EXERCISE 71_123

Part (Aa)

The roots of the quadratic equation $a_1 x^2 + b_1 x + c_1 = 0$ are $(\alpha, k\alpha)$ and the roots of the equation $a_2 x^2 + b_2 x + c_2 = 0$ are $(\beta, k\beta)$. Show that

$$a_1 b_2^{\,2} c_1 = a_2 b_1^{\,2} c_2$$

Part (Ba)

Show that the necessary and sufficient conditions that the roots of the equation $a x^2 + b x + c = 0$ are real and greater than 1 are

$$b^2 = 4ac > 0 \qquad b / a < -2 \qquad (b + c) / a > -1$$

Part (Ca)

If a, b and c are real constants and $c \neq 0$, show that the roots of the quadratic equation are real and unequal

$$(x-a)(x-b)=c^2$$

If α and β are the roots of this equation, find the equation whose roots are α/β and β/α.

Part (Da)

Find the values of k for which the quadratic equation is a perfect square

$$(x+1)(x+4)+k(x-1)(x-4)$$

Part (Ea)

(α,β,γ) are the roots of the cubic equation

$$x^3+bx^2+cx+d=0$$

and

$$\alpha^2+\beta^2+\gamma^2=14 \qquad \alpha^3+\beta^3+\gamma^3=20 \qquad \alpha^4+\beta^4+\gamma^4=98$$

Determine all the possible values of a, b, and c.

Find a set of possible integer values of the roots (α,β,γ).

Answer Part (A)

$$a_1x^2 + b_1x + c_1 = 0$$
$$a_2x^2 + b_2x + c_2 = 0$$

$$\alpha + k\alpha = \alpha\left(1+k\right) = -b_1 / a_1 \qquad k\alpha^2 = c_1 / a_1$$

$$\alpha^2 = \frac{b_1^{\,2}}{a_1^{\,2}\left(1+k\right)^2} \quad k\alpha^2 = \frac{k\,b_1^{\,2}}{a_1^{\,2}\left(1+k\right)^2} = \frac{c_1}{a_1} \quad \frac{k}{\left(1+k\right)^2} = \frac{a_1\,c_1}{b_1^{\,2}} = \frac{a_2\,c_2}{b_2^{\,2}}$$

$$a_1\,b_2^{\,2}\,c_1 = a_2\,b_1^{\,2}\,c_2 \qquad QED$$

Answer Part (B)

The roots of the quadratic equation $a x^2 + b x + c = 0$ are

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \alpha > \beta > 1$$

For α and β to be real then $b^2 - 4ac > 0$ otherwise the roots will have a non-zero imaginary part.

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} > 1$$

For $\beta > 1$ $\sqrt{b^2 - 4ac} < -(2a + b)$

$$b^2 - 4ac < 4a^2 + 2ab + b^2$$

$$\frac{b + c}{a} > -1$$

$\alpha > \beta > 1$ therefore $\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} > 2 \quad b / a < -2$

For equal roots $\alpha = \beta = -b / 2a > 1 \quad b / a < -2$ *QED*

Answer Part (C)

$$(x-a)(x-b)=c^2$$
$$x^2-(a+b)x+ab-c^2=0$$

Solve the quadratic equation

$$x=\frac{(a+b)\pm\sqrt{(a+b)^2-4\left(ab-c^2\right)}}{2}$$

If the roots are unequal, then $\sqrt{(a+b)^2-4\left(ab-c^2\right)}>0$

$$\sqrt{(a+b)^2-4\left(ab-c^2\right)}>0$$
$$(a+b)^2-4\left(ab-c^2\right)>0$$
$$a^2+b^2+2ab-4ab+4c^2>0$$
$$(a-b)^2+4c^2>0 \quad (a-b)^2>0 \quad 4c^2>0 \quad \Rightarrow \quad \text{roots are real and unequal}$$

The equation whose roots are α / β and β / α is $\left(x-\alpha / \beta\right)\left(x-\beta / \alpha\right)=0$

$$\left(x-\alpha / \beta\right)\left(x-\beta / \alpha\right)=0$$

$$x^2-\left(\alpha / \beta+\beta / \alpha\right)x+1=0$$

$$x^2-\left(\frac{\alpha^2+\beta^2}{\alpha\beta}\right)x+1=0$$

$$\alpha^2+\beta^2=\left(\alpha+\beta\right)^2-2\alpha\beta$$

$$\alpha+\beta=a+b \quad \alpha\beta=ab-c^2$$

$$x^2-\left(\frac{\left(a+b\right)^2-2\left(ab-c^2\right)}{ab-c^2}\right)x+1=0$$

$$\left(a+b\right)^2-2\left(ab-c^2\right)=a^2+b^2+2c^2$$

$$\left(ab-c^2\right)x^2-\left(a^2+b^2+2c^2\right)+\left(ab-c^2\right)=1$$

QED

Answer Part (D)

$$f(x) = (x+1)(x+4) + k(x-1)(x-4)$$
$$f(x) = (1+k)x^2 + 5(1-k)x + 4(1+k)$$

A perfect square has equal roots $\alpha = \beta$

$$\alpha + \beta = 2\alpha = \frac{-5(1-k)}{(1+k)} \qquad \alpha\beta = \alpha^2 = \frac{4(1+k)}{(1+k)} = 4 \quad \alpha = \pm 2$$
$$k = \frac{-(2\alpha+5)}{2\alpha-5}$$
$$\alpha = 2 \quad k = 9 \qquad \alpha = -2 \quad k = 1/9$$
$$k = 9 \quad f(x) = 10(x^2 - 4x + 4) = 10(x-2)^2$$
$$k = 1/9 \quad f(x) = (10/9)x^2 + 5(8/9)x + 4(10/9)$$
$$f(x) = (10/9)(x+2)^2$$

Answer Part (E)

$$x^3 + bx^2 + cx + d = 0$$

$$\alpha^2 + \beta^2 + \gamma^2 = 14 \qquad \alpha^3 + \beta^3 + \gamma^3 = 20 \qquad \alpha^4 + \beta^4 + \gamma^4 = 98$$

$$(1) \quad \alpha + \beta + \gamma = -b \qquad (2) \quad \alpha\beta + \alpha\gamma + \beta\gamma = c \qquad (3) \quad \alpha\beta\gamma = -d$$

$$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma) \qquad b^2 = 14 + 2c$$

$$(4) \quad c = \tfrac{1}{2}\left(b^2 - 14\right)$$

substitute the roots into the cubic equation and then add the 3 equations and use eq(4)

$$\alpha^3 + b\alpha^2 + c\alpha + d = 0$$

$$\beta^3 + b\beta^2 + c\beta + d = 0$$

$$\gamma^3 + b\gamma^2 + c\gamma + d = 0$$

$$\left(\alpha^3 + \beta^3 + \gamma^3\right) + b\left(\alpha^2 + \beta^2 + \gamma^2\right) + c(\alpha + \beta + \gamma) + 3d = 0$$

$$20 + 14b - \frac{b}{2}\left(b^2 - 14\right) + 3d = 0$$

$$(5) \quad b^3 - 42b - 40 - 6d = 0$$

$$x^3 + bx^2 + cx + d = 0 \quad \Rightarrow \quad x^4 + bx^3 + cx^2 + dx = 0$$

$$\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha = 0$$

$$\beta^4 + b\beta^3 + c\beta^2 + d\beta = 0$$

$$\gamma^4 + b\gamma^3 + c\gamma^2 + d\gamma = 0$$

Add the 3equations and use $\alpha^2+\beta^2+\gamma^2=14 \qquad \alpha^3+\beta^3+\gamma^3=20 \qquad \alpha^4+\beta^4+\gamma^4=98$

$98+20b+14c-bd=0$ replace c using eq(4)

$7b^2+20b-bd=0 \quad \Rightarrow \quad b=0$ is a solution and if $b\neq 0$

(6) $d=7b+20$

substitute (6) into (5)

$b^3-84b-160=0$ let the roots be $\left(b_1,b_2,b_3\right)$

$b_1+b_2+b_3=0 \quad b_1b_2b_3=160 \quad \Rightarrow \left(b_1,b_2,b_3\right)=\left(10,-8,-2\right)$

All possible values of b are $\left(-8,-2,0,10\right)$

From eq(6) $b\neq 0 \quad b=-8\Rightarrow d=-36 \quad b=-2\Rightarrow d=6 \quad b=10\Rightarrow d=90$

From eq(5) $b=0 \quad \Rightarrow \quad d=-20/3$

From eq(4) $\quad b=-8\Rightarrow c=25 \quad b=-2\Rightarrow c=-5 \quad b=0\Rightarrow c=-7 \quad b=10\Rightarrow c=43$

Consider the set of values $\quad b=-2 \quad c=-5 \quad d=6$

$x^3+bx^2+cx+d=0$

$x^3-2x^2-5x+6=0$ set of integer values for the roots are $\left(1,-2,3\right)$

$\left(x-1\right)\left(x+2\right)\left(x-3\right)=0$

QED