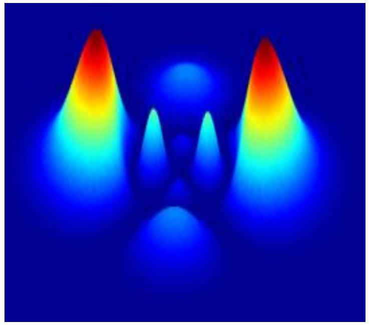


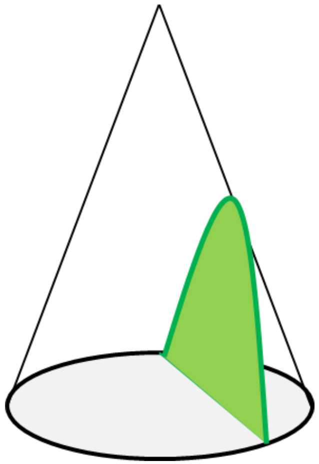
MATHEMATICS EXTENSION 2

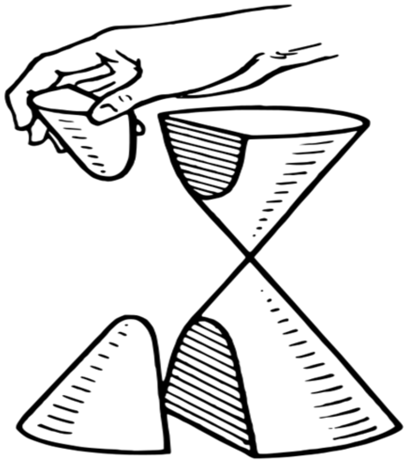
4 UNIT MATHEMATICS

TOPIC 3: CONICS

3.2 THE HYPERBOLA

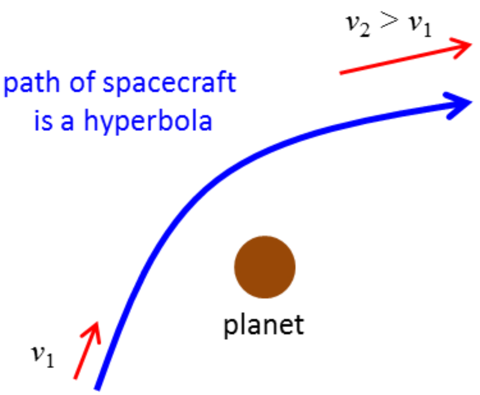
A **hyperbola** is **an open** curve with two branches, the intersection of a plane with both halves of a double cone. The plane does not have to be parallel to the axis of the cone.



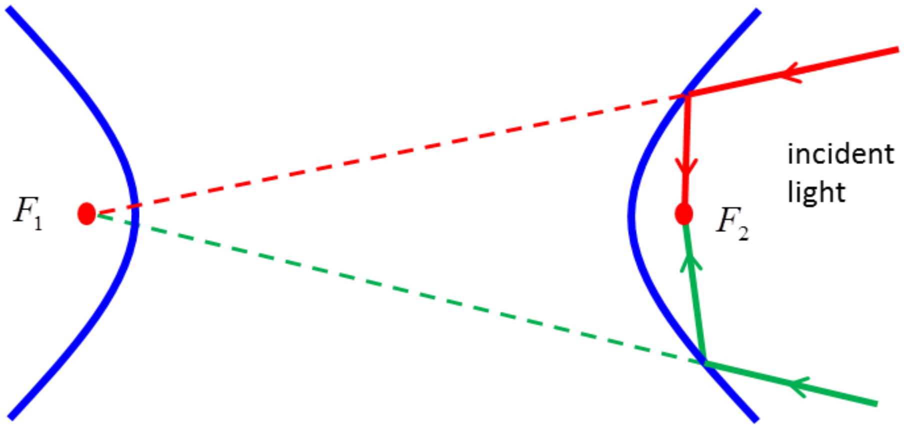


https://en.wikipedia.org/wiki/Hyperbola

The orbit of a spacecraft can sometimes be a **hyperbola**. A spacecraft can use the gravity of a planet to alter its path and propel it at higher speed away from the planet and back out into space using a technique called **gravitational slingshot effect**.



When light is directed towards one focus of a hyperbolic reflector, the light is deflected to the other focus. This property can be useful in collecting light from stars. If a set of stars are roughly equidistant from the Earth, a hyperbolic reflector can reflect light rays from these stars to one of its foci.



An **hyperbola** can be defined as the locus of all points that satisfy the equation

$$\frac{(x-x_1)^2}{a^2}-\frac{(y-y_1)^2}{b^2}=1$$

Variables: (x, y) the coordinates of any point on the ellipse

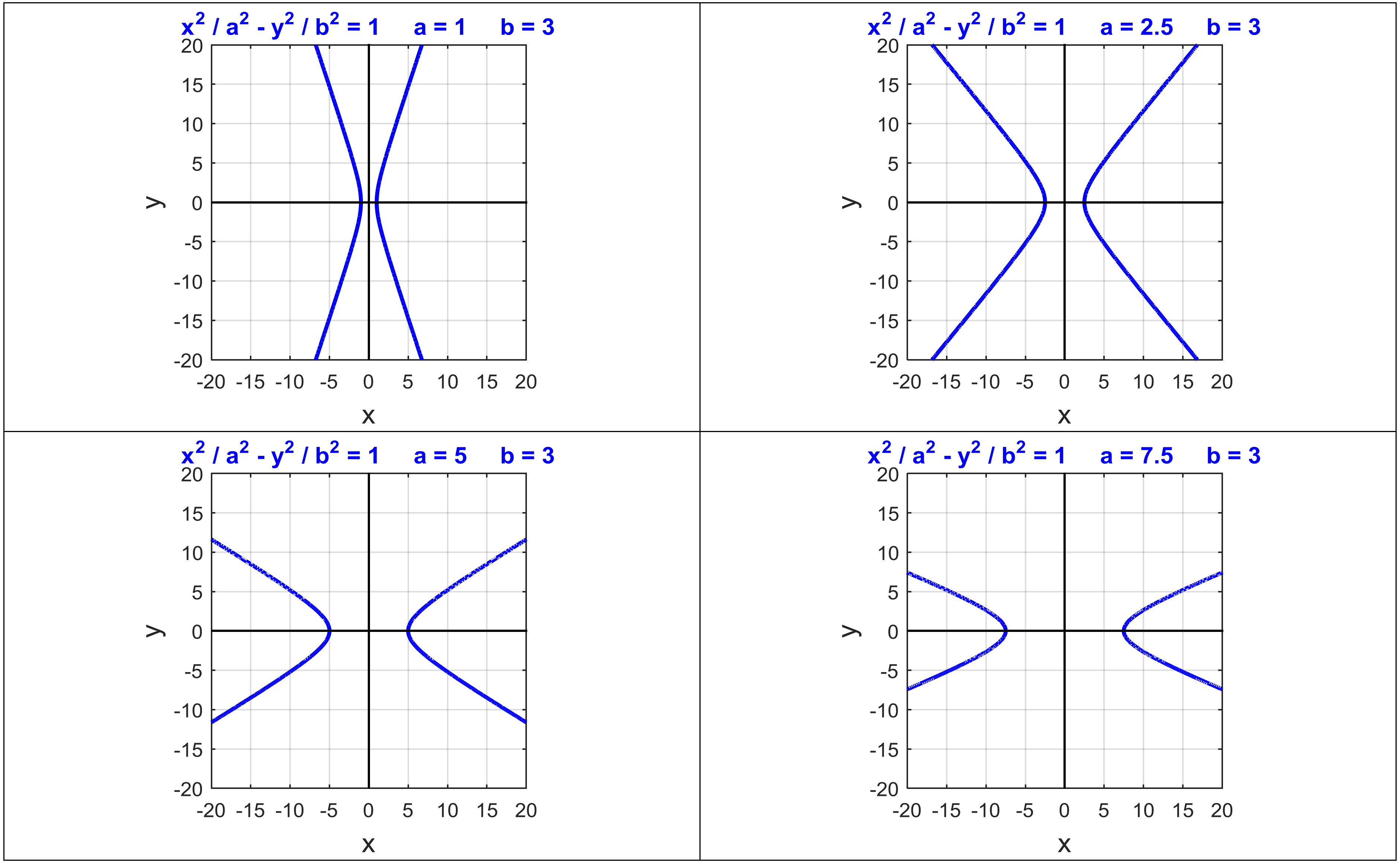
Constants: (x_1, x_2) the coordinates of the ellipse's centre

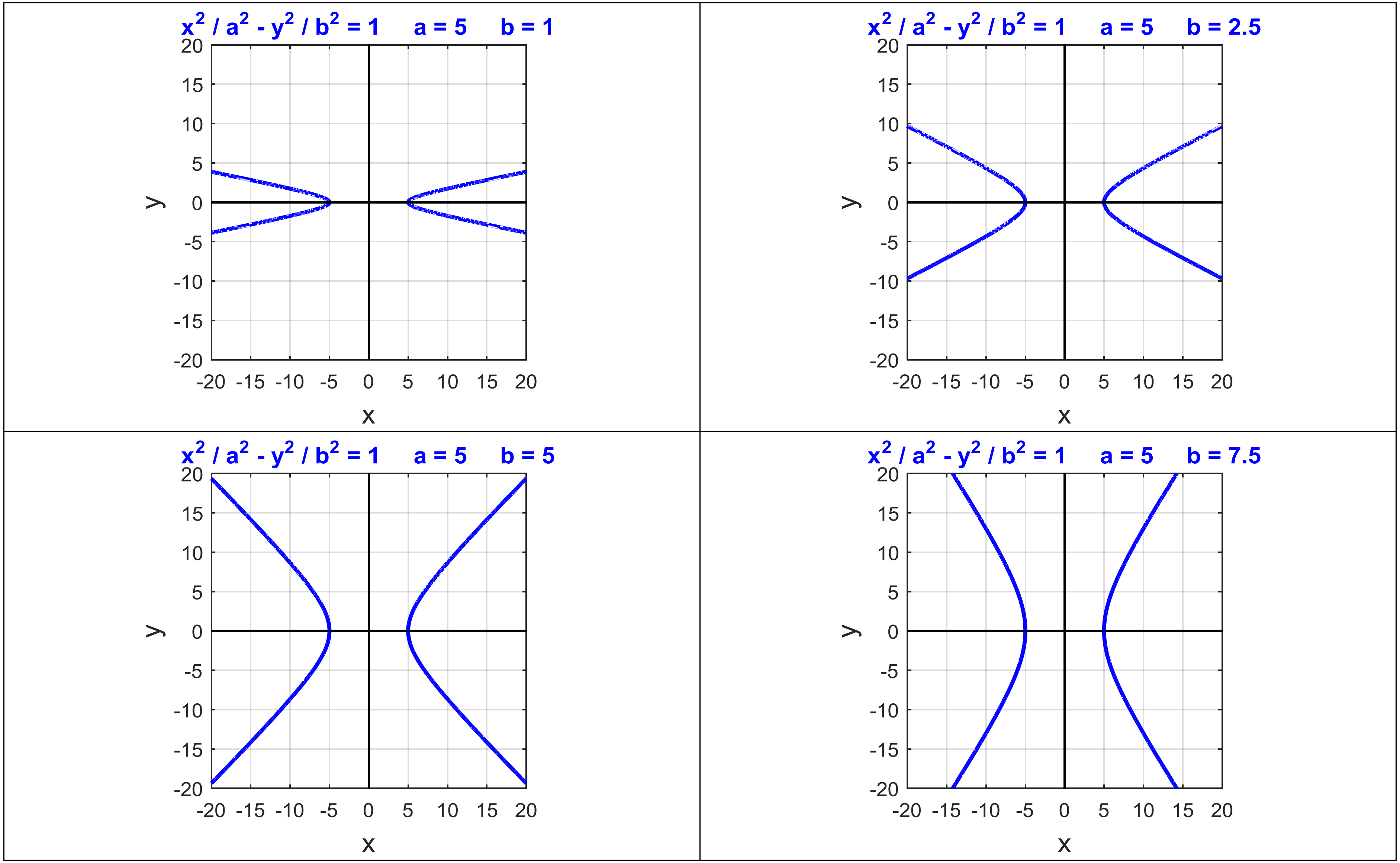
a, b hyperbola parameters

The equations for a hyperbola centred on the **origin** (0, 0) is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

The hyperbola is symmetrical with respect to both the X-axis and Y-axis. The vertices of a hyperbola have the Cartesian coordinates $A_1(-a, 0)$ and $A_2(a,0)$. Carefully examine the following graphs and take note of the position of the vertices A_1 and A_2 and how the shape of the hyperbola varies for the different values of a, b and b/a.


x2 / a2 - y2 / b2 = 1 a = 1 b = 3
x2 / a2 - y2 / b2 = 1 a = 2.5 b = 3
x2 / a2 - y2 / b2 = 1 a = 5 b = 3
x2 / a2 - y2 / b2 = 1 a = 7.5 b = 3
20 15 10 5 0 -5 -10 -15 -20
-20 -15 -10 -5 0 5 10 15 20
x
y

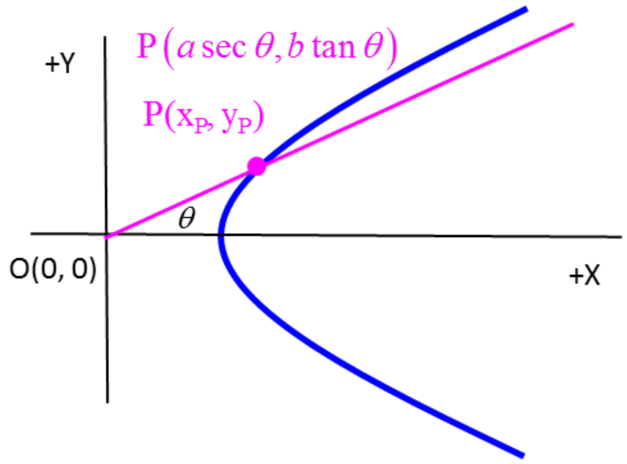

x2 / a2 - y2 / b2 = 1 a = 5 b = 1
x2 / a2 - y2 / b2 = 1 a = 5 b = 2.5
x2 / a2 - y2 / b2 = 1 a = 5 b = 5
x2 / a2 - y2 / b2 = 1 a = 5 b = 7.5
x
y
-20 -15 -10 -5 0 5 10 15 20

As seen from the equation and graph of a hyperbola it is a multi-valued function. For each value of x there are two y values.

The equation for an hyperbola with centre (0, 0) with can also be given in **parametric form**

$$x = a\sec(\theta)$$
$$y = b\tan(\theta)$$

where θ is an angle which ranges from 0 to 2π radians.



$$x = a\sec(\theta) \quad y = b\tan(\theta)$$
$$x^2/a^2 = \sec^2(\theta) \quad y^2/b^2 = \tan^2(\theta)$$
$$x^2/a^2 - y^2/b^2 = \sec^2(\theta) - \tan^2(\theta) = 1$$
$$\sec^2(\theta) = 1 + \tan^2(\theta)$$

The equation for the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

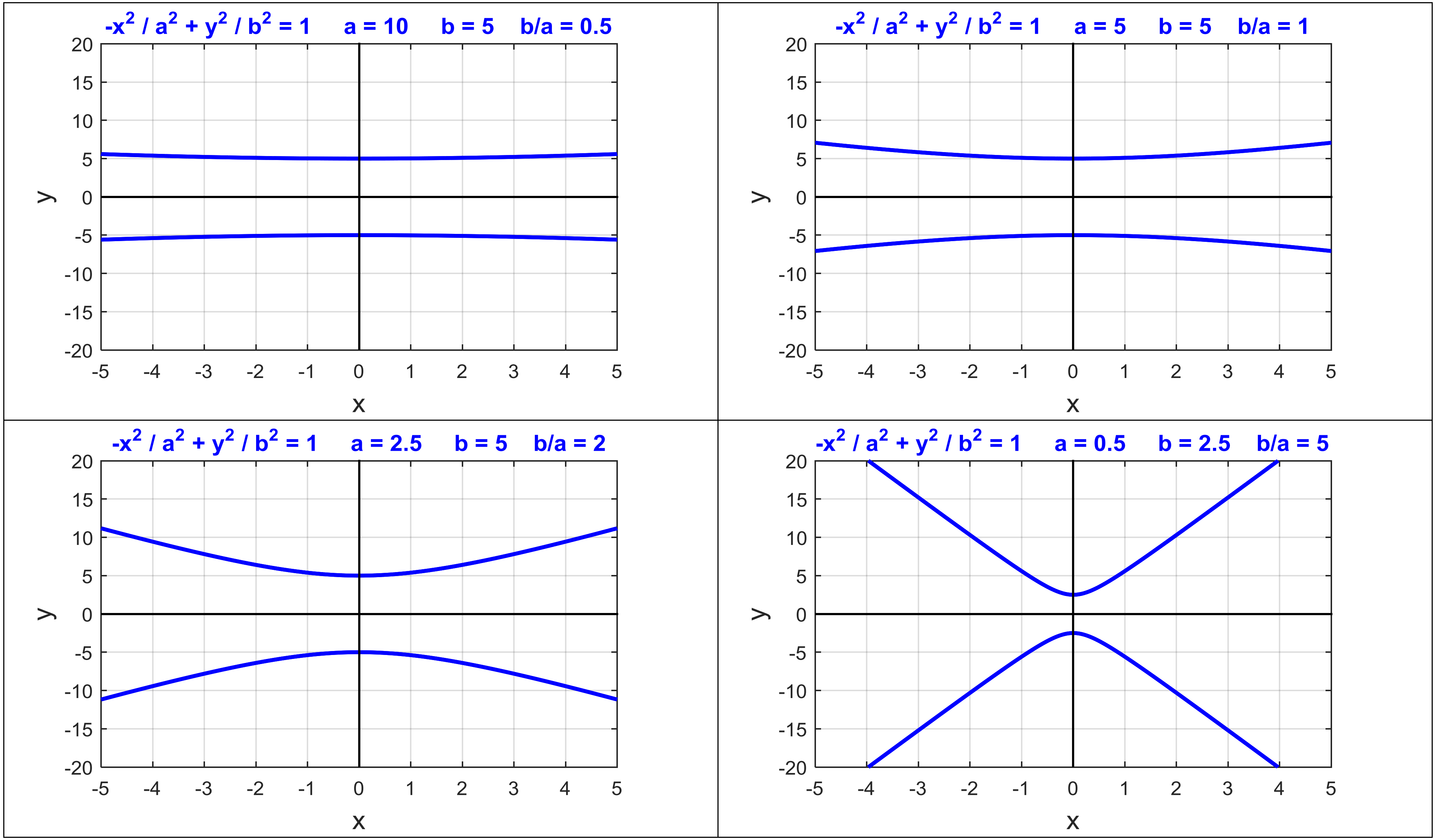
has the vertices $A_1(-a, 0)$ and $A_2(a,0)$ since $y=0 \quad \Rightarrow \quad x=\pm a$

The equation for the hyperbola which has the vertices on the Y-axis: $B_1(0, -b,)$ and $B_2(0, a)$ since $x=0 \quad \Rightarrow \quad y=\pm b$ is

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Carefully examine the graphs below for hyperbolas with different values of b/a.

Note: the X-axis and Y-axis have different scaling.



-x2 / a2 + y2 / b2 = 1 a = 10 b = 5 b/a = 0.5
-x2 / a2 + y2 / b2 = 1 a = 5 b = 5 b/a = 1
-x2 / a2 + y2 / b2 = 1 a = 2.5 b = 5 b/a = 2
-x2 / a2 + y2 / b2 = 1 a = 0.5 b = 2.5 b/a = 5
x
y

From viewing the above plots of the hyperbola, it is obvious that a hyperbola is actually two separate curves in mirror image. Some of the important terms associated with the graph of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are:

- The **vertices** where the hyperbola cuts either the X-axis are $A_1(-a, 0)$ and $A_2(a, 0)$.

  These are the points where the curve makes its sharpest turn.

  y must be real $\Rightarrow \quad y=\pm b\sqrt{\frac{x^2}{a^2}-1} \quad \Rightarrow \quad |x|\ge a$

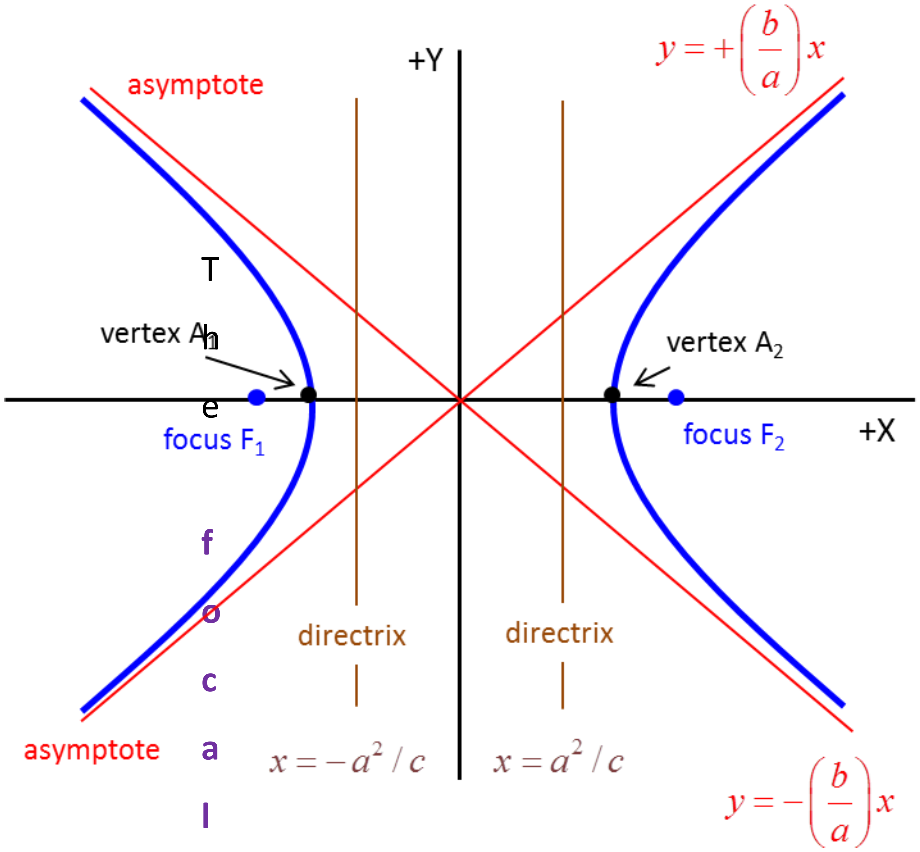
- The **asymptotes** show where the curve would go if continued indefinitely in each of the four directions. For large values of $x \;\Rightarrow\quad y=\pm b\sqrt{\frac{x^2}{a^2}-1}\rightarrow\pm\left(\frac{b}{a}\right)x \;\Rightarrow$

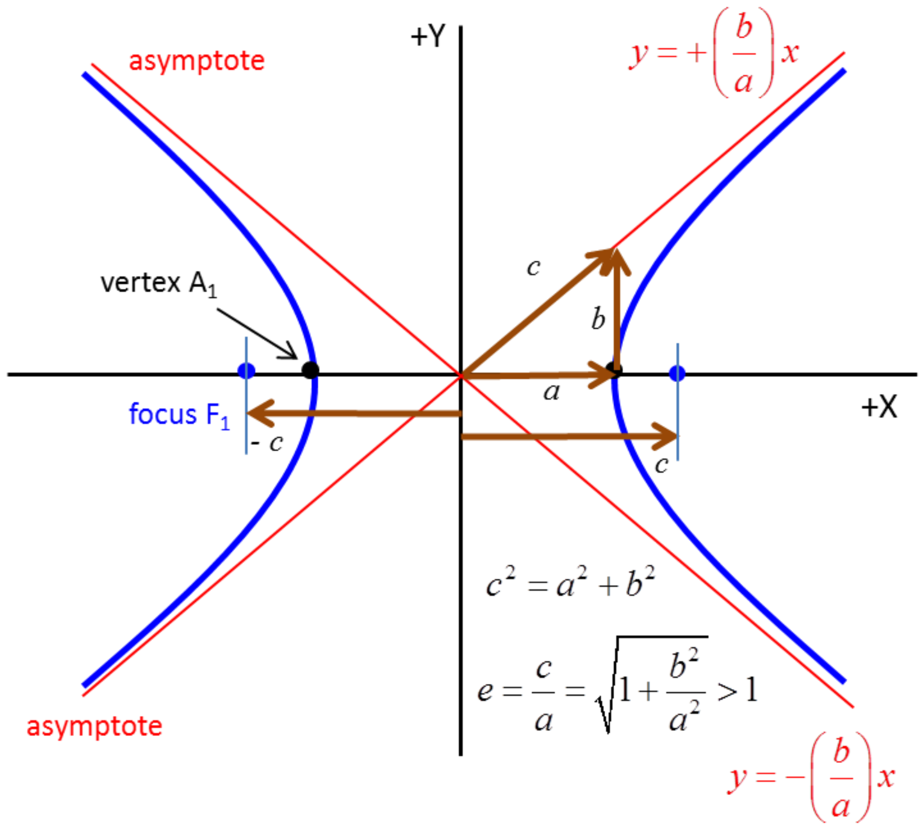
  equations for asymptotes are: $y=+\left(\frac{b}{a}\right)x \quad y=-\left(\frac{b}{a}\right)x$

  When $x = a$ then the $y = + b$ and $y = - b$ on the asymptote $y=+\left(\frac{b}{a}\right)x$

  When $x = - a$ then the $y = + b$ and $y = - b$ on the asymptote $y=-\left(\frac{b}{a}\right)x$

We define the distance c as $c^2 = a^2 + b^2$ $\qquad c = \sqrt{a^2 + b^2}$





- The two **focal points** are F_1 $(-c, 0)$ and F_2 $(c, 0)$.
- The two vertical straight lines $x = -c^2/a$ and $x = +c^2/a$ are each called a **directrix**.
- The X-axis and the Y-axis are both an **axis of symmetry.**

- Eccentricity e shows how "uncurvy" (varying from being a circle) the hyperbola is.

  The eccentricity e is given by the formula

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} = \sqrt{1 + \frac{b^2}{a^2}} > 1$$

Let

distance from a focus F to a point P on the hyperbola = d_{FP}

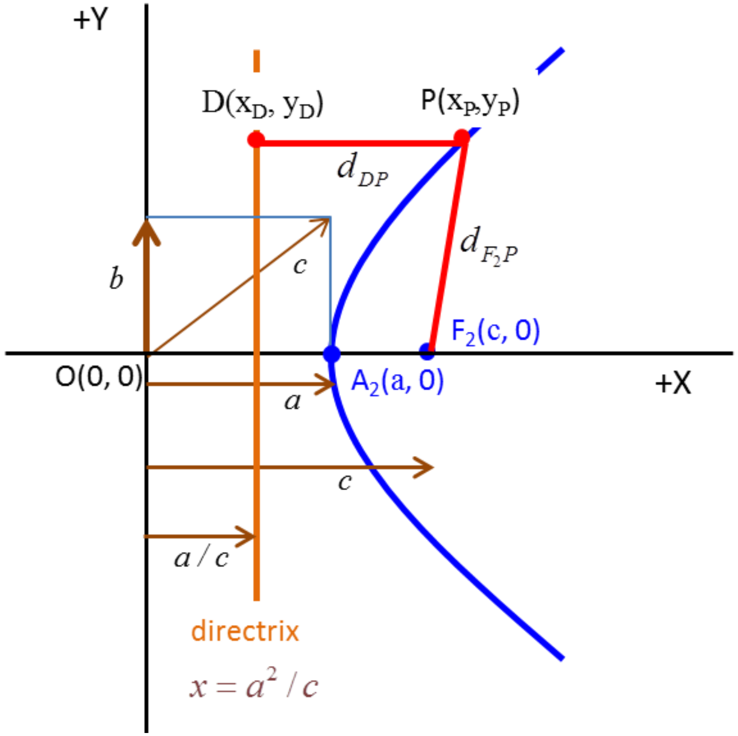
distance from the same point P to the point D on the directrix which has the same ordinate as P (the line DP parallel to the X-axis or perpendicular to the Y-axis) = d_{DP}

then a hyperbola can be defined as the set of points P such that the ratio

$\frac{d_{FP}}{d_{DP}} = e > 1$ where e is a constant called the **eccentricity**

$$c^2 = a^2 + b^2$$

$$e = \frac{c}{a} = \sqrt{1 + \frac{b^2}{a^2}} > 1$$



For any point P on the hyperbola:

$$\frac{d_{F_2P}}{d_{DP}} = e > 1$$

When P corresponds to the vertex A_2 then

$$d_{F_2P} = c - a \quad d_{DP} = a - a^2 / c$$

$$\frac{d_{F_2P}}{d_{DP}} = \frac{c - a}{a - a^2 / c} = \frac{c^2 - c\,a}{c\,a - a^2} = \frac{c(c-a)}{a(c-a)}$$

$$\frac{d_{F_2P}}{d_{DP}} = \frac{c}{a} = e > 1$$

Consider the three point: P(x_P, y_P) any point on the hyperbola; the focus F_2(c, 0); and D(a^2/c, y_P)

The distance from the focus F_2 to the point P = d_{F_2P}

The distance from the point D to the point P = d_{DP}

Using the formula for the distance between two points

$$d_{F_2P} = \sqrt{(x_P - c)^2 + {y_P}^2} \qquad d_{DP} = x_P - a^2/c$$

For a hyperbola

$$\frac{d_{F_2P}}{d_{DP}} = e = \frac{c}{a}$$

Combining these two relationships gives the Cartesian equation for a hyperbola

$$\left(\frac{d_{F_2P}}{d_{DP}}\right)^2 = \frac{c^2}{a^2} = \frac{(x_P - c)^2 + {y_P}^2}{\left(x_P - a^2/c\right)^2}$$

$$a^2 {x_P}^2 - 2a^2 c\, x_P + a^2 c^2 + a^2 {y_P}^2 = c^2 {x_P}^2 - 2a^2 c\, x_P + a^4$$

$$\left(a^2 - c^2\right) {x_P}^2 + a^2 {y_P}^2 = a^2\left(a^2 - c^2\right)$$

$$c^2 = a^2 + b^2 \qquad a^2 - c^2 = -b^2$$

$$-b^2 {x_P}^2 + a^2 {y_P}^2 = -a^2 b^2$$

$$\frac{{x_P}^2}{a^2} - \frac{{y_P}^2}{b^2} = 1$$

Derivation of the Cartesian form for a hyperbola from the locus of points, the difference of whose distances from two the two focal points is constant and equal to $2a$.

A hyperbola is a conic section defined as the locus of all points P in the plane the difference of whose distances d_{F_1P} and d_{F_2P} from two fixed points (the foci F_1 and F_2) separated by a distance $2c$ is a given positive constant k

$$\left|d_{F_1P} - d_{F_2P}\right| = k$$

Letting P fall on the left vertex $A_1(-a, 0)$ requires that

$$\left|d_{F_1P} - d_{F_2P}\right| = k = \left|c - a - (c + a)\right| = 2a$$

Let the point P(x_P, y) be on the hyperbola then using the definition of the hyperbola $\left|d_{F_1P} - d_{F_2P}\right| = 2a$ we can derive the equation of the hyperbola.

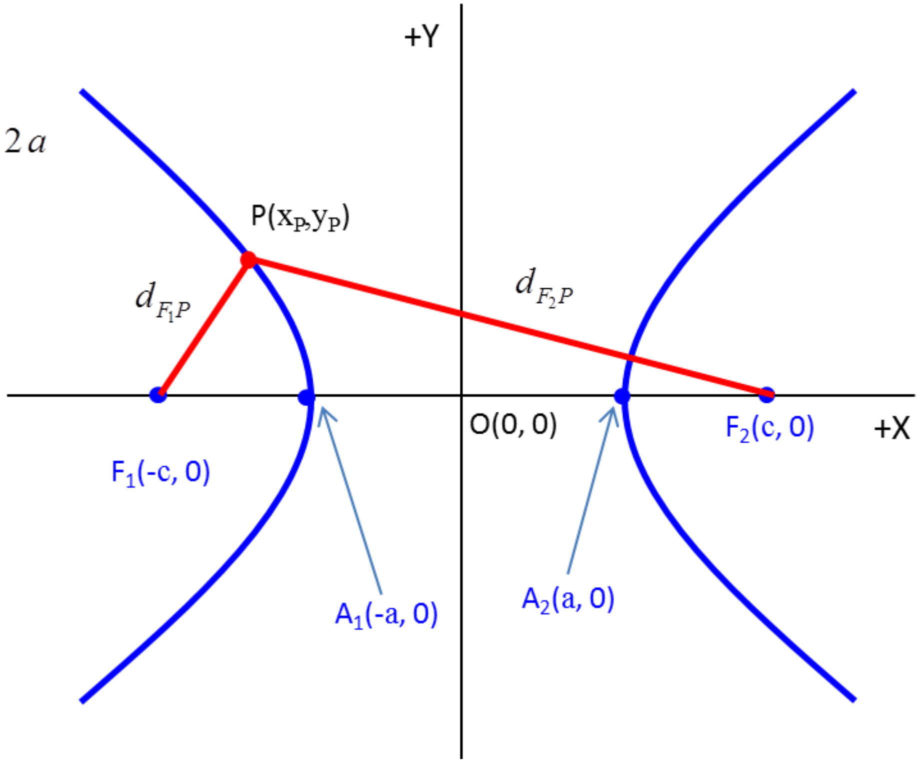
$$d_{F_1P} = \sqrt{(x_P + c)^2 + y_P{}^2} \qquad d_{F_1P}{}^2 = (x_P + c)^2 + y_P{}^2$$

$$d_{F_2P} = \sqrt{(x_P - c)^2 + y_P{}^2} \qquad d_{F_2P}{}^2 = (x_P - c)^2 + y_P{}^2$$

hyperbola $\quad \left|d_{F_1P} - d_{F_2P}\right| = 2a$

$c^2 = a^2 + b^2$

eccentricity $\quad e = \dfrac{c}{a}$



$$\sqrt{(x_P + c)^2 + y_P{}^2} - \sqrt{(x_P - c)^2 + y_P{}^2} = 2a$$

$$\sqrt{(x_P + c)^2 + y_P{}^2} = \sqrt{(x_P - c)^2 + y_P{}^2} + 2a$$

$$(x_P + c)^2 + y_P{}^2 = (x_P - c)^2 + y_P{}^2 + 4a\sqrt{(x_P - c)^2 + y_P{}^2} + 4a^2$$

$$x_P{}^2 + 2c\,x_P + c^2 + y_P{}^2 - x_P{}^2 + 2c\,x_P - c^2 - y_P{}^2 - 4a^2 = 4a\sqrt{(x_P - c)^2 + y_P{}^2}$$

$$(c/a)\,x_P - a = \sqrt{(x_P - c)^2 + y_P{}^2}$$

$$(c/a)^2\,x_P{}^2 - 2c\,x_P + a^2 = (x_P - c)^2 + y_P{}^2$$

$$(c/a)^2\,x_P{}^2 - 2c\,x_P - (x_P - c)^2 - y_P{}^2 = -a^2$$

$$\left((c/a)^2 - 1\right)x_P{}^2 - c^2 - y_P{}^2 = -a^2$$

$$\left(\frac{a^2 - c^2}{a^2}\right)x_P{}^2 + y_P{}^2 = a^2 - c^2 \qquad c^2 = a^2 + b^2 \qquad a^2 - c^2 = -b^2$$

$$\left(-b^2/a^2\right)x_P{}^2 + y_P{}^2 = -b^2$$

$$\frac{x_P{}^2}{a^2} - \frac{y_P{}^2}{b^2} = 1$$

$\Rightarrow$ equation of hyperbola $\quad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Equation of tangents to a hyperbola

The equation of a hyperbola with its centre at the origin is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The equation of the **tangent** to the hyperbola can be expressed as

$$y = M_1 x + B_1$$

where M_1 is the gradient and B_1 is the intercept of the straight line and the equation of the normal can be expressed as

$$y = M_2 x + B_2$$

where M_2 is the gradient and B_2 is the intercept of the straight line and

$M_1 M_2 = -1$ since the tangent and normal are perpendicular to each other.

The first derivative dy/dx at a point P$\left(x_P, y_P\right)$ on the hyperbola gives the gradient of the tangent to the ellipse at that point. The implicit differentiation of the equation for the hyperbola

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \left(\frac{b}{a}\right)^2 \frac{x}{y}$$

The gradient M_1 of the tangent at the point P$\left(x_P, y_P\right)$ is

$$M_1 = \left(\frac{b^2}{a^2}\right)\frac{x_P}{y_P}$$

and the intercept B_1 is

$$B_1 = y_P - M_1\, x_P = y_P - \left(\frac{b^2}{a^2}\right)\left(\frac{{x_P}^2}{y_P}\right)$$

The tangent crosses the X-axis ($y = 0$) at the point T$\left(x_T, 0\right)$ where

$$x_T = \frac{a^2}{x_P}$$

Proof

$$y_T = 0 \quad \Rightarrow \quad x_T = -B_1 / M_1 = -\left(\frac{y_P - \left(b^2/a^2\right)\left(x_P{}^2/y_P\right)}{\left(b^2/a^2\right)\left(x_P/y_P\right)}\right)$$

$$x_T = -\frac{a^2}{x_P}\left(\frac{y_P{}^2}{b^2} - \frac{x_P{}^2}{a^2}\right) \qquad \frac{y_P{}^2}{b^2} - \frac{x_P{}^2}{a^2} = -1$$

$$x_T = \frac{a^2}{x_P}$$

Equation of a normal to an ellipse

$$y = M_2 x + B_2$$

$$M_2 = \frac{-1}{M_1} = -\left(\frac{a^2}{b^2}\right)\frac{y_P}{x_P}$$

Intercept of normal B_2

$$B_2 = y_P + \left(\frac{a^2}{b^2}\right)y_P = y_P\left(\frac{a^2 + b^2}{b^2}\right)$$

The normal crosses the X-axis ($y = 0$) at the point N$\left(x_N, 0\right)$where

$$x_N = \left(\frac{a^2 + b^2}{a^2}\right) x_P$$

Proof

$$y_N = 0 \quad \Rightarrow \quad x_N = -B_2 / M_2 = -\left(\frac{y_P\left(a^2 + b^2\right)}{b^2}\right)\left(\frac{b^2}{-a^2}\right)\left(\left(\frac{x_P}{y_P}\right)\right)$$

$$x_N = \frac{\left(a^2 + b^2\right)}{a^2} x_P$$

There is no need to remember these formulae for the tangent and normal to the ellipse. It is best to derive the equations from first principles.

Examples Consider the ellipse $9x^2 - 25y^2 - 225 = 0$

Verify all the numerical values and information in the following two figures for

x_P = 10.0 and x_P = 7.5.

---

a = 5 b = 3 c = 5.83

P(x, y) = (10, 5.196)

A_1(x, y) = (-5, 0) A_2(x, y) = (5, 0)

F_1(x, y) = (-5.83, 0) F_2(x, y) = (5.83, 0)

D = (4.29, 5.196)

eccentricity e = 1.17

directrices 1: x = -4.29 directrices 2: x = 4.29

slope tangent M_1 = 0.693 slope normal M_2 = -1.44

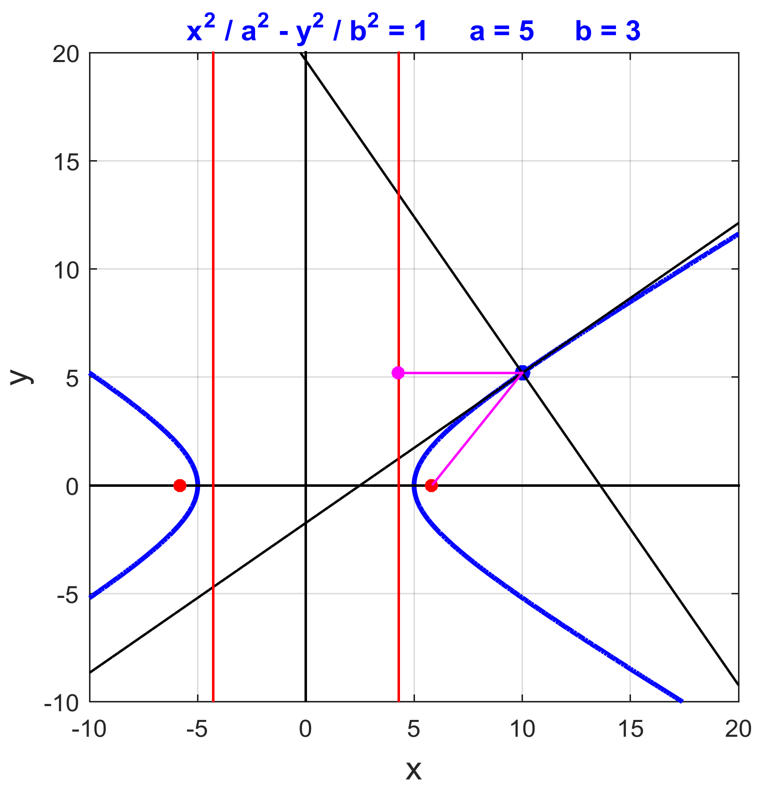
intercept tangent B_1 = -1.73 intercept normal B_2 = 19.6

T tangent cross X-axis: x_T = 2.5 N normal cross X-axis: x_N = 13.6

distances: PF_1 = 16.7 PF_2 = 6.66 $|PF_1 - PF_2|$ = 10

distances: PF_2 = 6.66 PD = 5.71 PF_2 / PD = 1.17

$x_P^2 / a^2 - y_P^2 / b^2 = 1$



$a = 5$ $b = 3$ $c = 5.83$

$P(x, y) = (7.5, 3.354)$

$A_1(x, y) = (-5, 0)$ $A_2(x, y) = (5, 0)$

$F_1(x, y) = (-5.83, 0)$ $F_2(x, y) = (5.83, 0)$

$D = (4.29, 3.354)$

eccentricity $e = 1.17$

directrices 1: $x = -4.29$ directrices 2: $x = 4.29$

slope tangent $M_1 = 0.805$ slope normal $M_2 = -1.24$

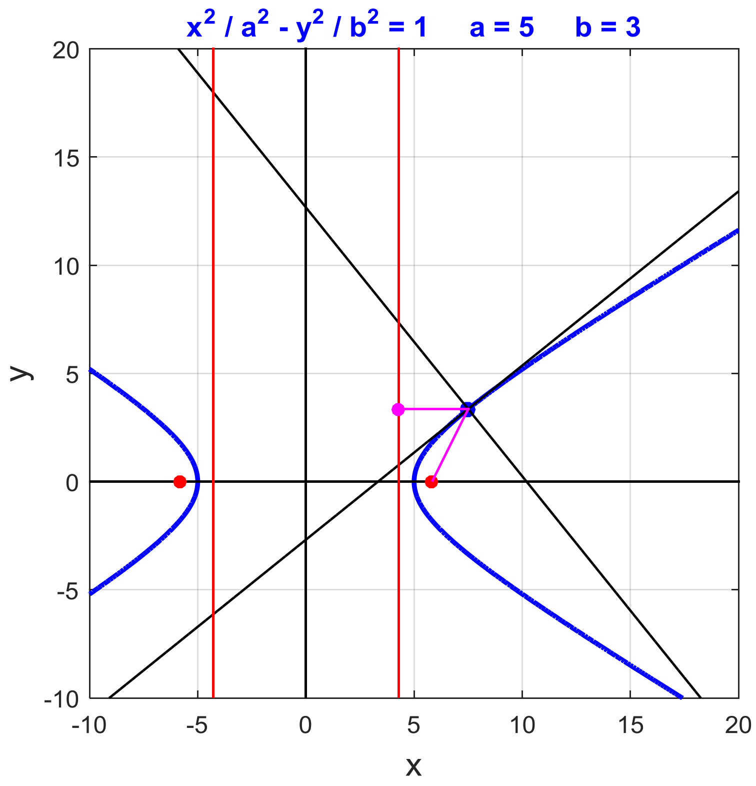
intercept tangent $B_1 = -2.68$ intercept normal $B_2 = 12.7$

T tangent cross X-axis: $x_T = 3.33$ N normal cross X-axis: $x_N = 10.2$

distances: $PF_1 = 13.7$ $PF_2 = 3.75$ $|PF_1 - PF_2| = 10$

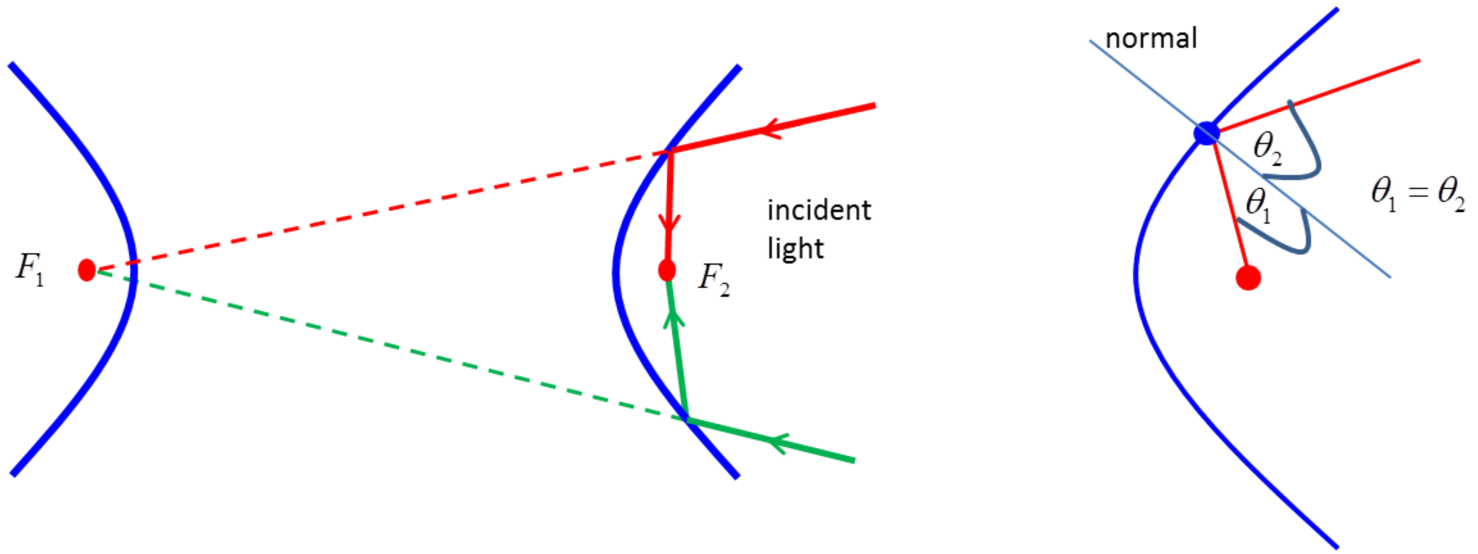
distances: $PF_2 = 3.75$ $PD = 3.21$ $PF_2 / PD = 1.17$

$x_P^2 / a^2 - y_P^2 / b^2 = 1$



Reflective Properties of Hyperbolas

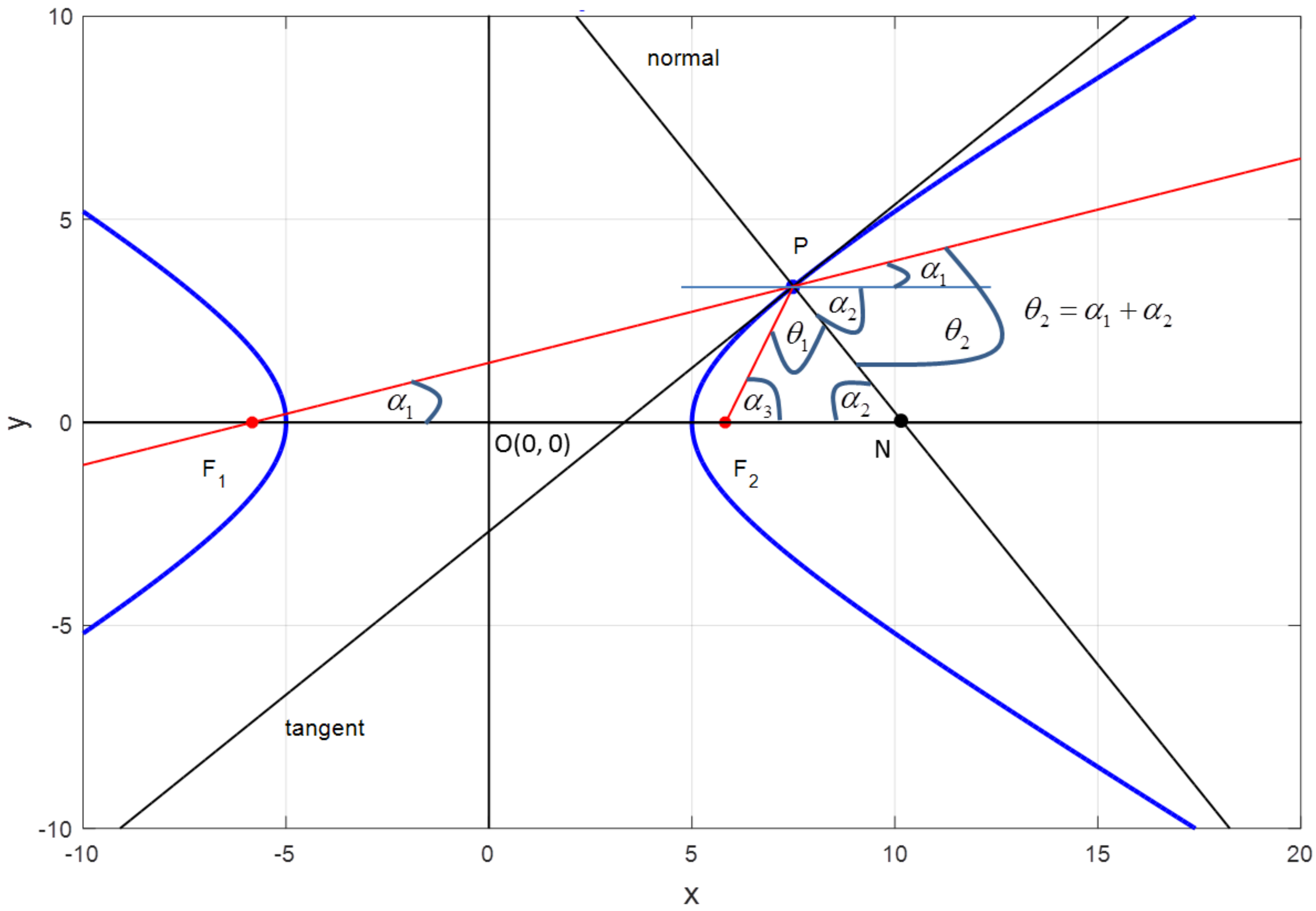
Rays directed toward the focus of hyperbola are reflected at the hyperbolic mirror to the other focus of hyperbola.



The angles θ_1 and θ_2 between the normal line and the straight lines drawn from the hyperbola focus to the given point are equal $\theta_1 = \theta_2$.

This is an example of the Law of Reflection: the angle of incidence is equal to the angle of reflection measured from the normal.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad c^2=a^2+b^2 \qquad F_1(-c,0) \quad F_2(c,0) \quad P(x_P,y_P) \quad N\left(\frac{a^2+b^2}{a^2},0\right)$$



$$\tan\alpha_1=\frac{y_P}{x_P+c} \qquad \tan\alpha_2=\frac{y_P}{x_N-x_P} \qquad x_N=\frac{a^2+b^2}{a^2} \qquad \tan\alpha_3=\frac{y_P}{x_P-c}$$

$$\theta_1=180-\alpha_2-\alpha_3$$

$$\theta_2=\alpha_1+\alpha_2 \quad \Rightarrow \quad \theta_1=\theta_2$$

Example

For the hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$ and where P is a point on the hyperbola with x_P = 7.500 verify the information and calculations shown below.

Calculations

$a = 5 \quad b = 3 \quad c = 5.831$

Point P: $x_P = 7.500 \quad y_P = 3.354$

Point N normal cuts X-axis: $x_N = 10.200 \quad y\mathrm{N} = 0$

focal points: $x_{F_1} = -5.831 \quad y_{F_1} = 0 \quad x_{F_2} = 5.831 \quad y_{F_2} = 0$

Angles:

$$\alpha_1 = 14.1226^\circ \qquad \alpha_2 = 51.1665^\circ \qquad \alpha_3 = 63.5444^\circ$$
$$\theta_1 = 65.289^\circ \qquad \theta_2 = 65.2891^\circ = \theta_1$$

Compare the calculated values with the above graph.

Note: It is always a good idea to label an angle using a lower case Greek letter.