DM545/DM871

Linear and Integer Programming

Lecture 5
Duality

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Outline

Outline

Dual Problem

Dual variables $\boldsymbol{y}$ in one-to-one correspondence with the constraints:

Primal problem:

$$\begin{aligned} \max \quad z &= \boldsymbol{c}^T \boldsymbol{x} \\ A\boldsymbol{x} &\leq \boldsymbol{b} \\ \boldsymbol{x} &\geq 0 \end{aligned}$$

Dual Problem:

$$\begin{aligned} \min \quad w &= \boldsymbol{b}^T \boldsymbol{y} \\ A^T\boldsymbol{y} &\geq \boldsymbol{c} \\ \boldsymbol{y} &\geq 0 \end{aligned}$$

Bounding approach

$$
\begin{aligned}
z^* = \max\ 4x_1 &+ x_2 + 3x_3 \\
x_1 + 4x_2 & \leq 1 \\
3x_1 + x_2 + x_3 & \leq 3 \\
x_1, x_2, x_3 & \geq 0
\end{aligned}
$$

a feasible solution is a lower bound but how good?
By tentatives:

$$
\begin{aligned}
(x_1, x_2, x_3) = (1, 0, 0) &\rightsquigarrow z^* \geq 4 \\
(x_1, x_2, x_3) = (0, 0, 3) &\rightsquigarrow z^* \geq 9
\end{aligned}
$$

What about upper bounds?

$$
\begin{array}{rcrl}
 & & 2 \cdot (\ \ x_1 + 4x_2 \qquad) & \leq 2 \cdot 1 \\
 & & +\, 3 \cdot (\ 3x_1 + x_2 + x_3) & \leq 3 \cdot 3 \\
\hline
4x_1 + x_2 + 3x_3 & \leq & 11x_1 + 11x_2 + 3x_3 & \leq \ 11 \\
\\
\boldsymbol{c}^T\boldsymbol{x} & \leq & \boldsymbol{y}^T A\boldsymbol{x} & \leq \boldsymbol{y}^T\boldsymbol{b}
\end{array}
$$

Hence $z^* \leq 11$. Is this the best upper bound we can find?

multipliers $y_1, y_2 \geq 0$ that preserve sign of inequality

$$
\begin{array}{rl}
y_1 \cdot (\ x_1 + 4x_2 \qquad\qquad) & \leq \ y_1(1) \\
y_2 \cdot (\ 3x_1 + x_2 + \qquad x_3) & \leq \ y_2(3) \\
\hline
(y_1 + 3y_2)x_1 + (4y_1 + y_2)x_2 + y_2x_3 & \leq y_1 + 3y_2
\end{array}
$$

Coefficients

$$
\begin{array}{rl}
y_1 + 3y_2 & \geq 4 \\
4y_1 + y_2 & \geq 1 \\
y_2 & \geq 3
\end{array}
$$

$z = 4x_1 + x_2 + 3x_3 \leq (y_1 + 3y_2)x_1 + (4y_1 + y_2)x_2 + y_2x_3 \leq y_1 + 3y_2$

then to attain the best upper bound:

$$
\begin{array}{rrl}
\min & y_1 + 3y_2 & \\
& y_1 + 3y_2 & \geq 4 \\
& 4y_1 + y_2 & \geq 1 \\
& y_2 & \geq 3 \\
& y_1, y_2 & \geq 0
\end{array}
$$

Multipliers Approach

$$
\begin{array}{c}
\pi_1 \\ \vdots \\ \pi_m \\ \pi_{m+1}
\end{array}
\left[\begin{array}{cccc|cccc|c|c}
a_{11} & a_{12} & \dots & a_{1n} & a_{1,n+1} & a_{1,n+2} & \dots & a_{1,m+n} & 0 & b_1 \\
\vdots & \ddots & & & & & & & & \\
a_{m1} & a_{m2} & \dots & a_{mn} & a_{m,n+1} & a_{m,n+2} & \dots & a_{m,m+n} & 0 & b_m \\
\hdashline
c_1 & c_2 & \dots & c_n & 0 & 0 & \dots & 0 & 1 & 0
\end{array}\right]
$$

Working columnwise, since at optimum $\bar{c}_k \leq 0$ for all $k = 1, \dots, n+m$:

$$
\left\{\begin{array}{ccccccccc}
\pi_1 a_{11} & + & \pi_2 a_{21} & \dots + & \pi_m a_{m1} & + \pi_{m+1} c_1 & \leq & 0 \\
\vdots & \ddots & & & & & & \vdots \\
\pi_1 a_{1n} & + & \pi_2 a_{2n} & \dots + & \pi_m a_{mn} & + \pi_{m+1} c_n & \leq & 0 \\
\hdashline
\pi_1 a_{1,n+1}, & & \pi_2 a_{2,n+1}, & \dots & \pi_m a_{m,n+1} & & \leq & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\pi_1 a_{1,n+m}, & & \pi_2 a_{2,n+m}, & \dots & \pi_m a_{m,n+m} & & \leq & 0 \\
\hdashline
 & & & & & \pi_{m+1} & = & 1 \\
\hdashline
\pi_1 b_1 & + & \pi_2 b_2 & \dots + & \pi_m b_m & & (\leq & 0)
\end{array}\right.
$$

(from the last row, $z = \boldsymbol{\pi b}$ so $\max z = \max \boldsymbol{\pi b}$)

$$
\begin{array}{rcrcrcl}
\max\ \pi_1 b_1 & + & \pi_2 b_2 & \ldots + & \pi_m b_m & & \\
\pi_1 a_{11} & + & \pi_2 a_{21} & \ldots + & \pi_m a_{m1} & \leq & -c_1 \\
\vdots & \ddots & & & & & \vdots \\
\pi_1 a_{1n} & + & \pi_2 a_{2n} & \ldots + & \pi_m a_{mn} & \leq & -c_n \\
& & & & \pi_1, \pi_2, \ldots \pi_m & \leq & 0
\end{array}
$$

$y = -\pi$

$$
\begin{array}{rcrcrcl}
\max\ (-y_1 b_1) & + & (-y_2 b_2) & \ldots + & (-y_m b_m) & & \\
(-y_1 a_{11}) & + & (-y_2 a_{21}) & \ldots + & (-y_m a_{m1}) & \leq & -c_1 \\
\vdots & \ddots & & & & & \vdots \\
(-y_1 a_{1n}) & + & (-y_2 a_{2n}) & \ldots + & (-y_m a_{mn}) & \leq & -c_n \\
& & & & -y_1, -y_2, \ldots - y_m & \leq & 0
\end{array}
$$

$$
\begin{array}{rl}
\min & w = \boldsymbol{b}^T \boldsymbol{y} \\
& A^T \boldsymbol{y} \geq \boldsymbol{c} \\
& \boldsymbol{y} \geq 0
\end{array}
$$

Example

$$
\begin{aligned}
\max\ 6x_1 &+ \ 8x_2 \\
5x_1 &+ 10x_2 \leq 60 \\
4x_1 &+ \ 4x_2 \ \leq 40 \\
& x_1, x_2 \geq \ 0
\end{aligned}
$$

$$
\left\{
\begin{array}{l}
5\pi_1 \ + \ 4\pi_2 \ + 6\pi_3 \leq 0 \\
10\pi_1 + \ 4\pi_2 \ + 8\pi_3 \leq 0 \\
1\pi_1 \ + \ 0\pi_2 \ + 0\pi_3 \leq 0 \\
0\pi_1 \ + \ 1\pi_2 \ + 0\pi_3 \leq 0 \\
0\pi_1 \ + \ 0\pi_2 \ + 1\pi_3 = 1 \\
60\pi_1 + 40\pi_2
\end{array}
\right.
$$

$$
\begin{aligned}
y_1 &= -\pi_1 \geq 0 \\
y_2 &= -\pi_2 \geq 0
\end{aligned}
$$

...

Duality Recipe

	Primal linear program	Dual linear program
Variables	$x_1, x_2, \ldots, x_n$	$y_1, y_2, \ldots, y_m$
Matrix	A	A^T
Right-hand side	$\mathbf{b}$	$\mathbf{c}$
Objective function	$\max \mathbf{c}^T\mathbf{x}$	$\min \mathbf{b}^T\mathbf{y}$
Constraints	ith constraint has $\leq$ $\geq$ $=$	$y_i \geq 0$ $y_i \leq 0$ $y_i \in \mathbb{R}$
	$x_j \geq 0$ $x_j \leq 0$ $x_j \in \mathbb{R}$	jth constraint has $\geq$ $\leq$ $=$

Outline

Symmetry

The dual of the dual is the primal:

Primal problem:

$$\begin{aligned} \max \quad z &= c^T x \\ Ax &\leq b \\ x &\geq 0 \end{aligned}$$

Dual Problem:

$$\begin{aligned} \min \quad w &= b^T y \\ A^T y &\geq c \\ y &\geq 0 \end{aligned}$$

Let's put the dual in the standard form

Dual problem:

$$\begin{aligned} \min \quad b^T y \quad &\equiv -\max -b^T y \\ -A^T y &\leq -c \\ y \quad &\geq 0 \end{aligned}$$

Dual of Dual:

$$\begin{aligned} -\min -c^T x & \\ -Ax \quad &\geq \quad -b \\ x \quad &\geq \quad 0 \end{aligned}$$

Weak Duality Theorem

As we saw the dual produces upper bounds. This is true in general:

Theorem (Weak Duality Theorem)

Given:

$$
\begin{aligned}
&(P)\ \max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0\}\\
&(D)\ \min\{\mathbf{b}^T\mathbf{y} \mid A^T\mathbf{y} \geq \mathbf{c}, \mathbf{y} \geq 0\}
\end{aligned}
$$

for any feasible solution $\mathbf{x}$ *of (P) and any feasible solution* $\mathbf{y}$ *of (D):*

$$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$$

Proof:
From (D) $c_j \leq \sum_{i=1}^m y_i a_{ij}\ \forall j$ and from (P) $\sum_{j=1}^n a_{ij}x_j \leq b_i\ \forall i$
From (D) $y_i \geq 0$ and from (P) $x_j \geq 0$

$$\sum_{j=1}^n c_j x_j \leq \sum_{j=1}^n \left(\sum_{i=1}^m y_i a_{ij}\right) x_j = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij}x_j\right) y_i \leq \sum_{i=1}^m b_i y_i$$

Strong Duality Theorem

Due to Von Neumann and Dantzig 1947 and Gale, Kuhn and Tucker 1951.

Theorem (Strong Duality Theorem)

Given:

$$(P)\ \max\{c^T x \mid Ax \leq b, x \geq 0\}$$
$$(D)\ \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

exactly one of the following occurs:

1. *(P) and (D) are both infeasible*
2. *(P) is unbounded and (D) is infeasible*
3. *(P) is infeasible and (D) is unbounded*
4. *(P) has feasible solution, then let an optimal be:* $\boldsymbol{x}^* = [x_1^*, \ldots, x_n^*]$
   (D) has feasible solution, then let an optimal be: $\boldsymbol{y}^* = [y_1^*, \ldots, y_m^*]$, *then:*

$$\boldsymbol{c}^T \boldsymbol{x}^* = \boldsymbol{b}^T \boldsymbol{y}^*$$

Proof:

- all other combinations of 3 possibilities (Optimal, Infeasible, Unbounded) for (P) and 3 for (D) are ruled out by weak duality theorem.

- we use the simplex method. (Other proofs independent of the simplex method exist, eg, Farkas Lemma and convex polyhedral analysis)

- The last row of the final tableau will give us

$$
\begin{aligned}
z = z^* + \sum_{k=1}^{n+m} \bar{c}_k x_k &= z^* + \sum_{j=1}^{n} \bar{c}_j x_j + \sum_{i=1}^{m} \bar{c}_{n+i} x_{n+i} \\
&= z^* + \bar{c}_B x_B + \bar{c}_N x_N
\end{aligned}
\tag{*}
$$

  In addition, $z^* = \sum_{j=1}^{n} c_j x_j^*$ (c_j, original values) because optimal value

- We define $\boxed{y_i^* = -\bar{c}_{n+i},}\ i = 1, 2, \ldots, m$

- We claim that $(y_1^*, y_2^*, \ldots, y_m^*)$ is a dual feasible solution satisfying $c^T x^* = b^T y^*$.

- Let's verify the claim:
  We substitute in (*): i) $z = \sum_{j=1}^{n} c_j x_j$; ii) $\bar{c}_{n+i} = -y_i^*$; and iii) $x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j$ for $i = 1, 2, \ldots, m$ ($n + i$ are the slack variables)

$$\sum_{j=1}^{n} c_j x_j = z^* + \sum_{j=1}^{n} \bar{c}_j x_j - \sum_{i=1}^{m} y_i^* \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right)$$
$$= \left(z^* - \sum_{i=1}^{m} y_i^* b_i \right) + \sum_{j=1}^{n} \left(\bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* \right) x_j$$

This must hold for every $(x_1, x_2, \ldots, x_n)$ hence:

$$z^* = \sum_{i=1}^{m} b_i y_i^* \qquad \Longrightarrow \quad y^* \text{ satisfies } c^T x^* = b^T y^*$$

$$c_j = \bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^*, j = 1, 2, \ldots, n$$

Since $\bar{c}_k \leq 0$ for every $k = 1, 2, \ldots, n+m$:

$$\bar{c}_j \leq 0 \rightsquigarrow \quad c_j - \sum_{i=1}^{m} y_i^* a_{ij} \leq 0 \rightsquigarrow \quad \sum_{i=1}^{m} y_i^* a_{ij} \geq c_j \qquad j = 1, 2, \ldots, n$$

$$\bar{c}_{n+i} \leq 0 \rightsquigarrow \quad y_i^* = -\bar{c}_{n+i} \geq 0, \qquad i = 1, 2, \ldots, m$$

$\Longrightarrow$ y^* is also dual feasible solution

Complementary Slackness Theorem

Theorem (Complementary Slackness)

A feasible solution x^* *for (P)*
A feasible solution y^* *for (D)*
Necessary and sufficient conditions for optimality of both:

$$\left(c_j - \sum_{i=1}^{m} y_i^* a_{ij}\right) x_j^* = 0, \quad j = 1, \dots, n$$

If $x_j^* \neq 0$ *then* $\sum y_i^* a_{ij} = c_j$ *(no surplus)*
If $\sum y_i^* a_{ij} > c_j$ *then* $x_j^* = 0$

Proof:

$$z^* = \boldsymbol{c}^T \boldsymbol{x}^* \leq \boldsymbol{y}^* A \boldsymbol{x}^* \leq \boldsymbol{b}^T \boldsymbol{y}^* = w^*$$

Hence from strong duality theorem:

$$\boldsymbol{c}\boldsymbol{x}^* - \boldsymbol{y}^* A \boldsymbol{x}^* = 0$$

In scalars

$$\sum_{j=1}^{n} \underbrace{(c_j - \sum_{i=1}^{m} y_i^* a_{ij})}_{\leq 0} \underbrace{x_j^*}_{\geq 0} = 0$$

Hence each term must be $= 0$

Proof in scalar form:

$$c_j x_j^* \leq \left(\sum_{i=1}^{m} a_{ij} y_i^* \right) x_j^* \quad j = 1, 2, \ldots, n \quad \text{from feasibility in D}$$

$$\left(\sum_{j=1}^{n} a_{ij} x_j^* \right) y_i^* \leq b_i y_i^* \quad i = 1, 2, \ldots, m \quad \text{from feasibility in P}$$

Summing in j and in i:

$$\sum_{j=1}^{n} c_j x_j^* \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i^* \right) x_j^* = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j^* \right) y_i^* \leq \sum_{i=1}^{m} b_i y_i^*$$

For the strong duality theorem the left hand side is equal to the right hand side and hence all inequalities become equalities.

$$\sum_{j=1}^{n} \underbrace{(c_j - \sum_{i=1}^{m} y_i^* a_{ij})}_{\leq 0} \underbrace{x_j^*}_{\geq 0} = 0$$

Duality - Summary

- Derivation:
  - Economic interpretation
  - Bounding Approach
  - Multiplier Approach
  - Recipe
  - Lagrangian Multipliers Approach (next time)
- Theory:
  - Symmetry
  - Weak Duality Theorem
  - Strong Duality Theorem
  - Complementary Slackness Theorem