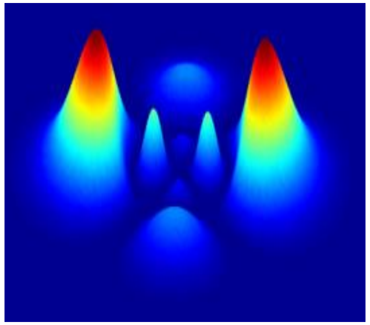


HSC MATHEMATICS: MATHEMATICS EXTENSION 1

3 UNIT MATHEMATICS

TOPIC 3 PROBABILITY

This section will give you a brief introduction to the theory and applications of probability.

Let $P(A)$ be the probability of an event called A occurring when a measurement or experiment is performed.

$$0 \le P(A) \le 1$$

If A is certain to happen then $P(A) = 1$

If A is certain to not happen then $P(A) = 0$

Consider "random experiments" such as coin tossing, throwing dice, drawing balls from an urn, lotteries, dealing cards in which the outcome of events are all **equally likely** outcomes. For example, if an experiment has N possible outcomes, all equally likely and N_s of these leads to "success", then, the probability of success is

Probability of success $\quad P(success) = \dfrac{N_s}{N}$

Example What is the probability of drawing an ace from a shuffled pack of cards?

$\Rightarrow$ There are 52 cards in total and there are 4 aces. $N = 52$ $N_S = 4$

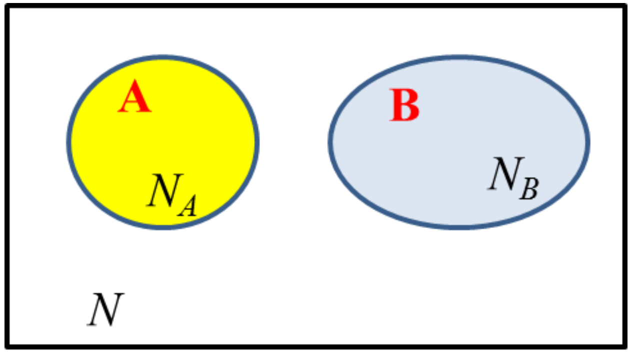
$$P(ace) = \frac{N_s}{N} = \frac{4}{52} = \frac{1}{13}$$

The event that A does **not** occur is written as $\overline{A}$. The events A and $\overline{A}$ are called **complementary**

$$P(A) + P(\overline{A}) = 1 \qquad P(\overline{A}) = 1 - P(A)$$

COMBINING PROABILITIES

Consider an experiment with N equally likely outcomes, involving two events A and B. However, before deciding how to combine probabilities it is necessary to know whether the two events A and B are **mutually exclusive** (A and B can't happen together) or not mutually exclusive.



MUTUALLY EXCLUSIVE EVENTS

Probability of A $\quad P(A)=\dfrac{N_A}{N}$

Probability of B $\quad P(B)=\dfrac{N_B}{N}$

Probability that of either event A **or** event B occurs is the **sum** of the probability of A and B

$$P(A \text{ or } B)=P(A+B)=P(A\cup B)=P(A)+P(B)=\frac{N_A+N_B}{N}$$

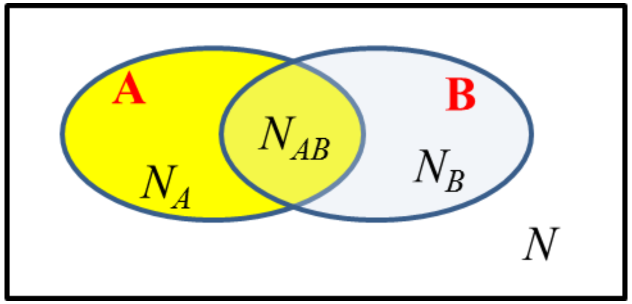
Example What is the probability of drawing an ace or a king from a shuffled pack of cards?

$$\Rightarrow \quad P(ace \text{ or } king)=P(ace\cup king)=\frac{1}{13}+\frac{1}{13}=\frac{2}{13}$$

For n mutually exclusive outcomes A_i (i = 1, 2, … , n), the probability of any one of the outcomes occurring is

$$P(A_1\cup\ldots\cup A_n)=\sum_i P(A_i)$$

NON MUTUALLY EXCLUSIVE EVENTS



Two events may occur together, for example, drawing an ace or a spade from a pack of card.

When two events are not mutually exclusive

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \text{ and } B) = \frac{N_A}{N} + \frac{N_B}{N} - \frac{N_{AB}}{N}$$

Example What is the probability of drawing an ace or a spade from a shuffled pack of cards?

$$\Rightarrow \quad P(ace) = \frac{1}{13} \qquad P(spade) = \frac{13}{52}$$

$P(ace \text{ or } spade)$ is not the sum of these values as the outcomes "ace" and "spade" are not exclusive; it is possible to have them both together by drawing the ace of spades.

$$P(ace \text{ of } spade) = \frac{1}{52}$$

$$P(ace \text{ or } spade) = P(ace) + P(spade) - P(ace \text{ of } spades) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Alternatively use the original definition of probability

$$P(ace \text{ or } spade) = \frac{\text{number of aces and spades}}{\text{total number of cards}} = \frac{4+12}{52} = \frac{16}{52} = \frac{4}{13}$$

MULTIPLYING PROBABILITIES

The probability of both A and B occurring is called the joint probability of A and B or the product of A and B.

$$P(A \text{ and } B) = P(AB) = P(A)\,P(B)$$

provided A is not affected by the outcome of B and B is not affected by the outcome of A, i.e. A and B must **independent**.

Example A card is drawn from two shuffled packs of cards. What is the probability that two aces are drawn?

$$P(ace) = \frac{N_{ace}}{N} = \frac{4}{52} = \frac{1}{13}$$

$$\Rightarrow \quad P(ace\ ace) = P(ace)\ P(ace) = \left(\frac{1}{13}\right)\left(\frac{1}{13}\right) = \frac{1}{169}$$

Example

A number is selected from the set {1 2 3 4 5 6 7 8 9}. Event A occurs if the number is even and event B occurs if the number is less than 6.

$N = 9$ Event A: even number {**2** **4** 6 8} $N_A = 4$ $P(A) = \dfrac{4}{9}$

Event B: < 6 {1 **2** 3 **4** 5} $N_B = 5$ $P(B) = \dfrac{5}{9}$

Events A and B are not mutually exclusive since {2 4} occur in both events.

What is the probability that events A and B both occur?

A and B {2 or 4} $\quad P(A \text{ and } B) = P(AB) = P(2 \text{ or } 4) = \dfrac{2}{9}$

What is the probability that events A or B both occur?

A or B $A \cup B$ {**2** **4** 6 8 1 3 5} $\quad P(A \text{ or } B) = P(A \cup B) = \dfrac{7}{9}$

Alternatively

$$P(A \; or \; B) = P(A \cup B) = P(A) + P(B) - P(A \text{ and} B) = \frac{4}{9} + \frac{5}{9} - \frac{2}{9} = \frac{7}{9}$$

NON-INDEPENDENT EVENTS

A card is draw from a single pack, then a second card is drawn without putting the first card back in the pack. What is the probability that I draw two aces?

$\Rightarrow$ This time the probability that I get an ace as the second card is affected by whether or not I removed an ace from the pack when I drew the first card. We use the notation $P(B \mid A)$ to denote the probability that B happens, given that we know that A happened. This is called a **conditional probability**.

$$P(A \text{ and } B) = P(B \mid A)\, P(A)$$

Hence, the probability of both cards being aces is

$$P(\,[second = ace] \text{ and } [\,first = ace]\,) = P(\,second = ace \mid first = ace\,)\, P(\,first = ace\,)$$

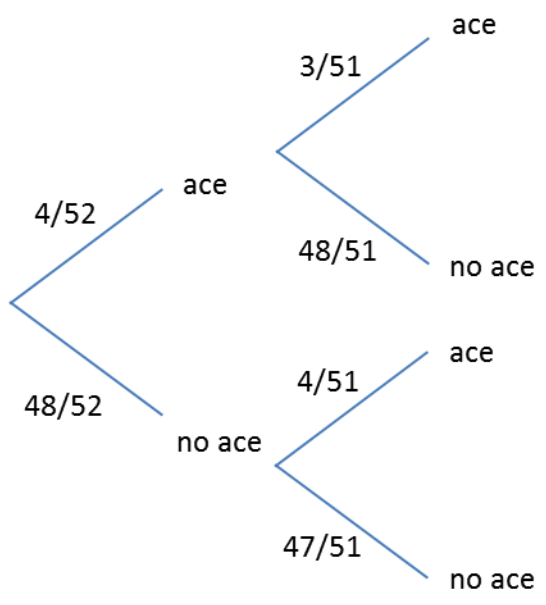
$$P(\,first = ace\,) = \frac{4}{52} = \frac{1}{13} \quad P(\,second = ace \mid first = ace\,) = \frac{3}{51}$$

$$P(\,[second = ace] \text{ and } [\,first = ace]\,) = \left(\frac{3}{51}\right)\left(\frac{1}{13}\right) = \left(\frac{1}{221}\right)$$

TREE DIAGRAMS

Tree diagrams are very useful to trace possible outcomes of two or more stages of an experiment and then to calculate the probabilities of certain final events.

Tree diagram for drawing two cards from a pack of card – probabilities of aces being drawn



2 aces $P_1 = (4/52)(3/51) = 3/663$

1 aces $P_2 = (4/52)(48/51) = 48/663$

1 aces $P_3 = (48/52)(4/51) = 48/663$

0 aces $P_4 = (48/52)(47/51) = 564/663$

$P_1 + P_2 + P_3 + P_4 = (3 + 48 + 48 + 564) / 663 = 1$
P(two aces) = 3/663
P(one ace) = (48+48)/663 = 96/663
P(no aces) = 564/663

What is the probability of ace being drawn as the second card when the first card was not an ace ?

$\Rightarrow$ from the tree diagram $P = 4/51$

Example

A large number of red and green balls are in an urn in the ratio of red to green of 3 to 7. A ball is selected and then returned, then another ball is selected and the process is repeated again. Find the probability for 3 balls being chosen such that (1) exactly two are red; (2) at least one is green; (3) three are all red or three are all green; and (4) one ball is white.

$$\Rightarrow \begin{array}{l} P(\text{Red}) = p = 3/10 \qquad P(\text{Green}) = q = 7/10 \\ P(\text{Red}) + P(\text{Green}) = 1 \quad \Rightarrow \quad P(\text{White}) = 0 \end{array}$$

You can draw a tree diagram to work out all the combination of events.

On each selection of a ball there are two possible outcomes R or G, therefore, the total number of outcomes for the drawing of the three balls is $N = (2)(2)(2) = 8$. The 8 outcomes and their probabilities are

$$\begin{array}{llll} P(RRR) = p^3 & P(RRG) = p^2 q & P(RGR) = p^2 q & P(RGG) = p\,q^2 \\ P(GGG) = q^3 & P(GGR) = p\,q^2 & P(GRG) = p\,q^2 & P(GRR) = p^2 q \end{array}$$

(1)
$$\begin{array}{l} P(2 \text{ red balls}) = P(RRG) + P(RGR) + P(GRR) \\ P(2 \text{ red balls}) = p^2 q + p^2 q + p^2 q = \left[(9)(7) + (9)(7) + (9)(7)\right]/1000 = 189/1000 \end{array}$$

(2) $P(\text{at least 1 G ball}) = 1 - P(RRR) = 1 - (3/10)^3 = 973/1000$

(3) $P(RRR) + P(WWW) = p^3 + q^3 = (3/10)^3 + (7/10)^3 = 370/1000$

(4) There are no white balls $P(W) = 0$

Notations

Intersection AND $\cap$

Union OR $\cup$

$$P(A \text{ and } B) = P(A \cap B) = P(B \mid A)\; P(A)$$

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example A pair of dice is thrown. What is the probability that they show a total of 3?

$\Rightarrow$ For one dice, the total number of outcomes is 6. When two dice are thrown, the total number of outcomes is N = (6)(6) = 36. There are only two ways of getting the sum of the dice to be equal to 3 {1 2} and {2 1} hence number of successful outcomes is N_s = 2. Therefore, the probability of a total of 3 is $P = N_s / N = 2/36 = 18$

Example A card is drawn randomly from a pack of 52 cards. What is the probability that is an odd-numbered card?

$\Rightarrow$ The total number of outcomes is N = 52. The odd cards of any suit are {1 3 5 7 9} and there are 4 suits {spades hearts diamonds clubs}. The number of successful outcomes is N_s = (5)(5) = 20. Hence the probability of an odd card is $P = N_s / N = 20/52 = 5/13$