ONLINE: MATHEMATICS EXTENSION 2

Topic 6 MECHANICS

6.61 MOTION IN A CIRCLE

COMPLEX NUMBERS

We can use complex numbers to mathematically investigate the motion of a particle in a circle.

Let the X axis be the real axis and the Y axis be the imaginary axis. The **position** of the particle is given by the complex number z as a vector

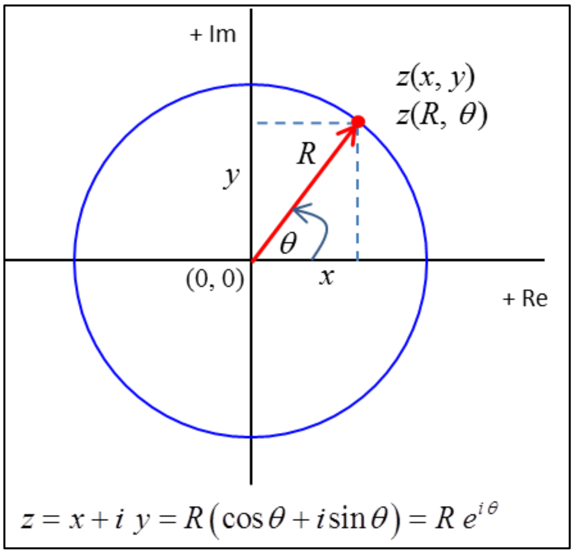
$$z = x + i\,y = R(\cos\theta + i\sin\theta) = R\,e^{i\theta}$$

where the angular displacement θ is a function of time $\theta = \theta(t)$. The magnitude of z is

$$|z| = R$$

and the angle of the complex vector z makes with the real axis is the argument of z

$$Arg(z) = \theta$$



For motion in a circle the radius is constant

$$\frac{dR}{dt} = \frac{d^2R}{dt^2} = 0 \qquad \dot{R} = \ddot{R} = 0$$

The **velocity** v of the particle is found by differentiating the position of the particle z with respect to time t.

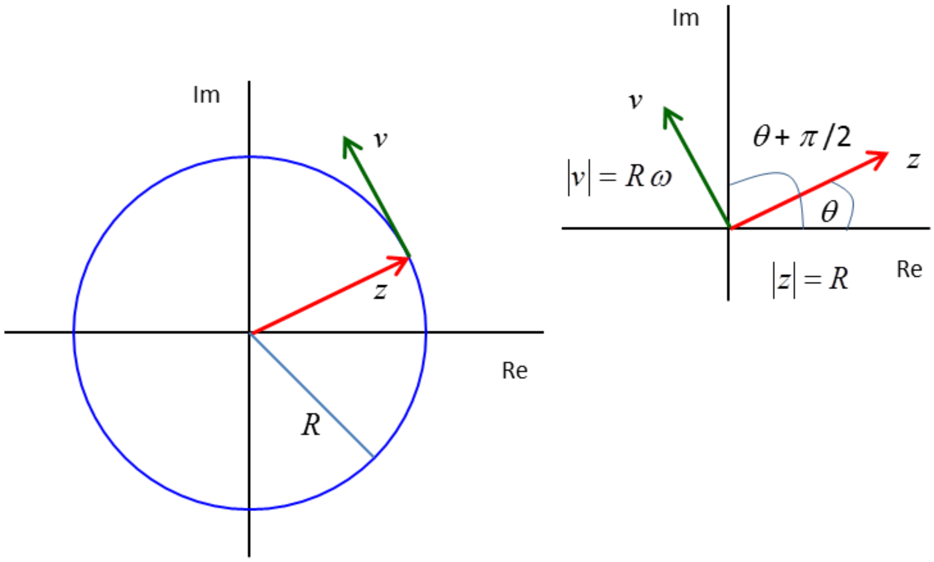
$$v = \frac{dz}{dt} = \dot{z}$$

$$v = \frac{d}{dt}\left(R\, e^{i\,\theta}\right) = R\left(i\,\frac{d\theta}{dt}\right)\, e^{i\,\theta} = i\,(R\,\omega)\, e^{i\,\theta}$$

$$v = i\,\omega\, z$$

where the **angular speed** is $\omega = \dfrac{d\theta}{dt} = \dot{\theta}$.

The magnitude of the velocity is $|v| = R\,\omega$. Multiplication by i produces a rotation of a complex vector by $\pi/2$ rad. Therefore z and $i\,z$ are at right angles to each other. Hence, the displacement vector z and velocity vector v are always perpendicular to each other.



$v = i\,\omega z = \omega\left(e^{i\pi/2}\right)\left(R\, e^{i\theta}\right) = R\,\omega\, e^{i(\theta+\pi/2)}$ $\qquad z = R\, e^{i\theta}$ $\qquad i = \cos(\pi/2) + i\,\sin(\pi/2) = e^{i\pi/2}$

The **acceleration** is found by differentiating the velocity with respect to time

$$a = \frac{dv}{dt} = \dot{v}$$

$$a = \frac{d}{dt}\left(i\,\omega\,z\right) = \frac{d}{dt}\left(i\,\omega\,R\,e^{i\,\theta}\right) = i\,R\left(\frac{d\omega}{dt}\,e^{i\,\theta} + \omega\,i\,\frac{d\theta}{dt}\,e^{i\,\theta}\right)$$

$$a = i\,\frac{d\omega}{dt}\,R\,e^{i\,\theta} - \omega^2\,R\,e^{i\,\theta}$$

$$a = \left(e^{i\,\pi/2}\right)\left(\frac{d\omega}{dt}\right)z - \omega^2\,z = \left(e^{i\,\pi/2}\right)\alpha z - \omega^2\,z \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

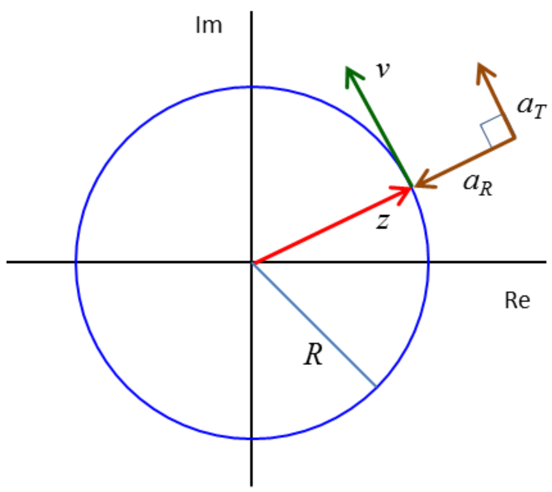
where the **angular acceleration** is $\alpha = \dfrac{d\omega}{dt}$.

The first term $\left(e^{i\,\pi/2}\right)\left(\dfrac{d\omega}{dt}\right)z$ corresponds to the change in speed of the particle and the direction is perpendicular to the displacement vector. This term is the tangential acceleration a_T. The magnitude of the tangential acceleration is

$$\left|a_T\right| = \left|\left(e^{i\,\pi/2}\right)\left(\frac{d\omega}{dt}\right)z\right| = R\,\frac{d\omega}{dt} = \frac{dv}{dt}$$

The second term $-\omega^2\,z$ corresponds to the change in direction and it is in a direction opposite to the displacement vector. This is the radial acceleration a_R. The magnitude of the radial acceleration is

$$\left|a_R\right| = \left|-\omega^2\,z\right| = \omega^2\,R = \frac{v^2}{R}$$



$$|z| = R$$

$$|v| = R\,\omega$$

$$\left|a_T\right| = \frac{dv}{dt} = \frac{d}{dt}\left(R\,\omega\right)$$

$$\left|a_R\right| = R\,\omega^2 = \frac{v^2}{R}$$