ONLINE: MATHEMATICS EXTENSION 2

Topic 6   MECHANICS

6.6   MOTION IN A CIRCLE

When a particle moves along a circular path (or curved path) its velocity must change even if its speed is constant, hence the particle must be accelerating. According to Newton's Second Law, the particle must be acted upon by a non-zero resultant force that produces the acceleration.

(1) $\vec{a} = \dfrac{d\vec{v}}{dt}$ instantaneous acceleration

(2) $\vec{a} = \dfrac{1}{m} \sum_i \vec{F_i}$ Newton's Second Law

Position, displacement and angular displacement

The equation of the circle with centre at (0, 0) and radius R is

(3) $x^2 + y^2 = R^2$

At any instant, the position of the particle is specified by its x and y **Cartesian coordinates**. The position of the particle can also be specified in terms of its **polar coordinates** (R, θ) where R is the radius of the circle and θ is the **angular displacement** of the particle. The angular displacement θ must be given in radians and is measured in an anticlockwise sense from the X axis as shown in figure (1).

$x^2 + y^2 = R^2$

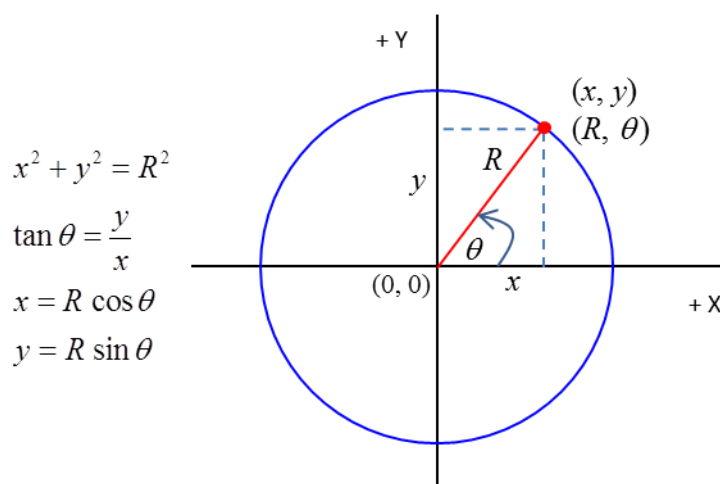$\tan \theta = \dfrac{y}{x}$

$x = R \cos \theta$

$y = R \sin \theta$

Fig. 1.   Particle moving in a circle. Its position is given in Cartesian coordinates (x, y) and in polar coordinates (R, θ).

Velocity v and angular velocity ω

When the particle is moving in a circle its displacement vector $\vec{s}$ has a constant magnitude R but its direction given by θ increases with time.

At time $t = t_1$ the particle is at the point P_1 and moves to a point P_2 in a time interval Δt.

$$x_1(t_1) = R\cos(\theta_1) \qquad y_1(t_1) = R\sin(\theta_1)$$
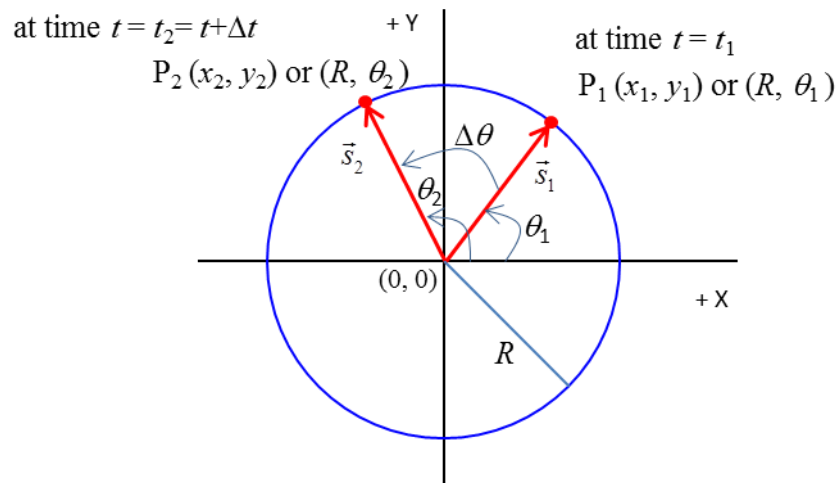$$x_2(t + \Delta t) = R\cos(\theta_2) \quad y_2(t + \Delta t) = R\sin(\theta_2)$$

Fig. 2.   Position of a particle at times t_1 and $t_1+\Delta t$.

The angular displacement changes by $\Delta\theta = \theta_2 - \theta_1$ in the time interval Δt. The rate of change of the angular displacement with time is called the **angular velocity** ω (Greek letter omega).

$$(4) \qquad \omega_{average} = \frac{\Delta\theta}{\Delta t} \qquad \text{average angular velocity} \quad [\text{rad.s}^{-1} \text{ radians/second}]$$

In the limit $\Delta t \rightarrow 0$ we can define the instantaneous angular velocity as

$$(5) \qquad \omega = \dot{\theta} = \frac{d\theta}{dt} \qquad \text{instantaneous angular velocity}$$

The instantaneous velocity is equal to the time rate of change of displacement. Therefore, the X and Y components of the particle's velocity are

$$v_x = \frac{dx}{dt} = \frac{d}{dt}(R\cos\theta) = -R\sin\theta\,\frac{d\theta}{dt}$$
$$(6) \qquad v_y = \frac{dy}{dt} = \frac{d}{dt}(R\sin\theta) = R\cos\theta\,\frac{d\theta}{dt}$$
$$v_x = -R\,\omega\sin\theta \qquad v_y = R\,\omega\cos\theta$$

The magnitude of the instantaneous velocity v is

$$v = \sqrt{v_x{}^2 + v_y{}^2} = \sqrt{\left(-R\,\omega\sin\theta\right)^2 + \left(R\,\omega\cos\theta\right)^2}$$

(7) $v = R\,\omega$

The direction of the instantaneous velocity is by the angle ϕ

$$\tan\phi = \frac{v_y}{v_x} = -\frac{\cos\theta}{\sin\theta} = -\frac{1}{\tan\theta}$$

But $\tan\theta$ is the slope of the displacement vector, therefore the velocity vector at all times is **perpendicular** to the displacement vector. The instantaneous velocity is always in the direction of the tangent to the circle at the instantaneous position of the particle. This velocity is known as the **tangential velocity** v_T. The is no radial motion, so the radial velocity v_R is zero.
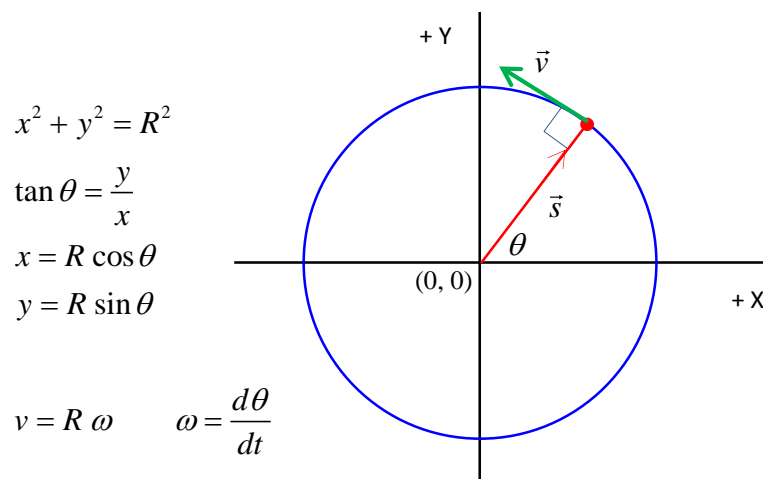
$$v = v_T = R\,\omega \quad v_R = 0$$

$x^2 + y^2 = R^2$

$\tan\theta = \dfrac{y}{x}$

$x = R\cos\theta$

$y = R\sin\theta$

$v = R\,\omega \qquad \omega = \dfrac{d\theta}{dt}$

Fig. 3.   The instantaneous velocity is at all times perpendicular to the displacement vector.

Acceleration a and angular acceleration α Forces

The instantaneous acceleration is the time rate of change of the velocity and the time rate of change of the angular velocity ω is known as the angular acceleration α.

$$\vec{a} = \frac{d\vec{v}}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \omega\frac{d\omega}{d\theta} = \frac{d\left(\frac{1}{2}\omega^2\right)}{d\theta}$$

The acceleration $\vec{a}$ can be resolved into its X and Y components or its radial $\vec{a}_R$ and tangential components $\vec{a}_T$.

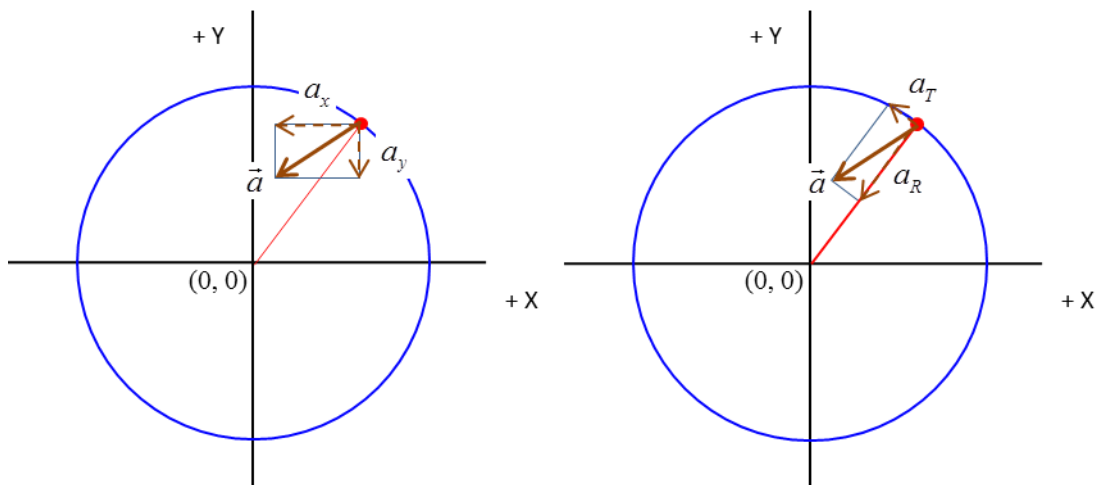$$\vec{a} = \vec{a}_x + \vec{a}_y = \vec{a}_R + \vec{a}_T$$

Fig. 4.   The acceleration $\vec{a}$ can be resolved into its X and Y components or its radial $\vec{a}_R$ and tangential components $\vec{a}_T$.

The X and Y components of the acceleration are found by differentiating the X and Y components of the velocity given in equation (6) with respect to time.

(8)
$$a_x = \dot{v}_x = \ddot{x} = -R\omega\cos\theta\frac{d\theta}{dt} - R\sin\theta\frac{d\omega}{dt} = -R\omega^2\cos\theta - R\frac{d\omega}{dt}\sin\theta$$

$$a_y = \dot{v}_y = \ddot{y} = -R\omega\sin\theta\frac{d\theta}{dt} + R\cos\theta\frac{d\omega}{dt} = -R\omega^2\sin\theta - R\frac{d\omega}{dt}\cos\theta$$

The radial and tangential components are easily found from equation (8) by taking

$$\theta = 0 \text{ rad} \quad \text{or} \quad \theta = \pi/2 \text{ rad}$$
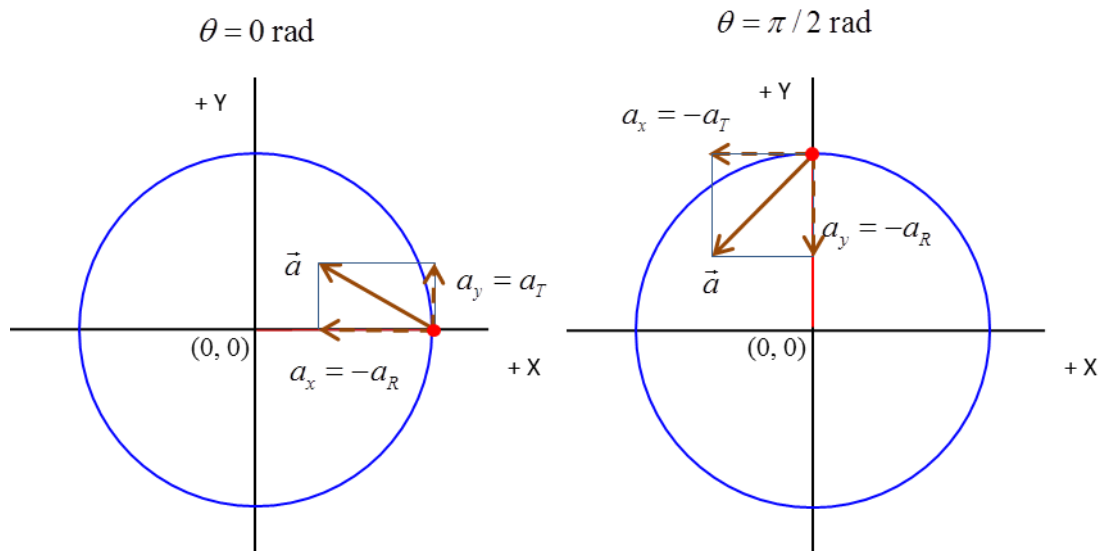
Fig. 5. The radial and tangential components of the acceleration can be calculated from the X and Y components when $\theta = 0$ rad or $\theta = \pi / 2$ rad.

$$\theta = 0 \text{ rad} \quad a_R = -a_x = +R\,\omega^2 \cos(0) + R\frac{d\omega}{dt}\sin(0)$$

$$a_R = R\,\omega^2 = \frac{v^2}{R} \qquad v = R\,\omega$$

$$\theta = \frac{\pi}{2} \text{ rad} \quad a_R = -a_y = +R\,\omega^2 \sin\left(\frac{\pi}{2}\right) + R\frac{d\omega}{dt}\cos\left(\frac{\pi}{2}\right)$$

$$a_R = R\,\omega^2 = \frac{v^2}{R} \qquad v = R\,\omega$$

$$\theta = 0 \text{ rad} \quad a_T = a_y = -R\,\omega^2 \sin(0) + R\frac{d\omega}{dt}\cos(\theta)$$

$$a_T = +R\frac{d\omega}{dt}$$

$$\theta = \frac{\pi}{2} \text{ rad} \quad a_T = -a_x = +R\,\omega^2 \cos\left(\frac{\pi}{2}\right) + R\frac{d\omega}{dt}\sin\left(\frac{\pi}{2}\right)$$

$$a_T = +R\frac{d\omega}{dt}$$

(9) $a_R = R\,\omega^2 = \dfrac{v^2}{R}$ radial acceleration

(10) $a_T = R\,\dfrac{d\omega}{dt}$ tangential acceleration

The magnitude of the tangential velocity is

$$(7) \qquad v_T = v = R\,\omega$$

and so the time rate of change of the speed of the particle gives the tangential acceleration a_T

$$a_T = \frac{dv_T}{dt} = \frac{d}{dt}\left(R\,\omega\right) = R\,\frac{d\omega}{dt}$$

which is the same result as given in equation (10).

The **radial acceleration** is also called the **normal component** a_N since it is always at right angles to the velocity of the particle. In Physics, it is mostly called the **centripetal acceleration** a_C. The centripetal acceleration represents the time rate of change which represents the change in **direction** of the velocity vector.

$$a_R \equiv a_N \equiv a_C$$

The particle moving in a circle always has a centripetal acceleration directed towards the centre of the circle. If the speed of the particle changes it has a tangential component of acceleration which is tangent to the circle. This tangential acceleration represents the time rate of change of the **magnitude** of the velocity vector.

The magnitude of the acceleration vector $\vec{a}$ is

$$(11) \qquad a = \sqrt{a_x{}^2 + a_y{}^2} = a = \sqrt{a_R{}^2 + a_T{}^2}$$

and its direction can be determined from

$$(12) \qquad \tan\theta_A = \frac{a_y}{a_x}$$

where θ_A is the angle of the acceleration vector measured with respect to the X axis.

As with any acceleration, there must be a resultant (net) force in the direction of the acceleration to produce it. For centripetal acceleration, this force is called the **centripetal force**. It is not a new kind of force, but merely a label for the force needed for circular motion. The centripetal force may be due to a spring, string, friction, gravitation, etc or a combination of these forces. The centripetal force is always directed towards the centre of the circle. There is no arrow for centripetal force in a free-body diagram (force diagram) as it is just a label for the force acting towards the centre of the circle. The word centripetal is derived from two Greek words meaning "seeking centre".

From Newton's Second Law (equation 2) and the equations for the acceleration (equations 9 and 10), the components of the force acting on the particle executing circular motion are:

$$(13) \quad F_R = m\,R\,\omega^2 = \frac{m\,v^2}{R} \qquad \text{radial or normal or centripetal force}$$

$$(14) \quad F_T = m\frac{dv}{dt} = m\,R\,\frac{d\omega}{dt} = m\,R\,\frac{d^2\theta}{dt} = m\,R\,\ddot{\theta} = m\,R\,\dot{\omega} \qquad \text{tangential force}$$
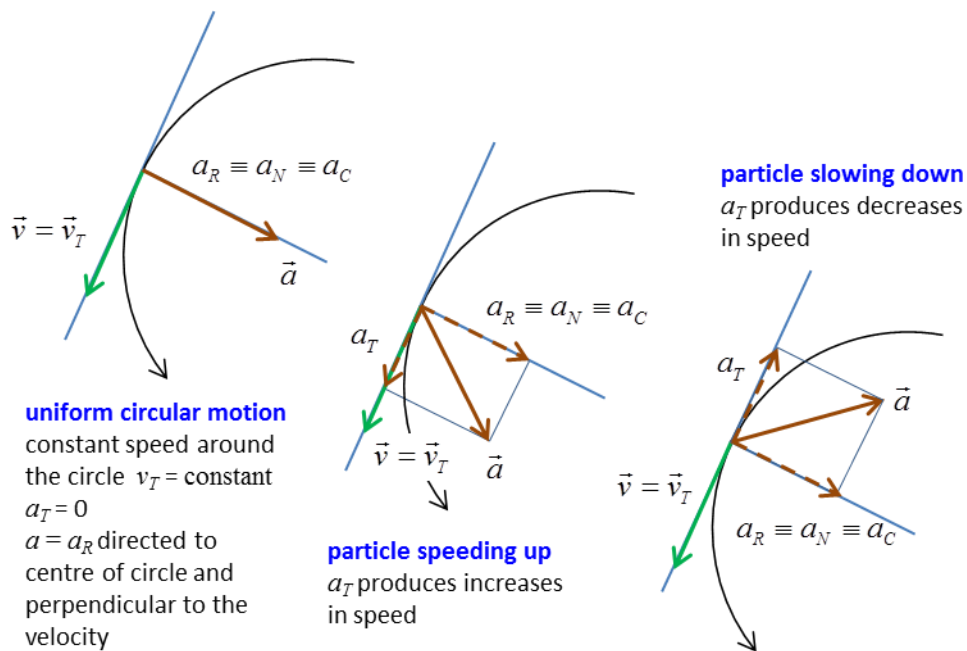

Fig. 6. Motion in a circle: velocity and acceleration vectors. The resultant force acting on the particle is in the same direction as the acceleration $\vec{a}$.

UNIFORM CIRCULAR MOTION

When the particle moves in a circle with constant speed, the motion is called uniform circular motion. The tangential acceleration is zero and the acceleration vector is directed towards the centre of the circle and perpendicular to the velocity vector as shown in figure (6).

$$a_T = 0 \quad \vec{a} = \vec{a}_C \qquad a = a_C \equiv a_R \equiv a_N$$

(9) $\quad a_C = R\,\omega^2 = \dfrac{v^2}{R}$ $\qquad$ centripetal (radial or normal) acceleration

In uniform circular motion, the **period** T is the time interval taken for the particle to travel one revolution (2π rad), that is, in the time interval T the particle travels a distance equal to the circumference of the circle $2\pi R$ with a constant speed.

Speed $\qquad\qquad\qquad v = \dfrac{\Delta s}{\Delta t} = \dfrac{2\pi R}{T}$

Angular speed $\qquad \omega = \dfrac{\Delta\theta}{\Delta t} = \dfrac{2\pi}{T}$

The number of revolutions per second is called the **frequency** f and the angular speed ω is often referred to as the **angular frequency**.

Period and frequency $\qquad T = \dfrac{1}{f}$

Angular frequency $\qquad \omega = \dfrac{2\pi}{T} = 2\pi f$

Centripetal acceleration $\qquad a_C = R\,\omega^2 = \dfrac{v^2}{R} = R\left(\dfrac{4\pi^2}{T^2}\right) = R\left(4\pi^2 f^2\right)$

We now consider an alternative derivation of the formula relating the centripetal acceleration to the speed of the particle.

$$\Delta\theta = \frac{\Delta s}{R} = \frac{\Delta v}{v}$$

$$\Delta v = \frac{v}{R}\Delta s \qquad \frac{\Delta v}{\Delta t} = \frac{v}{R}\frac{\Delta s}{\Delta t}$$

$$\text{limit } \Delta t \to 0$$

$$\frac{\Delta v}{\Delta t} \to a_C \qquad \frac{\Delta s}{\Delta t} \to v$$

$$a_C = \frac{v^2}{R}$$

The particle moves with a distance Δs along the arc of the circle in a time Δt and through an angle $\Delta\theta$.

$$\Delta\theta = \frac{\Delta s}{R}$$

The change in velocity

$$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$$

$$\vec{v}_2 = \vec{v}_1 + \Delta\vec{v}$$

$$v = |\vec{v}_1| = |\vec{v}_2|$$
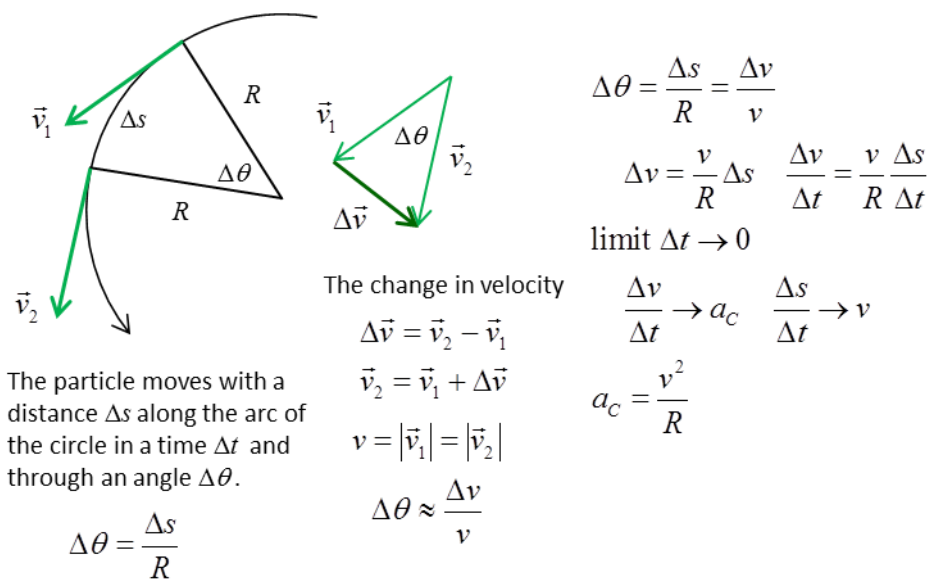
$$\Delta\theta \approx \frac{\Delta v}{v}$$

Fig. 7.   Finding the change in velocity and hence the centripetal acceleration.

So from figure (7), the centripetal acceleration is $a_C = \dfrac{v^2}{R}$ which is the same result as given in equation (9).

The X and Y components of the acceleration are

(8)
$$a_x = \dot{v}_x = \ddot{x} = -R\omega\cos\theta\frac{d\theta}{dt} - R\sin\theta\frac{d\omega}{dt} = -R\omega^2\cos\theta - R\frac{d\omega}{dt}\sin\theta$$

$$a_y = \dot{v}_y = \ddot{y} = -R\omega\sin\theta\frac{d\theta}{dt} + R\cos\theta\frac{d\omega}{dt} = -R\omega^2\sin\theta - R\frac{d\omega}{dt}\cos\theta$$

For uniform circular motion $\dfrac{d\omega}{dt} = 0$ hence the x and Y components reduce to

(15)
$$a_x = -\omega^2 x \qquad x = R\cos\theta$$
$$a_y = -\omega^2 y \qquad y = R\sin\theta$$

Equations (15) are simply the equations for **simple harmonic motion**. As the particles moves around the circle at a uniform speed

- The projection of the x position of the particle onto the X-axis moves with simple harmonic motion.
- The projection of the y position of the particle onto the Y-axis moves with simple harmonic motion.

Example

A bucket of water is swung in a vertical circle of radius R. The speed of the bucket at the top of the swing is v_T and the speed of the bucket at the bottom of the swing is v_B.

(a)   Find the force acting on the water by the bucket at the top of the swing.

(b)   Find the minimum speed v_T for the water to remain in the bucket.

(c)   Find the force exerted by the bucket on the water at the bottom of the swing.

(d)   Estimate the minimum speed at the top of the circle and the period of revolution that would keep you from getting wet if you swung the bucket of water in a vertical circle at a constant speed.

Solution

Draw **free-body diagrams** for the **water** at the top and bottom of the circle

water at top of swing

water at bottom of swing
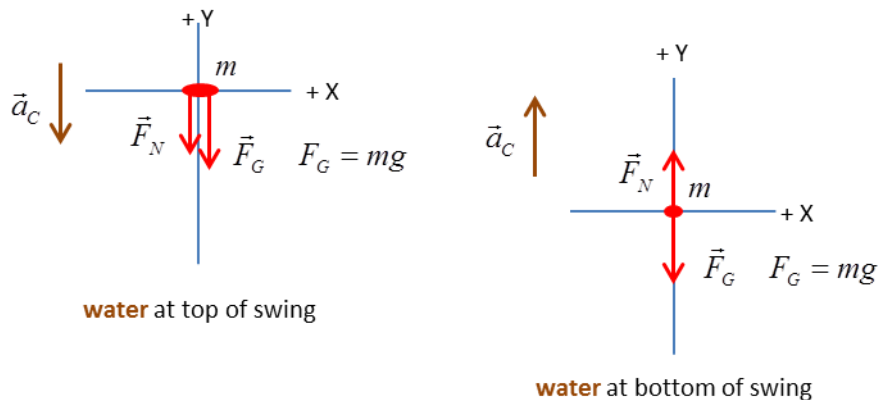
(a)

Top of swing: apply Newton's Second Law to the water

$$\sum F_x = 0 \qquad \sum F_y = -F_G - F_N = -\frac{m v_T^{\,2}}{R} \qquad F_G = m\,g$$

Find the normal force acting on the water

$$F_N = \frac{m v_T^{\,2}}{R} - m\,g$$

(b)

The bucket cannot exert an upward force on the water at the top. The minimum force it can exert is zero $F_N = 0$

$$0 = \frac{m\,v_{T\_min}^{\,2}}{R} - m\,g$$

$$v_{T\_min} = \sqrt{g\,R}$$

(c)

Bottom of swing: apply Newton's Second Law to the water

$$\sum F_x = 0 \qquad \sum Fy = -F_G + F_N = +\frac{m\,v_B^{\,2}}{R} \qquad F_G = m\,g$$

Find the normal force acting on the water

$$F_N = \frac{m\,v_B^{\,2}}{R} + m\,g$$

Note: There is no arrow for the centripetal force in the free-body diagrams. Centripetal force is not a kind of force but simply a label. When the whirling bucket is at the top of the circle, both gravity and the contact force of the bucket on the water contribute to the centripetal force. When the water is moving at its minimum speed at the top of the swing, the water is in free fall (acceleration = g) and the only force acting on the water is gravity ($m\,g$). At the bottom of the swing the normal force must be greater than the weight ($m\,g$) by enough to provide the necessary centripetal force to move the water in a circle.

(d)

Assume $R = 1.0$ m and $g = 9.8$ m.s^{-2}

$$v_{T\_min} = \sqrt{g\,R} = \sqrt{9.8} \text{ m.s}^{-1} \approx 3 \text{ m.s}^{-1}$$

$$T = \frac{2\pi R}{v_{T\_min}} \approx 2 \text{ s}$$

Example (Syllabus)

A string is 0.500 m long and will break if an object of mass exceeding 40.0 kg is hung vertically from it. An object of mass 2.00 kg is attached to one end of the string and it revolves around a horizontal circle with a uniform speed. Find the greatest angular velocity which may be imparted to the object without breaking the string.

$L = 0.500$ m $m_1 = 40.0$ kg $g = 9.80$ m.s^{-2}
$m = 2.00$ kg $\omega_{max} = ?$ rad.s^{-1} radius of circular motion $R = L = 0.500$ m

Max force $Fmax$ exerted on string before breaking

$$F_{max} = m_1\, g = (40)(9.8)\ \text{N} = 392\ \text{N}$$

Max centripetal force F_C exerted by string on object

$$F_C = F_{max}$$

$$m\, R\, \omega_{max}^{\,2} = m_1\, g$$

$$\omega_{max} = \sqrt{\frac{m_1\, g}{m\, R}} = \sqrt{\frac{(40)(9.8)}{(2)(0.5)}}\ \text{rad.s}^{-1} = 19.8\ \text{rad.s}^{-1}$$