EXERCISE 32_123 (HSC 2013/2)

Consider the equation $4x^2 - 25y^2 = 100$

Part (A) What type of curve does the equation correspond too? Is the eccentricity e of the curve: $e = 1$; e < 1, $e > 1$; $e = 0$?

Part (B) Give the Cartesian coordinates for the vertices and focal points. Calculate the eccentricity e of the curve?

Part (C) State the equations for the directrices and asymptotes of the curve.

Part (D) The point P on the curve has the X-coordinate $x_P = 10$ and $y_P > 0$. Where does the tangent to the curve at the point P cut the X-axis and Y-axis? Where does the normal to the tangent at the point P intersect the X-axis and the Y-axis

Part (E) Sketch the curve showing the vertices, focal points, the asymptotes, directrices, and the points where the tangent and the normal intersect the X-axis and Y-axis.

For your sketch: X-axis (-30 to +30) and Y-axis (-30 to +30).

Answer Part (A)

The equation $4x^2 - 25y^2 = 100$ corresponds to the curve of a **hyperbola**.

The **eccentricity** of hyperbolas is $e > 1$.

Answer Part (B)

A general expression for a hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The equation of the hyperbola $4x^2 - 25y^2 = 100$ can be re-written as

$$\frac{x^2}{5^2} - \frac{y^2}{2^2} = 1 \quad \Rightarrow \quad a = 5 \quad b = 2$$

The **vertices** of the hyperbola are $A_1(-a, 0)$ and $A_2(a, 0)$

$A_1(-5,0)$ and $A_2(5,0)$

The **focal length** c is $c^2 = a^2 + b^2 \quad c = \sqrt{a^2 + b^2} = \sqrt{25+4} = \sqrt{29} = 5.3852$

The **focal points** are $F_1(-c, 0)$ and $F_2(c, 0)$

$F_1(-\sqrt{29},0)$ and $F_2(\sqrt{29},0)$ or $F_1(-5.3852, 0)$ and $F_2(5.3852, 0)$

The **eccentricity** is $e = \frac{c}{a} = \frac{\sqrt{29}}{5} = 1.0770 > 1$

Answer Part (C)

The equations for the **directrices** are

$$x = \pm\frac{a^2}{c} \quad \Rightarrow \quad x = \frac{-25}{\sqrt{29}} = -4.6424 \qquad x = \frac{+25}{\sqrt{29}} = 4.6424$$

The equations for the **asymptotes** are

$$y = \pm\frac{b}{a}x \quad \Rightarrow \quad y = -\frac{2}{5}x = -0.4000\,x \qquad y = \frac{2}{5}x = 0.4000\,x$$

Answer Part (D)

The coordinates of the point P are (x_P, y_P)

$$x_P = 10 \quad y_P = \sqrt{4\left(\frac{100}{25}\right) - 1} = \sqrt{12} = 2\sqrt{3}$$

The equation of the straight line for the tangent is $y = M_1 x + B_1$

The gradient of the curve is given by the first derivative of the function

$$\left(\frac{2}{25}\right)x - \left(\frac{2}{4}\right)y\left(\frac{dy}{dx}\right) = 0 \qquad dy/dx = \left(\frac{4}{25}\right)\left(\frac{x}{y}\right)$$

The gradient M_1 at the point P

$$x_P = 10 \quad y_P = 2\sqrt{3} \quad M_1 = \left(\frac{4}{25}\right)\left(\frac{10}{2\sqrt{3}}\right) = \left(\frac{4}{5\sqrt{3}}\right) = 0.4619$$

The intercept B_1 of the tangent is

$$B_1 = y_P - M_1 x_P = 2\sqrt{3} - 10\left(\frac{4}{5\sqrt{3}}\right) = \frac{-2}{\sqrt{3}} = 1.1547$$

The tangent intersects the X-axis at the point T

$$y_T = 0 \quad x_T = -\frac{B_1}{M_1} = -\left(\frac{-2}{\sqrt{3}}\right)\left(\frac{5\sqrt{3}}{4}\right) = 2.5$$

The tangent intersects the Y-axis at the Point U

$$x_U = 0 \quad y_U = B_1 = \frac{-2}{\sqrt{3}} = -1.1547$$

The equation of the straight line for the normal is $y = M_2\,x + B_2$

where $M_2 = \dfrac{-1}{M_1} = -\left(\dfrac{5\sqrt{3}}{4}\right) = 2.1651$

The intercept B_2 of the normal is

$$B_2 = y_P - M_2\,x_P = 2\sqrt{3} + 10\left(\frac{5\sqrt{3}}{4}\right) = 14.5\sqrt{3} = 25.1147$$

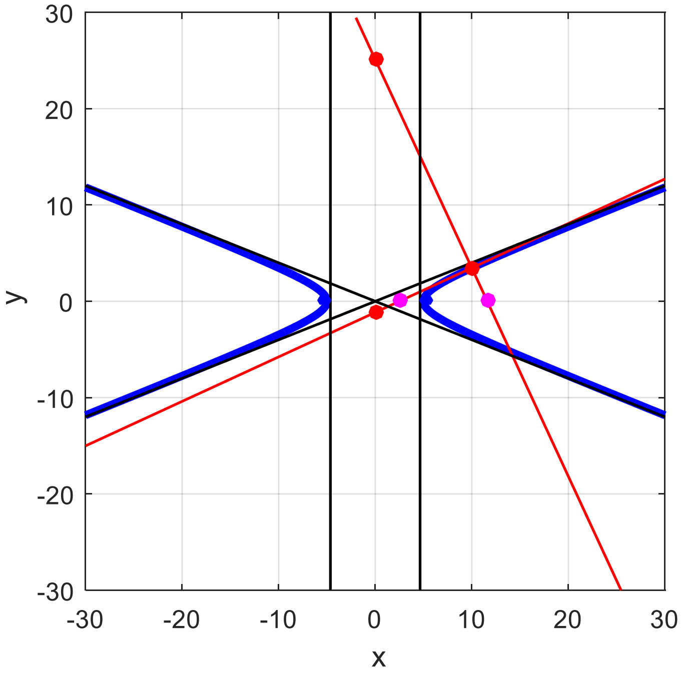
The normal intersects the X-axis at the point R

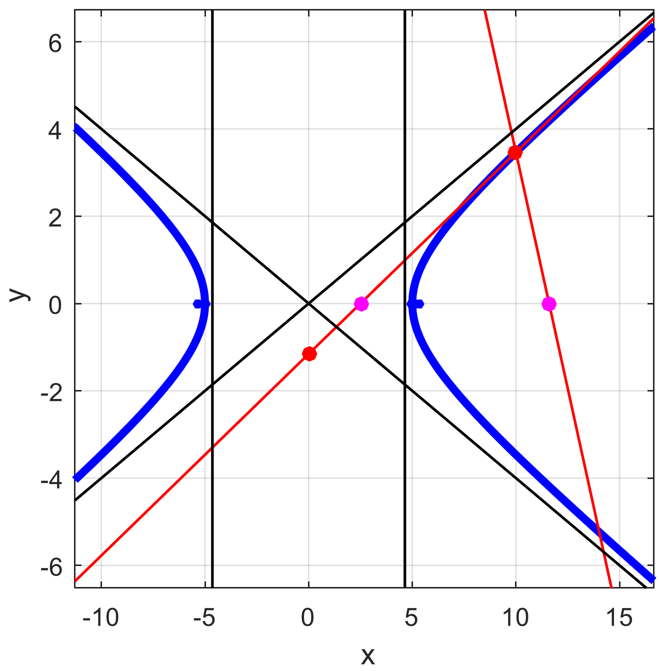
$$y_R = 0 \quad x_R = -\frac{B_2}{M_2} = -\left(14.5\sqrt{3}\right)\left(\frac{-4}{5\sqrt{3}}\right) = +11.600$$

The normal intersects the Y-axis at the Point S

$$x_S = 0 \quad y_S = B_2 = 14.5\sqrt{3} = 25.1147$$

Answer Part (E)





In the graphs, the vertices and focal points (blue dots) are very close to each other