# ONLINE: MATHEMATICS EXTENSION 2

## Topic 2 COMPLEX NUMBERS

## EXERCISE p2101

p001

(a) Specify the real and imaginary parts of the complex number $z$ and the complex conjugate of $z$

$$z = 55 - 22\,i$$

(b) Plot the complex number $z = -3 + 2\,i$ on an Argand diagram (complex plane) and determines its modulus and argument. Do the same for the complex conjugate of $z$.

(c) Convert the complex number $z = 3 - 4\,i$ to polar form and exponential form. Give the polar and exponential forms for the complex conjugate of $z$.

p002

Graph the complex numbers $z_1 = i$ and $z_2 = -i$ on Argand diagram. State the polar and exponential forms of these complex numbers.

p003

Find the rectangular, polar and exponentials form of the complex number

$$z = 6\angle\left(\frac{\pi}{3}\text{ rad}\right)$$

p004

Verify each of the following relationships

(a) $\frac{1}{2}(1+i)^2 = i$

(b) $\sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$

(c) $\sqrt{i} = e^{i\left(\frac{\pi}{4}\right)}$

(d) $\sqrt{-i} = \frac{1}{\sqrt{2}}(1-i)$

(e) $\sqrt{-i} = e^{i\left(-\frac{\pi}{4}\right)}$

p005

Rationalize the complex numbers

$$z_1 = \frac{2}{2+i} \qquad z_2 = \frac{5\,i}{1-2\,i}$$

p006

Find the simplest rectangular form of

$$z_1 = \left(1-i\right)^4 \qquad z_2 = \left(\sqrt{2}-i\right) - i\left(1-i\sqrt{2}\right) \qquad z_3 = \frac{10}{\left(1-i\right)\left(2-i\right)\left(3-i\right)}$$

p007

If $z = -1+i$ show that $z^7 = -8\left(1+i\right)$

# ANSWERS

a001

(a)

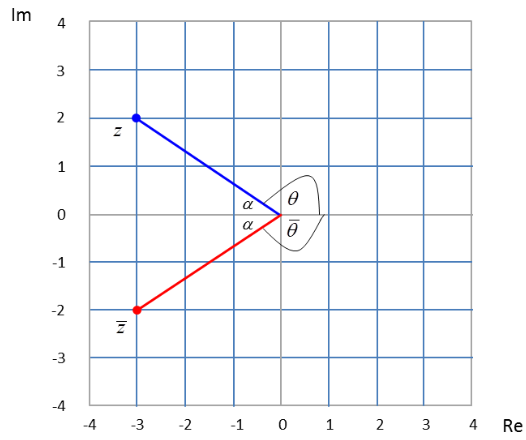
> Specify the real and imaginary parts of the complex number $z$ and the complex conjugate of z
> $$z = 55 - 22\,i$$

$$\mathrm{Re}(z) = 55 \quad \mathrm{Im}(z) = -22 \qquad \mathrm{Re}(\bar{z}) = 55 \quad \mathrm{Im}(\bar{z}) = 22$$

(b)

> Plot the complex number $z$ = -3 + 2 $i$ on an Argand diagram (complex plane) and determines its modulus and argument. Do the same for the complex conjugate of $z$.

$\bar{z}$ is a reflection of $z$ about the Re axis



$$z = x + i\ y \quad \bar{z} = x - i\ y$$

$$|z| = |\bar{z}| = \sqrt{(-3)^2 + 2^2} = 3.6056$$

$$\tan\alpha = \frac{2}{3} \quad \alpha = 0.5580\ \text{rad}$$

$$\theta = Arg(z) = (\pi - \alpha) = 2.5536\ \text{rad} = 146^{\circ}$$

$$\bar{\theta} = Arg(\bar{z}) = (-\pi + \alpha) = -2.5536\ \text{rad} = -146^{\circ}$$

(c)

Convert the complex number $z = 3 - 4\,i$ to polar form and exponential form. Give the polar and exponential forms for the complex conjugate of $z$.

$$z = x + i\ y \quad \bar{z} = x - i\ y \quad z = R\left(\cos\theta + i\ \sin\theta\right) = R\ e^{\,i\,\theta}$$

$$|z| = |\bar{z}| = \sqrt{3^2 + 4^2} = 5$$

$$\tan\alpha = \frac{4}{3} \quad \alpha = 0.9273\ \text{rad}$$

$$\theta = Arg(z) = -\alpha = -0.9273\ \text{rad} = -53^{\circ}$$

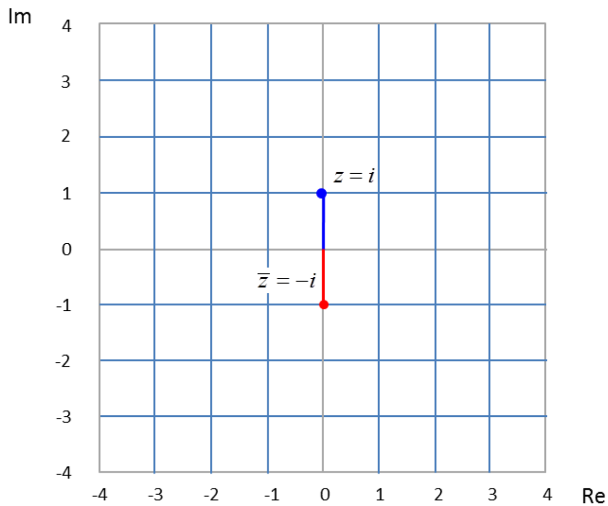
$$\bar{\theta} = Arg(\bar{z}) = \alpha = 0.9273\ \text{rad} = 53^{\circ}$$

$$z = 5\left[\cos(0.9273) - i\ \sin(0.9273)\right] = 5\ e^{\,i\,(-0.9273)}$$

$$\bar{z} = 5\left[\cos(0.9273) + i\ \sin(0.9273)\right] = 5\ e^{\,i\,(0.9273)}$$

a002

Graph the complex numbers $z_1 = i$ and $z_2 = -i$ on Argand diagram. State the polar and exponential forms of the complex numbers.




$$\theta_1 = \frac{\pi}{2} \quad z_1 = \cos\left(\pi/2\right) + i\,\sin\left(\pi/2\right) = e^{i\,\pi/2} = i$$

$$\theta_2 = -\frac{\pi}{2} \quad z_2 = \cos\left(-\pi/2\right) + i\,\sin\left(-\pi/2\right) = e^{i\,(-\pi/2)} = -i$$

a003

Find the rectangular, polar and exponentials form of the complex number

$$z = 6 \angle \left( \frac{\pi}{3} \text{ rad} \right)$$

$$R = 6 \quad \theta = \pi / 3$$

$$x = R\cos\theta = 6\cos(\pi/3) = 3 \qquad y = R\sin\theta = 6\sin(\pi/3) = 5.1962$$

$$z = 3 + i\,(5.1962)$$

$$z = 6\left[\cos(\pi/3) + i\sin(\pi/3)\right] = 6\ e^{\,i(\pi/3)}$$

a004

Verify each of the following relationships

(a) $\frac{1}{2}(1+i)^2 = i$

(b) $\sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$

(c) $\sqrt{i} = e^{i\left(\frac{\pi}{4}\right)}$

(d) $\sqrt{-i} = \frac{1}{\sqrt{2}}(1-i)$

(e) $\sqrt{-i} = e^{i\left(-\frac{\pi}{4}\right)}$

$$\frac{1}{2}(1+i)^2 = \frac{1}{2}(1+2\,i-1) = i$$

$$i = e^{\,i(\pi/2)} \quad \sqrt{i} = \left[e^{\,i(\pi/2)}\right]^{1/2} = e^{\,i(\pi/4)} = \cos(\pi/4) + i\,\sin(\pi/4) = \left(\frac{1}{\sqrt{2}}\right)(1+i)$$

$$-i = e^{\,i(-\pi/2)} \quad \sqrt{-i} = \left[e^{\,i(-\pi/2)}\right]^{1/2} = e^{\,i(\pi/4)} = \cos(-\pi/4) + i\,\sin(-\pi/4) = \left(\frac{1}{\sqrt{2}}\right)(1-i)$$

a005

Rationalize the complex numbers

$$z_1 = \frac{2}{2+i} \qquad z_2 = \frac{5\,i}{1-2\,i}$$

$$z_1 = \frac{2}{2+i} = \left(\frac{2}{2+i}\right)\left(\frac{2-i}{2-i}\right) = \frac{4-2\,i}{5} = \left(\frac{4}{5}\right) - \left(\frac{2}{5}\right) i$$

$$z_2 = \frac{5\,i}{1-2\,i} = \left(\frac{5\,i}{1-2\,i}\right)\left(\frac{1+2\,i}{1+2\,i}\right) = \frac{5\,i-10}{5} = -2+i$$

a006

Find the simplest rectangular form of

$$z_1 = (1-i)^4 \quad z_2 = \left(\sqrt{2}-i\right) - i\left(1 - i\sqrt{2}\right) \quad z_3 = \frac{10}{(1-i)(2-i)(3-i)}$$

$$z_1 = (1-i)^4$$

$$(1-i)^2 = 1 - i - i - 1 = -2\,i$$

$$(1-i)^4 = (-2\,i)^2 = -4$$

$$z_1 = (1-i)^4 = -4$$

$$z_2 = \left(\sqrt{2}-i\right) - i\left(1 - i\sqrt{2}\right) = \sqrt{2} - i - i - \sqrt{2} = -2\,i$$

$$z_3 = \frac{10}{(1-i)(2-i)(3-i)}$$

$$= \frac{10(1+i)(2+i)(3+i)}{(1-i)(2-i)(3-i)(1+i)(2+i)(3+i)} = \frac{(10)(10\,i)}{(2)(5)(10)} = i$$

a007

If $z = -1 + i$ show that $z^7 = -8(1+i)$

$$R = \sqrt{2} \quad \tan(|y/x|) = \tan(1) = \pi/4 \quad \theta = \pi - \pi/4 = 3\pi/4$$

$$z = 2^{1/2}\, e^{i(3\pi/4)} \qquad z^7 = 2^{7/2}\, e^{i(3\pi/4)7} = 8\, e^{i(21\pi/4)} = 8\, e^{i\left(\frac{16+5}{4}\pi\right)} = 8\, e^{i\left(\frac{5}{4}\pi\right)}$$

$$x = 8\cos(5\pi/4) = -8 \quad y = 8\sin(5\pi/4) = -8$$

$$z^7 = -8(1+i)$$