ONLINE: MATHEMATICS EXTENSION 2

Topic 2 COMPLEX NUMBERS

2.2 ARITHMETIC OPERATIONS WITH COMPLEX VARIABLES

You need to gain the ability to add, subtract, multiply and divide complex numbers.

The addition, subtraction, multiplication or division of two complex variables z_1 and z_2

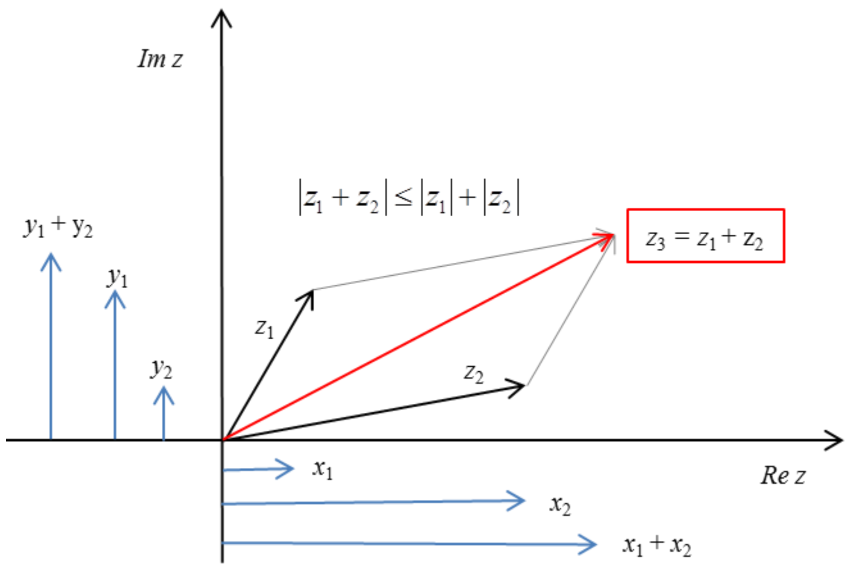
$$z_1 = x_1 + i\ y_1 \qquad z_2 = x_2 + i\ y_2 \qquad x_1, y_1, x_2, y_2 \quad \text{real numbers}$$

produces other complex variables

Addition

$$z_3 = z_1 + z_2 = x_1 + i\ y_1 + x_2 + i\ y_2$$
$$z_3 = (x_1 + x_2) + i\,(y_1 + y_2)$$

add real parts add imaginary parts



The vector representing $z_3 = z_1 + z_2$ corresponds to the diagonal of a parallelogram with the vectors z_1 and z_2 as adjacent sides.

From the parallelogram, it is obvious that

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Subtraction

$$z_4 = z_1 - z_2 = x_1 + i\ y_1 - (x_2 + i\ y_2)$$

$$z_4 = (x_1 - x_2) + i\ (y_1 - y_2)$$

subtract real parts   subtract imaginary parts

For subtraction using the Argand diagram simply add the vectors z_1 and $-z_2$

$$z_4 = z_1 - z_2 = z_1 + (-z_2)$$

Multiplication

$$z_5 = z_1\ z_2 = (x_1 + i\ y_1)(x_2 + i\ y_2) = (x_1\ x_2 - y_1\ y_2) + i\ (x_1\, y_2 + x_2\, y_1)$$

Division

$$z_6 = \frac{z_1}{z_2} = \frac{z_1\,\overline{z}_2}{z_2\ \overline{z}_2}$$

Multiply both the numerator and denominator by the complex conjugate of the denominator so that you are now dividing by the real number $z_2\ \overline{z}_2$.

Multiplication or division of two complex numbers is accomplished most easily when they are in exponential form.

$$z_5 = z_1\, z_2 = \left(R_1\, e^{i\,\theta_1}\right)\left(R_2\, e^{i\,\theta_2}\right) = R_1\, R_2\, e^{i\,(\theta_1+\theta_2)} = R_1\, R_2\, \angle\left(\theta_1+\theta_2\right)$$
$$= R_1\, R_2\left[\cos\left(\theta_1+\theta_2\right)+i\,\sin\left(\theta_1+\theta_2\right)\right]$$

$$z_6 = \frac{z_1}{z_2} = \left(\frac{R_1}{R_2}\right) e^{i\,(\theta_1-\theta_2)} = \left(\frac{R_1}{R_2}\right)\angle\left(\theta_1-\theta_2\right)$$
$$= R_1\, R_2\left[\cos\left(\theta_1-\theta_2\right)+i\,\sin\left(\theta_1-\theta_2\right)\right]$$

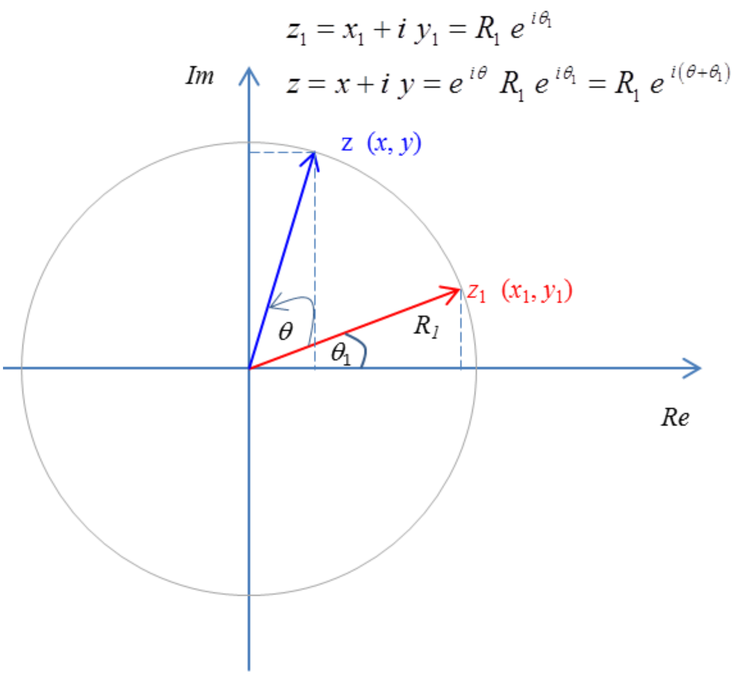
A complex number can be shown as a vector on an Argard diagram.

$$z_1 = x_1 + i\; y_1 = R_1\; e^{\,i\theta_1} \qquad R_1 = \sqrt{x_1^{\;2}+y_1^{\;2}} \qquad \tan\theta_1 = \frac{y_1}{x_1}$$

We will consider the result of multiplying the complex number z_1 by the complex number $e^{\,i\theta}$

$$z = e^{\,i\theta}\; R_1\; e^{\,i\theta_1} = R_1\; e^{\,i(\theta+\theta_1)}$$

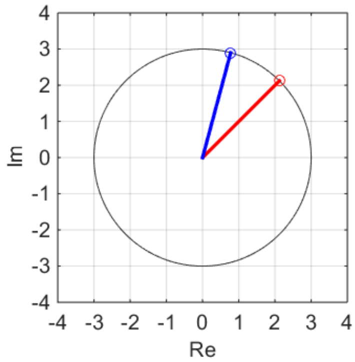
This means that the vector for z_1 is rotated anticlockwise about the origin through an angle θ to produce the vector for the new vector z.



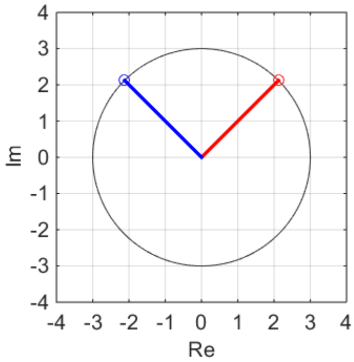
$$z = e^{i\theta}\, z_1 = R_1\, e^{i\theta_1} = R_1\, e^{i(\theta+\theta_1)}$$

Let $z_1 = 3e^{i(\pi/4)} = 3\left[\cos\left(\pi/4\right) + i\sin\left(\pi/4\right)\right] = 2.1213(1+i)$

$$\theta = \pi/6 \quad z = 3\left(e^{i(\pi/6+\pi/4)}\right) = 3e^{i(5\pi/12)}$$
$$= 0.7765 + i\,(2.8978)$$

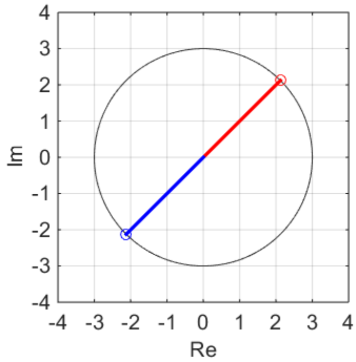


$$\theta = \pi/2 \quad z = \left(e^{i(\pi/2)}\right)\left(3e^{i(\pi/4)}\right)$$
$$z = \left[\cos\left(\pi/2\right) + i\sin\left(\pi/2\right)\right]\left(3e^{i(\pi/4)}\right)$$
$$z = i\left(3e^{i(\pi/4)}\right) = -2.1213 + i\,(2.1213)$$

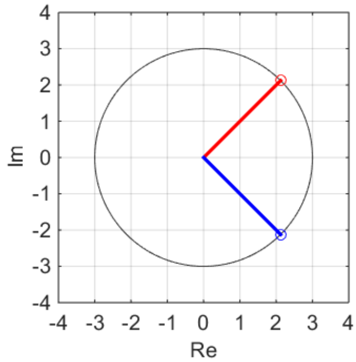


Multiplication by i produces an anticlockwise rotation by $(\pi/2)$ rad.

$$\theta = \pi \quad z = \left(e^{i\pi}\right)\left(3e^{i(\pi/4)}\right)$$
$$z = \left[\cos(\pi) + i\sin(\pi)\right]\left(3e^{i(\pi/4)}\right)$$
$$z = -\left(3e^{i(\pi/4)}\right) = -\left[2.1213 + i\,(2.1213)\right] = -z_1$$



$$\theta = 3\pi/2 \quad z = \left(e^{i(3\pi/2)}\right)\left(3e^{i(\pi/4)}\right)$$
$$z = \left(e^{i(2\pi-\pi/2)}\right)\left(3e^{i(\pi/4)}\right)$$
$$z = \left(e^{i(-\pi/2)}\right)\left(3e^{i(\pi/4)}\right)$$
$$z = \left[\cos\left(-\pi/2\right) + i\sin\left(-\pi/2\right)\right]\left(3e^{i(\pi/4)}\right)$$
$$z = -i\left(3e^{i(\pi/4)}\right) = 2.1213 - i\,(2.1213)$$



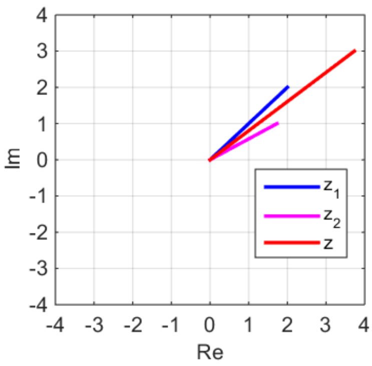
Clockwise rotation of z_1 through $(\pi/2)$ rad.

Example

Consider the two vectors $z_1 = 2 + 2\,i \qquad z_2 = \sqrt{3} + i$

Find $\quad z = z_1 + z_2$

$$z = z_1 + z_2 = \left(2 + \sqrt{3}\right) + \left(2 + 1\right)i = 3.7321 + 3.0000\,i$$



Find the magnitudes and arguments of z_1, z_2 and z

Know $\quad z = x + i\,y \quad |z| = R = \sqrt{x^2 + y^2} \qquad Arg(z) = \theta = a\tan\left(\dfrac{y}{x}\right)$

$$|z_1| = R_1 = \sqrt{2^2 + 2^2} = 2.8284 \qquad Arg(z) = \theta_1 = a\tan\left(\frac{2}{2}\right) = 0.7854 \text{ rad} = 45^{\circ}$$

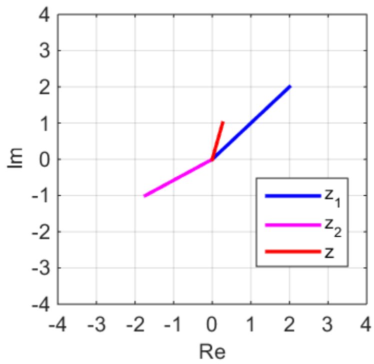
$$|z_2| = R_2 = \sqrt{\sqrt{3} + 1^2} = 2.0000 \qquad Arg(z) = \theta_2 = a\tan\left(\frac{1}{\sqrt{3}}\right) = 0.5236 \text{ rad} = 30^{\circ}$$

$$|z| = R = \sqrt{3.7321^2 + 3^2} = 4.7883 \qquad Arg(z) = \theta = a\tan\left(\frac{3}{3.7321}\right) = 0.6771 \text{ rad} = 38.7940^{\circ}$$

Find $z = z_1 - z_2$

$$z_1 = 2\,x + 2\,i \qquad -z_2 = -\left(\sqrt{3}\,x + i\right)$$

$$z = z_1 - z_2 = z_1 + (-z_2) = \left(2 - \sqrt{3}\right) + \left(2 - 1\right)i$$
$$= 0.28284 + 1.0000\,i$$



- $|z| = |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ hence the magnitude of the complex number $|z_1 - z_2|$ is equal to the distance between the two points $z_1(x_1, y_1)$ and $z_2\ (x_2, y_2)$ on the Argand diagram $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.
- $\theta = Arg\left(z_1 - z_2\right) = \text{atan}\left(\dfrac{y_2 - y_1}{x_2 - x_1}\right)$ but $\left(\dfrac{y_2 - y_1}{x_2 - x_1}\right)$ is the slope of the line joining the two points $z_1(x_1, y_1)$ and $z_2\ (x_2, y_2)$ on the Argand diagram and so, $\theta = Arg\left(z_1 - z_2\right)$ is equal to the angle this line makes with the horizontal direction (line parallel to the real axis) as measured in an anticlockwise sense with respect to the horizontal.

Find $z = z_1\, z_2$

$$z = x + i\,y \qquad z_1 = x_1 + i\,y_1 \qquad z_2 = x_2 + i\,y_2$$

Know $$z = z_1\, z_2 = \left(x_1\, x_2 - y_1\, y_2\right) + i\left(x_1 y_2 + x_2 y_1\right)$$

$$|z| = \sqrt{x^2 + y^2} \qquad Arg(z) = \theta = a\tan\left(\frac{y}{x}\right)$$

$$z = \left((2)\left(\sqrt{3}\right) - (2)(1)\right) + i\left((2)(1) + \left(\sqrt{3}\right)(2)\right) = 1.4641 + 5.4641\, i$$

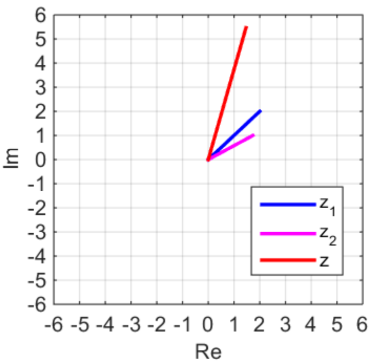
$$|z| = R = \sqrt{1.4641^2 + 5.4641^2} = 5.6569$$

$$Arg(z) = \theta = a\tan\left(\frac{5.4641}{1.4641}\right) = 1.3090 \text{ rad} = 75^{\circ}$$

Know $$z = R\,e^{i\,\theta} = \left(R_1\, e^{i\,\theta_1}\right)\left(R_2\, e^{i\,\theta_2}\right) = R_1\, R_2\; e^{i\,(\theta_1 + \theta_2)}$$

$$z = (2.8284)(2)\,e^{\,i\,(0.7854+0.5236)} = 5.6569\,e^{\,i\,(1.3090)}$$

$$|z| = R = 5.6569 \qquad Arg(z) = \theta = 1.3090 \text{ rad} = 75^{\circ} \qquad \left(75^{\circ} = 45^{\circ} + 30^{\circ}\right)$$



Find $\quad z = \dfrac{z_1}{z_2}$

$$z = x + i\,y \quad z_1 = x_1 + i\,y_1 \quad z_2 = x_2 + i\,y_2$$

Know $\quad z = \dfrac{z_1}{z_2}\dfrac{\overline{z}_2}{\overline{z}_2} = \dfrac{z_1\,\overline{z}_2}{\left|\overline{z}_2\right|^2} = \dfrac{\left(x_1 x_2 + y_1 y_2\right) + i\left(-x_1 y_2 + x_2 y_1\right)}{{x_2}^2 + {y_2}^2}$

$$|z| = \sqrt{x^2 + y^2} \qquad Arg(z) = \theta = a\tan\left(\frac{y}{x}\right)$$

$$x_1 = 2 \quad y_1 = 2 \quad x_2 = \sqrt{3} \quad y^2 = 1 \quad |z_2| = 2$$

$$z = \frac{\left(2\sqrt{3} + 2\right) + i\left(-2 + 2\sqrt{3}\right)}{4} = 1.3660 + 0.3660$$

$$|z| = R = \sqrt{1.3660^2 + 0.3660^2} = 1.4142$$

$$Arg(z) = \theta = a\tan\left(\frac{0.3660}{1.3660}\right) = 0.2618 \text{ rad} = 15^{\circ}$$

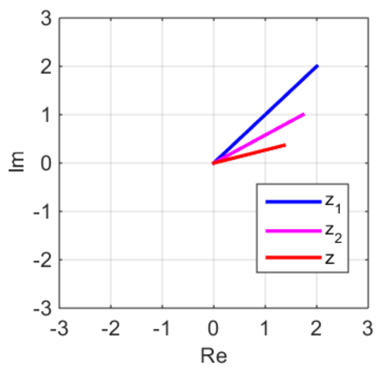
Know $\quad z = R\,e^{i\,\theta} = \left(R_1\,e^{i\,\theta_1}\right)\left(R_2\,e^{i\,\theta_2}\right) = R_1\,R_2\;e^{i\,(\theta_1+\theta_2)}$

$$R_1 = 2.8284 \qquad \theta_1 = 0.7854 \text{ rad} = 45^{\circ}$$

$$R_2 = \sqrt{\sqrt{3} + 1^2} = 2.0000 \qquad \theta_2 = 0.5236 \text{ rad} = 30^{\circ}$$

$$z = \frac{z_1}{z_2} = \left(\frac{R_1}{R_2}\right)e^{\,i\,(\theta_1-\theta_2)} = \left(\frac{2.8284}{2}\right)e^{\,i\,(0.7854-0.5236)} = 1.4142\;e^{\,i\,(0.2618)}$$

$$|z| = R = 1.4142 \qquad Arg(z) = \theta = 0.2618 \text{ rad} = 15^{\circ} \qquad \left(15^{\circ} = 45^{\circ} - 30^{\circ}\right)$$



Modulus and Complex conjugate relationships

- $\left|z_1\, z_2\right| = \left|z_1\right|\left|z_2\right| \qquad \arg\left(z_1\, z_2\right) = \arg\left(z_1\right) + \arg\left(z_2\right)$

  Proof

  $$z_1 = R_1\, e^{\,i\,\theta_1} \qquad z_2 = R_2\, e^{\,i\,\theta_2} \qquad z_1\, z_2 = \left(R_1\, e^{\,i\,\theta_1}\right)\left(R_2\, e^{\,i\,\theta_2}\right) = R_1\, R_2\, e^{\,i\,(\theta_1+\theta_2)}$$
  $$\left|z_1\right| = R_1 \quad \left|z_2\right| = R_2 \quad \left|z_1\, z_2\right| = R_1\, R_2 = \left|z_1\right|\left|z_2\right|$$
  $$\arg\left(z_1\right) = \theta_1 \quad \arg\left(z_2\right) = \theta_2 \quad \arg\left(z_1\, z_2\right) = \theta_1 + \theta_2 = \arg\left(z_1\right) + \arg\left(z_2\right)$$

- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{\left|z_1\right|}{\left|z_2\right|} \qquad \arg\left(\dfrac{z_1}{z_2}\right) = \arg\left(z_1\right) - \arg\left(z_2\right)$

  Proof

  $$z_1 = R_1\, e^{\,i\,\theta_1} \qquad z_2 = R_2\, e^{\,i\,\theta_2} \qquad \frac{z_1}{z_2} = \frac{R_1\, e^{\,i\,\theta_1}}{R_2\, e^{\,i\,\theta_2}} = \frac{R_1}{R_2}\, e^{\,i\,(\theta_1-\theta_2)}$$
  $$\left|z_1\right| = R_1 \quad \left|z_2\right| = R_2 \quad \left|\frac{z_1}{z_2}\right| = \frac{R_1}{R_2} = \frac{\left|z_1\right|}{\left|z_2\right|}$$
  $$\arg\left(z_1\right) = \theta_1 \quad \arg\left(z_2\right) = \theta_2 \quad \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg\left(z_1\right) - \arg\left(z_2\right)$$

- $\left|z^n\right| = \left|z\right|^n \qquad \arg\left(z^n\right) = n\arg\left(z\right)$

  Proof
  $$z = Re^{\,i\,\theta} \qquad z^n = R^n e^{\,i\,n\,\theta}$$
  $$\left|z\right|^n = R^n = \left|z\right|^n \qquad \arg\left(z^n\right) = n\,\theta = n\arg\left(z\right)$$

- $z_1^* + z_2^* = \left(z_1 + z_2\right)^* \qquad \left(z_1\, z_2\right)^* = z_1^*\, z_2^*$ * used for complex conjugate

  Proof

  $$z_1 = x_1 + i\, y_1 \qquad z_2 = x_2 + i\, y_2 \quad z_1 + z_2 = \left(x_1 + x_2\right) + i\left(y_1 + y_2\right)$$
  $$z_1^* = x_1 - i\, y_1 \qquad z_2^* = x_2 - i\, y_2 \quad \left(z_1 + z_2\right)^* = \left(x_1 + x_2\right) - i\left(y_1 + y_2\right)$$
  $$z_1^* + z_2^* = \left(z_1 + z_2\right)^*$$
  $$z_1 = R_1\, e^{\,i\,\theta_1} \quad z_1^* = R_1\, e^{-i\,\theta_1} \qquad z_2 = R_2\, e^{\,i\,\theta_2} \quad z_2^* = R_2\, e^{-i\,\theta_1 2}$$
  $$\left(z_1\, z_2\right)^* = R_1 R_2\, e^{-i\,(\theta_1+\theta_2)} = \left(R_1\, e^{-i\,\theta_1}\right)\left(R_2\, e^{-i\,\theta_1 2}\right) = z_1^*\, z_2^*$$

QUADRATIC FUNCTION

A quadratic function has the general form

$$a\,x^2 + b\,x + c = 0 \qquad a \neq 0$$

and its graph is a **parabola**.

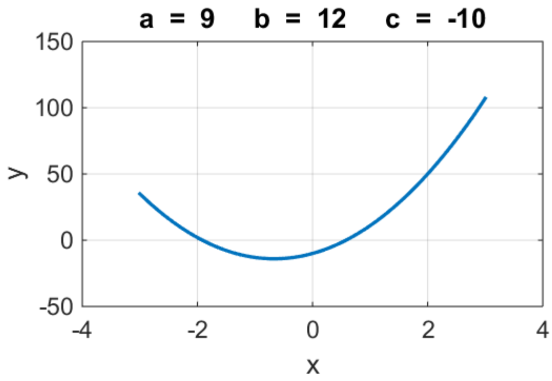
If there are real values of *x* for which the quadratic function $a\,x^2 + b\,x + c = 0$ then the curve will intersect the X axis at the values of *x* given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}$$

Two real roots when the parabola crosses the X axis twice

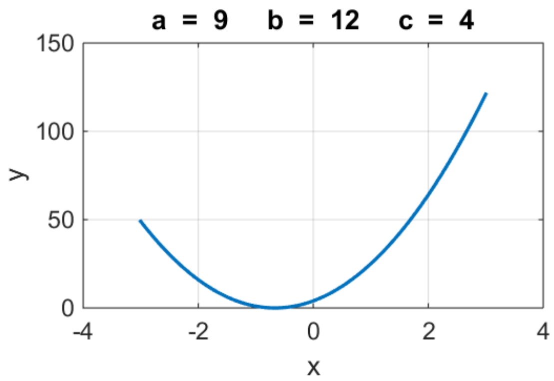
$x_1 = 0.5806$

$x_2 = -1.9139$



When the parabola touches the X axis at one point, there is only one real root

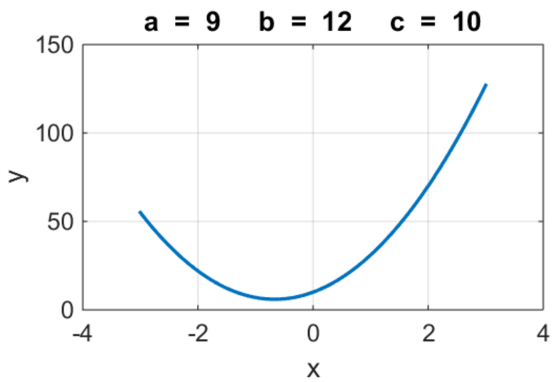
$x_1 = x_2 = -0.6667$



The are no real roots when the parabola does not cut the X axis. The **two** roots are now **imaginary**

$x_1 = -\,0.6667 + i\,(0.8165)$

$x_2 = -1.9139 - i\,(0.8165)$



Example

0ABC is a square on an Argand diagram. The point A is given by the ordered pair (has the ordered $\left(\sqrt{3},1\right)$. Find the complex numbers for the points B and C.

$$z_A = x_A + i\ y_A = \sqrt{3} + i$$

$$\left|z_A\right| = R_A = \sqrt{x_A{}^2 + y_A{}^2} = \sqrt{\sqrt{3}^2 + 1^2} = 2$$

$$Arg(z_A) = \theta_A = a\tan\left(\frac{1}{\sqrt{3}}\right) = 0.5236 \text{ rad} = 30^{\circ}$$

The angle between 0A and OC must be 90°. Multiplication by *i* produces an anticlockwise rotation of 90° (π/2 rad). Therefore, we can determine the coordinates of the point C from

$$z_C = i\ z_A = -1 + \sqrt{3}\ i$$

Hence the order pair for C is $\left(-1,\sqrt{3}\right)$ or $\left(-1, 1.7321\right)$.

$$\left|z_C\right| = R_C = \sqrt{x_C{}^2 + y_C{}^2} = \sqrt{1^2 + \sqrt{3}^2} = 2$$

$$Arg(z_C) = \theta_C = \pi - a\tan\left(\left|\frac{-\sqrt{3}}{1}\right|\right) = 2.0944 \text{ rad} = 120^{\circ}$$

0B must correspond to the diagonal of the square whose side length is $\left|z\right| = R_A = 2$.

Hence, $\left|z_B\right| = R_B = \sqrt{2^2 + 2^2} = 2.8284$

The angle between the side 0A and the diagonal OB is 45° (π/4 rad). Therefore,

$$\theta_B = \theta_A + 45^{\circ} = 30^{\circ} + 45^{\circ} = 1.3090 \text{ rad}$$

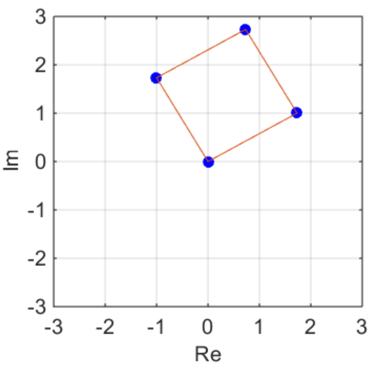
Hence the complex number z_B is

$$z_B = 2.8284\, e^{\,i\,(1.3090)}$$

$$x_B = R_B \cos\theta = \left(2.8284\right)\cos\left(1.3090\right) = 0.7320$$

$$y_B = R_B \sin\theta = \left(2.8284\right)\sin\left(1.3090\right) = 2.7320$$

Hence the order pair for B is $\left(0.7320, 2.7320\right)$.



The square is now rotated about O through an angle of 60° ($\pi/3$ rad) in a clockwise direction. Find the ordered pairs for the vertices of the square A_1, B_1 and C_1.

know multiplication of a complex number z by the complex number $e^{-i\left(\frac{\pi}{3}\right)}$ corresponds to a rotation of the vector representing z on an Argand diagram through an angle (π / 3) rad in a clockwise direction.

$$z_1 = e^{-i\left(\frac{\pi}{3}\right)} z = e^{-i\left(\frac{\pi}{3}\right)} R e^{i\theta} = R e^{i\left(\theta - \frac{\pi}{3}\right)}$$

The new vertices of the square A_1, B_1 and C_1 are determined by the rotation of the vectors representing z_A, z_B and z_C by ($\pi/3$) rad

$$z_{A1} = R_A e^{i\left(\theta_A - \frac{\pi}{3}\right)} = 2\, e^{i\left(0.5236 - \frac{\pi}{3}\right)} = 2\, e^{i(-0.5236)}$$

$$z_{B1} = R_B e^{i\left(\theta_B - \frac{\pi}{3}\right)} = 2.8284\, e^{i\left(1.3090 - \frac{\pi}{3}\right)} = 2.8284\, e^{i(0.2618)}$$

$$z_{C1} = R_C e^{i\left(\theta_C - \frac{\pi}{3}\right)} = 2\, e^{i\left(2.0944 - \frac{\pi}{3}\right)} = 2\, e^{i(1.0472)}$$

$$\theta_{A1} = -0.5236 \text{ rad} = -30^{\circ} \quad \theta_{B1} = 0.2618 \text{ rad} = 15^{\circ} \quad \theta_{C1} = 1.0472 \text{ rad} = 60^{\circ}$$

know The rectangular form of the complex numbers is found from

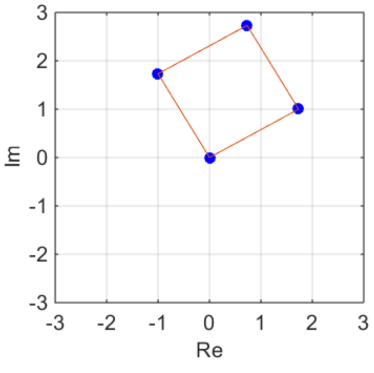
$$x = R\cos\theta \quad y = R\sin\theta$$

$$x_{A1} = 2\cos(-0.5236) = 1.7320 \quad y_{A1} = 2\sin(-0.5236) = -1.0000$$

$$x_{B1} = 2.8284\cos(0.2618) = 2.7320 \quad y_{B1} = 2.8284\sin(0.2618) = 0.7320$$

$$x_{C1} = 2\cos(1.0472) = 1.0000 \quad y_{C1} = 2\sin(1.0472) = 1.7321$$

The order pairs are A_1 (1.7320, -1.0000), B_1 (2.7320, 0.7320), C_1 (1.0000, 1.7321).



rotation by

60° clockwise →