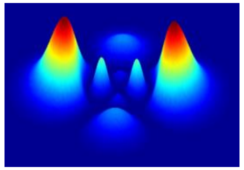


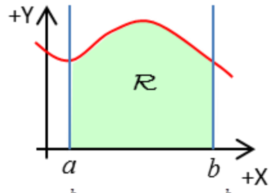
MATHEMATICS EXTENSION 2

4 UNIT MATHEMATICS

TOPIC 5: VOLUMES

5.1 AREA UNDER A CURVE

Consider a continuous $f(x)$ in the closed interval $[a, b]$. The area A under the curve is the region $\mathcal{R}$ bounded by the function $f(x)$ and the X axis $(y = 0)$ and between the vertical lines $x = a$ and $x = b$.

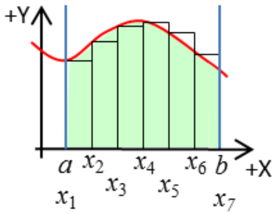


$A = \int_a^b f(x)\,dx = \int_a^b y\,dx$

We can set up a procedure for finding the value of the area A. Select a set of N equally spaced points along the X axis in the interval $[a, b]$ where

$$a = x_1, x_2, \cdots, x_{N-1}, x_N = b \qquad \Delta x = x_{k+1} - x_k \quad k = 1, 2, 3, \cdots, N-1$$

The interval $[a, b]$ is divided into $N-1$ subintervals. The area under the curve can be approximated by summing the area of $N-1$ rectangular strips of width Δx and height $f(x_k)$ with the area of the k^{th} rectangular strip equal to $A_k = f(x_k)\Delta x$.



$A \approx \sum_{k=1}^{N-1} A_k = \sum_{k=1}^{N-1} f(x_k)\,\Delta x = \sum_{k=1}^{N-1} y_k\,\Delta x$

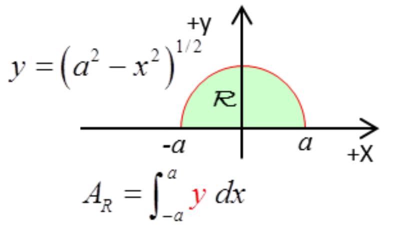
Hence the area A is the summation of the area of the $N-1$ strips

$$A \approx \sum_{k=1}^{N-1} A_k = \sum_{k=1}^{N-1} f(x_k)\,\Delta x = \sum_{k=1}^{N-1} y_k\,\Delta x$$

The approximation becomes better as $\Delta x \to 0$ and $N \to \infty$. The summation in the limit as $\Delta x \to 0$ is called the **definite integral of** $f(x)$ **from *a* to *b***.

$$A = \int_a^b f(x)\,dx = \int_a^b y\,dx$$ **definite integral**

Example **Find the area of a circle by the evaluation of a definite integral**



Solution

The equation of a circle with radius a is $x^2 + y^2 = a^2$

Sketch the upper hemisphere of the circle which is given by the function

$$f(x) = y = \left(a^2 - x^2\right)^{1/2}$$

The area of the hemisphere A_R is

$$A_R = \int_{-a}^{a} y\, dx = \int_{-a}^{a} \left(a^2 - x^2\right)^{1/2}\, dx$$

$$x = a\sin(\theta) \qquad dx = a\cos(\theta)\, d\theta \qquad \left(a^2 - x^2\right)^{1/2} = a\left(1 - \sin^2(\theta)\right) = a\cos(\theta)$$

$$x = a \to \theta = \pi/2 \qquad x = -a \to \theta = -\pi/2$$

$$A_R = \int_{-\pi/2}^{\pi/2} a^2 \cos^2(\theta)\, d\theta \qquad \cos^2(\theta) = \left(\tfrac{1}{2}\right)\left(\cos(2\theta) + 1\right)$$

$$A_R = \frac{a^2}{2}\int_{-\pi/2}^{\pi/2} \left(\cos(2\theta) + 1\right)\, d\theta$$

$$A_R = \frac{a^2}{2}\,\pi$$

The area of the circle A is $A = 2\,A_R$

$$A = \pi a^2$$ ***QED***