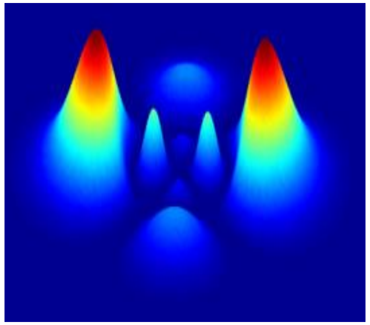


MATHEMATICS EXTENSION 2

4 UNIT MATHEMATICS

GEOMETRY and TRIGONOMETRY

Essential Background Topic

The topics of geometry and trigonometry are essential in the study of most of mathematics and is a fundamental topic in mathematics, physics, chemistry, engineering etc.

1 CIRCLE

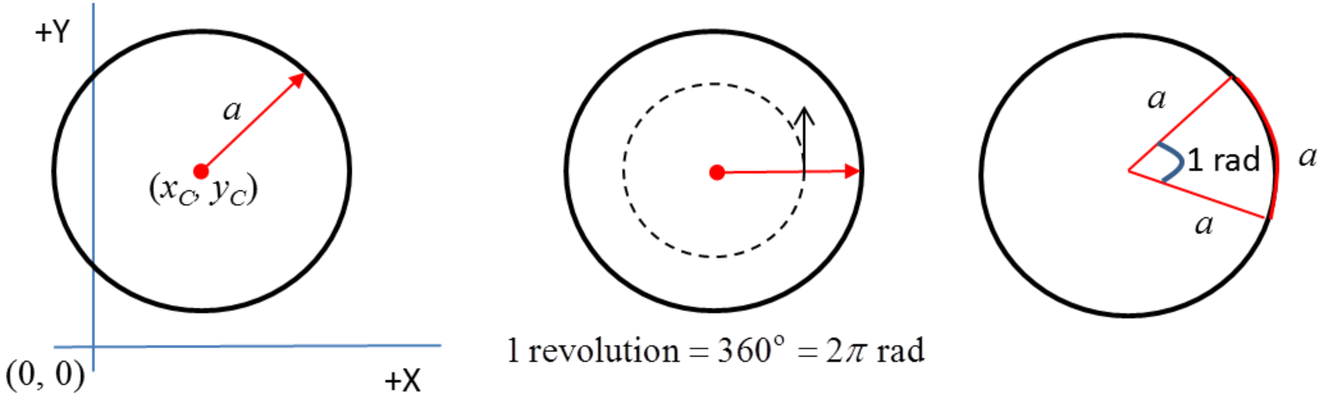
Equation of a circle: centre (x_C, y_C) and radius a $\quad (x - x_C)^2 + (y - y_C)^2 = a^2$

Circumference $\quad C = 2\pi a$

Area $\quad A = \pi a^2$

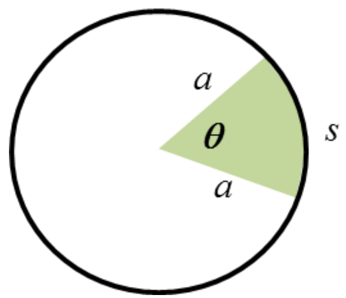
In many scientific and engineering calculations **radians** are used in preference to degrees in the measurement of angles. An angle of one radian is subtended by an arc having the same length as the radius.

$$1 \text{ revolution} = 360^\circ = 2\pi \text{ rad}$$



The **length of an arc** s of a circle which subtends an angle θ is

$$s = a\theta$$



The ratio of the area of the sector to the area of the full circle is the same as the ratio of the angle θ to the angle in a full circle. The full circle has area πa^2. Therefore

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{2\pi}$$

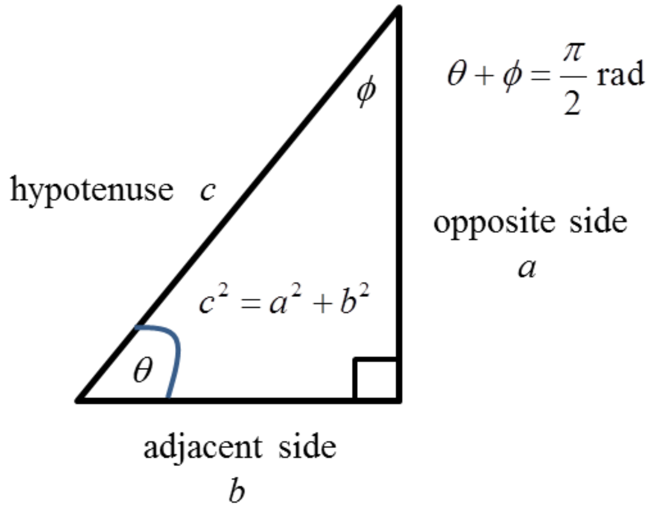
$$\text{area of sector} = \frac{a^2}{2}\theta$$

2 TRIANGLE

The Theorem of Pythagoras

$$c^2 = a^2 + b^2$$

Trigonometrical ratios in a right-angled triangle



Sine ratio $\sin(\theta) = \dfrac{a}{c}$

Cosine ratio $\cos(\theta) = \dfrac{b}{c}$

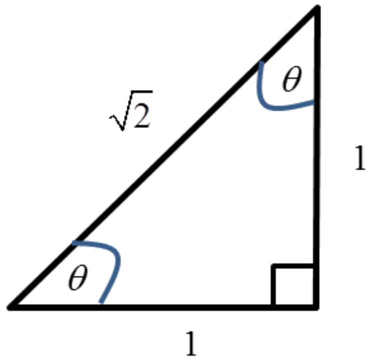
Tangent ratio $\tan(\theta) = \dfrac{a}{b}$

Cosecant ratio $\text{cosec}(\theta) = \dfrac{1}{\sin(\theta)} = \dfrac{c}{a}$

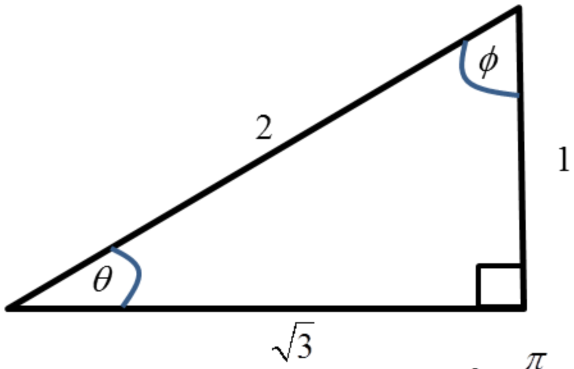
Scant ratio $\sec(\theta) = \dfrac{1}{\cos(\theta)} = \dfrac{c}{b}$

Cotangent ratio $\cot(\theta) = \dfrac{1}{\tan(\theta)} = \dfrac{b}{a}$

θ or ϕ	0	30^o $\pi/6$ rad	45^o $\pi/4$ rad	60^o $\pi/3$ rad	90^o $\pi/2$ rad
sin	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞



$$\theta = 45^\circ = \frac{\pi}{4}\text{ rad}$$

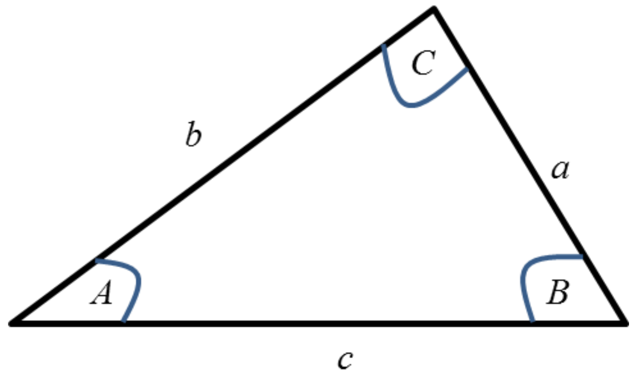


$$\theta = 30^\circ = \frac{\pi}{6}\text{ rad}$$

$$\phi = 60^\circ = \frac{\pi}{3}\text{ rad}$$

Law of Sines

$$\frac{a}{\sin(A)}=\frac{b}{\sin(B)}=\frac{c}{\sin(C)} \qquad \frac{\sin(A)}{a}=\frac{\sin(B)}{b}=\frac{\sin(C)}{c}$$



Law of Cosines

$$c^2=a^2+b^2-2ab\cos(C) \quad \cos(C)=\frac{a^2+b^2-c^2}{2ab}$$

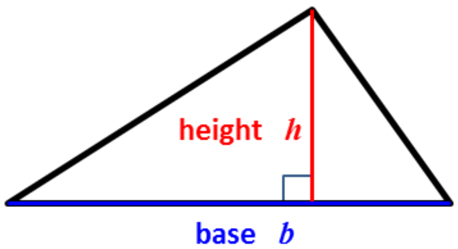
$$a^2=b^2+c^2-2bc\cos(A) \quad \cos(A)=\frac{b^2+c^2-a^2}{2bc}$$

$$b^2=a^2+c^2-2ac\cos(B) \quad \cos(B)=\frac{a^2+c^2-b^2}{2ac}$$

Law of Tangents

$$\frac{a-b}{a+b}=\frac{\tan\left(\frac{a-b}{2}\right)}{\tan\left(\frac{a+b}{2}\right)}$$

Area



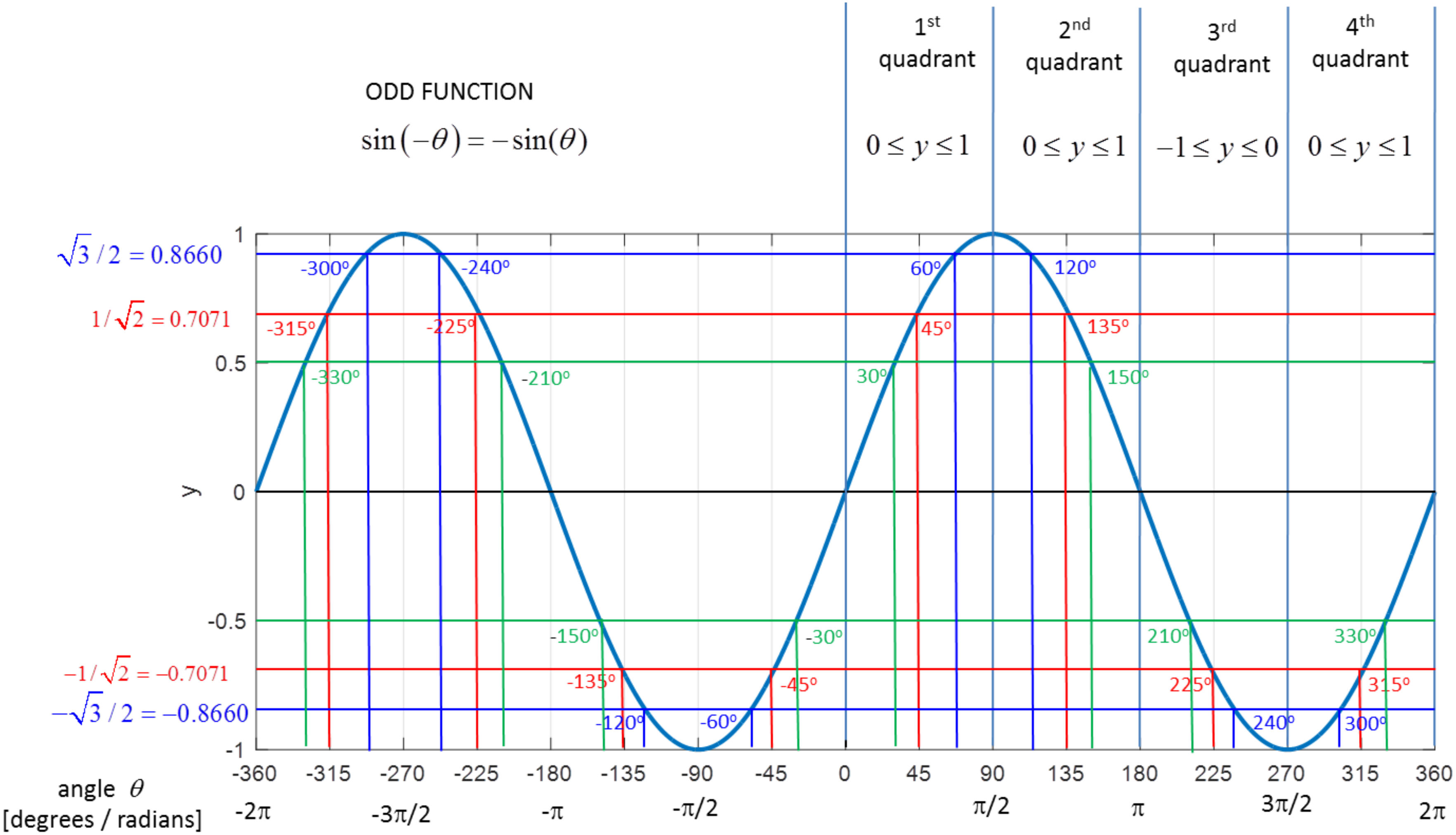
$$A=\tfrac{1}{2}(base)(height)$$

$$A=\tfrac{1}{2}bh$$

3 TRIGONOMETRIC FUNCTIONS

Knowledge of the trigonometric functions is vital in very many fields of engineering, mathematics and physics.

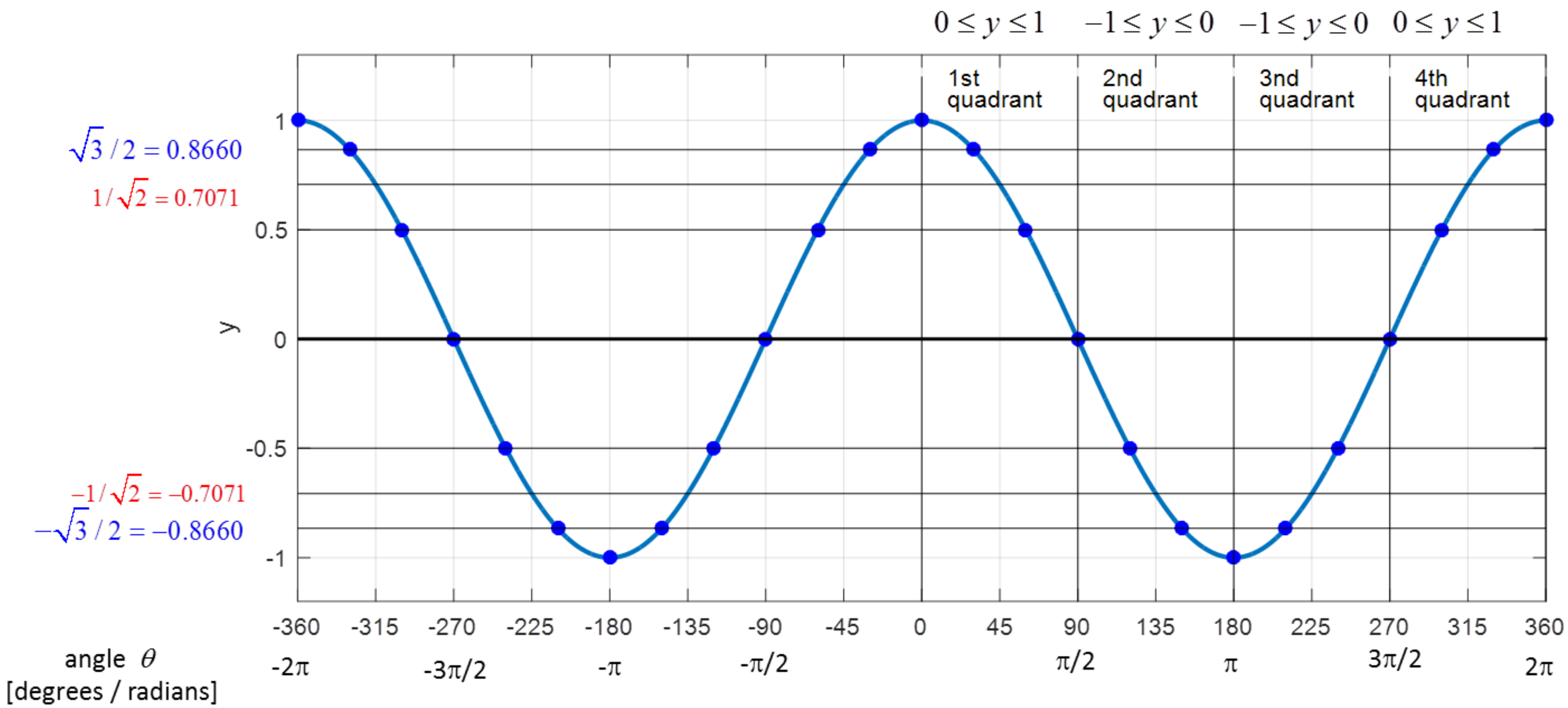
Sine function $y = \sin(\theta)$



Cosine function $y = \cos(\theta)$

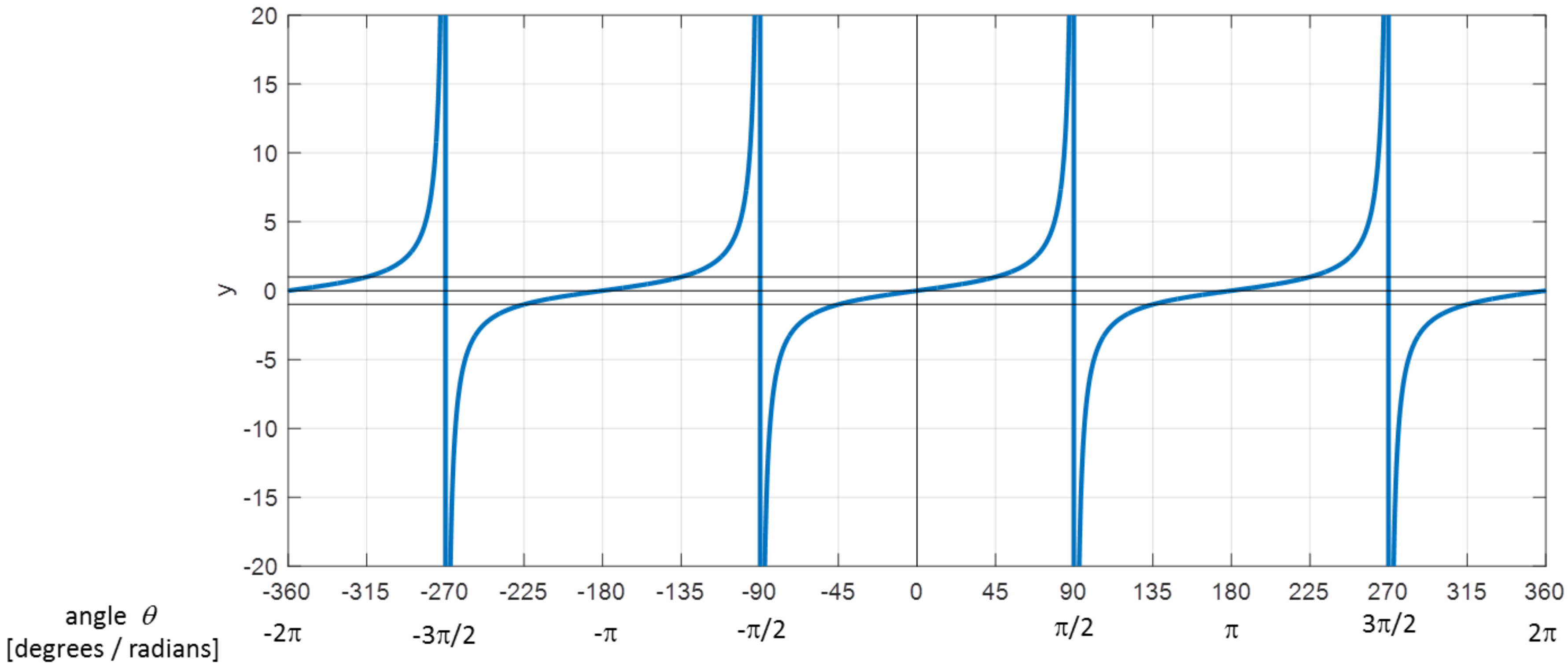
EVEN FUNCTION $\cos(-\theta) = \cos(\theta)$

spacing between blue dots is 30°



Tangent function $y = \tan(\theta)$

ODD FUNCTION $\tan(-\theta) = -\tan(\theta)$



4 TRIGONOMETRIC IDENTITIES and EQUATIONS

$$\sin^2(\theta)+\cos^2(\theta)=1$$
$$\sec^2(\theta)=1+\tan^2(\theta)$$
$$\operatorname{cosec}^2(\theta)=1+\cot^2(\theta)$$

$$\sin(x)=\sqrt{1-\cos^2(x)} \qquad \cos(x)=\sqrt{1-\sin^2(x)}$$
$$\tan^2(x)+1=\frac{\sin^2(x)}{\cos^2(x)}+1=\frac{\sin^2(x)+\cos^2(x)}{\cos^2(x)}=\frac{1}{\cos^2(x)}$$

$$\sin(\theta+\phi)=\sin(\theta)\cos(\phi)+\cos(\theta)\sin(\phi)$$
$$\sin(\theta-\phi)=\sin(\theta)\cos(\phi)-\cos(\theta)\sin(\phi)$$

$$\cos(\theta+\phi)=\cos(\theta)\cos(\phi)-\sin(\theta)\sin(\phi)$$
$$\cos(\theta-\phi)=\cos(\theta)\cos(\phi)+\sin(\theta)\sin(\phi)$$

$$\tan(\theta+\phi)=\frac{\tan(\theta)+\tan(\phi)}{1-\tan(\theta)\tan(\phi)}$$

$$\tan(\theta-\phi)=\frac{\tan(\theta)-\tan(\phi)}{1+\tan(\theta)\tan(\phi)}$$

$$\sin(\theta)-\sin(\phi)=2\sin\left(\frac{\theta-\phi}{2}\right)\cos\left(\frac{\theta+\phi}{2}\right)$$

$$\cos(\theta)-\cos(\phi)=-2\sin\left(\frac{\theta-\phi}{2}\right)\sin\left(\frac{\theta+\phi}{2}\right)$$

Double angle formulae

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 \quad \cos^2(x) = \tfrac{1}{2}\left(1 + \cos(2x)\right)$$

$$\tan(2x) = \frac{\sin(2x)}{\cos(2x)} = \frac{2\sin(x)\cos(x)}{\cos^2(x) - \sin^2(x)} \qquad \div\cos^2(x)$$

$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = \cos^2(x)\left(1 - \frac{\sin^2(x)}{\cos^2(x)}\right) = \left(\frac{1}{\sec^2(x)}\right)\left(1 - \tan^2(x)\right)$$

$$\cos(2x) = \frac{1 - \tan^2(x)}{1 + \tan^2(x)}$$

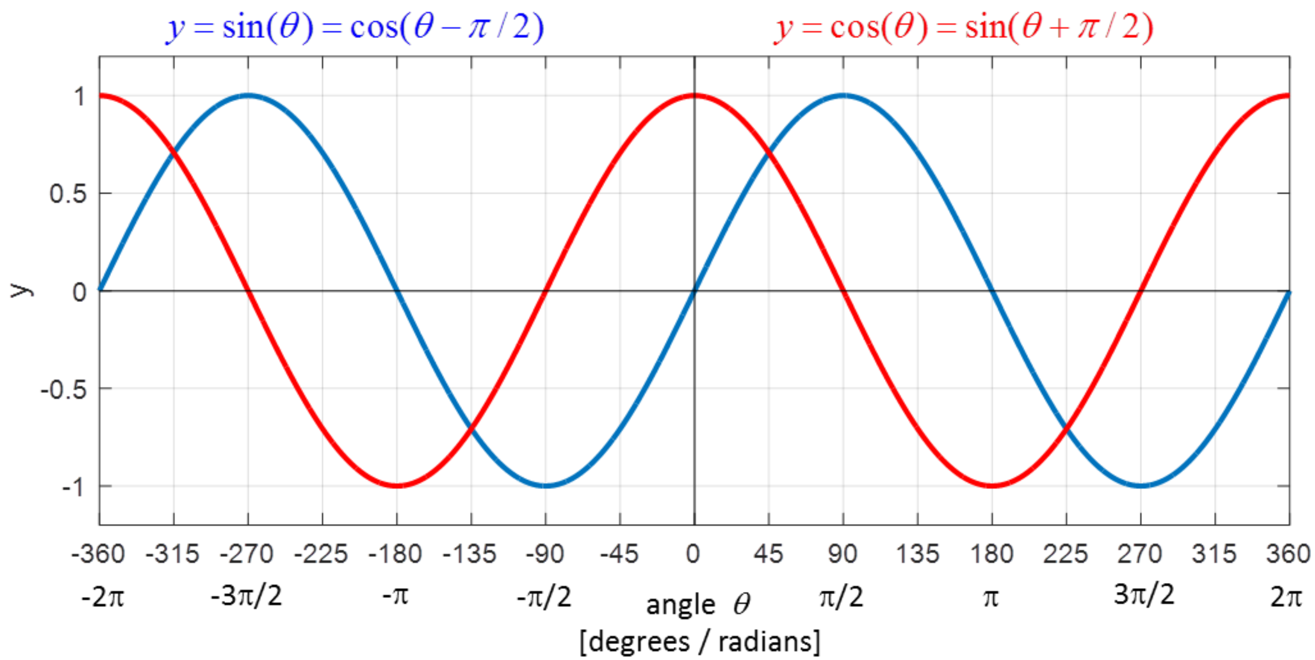
$$\sin(2x) = 2\sin(x)\cos(x) = \frac{2\sin(x)\cos^2(x)}{\cos(x)} = \frac{2\tan x}{\sec^2(x)}$$

$$\sin(2x) = \frac{2\tan x}{1 + \tan^2(x)}$$

The substitution $t = \tan(x/2)$ is often a useful one for integration of trigonometric functions because we can express

$$\sin(x) = \frac{2t}{1+t^2} \quad \cos(x) = \frac{1-t^2}{1+t^2} \quad dx = \frac{2\,dt}{1+t^2}$$

5 SINE and COSINE FUNCTIONS



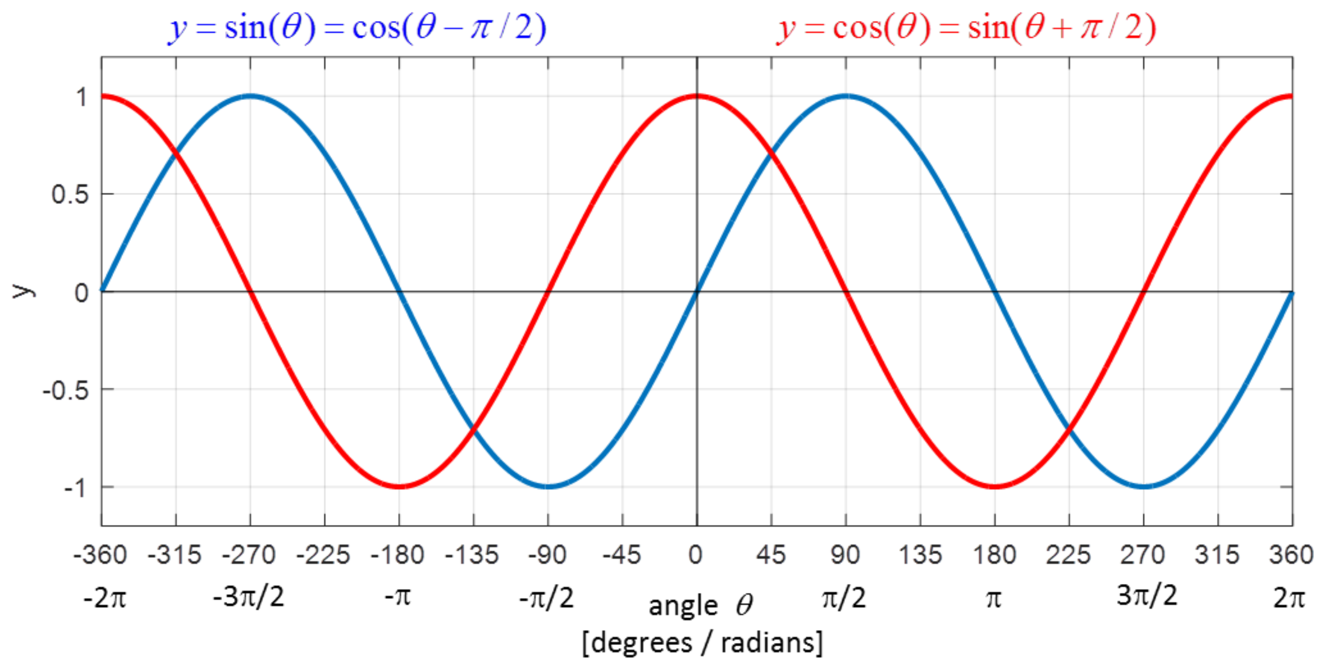
$$\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$$

$\phi = 90^{\circ} = \pi/2 \text{ rad} \quad \sin(\theta + \pi/2) = \cos(\theta)$ $\Rightarrow$ sine curve shifted to left through π/2 rad

$\phi = -90^{\circ} = -\pi/2 \text{ rad} \quad \sin(\theta - \pi/2) = -\cos(\theta)$ $\Rightarrow$ sine curve shifted to right through π/2 rad

$\phi = 180^{\circ} = \pi \text{ rad} \quad \sin(\theta + \pi) = -\sin(\theta)$ $\Rightarrow$ sine curve shifted to left through π rad

$\phi = -180^{\circ} = -\pi \text{ rad} \quad \sin(\theta - \pi) = -\sin(\theta)$ $\Rightarrow$ sine curve shifted to right through π rad



$\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$

$\phi = 90^{\circ} = \pi/2 \text{ rad} \quad \cos(\theta+\pi/2) = -\sin(\theta)$ $\Rightarrow$ cosine curve shifted to left through π/2 rad

$\phi = -90^{\circ} = -\pi/2 \text{ rad} \quad \cos(\theta-\pi/2) = \sin(\theta)$ $\Rightarrow$ cosine curve shifted to right through π/2 rad

$\phi = 180^{\circ} = \pi \text{ rad} \quad \cos(\theta+\pi) = -\cos(\theta)$ $\Rightarrow$ cosine curve shifted to left through π rad

$\phi = -180^{\circ} = -\pi \text{ rad} \quad \cos(\theta-\pi) = -\cos(\theta)$ $\Rightarrow$ cosine curve shifted to right through π rad