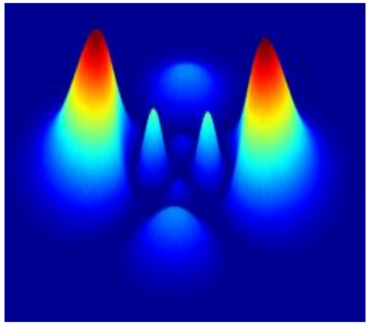


MATHEMATICS EXTENSION 2

4 UNIT MATHEMATICS

TOPIC 7: POLYNOMIALS

7.4 FINDING THE ROOTS OF POLYNOMIALS

HALF-INTERVAL METHOD and NEWTON'S METHOD

HALF-INTERVAL METHOD

Suppose we have two values x_1 and x_2 such that $P(x_1)P(x_2) < 0$ and since $P(x)$ is a continuous function, there is a root of $P(x)$ in the interval $x_1 < x < x_2$. Now calculate the value of x_3 the mid-point in the interval x_1 to x_2.

$$x_3 = \tfrac{1}{2}\left(x_1 + x_2\right)$$

If $P(x_3) = 0$ then x3 is the desired root.

If $P(x_3)P(x_2) < 0$ then replace x_1 by x_3

If $P(x_3)P(x_1) < 0$ then replace x_2 by x_3

and repeat the process until you get a reasonable of the value for the root.

This process is very tedious because you have to calculate $P(x)$ many times. It is best to use a spreadsheet and not a calculator. The example below shows how to use a spreadsheet.

Example Find the roots of the equation $P(x) = x^3 - 2x^2 - x + 2 = 0$

Enter values for x1 and x2 only						
	x1	p(x1)	x2	p(x2)	p(x1)*p(x2) < 1	x3
start values	0.0000	2.0000	1.3000	-0.4830	-0.9660	0.6500
x3 --> x1	0.6500	0.7796	1.3000	-0.4830	-0.3766	0.9750
x3 --> x1	0.9750	0.0506	1.3000	-0.4830	-0.0244	1.1375
x3 --> x2	0.9750	0.0506	1.1375	-0.2535	-0.0128	1.0563
x3 --> x2	0.9750	0.0506	1.0563	-0.1092	-0.0055	1.0156
x3 --> x2	0.9750	0.0506	1.0156	-0.0310	-0.0016	0.9953
x3 --> x1	0.9953	0.0094	1.0156	-0.0310	-0.0003	1.0055
x3 --> x2	0.9953	0.0094	1.0055	-0.0109	-0.0001	1.0004
x3 --> x2	0.9953	0.0094	1.0004	-0.0008	0.0000	0.9979
x3 --> x1	0.9979	0.0043	1.0004	-0.0008	0.0000	0.9991
x3 --> x1	0.9991	0.0018	1.0004	-0.0008	0.0000	0.9998

First root is x = +1.

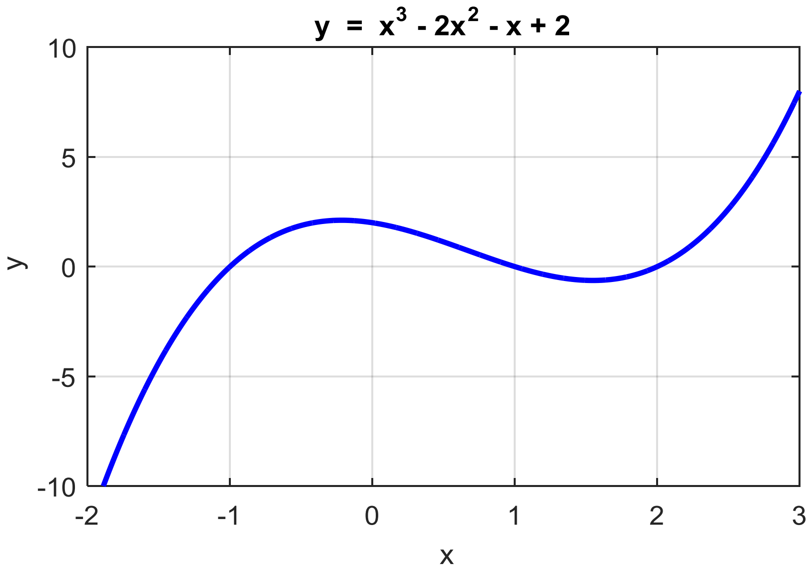
Enter values for x1 and x2 only						
	x1	p(x1)	x2	p(x2)	p(x1)*p(x2) < 1	x3
start values	1.7000	-0.5670	2.4000	1.9040	-1.0796	2.0500
x3 --> x2	1.7000	-0.5670	2.0500	0.1601	-0.0908	1.8750
x3 --> x1	1.8750	-0.3145	2.0500	0.1601	-0.0504	1.9625
x3 --> x1	1.9625	-0.1069	2.0500	0.1601	-0.0171	2.0063
x3 --> x2	1.9625	-0.1069	2.0063	0.0189	-0.0020	1.9844
x3 --> x1	1.9844	-0.0459	2.0063	0.0189	-0.0009	1.9953
x3 --> x1	1.9953	-0.0140	2.0063	0.0189	-0.0003	2.0008
x3 --> x2	1.9953	-0.0140	2.0008	0.0023	0.0000	1.9980

Second root is $x = 2$

Enter values for x1 and x2 only						
	x1	p(x1)	x2	p(x2)	p(x1)*p(x2) < 1	x3
start values	-2.0000	-12.0000	-0.5000	1.8750	-22.5000	-1.2500
x3 --> x1	-1.2500	-1.8281	-0.5000	1.8750	-3.4277	-0.8750
x3 --> x2	-1.2500	-1.8281	-0.8750	0.6738	-1.2318	-1.0625
x3 --> x1	-1.0625	-1.0625	-0.8750	0.6738	-0.7159	-0.9688
x3 --> x2	-1.0625	-0.3948	-0.9688	0.1826	-0.0721	-1.0156
x3 --> x1	-1.0156	-0.0950	-0.9688	0.1826	-0.0173	-0.9922
x3 --> x2	-1.0156	-0.0950	-0.9922	0.0466	-0.0044	-1.0039
x3 --> x1	-1.0039	-0.0235	-0.9922	0.0466	-0.0011	-0.9980
x3 --> x2	-1.0039	-0.0235	-0.9980	0.0117	-0.0003	-1.0010
x3 --> x1	-1.0010	-0.0059	-0.9980	0.0117	-0.0001	-0.9995
x3 --> x2	-1.0010	-0.0059	-0.9995	0.0029	0.0000	-1.0002

Third root is x = -1

The three roots are (-1, 1, 2)



Go to the online active: http://www.physics.usyd.edu.au/teach_res/hsp/sim/sim_poly.htm

POLYNOMIALS (view activity questions)

The equation of a polynomial of degree 4 can be written as

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

Investigate the graphs of polynomial up to degree 4 by changing the a coefficents.

Polynomial Coefficients

$a_0 =$ 2

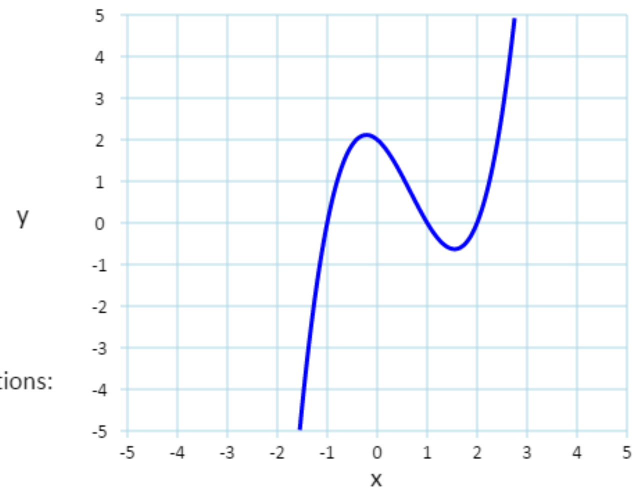
$a_1 =$ -1

$a_2 =$ -2

$a_3 =$ 1

$a_4 =$ 0

Place mouse in graph area for X,Y positions:



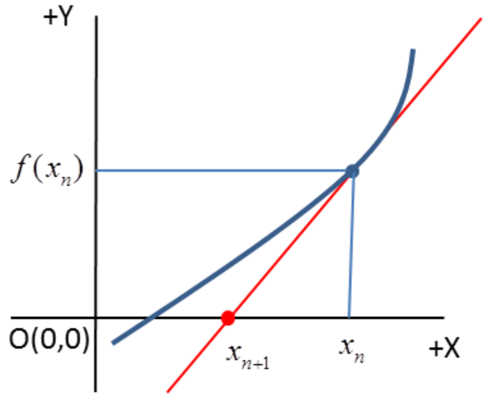
a_0 2 a_1 -1 a_2 -2 a_3 1 a_4 0

x_{max} 5 y_{max} 5 PLOT

NEWTON'S METHOD

Suppose that x_n is close to a root of $f(x)=0$. We can make an improved estimate of the root $f(x)=0$ by Newton's Method which involves $f(x)$ and $f'(x)$ as shown in the figure

The tangent to the curve intersects the X-axis at x_{n+1} at a point which should be closer to the root than x_n. The gradient $f'(x_n)$ of the tangent to the curve is approximated by



$$f'(x_n)=\frac{0-f(xn)}{x_{n+1}-x_0}$$

Rearranging this equation, we can the new estimate x_{n+1}

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

This method may fail if the function has a point of inflection, or other bad behaviour near the root of the function.

Example Find the roots of the equation by Newton's Method $P(x) = x^3 - 2x^2 - x + 2 = 0$

Solution This method is much easier to estimate the roots than the half-interval method. Again, it is a simple matter to perform the repeated calculations in a spreadsheet.

Starting value x = -2

Root x = -1

Newton's Method				
n	xn	f(xn)	f'(xn)	xn+1
1	**-2.0000**	-12.0000	19	-1.36842
2	-1.36842	-2.9392	10.09141	-1.07716
3	-1.07716	-0.4932	6.789494	-1.00452
4	-1.00452	-0.0272	6.045263	-1.00002
5	-1.00002	-0.0001	6.000169	-1
6	-1	0.0000	6	-1

Starting value x = 0.5

Root x = 1

Newton's Method				
n	xn	f(xn)	f'(xn)	xn+1
1	**0.5000**	1.1250	-2.25	1
2	1	0.0000	-2	1
3	1	0.0000	-2	1
4	1	0.0000	-2	1
5	1	0.0000	-2	1
6	1	0.0000	-2	1

Starting value x = 3

Root x = 2

Newton's Method				
n	xn	f(xn)	f'(xn)	xn+1
1	**3.0000**	8.0000	14	2.428571
2	2.428571	2.0991	6.979592	2.12782
3	2.12782	0.4509	4.07157	2.017076
4	2.017076	0.0524	3.137486	2.000375
5	2.000375	0.0011	3.003	2
6	2	0.0000	3.000001	2