INTEGER PROGRAMMING: BRANCH AND BOUND

LP relaxation

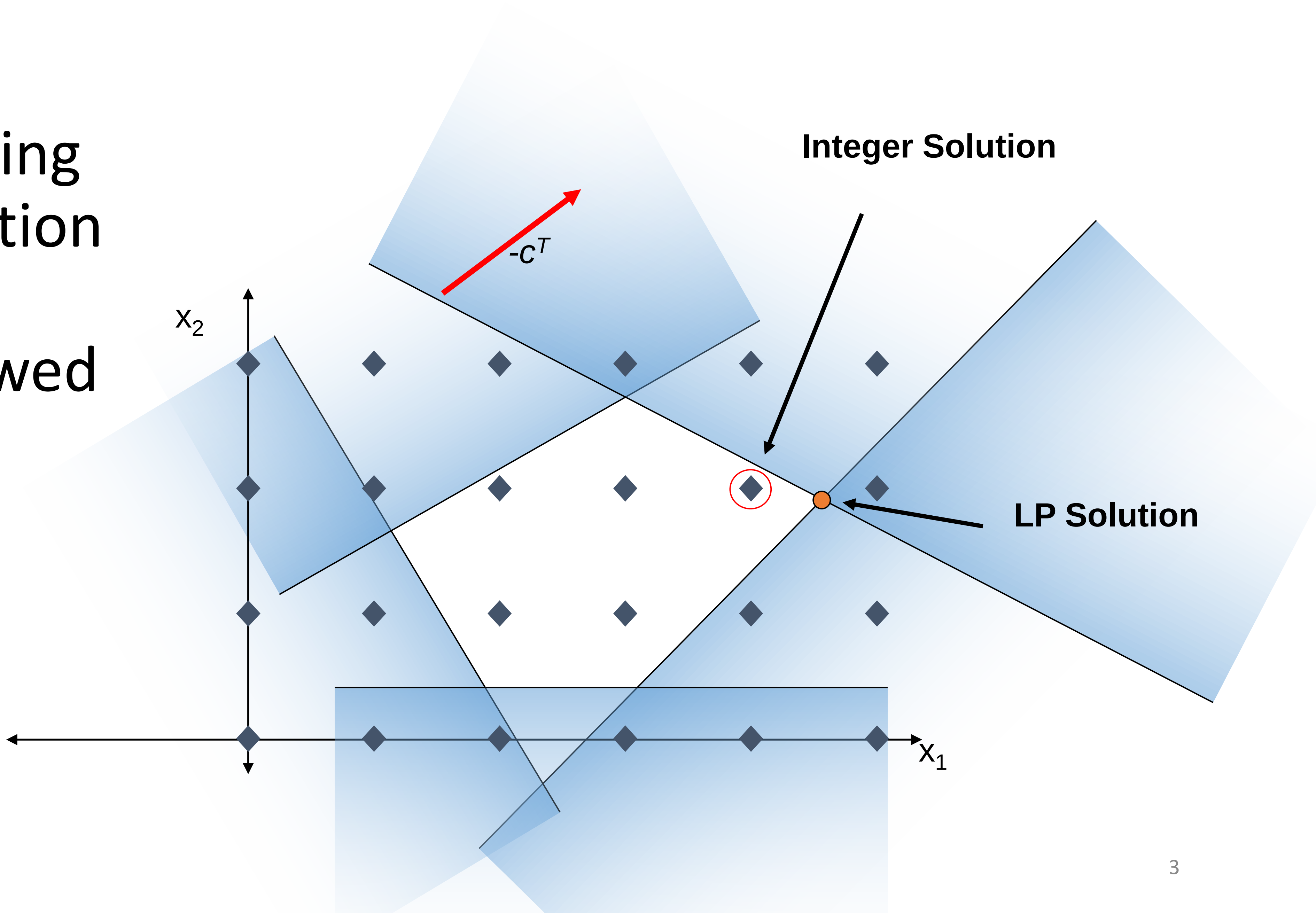
- For any IP we can generate an LP (called the ***LP relaxation***) from the IP by taking the same objective function and same constraints but with the "integer requirement" on variables relaxed
- That is, the variables are allowed to be continuous and bounded by appropriate constraints

e.g., $x_i \in \{0,1\}$ can be replaced by the two continuous constraints $x_i \geq 0$ and $x_i \leq 1$

Binary but continuous

solving an IP

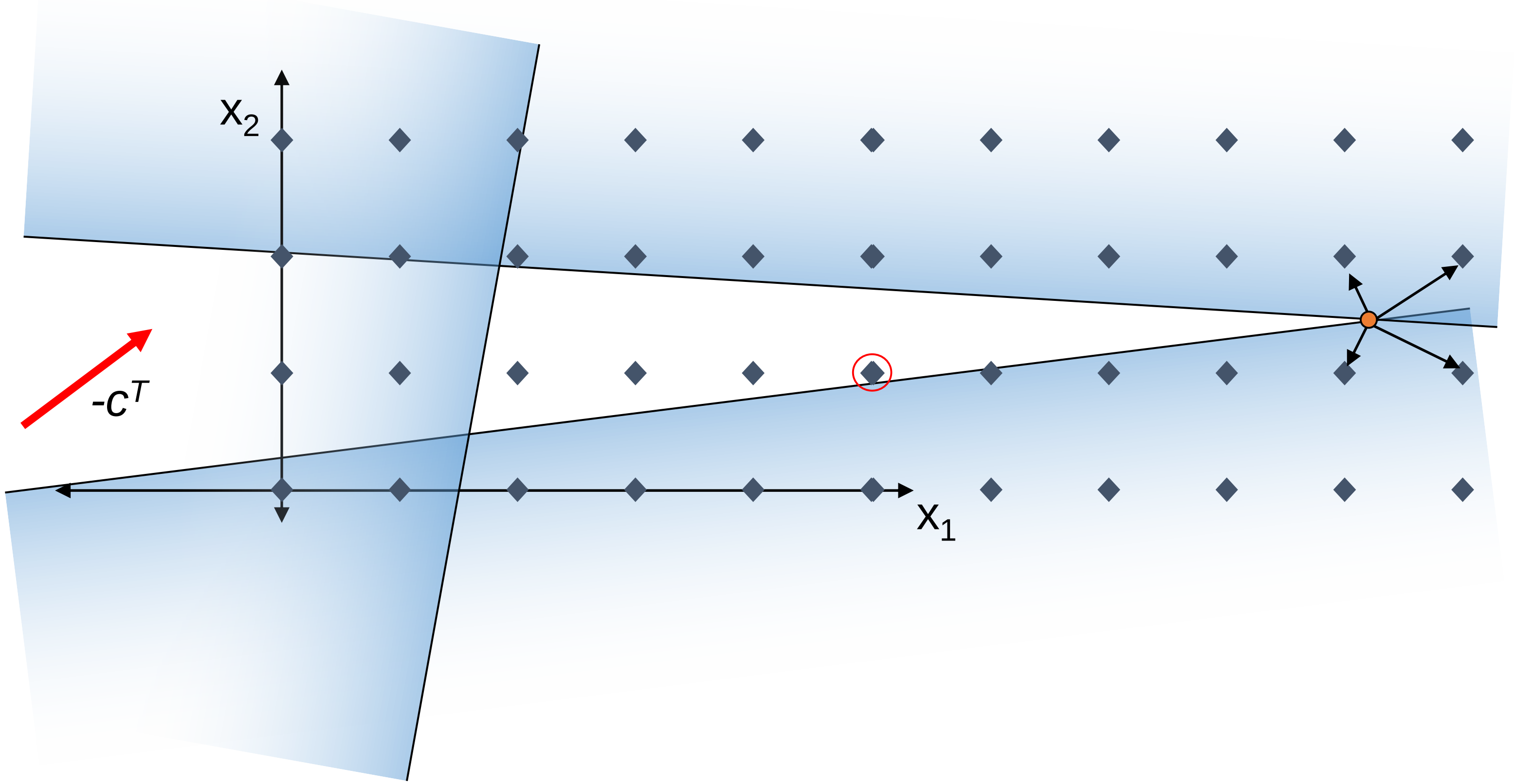
How about solving the IP minimization problem by LP relaxation followed by rounding?



solving an IP: LP rounding

In general, rounding an LP solution does not work

Rounding can be arbitrarily far away from integer solution



But, the LP provides a lower bound on the IP minimization

solving an IP

Complete enumeration: systematically consider all possible values of the decision variables

e.g., if there are n binary variables, there are 2^n different combinations of decision variable values

Usual idea: iteratively break the problem in two

solving an IP: complete enumeration

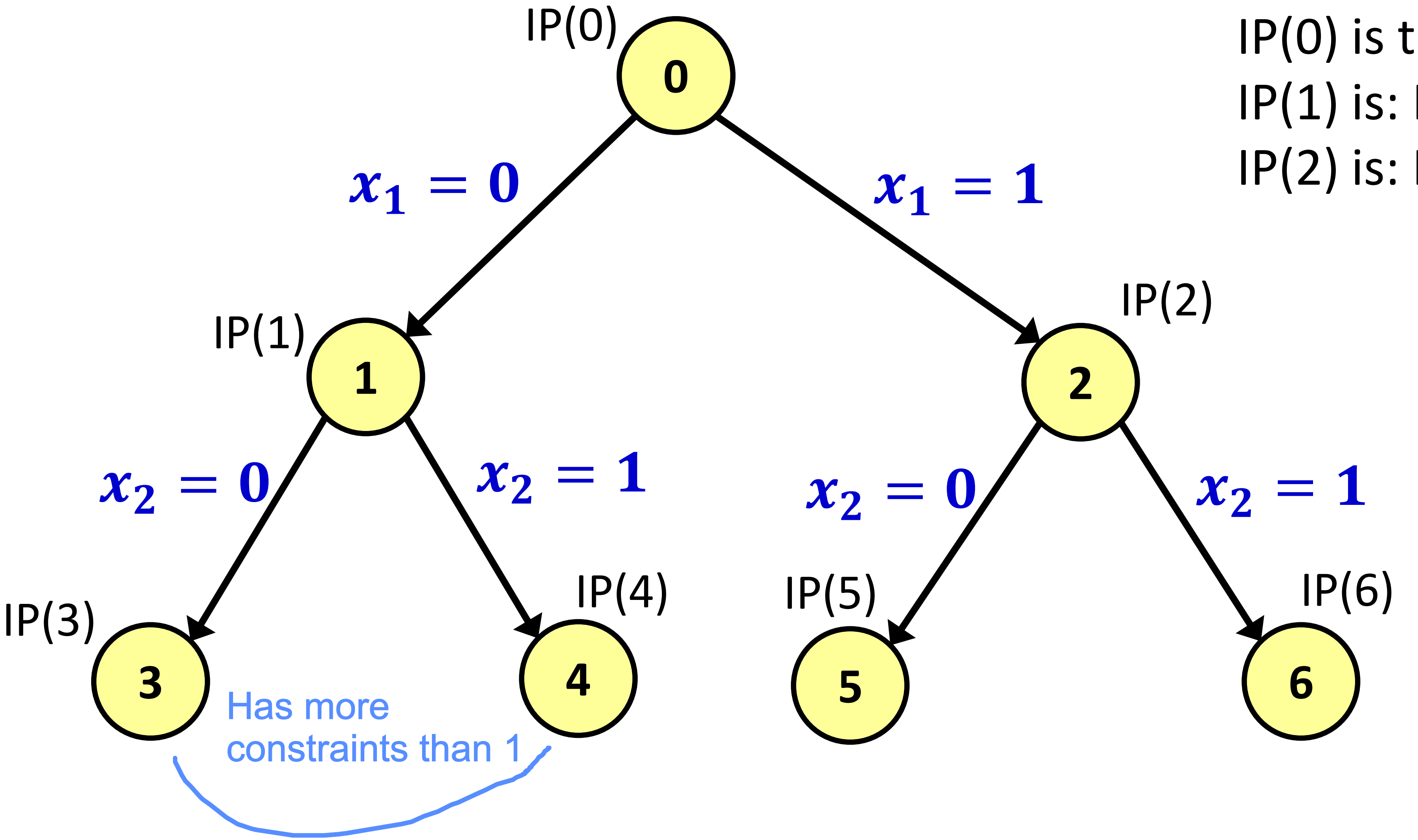
Enumeration tree is a node tree showing original problem plus more constraints

Consider a BP problem

→ iteratively break the problem in two:

- consider separately the case that $x_1 = 0$ and $x_1 = 1$
  - these ***sub-problems*** are called ***nodes***
  - each node represents the original problem plus additional constraints
- the resulting nodes are split again on x_2, then x_3, etc.
- the complete series of nodes and constraint ***branches*** form an ***enumeration tree***

solving an IP: enumeration tree



IP(0) is the original IP problem

IP(1) is: IP(0) + constraint: $x_1 = 0$

IP(2) is: IP(0) + constraint: $x_1 = 1$

Nodes 1 and 2 are ***children*** of node 0

Node 0 is called the root node and is the ***parent*** of nodes 1 and 2

Final nodes of tree (they have no children) are called ***leaves***

Note: If there are n binary variables, there are 2^n leaves in the enumeration tree

solving an IP: complete enumeration

Suppose that we could evaluate 1 billion solutions per second

Let n = number of binary variables

Problem size	Approximate solution times
n = 30	1 second
n = 40	17 minutes
n = 50	12 days
n = 60	31 years
n = 70	31,000 years

This is a problem… complete enumeration is not reasonable.

solving an IP: complete enumeration

LP relaxation and rounding

- no guarantee on the quality of the solution
- but, it is fast
- does provide a “lower bound” on IP minimization

Complete enumeration

- guarantee on quality of solution → *optimal*
- but, it is slow… *really*, slow

solving an IP: branch-and-bound

However, we can combine the two approaches LP relaxation and enumeration:

1. **Solve LP Relaxation to get fractional solution**
2. **Create two branches (sub-problems) by adding constraints**
3. **Use the “lower bound” property of LP solutions to eliminate parts of the tree**

The basic concept underlying the branch-and-bound technique is to *divide and conquer*.

solving an IP: branch-and-bound

The dividing (***branching***) is done by partitioning the entire set of feasible solutions into smaller and smaller subsets.

The conquering (***fathoming***) is done partially by:

i. giving a bound for the best solution in the branch
ii. discarding the branch if the bound indicates that it can not contain an optimal solution

branch & bound: terminology

Incumbent solution

- an IP feasible solution that is the best solution so far in the B&B search
- its objective value provides and *upper bound* to the minimization problem

Lower bound

- a solution to an “easier” problem such as LP relaxation
- ideally, you are looking for “tight” lower bounds – those that are closer to the optimal objective value

Fathoming

- dismiss nodes (and thus whole branches of the tree!) if further branching will yield no useful information

branch & bound

1. Solve LP relaxation for lower bound on cost for current branch
   - If solution exceeds upper bound (incumbent value), branch is terminated
   - Else: if solution is integer, replace incumbent solution and update upper bound
2. Create two branched problems by adding constraints to the problem
   - Select integer variable with fractional LP solution
   - Add constraints for this variable
3. Repeat until no branches remain, return optimal solution

major steps in B&B algorithm

1. Initialization
2. Check Stopping Condition
3. Explore Active Subproblem (Fathoming)
4. Branching
5. Bounding

step 1: initialization

- Let the node index $i = 0$
- Make node 0 (root node) *active*
- If any feasible IP solutions are known, let the best one be the incumbent solution. Otherwise, let $z^* = \infty$

step 2: check stopping condition

If there are no active nodes, then: **stop**.

- If there is an incumbent solution it is optimal
- If not, the IP is infeasible

Else (if there are any active nodes)

Select active node i and attempt to solve the associated subproblem LP_i

step 3: explore active subproblem

If LP_i is infeasible then
    fathom node i and go to Step 2
else if LP_i solution is IP feasible then
    If $z_i < z^*$ then
        Let $z^* = z_i$
        LP_i solution is the new incumbent solution
    Fathom node i and go to Step 2
else if $z_i \geq z^*$ then
    fathom node i and go to Step 2
else continue to Step 4

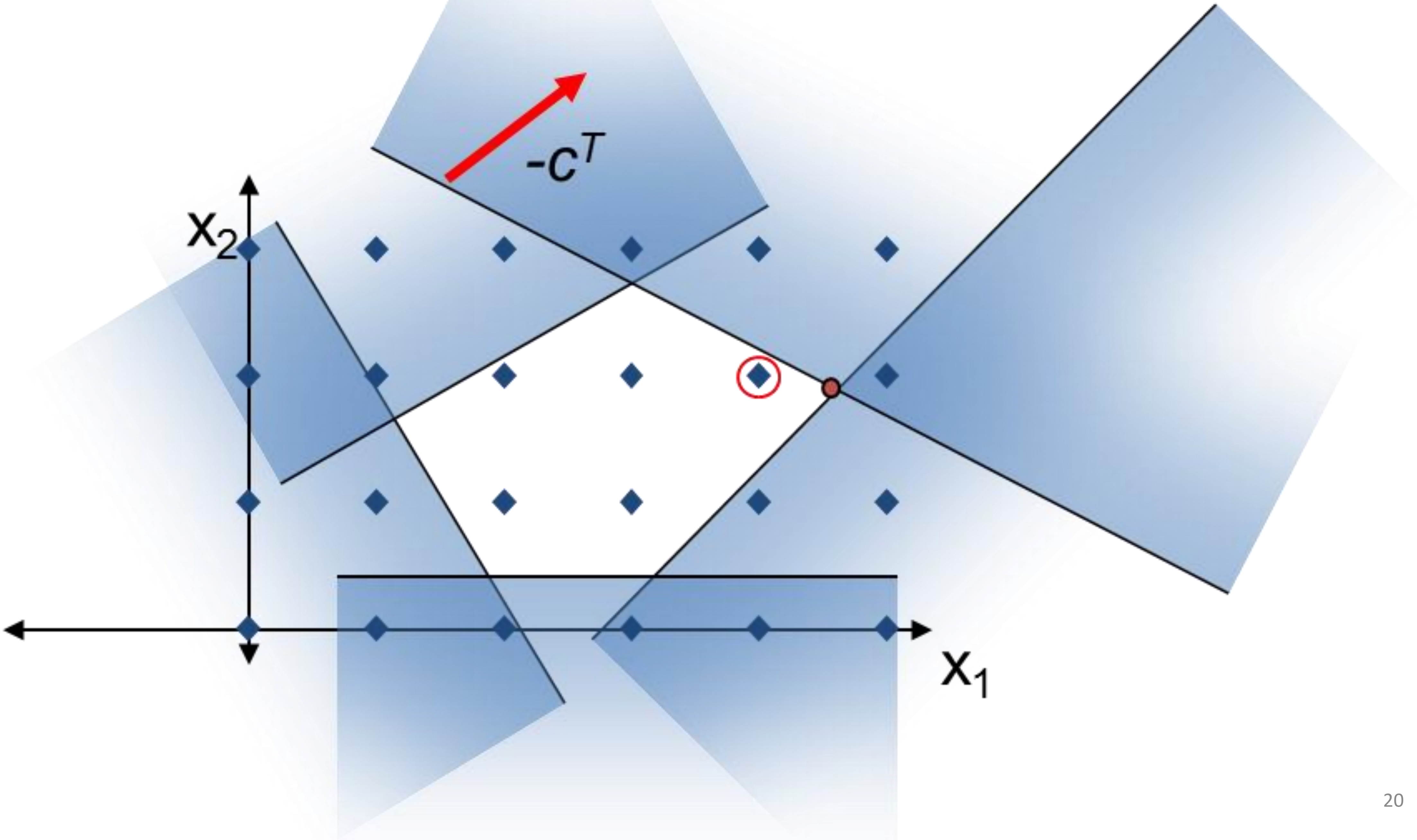
step 4: branching

- Select a integer variable x, that has fractional value v in the optimal solution to LP_i
  - Create node $i+1$ (down branch)

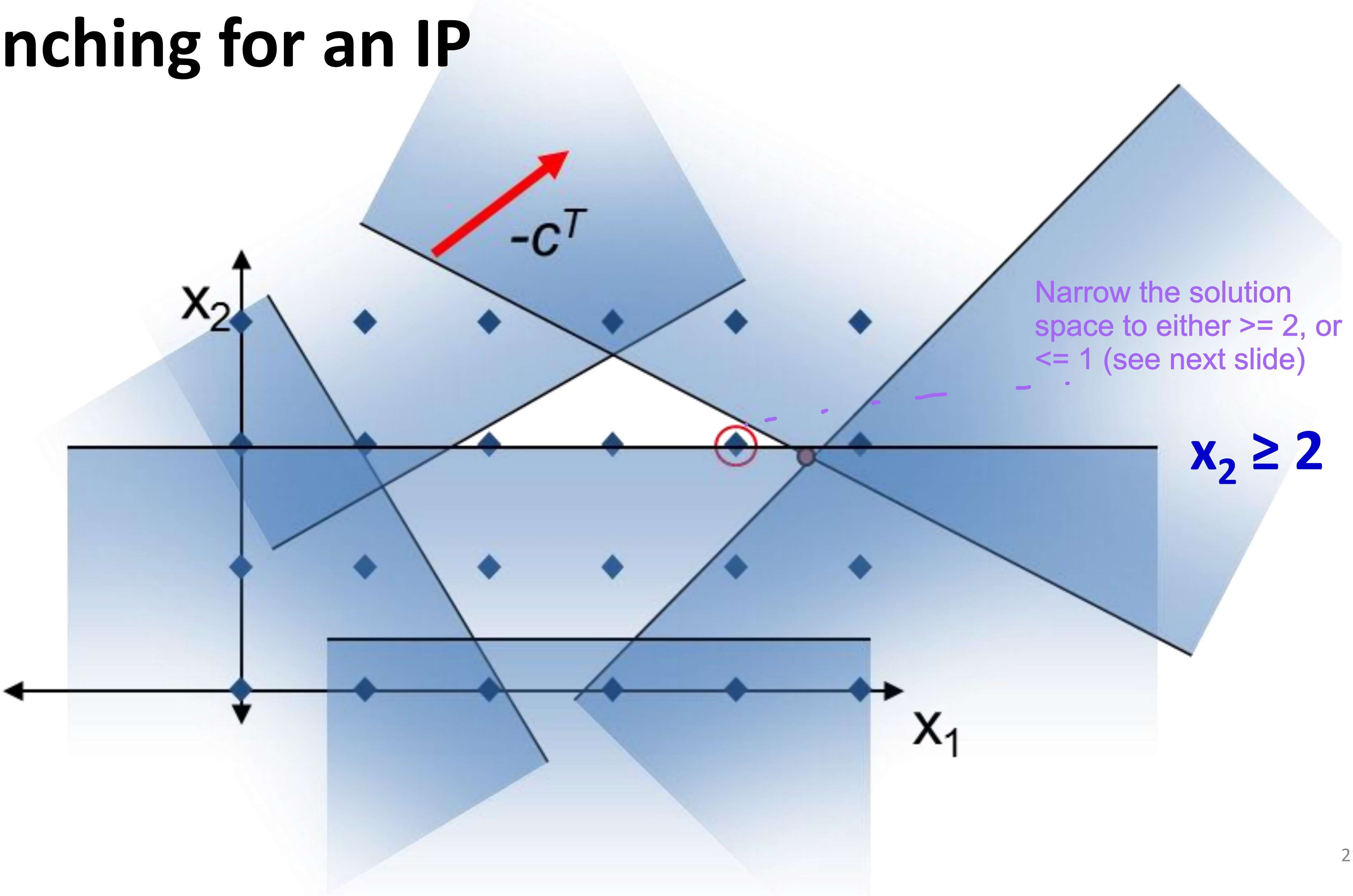
    i.e., let the associated subproblem LP_{i+1} be LP_i subject to $x \le \lfloor v \rfloor$ Down
  - Create node $i+2$ (up branch):

    i.e., let the associated subproblem LP_{i+2} be LP_i subject to $x \ge \lceil v \rceil$ Up
- Let the lower bound at nodes i, $i+1$, and $i+2$ be z_i
- Make nodes $i+1$ and $i+2$ active Active now,
- Make node i inactive Not active now
- Go to Step 2

branching for an IP



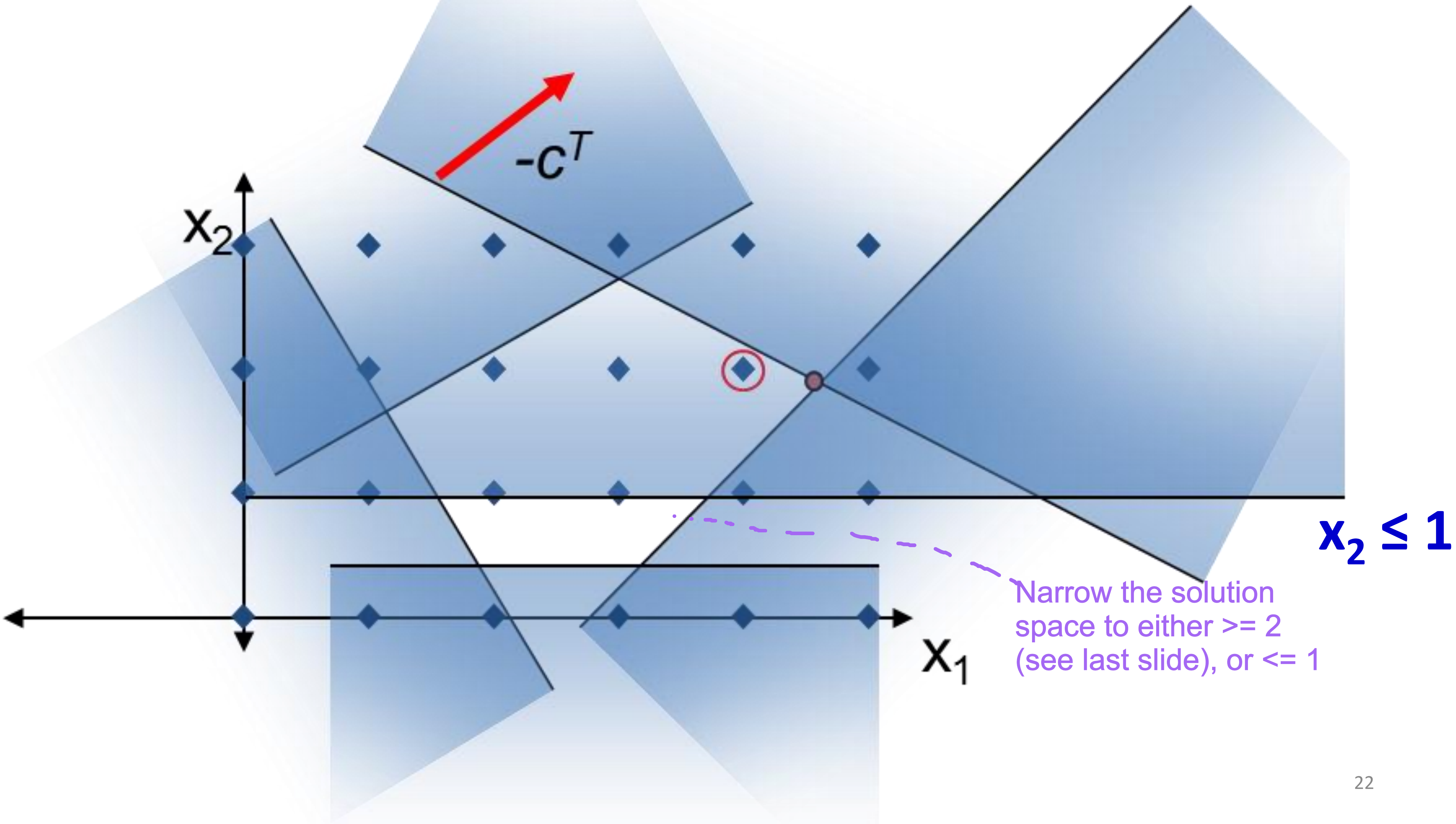
branching for an IP



Narrow the solution space to either >= 2, or <= 1 (see next slide)

$x_2 \geq 2$

branching for an IP



bounding

- A valid lower bound on the optimal objective function value can be found by taking the minimum lower bound over all active nodes.
- If there are no active nodes and there is an incumbent solution, then the incumbent solution provides both a lower and upper bound.

Finished when upper and lower bound meet!

Integer programming: Branch and Bound Example

B&B example: An IP problem with 4 integer variables

$$x_1, x_2, x_3, x_4 \in \{0,1\}$$

Minimize z[I,P]

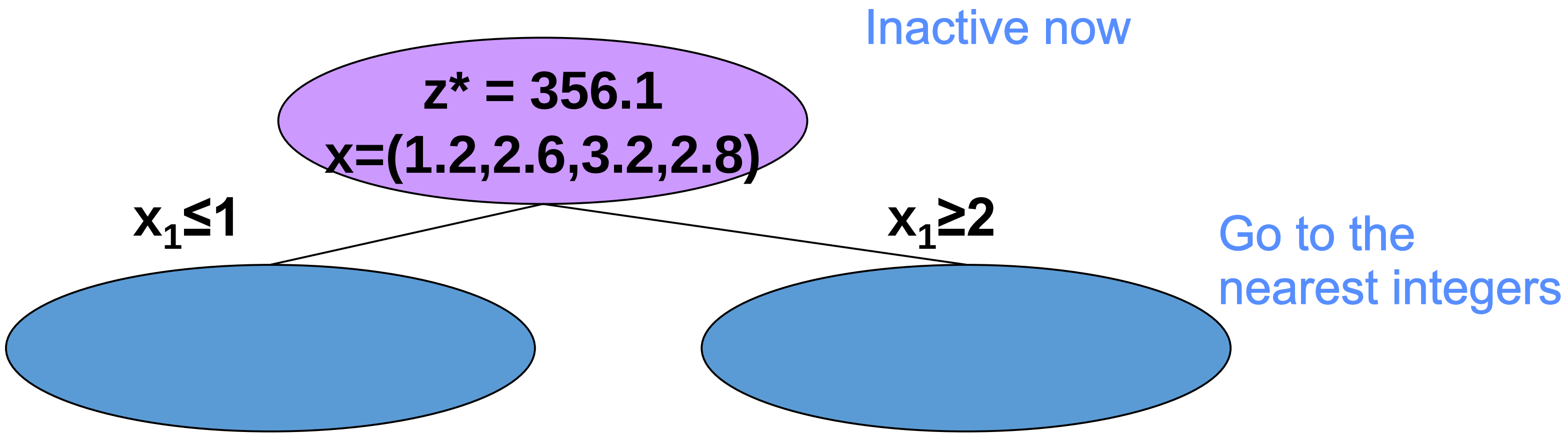
Let z_{IP} denote the value for which we wish to minimize, i.e., $z_{\text{IP}} = \min f(x)$

Let z^* denote the optimal objective value for any LP relaxation of the IP problem or sub-problem

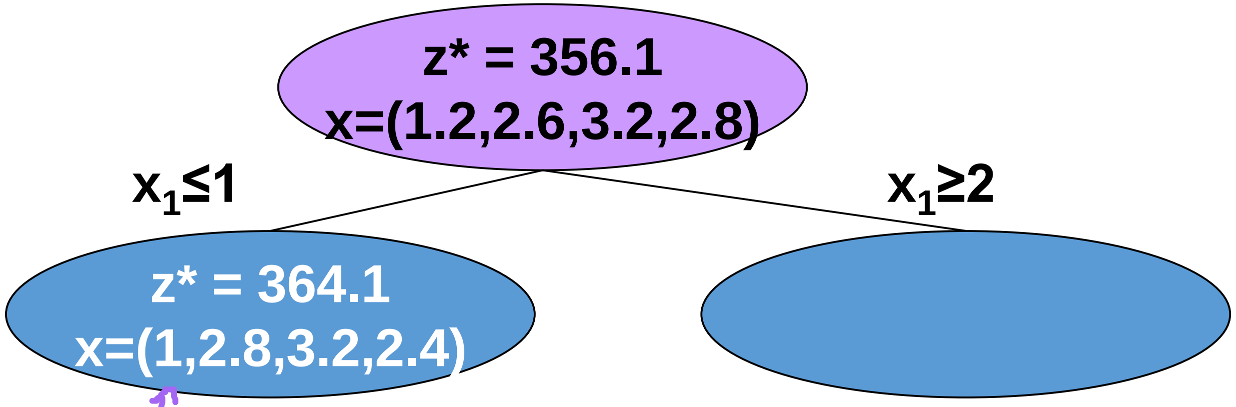
Z* is optimal objective

Initial LP (root node)

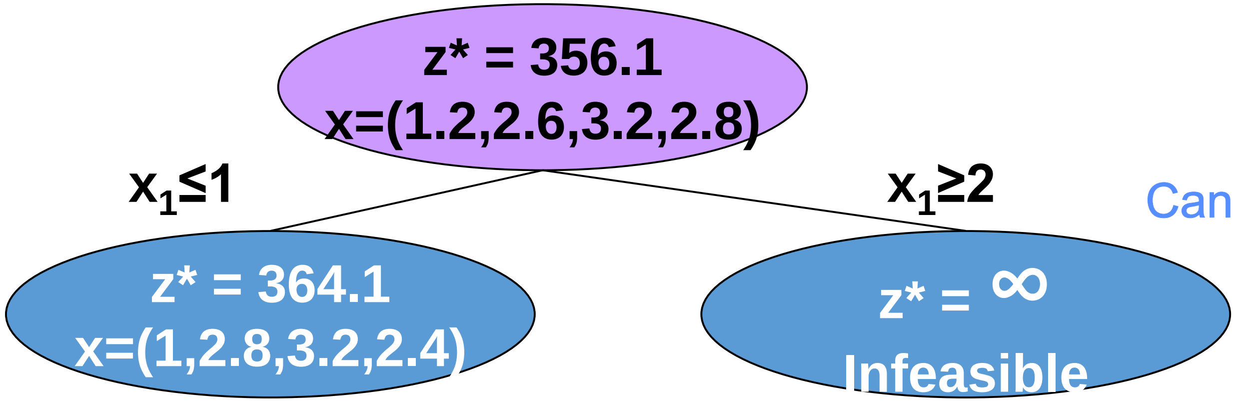
z* = 356.1
x=(1.2,2.6,3.2,2.8)


Inactive now
z* = 356.1
x=(1.2,2.6,3.2,2.8)
x1≤1
x1≥2
Go to the
nearest integers

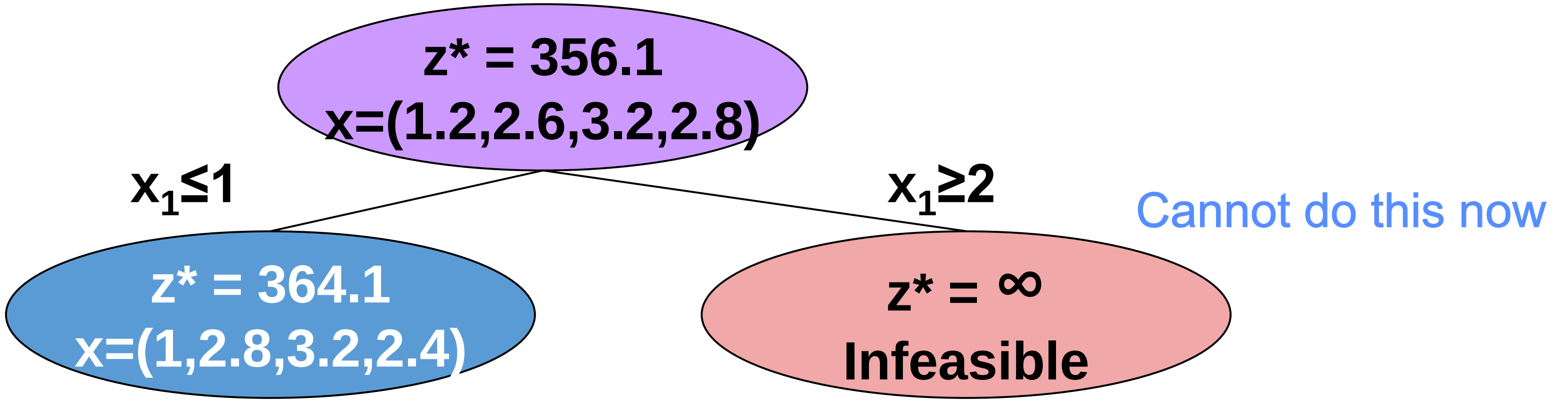
Solutions are x



Found integer solution, which is good!

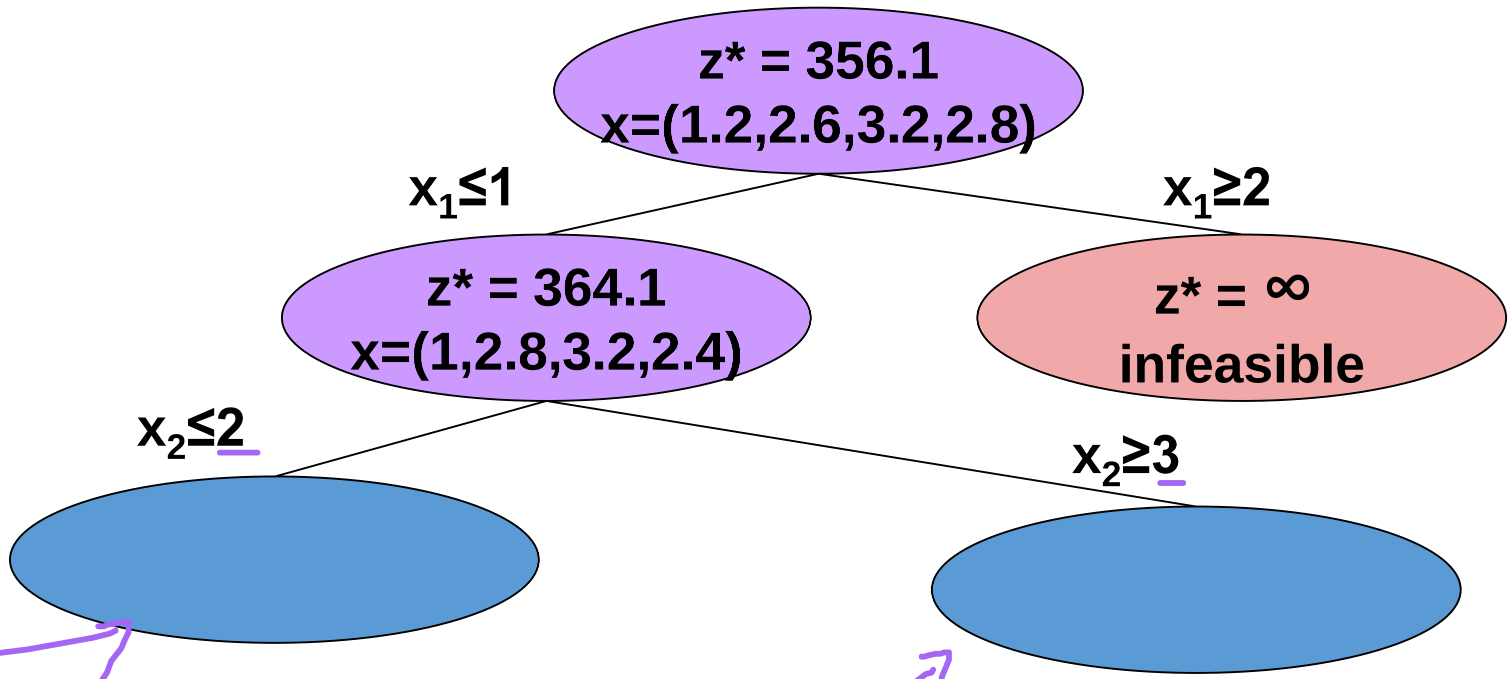


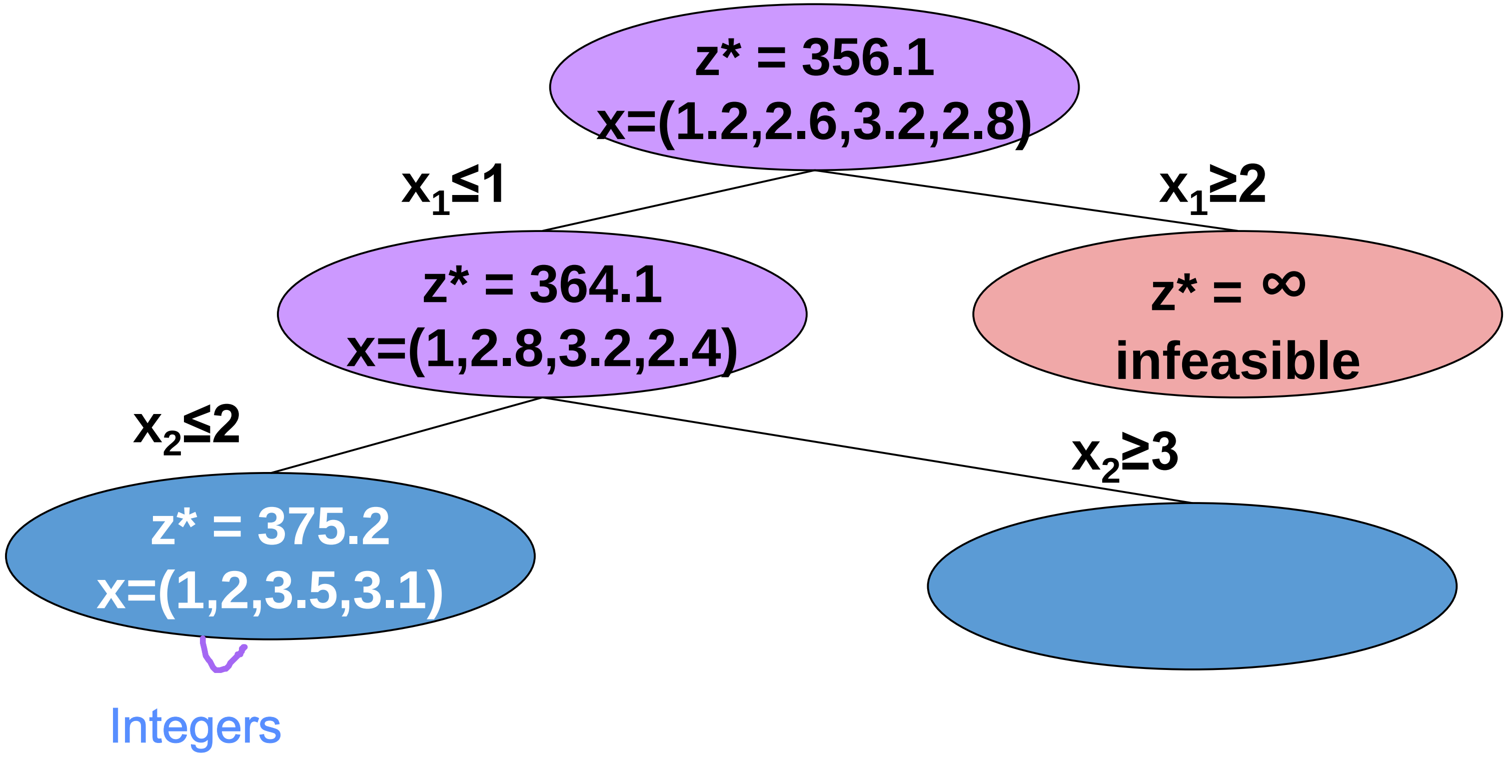
Cannot do this now

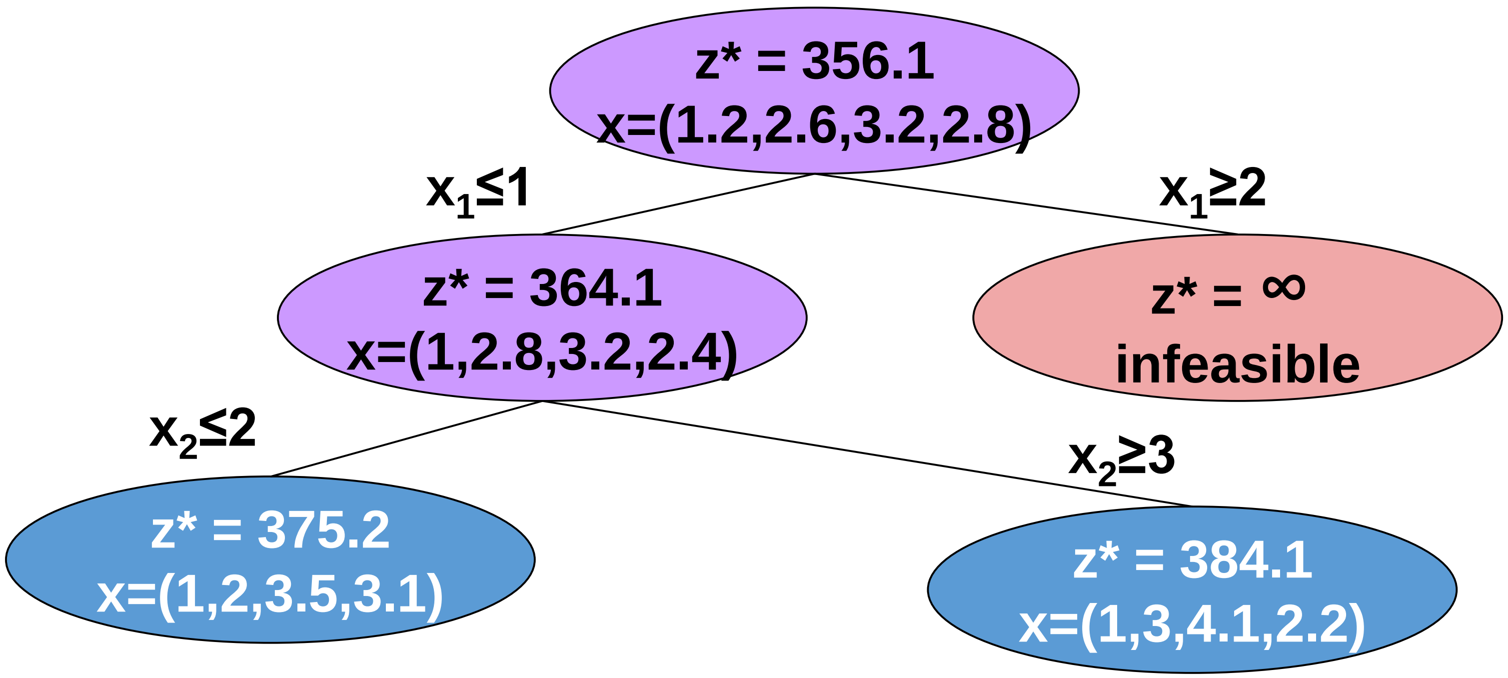

z* = 356.1
x=(1.2,2.6,3.2,2.8)
x1≤1
x1≥2
z* = 364.1
x=(1,2.8,3.2,2.4)
z* = ∞
Infeasible
Cannot do this now

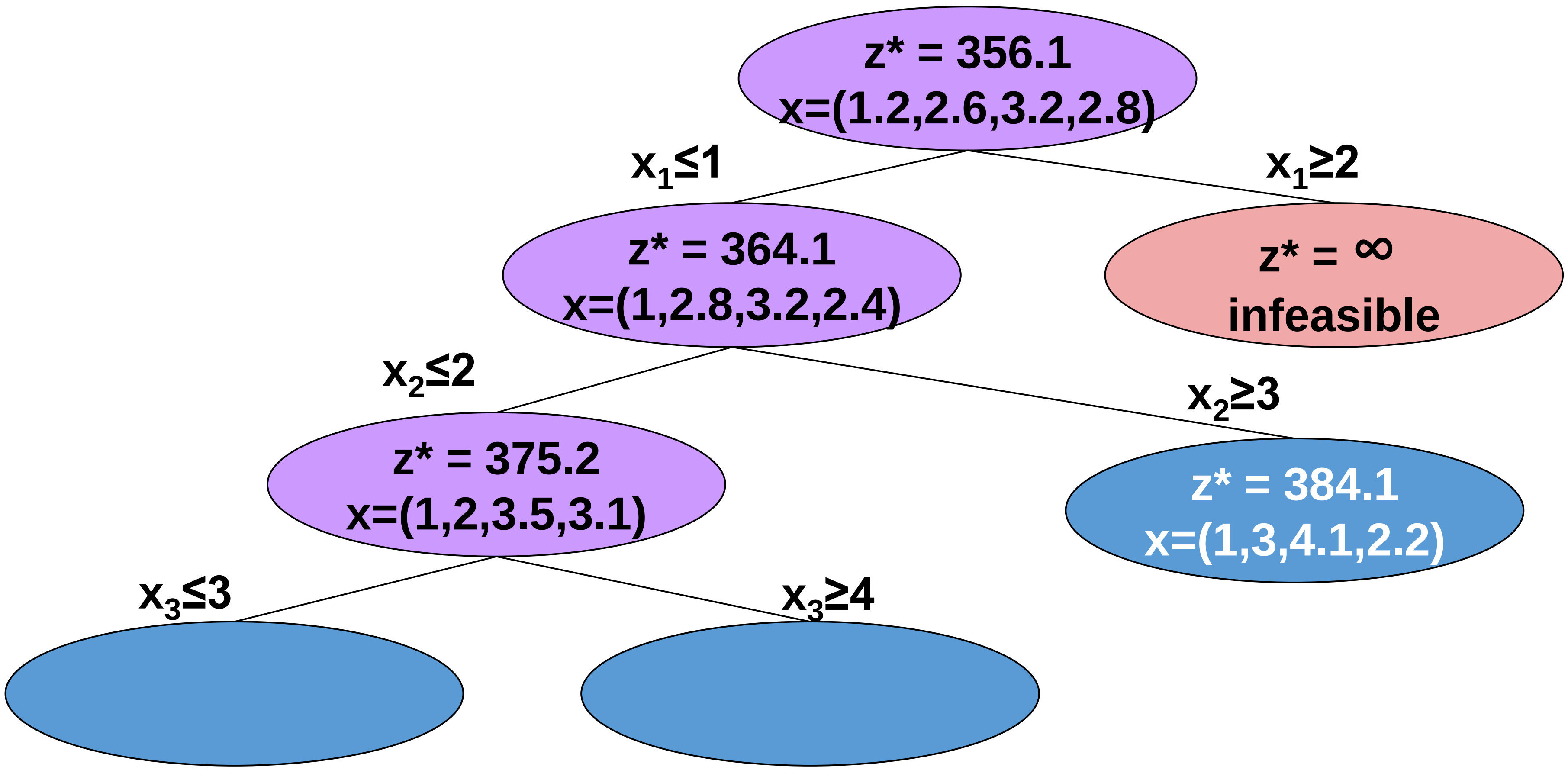
Branch to x[1] + 1
and x[1] - 1

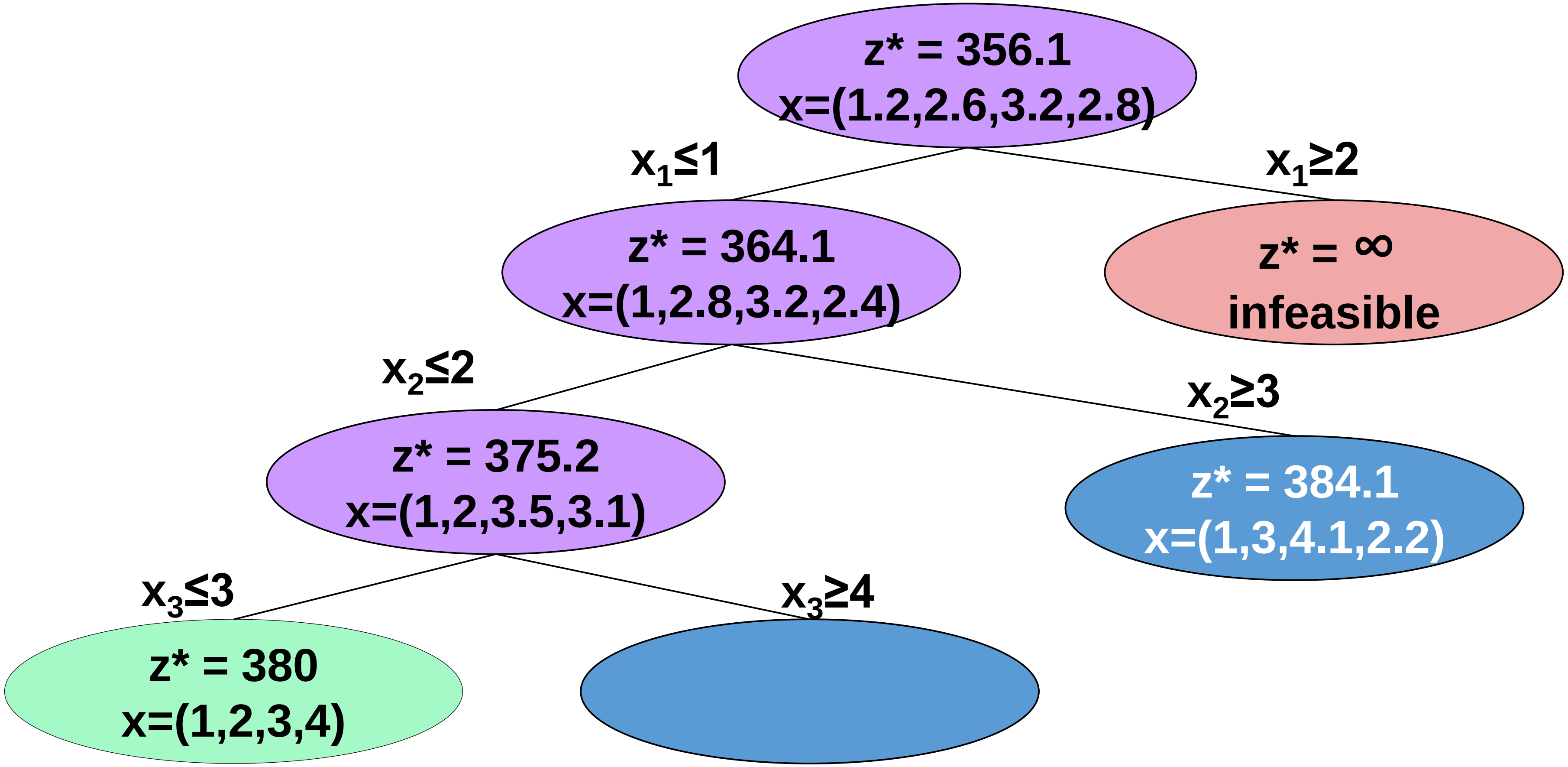
(2, 3)




z* = 356.1
x=(1.2,2.6,3.2,2.8)
x1≤1
x1≥2
z* = 364.1
x=(1,2.8,3.2,2.4)
z* = ∞
infeasible
x2≤2
x2≥3
z* = 375.2
x=(1,2,3.5,3.1)
Integers

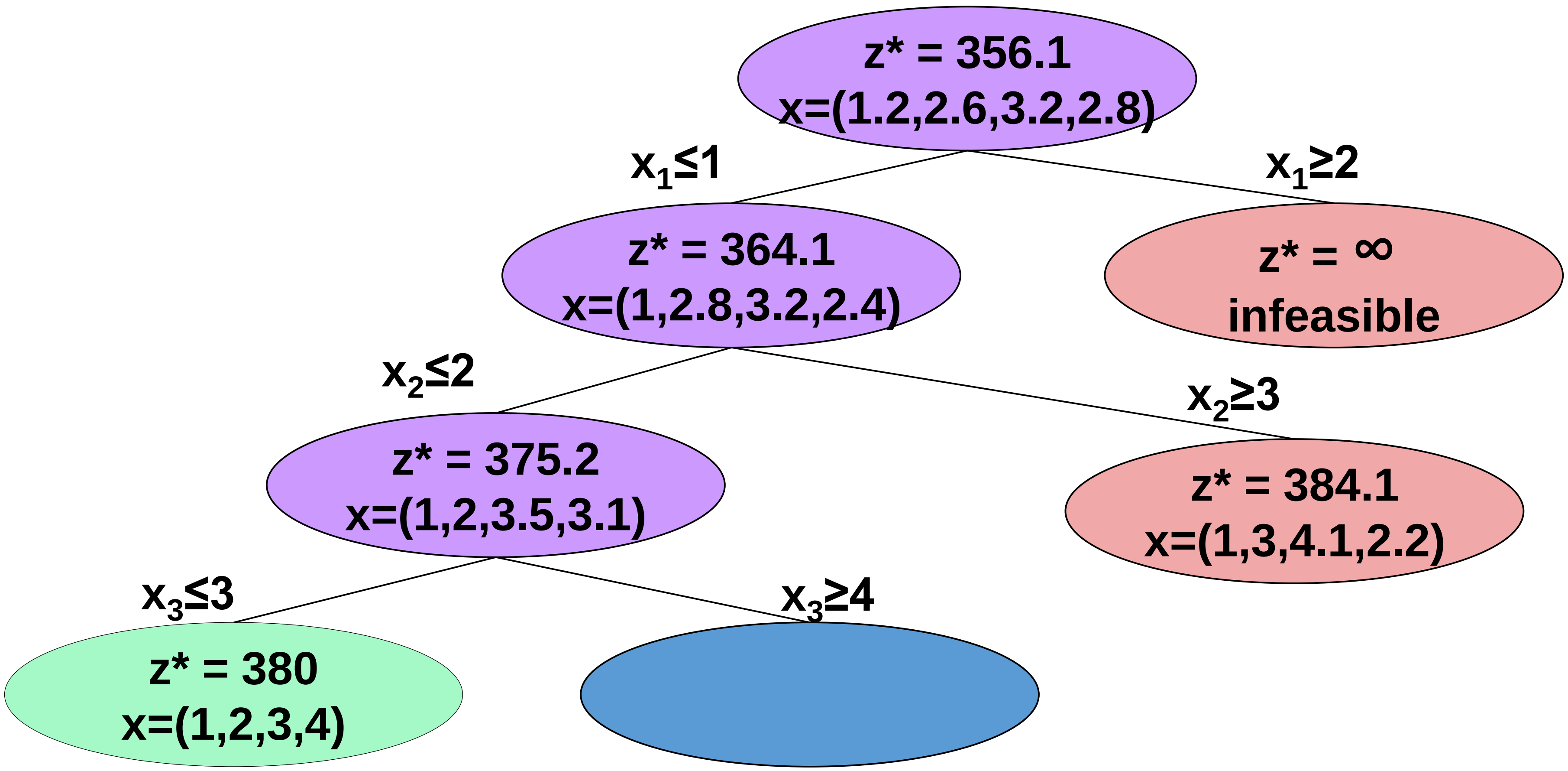

z* = 356.1
x=(1.2,2.6,3.2,2.8)
$x_1 \leq 1$
$x_1 \geq 2$
z* = 364.1
x=(1,2.8,3.2,2.4)
z* = ∞
infeasible
$x_2 \leq 2$
$x_2 \geq 3$
z* = 375.2
x=(1,2,3.5,3.1)
z* = 384.1
x=(1,3,4.1,2.2)

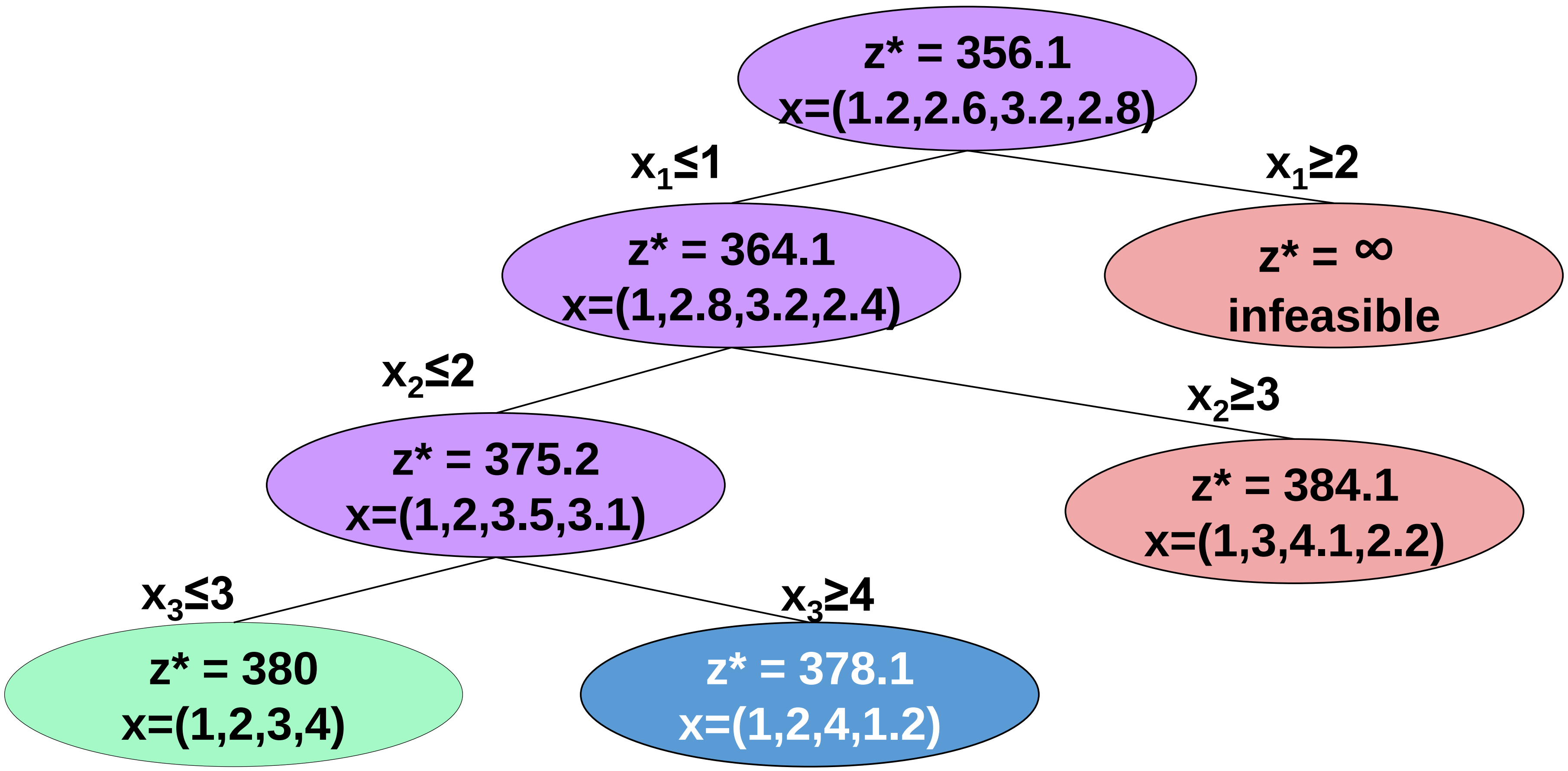

z* = 356.1
x=(1.2,2.6,3.2,2.8)
$x_1 \leq 1$
$x_1 \geq 2$
z* = 364.1
x=(1,2.8,3.2,2.4)
z* = ∞
infeasible
$x_2 \leq 2$
$x_2 \geq 3$
z* = 375.2
x=(1,2,3.5,3.1)
z* = 384.1
x=(1,3,4.1,2.2)
$x_3 \leq 3$
$x_3 \geq 4$



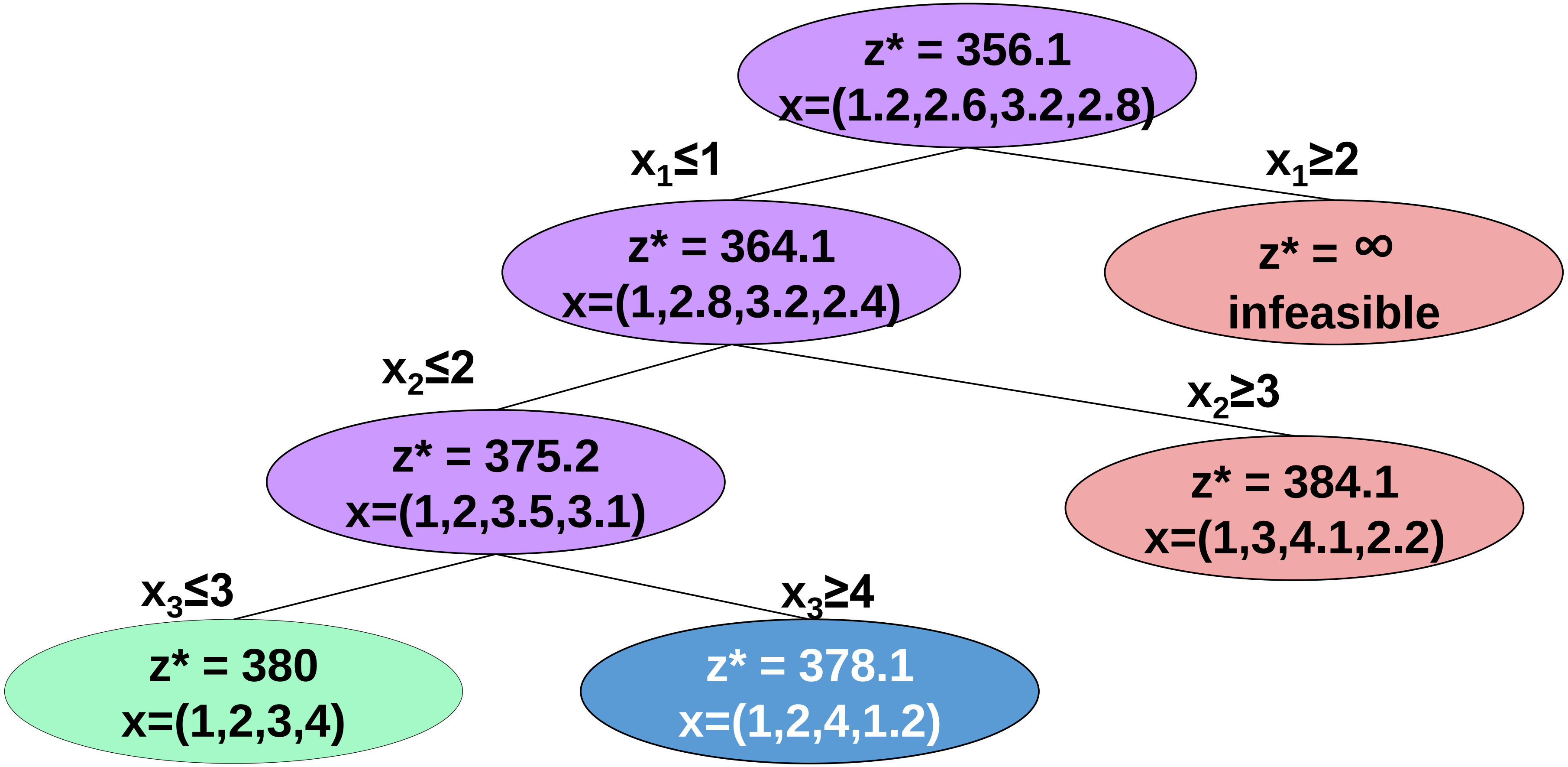
Now have found all integer solution, which is the goal. So now it is an incumbent.

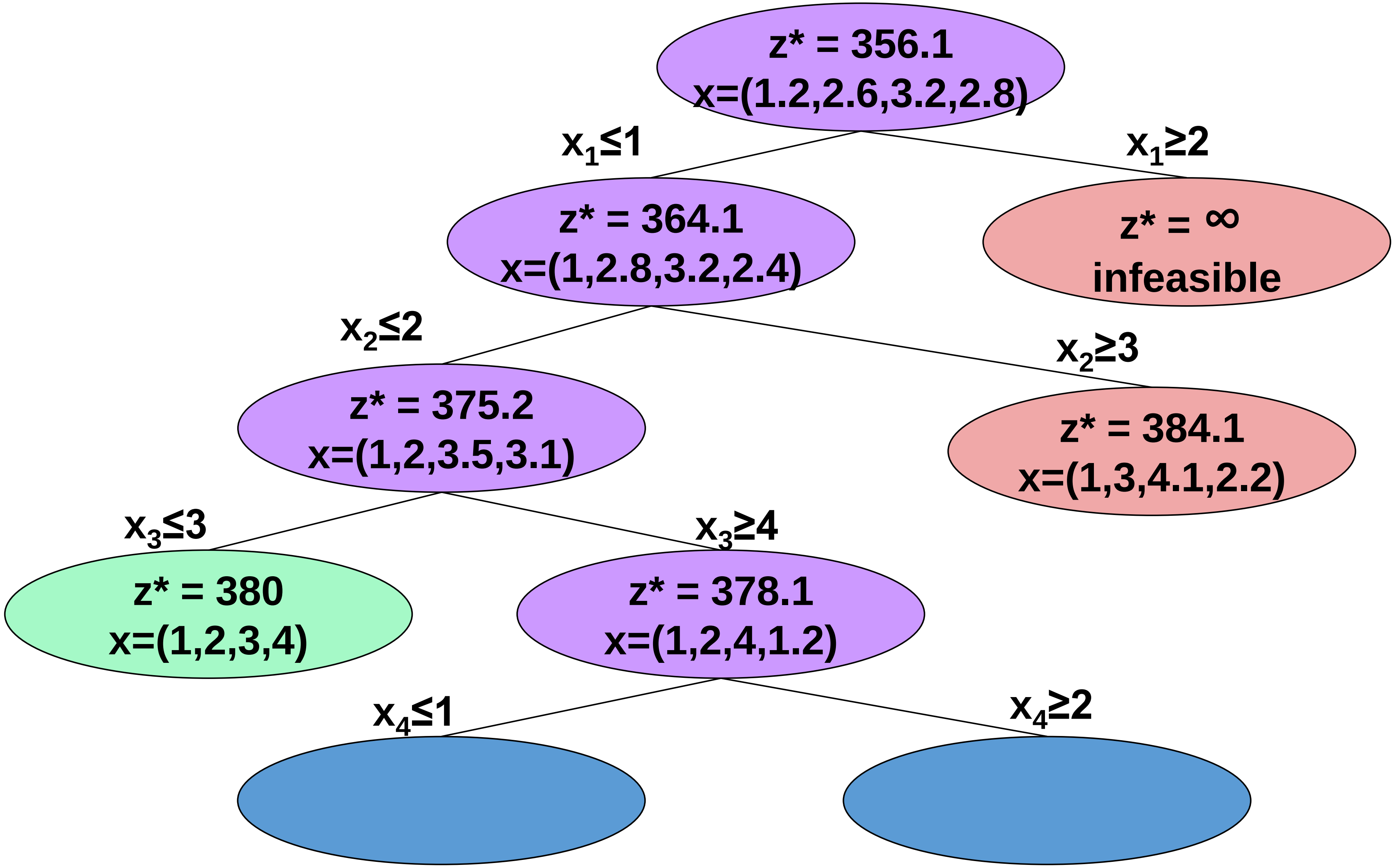
That solution is now the upper bound because minimization problem

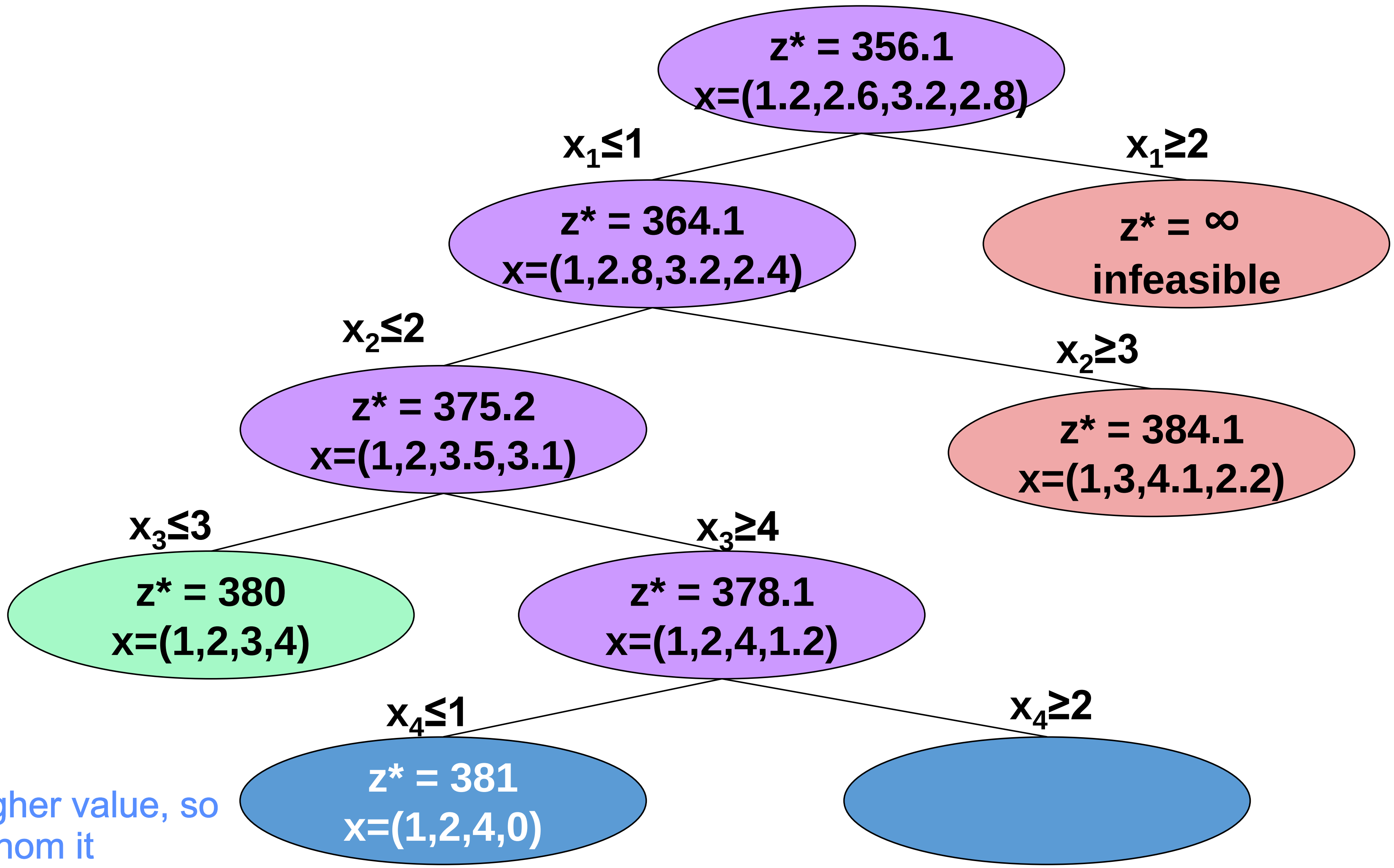

z* = 356.1
x=(1.2,2.6,3.2,2.8)
x1≤1
x1≥2
z* = 364.1
x=(1,2.8,3.2,2.4)
z* = ∞
infeasible
x2≤2
x2≥3
z* = 375.2
x=(1,2,3.5,3.1)
z* = 384.1
x=(1,3,4.1,2.2)
x3≤3
x3≥4
z* = 380
x=(1,2,3,4)



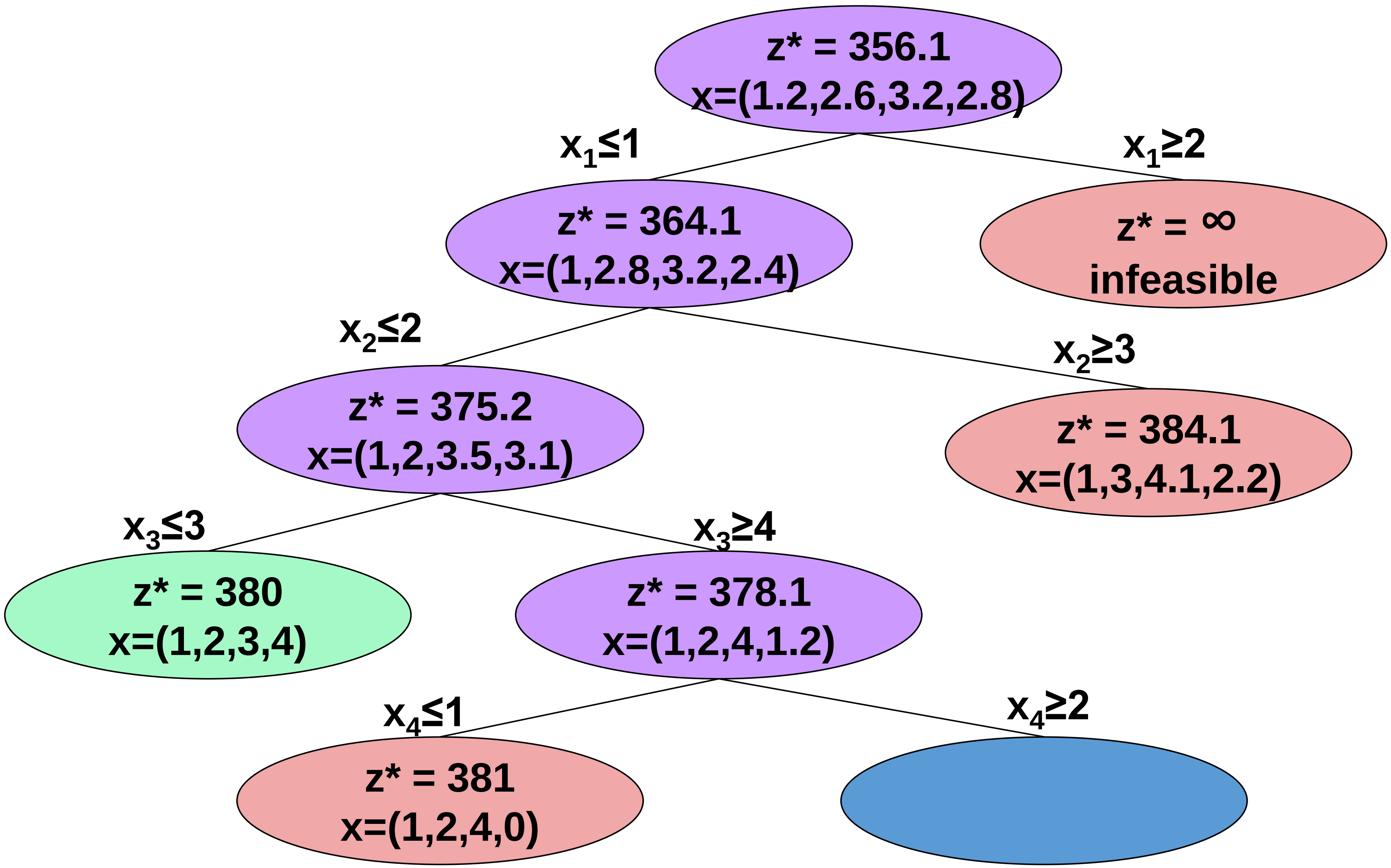
Not integer, so bound to next


z* = 356.1
x=(1.2,2.6,3.2,2.8)
$x_1 \le 1$
$x_1 \ge 2$
z* = 364.1
x=(1,2.8,3.2,2.4)
z* = ∞
infeasible
$x_2 \le 2$
$x_2 \ge 3$
z* = 375.2
x=(1,2,3.5,3.1)
z* = 384.1
x=(1,3,4.1,2.2)
$x_3 \le 3$
$x_3 \ge 4$
z* = 380
x=(1,2,3,4)
z* = 378.1
x=(1,2,4,1.2)

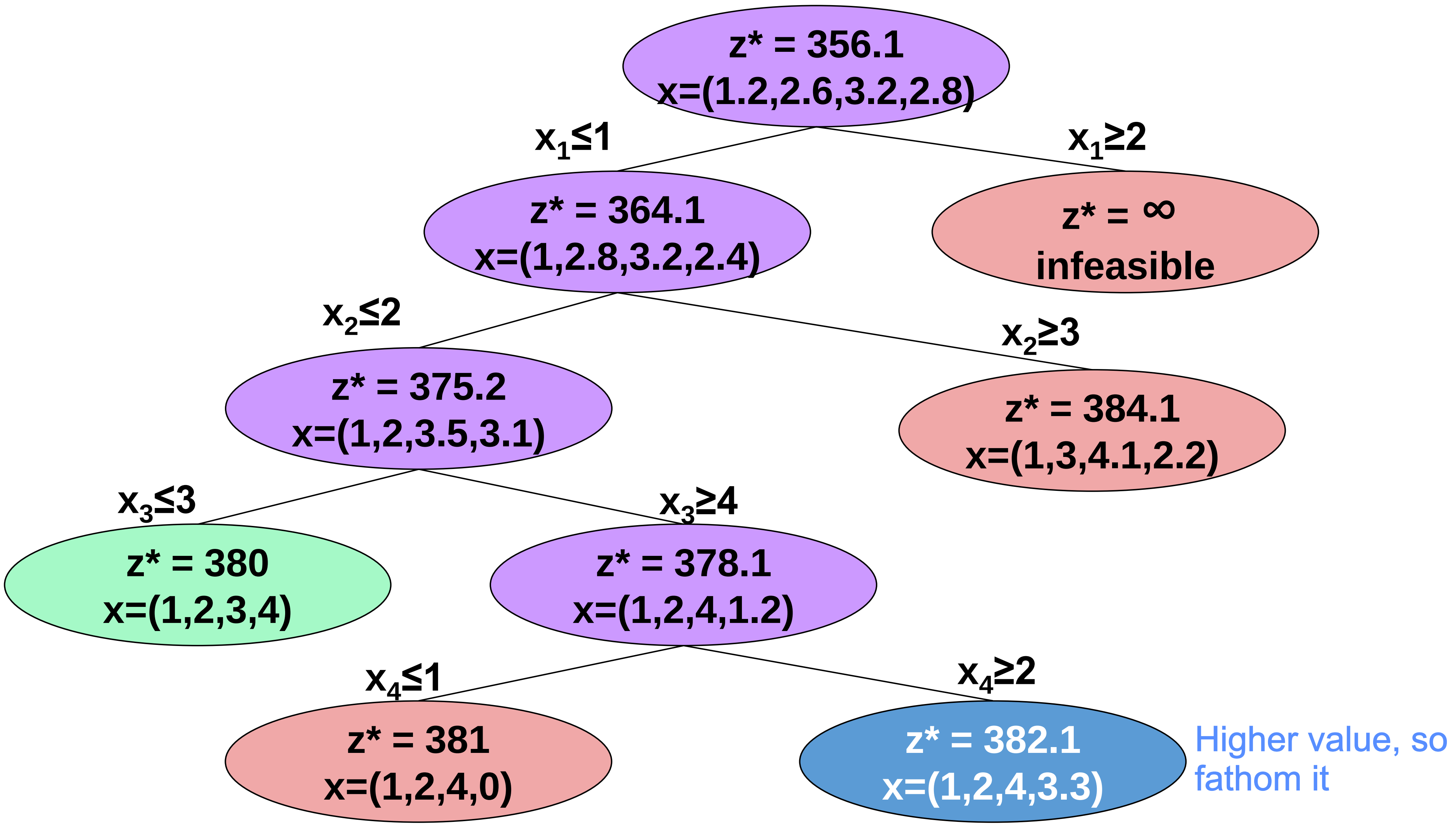

z* = 356.1
x=(1.2,2.6,3.2,2.8)
x1≤1
x1≥2
z* = 364.1
x=(1,2.8,3.2,2.4)
z* = ∞
infeasible
x2≤2
x2≥3
z* = 375.2
x=(1,2,3.5,3.1)
z* = 384.1
x=(1,3,4.1,2.2)
x3≤3
x3≥4
z* = 380
x=(1,2,3,4)
z* = 378.1
x=(1,2,4,1.2)
x4≤1
x4≥2



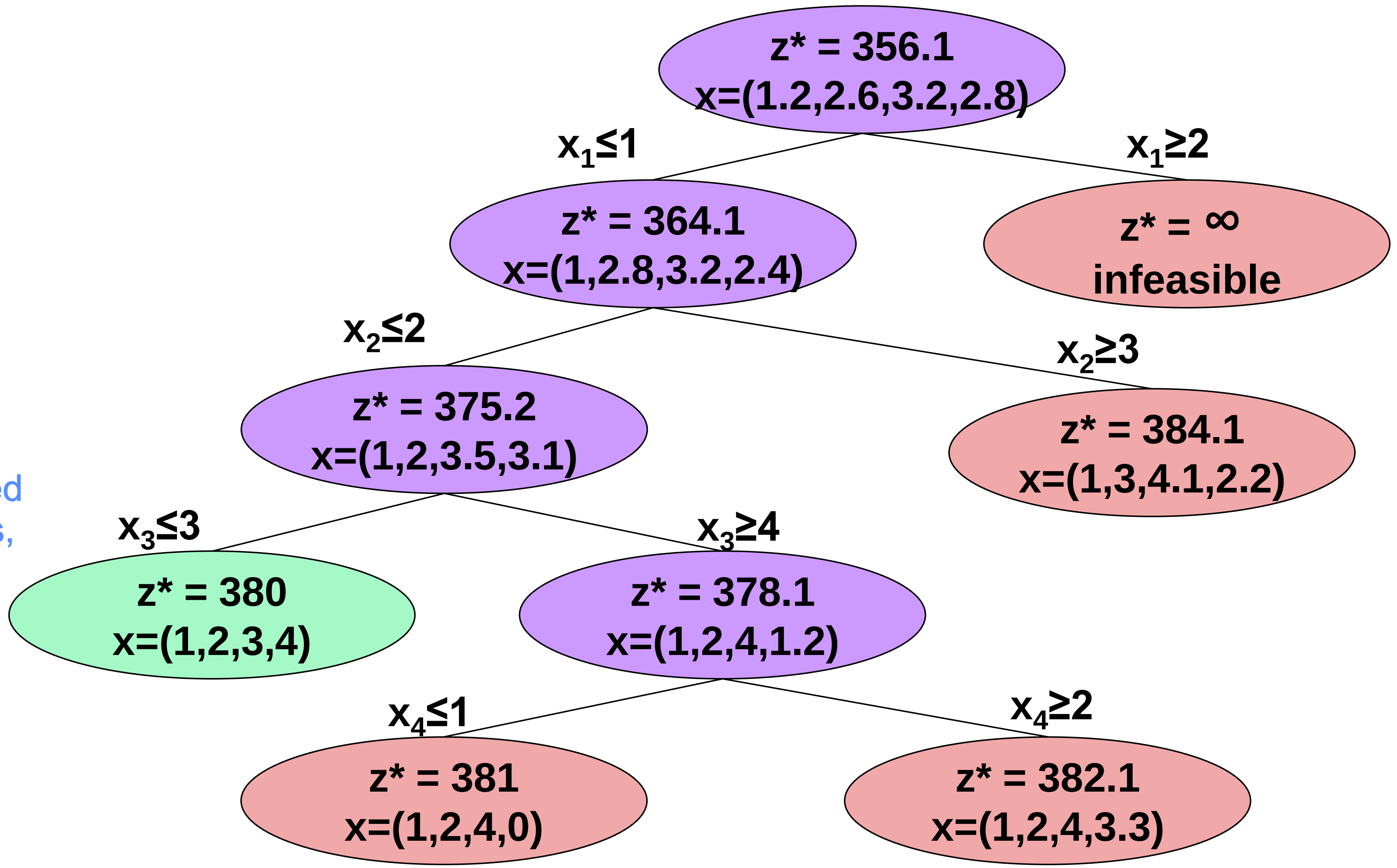
Higher value, so fathom it



z* = 356.1
x=(1.2,2.6,3.2,2.8)
$x_1 \leq 1$
$x_1 \geq 2$
z* = 364.1
x=(1,2.8,3.2,2.4)
z* = ∞
infeasible
$x_2 \leq 2$
$x_2 \geq 3$
z* = 375.2
x=(1,2,3.5,3.1)
z* = 384.1
x=(1,3,4.1,2.2)
$x_3 \leq 3$
$x_3 \geq 4$
z* = 380
x=(1,2,3,4)
z* = 378.1
x=(1,2,4,1.2)
$x_4 \leq 1$
$x_4 \geq 2$
z* = 381
x=(1,2,4,0)


z* = 356.1
x=(1.2,2.6,3.2,2.8)
x1≤1
x1≥2
z* = 364.1
x=(1,2.8,3.2,2.4)
z* = ∞
infeasible
x2≤2
x2≥3
z* = 375.2
x=(1,2,3.5,3.1)
z* = 384.1
x=(1,3,4.1,2.2)
x3≤3
x3≥4
z* = 380
x=(1,2,3,4)
z* = 378.1
x=(1,2,4,1.2)
x4≤1
x4≥2
z* = 381
x=(1,2,4,0)
z* = 382.1
x=(1,2,4,3.3)
Higher value, so fathom it

Since fathomed
two sub nodes,
this is solution



Integer programming: Formulations and Valid Inequalities

solving an IP: complete enumeration

Suppose that we could evaluate **1 trillion** solutions per second, and instantaneously eliminate about **99.9999999%** of all solutions as not worth considering

Let n = number of binary variables

Problem size	Approximate solution times
n = 70	1 second
n = 80	17 minutes
n = 90	12 days
n = 100	31 years
n = 110	31,000 years

Branch and bound eliminates 99% of solutions

but still could be very long!

How to solve large integer programs fast?

Eliminate much more than 99.999999999999999999999% of the solutions without having to evaluate them!

B&B plus…

Regular algorithm does a good job, but B&B plus will do much better

- The basic branch-and-bound algorithm certainly provides a vast improvement on the complete enumeration tree for an IP problem
- However, in practice we are still faced with an *enormous computational challenge*!
- There are few things we can do about this:
    - Branching strategies
    - Improve our IP formulation as best as possible (valid inequalities)
    - Employ "branch-and-cut" with Gomory Cuts
    - Cheat…

Solutions to B&B plus

Limits the solution space drastically by adding a new inequality

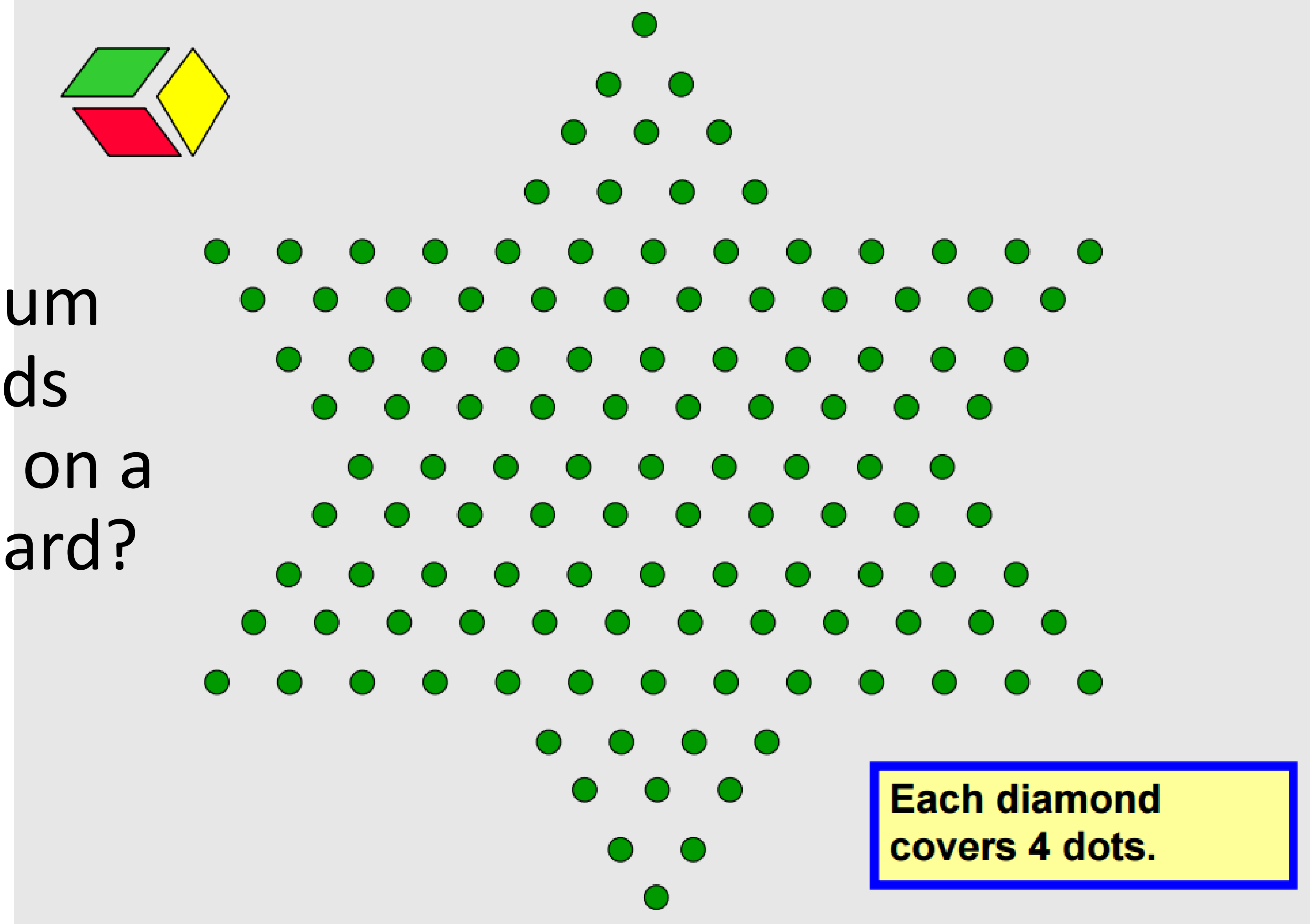
importance of good IP formulation

- Computational experiments suggest that the IP formulation *crucially influences* solution time and sometimes solvability
- To solve an IP efficiently, we want $\{x : Ax \leq b\}$ to be close to the convex hull of the feasible integer solutions
  - Finding the exact convex hull is not usually possible*
- A more tractable task: introduce ***valid inequalities***
  - *Valid inequality for an IP*: any constraint that does not eliminate any feasible integer solutions
  - Valid inequalities are also called ***cutting planes*** or simply ***cuts***

*unless you can formulate the problem as a minimum cost network flow problem!

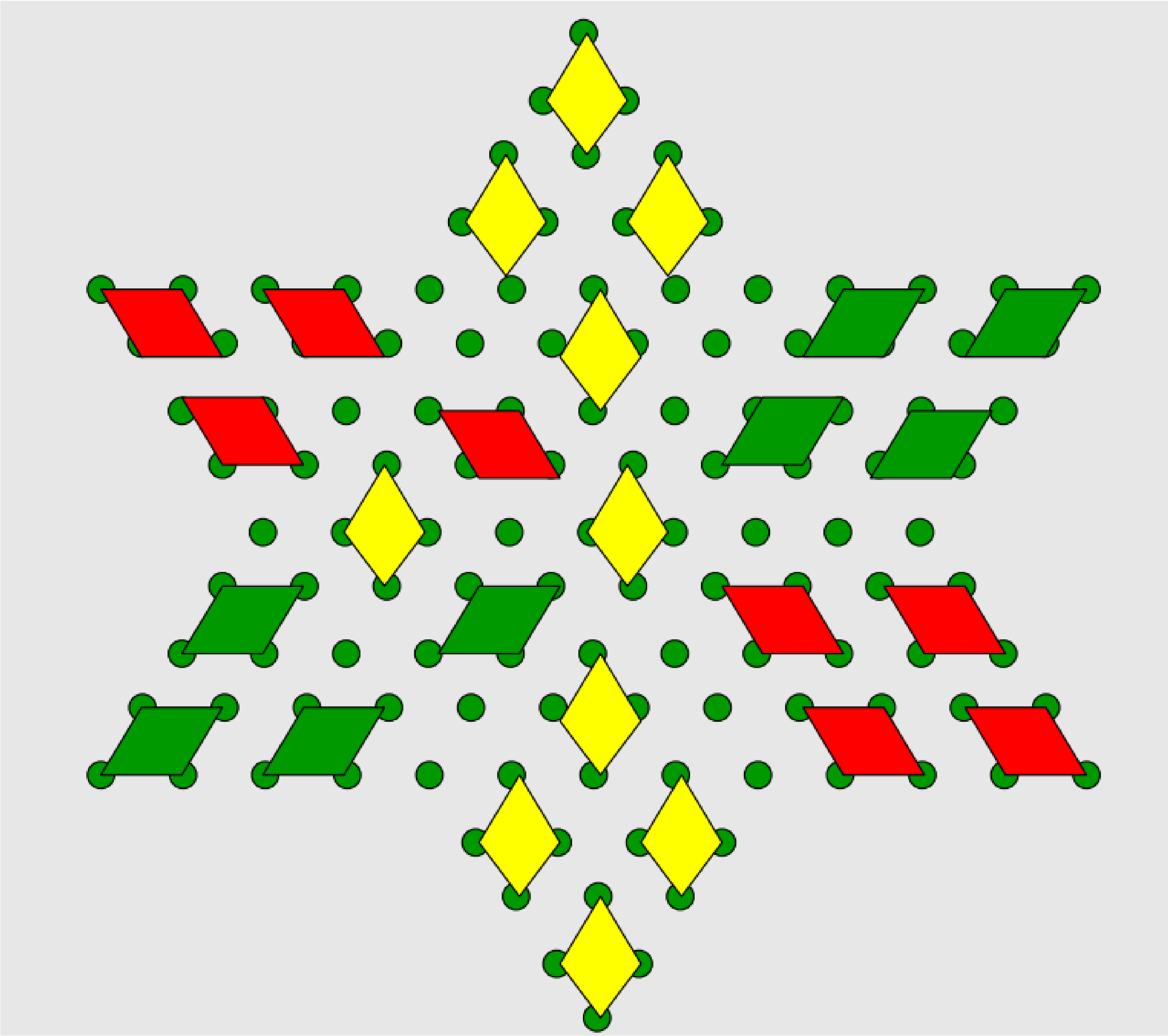
IP formulation example* with valid inequalities

What is the maximum number of diamonds that can be packed on a Chinese checkerboard?

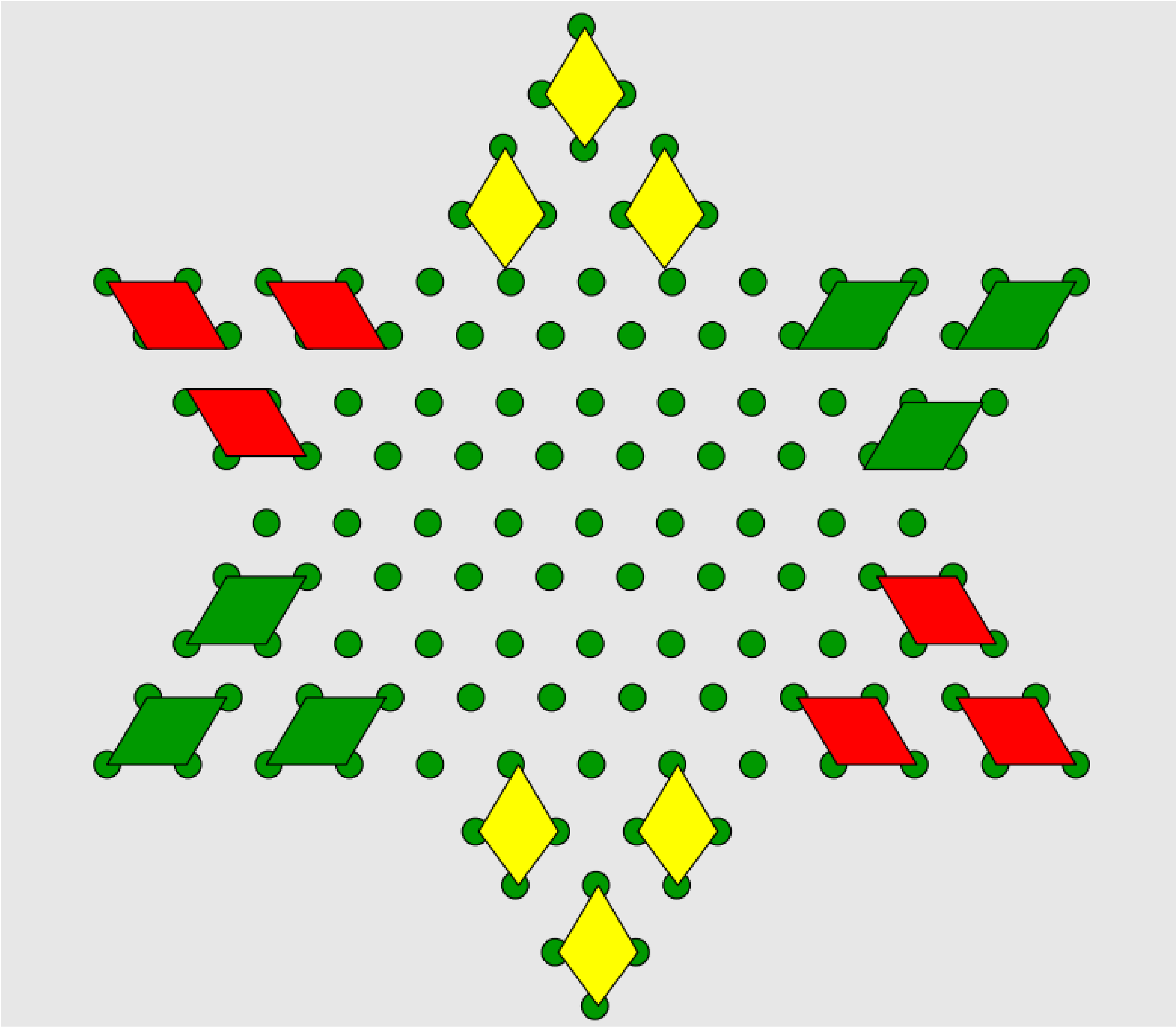


*Example adapted from MIT Open Courseware (Optimization Methods) at https://ocw.mit.edu/

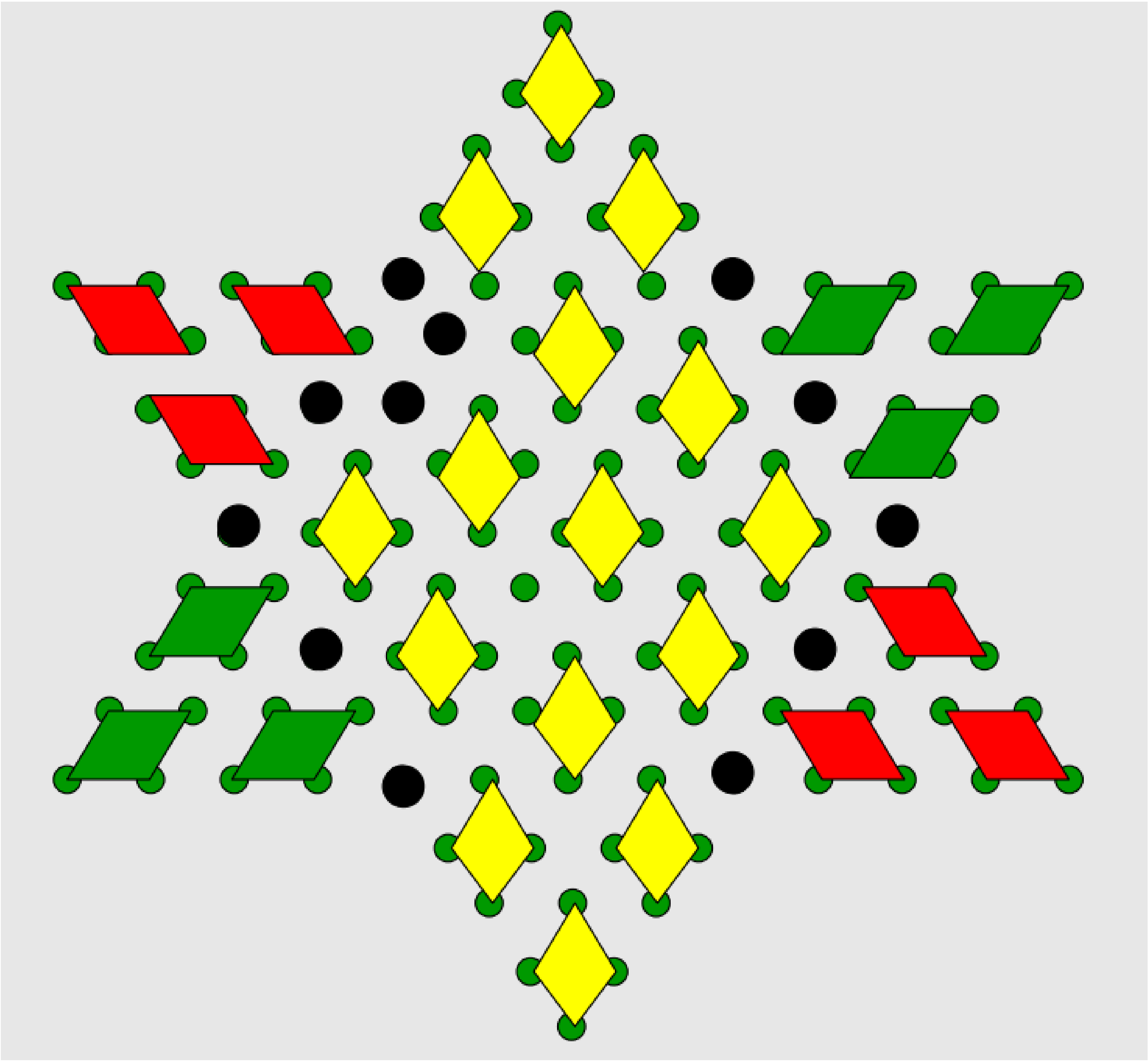
The diamonds are not permitted to overlap, or even to share a single circle.



What is the best packing you can find?



Here is one optimal packing solution.



set packing problem

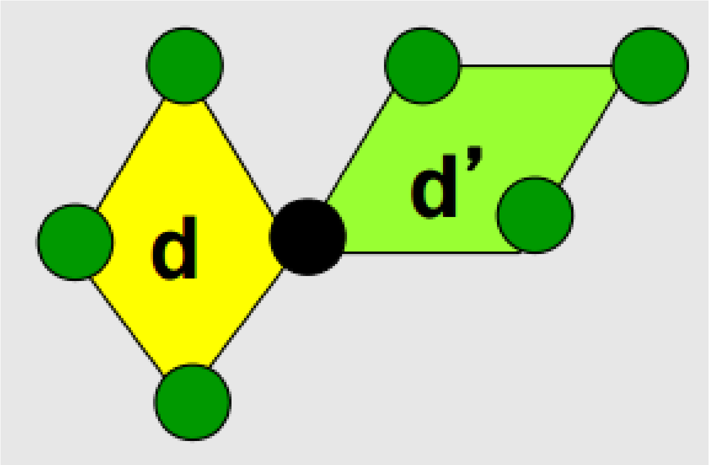
- Let D be the collection of diamonds

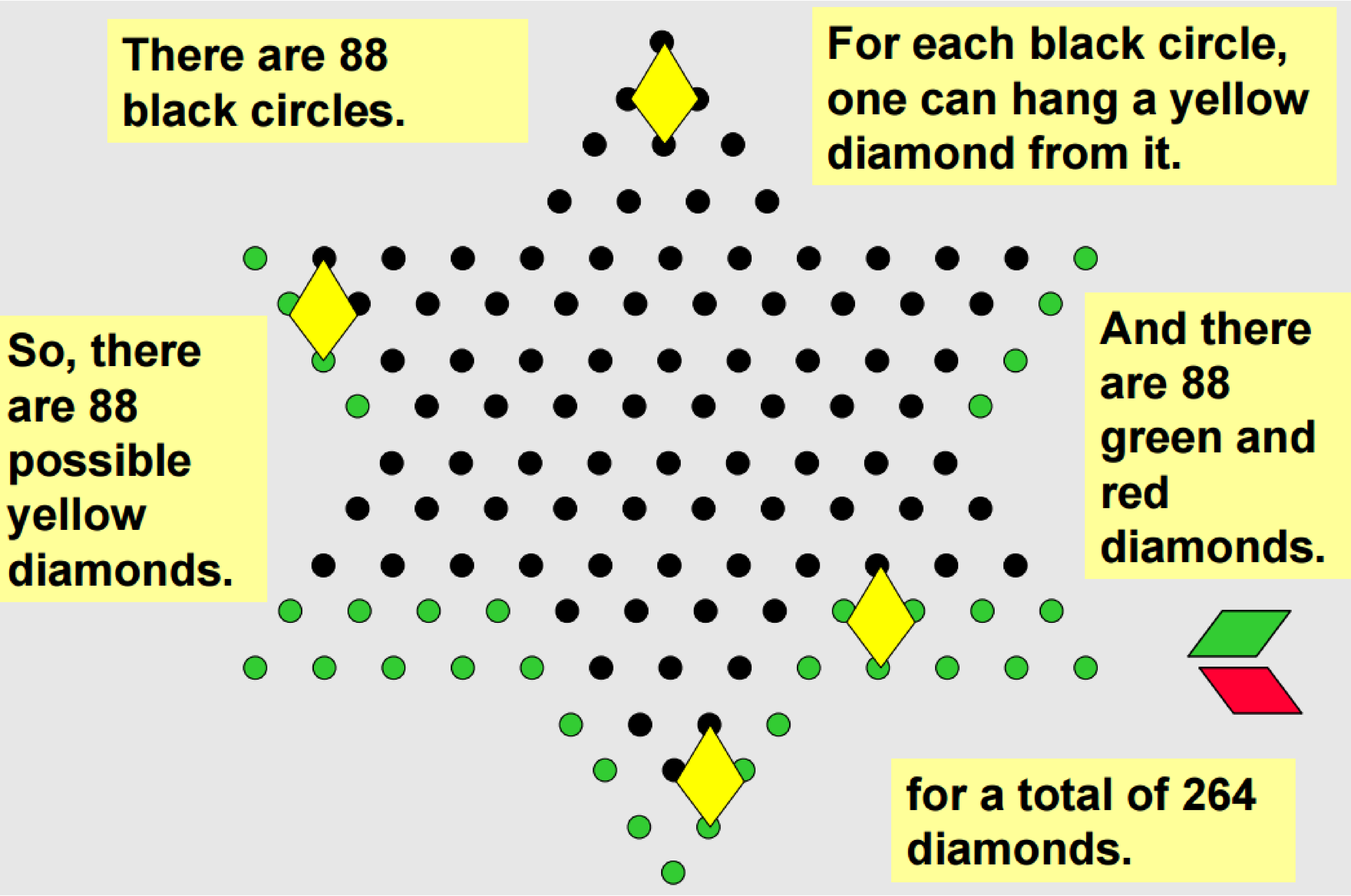
  (note: there are 264 possible diamonds, see next slide)

- Decision variables: $x_d \in \{0,1\}\ \forall d \in D$

$$x_d = \begin{cases} 1, & \text{if diamond } d \text{ is selected} \\ 0, & \text{if diamond } d \text{ is not selected} \end{cases}$$

- Need to ensure no overlap
- Thus, we want: $x_d + x_{d'} \leq 1$ for all possible diamond pairs d and d' that have at least one point in common
- Let the set O denote all such pairs




There are 88 black circles.
For each black circle, one can hang a yellow diamond from it.
So, there are 88 possible yellow diamonds.
And there are 88 green and red diamonds.
for a total of 264 diamonds.

set packing problem: first IP formulation

$$\max \sum_{d \in D} x_d$$

$$\text{s.t. } x_d + x_{d'} \leq 1 \; \forall (d, d') \in O$$

$$x_d \in \{0, 1\} \quad \forall d \in D$$

This formulation is terrible for B&B!

$\boldsymbol{z^*_{\mathrm{IP}} = 27}$**, but** $\boldsymbol{z^*_{LP} = 132}$

and the LP optimum solution: $x_d = {}^1/_2 \; \forall d \in D$

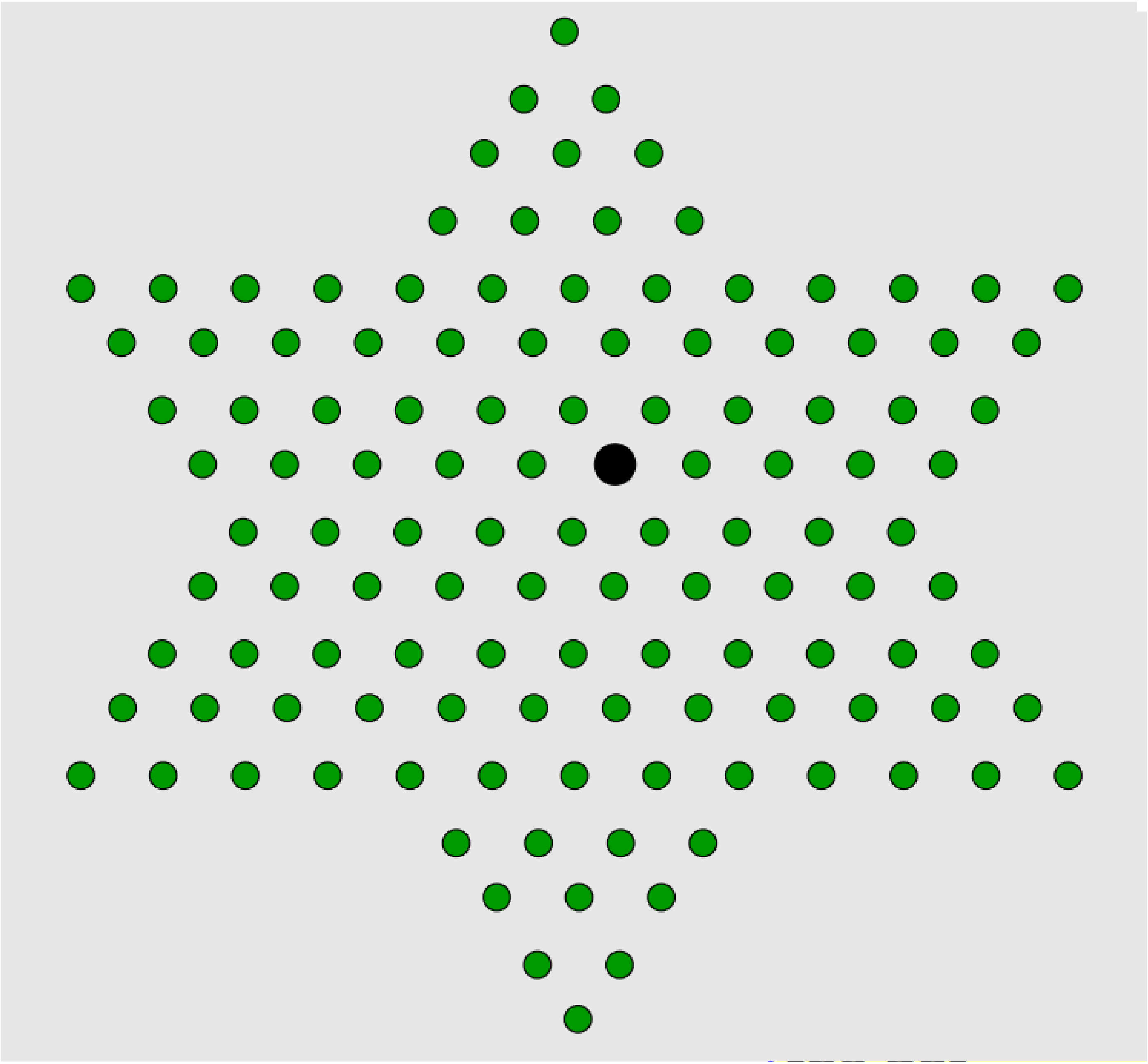
Useless information produced for optimal.

Add valid inequalities to limit solution space

B&B would take much more than **3 billion years** to solve this problem on the fastest computer unless it can add valid inequalities!

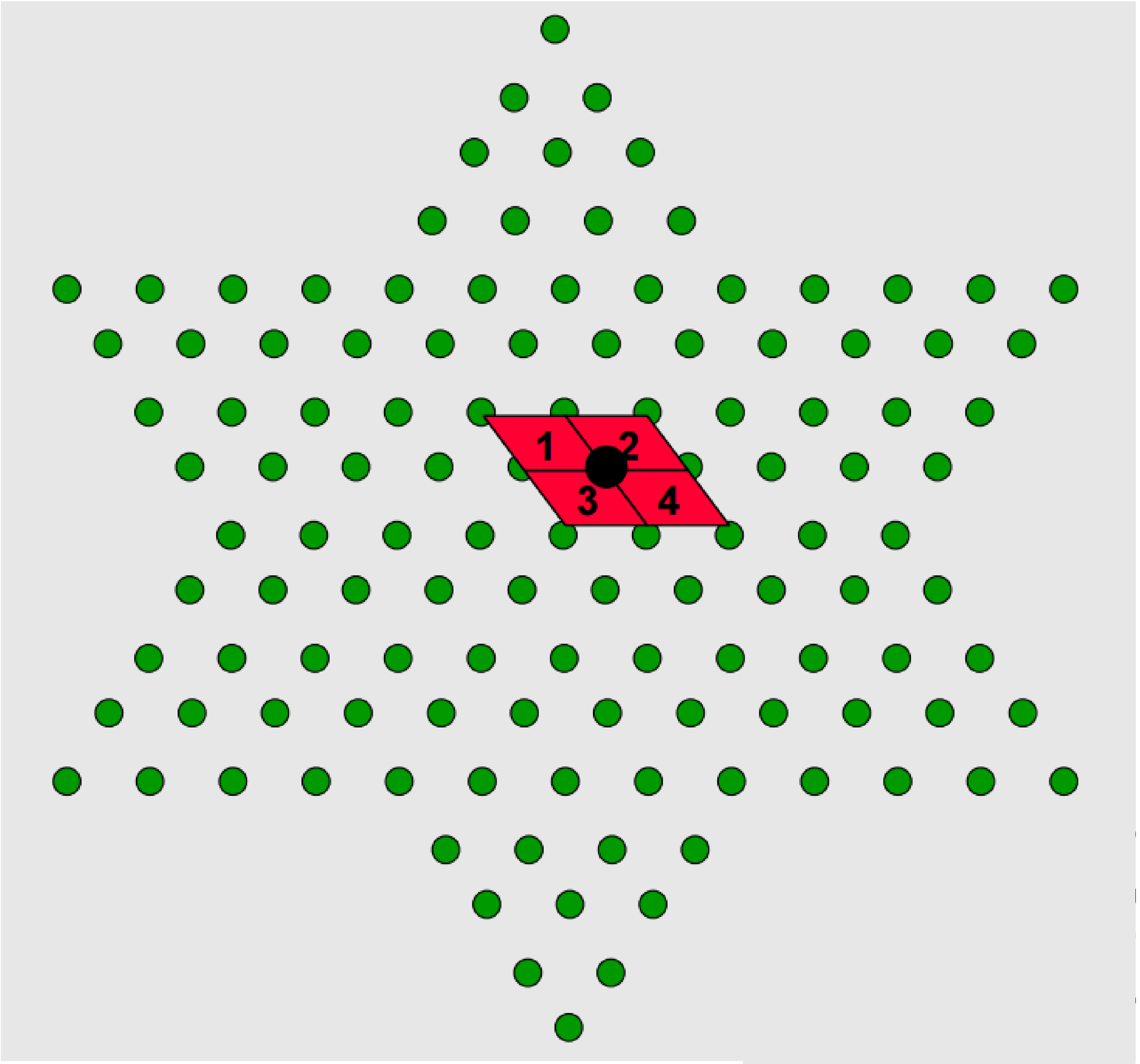
An improved IP formulation

For every dot, choose at most one of the diamonds containing the dot.



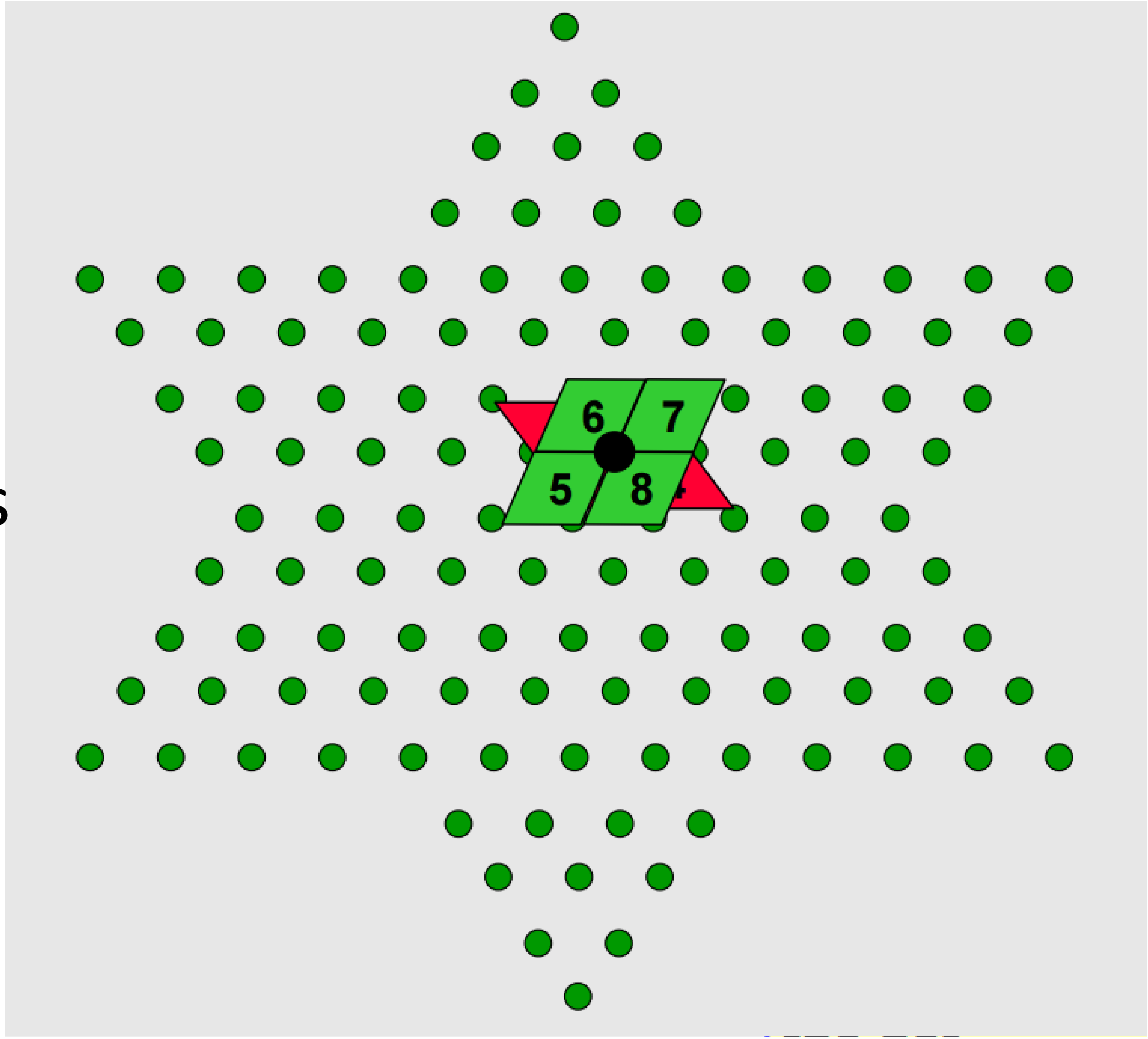
An improved IP formulation

For every dot, choose at most one of the diamonds containing the dot.



An improved IP formulation

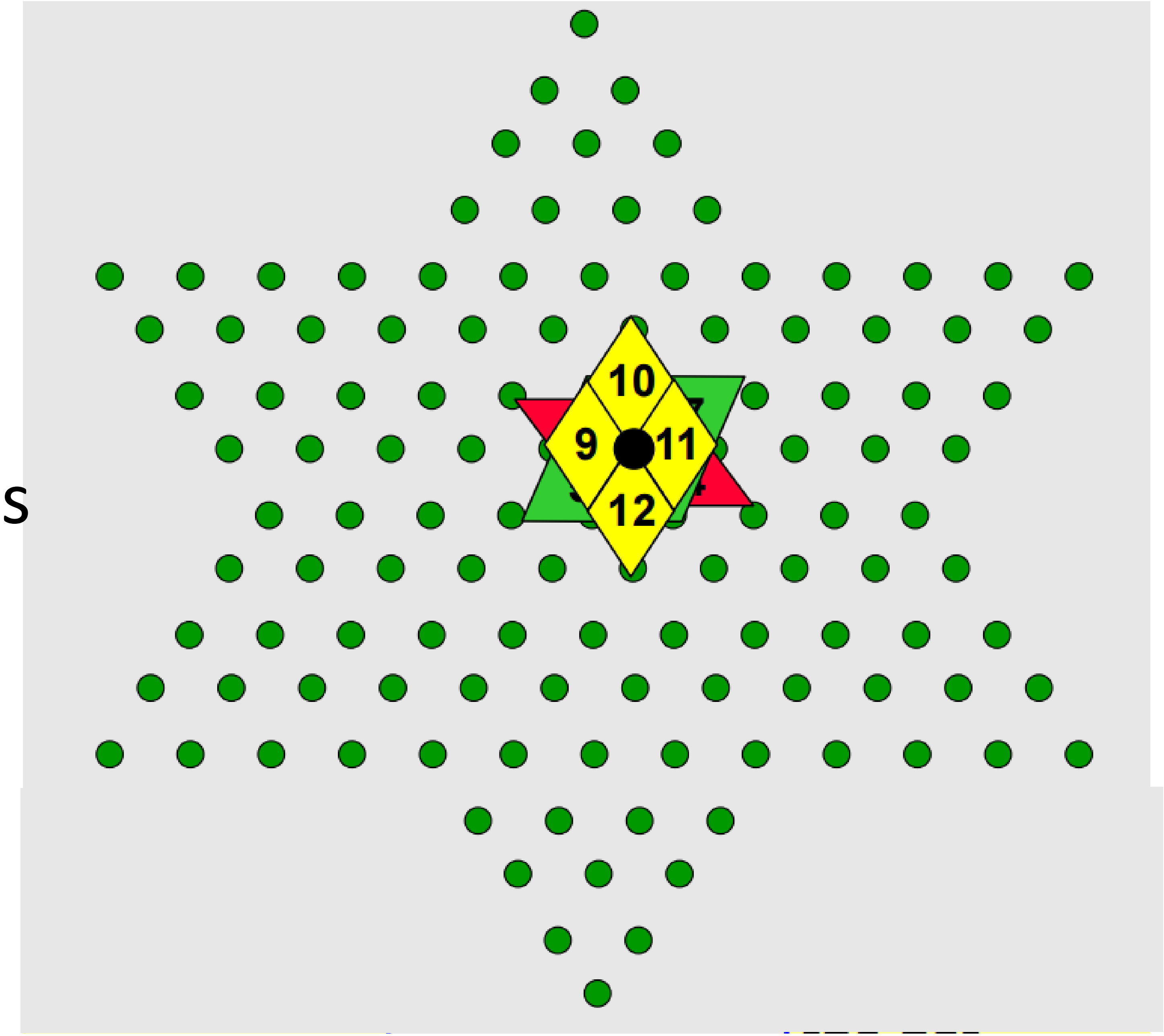
For every dot, choose at most one of the diamonds containing the dot.



An improved IP formulation

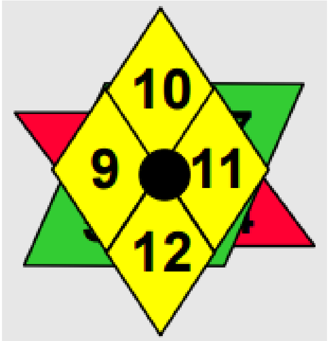
For every dot, choose at most one of the diamonds containing the dot.

At most 1 of the 12 diamonds may be selected.



set packing problem: an improved IP constraint

Let $D(c)$ be the set of diamonds that include circle c.



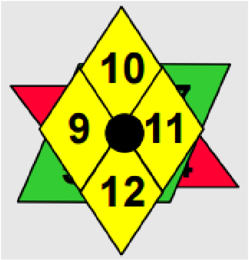
$$\sum_{d \in D(c)} x_d \leq 1 \quad \text{for each circle c}$$

Example constraint for one circle (in the interior):

$$x_1 + x_2 + x_3 + \ldots + x_{12} \leq 1$$

set packing problem: an improved IP constraint

$$x_1 + x_2 + x_3 + \ldots + x_{12} \leq 1$$



Note: $x_j = {}^1/_{12}$ would be LP feasible, but $x_j = {}^1/_2$ would not be feasible for all such circles!

In this constraint, we've combined 66 different constraints from our previous formulation:

New constraints tot
66

$$x_1 + x_2 \leq 1 \quad x_1 + x_3 \leq 1 \quad x_1 + x_4 \leq 1$$
$$x_1 + x_5 \leq 1 \quad x_1 + x_6 \leq 1 \quad x_1 + x_7 \leq 1$$
$$x_1 + x_8 \leq 1 \quad \ldots \quad x_{11} + x_{12} \leq 1$$

set packing problem: an improved IP formulation

$$\max \sum_{d \in D} x_d$$

$$\text{s.t.} \quad \sum_{d \in D(c)} x_d \leq 1 \quad \text{for each circle c}$$

$$x_d \in \{0, 1\} \quad \forall d \in D$$

New formulation

$$\boldsymbol{z^*_{\mathrm{LP}} = 27.5 \rightarrow z^*_{\mathrm{IP}} = 27}$$

Close to the optimal and very fast!

B&B solution time: **0.001 seconds!** (which, FYI, is considerably better than 3 billion years)

valid inequalities question

$$\begin{aligned} \max\ & 22x_1 + 19x_2 + 16x_3 + 12x_4 + 11x_5 + 8x_6 \\ \text{s.t. }\ & 7x_1 + 6x_2 + 5x_3 + 4x_4 + 4x_5 + 3x_6 \leq 14 \\ & x_j \in \{0, 1\} \quad \text{for } j = 1, \ldots, 6 \end{aligned}$$

1. Why can at most two of x_1, x_2, x_3 be set to 1?
2. How can you write this as a valid inequality?
3. What are some more valid inequalities you could come up with?

valid inequalities question

$$\begin{aligned} \max \ & 22x_1 + 19x_2 + 16x_3 + 12x_4 + 11x_5 + 8x_6 \\ \text{s.t.} \ & 7x_1 + 6x_2 + 5x_3 + 4x_4 + 4x_5 + 3x_6 \leq 14 \\ & x_j \in \{0,1\} \quad \text{for } j = 1, \ldots, 6 \end{aligned}$$

Some valid inequalities:

$$\begin{array}{lll} x_1 + x_2 + x_3 \leq 2 & x_1 + x_2 + x_5 \leq 2 & x_1 + x_3 + x_4 \leq 2 \\ x_1 + x_2 + x_4 \leq 2 & x_1 + x_2 + x_6 \leq 2 & \text{etc.} \end{array}$$

A really good constraint: $x_1 + x_2 + x_3 + x_4 \leq 2$

integer programming and B&B summary

- **IP dramatically improves the modeling capability**
  - yes/no decisions, contingent decisions
  - logical constraints
  - fixed costs and piecewise linear cost functions
- **Not as easy to model, and not as easy to solve**
- **Branch and Bound**
  - general technique based on divide and conquer
  - “implicit enumeration”
- **Cutting planes (valid inequalities)**
  - used to help improve B&B
  - there are “automatic” and “logical” ways of implementing
  - Key takeaway: a good IP formulation is crucial for practical efficiency