

# MIN-COST NETWORK FLOW

## PROBLEM EXAMPLES:

### Shortest Path Problem

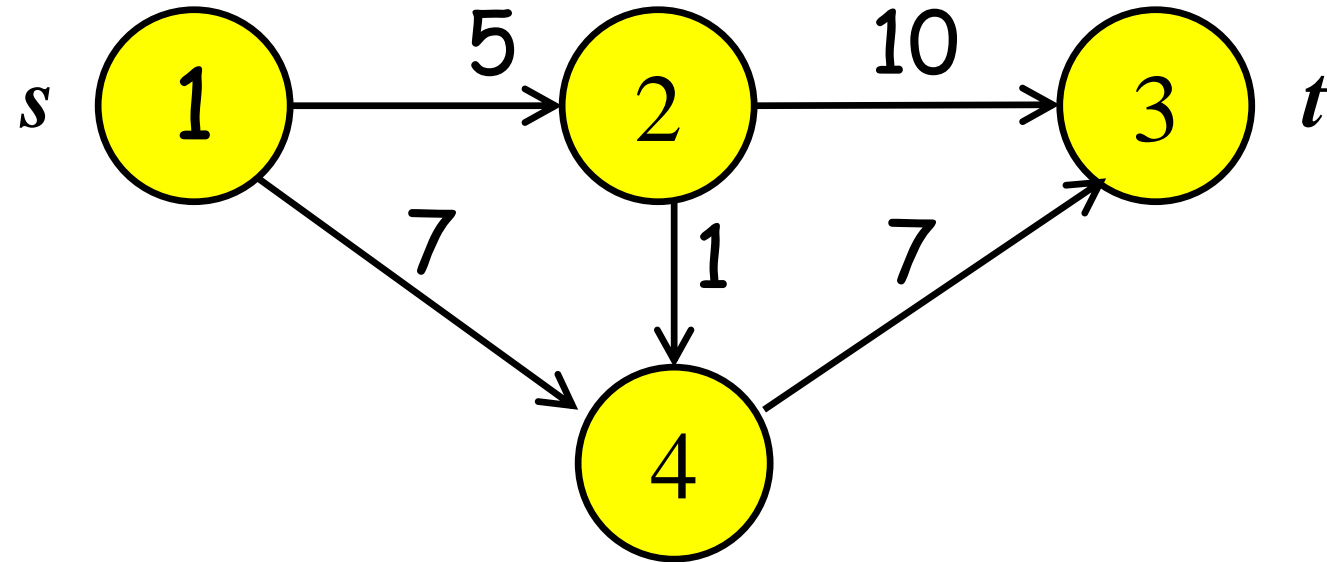
# Shortest Path Problem (MCNFP Formulation)

- Defined on a network with two special nodes:  $s$  and  $t$
- A *path* from  $s$  to  $t$  is an alternating sequence of distinct nodes and distinct arcs starting at  $s$  and ending at  $t$ :  
 $s, (s, n_1), n_1, (n_1, n_2), \dots, (n_i, n_j), n_j, (n_j, t), t$ 
  - There are *directed* and *undirected* paths
  - A network is *connected* if an undirected path exists between any pairs of nodes
  - If  $s = t$ , the path is called a *cycle*
  - A connected network without cycles is called a *tree*

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 $s, (s, n_1), n_1, (n_1, n_2), \dots, (n_i, n_j), n_j, (n_j, t), t$
- Find a minimum-cost directed path from  $s$  to  $t$

# shortest path example



1,(1,2),2,(2,3),3

Length = 15

1,(1,2),2,(2,4),4,(4,3)

Length = 13

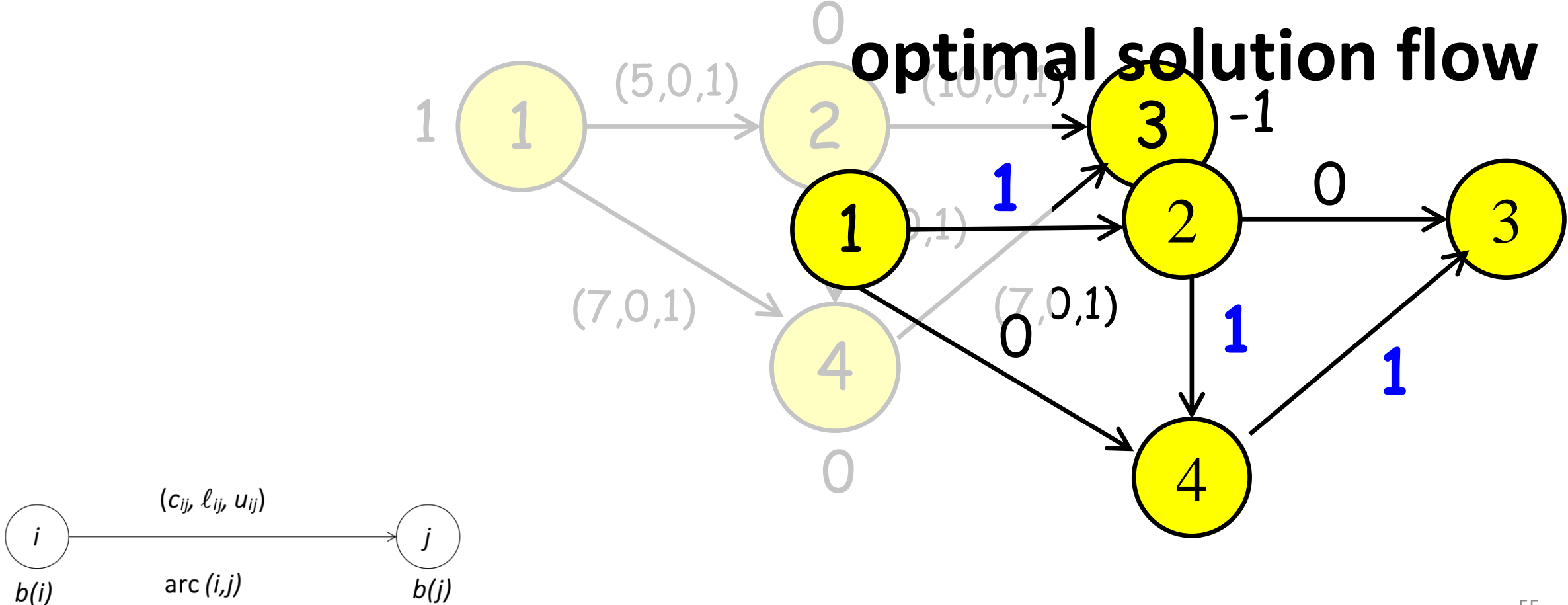
1,(1,4),4,(4,3),3

Length = 14

# MCNFP formulation of shortest path problem

- *Source* node  $s$  has a supply of 1
- *Terminal* node  $t$  has a demand of 1
- All other nodes are transshipment nodes
- Each arc has capacity 1
- Tracing the unit of flow from  $s$  to  $t$  gives a path from  $s$  to  $t$

# shortest path as MCNFP



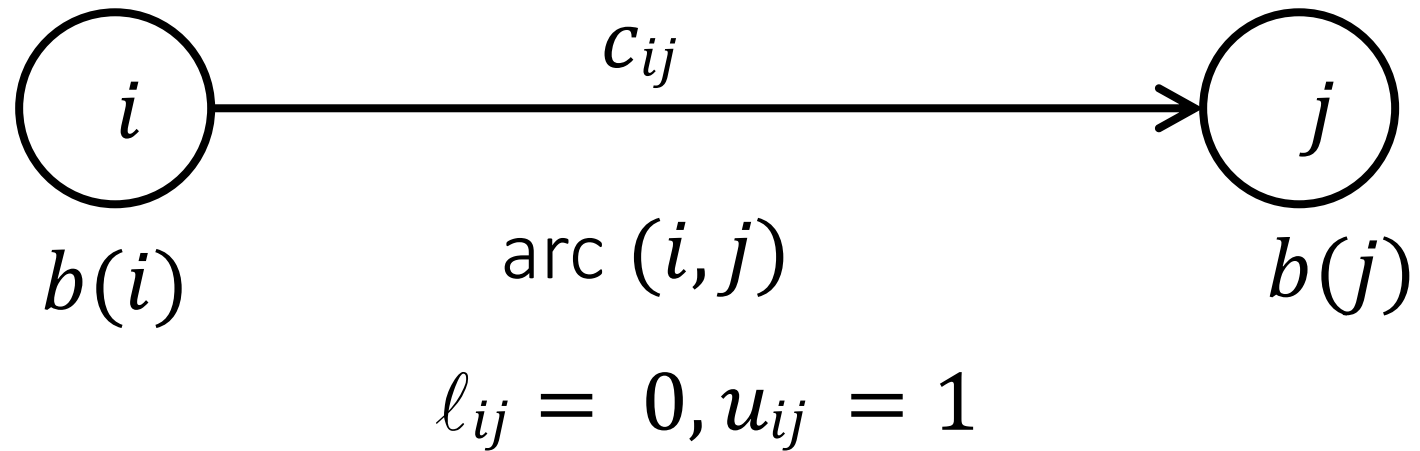
# shortest path example

In a rural area of Texas, there are six farms connected by small roads. The distances in miles between the farms are given in the table.

What is the minimum distance to get from Farm 1 to Farm 6?

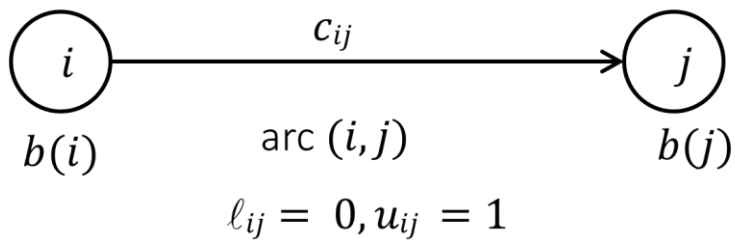
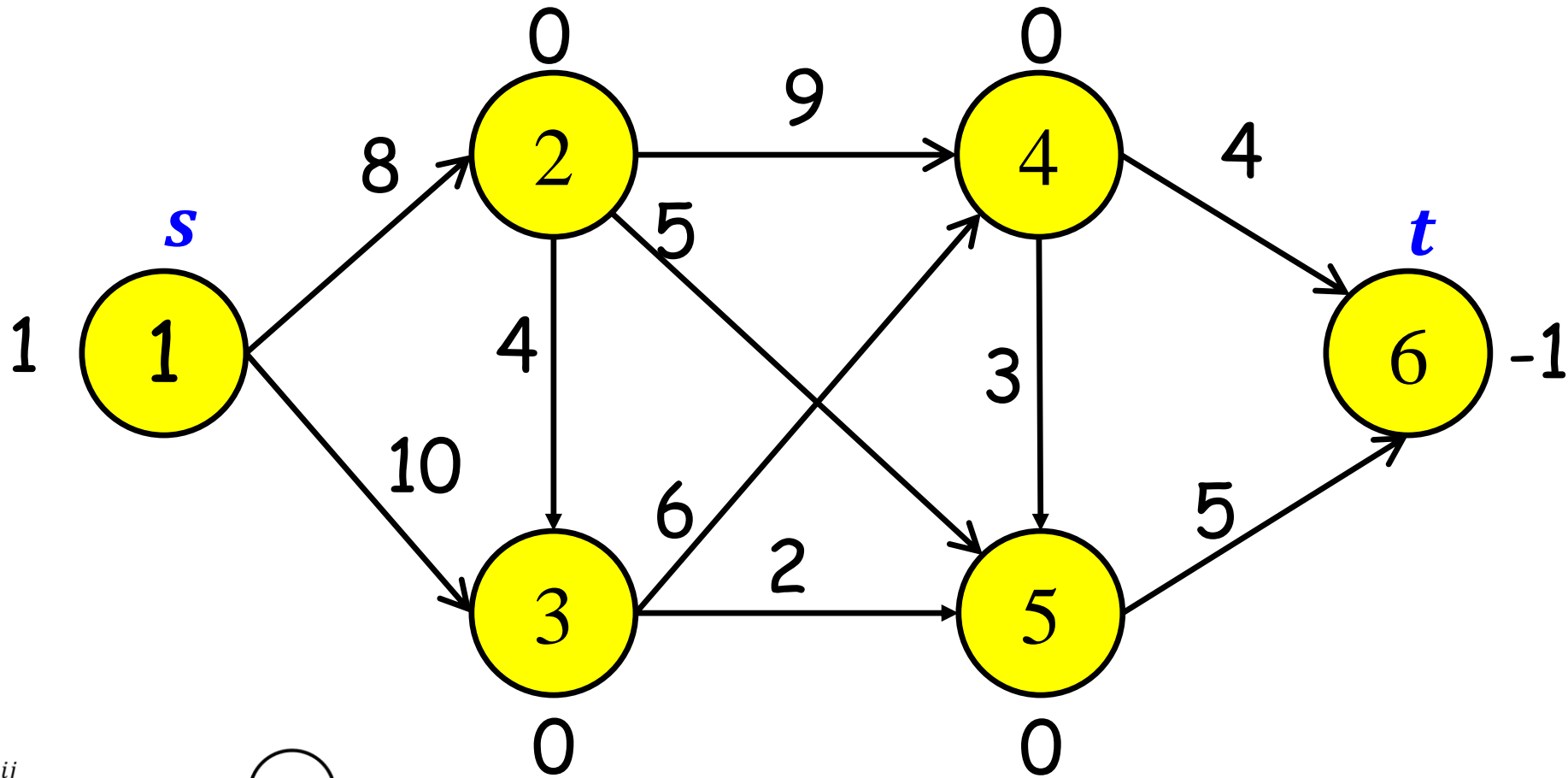
From Farm	To Farm	Distance
1	2	8
1	3	10
2	3	4
2	4	9
2	5	5
3	4	6
3	5	2
4	5	3
4	6	6
5	6	5

# graphical network flow formulation





# formulation as shortest path



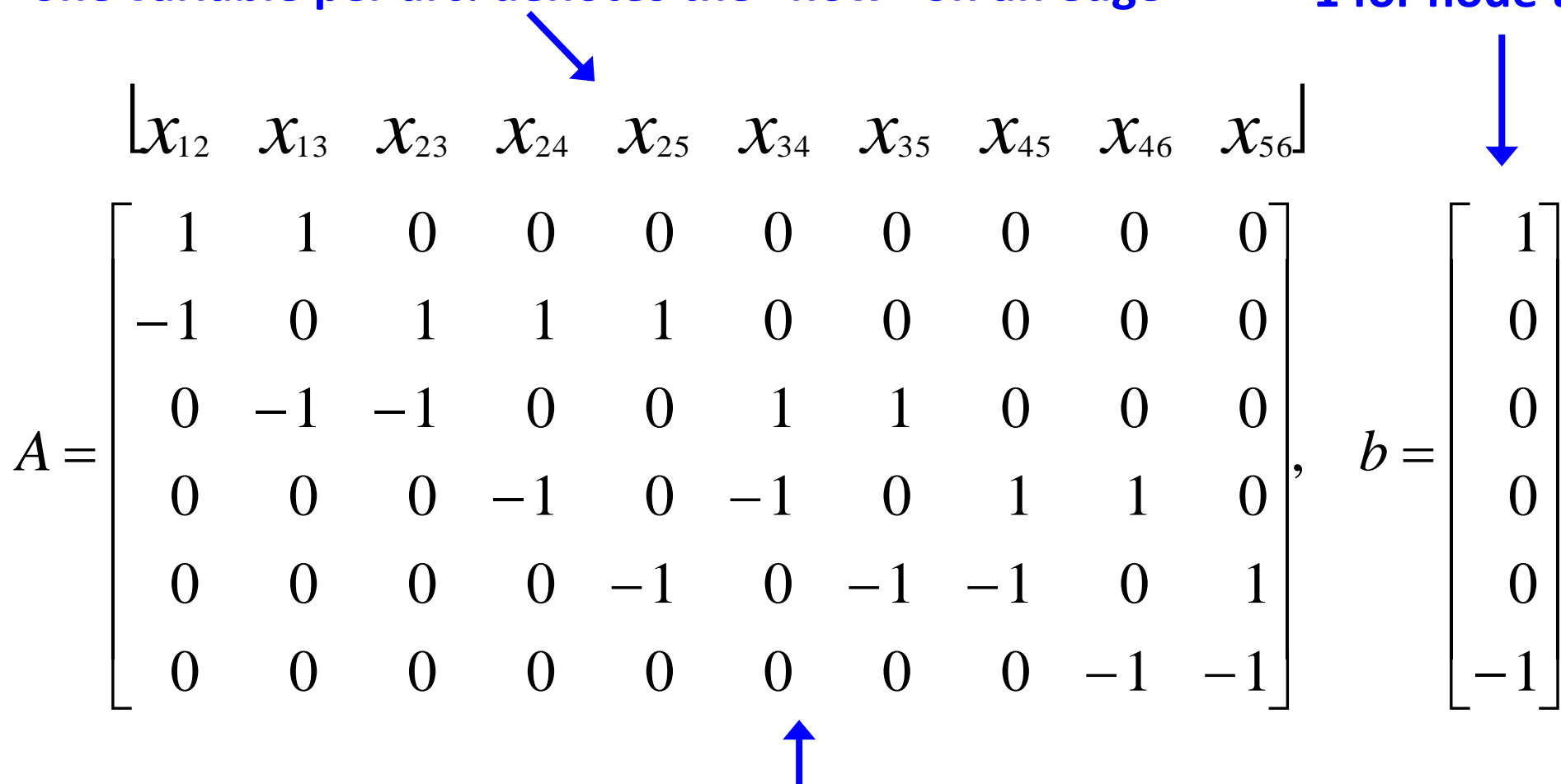
# matrix representation

node requirements:

+1 for node s;

-1 for node t

one variable per arc: denotes the “flow” on an edge



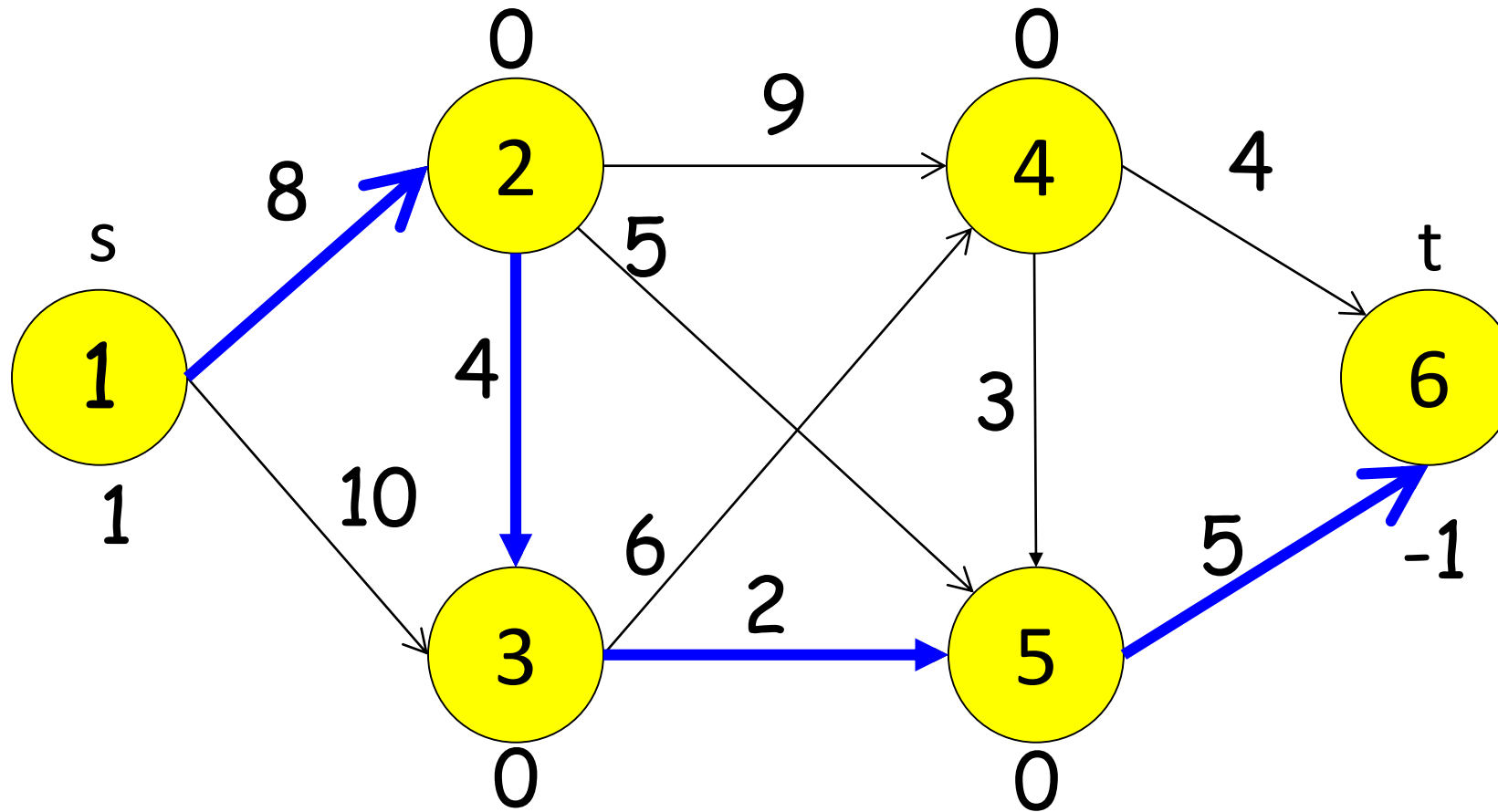
The diagram illustrates the construction of the node-arc incidence matrix  $A$  and the node requirements vector  $b$ . At the top, a list of arc variables  $x_{12}, x_{13}, x_{23}, x_{24}, x_{25}, x_{34}, x_{35}, x_{45}, x_{46}, x_{56}$  is shown, each enclosed in a box. A blue arrow points from the text "one variable per arc: denotes the 'flow' on an edge" to the  $x_{24}$  variable. Another blue arrow points from the text "node requirements: +1 for node s; -1 for node t" to the vector  $b$ . The matrix  $A$  is a 6x10 matrix where each column corresponds to an arc and each row to a node. The entries are 1, -1, or 0, representing the flow balance at each node for each arc. The vector  $b$  is a 6x1 column vector with values 1, 0, 0, 0, 0, -1, representing the net flow requirement at each node.

$$A = \begin{bmatrix} x_{12} & x_{13} & x_{23} & x_{24} & x_{25} & x_{34} & x_{35} & x_{45} & x_{46} & x_{56} \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

node-arc incidence matrix stipulates the “flow-balance constraints”

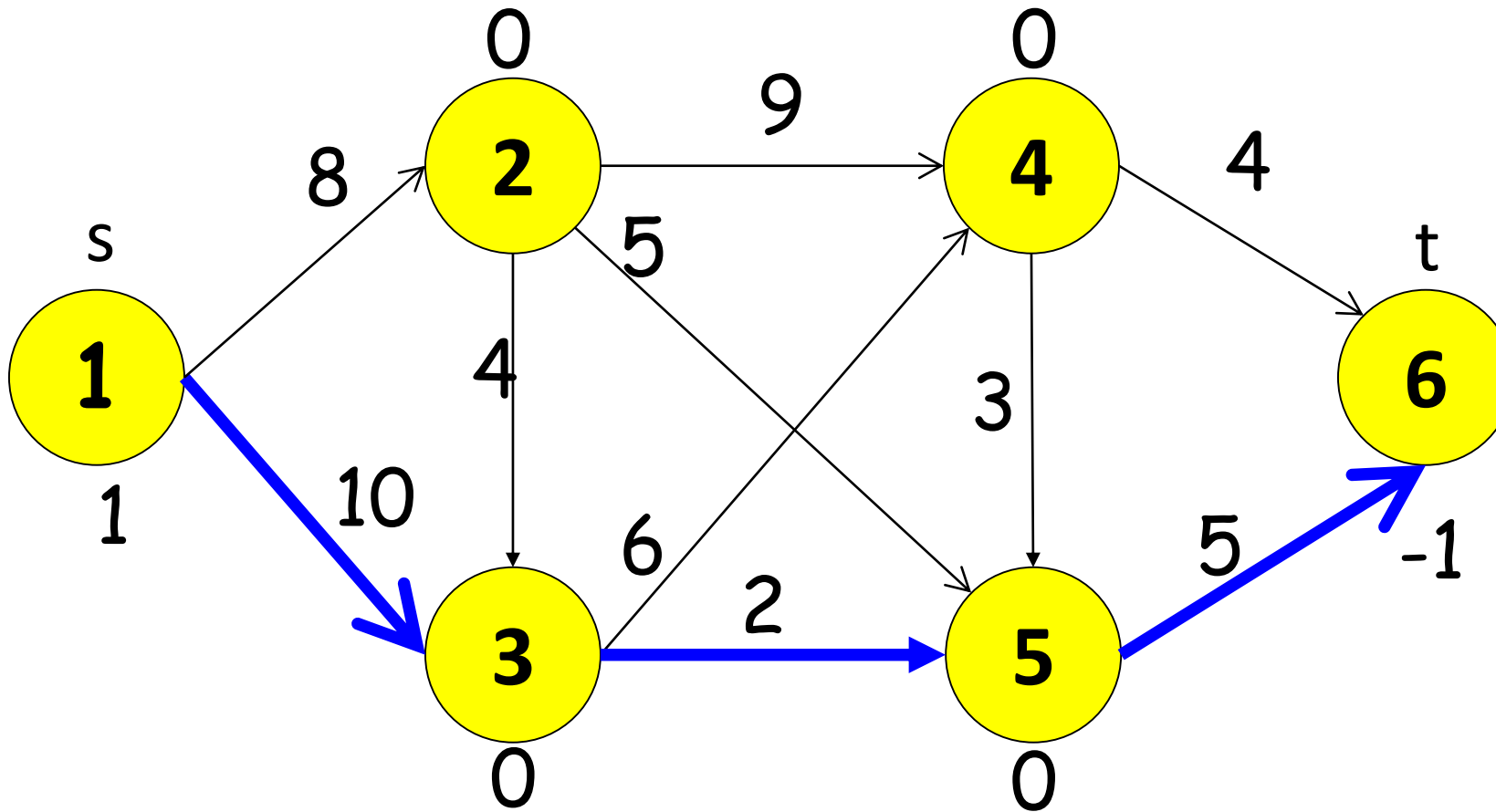
- every column relates to an arc
- every row relates to a node

# “greedy” solution



$x_{13} = x_{23} = x_{35} = x_{56} = 1$ ,  $x_{ij} = 0$  for all other arcs.  
Objective function value = 19.

# shortest path: optimal solution



$x_{13} = x_{35} = x_{56} = 1$ ,  $x_{ij} = 0$  for all other arcs.  
Objective function value = 17.