

DM545/DM871  
Linear and Integer Programming

Lecture 9  
**IP Modeling**  
**Formulations, Relaxations**

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## 1. Formulations

- Uncapacited Facility Location
- Alternative Formulations

## 2. Relaxations

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# Uncapacitated Facility Location (UFL)

## Given:

- depots  $N = \{1, \dots, n\}$
- clients  $M = \{1, \dots, m\}$
- $f_j$  fixed cost to use depot  $j$
- transport cost for all orders  $c_{ij}$

**Task:** Which depots to open and which depots serve which client

**Variables:**  $y_j = \begin{cases} 1 & \text{if depot opened} \\ 0 & \text{otherwise} \end{cases}$ ,  $x_{ij}$  fraction of demand of  $i$  satisfied by  $j$

## Objective:

$$\min \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} f_j y_j$$

## Constraints:

$$\sum_{j=1}^n x_{ij} = 1$$

$$\forall i = 1, \dots, m$$

$$\sum_{i \in M} x_{ij} \leq m y_j$$

$$\forall j \in N$$

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## Definition (Formulation)

A polyhedron  $P \subseteq \mathbb{R}^{n+p}$  is a **formulation** for a set  $X \subseteq \mathbb{Z}^n \times \mathbb{R}^p$  if and only if  $X = P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$

That is, if it does not leave out any of the solutions of the feasible region  $X$ .

There are **infinitely many** formulations

## Definition (Convex Hull)

Given a set  $X \subseteq \mathbb{Z}^n$  the **convex hull** of  $X$  is defined as:

$$\text{conv}(X) = \left\{ \mathbf{x} : \mathbf{x} = \sum_{i=1}^t \lambda_i \mathbf{x}^i, \quad \sum_{i=1}^t \lambda_i = 1, \quad \lambda_i \geq 0, \quad \text{for } i = 1, \dots, t, \right. \\ \left. \text{for all finite subsets } \{\mathbf{x}^1, \dots, \mathbf{x}^t\} \text{ of } X \right\}$$

### Proposition

$\text{conv}(X)$  is a polyhedron (ie, representable as  $Ax \leq b$ )

### Proposition

Extreme points of  $\text{conv}(X)$  all lie in  $X$

Hence:

$$\max\{c^T x : x \in X\} \equiv \max\{c^T x : x \in \text{conv}(X)\}$$

However it might require exponential number of inequalities to describe  $\text{conv}(X)$

What makes a formulation better than another?

$$X \subseteq \text{conv}(X) \subseteq P_2 \subset P_1$$

$P_2$  is better than  $P_1$

### Definition

Given a set  $X \subseteq \mathbb{R}^n$  and two formulations  $P_1$  and  $P_2$  for  $X$ ,  $P_2$  is a better formulation than  $P_1$  if  $P_2 \subset P_1$



## Example

 $P_1 = \text{UFL with } \sum_{i \in M} x_{ij} \leq my_j \quad \forall j \in N$  $P_2 = \text{UFL with } x_{ij} \leq y_j \quad \forall i \in M, j \in N$ 

$$P_2 \subset P_1$$

- $P_2 \subseteq P_1$  because summing  $x_{ij} \leq y_j$  over  $i \in M$  we obtain  $\sum_{i \in M} x_{ij} \leq my_j$
- $P_2 \subset P_1$  because there exists a point in  $P_1$  but not in  $P_2$ :  $m = 6 = 3 \cdot 2 = k \cdot n$

$$x_{10} = 1, x_{20} = 1, x_{30} = 1,$$

$$x_{41} = 1, x_{51} = 1, x_{61} = 1$$

$$\sum_i x_{i0} \leq 6y_0 \quad y_0 = 1/2$$

$$\sum_i x_{i1} \leq 6y_1 \quad y_1 = 1/2$$

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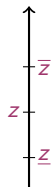
$$z = \max\{c(\mathbf{x}) : \mathbf{x} \in X \subseteq \mathbb{Z}^n\}$$

How can we prove that  $\mathbf{x}^*$  is optimal?

$\bar{z}$  is UB

$\underline{z}$  is LB

stop when  $\bar{z} - \underline{z} \leq \epsilon$



- **Primal bounds** (here lower bounds): every feasible solution gives a primal bound, it may be easy or hard to find, heuristics
- **Dual bounds** (here upper bounds): Relaxations

Optimality gap (SCIP):

- If primal and dual bounds have opposite signs, the gap is "Infinity".
- If primal and dual bounds have the same sign, the gap is

$$\frac{|pb - db|}{\min(|pb|, |db|)}$$

decreases monotonously during the solving process.

## Proposition

Given:  $(IP) \ z = \max\{c(\mathbf{x}) : \mathbf{x} \in X \subseteq \mathbb{R}^n\}$   
a relaxation of it is:  $(RP) \ z^R = \max\{f(\mathbf{x}) : \mathbf{x} \in T \subseteq \mathbb{R}^n\}$  if:

- (i)  $X \subseteq T$  or
- (ii)  $f(\mathbf{x}) \geq c(\mathbf{x}) \ \forall \mathbf{x} \in X$

In other terms:

$$\max_{\mathbf{x} \in T} f(\mathbf{x}) \geq \left\{ \begin{array}{l} \max_{\mathbf{x} \in T} c(\mathbf{x}) \\ \max_{\mathbf{x} \in X} f(\mathbf{x}) \end{array} \right\} \geq \max_{\mathbf{x} \in X} c(\mathbf{x})$$

- $T$ : candidate solutions;
- $X \subseteq T$  feasible solutions;
- $f(\mathbf{x}) \geq c(\mathbf{x}) \ \forall \mathbf{x} \in X$

How to construct relaxations?

1.  $IP : \max\{\mathbf{c}^T \mathbf{x} : \mathbf{x} \in P \cap \mathbb{Z}^n\}, \quad P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$   
 $LP : \max\{\mathbf{c}^T \mathbf{x} : \mathbf{x} \in P\}$

Better formulations give better bounds ( $P_1 \subseteq P_2$ )

## Proposition

- (i) *If a relaxation LP is infeasible, the original problem IP is infeasible.*
- (ii) *Let  $\mathbf{x}^*$  be optimal solution for LP. If  $\mathbf{x}^* \in X$  and  $f(\mathbf{x}^*) = c(\mathbf{x}^*)$  then  $\mathbf{x}^*$  is optimal for IP.*

2. **Combinatorial relaxations** to easy problems that can be solved rapidly  
Eg: TSP to Assignment problem Eg: Symmetric TSP to 1-tree

### 3. Lagrangian relaxation

$$IP : \quad z = \max\{\mathbf{c}^T \mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in X \subseteq \mathbb{Z}^n\}$$

$$LR : \quad z(\mathbf{u}) = \max\{\mathbf{c}^T \mathbf{x} + \mathbf{u}(\mathbf{b} - A\mathbf{x}) : \mathbf{x} \in X\}$$

$$z(\mathbf{u}) \geq z \quad \forall \mathbf{u} \geq 0$$

### 4. Duality:

#### Definition

Two problems:

$$z = \max\{c(\mathbf{x}) : \mathbf{x} \in X\} \quad w = \min\{w(\mathbf{u}) : \mathbf{u} \in U\}$$

form a **weak-dual pair** if  $c(\mathbf{x}) \leq w(\mathbf{u})$  for all  $\mathbf{x} \in X$  and all  $\mathbf{u} \in U$ .

When  $z = w$  they form a **strong-dual pair**

### Proposition

$z = \max\{\mathbf{c}^T \mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in \mathbb{Z}_+^n\}$  and  $w^{LP} = \min\{\mathbf{u}^T \mathbf{b} : A^T \mathbf{u} \geq \mathbf{c}, \mathbf{u} \in \mathbb{R}_+^m\}$   
(ie, dual of linear relaxation) form a weak-dual pair.

### Proposition

Let  $IP$  and  $D$  be weak-dual pair:

- (i) If  $D$  is unbounded, then  $IP$  is infeasible
- (ii) If  $\mathbf{x}^* \in X$  and  $\mathbf{u}^* \in U$  satisfy  $\mathbf{c}(\mathbf{x}^*) = w(\mathbf{u}^*)$  then  $\mathbf{x}^*$  is optimal for  $IP$  and  $\mathbf{u}^*$  is optimal for  $D$ .

The advantage is that we do not need to solve an LP like in the LP relaxation to have a bound, any feasible dual solution gives a bound.

Weak pairs:

Matching:  $z = \max\{1^T \mathbf{x} : A\mathbf{x} \leq 1, \mathbf{x} \in \mathbb{Z}_+^m\}$

V. Covering:  $w = \min\{1^T \mathbf{y} : A^T \mathbf{y} \geq 1, \mathbf{y} \in \mathbb{Z}_+^n\}$

Proof: consider LP relaxations, then  $z \leq z^{LP} = w^{LP} \leq w$ .  
(strong when graphs are bipartite)

Weak pairs:

S. Packing:  $z = \max\{1^T \mathbf{x} : A\mathbf{x} \leq 1, \mathbf{x} \in \mathbb{Z}_+^n\}$

S. Covering:  $w = \min\{1^T \mathbf{y} : A^T \mathbf{y} \geq 1, \mathbf{y} \in \mathbb{Z}_+^m\}$