ONLINE: MATHEMATICS EXTENSION 2

Topic 6 MECHANICS

EXERCISE p6503

Consider the vertical motion an object of mass m near the Earth's surface.The object was released with an initially velocity v_0. The resistive force due acting on the object is of the form $F_R = -\alpha v^2$ and directed in the opposite direction to the motion. Derive the following result and comment on the acceleration a, velocity v and displacement x as $t \to \infty$? Down is the positive direction.

Object falling $v > 0$ only $\quad a = g - \dfrac{\alpha}{m} v^2$

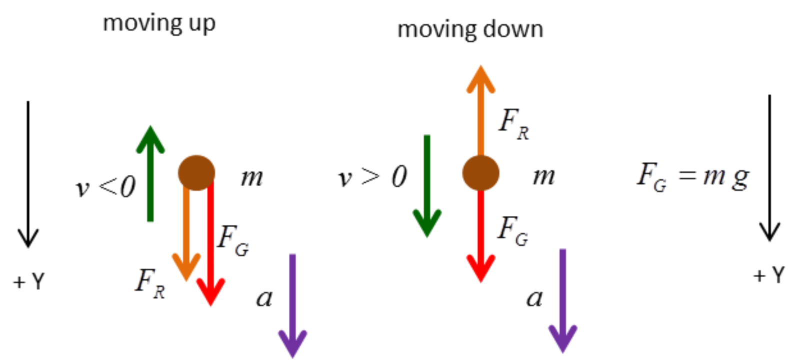
$$v_T = \sqrt{\frac{m\,g}{\alpha}} \qquad v = v_T \left(\frac{(v_0 + v_T) + (v_0 - v_T)\, e^{-\frac{2g}{v_T}t}}{(v_0 + v_T) - (v_0 - v_T)\, e^{-\frac{2g}{v_T}t}} \right) \qquad x = \left(\frac{V_T^{\,2}}{2g} \right) \log_e \left(\frac{v_T^{\,2} - v_0^{\,2}}{v_T^{\,2} - v^2} \right)$$

Object rising $v < 0$ only $\quad a = g + \dfrac{\alpha}{m} v^2$

$$v = v_T \,\tan\left[\operatorname{atan}\left(\frac{v_0}{v_T} \right) + \left(\frac{g}{v_T} \right) t \right] \qquad x = \left(\frac{v_T^{\,2}}{2g} \right) \log_e \left(\frac{v_T^{\,2} + v^2}{v_T^{\,2} + v_0^{\,2}} \right)$$

Solution

The forces acting on the object are the gravitational force F_G (weight) and the resistive force F_R. In our frame of reference, we will take down as the positive direction.



Newton's Second Law

$$m\,a = m\,g + F_R \qquad\qquad m\,a = m\,g - F_R$$

The equation of motion of the object is determined from Newton's Second Law.

$$m\,a = m\frac{dv}{dt} = F_G - F_R = mg - \alpha v^2 \left(v/|v|\right) \qquad a = g - \frac{\alpha}{m} v^2 \left(v/|v|\right)$$

where a is the acceleration of the object at any instance.

The initial conditions are $\quad t = 0 \quad v = v_0 \quad x = 0 \quad a = g - (\alpha/m)\, {v_0}^2 \left(\frac{v_0}{|v_0|}\right)$

When $a = 0$, the velocity is constant $v = v_T$ where v_T is the **terminal velocity**

$$0 = m\,g - \alpha\, {v_T}^2 \qquad {v_T}^2 = \frac{m\,g}{\alpha}$$

$$v_T = \sqrt{\frac{m\,g}{\alpha}}$$

We start with the equation of motion then integrate this equation where the limits of the integration are determined by the initial conditions ($t = 0$ and $v = v_0$) and final conditions (t and v)

Since the acceleration depends upon v^2 it a more difficult problem then for the linear resistive force example. We have to do separate analytical calculations for the motion when the object is falling or rising.

Velocity of the object is always positive (falling object) $v_0 \geq 0 \quad v \geq 0$

Equation of motion

$$a = g - \frac{\alpha}{m} v^2$$

$$a = \frac{dv}{dt} = g - \left(\frac{\alpha}{m}\right) v^2$$

$$dt = \frac{dv}{g - \left(\frac{\alpha}{m}\right) v^2} = \frac{dv}{\left(\frac{\alpha}{m}\right)\left(\left(\frac{m\,g}{\alpha}\right) - v^2\right)} \qquad {v_T}^2 = \frac{m\,g}{\alpha}$$

$$-\left(\frac{\alpha}{m}\right) dt = \frac{dv}{v^2 - {v_T}^2} = \left(\frac{1}{2 v_T}\right)\left(\frac{1}{v - v_T} - \frac{1}{v + v_T}\right) dv$$

$$-\left(2\sqrt{\frac{m\,g}{\alpha}}\right)\left(\frac{\alpha}{m}\right) dt = \left(\frac{1}{v - v_T} - \frac{1}{v + v_T}\right) dv$$

$$-\sqrt{\frac{4\,\alpha\,g}{m}}\, dt = \left(\frac{1}{v - v_T} - \frac{1}{v + v_T}\right) dv$$

$$-\sqrt{\frac{4\,\alpha\,g}{m}} \int_0^t dt = \int_{v_0}^{v} \left(\frac{1}{v - v_T} - \frac{1}{v + v_T}\right) dv$$

$$-\sqrt{\frac{4\,\alpha\,g}{m}}\, t = \left[-\log_e\left(v - v_T\right) - \log_e\left(v + v_T\right)\right]_{v_0}^{v}$$

$$\sqrt{\frac{4\alpha g}{m}}t = \Big[\log_e\left(v - v_T\right) + \log_e\left(v + v_T\right)\Big]_{v_0}^{v}$$

$$\sqrt{\frac{4\alpha g}{m}}t = \left[\log_e\left(\frac{v - v_T}{v_0 - v_T}\right) + \log_e\left(\frac{v - v_T}{v_0 - v_T}\right)\right]$$

$$-\sqrt{\frac{4\alpha g}{m}}t = \log_e\left\{\left(\frac{v - v_T}{v_0 - v_T}\right)\left(\frac{v_0 + v_T}{v + v_T}\right)\right\}$$

$$\left(\frac{v - v_T}{v_0 - v_T}\right)\left(\frac{v_0 + v_T}{v + v_T}\right) = e^{-\sqrt{\frac{4\alpha g}{m}}t} \qquad \sqrt{\frac{4\alpha g}{m}} = \sqrt{\frac{4\alpha g^2}{mg}} = \sqrt{\frac{4g^2}{{v_T}^2}} = \frac{2g}{v_T}$$

$$v - v_T = \left(v + v_T\right)\left(\frac{v_0 - v_T}{v_0 + v_T}\right)e^{-\frac{2g}{v_T}t} = \left(v + v_T\right)K \qquad K = \left(\frac{v_0 - v_T}{v_0 + v_T}\right)e^{-\frac{2g}{v_T}t}$$

$$v\left(1 - K\right) = v_T\left(1 + K\right) \qquad v = v_T\left(\frac{1 + K}{1 - K}\right)$$

$$v = v_T\left(\frac{1 + \left(\frac{v_0 - v_T}{v_0 + v_T}\right)e^{-\frac{2g}{v_T}t}}{1 - \left(\frac{v_0 - v_T}{v_0 + v_T}\right)e^{-\frac{2g}{v_T}t}}\right)$$

$$v = v_T\left(\frac{\left(v_0 + v_T\right) + \left(v_0 - v_T\right)e^{-\frac{2g}{v_T}t}}{\left(v_0 + v_T\right) - \left(v_0 - v_T\right)e^{-\frac{2g}{v_T}t}}\right)$$

valid only if $v_0 \geq 0 \quad v \geq 0$

We can now calculate the displacement x as a function of velocity v

$$a = \frac{dv}{dt} = \frac{v\,dv}{dx} = g - \left(\alpha / m\right) v^2$$

$$\frac{v\,dv}{dx} = \left(\alpha / m\right)\left(m\,g / \alpha - v^2\right) \qquad {v_T}^2 = m\,g / \alpha$$

$$dx = \left(\frac{m}{\alpha}\right)\frac{v\,dv}{\left({v_T}^2 - v^2\right)} = \left(\frac{-{V_T}^2}{2\,g}\right)\frac{\left(-2v\right)dv}{\left({v_T}^2 - v^2\right)}$$

$$\int_0^x dx = \left(\frac{-{v_T}^2}{2\,g}\right)\int_{v_0}^{v} \frac{\left(-2v\right)}{\left({v_T}^2 - v^2\right)}\,dv$$

$$x = \left(\frac{-{v_T}^2}{2\,g}\right)\left[\log_e\left({v_T}^2 - v^2\right)\right]_{v_0}^{v}$$

$$x = \left(\frac{{v_T}^2}{2\,g}\right)\log_e\left(\frac{{v_T}^2 - {v_0}^2}{{v_T}^2 - v^2}\right)$$

valid only if $v_0 \geq 0 \quad v \geq 0$

We can now investigate what happens as time $t \to \infty$

$$v(t \to \infty) = v_T \left(\frac{(v_0 + v_T) + 0}{(v_0 + v_T) - 0} \right) \qquad e^{-\frac{2g}{v_T}t} \to 0$$

$$v(t \to \infty) = v_T$$

In falling, the object will finally reach a constant velocity v_T ($a = 0$) which is known as the terminal velocity.

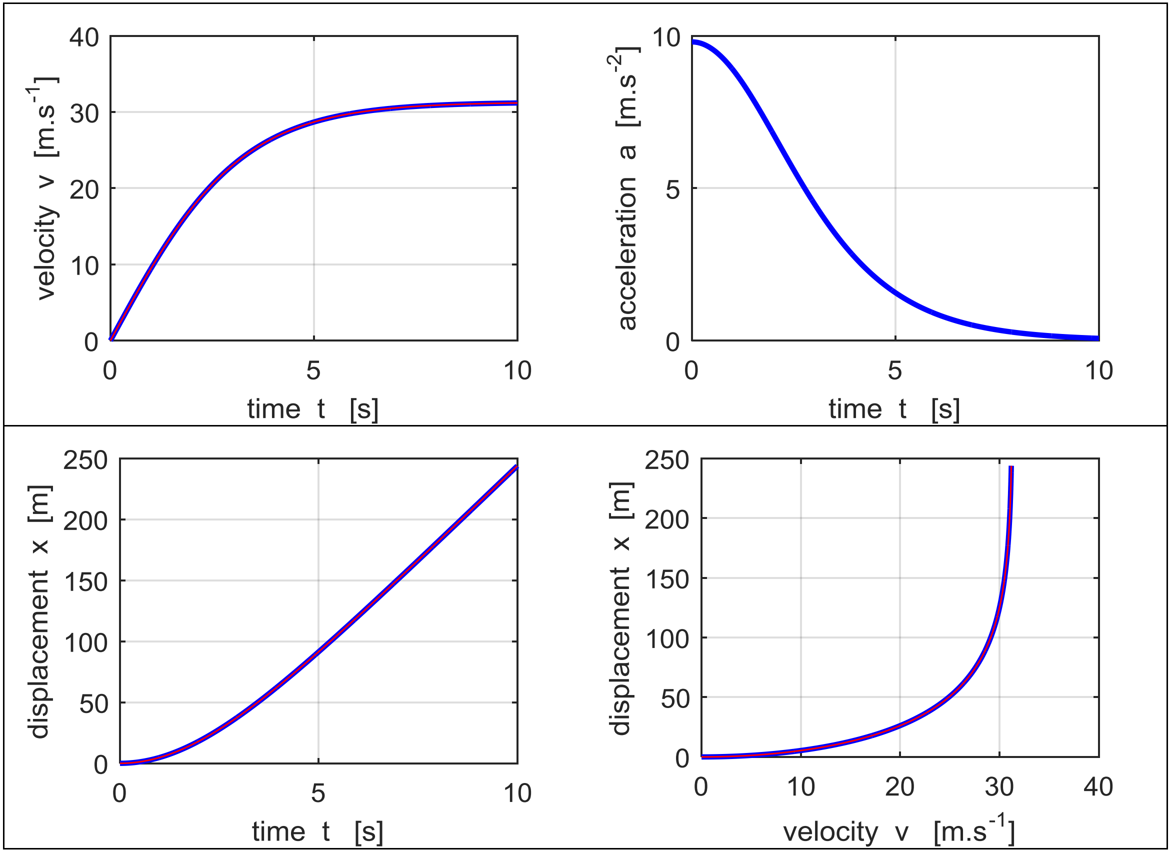
$$t \to \infty \quad v \to v_T \quad v_T - v \to 0 \quad \frac{1}{v_T - v} \to \infty$$

$$x = \left(\frac{V_T^{\ 2}}{2g} \right) \log_e \left(\frac{v_T^{\ 2} - v_0^{\ 2}}{v_T^{\ 2} - v^2} \right) \to \infty$$

In falling, as time t increases the objects displacement x just gets larger and larger.

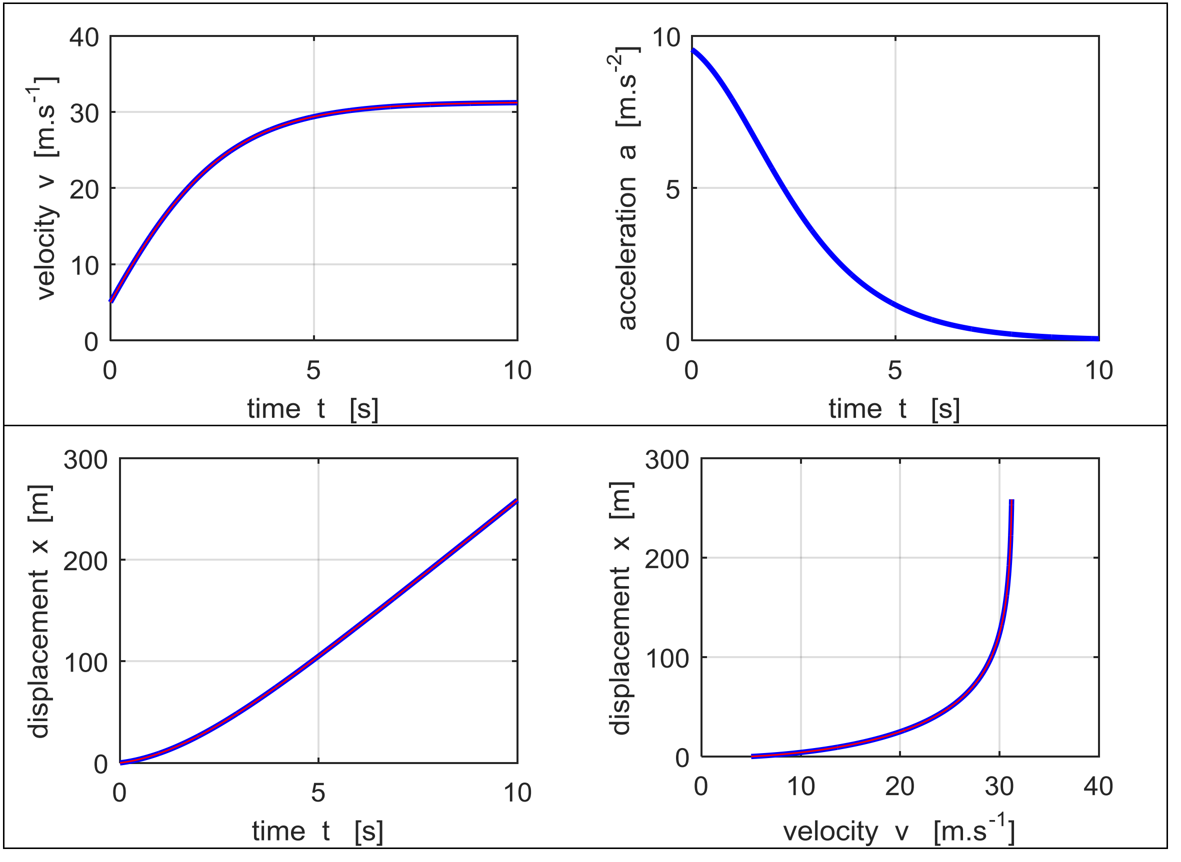
Example Small rock dropped from rest:

$$m = 0.010 \text{ kg} \quad \alpha = 1.00\times10^{-4} \text{ kg.m}^{-1} \quad v_0 = 0 \text{ m.s}^{-1} \quad \Rightarrow \; v_T = 31.3 \text{ m.s}^{-1}$$



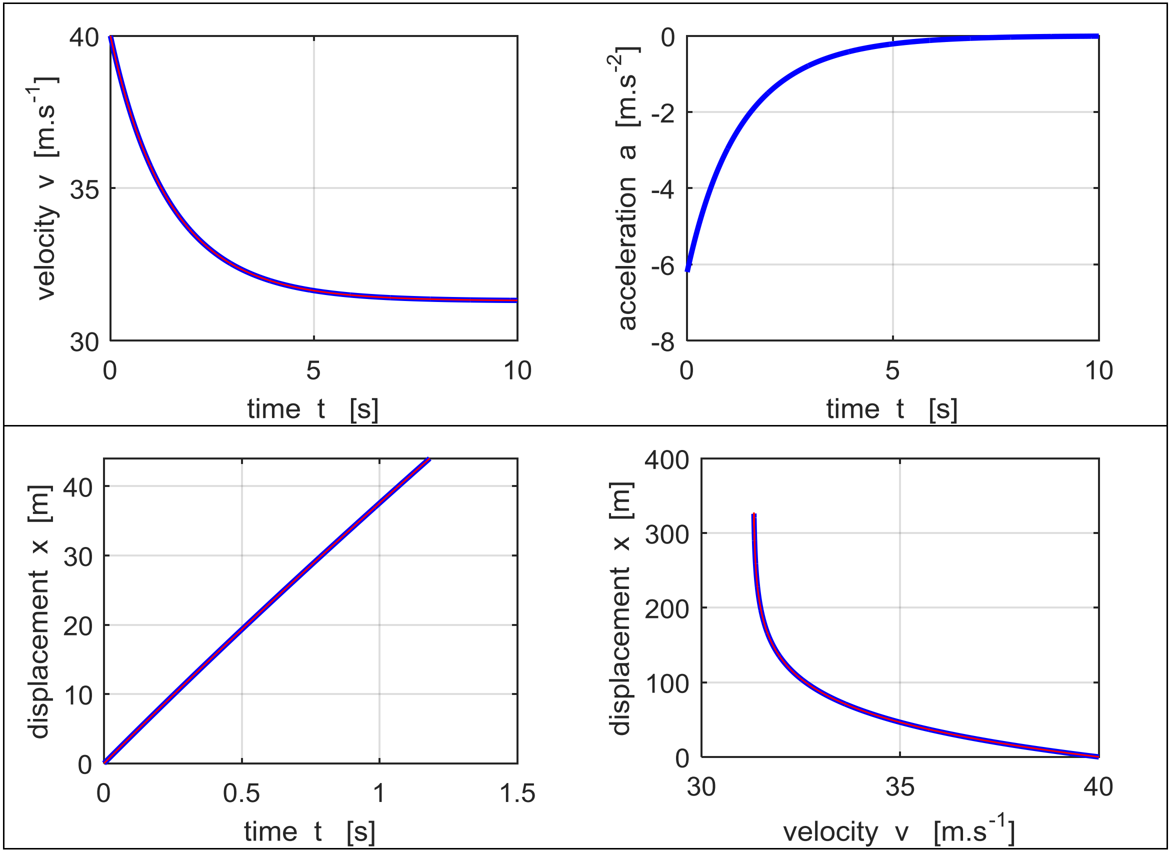
Example Small rock thrown vertically downward ($v < v_T$)

$$m = 0.010\ \text{kg} \quad \alpha = 1.00\times10^{-4}\ \text{kg.m}^{-1} \quad v_0 = +5.00\ \text{m.s}^{-1} \;\Rightarrow\; v_T = 31.3\ \text{m.s}^{-1}$$



Example Small rock thrown vertically downward ($v > v_T$)

$$m = 0.010 \text{ kg} \quad \alpha = 1.00\times10^{-4} \text{ kg.m}^{-1} \quad v_0 = +40.0 \text{ m.s}^{-1} \quad \Rightarrow \quad v_T = 31.3 \text{ m.s}^{-1}$$



For problems in which the object is projected vertically upward, you have to divide the problem into two parts. (1) Calculate the time to reach its maximum height and calculate the maximum height reached for the upward motion. (2) Reset the initial conditions to the position at maximum height where the initial velocity becomes $v_0 = 0$ and do the calculations for the downward movement of the object.

Velocity of the object negative and moving up $v_0 < 0 \quad and \quad v < 0$

Equation of motion

$$a = g + \frac{\alpha}{m} v^2 \qquad \text{valid only if} \quad v_0 < 0 \quad and \quad v < 0$$

Note: up is the positive direction

$$a = \frac{dv}{dt} = g + \left(\frac{\alpha}{m}\right)v^2$$

$$dt = \frac{dv}{g + \left(\frac{\alpha}{m}\right)v^2} = \frac{dv}{\left(\frac{\alpha}{m}\right)\left(\left(\frac{m\,g}{\alpha}\right) + v^2\right)} \qquad {v_T}^2 = \frac{m\,g}{\alpha}$$

$$\left(\frac{\alpha}{m}\right)dt = \frac{dv}{v^2 + {v_T}^2}$$

$$\left(\frac{\alpha}{m}\right)\int_0^t dt = \int_{v_0}^{v} \frac{dv}{v^2 + {v_T}^2}$$

Standard Integral $\displaystyle \int \frac{dx}{a^2 + x^2} = \left(\frac{1}{a}\right)\operatorname{atan}\left(\frac{x}{a}\right) + C$

$$\left(\frac{\alpha}{m}\right)t = \left(\frac{1}{v_T}\right)\left[\operatorname{atan}\left(\frac{v}{v_T}\right)\right]_{v_0}^{v} \qquad \left(\frac{m}{\alpha\,v_T}\right) = \left(\frac{m}{\alpha\,v_T}\frac{g}{g}\right) = \left(\frac{v_T}{g}\right)$$

$$t = \left(\frac{v_T}{g}\right)\left[\operatorname{atan}\left(\frac{v}{v_T}\right)\right]_{v_0}^{v} = \left(\frac{v_T}{g}\right)\left[\operatorname{atan}\left(\frac{v}{v_T}\right) - \operatorname{atan}\left(\frac{v_0}{v_T}\right)\right]$$

The time tup to reach maximum height occurs when $v = 0$

$$t_{up} = \left(\frac{v_T}{g}\right)\left[\operatorname{atan}\left(\frac{0}{v_T}\right) - \operatorname{atan}\left(\frac{v_0}{v_T}\right)\right] = -\left(\frac{v_T}{g}\right)\operatorname{atan}\left(\frac{v_0}{v_T}\right) \qquad v_0 < 0$$

The velocity v as a function of time t is

$$\operatorname{atan}\left(\frac{v}{v_T}\right) = \operatorname{atan}\left(\frac{v_0}{v_T}\right) + \left(\frac{g}{v_T}\right)t$$

$$v_0 < 0 \quad and \quad v < 0$$

$$v = v_T \, \tan\left[\operatorname{atan}\left(\frac{v_0}{v_T}\right) + \left(\frac{g}{v_T}\right)t\right] \qquad \operatorname{atan}\theta \equiv \tan^{-1}\theta$$

The displacement x as a function of velocity v is

$$a = \frac{dv}{dt} = \frac{v\,dv}{dx} = g + (\alpha / m)\,v^2$$

$$\frac{v\,dv}{dx} = (\alpha / m)\left(m\,g / \alpha - v^2\right) \qquad {v_T}^2 = m\,g / \alpha$$

$$dx = \left(\frac{m}{\alpha}\right)\frac{v\,dv}{\left({v_T}^2 + v^2\right)} = \left(\frac{{v_T}^2}{2\,g}\right)\frac{(2v)\,dv}{\left({v_T}^2 + v^2\right)}$$

$$\int_0^x dx = \left(\frac{{v_T}^2}{2\,g}\right)\int_{v_0}^{v} \frac{(2v)}{\left({v_T}^2 + v^2\right)}\,dv \qquad\qquad v_0 < 0 \quad and \quad v < 0$$

$$x = \left(\frac{{v_T}^2}{2\,g}\right)\left[\log_e\left({v_T}^2 + v^2\right)\right]_{v_0}^{v}$$

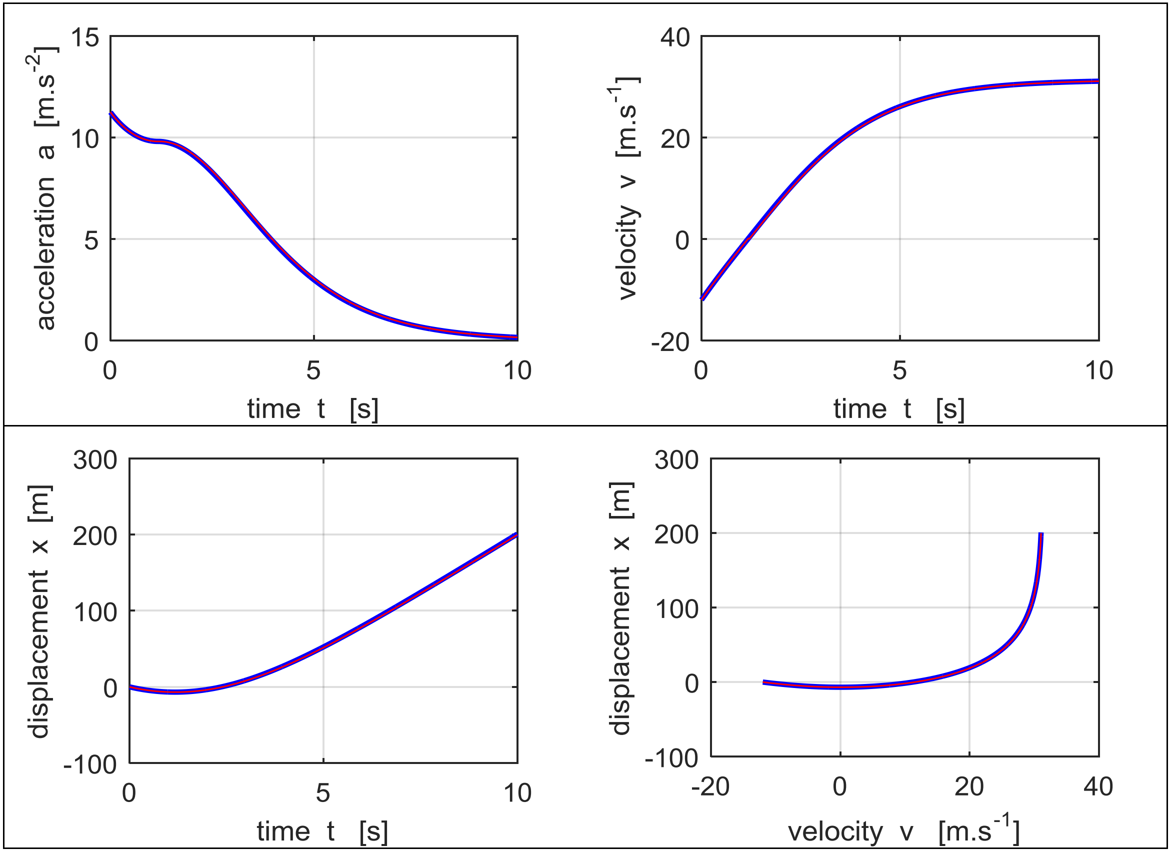
$$x = \left(\frac{{v_T}^2}{2\,g}\right)\log_e\left(\frac{{v_T}^2 + v^2}{{v_T}^2 + {v_0}^2}\right)$$

The maximum height x_{up} reached by the object occurs when $v = 0$

$$x_{up} = \left(\frac{{v_T}^2}{2\,g}\right)\log_e\left(\frac{{v_T}^2}{{v_T}^2 + {v_0}^2}\right)$$

Example Small rock thrown vertically upward ($v_0 < 0$ $v_0 = -u$ $u > 0$)

$$m = 0.010 \text{ kg} \quad \alpha = 1.00\times10^{-4} \text{ kg.m}^{-1} \quad v_0 = -12.0 \text{ m.s}^{-1} \quad \Rightarrow \quad v_T = 31.3 \text{ m.s}^{-1}$$



The terminal velocity v_T is

$$v_T^{\,2} = m\,g / \alpha$$
$$v_T = \sqrt{m\,g/\alpha} = \sqrt{\left(10^{-2}\right)(9.8)/\left(10^{-4}\right)}\ \text{m.s}^{-1}$$
$$v_T = 31.31\ \ \text{m.s}^{-1}$$

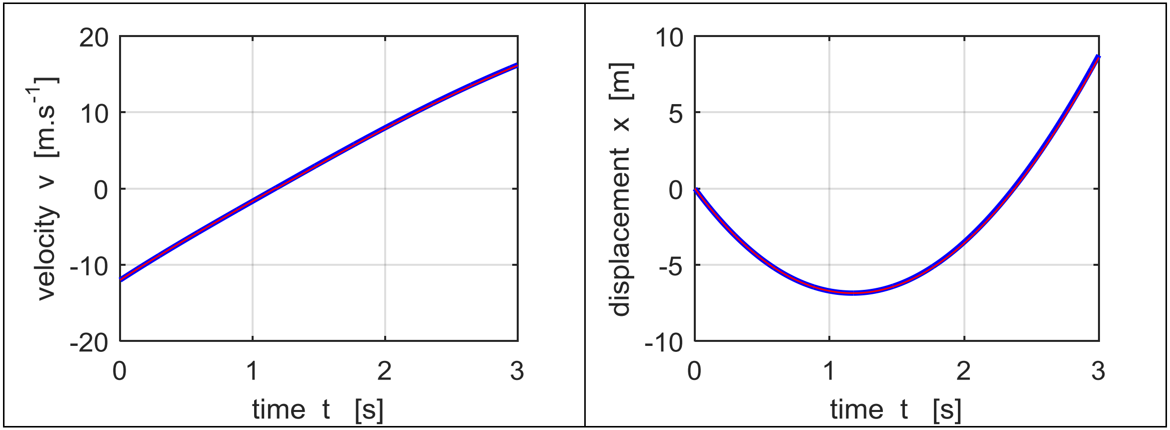
When $v = 0$ the object reaches its maximum height x_{up} (up is negative)

$$x_{up} = \left(\frac{v_T^{\,2}}{2\,g}\right)\log_e\left(\frac{v_T^{\,2}}{v_T^{\,2}+v_0^{\,2}}\right)$$
$$x_{up} = -6.855\ \text{m}$$

The time t_{up} to reach the maximum height

$$t_{up} = -\left(\frac{v_T}{g}\right)\text{atan}\left(\frac{v_0}{v_T}\right)$$
$$t_{up} = 1.169\ \text{s}$$

The calculations agree with the values for t_{up} and x_{up} determined from the graphs.



From the graphs:

$x = 0$ time $t = 2.275$ s velocity $v = 11.21$ m.s^{-1}

Time to fall from max height to origin $x = 0$ $t_{down} = (2.375 - 1.169)\text{ s} = 1.206\text{ s}$

takes slight longer to fall then rise to and from origin to max height

Launch speed = 12.00 m.s^{-1} slightly greater than return speed = 11.21 m.s^{-1}

In the absence of any resistive forces $a = g$ $v = v_0 + a\,t$ $v^2 = v_0^{\,2} + 2\,a\,s$

At maximum height $v = 0$ $t_{up} = 1.2245\,\text{s}$ $x_{up} = -7.3469\,\text{m}$