ONLINE: MATHEMATICS EXTENSION 2

Topic 6 MECHANICS

EXERCISE p6101

Consider the Earth as a sphere of radius R_E. At a radial displacement r with respect to the centre of the Earth where $r >> R_{E,}$ the acceleration due to gravity a is directed to the centre of the Earth and inversely proportional to the square of the displacement r. Derive the following results for an object projected vertically with an initial speed v_0 from the surface of the Earth. The magnitude of the acceleration due to gravity at the Earth's surface is g.

The minimum speed for the object to escape from the gravitational field of the Earth and never return is called the **escape speed** v_{esc} where

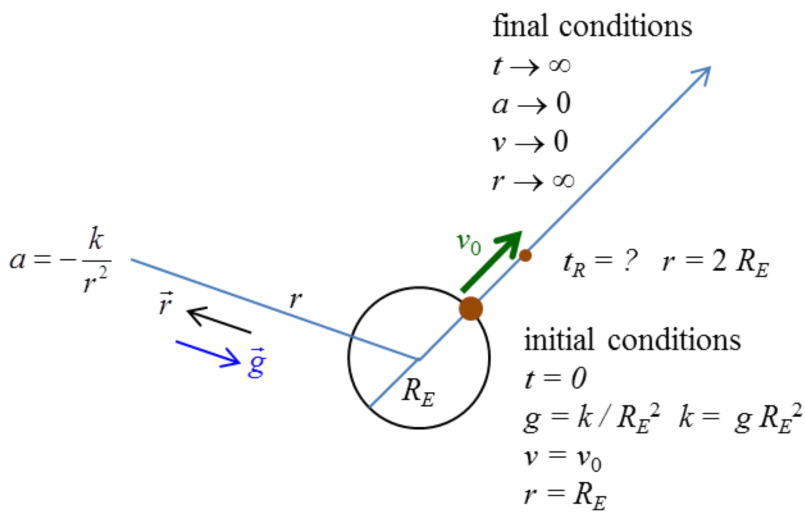
$$v_{esc} = \sqrt{2\, g\, R_E}$$

The time t_R for the object to reach a distance R_E above the Earth's surface is

$$t_R = \frac{1}{3}\left(\sqrt{\frac{R_E}{g}}\right)\left(4 - \sqrt{2}\right)$$

Solution

Step 1: Think about how to approach the problem
Step 2: Draw an annotated diagram of the physical situation
Step 3: What do you know about displacement, velocity and acceleration?



Acceleration a

$$a = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)$$

$$a = \frac{d}{dr}\left(\tfrac{1}{2}v^2\right) = -\frac{k}{r^2}$$ where k = constant of proportionality

Integrating the acceleration a with respect to r where the limits of the integration are determined by the initial conditions ($v = v_0 \quad r = R_E$) and final conditions ($v = 0 \quad r = \infty$)

$$a = \frac{dv}{dr}\left(\tfrac{1}{2}v^2\right) = -\frac{k}{r^2} \qquad g = \left|-\frac{k}{R_E{}^2}\right| = \frac{k}{R_E{}^2} \qquad k = g\,R_E{}^2$$

$$\int_{v_0}^{0} d\left(\tfrac{1}{2}v^2\right) = \int_{R_E}^{\infty}\left(-\frac{gR_E{}^2}{r^2}\right)dr$$

$$\left[\tfrac{1}{2}v^2\right]_{v_0}^{0} = \left[\frac{gR_E{}^2}{r}\right]_{R_E}^{\infty}$$

$$-\tfrac{1}{2}v_0{}^2 = 0 - \frac{gR_E{}^2}{R_E}$$

$$v_0 = \sqrt{2gR_E}$$

Hence, the escape velocity is $v_{esc} = \sqrt{2gR_E}$ ***QED***

The integration of the acceleration gives the velocity v as a function of r

$$v = \sqrt{2gR_E^{\,2}}\,(r)^{-1/2}$$

The velocity v is the time derivative of the displacement r

$$v = \frac{dr}{dt} = \sqrt{2gR_E^{\,2}}\,(r)^{-1/2}$$

To find the time interval t_R it takes for the displacement of the object to go from $r = R_E$ to $r = 2\,R_E$ is found by integration of the above equation where the lower limits are ($t = 0$ $\;r = R_E$) and the upper limits are ($t = t_R$ $\;r = 2\,R_E$)

$$dt = \left(\frac{1}{\sqrt{2g}\ R_E} \right) (r)^{1/2}\, dr$$

$$\int_0^{t_R} dt = \int_{R_E}^{2R_E} \left(\frac{1}{\sqrt{2g}\ R_E} \right) (r)^{1/2}\, dr$$

$$t_R = \left(\frac{1}{\sqrt{2g}\ R_E} \right) \left(\frac{2}{3} \right) \left[r^{3/2} \right]_{R_E}^{2R_E}$$

$$t_R = \left(\frac{1}{\sqrt{2g}\ R_E} \right) \left(\frac{2}{3} \right) \left(2\sqrt{2}\ R_E{}^{3/2} - R_E{}^{3/2} \right)$$

$$t_R = \left(\frac{1}{3} \right) \left(\sqrt{\frac{R_E}{g}} \right) \left(\sqrt{2} \right) \left(2\sqrt{2} - 1 \right)$$

$$t_R = \left(\frac{1}{3} \right) \left(\sqrt{\frac{R_E}{g}} \right) \left(4 - \sqrt{2} \right)$$

QED