

DM545/DM871  
Linear and Integer Programming

Lecture 4  
**Exception Handling and Initialization**

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# Simplex: Exception Handling, Overview

Solution of an LP problem:

- a.  $F \neq \emptyset$  and  $\nexists$  solution
- b.  $F \neq \emptyset$  and  $\exists$  solution
  - i) **one solution**
  - ii) infinite solutions
- c.  $F = \emptyset$

Handling exceptions in the Simplex Method

- 1. Unboundedness
- 2. More than one solution
- 3. Degeneracies
  - benign
  - cycling
- 4. Infeasible starting  
Phase I + Phase II

# Outline

Exception Handling  
Initialization

1. Exception Handling

2. Initialization

# Outline

Exception Handling  
Initialization

1. Exception Handling

2. Initialization

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ & x_2 \leq 5 \\ -x_1 + x_2 & \leq 1 \\ x_1, x_2 & \geq 0 \end{aligned}$$

- Initial tableau

	x1	x2	x3	x4	-z	b
x3	0	1	1	0	0	5
x4	-1	1	0	1	0	1
	2	1	0	0	1	0

- $x_2$  entering,  $x_4$  leaving

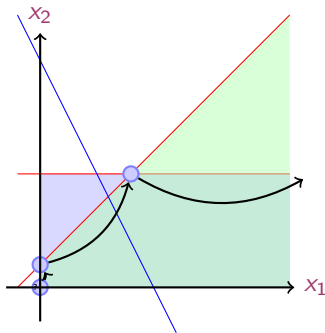
	x1	x2	x3	x4	-z	b
II'=II-I'	1	0	1	-1	0	4
I'=I	-1	1	0	1	0	1
III'=III-I'	3	0	0	-1	1	-1

$-x_1 + x_2 + x_4 = 1$ ,  $x_1$  can increase without restriction,  $\theta = \min\{\frac{b_i}{a_{is}} : a_{is} > 0, i = 1 \dots, n\}$

- $x_1$  entering,  $x_3$  leaving

	$x_1$	$x_2$	$x_3$	$x_4$	$-z$	$b$
I'=I	1	0	1	-1	0	4
II'=II+I'	0	1	1	0	0	5
III'=III-3I'	0	0	-3	2	1	-13

$x_4$  was already in basis but for both I and II ( $x_2 + x_3 + 0x_4 = 5$ ),  $x_4$  can increase arbitrarily



$$\begin{aligned}
 \max \quad & x_1 + x_2 \\
 & 5x_1 + 10x_2 \leq 60 \\
 & 4x_1 + 4x_2 \leq 40 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

- Initial tableau

	x1	x2	x3	x4	-z	b
x3	5	10	1	0	0	60
x4	4	4	0	1	0	40
	1	1	0	0	1	0

- $x_2$  enters,  $x_3$  leaves

	x1	x2	x3	x4	-z	b
I'=I/10	1/2	1	1/10	0	0	6
II'=II-4Ix4	2	0	-2/5	1	0	16
III'=III-I	1/2	0	-1/6	0	1	-6

- $x_1$  enters,  $x_4$  leaves

	$x_1$	$x_2$	$x_3$	$x_4$	$-z$	$b$
I' = I - II'/2	0	1	1/5	-1/4	0	2
II' = II/2	1	0	-1/5	1/2	0	8
III' = III - II'/2	0	0	0	-1/4	1	-10

$$\mathbf{x} = (8, 2, 0, 0), z = 10$$

nonbasic variables typically have reduced costs  $\neq 0$ . Here  $x_3$  has r.c. = 0. Let's make it enter the basis

- $x_3$  enters,  $x_2$  leaves

	$x_1$	$x_2$	$x_3$	$x_4$	$-z$	$b$
I' = 5I	0	5	1	-5/4	0	10
II' = II + I'/5	1	1	0	4	0	10
III' = III	0	0	0	-1/4	1	-10

$$\mathbf{x} = (10, 0, 10, 0), z = 10$$

There are 2 optimal solutions  $\rightsquigarrow$  all their convex combinations are optimal solutions (from the proof of the fundamental theorem of LP)  $\rightsquigarrow$



$$\mathbf{x} = \sum_{i=1}^2 \alpha_i \mathbf{x}_i$$

$$\alpha_i \geq 0$$

$$\sum_{i=1}^2 \alpha_i = 1$$

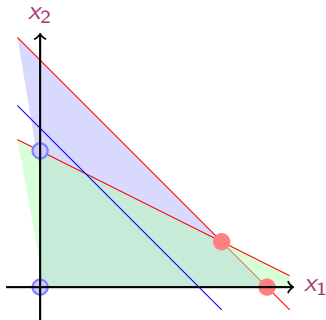
$$\mathbf{x}_1^T = [8, 2, 0, 0]$$

$$\mathbf{x}_2^T = [10, 0, 10, 0]$$

$$\alpha_1 = \alpha$$

$$\alpha_2 = 1 - \alpha$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} 8 \\ 2 \\ 0 \\ 0 \end{bmatrix} + (1 - \alpha) \begin{bmatrix} 10 \\ 0 \\ 10 \\ 0 \end{bmatrix}$$



$$x_1 = 8\alpha + 10(1 - \alpha)$$

$$x_2 = 2\alpha$$

$$x_3 = 10(1 - \alpha)$$

$$x_4 = 0$$

$$\begin{aligned} \max \quad & x_2 \\ -x_1 + x_2 \leq & 0 \\ x_1 \leq & 2 \\ x_1, x_2 \geq & 0 \end{aligned}$$

- Initial tableau

	x1	x2	x3	x4	-z	b
x3	-1	1	1	0	0	0
x4	1	0	0	1	0	2
	0	1	0	0	1	0

$b_i = 0$  (one basic var. is zero) might lead to cycling

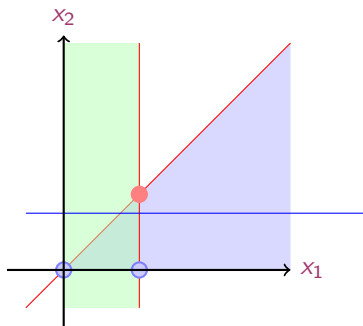
- degenerate pivot step: not improving, the entering variable stays at zero

	x1	x2	x3	x4	-z	b
	-1	1	1	0	0	0
	1	0	0	1	0	2
	1	0	-1	0	1	0

- now nondegenerate:

	$x_1$	$x_2$	$x_3$	$x_4$	$-z$	$b$
	0	1	0	1	0	2
	1	0	0	1	0	2
	0	0	-1	-1	1	-2

$$x_1 = 2, x_2 = 2, z = 2$$



$\geq n + 1$  constraints meet at a vertex

Def: An **improving variable** is one with positive reduced cost

Def: A **degenerate iteration** is one in which the objective function does not increase.

Def: The simplex method **cycles** if the same tableau appears in two iterations.

Degenerate conditions may appear often in practice but cycling is rare.  
(see compendium for the smallest possible example)

## Theorem

*If the simplex fails to terminate, then it must cycle.*

Proof:

- there is a finite number of basis and simplex chooses to always increase the objective value
- hence the only situation for not terminating is that a basis appears again and iterations in between are degenerate. Two tableaux with the same basis are the same (related to uniqueness of basic solutions)

Some pivoting rules can prevent the occurrence of cycling altogether.

So far we chose an **arbitrary improving variable** to enter. Rules for breaking ties in selecting **entering** improving variables (more important than selecting leaving variables)

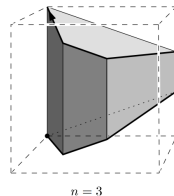
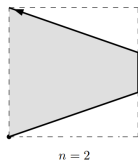
- **Largest Coefficient**: the improving var with largest coefficient in last row of the tableau.  
Original Dantzig's rule, can cycle
- **Largest increase**: absolute improvement:  $\operatorname{argmax}_j \{c_j \theta_j\}$   
computationally more costly
- **Steepest edge** the improving var that if entering in the basis moves the current basic feasible sol in a direction closest to the direction of the vector  $\mathbf{c}$  (ie, maximizes the cosine of the angle between the two vectors):

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta \quad \Rightarrow \quad \max_{\mathbf{x}_{\text{new}}} \frac{\mathbf{c}^T (\mathbf{x}_{\text{new}} - \mathbf{x}_{\text{old}})}{\|\cancel{\mathbf{c}}\| \|\mathbf{x}_{\text{new}} - \mathbf{x}_{\text{old}}\|}$$

- **Bland's rule (smallest-subscript rule)** chooses the improving var with the lowest index and, if there are more than one leaving variable, the one with the lowest index.  
Prevents cycling but is slow (no smart choice for entering variable)
- **Random edge** select var uniformly at random among the improving ones and the leaving ones among those selected by the ratio test
- **Perturbation method**: perturb values of  $b_i$  terms to avoid  $b_i = 0$ , which must occur for cycling.  
To avoid cancellations:  $0 < \epsilon_m \ll \epsilon_{m-1} \ll \dots \ll \epsilon_1 \ll 1$   
It affects the choice of the leaving variable  
Can be shown to be the same as lexicographic method, which prevents cycling

# Efficiency of Simplex Method

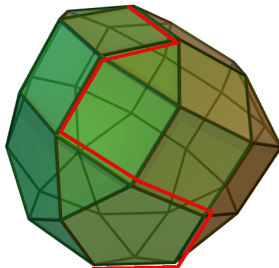
- Trying all points is  $\approx 4^m$
- In practice between  $2m$  and  $3m$  iterations
- Klee and Minty 1978 constructed an example that requires  $2^n - 1$  iterations in  $\mathbb{R}^n$ :



- random shuffle of indexes + lowest index for entering + lexicographic for leaving: expected iterations  $< e^{C\sqrt{n \ln n}}$

# Efficiency of Simplex Method

- unknown if there exists a pivot rule that leads to polynomial time.
- Clairvoyant's rule: shortest possible sequence of steps  
Hirsh conjecture  $O(n - d)$  for an  $n$ -facet polytope in  $d$ -dimensional Euclidean space but best known  $n^{1+\ln n}$



- smoothed complexity: slight random perturbations of worst-case inputs  
D. Spielman and S. Teng (2001), *Smoothed analysis of algorithms: why the simplex algorithm usually takes polynomial time*  
 $O(\max\{n^5 \log^2 m, n^9 \log^4 n, n^3 \sigma^{-4}\})$



# Outline

Exception Handling  
**Initialization**

1. Exception Handling

2. Initialization

# Initial Infeasibility

$$\begin{aligned}\max \quad & x_1 - x_2 \\ & x_1 + x_2 \leq 2 \\ & 2x_1 + 2x_2 \geq 5 \\ & x_1, x_2 \geq 0\end{aligned}$$

$$\begin{aligned}\max \quad & x_1 - x_2 \\ & x_1 + x_2 + x_3 = 2 \\ & -2x_1 - 2x_2 + x_4 = -5 \\ & x_1, x_2, x_3, x_4 \geq 0\end{aligned}$$

- Initial tableau

	x1	x2	x3	x4	-z	b
x3	1	1	1	0	0	2
x4	-2	-2	0	1	0	-5
	1	-1	0	0	1	0

~> we do not have an initial basic feasible solution!!

In general finding any feasible solution is difficult as finding an optimal solution, otherwise we could do binary search

## Auxiliary Problem (I Phase of Simplex)

We introduce auxiliary variables:

$$\begin{aligned}
 w^* &= \max -x_5 \equiv \min x_5 \\
 x_1 + x_2 + x_3 &= 2 \\
 2x_1 + 2x_2 - x_4 + x_5 &= 5 \\
 x_1, x_2, x_3, x_4, x_5 &\geq 0
 \end{aligned}$$

if  $w^* = 0$  then  $x_5 = 0$  and the two problems are equivalent

if  $w^* > 0$  then not possible to set  $x_5$  to zero.

- Initial tableau

	x1	x2	x3	x4	x5	-z	-w	b
	1	1	1	0	0	0	0	2
	2	2	0	-1	1	0	0	5
z	1	-1	0	0	0	1	0	0
w	0	0	0	0	-1	0	1	0

Keep  $z$  always in basis

- we reach a canonical form simply by letting  $x_5$  enter the basis:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$-z$	$-w$	$b$
I	1	1	1	0	0	0	0	2
II	2	2	0	-1	1	0	0	5
z	1	-1	0	0	0	1	0	0
IV+II	2	2	0	-1	0	0	1	5

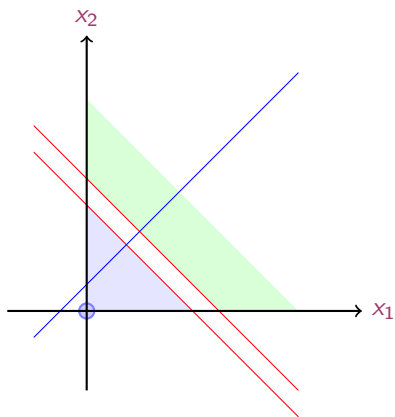
now we have a basic feasible solution!

- $x_1$  enters,  $x_3$  leaves

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$-z$	$-w$	$b$
I	1	1	1	0	0	0	0	2
II-2I'	0	0	-2	-1	1	0	0	1
III-I'	0	-2	-1	0	0	1	0	-2
IV-2I'	0	0	-2	-1	0	0	1	1

$w^* = -1$  then no solution with  $x_5 = 0$  exists then no feasible solution to initial problem

$$\begin{aligned} \max \quad & x_1 - x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 2 \\ & 2x_1 + 2x_2 \geq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$



# Initial Infeasibility - Another Example

$$\begin{aligned}\max \quad & x_1 - x_2 \\ & x_1 + x_2 \leq 2 \\ & 2x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0\end{aligned}$$

$$\begin{aligned}\max \quad & x_1 - x_2 \\ & x_1 + x_2 + x_3 = 2 \\ & 2x_1 + 2x_2 - x_4 = 2 \\ & x_1, x_2, x_3, x_4 \geq 0\end{aligned}$$

Auxiliary problem (I phase):

$$\begin{aligned}w = \max \quad & -x_5 \equiv \min x_5 \\ & x_1 + x_2 + x_3 = 2 \\ & 2x_1 + 2x_2 - x_4 + x_5 = 2 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0\end{aligned}$$

- Initial tableau

	x1	x2	x3	x4	x5	-z	-w	b
	1	1	1	0	0	0	0	2
	2	2	0	-1	1	0	0	2
z	1	-1	0	0	0	1	0	0
w	0	0	0	0	-1	0	1	0

~> we do not have an initial basic feasible solution.

- set in canonical form:

	x1	x2	x3	x4	x5	-z	-w	b
	1	1	1	0	0	0	0	2
	2	2	0	-1	1	0	0	2
z	1	-1	0	0	0	1	0	0
IV+II	2	2	0	-1	0	0	1	2

- $x_1$  enters,  $x_5$  leaves

	x1	x2	x3	x4	x5	-z	-w	b
	0	0	1	1/2	-1/2	0	0	1
	1	1	0	-1/2	1/2	0	0	1
z	0	-2	0	1/2	-1/2	1	0	-1
w	0	0	0	0	-1	0	1	0

$w^* = 0$  hence  $x_5 = 0$  we have a starting feasible solution for the initial problem.

- (II phase) We keep only what we need:

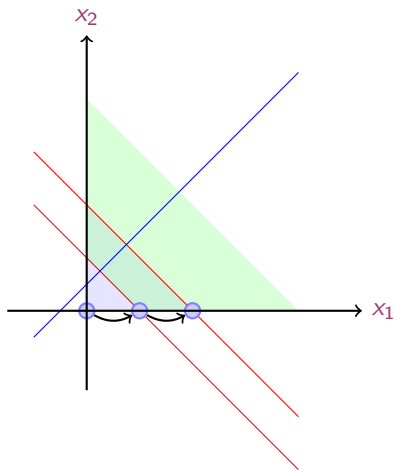
	x1	x2	x3	x4	-z	b
	0	0	1	1/2	0	1
	1	1	0	-1/2	0	1
	z	0	-2	0	1/2	1

- 
- |  | x1 | x2 | x3 | x4 | -z | b |
|--|----|----|----|----|----|---|
|  |    |    |    |    |    |   |
|  | 0  | 0  | 2  | 1  | 0  | 2 |
|  | 1  | 1  | 1  | 0  | 0  | 2 |
|  | z  | 0  | -2 | -1 | 0  | 1 |

Optimal solution:  $x_1 = 2, x_2 = 0, x_3 = 0, x_4 = 2, z = 2$ .



$$\begin{aligned} \max \quad & x_1 - x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 2 \\ & 2x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$



# In Dictionary Form

$$\begin{aligned} \max \quad & x_1 - x_2 \\ & x_1 + x_2 \leq 2 \\ & 2x_1 + 2x_2 \geq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{array}{rcl} x_3 & = & 2 - x_1 - x_2 \\ x_4 & = & -5 + 2x_1 + 2x_2 \\ \hline z & = & x_1 + x_2 \end{array}$$

sol. infeasible

We introduce corrections of infeasibility

$$\begin{aligned} \max \quad & -x_0 \equiv \min \quad x_0 \\ & x_1 + x_2 \leq 2 \\ & 2x_1 + 2x_2 - x_0 \geq 5 \\ & x_1, x_2, x_0 \geq 0 \end{aligned}$$

$$\begin{array}{rcl} x_3 & = & 2 - x_1 - x_2 \\ x_4 & = & -5 + 2x_1 + 2x_2 + x_0 \\ \hline z & = & \phantom{-5 + 2x_1 + 2x_2 +} - x_0 \end{array}$$

It is still infeasible but it can be made feasible by letting  $x_0$  enter the basis

which variable should leave?

the most infeasible: the var with the  $b$  term whose negative value has the largest magnitude

# Simplex: Exception Handling, Summary

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  - i) **one solution**
  - ii) infinite solutions
- c.  $F = \emptyset$

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