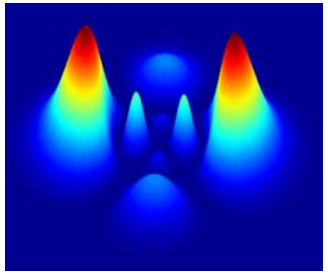


MATHEMATICS EXTENSION 2

TOPIC 5: VOLUMES

Exercise vol5_p004

Find the volumes of the solids of revolution for the function $y = x/2$ and bounded by the X-axis and the vertical lines x_a = 2 and x_b = 4 for the following axes of rotation

(A) X-axis $y_R = 0$

(B) Y-axis $x_R = 0$

(C) $y_R = -2$

(D) $y_R = +2$

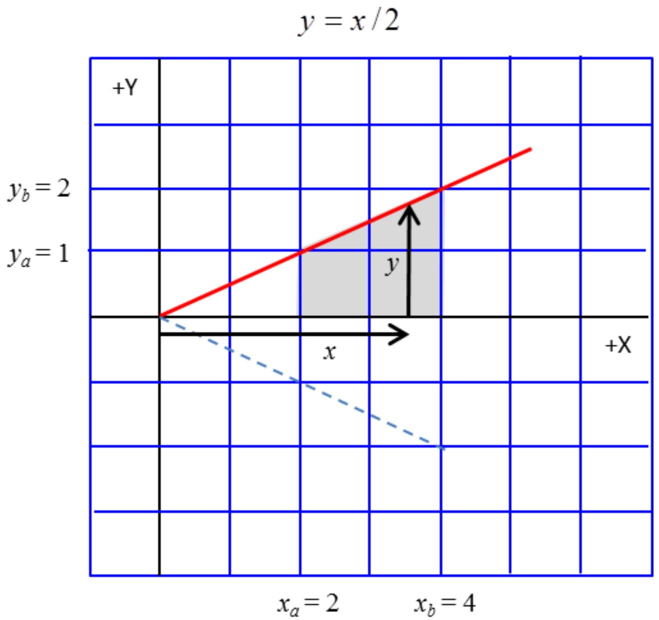
Solution

(A) **rotation around X-axis**

Volume of solid of revolution about the X-axis is

$$V = \pi \int_{x_a}^{x_b} y^2 \, dx \qquad \textbf{Disk Method}$$

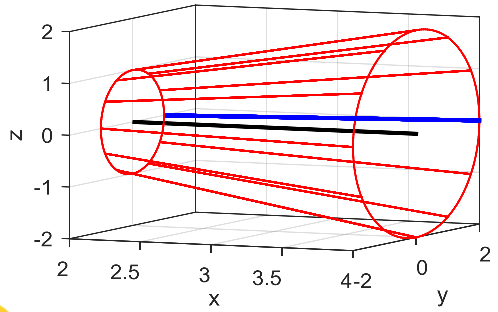
The limits of integration are $x_a = 2$ and $x_b = 4$



The function $y = f(x) \geq 0$ in the interval [2 4] is

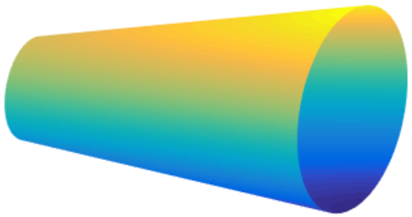
$$y = x/2 \qquad y^2 = x^2/4$$

The volume of the cone is



$$V = \frac{\pi}{4} \int_2^4 x^2 \, dx = \frac{\pi}{4} \left[\tfrac{1}{3} x^3 \right]_2^4 = \frac{\pi}{12} [64 - 8]$$

$$V = \frac{14\pi}{3}$$



(B) **rotation around Y-axis**

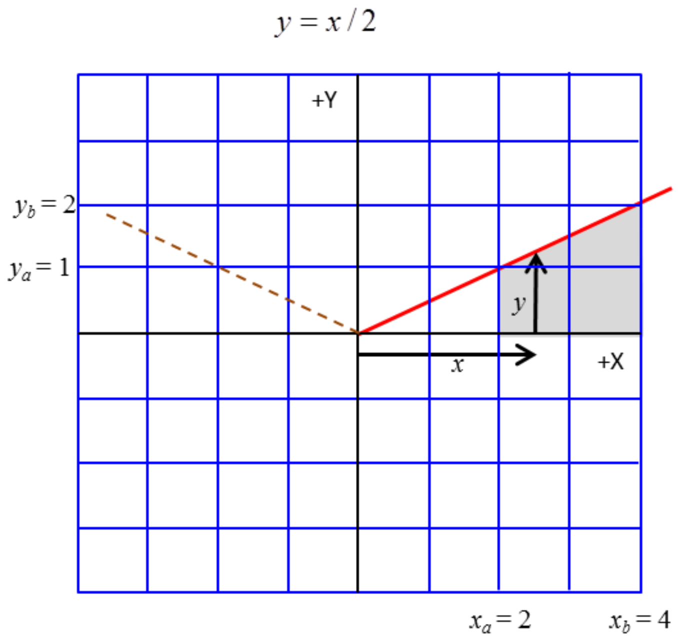
Volume of solid of revolution about the Y-axis is

$$V = 2\pi \int_{x_a}^{x_b} y\, x\, dx$$ **Cylindrical shell method**

The limits of integration are $x_a = 2$ and $x_b = 4$

The function $y = f(x) \geq 0$ in the interval [2 4] is

$$y = x/2$$



The volume of the cone is

$$V = 2\pi \int_2^4 \tfrac{1}{2} x^2\, dx = \pi \left[\tfrac{1}{3} x^3 \right]_2^4 = \frac{\pi}{3}[64 - 8]$$

$$V = \frac{56\pi}{3}$$

(C) **rotation about the horizontal line y_R = -2**

There are a number of ways in which this type of problem can be solved. The method we will use starts with

$$V = \int_{x_a}^{x_b} A(x)\,dx$$

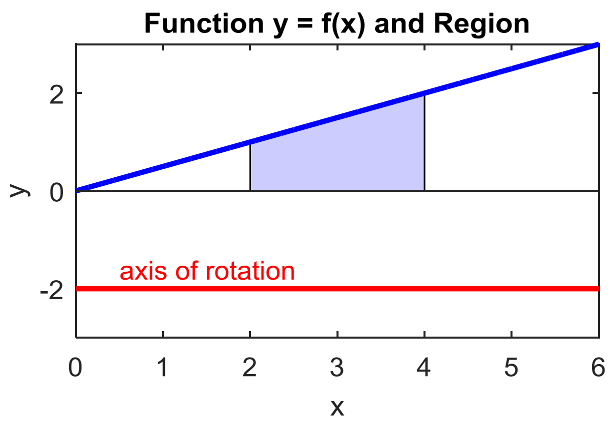
The solid is generated by a rotation through 360°, therefore the cross-sections of the solid of revolution will be circles, hence

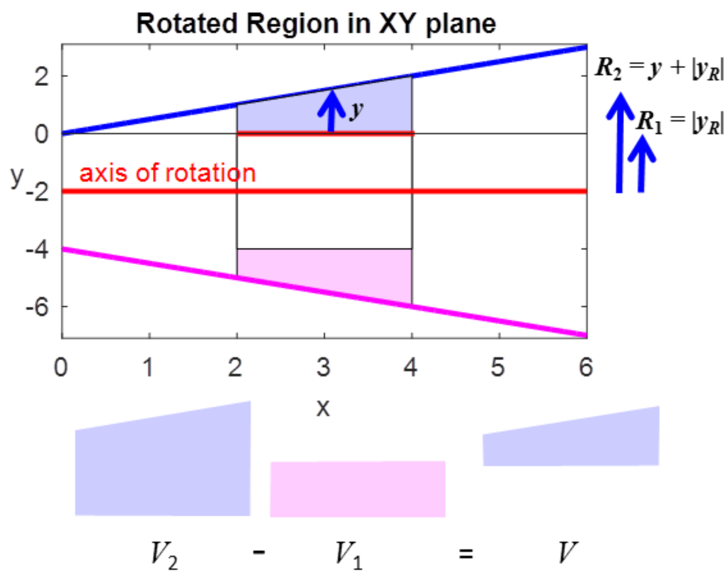
$$A(x) = \pi\, R(x)^2$$

and the volume V is

$$V = \pi \int_{x_a}^{x_b} R(x)^2\,dx$$

You need to carefully draw a set of sketches to clearly identify the region, the axis of rotation and determine the radius $R(x)$.





The function is $y = 0.5x \quad 2 \le x \le 4$ and the axis of rotation is $y_R = -2$. The rotation of the function around the axis of rotation generates a volume V_2.

$$R_2(x) = y + |y_R| = 0.5x + 2 \quad R_2(x)^2 = x^2/4 + 2x + 4 \quad x_a = 2 \quad x_b = 4$$

$$V_2 = \pi \int_2^4 \left(x^2/4 + 2x + 4 \right) dx$$

$$V_2 = \pi \left[\tfrac{1}{12} x^3 + x^2 + 4x \right]_2^4$$

$$V_2 = \left(\tfrac{1}{12}\left(4^3 - 2^3\right) + \left(4^2 - 2^2\right) + 4(4-2) \right) \pi$$

$$V_2 = \frac{74}{3}\pi$$

We need to subtract the volume V_1 generated by revolving the interval $2 \le x \le 4$ along the X-axis (y = 0) around the axis of rotation

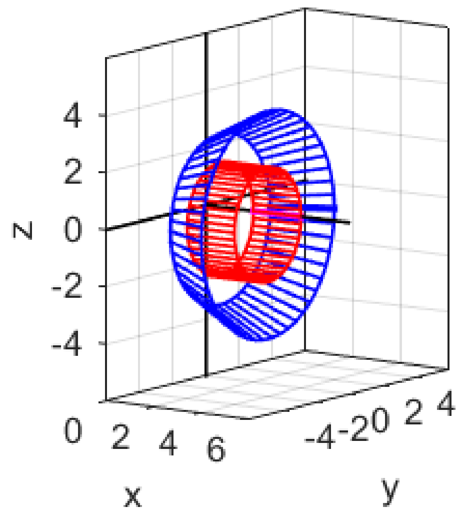
$$R_1(x) = |y_R| = 2 \quad R_1(x)^2 = 4 \quad x_a = 2 \quad x_b = 4$$

$$V_1 = \pi \int_2^4 4\, dx$$

$$V_1 = 4\pi \left[x \right]_2^4$$

$$V_1 = 8\pi$$

The volume V of the solid of revolution of the region is thus



$$V = V_2 - V_1 = \left(\frac{74}{3} - 8 \right) \pi$$

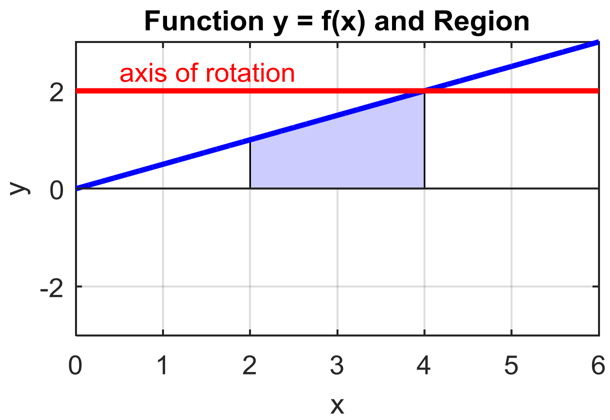
$$V = \frac{50}{3}\pi$$

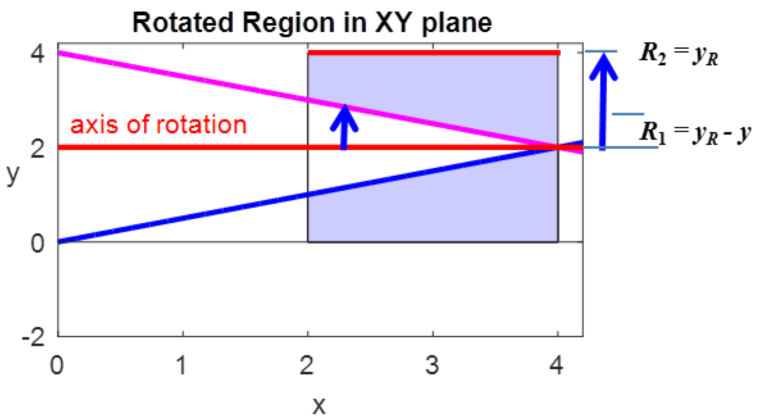
(D) rotation about the horizontal line y_R = +2

The solid is generated by a rotation through 360°, therefore the cross-sections of the solid of revolution will be circles, hence the volume of the rotated region can be found by evaluating the definite integral

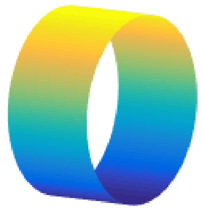
$$V = \pi \int_{x_a}^{x_b} R(x)^2 \, dx$$

You need to carefully draw a set of sketches to clearly identify the region, the axis of rotation and determine the radius $R(x)$.





The function is $y = 0.5x \quad 2 \le x \le 4$ and the axis of rotation is $y_R = +2$.



The volume V_2 generated by revolving the interval $2 \le x \le 4$ along the X-axis (y = 0) around the axis of rotation is

$$R_2(x) = |y_R| = 2 \quad R_2(x)^2 = 4 \quad x_a = 2 \quad x_b = 4$$

$$V_2 = \pi \int_2^4 4\,dx$$

$$V_2 = 4\pi \left[\, x \right]_2^4$$

$$V_2 = 8\pi$$

We need to subtract the volume V_1 generated by the rotation of the function around the axis of rotation

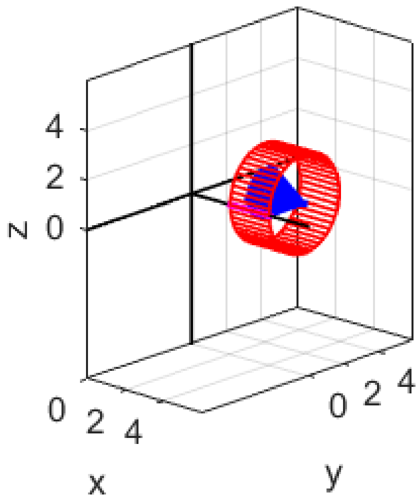
$$R_1(x) = y_R - y = 2 - 0.5x \quad R(x)^2 = x^2/4 - 2x + 4 \quad x_a = 2 \quad x_b = 4$$

$$V_1 = \pi \int_2^4 \left(x^2/4 - 2x + 4\right) dx$$

$$V_1 = \pi \left[\tfrac{1}{12} x^3 - x^2 + 4x \right]_2^4$$

$$V_1 = \left(\tfrac{1}{12}\left(4^3 - 2^3\right) - \left(4^2 - 2^2\right) + 4(4-2) \right)\pi$$

$$V_1 = \frac{2}{3}\pi$$



The volume V of the solid of revolution of the region is thus

$$V = V_2 - V_1 = \left(8 - \frac{2}{3}\right)\pi$$

$$V = \frac{22}{3}\pi$$