EXERCISE 33_123 (HSC 2013/12d)

Consider the equation $xy = k^2$ k is a constant

The origin of the Cartesian coordinate system is O(0, 0). The two points P(x_P, y_P) and Q(x_Q, y_Q) lie on the curve and $x_P = k\,p$ and $x_Q = k\,q$.

Part (A) What type of curve does the equation correspond too?

How are the distances of a vertex a and focal point c from the origin related to the constant k? Show that the eccentricity e is $e = \sqrt{2}$.

Part (B) State the Cartesian coordinates for the vertices and focal points of the curve in terms of k.

Part (C) State the equations for the asymptotes, axes of symmetry, and the directrices of the curve.

Part (D) The tangent to the curve at the point P cuts the X-axis and Y-axis at the points T and U respectively.

Show that the equation of the tangent at the point P is $x + p^2\,y = 2k\,p$.

Show that the points O, T, and U are on a circle with centre P.

Part (E) The tangent to the curve at the point Q cuts the X-axis and Y-axis at the points R and S respectively. Show that UR is parallel to PQ.

Answer Part (A)

Review notes on rectangular hyperbola

Sketch diagrams of the curve and label key features before answering the questions

vertices $A_1 \quad A_2$

focal points $F_1 \quad F_2$

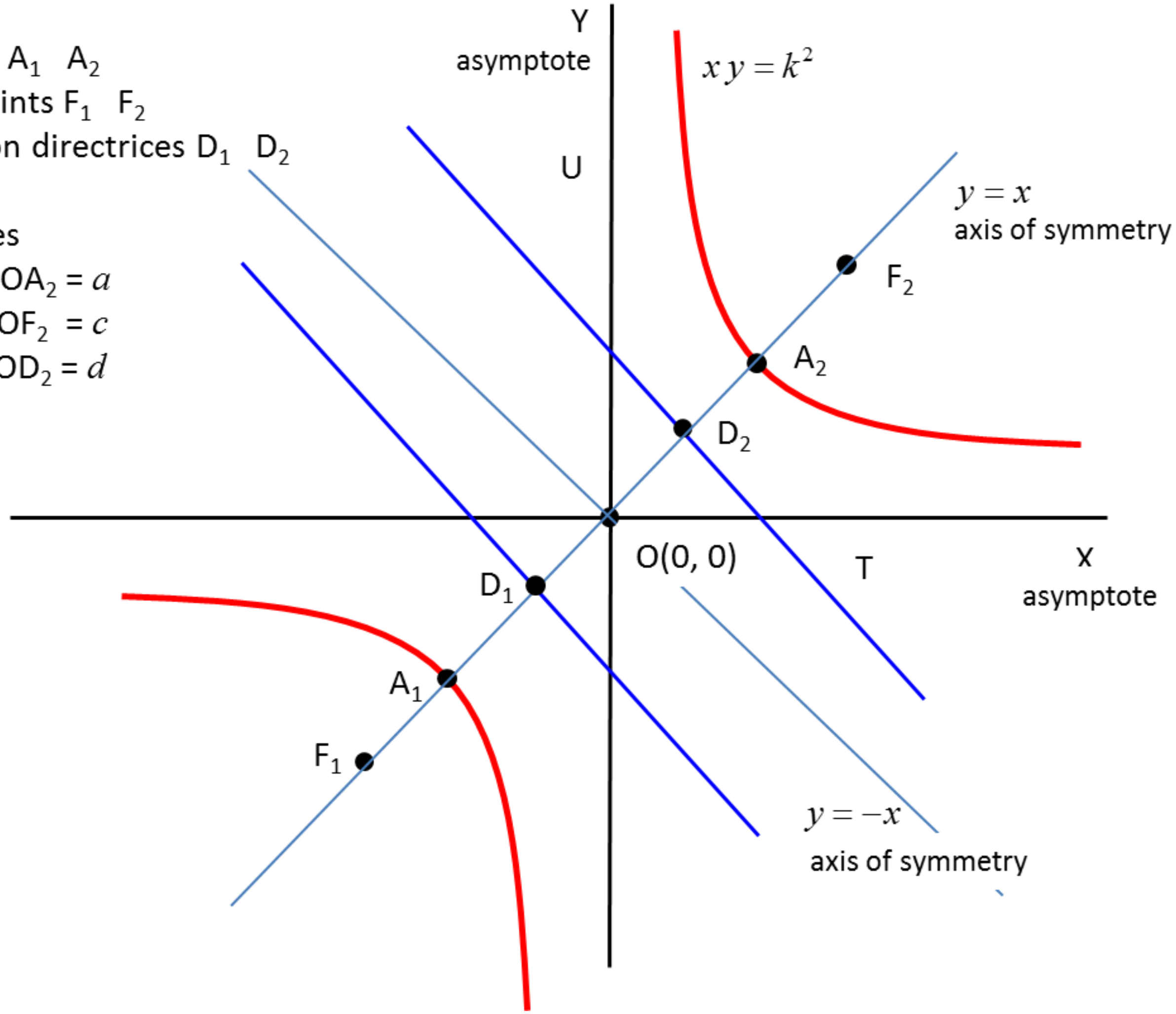
points on directrices $D_1 \quad D_2$

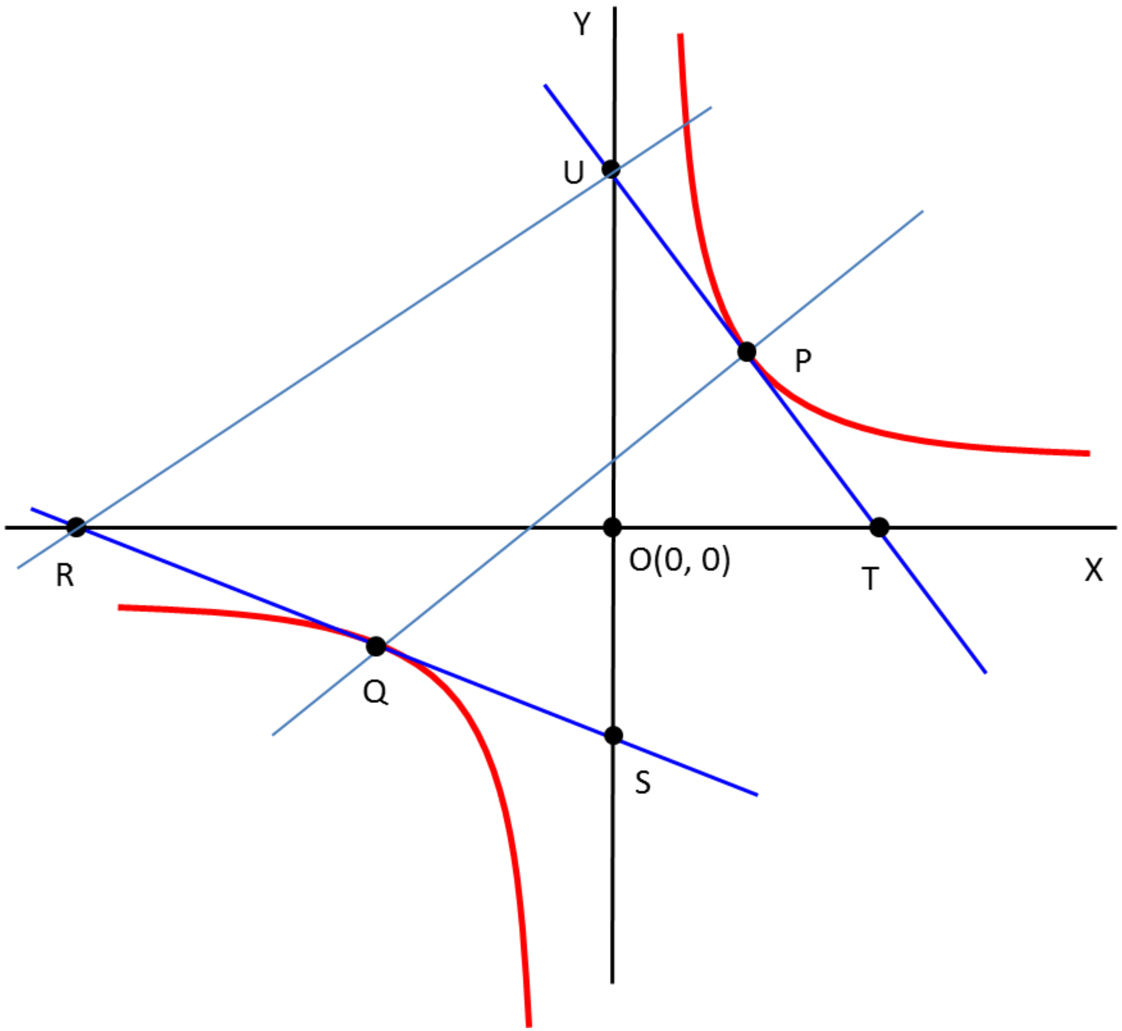
distances

$OA_1 = OA_2 = a$

$OF_1 = OF_2 = c$

$OD_1 = OD_2 = d$




Y
U
P
R
O(0, 0)
T
X
Q
S

The equation for the curve of a **rectangular hyperbola** with the **openings in the first and third quadrants** is

$$x\,y = \frac{a^2}{2}$$ where a is the distance from a vertex to the origin.

Therefore the equation $x\,y = k^2$ is a rectangular hyperbola opening in the first and third quadrants where

$$k = \frac{a}{\sqrt{2}} \qquad a = \sqrt{2}\,k$$

The **focal length** c for the rectangular hyperbola is

$$a = b \quad c^2 = a^2 + b^2 = 2a^2 \quad c = \sqrt{2}\,a \quad a = \sqrt{2}\,k \quad c = 2k$$

The **eccentricity** of hyperbolas is given by

$$e = \frac{c}{a} = \frac{2k}{\sqrt{2}\,k} = \sqrt{2}$$

Note: The syllabus and examination questions often use c as the focal length and as an arbitrary constant. In my notes c is used as the focal length and k is used as an arbitrary variable.

Answer Part (B)

The distances from the origin to the vertices A_1 and A_2 is a, therefore **vertices** of the hyperbola are $A_1\left(-a/\sqrt{2}, -a/\sqrt{2}\right)$ and $A_2\left(a/\sqrt{2}, a/\sqrt{2}\right)$

$$a = \sqrt{2}\,k \qquad k = a/\sqrt{2}$$

$A_1(-k, -k)$ and $A_2(k, k)$

Alternatively, the vertices are given by the intersection of the hyperbola $xy = k^2$ and the straight line x = y.

$$x = y \qquad xy = k^2 \quad \Rightarrow \quad x = \pm k \qquad y = \pm k$$

For a rectangular hyperbola a = b and the focal length c is $c^2 = a^2 + b^2 = 2a^2 \qquad c = \sqrt{2}\,a$

The focal length is c where $c = \sqrt{2}\,a = 2k$, therefore, the coordinates of the focal points are

$F_1\left(-c/\sqrt{2}, -c/\sqrt{2}\right)$ or $F_1\left(-\sqrt{2}\,k, -\sqrt{2}\,k\right)$

$F_2\left(c/\sqrt{2}, c/\sqrt{2}\right)$ or $F_2\left(\sqrt{2}\,k, \sqrt{2}\,k\right)$

Answer Part (C)

The equations for the **asymptotes** are

First quadrant +X axis and + Y axis

Third quadrant -X axis and –Y axis

X-axis $y = 0$ Y-axis $x = 0$

The rectangular hyperbola has symmetrical openings in the first and third quadrants. Therefore, there are two axes of symmetry

The lines $y = x$ and $y = -x$

The directrices must be lines that are parallel to the axis of symmetry $y = -x$ and the distance d from this line to the axis of symmetry is

$$d = \frac{a^2}{c} \quad a = \sqrt{2}\,k \quad c = 2k \quad d = k$$

Let the equations for the **directrices** be of the form $y = -x \pm B$ with one directrix passing through the point $D_1\left(-k/\sqrt{2},\ -k/\sqrt{2}\right)$ and the other through D_2 $\left(k/\sqrt{2},\ k/\sqrt{2}\right)$, therefore , $B = \sqrt{2}\,k$. Hence, the equations for the two directrices are

$$y = -x + \sqrt{2}\,k \quad \text{and} \quad y = -x - \sqrt{2}\,k\,.$$

Answer Part (D)

The coordinates of the point P are (x_P, y_P)

$$x_P = k\,p \qquad y_P = \frac{k^2}{k\,p} = \frac{k}{p}$$

The equation of the straight line for the tangent is $y = M_1\,x + B_1$

The gradient of the curve is given by the first derivative of the function $x\,y = k^2$

$$y + x\left(\frac{dy}{dx}\right) = 0 \qquad dy/dx = -\left(\frac{y}{x}\right) = -\left(\frac{k^2}{x^2}\right)$$

The gradient M_1 at the point P

$$x_P = k\,p \qquad y_P = \frac{k}{p} \qquad M_1 = -\left(\frac{y_P}{x_P}\right) = -\frac{1}{p^2}$$

The intercept B_1 of the tangent is

$$B_1 = y_P - M_1\,x_P = \frac{k}{p} + \left(\frac{1}{p^2}\right)(k\,p) = \frac{2\,k}{p}$$

Hence, the equation of the tangent is

$$x + p^2\, y = 2\, k\, p$$

The tangent intersects the X-axis at the point T

$$y_T = 0 \quad x_T = 2k\, p$$

The tangent intersects the Y-axis at the Point U

$$x_U = 0 \quad y_U = \frac{2k}{p}$$

If the points O, T and U lie on a circle with centre P then the distance OP, TP and UP must be equal

$$d_{OP}{}^2 = x_P{}^2 + y_P{}^2 = k^2\, p^2 + \frac{k^2}{p^2}$$

$$d_{TP}{}^2 = \left(x_P - x_T\right)^2 + y_P{}^2 = \left(k\, p - 2k\, p\right)^2 + \frac{k^2}{p^2} = k^2\, p^2 + \frac{k^2}{p^2}$$

$$d_{UP}{}^2 = x_P{}^2 + \left(y_P - y_U\right)^2 = k^2\, p^2 + \left(\frac{k}{p} - \frac{2k}{p}\right)^2 = k^2\, p^2 + \frac{k^2}{p^2}$$

Therefore, all the points O, T and U lie on a circle with centre P

Answer Part (E)

From part (D), the equation of tangent and coordinates of the points R and S are

$$x + q^2 y = 2\,k\,q$$

The tangent intersects the X-axis at the point R

$$y_R = 0 \quad x_R = 2kq$$

The tangent intersects the Y-axis at the Point S

$$x_S = 0 \quad y_S = \frac{2k}{q}$$

The slope of the line PQ is

$$m_{PQ} = \frac{k/p - k/q}{k\,p - k\,q} = \frac{-1}{p\,q}$$

The slope of the line UR is

$$m_{UR} = \frac{2k/p - 0}{0 - 2k\,q} = \frac{-1}{p\,q}$$

The slopes are equal, hence the two lines are parallel.