ONLINE: MATHEMATICS EXTENSION 2

Topic 2 COMPLEX NUMBERS

2.3 Powers and Roots of Complex Numbers

DE MOIVRE'S THEOREM

This is an important and useful theorem in complex number theory and can be derived from the product of two complex numbers in polar and exponential form.

$$z = \mathrm{R}\, e^{i\,\theta} = R\left(\cos\theta + i\,\sin\theta\right)$$

We can very easily compute the value of the complex number z^n when z is expressed in exponential form.

$$z^2 = \left(R\,e^{i\,\theta}\right)\left(R\,e^{i\,\theta}\right) = R^2\, e^{i\,(2\theta)} = R^2\left[\cos\left(2\,\theta\right) + i\,\sin\left(2\,\theta\right)\right]$$

Multiplying by z once more gives

$$z^3 = z\,z^2 = \left(R\,e^{i\,\theta}\right)\left(R^2\,e^{i\,(2\theta)}\right) = R^3\, e^{i\,(3\theta)} = R^3\left[\cos\left(3\theta\right) + i\,\sin\left(3\theta\right)\right]$$

so we can generalize the result

$$z^n = R^n\, e^{i\,(n\theta)} = R^n\left[\cos\left(n\,\theta\right) + i\,\sin\left(n\,\theta\right)\right]$$

If we take the special where $R = 1$, we obtain **de Moivre's theorem**

$$\left[\cos\theta + i\,\sin\theta\right]^n = \cos\left(n\,\theta\right) + i\,\sin\left(n\,\theta\right) \qquad n \quad \text{real}$$

Application to trigonometric formulae

Multiple angle formula for the sine and cosine functions can be easily derived from de Moivre's theorem, for example $\sin\left(3\theta\right)$ and $\cos\left(3\theta\right)$

$$\begin{aligned}\cos\left(3\theta\right) + i\,\sin\left(3\theta\right) &= \left(\cos\theta + i\,\sin\theta\right)^3 \\ &= \cos^3\theta - 3\cos\theta\,\sin^2\theta + 3i\,\sin\theta\,\cos^2\theta - i\,\sin^3\theta \\ &= \left(4\cos^3\theta - 3\cos\theta\right) = i\left(3\sin\theta - 4\sin^3\theta\right) \qquad \sin^2\theta = 1-\cos^2\theta \quad \cos^2\theta = 1-\sin^2\theta\end{aligned}$$

Equating the real and imaginary parts

$$\cos\left(3\theta\right) = 4\cos^3\theta - 3\cos\theta$$
$$\sin\left(3\theta\right) = 3\sin\theta - 4\sin^3\theta$$

We can also use de Moivre's theorem to obtain expressions for $\cos^n\theta$ and $\sin^n\theta$.

$$
\begin{aligned}
&z = \cos\theta + i\sin\theta \\
&z^n = [\cos\theta + i\sin\theta]^n = \cos(n\theta) + i\sin(n\theta) \\
&z^{-1} = [\cos\theta + i\sin\theta]^{-1} = \cos\theta - i\sin\theta \\
&z + z^{-1} = 2\cos\theta \qquad z - z^{-1} = 2\,i\sin\theta \\
&z^n + z^{-n} = 2\cos(n\theta) \qquad z^n - z^{-n} = 2i\sin(n\theta)
\end{aligned}
$$

For example, let $n = 3$

$$
\begin{aligned}
&\left(z+z^{-1}\right)^3 = \left(z^3+z^{-3}\right) + 3\left(z+z^{-1}\right) \qquad z\,z^{-1} = 1 \\
&(2\cos\theta)^3 = 2\cos(3\theta) + 3(2\cos\theta) \\
&8\cos^3\theta = 2\cos(3\theta) + 6\cos\theta \\
&\cos^3\theta = \frac{1}{4}\cos(3\theta) + \frac{3}{4}\cos\theta
\end{aligned}
$$

PROOF BY INDUCTION

- In proof by induction show that the statement is correct for $n = 1$.
- Assume the statement is true for n.
- Show that the statement is true for n+1 which is a statement with n replaced by n+1.

Prove $[\cos\theta + i\sin\theta]^n = \cos(n\theta) + i\sin(n\theta)$ by induction

$n = 1 \quad LHS = \cos\theta + i\sin\theta \quad RHS = \cos\theta + i\sin\theta \quad \Rightarrow \quad LHS = RHS$
statement is true for $n = 1$

Assume that $[\cos\theta + i\sin\theta]^n = \cos(n\theta) + i\sin(n\theta)$ is true

LHS for n+1

$$
\begin{aligned}
LHS &= [\cos\theta + i\sin\theta]^{n+1} = [\cos\theta + i\sin\theta]^n[\cos\theta + i\sin\theta] \\
&= [\cos(n\theta) + i\sin(n\theta)][\cos\theta + i\sin\theta] \\
&= [\cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta] + i[\sin(n\theta)\cos\theta + \cos(n\theta)\sin\theta]
\end{aligned}
$$

We can use the trigonometric identities

$$
\begin{aligned}
&\cos(A+B) = \cos A\cos B - \sin A\sin B \\
&\sin(A+B) = \sin A\cos B + \cos A\sin B
\end{aligned}
$$

$$LHS = [\cos((n+1)\theta) + i\sin((n+1)\theta)]$$

This is the statement with n replaced by n+1 on the *RHS* $\Rightarrow$ statement is true

ROOTS OF COMPLEX NUMBERS

Another important application of de Moivre's theorem is to calculate the roots of a number.

Suppose we require the n^{th} roots (n = 1, 2, 3, …) of z.

$$z = R\mathrm{e}^{i\theta}$$

We can add any integer multiple of 2π to θ without changing the number

$$z = R\mathrm{e}^{i(\theta + 2\pi k)} \qquad k \text{ is an integer}$$

Then

$$z^{\frac{1}{n}} = R\mathrm{e}^{i\left(\frac{\theta + 2\pi k}{n}\right)}$$

and we allow k to take the values 0, 1, 2, … , (n-1).

Example

Find the fifth roots of $z = \sqrt{3} + i$

Express $z = x + i\,y$ in exponential form $z = R\,e^{i\theta}$

$$x = \sqrt{3} \qquad y = 1 \qquad R = \sqrt{x^2 + y^2} = 2 \qquad \theta = a\tan\left(\frac{y}{x}\right) = a\tan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$z = 2e^{i\left(\frac{\pi}{6}\right)} = 2e^{i\left(\frac{\pi}{6} + 2\pi k\right)} \qquad k = 0,1,2,3,4$$

$$z^{1/5} = 2^{1/5}\,e^{i\left(\frac{\pi + 12\pi k}{30}\right)}$$

Since k = 0, 1, 2, 3, 4 we have **5 distinct roots**

$$k = 0 \qquad z^{1/5} = 2^{1/5}\,e^{i\left(\frac{\pi}{30}\right)} = 2^{1/5}\left[\cos\left(\frac{\pi}{30}\right) + i\sin\left(\frac{\pi}{30}\right)\right]$$

$$k = 1 \qquad z^{1/5} = 2^{1/5}\,e^{i\left(\frac{13\pi}{30}\right)} = 2^{1/5}\left[\cos\left(\frac{13\pi}{30}\right) + i\sin\left(\frac{13\pi}{30}\right)\right]$$

$$k = 2 \qquad z^{1/5} = 2^{1/5}\,e^{i\left(\frac{25\pi}{30}\right)} = 2^{1/5}\left[\cos\left(\frac{25\pi}{30}\right) + i\sin\left(\frac{25\pi}{30}\right)\right]$$

$$k = 3 \qquad z^{1/5} = 2^{1/5}\,e^{i\left(\frac{37\pi}{30}\right)} = 2^{1/5}\left[\cos\left(\frac{37\pi}{30}\right) + i\sin\left(\frac{37\pi}{30}\right)\right]$$

$$k = 4 \qquad z^{1/5} = 2^{1/5}\,e^{i\left(\frac{49\pi}{30}\right)} = 2^{1/5}\left[\cos\left(\frac{49\pi}{30}\right) + i\sin\left(\frac{49\pi}{30}\right)\right]$$

Example

Find the square root of a complex number $z = x + i\,y$

Express the complex number in exponential form $z = R\,e^{i\theta}$

$$\sqrt{z} = z^{1/2} = R\,e^{i\left(\frac{\theta+2\pi k}{2}\right)} \qquad k = 0, 1, \ldots \qquad \text{since there are two roots}$$

Express the result in rectangular form

$$z = \sqrt{3}x + i \qquad x = \sqrt{3} \qquad y = 1 \qquad R = \sqrt{x^2 + y^2} = 2 \qquad \theta = a\tan\left(\frac{y}{x}\right) = a\tan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$z^{1/2} = 2^{1/2}\,e^{i\left(\frac{\pi/6+2\pi k}{2}\right)} = 2^{1/2}\,e^{i\left(\frac{\pi}{12}+\pi k\right)} \qquad k = 0,\ 1$$

$$z_1^{\ 1/2} = 2^{1/2}\,e^{i(\pi/12)} \qquad z_2^{\ 1/2} = 2^{1/2}\,e^{i\left(\frac{\pi}{12}+\pi\right)}$$

$$z_1^{\ 1/2} = 2^{1/2}\left[\cos(\pi/12) + i\sin(\pi/12)\right]$$

$$z_2^{\ 1/2} = -2^{1/2}\left[\cos(\pi/12) + i\sin(\pi/12)\right]$$

$$z_1^{\ 1/2} = 1.3360 + i\,(0.3660)$$

$$z_2^{\ 1/2} = -\left[1.3360 + i\,(0.3660)\right]$$

Alternative procedure

$$z^{1/2} = w^2 = \sqrt{3}\,x + i$$

$$w = a + i\,b \qquad w^2 = (a + i\,b)^2 = \left(a^2 - b^2\right) + i\,(2\,a\,b)$$

$$a^2 - b^2 = \sqrt{3} \qquad 2\,a\,b = 1$$

Using the magnitudes of $\left|w^2\right|$ and $|w|$

$$\left|w^2\right| = |w|\,|w| \qquad \sqrt{3+1} = 2 = a^2 + b^2$$

$$a^2 + b^2 = 2 \qquad a^2 - b^2 = \sqrt{3} \quad \Rightarrow \quad 2a^2 = 2 + \sqrt{3}$$

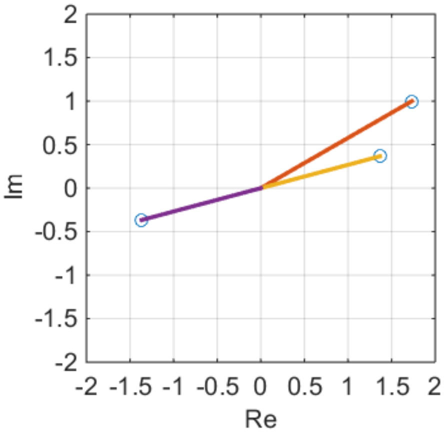
$$a = \pm\sqrt{\frac{2+\sqrt{3}}{2}} = \pm 1.3660$$

$$b = \frac{\pm 1}{2a} = \frac{\pm 1}{2(1.3660)\,a} = \pm 0.3660$$

Therefore, the two solutions are

$$w_1 = 1.3660 + i\,(0.3660) \qquad w_2 = -\left[1.3660 + i\,(0.3660)\right]$$

The point z and the two roots $z^{1/2}$ are shown on the Argand diagram



COMPLEX n^{th} ROOTS OF ± 1

We can write the number 1 as a complex number

$$z = 1 = e^{i(2\pi k)} \quad k = 0, 1, 2, 3, \ldots$$

We can now find the complex numbers w corresponding to the n^{th} roots of the number 1

$$w = z^{\frac{1}{n}} = e^{i\left(\frac{2\pi k}{n}\right)} \quad k = 0, 1, 2, 3, \ldots, (n-1)$$

The n roots can be shown on an Argand diagram. The roots are equally spaced around the circumference of the unit circle with centre (0,0). The angular spacing between each root is

$$\text{angular spacing between each root} = \frac{2\pi}{n} \qquad n = 1, 2, 3,$$

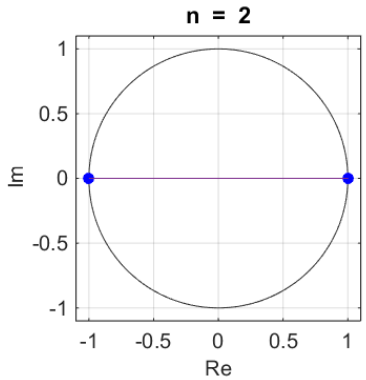
The lines joining the roots forms the shape of a regular n-sided polygon.

$n = 2$ ($k = 0$ and 1) two roots

$$k = 0 \quad w = z^{\frac{1}{2}} = e^{i(0)} = 1$$

$$k = 1 \quad w = z^{\frac{1}{2}} = e^{i(\pi)} = -1$$

angular spacing between roots = π rad (180°)



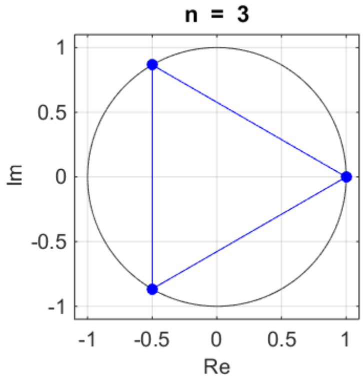
$n = 3$ $(k = 0, 2, 3)$ three roots

$$z^{\frac{1}{3}} = e^{i\left(\frac{2\pi k}{3}\right)} \quad k = 0, 1, 2$$

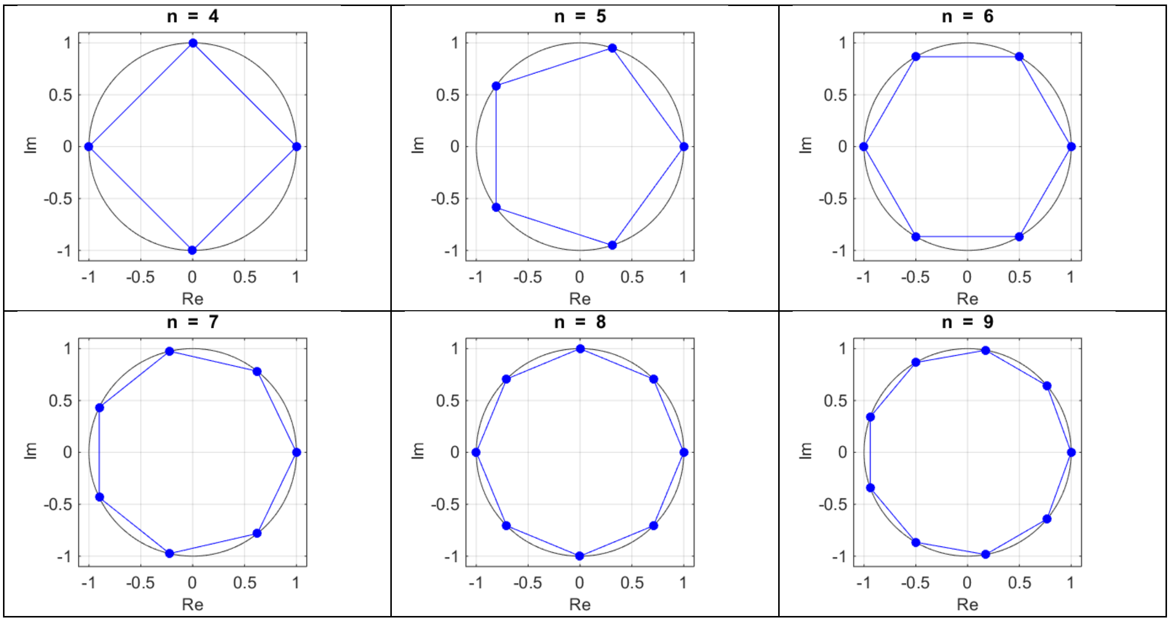
$$k = 0 \quad w = z^{\frac{1}{3}} = e^{i(0)} = 1$$

$$k = 1 \quad w = z^{\frac{1}{3}} = e^{i(2\pi/3)}$$

$$k = 2 \quad w = z^{\frac{1}{3}} = e^{i(4\pi/3)}$$



angular spacing between roots = $2\pi/3$ rad (120^o)



n^{th} roots of ±1 are equally spaced around the unit circle $\Delta\theta = \dfrac{2\pi}{n}\,\text{rad} = \dfrac{360^o}{n}$ with centre 0 and so from the vertices of a regular n-sided polygon.