EXERCISE 44_123

Part (Aa)

Evaluate the integral $\int_{1}^{\sqrt{3}} \frac{1+x}{x^2\left(1+x^2\right)}\,dx$

Part (Ba)

Evaluate the integral $\int_{0}^{1} \left(e^x - 1\right)^{1/2} dx$

Part (C)

Evaluate $I = \int \frac{dx}{x^2\sqrt{1+x}}$

Answer Part (A)

$$I = \int_1^{\sqrt{3}} \frac{1+x}{x^2\left(1+x^2\right)}dx$$

$$\frac{1+x}{x^2\left(1+x^2\right)} = \frac{Ax+B}{x^2} + \frac{Cx+D}{\left(1+x^2\right)}$$

$$N = 1+x = (Ax+B)\left(1+x^2\right) + (Cx+D)\left(x^2\right) = (A+C)x^3 + (B+D)x^2 + Ax + B$$

$$A+C=0 \quad B+D=0 \quad A=1 \quad B=1 \quad \Rightarrow \quad C=-1 \quad D=-1$$

$$\frac{1+x}{x^2\left(1+x^2\right)} = \frac{1}{x} + \frac{1}{x^2} - \frac{x}{1+x^2} - \frac{1}{1+x^2}$$

$$I = \int_1^{\sqrt{3}} \left(\frac{1}{x} + \frac{1}{x^2} - \frac{x}{1+x^2} - \frac{1}{1+x^2}\right)dx$$

$$\int \frac{dx}{x} = \log_e(x) \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$I = \left[\log_e(x) - \frac{1}{x} - \tfrac{1}{2}\log_e\left(1+x^2\right) - \tan^{-1}(x)\right]_1^{\sqrt{3}}$$

$$I = \log_e\left(\sqrt{3}\right) - \frac{1}{\sqrt{3}} + 1 - \tfrac{1}{2}\log_e(2) - \tan^{-1}\left(\sqrt{3}\right) + \tan^{-1}(1)$$

$$\tan^{-1}\left(\sqrt{3}\right) = \pi/3 \quad \tan^{-1}(1) = \pi/4$$

$$I = 1 - \frac{1}{\sqrt{3}} + \log_e\left(\sqrt{\frac{3}{2}}\right) - \frac{\pi}{12}$$

Answer Part (B)

$$I = \int_0^1 \left(e^x - 1\right)^{1/2} dx$$

$$u^2 = e^x - 1 \qquad 2u\,du = e^x\,dx \qquad dx = \frac{2u}{1+u^2} \quad x = 0 \to u = 0 \quad x = 1 \to u = \sqrt{e-1}$$

$$I = 2\int_0^{\sqrt{e-1}} \frac{u^2}{1+u^2}\,du$$

$$\frac{u^2}{1+u^2} = A + \frac{B}{1+u^2} = \frac{A+u^2+B}{1+u^2} \quad A = 1 \quad B = -1 \quad \frac{u^2}{1+u^2} = 1 - \frac{1}{1+u^2}$$

$$I = 2\int_0^{\sqrt{e-1}} \left(1 - \frac{1}{1+u^2}\right) du$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$I = 2\left[u - \tan^{-1}(u)\right]_0^{\sqrt{e-1}}$$

$$I = 2\left(\sqrt{e-1} - \tan^{-1}\left(\sqrt{e-1}\right)\right)$$

Answer Part (C)

$$I = \int \frac{dx}{x^2\sqrt{1+x}}$$

$$u^2 = 1+x \quad 2u\,du = dx \quad x^2 = \left(u^2-1\right)^2 = \left(1-u^2\right)^2 \quad \sqrt{1+x} = u$$

$$I = 2\int \frac{du}{\left(1-u^2\right)^2}$$

$$u = \cos\theta \quad du = -\sin\theta\,d\theta \quad 1-u^2 = \sin^2\theta$$

$$I = -2\int \frac{d\theta}{\sin^3\theta}$$

$$t = \tan\left(\theta/2\right) \quad dt = \frac{1}{2}\left(1+\tan^2(\theta/2)\right)d\theta = \frac{1}{2}\left(1+t^2\right)d\theta \quad d\theta = \frac{2}{1+t^2}dt$$

$$\sin\theta = \frac{2t}{1+t^2} \quad \frac{1}{\sin^3\theta} = \frac{\left(1+t^2\right)^3}{8t^3}$$

$$I = -2\int \left(\frac{\left(1+t^2\right)^3}{8t^3}\right)\left(\frac{2}{1+t^2}\right)dt$$

online review of integration and trig functions (see pages 11-12)

$$I=-\frac{1}{2}\int\left(\frac{1+2t^2+t^4}{t^3}\right)dt=-\frac{1}{2}\int\left(t^{-3}+2t^{-1}+t\right)dt$$

$$I=\left(\frac{1}{4t^2}-\log_e(t)-\frac{1}{4}t^2\right)+K$$

$$\cos\theta=\frac{1-t^2}{1+t^2}\quad \cos\theta+\cos\theta t^2=1-t^2\quad t^2=\frac{1-\cos\theta}{1+\cos\theta}$$

$$I=\left(\frac{1+\cos\theta}{4(1-\cos\theta)}-\log_e\left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)-\frac{1}{4}\left(\frac{1-\cos\theta}{1+\cos\theta}\right)\right)+K$$

$$u=\cos\theta$$

$$I=\left(\frac{1+u}{4(1-u)}-\frac{1}{4}\left(\frac{1-u}{1+u}\right)-\log_e\left(\sqrt{\frac{1-u}{1+u}}\right)\right)+K$$

$$I=\left(\frac{u}{\left(1-u^2\right)}-\frac{1}{2}\log_e\left(\frac{1-u}{1+u}\right)\right)+K$$

$$u=\sqrt{1+x}$$

$$I=-\frac{\sqrt{1+x}}{x}-\frac{1}{2}\log_e\left(\frac{1-\sqrt{1+x}}{1+\sqrt{1+x}}\right)+K$$