Technion - Israel Institute of Technology

Numerical Methods in Aeronautical Engineering (086172)

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1.1 a

Develop the completely implicit for the numerical solution of the PDE : $\dfrac{\partial u}{\partial t} = \dfrac{\partial u^2}{\partial x^2}$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \quad \frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) \tag{1.1}$$

Express $u(x,t)$ via Taylor series w.r.t time :

$$\begin{aligned} u(x,t+\delta t) &= u(x,t) + \delta t \frac{\partial u}{\partial t} + \frac{\delta t^2}{2!}\frac{\partial^2 u}{\partial t^2} + \frac{\delta t^3}{3!}\frac{\partial^3 u}{\partial t^3} + \dots \\ u(x,t+\delta t) &= (1 + \delta t \frac{\partial}{\partial t} + \frac{\delta t^2}{2!}\frac{\partial^2}{\partial t^2} + \frac{\delta t^3}{3!}\frac{\partial^3}{\partial t^3} + \dots) u(x,t) \end{aligned} \tag{1.2}$$

Exponent series identity :

$$u(x,t+\delta t) = u(x,t) \sum_{n=1}^{\infty} (\frac{\delta t}{n!} \cdot \frac{\partial}{\partial t})^n = exp(\delta t \frac{\partial}{\partial t}) u(x,t) \tag{1.3}$$

Substitute with 1.1 inside :

$$u(x,t+\delta t) = u(x,t) \sum_{n=1}^{\infty} (\frac{\delta t}{n!} \cdot \frac{\partial^2}{\partial x^2})^n = exp(\delta t \frac{\partial}{\partial x^2}) u(x,t) \tag{1.4}$$

In order to gain a linear operator <u>instead</u> of the the derivative term, we'll develop :

$$hD_x[\delta] = 2sinh^{-1}(\frac{\delta_x}{2}) \quad \Rightarrow \quad \frac{\partial}{\partial x} = \frac{2}{h}sinh^{-1}(\frac{\delta_x}{2}) \quad \Rightarrow \quad (\frac{\partial}{\partial x})^n = (\frac{2}{h}sinh^{-1}(\frac{\delta x}{2}))^n \quad (1.5)$$

Here :

$$u(x, t + \delta t) = exp(\delta t D_x^2) = exp\Big(\frac{4\delta t}{h^2} \cdot (sinh^{-1}(\frac{\delta x}{2}))^2\Big)u(x, t) \quad (1.6)$$

$Sinh^{-1}$ series identity :

$$sinh^{-1}(\frac{\delta x}{2}) = (\frac{\delta x}{2}) - \frac{1}{6}(\frac{\delta x}{2})^3 + \frac{3}{40}(\frac{\delta x}{2})^5 + \dots \quad (1.7)$$

Re-open as exponential identity :

$$\begin{aligned} u(x, t + \delta t) &= exp(\delta t D_x^2) = exp\Big(\frac{4\delta t}{h^2} \cdot ((\frac{\delta x}{2}) - \frac{1}{6}(\frac{\delta x}{2})^3 + \dots)^2\Big)u(x, t) \\ u(x, t + \delta t) &= \Big[\sum_{n=0}^{\infty} \frac{1}{n!}(\frac{4\delta t}{h^2}\Big((\frac{\delta x}{2}) - \frac{1}{6}(\frac{\delta x}{2})^3 + \dots\Big)^2)^n\Big]u(x, t) \end{aligned} \quad (1.8)$$

Neglecting higher terms from 2nd order, we get :

$$\begin{aligned} (x, t + \delta t) &= \Big[\sum_{n=0}^{1} \frac{1}{n!}(\frac{4\delta t}{h^2}\Big((\frac{\delta x}{2}) + O(h^2)\Big)^2)^n\Big]u(x, t) \\ u(x, t + \delta t) &\approx \Big[1 + \frac{4\delta t}{h^2}(\frac{\delta x}{2})^2\Big]u(x, t) \quad (1.9) \\ u(x, t + \delta t) &= u(x, t) + \frac{\delta t}{h^2} \cdot \delta_x^2\Big(u(x, t)\Big) \quad (1.10) \end{aligned}$$

Recall relations from the linear operators table :

$$\delta_x^2\Big[f(x)\Big] = \delta_x\Big[f(x)\Big] \quad \Rightarrow \quad \delta_x[\delta_x] = \delta_x^2 = \delta_x \quad (1.11)$$

Central F.D development :

$$\begin{aligned} \delta^2\Big[f(x + \frac{h}{2}) - f(x - \frac{h}{2})\Big] &= \delta\Big[f(x + \frac{h}{2}) - f(x - \frac{h}{2})\Big] \\ = \delta\Big[f(x + \frac{h}{2})\Big] - \delta\Big[f(x - \frac{h}{2})\Big] &= f(x + h) - 2f(x) + f(x - h) \end{aligned} \quad (1.12)$$

And implement on 1.10 :

$$u(x,t+\delta t) = u(x,t) + \frac{4\delta t}{h^2}\cdot(\frac{\delta_x}{2})^2\Big(u(x,t)\Big) = u(x,t) + \frac{\delta t}{h^2}(\delta_x^2\Big[u(x,t)\Big]) \tag{1.13}$$

General term for the θ method :

$$\frac{\partial u}{\partial t} = \big[\theta(\frac{\partial^2 u}{\partial x^2})_{j+1} + (1-\theta)(\frac{\partial^2 u}{\partial x^2})_j\big] \tag{1.14}$$

Set it for the completely implicit method ($\theta = 1$) :

$$(\frac{\partial u}{\partial t})_{j+1} = (\frac{\partial^2 u}{\partial x^2})_{j+1} \tag{1.15}$$

Finally, denote $R = \frac{\delta t}{h^2}$ and we get the method :

$$u(x,t+\delta t) = u(x,t) + R\big[u(x-h,t+\delta t) - 2u(x,t+\delta t) + u(x+h,t+\delta t)\big] \tag{1.16}$$

Which can be written numerically as :

$$\underline{\underline{u_{i,j+1}}} = u_{i,j} + R\big[u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}\big]$$

1.2 b

Consider the nonlinear PDE : $\displaystyle \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Bigg[F(u)\frac{\partial u}{\partial x}\Bigg].$
Develop the completely implicit method for this equation numerically :

$$\frac{\partial}{\partial x} = \frac{2}{h}\Big(sinh^{-1}(\frac{\delta_x}{2})\Big) = \frac{2}{h}\Big((\frac{\delta_x}{2}) - \frac{1}{6}(\frac{\delta_x}{2})^3 + \frac{3}{40}(\frac{\delta_x}{2})^5 + \dots\Big) \tag{1.17}$$

neglecting H.O.T $O(h^2)$:

$$\frac{\partial}{\partial x} = \frac{2}{h}\Big((\frac{\delta_x}{2}) + O(h^2)\Big) = \frac{\delta_x}{h} \tag{1.18}$$

Here, using cetral F.D for 1st derivative :

$$
\begin{aligned}
(\frac{\partial u}{\partial t})_{i,j+1} = \frac{\delta_x}{h}\Big[F(u)\frac{\partial u}{\partial x}\Big]_{i,j+1} &= \frac{\delta_x}{h}\Big[F_{i+\frac{1}{2},j+1}(\frac{\partial u}{\partial x})_{i+\frac{1}{2},j+1} - F_{i-\frac{1}{2},j+1}(\frac{\partial u}{\partial x})_{i-\frac{1}{2},j+1}\Big] \\
&= \frac{\delta_x}{h}\Big[F_{i+\frac{1}{2},j+1}(\frac{u_{i+1,j+1}-u_{i,j}}{\delta_x}) - F_{i-\frac{1}{2},j+1}(\frac{u_{i,j+1}-u_{i-1,j+1}}{\delta_x})\Big] \\
&= \frac{1}{h}\Big[F_{i+\frac{1}{2},j+1}u_{i+1,j+1} - (F_{i+\frac{1}{2},j+1}u_{i,j} + F_{i-\frac{1}{2},j+1}) + F_{i-\frac{1}{2},j+1}u_{i-1,j+1}\Big]
\end{aligned}
$$

Finally, substituting indices and plug R ratio :

$$\underline{\underline{u_{i,j+1}}} = u_{i,j} + \Big[F_{i+\frac{1}{2},j+1}u_{i+1,j+1} - (F_{i+\frac{1}{2},j+1}u_{i,j} + F_{i-\frac{1}{2},j+1}) + F_{i-\frac{1}{2},j+1}u_{i-1,j+1}\Big] \tag{1.19}$$

Provide that we want to eliminate half h indices, we'll average $F(u)$:

$$F\big[u_{i\pm\frac{1}{2},j+1}\big] = \frac{1}{2}\big[F(u_{i\pm 1,j+1}) + F(u_{i,j+1})\big] = \frac{1}{2}F\big[u_{i\pm 1,j+1} + u_{i,j+1}\big] \tag{1.20}$$

And plug back into the final solution :

$$\underline{\underline{u_{i,j+1}}} = u_{i,j} + \Big[u^2_{i+1,j+1} - 2u^2_{i,j} + u^2_{i-1,j+1}\Big] \tag{1.21}$$

1.3 c

Find the local truncation error for the fully implicit method when solving: $\dfrac{\partial u}{\partial t} = \dfrac{\partial u^2}{\partial x^2}$:

The PDE was brought into :

$$u(x, t+\delta t) = exp(\delta t D_x^2) = exp\Big(\frac{4\delta t}{h^2}\cdot(sinh^{-1}(\frac{\delta x}{2}))^2\Big)u(x,t) \tag{1.22}$$

And the hyperbolic sinus expansion :

$$
\begin{aligned}
sinh^{-1}(\frac{\delta x}{2}) &= (\frac{\delta x}{2}) - \frac{1}{6}(\frac{\delta x}{2})^3 + \frac{3}{40}(\frac{\delta x}{2})^5 + \dots \\
sinh^{-1}(\frac{\delta x}{2}) &= \Big[\sum_{n=0}^{\infty}\frac{1}{n!}(\frac{4\delta t}{h^2}\Big((\frac{\delta x}{2}) + O(h^2)\Big)^2)^n\Big]
\end{aligned}
\tag{1.23}
$$

Inevitably, a well thought out reduction had to be handled for a practical solution :

$$sinh^{-1}(\frac{\delta x}{2}) = \Big[\sum_{n=0}^{1} \frac{1}{n!}(\frac{4\delta t}{h^2}\Big((\frac{\delta x}{2}) + O(h^2)\Big)^2)^n\Big]u(x,t)$$

Thus :

$$sinh^{-1}(\frac{\delta x}{2}) = \Big[1 + \frac{4\delta t}{h^2}(\frac{\delta x}{2})\Big] + \underline{\underline{O(h^2)^2}} \tag{1.24}$$

Add it to the time derivative local truncation, and we get :

$$u_{i,j+1} = u_{i,j} + R\big[u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}\big] + \underline{\underline{O(k) + O(h^2)^2}} \tag{1.25}$$

$-fin-$