



MATHEMATICS EXTENSION 2

4 UNIT MATHEMATICS

TOPIC 4: INTEGRATION

4.5 METHODS OF INTEGRATION: INTEGRATION BY PARTS

One of the most useful and powerful integration **methods is integration by parts**.

Consider the differentiation of the function $y(x) = u(x) v(x)$

$$y(x) = u(x) v(x) \quad y = u v \quad \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

Now integrate this derivative with respect to x

$$\begin{aligned} \int \frac{dy}{dx} dx &= \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx \\ \int dy &= \int d(uv) = uv = \int u dv + \int v du \\ \int u dv &= uv - \int v du \end{aligned}$$

This is often useful for transforming an integral that can't be integrated into one which may be integrable. **You have to be careful to make the correct choice for u and v .**

Example 1 $I = \int x \cos(x) dx$

$$I = \int x \cos(x) dx$$

$$u = x \quad du = dx$$

$$dv = \cos(x) dx \quad v = \sin(x)$$

$$\int u dv = u v - \int v du$$

$$I = \int x \cos(x) dx = x \sin(x) - \int \sin(x) du + C$$

$$I = x \sin(x) + \cos(x) + C$$

check the answer by differentiation

Example 2 $I = \int \log_e(x) dx$

$$I = \int \log_e(x) dx$$

$$u = \log_e(x) \quad du = \left(\frac{1}{x}\right) dx$$

$$dv = dx \quad v = x \quad \int u dv = u v - \int v du$$

$$I = \int \log_e(x) dx = x \log_e(x) - \int x \left(\frac{1}{x}\right) dx + C$$

$$I = x \log_e(x) - x + C \quad \log_e(x) \equiv \ln(x)$$

check the answer by differentiation

Example 3 $I = \int x \sqrt{4x+1} dx$

$$I = \int \cos^{-1}(x) dx$$

$$u = \cos^{-1}(x) \quad du = \left(\frac{1}{\sqrt{1-x^2}} \right) dx$$

$$dv = dx \quad v = x$$

$$\int u dv = u v - \int v du$$

$$I = \int \cos^{-1}(x) dx = x \cos^{-1}(x) - \int x \left(\frac{1}{\sqrt{1-x^2}} \right) dx + C$$

$$\int x \left(\frac{1}{\sqrt{1-x^2}} \right) = \int x (1-x^2)^{-1/2} dx = - (1-x^2)^{1/2}$$

$$I = \int \cos^{-1}(x) dx = x \cos^{-1}(x) + (1-x^2)^{-1/2} + C$$

$$I = \int \cos^{-1}(x) dx = x \cos^{-1}(x) + \sqrt{1-x^2} + C$$

check the answer by differentiation

REDUCTION METHOD

It is often necessary to repeat the use of integration by parts to give **recurrence relationships**.

Example 4

$$I_n = \int x^n e^x dx$$

$$u = x^n \quad du = n x^{n-1} dx$$

$$dv = e^x dx \quad v = e^x$$

$$\int u dv = u v - \int v du$$

$$I_n = \int x^n e^x dx = x^n e^x - \int e^x n x^{n-1} dx$$

$$I_n = \int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$$

$$I_n = x^n e^x - n I_{n-1}$$

recurrence relationship or **reduction formula**

$$I_0 = \int x^0 e^x dx = e^x$$

$$I_1 = x^1 e^x - 1 I_0 = x e^x - e^x$$

$$I_2 = x^2 e^x - 2 I_1 = x^2 e^x - 2x e^x + 2e^x$$

etc

Example 5 $I_n = \int \sin^n(x) dx$

$$I_n = \int \sin^n(x) dx$$

$$u = \sin^{n-1}(x) \quad du = (n-1)\cos(x)\sin^{n-2}(x) dx$$

$$v dx = \sin(x) \quad v = -\cos(x)$$

$$I_n = \int \sin^{n-1}(x) \sin(x) dx = -\cos(x) \sin^{n-1}(x) - \int (-\cos(x))(n-1)\cos(x)\sin^{n-2}(x) dx$$

$$I_n = -\cos(x) \sin^{n-1}(x) + (n-1) \int \cos^2(x) \sin^{n-2}(x) dx \quad \cos^2(x) = 1 - \sin^2(x)$$

$$I_n = -\cos(x) \sin^{n-1}(x) + (n-1) \int \sin^{n-2}(x) dx - (n-1) \int \sin^n(x) dx$$

$$I_n = -\cos(x) \sin^{n-1}(x) + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n = \left(\frac{-1}{n} \right) (\cos(x) \sin^{n-1}(x)) + \left(\frac{n-1}{n} \right) I_{n-2}$$

