otimes_{k^{Q_0}} \cdots k^{Q_1}$. Each factor is a k^{Q_0}. It is still a bimodule by the remaining left action on the leftmost factor and the right action on the rightmost factor.*

This means we can ask for the cyclic part as $\bigoplus_{d=0}^{\infty} \bigoplus_{a \in Q_0} e_a k[Q]_d e_a$ with e_a being the idempotent in $k^{Q_0} \subset k[Q]_0$ for projection to paths that start/end at vertex a depending on whether it acts on the right or left respectively. That is a subalgebra of $k[Q]$ with basis given by paths that start and end at the same vertex. It breaks up as a direct sum of subalgebras for each vertex with the product between different summands being 0.

35.3 Quiver with Potential

35.3.1 Definition (Potential) *An element W of the cyclic path algebra $k[Q]_{cyc} \subseteq k[Q]$ (equality possible if there are no edges connecting different vertices).*

35.3.2 Definition (Cyclic Derivative) *Let $a \in k[Q]_{cyc}$ be a monomial. This means it is given as $a_1 \cdots a_d$ where all the $a_i \in k_1^Q$ are basis vector vectors for a single edge. The tensor product over k^{Q_0} has been omitted in favor of the multiplication notation. The endpoints all matchup for this to not give the 0 element. Because it is cyclic we can also form nonzero monomials $a_k a_{k+1} \cdots a_d a_1 \cdots a_{k-1}$ because the endpoints of a_d and a_1 still match up to give a nonzero product.*

So for every $\xi \in (k^{Q_1})^*$ define ∂_ξ as

$$\partial_\xi (a_1 \cdots a_d) \equiv \sum_{k=1}^{d} \xi(a_k) a_{k+1} \cdots a_d a_1 \cdots a_{k-1}$$

and extending linearly to all of $k[Q]_{cyc}$.

35.3.3 Definition (Jacobi Algebra) *Let $J(Q, W)$ be the quotient of the path algebra by the two-sided ideal generated by $\partial_\xi(W)$ as ξ varies through $(k^{Q_1})^*$. Since there is linearity in ξ we can use the ideal generated by $\partial_{\xi_a}(W)$ where ξ_a is the dual basis vector for the basis vector $a \in k^{Q_1}$.*

35.3.4 Lemma *If two potentials are cyclically equivalent, then their cyclic derivatives coincide.*

Two potentials being cyclically equivalent is equivalent to having the same universal trace.

The image of the noncyclic part of $k[Q]$ in the universal trace space is 0

$$\begin{tikzcd}
k[Q]_{cyc} \arrow[r] & k[Q] \arrow[d] \\
& k[Q]/([kQ,kQ])
\end{tikzcd}$$

This means that even though we originally defined potentials to be in $k[Q]_{cyc}$ we only ever care about their images in the quotient. We could also say they are defined as elements of $k[Q]$ without the cyclic restriction even though we did not define cyclic derivatives on all of $k[Q]$. Any lift from the quotient will do and will give something of the form $W_2 + J$ where J is noncyclic and will disappear in the quotient, and W_2 will be cyclically equivalent to the original W_1.

Proof Two potentials are called cyclically equivalent if their difference is in the span of $a_1 \cdots a_d - a_2 \cdots a_d a_1$.

$$\begin{aligned}
W_1 - W_2 &= \sum_{a_\bullet} c_{a_\bullet} [a_1, a_2 \cdots a_d] \\
\partial_\xi W_1 - \partial_\xi W_2 &= \sum c_{a_\bullet} \partial_\xi [a_1, a_2 \cdots a_d] \\
\partial_\xi [a_1, a_2 \cdots a_d] &= 0
\end{aligned}$$

Cyclic equivalence means that their difference is a commutator so them having the same universal trace is immediate. Conversely let I_1 be the span of commutators of the form $[a_1, a_2 \cdots a_n]$ where the first input must be of degree 1. We want to show this actually gives all of the commutator subalgebra without this restriction.

$$\begin{aligned}
x_1 \equiv a_1 \cdots a_d - a_2 \cdots a_d a_1 &\in I_1 \\
x_2 \equiv a_2 \cdots a_d a_1 - a_3 \cdots a_d a_1 a_2 &\in I_1 \\
x_k \equiv a_k \cdots a_d a_1 \cdots a_{k-1} - a_{k+1} \cdots a_d a_1 a_2 \cdots a_k &\in I_1 \\
\sum x_i = a_1 \cdots a_d - a_{k+1} \cdots a_d a_1 \cdots a_k &\in I_1 \\
\sum x_i = [a_1 \cdots a_k, a_{k+1} \cdots a_d] &
\end{aligned}$$

If $a_1 \cdots a_d$ is noncyclic monomial, $a_d a_1 = 0$

$$\begin{aligned}
a_1 \cdots a_d &= a_1 \cdots a_d - a_2 \cdots a_d a_1 \\
&= [a_1, a_2 \cdots a_d]
\end{aligned}$$

These form a basis for the complement of $k[Q]_{cyc}$ in $k[Q]$ so the vanishing follows.

35.3.5 Example *Let Q be the quiver with 1 vertex and 3 edges a, b and γ. The path algebra is the $k < a, b, \gamma >$ which is free "polynomial" algebra on three noncommutative generators. Let the potential be $W = \gamma\big[a, b\big]$.*

$$\begin{aligned} \partial_{\gamma^*} W &= \big[a, b\big] \\ \partial_{a^*} W &= b\gamma - \gamma b \\ \partial_{b^*} W &= \gamma a - a\gamma \end{aligned}$$

Therefore the Jacobian algebra is the commutative polynomial algebra on three generators $k[a, b, \gamma]$.

35.3.6 Definition (Ginzburg Cubic Potential) *The above is an example of a more general procedure where we start with a quiver Q and come up with a new quiver Q' and a potential W for that quiver which is cubic in the arrows of Q'.*

Q' has the same vertices as Q but in addition there are reverse arrows as well. If we only did this, it would be $\overline{Q}$ of definition 32.1.9. But in addition we will also have self loops on each vertex ω_i. From there we can define

$$W = \sum_{a \in Q'_1} [a, a^\dagger] \sum_{i \in Q'_0} \omega_i$$

This procedure applied to the 1 vertex, 1 loop quiver gives the above example.

35.3.7 Definition (Ginzburg dg-algebra) *Say we start with a quiver Q and a potential W for it. We again build a Q' from this in the same way, but now we will grade the arrows so $k[Q']$ is a graded algebra in a cohomological direction as well as path length. The original arrows will be there and with degree 0. Again the reversed arrows will be present but this time $a^\dagger$ will have degree -1. The self loops ω_i will have degree -2.*

Endow the path algebra with a differential of degree $+1$ as follows

$$\begin{aligned} da &= 0 \\ da^\dagger &= \partial_{\xi_a} W \\ d\omega_i &= e_i(\sum [a, a^\dagger])e_i \end{aligned}$$

The differentials of every path of length 1 above gives a linear combination of paths with the same source and target. The differentials do not necessarily play with path length.

Taking cohomology of this cochain complex in degree 0 takes the usual $k[Q']$ and quotients by the image of the $da^\dagger$ so it quotients by the ideal generated by $\partial_{\xi_a} W$ via $d(\gamma_1 a^\dagger \gamma_2) = \gamma_1(\partial_{\xi_a} W)\gamma_2$ as desired for $J(Q', W)$. Here γ_1 and γ_2 are arbitrary elements of $k[Q']$ which are closed by construction, and they are in degree 0.

35.3.8 Theorem (Keller-van den Bergh, Amiot) *The Ginzburg dg-algebra $\Gamma_{Q',W}$ is a 3-CY algebra. It's derived category of modules has a t-structure whose heart is equivalent to the ordinary (non-derived) category of modules for the Jacobi algebra. This is the connection back to definition 35.1.1.*

35.4 Moduli Spaces/Stacks

35.4.1 Moduli Stack

35.4.1 Definition ($M_{Q,\vec{d}}$) *For $\vec{d}$ a dimension vector on Q_0. Define the global quotient stack $M_{Q,\vec{d}}$ as $A_{Q,\vec{d}}$ quotiented by the action of $G_{Q,\vec{d}}$. Here $A_{Q,\vec{d}}$ is the product of $Hom(\mathbb{C}^{d(i)}, \mathbb{C}^{d(j)})$ as ij varies over all arrows in Q_1. $G_{Q,\vec{d}}$ acts by change of basis on the vector spaces at each of the vertices. So it acts by left inverse and right multiplications on the appropriate factors of the product. The inverse is as usual to make the action a right action.*

This is bad in that it is a stack, but good in that it is smooth in the sense of smooth stack.

In general a stack of objects in a suitably algebraic category $\mathcal{C}$ is smooth when the homological dimension is at most 1. Here we are looking at the stack of objects for $k[Q]-mod$. See chapter 31

35.4.2 Moduli Space

35.4.2 Definition ($X_{Q,\vec{d}}$) *Do the affinization. $Spec(\Gamma(A_{Q,\vec{d}})^{G_{Q,\vec{d}}})$. The points of these are in bijection with semisimple $\mathbb{C}Q$ representations.*

This is good because it is an affine scheme, but it is bad in that it is very much nonsmooth.

35.4.3 Link between them

There is a Jordan-Holder map $M \to X$. On any particular point, we send a representation of $\mathbb{C}[Q]$ of that prescribed dimension vector, take it's Jordan-Holder filtration, then direct sum all of the quotients to give a semisimple representation of $\mathbb{C}[Q]$. This is what it does on the closed points and one can further describe it's interaction with the topologies on both.

35.4.4 Use of Potential

On $A_{Q,\vec{d}}$ we have a function $W_{\vec{d}}$ which acts by sending a representation ρ to $\mathrm{tr}_\rho(W)$. W is a cyclic element of the path algebra so under the representation we get a sum of terms each of which is in $End(V_i)$ for some i vertex of the quiver. There can be many such terms, but what is important is that they each connect the same vector space to itself. The invariance under $G_{Q,\vec{d}}$ means that this descends to the moduli stack. It is an element of $\Gamma(A_{Q,\vec{d}})^{G_{Q,\vec{d}}}$ which the moduli space is taking $Spec$ of.

35.4.3 Definition (Sheaf of vanishing cycles) *Saying this is a function on a smooth stack or a nonsmooth affine scheme is difficult so instead the cleaner construction is to construct the sheaf of vanishing cycles. This will be a perverse sheaf (so in the derived category rather than being a sheaf by itself) which captures the information about $f^{-1}(0)$ for something we want to call a function f on the scheme or stack.*

35.4.4 Lemma (Vanishing cycles with a linear part) *Let f be a function on $Y \times A^n$ such that it is linear in the second factor. This means it can be written as $f = \sum z_i f_i$ where z_i are coordinates on A^n and the f_i are functions on Y. There is the projection π_2 onto the Y factor. In Y we can ask for the zero set of all the f_i.*

There is a natural translation of functors $\pi_{2,!}\phi_f \pi^*(-) \to i_* i^*(-) \otimes H^\bullet_{c.s.}(A^n)$ where ϕ_f is the vanishing cycle functor and $H_{c.s.}$ stands for compactly supported cohomology. Being a natural transformation comes from abstract nonsense, then the work is to show this is an isomorphism.

35.4.5 Corollary

$$H^\bullet_{c.s.}(Y \times A^n, \phi_f \underline{\mathbb{Q}}) \quad \simeq \quad H_{c.s.}(Z \times A^n, \underline{\mathbb{Q}})$$

35.4.6 Corollary *Let (Q, W) come from the Ginzburg cubic construction.*

$$\begin{aligned} H_\bullet(M, \phi_{\mathrm{tr}\, W} IC\underline{\mathbb{Q}}) &\simeq (H^\bullet_{c.s.}(M, \phi_{\mathrm{tr}\, W}\underline{\mathbb{Q}})^* \\ H^\bullet_{c.s.}(M, \phi_{\mathrm{tr}\, W}\underline{\mathbb{Q}}) &\simeq H^\bullet_{c.s.}(Rep\Pi_Q, \underline{\mathbb{Q}}) \otimes L^{(d,d)_{Euler}} \end{aligned}$$

where Π_Q is the preprojective algebra $(\mathbb{C}\overline{Q})/I_0$ from definition 32.1.11.

35.5 BPS Lie Algebra

35.6 DT Analog

Define an analog of DT invariants for the data of quiver, potential and a finite field equipped with a multiplicative character $(F_q, +) \to (\overline{\mathbb{Q}_\ell}^*, \times)$.

35.6.1 Definition

$$DT(Q, W, F_q, \psi)_{\vec{d}} \quad \equiv \quad \sum_\rho \psi\, \mathrm{tr}_\rho W$$

where the sum is over absolutely indecomposable representations of $k[Q]$ with dimension vector $\vec{d}$. If we let ψ take everything to 1 this is the count $A_{Q,\vec{d}}(q)$ from the Kac polynomial but interpreted through the inclusion $\mathbb{N} \to \overline{\mathbb{Q}_\ell}$. Then combine the results for different $\vec{d}$ into a generating series.

Chapter 36

Poisson Geometry Appendix

36.1 Poisson

36.1.1 Definition (Poisson Structure) *Let us set up the notation as $\{,\}_M$ as the Poisson bracket on $C^\infty(M)$. Similarly let π_M be the associated bivector field. It must satisfy skew-commutativity and the Jacobi identity. This amounts to a condition on Schouten bracket of π_M.*

36.1.2 Definition (Poisson Lie Algebroid definition 37.2.1)

$$\begin{array}{ccc} T^*M & & TM \\ & M & \end{array}$$

36.1.3 Definition (Symplectic Realization) *A symplectic realization of a Poisson manifold (P, π) is a symplectic manifold (Y, ω) equipped with a Poisson map $(Y, \omega^{-1}) \to (P, \pi)$ that is a surjective submersion.*

36.1.4 Definition (Differentiable Groupoid) *Let X_0 and X_1 be the set of objects and morphisms. Demand that X_0 is a submanifold of X_1 given by the identity morphisms. Demand that the composition/inverse are smooth maps and that the source and target maps are submersions.*

36.1.5 Definition (Symplectic Groupoid) *A Lie groupoid whose manifold of morphisms X_1 is equipped with a symplectic structure $\omega \in \Omega^2(X_1)$ that behaves under the three projections $X_2 \to X_1$ where X_2 is the space of composable pairs of morphisms.*

$$\delta\omega \equiv pr_1^*\omega - \odot^*\omega + pr_2^*\omega = 0$$

pr_i are pullbacks along the projection maps to the first and second of the pair. $\odot$ is the map $X_2 \to X_1$ given by composition.

One can phrase this as a Lagrangian condition in $(X_1, \omega) \times (X_1, -\omega) \times (X_1, \omega)$ where the outer terms represent the factors of the composable pair and the middle is the composite.

36.1.6 Definition (Poisson Dual Pairs)

36.1.7 Lemma (Symplectic Groupoid gives Symplectic Realization) *The inversion map is anti-symplectic. The fixed points of this is precisely X_0 realized as the identity maps. This gives that X_0 is Lagrangian. Then we can see that $pr_i^* C^\infty(X_0)$ centralize each other in $C^\infty(X_1)$. Using definition 36.1.6, this then shows that X_0 is Poisson in a unique way to make the source/target maps Poisson/anti-Poisson respectively.*

36.1.8 Example (Lie-Poisson structure) $X_1 = T^*G$ $X_0 = \mathfrak{g}^*$.

36.1.9 Example (Trivial Groupoid) *Let the space of objects be G and the fibers be trivial. Then we still get $X_1 = T^*G$ as a symplectic groupoid. But now the condition for composability is to project to the same point in G and the operation is addition in the fiber direction. The Poisson structure on $X_0 = G$ is the 0 Poisson structure.*

36.1.10 Example (Fundamental Groupoids) *Let X be a symplectic manifiold. Build the space $(X, \omega) \times (X, -\omega)$ as the symplectic groupoid where the composition operation just removes the middle point so $(p, q) \odot (q, r) = (p, r)$. The spaces of homotopy/homology classes of paths with fixed endpoints map to this space and we can lift the structure up to these.*

36.1.11 Definition (Integrable Poisson Manifold) *Those Poisson manifolds that arise as X_0 of a symplectic groupoid.*

36.1.12 Definition (Symplectic Double Groupoid)

36.2 Shifted Poisson Structures

36.2.1 Operadic

36.2.1 Theorem (Formality)

36.2.2 Safranov

36.3 Quasi-Poisson

Chapter 37

Generalized Geometry

37.1 $T \bigoplus T^*$

37.2 Courant Algebroid

37.2.1 Definition (Lie Algebroid) *A Lie algebroid is like a Lie algebra varying over a manifold M.*

There is a bracket on the sections of E. For $X, Y \in \Gamma(E)$ and f a smooth function on M, the following relations hold:

$$\begin{aligned} \big[X, fY\big]_E &= (\rho(X)f) \cdot Y + f\big[X, Y\big]_E \\ \rho(\big[X, Y\big]_E) &= \big[\rho(X), \rho(Y)\big]_{TM} \end{aligned}$$

37.2.2 Example (Lie algebra) *Let M be a point. The anchor map is trivial . The first condition becomes linearity over f which is just linearity over the constant functions. The second condition becomes vacuous equality in the tangent bundle to the point.*

37.2.3 Example (Bundle of Lie algebras over M) *Take the anchor map to be 0. The first relation becomes linearity over $C^\infty(M)$. The second condition becomes $0 = 0$. The anchor map is precisely what measures the failure of a Lie algebroid to just be a bundle of Lie algebras.*

37.2.4 Example (Tangent Bundle) *$E = TM$, The bracket in E is the bracket of vector fields and the anchor map is the identity.*

37.2.5 Example (Poisson Lie Algebroid) *For a Poisson manifold P, we construct an algebroid $E = T^*P$. The bracket is given by*

$$\big[\alpha, \beta\big] = L_{\pi\alpha}\beta - L_{\pi\beta}\alpha - d\pi(\alpha \wedge \beta)$$

37.2.6 Example (Lie Bialgebroid)

37.2.7 Definition (Dirac Structure) *Combine Poisson and symplectic structures into one notion as sub-bundles of $TM \bigoplus T^*M$. The special cases are given by thinking of the graphs of bundle maps $TM \to T^*M$ or vice versa.*

We don't get all sub-bundles. We get isotropic ones under the pairing of the dual fibers.

So from this we demand that the sub-bundle be maximal isotropic and satisfy some integrability conditions. This is what is known as a Dirac structure.

37.2.8 Example (Foliation) *We already saw how Dirac structures are simultaneous generalizations of Poisson and symplectic structures. Another possibility is to take a foliation $\mathcal{F}$. Then the annihilator $\mathcal{F}^{\perp}$. Let $\mathcal{F}$ to be an involutive foliation.*

Form $\mathcal{F} \bigoplus \mathcal{F}^{\perp}$

37.2.9 Definition (Courant Algebroid) *A Courant algebroid consists of a vector bundle $E \to M$ such that on sections of E there is a skew-symmetric bracket operation, an anchor map ρ to TM and on each fiber there is a non-degenerate inner product. The bracket sort of acts like a Lie-bracket but it fails the Jacobi identity by an exact term.*

$$\begin{aligned}
\big[\phi, \big[\psi, \chi\big]\big] + cyclic &= (\kappa^{-1}\rho^T d)T(\phi, \psi, \chi) \\
T(\phi, \psi, \chi) &\equiv \frac{1}{2}\langle \big[\phi, \psi\big], \chi\rangle + cyclic \\
D &\equiv (\kappa^{-1}\rho^T d)
\end{aligned}$$

where κ is the map $E \to E^*$ coming from the non-degenerate inner product. ρ^T is the transpose of ρ so it is a map $T^*M \to E^*$. d is the deRham differential. $T(\phi, \psi, \chi)$ becomes a function on M so dT is a section of T^*M. The entire equation is about sections of E.

37.2.10 Definition (Double Brackets) *We can define an alternative bracket which satisfies the Jacobi identity without such a T term, but at the expense of skew symmetry.*

Insert definition of double brackets.

What was this equation for??

$$\begin{aligned}
\big[\big[\phi + \psi, \phi + \psi\big]\big] &= \frac{1}{2}D\langle \phi \mid \phi\rangle + \frac{1}{2}D\langle \psi \mid \psi\rangle + \big[\big[\phi, \psi\big]\big] + \big[\big[\psi, \phi\big]\big] \\
&= \frac{1}{2}D\langle \phi + \psi \mid \phi + \psi\rangle
\end{aligned}$$

37.2.11 Example (Standard Courant Algebroid) *Let $E \equiv TM \bigoplus T^*M$.*

The inner product is the obvious one

$$\langle X + \xi \mid Y + \eta\rangle = \eta(X) + \xi(Y)$$

The bracket is as follows

$$\left[X + \xi \mid Y + \eta\right] \quad = \quad \left[X, Y\right] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi + \frac{1}{2} d(\eta(X) - \xi(Y))$$

If we look at this only as a bundle, then this is the Pontryagin bundle. The extended action of classical mechanics is a functional of curves $\mathbb{R} \to P$ where P is a bundle over $\mathbb{R}$ with $TM \bigoplus T^*M$ as fibers. Here M is the configuration space. The coordinate for the source $\mathbb{R}$ is λ while the base $\mathbb{R}$ of P corresponds to the actual time t. The fiber Courant algebroid has coordinates $q \in M$, $p \in T_q^*M$ and $v \in T_qM$.

37.2.12 Definition (Courant Sigma Model)

37.2.13 Lemma (Boundaries of Courant Sigma Model) *Canntaneo-Qiu-Zabzine*

37.3 Bismut Connection