ONLINE: MATHEMATICS EXTENSION 2

Topic 6 MECHANICS

EXERCISE p6201

Two stones are thrown simultaneously from the same point in the same upward direction at the angle α with respect to the horizontal. The initial velocities of the two stones are u_1 and u_2 where $u_1 < u_2$.The slower stone hits the ground at a point P on the same level as the point of projection. At that instant the faster stone just clears a wall of height h above the level of projection and its downward path makes an angle β with the horizontal.

(a) Show that the line joining them has an inclination with respect to the horizontal that is independent of time when the stones are still both in flight.

(b) What is the horizontal displacement from the point P to the foot of the wall in terms of α and h?

(c) Show that $u_2\left(\tan\alpha + \tan\beta\right) = 2u_1\tan\alpha$

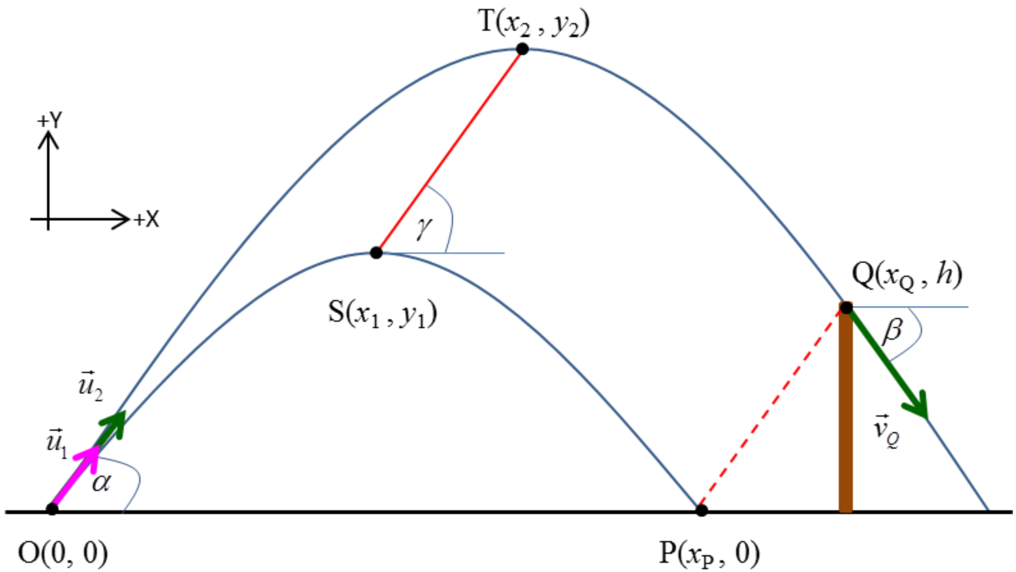
(d) If $\alpha = 2\beta$ show that $u_1 < \frac{3}{4}u_2$

Solution

Step 1: Think about how to approach the problem

Step 2: Draw an annotated diagram of the physical situation

Step 3: What do you know about projectile motion and motion with a constant acceleration?



The origin is at 0(0, 0).

Stone 1 has initial velocity u_1

$$u_{1x} = u_1 \cos\alpha \quad u_{1y} = u_1 \sin\alpha \quad \tan\alpha = \left(u_{1y} / u_{1y}\right)$$

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Stone 2 has initial velocity u_2 $u_2 > u_1$

$$u_{2x} = u_2 \cos\alpha \quad u_{2y} = u_1 \sin\alpha \quad \tan\alpha = \left(u_{2y} / u_{2y}\right)$$

Stone 1 lands at the point P(x_P, 0). At this instance stone 2 is at the point Q(x_Q, h).

The velocity of stone 2 at the point Q is $\vec{v}_Q$

$$v_{Qx} = v_Q \cos\alpha \quad u_{Qy} = v_Q \sin\alpha \quad \tan\beta = \left(v_{Qy} / v_{Qx}\right)$$

(a)

When stone 1 is at the point S(x_1, y_1) then stone 2 is at the point T(x_2, y_2).

The inclination of the line joining the stones 1 and 2 makes an angle γ with the horizontal where

$$\tan\gamma = \frac{y_2 - y_1}{x_2 - x_1}$$

Therefore, we need to find the coordinates of the points S(x_1, y_1) and T(x_2, y_2) at some instance t.

$$a_{1x} = 0 \quad a_{1y} = -g \quad v_{1y} = 0$$

Motion with constant acceleration $v = u + a\,t \quad v^2 = u^2 + 2a\,s \quad s = u\,t + \tfrac{1}{2}a\,t^2$

$$x_1 = u_{1x}\, t = \left(u_1 \cos\alpha\right)t \quad y_1 = u_{1y}\, t + \tfrac{1}{2}\left(-g\right)t^2 = \left(u_1 \sin\alpha\right)t - \tfrac{1}{2} g\, t^2$$

$$x_2 = \left(u_2 \cos\alpha\right)t \quad y_2 = \left(u_2 \sin\alpha\right)t - \tfrac{1}{2} g\, t^2$$

$$y_2 - y_1 = \left(u_2 \sin\alpha\right)t - \tfrac{1}{2} g\, t^2 - \left(\left(u_1 \sin\alpha\right)t - \tfrac{1}{2} g\, t^2\right) = \left(u_2 - u_1\right)\sin\alpha\, t$$

$$x_2 - x_1 = \left(u_2 \cos\alpha\right)t - \left(u_1 \cos\alpha\right)t = \left(u_2 - u_1\right)\cos\alpha\, t$$

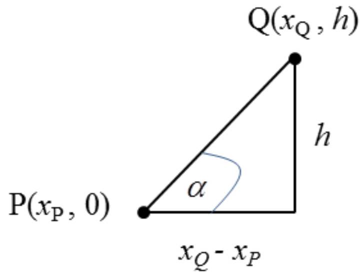
$$\tan\gamma = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\left(u_2 - u_1\right)\sin\alpha\, t}{\left(u_2 - u_1\right)\cos\alpha\, t} = \tan\alpha$$

$$\gamma = \alpha$$

The inclination angle is α hence, the inclination is independent of time.

(b)

The inclination angle of the lining joining the stones is independent of time or position



$$\tan\alpha = \frac{h}{x_Q - x_P}$$

The distance from the landing point of stone 1 at the point P to the foot of the wall is

$$x_Q - x_P = h\tan\alpha$$

(c) Show that $u_2\left(\tan\alpha+\tan\beta\right)=2u_1\tan\alpha$

Rearranging this equation gives (A) $\tan\beta=\left(\dfrac{2u_1}{u_2}-1\right)\tan\alpha$

Using the equations for constant acceleration at the time t that stone 1 hits the ground at the point P

Stone 1 (vertical motion)

$$s=ut+\tfrac{1}{2}at^2 \qquad 0=u_1\sin\alpha\, t-\tfrac{1}{2}gt^2 \qquad t=\frac{2u_1\sin\alpha}{g}$$

Stone 2 $\quad v_{2x}=u_2\cos\alpha$

$$v=u+at \qquad v_{2y}=u_2\sin\alpha-g\left(\frac{2u_1\sin\alpha}{g}\right)$$

$$v_{2y}=\left(u_2-2u_1\right)\left(\sin\alpha\right)$$

$$\tan\beta=-\frac{v_{2y}}{v_{2x}}=-\frac{\left(u_2-2u_1\right)\left(\sin\alpha\right)}{u_2\cos\alpha}$$

$$\tan\beta=\left(2\frac{u_1}{u_2}-1\right)\tan\alpha$$

Which is the result we need to show as given by equation (A). **QED**

(d)

$$\beta = \alpha / 2$$

$$\tan\beta = \left(2\frac{u_1}{u_2} - 1\right)\tan\alpha$$

$$\tan(\alpha/2) = \left(2\frac{u_1}{u_2} - 1\right)\tan(\alpha/2 + \alpha/2)$$

$$\tan(\alpha/2) = \left(2\frac{u_1}{u_2} - 1\right)\left(\frac{\tan(\alpha/2) + \tan(\alpha/2)}{1 - \tan^2(\alpha/2)}\right)$$

let $\quad z = \tan(\alpha/2) \qquad K = \left(2\dfrac{u_1}{u_2} - 1\right)$

$$z - z^3 = 2Kz \qquad z^2 = 1 - 2K$$

But $\;z = \tan(\alpha/2)\;$ must be a real quantity

$$2K < 1$$

$$2\left(2\frac{u_1}{u_2} - 1\right) < 1$$

$$4\frac{u_1}{u_2} < 3$$

$$u_1 < \tfrac{3}{4}u_2$$

QED