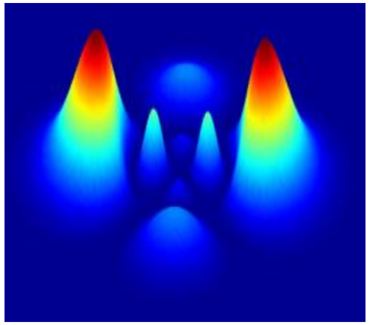

MATHEMATICS EXTENSION 2

4 UNIT MATHEMATICS

TOPIC 7: POLYNOMIALS

7.5 FUNDAMENTAL THEOREM OF ALGEBRA

COMPLEX ROOTS AND MULTIPLE ROOTS

Fundamental Theorem of algebra: every polynomial can be factored (over the real numbers) into a product of linear factors and irreducible quadratic factors.

The Fundamental Theorem of Algebra was first proved by **Carl Friedrich Gauss** (1777-1855).

Whenever a polynomial has been factored into only **linear** and **irreducible quadratics**, then it has been **factored completely**, since both linear factors and irreducible quadratics cannot be factored any further over the real numbers.

There are **no** general rules for factoring completely a polynomial when the number of degrees $n > 4$.

Is factorization unique? Yes, and No!

$$x^3 - x^2 = x^2\left(x-1\right) \quad \Rightarrow \quad \text{factors are } \; x^2 \; \text{ and } \; \left(x-1\right)$$

$$6x^3 - x^2 = 6x^2\left(x-1/6\right) = x^2\left(6x-1\right) \qquad \text{what are the factors?}$$

We can solve the problem by insisting on the factors have leading coefficient 1, and the leading coefficient of the original polynomial is written in front of the factors

$$6x^3 - x^2 = 6x^2\left(x-1/6\right)$$

MULTIPLE ROOTS

Example Find all real roots and their multiplicity of the polynomial

$$P(x) = (x-5)^3 (3x+4)^2 \left(x^2+2\right)^2 \left(x+\pi^2\right)^4$$

Solution

$x = 5$ multiplicity = 3

$x = 4/3$ multiplicity = 2

$x^2 + 2 = 0$ no real roots multiplicity = 2

$x = -\pi^2$ multiplicity = 4

degree of polynomial n = 3 + 2 + 2 + 2 + 4 = 13

Consider the polynomial $P(x)$ with the root α $x = \alpha$ $(x-\alpha) = 0$ with multiplicity *r*, then, the polynomial dP/dx has multiplicity of $(r-1)$

$$P(x) = (x-\alpha)^r S(x)$$
$$dP/dx = r(x-\alpha)^{r-1} S + (x-\alpha)^r dS/dx = (x-\alpha)^{r-1}\left(rS + (x-\alpha)dS/dx\right)$$

$\Rightarrow$ multiplicity = $(r-1)$ for dP/dx

$(x-\alpha)$ is called a factor of $P(x)$ of **order** *r*.

COMPLEX NUMBERS

Another statement of the **Fundamental Theorem of Algebra**:

Every polynomial equation of degree n with complex coefficients has n roots in the complex numbers.

Note: a **real number** is also complex a number with its imaginary part equal to zero.

For all polynomials with **real coefficients**, complex roots with non-zero imaginary parts $y \neq 0$ always occur as **conjugate pairs**.

If $\alpha = x + i\,y$ then $\beta = x - i\,y$ is also a root.

If $\beta = x - i\,y$ then $\alpha = x + i\,y$ is also a root.

Polynomial: real coefficients degree n	Number of roots	Possible combinations (complex roots $y \neq 0$
1	1	1 real root
2	2	2 real roots or 2 complex roots
3	3	3 real roots or 1 real root and 2 complex roots
4	4	4 real roots or 2 real roots and 2 complex roots or 4 complex roots

When the degree is odd (1, 3, 5, …) there is at least one real root !

Example (HSC 2013/11C)

Factorize the polynomials

$$P(z) = z^2 + 4z - 5 \qquad Q(z) = z^2 + 4z + 5 \qquad R(z) = z^2 + 4i\,z + 5$$

Solution

We need to find the roots of each polynomial. The degree of each polynomial is n =2, therefore in each case there are two roots.

$$P(z) = z^2 + 4z - 5 = 0 \quad \Rightarrow \quad z = 1 \quad z = -5$$

The roots are $\alpha = 1 \quad \beta = -5$ **both roots are real** and the factors are

$$(z-1) \quad (z+5) \qquad P(z) = (z-1)(z+5)$$

$$Q(z) = z^2 + 4z + 5 = 0$$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad a = 1 \quad b = 4 \quad c = 5$$

$$z = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i$$

The two roots are **conjugate pairs** $\alpha = -2 + i \quad \beta = -2 - i$ and the factors are

$$(z+2-i) \quad (z+2+i) \qquad Q(z) = (z+2-i)(z+2+i)$$

$$R(z) = z^2 + 4i\,z + 5 = 0$$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad a = 1 \quad b = 4i \quad c = 5$$

$$z = \frac{-4i \pm \sqrt{-16 - 20}}{2} = -2i \pm 3i \qquad z = i \qquad z = -5i$$

The two roots do not form a conjugate pair because **all the coefficients are not real**

$\alpha = i \quad \beta = -5i$ and the factors are

$$(z - i) \quad (z + 5i) \qquad R(z) = (z - i)(z + 5i)$$