Technion - Israel Institute of Technology

Numerical Methods in Aeronautical Engineering (086172)

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1

Suppose we want to solve the following linear algebraic system using Jacobi's method:

$$\sum_{j=1}^{N} a_{ij}x_j = b_i, \quad i=1,2...N$$

Prove that if $\sum_{j=1,\ j\neq i}^{N} \left|a_{ij}\right| \le \left|a_{ii}\right|, \quad i=1,2...N$ iterative convergence is assured. (This condition is known as Diagonal Dominance).

Let us decompose matrix A into 3 components (L - Lower, D - Diagonal, U - Upper) :

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \ddots & \ddots & \cdots \\ \vdots & \ddots & \ddots & \cdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_N \end{bmatrix} \quad \rightarrow \quad \underline{A}\bar{x} = \bar{b} \quad \Leftrightarrow \quad (\underline{L} + \underline{D} + \underline{U})\bar{x} = \bar{b} \qquad (1.1)$$

Isolating the diagonal, and express as Jacobi method :

$$\underline{D}\bar{x}^{(n+1)} = \bar{b} - (\underline{L} + \underline{U})\bar{x}^{(n)} \quad \rightarrow \quad \bar{x}^{(n+1)} = \underline{D}^{-1}(\bar{b} - (\underline{L} + \underline{U})\bar{x}^{(n)})$$

$$\text{Equivalently,} \quad x_i^{(n+1)} = \frac{1}{a_{ii}}(b_i - \sum_{j=1\neq i}^{N} a_{ij}x_j^{(n)}) = \frac{b_i}{a_{ii}} - \sum_{j=1\neq i}^{N} \frac{a_{ij}}{a_{ii}}x_j^{(n)} \ , \quad i = 1, 2, ..., N \qquad (1.2)$$

Constant elements (b_i/a_{ii}) are subtracted between iterations, so we're interested in :

$$\underline{C} = \sum_{j=1 \neq i}^{N} \frac{a_{ij}}{a_{ii}}, \qquad i = 1, 2, ..., N \tag{1.3}$$

Using Gershgorin theorem to asses C's eigenvalues :

$$\rho(\underline{C}) = \max_{1 \leq s \leq N} |\lambda_s| \leq S = \max_{1 \leq i \leq N} S_i = \max_{1 \leq i \leq N} \sum_{j=1 \neq i}^{N} \left| \frac{a_{ij}}{a_{ii}} \right|$$

$$\text{Let } a_{ii} \text{ satisfy:} \quad \sum_{j=1 \neq i}^{N} |a_{ij}| \leq |a_{ii}| \quad \text{such that :} \quad \sum_{j=1 \neq i}^{N} \left| \frac{a_{ij}}{a_{ii}} \right| \leq 1 \tag{1.4}$$

$$\text{here :} \quad \rho(\underline{C}) = \max_{1 \leq i \leq N} \sum_{j=1 \neq i}^{N} \left| \frac{a_{ij}}{a_{ii}} \right| = \sum_{j=1 \neq i}^{N} \left| \frac{a_{ij}}{a_{ii}} \right| \leq 1 \tag{1.5}$$

Since a_{ii} is diagonally dominant, the spectral radius is less than or equal to 1, at every row.

2

Use Brauer's theorem to investigate how boundary conditions influence the stability of <u>Crank-Nicholson's</u> method for the numerical solution of the following problem:

$$0 \leq x \leq 1 : \quad \frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}$$

subject to the initial and boundary conditions:

$$U(x,0) = U_0(x) \text{ where } U_0(x) \text{ is a given function}$$

$$x = 0, t \geq 0, \frac{\partial U}{\partial x} = K_1(U - v_1)$$

$$x = 1, t \geq 0, \frac{\partial U}{\partial x} = -K_2(U - v_2)$$

where $K_1 > 0, K_2 > 0, v_1, v_2$ are constants.

Let us write the PDE under C-N using central finite differences :

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Big[(\frac{\partial^2 u}{\partial x^2})_j + (\frac{\partial^2 u}{\partial x^2})_{j+1}\Big] \tag{2.1}$$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{2}(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}) \tag{2.2}$$

Define $R \equiv \frac{k}{h^2}$, we get :

$$-\frac{R}{2}u_{i-1,j+1} + u_{i,j+1}(1+R) - \frac{R}{2}u_{i+1,j+1} = \frac{R}{2}u_{i-1,j} + u_{i,j}(1-R) + \frac{R}{2}u_{i+1,j} \tag{2.3}$$

Both sides' conditions are defined by derivatives, such that index becomes - i $\in$ [0, N+1] :

$$\frac{\partial u}{\partial x}\Big|_{x=0}^{t\geq 0} = \frac{u_{1,j} - u_{-1,j}}{2h} = K_1(u_{0,j} - v_1), \quad \Rightarrow \quad \begin{cases} u_{-1,j} = u_{1,j} - 2hK_1(u_{0,j} - v_1) \\ u_{-1,j+1} = u_{1,j+1} - 2hK_1(u_{0,j+1} - v_1) \end{cases}$$

$$\frac{\partial u}{\partial x}\Big|_{x=1}^{t\geq 0} = \frac{u_{N+2,j} - u_{N,j}}{2h} = -K_2(u_{N+1,j} - v_2), \quad \Rightarrow \quad \begin{cases} u_{N+2,j} = u_{N,j} - 2hK_2(u_{N+1,j} - v_2) \\ u_{N+2,j+1} = u_{N,j+1} - 2hK_2(u_{N+1,j+1} - v_2) \end{cases}$$

Check the first and last rows :

$$\begin{aligned}
i = 0 \ : \quad & -\frac{R}{2}u_{-1,j+1} + (1+R)u_{0,j+1} - \frac{R}{2}u_{1,j+1} = \frac{R}{2}u_{-1,j} + (1-R)u_{0,j} + \frac{R}{2}u_{1,j} \\
& \text{LHS :} \quad -\frac{R}{2}\Big(u_{1,j+1} - 2hK_1(u_{0,j+1} - v_1)\Big) + (1+R)u_{0,j+1} - \frac{R}{2}u_{1,j+1} \\
& \text{LHS :} \quad \Rightarrow \quad \Big(1 + R(1+hK_1)\Big)u_{0,j+1} - Ru_{1,j+1} - (RhK_1v_1) \\
& \text{RHS :} \quad \frac{R}{2}\Big(u_{1,j} - 2hK_1(u_{0,j} - v_1)\Big) + (1-R)u_{0,j} + \frac{R}{2}u_{1,j} \\
& \text{RHS :} \quad \Rightarrow \quad \Big(1 - R(1+hK_1)\Big)u_{0,j} + Ru_{1,j} + (RhK_1v_1)
\end{aligned}$$

$$\begin{aligned}
i = N+1 \ : \quad & -\frac{R}{2}u_{N,j+1} + (1+R)u_{N+1,j+1} - \frac{R}{2}u_{N+2,j+1} = \frac{R}{2}u_{N,j} + (1-R)u_{N+1,j} + \frac{R}{2}u_{N+2,j} \\
& \text{LHS :} \quad -\frac{R}{2}u_{N,j+1} + (1+R)u_{N+1,j+1} - \frac{R}{2}\Big(u_{N,j+1} - 2hK_2(u_{N+1,j+1} - v_2)\Big) \\
& \text{LHS :} \quad -Ru_{N,j+1} + \Big(1 + R(1+hK_2))u_{N+1,j+1} - (RhK_2v_2) \\
& \text{RHS :} \quad \frac{R}{2}u_{N,j} + (1-R)u_{N+1,j+1} + \frac{R}{2}\Big(u_{N,j} - 2hK_2(u_{N+1,j} - v_2)\Big) \\
& \text{RHS :} \quad Ru_{N,j} + \Big(1 - R(1+hK_2)\Big)u_{N+1,j} + (RhK_2v_2)
\end{aligned}$$

Matrix presentation :

$$\textbf{LHS}: \begin{bmatrix} 1+R(1+hK_1) & -R & 0 & \cdots & \cdots & 0 \\ -\frac{R}{2} & 1+R & -\frac{R}{2} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -\frac{R}{2} & 1+R & -\frac{R}{2} \\ 0 & \cdots & \cdots & 0 & -R & 1+R(1+hK_2) \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ \vdots \\ u_{N+1} \end{bmatrix}_{j+1} - \begin{bmatrix} RhK_1v_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ RhK_2v_2 \end{bmatrix}$$

$$\textbf{RHS}: \begin{bmatrix} 1-R(1+hK_1) & R & 0 & \cdots & \cdots & 0 \\ \frac{R}{2} & 1-R & \frac{R}{2} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \frac{R}{2} & 1-R & \frac{R}{2} \\ 0 & \cdots & \cdots & 0 & R & 1-R(1+hK_2) \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ \vdots \\ u_{N+1} \end{bmatrix}_{j} + \begin{bmatrix} RhK_1v_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ RhK_2v_2 \end{bmatrix}$$

$$\text{Define}: \Lambda = \begin{bmatrix} -R(1+hK_1) & R & 0 & \cdots & 0 \\ \frac{R}{2} & -R & \frac{R}{2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \frac{R}{2} & -R & \frac{R}{2} \\ 0 & \cdots & 0 & R & -R(1+hK_2) \end{bmatrix} \Rightarrow \begin{cases} A_{LHS} = I - M \\ A_{RHS} = I + M \end{cases}$$

$$\text{Extract}: \quad \bar{\mathbf{u}}_{j+1} = A_{LHS}^{-1} A_{RHS}\ \mathbf{u}_j + A_{LHS}^{-1}\ 2\bar{b}$$

Given derivative initial conditions, our matrix is not a **c-a-b** matrix, and thus no analytic expression can be used. We shall therefore look for A_{LHS}^{-1} and A_{RHS} eigenvalues :

$$\text{Similar to}: \quad f_k(\underline{A})\bar{v} = f_k(\lambda)\bar{v} \quad \rightarrow \quad \bar{v} = f_k(\underline{A})^{-1} f_k(\lambda)\bar{v}$$

$$\text{Polynomial matrices}: \quad f(A_{LHS})^{-1} f(A_{RHS})\ \bar{v} = f(\lambda_{LHS})^{-1}\ f(\lambda_{RHS})\ \bar{v} = \frac{f(\lambda_{RHS})}{f(\lambda_{LHS})}\ \bar{v}$$

$$\text{Extract } (A_{LHS}^{-1} A_{RHS}) \text{ eigenvalues}: \quad \mu_i(\lambda) = \left| \frac{I_i - \Lambda_i}{I_i + \Lambda_i} \right| \le 1 \quad \forall \quad i$$

$$|1-\Lambda_i| \le |1+\Lambda_i| \ \rightarrow\ 1-\Lambda_i \le \pm(1+\Lambda_i) \quad \Rightarrow \quad \Lambda_i \ge 0 \quad \forall \quad i$$

Using Brauer's theorem, each A matrix produces 3 circles from rows [1, 1< i <N, N] :

$$\begin{aligned}
(i=1) \quad & \left|\lambda + R(1+hK_1)\right| \le R \quad \rightarrow \quad -R \le \lambda + R(1+hK_1) \le R \\
& -R(2+hK_1) \le \lambda \le RhK_1 \quad \Rightarrow \quad \{h, K_1, R\} > 0,\ |\lambda| > 0 \\
(1<i<N) \quad & |\lambda + R| \le R \quad \rightarrow \quad -R \le \lambda + R \le R \\
& -2R \le \lambda \le 0 \quad \Rightarrow \quad R > 0, \quad |\lambda| \ge 0 \\
(i=N) \quad & \left|\lambda + R(1+hK_2)\right| \le R \quad \rightarrow \quad -R \le \lambda + R(1+hK_2) \le R \\
& -R(2+hK_2) \le \lambda \le RhK_2 \quad \Rightarrow \quad \{h, K_2, R\} > 0,\ |\lambda| > 0
\end{aligned}$$

All λ_i are greater than zero, and thus satisfy stability criterion of eigenvalues $\mu_i \le 1$. Thus the *Crank-Nicolson* method is unconditionally stable, and is variables independent.

3

For the solution of the convection-diffusion equation:

$$-u\frac{\partial w}{\partial x} + D\frac{\partial^2 w}{\partial x^2} = 0 \quad (I)$$

(where u is the velocity and D is the diffusion coefficient, both of which are constant) the following numerical method is proposed:

$$w_{i+1} - \left(1 + e^{uh/D}\right)w_i + e^{uh/D}w_{i-1} = 0 \quad (II)$$

which can be solved iteratively using

$$w_i^{(n+1)} = \frac{1}{\left(1+e^{uh/D}\right)}\left[w_{i+1}^{(n)} + e^{uh/D}w_{i-1}^{(n)}\right], \quad i = 0,1,2,\ldots. \qquad (III)$$

(a) Show that under certain conditions equation *(II)* reduces to the finite difference equation that would be obtained by writing *(I)* using central differences.

(b) Prove that the iterative procedure *(III)* will not diverge.

(a) We'll express the exponent by using first order approximation of power series :

$$e^{\frac{uh}{D}} = \sum_{n=0}^{\infty}(\frac{uh}{D})^n = \sum_{n=0}^{1}(\frac{uh}{D})^n + \text{H.O.T} = 1 + \frac{uh}{D}$$

$$\text{Equation} \quad (II): \qquad w_{i+1} - (1+e^{\frac{uh}{D}})w_i + e^{\frac{uh}{D}}w_{i-1} = 0 \tag{3.1}$$

$$\text{Plug inside :} \quad w_{i+1} - (2+\frac{uh}{D})w_i + (1+\frac{uh}{D})w_{i-1} = 0 \tag{3.2}$$

$$\backslash \cdot D \quad \rightarrow \quad Dw_{i+1} - (2D+uh)w_i + (D+uh)w_{i-1} = 0$$

$$\text{Sort by coefficients :} \qquad -uh(w_i - w_{i-1}) + D(w_{i+1} - 2w_i + w_{i-1}) = 0 \quad \backslash \cdot \frac{1}{h^2}$$

$$\text{We get :} \qquad -u\cdot(\frac{w_i - w_{i-1}}{h}) + D\cdot(\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}) = 0 \tag{3.3}$$

The final equation is comprised of 2nd central F.Ds and 1st order backward F.Ds. To ensure reduction into central F.Ds of both derivatives, we must nullify u such as :

$$-\cancelto{0}{u}\cdot(\frac{w_i - w_{i-1}}{h}) + D\cdot(\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}) = 0$$

$$-\cancelto{0}{u}\cdot\frac{\partial u}{\partial x} + D\frac{\partial^2 u}{\partial x^2} = 0 \quad \Rightarrow \quad D\frac{\partial^2 u}{\partial x^2} = 0 \tag{3.4}$$

(b) Using (Eqn. 3.2) in its iterative form (III) :

$$w_i^{(n+1)} = \frac{1}{(1+e^{\frac{uh}{D}})}\left[w_{i+1}^{(n)} + e^{\frac{uh}{D}}w_{i-1}^{(n)}\right]$$

$$w_i^{(n+1)} = \frac{e^{\frac{uh}{D}}}{(1+e^{\frac{uh}{D}})}w_{i-1}^{(n)} + \frac{1}{(1+e^{\frac{uh}{D}})}w_{i+1}^{(n)}$$

$$\text{as vector} \;\; : \quad w_i^{(n+1)} = \mathbf{c}\cdot w_{i-1}^{(n)} + \mathbf{b}\cdot w_{i+1}^{(n)}$$

For convenience, let us nullify constant i.c. and b.c such that :

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ \vdots \\ w_N \end{bmatrix}^{(n+1)} = \begin{bmatrix} 0 & b & 0 & \cdots & 0 \\ c & 0 & b & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b \\ 0 & \cdots & \cdots & c & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ \vdots \\ w_N \end{bmatrix}^{(n)} + \begin{bmatrix} c\,\cancelto{0}{w_0} \\ 0 \\ \vdots \\ 0 \\ b\,\cancelto{0}{w_{N+1}} \end{bmatrix}$$

Using Gershgorin theorem ($A \equiv$ matrix $\uparrow$), and asking :

$$\rho(\underline{A}) = \max_{1\leq s\leq N} |\lambda_s| \leq S = \max_{1\leq i\leq N} S_i = \max \{\ b,\ c,\ b+c\ \}$$

$$(i) \qquad |\lambda - 0| \leq b \quad \rightarrow \quad |\lambda|\ \leq \frac{1}{(1+e^{\frac{uh}{D}})} \ \rightarrow \leq 1$$

$$(ii) \qquad |\lambda - 0| \leq c \quad \rightarrow \quad |\lambda|\ \leq \frac{1}{(1+e^{-\frac{uh}{D}})} \ \rightarrow \leq 1$$

$$(iii) \qquad |\lambda - 0| \leq b + c \quad \rightarrow \quad |\lambda| \leq \frac{1}{(1+e^{\frac{uh}{D}})} + \frac{e^{\frac{uh}{D}}}{(1+e^{\frac{uh}{D}})} \ \rightarrow = 1$$

Maximal eigenvalue is not greater than 1, and thus $\rho(A)\ \leq\ 1$ is fulfilled without constraints on any of the parameters.

4

Suppose we want to solve the following problem using Crank-Nicholson's method:

$$0 \le x \le 1: \quad \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

$$U(0,t) = U(1,t) = 0 \text{ and } U(x,0) = U0(x)$$

Instead of using a tridiagonal solver for the set of algebraic equations we get, we will use the following iterative method:

$$u_{i,j+1}^{(n+1)} = \frac{R}{2}\left(u_{i-1,j+1}^{(n)} - 2u_{i,j+1}^{(n)} + u_{i+1,j+1}^{(n)}\right) + \frac{R}{2}\left(u_{i-1,j} + u_{i+1,j}\right) + (1-R)u_{i,j}$$

where $R = k/h^2$, $i = 1,2,3....N$, $j = 0,1,2,3......$ and (n) represents the iterative index. Prove (mathematically) that the unconditional stability of the Crank-Nicholson method is lost due to a restrictive condition on R for iterative convergence.

Let us write the PDE under C-N using central finite differences :

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{2}\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}\right)$$

Define $R \equiv \frac{k}{h^2}$, we get :

$$-\frac{R}{2}u_{i-1,j+1} + u_{i,j+1}(1+R) - \frac{R}{2}u_{i+1,j+1} = \frac{R}{2}u_{i-1,j} + u_{i,j}(1-R) + \frac{R}{2}u_{i+1,j}$$

Sort iteratively as required :

$$u_{i,j+1}^{(n+1)} = \frac{R}{2}\left(u_{i-1,j+1}^{(n)} - 2u_{i,j+1}^{(n)} + u_{i+1,j+1}^{(n)}\right) + \frac{R}{2}u_{i-1,j} + u_{i,j}(1-R) + \frac{R}{2}u_{i+1,j}$$

i.c and b.c : $u(0,t) = u_{0,j} = 0, \qquad u(x_f,t) = u_{N+1,j} = 0, \qquad u(x,0) = u_{i,0} = U0$

We get the following system where only the first matrix is iteration dependent :

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_N \end{bmatrix}^{(n+1)} = \begin{bmatrix} -R & \frac{R}{2} & 0 & \cdots & 0 \\ \frac{R}{2} & -R & \frac{R}{2} & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{R}{2} \\ 0 & \cdots & 0 & \frac{R}{2} & -R \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_N \end{bmatrix}^{(n)} + \begin{bmatrix} 1-R & \frac{R}{2} & 0 & \cdots & 0 \\ \frac{R}{2} & 1-R & \frac{R}{2} & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{R}{2} \\ 0 & \cdots & 0 & \frac{R}{2} & 1-R \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_N \end{bmatrix}^{(n=1)}$$

Using Brauer theorem to check stability criterion :

$$\begin{aligned}
\rho(\underline{A}) = \max_{1\le s\le N}|\lambda_s| \le \max_{1\le i\le N} S_i \le 1 \\
(i=1) \quad |\lambda + R| \le \frac{R}{2} \quad \rightarrow \quad -\frac{R}{2} \le \lambda + R \le \frac{R}{2} \\
-\frac{3}{2}R \le \lambda \le -\frac{R}{2} \quad \Rightarrow_{|\lambda|\le 1} \quad R \le \{\frac{2}{3}, 2\} \\
(1 < i < N) \quad |\lambda + R| \le R \quad \rightarrow \quad -R \le \lambda + R \le R \\
-2R \le \lambda \le 0 \quad \Rightarrow_{|\lambda|\le 1} \quad R \le \{\frac{1}{2}\}
\end{aligned}$$

For stability analysis, one must take the strictest (=minimal) condition, here $R \le \frac{1}{2}$.

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