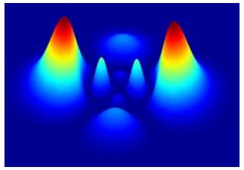


# MATHEMATICS EXTENSION 2

## 4 UNIT MATHEMATICS

## TOPIC 5: VOLUMES

## 5.3 SOLIDS OF REVOLUTION

In Topic 5.2, the volume $V$ of a solid aligned along the X-axis was given by the equation

$$V = \int_{x_a}^{x_b} A(x)\,dx$$

Solid figures can also be produced by rotating bounded regions in the XY plane in space through $360^o$. The solid generated by the rotation is called a **solid of revolution**.

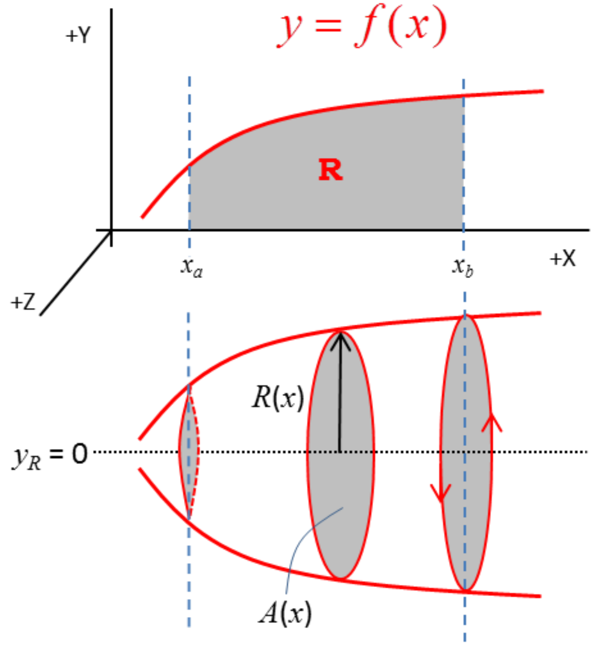
We will only consider solids of revolution that are generated by rotations about axes that are parallel to the **X-ax**is or the **Y-axis** (coordinates axes).

# DISK METHOD

**ROTATIONS ABOUT THE X-AXIS ( $y_R = 0$ )**

Let $y = f(x)$ be a single-valued continuous function where $f(x) \geq 0$ in the interval $x_a \leq x \leq x_b$. Consider the region ***R*** bounded by the function $y = f(x)$ and the X-axis ($y_R = 0$) for the interval $x_a \leq x \leq x_b$. When this region **R** is rotated about X-axis through the $360^{\circ}$ rotation, a solid of revolution is generated. The volume $V$ of the solid of revolution is given by

$$V = \int_{x_a}^{x_b} A(x)\,dx$$ **rotation about X-axis**




The solid generated by the rotation must have a circular cross-section with radius $R(x)$. Therefore, the cross-sectional area $A(x)$ is given by

$$A(x) = \pi\,R(x)^2 \quad R(x) = y \quad A(x) = \pi\,y^2$$

The volume $V$ of the solid of revolution is

$$V = \pi \int_{x_a}^{x_b} R(x)^2\,dx = \pi \int_{x_a}^{x_b} y^2\,dx$$ **disk method – rotation about X-axis**

This is called the **disk method**.

In the disk method, we sum up the volumes of an infinite number of infinitesimally thin circular disks to find the total volume of a solid. The solid has been decomposed into stacked circular disks, and by integrating the disk volumes we obtain the total volume.

## ROTATIONS ABOUT THE Y-AXIS ( $x_R = 0$ )

Let $x = g(y)$ be a single-valued continuous function where $g(y) \ge 0$ in the interval $y_a \le y \le y_b$. Consider the region $\boldsymbol{R}$ bounded by the function $x = g(y)$ and the Y-axis ($x_R = 0$) for the interval $y_a \le y \le y_b$. When this region **R** is rotated about Y-axis through the 360° rotation, a solid of revolution is generated. The volume $V$ of the solid of revolution is given by

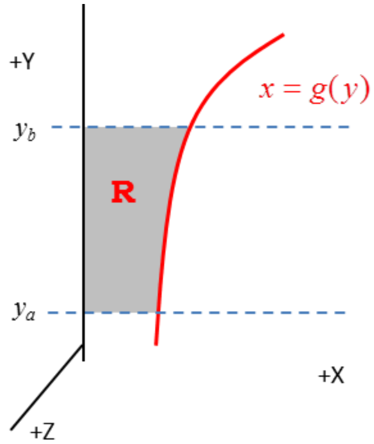
$$V = \int_{y_a}^{y_b} A(y)\, dy \qquad \textbf{rotation about Y-axis}$$

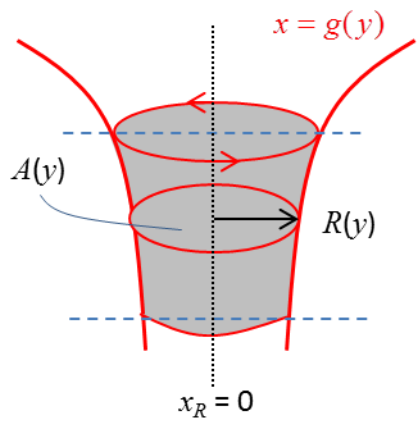
The solid generated by the rotation must have a circular cross-section with radius $R(y)$. Therefore, the cross-sectional area $A(y)$ is given by

$$A(y) = \pi R(y)^2 \quad R(y) = x \quad A(x) = \pi x^2$$

The volume $V$ of the solid of revolution is

$$V = \pi \int_{y_a}^{y_b} R(y)^2\, dy = \pi \int_{y_a}^{y_b} x^2\, dy \qquad \textbf{disk method – rotation about Y-axis}$$



# CYLINDRICAL SHELL METHOD

In another approach to find the volume $V$, we decompose the solid into hollow concentric shells or rings which are infinitesimally thin. For a rotation about the X-axis ( $y_R = 0$ )

The area of each ring is $A(x_k) = 2\pi y_k \, \Delta y$ (circumference x width)

The volume of each the cylindrical shell is

$$V(x_k) = (2\pi \, y \, \Delta y) \, x_k$$ (area x height)

The total volume $V$ of the solid of revolution is

$$V = \sum_{k=1}^{N} V(x_k) = \sum_{k=1}^{N} (2\pi \, x_k \, y) \Delta y$$

As the thickness of the shells becomes infinitely thin

$$N \to \infty \quad \Delta y \to 0 \quad \sum \; \to \int$$

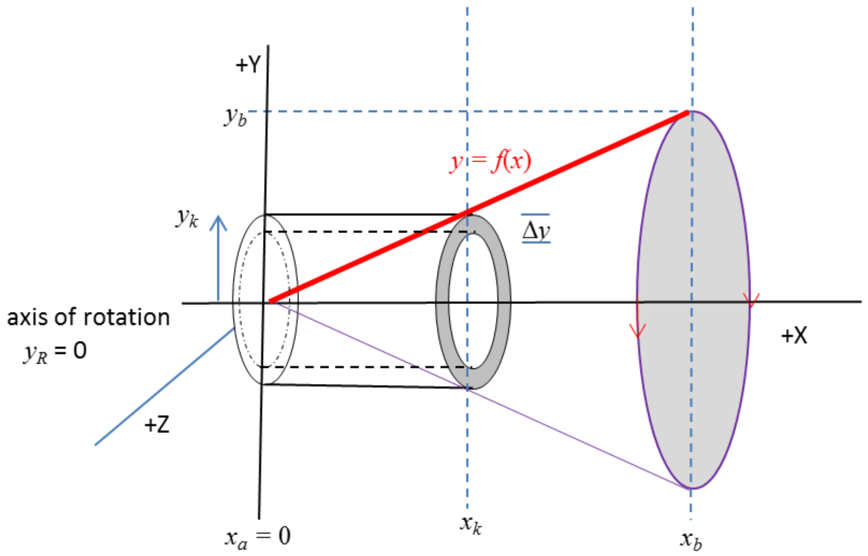
$$V = 2\pi \int_0^{y_b} x \, y \, dy$$

**cylindrical shell method**

$$V = 2\pi \int_0^{y_b} g(y) \, y \, dy$$

**rotation about X-axis**

***x = g(y)***



For a rotation about the Y-axis ( $x_R = 0$ )

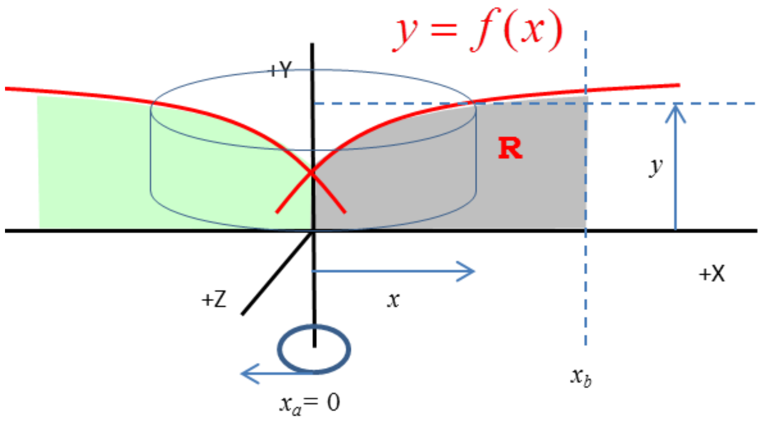
$$V = 2\pi \int_0^{x_b} y\ x\ dx$$

$$V = 2\pi \int_0^{x_b} f(x)\ x\ dx$$

**cylindrical shell method**

**rotation about Y-axis**

***y* = *f*(*x*)**

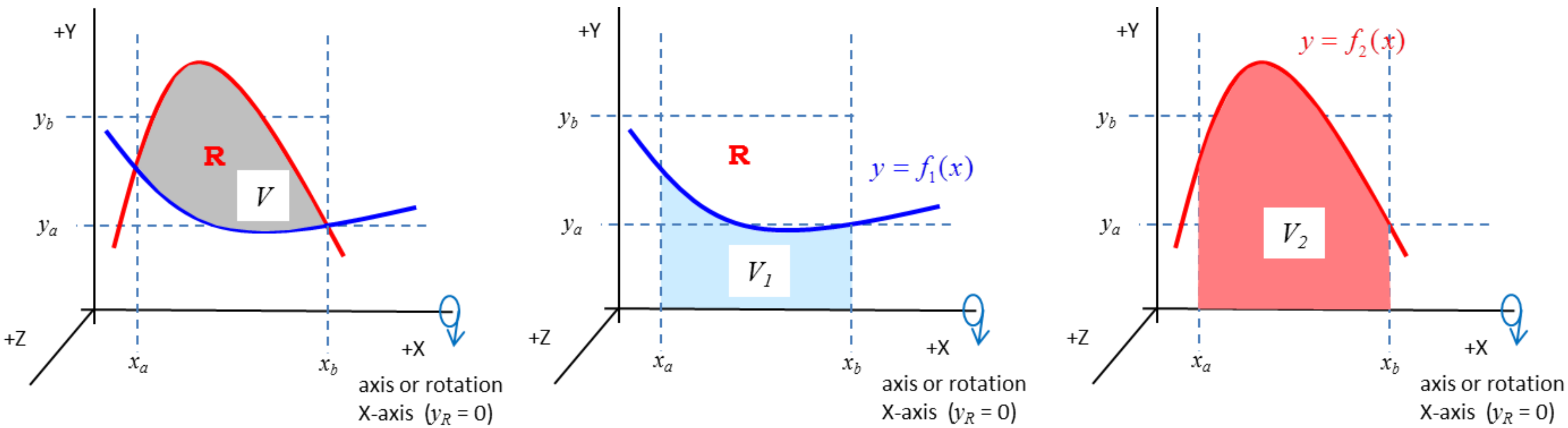


## Volumes generated by the rotation of bounded regions

When the region **R** between two functions defined in the XY plane is rotated, a solid of revolution can be produced. Let $y = f_1(x)$ and $y = f_2(x)$ be two functions such that $0 \le f_1(x) \le f_2(x)$ in the interval $x_a \le x \le x_b$. The volume $V$ of revolution of the region **R** is the difference between the volume $V_1$ of revolution of $f_1(x)$ and volume $V_2$ of revolution of $f_2(x)$.

$$V = V_2 - V_1$$

$$V_1 = \pi \int_{x_a}^{x_b} f_1(x)^2 \, dx \qquad V_2 = \pi \int_{x_a}^{x_b} f_2(x)^2 \, dx$$



## Solids of revolutions about lines parallel to a coordinate axis

We can also find the volume of revolution when the region is revolved about a line parallel to a coordinate axis. To do such calculations it is necessary to draw a "good quality" sketch of the region to be rotated and the axis of rotation $y_R$ = constant.

The volume $V$ of the solid of revolution for the region **R** bounded by $y = f(x)$, the X-axis ($y$ = 0) and the vertical lines $x_a$ and $x_b$ about the line $y_R$ ($y_R$ < 0) is

$$V = V_2 - V_1$$

$$V_1 = \pi \int_{x_a}^{x_b} R_1(x)^2\, dx \qquad V_2 = \pi \int_{x_a}^{x_b} R_2(x)^2\, dx$$

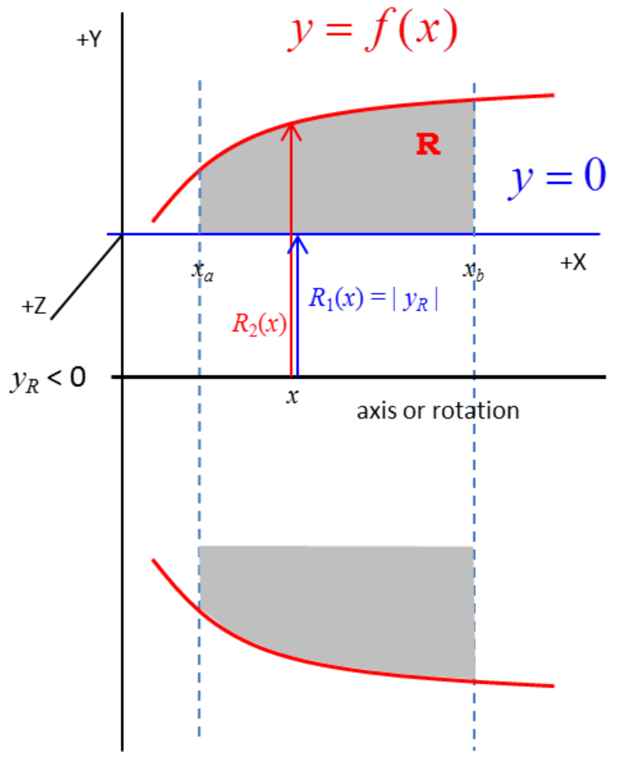
where

$$R_2(x) = f(x) - y_R = f(x) + |y_R|$$
$$R_1(x) = -y_R = |y_R| \quad y_R < 0$$

For the rotation about an axis of rotation when $y_R \geq f(x) \quad x_a \leq x \leq x_b$

$$R_2(x) = y_R \quad y_R \geq f(x)$$
$$R_1(x) = f(x)$$

For rotations about lines parallel to the Y-axis, then the x and y values are simply interchanged.




The best approach to mastering volume calculations for the HSC examination is to do lots of practice questions. Always start with a good diagram and work from first principles. Dot "blindly" apply an equation. Carefully choose the best method for the calculation.