HSC MATHEMATICS: MATHEMATICS EXTENSION 1

3 UNIT MATHEMATICS

TOPIC 17   BINOMIAL THEOREM

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n – factorial $n!$

For positive integers n the product of the first n natural numbers is called n – factorial and is written

$$n! = (1)(2)(3) \ldots (n-1)(n)$$

$$0! = 1 \quad 1! = 1 \quad 2! = (1)(2) = 2 \quad 7! = (1)(2)(3)(4)(5)(6)(7) = 5040$$

A **binomial** is a polynomial of two terms such as $2x + 5y$ or $2 - (a+b)^3$.

The **Binomial Theorem** is a quick way of expanding a binomial expression that has been raised to some power n, for example, $(5x+6)^{12}$.

Consider the polynomial $(x+y)^n$ where n is an integer $n=1, 2, 3, \ldots$ This polynomial can be expanded using the **Binomial Theorem** in terms of the

binomial coefficients $\quad {}^{n}C_k \equiv \begin{pmatrix} n \\ k \end{pmatrix} \qquad n=1, 2, 3, \ldots \quad 0 \le k \le n$

$\begin{pmatrix} n \\ k \end{pmatrix}$ is read as 'n over k'

$${}^{n}C_k = \begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!} \qquad 0 \le k \le n \qquad {}^{n}C_k = \begin{pmatrix} n \\ k \end{pmatrix} = {}^{n}C_{n-k} = \begin{pmatrix} n \\ n-k \end{pmatrix} = \frac{n!}{k!(n-k)!}$$

$${}^{n}C_0 = \begin{pmatrix} n \\ 0 \end{pmatrix} = 1 \quad {}^{0}C_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 1 \quad {}^{n}C_n = \begin{pmatrix} n \\ n \end{pmatrix} = 1$$

$${}^{4}C_0 = \begin{pmatrix} 4 \\ 0 \end{pmatrix} = 1 \quad {}^{4}C_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \frac{(4)(3)(2)(1)}{(1)(3)(2)(1)} = 4 = {}^{4}C_3 \quad {}^{4}C_2 = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \frac{(4)(3)(2)(1)}{(1)(2)(2)(1)} = 6$$

$${}^{4}C_3 = \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \frac{(4)(3)(2)(1)}{(1)(2)(3)(1)} = 4 = {}^{4}C_1 \quad {}^{4}C_4 = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 1$$

Pascal's triangle

An easy way to calculate the binomial coefficients is to use **Pascal's Triangle** where the binomial coefficients are arranged in a triangle. Each interior number is the sum of the nearest two numbers in the previous row.

$$1 \qquad\qquad {}^{0}C_0$$

$$1\ 1 \qquad\qquad {}^{1}C_0\ {}^{1}C_1$$

$$1\ 2\ 1 \qquad\qquad {}^{2}C_0\ {}^{2}C_1\ {}^{2}C_2$$

$$1\ 3\ 3\ 1 \qquad\qquad {}^{3}C_0\ {}^{3}C_1\ {}^{3}C_2\ {}^{3}C_3$$

$$1\ 4\ 6\ 4\ 1 \qquad\qquad {}^{4}C_0\ {}^{4}C_1\ {}^{4}C_2\ {}^{4}C_3\ {}^{4}C_4$$

$$1\ 5\ 10\ 10\ 5\ 1 \qquad\qquad {}^{5}C_0\ {}^{5}C_1\ {}^{5}C_2\ {}^{5}C_3\ {}^{5}C_4\ {}^{5}C_5$$

$$1\ 6\ 15\ 20\ 15\ 6\ 1 \qquad {}^{6}C_0\ {}^{6}C_1\ {}^{6}C_2\ {}^{6}C_3\ {}^{6}C_4\ {}^{6}C_5\ {}^{6}C_6$$

adding adjacent numbers gives
the number in a row below e.g.
$$5 + 10 \rightarrow 15$$

$$\binom{6}{6} = 1 \qquad \binom{6}{5} = \frac{(6)(5)(4)(3)(2)(1)}{(5)(4)(3)(2)(1)(1)} = 6 \qquad \binom{6}{4} = \frac{(6)(5)(4)(3)(2)(1)}{(4)(3)(2)(1)(1)(2)} = 15$$

$$\binom{6}{3} = \frac{(6)(5)(4)(3)(2)(1)}{(3)(2)(1)(1)(2)(3)} = 20 \qquad \binom{6}{2} = \frac{(6)(5)(4)(3)(2)(1)}{(2)(1)(1)(2)(3)(4)} = 15$$

$$\binom{6}{1} = \frac{(6)(5)(4)(3)(2)(1)}{(1)(1)(2)(3)(4)(5)} = 6 \qquad \binom{6}{0} = 1$$

The construction of a Pascal Triangle is based upon the relationship between binomial coefficients

$$^n C_k = {}^{n-1} C_{k-1} + {}^{n-1} C_k \qquad 1 \le k \le n-1$$

This is known as **Pascal's Triangle Identity**.

Example $\quad LHS = {}^6 C_2 = 15 \quad RHS = {}^5 C_1 + {}^5 C_2 = 5 + 10 = 15 \quad \Rightarrow LHS = RHS$

Proof $\quad {}^{n}C_{k} = {}^{n-1}C_{k-1} + {}^{n-1}C_{k} \qquad 1 \le k \le n-1$

$${}^{n}C_{k} = \frac{n!}{k!(n-k)!} \qquad 1 \le k \le n$$

$$k = 1 \quad {}^{n}C_{1} = \frac{n!}{1!(n-1)!} = n \qquad {}^{n-1}C_{0} + {}^{n-1}C_{1} = 1 + \frac{(n-1)!}{1!(n-2)!} = 1 + n - 1 = n$$

QED

$k \ge 2$

$$\begin{aligned}
{}^{n-1}C_{k-1} + {}^{n-1}C_{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{(k)!(n-k)!} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!}\left(\frac{1}{n-k} + \frac{1}{k}\right) = \frac{(n-1)!}{(k-1)!(n-k)!}\left(\frac{k+n-k}{(k)(n-k)}\right) \\
&= \frac{n!}{k!(n-k)!} = {}^{n}C_{k} \qquad \text{\textit{QED}}
\end{aligned}$$

The coefficients of the variables x and y in the expansion of $(x+y)^{n}$ are called the binomial coefficients. The $(k+1)^{\text{th}}$ binomial coefficient of order n (n a positive integer) is

$${}^{n}C_{k} = \binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \textbf{(k+1)}^{\textbf{th}} \textbf{ binomial coefficient}$$

${}^{n}C_{k} \equiv \binom{n}{k}$ gives the number of combinations of n things k at a time.

Binomial Theorem

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}\,y + \binom{n}{2}x^{n-2}\,y^2 + \ldots + \binom{n}{n-1}x\,y^{n-1} + y^n$$

$$(x+y)^n = \sum_{k=0}^{k=n}\binom{n}{k}x^{n-k}\,y^k \equiv \sum_{k=0}^{k=n}{}^nC_k\,x^{n-k}\,y^k$$

Example   Expand the binomial expression $(a+b)^6$

$\Rightarrow$ Construct a Pascal Triangle to find the binomial coefficients

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

$\Leftarrow$

The sum of the binomial coefficients is equal to 2^n and is obtained by setting $x=y=1$

$$n=6 \quad (1+1)^6 = 2^6 = \sum_{k=0}^{k=n}\binom{n}{k} = 1+6+15+20+15+6+1 = 64$$

The binomial theorem is proved by **induction**. That is, it is shown to hold for $n = 1$, and further shown that if it holds for any given value of n then it also holds for the next higher value of n.

$$(x + y)^n = \sum_{k=0}^{k=n} \binom{n}{k} x^{n-k} y^k \quad \text{Suppose} \quad (x + y)^n = \sum_{k=0}^{k=n} \binom{n}{k} x^{n-k} y^k \text{ is true. Now consider}$$

$$(x + y)^{n+1} = (x + y)(x + y)^n$$

$$= (x + y) \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n+1-k} y^k + \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^{j+1}$$

$$= \binom{n}{0} x^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^{n+1-k} y^k + \binom{n}{n} y^{n+1} + \sum_{j=0}^{n-1} \binom{n}{j} x^{n-j} y^{j+1}$$

Let $j = k - 1 \quad k = j + 1$

$$(x + y)^{n+1} = x^{n+1} + y^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^{n+1-k} y^k + \sum_{k=1}^{n} \binom{n}{k-1} x^{n+1-k} y^k$$

$$= x^{n+1} + y^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] x^{n+1-k} y^k$$

Using Pascal's Identity $\quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

$$(x + y)^{n+1} = x^{n+1} + y^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^{n+1-k} y^k = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k \qquad \textit{QED}$$

Example (a) Use the binomial theorem to expand $(2+3x)^5$.

(b) What is the 4th term in the expansion for increasing powers of x?

(c) Find the largest coefficient in the expansion.

(d) Expand $(2-3x)^5$

(e) Use the binomial theorem to **differentiate** the function $y=(2+3x)^5$

(f) Use the binomial theorem to **integrate** $y=(2+3x)^5$ from $x=0$ to $x=1$.

$\Rightarrow$ (a)

Always write down the binomial theorem to start answering the question

$$(x+y)^5 = \sum_{k=0}^{k=5} \binom{5}{k} x^{n-k} y^k$$

$$(2+3x)^5 = \sum_{k=0}^{k=5} \binom{5}{k} (2)^{5-k} (3)^k x^k$$

Use Pascal's triangle to give the binomial coefficients $\binom{5}{k}$ $\rightarrow$ 1 5 10 10 5 1

$$(2+3x)^5 = (1)(2^5)(1) + (5)(2^4)(3)x + (10)(2^3)(3^2)x^2 + (10)(2^2)(3^3)x^3 + (5)(2)(3^4)x^4 + (1)(3^5)x^5$$

$$= 32 + 240x + 720x^2 + 1080x^3 + 810x^4 + 243x^5$$

(b) increasing powers of x: x^0 x^1 x^2 x^3 x^4 x^5 4th term $\Rightarrow k = 3$

4th term $t_4 = \dbinom{5}{3}(2)^{5-3}(3)^3 x^3 = 1080 x^3$ in agreement with part (a)

(c) Let the coefficients be represented by a_k. Since x and y are positive, we can check whether the coefficients are increasing or decreasing by considering the ratio a_{k+1}/a_k $0 \le k < n$

If $a_{k+1}/a_k > 1$ $a_{k+2}/a_{k+1} < 1$ $a_k < a_{k+1} > a_{k+2}$ $\Rightarrow$ a_{k+1} is the largest coefficient

$$a_{k+1}/a_k = \dfrac{\dbinom{5}{k+1}(2^{5-k-1})(3^{k+1})}{\dbinom{5}{k}(2^{5-k})(3^k)} = \dfrac{(5-k)}{(k+1)}\left(\dfrac{3}{2}\right) > 1 \Rightarrow k < 2.6 \quad k = 2 \quad k+1 = 3$$

Therefore $a_3 = 1080$ is the largest coefficient in agreement with part (a).

(d) $(x+y)^5 = \displaystyle\sum_{k=0}^{k=5}\binom{5}{k} x^{n-k} y^k$ $(2-3x)^5 = \displaystyle\sum_{k=0}^{k=5}\binom{5}{k}(2)^{5-k}(-1)^k (3)^k x^k$

$(2-3x)^5 = (1)(2^5)(1) - (5)(2^4)(3)x + (10)(2^3)(3^2)x^2 - (10)(2^2)(3^3)x^3 + (5)(2)(3^4)x^4 - (1)(3^5)x^5$

$\qquad = 32 - 240x + 720x^2 - 1080x^3 + 810x^4 - 243x^5$

(e)

$$y = \left(2 + 3x\right)^5 \quad dy/dx = (5)(3)(2+3x)^4$$

$$(x+y)^4 = \sum_{k=0}^{k=4} \binom{4}{k} x^{n-k} y^k \quad \text{Pascal's triangle} \quad n=4 \quad \rightarrow 1\ 4\ 6\ 4\ 1$$

$$dy/dx = (15)\left(2^4 + (4)(2)^3(3x) + (6)(2)^2(3x)^2 + (4)(2)(3x)^3 + (3x)^4\right)$$

$$dy/dx = 240 + 1440x + 3240x^2 + 3240x^3 + 1215x^4$$

$$y = \left(2+3x\right)^5 = 32 + 240x + 720x^2 + 1080x^3 + 810x^4 + 243x^5$$

$$dy/dx = 240 + 1440x + 3240x^2 + 3240x^3 + 1215x^4 \quad QED$$

(f)

$$y = \left(2+3x\right)^5 \quad I = \int_0^1 \left(2+3x\right)^5\, dx$$

$$I = \left(\frac{1}{18}\right)\left[\left(2+3x\right)^6\right]_0^1 = \left(\frac{1}{18}\right)\left(5^6 - 2^6\right) = 864.5$$

$$y = \left(2+3x\right)^5 = 32 + 240x + 720x^2 + 1080x^3 + 810x^4 + 243x^5$$

$$I = \int_0^1 \left(32 + 240x + 720x^2 + 1080x^3 + 810x^4 + 243x^5\right)dx$$

$$I = \left[32x + 120x^2 + 240x^3 + 270x^4 + 162x^5 + 40.5x^6\right]_0^1$$

$$I = 32 + 120 + 240 + 270 + 162 + 40.5 = 864.5 \quad QED$$

$\Leftarrow$

Example Find the coefficients of x^4 and x^3 in the expansion of $\left(2x - \dfrac{1}{x}\right)^6$

Check your answer by expanding the expression using the binomial theorem.

$\Rightarrow$

$$(x+y)^6 = \sum_{k=0}^{k=6} \binom{6}{k} x^{6-k} y^k \qquad \text{Pascal's triangle} \qquad n=6 \quad \rightarrow 1\ 6\ 15\ 20\ 15\ 6\ 1$$

$$\left(2x - \frac{1}{x}\right)^6 = \sum_{k=0}^{k=6} \binom{6}{k}(2)^{6-k}\left(x^{6-k}\right)(-1)^k x^{-k} = \sum_{k=0}^{k=6} \binom{6}{k}(-1)^k (2)^{6-k}\left(x^{6-2k}\right)$$

For the term in x^4 we require $6-2k=4 \Rightarrow k=1$, therefore, the coefficient $a_{k=1}$ is

$$a_{k=1} = \binom{6}{1}(2)^5(-1) = -(6)(32) = -192$$

For the term in x^3 we require $6-2k=3 \Rightarrow k=1.5$, k is not an integer, there is no term in x^3

$$\left(2x - \frac{1}{x}\right)^6 = 64x^6 - 192x^4 + 240x^2 - 160 + 60/x^2 - 12/x^4 + 1/x^6$$

$\Leftarrow$

Example Find the term in the expansion of $(3 + 5x)^{20}$ with the greatest coefficient.

$\Rightarrow$

Always write down the binomial theorem to start answering the question

$$(x + y)^{20} = \sum_{k=0}^{k=20} \binom{20}{k} x^{n-k} y^k \qquad (3 + 5x)^{20} = \sum_{k=0}^{k=20} \binom{20}{k} (3)^{20-k} (5)^k x^k$$

In the expansion of $(x + y)^n$ where x and y are both positive, then successive coefficients in the expansion get larger and then smaller. Therefore, the ratio R of the $(k+1)^{\text{th}}$ coefficient to the kth coefficient will exceed one or be equal to one until the largest term is reached.

If $\quad a_{k+1} / a_k > 1 \qquad a_{k+2} / a_{k+1} < 1 \qquad a_k < a_{k+1} > a_{k+2} \quad \Rightarrow \quad a_{k+1}$ is the largest coefficient

$$R = \frac{\dfrac{20!}{(k+1)!(20-k-1)!} 3^{20-k-1} 5^{k+1}}{\dfrac{20!}{k!(20-k)!} 3^{20-k} 5^k} = \left(\frac{5}{3}\right)\left(\frac{1}{k+1}\right)(20-k) \geq 1$$

$$100 - 5k \geq 3k + 3 \quad 8k \leq 97 \quad k \leq 12.125 \quad \Rightarrow \quad k = 12 \quad k + 1 = 13$$

The largest term is term is $\binom{20}{13} 3^7 5^{13} x^{13}$

$\Leftarrow$

Example

(a) Expand $(x+1)^n$ substitute $x=1$

(b) Expand $(x-1)^n$ substitute $x=1$ hence show that

$$^nC_1 +\,^nC_3 +\,^nC_5 +\ldots = \,^nC_0 +\,^nC_2 +\,^nC_4 +\ldots$$

(c) Show that $2^{n-1} = \,^nC_1 +\,^nC_3 +\,^nC_5 +\ldots = \,^nC_0 +\,^nC_2 +\,^nC_4 +\ldots$

(d) Prove $^{n+1}C_k = \,^nC_k +\,^nC_{k-1}$ which gives the Pascal's triangle identity

(e) Show that $\displaystyle \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

$\Rightarrow$

Always write down the binomial theorem to start answering the question

$$(x+y)^n = \sum_{k=0}^{k=n} \binom{n}{k} x^{n-k}\, y^k$$

(a) $\displaystyle (x+1)^n = \sum_{k=0}^{k=n} \binom{n}{k} x^{n-k}$ $x=1 \;\Rightarrow\; 2^n = \sum_{k=0}^{k=n} \binom{n}{k} = \,^nC_0 +\,^nC_1 +\,^nC_2 +\ldots +\,^nC_n$

(b) $\displaystyle (x-1)^n = \sum_{k=0}^{k=n} \binom{n}{k} (-1)^k\, x^{n-k}$

$$x = 1 \quad \Rightarrow \quad 0 = \sum_{k=0}^{k=n} \binom{n}{k}(-1)^k = {}^n C_0 - {}^n C_1 + {}^n C_2 - {}^n C_3 \ldots + (-1)^n \, {}^n C_n$$

$$^n C_1 + {}^n C_3 + {}^n C_5 + \ldots = {}^n C_0 + {}^n C_2 + {}^n C_4 + \ldots$$

(c) Add the results from parts (a) and (b)

$$2^n = 2\left({}^n C_0 + {}^n C_2 + {}^n C_4 + \ldots \right) \quad \Rightarrow$$

$$2^{n-1} = {}^n C_0 + {}^n C_2 + {}^n C_4 + \ldots = {}^n C_1 + {}^n C_3 + {}^n C_5 + \ldots$$

(d) $(x+1)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k}$ coefficient for x^{n+1-k} is $\binom{n+1}{k}$

$$(x+1)^{n+1} = (x+1)(x+1)^n = (x+1)\sum_{k=0}^{n}\binom{n}{k}x^{n-k} = \sum_{k=0}^{n}\binom{n}{k}x^{n+1-k} + \sum_{k=0}^{n}\binom{n}{k}x^{n-k}$$

coefficient for x^{n+1-k} is $\binom{n}{k} + \binom{n}{k-1}$

Therefore, $^{n+1}C_k = {}^n C_k + {}^n C_{k-1}$ $\binom{n}{k} \equiv {}^n C_k$

(e) $\quad (x+1)^{2n} = \displaystyle\sum_{k=0}^{k=2n} \binom{2n}{k} x^{2n-k}$

We are interested in the term $\binom{2n}{n} x^n$ when $k = n$.

Also, $\quad (x+1)^n = \displaystyle\sum_{k=0}^{k=n} \binom{n}{k} x^{n-k} \qquad (x+1)^n (x+1)^n = \displaystyle\sum_{k=0}^{k=n} \binom{n}{k} x^{n-k} \displaystyle\sum_{k=0}^{k=n} \binom{n}{k} x^{n-k}$

The term in x^n from the expansion expressed as a product is

$$\sum_{k=0}^{n} \left[\binom{n}{k} x^{n-k} \binom{n}{n-k} x^k \right]$$

x^n is formed by taking the product of terms in x^{n-k} from the first expression and x^k from the second expression in the product.

$$\binom{n}{k} = \binom{n}{n-k} \qquad \sum_{k=0}^{n} \left[\binom{n}{k} x^{n-k} \binom{n}{n-k} x^k \right] = \sum_{k=0}^{n} \left[\binom{n}{k}^2 \right] x^n$$

Hence, equating the coefficients of the terms for x^n

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

$\Longleftarrow$