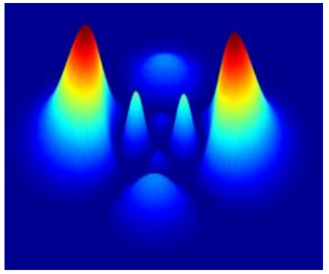


MATHEMATICS EXTENSION 2

TOPIC 5: VOLUMES

Exercise vol5_p006

Find the volumes of the solids of revolution for the region bounded by the function $y = 2\sqrt{x}$, the X-axis and the vertical lines $x_a = 0$ and $x_b = 4$ for the following axes of rotation

(A) X-axis $y_R = 0$

(B) Y-axis $x_R = 0$

(C) $y_R = -2$

Solution

(A) **rotation around X-axis**

Volume of solid of revolution about the X-axis is

$$V = \pi \int_{x_a}^{x_b} y^2\, dx$$ **Disk Method**

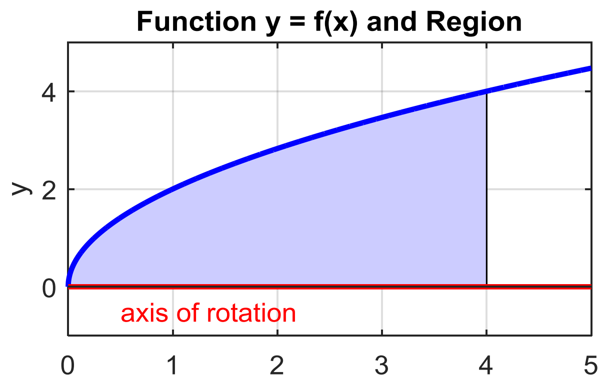
The limits of integration are $x_a = 0$ and $x_b = 4$

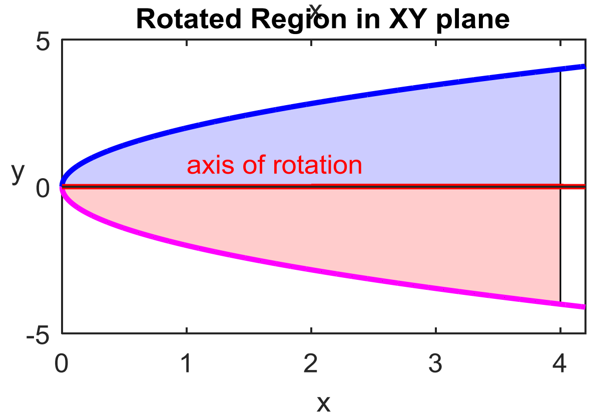
The function $y = f(x) \geq 0$ in the interval [0 4] is

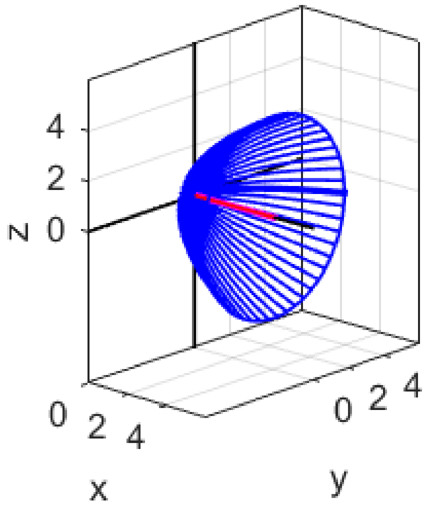
$$y = 2\sqrt{x} \qquad y^2 = 4x$$

The volume of the cone is

$$V = 4\pi \int_0^4 x\, dx = 4\pi \left[\tfrac{1}{2}x^2\right]_0^4 = 32\pi$$







We can also find the volume of solid of revolution about the X-axis using the cylindrical shell method

$$V = 2\pi \int_{y_a}^{y_b} y\,x\,dy \qquad \textbf{Cylindrical Shell Method}$$

The limits of integration are $y_a = 0$ and $y_b = 4$ and the function is

$$y = 2\sqrt{x} \qquad x = \frac{y^2}{4} \qquad x\,y = \frac{y^3}{4}$$

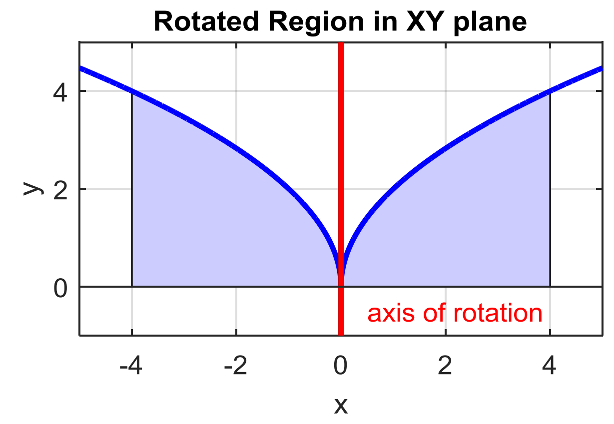
The volume V of the solid of revolution is

$$V = 2\pi \int_0^4 \frac{y^3}{4}\,dy = 2\pi\left[\tfrac{1}{4}\,y^3\right]_0^4$$

$$V = 32\pi$$

(B) rotation around Y-axis

Volume of solid of revolution about the Y-axis is $V = 2\pi \int_{x_a}^{x_b} y\,x\,dx$



Cylindrical Shell Method

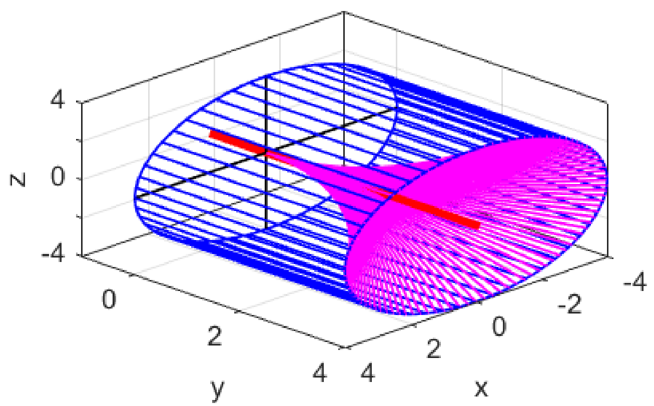
The limits of integration are $x_a = 0$ and $x_b = 4$. The function $y = f(x) \geq 0$ in the interval [0 4] is

$$y = 2\sqrt{x} \qquad x\,y = 2x^{3/2}$$

The volume of the solid of revolution is

$$V = 2\pi \int_0^4 2x^{3/2}\,dx = 4\pi\left[\tfrac{2}{5}x^{5/2}\right]_0^4 = \frac{8\pi}{5}[32] = \frac{256\pi}{5}$$

We can also find the volume of revolution using the **Disk Method** but it is a little more difficult. V_2 is the volume of revolution for the rotation of the line $x_b = 4$ in the interval $0 \leq y \leq 4$



$$V_2 = \pi \int_{y_a}^{y_b} x^2\,dy = \pi \int_{y_a}^{y_b} x^2\,dy = \pi \int_0^4 4^2\,dy = 16\pi\left[y\right]_0^4 = 64\pi$$

V_1 is the volume of revolution of the region bounded by the function and the Y-axis

$$V_2 = \pi \int_{y_a}^{y_b} x^2\,dy = \pi \int_{y_a}^{y_b} \left(y^4/16\right)dy = \frac{1}{16}\pi\left[\tfrac{1}{5}y^5\right]_0^4 = \frac{64}{5}\pi$$

The volume of revolution of the region is

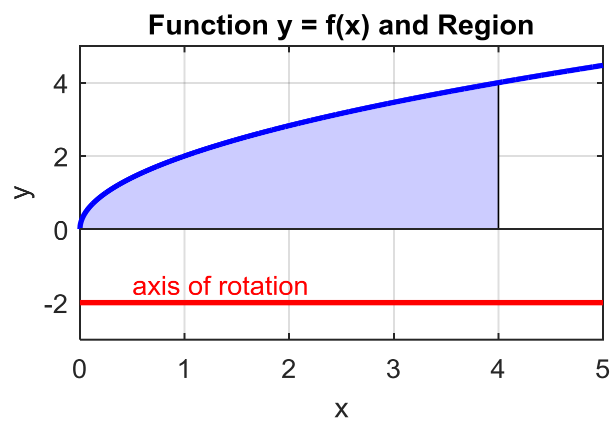
$$V = V_2 - V_1 = \left(64 - \frac{64}{5}\right) = \frac{256\pi}{5}$$

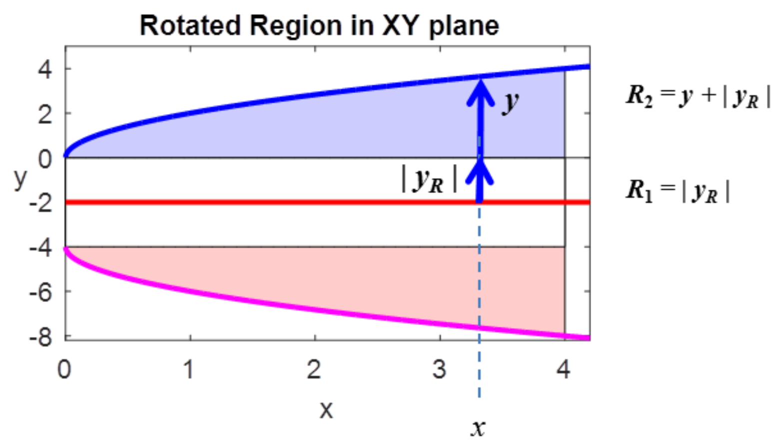
(C) rotation about the horizontal line y_R = -2

The solid is generated by a rotation through 360°, therefore the cross-sections of the solid of revolution will be circles, hence, the volume of the rotated region can be found by evaluating the definite integral

$$V = \pi \int_{x_a}^{x_b} R(x)^2 \, dx$$

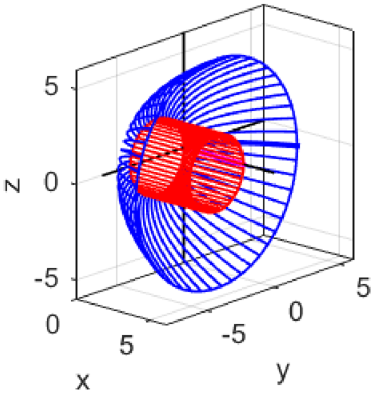
You need to carefully draw a set of sketches to clearly identify the region, the axis of rotation and determine the radius $R(x)$.





The function is $y = 2\sqrt{x} \quad 0 \le x \le 4$ and the axis of rotation is $y_R = -2$.

The volume V_2 generated by revolving the function $y = 2\sqrt{x}$ in interval $0 \le x \le 4$ around the around the axis of rotation y_R = +2 is



$$R_2(x) = y + |y_R| = 2 \quad R_2(x)^2 = 4x + 8x^{1/2} + 4 \quad x_a = 2 \quad x_b = 4$$

$$V_2 = \pi \int_2^4 \left(4x + 8x^{1/2} + 4\right) dx$$

$$V_2 = \pi \left[2x^2 + \frac{16}{3} x^{3/2} + 4x \right]_0^4$$

$$V_2 = \frac{272}{3}\pi$$

We need to subtract the volume V_1 generated by the rotation of the X-axis in the interval $0 \le x \le 4$ around the axis of rotation

$$R_1(x) = 2 \quad R(x)^2 = 4 \quad x_a = 2 \quad x_b = 4$$

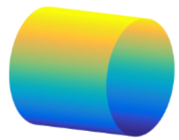
$$V_1 = 4\pi \int_0^4 dx$$

$$V_1 = 4\pi [x]_0^4$$

$$V_1 = 16\pi = \frac{48}{3}\pi$$

The volume V of the solid of revolution of the region is thus

$$V = V_2 - V_1 = (272 - 48)\frac{\pi}{3}$$



$$V = \frac{224\pi}{3}$$