ONLINE: MATHEMATICS EXTENSION 2

Topic 2 COMPLEX NUMBERS

EXERCISE p2201

p001

Given the two complex numbers $z_1 = 2\angle 30^o$ $z_2 = 3\angle 60^o$

find the real part, imaginary part, modulus and argument for each of the following complex numbers:

$$z_1 \quad \bar{z}_1 \quad z_2 \quad \bar{z}_2 \quad z_1 + z_2 \quad z_1 + \bar{z}_2 \quad z_1 - z_2 \quad z_1 - \bar{z}_2 \quad z_1 z_2 \quad z_1 \bar{z}_2 \quad z_1 / z_2 \quad z_1 / \bar{z}_2$$

Locate the results on Argand diagrams.

p002

Given the two complex numbers $z_1 = 3\angle\left(3\pi/4\right) \quad z_2 = 2\angle\left(-5\pi/6\right)$

find the real part, imaginary part, modulus and argument for each of the following complex numbers:

$$z_1 \quad \overline{z}_1 \quad z_2 \quad \overline{z}_2 \quad z_1 + z_2 \quad z_1 - z_2 \quad z_1\, z_2 \quad z_1 / z_2$$

Locate the results on Argand diagrams.

p003

Prove that $$\ln\left[\frac{1}{\sqrt{2}}\left(1+i\right)\right] = \frac{i\,\pi}{4}$$

p004

Prove that $$\ln(a + i\,b) = \ln\left(\left|a + i\,b\right|\right) + i\ \text{atan}\left(\frac{\text{b}}{\text{a}}\right)$$

p005

Express the following in rectangular form for k = 0, 2, ,3, .., , 8.

$$z_k = \mathrm{e}^{i\,(k\pi/4)} \qquad k = 0,1,3,\ldots,8$$

Plot the complex numbers on an Argand diagram.

p006

Prove the following using the properties of complex numbers

$$\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$$

$$\sin(\theta+\phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$$

p007

Form the product $z_1\, z_2$ and the quotient z_1 / z_2 for

$$z_1 = 3\left[\cos(\pi/6) + i\sin(\pi/6)\right] \quad z_2 = 1.5\left[\cos(4\pi/3) + i\sin(4\pi/3)\right]$$

ANSWERS

a001

(a)

Given the two complex numbers $z_1 = 2\angle 30^o \quad z_2 = 3\angle 60^o$

find the real part, imaginary part, modulus and argument for each of the following complex numbers:

$z_1 \quad \overline{z}_1 \quad z_2 \quad \overline{z}_2 \quad z_1 + z_2 \quad z_1 + \overline{z}_2 \quad z_1 - z_2 \quad z_1 - \overline{z}_2 \quad z_1\, z_2 \quad z_1\, \overline{z}_2 \quad z_1 / z_2 \quad z_1 / \overline{z}_2$

Locate the results on Argand diagrams.

know

Rectangular form $\quad z = x + i\ y$

Polar form $z = R\left(\cos\theta + i\sin\theta\right)$

Exponential form $\quad z = R\,e^{i\theta}$

Modulus $\quad |z| = \sqrt{z\,\overline{z}} = R = \sqrt{x^2 + y^2}$

Argument $\tan\theta = \dfrac{\operatorname{Im} z}{\operatorname{Re} z} = \dfrac{y}{x}$ $\qquad \theta = a\tan\left(\dfrac{y}{x}\right) \equiv \tan^{-1}\left(\dfrac{y}{x}\right)$

Real part $x = R\cos\theta$

Imaginary part $y = R\sin\theta$

$$z_1 + z_2 = \left(x_1 + x_2\right) + i\left(y_1 + y_2\right)$$

$$z_1 - z_2 = \left(x_1 - x_2\right) + i\left(y_1 - y_2\right)$$

$$z_1\, z_2 = \left(x_1\, x_2 - y_1\, y_2\right) + i\left(x_1\, y_2 + x_2\, y_1\right)$$

$$\frac{z_1}{z_2} = \frac{z_1\,\overline{z}_2}{z_2\,\overline{z}_2}$$

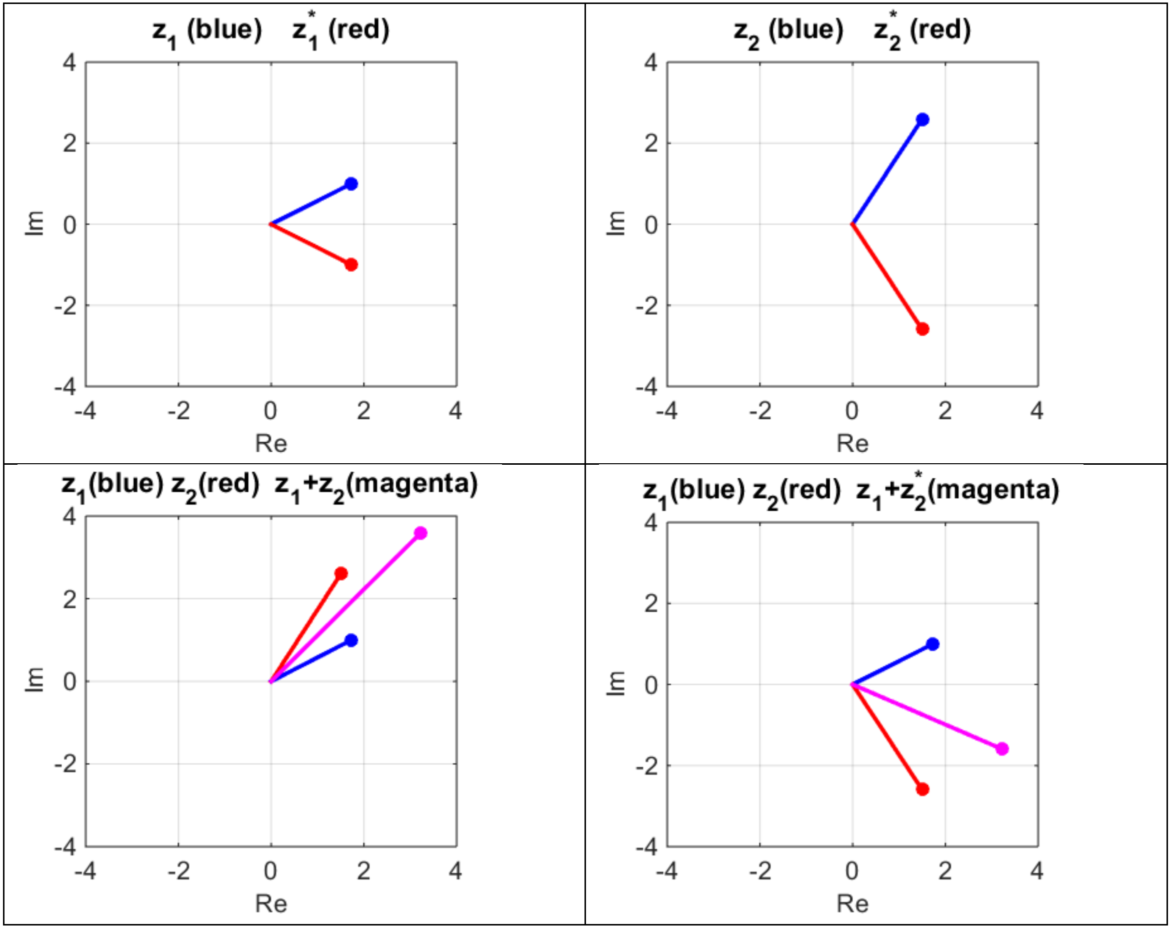
$$z_1\, z_2 = R_1\, R_2\, e^{i\left(\theta_1 + \theta_2\right)} = R_1\, R_2\, \angle\left(\theta_1 + \theta_2\right)$$
$$= R_1\, R_2\left[\cos\left(\theta_1 + \theta_2\right) + i\,\sin\left(\theta_1 + \theta_2\right)\right]$$

$$\frac{z_1}{z_2} = \left(\frac{R_1}{R_2}\right) e^{i\left(\theta_1 - \theta_2\right)} = \left(\frac{R_1}{R_2}\right)\angle\left(\theta_1 - \theta_2\right)$$
$$= R_1\, R_2\left[\cos\left(\theta_1 - \theta_2\right) + i\,\sin\left(\theta_1 - \theta_2\right)\right]$$

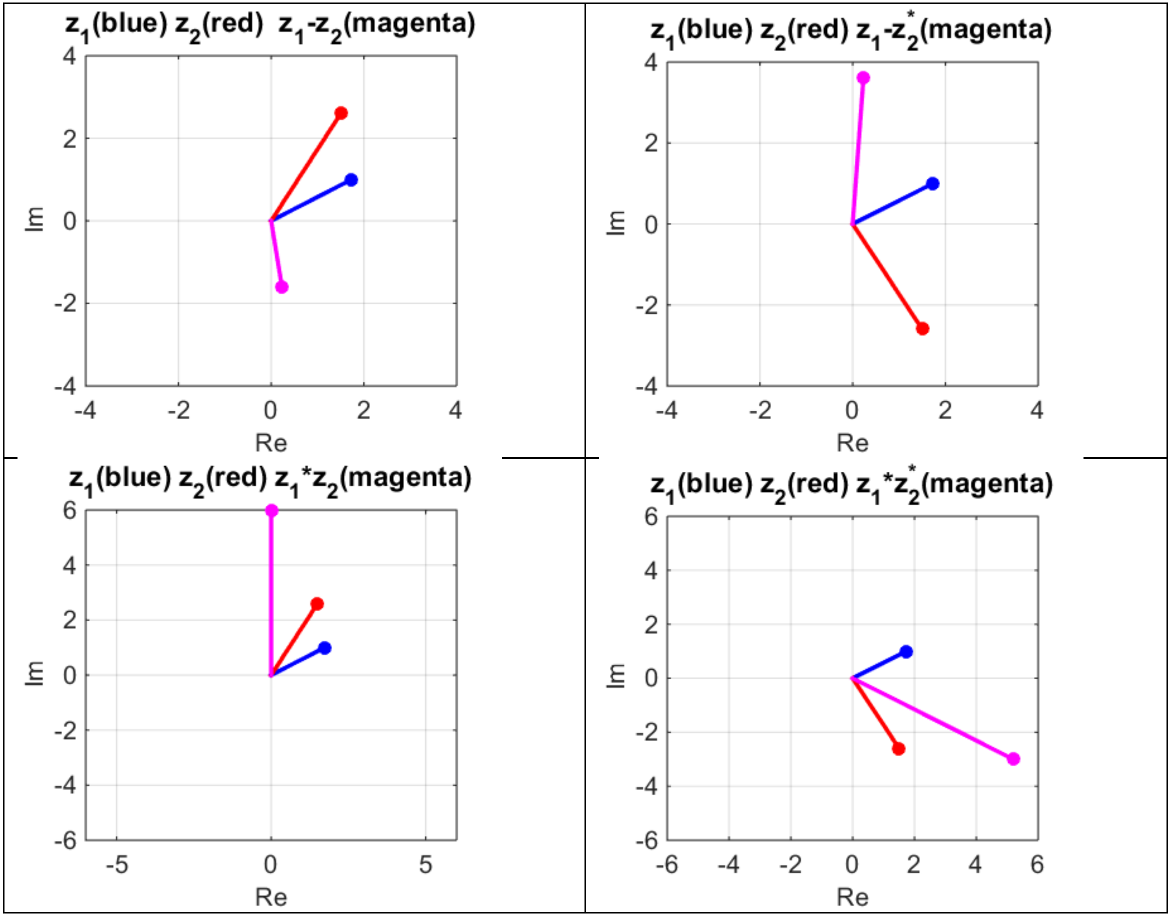
$$|z_1| = R_1 = 2 \quad \theta_1 = 30^{\circ} = \pi/6 \text{ rad} \quad x_1 = 2\cos(\pi/6) = 1.7321 \quad y_1 = 2\sin(\pi/6) = 1.0000$$

$$|z_2| = R_1 = 3 \quad \theta_2 = 60^{\circ} = \pi/3 \text{ rad} \quad x_2 = 3\cos(\pi/3) = 1.5000 \quad y_2 = 3\sin(\pi/3) = 2.5981$$

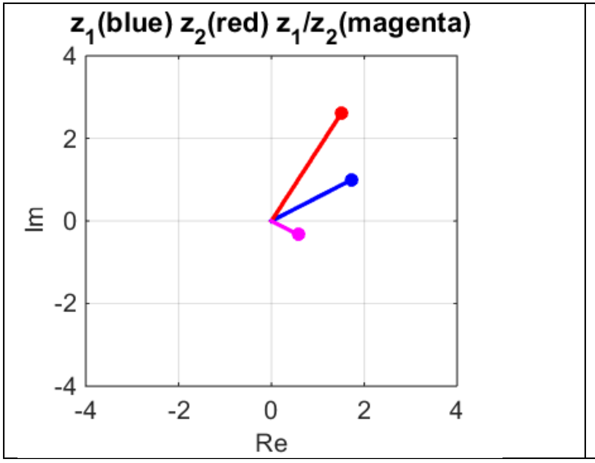
z	**Re(*z*) = *x***	**Im(*z*) = *y***	**|*z*| = *R***	***Arg*(*z*) = *θ* rad**	***Arg*(*z*) = *θ* deg**
z_1	1.7321	1.0000	2.0000	$\pi/6$	30
$\overline{z}_1$	1.7321	-1.0000	2.0000	- $\pi/6$	- 30
z_2	1.5000	2.5981	3.0000	$\pi/3$	60
$\overline{z}_2$	1.5000	-2.5981	3.0000	- $\pi/3$	- 60
$z_1 + z_2$	3.2321	3.5981	4.8366	0.8389	48.07
$z_1 + \overline{z}_2$	3.2321	-1.5981	3.6056	-0.4592	- 26.31
$z_1 - z_2$	0.2321	-1.5981	1.6148	-1.4266	- 81.74
$z_1 - \overline{z}_2$	0.2321	3.5981	3.6056	1.5064	86.31
$z_1 z_2$	0	6.0000	6.0000	$\pi/2$	90
$z_1 \overline{z}_2$	5.1962	-3.0000	6.0000	- $\pi/6$	- 30
z_1 / z_2	0.5774	-0.3333	0.6667	- $\pi/6$	- 30
$z_1 / \overline{z}_2$	0	0.6667	0.6667	$\pi/2$	90

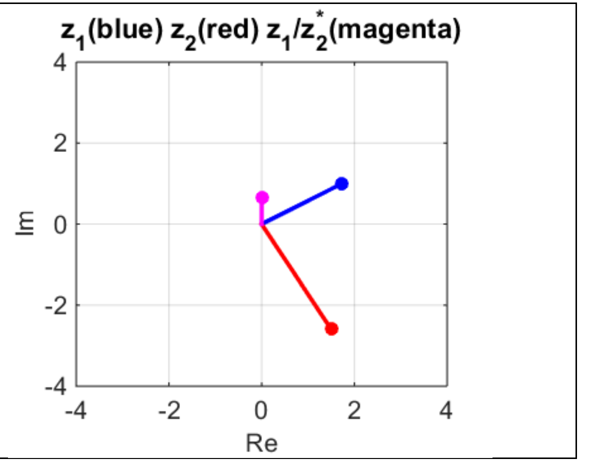


z_1 (blue) z_1^* (red)
z_2 (blue) z_2^* (red)
z_1(blue) z_2(red) z_1+z_2(magenta)
z_1(blue) z_2(red) $z_1+z_2^*$(magenta)
Im
Re



z1(blue) z2(red) z1-z2(magenta)
z1(blue) z2(red) z1-z2*(magenta)
z1(blue) z2(red) z1*z2(magenta)
z1(blue) z2(red) z1*z2*(magenta)
Im
Re


z1(blue) z2(red) z1/z2(magenta)
Im
Re
4
2
0
-2
-4
-4
-2
0
2
4


z1(blue) z2(red) z1/z2*(magenta)
Im
Re
4
2
0
-2
-4
-4
-2
0
2
4

a002

Given the two complex numbers $z_1 = 3\angle(3\pi/4) \quad z_2 = 2\angle(-5\pi/6)$

find the real part, imaginary part, modulus and argument for each of the following complex numbers:

$$z_1 \quad \overline{z}_1 \quad z_2 \quad \overline{z}_2 \quad z_1 + z_2 \quad z_1 - z_2 \quad z_1 z_2 \quad z_1 / z_2$$

Locate the results on Argand diagrams.

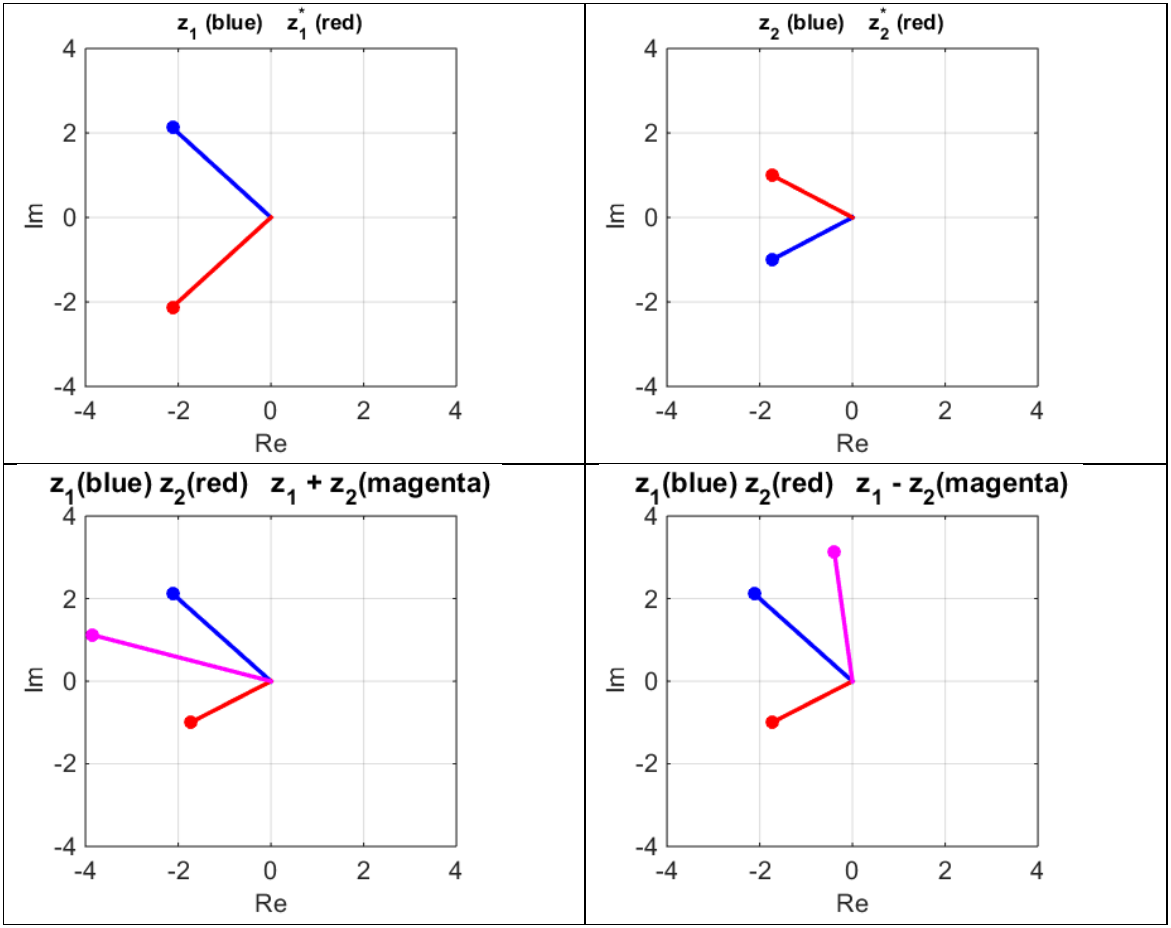
$|z_1| = R_1 = 3 \quad \theta_1 = 135^{\circ} = 3\pi/4 \text{ rad}$

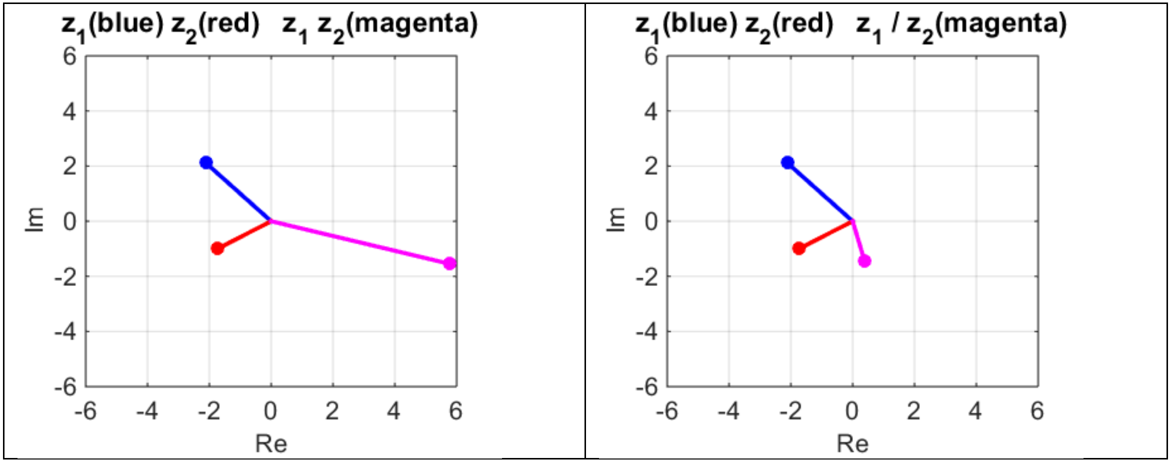
$x_1 = 3\cos(3\pi/4) = -2.1213 \quad y_1 = 3\sin(3\pi/4) = 2.1213$

$|z_2| = R_1 = 2 \quad \theta_2 = -150^{\circ} = -5\pi/6 \text{ rad}$

$x_2 = 2\cos(-5\pi/6) = -1.7321 \quad y_2 = 2\sin(-5\pi/6) = -1.0000$

z	**Re(z) = *x***	**Im(z) = *y***	**|z| = *R***	***Arg*(z) = *θ* rad**	***Arg*(z) = *θ* deg**
z_1	-2.1213	2.21213	3.0000	- $3\pi/4$	135.00
$\overline{z}_1$	-2.1213	-2.1213	3.0000	- $\pi/4$	- 45.00
z_2	-1.7321	-1.0000	2.0000	- $5\pi/6$	-150.00
$\overline{z}_2$	-1.7321	-2.5981	1.0000	$\pi/4$	45.00
$z_1 + z_2$	-3.8534	1.1213	4.0132	2.8584	163.78
$z_1 - z_2$	-0.3893	3.1455	3.1455	1.6949	97.11
$z_1 z_2$	5.7956	-1.5529	6.0000	- 0.2618	-15.00
z_1 / z_2	0.3882	-1.4489	1.5000	-1.3090	- 75.00


z1 (blue) z1* (red)
z2 (blue) z2* (red)
z1(blue) z2(red) z1 + z2(magenta)
z1(blue) z2(red) z1 - z2(magenta)
Im
Re
4
2
0
-2
-4



z1(blue) z2(red) z1 z2(magenta)
z1(blue) z2(red) z1 / z2(magenta)
Im
Re
6
4
2
0
-2
-4
-6

a003

Prove that $\ln\left[\frac{1}{\sqrt{2}}\left(1+i\right)\right]=\frac{i\,\pi}{4}$

$$z=\frac{1}{\sqrt{2}}\left(1+i\right) \quad \left|z\right|=1=R \qquad \theta=Arg\left(z\right)=\operatorname{atan}\left(1\right)=\pi/4$$

$$z=R\,e^{\,i\,\theta}=e^{\,i\,(\pi/4)}$$

$$\ln\left(\frac{1}{\sqrt{2}}\left(1+i\right)\right)=\ln\left(e^{\,i\,(\pi/4)}\right)=i\,\left(\pi/4\right)$$

a004

Prove that $\ln(a+i\,b)=\ln\left(\left|a+i\,b\right|\right)+i\ \mathrm{atan}\left(\dfrac{\mathrm{b}}{\mathrm{a}}\right)$

$$z=\left(a+i\,b\right)\quad \left|z\right|=\left|a+i\,b\right|=R\qquad \theta=Arg\left(z\right)=\mathrm{atan}\left(b/a\right)$$

$$z=R\,e^{i\,\theta}=\left|a+i\,b\right|e^{\,i\left(\mathrm{atan}\left(b/a\right)\right)}$$

$$\ln\left(a+i\,b\right)=\ln\left(\left|a+i\,b\right|e^{\,i\left(\mathrm{atan}\left(b/a\right)\right)}\right)$$

$$=\ln\left(\left|a+i\,b\right|\right)+\ln\left(e^{\,i\left(\mathrm{atan}\left(b/a\right)\right)}\right)$$

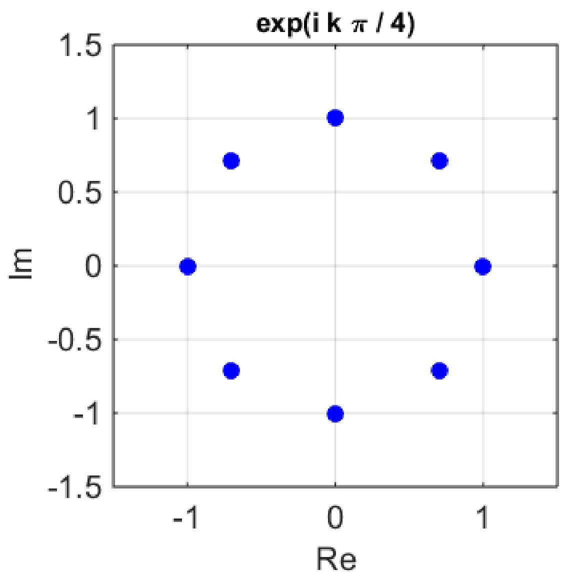
$$=\ln\left(\left|a+i\,b\right|\right)+i\ \mathrm{atan}\left(b/a\right)$$

a005

Express the following in rectangular form for k = 0, 2, ,3, .., , 8.

$$z_k = \mathrm{e}^{i(k\pi/4)} \qquad k = 0, 1, 3, \ldots, 8$$

Plot the complex numbers on an Argand diagram.



$$z_k = e^{i(k\pi/4)} \qquad k = 0,1,3,\ldots,8$$

$$|z_k| = 1 \qquad \theta_k = Arg(z_k) = k\pi/4 \qquad k = 0,1,3,\ldots,8$$

$$\mathrm{Re}(z_k) = x_k = \cos(k\pi/4) \qquad \mathrm{Im}(z_k) = y_k = \sin(k\pi/4)$$

$$z_k = x_k + i\ y_k = \cos(k\pi/4) + i\sin(k\pi/4)$$

$$z_0 = 1$$

$$z_1 = x_1 + i\ y_1 = \cos(\pi/4) + i\sin(\pi/4) = \frac{1}{\sqrt{2}}(1+i)$$

$$z_2 = x_2 + i\ y_2 = \cos(2\pi/4) + i\sin(2\pi/4) = i$$

$$z_3 = x_3 + i\ y_3 = \cos(3\pi/4) + i\sin(3\pi/4) = \frac{1}{\sqrt{2}}(-1+i)$$

$$z_4 = x_4 + i\ y_4 = \cos(4\pi/4) + i\sin(4\pi/4) = -1$$

$$z_5 = x_5 + i\ y_5 = \cos(5\pi/4) + i\sin(5\pi/4) = \frac{-1}{\sqrt{2}}(1+i)$$

$$z_6 = x_6 + i\ y_6 = \cos(6\pi/4) + i\sin(6\pi/4) = -i$$

$$z_7 = x_7 + i\ y_7 = \cos(7\pi/4) + i\sin(7\pi/4) = \frac{1}{\sqrt{2}}(1-i)$$

$$z_8 = x_8 + i\ y_8 = \cos(8\pi/4) + i\sin(8\pi/4) = 1$$

a006

Prove the following using the properties of complex numbers

$$\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$$

$$\sin(\theta+\phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$$

$$z_1 = \cos(\theta) + i\sin(\theta) = e^{i\theta}$$

$$z_2 = \cos(\phi) + i\sin(\phi) = e^{i\phi}$$

$$z_1 z_2 = \left[\cos(\theta) + i\sin(\theta)\right]\left[\cos(\phi) + i\sin(\phi)\right] = e^{i(\theta+\phi)}$$

$$\left[\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)\right] + i\left[\cos(\theta)\sin(\phi) - \sin(\theta)\cos(\phi)\right]$$

$$= \cos(\theta+\phi) + i\sin(\theta+\phi)$$

$$\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$$

$$\sin(\theta+\phi) = \cos(\theta)\sin(\phi) - \sin(\theta)\cos(\phi)$$

a007

Form the product $z_1\, z_2$ and the quotient z_1 / z_2 for

$$z_1 = 3\left[\cos\left(\pi/6\right) + i\,\sin\left(\,\pi/6\right)\right] \quad z_2 = 1.5\left[\cos\left(4\pi/3\right) + i\,\sin\left(\,4\pi/3\right)\right]$$

$$z_1 = 3\left[\cos\left(\pi/6\right) + i\,\sin\left(\,\pi/6\right)\right] \quad z_2 = 1.5\left[\cos\left(4\pi/3\right) + i\,\sin\left(\,4\pi/3\right)\right]$$

$$z_1 = 3\,e^{\,i\,(\pi/6)} \qquad z_2 = 1.5\,e^{\,i\,(4\pi/3)}$$

$$z_1\,z_2 = 4.5\,e^{\,i\,(\pi/6+4\pi/3)} = 4.5\,e^{\,i\,(3\pi/2)}$$

$$= 4.5\left[\cos\left(3\pi/2\right) + i\,\sin\left(3\pi/2\right)\right] = -4.5\,i$$

$$z_1 / z_2 = 2\,e^{\,i\,(\pi/6-4\pi/3)} = 2\,e^{\,i\,(-7\pi/3)}$$

$$= 2\left[\cos\left(-7\pi/3\right) + i\,\sin\left(-7\pi/3\right)\right] = -1.7321 + i = \sqrt{3} + i$$