Technion - Israel Institute of Technology

Numerical Methods in Aeronautical Engineering (086172)

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1

Suppose we want to solve the following linear algebraic system using Jacobi's method:

$$\sum_{j=1}^{N} a_{ij}x_j = b_i, \quad i = 1,2...N$$

Prove that if $\sum_{j=1,\, j\neq i}^{N} |a_{ij}| \leq |a_{ii}|, \quad i = 1,2...N$ iterative convergence is assured. (This condition is known as Diagonal Dominance).

Let us decompose matrix A into 3 components (L - Lower, D - Diagonal, U - Upper) :

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \ddots & \ddots & \cdots \\ \vdots & \ddots & \ddots & \cdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_N \end{bmatrix} \quad \rightarrow \quad \underline{A}\bar{x} = \bar{b} \quad \Leftrightarrow \quad (\underline{L} + \underline{D} + \underline{U})\bar{x} = \bar{b} \tag{1.1}$$

Isolating the diagonal, and express as Jacobi method :

$$\underline{D}\bar{x}^{(n+1)} = \bar{b} - (\underline{L} + \underline{U})\bar{x}^{(n)} \quad \rightarrow \quad \bar{x}^{(n+1)} = \underline{D}^{-1}(\bar{b} - (\underline{L} + \underline{U})\bar{x}^{(n)})$$

Equivalently, $$x_i^{(n+1)} = \frac{1}{a_{ii}}(b_i - \sum_{j=1\neq i}^{N} a_{ij}x_j^{(n)}) = \frac{b_i}{a_{ii}} - \sum_{j=1\neq i}^{N} \frac{a_{ij}}{a_{ii}} x_j^{(n)} \;, \quad i = 1, 2, ..., N \tag{1.2}$$

Constant elements (b_i/a_{ii}) are subtracted between iterations, so we're interested in :

$$\underline{C} = \sum_{j=1\neq i}^{N} \frac{a_{ij}}{a_{ii}}, \qquad i = 1, 2, ..., N \tag{1.3}$$

Using Gershgorin theorem to asses C's eigenvalues :

$$\rho(\underline{C}) = \max_{1\leq s\leq N} |\lambda_s| \leq S = \max_{1\leq i\leq N} S_i = \max_{1\leq i\leq N} \sum_{j=1\neq i}^{N} \left|\frac{a_{ij}}{a_{ii}}\right|$$

$$\text{Let } a_{ii} \text{ satisfy: } \quad \sum_{j=1\neq i}^{N} |a_{ij}| \leq |a_{ii}| \quad \text{such that :} \quad \sum_{j=1\neq i}^{N} \left|\frac{a_{ij}}{a_{ii}}\right| \leq 1 \tag{1.4}$$

$$\text{here :} \quad \rho(\underline{C}) = \max_{1\leq i\leq N} \sum_{j=1\neq i}^{N} \left|\frac{a_{ij}}{a_{ii}}\right| = \sum_{j=1\neq i}^{N} \left|\frac{a_{ij}}{a_{ii}}\right| \leq 1 \tag{1.5}$$

Since a_{ii} is diagonally dominant, the spectral radius is less than or equal to 1, at every row.

2

Use Brauer's theorem to investigate how boundary conditions influence the stability of Crank-Nicholson's method for the numerical solution of the following problem:

$$0 \leq x \leq 1 : \quad \frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}$$

subject to the initial and boundary conditions:

$U(x,0) = U_0(x)$ where $U_0(x)$ is a given function

$$x = 0, t \geq 0, \frac{\partial U}{\partial x} = K_1(U - v_1)$$

$$x = 1, t \geq 0, \frac{\partial U}{\partial x} = -K_2(U - v_2)$$

where $K_1 > 0, K_2 > 0, v_1, v_2$ are constants.

Let us write the PDE under C-N using central finite differences :

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Big[(\frac{\partial^2 u}{\partial x^2})_j + (\frac{\partial^2 u}{\partial x^2})_{j+1}\Big] \tag{2.1}$$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{2}(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}) \tag{2.2}$$

Define $R \equiv \frac{k}{h^2}$, we get :

$$-\frac{R}{2}u_{i-1,j+1} + u_{i,j+1}(1+R) - \frac{R}{2}u_{i+1,j+1} = \frac{R}{2}u_{i-1,j} + u_{i,j}(1-R) + \frac{R}{2}u_{i+1,j} \tag{2.3}$$

Both sides' conditions are defined by derivatives, such that index becomes - i $\in$ [0, N+1] :

$$\frac{\partial u}{\partial x}\Big|_{x=0}^{t\geq 0} = \frac{u_{1,j} - u_{-1,j}}{2h} = K_1(u_{0,j} - v_1), \quad \Rightarrow \quad \begin{cases} u_{-1,j} = u_{1,j} - 2hK_1(u_{0,j} - v_1) \\ u_{-1,j+1} = u_{1,j+1} - 2hK_1(u_{0,j+1} - v_1) \end{cases}$$

$$\frac{\partial u}{\partial x}\Big|_{x=1}^{t\geq 0} = \frac{u_{N+2,j} - u_{N,j}}{2h} = -K_2(u_{N+1,j} - v_2), \quad \Rightarrow \quad \begin{cases} u_{N+2,j} = u_{N,j} - 2hK_2(u_{N+1,j} - v_2) \\ u_{N+2,j+1} = u_{N,j+1} - 2hK_2(u_{N+1,j+1} - v_2) \end{cases}$$

Check the first and last rows :

$$\begin{aligned}
i = 0 \ : \quad & -\frac{R}{2}u_{-1,j+1} + (1+R)u_{0,j+1} - \frac{R}{2}u_{1,j+1} = \frac{R}{2}u_{-1,j} + (1-R)u_{0,j} + \frac{R}{2}u_{1,j} \\
& \text{LHS :} \quad -\frac{R}{2}\Big(u_{1,j+1} - 2hK_1(u_{0,j+1} - v_1)\Big) + (1+R)u_{0,j+1} - \frac{R}{2}u_{1,j+1} \\
& \text{LHS :} \quad \Rightarrow \quad \Big(1 + R(1 + hK_1)\Big)u_{0,j+1} - Ru_{1,j+1} - (RhK_1v_1) \\
& \text{RHS :} \quad \frac{R}{2}\Big(u_{1,j} - 2hK_1(u_{0,j} - v_1)\Big) + (1-R)u_{0,j} + \frac{R}{2}u_{1,j} \\
& \text{RHS :} \quad \Rightarrow \quad \Big(1 - R(1 + hK_1)\Big)u_{0,j} + Ru_{1,j} + (RhK_1v_1) \\
i = N+1 \ : \quad & -\frac{R}{2}u_{N,j+1} + (1+R)u_{N+1,j+1} - \frac{R}{2}u_{N+2,j+1} = \frac{R}{2}u_{N,j} + (1-R)u_{N+1,j} + \frac{R}{2}u_{N+2,j} \\
& \text{LHS :} \quad -\frac{R}{2}u_{N,j+1} + (1+R)u_{N+1,j+1} - \frac{R}{2}\Big(u_{N,j+1} - 2hK_2(u_{N+1,j+1} - v_2)\Big) \\
& \text{LHS :} \quad -Ru_{N,j+1} + \Big(1 + R(1 + hK_2))u_{N+1,j+1} - (RhK_2v_2) \\
& \text{RHS :} \quad \frac{R}{2}u_{N,j} + (1-R)u_{N+1,j+1} + \frac{R}{2}\Big(u_{N,j} - 2hK_2(u_{N+1,j} - v_2)\Big) \\
& \text{RHS :} \quad Ru_{N,j} + \Big(1 - R(1 + hK_2)\Big)u_{N+1,j} + (RhK_2v_2)
\end{aligned}$$

Matrix presentation :

$$\textbf{LHS}: \begin{bmatrix} 1+R(1+hK_1) & -R & 0 & \cdots & \cdots & 0 \\ -\frac{R}{2} & 1+R & -\frac{R}{2} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -\frac{R}{2} & 1+R & -\frac{R}{2} \\ 0 & \cdots & \cdots & 0 & -R & 1+R(1+hK_2) \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ \vdots \\ u_{N+1} \end{bmatrix}_{j+1} - \begin{bmatrix} RhK_1v_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ RhK_2v_2 \end{bmatrix}$$

$$\textbf{RHS}: \begin{bmatrix} 1-R(1+hK_1) & R & 0 & \cdots & \cdots & 0 \\ \frac{R}{2} & 1-R & \frac{R}{2} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \frac{R}{2} & 1-R & \frac{R}{2} \\ 0 & \cdots & \cdots & 0 & R & 1-R(1+hK_2) \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ \vdots \\ u_{N+1} \end{bmatrix}_{j} + \begin{bmatrix} RhK_1v_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ RhK_2v_2 \end{bmatrix}$$

$$\text{Define}: \quad \mathbf{M} = \begin{bmatrix} -R(1+hK_1) & R & 0 & \cdots & 0 \\ \frac{R}{2} & -R & \frac{R}{2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \frac{R}{2} & -R & \frac{R}{2} \\ 0 & \cdots & 0 & R & -R(1+hK_2) \end{bmatrix} \Rightarrow \begin{cases} A_{LHS} = I - M \\ A_{RHS} = I + M \end{cases}$$

$$\text{Extract}: \quad \bar{\mathbf{u}}_{j+1} = A_{LHS}^{-1} A_{RHS}\ \mathbf{u}_j + A_{LHS}^{-1}\ 2\bar{b}$$

Given derivative initial conditions, our matrix is not a **c-a-b** matrix, and thus no analytic expression can be used. We shall therefore look for $(A_{RHS}^{-1} A_{LHS})$ eigenvalues :

$$\text{Similar to}: \quad f_k(\underline{A})\bar{v} = f_k(\lambda)\bar{v} \quad \to \quad \bar{v} = f_k(\underline{A})^{-1} f_k(\lambda)\bar{v}$$

$$\text{Polynomial matrices}: \quad f(A_{LHS})^{-1} f(A_{RHS})\ \bar{v} = f(\lambda_{LHS})^{-1}\ f(\lambda_{RHS})\ \bar{v} = \frac{f(\lambda_{RHS})}{f(\lambda_{LHS})}\ \bar{v}$$

$$\text{Denote } (A_{LHS}^{-1} A_{RHS}) \text{ eigenvalues}: \quad \mu_i(\xi) = \frac{f(\xi_{RHS})}{f(\xi_{LHS})} = \frac{1+\xi}{1-\xi}, \qquad |\mu_i| \le 1 \quad \forall \quad i$$

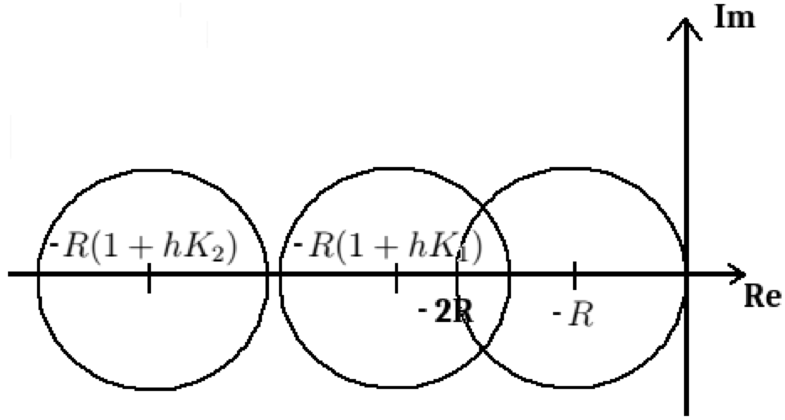
Since M appears in both A_{LHS} and A_{RHS}, eigenvalues must be less than or equal to one :

$$\left|\frac{1+\xi}{1-\xi}\right| \leq 1 \quad \rightarrow \quad \xi \leq 0$$

Using Brauer's theorem, M produces 3 different circles from rows [1, 1< i <N, N] :

$$\begin{array}{lll}
(i) & \big|\xi + R(1+hK_1)\big| \leq R & \rightarrow \quad -R(2+hK_1) \leq \xi \leq -RhK_1 \\
(ii) & |\xi + R| \leq R & \rightarrow \quad -2R \leq \xi \leq 0 \\
(iii) & \big|\xi + R(1+hK_2)\big| \leq R & \rightarrow \quad -R(2+hK_2) \leq \xi \leq -RhK_2
\end{array}$$

For the sake of convenience and visualization I set $(K_2 > K_1 > h > 0)$:



All <u>centers</u> of circles are at least a radius (=R) away from the origin, and thus satisfy :

$$|\xi| \leq 0 \quad \Rightarrow \quad \mu_i(\lambda) = \frac{1+\xi}{1-\xi} \leq 1 \quad \text{(Spectral radius)}$$

3

For the solution of the convection-diffusion equation:

$$-u\frac{\partial w}{\partial x}+D\frac{\partial^2 w}{\partial x^2}=0 \quad (I)$$

(where *u* is the velocity and *D* is the diffusion coefficient, both of which are constant) the following numerical method is proposed:

$$w_{i+1}-\left(1+e^{uh/D}\right)w_i+e^{uh/D}w_{i-1}=0 \quad (II)$$

which can be solved iteratively using

$$w_i^{(n+1)}=\frac{1}{\left(1+e^{uh/D}\right)}\left[w_{i+1}^{(n)}+e^{uh/D}w_{i-1}^{(n)}\right], \quad i=0,1,2,.... \qquad (III)$$

(a) Show that under certain conditions equation *(II)* reduces to the finite difference equation that would be obtained by writing *(I)* using central differences.
(b) Prove that the iterative procedure *(III)* will not diverge.

(a) We'll express the exponent by using first order approximation of power series :

$$e^{\frac{uh}{D}}=\sum_{n=0}^{\infty}(\frac{uh}{D})^n=\sum_{n=0}^{1}(\frac{uh}{D})^n+\text{H.O.T}=1+\frac{uh}{D}$$

$$\text{Equation} \quad (II): \qquad w_{i+1}-(1+e^{\frac{uh}{D}})w_i+e^{\frac{uh}{D}}w_{i-1}=0 \qquad (3.1)$$

$$\text{Plug inside :} \quad w_{i+1}-(2+\frac{uh}{D})w_i+(1+\frac{uh}{D})w_{i-1}=0 \qquad (3.2)$$

$$\backslash\cdot D \quad \rightarrow \quad Dw_{i+1}-(2D+uh)w_i+(D+uh)w_{i-1}=0$$

$$\text{Sort by coefficients :} \qquad -uh(w_i-w_{i-1})+D(w_{i+1}-2w_i+w_{i-1})=0 \quad \backslash\cdot\frac{1}{h^2}$$

$$\text{We get :} \qquad -u\cdot(\frac{w_i-w_{i-1}}{h})+D\cdot(\frac{w_{i+1}-2w_i+w_{i-1}}{h^2})=0 \qquad (3.3)$$

The final equation is comprised of 2nd central F.Ds and 1st order backward F.Ds. To ensure reduction into central F.Ds of both derivatives, we must nullify u such as :

$$-\cancel{u}^{0}\cdot(\frac{w_i-w_{i-1}}{h})+D\cdot(\frac{w_{i+1}-2w_i+w_{i-1}}{h^2})=0$$

$$-\cancel{u}^{0}\cdot\frac{\partial u}{\partial x}+D\frac{\partial^2 u}{\partial x^2}=0 \quad \Rightarrow \quad D\frac{\partial^2 u}{\partial x^2}=0 \qquad (3.4)$$

(b) Using (Eqn. 3.2) in its iterative form (III) :

$$
\begin{aligned}
w_i^{(n+1)} &= \frac{1}{(1+e^{\frac{uh}{D}})}\left[w_{i+1}^{(n)} + e^{\frac{uh}{D}}\, w_{i-1}^{(n)}\right] = 0 \\
w_i^{(n+1)} &= \frac{e^{\frac{uh}{D}}}{(1+e^{\frac{uh}{D}})} w_{i-1}^{(n)} + \frac{1}{(1+e^{\frac{uh}{D}})} w_{i+1}^{(n)} = 0 \\
\text{as vector} \ : \quad w_i^{(n+1)} &= \mathbf{c}\cdot w_{i-1}^{(n)} + \mathbf{b}\cdot w_{i+1}^{(n)} = 0
\end{aligned}
$$

For convenience, let us nullify constant i.c. and b.c such that :

$$
\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ \vdots \\ w_N \end{bmatrix}^{(n+1)} = \begin{bmatrix} 0 & b & 0 & \cdots & 0 \\ c & 0 & b & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b \\ 0 & \cdots & \cdots & c & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ \vdots \\ w_N \end{bmatrix}^{(n)} + \begin{bmatrix} c\, \cancelto{0}{w_0} \\ 0 \\ \vdots \\ 0 \\ b\, \cancelto{0}{w_{N+1}} \end{bmatrix}
$$

Using Gershgorin theorem ($A \equiv$ matrix $\uparrow$) :

$$
\begin{aligned}
\rho(\underline{A}) = \max_{1\le s\le N} |\lambda_s| \le S &= \max_{1\le i\le N} S_i = \max\ \{\ b,\ c,\ b+c\ \} \\
(i) \qquad |\lambda - 0| \le b \quad &\rightarrow \quad |\lambda| \ \le \frac{1}{(1+e^{\frac{uh}{D}})} \ \rightarrow \le 1 \\
(ii) \qquad |\lambda - 0| \le c \quad &\rightarrow \quad |\lambda| \ \le \frac{1}{(1+e^{-\frac{uh}{D}})} \ \rightarrow \le 1 \\
(iii) \qquad |\lambda - 0| \le b+c \quad &\rightarrow \quad |\lambda| \le \ \frac{1}{(1+e^{\frac{uh}{D}})} + \frac{e^{\frac{uh}{D}}}{(1+e^{\frac{uh}{D}})} \ \rightarrow = 1
\end{aligned}
$$

Maximal eigenvalue is not greater than 1, and thus $\rho(A) \ \le \ 1$ is fulfilled without constraints on any of the parameters.

4

Suppose we want to solve the following problem using Crank-Nicholson's method:

$$0 \le x \le 1: \quad \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

$$U(0,t) = U(1,t) = 0 \text{ and } U(x,0) = U0(x)$$

Instead of using a tridiagonal solver for the set of algebraic equations we get, we will use the following iterative method:

$$u_{i,j+1}^{(n+1)} = \frac{R}{2}\left(u_{i-1,j+1}^{(n)} - 2u_{i,j+1}^{(n)} + u_{i+1,j+1}^{(n)}\right) + \frac{R}{2}\left(u_{i-1,j} + u_{i+1,j}\right) + \left(1-R\right)u_{i,j}$$

where $R = k/h^2$, $i = 1,2,3....N$, $\quad j = 0,1,2,3......$ and (n) represents the iterative index. Prove (mathematically) that the unconditional <u>stability</u> of the Crank-Nicholson method is lost due to a restrictive condition on R for <u>iterative convergence</u>.

Let us write the PDE under C-N using central finite differences :

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{2}\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}\right)$$

Define $R \equiv \frac{k}{h^2}$, we get :

$$-\frac{R}{2}u_{i-1,j+1} + u_{i,j+1}(1+R) - \frac{R}{2}u_{i+1,j+1} = \frac{R}{2}u_{i-1,j} + u_{i,j}(1-R) + \frac{R}{2}u_{i+1,j}$$

Sort iteratively as required :

$$u_{i,j+1}^{(n+1)} = \frac{R}{2}\left(u_{i-1,j+1}^{(n)} - 2u_{i,j+1}^{(n)} + u_{i+1,j+1}^{(n)}\right) + \frac{R}{2}u_{i-1,j} + u_{i,j}(1-R) + \frac{R}{2}u_{i+1,j}$$

$$-fin-$$