

DM545/DM871  
Linear and Integer Programming

Lecture 13  
Network Flows, Cntd

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# Outline

Duality  
Network Simplex  
Final Remarks

1. Duality in Network Flow Problems

2. Network Simplex

3. Final Remarks

	$x_{e_1}$	$x_{e_2}$	$\dots$	$x_{ij}$	$\dots$	$x_{e_m}$		
	$c_{e_1}$	$c_{e_2}$	$\dots$	$c_{ij}$	$\dots$	$c_{e_m}$		
1	1	.	$\dots$	.	$\dots$	.	=	$b_1$
2	.	.	$\dots$	.	$\dots$	.	=	$b_2$
$\vdots$	$\vdots$	$\ddots$					=	$\vdots$
$i$	-1	.	$\dots$	1	$\dots$	.	=	$b_i$
$\vdots$	$\vdots$	$\ddots$					=	$\vdots$
$j$	.	.	$\dots$	-1	$\dots$	.	=	$b_j$
$\vdots$	$\vdots$	$\ddots$					=	$\vdots$
$n$	.	.	$\dots$	.	$\dots$	.	=	$b_n$
$e_1$	1						$\leq$	$u_1$
$e_2$		1					$\leq$	$u_2$
$\vdots$	$\vdots$	$\ddots$					$\leq$	$\vdots$
$(i,j)$				1			$\leq$	$u_{ij}$
$\vdots$	$\vdots$	$\ddots$					$\leq$	$\vdots$
$e_m$						1	$\leq$	$u_m$

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# Shortest Path - Dual LP

$$z = \min \sum_{ij \in A} c_{ij} x_{ij}$$

$$\sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = 1 \quad \text{for } i = s \quad (\pi_s)$$

$$\sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = 0 \quad \forall i \in V \setminus \{s, t\} \quad (\pi_i)$$

$$\sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = -1 \quad \text{for } i = t \quad (\pi_t)$$

$$x_{ij} \geq 0 \quad \forall ij \in A$$

Dual problem:

$$g^{LP} = \max \pi_s - \pi_t$$

$$\pi_i - \pi_j \leq c_{ij} \quad \forall ij \in A$$

Hence, the shortest path can be found by potential values  $\pi_i$  on nodes such that  $\pi_s = z, \pi_t = 0$  and  $\pi_i - \pi_j \leq c_{ij}$  for  $ij \in A$

# Maximum $(s, t)$ -Flow

Adding a backward arc from  $t$  to  $s$ :

$$z = \max x_{ts}$$

$$\sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = 0$$

$$\forall i \in V \quad (\pi_i)$$

$$x_{ij} \leq u_{ij}$$

$$\forall ij \in A \quad (w_{ij})$$

$$x_{ij} \geq 0$$

$$\forall ij \in A$$

Dual problem:

$$g^{LP} = \min \sum_{ij \in A} u_{ij} w_{ij}$$

$$\pi_i - \pi_j + w_{ij} \geq 0$$

$$\forall ij \in A$$

$$\pi_t - \pi_s \geq 1$$

$$w_{ij} \geq 0$$

$$\forall ij \in A$$

	$x_{e_1}$	$x_{e_2}$	$\dots$	$x_{ij}$	$\dots$	$x_{e_m}$		
	$c_{e_1}$	$c_{e_2}$	$\dots$	$c_{ij}$	$\dots$	$c_{e_m}$		
1	1	.	$\dots$	.	$\dots$	.	=	$b_1$
2	.	.	$\dots$	.	$\dots$	.	=	$b_2$
$\vdots$	$\vdots$	$\ddots$					=	$\vdots$
$i$	-1	.	$\dots$	1	$\dots$	.	=	$b_i$
$\vdots$	$\vdots$	$\ddots$					=	$\vdots$
$j$	.	.	$\dots$	-1	$\dots$	.	=	$b_j$
$\vdots$	$\vdots$	$\ddots$					=	$\vdots$
$n$	.	.	$\dots$	.	$\dots$	.	=	$b_n$
$e_1$	1						$\leq$	$u_1$
$e_2$		1					$\leq$	$u_2$
$\vdots$	$\vdots$	$\ddots$					$\leq$	$\vdots$
$(i,j)$				1			$\leq$	$u_{ij}$
$\vdots$	$\vdots$	$\ddots$					$\leq$	$\vdots$
$e_m$						1	$\leq$	$u_m$

$$g^{LP} = \min \sum_{ij \in A} u_{ij} w_{ij} \quad (1)$$

$$\pi_i - \pi_j + w_{ij} \geq 0 \quad \forall ij \in A \quad (2)$$

$$\pi_t - \pi_s \geq 1 \quad (3)$$

$$w_{ij} \geq 0 \quad \forall ij \in A \quad (4)$$

- Without (3) all potentials would go to 0.
- Keep  $w$  low because of objective function
- Keep all potentials low  $\rightsquigarrow$  (3)  $\pi_s = 0, \pi_t = 1$
- Cut  $C$ : on left =1 on right =0. Where is the transition?
- Vars  $w$  identify the cut  $\rightsquigarrow \pi_j - \pi_i + w_{ij} \geq 0 \rightsquigarrow w_{ij} = 1$

$$w_{ij} = \begin{cases} 1 & \text{if } ij \in C \\ 0 & \text{otherwise} \end{cases}$$

for those arcs that minimize the cut capacity  $\sum_{ij \in A} u_{ij} w_{ij}$

- Complementary slackness:  $w_{ij} = 1 \implies x_{ij} = u_{ij}$



## Theorem

A strong dual to the max  $(st)$ -flow is the minimum  $(st)$ -cut problem:

$$\min_X \left\{ \sum_{ij \in A: i \in X, j \notin X} u_{ij} : s \in X \subset V \setminus \{t\} \right\}$$

# Max Flow Algorithms

## Optimality Condition

- Ford Fulkerson augmenting path algorithm  $O(m|x^*|)$
- Edmonds-Karp algorithm (augment by shortest path) in  $O(nm^2)$
- Dinic algorithm in layered networks  $O(n^2m)$
- Karzanov's push relabel  $O(n^2m)$

# Min Cost Flow - Dual LP

$$\begin{aligned}
 & \min \sum_{ij \in A} c_{ij} x_{ij} \\
 & \sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = b_i & \forall i \in V & \quad (\pi_i) \\
 & x_{ij} \leq u_{ij} & \forall ij \in A & \quad (w_{ij}) \\
 & x_{ij} \geq 0 & \forall ij \in A &
 \end{aligned}$$

Dual problem:

$$\max \sum_{i \in V} b_i \pi_i - \sum_{ij \in E} u_{ij} w_{ij} \tag{1}$$

$$-c_{ij} + \pi_i - \pi_j \leq w_{ij} \quad \forall ij \in E \tag{2}$$

$$w_{ij} \geq 0 \quad \forall ij \in A \tag{3}$$

	$x_{e_1}$	$x_{e_2}$	$\dots$	$x_{ij}$	$\dots$	$x_{e_m}$		
	$c_{e_1}$	$c_{e_2}$	$\dots$	$c_{ij}$	$\dots$	$c_{e_m}$		
1	1	.	$\dots$	.	$\dots$	.	=	$b_1$
2	.	.	$\dots$	.	$\dots$	.	=	$b_2$
$\vdots$	$\vdots$	$\ddots$					=	$\vdots$
$i$	-1	.	$\dots$	1	$\dots$	.	=	$b_i$
$\vdots$	$\vdots$	$\ddots$					=	$\vdots$
$j$	.	.	$\dots$	-1	$\dots$	.	=	$b_j$
$\vdots$	$\vdots$	$\ddots$					=	$\vdots$
$n$	.	.	$\dots$	.	$\dots$	.	=	$b_n$
$e_1$	1						$\leq$	$u_1$
$e_2$		1					$\leq$	$u_2$
$\vdots$	$\vdots$	$\ddots$					$\leq$	$\vdots$
$(i,j)$				1			$\leq$	$u_{ij}$
$\vdots$	$\vdots$	$\ddots$					$\leq$	$\vdots$
$e_m$						1	$\leq$	$u_m$

- When is the set of feasible solutions  $\mathbf{x}, \boldsymbol{\pi}, \mathbf{w}$  optimal?
- define reduced costs  $\bar{c}_{ij} = c_{ij} - \pi_i + \pi_j$ , hence (2) becomes  $-\bar{c}_{ij} \leq w_{ij}$
- $u_e = \infty$  then  $w_e = 0$  (from obj. func) and  $\bar{c}_{ij} \geq 0$  (from 2)
- $u_e < \infty$  then  $w_e \geq 0$  and  $w_e \geq -\bar{c}_{ij}$  then  $w_e = \max\{0, -\bar{c}_{ij}\}$ , hence  $w_e$  is determined by others and irrelevant
- Complementary slackness th. for optimal solutions:  
 each primal variable  $\times$  the corresponding dual slack must be equal 0, ie,  $x_e(\bar{c}_e + w_e) = 0$ ;
  - $x_e > 0$  then  $-\bar{c}_e = w_e = \max\{0, -\bar{c}_e\}$ ,  
 $x_e > 0 \implies -\bar{c}_e \geq 0$  or equivalently (by negation)  $\bar{c}_e > 0 \implies x_e = 0$
 each dual variable  $\times$  the corresponding primal slack must be equal 0, ie,  $w_e(x_e - u_e) = 0$ ;
  - $w_e > 0$  then  $x_e = u_e$   
 $-\bar{c}_e > 0 \implies x_e = u_e$  or equivalently  $\bar{c}_e < 0 \implies x_e = u_e$

Hence:

$$\bar{c}_e > 0 \text{ then } x_e = 0$$

$$\bar{c}_e < 0 \text{ then } x_e = u_e \neq \infty$$

# Min Cost Flow Algorithms

The conditions derived can be used to define a solution approach for the minimum cost flow problem.

Directed cycle  $\equiv$  circuit

Note that if a set of potentials  $\pi_i, i \in V$  are given, and the cost of a circuit wrt. the reduced costs for the edges ( $\bar{c}_{ij} = c_{ij} + \pi_j - \pi_i$ ) are calculated, the cost remains the same as the original costs because the potentials are “telescoped” to 0.

## Theorem (Optimality conditions)

*Let  $\mathbf{x}$  be feasible flow in  $N(V, A, l, \mathbf{u}, \mathbf{b})$  then  $\mathbf{x}$  is min cost flow in  $N$  iff  $N(\mathbf{x})$  contains no directed cycle of negative cost.*

Note that a (directed) circuit with negative cost in  $N(\mathbf{x})$  corresponds to a negative cost cycle in  $N$ , if costs are added for forward edges and subtracted for backward edges.

- Cycle canceling algorithm with Bellman Ford Moore for negative cycles  $O(nm^2UC)$ ,  
 $U = \max |u_e|, C = \max |c_e|$
- Build up algorithms  $O(n^2mM)$ ,  $M = \max |b(v)|$

# Outline

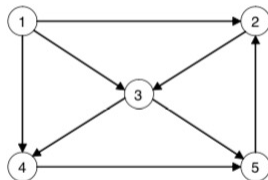
Duality  
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# Min Cost Flow



$$A = \begin{bmatrix} 1 & 1 & 1 & & & & & & \\ -1 & & & 1 & & & & & -1 \\ & -1 & & -1 & 1 & 1 & & & \\ & & -1 & & -1 & & 1 & & \\ & & & & & -1 & -1 & 1 & \\ & & & & & & & & \end{bmatrix}$$

$$\begin{aligned} \min \quad & 10x_{12} + 8x_{13} + x_{14} + 2x_{23} + x_{34} + 4x_{35} + 12x_{45} - 7x_{52} \\ & x_{12} + x_{13} + x_{14} = 10 \\ & -x_{12} + x_{23} - x_{52} = 4 \\ & -x_{13} - x_{23} + x_{34} + x_{35} = 0 \\ & -x_{14} - x_{34} + x_{45} = -6 \\ & -x_{35} - x_{45} + x_{52} = -8 \\ & x \geq 0 \end{aligned}$$

- $A$  is not full-rank: adding all rows  $\rightsquigarrow$  null vector, i.e., the rows of  $A$  are not linearly indep.
- Since we assume that total supply equal total demand, i.e.,  $\sum_{i \in V} b_i = 0$  then  $\text{rank}[A] = \text{rank}[A \ b]$ .
- Hence, one of the equations can be canceled.



- assume network  $N$  is connected
- **cycle**: here, a set of arcs forming a closed path (i.e., a path in which the first and the last node of the path coincide) when ignoring their orientation
- **spanning tree**: here, a tree that reaches every node (it coincides with the classical notion of spanning tree if one disregards arc orientation).

### Theorem (Spanning Trees)

For an undirected graph  $D' = (N, A')$ , the following are equivalent:

- (a)  $G' = (N, E)$  is a tree (acyclic and connected);
- (b)  $G' = (N, E)$  is acyclic and has  $n - 1$  arcs; and
- (c)  $G' = (N, E)$  is connected and has  $n - 1$  arcs.

Since we know that the matrix  $A$  is not full-rank, a basis of  $A$  consists of only  $n - 1$  linearly independent columns of  $A$ . These columns correspond to a collection of arcs of the flow network.

### Theorem

*Given a connected flow network, letting  $A$  be its incidence matrix, a submatrix  $B$  of size  $(n - 1) \times (n - 1)$  is a **basis** of  $A$  **if and only if** the arcs associated with the columns of  $B$  form a **spanning tree**.*

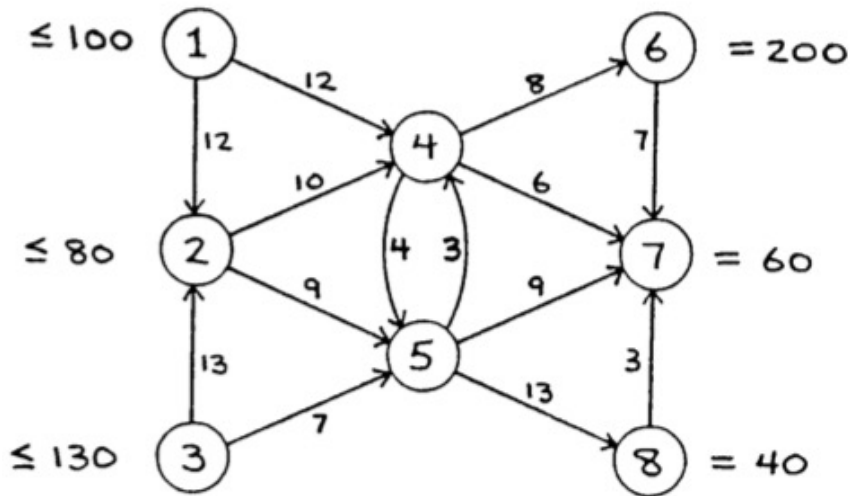
Proof:

if columns from  $A$  correspond to a spanning tree  $\implies$  they are lin. indep.,  $B$  is upper triangular  
if a subset of columns of  $A$  are a basis  $\implies$  they are  $n - 1$  and acyclic

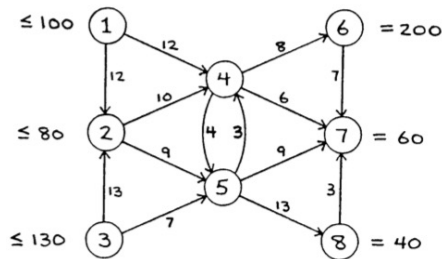
Hence, all basic feasible solutions explored by the simplex algorithm are spanning trees of the flow network.

As for any LP, also in min-cost flow problems there are feasible, infeasible and degenerate bases.  
(feasible if  $\mathbf{x}_B = A_B^{-1} \mathbf{b} \geq 0$ ).

# Example



# Example

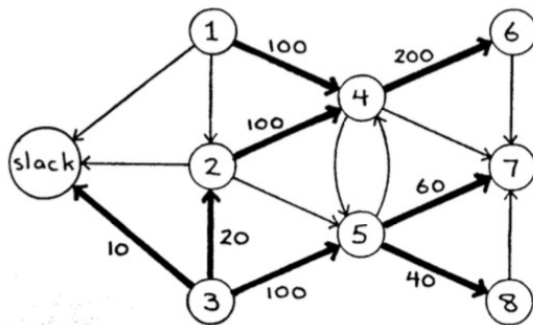


$$12x_{12} + 12x_{14} + 10x_{24} + 9x_{25} + 13x_{32} + 7x_{35} + 4x_{45} + 8x_{46} + 6x_{47} + 3x_{54} + 9x_{57} + 13x_{58} + 7x_{67} + 3x_{87}$$

$$\begin{array}{rcl}
 +x_{12} + & x_{14} & +s_1 = 100 \\
 -x_{12} & +x_{24} + x_{25} - x_{32} & +s_2 = 80 \\
 & +x_{32} + x_{35} & +s_3 = 130 \\
 -x_{14} - x_{24} & +x_{45} + x_{46} + x_{47} - x_{54} & = 0 \\
 & -x_{25} - x_{35} - x_{45} & +x_{54} + x_{57} + x_{58} = 0 \\
 & -x_{46} & +x_{67} = -200 \\
 & -x_{47} - x_{57} & -x_{67} - x_{87} = -60 \\
 & & -x_{58} + x_{87} = -40
 \end{array}$$

$$x_{12}, x_{14}, x_{24}, x_{25}, x_{32}, x_{35}, x_{45}, x_{46}, x_{47}, x_{54}, x_{57}, x_{58}, x_{67}, x_{87}, s_1, s_2, s_3 \geq 0$$

# Example



$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

- solve  $Bx_B = b$  in value of variables to check feasibility; easy because of structure or because done by updates.
- solve  $\pi^T B = c_B^T$  in  $\pi$  (dual potential variables to derive reduced costs); easy because of structure of  $B$ .
- calculate  $\bar{c}_{ij} = c_{ij} + \pi_j - \pi_i$

$$\pi_1 - \pi_4 = 12$$

$$\pi_2 - \pi_4 = 10$$

$$\pi_3 - \pi_2 = 13$$

$$\pi_3 - \pi_5 = 7$$

$$\pi_4 - \pi_6 = 8$$

$$\pi_5 - \pi_7 = 9$$

$$\pi_5 - \pi_8 = 13$$

$$\pi_3 = 0$$

$$\pi_3 = 0 \text{ and } \pi_3 - \pi_5 = 7 \Rightarrow \pi_5 = -7$$

$$\pi_5 = -7 \text{ and } \pi_5 - \pi_8 = 13 \Rightarrow \pi_8 = -20$$

$$\pi_5 = -7 \text{ and } \pi_5 - \pi_7 = 9 \Rightarrow \pi_7 = -16$$

$$\pi_3 = 0 \text{ and } \pi_3 - \pi_2 = 13 \Rightarrow \pi_2 = -13$$

$$\pi_2 = -13 \text{ and } \pi_2 - \pi_4 = 10 \Rightarrow \pi_4 = -23$$

$$\pi_4 = -23 \text{ and } \pi_4 - \pi_6 = 8 \Rightarrow \pi_6 = -31$$

$$\pi_4 = -23 \text{ and } \pi_1 - \pi_4 = 12 \Rightarrow \pi_1 = -11$$

$$d_{12} = c_{12} - \pi_1 + \pi_2 = 12 - (-11) + (-13) = 10$$

$$d_{25} = c_{25} - \pi_2 + \pi_5 = 9 - (-13) + (-7) = 15$$

$$d_{45} = c_{45} - \pi_4 + \pi_5 = 4 - (-23) + (-7) = 20$$

$$d_{54} = c_{54} - \pi_5 + \pi_4 = 3 - (-7) + (-23) = -13$$

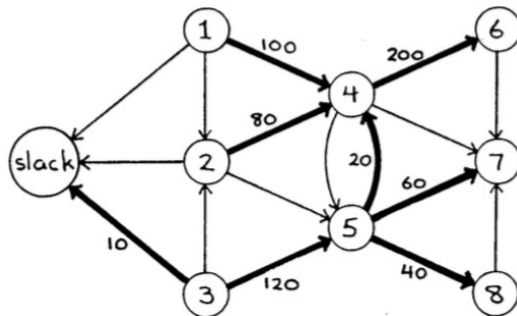
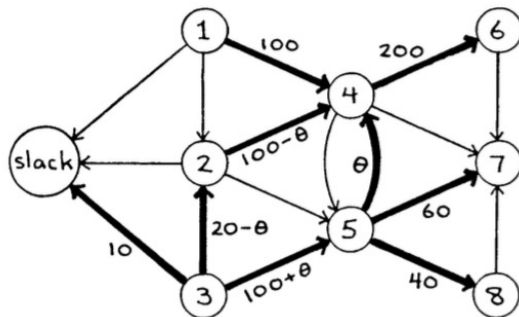
$$d_{47} = c_{47} - \pi_4 + \pi_7 = 6 - (-23) + (-16) = 13$$

$$d_{67} = c_{67} - \pi_6 + \pi_7 = 7 - (-31) + (-16) = 22$$

$$d_{87} = c_{87} - \pi_8 + \pi_7 = 3 - (-20) + (-16) = 7$$

$$d_1 = 0 - \pi_1 = -(-11) = 11$$

$$d_2 = 0 - \pi_2 = -(-13) = 13$$



How much can we increase the flow  $\theta$  through  
(54)?

Until (32) reaches zero

- It can be proved that, because the basis corresponds to a tree, the equations can always be solved by simple substitution.
- The order of substitution can always be found by “walking around the tree”.
- Efficient implementations further reduce the cost of determining  $\pi$  by updating it as they walk around the tree, rather than computing it anew at each iteration.
- When the network simplex steps are to be carried out by a computer, it is not so obvious how
- A few concise and clever data structures are used to represent the basis tree in a way that allows the walk around the tree and finding the circuit induced by the entering arc efficiently.
- The data structures can themselves be efficiently updated as the tree changes from iteration to iteration.



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- 
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  - DM872 Matematisk optimering i praksis (5 ECTS, efterår)
  - DM878 Visualisering (5 ECTS, efterår)
  - MM856 Grafteori (10 ECTS, efterår)
- 
- DM870 Data mining and machine learning (10 ECTS, forår)
  - DM887 Reinforcement learning (10 ECTS, forår)
  - AI505+ IAAI501 Optimering (7.5 + 2.5 ECTS, forår)
- 
- 30 ECTS valgfag
- 
- 30 ECTS kandidatspeciale
-

# Bachelor and Master projects

- Ideas for student projects:  
<https://imada.sdu.dk/u/march/Blog/references/2022/04/20/projects.html>
- But you can also come with your ideas