ONLINE: MATHEMATICS EXTENSION 2

Topic 6 MECHANICS

EXERCISE p6401

Consider an object of mass *m* initially moving with a velocity v_0. It then encounters a resistive force of the form $F_R = -\beta v$ and directed in the opposite direction to the motion. Show that the velocity *v* and displacement *x* of the object as functions of time *t* are

$$v = v_0\, e^{-\left(\beta/m\right)t} \qquad\qquad x = \left(\frac{m v_0}{\beta}\right)\left(1 - e^{(-\beta/m)t}\right)$$

What are the values of the velocity and displacement as $t \to \infty$?

Plot velocity and displacement time graphs for the parameters

$v_0 = 10 \quad m = 2 \quad \beta = 5$

$v_0 = 5 \quad m = 2 \quad \beta = 5$

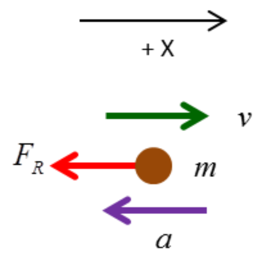
$v_0 = 10 \quad m = 4 \quad \beta = 5$

$v_0 = 10 \quad m = 2 \quad \beta = 10$

Comment on the physical significant of these changes in parameters.

Solution

The force acting on the object is the resistive force F_R. In our frame of reference, we will take to the right as the positive direction.



The equation of motion of the object is determined from Newton's Second Law.

$$m\,a = m\,\frac{dv}{dt} = F_R = -\beta\,v$$

where a is the acceleration of the object at any instance.

The initial conditions are $t = 0 \quad v = v_0 \quad x = 0 \quad a = -(\beta / m)\, v_0$

We start with the equation of motion then integrate this equation where the limits of the integration are determined by the initial conditions (t = 0 and v = v_0) and final conditions (t and v)

$$\frac{dv}{dt} = -\left(\frac{\beta}{m}\right)v$$

$$\frac{dv}{v} = -\left(\frac{\beta}{m}\right)dt$$

$$\int_{v_0}^{v} \frac{dv}{v} = \int_0^t -\left(\frac{\beta}{m}\right)dt$$

$$\left[\log_e(v)\right]_{v_0}^{v} = -\left(\frac{\beta}{m}\right)t$$

$$\log_e\left(\frac{v}{v_0}\right) = -\left(\frac{\beta}{m}\right)t$$

$$v = v_0\, e^{(-\beta/m)\,t}$$

We can now calculate the displacement x as a function of time t

$$v = \frac{dx}{dt} \qquad dx = v\, dt$$

$$v = v_0\, e^{(-\beta/m)\,t}$$

$$\int_0^x dx = \int_{v_0}^{v} v_0\, e^{(-\beta/m)\,t}\; dt$$

$$x = -\left(\frac{m\, v_0}{\beta}\right)\left[e^{(-\beta/m)\,t}\right]_0^t$$

$$x = \left(\frac{m\, v_0}{\beta}\right)\left(1 - e^{(-\beta/m)\,t}\right)$$

The velocity v also can be given as a function of x

$$a = \frac{dv}{dt} = v\frac{dv}{dx} = -\frac{\beta}{m}v \qquad\qquad dv = -\frac{\beta}{m}dx$$

$$\int_{v_0}^{v} dv = \left(-\frac{\beta}{m}\right)\int_0^x dx \qquad\qquad v = v_0 - \frac{\beta}{m}x$$

The graph of v vs x is a straight line.

When $v = 0$ the stopping distance is $x_{stopping} = \dfrac{m\, v_0}{\beta}$

We can also derive the result from

$$a = v\frac{dv}{dx} = -\left(\frac{\beta}{m}\right)v$$

$$dv = -\left(\frac{\beta}{m}\right)dx$$

$$\int_{v_0}^{v} dv = -\left(\frac{\beta}{m}\right)\int_0^x dx$$

$$-\left(\frac{\beta}{m}\right)x = v - v_0$$

$$x = \left(\frac{m\,v_0}{\beta}\right)\left(1 - e^{(-\beta/m)\,t}\right)$$

We can now investigate what happens when $t \to \infty$

$$t \to \infty \quad v = v_0\, e^{(-\beta/m)\,t} \to 0$$

$$t \to \infty \quad x = \left(\frac{m\,v_0}{\beta}\right)\left(1 - e^{(-\beta/m)\,t}\right) \to \frac{m\,v_0}{\beta}$$

The object keeps moving till $v = 0$, which happens only in the limit $t \to \infty$. Then the object stop at the position $x = \dfrac{m\,v_0}{\beta}$.

We can define a **time constant** τ

$$\tau = \frac{\beta}{m}$$

The velocity and displacement can be expressed as

$$v = v_0 \, e^{-t/\tau}$$

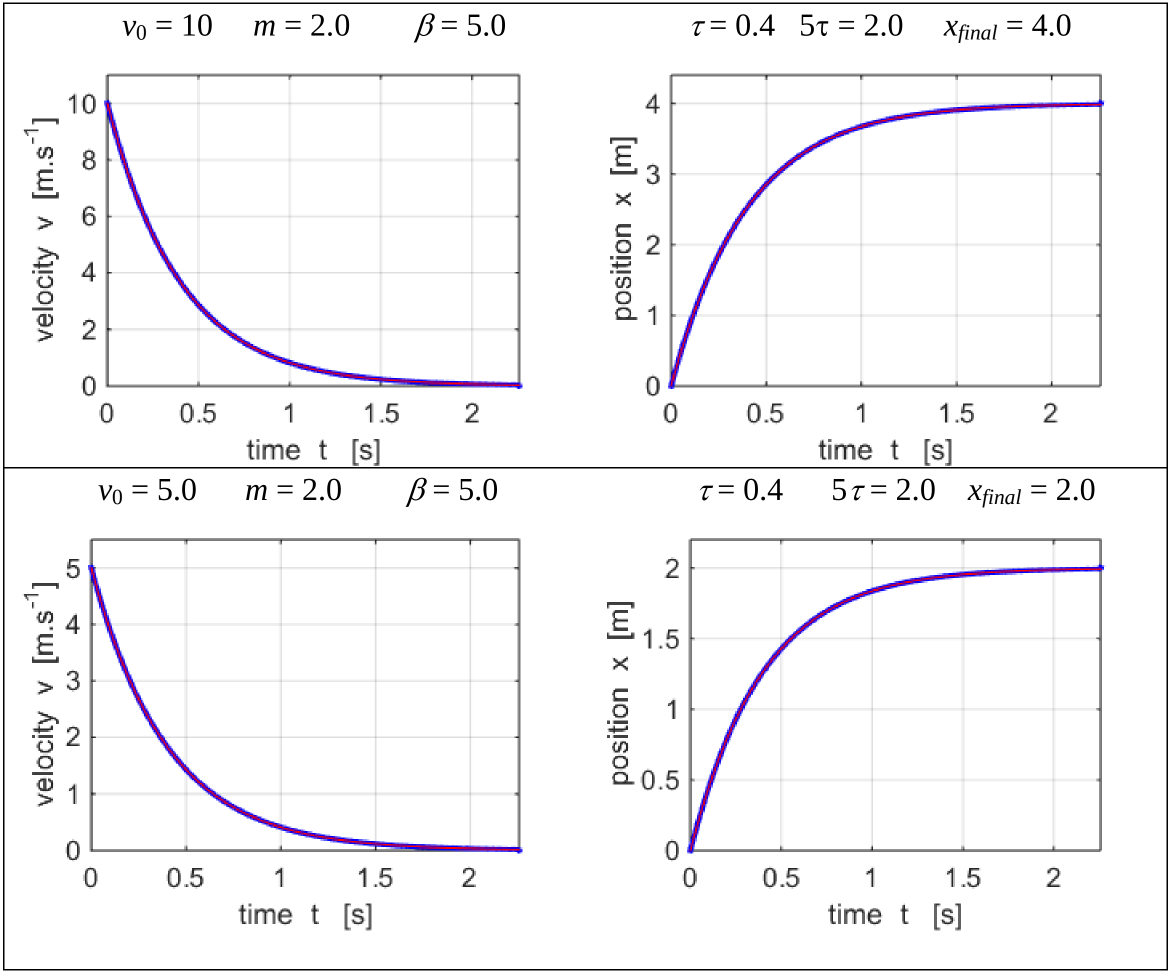
$$x = \left(\frac{m\,v_0}{\beta}\right)\left(1 - e^{-t/\tau}\right)$$

After a time of about 5τ, the particle will stop when the speed of the particle becomes zero

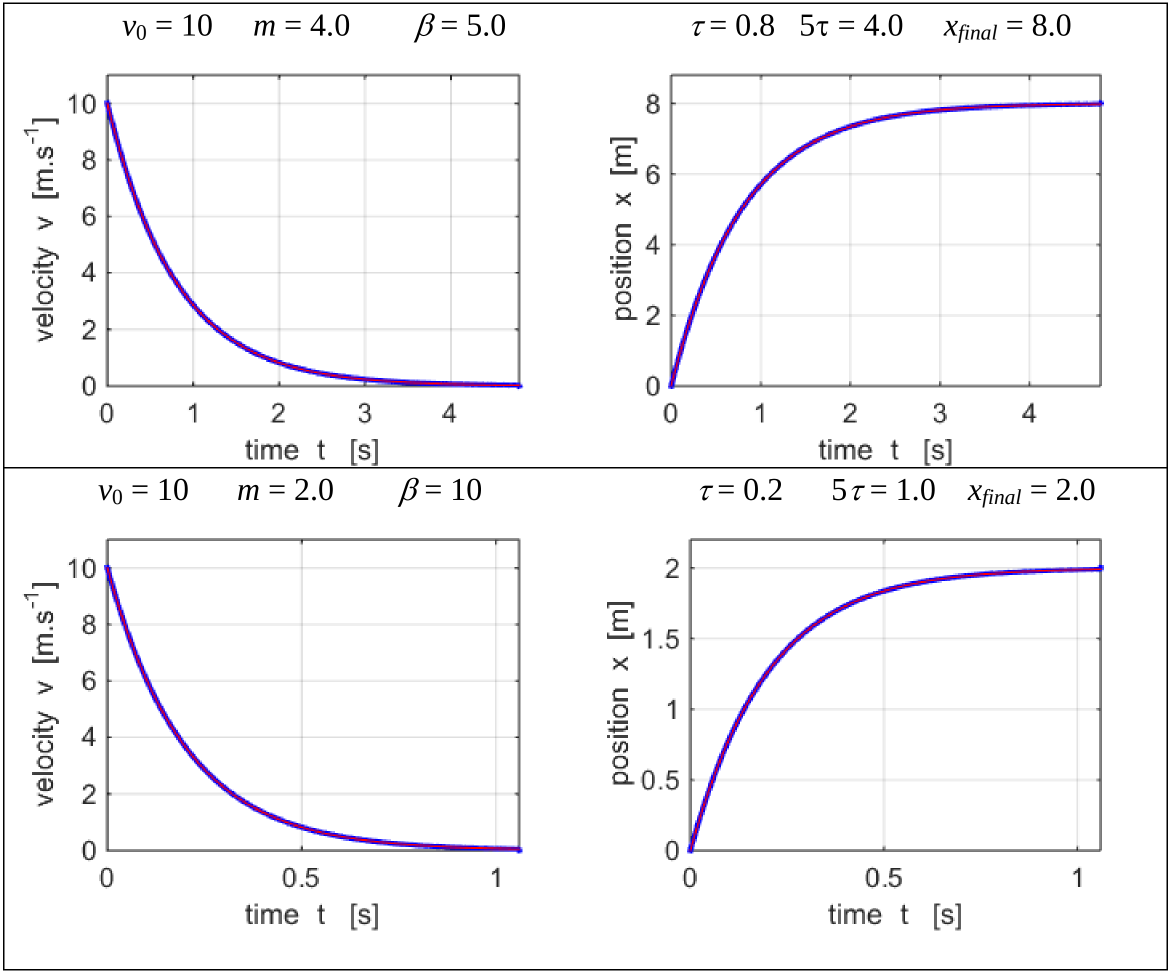
$$x_{final} = \left(\frac{m\,v_0}{\beta}\right) \qquad \text{stopping time} \sim 5\ \tau$$

The stopping time is independent of the initial velocity but the greater the initial velocity the greater the stopping distance. The larger the constant β, the shorter the stopping distance and quicker it stops and the larger the mass, the greater the stopping distance and it takes a longer time to stop the object.

The following graphs show the velocity and displacement as functions of time for varying values of v_0, m and β.



v0 = 10 m = 2.0 β = 5.0
τ = 0.4 5τ = 2.0 xfinal = 4.0
velocity v [m.s-1]
time t [s]
position x [m]
time t [s]
v0 = 5.0 m = 2.0 β = 5.0
τ = 0.4 5τ = 2.0 xfinal = 2.0
velocity v [m.s-1]
time t [s]
position x [m]
time t [s]



v_0 = 10 m = 4.0 β = 5.0
τ = 0.8 5τ = 4.0 x_{final} = 8.0
velocity v [m.s^{-1}]
time t [s]
position x [m]
time t [s]
v_0 = 10 m = 2.0 β = 10
τ = 0.2 5τ = 1.0 x_{final} = 2.0
velocity v [m.s^{-1}]
time t [s]
position x [m]
time t [s]