HSC MATHEMATICS: MATHEMATICS EXTENSION 1 (3 UNIT)

TOPIC 18 PERMUTATIONS COMBINATIONS PROBABILITY

EXERCISE ex3u18125

Click Question for answer and click Answer to return to question

Question 1

The letters A A A B B C E are to be arranged in straight line. Calculate the following

(A) The total number of arrangements.

(B) The number of arrangements with the B's at the end.

(C) The number of arrangements with the B's together.

Question 2

Three different physics books, two different chemistry books, one mathematics book and one biology book are to be arranged on a shelf. Calculate the following

(A) The total number of distinct arrangements of the books.

(B) The number of arrangements with the chemistry books at both ends.

(C) The number of arrangements with a physics book at each end.

(D) If the physics books must be placed together in **any order**, what is the number of arrangements?

(E) If the physics books must be together and the chemistry books together, what is the number of arrangements?

Question 3 HSC EXT1 2011/2e

There are 50 different songs on a CD that can be arranged in any order.

(A) How many permutations are there for the 50 songs?

(B) If the five favourite songs are tracks 1, 2, 3, 4, 5 (in any order), then, how many arrangements are there for the 50 songs?

Question 4

Consider the word **PARALLEL**

(A) How many ordered arrangements can be made from all the letters?

(B) How many ordered arrangements have the letter **L** all together?

(C) If the letters are randomly selected and arranged in a straight line, what is the probability that the **L**’s are not together?

Question 5

There are 18 people to select from for a team of 7 to play rugby.

(A) How many ways can the 7 players be selected?

(B) Two players, the captain and vice-captain must be included in the team, how many ways can the 7 players be selected?

(C) Two of the people have the flu and can’t be picked in the team, how many ways can the 7 players be selected?

Question 6

A choir has 32 members with 15 males and 17 females.

(A) In how many ways can a leadership team be selected of five members if there is at least three females?

(B) In how many ways can a leadership team be selected of five members if at least one male is selected?

Question 7

How many ways can 12 pieces of fruit be distributed to 3 children, if each child receives an even number of fruits?

This question is best answered using a spreadsheet.

The MS EXCEL expression for the binomial coefficient ${}^{n}C_{k}$ is =combin(n,k)

Question 8

How many combinations and permutations are of the letters

M A T H S and **M A T S S**

if all 5 letters are selected and if only 3 letters are selected?

Question 9

Consider the words **ARRANGE CARROTS**

(A) How many permutations can be made if all letters are selected (ignore spaces)?

(B) How many permutations can be made if all letters are selected so that the arrangement must start and finish with the letter **A**?

(C) How many permutations can be made if all letters are selected so that the arrangement must start with the letters **ARRA**?

Question 10

If the numbers 3 4 6 7 8 9 are randomly arranged in a straight line to give a six digit number, what is the probability that the number is

(A) an odd number ?

(B) an even number less than 500000 ?

(C) an odd number greater than 500000 ?

Answer 1 **A A A B B C E**

(A) 7 letters $n=7$ 3x A $n_A=3$ 3xB $n_B=2$ Total number of arrangements $N=\dfrac{{}^7P_7}{n_A!\,n_B!}=\dfrac{n!}{n_A!\,n_B!}=\dfrac{7!}{3!\,2!}=420$

(B)

Grouping of letters {B} {A A A C E} {B}

Ways of grouping the Bs at the ends $N_1=1$ Ways of grouping {A A A C E} $N_2=\dfrac{5!}{3!}=20$

Number of arrangements $N=N_1\,N_2=20$

(C)

Grouping of letters {B B} {A} {A} {A} {C} {E}

Ways of grouping the letters $N=\dfrac{6!}{3!}=120$

Answer 2

(A) There are 7 books to be arranged $n = 7$

The number of distinct arrangements of the book $N = n! = 7! = 5040$

(B) Chemistry book at each end {C} {P1 P2 P3 M1 B1} {C}

Arrangements of chemistry books at each end $N_1 = 2$

Arrangements of the set of 5 book {P1 P2 P3 M B} $N_2 = 5! = 120$

Arrangements with chemistry books at the ends $N = N_1\ N_2 = (2)(120) = 240$

(C) Physics book at each end {P} {P C1 C2 M1 B1} {P}

Arrangements of physics books at each end $N_1 = {}^3P_2 = \dfrac{3!}{1!} = 6$

Arrangements of the set of 5 book {P C1 C2 M1 B1} $N_2 = 5! = 120$

Arrangements with physics books at the ends $N = N_1\ N_2 = (6)(120) = 720$

(D)

Arrangements of physics book {P1 P2 P3} in any order $N_1 = 1$

Arrangements of 4 books {C1 C2 M1 B1} $N_2 = 4! = 24$

Arrangements of the two sets of books {P1 P2 P3} and {C1 C2 M1 B1} $N_3 = 2$

Arrangements with grouping of the physics books together in any order $N = N_1\ N_2\ N_3 = (1)(24)(2) = 48$

(E)

Sets of books to be arranged {P1 P2 P3} {C1 C2} {M1} {B1}

Arrangements of the 3 physics books {P1 P2 P3} $N_1 = 3! = 6$

Arrangements of the 2 chemistry books {C1 C2} $N_2 = 2! = 2$

Arrangements of the four sets of books {P1 P2 P3} {C1 C2} {M1} {B1} $N_3 = 4! = 24$

Arrangements with grouping of the physics books together in any order $N = N_1\ N_2\ N_3 = (6)(2)(24) = 188$

Answer 3

(A)

The number of permutations of n distinct objects is $n!$

Number of arrangements of the 50 songs = 50!

(B)

5 songs are arranged first then 45 songs are arranged

Number of arrangements of the 50 songs = (5!) (45!)

Answer 4

P A R A L L E L

Number of letters and repetition of letters (2xA & 3xL) $n = 8 \quad n_1 = 2 \quad n_2 = 3$

(A)

Number of permutations of all the letters $N_{total} = \dfrac{n!}{n_1!\, n_2!} = \dfrac{8!}{2!\,3!} = 3360$

(B) all the L's together

Subsets of 6 letters {L L L} {P} {R} {E} {A} {A} $n = 6 \quad n_1 = 2$

Number of permutations $N = \dfrac{n!}{n_1!} = \dfrac{6!}{2!} = 360$

(C)

Probability that all the L's are together $\text{Prob}\left(LLL\right) = N \,/\, N_{total} = 360 / 3360 = 3/28$

Probability that all the L's are **not** together $\text{Prob}\left(no\ LLL\right) = 1 - N \,/\, N_{total} = 1 - 3/28 = 25/28$

Answer 5

People 18 Team 7

(A)

Number of combinations of 7 people selected from 18 $\quad {}^{18}C_7 = \dfrac{18!}{7!\,(18-7)!} = 31824$

(B)

Two people included in team: people 16 Team 5

Number of combinations of 5 people selected from 16 $\quad {}^{16}C_5 = \dfrac{16!}{5!\,(16-5)!} = 4368$

(C)

Two people **not** included in team: people 16 Team 7

Number of combinations of 7 people selected from 16 $\quad {}^{16}C_7 = \dfrac{16!}{7!\,(16-7)!} = 11440$

Answer 6

(A)

The leadership team can have 3 or 4 or 5 females selected

Number of ways of selecting 3 females $n_1 = \left({}^{17}C_3\right)\left({}^{15}C_2\right) = 71400$

Number of ways of selecting 4 females $n_2 = \left({}^{17}C_4\right)\left({}^{15}C_1\right) = 35700$

Number of ways of selecting 5 females $n_3 = \left({}^{17}C_5\right)\left({}^{15}C_0\right) = 6188$

Numbers of ways of selecting 3 or more females $n = n_1 + n_2 + n_3 = 71400 + 35700 + 6188 = 113288$

(B)

The total number of ways of selecting 5 members from 32 is $N_{total} = {}^{32}C_5 = 201376$

If no males are selected then the 5 females are selected in $n_3 = \left({}^{17}C_5\right)\left({}^{15}C_0\right) = 6188$ ways

The number of ways in which at least one male is be selected is $N = N_{total} - n_3 = 201376 - 6188 = 195188$

Answer 7

How many ways can 12 pieces of fruit be distributed to 3 children, if each child receives an even number of fruits?

Step 1: determine the ways in which the fruits can be divided so that each child gets an even number

Step 2: determine the ways child #1 gets the fruit, then child #2 and then child # 3 using the formula ${}^{n}C_{k}$

Step 3: multiple together the number of ways for each child $N = \left({}^{n_1}C_{k_1}\right)\left({}^{n_2}C_{k_2}\right)\left({}^{n_3}C_{k_3}\right)$

Step 4: add all the ways for each distribution $N_{total} = \sum N$

distributions			child #1		child #2		child #3		child #1	child #2	child #3	ways
child #1	child #2	child # 3	n1	k1	n2	k2	n3	k3	nCk	nCk	nCk	N
2	2	8	12	2	10	2	8	8	66	45	1	2970
2	8	2	12	2	10	8	2	2	66	45	1	2970
8	2	2	12	8	4	2	2	2	495	6	1	2970
4	2	6	12	4	8	2	6	6	495	28	1	13860
4	6	2	12	4	8	6	2	2	495	28	1	13860
2	4	6	12	2	10	4	6	6	66	210	1	13860
2	6	4	12	2	10	6	4	4	66	210	1	13860
6	2	4	12	6	6	2	4	4	924	15	1	13860
6	4	2	12	6	6	4	2	2	924	15	1	13860
4	4	4	12	4	8	4	4	4	495	70	1	34650
											sum -->	**126720**

The total number of ways the fruit can be divided amongst the children is **126720**

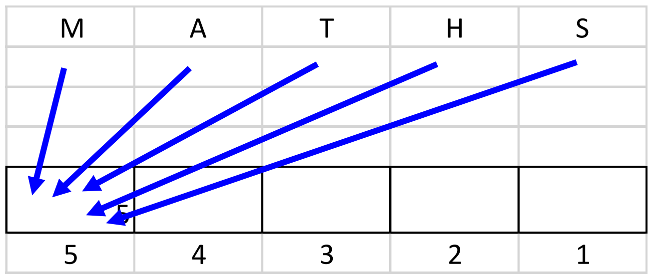
Answer 8

The number of permutations (ordered arrangements) of the letters M A T H S are:

MATHS MATSH MAHTS MAHST MASHT MASTH MTAHS MTASH MTHAS MTHSA MTSHA MTSAH MHTAS MHTSA MHATS MHAST MHSAT MHSTA MSTHA MSTAH MSHTA MSHAT MSAHT MSATH
AMTHS AMTSH AMHTS AMHST AMSHT AMSTH ATMHS ATMSH ATHMS ATHSM ATSHM ATSMH AHTMS AHTSM AHMTS AHMST AHSMT AHSTM ASTHM ASTMH ASHTM ASHMT ASMHT ASMTH
TAMHS TAMSH TAHMS TAHSM TASHM TASMH TMAHS TMASH TMHAS TMHSATMSHA TMSAH THMAS THMSA THAMS THASM THSAM THSMA TSMHA TSMAH TSHMA TSHAM TSAHM TSAMH
HATMS HATSM HAMTS HAMST HASMT HASTM HTAMS HTASM HTMAS HTMSA HTSMA HTSAM HMTAS HMTSA HMATS HMAST HMSAT HMSTA HSTMA HSTAM HSMTA HSMAT HSAMT HSATM
SATHM SATMH SAHTM SAHMT SAMHT SAMTH STAHM STAMH STHAM STHMA STMHA STMAH SHTAM SHTMA SHATM SHAMT SHMAT SHMTA SMTHA SMTAH SMHTA SMHAT SMAHT SMATH

If you count them, there are 120 permutations. Why?

Think of each of the 5 letters as 5 objects to be placed into one of 5 boxes with no repetitions



5 letters can go into first box, 4 letters into the second box, 3 letters into the third box, 2 letters into the fourth box and the last letter into the fifth box. So the number of permutations or ordered arrangements N_P is

$$N_P = (5)(4)(3)(2)(1) = 5! = 120$$

Let the symbol n represent the number of objects and k represent the number of boxes (or number of selections). The number of **permutations** N_P is given by

$$N_P = {}^nP_k = \frac{n!}{(n-k)!}$$

In our example $n = 5 \quad k = 5 \quad {}^5P_5 = \frac{5!}{0!} = 5! = 120 \qquad 0! = 1$

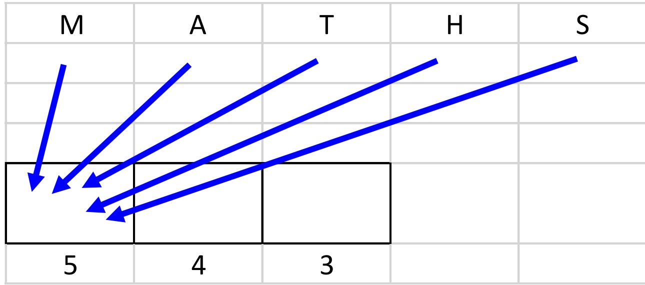
If the order of the objects (letters) does not matter, how many ways can we arrange the objects? 5 objects can be arranged in 5! ways. The number of unordered arrangements is called the number of **combinations** N_C

$N_C = \frac{5!}{5!} = 1$ there is only 1 unordered arrangement of all 5 letters

So in general, the relationship between the number of permutations (ordered selection) and the number of combinations (unordered) is

$${}^nC_k = \frac{{}^nP_k}{k!}$$

We will now consider selecting 3 of the letters, i.e. placing any one of the 5 objects into 3 boxes



$$n = 5 \quad k = 3$$

$$N_P = (5)(4)(3) = 60$$

Number of permutations (ordered arrangements) $\quad N_P = {}^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = (5)(4)(3) = 60$

The three objects can be arranged in 3! = (3)(2)(1) ways, hence, the number of combinations (unordered arrangements) is

$${}^5C_3 = \frac{{}^5P_3}{3!} = \frac{60}{6} = 10$$

The 10 combinations are THS AHS ATS ATH MHS MTS MTH MAS MAH MAT

There are 10 combinations of three distinct letters. The three letters can be arranged in (3)(2)(1) = 3! = 6 ways, therefore, the number of permutations is (10)(6) = 60.

What about the case where some of the letters (objects) are the same?

MAT**HS** and MAT**SH** are two distinct permutations. What about MAT**SS**? Interchanging the S and S makes no difference, there is only one permutation. In general for n objects and k selections, with n_1 objects the same, n_2 objects the same and so on, the number of permutations N_P is

$$N_P = \frac{{}^nP_k}{n_1!\ n_2!\dots}$$

For the letters MATSS $n = 5$ $n_1 = 2$ with 3 selections $k = 3$ the number of permutations is

$$N_P = \frac{{}^5P_3}{2!} = \frac{5!}{(2!)\,(2!)} = 30$$

It is a more difficult question to calculate the number of arrangements when some of the letters are the same and not all the letters are selected $(k < n)$. We have to consider carefully all the cases for the repeated letters (objects) separately.

Consider the case of selecting 3 letters from MAT**SS**

zero S's {MAT} combinations $N_C = 1$

The three letters can be arranged in 3! ways so the number of permutations N_P is

$N_P = (1)(3!) = 6$

one S {Sxx} combinations $N_C = {}^3C_2 = \dfrac{3!}{2!\,1!} = 3$

The three letters can be arranged in 3! ways so the number of permutations N_P is

$N_P = (3)(3!) = 18$

Two S's {SSx} combinations $N_C = {}^3C_1 = \dfrac{3!}{1!\ 2!} = 3$

The three letters can be arranged in 3!/2! = 3 ways so the number of permutations N_P is

$N_P = (3)(3) = 9$

Total number of combinations = (1 + 3 + 3) = 7 TSS ASS ATS MSS MTS MAS MAT

Total number of permutations = (6 + 18 + 9) = 33

Answer 9

A R R A N G E C A R R O T S

Number of letters $n = 14$ repetition of letters $n_A = 3$ $n_R = 3$

(A)

Number of permutations $N = \dfrac{14!}{(3!)\,(4!)} = 605404800$

(B) **A** xxxxxxxxxxxx **A**

Number of ways to arrange the A's at the ends $N_1 = (3)(2) = 6$

Permutations of remaining letters **RRNGECARROTS** $n = 12$ repetition of letters $n_R = 4$

Number of permutations $N_2 = \dfrac{12!}{4!} = 19958400$

Total number of permutations $N = N_1\ N_2 = (6)\ (19958400) = 119750400$

(C) **ARRA** xxxxxxxxxx

Number of ways to arrange the A's $N_1 = (3)(2) = 6$

Number of ways to arrange the R's $N_2 = (4)(3) = 12$

Permutations of remaining letters **NGECARROTS** $n = 10$ repetition of letters $n_R = 2$

Number of permutations $N_3 = \dfrac{10!}{2!} = 1814400$

Total number of permutations $N = N_1\ N_2\ N_3 = (6)(12)(1814400) = 130636800$

Question 10

{3 4 6 7 8 9} $n = 6$

(A)

Total number of permutations $N = n! = 6! = 720$

(B)

Odd number x x x x x {3 7 9}

Number of permutations for an odd number $N = (5!)\ (3) = 360$

$\text{Prob(odd number)} = \frac{360}{720} = \frac{1}{2}$ obviously correct since we have three odd numbers {3 7 9} and three even numbers (4 6 8}

(C)

Odd number < 500000

The format of the numbers are of the form

4 xxxx {6 8} permutations $N_1 = (1)(4!)(2) = 48$

3 xxxx {4 6 8} permutations $N_2 = (1)(4!)(3) = 72$

Number of permutations $N = N_1 + N_2 = 48 + 72 = 120$

$$\text{Prob(even number} < 500000) = \frac{120}{720} = \frac{1}{6}$$

(D)

odd number > 500000

The format of the numbers are of the form

6 xxxx {3 7 9} permutations $N_1 = (1)(4!)(3) = 72$

7 xxxx {3 9} permutations $N_2 = (1)(4!)(2) = 48$

8 xxxx {3 7 9} permutations $N_3 = (1)(4!)(3) = 72$

9 xxxx {3 7} permutations $N_4 = (1)(4!)(2) = 48$

Number of permutations $N = N_1 + N_2 + N_3 + N_4 = 72 + 48 + 72 + 48 = 240$

$$\text{Prob(odd number} > 500000) = \frac{240}{720} = \frac{1}{3}$$