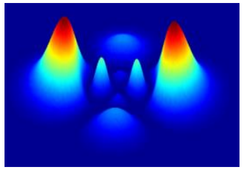


MATHEMATICS EXTENSION 2

4 UNIT MATHEMATICS

TOPIC 5: VOLUMES

5.5 SOLIDS OF REVOLUTION (Part 3)

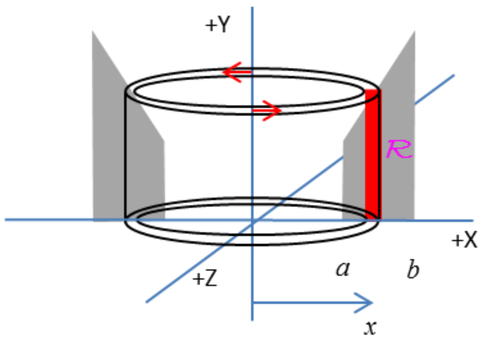
Cylindrical Shell Method

Let $f(x)$ be a continuous functions such that

$$f(x) \geq 0 \quad a \leq x \leq b \quad a \geq 0$$

The region $\mathcal{R}$ is the area bounded by the function $y = f(x)$ and the X-axis in the interval $[a, b]$. Revolving of the region $\mathcal{R}$ about the Y-axis generates a solid called a **solid of revolution**. Consider a thin strip of the region $\mathcal{R}$ at position x and of width Δx and height $y = f(x)$. When this strip is rotated around the Y-axis it sweeps out a cylindrical shell of radius x, thickness Δx and height $y = f(x)$. The volume of this thin cylindrical shell is

$$V_{shell} = (2\pi\, x\, \Delta x)\, y$$



By letting $\Delta x \to 0$ and summing the contribution of each shell for x in the interval $a \leq x \leq b$, the volume V of the solid of revolution is given by the definite integral

$$V = 2\pi \int_a^b x\, y\, dx$$

cylindrical shell formula

Example

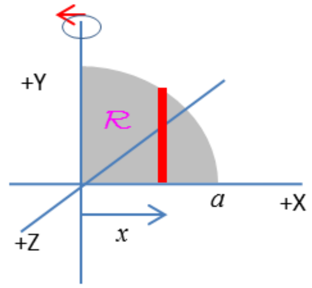
Consider the region $\mathcal{R}$ formed by the part of a circle of radius a in the first quadrant and the X-axis. Revolution about the Y-axis of the region $\mathcal{R}$ produces a solid hemisphere. Find the volume V of the hemisphere by the cylindrical shell method and the disk method.

How to approach the problem:
Sketch the function and the solid.
Give the equations for the shape of the solid. Find the upper and lower limits for the bounded region.
Evaluation the definite integral to find the volume.

Solution

Volume of solid of revolution about the Y-axis using the **cylindrical shell method** is

$$V = 2\pi \int_a^b x\, y\, dx$$



The equation of the circle is $\quad x^2 + y^2 = a^2$

For the region of the circle in the first quadrant

$$y = \left(a^2 - x^2\right)^{1/2}$$

and the limits of integration are 'a' = 0 and 'b' = 0

The volume is

$$V = 2\pi \int_0^a x\, y\, dx = \int_0^a x\left(a^2 - x^2\right)^{1/2} dx$$

$$u = a^2 - x^2 \quad du = -2\,x\,dx \quad x\,dx = \left(-1/2\right) du$$

$$x = 0 \quad u = a^2 \qquad x = a \quad u = 0$$

$$V = -\pi \int_{a^2}^0 u^{1/2} du = \pi \int_0^{a^2} u^{1/2} du$$

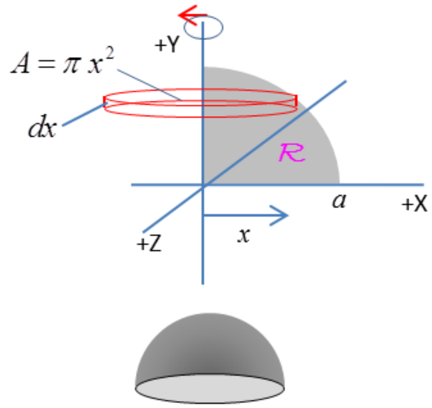
$$V = \left(\frac{2\pi}{3}\right) a^3$$

QED

Volume of solid of revolution about the Y-axis using the **disk method** is

$$V = \pi \int_a^b x^2 \, dy$$

The equation of the circle is $\quad x^2 + y^2 = a^2$



For the region of the circle in the first quadrant

$$x^2 = \left(a^2 - y^2\right)$$

and the limits of integration are $'a' = 0$ and $'b' = 0$

The volume is

$$V = \pi \int_0^a \left(a^2 - y^2\right)^2 dy$$

$$V = \pi \left[a^2 y - \tfrac{1}{3} y^3\right]_0^a = \pi \left(a^3 - \tfrac{1}{3} a^3\right)$$

$$V = \left(\frac{2\pi}{3}\right) a^3$$

QED