DM545/DM871
Linear and Integer Programming

Lecture 12
Network Flows

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Outline

1. Well Solved Problems

2. (Minimum Cost) Network Flows

3. Application Example

Outline

Separation problem

$\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{x} \in X\} \equiv \max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{x} \in \text{conv}(X)\}$
$X \subseteq \mathbb{Z}^n$, P a polyhedron $P \subseteq \mathbb{R}^n$ and $X = P \cap \mathbb{Z}^n$

Definition (Separation problem for a COP)

Given $\boldsymbol{x}^* \in P$; is $\boldsymbol{x}^* \in \text{conv}(X)$? If not find an inequality $\boldsymbol{ax} \leq \boldsymbol{b}$ satisfied by all points in X but violated by the point $\boldsymbol{x}^*$.

(Farkas' lemma states the existence of such an inequality.)

Properties of Easy Problems

Four properties that often go together:

Definition

(i) Efficient optimization property: $\exists$ a polynomial algorithm for $\max\{cx : x \in X \subseteq \mathbb{R}^n\}$

(ii) Strong duality property: $\exists$ strong dual D $\min\{w(u) : u \in U\}$ that allows to quickly verify optimality

(iii) Efficient separation problem: $\exists$ efficient algorithm for separation problem

(iv) Efficient convex hull property: a compact description of the convex hull is available

Example:
  If explicit convex hull   strong duality holds
                             efficient separation property (just description of $\text{conv}(X)$)

Theoretical analysis to prove results about

- strength of certain inequalities that are facet defining
  2 ways

- descriptions of convex hull of some discrete $X \subseteq \mathbb{Z}^*$
  several ways, we see one next

Example

Let

$$X = \{(x, y) \in \mathbb{R}_+^m \times \mathbb{B}^1 : \sum_{i=1}^m x_i \leq my, x_i \leq 1 \text{ for } i = 1, \ldots, m\}$$

$$P = \{(x, y) \in \mathbb{R}_+^m \times \mathbb{R}^1 : x_i \leq y \text{ for } i = 1, \ldots, m, y \leq 1\}$$

.

Polyhedron P describes $\text{conv}(X)$

Totally Unimodular Matrices

When the LP solution to this problem

$$IP : \max\{\boldsymbol{c}^T\boldsymbol{x} : A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \in \mathbb{Z}_+^n\}$$

with all data integer will have integer solution?

$$\begin{bmatrix} & & \vdots & \vdots & \vdots \\ A_N & A_B & 0 & \boldsymbol{b} \\ & & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{c}_N^T & \boldsymbol{c}_B^T & 1 & 0 \end{bmatrix}$$

$A_B x_B + A_N x_N = b$

$\boldsymbol{x}_N = 0 \rightsquigarrow A_B\boldsymbol{x}_B = \boldsymbol{b}$,
A_B $m \times m$ non singular matrix
$\boldsymbol{x}_B \geq 0$

Cramer's rule for solving systems of linear equations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \qquad x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad \boldsymbol{x} = A_B^{-1}\boldsymbol{b} = \frac{A_B^{adj}\boldsymbol{b}}{\det(A_B)}$$

Definition

- A square integer matrix B is called unimodular (UM) if $\det(B) = \pm 1$
- An integer matrix A is called totally unimodular (TUM) if every square, nonsingular submatrix of A is UM

Proposition

- *If A is TUM then all vertices of $R_1(A) = \{x : Ax = b, x \geq 0\}$ are integer if b is integer*
- *If A is TUM then all vertices of $R_2(A) = \{x : Ax \leq b, x \geq 0\}$ are integer if b is integer.*

Proof: if A is TUM then $\begin{bmatrix} A & I \end{bmatrix}$ is TUM

Any square, nonsingular submatrix C of $\begin{bmatrix} A & I \end{bmatrix}$ can be written as

$$C = \begin{bmatrix} B & 0 \\ D & I_k \end{bmatrix}$$

where B is square submatrix of A. Hence $\det(C) = \det(B) = \pm 1$

Proposition

The transpose matrix A^T of a TUM matrix A is also TUM.

Theorem (Sufficient condition)

An integer matrix A is TUM if

1. *$a_{ij} \in \{0, -1, +1\}$ for all i, j*
2. *each column contains at most two non-zero coefficients ($\sum_{i=1}^{m} |a_{ij}| \leq 2$)*
3. *if the rows can be partitioned into two sets I_1, I_2 such that:*
   - *if a column has 2 entries of same sign, their rows are in different sets*
   - *if a column has 2 entries of different signs, their rows are in the same set*

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Proof: by induction

Basis: one matrix of one element $\{0, +1, -1\}$ is TUM

Induction: let C be of size k.

If C has column with all 0s then it is singular.

If a column with only one 1 then expand on that by induction

If 2 non-zero in each column then

$$\forall j : \sum_{i \in I_1} a_{ij} = \sum_{i \in I_2} a_{ij}$$

but then a linear combination of rows is zero and $\det(C) = 0$

Other matrices with integrality property:

- TUM
- Balanced matrices
- Perfect matrices
- Integer vertices

Defined in terms of forbidden substructures that represent fractionating possibilities.

Proposition

A is always TUM if it comes from

- *node-edge incidence matrix of undirected bipartite graphs (ie, no odd cycles) ($I_1 = U, I_2 = V, B = (U, V, E)$)*
- *node-arc incidence matrix of directed graphs ($I_2 = \emptyset$)*

Eg: Shortest path, max flow, min cost flow, bipartite weighted matching

Summary

1. Well Solved Problems

2. (Minimum Cost) Network Flows

3. Application Example

Outline

Terminology

Network:
- directed graph $D = (V, A)$
- arc, directed link, from tail to head
- lower bound $l_{ij} > 0$, $\forall ij \in A$, capacity $u_{ij} \geq l_{ij}$, $\forall ij \in A$
- cost c_{ij}, linear variation (if $ij \notin A$ then $l_{ij} = u_{ij} = 0$, $c_{ij} = 0$)
- balance vector $b(i)$, $b(i) > 0$ supply node (source), $b(i) < 0$ demand node (sink, tank), $b(i) = 0$ transhipment node (assumption $\sum_i b(i) = 0$)
- $N = (V, A, l, u, b, c)$

Network Flows
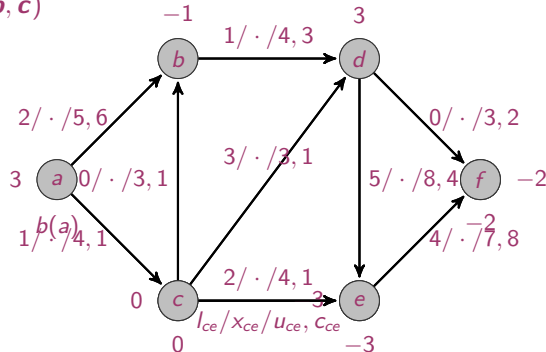
Flow $\boldsymbol{x} : A \to \mathbb{R}$
balance vector of $\boldsymbol{x}$: $b_{\boldsymbol{x}}(v) = \sum_{vu \in A} x_{vu} - \sum_{wv \in A} x_{wv}$, $\forall v \in V$

$$b_{\boldsymbol{x}}(v) \begin{cases} > 0 & \text{source} \\ < 0 & \text{sink/target/tank} \\ = 0 & \text{balanced} \end{cases}$$
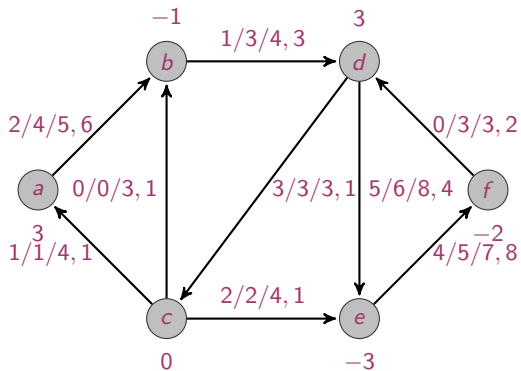
(generalizes the concept of path with $b_{\boldsymbol{x}}(v) = \{0, 1, -1\}$)

feasible $l_{ij} \le x_{ij} \le u_{ij}$, $b_{\boldsymbol{x}}(i) = b(i)$
cost $\boldsymbol{c}^T \boldsymbol{x} = \sum_{ij \in A} c_{ij} x_{ij}$ (varies linearly with $\boldsymbol{x}$)

If iji is a 2-cycle and all $l_{ij} = 0$, then at least one of x_{ij} and x_{ji} is zero.

Example

Feasible flow of cost 109

Minimum Cost Network Flows

Find cheapest flow through a network in order to satisfy demands at certain nodes from available supplier nodes.

Variables:

$$x_{ij} \in \mathbb{R}_0^+$$

Objective:

$$\min \sum_{ij \in A} c_{ij} x_{ij}$$

$$\min \mathbf{c}^T \mathbf{x}$$
$$N\mathbf{x} = \mathbf{b}$$
$$\mathbf{l} \leq \mathbf{x} \leq \mathbf{u}$$

Constraints: mass balance + flow bounds

$$\sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = b(i) \quad \forall i \in V$$

$$l_{ij} \leq x_{ij} \leq u_{ij}$$

N node arc incidence matrix

(assumption: all values are integer, we can multiply if rational)

	x_{e_1}	x_{e_2}	$\ldots$	x_{ij}	$\ldots$	x_{e_m}		
	c_{e_1}	c_{e_2}	$\ldots$	c_{ij}	$\ldots$	c_{e_m}		
1	1	.	$\ldots$	.	$\ldots$	.	$=$	b_1
2	.	.	$\ldots$	.	$\ldots$	.	$=$	b_2
$\vdots$	$\vdots$	$\ddots$					$=$	$\vdots$
i	-1	.	$\ldots$	1	$\ldots$	.	$=$	b_i
$\vdots$	$\vdots$	$\ddots$					$=$	$\vdots$
j	.	.	$\ldots$	-1	$\ldots$	.	$=$	b_j
$\vdots$	$\vdots$	$\ddots$					$=$	$\vdots$
n	.	.	$\ldots$	.	$\ldots$	.	$=$	b_n
e_1	1						$\leq$	u_1
e_2		1					$\leq$	u_2
$\vdots$	$\vdots$	$\ddots$					$\leq$	$\vdots$
(i,j)				1			$\leq$	u_{ij}
$\vdots$	$\vdots$	$\ddots$					$\leq$	$\vdots$
e_m						1	$\leq$	u_m

Reductions/Transformations

Lower bounds

Let $N = (V, A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{b}, \boldsymbol{c})$
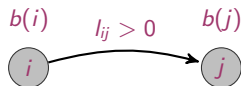
$$N' = (V, A, \boldsymbol{l}', \boldsymbol{u}', \boldsymbol{b}', \boldsymbol{c})$$
$$b'(i) = b(i) - l_{ij}$$
$$b'(j) = b(j) + l_{ij}$$
$$u'_{ij} = u_{ij} - l_{ij}$$
$$l'_{ij} = 0$$

$$\boldsymbol{c}^T \boldsymbol{x}$$

$$\boldsymbol{c}^T \boldsymbol{x}' + \sum_{ij \in A} c_{ij} l_{ij}$$
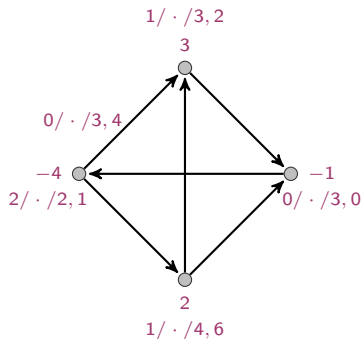
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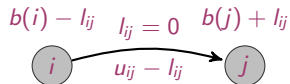
Undirected arcs
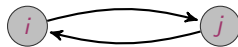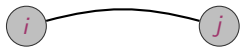
Vertex splitting

If there are bounds and costs of flow passing through vertices where $b(v) = 0$ (used to ensure that a node is visited):

$N = (V, A, l, u, c, l^*, u^*, c^*)$
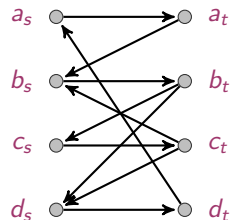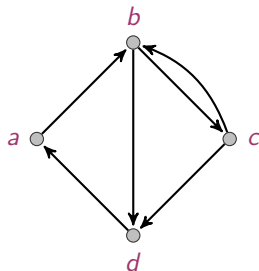
From D to D_{ST} as follows:

$$\forall v \in V \quad \leadsto v_s, v_t \in V(D_{ST}) \text{ and } v_s v_t \in A(D_{ST})$$
$$\forall xy \in A(D) \leadsto x_t y_s \in A(D_{ST})$$
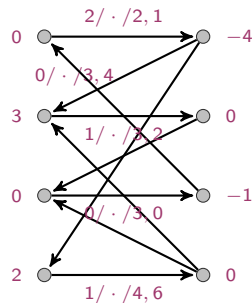
$$\forall v \in V \text{ and } v_s v_t \in A_{ST} \rightsquigarrow h'(v_s v_t) = h^*(v), \ h^* \in \{l^*, u^*, c^*\}$$
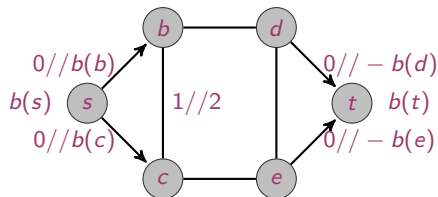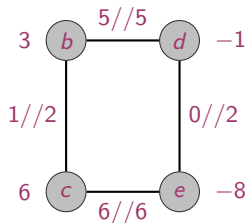$$\forall xy \in A \text{ and } x_t y_s \in A_{ST} \rightsquigarrow h'(x_t y_s) = h(x, y), h \in \{l, u, c\}$$

If $b(v) = 0$, then $b'(v_s) = b'(v_t) = 0$

If $b(v) < 0$, then $b'(v_s) = 0$ and $b'(v_t) = b(v)$

If $b(v) > 0$, then $b'(v_s) = b(v)$ and $b'(v_t) = 0$

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(s, t)-**flow:**

$$b_x(v) = \begin{cases} k & \text{if } v = s \\ -k & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}, \qquad |x| = |b_x(s)|$$

$b(s) = \sum_{v:b(v)>0} b(v) = M$

$b(t) = \sum_{v:b(v)<0} b(v) = -M$

$\exists$ feasible flow in $N \iff \exists (s, t)$-flow in N_{st} with $|x| = M \iff$ max flow in N_{st} is M
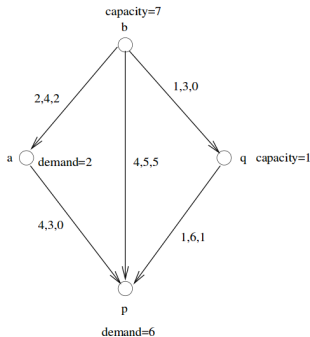
Residual Network

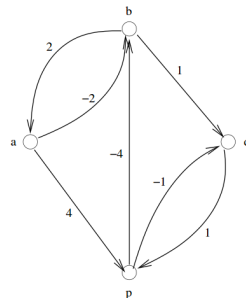**Residual Network** $N(x)$: given that a flow x already exists, how much flow excess can be moved in G?

Replace arc $ij \in N$ with arcs:

	residual capacity	cost
ij :	$r_{ij} = u_{ij} - x_{ij}$	c_{ij}
ji :	$r_{ji} = x_{ij}$	$-c_{ij}$

$(N, \boldsymbol{c}, \boldsymbol{u}, \boldsymbol{x})$　　　　$(N(\boldsymbol{x}), \boldsymbol{r}, \boldsymbol{c}')$

Special cases

Shortest path problem path of minimum cost from s to t with costs $\lessgtr 0$
$$b(s) = 1, b(t) = -1, b(i) = 0$$
if to any other node? $b(s) = (n-1), b(i) = 1, u_{ij} = n-1$

Max flow problem incur no cost but restricted by bounds
steady state flow from s to t
$$b(i) = 0 \; \forall i \in V, \qquad c_{ij} = 0 \; \forall ij \in A \qquad ts \in A$$
$$c_{ts} = -1, \qquad u_{ts} = \infty$$

Assignment problem min weighted bipartite matching,
$$|V_1| = |V_2|, A \subseteq V_1 \times V_2$$
$$c_{ij}$$
$$b(i) = 1 \; \forall i \in V_1 \qquad b(i) = -1 \; \forall i \in V_2 \qquad u_{ij} = 1 \; \forall ij \in A$$

Special cases

Transportation problem/Transhipment distribution of goods, warehouses-costumers
$$|V_1| \neq |V_2|, \qquad u_{ij} = \infty \text{ for all } ij \in A$$

$$\min \sum c_{ij} x_{ij}$$
$$\sum_i x_{ij} \geq b_j \qquad\qquad \forall j \in V_2$$
$$\sum_j x_{ij} \leq a_i \qquad\qquad \forall i \in V_1$$
$$x_{ij} \geq 0$$

if $\sum a_i = \sum b_i$ then $\geq / \leq$ become $=$
if $\sum a_i > \sum b_i$ then add dummy tank nodes
if $\sum a_i < \sum b_i$ then infeasible

Multi-commodity flow problem ship several commodities using the same network, different origin destination pairs separate mass balance constraints, share capacity constraints, min overall flow

$$\min \sum_k \boldsymbol{c}^k \boldsymbol{x}^k$$
$$N \boldsymbol{x}^k \geq \boldsymbol{b}^k \qquad \forall k$$
$$\sum_k \boldsymbol{x}_{ij}^k \leq \boldsymbol{u}_{ij} \qquad \forall ij \in A$$
$$0 \leq \boldsymbol{x}_{ij}^k \leq \boldsymbol{u}_{ij}^k$$

What is the structure of the matrix now? Is the matrix still TUM?

Outline

Ship loading problem

Plenty of applications. See Ahuja Magnanti Orlin, Network Flows, 1993

- A cargo company (eg, Maersk) uses a ship with a capacity to carry at most r units of cargo.

- The ship sails on a long route (say from Southampton to Alexandria) with several stops at ports in between.

- At these ports cargo may be unloaded and new cargo loaded.

- At each port there is an amount b_{ij} of cargo which is waiting to be shipped from port i to port $j > i$

- Let f_{ij} denote the income for the company from transporting one unit of cargo from port i to port j.

- The goal is to plan how much cargo to load at each port so as to maximize the total income while never exceeding ship's capacity.
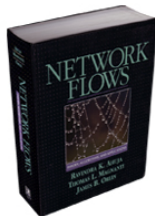
Application Example: Modeling

- n number of stops including the starting port and the terminal port.

- $N = (V, A, l \equiv 0, u, c)$ be the network defined as follows:

  - $V = \{v_1, v_2, ..., v_n\} \cup \{v_{ij} : 1 \leq i < j \leq n\}$

  - $A = \{v_1 v_2, v_2 v_3, ... v_{n-1} v_n\} \cup \{v_{ij} v_i, v_{ij} v_j : 1 \leq i < j \leq n\}$

  - capacity: $u_{v_i v_{i+1}} = r$ for $i = 1, 2, ..., n-1$ and all other arcs have capacity ∞.

  - cost: $c_{v_{ij} v_i} = -f_{ij}$ for $1 \leq i < j \leq n$ and all other arcs have cost zero (including those of the form $v_{ij} v_j$)

  - balance vector: $b(v_{ij}) = b_{ij}$ for $1 \leq i < j \leq n$ and the balance vector of $b(v_i) = -b_{1i} - b_{2i} - ... - b_{i-1,i}$ for $i = 1, 2, ..., n$

Application Example: Modeling
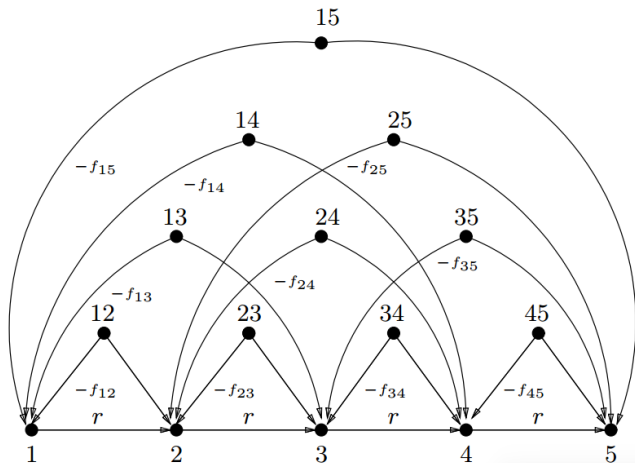
Application Example: Modeling

Claim: the network models the ship loading problem.

- suppose that $t_{12}, t_{13}, ..., t_{1n}, t_{23}, ..., t_{n-1,n}$ are cargo numbers, where t_{ij} ($\leq b_{ij}$) is the amount of cargo the ship will transport from port i to port j and that the ship is never loaded above capacity.

- total income is

$$I = \sum_{1 \leq i < j \leq n} t_{ij} f_{ij}$$

- Let x be the flow in N defined as follows:
    - flow on an arc of the form $v_{ij} v_i$ is t_{ij}
    - flow on an arc of the form $v_{ij} v_j$ is $b_{ij} - t_{ij}$
    - flow on an arc of the form $v_i v_{i+1}$, $i = 1, 2, ..., n-1$, is the sum of those t_{ab} for which $a \leq i$ and $b \geq i + 1$.

- since t_{ij}, $1 \leq i < j \leq n$, are legal cargo numbers then x is feasible with respect to the balance vector and the capacity restriction.

- the cost of x is $-I$.

Application Example: Modeling

- Conversely, suppose that x is a feasible flow in N of cost J.

- we construct a feasible cargo assignment $s_{ij}, 1 \leq i < j \leq n$ as follows:

  - let s_{ij} be the value of x on the arc $v_{ij} v_i$.

- income $-J$