DM545/DM871

Linear and Integer Programming

Lecture

Cutting Planes

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Outline

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Valid Inequalities

- IP: $z = \max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{x} \in X\},\ X = \{\boldsymbol{x} : A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \in \mathbb{Z}_+^n\}$
- Proposition: $\text{conv}(X) = \{\boldsymbol{x} : \tilde{A}\boldsymbol{x} \leq \tilde{\boldsymbol{b}}, \boldsymbol{x} \geq 0\}$ is a polyhedron
- LP: $z = \max\{\boldsymbol{c}^T\boldsymbol{x} : \tilde{A}\boldsymbol{x} \leq \tilde{\boldsymbol{b}}, \boldsymbol{x} \geq 0\}$ would be the best formulation
- $\tilde{\boldsymbol{a}}\boldsymbol{x} \leq \tilde{\boldsymbol{b}}$ facet defining inequalities
- Key idea: try to approximate the best formulation.

Definition (Valid inequalities)

$\boldsymbol{a}\boldsymbol{x} \leq \boldsymbol{b}$ is a valid inequality for $X \subseteq \mathbb{R}^n$ if $\boldsymbol{a}\boldsymbol{x} \leq \boldsymbol{b}\ \forall \boldsymbol{x} \in X$

Which are useful inequalities? and how can we find them? How can we use them?

Example: Pre-processing

- $X = \{(x, y) : x \leq 999y; \quad 0 \leq x \leq 5, \quad y \in \mathbb{B}^1\}$

$$x \leq 5y$$

- $X = \{x \in \mathbb{Z}^4_+ : 13x_1 + 20x_2 + 11x_3 + 6x_4 \geq 72\}$

$$2x_1 + 2x_2 + x_3 + x_4 \geq \frac{13}{11}x_1 + \frac{20}{11}x_2 + x_3 + \frac{6}{11}x_4 \geq \frac{72}{11} = 6 + \frac{6}{11}$$

$$2x_1 + 2x_2 + x_3 + x_4 \geq 7$$

- Capacitated facility location:

$$\sum_{i \in M} x_{ij} \leq b_j y_j \quad \forall j \in N \qquad\qquad x_{ij} \leq b_j y_j$$

$$\sum_{j \in N} x_{ij} = a_i \quad \forall i \in M \qquad\qquad x_{ij} \leq a_i$$

$$x_{ij} \geq 0, \quad y_j \in B^n \qquad\qquad x_{ij} \leq \min\{a_i, b_j\} y_j$$

Converting Weak to Strong MIP Formulations

Strong formulations $\equiv$ better, tighter formulations

Detection possible from the log output of a solver.

Possible actions:

1. Add cuts to existing models
   - Combining constraints
   - Using a graph representation (clique cuts)
   - Using a disjunctive approach

   $\rightsquigarrow$ Many found automatically by the solver in pre-solver phase

2. (Change the model)

3. (Change the algorithm, eg, column generation)

(Lazy) constraints $\neq$ cuts

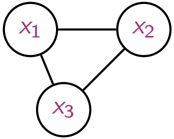
Add cuts to the existing model

$$
\begin{aligned}
\text{maximize } & x_1 + x_2 + x_3 \\
\text{subject to } & x_1 + x_2 \leq 1 \\
& x_2 + x_3 \leq 1 \\
& x_1 + x_3 \leq 1 \\
& x_i \in \{0, 1\} \qquad i = 1, 2, 3
\end{aligned}
$$

Combine and round constraints:

$$
\begin{aligned}
2x_1 + 2x_2 + 2x_3 &\leq 3 \\
x_1 + x_2 + x_3 &\leq \frac{3}{2} \\
x_1 + x_2 + x_3 &\leq 1
\end{aligned}
$$

Create a conflict graph; at most one binary in a clique can be 1



$$x_1 + x_2 + x_3 \leq 1$$

Chvátal-Gomory cuts

- $X \in P \cap \mathbb{Z}^n_+, \quad P = \{\boldsymbol{x} \in \mathbb{R}^n_+ : A\boldsymbol{x} \leq \boldsymbol{b}\}, \quad A \in \mathbb{R}^{m \times n}$
- $\boldsymbol{u} \in \mathbb{R}^m_+$, $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots \boldsymbol{a}_n\}$ columns of A

CG procedure to construct valid inequalities

1) $\displaystyle\sum_{j=1}^{n} \boldsymbol{u}^T \boldsymbol{a}_j x_j \leq \boldsymbol{u}^T \boldsymbol{b}$ valid: $\boldsymbol{u} \geq 0$

2) $\displaystyle\sum_{j=1}^{n} \lfloor \boldsymbol{u}^T \boldsymbol{a}_j \rfloor x_j \leq \boldsymbol{u}^T \boldsymbol{b}$ valid: $\boldsymbol{x} \geq 0$ and $\sum \lfloor \boldsymbol{u}^T \boldsymbol{a}_j \rfloor x_j \leq \sum \boldsymbol{u}^T \boldsymbol{a}_j x_j$

3) $\displaystyle\sum_{j=1}^{n} \lfloor \boldsymbol{u}^T \boldsymbol{a}_j \rfloor x_j \leq \lfloor \boldsymbol{u}^T \boldsymbol{b} \rfloor$ valid for X since $\boldsymbol{x} \in \mathbb{Z}^n$

Theorem

by applying this CG procedure a finite number of times every valid inequality for X can be obtained

However not all the constraints generated by $\boldsymbol{u} \in \mathbb{R}^m_+$ are tightenings.

Cutting Plane Algorithms

- $X \in P \cap \mathbb{Z}^n_+$
- a family of valid inequalities $\mathcal{F} : \boldsymbol{a}^T\boldsymbol{x} \leq b, (\boldsymbol{a}, b) \in \mathcal{F}$ for X
- we do not find them all a priori, only interested in those close to optimum

Cutting Plane Algorithm

Init.: $t = 0, P^0 = P$

Iter. t: Solve $\bar{z}^t = \max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{x} \in P^t\}$
let $\boldsymbol{x}^t$ be an optimal solution
if $\boldsymbol{x}^t \in \mathbb{Z}^n$ stop, $\boldsymbol{x}^t$ is opt to the IP
if $\boldsymbol{x}^t \notin \mathbb{Z}^n$ solve separation problem for $\boldsymbol{x}^t$ and $\mathcal{F}$
if $(\boldsymbol{a}^t, b^t)$ is found with $\boldsymbol{a}^t\boldsymbol{x}^t > b^t$ that cuts off x^t

$$P^{t+1} = P \cap \{\boldsymbol{x} : \boldsymbol{a}^i\boldsymbol{x} \leq b^i, i = 1, \ldots, t\}$$

else stop (P^t is in any case an improved formulation)

Gomory's fractional cutting plane algorithm

Cutting plane algorithm + Chvátal-Gomory cuts

- $\max\{\boldsymbol{c}^T\boldsymbol{x} : A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0, \boldsymbol{x} \in \mathbb{Z}^n\}$
- Solve LPR to optimality

$$\left[\begin{array}{c:c:c:c} I & \bar{A}_N = A_B^{-1}A_N & 0 & \bar{b} \\ \hdashline \bar{c}_B & \bar{c}_N(\leq 0) & 1 & -\bar{d} \end{array}\right]$$

$$x_{B_u} = \bar{b}_u - \sum_{j\in N} \bar{a}_{uj}x_j, \quad u = 1..m$$
$$z = \bar{d} + \sum_{j\in N} \bar{c}_j x_j$$

- If basic optimal solution to LPR is not integer then $\exists$ some row u: $\bar{b}_u \notin \mathbb{Z}^1$.
  The Chvatál-Gomory cut applied to this row is:

$$x_{B_u} + \sum_{j\in N} \lfloor \bar{a}_{uj} \rfloor x_j \leq \lfloor \bar{b}_u \rfloor$$

(B_u is the index in the basis B corresponding to the row u) (cntd)

- Eliminating $x_{B_u} = \bar{b}_u - \sum_{j \in N} \bar{a}_{uj} x_j$ in the CG cut we obtain:

$$\sum_{j \in N} (\underbrace{\bar{a}_{uj} - \lfloor \bar{a}_{uj} \rfloor}_{0 \leq f_{uj} < 1}) x_j \geq \underbrace{\bar{b}_u - \lfloor \bar{b}_u \rfloor}_{0 < f_u < 1}$$

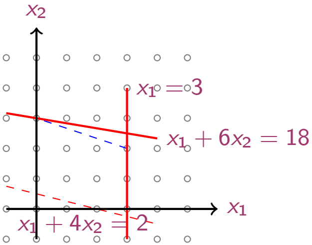
$$\sum_{j \in N} f_{uj} x_j \geq f_u$$

$f_u > 0$ or else u would not be row of fractional solution. It implies that x^* in which $x^*_N = 0$ is cut out!

(theoretically it terminates after a finite number of iterations, but in practice not successful.)

Example

$$
\begin{aligned}
\max\ & x_1 + 4x_2 \\
& x_1 + 6x_2 \leq 18 \\
& x_1 \qquad\quad \leq 3 \\
& x_1, x_2 \geq 0 \\
& x_1, x_2 \text{integer}
\end{aligned}
$$




```
|   | x1 | x2 | x3 | x4 | -z |  b |
|---+----+----+----+----+----+----|
|   |  1 |  6 |  1 |  0 |  0 | 18 |
|   |  1 |  0 |  0 |  1 |  0 |  3 |
|---+----+----+----+----+----+----|
|   |  1 |  4 |  0 |  0 |  1 |  0 |
```

```
|   | x1 | x2 | x3   | x4   | -z |    b |
|---+----+----+------+------+----+------|
|   |  0 |  1 | 1/6  | -1/6 |  0 | 15/6 |
|   |  1 |  0 | 0    | 1    |  0 |    3 |
|---+----+----+------+------+----+------|
|   |  0 |  0 | -2/3 | -1/3 |  1 |  -13 |
```

$x_2 = 5/2, x_1 = 3$

Optimum, not integer

- We take the first row: | | 0 | 1 | 1/6 | -1/6 | 0 | 15/6 |

- CG cut $\sum_{j \in N} f_{uj} x_j \geq f_u \quad \rightsquigarrow \quad \frac{1}{6}x_3 + \frac{5}{6}x_4 \geq \frac{1}{2}$

- Let's verify that it is a CG cut:

$$
\begin{array}{l}
1/6\ (x_1 + 6x_2 \leq 18) \\
5/6\ (x_1 \qquad\quad \leq 3) \\
\hline
\qquad x_1 + x_2 \leq 3 + 5/2 = 5.5
\end{array}
$$

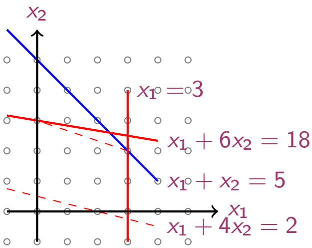
  since x_1, x_2 are integer $x_1 + x_2 \leq 5$. And it leaves out x^*.

- Let's see how it looks in the space of the original variables: from the first tableau:

$$
\begin{aligned}
x_3 &= 18 - 6x_2 - x_1 \\
x_4 &= 3 - x_1
\end{aligned}
$$

$$
\frac{1}{6}(18 - 6x_2 - x_1) + \frac{5}{6}(3 - x_1) \geq \frac{1}{2} \quad \rightsquigarrow \quad x_1 + x_2 \leq 5
$$

- Graphically:



- Let's continue:

	x1	x2	x3	x4	x5	-z	b
	0	0	-1/6	-5/6	1	0	-1/2
	0	1	1/6	-1/6	0	0	5/2
	1	0	0	1	0	0	3
	0	0	-2/3	-1/3	0	1	-13

We need to apply dual-simplex (will always be the case, why?)

ratio rule: $\min\{|\frac{c_j}{a_{ij}}| \; : \; a_{ij} < 0\}$

- After the dual simplex iteration:

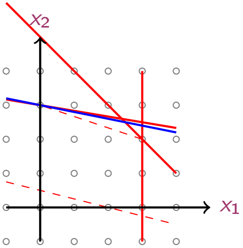
```
|   | x1 | x2 | x3   | x4 | x5   | -z | b     |
|---+----+----+------+----+------+----+-------|
|   |  0 |  0 | 1/5  |  1 | -6/5 |  0 | 3/5   |
|   |  0 |  1 | 1/5  |  0 | -1/5 |  0 | 13/5  |
|   |  1 |  0 | -1/5 |  0 | 6/5  |  0 | 12/5  |
|---+----+----+------+----+------+----+-------|
|   |  0 |  0 | -3/5 |  0 | -2/5 |  1 | -64/5 |
```

We can choose any of the three rows.

Let's take the third: CG cut:
$\frac{4}{5}x_3 + \frac{1}{5}x_5 \geq \frac{2}{5}$

- In the space of the original variables:

$$4(18 - x_1 - 6x_2) + (5 - x_1 - x_2) \geq 2$$
$$x_1 + 5x_2 \leq 15$$



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