FMI, Computer Science, Master
Advanced Logic for Computer Science

Seminar 1

(S1.1) Consider the first-order language $\mathcal{L}_{ar} = (\dot{<}; \dot{+}, \dot{\times}, \dot{S}; \dot{0})$ (the language of arithmetics) and the $\mathcal{L}_{ar}$-structure $\mathcal{N} = (\mathbb{N}, <, +, \cdot, S, 0)$.

(i) Let $x, y \in V$ with $x \neq y$ and $t = \dot{S}x\dot{\times}\dot{S}\dot{S}y = \dot{\times}(\dot{S}x, \dot{S}\dot{S}y)$. Evaluate $t^{\mathcal{N}}(e)$, where $e : V \to \mathbb{N}$ is an assignment verifying $e(x) = 3$ and $e(y) = 7$.

(ii) Let $\varphi = x\dot{<}\dot{S}y \to (x\dot{<}y \vee x = y) = \dot{<}(x, \dot{S}y) \to (\dot{<}(x, y) \vee x = y)$. Prove that $\mathcal{N} \vDash \varphi[e]$ for all $e : V \to \mathbb{N}$.

Proof. (i) For any assignment $e : V \to \mathbb{N}$, we have that

$$\begin{aligned} t^{\mathcal{N}}(e) &= \dot{\times}^{\mathcal{N}}((\dot{S}x)^{\mathcal{N}}(e), (\dot{S}\dot{S}y)^{\mathcal{N}}(e)) = (\dot{S}x)^{\mathcal{N}}(e) \cdot (\dot{S}\dot{S}y)^{\mathcal{N}}(e) \\ &= \dot{S}^{\mathcal{N}}(x^{\mathcal{N}}(e)) \cdot \dot{S}^{\mathcal{N}}((\dot{S}y)^{\mathcal{N}}(e)) = S(e(x)) \cdot S(\dot{S}^{\mathcal{N}}(y^{\mathcal{N}}(e))) \\ &= S(e(x)) \cdot S(S(e(y))). \end{aligned}$$

Hence, if $e(x) = 3$ and $e(y) = 7$, then

$$t^{\mathcal{N}}(e) = S(3) \cdot S(S(7)) = 4 \cdot 9 = 36.$$

(ii) For any assignemnt $e : V \to \mathbb{N}$, we have that

$$\begin{aligned} \mathcal{N} \vDash \varphi[e] &\iff \mathcal{N} \nvDash (\dot{<}(x, \dot{S}y))[e] \text{ or } \mathcal{N} \vDash (\dot{<}(x, y) \vee x = y)[e] \\ &\iff \dot{<}^{\mathcal{N}}(e(x), S(e(y)) \text{ is not satisfied or} \\ & \mathcal{N} \vDash (\dot{<}(x, y))[e] \text{ or } \mathcal{N} \vDash (x = y)[e] \\ &\iff < (e(x), S(e(y)) \text{ is not satisfied or } < (e(x), e(y)) \\ & \text{ or } e(x) = e(y) \\ &\iff e(x) \geq S(e(y)) \text{ or } e(x) < e(y) \text{ or } e(x) = e(y) \\ &\iff e(x) \geq e(y) + 1 \text{ or } e(x) < e(y) \text{ or } e(x) = e(y). \end{aligned}$$

Hence, $\mathcal{N} \vDash \varphi[e]$ for all $e : V \to \mathbb{N}$.

We usually write

$$\begin{aligned} \mathcal{N} \vDash \varphi[e] &\iff \mathcal{N} \nvDash (\dot{<}(x, \dot{S}y))[e] \text{ or } \mathcal{N} \vDash (\dot{<}(x, y) \vee x = y)[e] \\ &\iff e(x) \geq S(e(y)) \text{ or } e(x) < e(y) \text{ or } e(x) = e(y) \\ &\iff e(x) \geq e(y) + 1 \text{ or } e(x) < e(y) \text{ or } e(x) = e(y). \end{aligned}$$

□

Notation. *Let $\mathcal{L}$ be a first-order language. For any variables x, y with $x \neq y$, $\mathcal{L}$-structure $\mathcal{A}$, $e : V \to A$ and $a, b \in A$, we have that:*

$$(e_{y\leftarrow b})_{x\leftarrow a} = (e_{x\leftarrow a})_{y\leftarrow b}.$$

In this case, we denote their common value with$e_{x\leftarrow a,y\leftarrow b}$. Thus,

$$e_{x\leftarrow a,y\leftarrow b} : V \to A, \quad e_{x\leftarrow a,y\leftarrow b}(v) = \begin{cases} e(v) & \textit{dacă } v \neq x \textit{and } v \neq y \\ a & \textit{dacă } v = x \\ b & \textit{dacă } v = y. \end{cases}$$

(S1.2) Let $\mathcal{L}$ be a first-order language. Prove that for any formulas φ, ψ and any distinct variables x, y,

(i) $\neg\exists x\varphi \mathrel{\text{⫤}}\!\!\models \forall x\neg\varphi$;

(ii) $\forall x(\varphi \land \psi) \mathrel{\text{⫤}}\!\!\models \forall x\varphi \land \forall x\psi$;

(iii) $\exists y\forall x\varphi \models \forall x\exists y\varphi$;

(iv) $\forall x(\varphi \to \psi) \models \forall x\varphi \to \forall x\psi$.

Proof. Let $\mathcal{A}$ be an $\mathcal{L}$-structure and $e : V \to A$ be an $\mathcal{A}$-assignment.

(i) We know that "$\exists x$" is an abbreviation for "$\neg\forall x\neg$".

$\mathcal{A} \models (\neg\exists x\varphi)[e] \iff \mathcal{A} \models (\neg\neg\forall x\neg\varphi)[e] \iff$ it is not true that $\mathcal{A} \models (\neg\forall x\neg\varphi)[e] \iff$ it is not true that it is not true that $\mathcal{A} \models (\forall x\neg\varphi)[e] \iff \mathcal{A} \models (\forall x\neg\varphi)[e]$.

(ii) $\mathcal{A} \models (\forall x(\varphi \land \psi))[e] \iff$ for all $a \in A$, we have that $\mathcal{A} \models (\varphi \land \psi)[e_{x\leftarrow a}] \iff$ for all $a \in A$, we have that $\mathcal{A} \models \varphi[e_{x\leftarrow a}]$ and $\mathcal{A} \models \psi[e_{x\leftarrow a}] \iff \big($for all $a \in A$, we have that $\mathcal{A} \models \varphi[e_{x\leftarrow a}]\big)$ and $\big($for all $a \in A$, we have that $\mathcal{A} \models \psi[e_{x\leftarrow a}]\big) \iff \mathcal{A} \models (\forall x\varphi)[e]$ and $\mathcal{A} \models (\forall x\psi)[e] \iff \mathcal{A} \models (\forall x\varphi \land \forall x\psi)[e]$.

(iii) Using the hypothesis that $x \neq y$, we get that $\mathcal{A} \models (\exists y\forall x\varphi)[e] \iff$ there exists $b \in A$ such that for all $a \in A$ we have that $\mathcal{A} \models \varphi[(e_{y\leftarrow b})_{x\leftarrow a}]$, hence

$$\mathcal{A} \models (\exists y\forall x\varphi)[e] \iff \text{there exists } b \in A \text{ s.t. for all } a \in A,\ \mathcal{A} \models \varphi[e_{x\leftarrow a,y\leftarrow b}] \quad (*).$$

and $\mathcal{A} \models (\forall x\exists y\varphi)[e] \iff$ for all $c \in A$ there exists $d \in A$ such that $\mathcal{A} \models \varphi[(e_{x\leftarrow c})_{y\leftarrow d}]$, hence

$$\mathcal{A} \models (\forall x\exists y\varphi)[e] \iff \text{for all } c \in A \text{ there exists } d \in A \text{ s.t. } \mathcal{A} \models \varphi[e_{x\leftarrow c,y\leftarrow d}] \quad (**).$$

We know (*) and we wish to show (**). Let $c \in A$. We wish to get $d \in A$ such that $\mathcal{A} \models \varphi[e_{x\leftarrow c,y\leftarrow d}]$.

Let b satisfy (*) and take $d := b$. Then, for all $a \in A$ we have that $\mathcal{A} \models \varphi[e_{x\leftarrow a,y\leftarrow d}]$. In particular, letting $a := c$, we get $\mathcal{A} \models \varphi[e_{x\leftarrow c,y\leftarrow d}]$, as needed.

(iv) We have that $\mathcal{A} \vDash (\forall x(\varphi \to \psi))[e] \iff$ for all $a \in A$, $\mathcal{A} \vDash (\varphi \to \psi)[e_{x \leftarrow a}] \iff$ for all $a \in A$, $\varphi^{\mathcal{A}}(e_{x\leftarrow a}) \boldsymbol{\to} \psi^{\mathcal{A}}(e_{x\leftarrow a}) = 1 \iff$

$$(*) \quad \text{for all } a \in A,\ \varphi^{\mathcal{A}}(e_{x\leftarrow a}) \leq \psi^{\mathcal{A}}(e_{x\leftarrow a}).$$

We obtain similarly that $\mathcal{A} \vDash (\forall x\varphi \to \forall x\psi)[e] \iff$

$$(**) \quad \text{for all } a \in A,\ (\forall x\varphi)^{\mathcal{A}}(e) \leq (\forall x\psi)^{\mathcal{A}}(e).$$

We assume (*) and we have to prove (**).

If $(\forall x\varphi)^{\mathcal{A}}(e) = 0$, (**) is obvious. Suppose that $(\forall x\varphi)^{\mathcal{A}}(e) = 1$, that is

$$(***) \quad \text{for all } b \in A,\ \varphi^{\mathcal{A}}(e_{x\leftarrow b}) = 1.$$

We need to prove that $(\forall x\psi)^{\mathcal{A}}(e) = 1$, that is

$$\text{for all } c \in A,\ \psi^{\mathcal{A}}(e_{x\leftarrow c}) = 1.$$

Let $c \in A$. By (*), we have that $\varphi^{\mathcal{A}}(e_{x\leftarrow c}) \leq \psi^{\mathcal{A}}(e_{x\leftarrow c})$ and, by (***), that $\varphi^{\mathcal{A}}(e_{x\leftarrow c}) = 1$. Hence, $\psi^{\mathcal{A}}(e_{x\leftarrow c}) = 1$, as needed.

□

(S1.3) Let x, y be distinct variables. Give examples of first-order languages $\mathcal{L}$ and formulas φ, ψ of $\mathcal{L}$ such that:

(i) $\forall x(\varphi \vee \psi) \not\vDash \forall x\varphi \vee \forall x\psi$;

(ii) $\exists x\varphi \wedge \exists x\psi \not\vDash \exists x(\varphi \wedge \psi)$;

(iii) $\forall x\exists y\varphi \not\vDash \exists y\forall x\varphi$.

Proof. Consider $\mathcal{L}_{ar} = (\dot{<}, \dot{+}, \dot{\times}, \dot{S}, \dot{0})$, the $\mathcal{L}_{ar}$-structure $\mathcal{N} := (\mathbb{N}, <, +, \cdot, S, 0)$ and $e : V \to \mathbb{N}$ be an arbitrary assignment (we take, for example, $e(v) := 7$ for all $v \in V$).

(i) Let $\dot{2} := \dot{S}\dot{S}\dot{0}$, $\varphi := x\dot{<}\dot{2}$ and $\varphi := \neg(x\dot{<}\dot{2})$. Then

$$\mathcal{N} \vDash \forall x(\varphi \vee \psi)[e].$$

On the other hand,

(a) $\mathcal{N} \vDash (\forall x\varphi)[e] \iff$ for all $n \in \mathbb{N}$, we have that $\mathcal{N} \vDash \varphi[e_{x\leftarrow n}] \iff$ for all $n \in \mathbb{N}$, we have that $n < 2$, which is false (take $n := 3$, for example). Hence, $\mathcal{N} \not\vDash (\forall x\varphi)[e]$.

(b) $\mathcal{N} \vDash (\forall x\psi)[e] \iff$ for all $n \in \mathbb{N}$, we have that $\mathcal{N} \vDash \psi[e_{x\leftarrow n}] \iff$ for all $n \in \mathbb{N}$, we have that $n \geq 2$, which is false (take $n := 1$, for example). Hence, $\mathcal{N} \nvDash (\forall x\psi)[e]$.

It follows that

$$\mathcal{N} \nvDash (\forall x\varphi \vee \forall x\psi)[e].$$

(ii) Let $\dot{2} := \dot{S}\dot{S}\dot{0}$, $\varphi := x\dot{<}\dot{2}$ and $\psi := \neg(x\dot{<}\dot{2})$. Then

(a) $\mathcal{N} \vDash (\exists x\varphi)[e] \iff$ there exists $n \in \mathbb{N}$ such that $\mathcal{N} \vDash \varphi[e_{x\leftarrow n}] \iff$ there exists $n \in \mathbb{N}$ such that $n < 2$, which is true (take $n := 1$, for example). Hence, $\mathcal{N} \vDash (\exists x\varphi)[e]$.

(b) $\mathcal{N} \vDash (\exists x\psi)[e] \iff$ there exists $n \in \mathbb{N}$ such that $\mathcal{N} \vDash \psi[e_{x\leftarrow n}] \iff$ there exists $n \in \mathbb{N}$ such that $n \geq 2$, which is true (take $n := 3$, for example). Hence, $\mathcal{N} \vDash (\exists x\psi)[e]$.

It follows that

$$\mathcal{N} \vDash (\exists x\varphi \wedge \exists x\psi)[e].$$

On the other hand, $\mathcal{N} \vDash \exists x(\varphi \wedge \psi)[e] \iff$ there exists $n \in \mathbb{N}$ such that $\mathcal{N} \vDash (\varphi \wedge \psi)[e_{x\leftarrow n}] \iff$ there exists $n \in \mathbb{N}$ such that $n < 2$ and $n \geq 2$, which is false. Thus,

$$\mathcal{N} \nvDash \exists x(\varphi \wedge \psi)[e].$$

(iii) Let $\varphi := x\dot{<}y$. Then

$$\begin{aligned}
\mathcal{N} \vDash (\forall x\exists y\varphi)[e] \quad &\iff \quad \text{for all } n \in \mathbb{N}, \text{ we have that } \mathcal{N} \vDash (\exists y\varphi)[e_{x\leftarrow n}] \\
&\iff \quad \text{for all } n \in \mathbb{N} \text{ there exists } m \in \mathbb{N} \text{ such that } \mathcal{N} \vDash \varphi[e_{x\leftarrow n, y\leftarrow m}] \\
&\iff \quad \text{for all } n \in \mathbb{N} \text{ there exists } m \in \mathbb{N} \text{ such that } n < m,
\end{aligned}$$

which is true (take $m := n + 1$, for example). Hence,

$$\mathcal{N} \vDash (\forall x\exists y\varphi)[e].$$

On the other hand,

$$\begin{aligned}
\mathcal{N} \vDash (\exists y\forall x\varphi)[e] \quad &\iff \quad \text{there exists } m \in \mathbb{N} \text{ such that } \mathcal{N} \vDash (\forall x\varphi)[e_{y\leftarrow m}] \\
&\iff \quad \text{there exists } m \in \mathbb{N} \text{ such that for all } n \in \mathbb{N} \\
&\qquad\quad \text{we have that } \mathcal{N} \vDash \varphi[e_{x\leftarrow n, y\leftarrow m}] \\
&\iff \quad \text{there exists } m \in \mathbb{N} \text{ such that for all } n \in \mathbb{N}, \text{ we have that } n < m,
\end{aligned}$$

which is false. It follows that

$$\mathcal{N} \nvDash (\exists y\forall x\varphi)[e].$$

□