FMI, Computer Science, Master
Advanced Logic for Computer Science

Seminar 3

(S3.1) Let $\mathcal{L}$ be a first-order language and φ be a sentence of $\mathcal{L}$ with the property that for all $m \in \mathbb{N}$,

$$\text{there exists a finite } \mathcal{L}\text{-structure } \mathcal{A} \text{ of cardinality } \geq m \text{ such that } \mathcal{A} \vDash \neg\varphi.$$

Prove that $\neg\varphi$ has an infinite model.

Proof. We apply Proposition 1.75 with $\Gamma = \{\neg\varphi\}$. $\square$

(S3.2) Let $\mathcal{L}_{Graf}$ be the language of graphs. Decide if the following affirmations are true or false:

(i) the class of graphs is axiomatizable;

(ii) the class of graphs is finitely axiomatizable;

(iii) the class of finite graphs is axiomatizable;

(iv) the class of finite graphs is finitely axiomatizable;

(v) the class of infinite graphs is axiomatizable;

(vi) the class of infinite graphs is finitely axiomatizable.

Proof. (i), (ii) are true (see slide 63 from the handouts). The class of graphs is axiomatized by the finite set $\Gamma := \{(IREFL), (SIM)\}$.
We apply in the sequel Proposition 1.77. The hypothesis (*) from this proposition is satisfied, as for any $m \in \mathbb{N}$ there exists a finite graph with at least m vertices. The class of finite (resp. infinite) graphs coincides with the class of finite (resp. infinite) models of Γ.
By Proposition 1.77.(ii), it follows that (iii) is false, hence (iv) is false, too.
By Proposition 1.77.(iii), it follows that (v) is true and that (vi) is false. $\square$

(S3.3) Let $\mathcal{L}$ be a first-order language, $\mathcal{K}$ be a class of $\mathcal{L}$-structures and $\mathcal{K}^c$ its complement in the class of all $\mathcal{L}$-structures. Prove that if both $\mathcal{K}$ and $\mathcal{K}^c$ are axiomatizable, then both of them are finitely axiomatizabile.

Proof. Let Γ, $\Delta \subseteq Sen_{\mathcal{L}}$ be such that $\mathcal{K} = Mod(\Gamma)$, $\mathcal{K}^c = Mod(\Delta)$. Suppose by contradiction that $\mathcal{K}$ is not finitely axiomatizable. We prove, with the help of the Compactness Theorem, that $\Gamma \cup \Delta$ is satisfiable. Let $\Sigma \subseteq \Gamma \cup \Delta$ be finite. Then $\Sigma \subseteq \Gamma_0 \cup \Delta$, where $\Gamma_0 \subseteq \Gamma$ is finite. Since $\mathcal{K} = Mod(\Gamma) \subseteq Mod(\Gamma_0)$ and $\mathcal{K} \neq Mod(\Gamma_0)$, we get that there exists $\mathcal{A}$ such that $\mathcal{A} \vDash \Gamma_0$ and $\mathcal{A} \in \mathcal{K}^c$. Since $\mathcal{A} \in \mathcal{K}^c$, we have that $\mathcal{A} \vDash \Delta$. Hence, $\mathcal{A} \vDash \Gamma_0 \cup \Delta$, so $\mathcal{A} \vDash \Sigma$.
Applying the Compactness Theorem, we get that $\Gamma \cup \Delta$ has a model $\mathcal{B}$. It follows that $\mathcal{B} \in \mathcal{K} \cap \mathcal{K}^c = \emptyset$, which is, obviously, a contradiction.
We prove similarly that $\mathcal{K}^c$ is finitely axiomatizable. □

(S3.4) Let $\mathcal{L}$ be a first-order language and Σ be a set of sentences satisfying

$$(*) \quad \text{for all } m \in \mathbb{N}, \Sigma \text{ has a finite model of cardinality } \geq m.$$

Prove that the class of finite models of Σ is not axiomatizable.

Proof. Let us denote with $\mathcal{T}$ the class of of finite models of Σ. Suppose by contradiction that $\mathcal{T}$ is axiomatizable and let $\Gamma \subseteq Sen_{\mathcal{L}}$ be such that $\mathcal{T} = Mod(\Gamma)$. Let

$$\Delta := \Gamma \cup \{\exists^{\geq n} \mid n \geq 1\}.$$

We prove that Δ is satisfiable with the help of the Compactness Theorem. Let Δ_0 be a finite subset of Δ. Then

$$\Delta_0 \subseteq \Gamma \cup \{\exists^{\geq n_1}, \dots, \exists^{\geq n_k}\} \text{ for some } k \in \mathbb{N}.$$

By (*), there exists $\mathcal{A} \in \mathcal{T}$ such that $|A| \geq \max\{n_1, \dots, n_k\}$. Then $\mathcal{A} \vDash \exists^{\geq n_i}$ for all $i = 1, \dots, k$ and $\mathcal{A} \vDash \Gamma$, since $\mathcal{T} = Mod(\Gamma)$. We get that $\mathcal{A} \vDash \Gamma \cup \{\exists^{\geq n_1}, \dots, \exists^{\geq n_k}\}$, so $\mathcal{A} \vDash \Delta_0$. Thus, Δ_0 is satisfiable.
Applying the Compactness Theorem, it follows that

$$\Delta = \Gamma \cup \{\exists^{\geq n} \mid n \geq 1\}$$

has a model $\mathcal{B}$.
Since $\mathcal{B} \vDash \Gamma$, we have that $\mathcal{B}$ is finite.
Since $\mathcal{B} \vDash \{\exists^{\geq n} \mid n \geq 1\}$, we have that $\mathcal{B}$ is infinite.
We have obtained a contradiction. □