EXERCISE 17_123

Part (Aa)

Consider the polynomial

$$y = 3x^4 - 8x^3 - 30x^2 + 72x + 27$$

Find the stationary points and indicate whether they are a maximum, minimum or a point of inflection.

Evaluate the polynomial at x = 0 and at each stationary point.

Sketch the polynomial and find where the curve cuts the X-axis.

Part (Ba)

Sketch the curve

$$y^2 = x^2\left(1 - x^2\right)$$

Showing its maximum width.

Find the total area and volume of the curve enclosed by the loops.

Answer Part (A)

Stationary points occur when $dy/dx = 0$

$$y = 3x^4 - 8x^3 - 30x^2 + 72x + 27$$
$$dy/dx = 12x^3 - 24x^2 - 60x + 72 = 0$$
$$x^3 - 2x^2 - 5x + 6 = 0$$

The roots of the cubic equation can be found from the relationships between the coefficients and the roots $\alpha + \beta + \gamma = -b/a \quad \alpha\beta\gamma = -d/a$

$$x^3 - 2x^2 - 5x + 6 = (x+2)(x-1)(x-3) = 0$$
$$dy/dx = 0 \quad \Rightarrow \quad x = -2 \quad x = 1 \quad x = 3$$

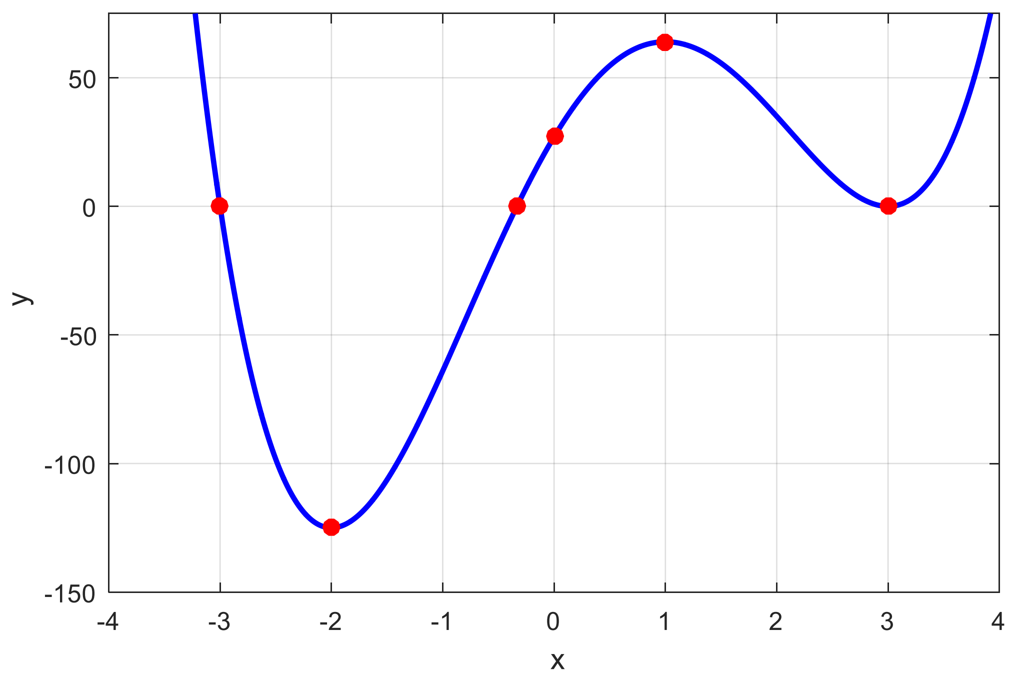
The type of stationary point is given by d^2y/dx^2

$$d^2y/dx^2 = 0 \quad \Rightarrow \quad \text{inflection point} \quad d^2y/dx^2 < 0 \quad \Rightarrow \quad \max \quad d^2y/dx^2 > 0 \quad \Rightarrow \quad \min$$

$$d^2y/dx^2 = 3x^2 - 4x - 5$$

$$x = -2 \quad d^2y/dx^2 = 15 > 0 \quad \Rightarrow \quad \min$$
$$x = 1 \quad d^2y/dx^2 = -6 > 0 \quad \Rightarrow \quad \max$$
$$x = 3 \quad d^2y/dx^2 = 10 > 0 \quad \Rightarrow \quad \min$$

$$x = 0 \Rightarrow y = 27 \quad x = -2 \Rightarrow y = -125 \quad x = 1 \Rightarrow y = 64 \quad x = 3 \Rightarrow y = 0$$



The polynomial cuts the X-axis (y = 0) at x = -3, x ≈ -0.33 and x = 3

Answer Part (B)

$$y^2 = x^2\left(1-x^2\right)$$

When $y = 0$ $x = 0$, $x = -1$ and $x = 1$

$$y = x\sqrt{1-x^2} \quad y \text{ is a real number} \quad \Rightarrow -1 \le x \le 1$$

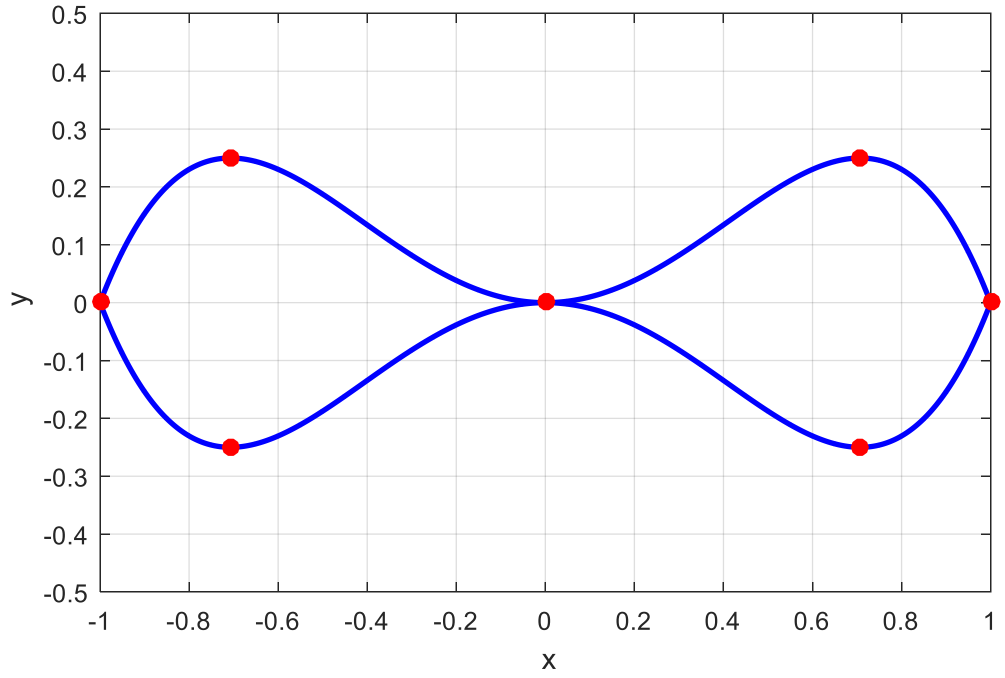
The stationary (turning) points occur when $dy/dx = 0$

$$y = \pm x\left(1-x^2\right)^{1/2}$$

$$dy/dx = \left(1-x^2\right)^{1/2} - x^2\left(1-x^2\right)^{-1/2} = 0$$

$$x^2 = 1/2 \quad x = \pm\frac{1}{\sqrt{2}} \quad \Rightarrow \quad y = \pm\frac{1}{2}$$

The maximum width of each loop of the curve is 1.



The total area A enclosed by the two loops is

$$A = 4\int_0^1 x\left(1-x^2\right)^{1/2} dx$$

$$x = \sin\theta \quad dx/d\theta = \cos\theta \quad dx = \cos\theta\, d\theta \quad \left(x = 0\ y = 0\right) \quad \left(x = 1\ y = \pi/2\right)$$

$$A = 4\int_0^{\pi/2} \sin\theta\cos^2\theta\ d\theta$$

$$A = \frac{-4}{3}\left[\cos^3\theta\right]_0^{\pi/2} = \left(\frac{-4}{3}\right)(0-1)$$

$$A = \frac{4}{3}$$

The volume V can be found by considering the rotation of the curve about the X-axis. The volume element generated can be divided into a series of cylinders of cross-sectional area πy^2 and width dx. The volume is found by adding the volumes of each element and as $dx \to 0$ the summation becomes the integral

$$V = 2\int_0^1 \pi y^2\,dx = 2\pi\int_0^1 x^2\left(1-x^2\right)^2 dx$$ the factor 2 is because we have two loops

$$x = \sin\theta \quad dx/d\theta = \cos\theta \quad dx = \cos\theta\,d\theta \quad \left(x=0\ y=0\right) \quad \left(x=1\ y=\pi/2\right)$$

$$V = 2\pi\int_0^{\pi/2}\left(\cos\theta\sin^2\theta - \cos\theta\,\sin^4\theta\right)d\theta$$

$$V = 2\pi\left[\frac{1}{3}\sin^3\theta - \frac{1}{5}\sin^5\theta\right]_0^{\pi/2} = 2\pi\left(\frac{1}{3}-\frac{1}{5}\right)$$

$$V = \frac{4\pi}{15}$$