Many problems in the mathematical analysis of particles moving under the influence of resistive forces, you start with the equation of motion. Then to find from the forces or acceleration you must then integrate the equation of motion to find velocities as functions of time and /or displacement and displacements as functions of time and /or velocity.

To be successful in this topic you need to be very familiar with the exponential and log functions and be competent in integrating functions.

BASIC KNOWLEDGE (need to know this information very well)

$$\log_e(y) \equiv \ln(y)$$

$$\log_e(A_1 \, A_2) = \log_e(A_1) + \log_e(A_2) \qquad \log_e(A_1 / A_2) = \log_e(A_1) - \log_e(A_2)$$

$$\log_e(A)^n = n \log_e(A)$$

$$y = A_1 e^{A_2 x} \qquad \log_e(y) \equiv \ln(y)$$

$$\log_e(y) = \log_e\left(A_1 e^{A_2 x}\right) = \log_e(A_1) + A_2 \, x$$

$$\int \frac{dx}{a+b\,x} = \frac{1}{b} \log_e(a+b\,x) + C$$

$$\int \frac{f'(x)\,dx}{f(x)} = \log_e(f(x)) + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C = \frac{1}{a} a\tan\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log_e\left(\frac{a+x}{a-x}\right) + C$$

$$v(t) = \frac{dx(t)}{dt} \qquad x(t) = \int v(t)\,dt$$

$$a(t) = \frac{dv(t)}{dt} \qquad v(t) = \int a(t)\,dt$$

$$a(v) = \frac{dv(t)}{dt} = v\frac{dv(x)}{dx} \quad dx = \frac{v}{a(v)}\,dv \qquad x = \int \frac{v}{a(v)}\,dv$$

$$a(x) = \frac{dv(t)}{dt} = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right) \qquad d\left(\tfrac{1}{2}v^2\right) = \int a(x)\,dx$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \qquad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\tan(A) = z \quad A = \operatorname{atan}(z) \equiv \tan^{-1}(z) \quad \tan^{-1}(\tan(A)) = A$$

MOTION UNDER THE INFLUENCE OF A RESISTIVE FORCE

We will consider the motion of a particle moving vertically. The forces acting on the particle are the gravitational force (weight) $\vec{F}_G$ and a resistive motion $\vec{F}_R$ opposing the motion of the particle. The direction of the resistive force $\vec{F}_R$ is always opposite to the direction of the velocity.
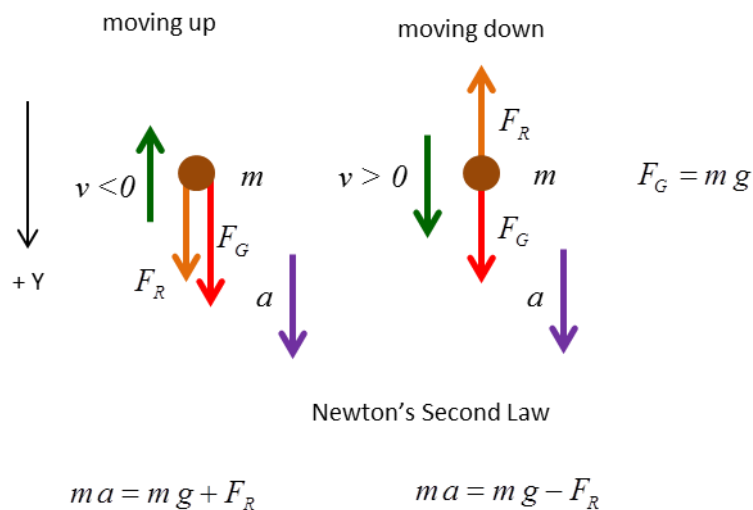

Fig. 1. Forces acting on a particles of mass *m* are the gravitational force F_G and the resistive force F_R opposing the motion of the object. The frame of reference is the + Y axis pointing to down.

The equation of motion of the particle can be derived from Newton's Second Law

(1) $\qquad \vec{a} = \frac{1}{m} \sum_i \vec{F}_i \qquad$ Newton's Second Law

(2) $\qquad a_{up} = \frac{F_G + F_R}{m} \qquad a_{down} = \frac{F_G - F_R}{m}$

Usually the resistive force F_R is a function of velocity v. Two very important examples are

(3) $F_R = -\beta v$

(4) $F_R = -\alpha v^2$

Consider the motion of an object falling vertically (+ direction is down). Then the resultant force F_{net} acting on the particle is

(5) $F_{net} = m g - F_R$

The net force acting on the particle can never be negative ($F_{net} \geq 0$), so the maximum value of the resistive force is $F_{R\_max} = m g$. When the net force acting on the particle is zero ($F_{net} = 0$), the acceleration must also be zero ($a = 0$), therefore the particle will move downward with a constant speed. This is the maximum speed at which the particle falls and is called the **terminal speed** v_T. The speed of the particle in this instance is the magnitude of the instantaneous velocity. The terminal velocity v_T depends upon the mass m of the particle when the resistive force is given by equations (2) and (3).

$$F_R = -\beta v \quad \beta v_T = m g \quad v_T = m \frac{g}{\beta} \quad v_T \propto m$$

$$F_R = -\alpha v^2 \quad \alpha v_T^2 = m g \quad v_T = \sqrt{m \frac{g}{\alpha}} \quad v_T \propto \sqrt{m}$$

The motion of falling objects is usually described with constant acceleration. This is only approximately true: air resistance, a friction which increases with increasing speed, acts against gravity, so the speed of falling objects tends toward a limit called terminal speed. The physics of air resistance applies to motion through any sort of fluid: gas or liquid. To a reasonable approximation, fluid resistance tends to depend on either the first power of the speed (a linear resistance) or the second power (a quadratic resistance).

At small speeds, fluid resistance is linear

$$(3) \qquad F_R = -\beta v$$

where β has the units of $N.m^{-1}.s = kg.s^{-1}$. The force βv is sometimes called the **viscous drag**. It arises mainly from the cohesion of the layers of the fluid.

For large speeds, fluid resistance is better described by a quadratic dependence on speed

$$(4) \qquad F_R = -\alpha v^2$$

where the units of α are $N.m^{-2}.s^2 = kg.m^{-1}$. This term is usually called **drag**. It is related to the momentum transfer between the moving object and the fluid through which it travels.

In general, both terms (and others, proportional to higher powers of v) may be present

$$(5) \qquad F_R = -\beta v - \alpha v^2$$

If $v \ll 1$, drop the second term or ignore the first term if $v \gg 1$.

The speeds of everyday objects are large enough that the effects of fluid resistance are due entirely to quadratic drag. Skydivers are slowed appreciably by the quadratic term and viscosity does nothing.