Curs 8 STLS II

Caractere de grup

$(G, +, 0)$ grup abelian

caracter: $\chi : (G, +, 0) \to (\mathbb{C}^\times, \cdot, 1)$ morfism

$\chi(0) = 1$!

$|G| = n \Rightarrow \chi^n(g) = \chi(ng) = \chi(0) = 1 \Rightarrow$

$\Rightarrow \chi(g)$ rădăcini ale lui 1

$\chi_{1,2}$ caractere $\Rightarrow \chi_1 \cdot \chi_2$ caracter

$\hat{G}$ grupul caracterelor lui G, cu acest produs

$\hat{\mathbb{Z}}_n$, $\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$

$y \in \mathbb{Z}$ $\chi_y(x) = e^{\frac{2\pi i x y}{n}}$ caracter al lui $\mathbb{Z}_n$

$\chi_y = \chi_z \Leftrightarrow y \equiv z \mod n$

$\hat{\mathbb{Z}}_n = \{\chi_x \mid x \in \mathbb{Z}_n\} \simeq \mathbb{Z}_n$

Th G grup abelian finit: $G \simeq \hat{G}$

Dem. G grup abelian finit $\Rightarrow G \simeq G_1 \times G_2 \times \dots \times G_m$ ciclice

$\hat{G}_i \simeq G_i$ $G \simeq G_1 \oplus G_2 \oplus \dots \oplus G_m$

$\forall g \in G$, $g = g_1 + g_2 + \dots + g_m$

χ_i caracter al lui G_i $(\chi_i \in \hat{G}_i)$. Definim

$\chi(g) = \chi_1(g_1) \cdot \chi_2(g_2) \cdot \dots \cdot \chi_m(g_m)$ caracter unic determinat $\square$

Caracterele lui $\mathbb{F}_2^m$.

$\boxed{e^{\pi i} = -1}$ r. Euler

$\hat{\mathbb{F}}_2 : \chi_y(x) = e^{\frac{2\pi i x y}{2}} = (-1)^{xy}$

$\mathbb{Z}_2$

$\hat{\mathbb{F}}_2^m : \chi_{\vec{y}}(\vec{x}) = (-1)^{x_1 y_1 + \dots + x_m y_m} = (-1)^{\vec{x} \cdot \vec{y}}$

produs scalar peste $\mathbb{F}_2^m$

$G = \{g_1, \dots, g_n\}$ grup abelian finit.

$\underset{sp. vect/\mathbb{C}}{\mathbb{V}} = \{ f : G \to \mathbb{C} \}$ de dim $= n$.

$e_i := "e_i(g_j) = \delta_{ij}" = \begin{cases} 1, & i=j \\ 0, & \text{altfel} \end{cases}$

baza canonică a lui $\mathbb{V}$.

$\langle f | h \rangle = \sum_{i=1}^{n} f^*(g_i)\, h(g_i)$ produs Hermitian

$* =$ conjugare complexă

$\| f \| = \sqrt{\langle f | f \rangle}$ Obs $\hat{G} \subseteq \mathbb{V}$

Th de ortogonalitate a caracterelor

$\langle \chi_i | \chi_j \rangle = \begin{cases} 0, & i \neq j \\ n, & i = j \end{cases}$ ortogonal, dar nu ortonormal

Dem.

$1 = |\chi(g)|^2 = \chi^*(g)\,\chi(g) \Rightarrow \chi^*(g) = \chi^{-1}(g)$ în $\mathbb{C}$.

$\langle \chi_i | \chi_j \rangle = \sum_{k=1}^{n} \chi_i^*(g_k)\,\chi_j(g_k) = \sum_{k=1}^{n} \chi_i^{-1}(g_k)\,\chi_j(g_k) =$

$= \sum_{k=1}^{n} (\chi_i^{-1}\chi_j)(g_k)$.

Dacă $i = j \Rightarrow \chi_j^{-1}\chi_i = 1$ caracter trivial $\Rightarrow n$

Dacă $i \neq j \Rightarrow \chi_j^{-1}\chi_i = \chi$ caracter netrivial

$\Downarrow$ caracter din grup $\exists g$ cu $\chi(g) \neq 1$

Fie $S = \sum_{k=1}^{n} \chi(g_k)$

Obs: $g_k \rightsquigarrow g_k + g$ permutare a lui G

$S = \sum_{k=1}^{n} \chi(g + g_k) = \chi(g) \sum_{k=1}^{n} \chi(g_k) = \chi(g)\, S$.

Deci $\boxed{S = 0}$.

Corolar $B_i = \frac{1}{\sqrt{n}} \chi_i$ formează o bază ortogonală a lui $\mathbb{V}$.
funcții

Concluzie: $X \in \mathbb{C}^{n \times n}$ matrice $X = (\chi_j(g_i))$

$X^{-1} = \frac{1}{n} X^*$, unde $X^* = \overline{X^T}$

$\frac{1}{\sqrt{n}} X$ matrice unitară !

Transformarea Fourier discretă

Orice funcție $f \in \mathbb{V}$ are o unică reprezentare în baza B corespunzătoare lui G.

ortonormală, $\frac{1}{\sqrt{n}} \chi_g$

$$f = \hat{f}_1 B_1 + \dots + \hat{f}_n B_n \; ; \quad \hat{f}_i \in \mathbb{C}$$

Definim $\hat{f} : G \to \mathbb{C}$ dată de $\hat{f}(g_i) = \hat{f}_i$

$(\hat{f} \in \mathbb{V})$ transformata Fourier discretă a lui f.

$(*)$ $\langle B_i | f \rangle = \hat{f}_i$ $\quad \hat{f}(g_i) = \langle B_i | f \rangle =$

$$= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \chi_i^*(g_k) f(g_k).$$

Exemplu $f : \mathbb{Z}_n \to \mathbb{C}$ $\quad \hat{f}(x) = \frac{1}{\sqrt{n}} \sum_{y \in \mathbb{Z}_n} e^{-\frac{2\pi i x y}{n}} f(y)$

transf. Fourier a lui f.

Exemplu $f : \mathbb{F}_2^m \to \mathbb{C}$ $\quad \hat{f}(\vec{x}) = \frac{1}{\sqrt{2^m}} \sum_{\vec{y} \in \mathbb{F}_2^m} (-1)^{\vec{x} \cdot \vec{y}} f(\vec{y})$

nr. elem. din grup

transf. Hadamard-Walsh

motiv: $W_2 \oplus W_2 \oplus \dots \oplus W_2$

$$\begin{pmatrix} \hat{f}(g_1) \\ \hat{f}(g_2) \\ \vdots \\ \hat{f}(g_n) \end{pmatrix} = \frac{1}{\sqrt{n}} \begin{pmatrix} \chi_1^*(g_1) & \chi_2^*(g_2) & \cdots & \chi_1^*(g_n) \\ \vdots & & & \vdots \\ \chi_n^*(g_1) & \cdots & \cdots & \chi_n^*(g_n) \end{pmatrix} \begin{pmatrix} f(g_1) \\ f(g_2) \\ \vdots \\ f(g_n) \end{pmatrix}$$

- transformata Fourier este o aplicatie liniară (ba chiar unitară)
- matricea corespunzătoare este $\frac{1}{\sqrt{n}} X^*$

$$\begin{pmatrix} f(g_1) \\ f(g_2) \\ \vdots \\ f(g_n) \end{pmatrix} = \frac{1}{\sqrt{n}} \begin{pmatrix} \chi_1(g_1) & \cdots & \chi_n(g_1) \\ \vdots & & \vdots \\ \chi_1(g_n) & \cdots & \chi_n(g_n) \end{pmatrix} \begin{pmatrix} \hat{f}(g_1) \\ \hat{f}(g_2) \\ \vdots \\ \hat{f}(g_n) \end{pmatrix}$$

- transformata Fourier inversă
- scriere în clar: $\tilde{f}(g_i) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \chi_k(g_i) f(g_k)$

Obs. $\boxed{\tilde{\hat{f}} = \hat{\tilde{f}} = f}$

$\wedge$ transf. directă
$\sim$ transf. inversă

$\boxed{\|f\| = \|\hat{f}\| = \|\tilde{f}\|}$ – identitatea lui Parseval

apl. lin. ale spațiului Hermitian care păstrează norma – cele unitare

deoarece $\frac{1}{\sqrt{n}} X$ unitar.

Î n $\mathbb{Z}_n$: $\chi_x(y) = \chi_y(x)$

transf. Fourier inversă : $\tilde{f}(x) = \frac{1}{\sqrt{n}} \sum_{y \in \mathbb{Z}_n} e^{\frac{2\pi i x y}{n}} f(y)$

Î n $\mathbb{F}_2^m$: $\tilde{f}(\vec{x}) = \frac{1}{\sqrt{2^m}} \sum_{y \in \mathbb{F}_2^m} (-1)^{\vec{x}\cdot\vec{y}} f(\vec{y}) = \hat{f}(\vec{x})$

aici transf. Fourier discretă este o involuție.

Def. $f : G \to \mathbb{C}$ s.n. periodică $\iff$

$\exists \underset{\text{perioadă}}{p} \in G \setminus \{0\}$ a.î. $\forall g \in G$ $f(g+p) = f(g)$

$$\hat{f}(g_i) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \chi_i^*(g_k) \cdot f(g_k) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \chi_i^*(g_k + p - p) \cdot f(g_k + p) =$$

$$= \chi_i^*(-p) \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \chi_i^*(g_k + p) f(g_k + p) = \underbrace{\chi_i^*(-p)}_{\chi_i(p)} \hat{f}(g_i)$$

$\Rightarrow$ Dacă f periodică de perioadă p şi i ales a.î. $\chi_i(p) \neq 1$, atunci $\hat{f}(g_i) = 0$.

Transformarea Fourier Cuantică

G grup finit, H spațiu Hilbert care se reprezintă pe G.

H bază $\{ |g\rangle \mid g \in G \}$

$|g_i\rangle = (0, \dots, 0, \underset{\text{poz } i}{1}, 0, \dots, 0)^T$

Stare: $c_1 |g_1\rangle + \dots + c_n |g_n\rangle$ $\quad \sum |c_i|^2 = 1$, $c_i \in \mathbb{C}$

adică $f : G \to \mathbb{C}$, $f(g_i) = c_i$ şi $\|f\| = 1$.

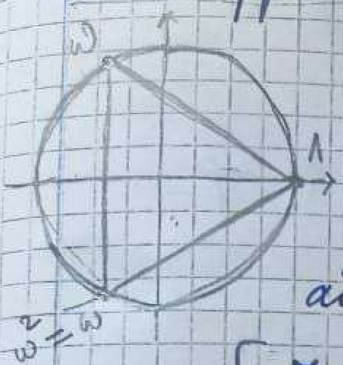
QFT (quantum Fourier transform):

$$\sum_{i=1}^{n} f(g_i) |g_i\rangle \rightsquigarrow \sum_{i=1}^{n} \hat{f}(g_i) |g_i\rangle$$

cu matricea unitară $\frac{1}{\sqrt{n}} X$ adică

$$\frac{1}{\sqrt{n}} \begin{pmatrix} \chi_1^*(g_1) & \dots & \chi_1^*(g_n) \\ \vdots & & \vdots \\ \chi_n^*(g_1) & \dots & \chi_n^*(g_n) \end{pmatrix}$$

Giuseppe Cardano



$z_1, z_2, z_3 \in \mathbb{C}$ $\quad \omega = \exp\left(\frac{2\pi i}{3}\right)$

$z^3 = 1$ are soluțiile $1, \omega, \omega^2$, $1+\omega+\omega^2 = 0$

Căutăm polinomul $P(z) = \eta_0 + \eta_1 z + \eta_2 z^2$.

a.î. $P(1) = z_0$, $P(\omega) = z_1$, $P(\omega^2) = z_2$

$$(★)\begin{cases} z_0 = \eta_0 + \eta_1 + \eta_2 \\ z_1 = \eta_0 + \eta_1 \omega + \eta_2 \omega^2 \\ z_2 = \eta_0 + \eta_1 \omega^2 + \eta_2 \omega \end{cases} \Leftrightarrow \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} = 3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

transpusa conjugatei

$$\Omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

unitară, $\Omega \Omega^* = I_3$

(★) devine
$$\begin{cases} 3\eta_0 = z_0 + z_1 + z_2 \\ 3\eta_1 = z_0 + z_1 \omega^2 + z_2 \omega \\ 3\eta_2 = z_0 + z_1 \omega + z_2 \omega^2 \end{cases}$$

η_0 = centrul de greutate al $\Delta z_0 z_1 z_2$

după translație cu $-\eta_0 \Rightarrow \eta_0 = 0$.

$$\begin{cases} z_0 = \eta_1 + \eta_2 \\ z_1 = \eta_1 \omega + \eta_2 \omega^2 \\ z_3 = \eta_1 \omega^2 + \eta_2 \omega \end{cases}$$

$\omega^2 + \omega = -1$

$\searrow$ $9\eta_1 \eta_2 = z_0^2 + z_1^2 + z_2^2 - z_0 z_1 - z_1 z_2 - z_0 z_2$

$z_0 + z_1 + z_2 = 0 \Rightarrow z_0^2 + z_1^2 + z_2^2 + 2z_1 z_2 + 2z_0 z_1 + 2z_0 z_2$

$\Rightarrow 3\eta_1 \eta_2 = -(z_0 z_1 + z_1 z_2 + z_0 z_2)$

$$z^3 + pz + q = 0$$ (se cunoaște schimbarea de variabilă: $ax^3+bx^2+cx+d=0$)

Viète:

$$p = z_0 z_1 + z_1 z_2 + z_0 z_2$$

$$\boxed{3\eta_1\eta_2 = -p}$$

$$(z-1)(z-\omega)(z-\omega^2) = z^3 - 1$$

$$z \xrightarrow{\text{înloc.}} -z \qquad (z+1)(z+\omega)(z+\omega^2) = z^3+1$$

deci dacă $z = \eta_1/\eta_2$ + omogenizare:

$$(\eta_1+\eta_2)(\eta_1+\omega\eta_2)(\eta_1+\omega^2\eta_2) = \eta_1^3+\eta_2^3$$

înmulțim $\omega \cdot$ $\omega^2 \cdot$ (nu schimbă nimic pt. $\omega^3=1$)

$$\overbrace{(\eta_1+\eta_2)}^{z_0}\overbrace{(\omega\eta_1+\omega^2\eta_2)}^{z_1}\overbrace{(\omega^2\eta_1+\omega\eta_2)}^{z_2} = \eta_1^3+\eta_2^3$$

$$z_0 z_1 z_2 = \eta_1^3 + \eta_2^3 = -q$$

$$\eta_1^3 + \eta_2^3 = -q$$ η_1^3, η_2^3 necunoscute

$$\eta_1^3\eta_2^3 = -\frac{p^3}{27}$$ se cunosc suma și produsul

$\Downarrow$

se poate construi o ec. care le are ca sol

Def. __rezolventa pătratică__: $x^2 + qx - \frac{p^3}{27} = 0$

Rez. ec. $\Rightarrow$ $\eta_1^3 = -\frac{q}{2} + \sqrt{\Delta}$

$\eta_2^3 = -\frac{q}{2} - \sqrt{\Delta}$

unde $\Delta = \frac{q^2}{4} + \frac{p^3}{27}$ discriminantul ecuației de grad 3.

- se calc. η_1

$$\eta_1\eta_2 = -\frac{p}{3} \Rightarrow \text{se calc. } \eta_2$$