ONLINE: MATHEMATICS EXTENSION 2

Topic 6 MECHANICS

EXERCISE p6501

A ball bearing was released from rest and dropped through a viscous liquid. The resistive force acting on the ball had magnitude $k\,v$ where k is a constant depending on the radius of the ball and the viscosity of the liquid and v is the velocity of the ball.

Find the following:

The terminal velocity v_T of the ball

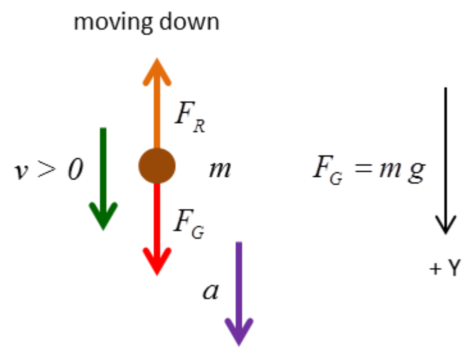
t as a function of v

v as a function of t

The time it takes for the ball to reach a speed equal to half its terminal speed

Solution

The forces acting on the ball as it falls through the liquid are the gravitational force F_G and the resistive force F_R. In our frame of reference, we will take down as the positive direction.



The equation of motion of the ball is determined from Newton's Second Law.

$$m\,a = m\,g - k\,v$$

where a is the acceleration of the ball at any instance and g is the acceleration due to gravity.

The initial conditions are $\quad t = 0 \quad a = g \quad v = 0$

As the ball falls, value of v increases until it reaches its terminal speed $v = v_T$ when the acceleration becomes zero

$$a = 0 \quad v = v_T = \text{constant}$$

$$0 = m\,g - k\,v_T$$

$$v_T = \frac{m\,g}{k}$$

We start with the equation of motion

$$m\frac{dv}{dt} = m\,g - k\,v$$

$$dt = \frac{m\,dv}{m\,g - k\,v} = \frac{dv}{g - (k/m)v}$$

then integrate this equation where the limits of the integration are determined by the initial conditions (t = 0 and v = 0) and final conditions (t and v)

$$\int_0^t dt = \int_0^v \frac{dv}{g - (k/m)v}$$

$$t = \int_0^v \frac{dv}{g - (k/m)v}$$

The integration can be done by making the substitution

$$u = g - (k/m)v$$

$$du = -(k/m)dv \quad dv = -(m/k)du$$

and the now limits of the integration are

$$v = 0 \rightarrow \quad u = g \quad v \rightarrow \quad u = g - (k/m)v$$

$$t = \left(-m/k\right)\int_{g}^{g-(k/m)v} \frac{du}{u}$$

$$t = \left(-m/k\right)\left[\log_e\left(u\right)\right]_{g}^{g-(k/m)v}$$

$$t = \left(-m/k\right)\left[\log_e\left(g-\left(k/m\right)v\right)-\log_e\left(g\right)\right]$$

$$t = \left(-m/k\right)\log_e\left(1-\left(k/mg\right)v\right)$$

The equation of the velocity as a function of time *t* is

$$e^{(-k/m)t} = 1-\left(k/m\,g\right)v$$

$$v = \left(\frac{m\,g}{k}\right)\left(1-e^{(-k/m)t}\right)$$

$$v = v_T\left(1-e^{(-k/m)t}\right)$$

The time to reach half the terminal velocity is

$$v = v_T/2$$

$$v_T/2 = v_T\left(1-e^{(-k/m)t}\right)$$

$$e^{(-k/m)t} = 1/2$$

$$\left(-k/m\right)t = \log_e(1/2)$$

$$t = \left(\frac{m}{k}\right)\log_e(2)$$