C10 – Model checking

Program Verification

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Overview

Model Checking

LTL Logic

CTL Logic

Model checking algorithm for CTL

Model Checking

The big picture

- Application domain: mostly concurrent, reactive systems
- Verify that a system satisfies a property
  1. Model the system in the model checker's description language. Call this model M.
  2. Express the property to be verified in the model checker's specification language. Call this formula ϕ.
  3. Run the model checker to show that M satisfies ϕ.
- Automatic for finite-state models

Model Checking - Models

A model of some system has

- A finite set of states
- A subset of states considered as the initial states
- A transition relation which, given a state, describes all the states that can be reached ”in one time step”

Model Checking - Specifications

We are interested in specifying behaviours of systems over time (use Temporal Logic).

Specifications are built from

- primitive properties of individual states
  - e.g., *is on*, *is off*, *is active*, *is reading*
- propositional connectives $\wedge, \vee, \neg, \rightarrow$
- temporal connectives
  - e.g., *At* **all times**, *the system is not simultaneously reading and writing.*
  - e.g., *If a request signal is asserted* **at some time**, *a corresponding grant signal will be asserted* **within 10 time units.**

Linear vs. Branching Time

Linear Time

- Considers paths (sequences of states)
- Questions of the form
  - *For all paths, does some path property hold?*
  - *Does there exist a path such that some path property holds?*

Branching Time

- Considers trees of possible future states from each initial state
- Questions can become more complex
  - *For all states reachable from an initial state, does there exist an onwards path to a state satisfying some property?*

LTL Logic

Syntax

LTL = Linear(-time) Temporal Logic

Assume some set *Atoms* of atomic propositions.

LTL formulas are defined by:

$$\varphi \;:=\; p \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid \mathbf{X}\,\varphi \mid \mathbf{F}\,\varphi \mid \mathbf{G}\,\varphi \mid \varphi\,\mathbf{U}\,\varphi$$

Pronunciation:

- $\mathbf{X}\,\varphi$ – neXt φ
- $\mathbf{F}\,\varphi$ – Future φ
- $\mathbf{G}\,\varphi$ – Globally φ
- $\varphi\,\mathbf{U}\,\psi$ – φ Until ψ

Precedence high-to-low:

- $\mathbf{X}, \mathbf{F}, \mathbf{G}, \neg$
- $\mathbf{U}$
- $\wedge, \vee$
- $\rightarrow$

Syntax

Example
The following are LTL formulas:

- $\mathbf{F}\ p \land \mathbf{G}\ q \rightarrow p\ \mathbf{U}\ r$
- $\mathbf{F}(p \rightarrow \mathbf{G}\ r) \lor \neg q\ \mathbf{U}\ p$
- $p\ \mathbf{U}\ (q\ \mathbf{U}\ r)$
- $\mathbf{G}\ \mathbf{F}\ p \rightarrow \mathbf{F}(q \lor s)$

Syntax

Example
The following are LTL formulas:

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- $p\ \mathbf{U}\ (q\ \mathbf{U}\ r)$
- $\mathbf{G}\ \mathbf{F}\ p \rightarrow \mathbf{F}(q \vee s)$

The following are not LTL formulas:

- $\mathbf{U}\ r$
- $q\ \mathbf{G}\ p$

Informal semantics

LTL formulas are evaluated at a position i along a path π through the system:

- An atomic proposition p holds if p is true in the state at position i.
- The propositional connectives $\neg, \wedge, \vee, \rightarrow$ have their usual meanings.
- Meaning of temporal connectives:
  - $\mathbf{X}\varphi$ holds if φ holds at the next position
  - $\mathbf{F}\varphi$ holds if there exists a future position where φ holds
  - $\mathbf{G}\varphi$ holds if φ holds in all future positions
  - $\varphi\mathbf{U}\psi$ holds if there exists a future position where ψ holds, and φ holds for all positions prior to that

A path is a sequence of states connected by transitions.

Informal semantics

Example

G *invariant*

- *invariant* is true for all future positions

Informal semantics

Example

$\mathbf{G}\ \mathit{invariant}$

- *invariant* is true for all future positions

$\mathbf{G}\ \neg(\mathit{read} \wedge \mathit{write})$

- In all future positions, it is not the case that *read* and *write*

Informal semantics

Example

$\mathbf{G}\ \mathit{invariant}$

- *invariant* is true for all future positions

$\mathbf{G}\ \neg(\mathit{read} \wedge \mathit{write})$

- In all future positions, it is not the case that *read* and *write*

$\mathbf{G}(\mathit{request} \rightarrow \mathbf{F}\ \mathit{grant})$

- At every point in the future, a *request* implies that there exists a future point where *grant* holds.

Informal semantics

Example

$\mathbf{G}(\mathit{request} \rightarrow (\mathit{request}\ \mathbf{U}\ \mathit{grant}))$

- At every point in the future, a *request* implies that there exists a future point where *grant* holds, and *request* holds up until that point.

Informal semantics

Example

$\mathbf{G}(\mathit{request} \rightarrow (\mathit{request}\ \mathbf{U}\ \mathit{grant}))$

- At every point in the future, a *request* implies that there exists a future point where *grant* holds, and *request* holds up until that point.

$\mathbf{G}\ \mathbf{F}\ \mathit{enabled}$

- In all future positions, there is a future position where *enabled* holds.

Informal semantics

Example

$\mathbf{G}(\mathit{request} \rightarrow (\mathit{request}\ \mathbf{U}\ \mathit{grant}))$

- At every point in the future, a *request* implies that there exists a future point where *grant* holds, and *request* holds up until that point.

$\mathbf{G}\ \mathbf{F}\ \mathit{enabled}$

- In all future positions, there is a future position where *enabled* holds.

$\mathbf{F}\ \mathbf{G}\ \mathit{enabled}$

- There is a future position, from which all future positions have *enabled* holding.

Transition Systems and Paths

A transition system (or model) $\mathcal{M} = (S, \rightarrow, L)$ consists of

- S a finite set of states
- $\rightarrow \subseteq S \times S$ a transition relation
- $L : S \rightarrow \mathcal{P}(Atoms)$ a labelling function

such that for all $s_1 \in S$ there exists $s_2 \in S$ with $s_1 \rightarrow s_2$ (*serial condition*).

Note:

- $Atoms$ is a fixed set of atomic propositions and $\mathcal{P}(Atoms)$ is the powerset of $Atoms$. Thus, $L(s)$ is just the set of atomic propositions that are true in state s.

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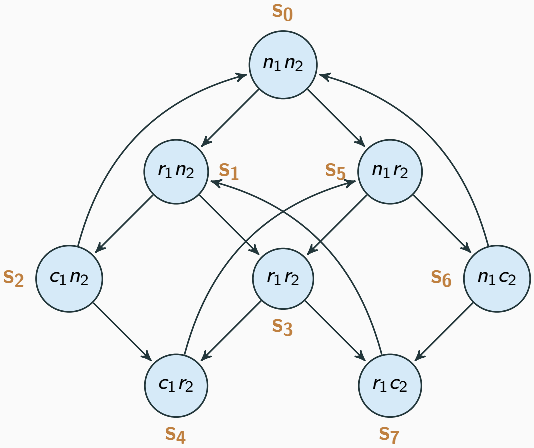
A path π in a transition system $\mathcal{M} = (S, \rightarrow, L)$ is an infinite sequence of states $s_0, s_1, s_2, \ldots$ such that for all $i \geq 0$, $s_i \rightarrow s_{i+1}$.

Paths are written as $\pi = s_0 \rightarrow s_1 \rightarrow s_2 \rightarrow \ldots$

We write π^i for the suffix starting at s_i. For example, π^3 is $s_3 \rightarrow s_4 \rightarrow \ldots$

Transition Systems and Paths

Example



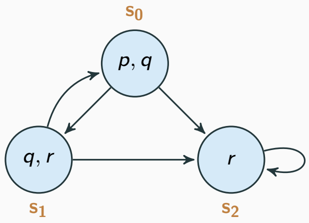
$Atoms = \{n_1, n_2, r_1, r_2, c_1, c_2\}$

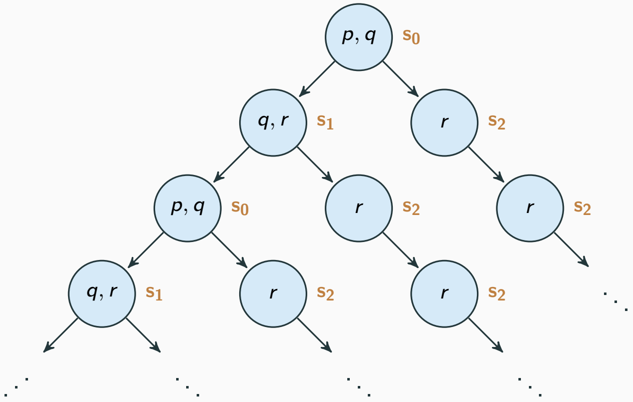
$\mathcal{M} = (S, \rightarrow, L)$ where

- $S = \{s_0, s_1, \ldots, s_7\}$
- $\rightarrow = \{(s_0, s_1), (s_0, s_5), \ldots\}$
- $L(s_0) = \{n_1, n_2\}$
- $L(s_1) = \{r_1, n_2\}$
- $\ldots$

Unwinding a Transition System

Visualise all computational paths from a given state s by unwinding the transition system to obtain an infinite computation tree.





Satisfaction relation

Let $\mathcal{M} = (S, \to, L)$ be a model and $\pi = s_0 \to s_1 \to \ldots$ be a path in $\mathcal{M}$.

"The path π satisfies the LTL formula φ"

$$\pi \models \varphi$$

$\pi \models \top$		
$\pi \not\models \bot$		
$\pi \models p$	iff	$p \in L(s_0)$
$\pi \models \varphi \wedge \psi$	iff	$\pi \models \varphi$ and $\pi \models \psi$
$\pi \models \varphi \vee \psi$	iff	$\pi \models \varphi$ or $\pi \models \psi$
$\pi \models \varphi \to \psi$	iff	$\pi \models \varphi$ implies $\pi \models \psi$

Satisfaction relation

$\pi \models \mathbf{X}\ \varphi$	iff	$\pi^1 \models \varphi$
$\pi \models \mathbf{F}\ \varphi$	iff	there exists $i \geq 0$ such that $\pi^i \models \varphi$
$\pi \models \mathbf{G}\ \varphi$	iff	for all $i \geq 0$ we have $\pi^i \models \varphi$
$\pi \models \varphi_1\ \mathbf{U}\ \varphi_2$	iff	there exists $i \geq 0$ such that $\pi^i \models \varphi_2$ and for all $j \in \{0, \ldots, i-1\}$ we have $\pi^j \models \varphi_1$

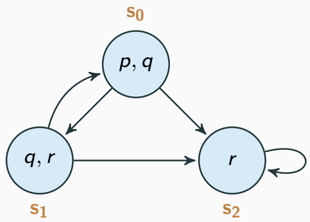
We write

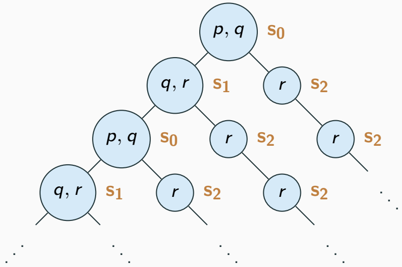
$$\mathcal{M}, s \models \varphi$$

if for every path π of a model $\mathcal{M}$ starting at state s we have $\pi \models \varphi$.

Satisfaction relation

Example





Satisfaction relation

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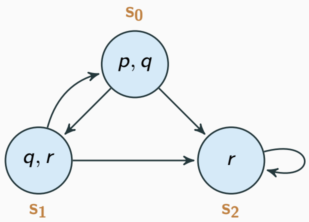
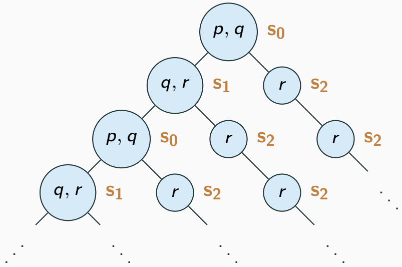
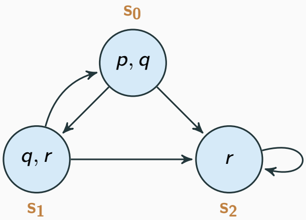


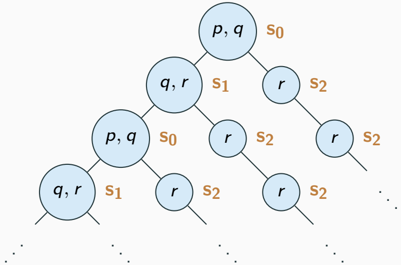


1. $\mathcal{M}, s_0 \models p \wedge q$

Satisfaction relation

Example

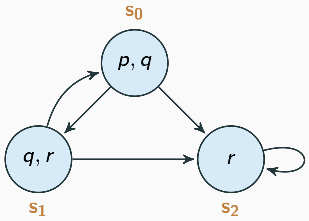


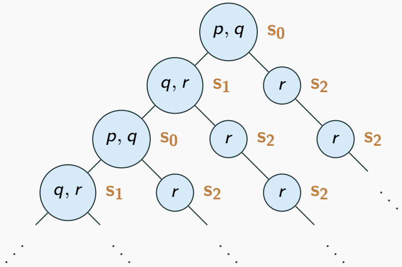


1. $\mathcal{M}, s_0 \models p \wedge q$
2. $\mathcal{M}, s_0 \models \neg r$

Satisfaction relation

Example





1. $\mathcal{M}, s_0 \models p \wedge q$
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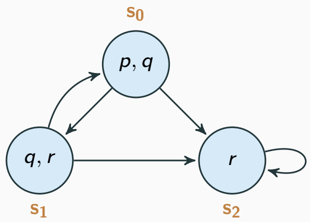
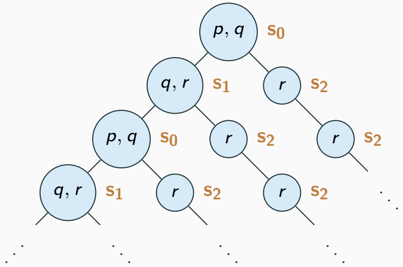
3. $\mathcal{M}, s_0 \models \mathbf{X}\ r$

Satisfaction relation

Example





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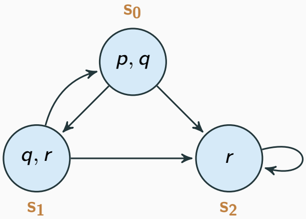
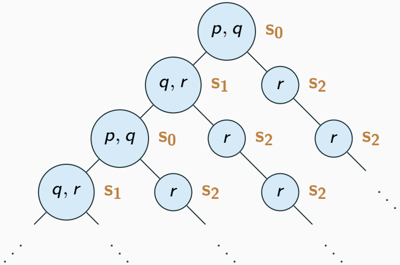
Satisfaction relation

Example





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Satisfaction relation

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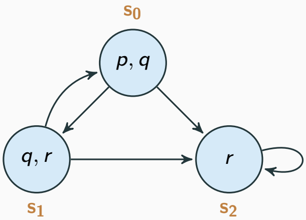

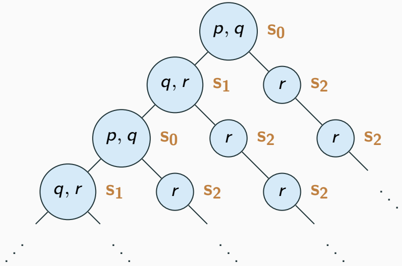

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Satisfaction relation

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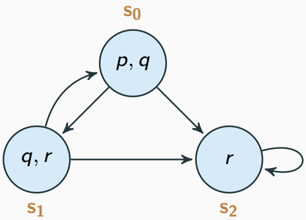

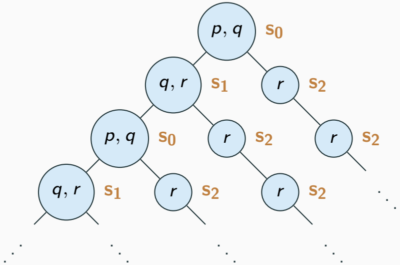

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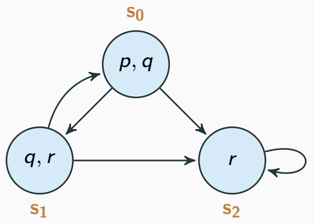
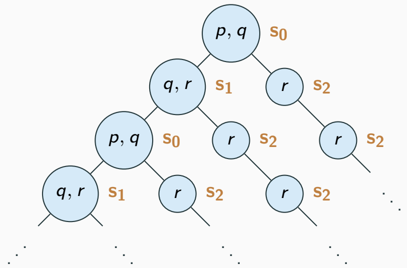
6. $\mathcal{M}, s_2 \models \mathbf{G}\ r$
7. $\mathcal{M}, s_0 \models$
   $\mathbf{F}\ (\neg q \wedge r) \rightarrow \mathbf{F}\ \mathbf{G}\ r$

Satisfaction relation

Example





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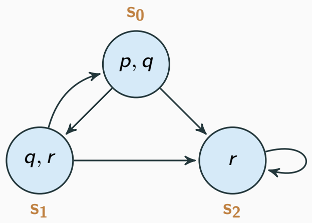
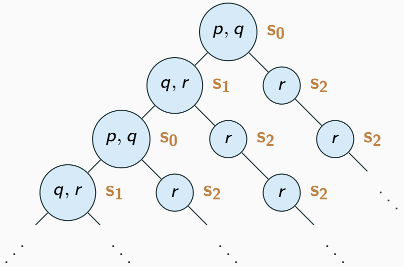
Satisfaction relation

Example





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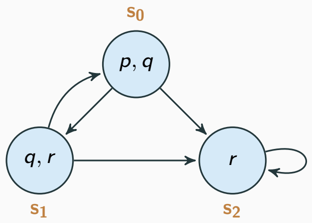
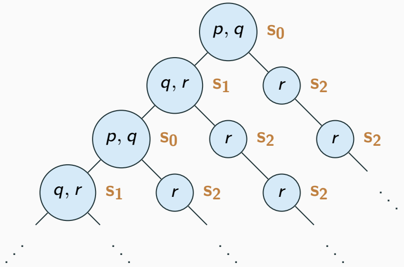
8. $\mathcal{M}, s_0 \not\models \mathbf{G}\ \mathbf{F}\ p$
9. $\mathcal{M}, s_0 \models \mathbf{G}\ \mathbf{F}\ p \to \mathbf{G}\ \mathbf{F}\ r$

Satisfaction relation

Example





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9. $\mathcal{M}, s_0 \models \mathbf{G}\ \mathbf{F}\ p \rightarrow \mathbf{G}\ \mathbf{F}\ r$
10. $\mathcal{M}, s_0 \not\models \mathbf{G}\ \mathbf{F}\ r \rightarrow \mathbf{G}\ \mathbf{F}\ p$

Equivalences

Two formulas are equivalent, denoted $\varphi \equiv \psi$, if they are satisfied by the same models.

Equivalences from Propositional Logic:

$$\neg(\varphi \wedge \psi) \equiv \neg\varphi \vee \neg\psi \qquad \neg(\varphi \vee \psi) \equiv \neg\varphi \wedge \neg\psi$$

Equivalences from LTL:

$$\neg\mathbf{X}\varphi \equiv \mathbf{X}\neg\varphi \qquad \neg\mathbf{G}\varphi \equiv \mathbf{F}\neg\varphi \qquad \neg\mathbf{F}\varphi \equiv \mathbf{G}\neg\varphi$$

Distributive laws:

$$G(\varphi \wedge \psi) \equiv G\varphi \wedge G\psi \qquad F(\varphi \vee \psi) \equiv F\varphi \vee F\psi$$

Equivalences

Inter-definitions:

$$\mathbf{F}\varphi \equiv \neg\mathbf{G}\neg\varphi \qquad \mathbf{G}\varphi \equiv \neg\mathbf{F}\neg\varphi \qquad \mathbf{F}\varphi \equiv \top\,\mathbf{U}\,\varphi$$

Idempotency:

$$\mathbf{FF}\varphi \equiv \mathbf{F}\varphi \qquad \mathbf{GG}\varphi \equiv \mathbf{G}\varphi$$

Some more surprising equivalences:

$$\mathbf{GFG}\varphi \equiv \mathbf{FG}\varphi \qquad \mathbf{FGF}\varphi \equiv \mathbf{GF}\varphi \qquad \mathbf{G}(\mathbf{F}\varphi \vee \mathbf{F}\varphi) \equiv \mathbf{GF}\varphi \vee \mathbf{GF}\psi$$

CTL Logic

Syntax

CTL = Computation Tree Logic

Assume some set *Atoms* of atomic propositions.

CTL formulas are defined by:

$$\varphi \;:=\; p \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid$$
$$\mathbf{AX}\varphi \mid \mathbf{EX}\varphi \mid \mathbf{AF}\varphi \mid \mathbf{EF}\varphi \mid \mathbf{AG}\varphi \mid \mathbf{EG}\varphi \mid A[\varphi\mathbf{U}\varphi] \mid E[\varphi\mathbf{U}\varphi]$$

Each temporal connective is a pair of a path quantifier:

- **A** – for all paths
- **E** – there exists a path

and an LTL-like temporal operator X, F, G, U.

Precedence (high-to low): $(\mathbf{AX}, \mathbf{EX}, \mathbf{AF}, \mathbf{EF}, \mathbf{AG}, \mathbf{EG}, \neg)$, $(\mathbf{AU}, \mathbf{EU})$, $(\wedge, \vee)$, $\rightarrow$

Syntax

Example
The following are CTL formulas:

- $\mathbf{AG}(q \rightarrow \mathbf{EG}\ r)$
- $\mathbf{EF}\ \mathbf{E}[r\ \mathbf{U}\ q]$
- $\mathbf{A}[p\ \mathbf{U}\ \mathbf{EF}\ r]$
- $\mathbf{EF}\ \mathbf{EG}\ p \rightarrow \mathbf{AF}\ r$

Syntax

Example
The following are CTL formulas:

- $\mathbf{AG}(q \to \mathbf{EG}\ r)$
- $\mathbf{EF}\ \mathbf{E}[r\ \mathbf{U}\ q]$
- $\mathbf{A}[p\ \mathbf{U}\ \mathbf{EF}\ r]$
- $\mathbf{EF}\ \mathbf{EG}\ p \to \mathbf{AF}\ r$

The following are not CTL formulas:

- $\mathbf{EF}\ F\ r$
- $\mathbf{A}\neg F \neg p$
- $G[r\ \mathbf{U}\ q]$
- $\mathbf{EF}(r\ \mathbf{U}\ q)$

Transition Systems and Paths

(This is the same as for LTL)

A transition system (or model) $\mathcal{M} = (S, \rightarrow, L)$ consists of

- S a finite set of states
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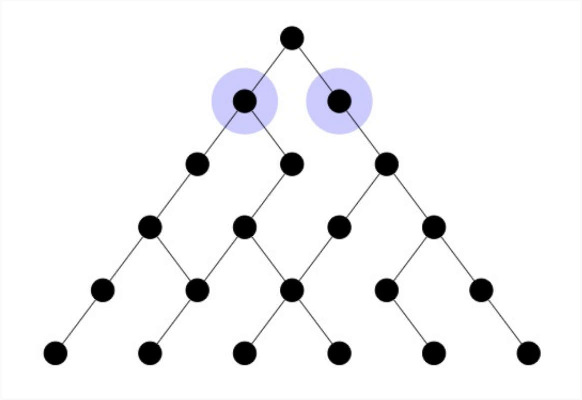
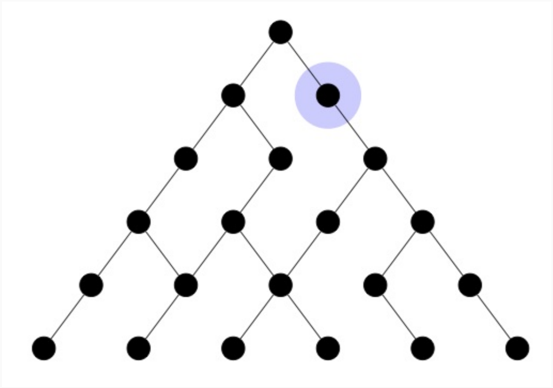
CTL in pictures



AX φ

For every next state, φ holds.

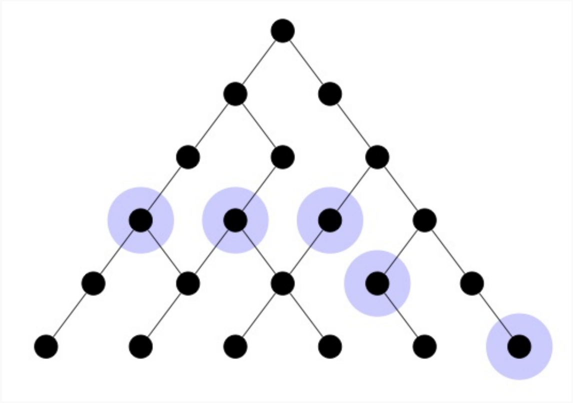
CTL in pictures



EX φ

There exists a next state where φ holds.

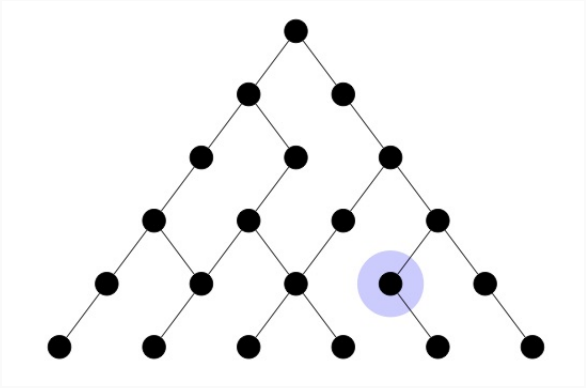
CTL in pictures



AF φ

For all paths, there exists a future state where φ holds.

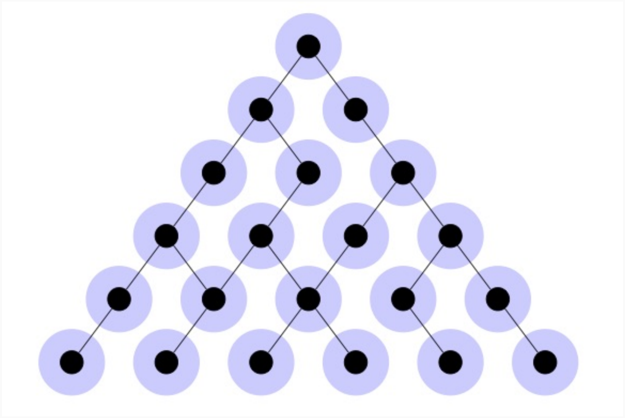
CTL in pictures



EF φ

There exists a path with a future state where φ holds.

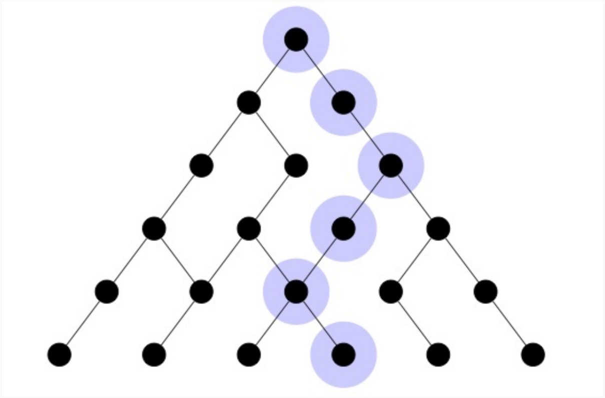
CTL in pictures



AG φ

For all paths, for all states along them, φ holds.

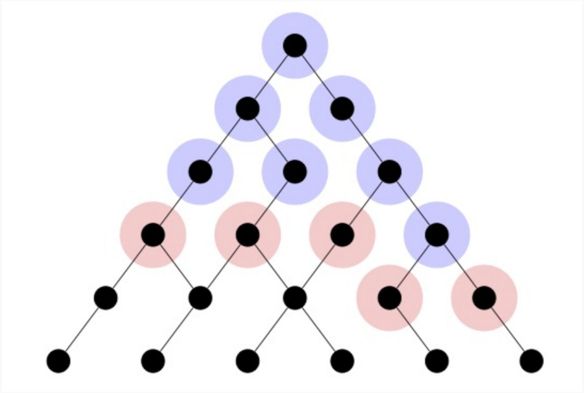
CTL in pictures



EG φ

There exists a path such that, for all states along it, φ holds.

CTL in pictures



$\mathbf{A}[\varphi \; \mathbf{U} \; \psi]$

For all paths, ψ eventually holds, and φ holds at all states earlier.

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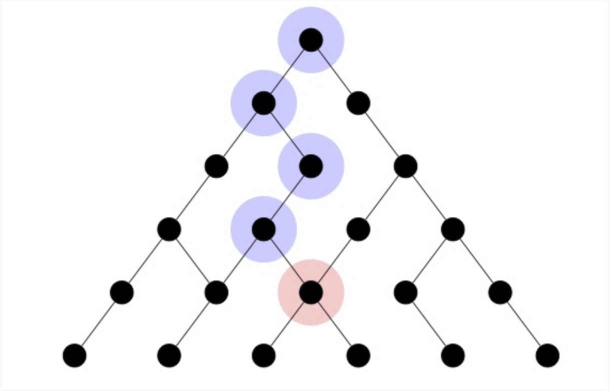

$\mathbf{E}[\varphi \, \mathbf{U} \, \psi]$

There exists a path wehre ψ eventually holds, and φ holds at all states earlier.

Informal semantics

Example

EF φ

- there exists a future state where eventually φ is true

AG AF φ

- for all future states, φ will eventually hold

AG$(\varphi \rightarrow$ **AF** $\psi)$

- for all future states, if φ holds, then ψ will eventually hold

Informal semantics

Example

$\mathbf{AG}(\varphi \rightarrow \mathbf{E}[\varphi \mathbf{U} \psi])$

- for all future states, if φ holds, then there is a future where ψ will eventually hold, and φ holds for all points in between

$\mathbf{AG}(\varphi \rightarrow \mathbf{EG}\, \psi)$

- for all future states, if φ holds, then there is a future where ψ always holds

$\mathbf{EF}\, \mathbf{AG}\, \varphi$

- there exists a possible state in the future from where φ is always true

Satisfaction Relation

"The state s of the model $\mathcal{M}$ satisfies the CTL formula φ"

$$\mathcal{M}, s \models \varphi$$

$\mathcal{M}, s \models \top$		
$\mathcal{M}, s \not\models \bot$		
$\mathcal{M}, s \models p$	iff	$p \in L(s)$
$\mathcal{M}, s \models \varphi \wedge \psi$	iff	$\mathcal{M}, s \models \varphi$ and $\mathcal{M}, s \models \psi$
$\mathcal{M}, s \models \varphi \vee \psi$	iff	$\mathcal{M}, s \models \varphi$ or $\mathcal{M}, s \models \psi$
$\mathcal{M}, s \models \varphi \rightarrow \psi$	iff	$\mathcal{M}, s \models \varphi$ implies $\mathcal{M}, s \models \psi$
$\mathcal{M}, s \models$ **AX** φ	iff	for all $s' \in S$ such that $s \rightarrow s'$ we have $\mathcal{M}, s' \models \varphi$
$\mathcal{M}, s \models$ **EX** φ	iff	there exists $s' \in S$ such that $s \rightarrow s'$ and $\mathcal{M}, s' \models \varphi$

Satisfaction Relation

Assume that $\pi = s_0 \to s_1 \to s_2 \to \dots$

$\mathcal{M}, s \models \mathbf{AF}\ \varphi$	iff	for all paths π such that $s_0 = s$
		there exists i such that $\mathcal{M}, s_i \models \varphi$
$\mathcal{M}, s \models \mathbf{EF}\ \varphi$	iff	there exists a path π such that $s_0 = s$ and
		there exists i such that $\mathcal{M}, s_i \models \varphi$
$\mathcal{M}, s \models \mathbf{AG}\ \varphi$	iff	for all paths π such that $s_0 = s$
		for all i, $\mathcal{M}, s_i \models \varphi$
$\mathcal{M}, s \models \mathbf{EG}\ \varphi$	iff	there exists a path π such that $s_0 = s$ and
		for all i, $\mathcal{M}, s_i \models \varphi$
$\mathcal{M}, s \models A[\varphi_1 \mathbf{U}\ \varphi_2]$	iff	for all paths π such that $s_0 = s$
		there exists i such that $\mathcal{M}, s_i \models \varphi_2$ and
		for all $j < i$, $\mathcal{M}, s_j \models \varphi_1$
$\mathcal{M}, s \models E[\varphi_1 \mathbf{U}\ \varphi_2]$	iff	there exists a path π such that $s_0 = s$ and
		there exists i such that $\mathcal{M}, s_i \models \varphi_2$ and
		for all $j < i$, $\mathcal{M}, s_j \models \varphi_1$

CTL - Equivalences

$$\begin{aligned}
\neg \mathbf{EX}\,\varphi &\equiv \mathbf{AX}\,\neg\varphi \\
\neg \mathbf{EF}\,\varphi &\equiv \mathbf{AG}\,\neg\varphi \\
\neg \mathbf{EG}\,\varphi &\equiv \mathbf{AF}\,\neg\varphi \\
\mathbf{AF}\,\varphi &\equiv \mathbf{A}[\top\,\mathbf{U}\,\varphi] \\
\mathbf{EF}\,\varphi &\equiv \mathbf{E}[\top\,\mathbf{U}\,\varphi] \\
\mathbf{A}[\varphi\,\mathbf{U}\,\psi] &\equiv \neg(\mathbf{E}[\neg\psi\,\mathbf{U}\,(\neg\varphi \wedge \neg\psi)] \vee \mathbf{EG}\,\neg\psi)
\end{aligned}$$

From these, one can show that the sets $\{\mathbf{AU}, \mathbf{EU}, \mathbf{EX}\}$ and $\{\mathbf{EU}, \mathbf{EG}, \mathbf{EX}\}$ are both adequate sets of temporal connectives.

Differences between LTL and CTL

LTL allows questions of the form:

- For all paths, does the LTL formula φ hold?
- Does there exist a path on which the LTL formula φ holds?
  (Ask whether $\neg\varphi$ holds on all paths.)

CTL allows mixing of path quantifiers:

- $\mathbf{AG}(p \to \mathbf{EG}\, q)$
  (For all paths, if p is true then there is a path on which q is always true.)
- Intuitively, in CTL we cannot fix a path π and talk about it.

Differences between LTL and CTL

However, some path properties are impossible to express in CTL.

The following formulas do not express the same thing:

- LTL: $G\ F\ p \rightarrow G\ F\ q$
  for all paths π, if p holds infinitely often on π, then q holds infinitely often on π
- CTL: $\mathbf{AG}\ \mathbf{AF}\ p \rightarrow \mathbf{AG}\ \mathbf{AF}\ q$
  if p holds infinitely often on all paths, then q holds infinitely often on all paths.

There exist fair refinements of CTL that address this issue to some extent.

Model checking algorithm for CTL

Model checking - The big picture

- An efficient algorithm to decide $\mathcal{M}, s \models \varphi$
- Typically without user interaction (fully automated)
- Different approaches: semantic, automata, tableau,...
- Provide a counterexample when $\mathcal{M}, s \not\models \varphi$.
  The counterexample provides a clue to what is wrong:
  - The system might be incorrect
  - The model might be too abstract (in need of refinement)
  - The specification might not be the intended one

State explosion problem

- The number of states grow exponentially with the number of system components.
- Different techniques to make state explosion less severe:
  - Abstraction
  - Induction
  - Partial order reduction
  - ...

CTL model checking

- We present an algorithm which, given a model and a CTL formula, outputs the set of states of the model that satisfy the formula.
- The algorithm handles CTL formulas containing the connectives

$$\{\bot, \neg, \wedge, \textbf{AF}, \textbf{EU}, \textbf{EX}\}$$

- Given an arbitrary CTL formula φ, we simply pre-process φ in order to write it in an equivalent form in terms of the above set of connectives.

The labelling algorithm

INPUT: a CTL model $\mathcal{M} = (S, \rightarrow, L)$ and a CTL formula φ

OUTPUT: the set of states of $\mathcal{M}$ which satisfy φ

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- **EX** ψ_1: label a state with **EX** ψ_1 if one of its successors is labelled with ψ_1

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- **EX** ψ_1: label a state with **EX** ψ_1 if one of its successors is labelled with ψ_1
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  - If any state *s* is labelled with ψ_1, label it with **AF** ψ_1
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- **EX** ψ_1: label a state with **EX** ψ_1 if one of its successors is labelled with ψ_1
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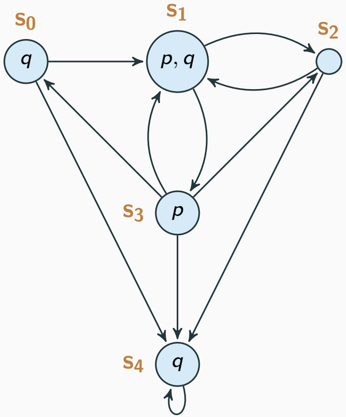
    - **Repeat:** label any state with **AF** ψ_1 if all its successor states are labelled with **AF** ψ_1, until there is no change.
- **E**$[\psi_1$ **U** $\psi_2]$:
    - If any state *s* is labelled with ψ_2, label it with **E**$[\psi_1$ **U** $\psi_2]$
    - **Repeat:** label any state with **E**$[\psi_1$ **U** $\psi_2]$ if it is labelled with ψ_1 and at least one of its successors is labelled with **E**$[\psi_1$ **U** $\psi_2]$, until there is no change.

The labelling algorithm

Example

$\mathbf{AG(q \to EG\ p)} \equiv \neg\mathbf{E}[\neg\bot\ \mathbf{U}\ (q \wedge \mathbf{AF}\neg p)]$

- Label q: $\{s_0, s_1, s_4\}$
- Label p: $\{s_1, s_3\}$



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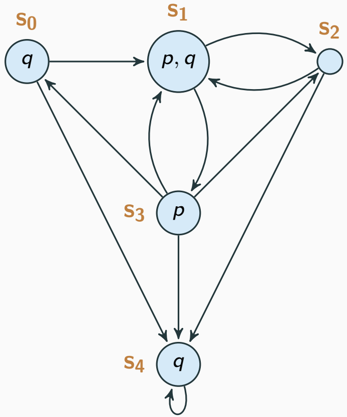

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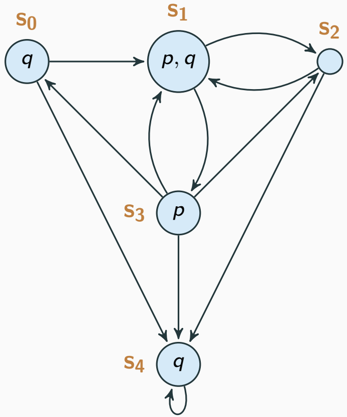

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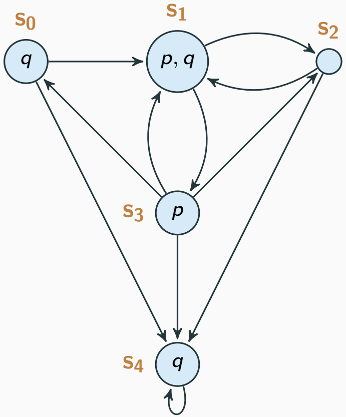

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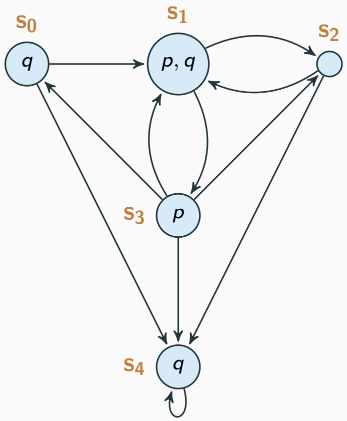

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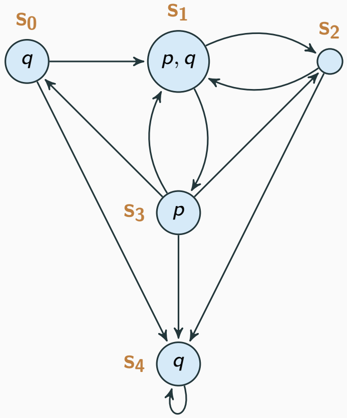

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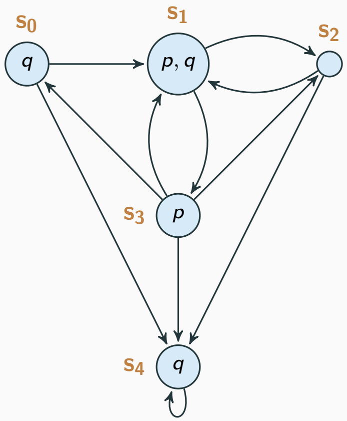

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  3. $\{s_0, s_4, s_2, s_1\}$
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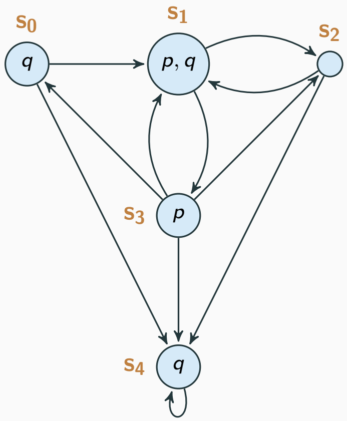

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$\mathbf{AG}(q \to \mathbf{EG}\ p) \equiv \neg\mathbf{E}[\neg\bot\ \mathbf{U}\ (q \land \mathbf{AF}\neg p)]$

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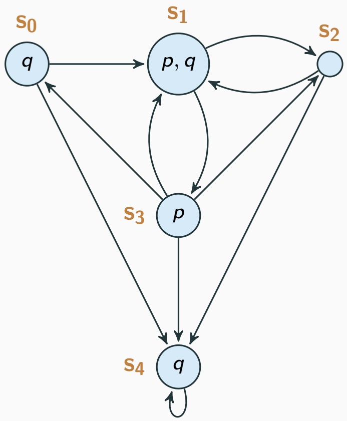

The state explosion problem

Although the labelling algorithm is linear in the size of the model, unfortunately the size of the model is itself more often than not exponential in the number of variables and the number of components of the system which execute in parallel.

For example, adding a boolean variable to a program will double the complexity of verifying a property of it.

The tendency of state spaces to become very large is known as the state explosion problem.

The state explosion problem

A lot of research had gone into finding ways to overcoming it:

- Efficient data structures, called ordered binary decision diagrams (OBDDs) which represent sets of states instead of individual states.
- Abstraction: that one may interpret a model abstractly, uniformly or for a specific property.
- Partial order reduction: for asynchronous systems, several interleavings of components traces may be equivalent as far as satisfaction of the formula to be checked is concerned.
- Induction: model-checking systems with (e.g.) large numbers of identical, or similar, components can often be implemented by "induction" on this number.
- ...

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- Lecture Notes on "Computer-Aided Reasoning for Software Engineering", University of Washington, Emina Torlak.