ONLINE: MATHEMATICS EXTENSION 2

Topic 2   COMPLEX NUMBERS

EXERCISE p2301

p001

Prove the following relationships

$$\cos^4(\theta) = \left(\frac{1}{8}\right)\left[\cos(4\theta) + 4\cos(2\theta) + 3\right]$$

$$\sin^4(\theta) = \left(\frac{1}{8}\right)\left[\cos(4\theta) - 4\cos(2\theta) + 3\right]$$

p002

Show that

$$\frac{\left(\sqrt{3}+i\right)^{6}\left(1+i\sqrt{3}\right)^{4}}{\left(\sqrt{3}-i\right)^{4}\left(1-i\sqrt{3}\right)^{3}}=8$$

p003

Find the cubic roots of 2. Plot the roots on an Argand diagram.

p004

Find the fourth roots of $3+2\,i$. Plot the roots on an Argand diagram.

p005

Prove the following

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \qquad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \qquad \tan\theta = -i\left(\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}\right)$$

p006

Using the results of exercise (5) show

$$\tan(\theta/2) = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$$

$$\sin(\theta/2) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos(\theta/2) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

p007

Solve the equation $z^5 - 1 = 0$

ANSWERS

a001

(a)

Prove the following relationships

$$\cos^4(\theta) = \left(\frac{1}{8}\right)\left[\cos(4\theta) + 4\cos(2\theta) + 3\right]$$

$$\sin^4(\theta) = \left(\frac{1}{8}\right)\left[\cos(4\theta) - 4\cos(2\theta) + 3\right]$$

$$\left[\cos(\theta) + i\sin(\theta)\right]^2 = e^{i(2\theta)} = \cos(2\theta) + i\sin(2\theta)$$

$$\left[\cos(\theta) + i\sin(\theta)\right]^2 = \left[\cos^2(\theta) - \sin^2(\theta)\right] + i\left[2\cos(\theta)\sin(\theta)\right]$$

Equating real and imaginary parts

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$\sin(2\theta) = 2\cos(\theta)\sin(\theta)$$

Binomial Theorem: a very useful formula to remember for the expansion a function of the form $(a + b)^n$ where $n = 1, 2, 3, \ldots$

$$(a+b)^n = a^n + \frac{n}{1!}a^{(n-1)}b^1 + \frac{n(n-1)}{2!}a^{(n-2)}b^2 + \frac{n(n-1)(n-2)}{3!}a^{(n-3)}b^3 + \ldots$$

$$\left(a+b\right)^4 = a^4 + 4\,a^3\,b^1 + \frac{(4)(3)}{(2)(1)}a^2\,b^2 + \frac{(4)(3)(2)}{(3)(2)(1)}a\,b^3 + \frac{(4)(3)(2)(1)}{(4)(3)(2)(1)}\,b^4$$

$$\left(a+b\right)^4 = a^4 + 4\,a^3\,b^1 + 6\,a^2\,b^2 + 4\,a\,b^3 + \,b^4$$

$$\left[\cos\left(\theta\right) + i\,\sin\left(\theta\right)\right]^4 = e^{\,i\,(4\theta)} = \cos\left(4\theta\right) + i\,\sin\left(4\theta\right)$$

$$\left[\cos\left(\theta\right) + i\,\sin\left(\theta\right)\right]^4$$

$$= \cos^4\left(\theta\right) + (i)(4)\cos^3\left(\theta\right)\sin\left(\theta\right) + (i)^2\,(6)\cos^2\left(\theta\right)\sin^2\left(\theta\right) + (i)^3(4)\cos\left(\theta\right)\sin^3\left(\theta\right) + (i)^4\sin^4\left(\theta\right)$$

$$= \left[\cos^4\left(\theta\right) + \sin^4\left(\theta\right) - (6)\cos^2\left(\theta\right)\sin^2\left(\theta\right)\right] + i\left[(4)\cos^3\left(\theta\right)\sin\left(\theta\right) - (4)\cos\left(\theta\right)\sin^3\left(\theta\right)\right]$$

Equating real and imaginary parts

$$\cos\left(4\theta\right) = \cos^4\left(\theta\right) + \sin^4\left(\theta\right) - (6)\cos^2\left(\theta\right)\sin^2\left(\theta\right)$$

$$\sin\left(4\theta\right) = (4)\cos^3\left(\theta\right)\sin\left(\theta\right) - (4)\cos\left(\theta\right)\sin^3\left(\theta\right)$$

$$\cos(4\theta) + 4\cos(2\theta) + 3$$

$$= \quad \cos^4(\theta) + \sin^4(\theta) - (6)\cos^2(\theta)\sin^2(\theta)$$

$$+ 4\left[\cos^2(\theta) - \sin^2(\theta)\right) + 3$$

$$= \cos^4(\theta) + \left[1 - \cos(\theta)\right]^2 - 6\cos^2(\theta)\left[1 - \cos^2(\theta)\right]$$

$$+ 8\cos^2(\theta) - 1$$

$$= 8\cos^4(\theta)$$

$$\cos^4(\theta) = \left(\frac{1}{8}\right)\left[\cos(4\theta) + 4\cos(2\theta) + 3\right]$$

$$\cos^4(\theta) = \left[1 - \sin^2(\theta)\right]^2 = 1 - 2\sin^2(\theta) + \sin^4(\theta)$$

$$= \cos^2(\theta) + \sin^2(\theta) - 2\sin^2(\theta) + \sin^4(\theta)$$

$$= \cos^2(\theta) - \sin^2(\theta) + \sin^4(\theta)$$

$$= \cos(2\theta) + \sin^4(\theta)$$

$$\sin^4(\theta) = \left(\frac{1}{8}\right)\left[\cos(4\theta) - 4\cos(2\theta) + 3\right]$$

a002

Show that

$$\frac{\left(\sqrt{3}+i\right)^6\left(1+i\sqrt{3}\right)^4}{\left(\sqrt{3}-i\right)^4\left(1-i\sqrt{3}\right)^3}=8$$

$$\frac{\left(\sqrt{3}+i\right)^6\left(1+i\sqrt{3}\right)^4}{\left(\sqrt{3}-i\right)^4\left(1-i\sqrt{3}\right)^3}=8$$

$$z_1=\sqrt{3}+i=2\,e^{i\pi/6}$$

$$z_1{}^6=2^6\,e^{i\pi}$$

$$z_2=1+i\sqrt{3}=2\,e^{i\pi/3}$$

$$z_2{}^4=2^4\,e^{i\,4\pi/3}$$

$$z_3=\sqrt{3}-i=2\,e^{-i\pi/6}$$

$$z_3{}^{-4}=2^{-4}\,e^{i\,2\pi/3}$$

$$z_4=1-\sqrt{3}i=2\,e^{-i\pi/3}$$

$$z_3{}^{-3}=2^{-3}\,e^{i\pi}$$

$$z_1\,z_2\,z_3\,z_4=2^{(6+4-4-3)}\,e^{i\left(\pi+4\pi/3+2\pi/3+\pi\right)}=2^3\,e^{i\left(\pi+4\pi/3+2\pi/3+\pi\right)}=8$$

a003

Find the cubic root of 2. Plot the roots on an Argand diagram.

$$\sqrt[3]{2}R = ?$$

$$z = 2\left[\cos(2\pi) + i\sin(2\pi)\right]$$

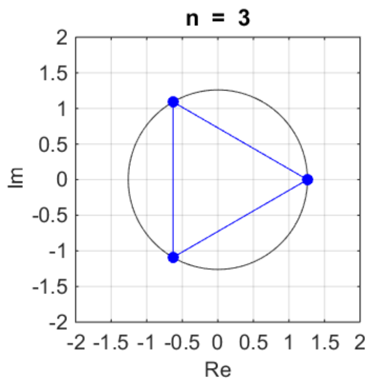
$$= 2\left[\cos(2\pi k) + i\sin(2\pi k)\right] \quad k = 0, 1, 2$$

$$w_k = z^{1/3} = 2^{1/3}\left[\cos(2\pi k/3) + i\sin(2\pi k/3)\right]$$

$$w_0 = 2^{1/3}\left[\cos(0) + i\sin(0)\right] = 2^{1/3}$$

$$w_1 = 2^{1/3}\left[\cos(2\pi/3) + i\sin(2\pi/3)\right] = 2^{1/3}\left[-1/2 + i\left(\sqrt{3}/2\right)\right]$$

$$w_2 = 2^{1/3}\left[\cos(4\pi/3) + i\sin(4\pi/3)\right] = 2^{1/3}\left[-1/2 - i\left(\sqrt{3}/2\right)\right]$$



a004

Find the fourth roots of $3+2\,i$. Plot the roots on an Argand diagram.

There are **four** roots.

$$\sqrt[4]{3+2\,i}=?$$

$$z=3+2\,i \qquad |z|=\sqrt{3^2+2^2}=13^{1/2} \qquad \theta=\arg(z)=\operatorname{atan}(2/3)=0.5880\text{ rad}$$

$$z=13^{1/2}\left[\cos(\theta)+i\sin(\theta)\right]=13^{1/2}\,e^{\,i\,\theta}=13^{1/2}\,e^{\,i\,(\theta+2\pi k)} \qquad k=0,1,2,3$$

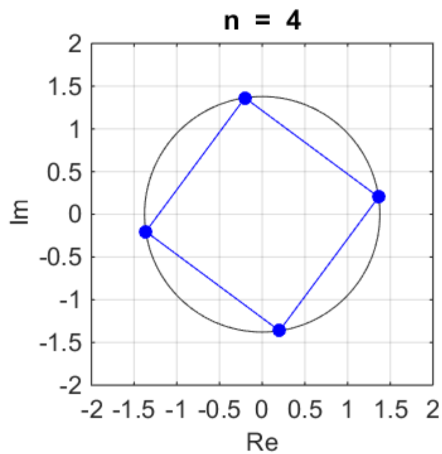
$$w_k=z^{1/4}=13^{1/8}\,e^{\,i\,(\theta/4+\pi k/2)} \qquad \theta/4=0.1651\text{ rad}$$

$$w_0=13^{1/8}\left[\cos(\theta/4)+i\sin(\theta/4)\right]=13^{1/8}\left[0.9892+i\,(0.1465)\right]$$

$$w_1=13^{1/8}\left[\cos(\theta/4+\pi/2)+i\sin(\theta/4+\pi/2)\right]=13^{1/8}\left[-0.1465+i\,(0.9892)\right]$$

$$w_2=13^{1/8}\left[\cos(\theta/4+\pi)+i\sin(\theta/4+\pi)\right]=13^{1/8}\left[-0.9892+i\,(-0.1465)\right]$$

$$w_3=13^{1/8}\left[\cos(\theta/4+3\pi/2)+i\sin(\theta/4+3\pi/2)\right]=13^{1/8}\left[0.1465+i\,(-0.9892)\right]$$


n = 4
Im
Re
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2

a005

Prove the following

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \qquad \sin\theta = \frac{1}{2\,i}\left(e^{i\theta} - e^{-i\theta}\right) \qquad \tan\theta = -i\left(\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \qquad \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta \qquad \sin\theta = \frac{1}{2\,i}\left(e^{i\theta} - e^{-i\theta}\right)$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = -i\left(\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}\right)$$

a006

Using the results of exercise (5) show

$$\tan(\theta/2) = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$$

$$\sin(\theta/2) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos(\theta/2) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = -i\left(\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}\right)$$

Replace θ by $\theta/2$

$$\tan(\theta/2) = -i\left(\frac{e^{i(\theta/2)} - e^{-i(\theta/2)}}{e^{i(\theta/2)} + e^{-i(\theta/2)}}\right) = -i\left(\frac{e^{i(\theta/2)} - e^{-i(\theta/2)}}{e^{i(\theta/2)} + e^{-i(\theta/2)}}\right)\left(\frac{e^{i(\theta/2)}}{e^{i(\theta/2)}}\right)$$

$$= -i\left(\frac{e^{i\theta} - 1}{e^{i\theta} + 1}\right) = -i\,\frac{z_1}{z_2}$$

$$z_1 = e^{i\theta} - 1 = (\cos\theta - 1) + i\sin\theta$$

$$z_2 = e^{i\theta} + 1 = (\cos\theta + 1) + i\sin\theta \quad \overline{z}_2 = e^{i\theta} + 1 = (\cos\theta + 1) - i\sin\theta$$

$$z = \frac{z_1}{z_2} = \frac{z_1\,\overline{z}_2}{z_2\,\overline{z}_2}$$

$$z_2\,\overline{z}_2 = (\cos\theta + 1)^2 + \sin^2\theta = \cos^2\theta + 2\cos\theta + \sin^2\theta = 2\left(1 + \cos\theta\right)$$

$$z = \frac{\left((\cos\theta - 1) + i\sin\theta\right)\left((\cos\theta + 1) - i\sin\theta\right)}{2\left(1 + \cos\theta\right)}$$

$$= \frac{(\cos\theta - 1)(\cos\theta + 1) - \sin^2\theta + i\left(-(\cos\theta - 1)\sin\theta + (\cos\theta + 1)\sin\theta\right)}{2\left(1 + \cos\theta\right)}$$

$$= \frac{\cos^2\theta + 1 - \sin^2\theta + i\left(-\sin\theta\cos\theta + \sin\theta + \sin\theta\cos\theta + \sin\theta\right)}{2\left(1 + \cos\theta\right)}$$

$$= \frac{i\sin\theta}{1 + \cos\theta}$$

$$\tan(\theta/2) = -i\,z$$

$$\tan(\theta/2) = \frac{\sin\theta}{1 + \cos\theta}$$

$$\tan\left(\theta/2\right) = \frac{\sin\theta}{1+\cos\theta} = \left(\frac{\sin\theta}{1+\cos\theta}\right)\left(\frac{1-\cos\theta}{1-\cos\theta}\right)$$

$$= \frac{\sin\theta\left(1-\cos\theta\right)}{1-\cos^2\theta} = \frac{\sin\theta\left(1-\cos\theta\right)}{\sin^2\theta}$$

$$= \frac{\left(1-\cos\theta\right)}{\sin\theta}$$

$$\tan^2(\theta/2) = \frac{(1-\cos\theta)^2}{\sin^2\theta}$$

$$\frac{\sin^2(\theta/2)}{\cos^2(\theta/2)} = \frac{(1-\cos\theta)^2}{\sin^2\theta}$$

$$\sin^2(\theta/2) = \left[\frac{(1-\cos\theta)^2}{\sin^2\theta}\right]\cos^2(\theta/2) = \left[\frac{(1-\cos\theta)^2}{\sin^2\theta}\right]\left(1-\sin^2(\theta/2)\right)$$

$$\sin^2(\theta/2)\left[1+\frac{(1-\cos\theta)^2}{\sin^2\theta}\right] = \left[\frac{(1-\cos\theta)^2}{\sin^2\theta}\right]$$

$$\sin^2(\theta/2)\left(\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta\right) = (1-\cos\theta)^2$$

$$\sin^2(\theta/2) = \frac{(1-\cos\theta)^2}{2(1-\cos\theta)} = \frac{(1-\cos\theta)}{2} = 1-\cos^2(\theta/2)$$

$$\sin(\theta/2) = \pm\sqrt{\frac{(1-\cos\theta)}{2}}$$

$$\cos(\theta/2) = \pm\sqrt{\frac{(1+\cos\theta)}{2}}$$

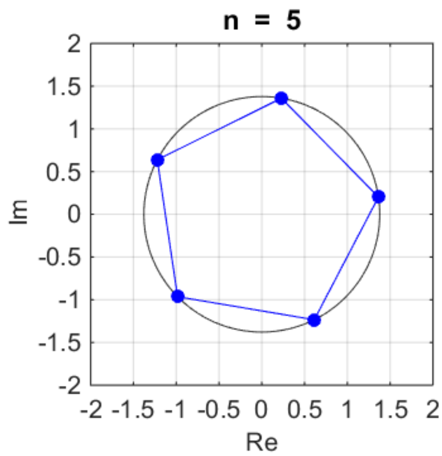
a007

Solve the equation $z^5 - 1 = 0$

There are **five** roots for z.

$$z^5 - 1 = 0$$
$$z^5 = 1 = \cos(2\pi k) + i \sin(2\pi k) \qquad k = 0,1,2,3,4$$
$$z_k = \cos(2\pi k/5) + i \sin(2\pi k/5)$$
$$z_0 = 1$$
$$z_1 = \cos(2\pi/5) + i \sin(2\pi/5)$$
$$z_2 = \cos(4\pi/5) + i \sin(4\pi/5)$$
$$z_3 = \cos(6\pi/5) + i \sin(6\pi/5)$$
$$z_4 = \cos(8\pi/5) + i \sin(8\pi/5)$$

When these five complex numbers are plotted on an Argand diagram, they will lie on the circle $x^2 + y^2 = 1$ and be equally spaced with angular separation equal to $2\pi/5$ rad = 72^o.


n = 5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
Im
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
Re