ONLINE: MATHEMATICS EXTENSION 2

Topic 6 MECHANICS

EXERCISE p6402

Consider an object of mass m initially moving with a velocity v_0. It then encounters a resistive force of the form $F_R = -\alpha v^2$ and directed in the opposite direction to the motion. Derive the following relationships:

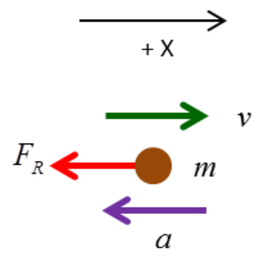
$$v = \frac{v_0}{1 + \left(\frac{\alpha v_0}{m}\right)t} \qquad x = \left(\frac{m}{\alpha}\right)\log_e\left(1 + \left(\frac{\alpha v_0}{m}\right)t\right)$$

$$a = -\left(\frac{\alpha v_0^{\,2}}{m}\right)\frac{1}{\left(1 + \left(\frac{\alpha v_0}{m}\right)t\right)^2} \qquad x = \left(\frac{m}{\alpha}\right)\log_e\left(\frac{v_0}{v}\right)$$

$$v = v_0\, e^{-\left(\frac{\alpha}{m}\right)x}$$

What are the values of the acceleration a, velocity v and displacement x as time $t \to \infty$?

Solution



The force acting on the object is the resistive force F_R. In our frame of reference, we will take to the right as the positive direction.

The equation of motion of the object is determined from Newton's Second Law.

$$m\,a = m\frac{dv}{dt} = F_R = -\alpha\,v^2$$

where a is the acceleration of the object at any instance.

The initial conditions are $t = 0 \quad v = v_0 \quad x = 0$

We start with the equation of motion then integrate this equation where the limits of the integration are determined by the initial conditions (t = 0 and $v = v_0$) and final conditions (t and v)

$$\frac{dv}{dt} = -\left(\frac{\alpha}{m}\right)v^2$$

$$\frac{dv}{v^2} = -\left(\frac{\alpha}{m}\right)dt$$

$$\int_{v0}^{v} v^{-2}\, dv = \int_0^t -\left(\frac{\alpha}{m}\right)dt$$

$$-\left[v^{-1}\right]_{v_0}^{v} = -\left(\frac{\alpha}{m}\right)t$$

$$\frac{-1}{v} + \frac{1}{v_0} = -\left(\frac{\alpha}{m}\right)t$$

$$\frac{1}{v} = \frac{1}{v_0} + \left(\frac{\alpha}{m}\right)t$$

$$\frac{1}{v} = \frac{1}{v_0}\left(1 + \left(\frac{\alpha\, v_0}{m}\right)t\right)$$

$$v = \frac{v_0}{\left(1 + \left(\frac{\alpha\, v_0}{m}\right)t\right)}$$

We can now calculate the displacement x as a functions of time t and velocity v

$$v = \frac{dx}{dt} = \frac{v_0}{\left(1 + \left(\frac{\alpha v_0}{m}\right) t\right)}$$

$$dx = \frac{v_0}{\left(1 + \left(\frac{\alpha v_0}{m}\right) t\right)} dt$$

$$\int_0^x dx = \int_{v_0}^{v} \frac{v_0}{\left(1 + \left(\frac{\alpha v_0}{m}\right) t\right)} dt$$

$$x = \left(\frac{m v_0}{\alpha v_0}\right) \left[\log_e \left(1 + \left(\frac{\alpha v_0}{m}\right) t\right)\right]_0^t$$

$$x = \left(\frac{m}{\alpha}\right) \left[\log_e \left(1 + \left(\frac{\alpha v_0}{m}\right) t\right)\right]$$

Alternatively

$$a = \frac{dv}{dt} = v\frac{dv}{dx} = -\left(\frac{\alpha}{m}\right)v^2$$

$$\frac{dv}{v} = -\left(\frac{\alpha}{m}\right)dx$$

$$\int_{v0}^{v}\frac{dv}{v} = \int_{0}^{x}-\left(\frac{\alpha}{m}\right)dx$$

$$\left[\log_e(v)\right]_{v_0}^{v} = -\left(\frac{\alpha}{m}\right)x$$

$$x = -\left(\frac{m}{\alpha}\right)\log_e\left(\frac{v}{v_0}\right)$$

$$\frac{v}{v_0} = \frac{1}{\left(1+\left(\frac{\alpha\, v_0}{m}\right)t\right)}$$

$$x = \left(\frac{m}{\alpha}\right)\log_e\left(1+\left(\frac{\alpha\, v_0}{m}\right)t\right)$$

The velocity *v* also can be given as a function of *x*

$$x = -\left(\frac{m}{\alpha}\right)\log_e\left(\frac{v}{v_0}\right)$$

$$\log_e\left(\frac{v}{v_0}\right) = -\left(\frac{\alpha}{m}\right)x$$

$$v = v_0\, e^{-\left(\frac{\alpha}{m}\right)x}$$

We can now investigate what happens when $t \to \infty$

$$v = \frac{v_0}{1 + \left(\dfrac{\alpha v_0}{m}\right)t} \qquad t \to \infty \quad v \to 0$$

$$a = -\left(\frac{\alpha v_0{}^2}{m}\right)\frac{1}{\left(1 + \left(\dfrac{\alpha v_0}{m}\right)t\right)^2} \qquad t \to \infty \quad a \to 0$$ ***a* becomes small very rapidly**

$$a\left(t \to \infty\right) = -\left(\frac{m}{\alpha}\right)\frac{1}{t^2} \to 0$$

$$x = \left(\frac{m}{\alpha}\right)\log_e\left(1 + \left(\frac{\alpha v_0}{m}\right)t\right) \qquad t \to \infty \quad x \to \text{bigger}$$

SURPRISING RESULT ***x* gets bigger and bigger with time**

Plots for the parameters: m = 2.0 kg α = 5.0 kg.m^{-1} g = 9.8 m.s^{-2} v_0 = 10.0 m.s^{-1}