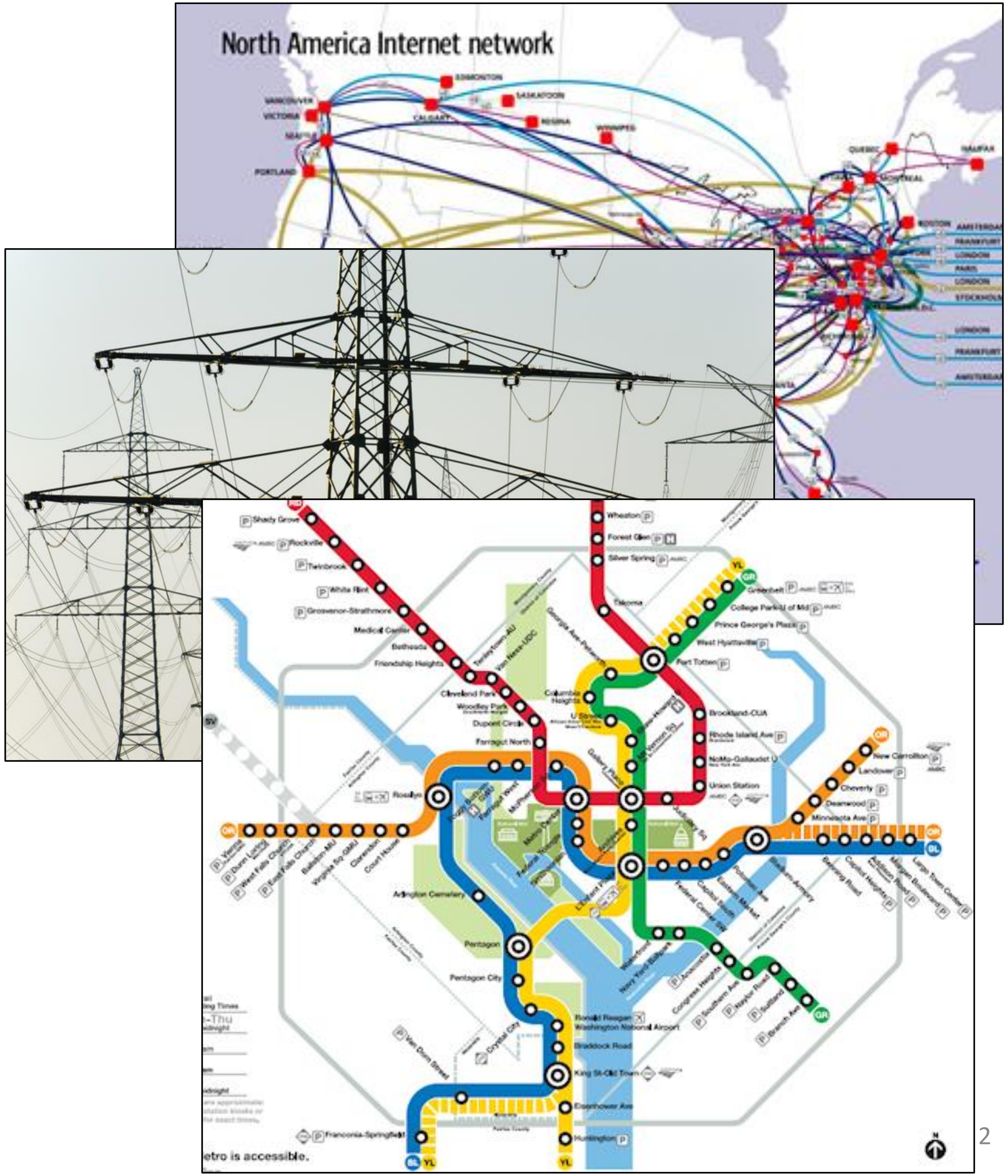
Network Science and Network Flow Optimization

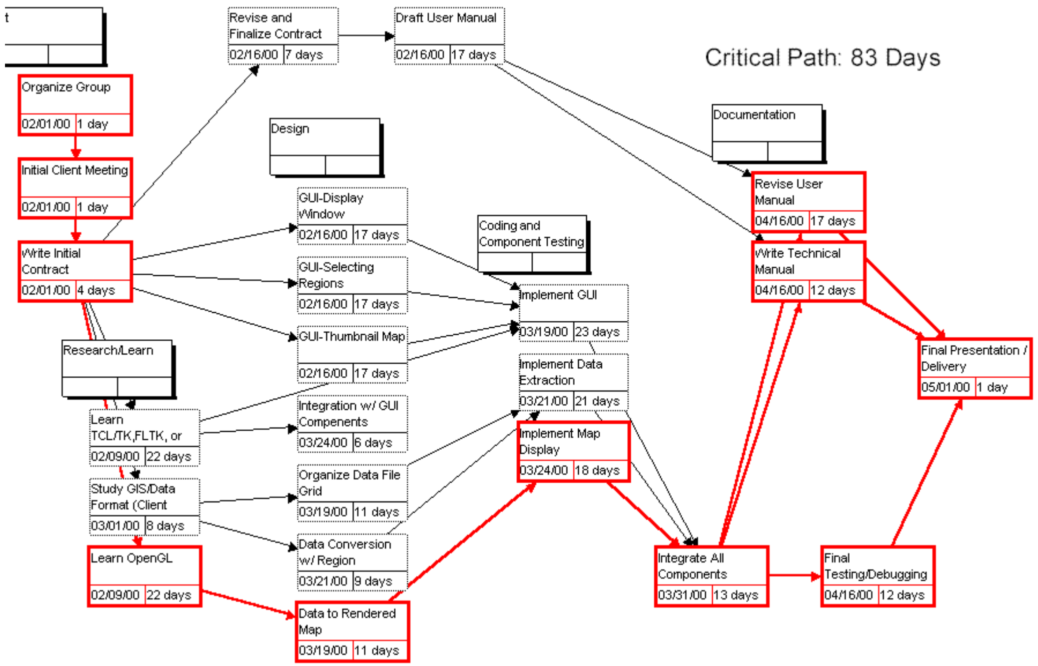
Network systems

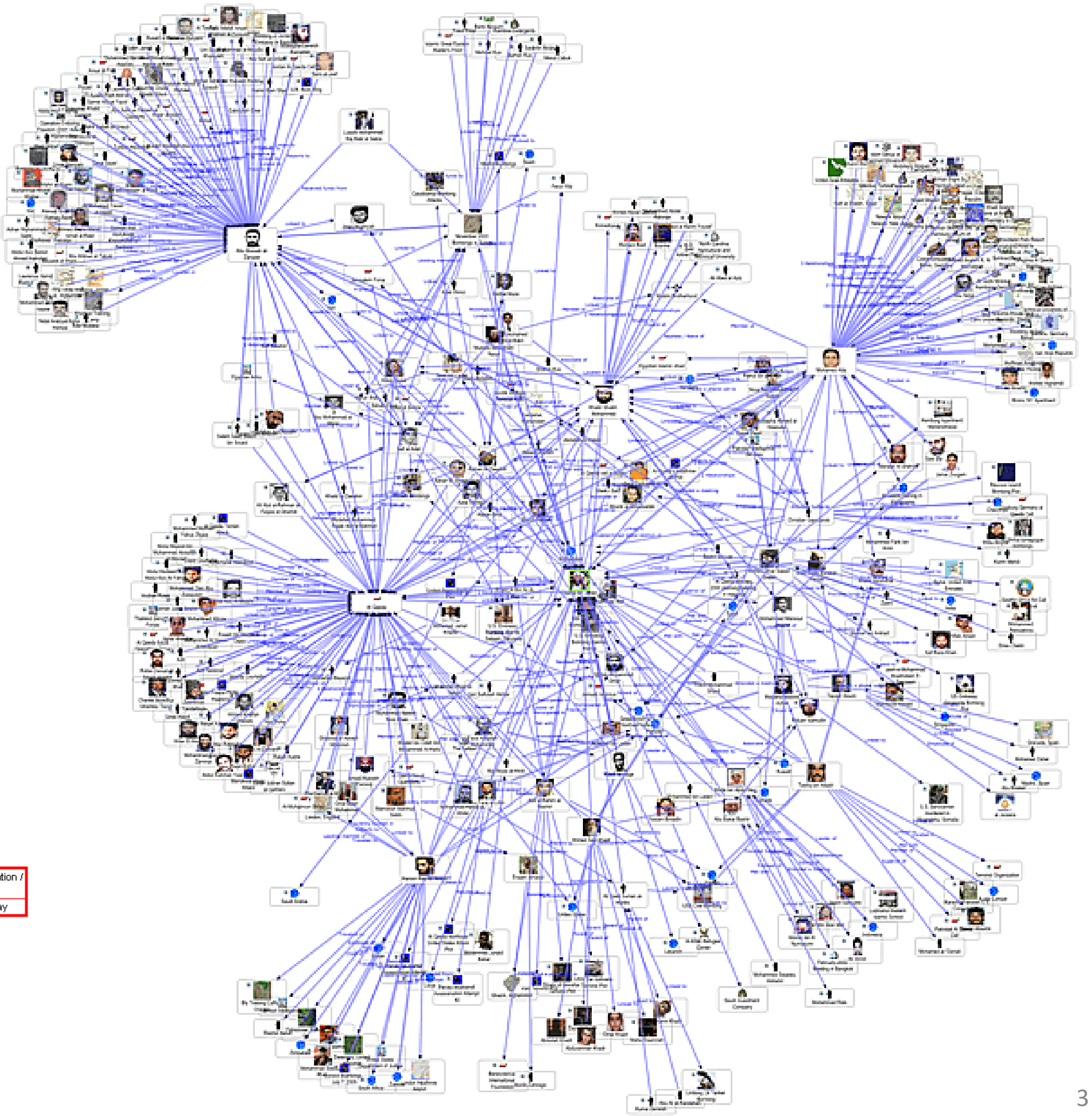
- Communication, e.g., internet, cellphones, satellite TV
- Utility systems, e.g., electric power networks, water distribution
- Transportation, e.g, highways, railways, airline routes



Network systems

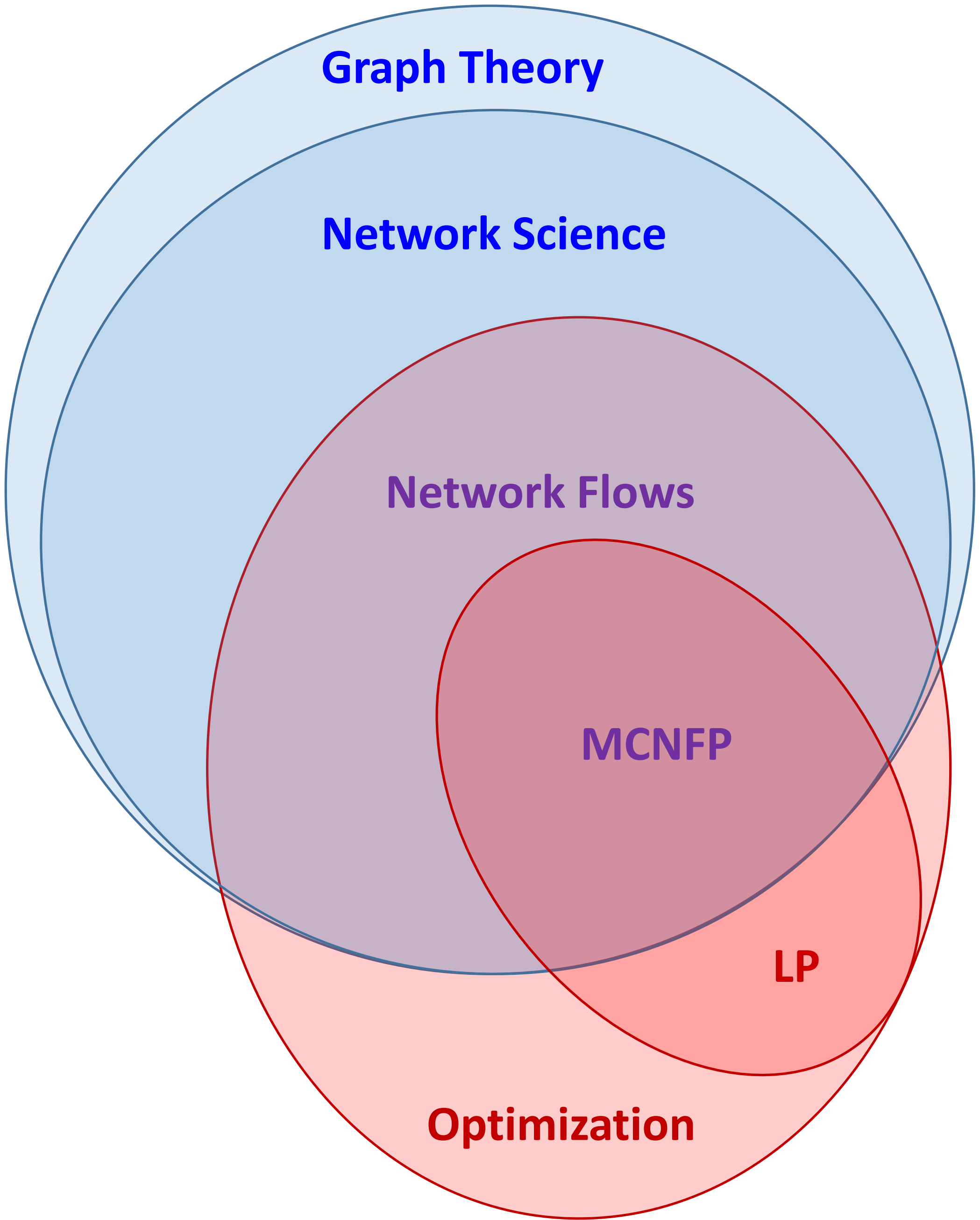
- Social networks
- Financial networks
- Project scheduling





Actually, let's back up a bit…

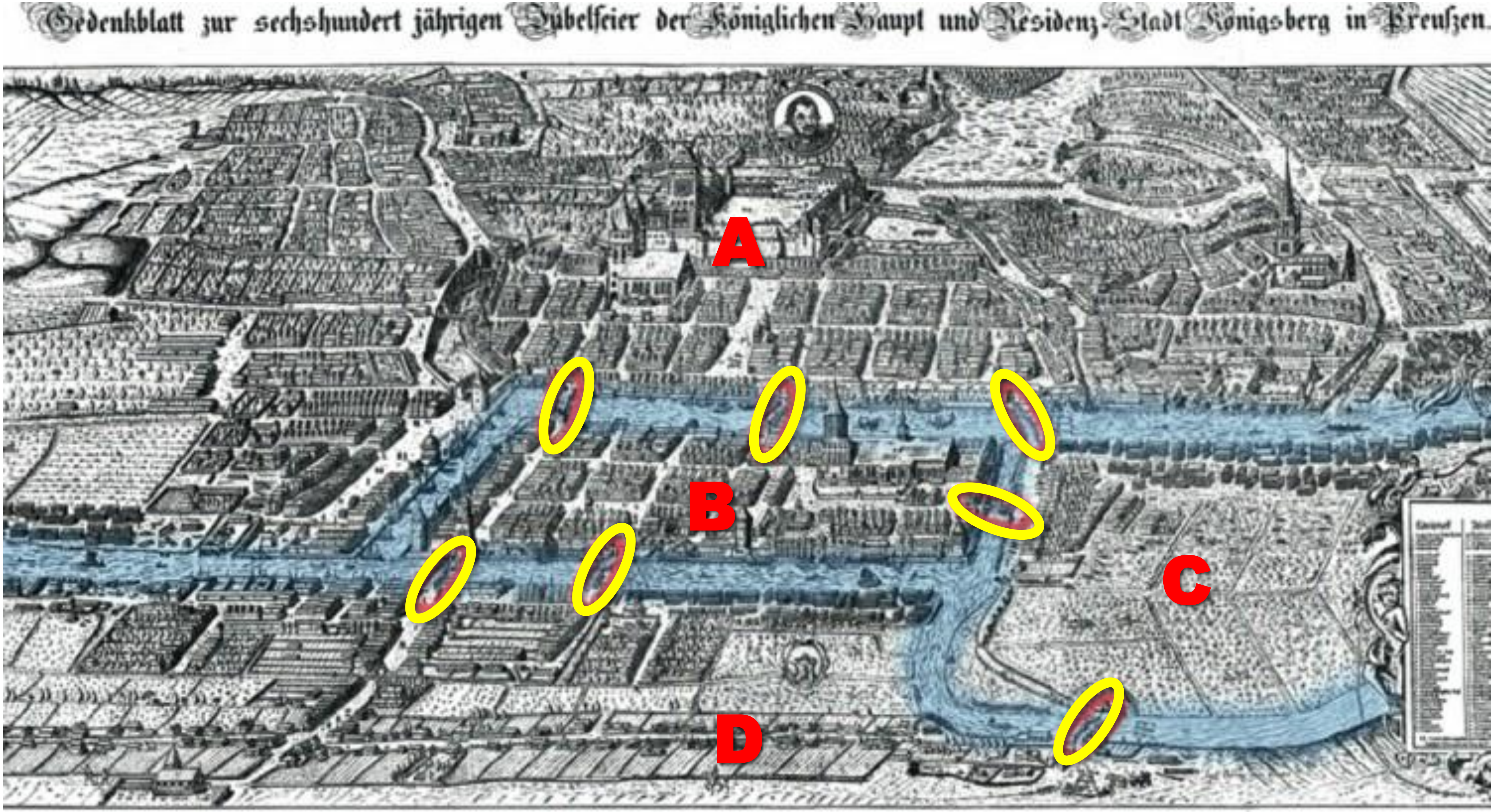
- Graph Theory, Network Science, Network Flow Optimization
  - Terms "graph" and "network" are synonymous
- Graph theory: branch of discrete math for proving theorems/developing algorithms for arbitrary graphs; provides mathematical tools of network science
- Network science: typically, more attributes associated with graph elements; real-world instances
- Network flows: subset of network science associated with optimization
- MCNFP: key LP problem type/formulation within network flow optimization



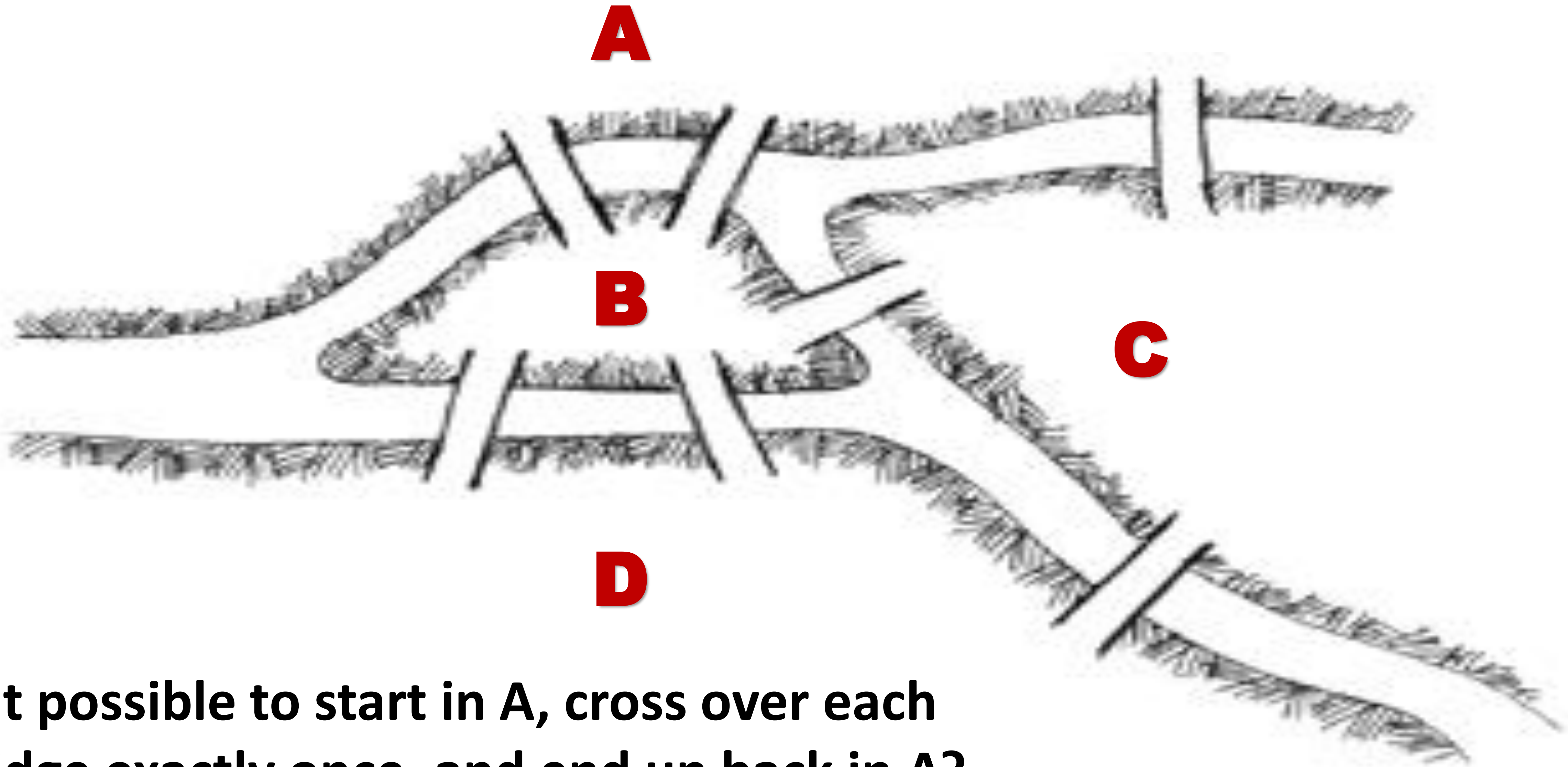
The Bridges of Koenigsberg: Euler 1736

- “Graph Theory” began in 1736
- Leonard Eüler
  - Visited Koenigsberg
  - People wondered whether it is possible to take a walk, end up where you started from, and cross each bridge in Koenigsberg exactly once
  - Generally, it was believed to be impossible

The Bridges of Koenigsberg: Euler 1736

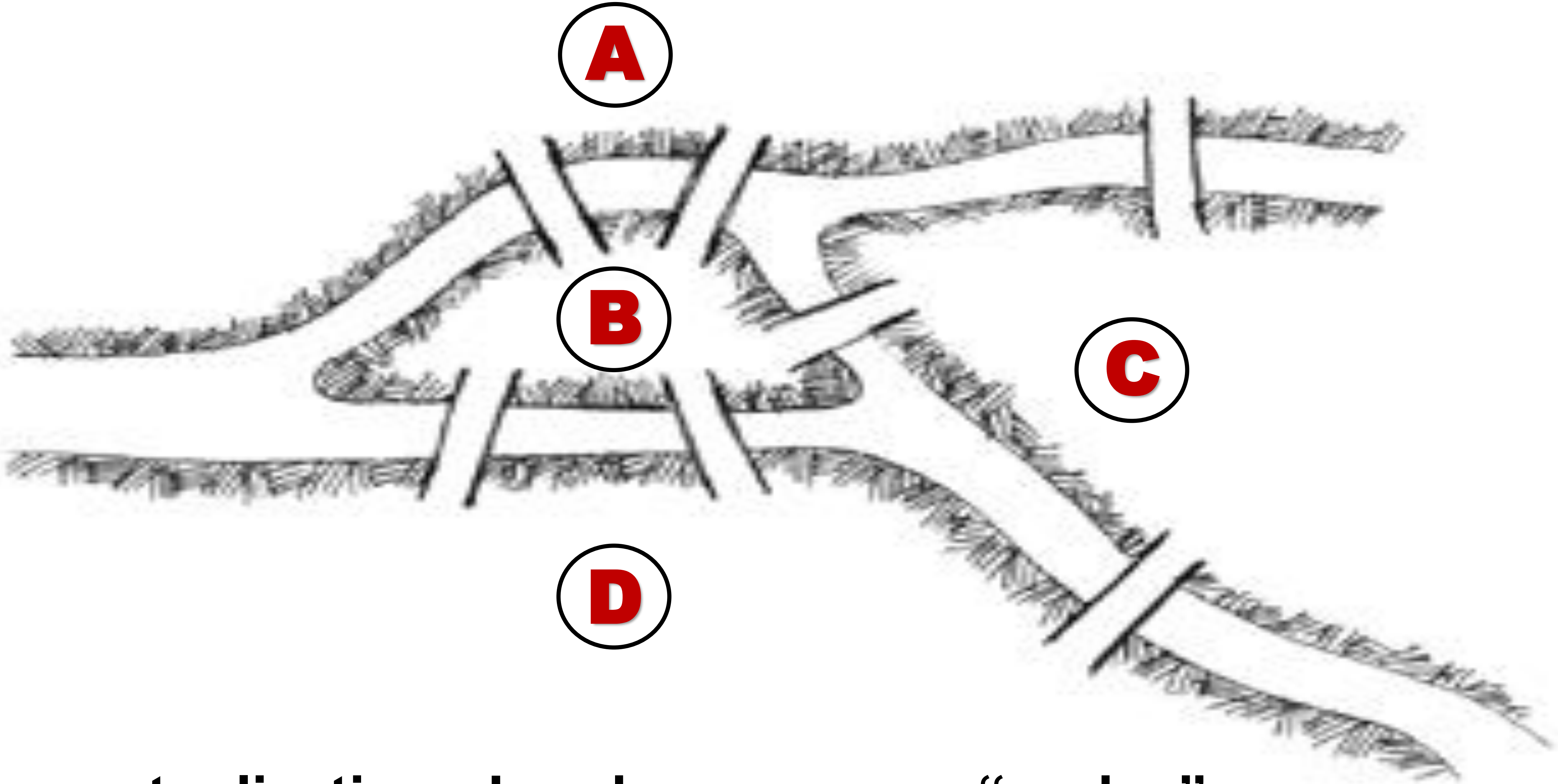


The Bridges of Koenigsberg: Euler 1736



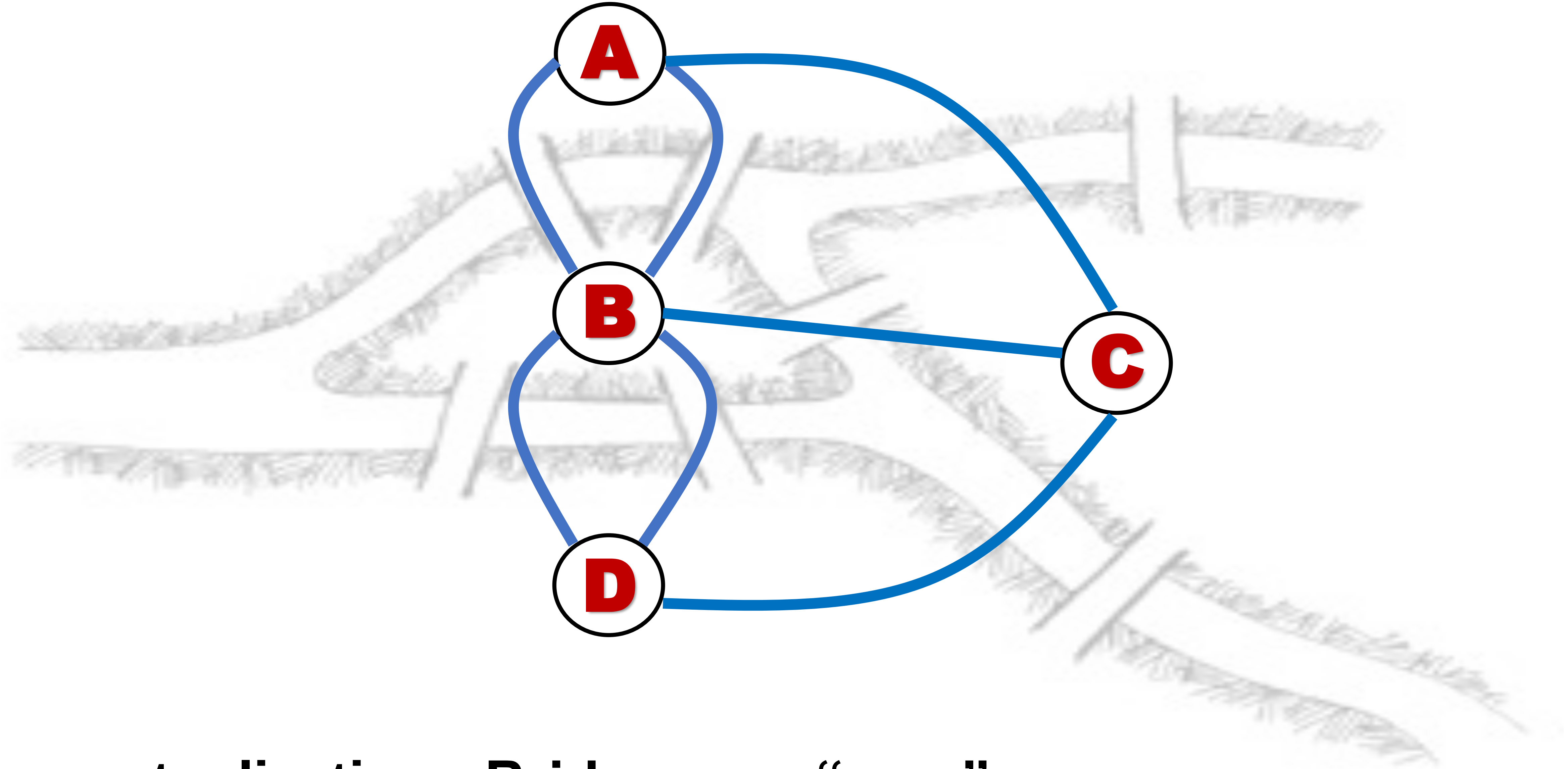
Is it possible to start in A, cross over each bridge exactly once, and end up back in A?

The Bridges of Koenigsberg: Euler 1736



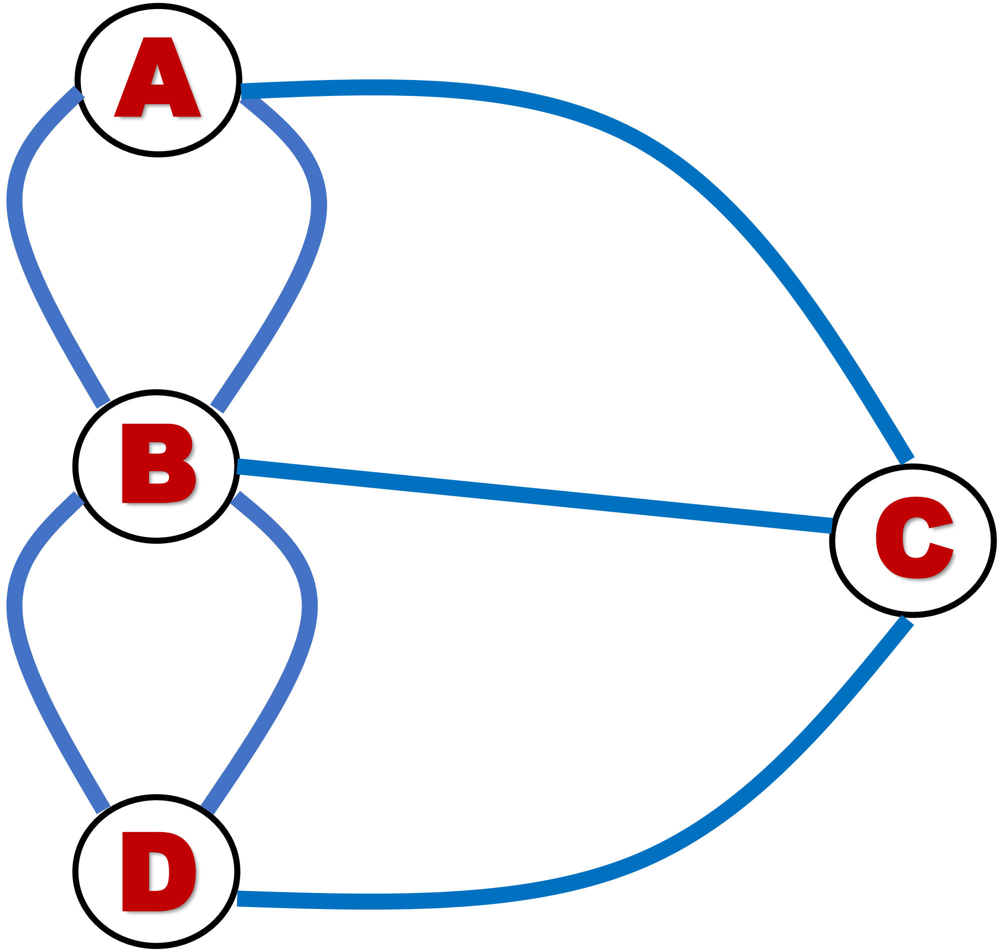
Conceptualization: Land masses are "nodes"

The Bridges of Koenigsberg: Euler 1736



Conceptualization: Bridges are “arcs”

The Bridges of Koenigsberg: Euler 1736



Can you walk, starting at A and ending at A, so as to pass through each arc exactly once?

Notation and Terminology

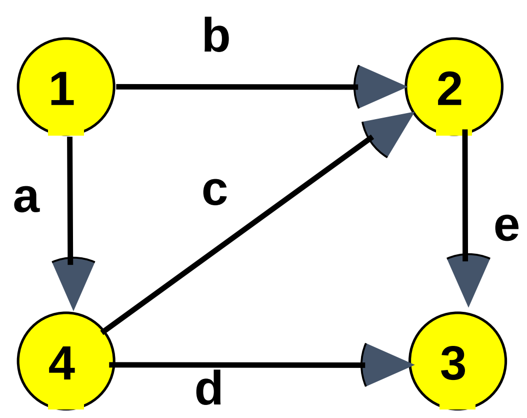
- A **network** or **graph** consists of points, and lines connecting pairs of points.
- The points are called **nodes** or vertices.
- The lines are called **arcs** or edges or links.
- The arcs may have a direction on them, in which case they are called **directed arcs.**
- If all the arcs in a network are directed, the network is a **directed network.**
- If all the arcs are undirected, the network is an **undirected network.**

Notation and Terminology

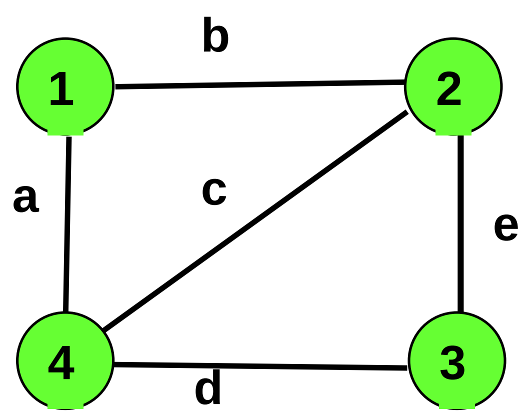
Graph or Network $G = (N, A)$

Node set $N = \{1, 2, 3, 4\}$

Arc Set $A = \{(1,2), (1,4), (2,3), (4,2), (4,3)\}$



A Directed Graph or
Directed Network

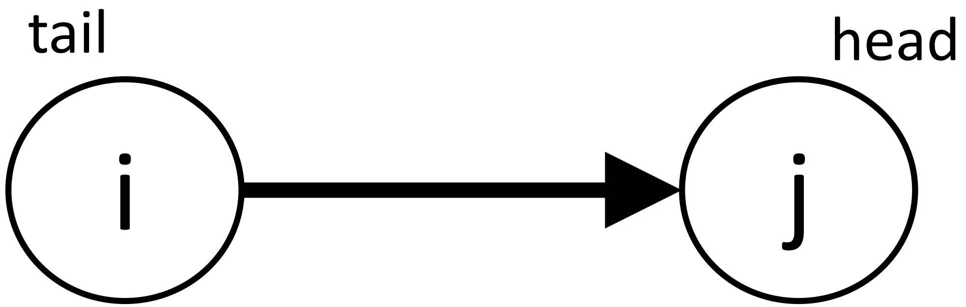


An Undirected Graph or
Undirected Network

In an undirected graph, $(i, j) = (j, i)$

Notation and Terminology

- ***Directed arc*** $(\boldsymbol{i}, \boldsymbol{j})$: i is "tail" and j is "head" node of the arc

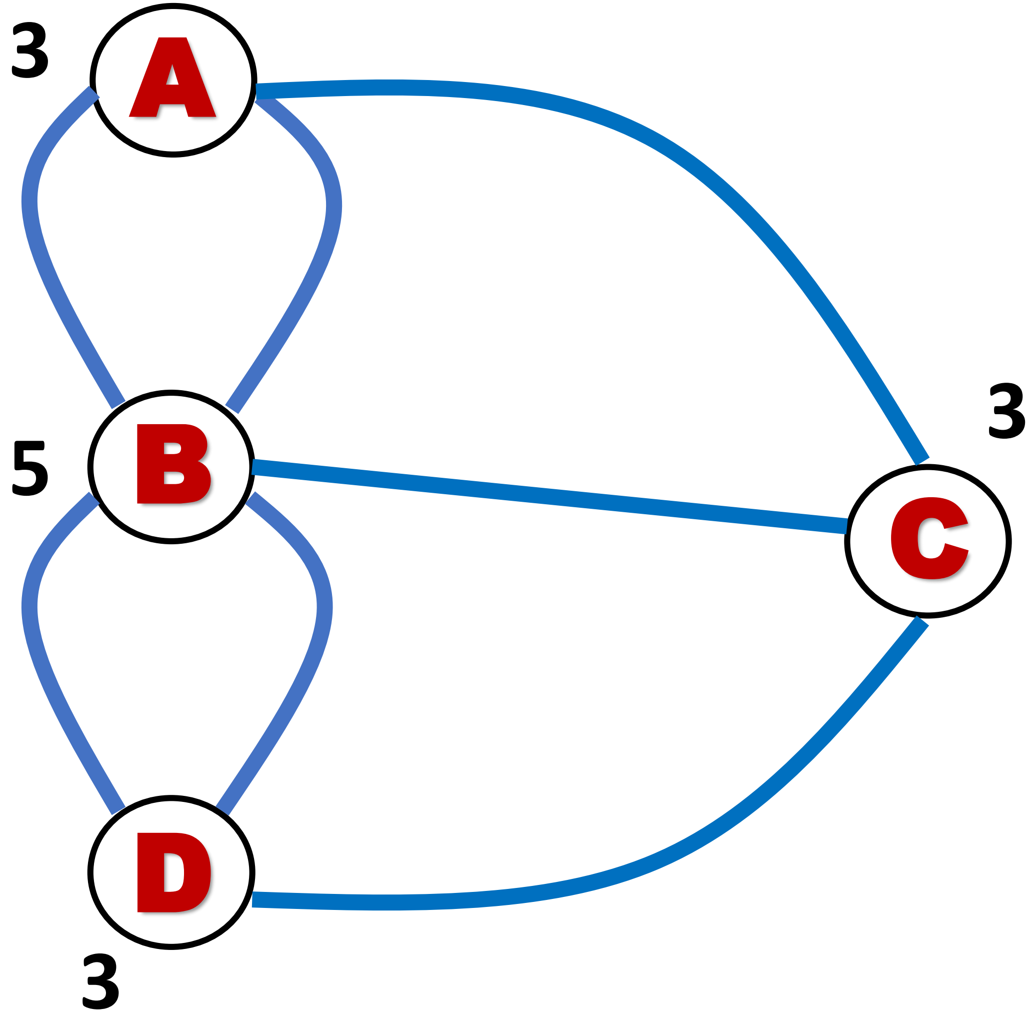


- Arc (i, j) is said to be ***incident*** to nodes i and j
- Two nodes connected by an arc (or any two arcs connected by a node) are said to be **adjacent**.
- **Degree** of a node: number of arcs incident on it
- For directed graphs:
  - **indegree** of a node: number of arcs leading into that node;
  - **outdegree:** number of arcs leading away from it

Notation and Terminology

- ***Subgraph*** $G' = (N', A')$ is a subgraph of G, if N' is subset of N and A' is subset of A
- ***Walk***: subgraph of G consisting of a sequence of nodes and arcs
- ***Directed walk***: oriented version of a walk
- **Path:** walk without any repetition of nodes; **directed path** is oriented and has no “backward” arcs
- A path that begins and ends at the same node is a **cycle** and may be either directed or undirected.
- A network is **connected** if there exists an undirected path between any pair of nodes.
- A connected network without cycles is called a **tree**

The Bridges of Koenigsberg: Euler 1736



Can you walk, starting at A and ending at A, so as to pass through each arc exactly once? **Eulerian cycle**

NETWORK FLOWS

APPLICATIONS

Network Flow Applications

- **Transportation Problem**. A commodity, which can be produced in different plants, needs to be shipped to different distribution centers. Given the cost of shipping a commodity between any two points, the capacity of each plant, and the demands at each distribution center, find the minimal cost shipping plan.
- **Assignment Problem**. There are people to be assigned to different tasks. Each individual is assigned to one job only and each job is performed by one person. Given the cost that each individual charges for performing each of the jobs, find a minimal cost assignment.
- **Shortest Path Problem.** Given a directed network and the length of each arc in this network, find a shortest between two given nodes.
- **Maximum Value Flow**. Given a directed network of roads that connects two cities and the capacities of these roads, find the maximum number of units (cars) that can be routed from one city to another.

Network Flow Applications

- **Minimum Cost Flow Problem (MCNFP).** Given a directed network with upper and lower capacities on each of its arcs, and given a set of external flows (positive or negative) that need to be routed through this network, find the minimal cost routing of the given flows through this network.
- **Multicommodity Flows.** Multiple commodities should be routed through a network subject to commodity-specific supplies, demands, and costs; however, the commodities are subject to the same capacity bounds. Extension of the MCNFP.
- **Generalized Flow.** Flow from one node increases or decreases before arriving at another node. This is useful for modeling dissipation (e.g., loss of heat, water evaporation, loss of pressure, seepage, detioration, spoilage) or positive gains (e.g., monetary growths from interest rates), and conversions of units (e.g., currency, raw materials to final goods, labor to production units)