Department of Mathematics and Computer Science
University of Southern Denmark, Odense

October 30, 2024
Marco Chiarandini

DM545/DM871 – Linear and integer programming

Sheet 4, Autumn 2024

Exercises with the symbol $^+$ are to be done at home before the class. Exercises with the symbol * will be tackled in class and should be at least read at home. The remaining exercises are left for self training after the exercise class.

Exercise 1$^+$

Solve the systems $\mathbf{y}^T E_1 E_2 E_3 E_4 = [1\ 2\ 3]$ and $E_1 E_2 E_3 E_4 \mathbf{d} = [1\ 2\ 3]^T$ with

$$E_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0.5 & 0 \\ 0 & 4 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_4 = \begin{bmatrix} -0.5 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Exercise 2* Sensitivity Analysis and Revised Simplex

A furniture-manufacturing company can produce four types of products using three resources.

- A bookcase requires three hours of work, one unit of metal, and four units of wood and it brings in a net profit of 19 Euro.
- A desk requires two hours of work, one unit of metal and three units of wood, and it brings in a net profit of 13 Euro.
- A chair requires one hour of work, one unit of metal and three units of wood and it brings in a net profit of 12 Euro.
- A bedframe requires two hours of work, one unit of metal, and four units of wood and it brings in a net profit of 17 Euro.
- Only 225 hours of labor, 117 units of metal and 420 units of wood are available per day.

In order to decide how much to make of each product so as to maximize the total profit, the managers solve an LP problem.

1) Write the mathematical programming formulation of the problem.
2) With the help of a computational environment such as Python for carrying out linear algebra operations, write the optimal tableau, which has x_1, x_3 and x_4 in basis.

   Do this task:

   - first, using the original simplex method
   - second, using the revised simplex. In this case, start by writing A_B, A_N, then calculate $A_B^{-1}A_N$, and the finally derive the optimal tableau and verify that the solution is indeed optimal.
3) What is the increase in price (reduced cost) that would make product x_2 worth to be produced?
4) What is the marginal value (shadow price) of an extra hour of work or amount of metal and wood?
5) Are all resources totally utilized, i.e. are all constraints "binding", or is there slack capacity in some of them? Answer this question in the light of the complementary slackness theorem.
6) From the economical interpretation of the dual why product x_2 is not worth producing? What is its imputed cost?

Perform a sensitivity analysis for the following variants:

7) The net profit brought in by each desk increases from 13 Euro to 15 Euro.
8) The availability of metal increases from 117 to 125 units per day
9) The company may also produce coffee tables, each of which requires three hours of work, one unit of metal, two units of wood and bring in a net profit of 14 Euro.
10) The number of chairs produced must be at most five times the numbers of desks

Exercise 3* Factory Planning and Machine Maintenance

A firm makes seven products $1, \ldots, 7$ on the following machines: 4 grinders, 2 vertical drills, 3 horizontal drills, 1 borer, and 1 planer.
Each product yields a certain contribution to the profit (defined as selling price minus cost of raw materials expressed in Euro/unit). These quantities (in Euro/unit) together with the production times (hours/unit) required on each process are given below.

product	1	2	3	4	5	6	7
profit	10	6	8	4	11	9	3
grinding	0.5	0.7	0	0	0.3	0.2	0.5
vdrill	0.1	0.2	0	0.3	0	0.6	0
hdrill	0.2	0	0.8	0	0	0	0.6
boring	0.05	0.03	0	0.07	0.1	0	0.08
planning	0	0	0.01	0	0.05	0	0.05

In the first month (January) and the five subsequent months certain machines will be down for maintenance. These machines will be:

January	1 grinder
February	2 hdrill
March	1 borer
April	1 vdrill
May	1 grinder
May	1 vdrill
June	1 planer
June	1 hdrill

There are marketing limitations on each product in each month. That is, in each month the amount sold for each product cannot exceed these values:

product	1	2	3	4	5	6	7
January	500	1000	300	300	800	200	100
February	600	500	200	0	400	300	150
March	300	600	0	0	500	400	100
April	200	300	400	500	200	0	100
May	0	100	500	100	1000	300	0
June	500	500	100	300	1100	500	60

It is possible to store products in a warehouse. The capacity of the storage is 100 units per product type per month. The cost is 0.5 Euro per unit of product per months. There are no stocks in the first month but it is desired to have a stock of 50 of each product type at the end of June.
The factory works 6 days a week with two shifts of 8 hours each day. (It can be assumed that each month consists of 24 working days.)
The factory wants to determine a production plan, that is, the quantity to produce, sell and store in each month for each product, that maximizes the total profit.

Task 1 Model the factory planning problem for the month of January as an LP problem.

Task 2 Model the multi-period (from January to June) factory planning problem as an LP problem. Use mathematical notation and indicate in general terms how many variables and how many constraints your model has.

Task 3 Instead of stipulating when each machine is down for maintenance, it is desired to find the best month for each machine to be down.
Each machine must be down for maintenance in one month of the six apart from the grinding machines, only two of which need be down in any six months.
Extend the model to allow it to make these extra decisions.

- How many variables did you need to add? What is the domain of these variables?
- Has the matrix of the problem a similar structure to the one of Task 2?
- Is the solution from Task 2 a valid solution to this problem? What information does it give about the solution of the present task?