ONLINE: MATHEMATICS EXTENSION 2

Topic 6 MECHANICS

EXERCISE p6502

Consider an object of mass m falling due to gravity. The object was released with an initially velocity v_0. The resistive force due to the medium the object falls through is of the form $F_R = -\beta v$ and directed in the opposite direction to the motion. Derive the following results

$$v_T = \frac{m g}{\beta}$$

$$a = \left(\frac{\beta}{m}\right)\left(v_T - v_0\right)e^{(-\beta/m)t} \qquad v = v_T + \left(v_0 - v_T\right)e^{(-\beta/m)t}$$

$$x = v_T\, t + \left(\frac{m}{\beta}\right)\left(v_T - v_0\right)e^{(-\beta/m)t} \qquad x = \left(\frac{m}{b}\right)\left(\left(v_0 - v\right) + v_T \log_e\left(\frac{v_T - v_0}{v_T - v}\right)\right)$$

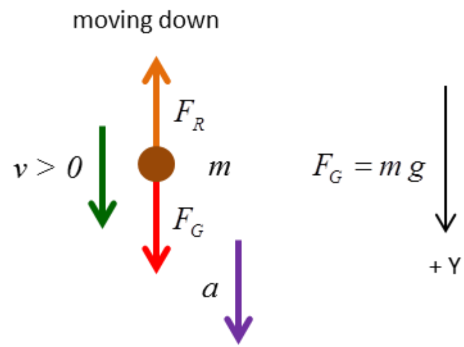
Comment on the acceleration a, velocity v and displacement x as $t \to \infty$?

Sketch graphs for acceleration a, velocity v and displacement x time graphs for

$v_0 > v_T$ $v_0 = 0$ and $v_0 < v_T$ where v_0 is the initial velocity.

Solution

The forces acting on the object are the gravitational force F_G (weight) and the resistive force F_R. In our frame of reference, we will take down as the positive direction.



The equation of motion of the object is determined from Newton's Second Law.

$$m\,a = m\frac{dv}{dt} = F_G - F_R = m\,g - \beta\,v$$

where a is the acceleration of the object at any instance.

The initial conditions are $\quad t = 0 \quad v = v_0 \quad x = 0 \quad a = g - \left(\frac{\beta}{m}\right)v_0$

When $a = 0$, the velocity is constant $v = v_T$ where v_T is the terminal velocity

$$0 = m\,g - \beta\,v_T$$

$$v_T = \frac{m\,g}{\beta}$$

We start with the equation of motion then integrate this equation where the limits of the integration are determined by the initial conditions ($t = 0$ and $v = v_0$) and final conditions (t and v)

$$a = \frac{dv}{dt} = g - \left(\frac{\beta}{m}\right)v = -\left(\frac{\beta}{m}\right)\left(v - \frac{m\,g}{\beta}\right)$$

$$u = v - \frac{m\,g}{\beta} \qquad du = dv \qquad -\left(\frac{\beta}{m}\right)dt = \frac{du}{u}$$

$$-\left(\frac{\beta}{m}\right)\int_0^t dt = \int_{u0}^{u} \frac{du}{u} \qquad -\left(\frac{\beta}{m}\right)t = \left[\log_e\left(u\right)\right]_{v_0 - \frac{m\,g}{\beta}}^{v - \frac{m\,g}{\beta}} = \log_e\left(\frac{v - \frac{m\,g}{\beta}}{v_0 - \frac{m\,g}{\beta}}\right)$$

$$\left(\frac{v - \frac{m\,g}{\beta}}{v_0 - \frac{m\,g}{\beta}}\right) = e^{(-\beta/m)\,t}$$

$$v = \frac{m\,g}{\beta} + \left(v_0 - \frac{m\,g}{\beta}\right)e^{(-\beta/m)\,t} \qquad v_T = \frac{m\,g}{\beta}$$

$$v = v_T + \left(v_0 - v_T\right)e^{(-\beta/m)\,t}$$

$$v_0 = 0 \quad \Rightarrow \quad v = v_T\left(1 - e^{(-\beta/m)\,t}\right)$$

$$v_0 = v_T \Rightarrow \quad v = v_T$$

$$v_0 < v_T \Rightarrow \quad v \text{ increases to } v_T$$

$$v_0 > v_T \Rightarrow \quad v \text{ decreases to } v_T$$

In every case, the velocity *v* tends towards the limiting value *vT*.

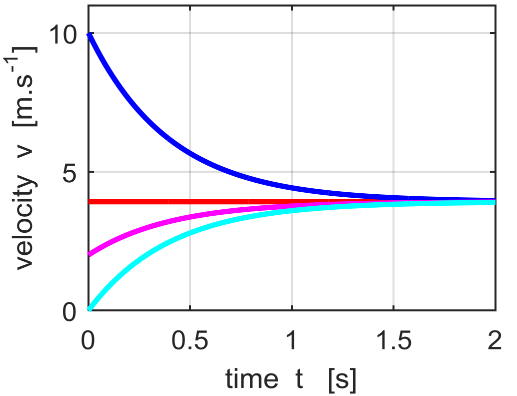
Plots of the velocity *v* as a function of time *t*

m = 2.00 kg

β = 5.00 kg.s^{-1}

g = 9.80 m.s^{-2}

v_T = 3.92 m.s^{-1}



Initial values for velocity v_0 [m.s^{-1}]

blue: 10 red: v_T magenta: 2 cyan: 0

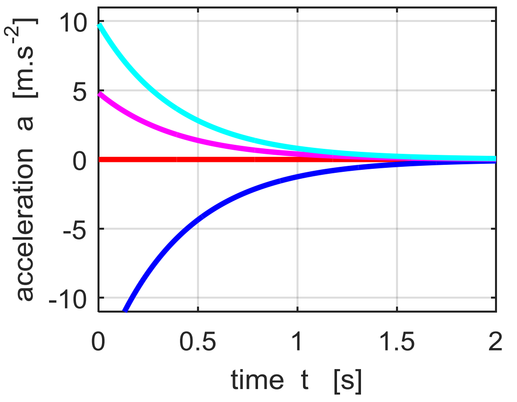
The acceleration a as a function of time t is

$$v = v_T + (v_0 - v_T)e^{(-\beta/m)t}$$

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(v_T + (v_0 - v_T)e^{(-\beta/m)t}\right)$$

$$a = (v_0 - v_T)\left(\frac{-\beta}{m}\right)e^{(-\beta/m)t}$$

$$a = (v_T - v_0)\left(\frac{\beta}{m}\right)e^{(-\beta/m)t}$$



$$v_0 = 0 \quad \Rightarrow \quad a = \left(\frac{\beta\, v_T}{m}\right)e^{(-\beta/m)t}$$

$$\Rightarrow \quad a = g\, e^{(-\beta/m)t}$$

$$v_0 = v_T \Rightarrow \quad a = 0$$

$$v_0 < v_T \Rightarrow \quad a > 0 \quad \text{and decreases to 0}$$

$$v_0 > v_T \Rightarrow \quad a < 0 \quad \text{and } a \text{ increases to 0}$$

$$t \to \infty \Rightarrow \quad a \to 0$$

Initial values for velocity v_0 [m.s^{-1}]

blue: 10 red: v_T magenta: 2 cyan: 0

We can now calculate the displacement x as a function of velocity t

$$v = v_T + (v_0 - v_T)\, e^{(-\beta/m)t}$$

$$v = \frac{dx}{dt} \qquad dx = v\, dt$$

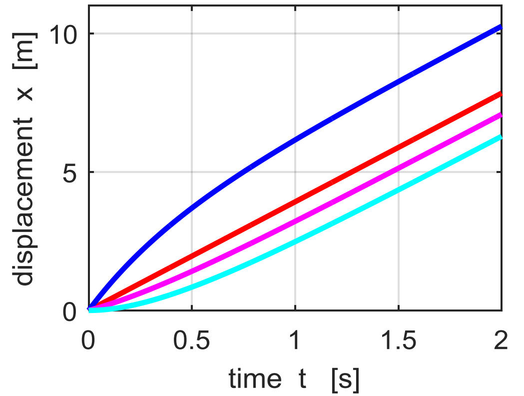
$$\int_0^x dx = \int_0^t \left(v_T + (v_0 - v_T)\, e^{(-\beta/m)t} \right) dt$$

$$x = \left[v_T\, t - \left(\frac{m}{\beta}\right)(v_0 - v_T)\, e^{(-\beta/m)t} \right]_0^t$$

$$x = v_T\, t - \left(\frac{m}{\beta}\right)(v_0 - v_T)\, e^{(-\beta/m)t} + \left(\frac{m}{\beta}\right)(v_0 - v_T)$$

$$x = v_T\, t + \left(\frac{m}{\beta}\right)(v_0 - v_T)\left(1 - e^{(-\beta/m)t}\right)$$

$$t \to \infty \qquad x \to v_T$$



$$v_0 = 0 \quad \Rightarrow \quad x = v_T \left(t + \left(\frac{m}{\beta}\right)\left(e^{(-\beta/m)t} - 1\right) \right)$$

Initial values for velocity v_0 [m.s^{-1}]

blue: 10 red: v_T magenta: 2 cyan: 0

So far we have only considered the case where the initial velocity was either zero or a positive quantity ($v_0 \geq 0$), i.e., the object was released from rest or projected downward. We will now consider the case where the object was project vertically upward ($v_0 < 0$). Note: in our frame of reference, the origin is taken as $x = 0$, the position of the object at time $t = 0$; down is the positive direction and up is the negative direction.

When the object is launched upward at time $t = 0$, the initial velocity has a negative value. Let u be the magnitude of the initial velocity v_0

$$v_0 < 0 \quad v_0 = -u \quad u > 0$$

Therefore, the equation for the velocity v as a function of time t can be expressed as

$$v = v_T + \left(v_0 - v_T\right) e^{(-\beta/m)t}$$
$$v = v_T - \left(u + v_T\right) e^{(-\beta/m)t}$$

We can now find the time t_{up} it takes for the object to rise to its maximum height x_{up} above the origin (remember: up is negative). At the highest point $v = 0$, therefore,

$$0 = v_T - \left(u + v_T\right) e^{(-\beta/m)t_{up}}$$
$$t_{up} = \left(\frac{m}{\beta}\right) \log_e \left(1 + \frac{u}{v_T}\right)$$

The maximum height x_{up} reached by the object in time $t = t_{up}$ is

$$x = v_T\, t + \left(\frac{m}{\beta}\right)\left(v_0 - v_T\right)\left(1 - e^{(-\beta/m)\,t}\right)$$

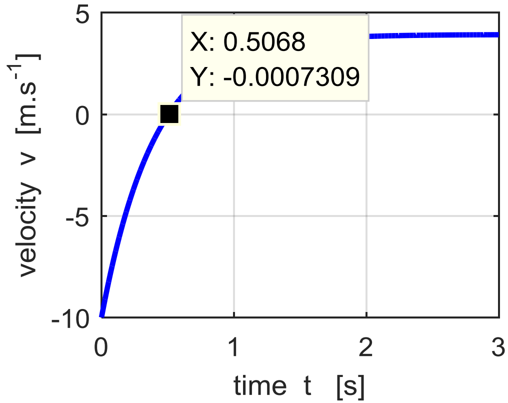
$$x_{up} = v_T\, t_{up} - \left(\frac{m}{\beta}\right)\left(u + v_T\right)\left(1 - e^{(-\beta/m)\,t_{up}}\right)$$

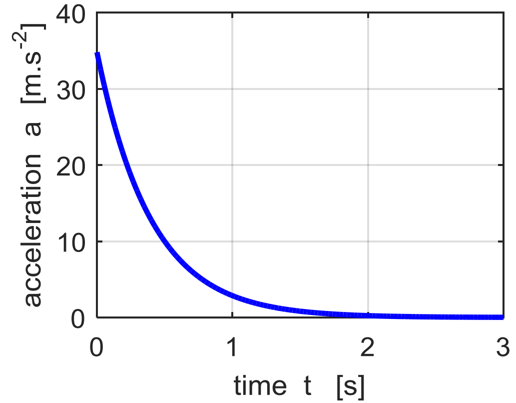
For the parameters

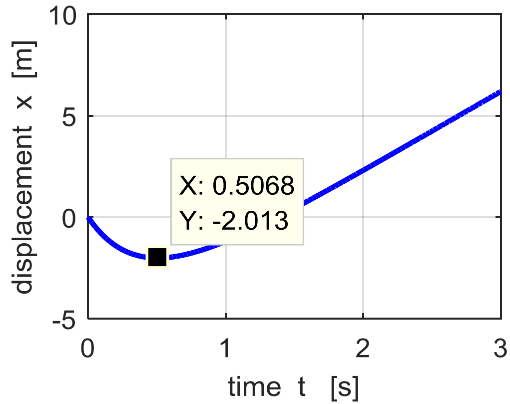
$m = 2.00$ kg $\beta = 5.00$ kg.s^{-1} $g = 9.8$ m.s^{-2} $u = 10$ m.s^{-1} $v_T = 3.92$ m.s^{-1}

The time to reach maximum height is $t_{up} = 0.507$ s

The max height h_{up} reached is $h_{up} = 2.013$ m $x_{up} = -\ 2.013$ m


X: 0.5068
Y: -0.0007309
velocity v [m.s-1]
time t [s]


acceleration a [m.s-2]
time t [s]


X: 0.5068
Y: -2.013
displacement x [m]
time t [s]

We can find the displacement x as a function of velocity v

$$a = v\frac{dv}{dx} = g - \left(\beta / m\right)v \qquad dx = \frac{v\,dv}{g - \left(\beta / m\right)v}$$

$$\left(\frac{1}{\beta / m}\right)dx = \frac{v\,dv}{\left(mg / \beta\right) - v} \qquad v_T = mg / \beta \qquad \left(\beta / m\right)dx = \frac{v\,dv}{v_T - v}$$

We can integrate this equation by a substitution method or an algebraic manipulation method.

Substitution Method

$$u = v_T - v \quad du = -dv \quad v = v_T - u \quad dv = -du \quad v_0 = v_T - u_0 \quad u_0 = v_T - v_0$$

$$\left(\beta / m\right)dx = \frac{-\left(v_T - u\right)}{u}du$$

$$\int_0^x \left(\beta / m\right)dx = \int_{u_0}^{u} \frac{-\left(v_T - u\right)}{u}du = \int_{u0}^{u}\left(1 - \frac{v_T}{u}\right)du$$

$$\left(\beta / m\right)x = \left[u - v_T \log_e\left(u\right)\right]_{u_0}^{u} = \left(u - u_0\right) - v_T \log_e\left(\frac{u}{u_0}\right)$$

$$\left(\beta / m\right)x = \left(v_0 - v\right) + v_T \log_e\left(\frac{v_T - v_0}{v_T - v}\right)$$

$$x = \left(\frac{m}{b}\right)\left(\left(v_0 - v\right) + v_T \log_e\left(\frac{v_T - v_0}{v_T - v}\right)\right)$$

Algebraic manipulation

$$\left(\beta / m\right) dx = \frac{v\,dv}{v_T - v}$$

$$\frac{v\,dv}{v_T - v} = v_T\left(\frac{-1}{v_T} + \frac{1}{v_T - v}\right)$$

$$\int_0^x \left(\beta / m\right) dx = v_T\left(\int_{v_0}^{vu}\left(\frac{-1}{v_T} + \frac{1}{v_T - v}\right) dv\right)$$

$$\left(\beta / m\right) x = v_T\left[\frac{-v}{v_T} - \log_e\left(v_T - v\right)\right]_{v_0}^{v}$$

$$\left(\beta / m\right) x = \left(\left(v_0 - v\right) + v_T \log_e\left(\frac{v_T - v_0}{v_T - v}\right)\right)$$

$$x = \left(\frac{m}{b}\right)\left(\left(v_0 - v\right) + v_T \log_e\left(\frac{v_T - v_0}{v_T - v}\right)\right)$$

QED

For the object projected up with an initial velocity $v_0 = - u$ where $u > 0$, the maximum height reached x_{up} occurs when $v = 0$

$$x = \left(\frac{m}{b}\right)\left(\left(v_0 - v\right) + v_T \log_e\left(\frac{v_T - v_0}{v_T - v}\right)\right)$$

$$x_{up} = \left(\frac{m}{b}\right)\left(\left(-u\right) + v_T \log_e\left(\frac{v_T + u}{v_T}\right)\right)$$

$$x_{up} = \left(\frac{m}{b}\right)\left(v_T \log_e\left(1 + \frac{u}{v_T}\right) - u\right)$$

Note: up is negative and down is positive in our frame of reference.

For the parameters

m = 2.00 kg β = 5.00 kg.s^{-1} g = 9.8 m.s^{-2} u = 10 m.s^{-1} v_T = 3.92 m.s^{-1}

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