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Linear and Integer Programming

Linear Programming

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Outline

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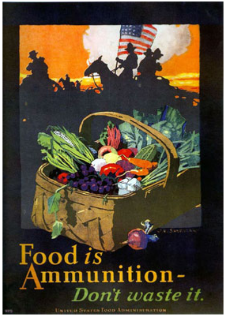
1. Introduction
   - Diet Problem
2. Solving LP Problems
   - Fourier-Motzkin method
3. Mathematical Programming
   - Definitions
   - Fundamental Theorem of LP
   - Gaussian Elimination

The Diet Problem (Blending Problems)

- Select a set of foods that will satisfy a set of daily nutritional requirements at minimum cost.
- Motivated in the 1930s and 1940s by US army.
- Formulated as a linear programming problem by George Stigler
- (programming intended as planning not computer code)



min cost/weight
subject to nutrition requirements:
eat enough but not too much of Vitamin A
eat enough but not too much of Sodium
eat enough but not too much of Calories
...

The Diet Problem

Suppose there are:

- 3 foods available: corn, milk, and bread, and
- there are restrictions on the number of calories (between 2000 and 2250) and the amount of Vitamin A (between 5,000 and 50,000)

Food	Cost per serving	Vitamin A	Calories
Corn	\$0.18	107	72
2% Milk	\$0.23	500	121
Wheat Bread	\$0.05	0	65

The Mathematical Model

Parameters (given data)

F := set of foods
N := set of nutrients

a_{ij} := amount of nutrient i in food j, $\forall i \in N$, $\forall j \in F$
c_j := cost per serving of food $j, \forall j \in F$
$F_{min,j}$:= minimum number of required servings of food $j, \forall j \in F$
$F_{max,j}$:= maximum allowable number of servings of food $j, \forall j \in F$
$N_{min,i}$:= minimum required level of nutrient $i, \forall i \in N$
$N_{max,i}$:= maximum allowable level of nutrient $i, \forall i \in N$

Decision Variables

x_j := number of servings of food j to purchase/consume, $\forall j \in F$

The Mathematical Model

Objective Function: Minimize the total cost of the food

$$\text{Minimize} \sum_{j \in F} c_j x_j$$

Constraint Set 1: For each nutrient $i \in N$, at least meet the minimum required level

$$\sum_{j \in F} a_{ij} x_j \geq N_{min,i}, \qquad \forall i \in N$$

Constraint Set 2: For each nutrient $i \in N$, do not exceed the maximum allowable level.

$$\sum_{j \in F} a_{ij} x_j \leq N_{max,i}, \qquad \forall i \in N$$

Constraint Set 3: For each food $j \in F$, select at least the minimum required number of servings

$$x_j \geq F_{min,j}, \qquad \forall j \in F$$

Constraint Set 4: For each food $j \in F$, do not exceed the maximum allowable number of servings.

$$x_j \leq F_{max,j}, \qquad \forall j \in F$$

The Mathematical Model

system of equalities and inequalities

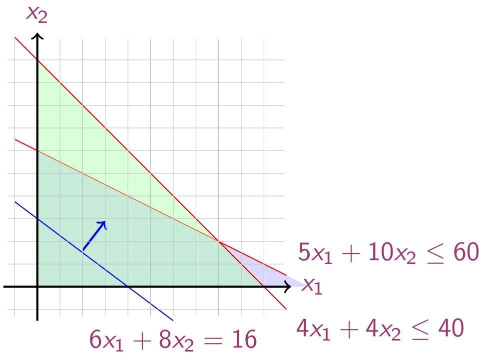
$$
\begin{aligned}
\min \quad & \sum_{j \in F} c_j x_j \\
\sum_{j \in F} a_{ij} x_j & \geq N_{min,i}, \qquad \forall i \in N \\
\sum_{j \in F} a_{ij} x_j & \leq N_{max,i}, \qquad \forall i \in N \\
x_j & \geq F_{min,j}, \qquad \forall j \in F \\
x_j & \leq F_{max,j}, \qquad \forall j \in F
\end{aligned}
$$

Mathematical Model

Machines/Materials A and B
Products 1 and 2

$$
\begin{aligned}
\max\ 6x_1 + \ 8x_2 & \\
5x_1 + 10x_2 &\leq 60 \\
4x_1 + \ 4x_2 &\leq 40 \\
x_1 &\geq \ 0 \\
x_2 &\geq \ 0
\end{aligned}
$$

Graphical Representation:



In Matrix Form

$$
\begin{array}{rl}
\max & c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n = z \\
\text{s.t.} & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n \leq b_1 \\
& a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n \leq b_2 \\
& \ldots \\
& a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n \leq b_m \\
& x_1, x_2, \ldots, x_n \geq 0
\end{array}
$$

$$
\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

$$
\begin{array}{rl}
\max & z = \mathbf{c}^T\mathbf{x} \\
& A\mathbf{x} \leq \mathbf{b} \\
& \mathbf{x} \geq 0
\end{array}
$$

Linear Programming

Abstract mathematical model:
Parameters, Decision Variables, Objective, Constraints
(+ Domains & Quantifiers)

The Syntax of a Linear Programming Problem

objective func.	$\max / \min\ c^T x$	$c \in \mathbb{R}^n$
constraints	s.t. $Ax \gtreqless b$	$A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$
	$x \geq 0$	$x \in \mathbb{R}^n, 0 \in \mathbb{R}^n$

Essential features: continuity, linearity (proportionality and additivity), certainty of parameters

- Any vector $x \in \mathbb{R}^n$ satisfying all constraints is a feasible solution.
- Each $x^* \in \mathbb{R}^n$ that gives the best possible value for $c^T x$ among all feasible x is an optimal solution or optimum
- The value $c^T x^*$ is the optimum value

Diet Problem — History

- The linear programming model consisted of 9 equations in 77 variables
- In 1944, Stigler guessed an near-optimal solution using a heuristic method
- In 1947, the National Bureau of Standards used the newly developed simplex method to solve Stigler's model.
  It took 9 clerks using hand-operated desk calculators 120 man days to solve for the optimal solution
- The original instance: https://developers.google.cn/optimization/lp/stigler_diet

AMPL Model

```
# diet.mod
set NUTR;
set FOOD;

param cost {FOOD} > 0;
param f_min {FOOD} >= 0;
param f_max { j in FOOD} >= f_min[j];
param n_min { NUTR } >= 0;
param n_max {i in NUTR } >= n_min[i];
param amt {NUTR,FOOD} >= 0;

var Buy { j in FOOD} >= f_min[j], <= f_max[j]

minimize total_cost: sum { j in FOOD } cost [j] * Buy[j];
subject to diet { i in NUTR }:
        n_min[i] <= sum {j in FOOD} amt[i,j] * Buy[j] <= n_max[i];
```

AMPL Model

```
# diet.dat
data;

set NUTR := A B1 B2 C ;
set FOOD := BEEF CHK FISH HAM MCH MTL SPG TUR;

param: cost f_min f_max :=
  BEEF 3.19 0 100
  CHK 2.59 0 100
  FISH 2.29 0 100
  HAM 2.89 0 100
  MCH 1.89 0 100
  MTL 1.99 0 100
  SPG 1.99 0 100
  TUR 2.49 0 100 ;

param: n_min n_max :=
   A 700 10000
   C 700 10000
   B1 700 10000
   B2 700 10000 ;
# %
```

```
param amt (tr):
          A C B1 B2 :=
   BEEF 60 20 10 15
   CHK 8 0 20 20
   FISH 8 10 15 10
   HAM 40 40 35 10
   MCH 15 35 15 15
   MTL 70 30 15 15
   SPG 25 50 25 15
   TUR 60 20 15 10 ;
```

Python Script

Model

```python
# Model diet.py
m = Model("diet")

# Create decision variables for the foods to buy
buy = {}
for f in foods:
    buy[f] = m.addVar(obj=cost[f], name=f)

# Nutrition constraints
for c in categories:
    m.addConstr(
      quicksum(nutritionValues[f,c] * buy[f] for f in foods) <= maxNutrition[c], name=c+'max')
    m.addConstr(
      quicksum(nutritionValues[f,c] * buy[f] for f in foods) >= minNutrition[c], name=c+'min')

# Solve
m.optimize()
```

Python Script
Data

```
from gurobipy import *

categories, minNutrition, maxNutrition = multidict({
  'calories': [1800, 2200],
  'protein': [91, GRB.INFINITY],
  'fat': [0, 65],
  'sodium': [0, 1779] })

foods, cost = multidict({
  'hamburger': 2.49,
  'chicken': 2.89,
  'hot dog': 1.50,
  'fries': 1.89,
  'macaroni': 2.09,
  'pizza': 1.99,
  'salad': 2.49,
  'milk': 0.89,
  'ice cream': 1.59 })
```

```
# Nutrition values for the foods
nutritionValues = {
  ('hamburger', 'calories'): 410,
  ('hamburger', 'protein'): 24,
  ('hamburger', 'fat'): 26,
  ('hamburger', 'sodium'): 730,
  ('chicken', 'calories'): 420,
  ('chicken', 'protein'): 32,
  ('chicken', 'fat'): 10,
  ('chicken', 'sodium'): 1190,
  ('hot dog', 'calories'): 560,
  ('hot dog', 'protein'): 20,
  ('hot dog', 'fat'): 32,
  ('hot dog', 'sodium'): 1800,
  ('fries', 'calories'): 380,
  ('fries', 'protein'): 4,
  ('fries', 'fat'): 19,
  ('fries', 'sodium'): 270,
  ('macaroni', 'calories'): 320,
  ('macaroni', 'protein'): 12,
  ('macaroni', 'fat'): 10,
  ('macaroni', 'sodium'): 930,
  ('pizza', 'calories'): 320,
  ('pizza', 'protein'): 15,
  ...
```

Outline

History of Linear Programming (LP)

System of linear equations

⇝ It is impossible to find out who knew what when first.
Just two “references”:

- Egyptians and Babylonians considered about 2000 B.C. the solution of special linear equations. But, of course, they described examples and did not describe the methods in "today’s style".
- What we call “Gaussian elimination” today has been explicitly described in Chinese “Nine Books of Arithmetic” which is a compendium written in the period 2000 B.C. to A.D. 9, but the methods were probably known long before that.
- Gauss, by the way, never described “Gaussian elimination”. He just used it and stated that the linear equations he used can be solved “per eliminationem vulgarem”

History of Linear Programming (LP)

- Origins date back to Newton, Leibnitz, Lagrange, etc.
- In 1827, Fourier described a variable elimination method for systems of linear inequalities, today often called Fourier-Motzkin elimination (Motzkin, 1937). It can be turned into an LP solver but inefficient.
- In 1932, Leontief (1905-1999) Input-Output model to represent interdependencies between branches of a national economy (1976 Nobel prize)
- In 1939, Kantorovich (1912-1986): Foundations of linear programming (Nobel prize in economics with Koopmans on LP, 1975) on Optimal use of scarce resources: foundation and economic interpretation of LP
- The math subfield of Linear Programming was created by George Dantzig, John von Neumann (Princeton), and Leonid Kantorovich in the 1940s.
- In 1947, Dantzig (1914-2005) invented the (primal) simplex algorithm working for the US Air Force at the Pentagon. (program=plan)

History of LP (cntd)

- In 1954, Lemke: dual simplex algorithm,
- In 1954, Dantzig and Orchard Hays: revised simplex algorithm
- In 1970, Victor Klee and George Minty created an example that showed that the classical simplex algorithm has exponential worst-case behavior.
- In 1979, L. Khachain found a new efficient algorithm for linear programming. It was terribly slow. (Ellipsoid method)
- In 1984, Karmarkar discovered yet another new efficient algorithm for linear programming. It proved to be a strong competitor for the simplex method. (Interior point method)

History of Optimization

- In 1951, Nonlinear Programming began with the Karush-Kuhn-Tucker Conditions
- In 1952, Commercial Applications and Software began
- In 1950s, Network Flow Theory began with the work of Ford and Fulkerson.
- In 1955, Stochastic Programming began
- In 1958, Integer Programming began by R. E. Gomory.
- In 1962, Complementary Pivot Theory

Outline

Fourier Motzkin elimination method

Has $Ax \leq b$ a solution? (Assumption: $A \in \mathbb{Q}^{m \times n}, \mathrm{b} \in \mathbb{Q}^n$)
Idea:

1. transform the system into another by eliminating some variables such that the two systems have the same solutions over the remaining variables.
2. reduce to a system of constant inequalities that can be easily decided

Let x_r be the variable to eliminate
Let $M = \{1 \ldots m\}$ indices of the constraints
For a variable j let's partition the rows of the matrix in

$$N = \{i \in M \mid a_{ij} < 0\}$$
$$Z = \{i \in M \mid a_{ij} = 0\}$$
$$P = \{i \in M \mid a_{ij} > 0\}$$

$$\begin{cases} x_r \geq b'_{ir} - \sum_{k=1}^{r-1} a'_{ik} x_k, & a_{ir} < 0 \\ x_r \leq b'_{ir} - \sum_{k=1}^{r-1} a'_{ik} x_k, & a_{ir} > 0 \\ \text{all other constraints} & i \in Z \end{cases} \qquad \begin{cases} x_r \geq A_i(x_1, \ldots, x_{r-1}), & i \in N \\ x_r \leq B_i(x_1, \ldots, x_{r-1}), & i \in P \\ \text{all other constraints} & i \in Z \end{cases}$$

Hence the original system is equivalent to

$$\begin{cases} \max\{A_i(x_1, \ldots, x_{r-1}), i \in N\} \leq x_r \leq \min\{B_i(x_1, \ldots, x_{r-1}), i \in P\} \\ \text{all other constraints} \quad i \in Z \end{cases}$$

which is equivalent to

$$\begin{cases} A_i(x_1, \ldots, x_{r-1}) \leq B_j(x_1, \ldots, x_{r-1}) & i \in N, j \in P \\ \text{all other constraints} & i \in Z \end{cases}$$

we eliminated x_r but:

$$\begin{cases} |N| \cdot |P| \text{ inequalities} \\ |Z| \text{ inequalities} \end{cases}$$

after d iterations if $|P| = |N| = m/2$ exponential growth: $(1/4^d)(m/2)^{2^d}$

Example

$$\begin{array}{rcrl}
-7x_1 & + & 6x_2 & \leq 25 \\
x_1 & - & 5x_2 & \leq 1 \\
x_1 & & & \leq 7 \\
-x_1 & + & 2x_2 & \leq 12 \\
-x_1 & - & 3x_2 & \leq 1 \\
2x_1 & - & x_2 & \leq 10
\end{array}$$

x_2 variable to eliminate
$N = \{2, 5, 6\}, Z = \{3\}, P = \{1, 4\}$
$|Z \cup (N \times P)| = 7$ constraints

By adding one variable and one inequality, Fourier-Motzkin elimination can be turned into an LP solver.

Outline

Outline

Definitions

- $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ closed interval
  $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ open interval

- column vector and matrices
  scalar product: $\mathrm{y}^T\mathrm{x} = \sum_{i=1}^{n} y_i x_i$

- $A\mathrm{x}$ column vector combination of the columns of A;
  $\mathrm{u}^T A$ row vector combination of the rows of A

- linear combination

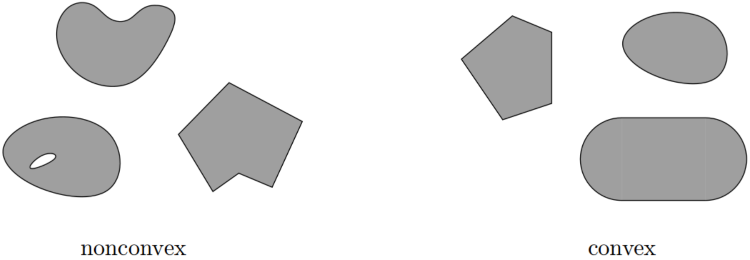
$$\begin{aligned} \mathrm{v}_1, \mathrm{v}_2 \ldots, \mathrm{v}_k &\in \mathbb{R}^n \\ \boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_k]^T &\in \mathbb{R}^k \end{aligned} \qquad \mathrm{x} = \lambda_1 \mathrm{v}_1 + \cdots + \lambda_k \mathrm{v}_k = \sum_{i=1}^{k} \lambda_i \mathrm{v}_i$$

  moreover:

$\boldsymbol{\lambda} \geq 0$	conic combination	
$\boldsymbol{\lambda}^T 1 = 1$	affine combination	$\left(\sum_{i=1}^{k} \lambda_i = 1\right)$
$\boldsymbol{\lambda} \geq 0$ and $\boldsymbol{\lambda}^T 1 = 1$	convex combination	

Definitions

- set S is linear (affine) independent if no element of it can be expressed as linear combination of the others
  Eg: $S \subseteq \mathbb{R}^n \implies$ max n lin. indep. (max $n+1$ aff. indep.)
- convex set: if $x, y \in S$ and $0 \leq \lambda \leq 1$ then $\lambda x + (1-\lambda)y \in S$



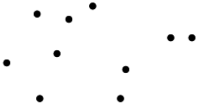
- convex function if its epigraph $\{(x,y) \in \mathbb{R}^2 : y \geq f(x)\}$ is a convex set or $f : X \to \mathbb{R}$ and if $\forall x, y \in X, \lambda \in [0,1]$ it holds that $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$

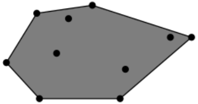
Definitions

- For a set of points $S \subseteq \mathbb{R}^n$

$$\begin{array}{rl} \text{lin}(S) & \text{linear hull (span)} \\ \text{cone}(S) & \text{conic hull} \\ \text{aff}(S) & \text{affine hull} \\ \text{conv}(S) & \text{convex hull} \end{array}$$

X





the convex hull of X

$$\boxed{\text{conv}(X) = \{\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n \mid x_i \in X,\ \lambda_1, \ldots, \lambda_n \geq 0 \ \text{ and } \sum_i \lambda_i = 1\}}$$

Definitions

- rank of a matrix for columns (= for rows)
  if (m, n)-matrix has rank $= \min\{m, n\}$ then the matrix is full rank
  if (n, n)-matrix is full rank then it is regular and admits an inverse

- $G \subseteq \mathbb{R}^n$ is an hyperplane if $\exists\, \mathrm{a} \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$:

$$G = \{\mathrm{x} \in \mathbb{R}^n \mid \mathrm{a}^T \mathrm{x} = \alpha\}$$

- $H \subseteq \mathbb{R}^n$ is an halfspace if $\exists\, \mathrm{a} \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$:

$$H = \{\mathrm{x} \in \mathbb{R}^n \mid \mathrm{a}^T \mathrm{x} \leq \alpha\}$$

  ($\mathrm{a}^T \mathrm{x} = \alpha$ is a supporting hyperplane of H)

Definitions

- a set $S \subset \mathbb{R}^n$ is a polyhedron if $\exists\, m \in \mathbb{Z}^+, A \in \mathbb{R}^{m\times n}, \mathrm{b} \in \mathbb{R}^m$:

  $$P = \{\mathrm{x} \in \mathbb{R}^n \mid A\mathrm{x} \leq \mathrm{b}\} = \bigcap_{i=1}^{m} \{\mathrm{x} \in \mathbb{R}^n \mid \mathrm{a}_{i,\cdot}^T \mathrm{x} \leq b_i\}$$

  i.e., a polyhedron $P \neq \mathbb{R}^n$ is determined by finitely many halfspaces

- a polyhedron P is a polytope if it is bounded: $\exists\, B \in \mathbb{R}, B > 0$:

  $$P \subseteq \{\mathrm{x} \in \mathbb{R}^n \mid \|\mathrm{x}\| \leq B\}$$

  ($\|\mathrm{x}\| = \sqrt{\sum_{i=1}^n x_i^2}$ is the Euclidean norm of the vector $\mathrm{x} \in \mathbb{R}$)

- a point x of a polyhedron P is said to be an extreme point or a vertex of P if it cannot be expressed as a strict convex combination of other two points of the polyhedron, i.e., if there exist no $\mathrm{y}, \mathrm{z} \in P$, $\mathrm{y} \neq \mathrm{z}$ and $\lambda \in (0, 1)$ such that $\mathrm{x} = \lambda\mathrm{y} + (1-\lambda)\mathrm{z}$

- every point of a polytope can be obtained as the convex combination of its vertices. (Minkowski-Weyl Theorem)

Definitions

- If A and b are made of rational numbers, $P = \{\mathrm{x} \in \mathbb{R}^n \mid A\mathrm{x} \leq \mathrm{b}\}$ is a rational polyhedron
- General optimization problem: $\max\{\varphi(\mathrm{x}) \mid \mathrm{x} \in F\}$, $\quad F$ is feasible region for x
- Note: if F is open, eg, $x < 5$ then: $\sup\{x \mid x < 5\}$
  sumpreum: least element of $\mathbb{R}$ greater or equal than any element in F
- $\arg\min\{f(i) \mid i \in I\}$ argument $i^* \in I$ such that $f(i^*) = \min\{f(i) \mid i \in I\}$

Definitions

- The inequality denoted by (a, α) is called a valid inequality for P if $ax \leq \alpha, \forall x \in P$.
  Note that (a, α) is a valid inequality if and only if P lies in the half-space $\{x \in \mathbb{R}^n \mid ax \leq \alpha\}$.
- A face of P is $F = \{x \in P \mid ax = \alpha\}$ where (a, α) is a valid inequality for P. Hence, it is the intersection of P with the hyperplane of a valid inequality. It is said to be proper if $F \neq \emptyset$ and $F \neq P$.
- If $F \neq \emptyset$ we say that the corresponding hyperplane supports P.
  If c is a non zero vector for which $\delta = \max\{c^T x \mid x \in P\}$ is finite,
  then the set $\{x \mid c^T x = \delta\}$ is called supporting hyperplane.
- A point x for which $\{x\}$ is a face is called a vertex of P and also a basic solution of $Ax \leq b$ ($0\,\text{dim}$ face)
- A facet is a maximal face distinct from P
  $cx \leq d$ is facet defining if $cx = d$ is a supporting hyperplane of P of $n - 1\,\text{dim}$

Linear Programming Problem

Input: a matrix $A \in \mathbb{R}^{m \times n}$ and column vectors $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

Task:

1. decide that $\{x \in \mathbb{R}^n; Ax \leq b\}$ is empty (prob. infeasible), or
2. find a column vector $x \in \mathbb{R}^n$ such that $Ax \leq b$ and $c^T x$ is max, or
3. decide that for all $\alpha \in \mathbb{R}$ there is an $x \in \mathbb{R}^n$ with $Ax \leq b$ and $c^T x > \alpha$ (prob. unbounded)

1. $F = \emptyset$
2. $F \neq \emptyset$ and $\exists$ solution
    1. one solution
    2. infinite solutions
3. $F \neq \emptyset$ and $\nexists$ solution

Outline

Fundamental Theorem of LP

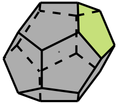
Theorem (Fundamental Theorem of Linear Programming)

Given:

$$\min\{c^T x \mid x \in P\} \text{ where } P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

If P is a bounded polyhedron and not empty and x^* is an optimal solution to the problem, then:

- *x^* is an extreme point (vertex) of P, or*
- *x^* lies on a face $F \subset P$ of optimal solutions*



Proof idea:

- assume x^* not a vertex of P then $\exists$ a ball around it still in P. Show that a point in the ball has better cost
- if x^* is not a vertex then it is a convex combination of vertices. Show that all points are also optimal.

Implications:

- the optimal solution is at the intersection of supporting hyperplanes.
- hence finitely many possibilities
- solution method: write all inequalities as equalities and solve all $\binom{m}{n}$ systems of linear equalities (n # variables, m # equality constraints)
- for each point we then need to check if feasible and if best in cost.
- each system is solved by Gaussian elimination
- Stirling approximation:

$$\binom{2m}{m} \approx \frac{4^m}{\sqrt{\pi m}} \text{ as } m \to \infty$$

Simplex Method

1. find a solution that is at the intersection of some n hyperplanes
2. try systematically to produce the other points by exchanging one hyperplane with another
3. check optimality, proof provided by duality theory

Outline

Gaussian Elimination

1. Forward elimination
   reduces the system to row echelon form by elementary row operations
   - multiply a row by a non-zero constant
   - interchange two rows
   - add a multiple of one row to another

   (or LU decomposition)

2. Back substitution (or reduced row echelon form - RREF)

Example

$$
\begin{array}{rcl}
2x + y - z = 8 & & (R1) \\
-3x - y + 2z = -11 & & (R2) \\
-2x + y + 2z = -3 & & (R3)
\end{array}
$$

```
|----+----+----+----+-----|
| R1 |  2 |  1 | -1 |   8 |
| R2 | -3 | -1 |  2 | -11 |
| R3 | -2 |  1 |  2 |  -3 |
|----+----+----+----+-----|
```

$$
\begin{array}{rcl}
2x + y - z = 8 & & (R1) \\
+ \frac{1}{2}y + \frac{1}{2}z = 1 & & (R2) \\
+ 2y + 1z = 5 & & (R3)
\end{array}
$$

```
|---------------+---+-----+------+---|
| R1'=1/2 R1    | 1 | 1/2 | -1/2 | 4 |
| R2'=R2+3/2xR1 | 0 | 1/2 | 1/2  | 1 |
| R3'=R3+R1     | 0 | 2   | 1    | 5 |
|---------------+---+-----+------+---|
```

$$
\begin{array}{rcl}
2x + y - z = 8 & & (R1) \\
+ \frac{1}{2}y + \frac{1}{2}z = 1 & & (R2) \\
- z = 1 & & (R3)
\end{array}
$$

```
|-------------+---+-----+------+---|
| R1'=R1      | 1 | 1/2 | -1/2 | 4 |
| R2'=2 R2    | 0 |   1 |    1 | 2 |
| R3'=R3-4xR2 | 0 |   0 |   -1 | 1 |
|-------------+---+-----+------+---|
```

$$
\begin{array}{rcl}
2x + y - z = 8 & & (R1) \\
+ \frac{1}{2}y + \frac{1}{2}z = 1 & & (R2) \\
- z = 1 & & (R3)
\end{array}
$$

```
|---------------+---+-----+---+-----|
| R1'=R1-1/2xR3 | 1 | 1/2 | 0 | 7/2 |
| R2'=R2+R3     | 0 |   1 | 0 |   3 |
| R3'=-R3       | 0 |   0 | 1 |  -1 |
|---------------+---+-----+---+-----|
```

$$
\begin{array}{rcl}
x = 2 & & (R1) \\
y = 3 & & (R2) \\
z = -1 & & (R3)
\end{array}
$$

```
|---------------+---+---+---+----+
| R1'=R1-1/2xR2 | 1 | 0 | 0 |  2 | => x=2
| R2'=R2        | 0 | 1 | 0 |  3 | => y=3
| R3'=R3        | 0 | 0 | 1 | -1 | => z=-1
|---------------+---+---+---+----+
```

LU Factorization

$$\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \\ -3 \end{bmatrix}$$

$$A\mathrm{x} = \mathrm{b}$$

$$\mathrm{x} = A^{-1}\mathrm{b}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$A = PLU$$

$$\mathrm{x} = A^{-1}\mathrm{b} = U^{-1}L^{-1}P^{T}\mathrm{b}$$

$$\mathrm{z}_1 = P^{T}\mathrm{b}, \quad \mathrm{z}_2 = L^{-1}\mathrm{z}_1, \quad \mathrm{x} = U^{-1}\mathrm{z}_2$$

In Python

```
In [1]: import scipy as sc

In [2]: A = sc.array([[2,1,−1],[−3,−1,2],[−2,1,2]])

In [3]: from scipy import linalg as sl

In [4]: P,L,U = sl.lu(A)

In [5]: print(P,L,U)
        [[0. 0. 1.]
         [1. 0. 0.]
         [0. 1. 0.]]
        [[ 1. 0. 0. ]
         [ 0.66666667 1. 0. ]
         [−0.66666667 0.2 1. ]]
        [[−3. −1. 2. ]
         [ 0. 1.66666667 0.66666667]
         [ 0. 0. 0.2 ]]
```

Polynomial time $O(n^2m)$ but needs to guarantee that all the numbers during the run can be represented by polynomially bounded bits

Summary