DM545/DM871 – Linear and integer programming

Sheet 8, Spring 2024

Exercises with the symbol $^+$are to be done at home before the class. Exercises with the symbol *will be tackled in class and should be at least read at home. The remaining exercises are left for self training after the exercise class.

Exercise 1$^+$ [DT97]

Suppose that in a minimum cost flow problem restrictions are placed on the total flow leaving a node k, i.e.

$$\underline{\theta}_k \leq \sum_{(k,j)\in E} x_{kj} \leq \overline{\theta}_k$$

Show how to modify these restrictions to convert the problem into a standard cost flow problem.

Exercise 2$^+$

The production plan of a factory for the next year is to produce d_t units of product per month t, $t = 1, \ldots, 12$. Each worker can produce k units of product in a month. The monthly salary is equal to s. Employing and firing personnel has costs: precisely, employing one person costs p while firing one costs q. Assuming that initially there are g_0 workers, determine the number of workers that must be present during every month such that the demand is always satisfied and the overall costs of salary, employment, and firing are minimized.

Exercise 3$^+$ Directed Chinese Postman Problem

Suppose a postman has to deliver mail along all the streets in a small town. Assume furthermore that on one-way streets the mail boxes are all on one side of the street, whereas for two-way streets, there are mail boxes on both sides of the street. For obvious reasons the postman wishes to minimize the distance he has to travel in order to deliver all the mail and return home to his starting point. Show how you can solve this problem using minimum cost flows. A similar model can be formulated for the Snow Plow problem or the Salt Spreding problem.

Exercise 4* Warehousing of Seasonal Products

A company manufactures multiple products. The products are seasonal with demand varying weekly, monthly, or quarterly. To use its work-force and capital equipment efficiently, the company wishes to "smooth" production, storing pre-season production to supplement peak-season production. The company has a warehouse with fixed capacity R that it uses to store all the products it produces. Its decision problem is to identify the production levels of all the products for every week, month, or quarter of the year that will permit it to satisfy the demands incurring the minimum possible production and storage costs.
We can represent this warehousing problem as a relevant generalization of the min cost network flow problem encountered in the course.
For simplicity, consider a situation in which the company makes two products and then it needs to schedule its production for each of the next four quarters of the year. Let d_j^1 and d_j^2 denote the demand for products 1 and 2 in quarter j. Suppose that the production capacity for the jth quarter is u_j^1 and u_j^2, and that the per unit cost of production for this quarter is c_j^1 and c_j^2. Let h_j^1 and h_j^2 denote the storage (holding) costs per unit of the two products from quarter j to quarter $j + 1$.
Represent graphically the network in the two products four periods case and write the Linear Programming formulation of the problem. Which network flows problem models this application? If all input data are integer, will the solution be integer?

Exercise 5* Scheduling on Uniform Parallel Machines

We consider scheduling a set J of jobs on M uniform parallel machines. Each job $j \in J$ has a processing requirement p_j (denoting the number of machine days required to complete the job), a release data r_j (representing the beginning of the day when job j becomes available for processing), and a deadline $d_j \geq r_j + p_j$ (representing the beginning of the day by which the job must be completed). We assume that a machine can work on only one job at a time and that each job can be processed by at most one machine at a time. However we allow preemptions (ie, we can interrupt a job and process it on different machines on different days). The scheduling problem is to determine a feasible schedule that completes all jobs before their due dates or to show that no such schedule exists.
Formulate the feasible scheduling problem as a maximum flow problem.
It may help to consider an example of the problem, for example, the one given in Table 1.

Job (j)	1	2	3	4
Processing time (p_j)	1.5	1.25	2.1	3.6
Release time (r_j)	3	1	3	5
Due date (d_j)	5	4	7	9

Table 1:

Exercise 6* Tanker Scheduling Problem

A steamship company has contracted to deliver perishable goods between several different origin-destination pairs. Since the cargo is perishable the customers have specified precise dates (ie, delivery dates) when the shipments must reach their destinations. (The cargoes may not arrive early or late). The steamship company wants to determine the minimum number of ships needed to meet the delivery dates of the shiploads.
Formulate this problem as a maximum flow problem modeling the example in Table 2 with four shipments. Each shipment is a full shipload with the characteristics shown in Table 2. For example, as specified by the first row in this figure, the company must deliver one shipload available at port A and destined for port C on day 3.

ship-ment	origin	desti-nation	delivery date
1	Port A	Port C	3
2	Port A	Port C	8
3	Port B	Port D	3
4	Port B	Port C	6

	C	D
A	3	2
B	2	3

	A	B
C	2	1
D	1	2

Table 2: Data for the tanker scheduling problem: Left shipment characteristics; Center, shipment transit times; Right return times.

Exercise 7 [Goe11]

A managing director has to launch the marketing of a new product. Several candidate products are at his disposal and he has to choose the best one. Hence, he let each of these products be analysed by a team made of an engineer and a trader who write a review together. The teams are made along the graph in Figure 1; each edge corresponds to a product and its endvertices to the engineer and trader examining it.

a) How many people at least does the managing director gather in order to have the report on all the products? (The report can be given by either the engineer or the trader.)

b) Assuming now that the report must be done jointly by an engineering and a trader, and that each engineer and trader can be occupied with only one candidate product, give a polynomial time algorithm to identify which products will for sure not have the possibility to obtain a report.

Exercise 8

Given the Network in Figure 2, determine the max flow from 1 to 7 and indicate the min cut.

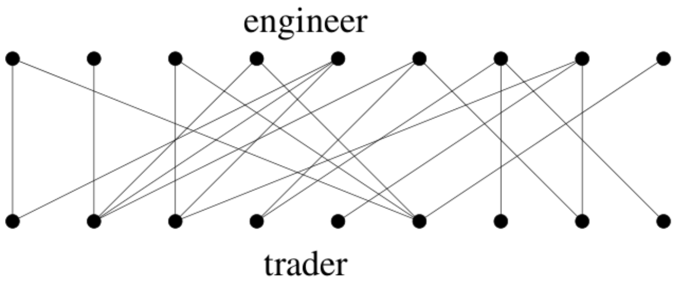
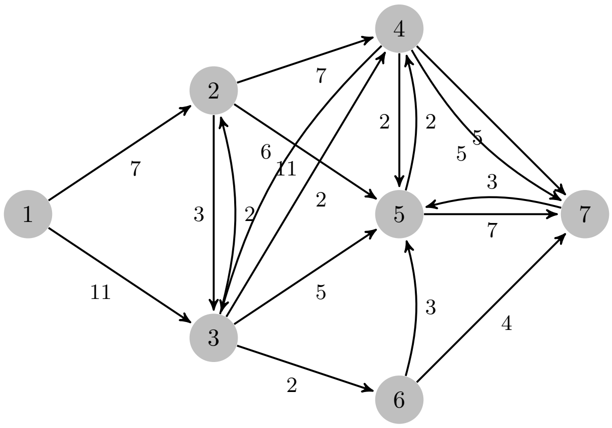


Figure 1:



footnotesize

```
\documentclass{standalone}
\usepackage{tikz}

%%% TIKZ STUFF %%%
\usetikzlibrary{arrows}
\tikzstyle{vertex}=[circle,fill=black!25,minimum size=20pt,inner sep=0pt]
\tikzstyle{selected vertex} = [vertex, fill=red!24]
\tikzstyle{edge} = [draw,thick,-]
\tikzstyle{arc} = [draw,thick,->,shorten >=1pt,>=stealth']
\tikzstyle{arcl} = [draw,thick,->,shorten >=1pt,>=stealth',bend left=25]
\tikzstyle{arcr} = [draw,thick,->,shorten >=1pt,>=stealth']
\tikzstyle{weight} = [font=\small]
\tikzstyle{selected edge} = [draw,line width=5pt,-,red!50]
\tikzstyle{ignored edge} = [draw,line width=5pt,-,black!20]
%%% TIKZ STUFF %%%


\begin{document}

\begin{tikzpicture}[scale=0.9, auto,swap]
    % First we draw the vertices
  \foreach \pos/\name in {{(0,3)/1}, {(3,1)/3}, {(3,5)/2},
                          {(6,0)/6}, {(6,3)/5}, {(6,6)/4}, {(9,3)/7}}
        \node[vertex] (\name) at \pos {$\name$};
    % Connect vertices with edges and draw weights
    \foreach \source/ \dest /\weight in {
      1/2/7, 1/3/11, 2/3/3, 3/6/2,
      2/4/7, 2/5/11, 3/4/2, 3/5/5,
      4/7/5, 4/5/2, 5/7/7, 6/7/4}
    \path[arcr] (\source) -- node[weight] {$\weight$} (\dest);
    \foreach \source/ \dest /\weight in {
      3/2/2, 4/3/6, 4/7/5, 5/4/2, 7/5/3, 6/5/3}
        \path[arcl,bend right] (\source) edge [bend right=15] node[weight] {$\weight$} (\dest);
\end{tikzpicture}
\end{document}
```

Figure 2: Find the maximum flow from 1 to 7. Numbers on arcs are capacity values. [In preparation for the exam, below the graph you find the excerpt of latex code to produce the picture. You can use it to experiment whether its use is fast enough for an exam session.]

Exercise 9

Consider the following IP problem:

$$\begin{aligned} \max \quad & 4x_1 + 7x_2 \\ s.t. \quad & x_1 + 3x_2 \leq 12 \\ & 4x_1 + 6x_2 \leq 27 \\ & 4x_1 + 2x_2 \leq 20 \\ & x_1, x_2 \geq 0, x_1, x_2 \in \mathbb{Z} \end{aligned} \tag{1}$$

Subtask a

Give a heuristic primal bound and describe how you determined it.

Subtask b

Write the LP relaxation (1lp) of (1) to obtain a dual bound. Explain the relation between the optimal solution of (1lp) and the optimal solution of (1).

Subtask c

Write the first simplex tableau of (1lp) and indicate which variables constitute a basic solution. Call s_1, s_2, s_3 the slack variables.

Subtask d

Explain which variable leaves the basis and which variable enters the basis in the first iteration of the simplex algorithm with largest coefficient pivot rule. Show that the answer would be the same if, instead, the largest increase pivot rule was used.

Subtask e

After a number of iterations the tableau is the following:

x1	x2	s1	s2	s3	-z	b
0	1	2/3	-1/6	0	0	7/2
1	0	-1	1/2	0	0	3/2
0	0	8/3	-5/3	1	0	7
0	0	-2/3	-5/6	0	1	-61/2

Argue that an optimal solution for (1lp) has been found and give for it the value of x_1 and x_2 together with its objective function value. Report the optimality gap for (1) at this stage.

Subtask e

Show how you can reconstruct the tableau at the previous point by just knowing that x_2, x_1 and s_3 are in basis and that:

$$A_B^{-1} = \begin{bmatrix} 2/3 & -1/6 & 0 \\ -1 & 1/2 & 0 \\ 8/3 & -5/3 & 1 \end{bmatrix}.$$

Subtask f

From the second row of the last tableau derive a Gomory cut and write it in the space of the original variables.

Argue shortly that the cut is a valid inequality for (1) and that it will make the current optimal solution of (1lp) infeasible.

Subtask g

Introduce the cut in the tableau and explain how the solution algorithm will continue. Indicate the new pivot and explain how you found it. (You do not need to carry out the simplex iteration.)

Subtask h

After the introduction of the cut the tableau of the optimal solution to the new LP problem is the following.

x1	x2	s1	s2	s3	s4	-z	b
0	1	2/3	0	0	-1/3	0	11/3
0	0	0	1	0	-2	0	1
0	0	8/3	0	1	-10/3	0	26/3
1	0	-1	0	0	1	0	1
0	0	-2/3	0	0	-5/3	1	-89/3

Explain how the solution process would continue from this stage by branch and bound. Define the next branching and indicate what can be done in each open node.

References

[DT97] George B. Dantzig and Mukund N. Thapa. *Linear Programming*. Springer, 1997.

[Goe11] Michel X. Goemans. Lecture notes on bipartite matching. http://www-math.mit.edu/~goemans/18433S11/matching-notes.pdf, 2011.