Faculty of Aerospace Engineering **Technion- Israel Institute of Technology**

Numerical Methods in Aerospace Engineering Spring Semester 2019

Examples Sheet 3

Question 1 **Please submit your solution to this question according to the instructions on the course's site.**

Question 1

A plate has the shape of a polygon; ABCDE. The coordinates (x, y) of the corners are:
$(x_A, y_A) = (0,0), (x_B, y_B) = (5,0), (x_C, y_C) = (5,2), (x_D, y_D) = (2,7), (x_E, y_E) = (0,7)$

Solve the heat equation: $\frac{\partial}{\partial x}\left(K_1 \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_2 \frac{\partial u}{\partial y}\right) = 0$ for the temperature of the plate u under the following boundary conditions:

$on\ AE : u = 1;\ on\ BC\ and\ CD : u = 0;\ on\ AB\ and\ DE : \partial u / \partial y = 0$.

The conduction coefficients K_1, K_2 are functions of u:

$K_1 = K_{10}(1 + \alpha u), \quad K_2 = K_{20}(1 + \alpha u)$.

Find the solutions for the following cases:

[1] $\alpha = 0, \quad K_{10} = K_{20} = 1$

[2] $\alpha = 0.1, \quad K_{10} = K_{20} = 1$

[3] $\alpha = 0.1, \quad K_{10} = 1, \quad K_{20} = 2$

[4] $\alpha = 0.1, \quad K_{10} = 1, \quad K_{20} = 1/2$

Question 2

Suppose we want to solve the following linear algebraic system using Jacobi's method:

$$\sum_{j=1}^{N} a_{ij} x_j = b_i, \quad i = 1,2...N$$

Prove that if $\sum_{j=1,\ j \neq i}^{N} |a_{ij}| \leq |a_{ii}|, \quad i = 1,2...N$ iterative convergence is assured. (This condition is known as Diagonal Dominance).

Question 3

Use Brauer's theorem to investigate how boundary conditions influence the stability of Crank-Nicholson's method for the numerical solution of the following problem:

$$0 \leq x \leq 1 : \quad \frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}$$

subject to the initial and boundary conditions:

$$U(x,0) = U_0(x) \text{ where } U_0(x) \text{ is a given function}$$

$$x = 0, t \geq 0, \frac{\partial U}{\partial x} = K_1(U - v_1)$$

$$x = 1, t \geq 0, \frac{\partial U}{\partial x} = -K_2(U - v_2)$$

where $K_1 > 0, K_2 > 0, v_1, v_2$ are constants.

Question 4

For the solution of the convection-diffusion equation:

$$-u\frac{\partial w}{\partial x} + D\frac{\partial^2 w}{\partial x^2} = 0 \quad (I)$$

(where u is the velocity and D is the diffusion coefficient, both of which are constant) the following numerical method is proposed:

$$w_{i+1} - \left(1 + e^{uh/D}\right)w_i + e^{uh/D}w_{i-1} = 0 \quad (II)$$

which can be solved iteratively using

$$w_i^{(n+1)} = \frac{1}{\left(1 + e^{uh/D}\right)}\left[w_{i+1}^{(n)} + e^{uh/D}w_{i-1}^{(n)}\right], \quad i = 0,1,2,.... \qquad (III)$$

(a) Show that under certain conditions equation *(II)* reduces to the finite difference equation that would be obtained by writing *(I)* using central differences.

(b) Prove that the iterative procedure *(III)* will not diverge.

Question 5

Suppose we want to solve the following problem using Crank-Nicholson's method:

$$0 \leq x \leq 1: \quad \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

$$U(0,t) = U(1,t) = 0 \text{ and } U(x,0) = U0(x)$$

Instead of using a tridiagonal solver for the set of algebraic equations we get, we will use the following iterative method:

$$u_{i,j+1}^{(n+1)} = \frac{R}{2}\left(u_{i-1,j+1}^{(n)} - 2u_{i,j+1}^{(n)} + u_{i+1,j+1}^{(n)}\right) + \frac{R}{2}\left(u_{i-1,j} + u_{i+1,j}\right) + \left(1 - R\right)u_{i,j}$$

where $R = k/h^2$, $i = 1,2,3....N$, $j = 0,1,2,3......$ and (n) represents the iterative index.

Prove (mathematically) that the unconditional stability of the Crank-Nicholson method is lost due to a restrictive condition on R for iterative convergence.

Good luck!!