FMI, Computer Science, Master
Advanced Logic for Computer Science

Seminar 4

(S4.1) Let $\mathcal{L}$ be a first-order language that contains two unary relation symbols S, T and one binary relation symbol R. Find Skolem normal forms for the following formulas of $\mathcal{L}$:

$$\begin{aligned} \chi &:= \exists y \forall x \exists v ((S(y) \vee R(x,v)) \rightarrow (T(v) \rightarrow S(y))) \\ \delta &:= \forall x \exists u \forall y \exists v \left((S(u) \rightarrow R(v,y)) \vee (S(v) \rightarrow T(x))\right). \end{aligned}$$

Proof. We have that

$$\begin{aligned} \chi^1 &= \forall x \exists v ((S(e) \vee R(x,v)) \rightarrow (T(v) \rightarrow S(e))) \\ & \quad \text{where } e \text{ is a new constant symbol} \\ \chi^2 &= \forall x ((S(e) \vee R(x,g(x))) \rightarrow (T(g(x)) \rightarrow S(e))) \\ & \quad \text{where } g \text{ is a new unary function symbol.} \end{aligned}$$

Since χ^2 is a universal sentence, it follows that $\chi^{Sk} = \chi^2$ is a Skolem normal form for χ.

$$\begin{aligned} \delta^1 &= \forall x \forall y \exists v \left((S(h(x)) \rightarrow R(v,y)) \vee (S(v) \rightarrow T(x))\right) \\ & \quad \text{where } h \text{ is a new unary function symbol} \\ \delta^2 &= \forall x \forall y \left((S(h(x)) \rightarrow R(n(x,y),y)) \vee (S(n(x,y)) \rightarrow T(x))\right) \\ & \quad \text{where } n \text{ is a new binary function symbol.} \end{aligned}$$

Since δ^2 is a universal sentence, it follows that $\delta^{Sk} = \delta^2$ is a Skolem normal form for δ. □

(S4.2) Let $\mathcal{M} = (W, R, V)$ be a model for ML_0 and w a state in $\mathcal{M}$. Prove that for every formula φ,

$$\mathcal{M}, w \Vdash \Box\varphi \quad \text{iff} \quad \text{for every } v \in W, Rwv \text{ implies } \mathcal{M}, v \Vdash \varphi.$$

Proof. We have that

$$\begin{array}{lll}
\mathcal{M}, w \Vdash \Box\varphi & \text{iff} & \mathcal{M}, w \Vdash \neg\Diamond\neg\varphi \\
& \text{iff} & \mathcal{M}, w \nVdash \Diamond\neg\varphi \\
& \text{iff} & \text{there does not exist } v \in W \text{ such that } (Rwv \text{ and } \mathcal{M}, v \Vdash \neg\varphi) \\
& \text{iff} & \text{for every } v \in W, \text{ we don't have that } (Rwv \text{ and } \mathcal{M}, v \Vdash \neg\varphi) \\
& \text{iff} & \text{for every } v \in W,\ Rwv \text{ is false or } \mathcal{M}, v \nVdash \neg\varphi \\
& \text{iff} & \text{for every } v \in W,\ Rwv \text{ is false or } \mathcal{M}, v \Vdash \varphi \\
& \text{iff} & \text{for every } v \in W,\ Rwv \text{ implies } \mathcal{M}, v \Vdash \varphi.
\end{array}$$

□

(S4.3) Consider the frame $\mathcal{F} = (W = \{w_1, w_2, w_3, w_4, w_5\}, R)$, where Rw_iw_j iff $j = i + 1$:

$$w_1 \longrightarrow w_2 \longrightarrow w_3 \longrightarrow w_4 \longrightarrow w_5$$

Let us choose a valuation V such that $V(p) = \{w_2, w_3\}$, $V(q) = \{w_1, w_2, w_3, w_4, w_5\}$ and $V(r) = \emptyset$. Consider the model $\mathcal{M} = (\mathcal{F}, V)$. Prove the following:

(i) $\mathcal{M}, w_1 \Vdash \Diamond\Box p$;

(ii) $\mathcal{M}, w_1 \nVdash \Diamond\Box p \to p$;

(iii) $\mathcal{M}, w_2 \Vdash \Diamond(p \wedge \neg r)$;

(iv) $\mathcal{M}, w_1 \Vdash q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond q)))$;

(v) $\mathcal{M} \Vdash \Box q$.

Proof. (i) $\mathcal{M}, w_1 \Vdash \Diamond\Box p$ iff there exists $v \in W$ such that Rw_1v and $\mathcal{M}, v \Vdash \Box p$.

Take $v := w_2$. As Rw_1w_2, it remains to prove that $\mathcal{M}, w_2 \Vdash \Box p$.

We have that

$$\begin{array}{lll}
\mathcal{M}, w_2 \Vdash \Box p & \text{iff} & \text{for every } u \in W,\ Rw_2u \text{ implies } \mathcal{M}, u \Vdash p. \\
& \text{iff} & \mathcal{M}, w_3 \Vdash p \text{ (since } w_3 \text{ is the unique } u \in W \text{ such that } Rw_2u) \\
& \text{iff} & w_3 \in V(p), \text{ which is true.}
\end{array}$$

(ii) Using classical propositional logic, we have that

$$\begin{array}{lll}
\mathcal{M}, w_1 \Vdash \Diamond\Box p \to p & \text{iff} & \mathcal{M}, w_1 \Vdash \neg\Diamond\Box p \vee p \\
& \text{iff} & \mathcal{M}, w_1 \Vdash \neg\Diamond\Box p \text{ or } \mathcal{M}, w_1 \Vdash p.
\end{array}$$

By (i), $\mathcal{M}, w_1 \Vdash \Diamond\Box p$, hence $\mathcal{M}, w_1 \nVdash \neg\Diamond\Box p$. Since $w_1 \notin V(p)$, it follows that $\mathcal{M}, w_1 \nVdash p$.

Thus, $\mathcal{M}, w_1 \nVdash \Diamond\Box p \to p$.

(iii) We have that

$$\begin{array}{lll}
\mathcal{M}, w_2 \Vdash \Diamond(p \wedge \neg r) & \text{iff} & \text{there exists } v \in W \text{ such that } Rw_2v \text{ and } \mathcal{M}, v \Vdash p \wedge \neg r \\
& \text{iff} & \mathcal{M}, w_3 \Vdash p \wedge \neg r \\
& & \text{since } w_3 \text{ is the unique } v \text{ such that } Rw_2v \\
& \text{iff} & \mathcal{M}, w_3 \Vdash p \text{ and } \mathcal{M}, w_3 \Vdash \neg r \\
& \text{iff} & \mathcal{M}, w_3 \Vdash p \text{ and } \mathcal{M}, w_3 \nVdash r \\
& \text{iff} & w_3 \in V(p) \text{ and } w_3 \notin V(r), \text{ which is true} \\
& & \text{by the definition of } V.
\end{array}$$

(iv) Let us denote

$$\varphi := q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond q))), \quad \psi := \Diamond(q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond q)))$$
$$\chi := \Diamond(q \wedge \Diamond(q \wedge \Diamond q)).$$

We have that

$$\begin{array}{lll}
\mathcal{M}, w_1 \Vdash \varphi & \text{iff} & \mathcal{M}, w_1 \Vdash q \text{ and } \mathcal{M}, w_1 \Vdash \psi \\
& \text{iff} & \mathcal{M}, w_1 \Vdash \psi \quad (\text{since } w_1 \in V(q), \text{ hence } \mathcal{M}, w_1 \Vdash q) \\
& \text{iff} & \text{there exists } v \in W \text{ such that } Rw_1v \text{ and } \mathcal{M}, v \Vdash q \wedge \chi \\
& \text{iff} & \mathcal{M}, w_2 \Vdash q \wedge \chi \\
& & \text{since } w_2 \text{ is the unique } v \in W \text{ such that } Rw_1v \\
& \text{iff} & \mathcal{M}, w_2 \Vdash \chi \quad (\text{since } w_2 \in V(q), \text{ hence } \mathcal{M}, w_2 \Vdash q) \\
& \text{iff} & \text{there exists } u \in W \text{ such that } Rw_2u \text{ and } \mathcal{M}, u \Vdash q \wedge \Diamond(q \wedge \Diamond q) \\
& \text{iff} & \mathcal{M}, w_3 \Vdash q \wedge \Diamond(q \wedge \Diamond q) \\
& & \text{since } w_3 \text{ is the unique } u \in W \text{ such that } Rw_2u \\
& \text{iff} & \mathcal{M}, w_3 \Vdash \Diamond(q \wedge \Diamond q) \\
& & \text{since } w_3 \in V(q), \text{ hence } \mathcal{M}, w_3 \Vdash q \\
& \text{iff} & \text{there exists } v' \in W \text{ such that } Rw_3v' \text{ and } \mathcal{M}, v' \Vdash q \wedge \Diamond q \\
& \text{iff} & \mathcal{M}, w_4 \Vdash q \wedge \Diamond q \\
& & \text{since } w_4 \text{ is the unique } v' \in W \text{ such that } Rw_3v' \\
& \text{iff} & \mathcal{M}, w_4 \Vdash \Diamond q \quad (\text{since } w_4 \in V(q), \text{ hence } \mathcal{M}, w_4 \Vdash q) \\
& \text{iff} & \text{there exists } u' \in W \text{ such that } Rw_4u' \text{ and } \mathcal{M}, u' \Vdash q \\
& \text{iff} & \mathcal{M}, w_5 \Vdash q \\
& & \text{since } w_5 \text{ is the unique } u' \in W \text{ such that } Rw_4u' \\
& \text{iff} & w_5 \in V(q), \text{ which is true.}
\end{array}$$

(v) Let $w \in W$ be arbitrary. We have that $\mathcal{M}, w \Vdash \Box q$ iff for every $v \in W$, Rwv implies $\mathcal{M}, v \Vdash q$ iff for every $v \in W$, Rwv implies $v \in V(q)$, which is true, since $V(q) = W$. □

(S4.4) Verify if the following formulas of ML_0 are satisfiable:

(i) $\Diamond p \wedge \Box\neg p$;

(ii) $\Diamond p \wedge \Diamond\neg p$.

Proof. (i) For any model $\mathcal{M} = (W, R, V)$ and state w in $\mathcal{M}$, we have that

$$\begin{array}{lll}
\mathcal{M}, w \Vdash \Diamond p \wedge \Box\neg p & \text{iff} & \mathcal{M}, w \Vdash \Diamond p \text{ and } \mathcal{M}, w \Vdash \Box\neg p \\
& \text{iff} & (*) \text{ and } (**),
\end{array}$$

where

(*) there exists $v \in W$ such that Rwv and $\mathcal{M}, v \Vdash p$,

(**) for every $u \in W$, Rwu implies $\mathcal{M}, u \Vdash \neg p$.

Assume that (*) and (**) are satisfied. Let $v \in W$ be such that Rwv and $\mathcal{M}, v \Vdash p$. Applying (**) with $u := v$, it follows that $\mathcal{M}, v \Vdash \neg p$, hence $\mathcal{M}, v \not\Vdash p$. We have obtained a contradiction. It follows that (*) and (**) can not be simultaneously true, hence $\mathcal{M}, w \not\Vdash \Diamond p \wedge \Box\neg p$.

Thus, $\Diamond p \wedge \Box\neg p$ is not satisfiable.

(ii) For any model $\mathcal{M} = (W, R, V)$ and state w in $\mathcal{M}$, we have that

$\mathcal{M}, w \Vdash \Diamond p \wedge \Diamond\neg p$ iff (*) and (**),

where

(*) there exists $v \in W$ such that Rwv and $\mathcal{M}, v \Vdash p$,

(**) there exists $u \in W$ such that Rwu and $\mathcal{M}, u \Vdash \neg p$.

Let $\mathcal{M}_0 = (W_0, R_0, V_0)$, where

$$W_0 = \{a, b\}, \quad R_0 = \{(a, a), (a, b)\}, \quad V_0(p) = \{a\}.$$

We prove that

$$\mathcal{M}_0, a \Vdash \Diamond p \wedge \Diamond\neg p.$$

We have that R_0aa and $\mathcal{M}_0, a \Vdash p$, hence (*) is satisfied with $w := a$ and $v := a$.

Furthermore, R_0ab and $\mathcal{M}_0, b \Vdash \neg p$, since $b \notin V_0(p)$, so $\mathcal{M}_0, b \not\Vdash p$. Thus (**) is satisfied with $w := a$ and $u := b$.

It follows that $\Diamond p \wedge \Diamond\neg p$ is satisfiable.

$\Box$