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DM545/DM871 – Linear and integer programming

Sheet 3, Autumn 2025

Solution:

Included.

Exercises with the symbol $^+$ are to be done at home before the class. Exercises with the symbol * will be tackled in class and should be at least read at home. The remaining exercises are left for self training after the exercise class.

Exercise 1$^+$

Show that the dual of $\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is $\min\{\mathbf{y}^T\mathbf{b} \mid \mathbf{y}^T A \geq \mathbf{c}\}$ using the bounding method.
Show that the dual of $\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is $\min\{\mathbf{y}^T\mathbf{b} \mid \mathbf{y}^T A \geq \mathbf{c}, \mathbf{y} \geq \mathbf{0}\}$ using the Lagrangian multipliers method.

Solution:

Given $\max\{c^T x \mid Ax = b, x \geq 0\}$ we search for multipliers $y \in \mathbb{R}^n$ such that $y^T Ax = y^T b$ (since we have equalities, the multipliers can be both positive or negative as we do not need to ensure the maintainance of the direction of the inequality). To ensure that we find an upper bound and hence have $c^T x \leq y^T Ax$, we impose $y^T A \geq c^T$ (since $x \geq 0$). Hence, the best upper bound will be given by solving $\min\{y^T b \mid y^T A \geq c^T\}$ (recalling from linear algebra that $(AB)^T = B^T A^T$, we can rewrite: $\min\{y^T b \mid A^T y \geq c\}$, which is the form we would obtain using the recipe method.)

$$\begin{array}{lrll} \min & z = \mathbf{c}^T\mathbf{x} & & \mathbf{c} \in \mathbb{R}^n \\ & A\mathbf{x} \leq \mathbf{b} & & A \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m \\ & \mathbf{x} \geq \mathbf{0} & & \mathbf{x} \in \mathbb{R}^n \end{array}$$

We want to create a relaxation by enlarging the feasibility region and ensuring that in the feasibility region the new objective function stays above $\mathbf{c}^T\mathbf{x}$ because we want to approximate the function from above. For feasible solutions to the original problem we have: $\mathbf{b} - A\mathbf{x} \geq \mathbf{0}$. Hence, when we relax this quantity in the objective function we multiply by nonnegative coefficients $\mathbf{y} \in \mathbb{R}^n_+$.

$$\max_{\mathbf{y}\in\mathbb{R}^m_+}\{\min_{\mathbf{x}\in\mathbb{R}^n_+}\{\mathbf{c}^T\mathbf{x} + \mathbf{y}^T(\mathbf{b} - A\mathbf{x})\}\}$$
$$\max_{\mathbf{y}\in\mathbb{R}^m_+}\{\min_{\mathbf{x}\in\mathbb{R}^n_+}\{(\mathbf{c}^T - \mathbf{y}^T A)\mathbf{x} + \mathbf{y}^T\mathbf{b}\}\}$$

The inner problem is easy to solve when we are trying to maximize the outer problem since this latter one we will try to make $A^T\mathbf{y} \leq \mathbf{c}$. Hence, we are left with the problem

$$\begin{array}{lrl} \max & \mathbf{b}^T\mathbf{y} & \\ & A^T\mathbf{y} & \leq \mathbf{c} \\ & \mathbf{y} \in \mathbb{R}^m_+ & \end{array}$$

Exercise 2*

Consider the following LP problem:

$$\begin{array}{ll} \max & 2x_1 + 3x_2 \\ & 2x_1 + 3x_2 \leq 30 \\ & x_1 + 2x_2 \geq 10 \\ & x_1 - x_2 \leq 1 \\ & x_2 - x_1 \leq 1 \\ & x_1 \geq 0 \end{array}$$

- Write the dual problem
- Using the optimality conditions derived from the theory of duality, and without using the simplex method, find the optimal solution of the dual knowing that the optimal solution of the primal is $(27/5, 32/5)$.

Solution:

The dual is:

$$\begin{aligned} \min \quad & 30y_1 + 10y_2 + y_3 + y_4 \\ & 2y_1 + y_2 + y_3 - y_4 \geq 2 \\ & 3y_1 + 2y_2 - y_3 + y_4 = 3 \\ & y_1, y_3, y_4 \geq 0 \\ & y_2 \leq 0 \end{aligned}$$

We use the complementary slackness theorem.

$$\begin{cases} 2y_1 + y_2 + y_3 - y_4 = 2 \\ 3y_1 + 2y_2 - y_3 + y_4 = 3 \\ y_2 = 0 \\ y_3 = 0 \end{cases}$$

The first because the corresponding variable of the primal is > 0, the second for the same reason or however because it is already tight by definition, the third and fourth equation are a consequence of the fact that substituting the value of the primal variables variables in the primal problem, the second and third constraints are binding. What we obtain is a linear system of four equations in four variables that we can solve to find the value of the variables of the dual problem.

Exercise 3

Consider the problem

$$\begin{aligned} \text{maximize} \quad & 5x_1 + 4x_2 + 3x_3 \\ \text{subject to} \quad & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Without applying the simplex method, how can you tell whether the solution $(2, 0, 1)$ is an optimal solution? Is it? [Hint: consider consequences of Complementary slackness theorem.]

Solution:

The primal problem:

$$\begin{aligned} \max \quad & 5x_1 + 4x_2 + 3x_3 \\ \text{s.t} \quad & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

The dual problem:

$$\begin{aligned} \min \quad & 5y_1 + 11y_2 + 8y_3 \\ \text{s.t} \quad & 2y_1 + 4y_2 + 3y_3 \geq 5 \\ & 3y_1 + y_2 + 4y_3 \geq 4 \\ & y_1 + 2y_2 + 2y_3 \geq 3 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

If the Complementary slackness theorem holds and the objective value for the primal and dual is the same, then the solution $(2, 0, 1)$ is optimal. We check that it does:
Since $x_1, x_3 > 0$ we have the two constraints:

$$2y_1 + 4y_2 + 3y_3 = 5$$
$$y_1 + 2y_2 + 2y_3 = 3$$

Now we can test which constraints are binding in the primal problem.
We can see that the second constraint is non-binding, then $y_2 = 0$ and we have the following system of equations:

$$2y_1 + 4y_2 + 3y_3 = 5$$
$$y_1 + 2y_2 + 2y_3 = 3$$
$$y_2 = 0$$

Then we have that:

$$2y_1 + 3y_3 = 5$$
$$y_1 + 2y_3 = 3$$

Solved as before will give $y_1 = 1, y_3 = 1$.
Then the solution is:
$y_1^* = 1, y_2^* = 0, y_3^* = 1$
We need to check if it is feasible. We have already checked the conditions for the primal problem and we have already checked the first and thrid constraint in the dual, so we need to check the second constraint of the dual. It needs to be non-binding since $x_2 = 0$:

$$3y_1^* + y_2^* + 4y_3^* > 4$$
$$\implies \quad 3 * 1 + 0 + 4 * 1 > 4$$
$$\implies \quad 7 > 4$$

So its non-binding.
We check that the objective values are the same:
For the primal: $5 * 2 + 4 * 0 + 3 * 1 = 13$
For the dual: $5 * 1 + 11 * 0 + 8 * 1 = 13$
Then the solution is optimal.

Exercise 4

Consider the following LP:

$$\begin{aligned} \min \quad & 3x_1 + 2x_2 - 4x_3 \\ & 2x_1 + x_2 + x_3 \geq 3 \\ & x_1 + x_2 + 2x_3 \leq 5 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Find the optimal solution knowing that the solution of the dual problem is $(u_1, u_2) = (10/3, 11/3)$.

Solution:

Rewrite:

$$\begin{aligned} \max \quad & -3x_1 - 2x_2 + 4x_3 \\ \text{s.t} \quad & -2x_1 - x_2 - x_3 \leq -3 \\ & x_1 + x_2 + 2x_3 \leq 5 \\ & x_i \geq 0 \qquad \forall i \in \{1, 2, 3\} \end{aligned}$$

The dual problem:

$$\begin{aligned} \min \quad & -3u_1 + 5u_2 \\ \text{s.t} \quad & -2u_1 + u_2 \geq -3 \\ & -u_1 + u_2 \geq -2 \\ & -u_1 + 2u_2 \geq 4 \\ & u_1, u_2 \geq 0 \end{aligned}$$

Now we want to find a solution to the primal. Since $u_1, u_2 > 0$ then we have the two constraints:

$$-2x_1 - x_2 - x_3 = -3$$
$$x_1 + x_2 + 2x_3 = 5$$

We check which constraints are binding in the dual:

We see that the second constraint is non-binding, hence $x_2 = 0$. And we have the following system of equations:

$$\begin{aligned} -2x_1 - x_2 - x_3 &= -3 \\ x_1 + x_2 + 2x_3 &= 5 \\ x_2 &= 0 \end{aligned}$$

Hence we have:

$$\begin{aligned} -2x_1 - x_3 &= -3 \\ x_1 + 2x_3 = 5 &\implies x_1 = 5 - 2x_3 \end{aligned}$$

Then we can insert x_1 in the first equation:

$$\implies -2(5 - 2x_3) - x_3 = -3 \implies -10 + 3x_3 = -3 \implies 3x_3 = 7 \implies x_3 = 7/3$$

And we can find x_1:

$$x_1 = 5 - 2x_3 = 5 - 2 * 7/3 = 1/3$$

Then the solution of the primal is: $x_1 = 1/3, x_2 = 0, x_3 = 7/3$

Exercise 5*

Consider the following problem:

$$\begin{array}{ll} \text{maximize} & z = x_1 - x_2 \\ \text{subject to} & x_1 + x_2 \leq 2 \\ & 2x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

In the ordinary simplex method this problem does not have an initial feasible basis. Hence, the method has to be enhanced by a preliminary phase to attain a feasible basis. Traditionally we talk about a *phase I–phase II* simplex method. In phase I an initial feasible solution is sought and in phase II the ordinary simplex is started from the initial feasible solution found.
There are two ways to carry out phase I.

- Solving an an auxiliary LP problem defined by introducing auxiliary variables and minimizing them in the objective. The solution of the auxiliary LP problem gives an initial feasible basis or a proof of infeasibility.
- Applying the dual simplex on a possibly modified problem to find a feasible solution. If the initial infeasible tableau of the original problem is not optimal then the objective function can be temporarily modified for this phase in order to make the initial tableau optimal although not feasible. Opposite to the primal simplex method, the dual simplex method iterates through infeasible basic solutions, while maintaining them optimal, and stops when a feasible solution is reached.

Dual Simplex: The strong duality theorem states that we can solve the primal problem by solving its dual. You can verify that applying the *primal simplex method* to the dual problem corresponds to the following method, called *dual simplex method* that works on the primal problem:

1. (Feasibility condition) select the leaving variable by picking the basic variable whose right-hand side term is negative, i.e., select i^* with $b_{i^*} < 0$.
2. (Optimality condition) pick the entering variable by scanning across the selected row and comparing ratios of the coefficients in this row to the corresponding coefficients in the objective row, looking for the largest negated. Formally, select j^* such that $j^* = \min\{|c_j/a_{i^*j}| \ : \ a_{i^*j} < 0\}$
3. Update the tableau around the pivot in the same way as with the primal simplex.
4. Stop if no right-hand side term is negative.

Duality can help us with the issue of initial feasible basic solutions. In the problem above, the initial tableau is infeasible and not optimal, hence we cannot apply the primal simplex nor the dual simplex. However, if the objective function was $w = -x_1 - x_2$, then we would have the conditions of infeasibility and optimality needed by the dual simplex. You can understand this also looking at the dual problem. With the new objective function the initial basic solution of the dual problem would be feasible and we could solve the problem solving the dual problem with the primal simplex. In contrast, with original objective function z the primal simplex has infeasible initial basis in both problems. So we can change temporarily the objective function z with w and apply the dual simplex method to the primal problem. When it stops we reached a feasible solution that is optimal with respect to w. We can then reintroduce the original objective function and continue iterating with the primal simplex. The phase I–phase II simplex method that uses the dual simplex is also called the *dual-primal simplex method*.
Apply the two versions of the phase I–phase II simplex method (that is, phase I is carried out with the auxiliary problem or with the dual simplex) to the problem above and verify that they lead to the same solutions.

Solution:

$$\begin{array}{rrcrcl} \max & x_1 & - & x_2 & = & z \\ & x_1 & + & x_2 & \leq & 2 \\ & 2x_1 & + & 2x_2 & \geq & 2 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

We put in equational standard form by introducing a slack variable $s_1 \geq 0$ and a surplus variable $s_2 \geq 0$:

$$\begin{array}{rrcrcrcl} \max & x_1 & - & x_2 & = & z & & \\ & x_1 & + & x_2 & + & s_1 & = & 2 \\ & 2x_1 & + & 2x_2 & - & s_2 & = & 2 \\ & & & & & x_1, x_2, s_1, s_2 & \geq & 0 \end{array}$$

This form is not canonical and therefore the first tableau does not have a feasible starting solution.

Auxiliary Problem Approach

We proceed by

- Phase I solving an auxiliary/augmented problem
- Phase II continuing with ordinary simplex

Phase I We introduce an auxiliary variable $a_1 \geq 0$ in the constraint that makes the infeasibility, thus yielding a canonical form:

$$\begin{array}{rrcrcrcrcl} \max & x_1 & - & x_2 & = & z & & & \\ & x_1 & + & x_2 & + & s_1 & & = & 2 \\ & 2x_1 & + & 2x_2 & - & s_2 & + & a_1 & = & 2 \\ & & & & & x_1, x_2, s_1, s_2, a_1 & & \geq & 0 \end{array}$$

Now we have a canonical form

x1	x2	s1	s2	a1	-z	b
1	1	1	0	0	0	2
2	2	0	-1	1	0	2
1	-1	0	0	0	1	0

This problem will have the same solution as the original one only when $a_1 = 0$. We can then solve

- an *augmented problem* by introducing the following objective function $\max w = x_1 - x_2 - Ma_1$, where M is a large enough constant or

- an *auxiliary* problem $\min w = a_1 = -\max(-a_1)$.

Let's take the auxiliary problem, if $w^* > 0$ then we will conclude that the feasibility region of the orginal problem is empty. Otherwise, if $w^* = 0$, then this implies that $a_1 = 0$ and we found a feasible solution. Let's proceed by setting up the tableau of the auxiliary problem

x1	x2	s1	s2	a1	-z	-w	b
1	1	1	0	0	0	0	2
2	2	0	-1	1	0	0	2
1	-1	0	0	0	1	0	0
0	0	0	0	-1	0	1	0

This is not in canonical form but it is easy to bring it to canonical form: just add the second row to the last one.

x1	x2	s1	s2	a1	-z	-w	b
1	1	1	0	0	0	0	2
2	2	0	-1	1	0	0	2
1	-1	0	0	0	1	0	0
2	2	0	-1	0	0	1	2

The variables s_1, a_1 give us a feasible basis now. It is not optimal. We proceed with the pivot operations. In this case it is worth noting that in the ratio rule, we do not consider the third row since that row corresponds to the orginal objective function and not to a constraint.
We make x_1 enter the basis and consequently a_1 goes out. The pivot is 2 and the new tableau:

	x1	x2	s1	s2	a1	-z	-w	b
R1'=R1-R2'	0	0	1	1/2	-1/2	0	0	1
R2'=R2/2	1	1	0	-1/2	1/2	0	0	1
R3'=R3-R2'	0	-2	0	1/2	-1/2	1	0	-1
R4'=R4-R2	0	0	0	0	-1	0	1	0

The tableau is optimal. One non basic variable has reduced cost null, which indicates that there are infinite solutions, but this is not relevant now. The relevant thing is that $w^* = 0$ hence the minimum of the auxiliary problem is 0 and hence there is a feasible solution for $a_1 = 0$. This concludes the Phase I of the algorithm since a feasible solution for the auxiliary problem is feasible also for the original problem.

Phase II We throw away the last row and the second last column from the tableau since we do not need them anymore.

x1	x2	s1	s2	a1	-z	b
0	0	1	1/2	-1/2	0	1
1	1	0	-1/2	1/2	0	1
0	-2	0	1/2	-1/2	1	-1

The tableau is not optimal. The basic solution corresponding to this tableau is feasible but not optimal. We bring s_2 in the basis and make s_1 leave. The new tableau is:

	x1	x2	s1	s2	a1	-z	b
R1'=2*R1	0	0	2	1	-1	0	2
R2'=R2+R1	1	1	0	0	1/2	0	2
R3'=R3-R1	0	-2	-1	0	0	1	-2

The tableau is now optimal. The optimal solution is $\mathbf{x} = (2, 0)$ and $z^* = 2$.

Dual-Primal Simplex Method

Phase I Let's write the dual of the problem above:

$$\begin{array}{rrcl} \max & x_1 - x_2 & = & z \\ & x_1 + x_2 & \leq & 2 \\ & 2x_1 + 2x_2 & \geq & 2 \\ & x_1, x_2 & \geq & 0 \end{array} \qquad \begin{array}{rrcl} \min & 2y_1' + 2y_2' & = & w \\ & y_1' + 2y_2' & \geq & 1 \\ & y_1' + 2y_2' & \geq & -1 \\ & y_1' & \geq & 0 \\ & y_2' & \leq & 0 \end{array} \xrightarrow[y_2'=-y_2]{y_1'=y_1}$$

$$\begin{array}{rrcl} \min & 2y_1 - 2y_2 & = & w \\ & y_1 - 2y_2 & \geq & 1 \\ & y_1 - 2y_2 & \geq & -1 \\ & y_1, y_2 & \geq & 0 \end{array}$$

If we put this LP problem in standard form:

$$\begin{array}{rrcl} \max & -2y_1 + 2y_2 & = & w \\ & -y_1 + 2y_2 & \leq & -1 \\ & -y_1 + 2y_2 & \leq & 1 \\ & y_1, y_2 & \geq & 0 \end{array}$$

and look at the tableau:

y1	y2	s1	s2	-z	b
-1	2	1	0	0	-1
-1	2	0	1	0	1
-2	2	0	0	1	0

we see that the initial tableau like for the primal problem is infeasible.
However, the dual problem has an advantage, if we change temporarily the objective function of the primal problem to $\eta = -x_1 - x_2$, the dual problm becomes:

$$\begin{array}{rrcl} \max & -x_1 - x_2 & = & \eta \\ & x_1 + x_2 & \leq & 2 \\ & 2x_1 + 2x_2 & \geq & 2 \\ & x_1, x_2 & \geq & 0 \end{array} \qquad \begin{array}{rrcl} \min & 2y_1 - 2y_2 & = & \gamma \\ & y_1 - 2y_2 & \geq & -1 \\ & y_1 - 2y_2 & \geq & -1 \\ & y_1, y_2 & \geq & 0 \end{array} \qquad \begin{array}{rrcl} \max & -2y_1 + 2y_2 & = & \gamma \\ & -y_1 + 2y_2 & \leq & 1 \\ & -y_1 + 2y_2 & \leq & 1 \\ & y_1, y_2 & \geq & 0 \end{array}$$

and the corresponding tableau has an easy feasible basic solution:

y1	y2	s1	s2	-z	b
-1	2	1	0	0	1
-1	2	0	1	0	1
-2	2	0	0	1	0

We can then solve to optimality with the primal simplex: the variable y_2 enters the basis and the variable s_2 exits. The new tableau becomes:

	y1	y2	s1	s2	-z	b
R1	-1	2	1	0	0	1
R2'=R2/2	-1/2	1	0	1/2	0	1/2
R3'=R3-R2	-1	0	0	-1	1	-1/2

and it is optimal. At this stage we can go back to the primal problem where we now have a feasible solution, change the objective function back to the original one and continue with the primal simplex.

We can do the same iteration on the primal but with the dual simplex. Let's write the tableau of the primal with the objective function temporarily changed and keeping the old objective as well:

x1	x2	s1	s2	-z	-e	b
1	1	1	0	0	0	2
-2	-2	0	1	0	0	-2
1	-1	0	0	1	0	0
-1	-1	0	0	0	1	0

As we see we have the conditions of the dual simplex satisfied, the tableau is optimal but not feasible. Let's make an iteration of the dual simplex. We choose the row with negative b term and the column with negative pivot that minimizes the ratio test: $|c/a|$. We choose the second row and the second column (again watch out that we do not consider the row of the old objective to decide the column). In other terms, we try to make the solution feasible while minimizing the loss in quality. The opertations to update the tableau remain the same as for the primal simplex. We obtan:

	x1	x2	s1	s2	-z	-e	b
R1'=R1-R2'	0	0	1	1/2	0	0	1
R2'=-1/2R2	1	1	0	-1/2	0	0	1
R3'=R3+R2'	2	0	0	-1/2	1	0	1
R4'=R4+R2'	0	0	0	-1/2	0	1	1

This tableau is optimal for the dual simplex, this means that a feasible solution for the primal problem has been found: $(0, 1, 1, 0)$. We can now proceed with the primal simplex.
Note that the considerations on the dual problem made above were just for explanation purposes, when solving our LP problem we do not need to write the dual form of it or its tableaux. Instead, we just need to switch from dual simplex to primal simplex always working on the original (the primal) formulation of the problem. The dual simplex method is simply a new way of picking the entering and leaving variables in a sequence of primal tableaux.

Phase II We can now remove the temporary objective function and the corresponding column and proceed with the primal simplex.

x1	x2	s1	s2	-z	b
0	0	1	1/2	0	1
1	1	0	-1/2	0	1
2	0	0	-1/2	1	1

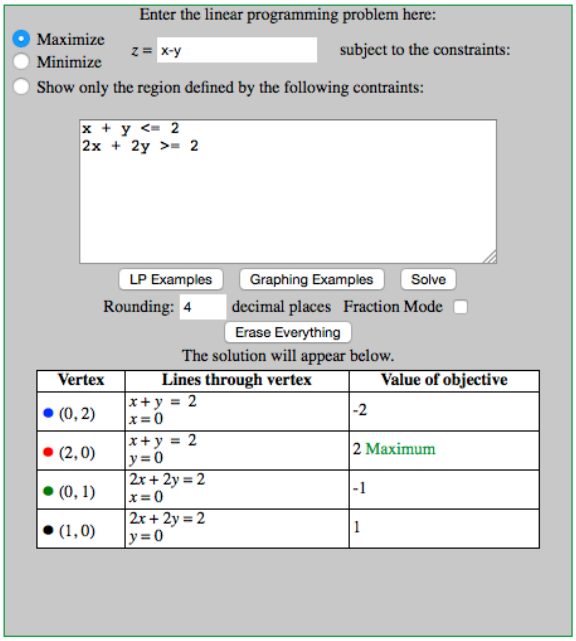
x_1 enters the basis and x_2 exits. The tableau is updated consequently:

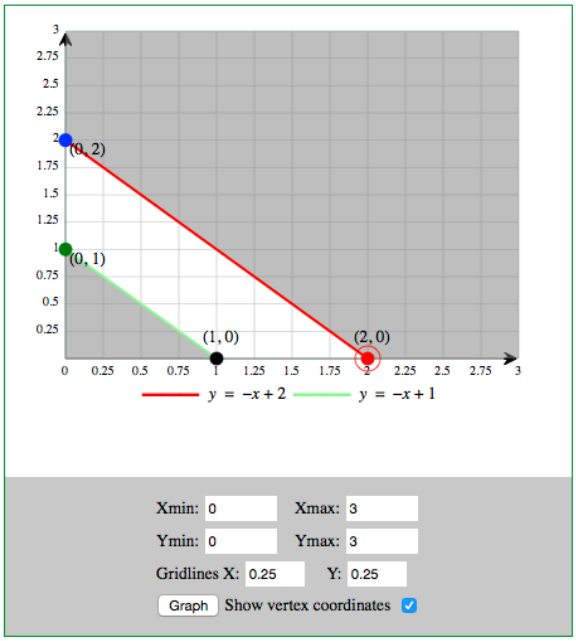
	x1	x2	s1	s2	-z	b
R1'=R1	0	0	1	1/2	0	1
R2'=R2	1	1	0	-1/2	0	1
R3'=R3-2*R2	0	-2	0	1/2	1	-1

A reduced cost is still positive, hence we make s_2 enters in the basis and s_1 leave. This leads to

	x1	x2	s1	s2	-z	b
R1'=2*R1	0	0	2	1	0	2
R2'=R2+R1	1	1	1	0	0	2
R3'=R3-R1	0	-2	-1	0	1	-2

The tableau is now optimal and the corresponding basic feasible solution is $\mathbf{x} = (2, 0)$ and has value $z^* = 2$.
We can visualize the problem using the LP Grapher tool linked from the course webpage:





Exercise 6*

Write the dual of the following problem

$$
\begin{aligned}
(P) \qquad \max & \sum_{j\in J}\sum_{i\in I} r_j x_{ij} & \\
& \sum_{j\in J} x_{ij} \le b_i & \forall i \in I \\
& \sum_{i\in I} x_{ij} \le d_j & \forall j \in J \\
& \sum_{i\in I} p_i x_{ij} = p_j \sum_{i\in I} x_{ij} & \forall j \in J \\
& x_{ij} \ge 0 & \forall i \in I, j \in J
\end{aligned}
$$

Solution:

There are three different sets of constraints. We introduce the dual variables $\alpha_i \geq 0$, for $i \in I$, for the first set; the dual variables $\beta_j \geq 0$, for $j \in J$, for the second set; and the dual variables $\gamma_j \in \mathbb{R}$, for $j \in J$ for the third set.
We then write the A matrix for the example in the picture, augmented with the b vector:

$$\begin{array}{c} \begin{matrix} x_{G1} & x_{G2} & x_{G3} & x_{G4} & x_{T1} & x_{T2} & x_{T3} & x_{T4} & \end{matrix} \\ \left[\begin{array}{cccccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & b_1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & b_2 \\ 1 & & & & 1 & & & & d_1 \\ & 1 & & & & 1 & & & d_2 \\ & & 1 & & & & 1 & & d_3 \\ & & & 1 & & & & 1 & d_4 \\ p_G - p1 & & & & p_T - p1 & & & & \\ & p_G - p2 & & & & p_T - p2 & & & \\ & & p_G - p3 & & & & p_T - p3 & & \\ & & & p_G - p4 & & & & p_T - p4 & \end{array}\right] \\ \begin{matrix} r_1 & r_2 & r_3 & r_4 & r_1 & r_2 & r_3 & r_4 & \end{matrix} \end{array}$$

and finally the dual from the columns of the A matrix in general terms:

$$\begin{aligned} (D) \qquad & \min \sum_{i \in I} \alpha_i b_i + \sum_{j \in J} \beta_j d_j & \\ & \alpha_i + \beta_j + (p_i - p_j)\gamma_j \geq r_j & \forall i \in I, j \in J \\ & \alpha_i \geq 0 & \forall i \in I \\ & \beta_j \geq 0 & \forall j \in J \\ & \gamma_j \in \mathbb{R} & \forall j \in J \end{aligned}$$