# Technion - Israel Institute of Technology

# 1 Theoretical Question

Given $y' = f(x, y)$ where $y(0) = 1$, and the numeric equation is :

$$y_{n+1} = y_n + h\big[\alpha f_n + \beta f_{n-1}\big] + R \tag{1.1}$$

$h$ is the integration step, $\alpha, \beta$ are constants.

## 1.1 Implicit / Explicit method ?

In implicit method, the desired calculation is dependent on calculations from the same step, that are not yet to be known. Contrarily, in explicit method, the calculation is dependent on previous known steps. As seen on RHS, all calculations $(n, n-1)$ satisfy $n_{RHS} < n + 1$, hence it's explicit, and multi-step. However, the first step would be initialized by a one-step method, preferably of the same local accuracy as that.

## 1.2 Find $\alpha, \beta, R$ using Taylor expansion

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n + ... \tag{1.2}$$

$$y_{n-1} = y_n - hy'_n + \frac{h^2}{2!}y''_n - \frac{h^3}{3!}y'''_n + ... \tag{1.3}$$

derive :

$$y'_{n+1} = f_{n+1} = y'_n + hy''_n + \frac{h^2}{2!}y'''_n + ... \tag{1.4}$$

$$y'_{n-1} = f_{n-1} = y'_n - hy''_n + \frac{h^2}{2!}y'''_n + ... \tag{1.5}$$

Plug it :

$$hy'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n = h\big[\alpha y'_n + \beta(y'_n - hy''_n + \frac{h^2}{2!}y'''_n)\big] + R \tag{1.6}$$

$$hy'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n = h\big[y'_n(\alpha+\beta) - y''_n(\beta h) + y'''_n\beta(\frac{h^2}{2!})\big] + R \tag{1.7}$$

Equating coefficients :

$$y_n: \quad 0 = 0, \qquad y'_n: \quad h = h(\alpha+\beta), \qquad y''_n: \quad \frac{h^2}{2} = -\beta h^2 \tag{1.8}$$

$$\alpha+\beta = 1, \qquad \beta = -\frac{1}{2}, \qquad \alpha = 1.5 \tag{1.9}$$

$$\frac{h^3}{3!}y'''_n = \frac{h^3}{3!}y'''_n \tag{1.10}$$

Plug and find $R$ :

$$\frac{h^3}{3!}y'''_n = hy'''_n\beta(\frac{h^2}{2!}) + R \quad \rightarrow \quad y'''_n h^3 \cdot (\frac{1}{3!} - \frac{\beta}{2}) = R \tag{1.11}$$

$$R_i = \frac{5}{12}h^3 y'''_n = \frac{5}{12}h^3 \cdot f''_n(\xi) \tag{1.12}$$

## 1.3 Predictor–corrector method

Find conditions for convergence for the following equation :

$$y_{n+1} = y_n + \frac{h}{24}\big[9f^k_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\big] + R = G(y_{n+1}) \tag{1.13}$$

Demand $|\frac{\partial G(y^{k+1}_{n+1}}{\partial y^k_{n+1}})| < 1$ and step must be $(h > 0)$ :

$$|G(y_{n+1})| \quad = \quad |\frac{\partial G(y^{k+1}_{n+1})}{\partial f(x_{n+1}, y^k_{n+1})}| \cdot |\frac{\partial f(x_{n+1}, y^k_{n+1})}{\partial y^k_{n+1}}| \quad = \quad \frac{9h}{24} \cdot |\frac{\partial f(x_{n+1}, y^k_{n+1})}{\partial y^k_{n+1}}| \tag{1.14}$$

$$0 < \quad \frac{9h}{24} \cdot |\frac{\partial f(x_{n+1}, y^k_{n+1})}{\partial y^k_{n+1}}| < 1 \tag{1.15}$$

And finally we get the constraint :

$$0 < h < (\frac{24}{9}) \cdot \frac{1}{|\frac{\partial f(x_{n+1}, y^k_{n+1})}{\partial y^k_{n+1}}|} \tag{1.16}$$

## 1.4 Method stability

The equation : $y_{n+1} = y_n + h\big[\alpha y_n' + \beta y_{n-1}'\big] + R,\qquad$ plug $f = -Ay$ :

$$y_{n+1} = y_n + h\big[-\alpha A y_n - \beta A y_{n-1}\big] \quad\to\quad y_{n+1} = y_n + \frac{1}{2}Ah\big[-3y_n + y_{n-1}\big] \tag{1.17}$$

Define : $t \equiv \frac{1}{2}Ah$, and $y_n = C\beta^n$ where $(C, \beta \neq 0)$ :

$$C\beta^{n+1} = C\beta^n + \frac{1}{2}Ah\big[-3C\beta^n + C\beta^{n-1}\big] \quad \cdot\backslash(\frac{1}{C\beta^{n-1}}) \tag{1.18}$$

$$\beta^2 = \beta + t[-3\beta + 1] \quad\to\quad \beta^2 + \beta(3t-1) - t = 0 \tag{1.19}$$

$$\beta_{1,2} = \frac{-3t + 1 \pm \sqrt{9t^2 - 2t + 1}}{2} \tag{1.20}$$

Solving using Taylor expansion of 1st approximation :

$$\beta_i(t) = \beta_i(0) + t\beta_i'(0) + O(h^2) \tag{1.21}$$

Such that :

$$\beta_{1,2}(0) = \frac{1 \pm 1}{2} \quad\to\quad (+)\quad \beta_1(0) = 1 \quad (-)\quad \beta_2(0) = 0 \tag{1.22}$$

And its derivative :

$$\beta_{1,2}'(0) = \frac{\partial}{\partial t}\cdot(\frac{u}{v}) = \frac{u'v - uv'}{v^2} = \frac{u'v}{v^2} \tag{1.23}$$

$$\beta_{1,2}'(0) = \frac{\partial}{\partial t}\cdot(\frac{u}{v}) = \frac{u'v - uv'}{v^2} = \frac{u'v}{v^2} = \frac{2\cdot(-3 \pm \frac{18t-2}{2\sqrt{9t^2-2t+1}})}{4}$$

$$\beta_{1,2}'(0) = \frac{1}{2}(-3 \pm \frac{9t-1}{\sqrt{9t^2-2t+1}}) = \frac{1}{2}(-3 \pm -1) \tag{1.24}$$

$$(+)\quad \beta_1(0) = -2 \quad (-)\quad \beta_2(0) = -1 \tag{1.25}$$

Plug back to (Eq. 1.21) :

$$(+)\quad \beta_1(t) = 1 - 2t, \qquad (-)\quad \beta_1(t) = 0 - t \tag{1.26}$$

Super-position using ($x_n = x_0 + nh \rightarrow h = \frac{x_n}{n}$) :

$$y_n = C_1\beta_1^n + C_2\beta_2^n = C_1(1-2t)^n + C_2(-t)^n \tag{1.27}$$

$$y_n = C_1(1-Ah)^n + C_2(-1)^n(\frac{1}{2}Ah)^n = C_1(1-\frac{Ax_n}{n})^n + C_2(-1)^n(\frac{\frac{1}{2}Ax_n}{n})^n \tag{1.28}$$

Approaching limit :

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} C_1e^{-Ax_n} + C_2(-1)^n(\frac{1}{2}\frac{Ax_n}{n})^n \tag{1.29}$$

We can see that the later element ($C_2$) causes oscillations, so stability would be promised when it diminishes : $|Ah| < 1$.

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