

DM545/DM871

Linear and Integer Programming

Introduction to Linear Programming Notation and Modeling

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Outline

1. Course Organization

2. Preliminaries

3. Introduction: Operations Research

Resource Allocation

Duality

Who is here?

82 in total registered in ItsLearning

DM545 (5 ECTS)

38, who??

- Math-economy
(Bachelor, 4th semester)
- Others?

Prerequisites

- Programming
- Linear Algebra

DM871 (5 ECTS)

44, who??

- Computer Science
(Master)
- Mathematics
(Bachelor/Master)
- Applied Mathematics
(Bachelor/Master)
- Data Science (Master)
- Others?

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Aims of the course

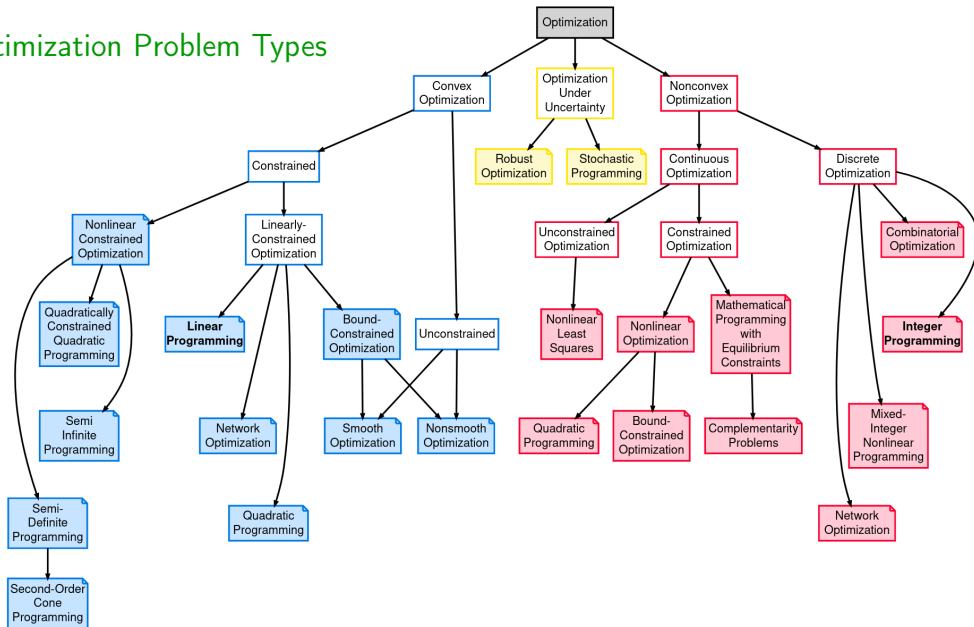
Learn about **mathematical optimization**:

- linear programming (continuous linear optimization)
- integer linear programming (discrete linear optimization)

Optimization is an important tool in **decision making** and in analyzing physical systems. In mathematical terms, an **optimization problem** is the problem of finding the **best solution** from the set of all **feasible solutions**.

The first step in the optimization process is constructing an appropriate **mathematical model**

Optimization Problem Types



Contents of the Course (aka Syllabus)

Linear Programming

- 1 Introduction - Linear Programming, Notation & Modeling
- 2 Linear Programming, Simplex Method
- 3 Exception Handling
- 4 Duality Theory
- 5 Sensitivity
- 6 Revised Simplex Method

Integer Linear Programming

- 7 Modeling Examples, Good Formulations, Relaxations
- 8 Well Solved Problems
- 9 Network Optimization Models (Max Flow, Min cost flow, Matching)
- 10 Cutting Planes & Branch and Bound
- 11 More on Modeling

Practical Information

Teacher: Marco Chiarandini (imada.sdu.dk/u/marco/)
Instructors: Matinus Herman Kiilerich Kristensen (M1),
Lena Gladbach (H1)

Schedule, alternative views:

- mitsdu.sdu.dk, SDU Mobile
- Official course description (læserplanen)
- ItsLearning
- <https://dm871.github.io/>

Schedule (8 weeks):

- Introductory classes: 28 hours (14 classes)
- Training classes: 16 hours (8 classes)

Communication Means

- ItsLearning \Leftrightarrow External Web Page
(link <https://dm871.github.io>)
- **Announcements** in ItsLearning
- Write to Marco (marco@imada.sdu.dk) or to instructor
- Ask peers

~> It is good to ask questions!!

~> Let me know if you think we should do things differently!

Sources — Reading Material

Main references:

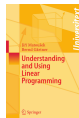
[LN] Lecture Notes (continously updated)

[F] M. Fischetti, [Introduction to Mathematical Optimization](#),
Independently published, 2019



Linear Programming:

[VA] R. Vanderbei. Linear Programming: Foundations and Extensions. Springer US, 2008



Integer Programming:

[Wo] L.A. Wolsey. Integer programming. John Wiley & Sons, New York, USA, 1998



Others

[HL] Frederick S Hillier and Gerald J Lieberman, Introduction to Operations Research, 9th edition, 2010

External Web Page is the main reference for list of contents (ie¹, syllabus, pensum).

It contains:

- slides
- list of topics and references
- exercises
- links
- tutorials for programming tasks

¹ie = id est, eg = exempli gratia, wrt = with respect to

- Two obligatory [24 h Take-Home Assignments](#), evaluation by internal censor
 - individual !!
 - exercises similar to previous 4 hour written exams
 - style: short answers about calculations and modeling. Differences between DM871 and DM545
 - (language: Danish and/or English)
- Final grade: overall evaluation but as starting point the average grade rounded up
- Tentative plan:
 - Test 1 (about weeks 5, 6, 7, 8) in week 9 or 10
 - Test 2 (about weeks 8, 9, 10, 11, 12) in week 13, Monday 25/3 or Tuesday 26/3 or week 14 Tuesday 2/4.

Which day? Which time range? (Pool coming in ItsLearning)

- Do the plus exercises in advance
- Prepare the starred exercises in advance to get out the most from the session
- Participate actively in class
- Try the others unsolved in class after the session
- Best if carried out in small groups
- (Starred) Exercises are examples of exam questions (but not only!)

Concepts from Linear Algebra

Linear Algebra:

manipulation of matrices and vectors with some theoretical background

Linear Algebra

- Matrices and vectors - Matrix algebra

- Dot (scalar, Euclidean inner) product

- Geometric insights

- Systems of Linear Equations - Row echelon form, Gaussian elimination

- Matrix inversion and determinants

- Rank and linear dependency

- gives you the ability to create new and useful artifacts
- allows you to have more control of your world as more and more of it becomes digital
- is just fun.

It can also help you to [understand math](#).

- listening to lectures
- watching an instructor work through a derivation
- working through numerical examples by hand
- learn [by doing](#), [interacting with Python](#).
- Python 3.8+
Pyomo, Python-MIP, PySCIPOpt: Python interfaces to SCIP Optimization Suite
(Commercial alternative Gurobi or Cplex \approx 100 000 Dkk)
- ipython, jupyter, jupyterLab (= interactive python)? Or Visual Code or Spyder3.

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Sets

- A **set** is a collection of objects. eg.: $A = \{1, 2, 3\}$
- $A = \{n \mid n \text{ is a whole number and } 1 \leq n \leq 3\}$
(\mid reads 'such that')
- $B = \{x \mid x \text{ is a student of this course}\}$
- $x \in A$
 x belongs to A
- set of no members: **empty set**, denoted \emptyset
- if a set S is a (**proper**) **subset** of a set T , we write $(S \subset T) \quad T \supseteq S$
 $\{1, 2, 5\} \subset \{1, 2, 4, 5, 6, 30\}$
- for two sets A and B , the **union** $A \cup B$ is $\{x \mid x \in A \text{ or } x \in B\}$
- for two sets A and B , the **intersection** $A \cap B$ is $\{x \mid x \in A \text{ and } x \in B\}$
 $A = \{1, 2, 3, 5\}$ and $B = \{2, 4, 5, 7\}$, then $A \cap B = \{2, 5\}$

Numbers

- set of real numbers: \mathbb{R}
- set of natural numbers: $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ (positive integers); \mathbb{N}_0 to include zero
- set of all integers: $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$; \mathbb{Z}_0^+ only positives and zero
- set of rational numbers: $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$
- set of complex numbers: \mathbb{C}
- absolute value (non-negative):

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0 \end{cases}$$

- the set \mathbb{R}^2 is the set of ordered pairs (x, y) of real numbers
(eg, coordinates of a point wrt a pair of axes, the [Cartesian plane](#))

Matrices and Vectors

- A **matrix** is a rectangular array of numbers or symbols. It can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- An $n \times 1$ matrix is a **column vector**, or simply a vector:

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- the set \mathbb{R}^n is the set of vectors $[x_1, x_2, \dots, x_n]^T$ of real numbers (eg, coordinates of a point wrt an n -dimensional space, the **Euclidean Space**)

- An **undirected graph** $G=(V,E)$ is a structure made of a pair of sets: V the set of **vertices** and $E \subseteq \{e = u, v \mid u, v \in V\}$ the set of **edges** (undirected links)
- A **directed graph (digraph)** $D = (V, A)$ is a pair of sets: V the set of **vertices** and $A \subseteq \{uv = (u, v) \mid u, v \in V\}$ the set of **arcs** (directed links)
- Graph drawing (2D embeddings)
- Labelled graphs and weighted graphs
- Walks, Paths, Cycles, Connectedness
- Problems on graphs: eg, determining intrinsic properties such as shortest **st**-path
- Matrix representations: **adjacency matrix**, **incidence matrix**

- adjacency set (neighborhood set) for $v \in V$: $N(v) = \{u \mid u \in V, uv \in E\}$; vertex degree (number of incident edges) $\delta(v)$; outdegree (number of outgoing edges), $\delta^+(v)$; indegree (number of incoming edges), $\delta^-(v)$;
- subgraph and induced subgraph $G' = G[V] \subseteq G$
- cut: set of edges that separates a connected graphs into two connected components $G[X]$, $G[Y]$: $C = \{xy \mid x \in X, y \in Y\}$
- bipartite graphs $G = (U, V, E)$, trees
- multi-graphs, hypergraphs, mixed graphs
- Graphs are the basic subject studied by graph theory. The word "graph" was first used in this sense by J. J. Sylvester in 1878 due to a direct relation between mathematics and chemical structure.

Elementary Algebra: the study of symbols and the rules for manipulating symbols. It differs from **arithmetic** in the use of abstractions, such as using letters to stand for numbers that are either unknown or allowed to take on many values

- collecting up terms: eg. $2a + 3b - a + 5b = a + 8b$
- multiplication of variables: eg:

$$a(-b) - 3ab + (-2a)(-4b) = -ab - 3ab + 8ab = 4ab$$

- expansion of bracketed terms: eg:

$$\begin{aligned} -(a - 2b) &= -a + 2b, \\ (2x - 3y)(x + 4y) &= 2x^2 - 3xy + 8xy - 12y^2 \\ &= 2x^2 + 5xy - 12y^2 \end{aligned}$$

- $a^r a^s = a^{r+s}$, $(a^r)^s = a^{rs}$, $a^{-n} = 1/a^n$,
 $a^{m/n} = (a^{1/n})^m$, $a^{1/n} = x \iff x^n = a$, $a^x = n \iff x = \log_a n$

- In Mathematics and Statistics, a **variable** is an alphabetic character representing a **value**, which is unknown. They are used in **symbolic** calculations.
Commonly given one-character names.
- in contrast, a **parameter** or **constant** or **given** is a known real number
- in contrast, in **Computer Science**, a **variable** is a storage location paired with an associated identifier, which contains a value that may be known or unknown.
Commonly given long, explanatory names.

- a **function** f on a set \mathcal{X} into a set \mathcal{Y} is a rule that assigns a **unique** element $f(x)$ in \mathcal{Y} to each element x in \mathcal{X} .

$$y = f(x)$$

y dependent
variable

x independent
variable

- a **linear function** has only sums and scalar multiplications, that is, for variable $x \in \mathbb{R}$ and scalars $a, b \in \mathbb{R}$:

$$f(x) := ax + b$$

(actually the one above is an affine function)

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What is Operations Research?

Operations Research (aka, Management Science, Analytics):
is the discipline that uses a **scientific approach to decision making**.

It seeks to determine how best to design and operate a system,
usually under conditions requiring the allocation of scarce resources,
by means of **mathematics** and **computer science**.

Quantitative methods for planning and analysis.

It encompasses a wide range of problem-solving techniques and methods applied in the pursuit of improved decision-making and efficiency:

- simulation,
- **mathematical optimization**,
- queueing theory and other stochastic-process models,
- Markov decision processes
- econometric methods,
- data envelopment analysis,
- neural networks,
- expert systems

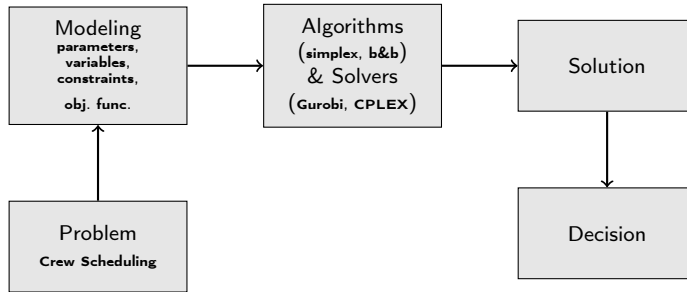
Some Examples ...

- Production Planning and Inventory Control
- Budget Investment
- Blending and Refining
- Manpower Planning
 - Crew Rostering (airline crew, rail crew, nurses)
- Packing Problems
 - Knapsack Problem
- Cutting Problems
 - Cutting Stock Problem
- Routing
 - Vehicle Routing Problem (trucks, planes, trains ...)
- Locational Decisions
 - Facility Location
- Scheduling/Timetabling
 - Examination timetabling/ train timetabling
- + many more

Common Characteristics

- Planning decisions must be made
- The problems relate to quantitative issues
 - Cheapest
 - Shortest route
 - Fewest number of people
- Not all plans are feasible - there are constraining rules
 - Limited amount of available resources
- It can be extremely difficult to figure out what to do

OR - The Process?



1. Observe the System
2. Formulate the Problem
3. Formulate Mathematical Model
4. Verify Model
5. Select Alternative
6. Show Results to Company
7. Implementation

Central Idea

Build a mathematical model describing exactly what one wants, and what the “rules of the game” are. However, **what is a mathematical model and how?**

- Find out exactly what the decision maker needs to know:
 - which investment?
 - which product mix?
 - which job j should a person i do?
- Define **Parameters**, that is the given, known elements of the problem.
- Define **Decision Variables** of suitable type (continuous, integer, binary) corresponding to the needs
- Formulate **Objective Function** computing the benefit/cost
- Formulate mathematical **Constraints** indicating the interplay between the different variables.

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Resource Allocation

In manufacturing industry, **factory planning**: find the best product mix.

Example

A factory makes two products **standard** and **deluxe**. Eg, yougurt, sleeping beds, etc.

A unit of **standard** gives a profit of 6(k) Dkk.

A unit of **deluxe** gives a profit of 8(k) Dkk.

The warming|grinding and cooling|polishing times in terms of hours per week for a **unit** of each type of product are given below:

	Standard	Deluxe
(Machine 1) Warming Grinding	5	10
(Machine 2) Cooling Polishing	4	4

Warming|Grinding capacity: 60 hours per week

Cooling|Polishing capacity: 40 hours per week

Q: How much of each product, **standard** and **deluxe**, should we produce to maximize the profit?

Mathematical Model

Decision Variables

$x_1 \geq 0$ units of product standard

$x_2 \geq 0$ units of product deluxe

Object Function

$\max 6x_1 + 8x_2$ maximize profit

Constraints

$5x_1 + 10x_2 \leq 60$ machine 1 capacity

$4x_1 + 4x_2 \leq 40$ machine 2 capacity

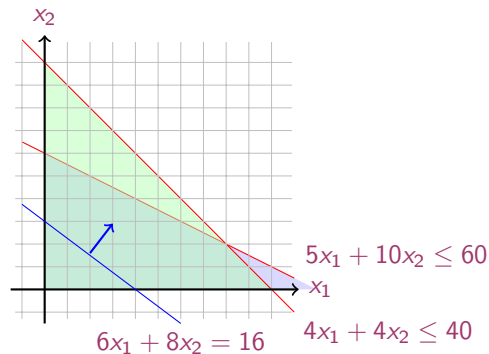
Mathematical Model

Machines/Materials A and B
Products 1 and 2

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

a_{ij}	1	2	b_i
A	5	10	60
B	4	4	40
c_j	6	8	

Graphical Representation:



Resource Allocation - General Model

Managing a production facility

$j = 1, 2, \dots, n$ products

$i = 1, 2, \dots, m$ materials

b_i units of raw material at disposal

a_{ij} units of raw material i to produce one unit of product j

σ_j market price of unit of j th product

ρ_i prevailing market value for material i

$c_j = \sigma_j - \sum_{i=1}^m \rho_i a_{ij}$ profit per unit of product j

x_j amount of product j to produce

$$\begin{aligned}
 &\max \quad c_1 x_1 + c_2 x_2 + c_3 x_3 + \dots + c_n x_n = z \\
 &\text{subject to} \quad a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \dots + a_{1n} x_n \leq b_1 \\
 &\quad \quad \quad a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \dots + a_{2n} x_n \leq b_2 \\
 &\quad \quad \quad \dots \\
 &\quad \quad \quad a_{m1} x_1 + a_{m2} x_2 + a_{m3} x_3 + \dots + a_{mn} x_n \leq b_m \\
 &\quad \quad \quad x_1, x_2, \dots, x_n \geq 0
 \end{aligned}$$

Notation

$$\begin{array}{ll}\max & c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n = z \\ \text{s.t.} & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n \leq b_2 \\ & \dots \\ & a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n \leq b_m \\ & x_1, x_2, \dots, x_n \geq 0\end{array}$$

$$\begin{array}{ll}\max & \sum_{j=1}^n c_jx_j \\ & \sum_{j=1}^n a_{ij}x_j \leq b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n\end{array}$$

In Matrix Form

$$\begin{aligned} \max \quad & c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n = z \\ \text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n \leq b_2 \\ & \dots \\ & a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n \leq b_m \\ & x_1, x_2, \dots, x_n \geq 0 \end{aligned}$$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{aligned} \max \quad & z = c^T x \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Our Numerical Example

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n \end{aligned}$$

$$\begin{aligned} \max \quad & c^T x \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

$$x \in \mathbb{R}^n, c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\max \quad \begin{bmatrix} 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2 \geq 0$$

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Resource Valuation problem: Determine the value of the raw materials in hand such that:
(i) it would be convenient selling and (ii) an outside company would be willing to buy them.

z_i value of a unit of raw material i

$\sum_{i=1}^m b_i z_i$ total expenses for buying (or opportunity cost, cost of having instead of selling)

ρ_i prevailing unit market value of material i

σ_j prevailing unit product price

Goal: for the outside company to minimize the total expenses;
(for the owing company the minimum amount of opportunity cost to accept for selling)

$$\min \sum_{i=1}^m b_i z_i \tag{1}$$

$$z_i \geq \rho_i, \quad i = 1 \dots m \tag{2}$$

$$\sum_{i=1}^m z_i a_{ij} \geq \sigma_j, \quad j = 1 \dots n \tag{3}$$

(2) otherwise selling to someone else and (3) otherwise not selling

Let

$$y_i = z_i - \rho_i$$

markup that the company would make by selling the raw material instead of producing.

$$\begin{aligned} \min \quad & \sum_{i=1}^m y_i b_i + \sum_i \cancel{\rho_i b_i} \\ & \sum_{i=1}^m y_i a_{ij} \geq c_j, \quad j = 1 \dots n \\ & y_i \geq 0, \quad i = 1 \dots m \end{aligned}$$

Dual Problem

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n \end{aligned}$$

Primal Problem