ONLINE: MATHEMATICS EXTENSION 2

Topic 6 MECHANICS

EXERCISE p6202

Prove that the range R of a projectile fired upward at an angle θ with initial velocity u is

$$R = \frac{u^2 \sin(2\theta)}{g}$$

where g is the acceleration due to gravity.

A garden sprinkler sprays water about its vertical axis at a constant speed of u. The direction of the spray varies between angles of 15^o to 60^o with respect to the horizontal.

Prove that from a fixed position O on level ground, the sprinkler will wet the surface of an annular region with centre O with an minimum radius of $u^2/2g$ and a maximum radius u^2/g .

Show that by locating the sprinkler appropriately relative to a rectangular garden bed of dimensions 6 m x 3 m, the entire bed may be water provided that

$$\frac{u^2}{2g} \geq 1 + \sqrt{7}$$

Solution

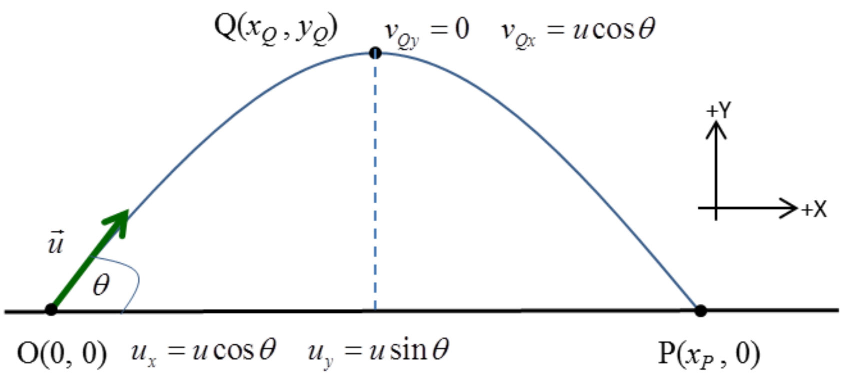
Step 1: Think about how to approach the problem

Step 2: Draw annotated diagrams of the physical situations

Step 3: What do you know about projectile motion and motion with a constant acceleration?

We can consider the motion in the +X direction to be independent of the motion in the Y direction and use the equation for constant acceleration

$$v = u + a\,t \qquad s = u\,t + \tfrac{1}{2}a\,t^2 \qquad v^2 = u^2 + 2\,a\,s$$



First, calculate the time t_Q it takes for the projectile to reach its maximum height at Q.

Horizontal motion: X direction	Vertical motion: Y direction
$v_{Qx} = u\cos\theta$ $x_Q = v_x\, t = u\cos\theta\, t$ $x_Q = (u\cos\theta)\left(\dfrac{u\sin\theta}{g}\right) = \dfrac{u^2\cos\theta\sin\theta}{g}$	$a_y = -g \quad v_{Qy} = 0$ $0 = u\sin\theta - g\, t$ $t = \dfrac{u\sin\theta}{g}$

The range of the projectile is $R = x_P = 2x_Q$

$$R = \frac{u^2\,(2)\cos\theta\sin\theta}{g} = \frac{u^2\,\sin(2\theta)}{g}$$

QED

The maximum range occurs when $\sin(2\theta) = 1 \quad \theta = 45^o$

$$R_{\max} = \frac{u^2}{g}$$

The minimum range is

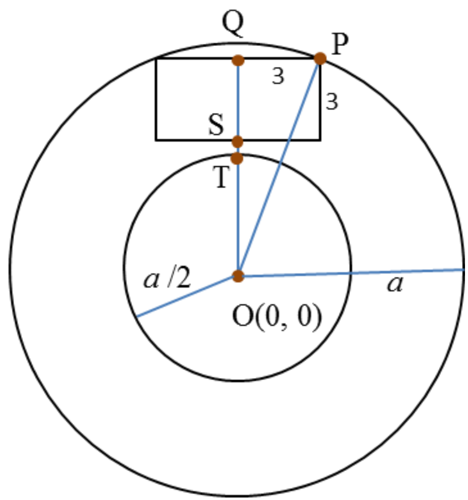
$$R(\theta = 15^o) = \frac{u^2\,\sin(30^o)}{g} = \frac{u^2}{2g}$$

$$R(\theta = 60^o) = \frac{u^2\,\sin(120^o)}{g} = \frac{0.866\,u^2}{g}$$

$$R_{\min} = \frac{u^2}{2g} \quad \theta = 15^o$$

QED

The rectangular garden bed 6 m x 3 m must fit into the region bounded by the two circles. Let *a* be the radius of the larger circle and *a*/2 be the radius of the smaller circle



$$a = \frac{u^2}{g} \qquad a/2 = \frac{u^2}{2g}$$

Hence, we require that the distance OT to be less than the distance OS.

We need to find the coordinates of the points P, Q, S and T.

Point P lies on the circle $x^2 + y^2 = a^2$

$$x_P = 3 \quad y_P = \sqrt{a^2 - 9}$$

Point Q $\quad x_Q = 0 \quad y_Q = \sqrt{a^2 - 9}$

Point S $\quad x_S = 0 \quad y_S = \sqrt{a^2 - 9} - 3$

Point T $\quad x_T = 0 \quad y_Q = a/2$

We require that OS ≥ OT

$$\sqrt{a^2-9}-3 \ge a/2$$
$$2\sqrt{a^2-9} \ge 6+a$$
$$4\left(a^2-9\right) \ge 36+12a+a^2$$
$$3a^2-12a-72 \ge 0$$
$$a^2-4a-24 \ge 0$$

We need to solve the quadratic equation $a^2-4a-24=0$ to find a

$$a=\left(\tfrac{1}{2}\right)\left(4+\sqrt{16+(4)(1)(24)}\right)=\left(\tfrac{1}{2}\right)\left(4+\sqrt{16+(4)(1)(4)\left(6\right)}\right)$$
$$a=\left(\tfrac{1}{2}\right)\left(4+4\sqrt{1+6}\right)=2+2\sqrt{7}$$

Hence

$$a/2=\frac{u^2}{2g} \ge 1+\sqrt{7}$$ ***QED***