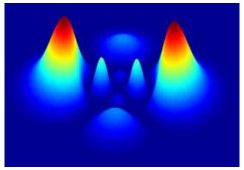
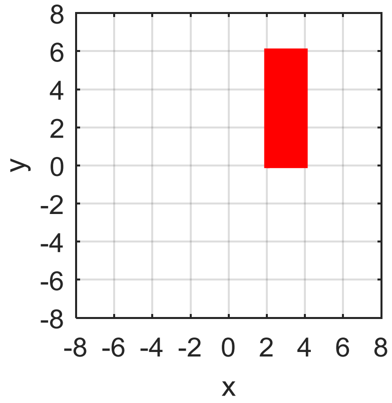


MATHEMATICS EXTENSION 2

TOPIC 5: VOLUMES

Exercise vol5_p001

Let $\mathcal{R}$ be the region enclosed by the rectangle with dimensions 2x6 as shown in the diagram. The XY plane coordinates of the corners are (2,0), (4,0), (4,6) and (2,6).



The volume V of a cylinder is

(1) $V = \pi R^2 H$ where R is its radius and H is the height.

Find the volumes V of the solids produced from the following rotations using equation (1) and by the evaluation of a definite integral.

(A) The region $\mathcal{R}$ is rotated about the lines

$x = 3$, $x = 2$, $x = 4$ and $x = 0$ (Y-axis).

(B) The region $\mathcal{R}$ is rotated about the lines

$y = 0$ (X-axis), $y = 6$ and $y = -2$.

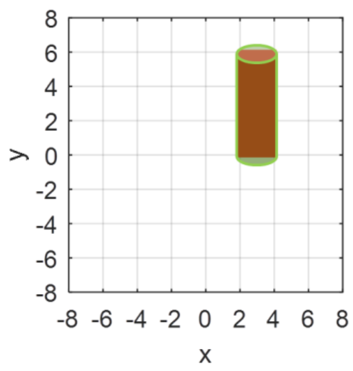
Solution

(A)

$x = 3$

$R = 1 \quad H = 6 \qquad V = \pi R^2\, H = 6\pi$

The solid of revolution is generated by rotating the line $x = 4$ in the interval $0 \le y \le 6$ about the axis $x = 3$. The volume V is



How to approach the problem:
Sketch the function and the solid.
Give the equations for the shape of the solid. Find the upper and lower limits for the bounded region.
Evaluation the definite integral to find the volume.

$$V = \int_{y_a}^{y_b} A(y)\, dy$$

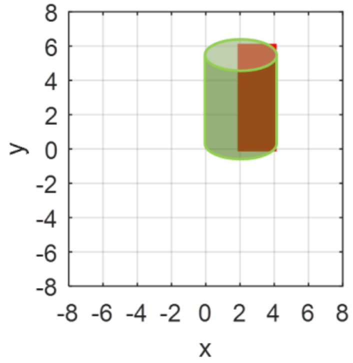
$$y_a = 0 \quad y_b = 6 \quad A(y) = \pi R^2 \quad R = 1 \quad \Rightarrow A(y) = \pi$$

$$V = \int_0^6 A(y)\, dy = \int_0^6 \pi\, dy = \pi\left[y\right]_0^6 = 6\pi$$

$x = 2$

$R = 2 \quad H = 6 \qquad V = \pi R^2\, H = 24\pi$

The solid of revolution is generated by rotating the line $x = 4$ in the interval $0 \le y \le 6$ about the axis $x = 2$. The volume V is



$$V = \int_{y_a}^{y_b} A(y)\, dy$$

$$y_a = 0 \quad y_b = 6 \quad A(y) = \pi R^2 \quad R = 2 \quad \Rightarrow A(y) = 4\pi$$

$$V = \int_0^6 A(y)\, dy = \int_0^6 4\pi\, dy = 4\pi\left[y\right]_0^6 = 24\pi$$

$x = 4$

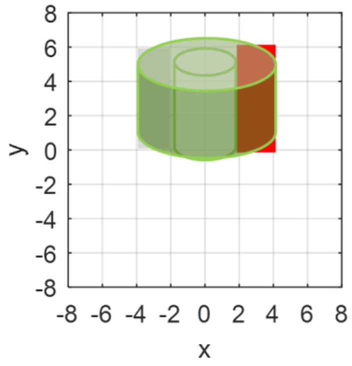
By symmetry, the volume of the cylinder is the same as the rotation about the line $x = 2 \quad \Rightarrow V = 24\pi$

$x = 0$

$$H = 6 \quad R_{out} = 4 \quad V_{out} = \pi R_{out}^{\ 2} H = 96\pi$$

$$R_{in} = 2 \quad V_{in} = \pi R_{in}^{\ 2} H = 24\pi$$

$$V = V_{out} - V_{in} = 72\pi$$



The volume of the solid of revolution by the method of cylindrical shells

$$V = \int_{x_a}^{x_b} (2\pi x)\, y\, dx = 2\pi \int_{x_a}^{x_b} x\, y\, dx$$

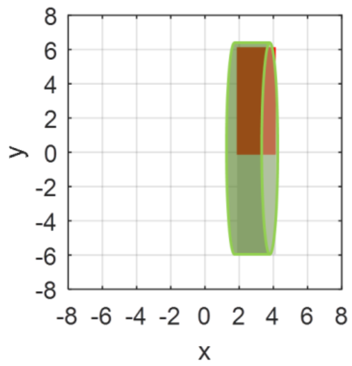
$$x_a = 2 \quad x_b = 4 \quad y = 6$$

$$V = 12\pi \int_2^4 x\ dx = 12\pi \left(\tfrac{1}{2}\right)\left[x^2\right]_2^4 = 6\pi(16-4) = 72\pi$$

(B)

$y = 0$

$$R = 6 \quad H = 2 \quad V = \pi R^2 H = 72\pi$$



The solid of revolution is generated by rotating the line $y = 6$ in the interval $2 \le x \le 4$ about the axis $y = 0$. The volume V is

$$V = \int_{x_a}^{x_b} A(x)\, dx$$

$$a = 2 \quad b = 4 \quad A(x) = \pi R^2 \quad R = 6 \quad \Rightarrow A(y) = 36\pi$$

$$V = \int_2^4 A(x)\, dx = \int_2^4 36\pi\, dy = 36\pi\left[x\right]_2^4 = 72\pi$$

$y = 6$

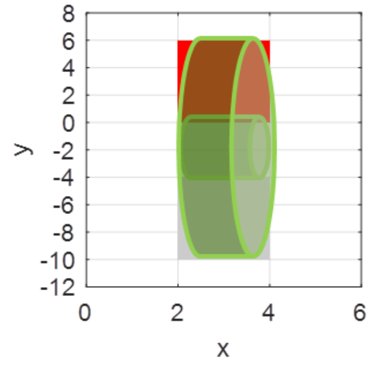
By symmetry, the volume of the cylinder is the same as the rotation about the line $y = 0 \quad \Rightarrow V = 72\pi$

$y = -2$

$$H = 2 \quad R_{out} = 8 \quad V_{out} = \pi R_{out}^{\ 2} H = 128\pi$$

$$R_{in} = 2 \quad V_{in} = \pi R_{in}^{\ 2} H = 8\pi$$

$$V = V_{out} - V_{in} = 120\pi$$



The volume of the solid of revolution by the method of cylindrical shells is

$$V = \int_{y_a}^{y_b} (2\ \pi\ y)\, x\ dy = 2\ \pi \int_a^b x\ y\ dy$$

$$y_a = -2 \quad y_b = 8 \quad x = 2$$

$$V = \ 4\pi \int_{-2}^{8} y\ \ dy = 4\pi\left(\tfrac{1}{2}\right)\left[y^2\right]_{-2}^{8} = 2\pi\left(64 - 4\right) = 120\pi$$