

DM545/DM871  
Linear and Integer Programming

## Branch and Bound

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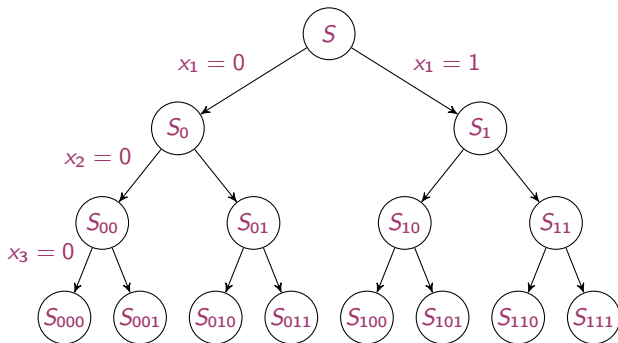
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## 1. Branch and Bound

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- Consider the problem  $z = \max\{\mathbf{c}^T \mathbf{x} : \mathbf{x} \in S\}$
- Divide and conquer: let  $S = S_1 \cup \dots \cup S_k$  be a decomposition of  $S$  into smaller sets, and let  $z^k = \max\{\mathbf{c}^T \mathbf{x} : \mathbf{x} \in S_k\}$  for  $k = 1, \dots, K$ . Then  $z = \max_k z^k$

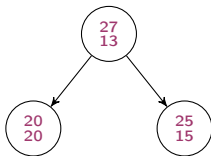
For instance if  $S \subseteq \{0, 1\}^3$  the enumeration tree is:



Let's consider a maximization problem

- Let  $\bar{z}^k$  be an upper bound on  $z^k$  (dual bound)
- Let  $\underline{z}^k$  be a lower bound on  $z^k$  (primal bound)
- $(\underline{z}^k \leq z^k \leq \bar{z}^k)$
- $\underline{z} = \max_k \underline{z}^k$  is a lower bound on  $z$
- $\bar{z} = \max_k \bar{z}^k$  is an upper bound on  $z$

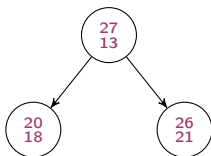
# Pruning (Fathoming)



$$\bar{z} = 25$$

$$\underline{z} = 20$$

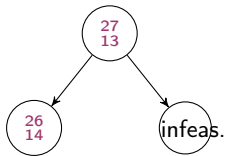
pruned by optimality



$$\bar{z} = 26$$

$$\underline{z} = 21$$

pruned by bounding

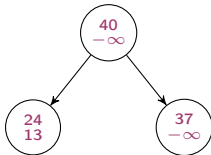


$$\bar{z} = 26$$

$$\underline{z} = 14$$

pruned by infeasibility

# Pruning



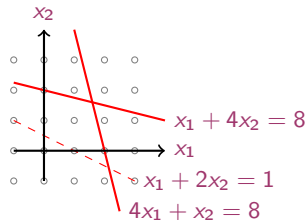
$$\bar{z} = 37$$

$$\underline{z} = 13$$

nothing to prune

# LP Based Branch & Bound: Example

$$\begin{aligned}
 \max \quad & x_1 + 2x_2 \\
 \text{s.t.} \quad & x_1 + 4x_2 \leq 8 \\
 & 4x_1 + x_2 \leq 8 \\
 & x_1, x_2 \geq 0, \text{ integer}
 \end{aligned}$$



- Solve LP

|              | x1 | x2   | x3 | x4   | -z | b  |
|--------------|----|------|----|------|----|----|
| I            | 1  | 4    | 1  | 0    | 0  | 8  |
| II           | 4  | 1    | 0  | 1    | 0  | 8  |
| III          | 1  | 2    | 0  | 0    | 1  | 0  |
|              | x1 | x2   | x3 | x4   | -z | b  |
| I'=I-II'     | 0  | 15/4 | 1  | -1/4 | 0  | 6  |
| II'=1/4II    | 1  | 1/4  | 0  | 1/4  | 0  | 2  |
| III'=III-II' | 0  | 7/4  | 0  | -1/4 | 0  | -2 |



- continuing

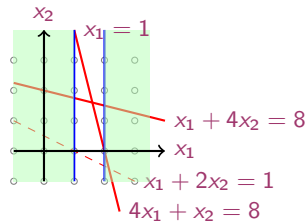
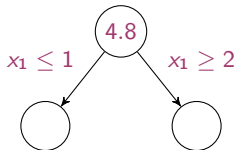
|                       | $x_1$ | $x_2$ | $x_3$   | $x_4$   | $-z$ | $b$         |
|-----------------------|-------|-------|---------|---------|------|-------------|
| I' = $4/15$ I         | 0     | 1     | $4/15$  | $-1/15$ | 0    | $24/15$     |
| II' = $II - 1/4$ I'   | 1     | 0     | $-1/15$ | $4/15$  | 0    | $24/15$     |
| III' = $III - 7/4$ I' | 0     | 0     | $-7/15$ | $-3/5$  | 1    | $-2 - 14/5$ |

$$x_2 = 1 + 3/5 = 1.6$$

$$x_1 = 8/5$$

The optimal solution will not be more than  $2 + 14/5 = 4.8$

- Both variables are fractional, we pick one of the two:



- Let's consider first the left branch:

|  | x1 | x2 | x3    | x4    | x5 | -z | b     |
|--|----|----|-------|-------|----|----|-------|
|  | 1  | 0  | 0     | 0     | 1  | 0  | 1     |
|  | 0  | 1  | 4/15  | -1/15 | 0  | 0  | 24/15 |
|  | 1  | 0  | -1/15 | 4/15  | 0  | 0  | 24/15 |
|  | 0  | 0  | -7/15 | -3/5  | 0  | 1  | -24/5 |

|              | x1 | x2 | x3    | x4    | x5 | b | -z    |
|--------------|----|----|-------|-------|----|---|-------|
| I' = I - III | 0  | 0  | 1/15  | -4/15 | 1  | 0 | -9/15 |
|              | 0  | 1  | 4/15  | -1/15 | 0  | 0 | 24/15 |
|              | 1  | 0  | -1/15 | 4/15  | 0  | 0 | 24/15 |
|              | 0  | 0  | -7/15 | -3/5  | 0  | 1 | -24/5 |

|                 | x1 | x2 | x3     | x4 | x5    | b | -z     |
|-----------------|----|----|--------|----|-------|---|--------|
| I' = -15/4I     | 0  | 0  | -1/4   | 1  | -15/4 | 0 | 9/4    |
| II' = II - 1/4I | 0  | 1  | 15/60  | 0  | -1/4  | 0 | 7/4    |
| III' = III + I  | 1  | 0  | 0      | 0  | 1     | 0 | 1      |
|                 | 0  | 0  | -37/60 | 0  | -9/4  | 1 | -90/20 |

always a  $b$  term negative  
after branching:

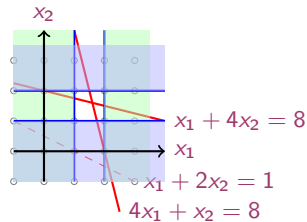
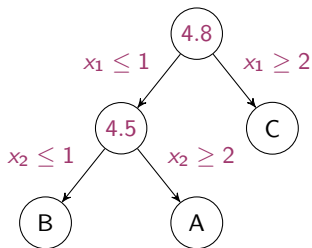
$$b_1 = \lfloor \bar{b}_3 \rfloor$$

$$\bar{b}_1 = \lfloor \bar{b}_3 \rfloor - b_3 < 0$$

Dual simplex:

$$\min_j \{ \left| \frac{c_j}{a_{ij}} \right| : a_{ij} < 0 \}$$

- Let's branch again



We have three open problems. Which one do we choose next?  
Let's take A.

|  | x1 | x2 | x3     | x4 | x5    | x6 | b | -z   |
|--|----|----|--------|----|-------|----|---|------|
|  | 0  | -1 | 0      | 0  | 0     | 1  | 0 | -2   |
|  | 0  | 0  | -1/4   | 1  | -15/4 |    | 0 | 9/4  |
|  | 0  | 1  | 15/60  | 0  | -1/4  |    | 0 | 7/4  |
|  | 1  | 0  | 0      | 0  | 1     |    | 0 | 1    |
|  | 0  | 0  | -37/60 | 0  | -9/4  |    | 1 | -9/2 |

|       | x1 | x2 | x3     | x4 | x5    | x6 | b | -z   |
|-------|----|----|--------|----|-------|----|---|------|
| III+I | 0  | 0  | 1/4    | 0  | -1/4  | 1  | 0 | -1/4 |
|       | 0  | 0  | -1/4   | 1  | -15/4 |    | 0 | 9/4  |
|       | 0  | 1  | 15/60  | 0  | -1/4  |    | 0 | 7/4  |
|       | 1  | 0  | 0      | 0  | 1     |    | 0 | 1    |
|       | 0  | 0  | -37/60 | 0  | -9/4  |    | 1 | -9/2 |

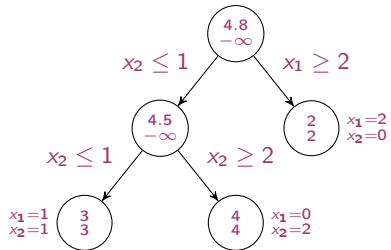
continuing we find:

$$x_1 = 0$$

$$x_2 = 2$$

$$OPT = 4$$

The final tree:



The optimal solution is 4.

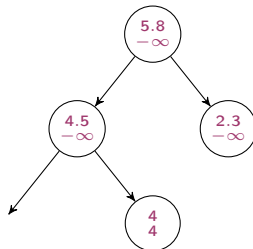
# Pruning

Pruning:

1. by optimality:  $z^k = \max\{c^T x : x \in S^k\}$

2. by bound  $\bar{z}^k \leq \underline{z}$

Example:



3. by infeasibility  $S^k = \emptyset$

## Bounding:

1. LP relaxation
2. Lagrangian relaxation
3. Combinatorial relaxation
4. Duality

## Branching:

$$S_1 = S \cap \{x : x_j \leq \lfloor \bar{x}_j \rfloor\}$$

$$S_2 = S \cap \{x : x_j \geq \lceil \bar{x}_j \rceil\}$$

thus the current optimum is not feasible in  $S_1$  and in  $S_2$ .

Which variable to choose?

Eg: Most fractional variable  $\arg \max_{j \in C} \min\{f_j, 1 - f_j\}$

**Choosing Node for Examination** from the list of active (or open):

- Depth First Search (a good primal sol. is good for pruning + easier to reoptimize by just adding a new constraint)
- Best Bound First: (eg. largest upper:  $\bar{z}^s = \max_k \bar{z}^k$  or largest lower - to die fast)
- Mixed strategies

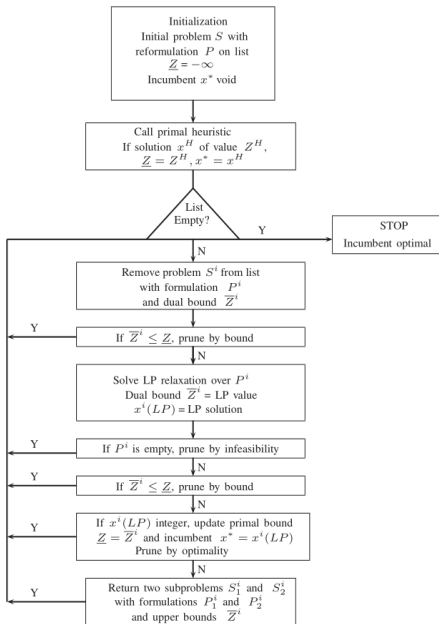
**Reoptimizing:** dual simplex

**Updating the Incumbent:** when new best feasible solution is found:

$$\underline{z} = \max\{\underline{z}, 4\}$$

**Store the active nodes:** bounds + optimal basis (remember the revised simplex!)





[Wolsey, 2021]

Lazy strategy: pruning, calculation, branching, queue insertion; open nodes are stored with the bound of their father

**Branch-and-Bound Algorithm ;**

```

begin
1.   $m := 1$ ;  $parent[1] := 0$ ;  $Q := \emptyset$ ;
     $z_{OPT} :=$  heuristic solution value (possibly  $+\infty$ );
2.  solve the continuous relaxation  $\min\{\mathbf{c}^T \mathbf{x} : \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq 0\}$ , and let
     $\mathbf{x}^*$  be the optimal solution found;
3.   $LB[1] := \mathbf{c}^T \mathbf{x}^*$ ;
4.  if ( $\mathbf{x}^*$  integer) and ( $\mathbf{c}^T \mathbf{x}^* < z_{OPT}$ ) then
    begin
         $\mathbf{x}_{OPT} = \mathbf{x}^*$ ;  $z_{OPT} := \mathbf{c}^T \mathbf{x}^*$ 
    end ;
5.  if  $LB[1] < z_{OPT}$  then
    begin
        choose the fractional branching variable  $x_h^*$  ;
         $vbranch[1] := h$ ;  $value[1] := x_h^*$ ;
         $Q := \{1\}$ 
    end ;
6.  while  $Q \neq \emptyset$  do /* process the active open nodes*/
    begin
7.      choose a node  $t \in Q$ , and set  $Q := Q \setminus \{t\}$  ;
8.       $h := vbranch[t]$ ;  $val := value[t]$  ;
9.      for  $child := 1$  to 2 do /* generate the children of node  $t$  */
      begin
10.          $m := m + 1$  ;
11.         if  $child = 1$ 
            then  $parent[m] := t$ 
            else  $parent[m] := -t$  ;
            define problem  $PL_m$  associated with node  $m$ 
            (constraints of  $PL_t$  plus  $x_h \leq \lfloor val \rfloor$  if  $child = 1$ ,
             or  $x_h \geq \lceil val \rceil$  if  $child = 2$ ) ;
12.         solve problem  $PL_m$ , and let  $\mathbf{x}^*$  be the optimal solution found ;
13.          $LB[m] := \mathbf{c}^T \mathbf{x}^*$ ;
14.         if ( $\mathbf{x}^*$  integer) and ( $\mathbf{c}^T \mathbf{x}^* < z_{OPT}$ ) then
            begin /* update the optimal solution */
                 $\mathbf{x}_{OPT} := \mathbf{x}^*$ ;  $z_{OPT} := \mathbf{c}^T \mathbf{x}^*$  ;
                 $Q := Q \setminus \{j \in Q : LB[j] \geq z_{OPT}\}$ 
            end ;
15.         if  $LB[m] < z_{OPT}$  then
            begin
                choose the fractional branching variable  $x_k^*$  ;
                 $vbranch[m] := k$ ;  $value[m] := x_k^*$  ;
                 $Q := Q \cup \{m\}$ 
            end
        end
    end
end .

```

Warning: this is for a minimization problem

[Fischetti, 2019]

Eager strategy: branching, calculation, pruning, queue insertion; open nodes are stored with their own bounds

- Preprocessor: constraint/problem/structure specific tightening bounds  
redundant constraints  
variable fixing: eg:  $\max\{\mathbf{c}^T \mathbf{x} : \mathbf{Ax} \leq \mathbf{b}, \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}\}$   
fix  $x_j = l_j$  if  $c_j < 0$  and  $a_{ij} > 0$  for all  $i$   
fix  $x_j = u_j$  if  $c_j > 0$  and  $a_{ij} < 0$  for all  $i$
- User defined branching priorities: establish the next variable to branch (not in gurobi)
- Special ordered sets SOS (or generalized upper bound GUB)

$$\sum_{j=1}^k x_j = 1 \quad x_j \in \{0, 1\}$$

instead of:  $S_0 = S \cap \{\mathbf{x} : x_j = 0\}$  and  $S_1 = S \cap \{\mathbf{x} : x_j = 1\}$

$\{\mathbf{x} : x_j = 0\}$  leaves  $k - 1$  possibilities

$\{\mathbf{x} : x_j = 1\}$  leaves only 1 possibility

hence tree unbalanced

here:  $S_1 = S \cap \{\mathbf{x} : x_{j_i} = 0, i = 1..r\}$  and  $S_2 = S \cap \{\mathbf{x} : x_{j_i} = 0, i = r + 1, .., k\}$ ,

$r = \min\{t : \sum_{i=1}^t x_{j_i}^* \geq \frac{1}{2}\}$

- Cutoff value: a user-defined primal bound to pass to the system.
- Simplex strategies: simplex is good for reoptimizing but for large models interior points methods may work best.
- Strong branching: extra work to decide more accurately on which variable to branch:
  1. choose a set  $C$  of fractional variables
  2. reoptimize for each of them (in case for limited iterations)
  3.  $\bar{z}_j^\downarrow, \bar{z}_j^\uparrow$  (dual bound of down and up branch)

$$j^* = \arg \min_{j \in C} \max\{\bar{z}_j^\downarrow, \bar{z}_j^\uparrow\}$$

ie, choose variable with most change in objective function, ie, largest decrease of dual bound, eg, largest decrease of UB for max problem

There are four common reasons why integer programs can require a significant amount of solution time:

1. There is lack of node throughput due to troublesome linear programming node solves.
2. There is lack of progress in the best integer solution, i.e., the primal bound.
3. There is lack of progress in the best dual bound.
4. There is insufficient node throughput due to numerical instability in the problem data or excessive memory usage.

For 2) or 3) the gap best feasible-dual bound is large:

$$\text{gap} = \frac{|\text{Primal bound} - \text{Dual bound}|}{\text{Primal bound} + \epsilon} \cdot 100$$

- heuristics for finding feasible solutions (generally NP-complete problem)
- find better lower bounds if they are weak: addition of cuts, stronger formulation, branch and cut
- Branch and cut: a B&B algorithm with cut generation at all nodes of the tree. (instead of reoptimizing, do as much work as possible to tighten)

Cut pool: stores all cuts centrally

Store for active node: bounds, basis, pointers to constraints in the cut pool that apply at the node

# Relative Optimality Gap

In CPLEX:

$$\text{gap} = \frac{|\text{best dual bound} - \text{best integer}|}{|\text{best integer} + 10^{-11}|}$$

Issues...

In SCIP, primal gap:

$$\text{gap} = \frac{db - pb}{|pb|}$$

Last standard avoids problem of non decreasing gap if we go through zero

|      |      |           |      |          |           |         |         |
|------|------|-----------|------|----------|-----------|---------|---------|
| 3186 | 2520 | -666.6217 | 4096 | 956.6330 | -667.2010 | 1313338 | 169.74% |
| 3226 | 2560 | -666.6205 | 4097 | 956.6330 | -667.2010 | 1323797 | 169.74% |
| 3266 | 2600 | -666.6201 | 4095 | 956.6330 | -667.2010 | 1335602 | 169.74% |

Elapsed real time = 2801.61 sec. (tree size = 77.54 MB, solutions = 2)

|         |      |           |      |           |           |         |         |
|---------|------|-----------|------|-----------|-----------|---------|---------|
| * 3324+ | 2656 |           |      | -125.5775 | -667.2010 | 1363079 | 431.31% |
| 3334    | 2668 | -666.5811 | 4052 | -125.5775 | -667.2010 | 1370748 | 431.31% |
| 3380    | 2714 | -666.5799 | 4017 | -125.5775 | -667.2010 | 1388391 | 431.31% |
| 3422    | 2756 | -666.5791 | 4011 | -125.5775 | -667.2010 | 1403440 | 431.31% |



Issues resolved by  
 MIPLIB standard (Koch et al. MIPLIB 2010, Math Prog Comput):

$$\text{gap}(pb, db) = \begin{cases} 0.0 & pb = db = 0 \\ -- & \text{no feasible solution found,} \\ & \text{problem reported as infeasible,} \\ & \text{or } db \leq 0 \leq pb \\ \frac{pb - db}{\inf\{|z|, z \in [db, pb]\}} \cdot 100 & \text{else} \end{cases} \quad (\text{minimization prob.})$$

The gap decreases monotonously during the solving process

In SCIP <https://www.scipopt.org/doc/html/FAQ.php#gapdefinition>

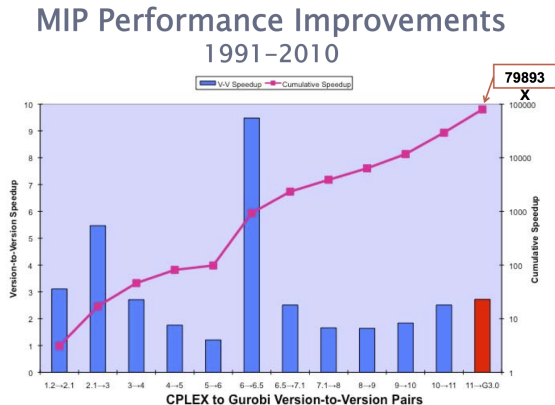
$$\text{gap}(pb, db) = \begin{cases} 0.0 & pb = db \\ \infty & pb \cdot db \leq 0 \\ \frac{|pb - db|}{\min\{|pb|, |db|\}} & \text{else}(pb \cdot db > 0) \end{cases}$$

# MILP Solvers Breakthroughs

Branch and Bound

We have seen Fractional Gomory cuts.

The introduction of Mixed Integer Gomory cuts in CPLEX was the major breakthrough of CPLEX 6.5 and produced the version-to-version speed-up given by the blue bars in the chart below



(source: R. Bixby. Mixed-Integer Programming: It works better than you may think. 2010. Slides on the net)

Speedup over the past 25 years:

Hardware 2 000 times

Software 2 000 000 times

We did not treat:

- LP: Dantzig Wolfe decomposition
- LP: Column generation
- LP: Delayed column generation
- IP: Branch and Price
- LP: Benders decompositions
- LP: Lagrangian relaxation

They are topics of DM872.

Solve linear programming problems with:

- Simple method tool <https://www.zweigmedia.com/simplex/simplex.php>
- In Python: `scipy.optimize.linprog`
- In Python: Python-MIP, `pyscipopt`, `gurobipy`

# Summary

Branch and Bound

## 1. Branch and Bound