NETWORK FLOWS: THE MINIMUM COST NETWORK FLOW PROBLEM

elements of the MCNFP

Defined on a network: $G = (N, A)$

where N is a set of n *nodes*: $\{1, 2, \ldots, n\}$

and A is a set of m arcs as a subset of $N \times N$

Each node i has an associated value $b(i)$

- $b(i) > 0$ → node i is a *supply* node with a supply of $b(i)$ units
- $b(i) < 0$ → node i is a *demand* node with a demand for $b(i)$ units of some commodity
- $b(i) = 0$ → node i is a *transshipment* node

elements of the MCNFP

The set A is a set of *arcs,* e.g. (i, j) for $i \in N, j \in N$
each of which may carry *flow of a commodity*

Decision variable: x_{ij} determines the units of flow on arc (i, j)

Arc (i, j) has certain characteristics:

- cost c_{ij} per unit of flow on arc (i, j)
- upper bound on flow of u_{ij} (capacity)
- lower bound on flow of ℓ_{ij} (usually 0)

example MCNFP

$$N = \{1, 2, 3, 4\}$$

$$b(1) = 5, b(2) = -2, b(3) = 0, b(4) = -3$$

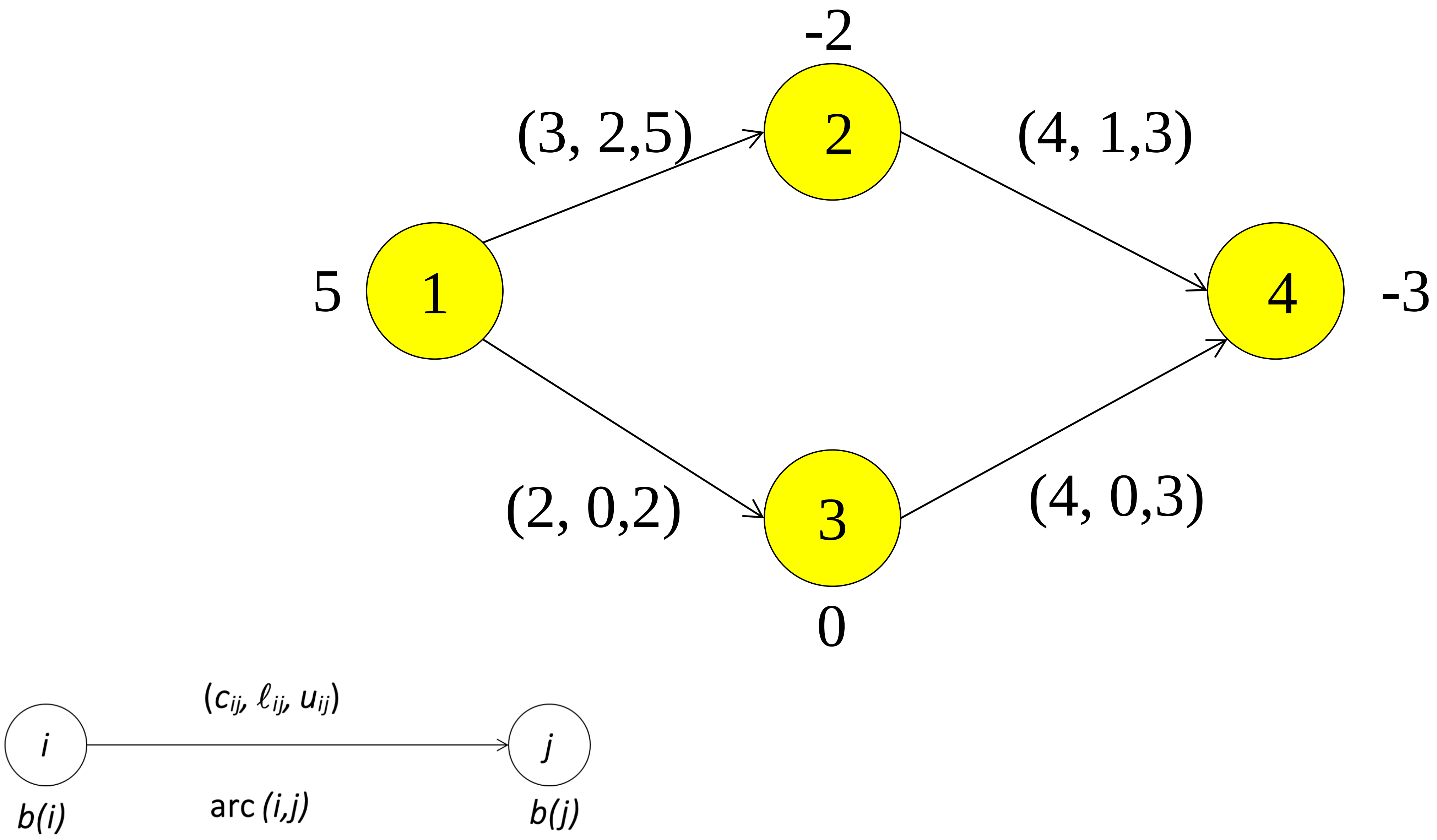
$$A = \{(1,2), (1,3), (2,4), (3,4)\}$$

$$c_{12} = 3, c_{13} = 2, c_{24} = 4, c_{34} = 4$$

$$\ell_{12} = 2, \ell_{13} = 0, \ell_{24} = 1, \ell_{34} = 0$$

$$u_{12} = 5, u_{13} = 2, u_{24} = 3, u_{34} = 3$$

example MCNFP



requirements for a feasible flow

- Flow on all arcs is within the allowable bounds:
  $\ell_{ij} \leq x_{ij} \leq u_{ij}$ for all arcs (i, j)

- Flow is balanced at all nodes:
  flow out of node i – flow into node i = $b(i)$

- MCNFP: find a minimum-cost feasible flow

MCNFP formulation

$$
\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in A} c_{ij} x_{ij} \\
\text{subject to} \quad & \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = b_i \qquad \forall i \in N \\
& l_{ij} \le x_{ij} \le u_{ij} \qquad \forall (i,j) \in A
\end{aligned}
$$

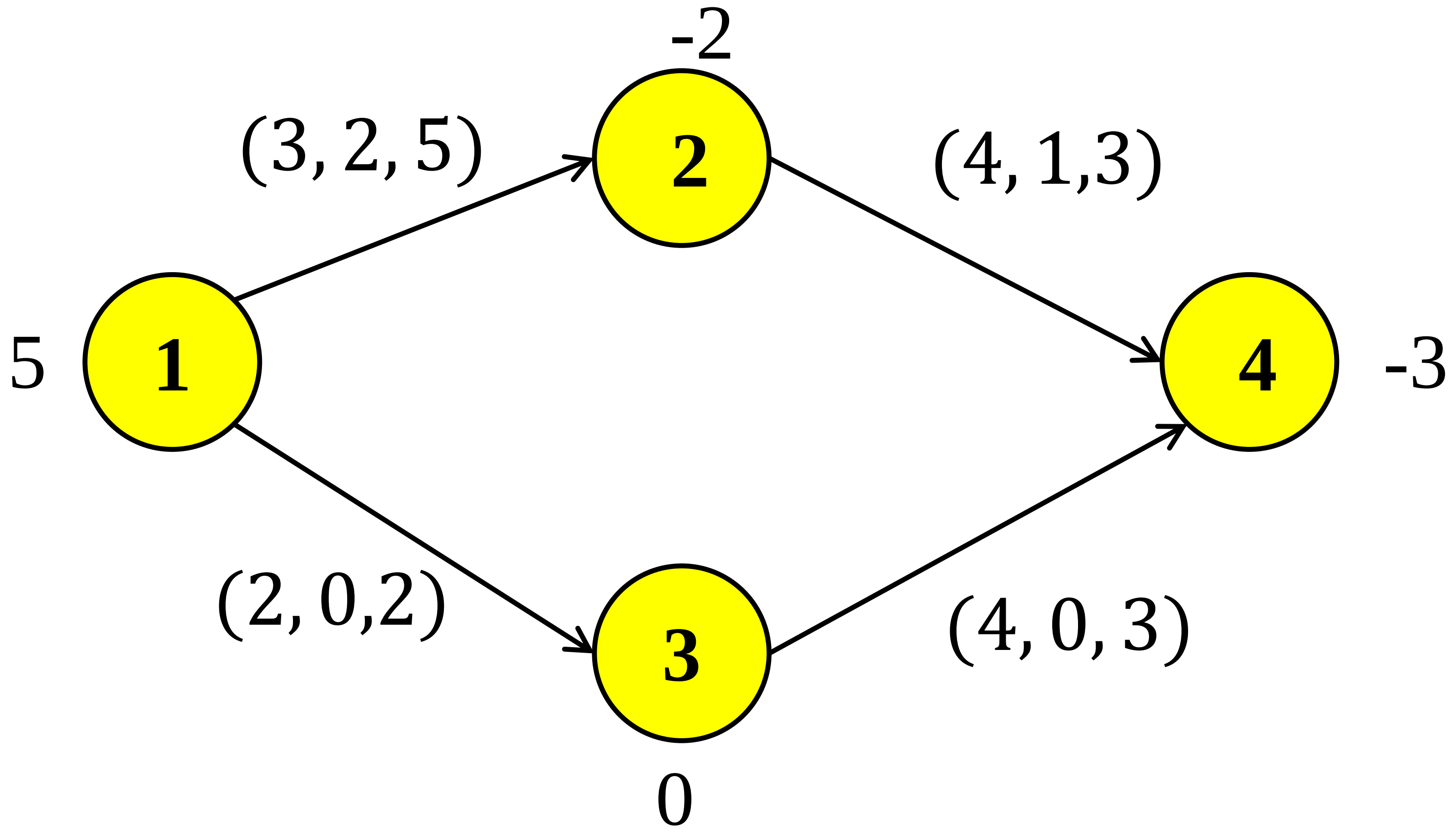
MCNFP example formulation (expanded)

$$
\begin{aligned}
\text{minimize}\ & 3x_{12} + 2x_{13} + 4x_{24} + 4x_{34} \\
\text{s.t.}\quad & x_{12} + x_{13} = 5 && \text{(node 1)} \\
& x_{24} - x_{12} = -2 && \text{(node 2)} \\
& x_{34} - x_{13} = 0 && \text{(node 3)} \\
& -x_{24} - x_{34} = -3 && \text{(node 4)} \\
& 2 \leq x_{12} \leq 5 \\
& 0 \leq x_{13} \leq 2 \\
& 1 \leq x_{24} \leq 3 \\
& 0 \leq x_{34} \leq 3
\end{aligned}
$$

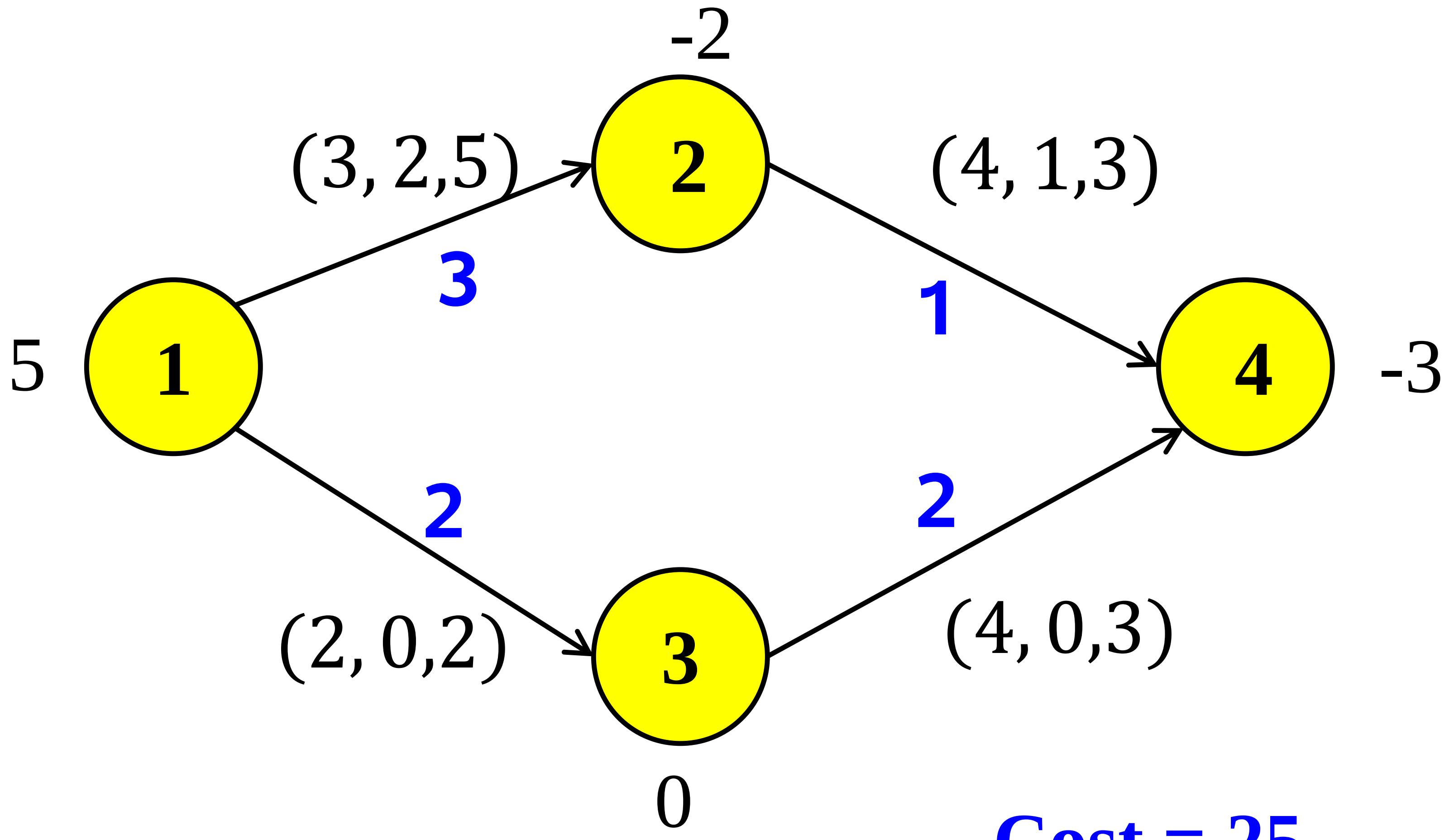
node arc incidence matrix

	x_{12}	x_{13}	x_{24}	x_{34}
1	**1**	**1**	**0**	**0**
2	**-1**	**0**	**1**	**0**
3	**0**	**-1**	**0**	**1**
4	**0**	**0**	**-1**	**-1**

example feasible solution



optimal solution for example



Cost = 25

Since,

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

it must hold that,

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Cramer's rule

Use determinants to solve $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

$$x_j = \frac{|B_j|}{|A|}$$

where,

$$B_j = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,j-1} & b_1 & a_{1,j+1} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2,j-1} & b_2 & a_{2,j+1} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n,j-1} & b_n & a_{n,j+1} & \dots & a_{nn} \end{bmatrix}$$

i.e., take the matrix A and replace column j with the vector b to form matrix B_j.

Cramer's rule to solve for x_{12}

Is $\left|B_{(1,2)}\right|$ an integer?

Does $|A| = \pm 1$?

$$x_{12} = \frac{\begin{vmatrix} 5 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -3 & 0 & -1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{vmatrix}} = \frac{\left|B_{(1,2)}\right|}{|A|}$$

total unimodularity

- A square, integer matrix is *unimodular* if its determinant is 1 or -1.
- An integer matrix A is called *totally unimodular* (TU) if every square, nonsingular submatrix of A is unimodular.

total unimodularity

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

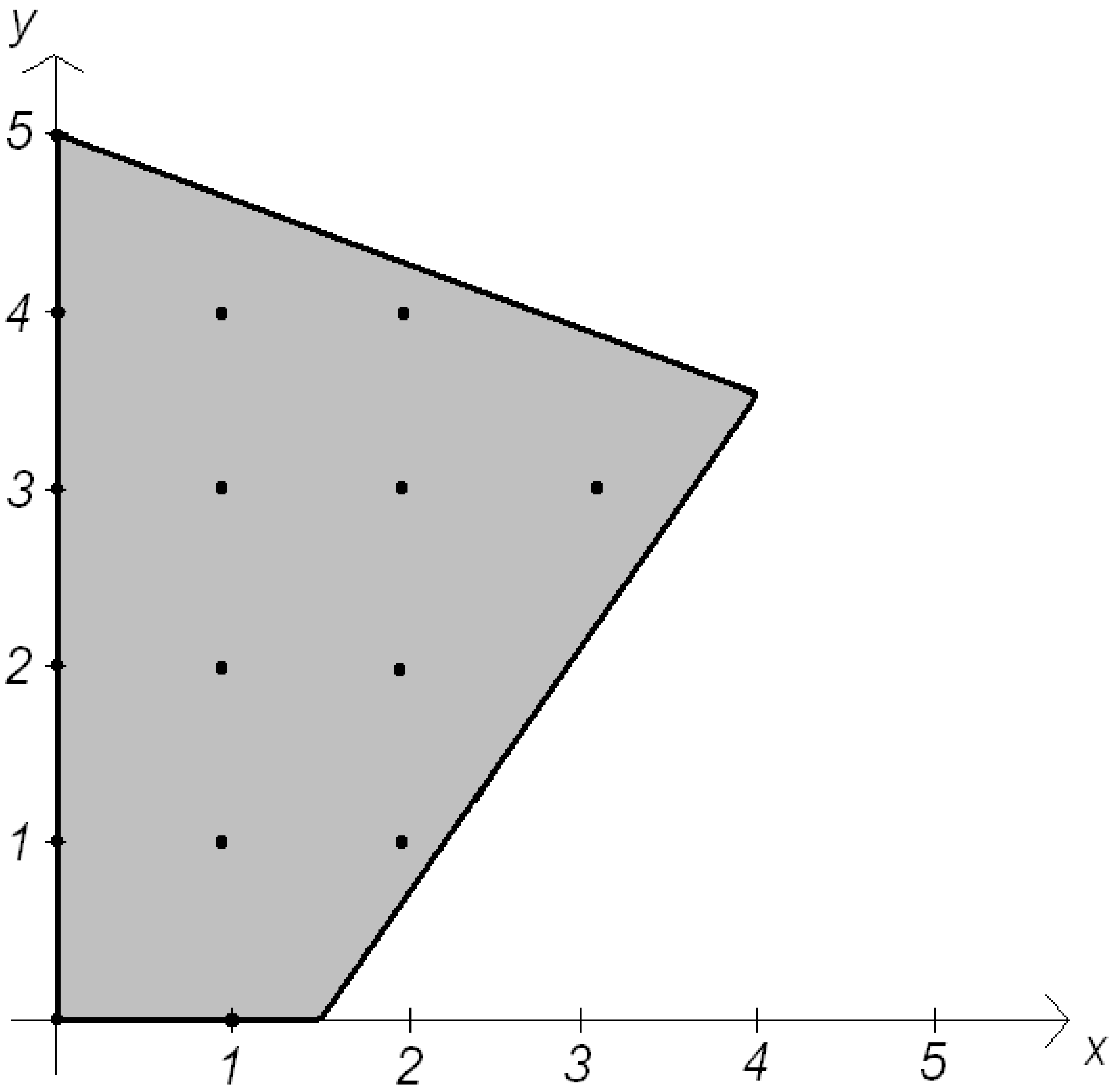
$$\text{Not TU}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

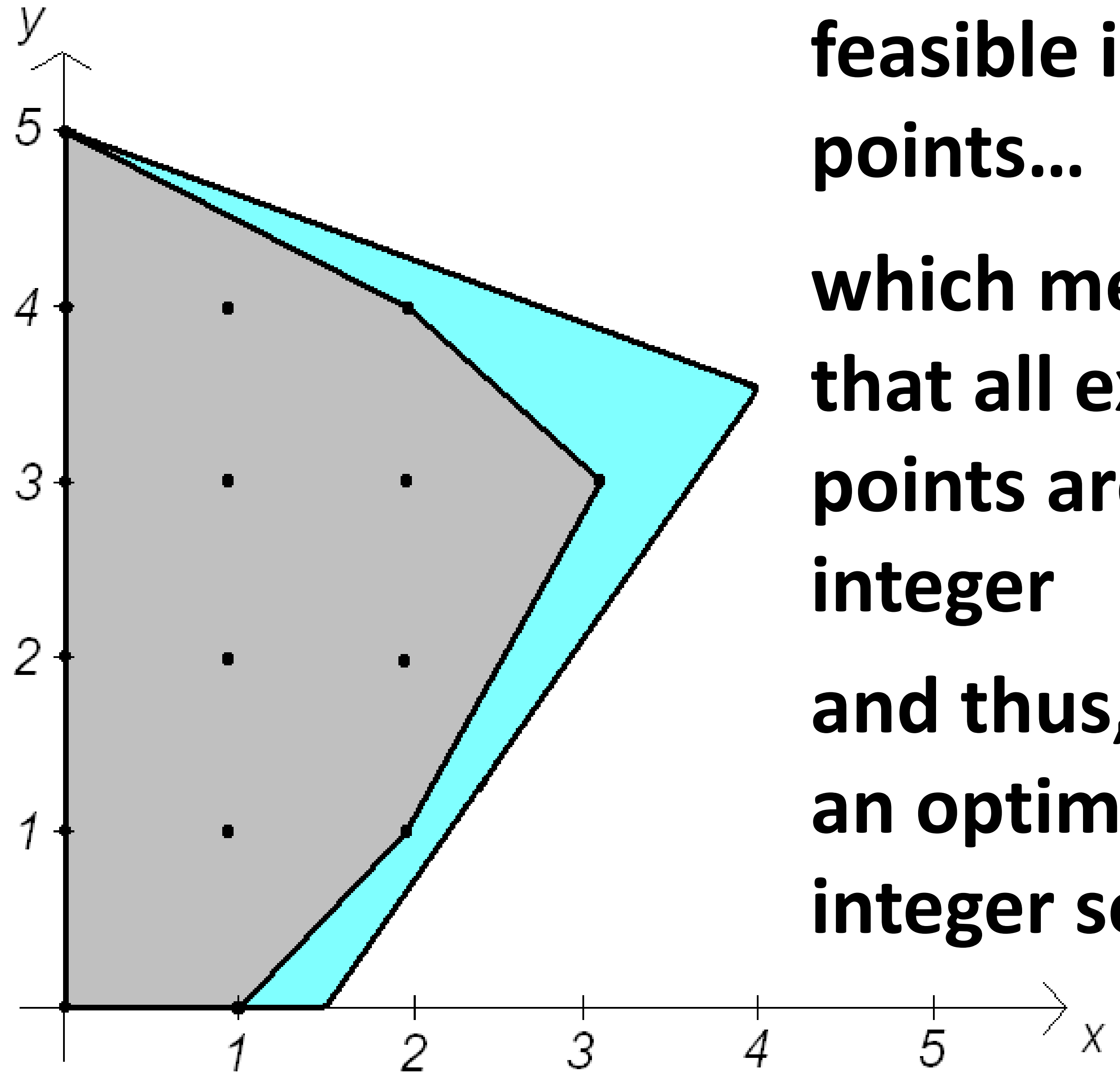
$$\text{TU}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \qquad \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

total unimodularity



total unimodularity



Convex hull of feasible integer points…

which means that all extreme points are integer

and thus, there is an optimal integer solution

total unimodularity theorem

An integer matrix A is TU if:

1. all entries are -1, 0 or 1
2. at most two non-zero entries appear in any column
3. the rows of A can be partitioned into two disjoint sets such that
   - if a column has two entries of the same sign, their rows are in different sets
   - if a column has two entries of different signs, their rows are in the same set

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

Properties (1) and (2) hold by definition of an node arc incidence matrix

Let M1 and M2 be a partition of rows.
In particular, let M1 = {A} and M2 = {}.
As such, property (3) holds.

Digression: what about the Transportation problem constraint matrix?

1	1	1	1	1	1																		
						1	1	1	1	1	1												
												1	1	1	1	1	1						
																		1	1	1	1	1	1
1						1						1						1					
	1						1						1						1				
		1						1						1						1			
			1						1						1						1		
				1						1						1						1	
					1						1						1						1

Properties (1) and (2) hold.

Let M1 = supply constraint rows.

and M2 = demand constraint rows.

As such, property (3) holds.

using Cramer's rule to solve for x_{12}

Is $\left|B_{(1,2)}\right|$ an integer?

Yes!

Does $|A| = \pm 1$?

Yes!

$$x_{12} = \frac{\begin{vmatrix} 5 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -3 & 0 & -1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{vmatrix}} = \frac{\left|B_{(1,2)}\right|}{|A|}$$

All MCNFP have TU constraint matrices. All MCNFP feasible solutions will be integer!

Min-Cost Network Flow Problem (MCNFP) Continued

Network Flow Problems

- There are a variety of “network flow” problems
- The MCNFP is one, common, and useful type
- Many problems (in general) can be represented as network flow problems and the MCNFP in particular
- The “flow” might be actual “things”
  - e.g. people, commodities, water, electricity in a physical system (transportation, pipelines, powergrid, etc.)
- Or it might represent other more abstract concepts
  - e.g., information, financial transactions, or logical decisions

MCNFP Examples

- Transportation problem (again)
- Multi-period planning (again)
- Shortest path problem
- Maximum flow problem

MCNFP formulation

$$\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} \\
\text{subject to} \quad & \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = b_i \quad && \forall i \in N \\
& l_{ij} \le x_{ij} \le u_{ij} && \forall (i,j) \in A
\end{aligned}$$

File: **MCNFP.txt**

```
# AMPL model for the Minimum Cost Network Flow Problem

# SETS AND PARAMETERS-------------------------------

set NODES;                         # nodes in the network
set ARCS within {NODES, NODES};    # arcs in the network

param b {NODES} default 0; # supply/demand for node i
param c {ARCS}  default 0; # cost of one unit flow on (i,j)
param l {ARCS}  default 0; # lower bound on flow on (i,j)
param u {ARCS}  default Infinity; # upper bound on (i,j)
```

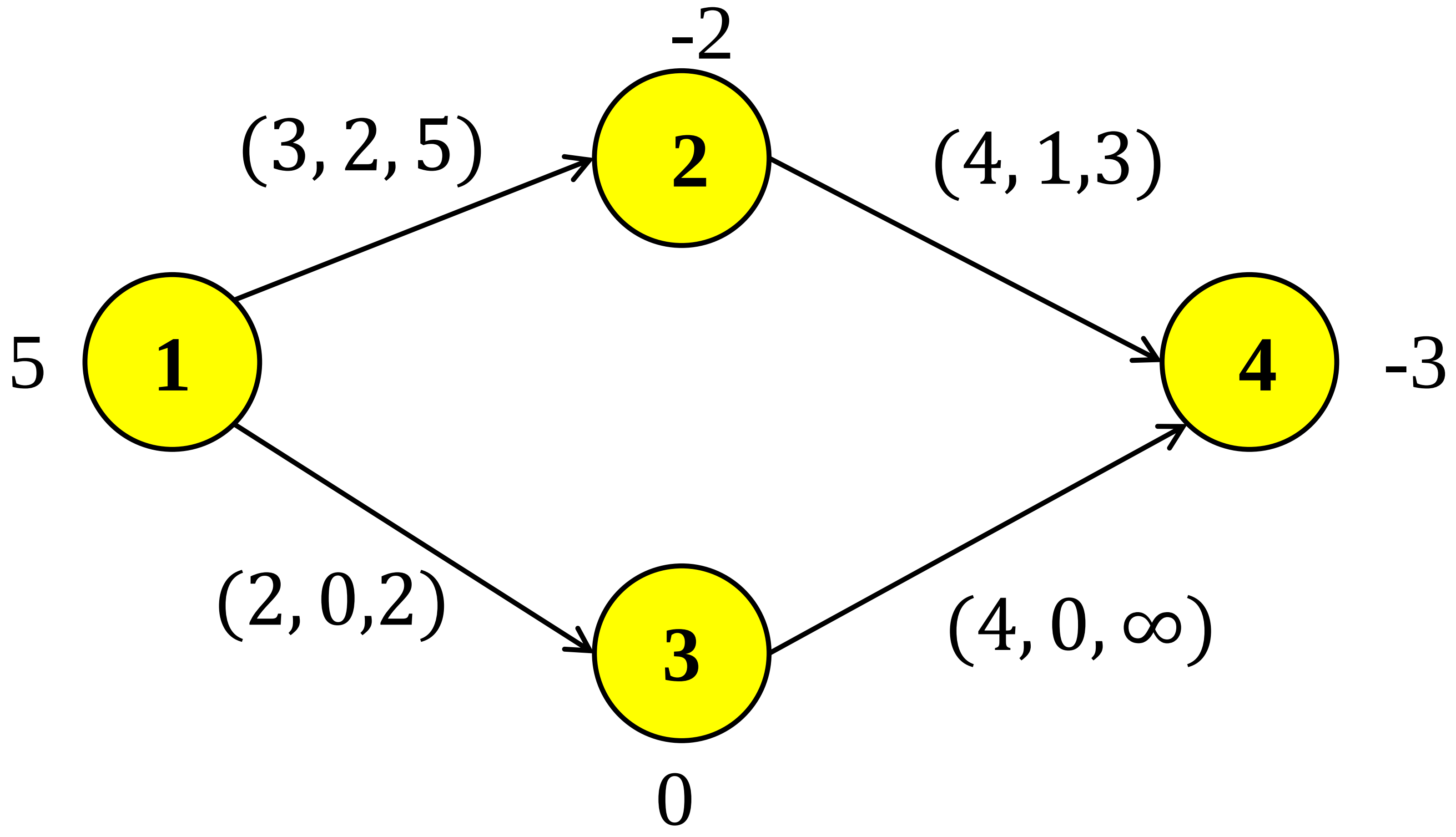
```
#DECISION VARIABLES --------------------
var x {ARCS};          # flow on arc (i,j)

#OBJECTIVE ---------------------------------
#minimize arc flow cost
minimize cost: sum{(i,j) in ARCS} c[i,j] * x[i,j];

#CONSTRAINTS --------------------------------
subject to flow_balance {i in NODES}:
     sum{j in NODES: (i,j) in ARCS} x[i,j]
        - sum{j in NODES: (j,i) in ARCS} x[j,i] = b[i];

subject to capacity {(i,j) in ARCS}:
            l[i,j] <= x[i,j] <= u[i,j];
```

example network



```
#MCNFP Problem - data file for MCNFP.txt
#note: default arc costs and lower bounds are 0
#      default arc upper bounds are infinity
#      default node requirements are 0

set NODES :=  1 2 3 4 ;
set ARCS := (1, 2) (1,3) (2,4) (3,4)  ;

param b :=   1  5
             2 -2
             3  0
             4 -3;

param:            c  l u :=
        [1,2] 3  2 5
        [1,3] 2  . 2
        [2,4] 4  1 3
        [3,4] 4  . .        ;
```

Min-Cost Network Flow Problem Examples: Transportation Problem (again!)

A mobile-home manufacturer channels its units through distribution centers located in Elkhart, Ind., Albany, N.Y., Camden, N.J., and Petersburg, Va.

The distribution centers have in inventory 30, 75, 60, and 35 mobile homes, respectively.

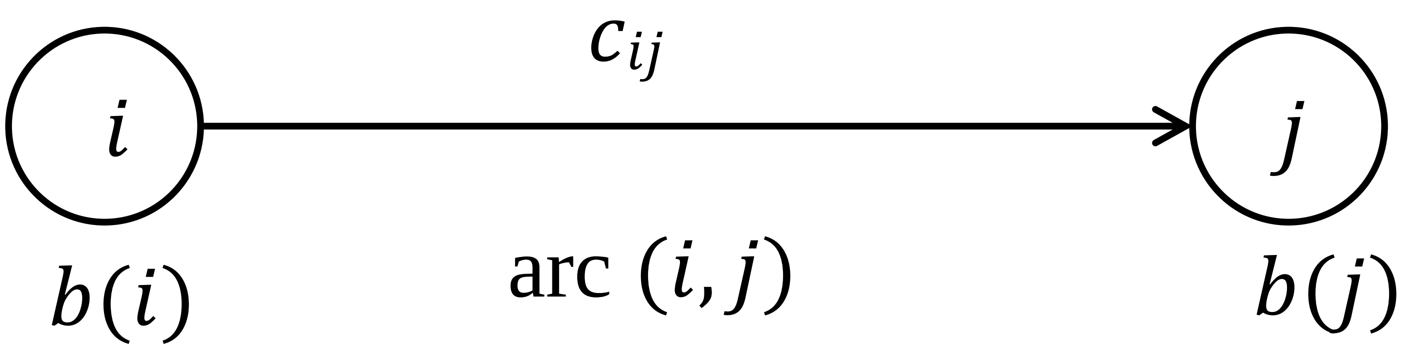
Demand at the dealerships:

Dealer	A	B	C	D	E	F
Units	25	40	15	25	50	45

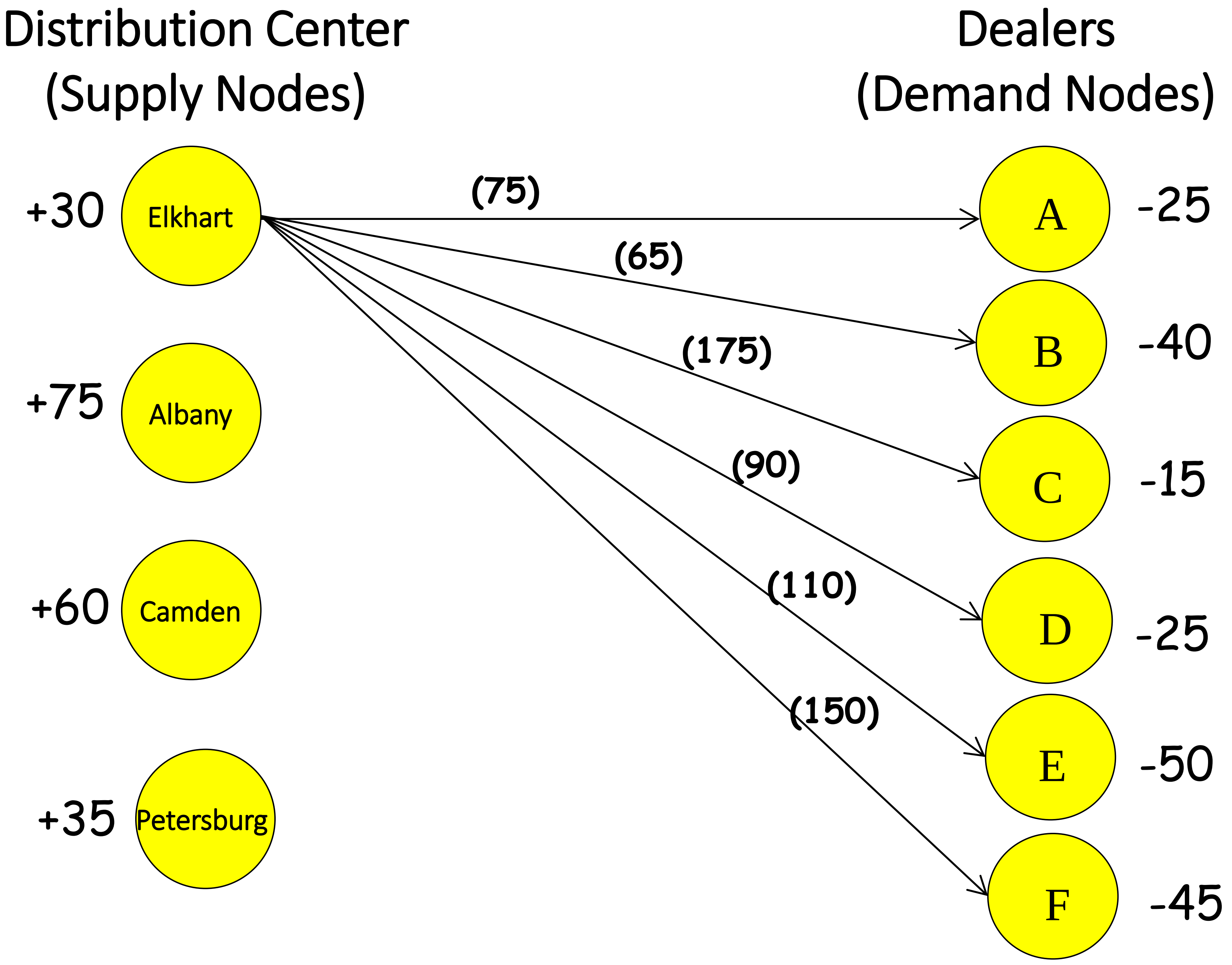
Transportation costs (dollars per unit) between each distribution center and the dealerships are:

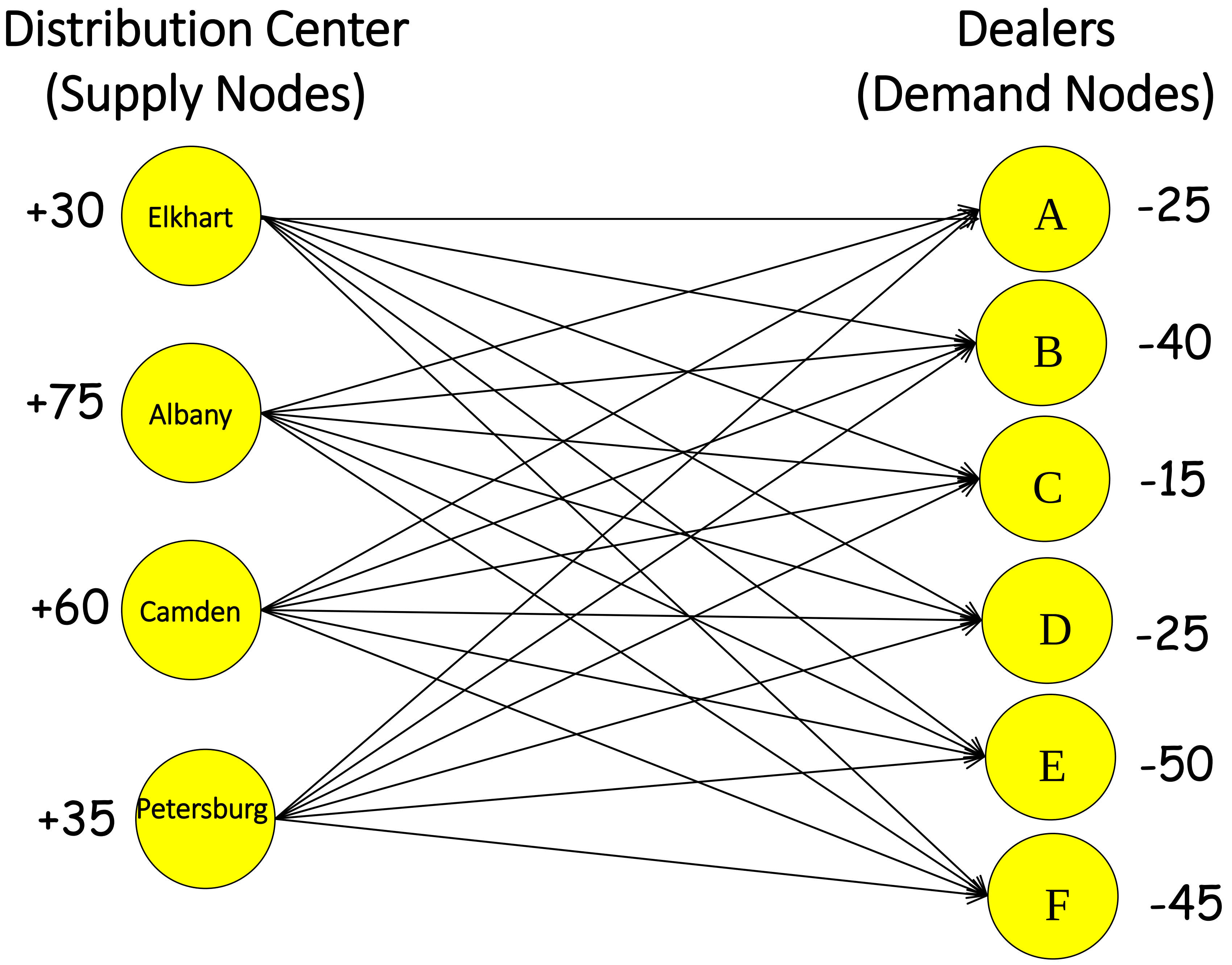
	A	B	C	D	E	F
Elkhart	75	65	175	90	110	150
Albany	90	30	45	50	105	130
Camden	40	55	35	80	70	75
Petersburg	95	150	100	115	55	55

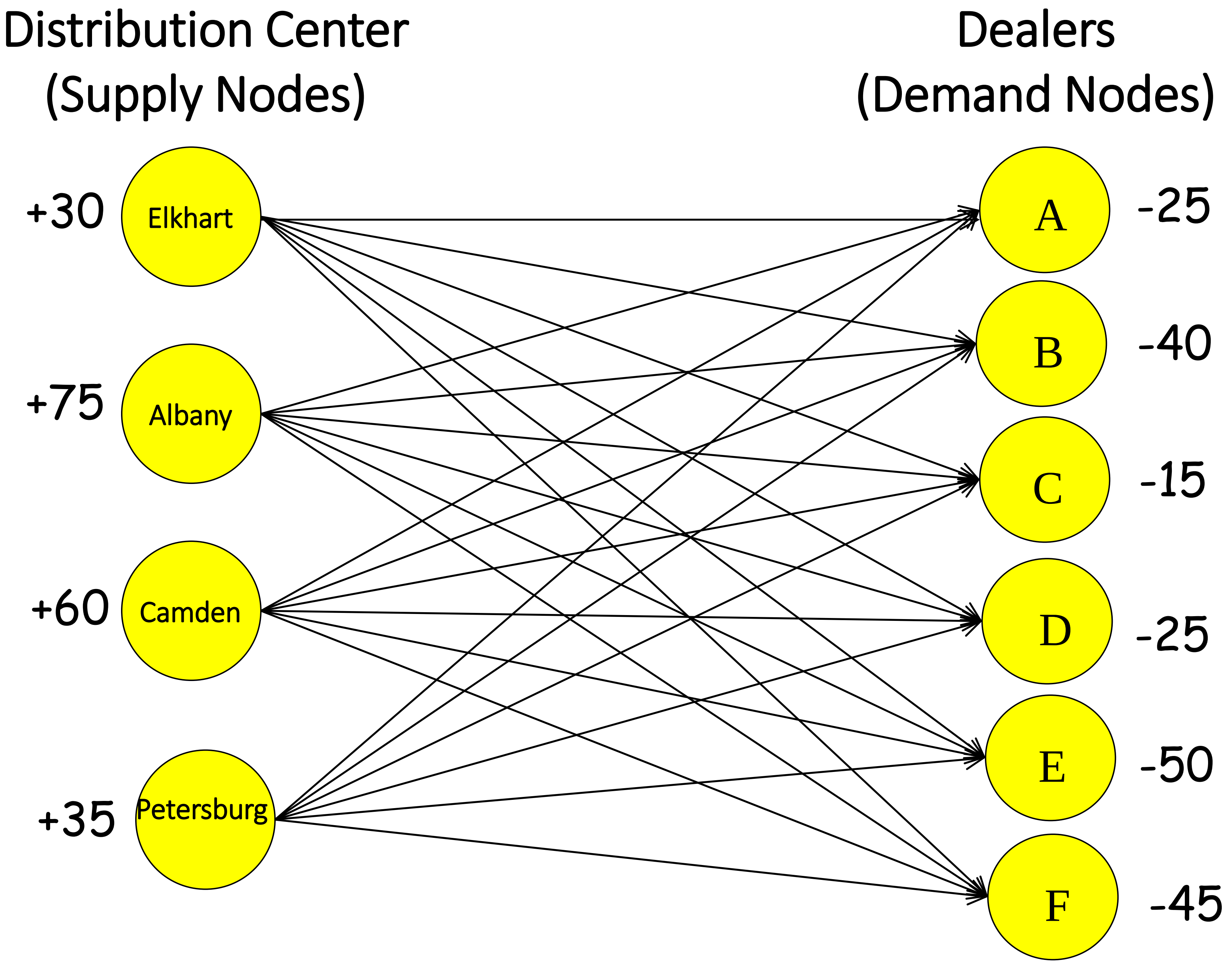
Transportation Problems (MCNFP Formulation)



$$\ell_{ij} = 0$$
$$u_{ij} = \infty$$

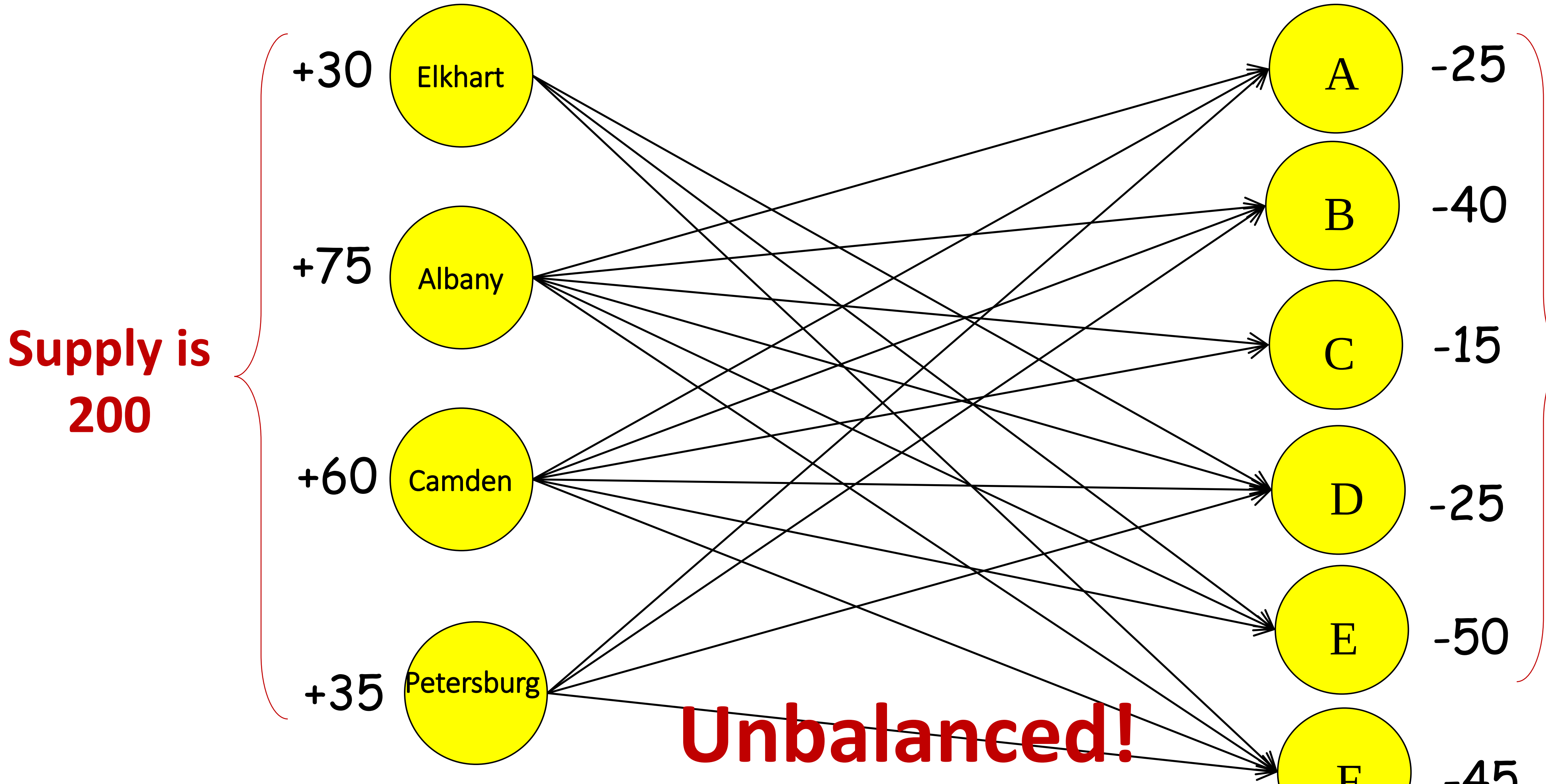

Distribution Center
(Supply Nodes)
Dealers
(Demand Nodes)
+30
Elkhart
+75
Albany
+60
Camden
+35
Petersburg
(75)
(65)
(175)
(90)
(110)
(150)
A
-25
B
-40
C
-15
D
-25
E
-50
F
-45

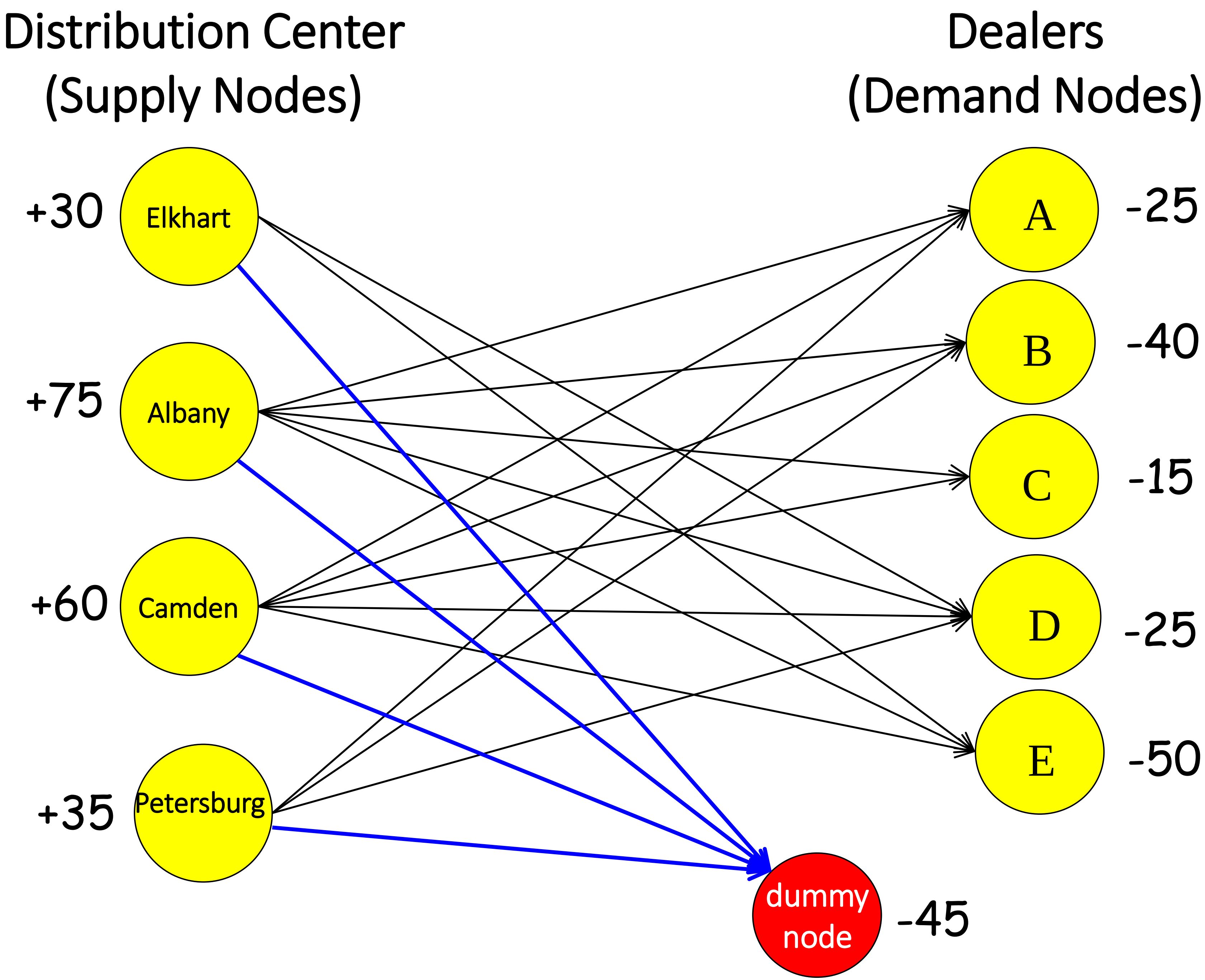

Distribution Center
(Supply Nodes)
Dealers
(Demand Nodes)
+30
Elkhart
+75
Albany
+60
Camden
+35
Petersburg
A
-25
B
-40
C
-15
D
-25
E
-50
F
-45


Distribution Center
(Supply Nodes)
Dealers
(Demand Nodes)
+30
Elkhart
+75
Albany
+60
Camden
+35
Petersburg
A
-25
B
-40
C
-15
D
-25
E
-50
F
-45

Distribution Center
(Supply Nodes)

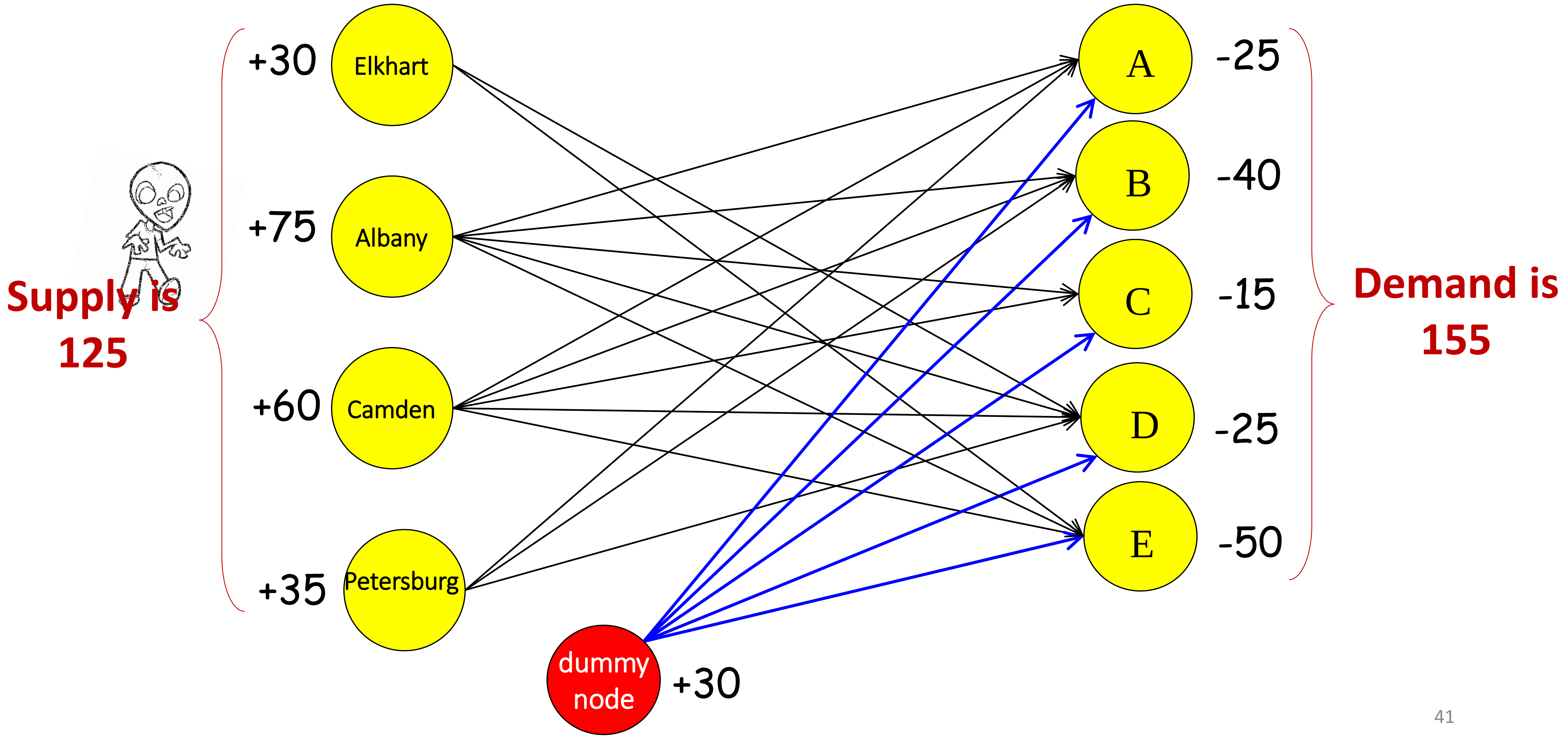
Dealers
(Demand Nodes)




Distribution Center
(Supply Nodes)
Dealers
(Demand Nodes)
+30
Elkhart
+75
Albany
+60
Camden
+35
Petersburg
A
-25
B
-40
C
-15
D
-25
E
-50
dummy
node
-45

Distribution Center (Supply Nodes)

Dealers (Demand Nodes)



Min-Cost Network Flow Problem Examples: Multi-period Planning (again!)

NSC produces steel.
Demand for the next four months is given:

Jan	Feb	Mar	Apr
2400	2200	2700	2500

NSC can meet demand by producing steel, by drawing from its inventory or by a combination. January inventory is 1000 tons.

Production costs vary from month to month as shown:

Jan	Feb	Mar	Apr
7400	7500	7600	7800

Inventory holding costs: $120

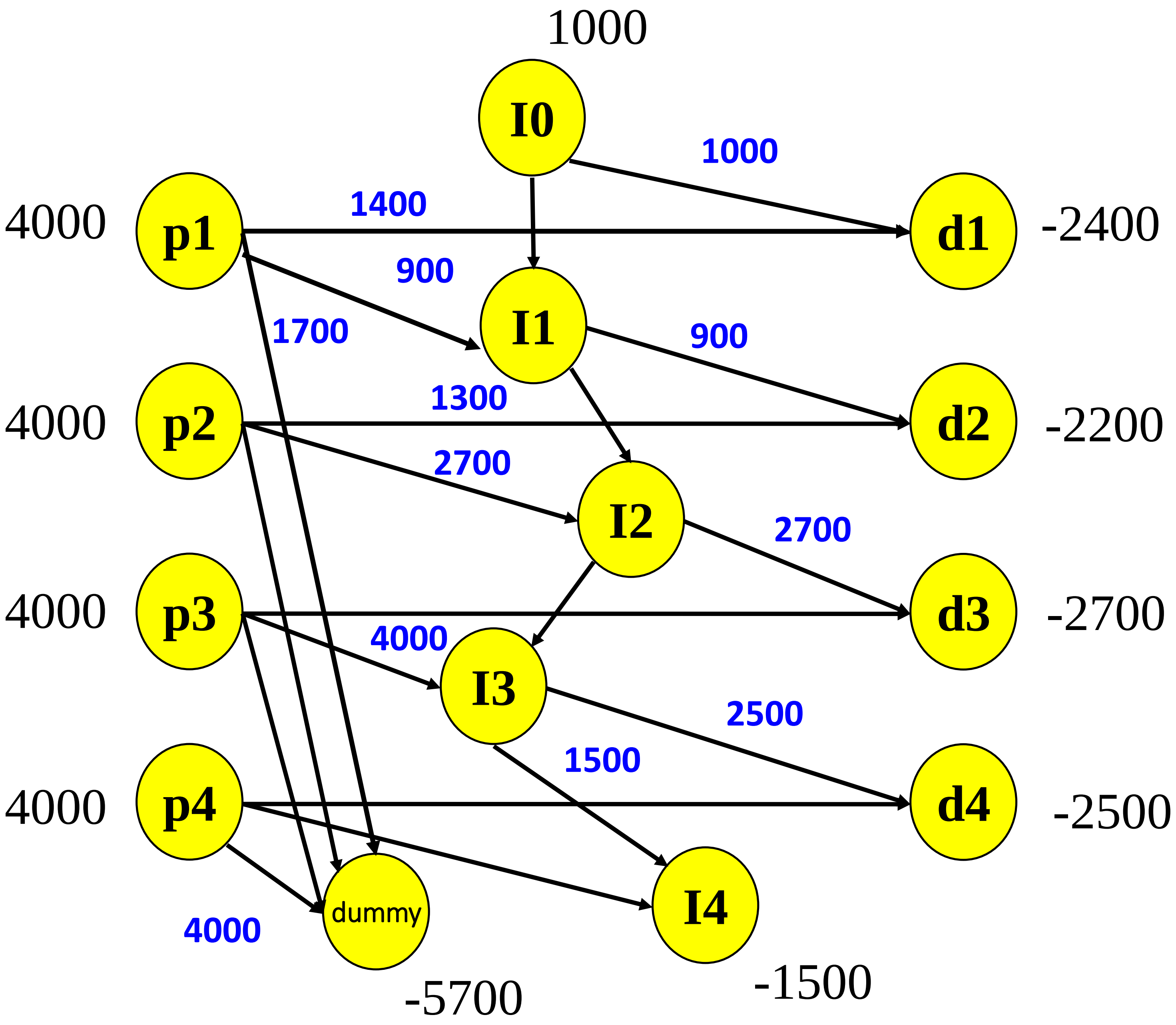
Monthly production capacity is 4000 tons.

Operations requires ending inventory for April: 1500 tons.

Formulate the NSC multi-period production planning problem as a network flow problem.

1. Draw the network.
2. Denote the node requirement values.
3. Clarify the arc flow upper and lower bounds and costs on arcs as needed.

MCNFP formulation of NSC problem



arc parameters

- All arcs have $\ell_{ij} = 0$ and $u_{ij} = \infty$
- Arcs (p_i, dummy) have cost 0.
- Arcs (I_i, d_{i+1}) and (I_i, I_{i+1}) have cost 120.

Production to meet demand

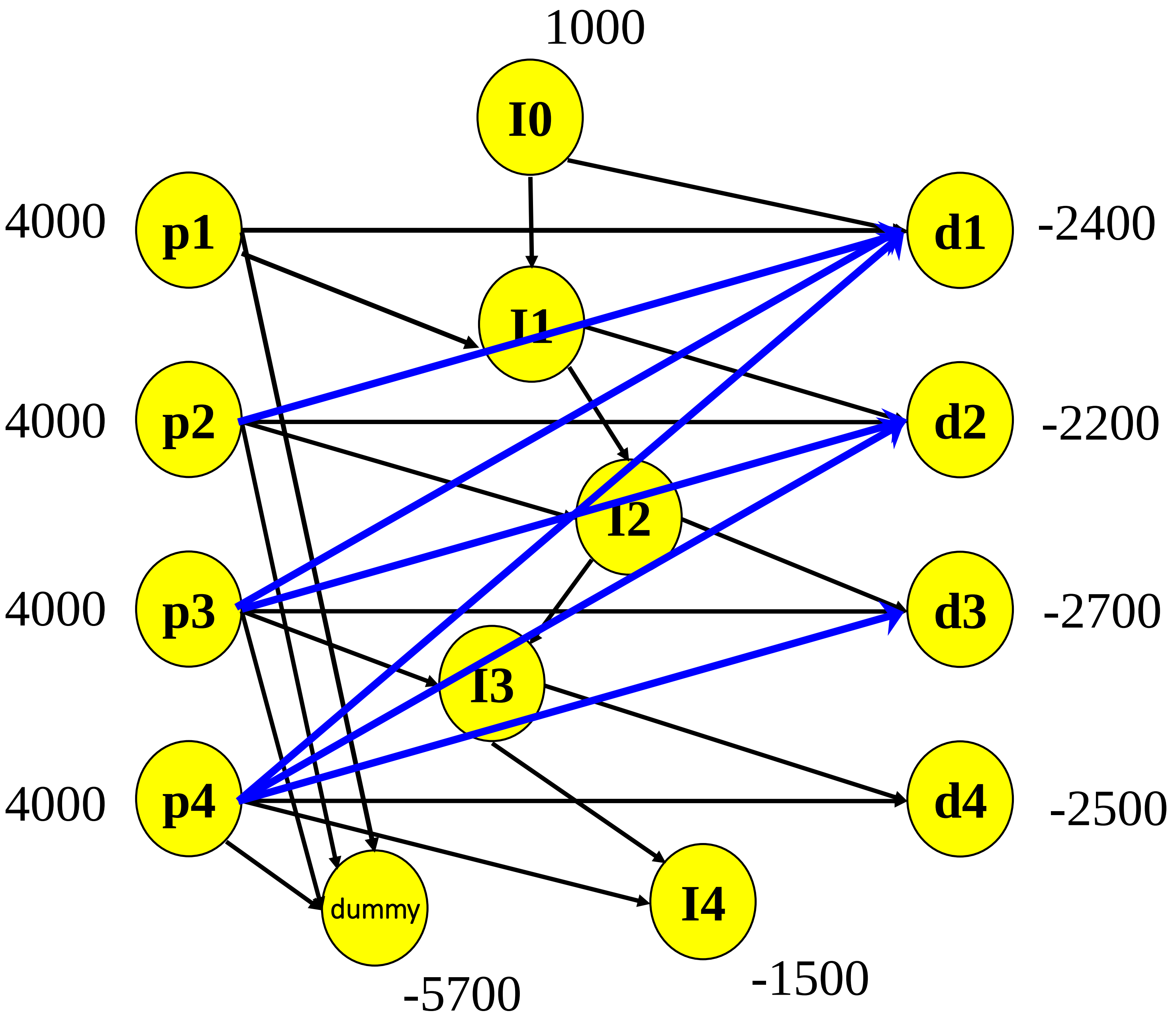
Arc	Cost
(p1,d1)	7400
(p2,d2)	7500
(p3,d3)	7600
(p4,d4)	7500

Production for extra-inventory

Arc	Cost
(p1,i1)	7400
(p2,i2)	7500
(p3,i3)	7600
(p4,i4)	7500

What if you want to allow backorders?

backorders


1000
I0
4000
p1
d1
-2400
I1
4000
p2
d2
-2200
I2
4000
p3
d3
-2700
I3
4000
p4
d4
-2500
dummy
I4
-5700
-1500

additional arc parameters with back orders

- All arcs have $\ell_{ij} = 0$ and $u_{ij} = \infty$

Production to meet demand (backorder of $200/ton/month cost included)

Arc	Cost
(p2,d1)	7700
(p3,d1)	8000
(p3,d2)	7800
(p4,d1)	8400
(p4,d2)	8200
(p4,d3)	8000

Min-Cost Network Flow Problem Examples: Shortest Path Problem

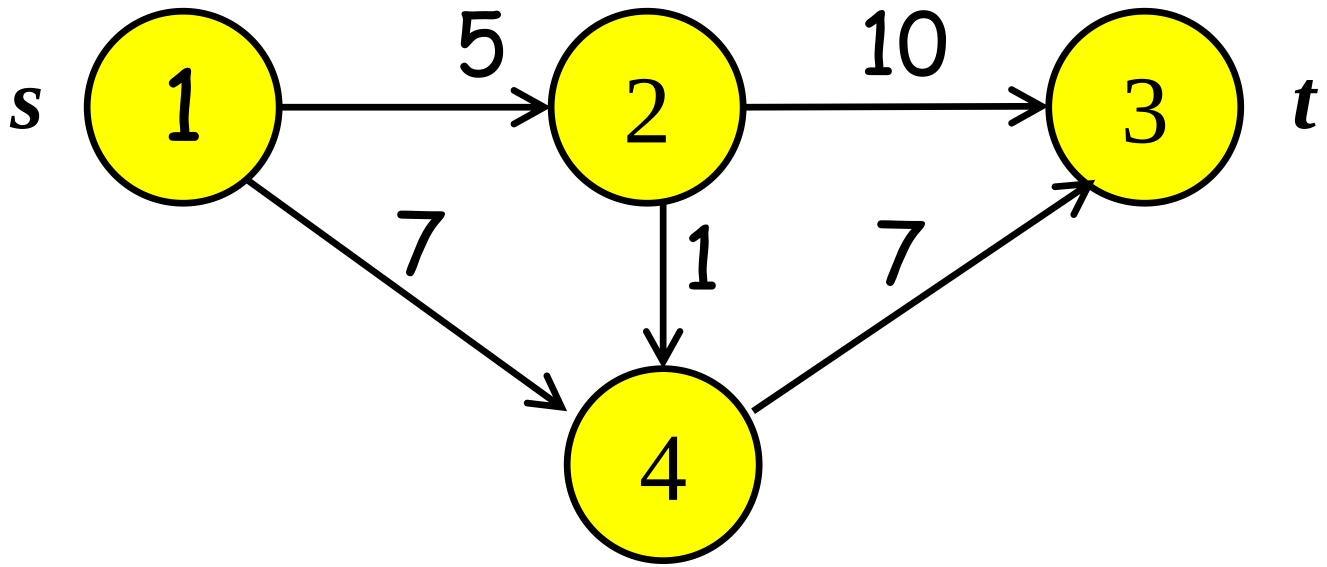
Shortest Path Problem (MCNFP Formulation)

- Defined on a network with two special nodes: s and t
- A *path* from s to t is an alternating sequence of distinct nodes and distinct arcs starting at s and ending at t:
  $s, (s, n_1), n_1, (n_1, n_2), \dots, (n_i, n_j), n_j, (n_j, t), t$
  - There are *directed* and *undirected* paths
  - A networks is *connected* if an undirected path exists between any pairs of nodes
  - If $s = t$, the path is called a *cycle*
  - A connected network without cycles is called a *tree*

Shortest Path Problem (MCNFP Formulation)

- Defined on a network with two special nodes: s and t
- A *path* from s to t is an alternating sequence of distinct nodes and distinct arcs starting at s and ending at t: $s, (s, n_1), n_1, (n_1, n_2), \ldots, (n_i, n_j), n_j, (n_j, t), t$
- Find a minimum-cost directed path from s to t

shortest path example



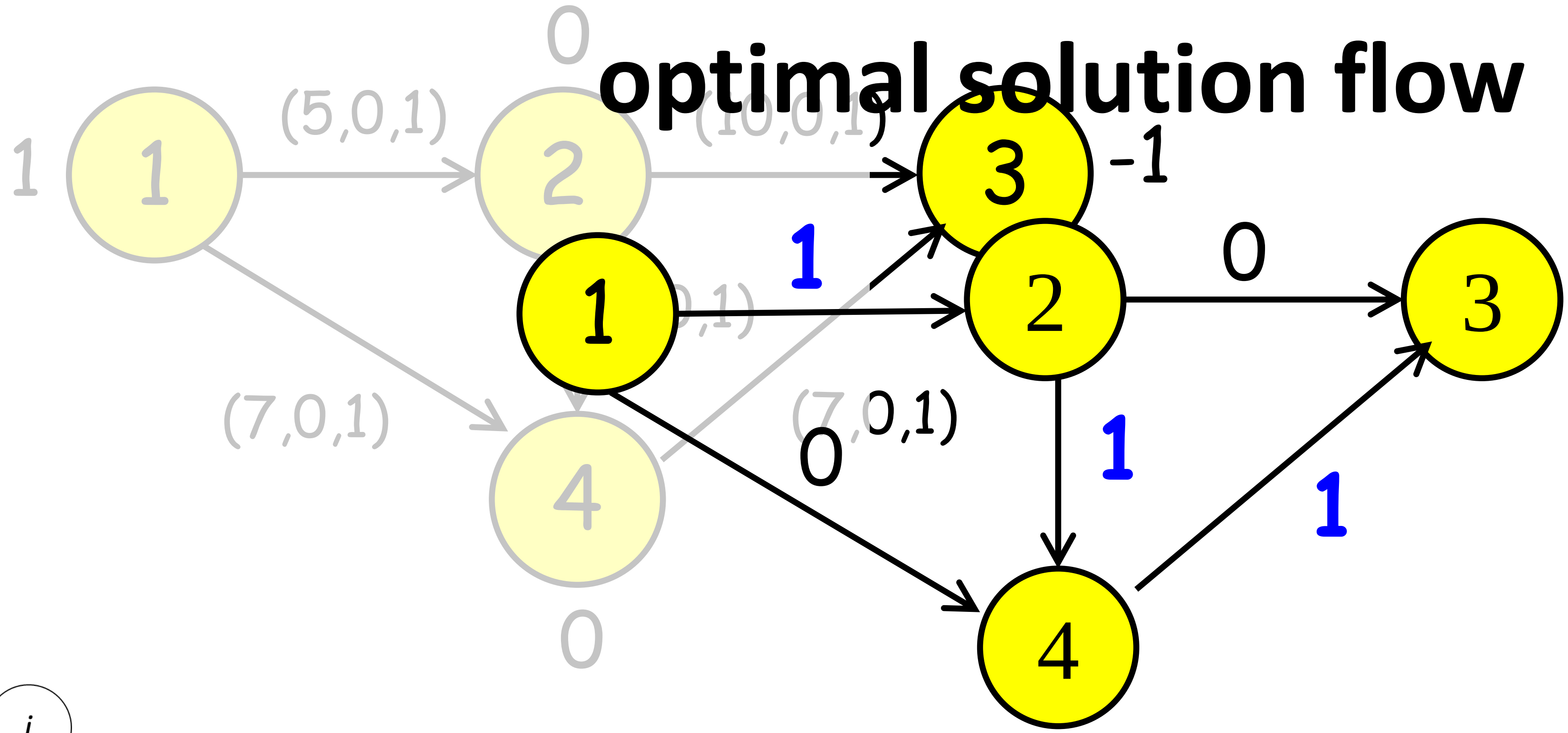
1,(1,2),2,(2,3),3	Length = 15
1,(1,2),2,(2,4),4,(4,3)	*Length = 13*
1,(1,4),4,(4,3),3	Length = 14

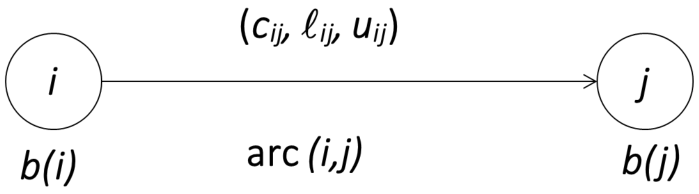
MCNFP formulation of shortest path problem

- *Source* node s has a supply of 1
- *Terminal* node t has a demand of 1
- All other nodes are transshipment nodes
- Each arc has capacity 1
- Tracing the unit of flow from s to t gives a path from s to t

shortest path as MCNFP

optimal solution flow





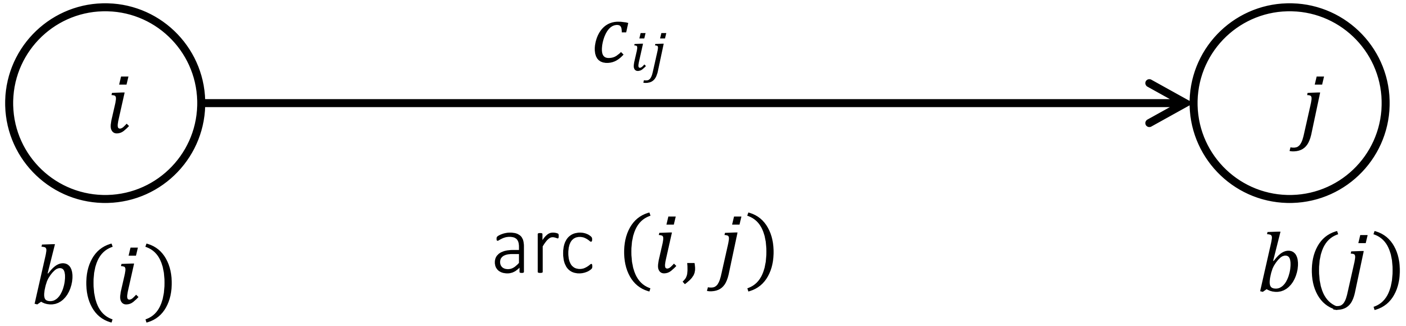
shortest path example

In a rural area of Texas, there are six farms connected by small roads. The distances in miles between the farms are given in the table.

What is the minimum distance to get from Farm 1 to Farm 6?

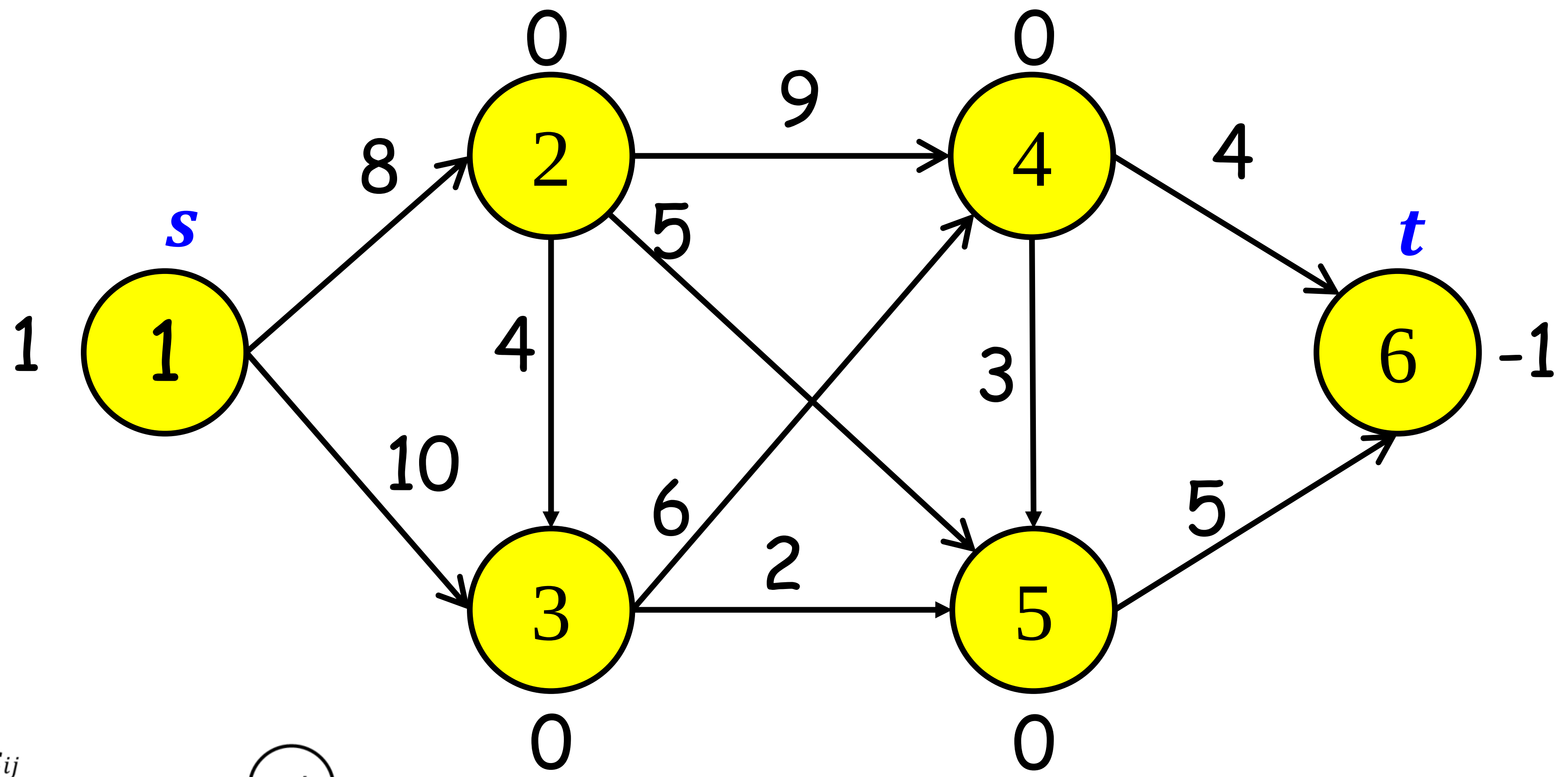
From Farm	To Farm	Distance
1	2	8
1	3	10
2	3	4
2	4	9
2	5	5
3	4	6
3	5	2
4	5	3
4	6	6
5	6	5

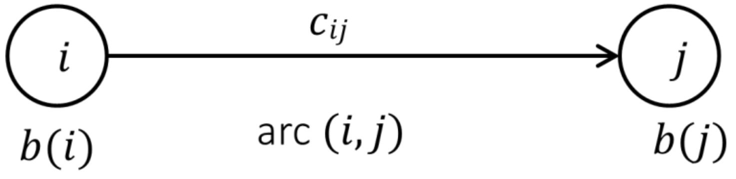
graphical network flow formulation



$$\ell_{ij} = 0, u_{ij} = 1$$

formulation as shortest path





$\ell_{ij} = 0, u_{ij} = 1$

matrix representation

node requirements:
+1 for node s;
-1 for node t

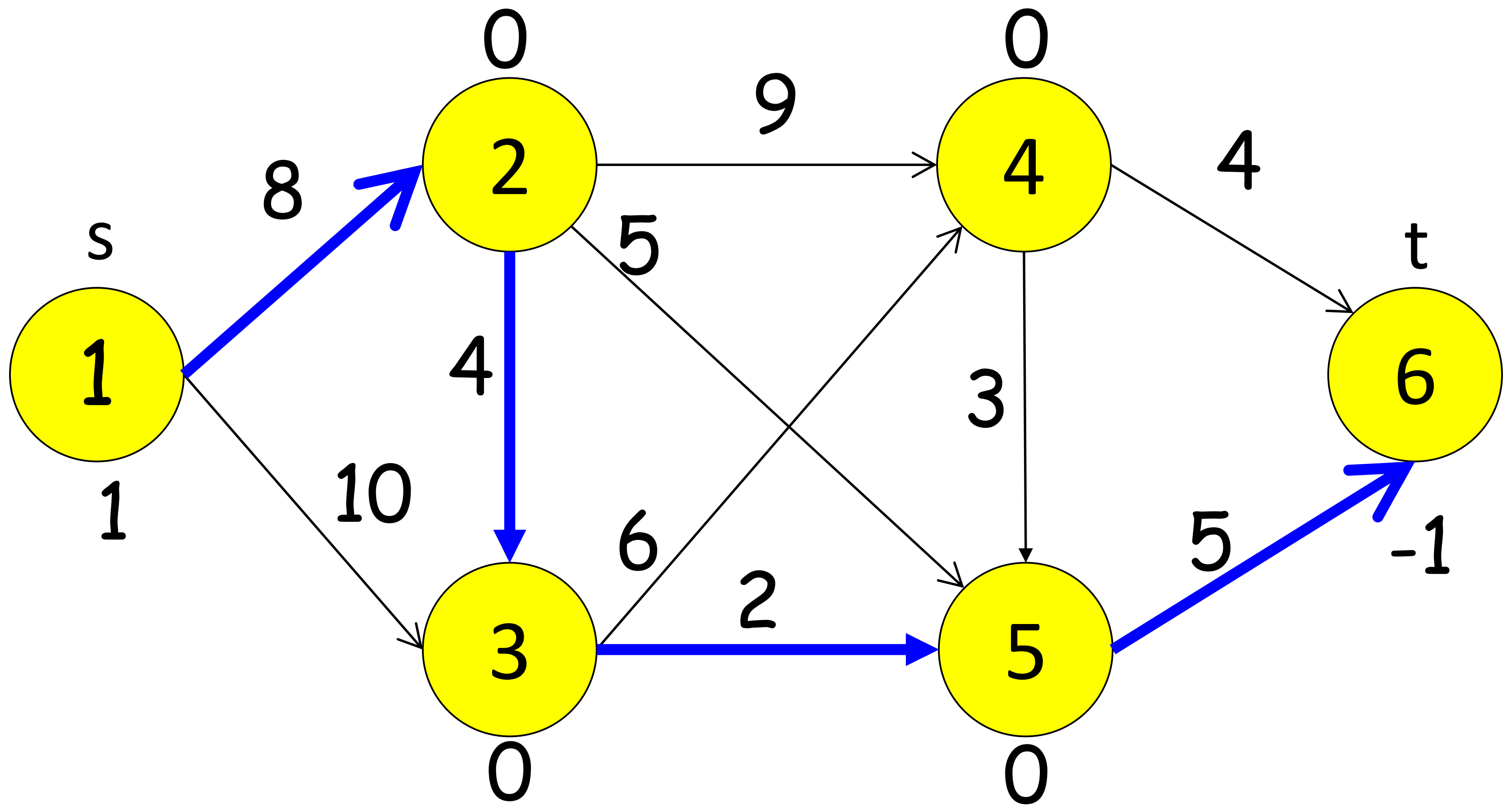
one variable per arc: denotes the "flow" on an edge

$$\begin{array}{c} \begin{array}{cccccccccc} x_{12} & x_{13} & x_{23} & x_{24} & x_{25} & x_{34} & x_{35} & x_{45} & x_{46} & x_{56} \end{array} \\ A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \end{array}$$

<u>node-arc incidence matrix</u> stipulates the "flow-balance constraints"

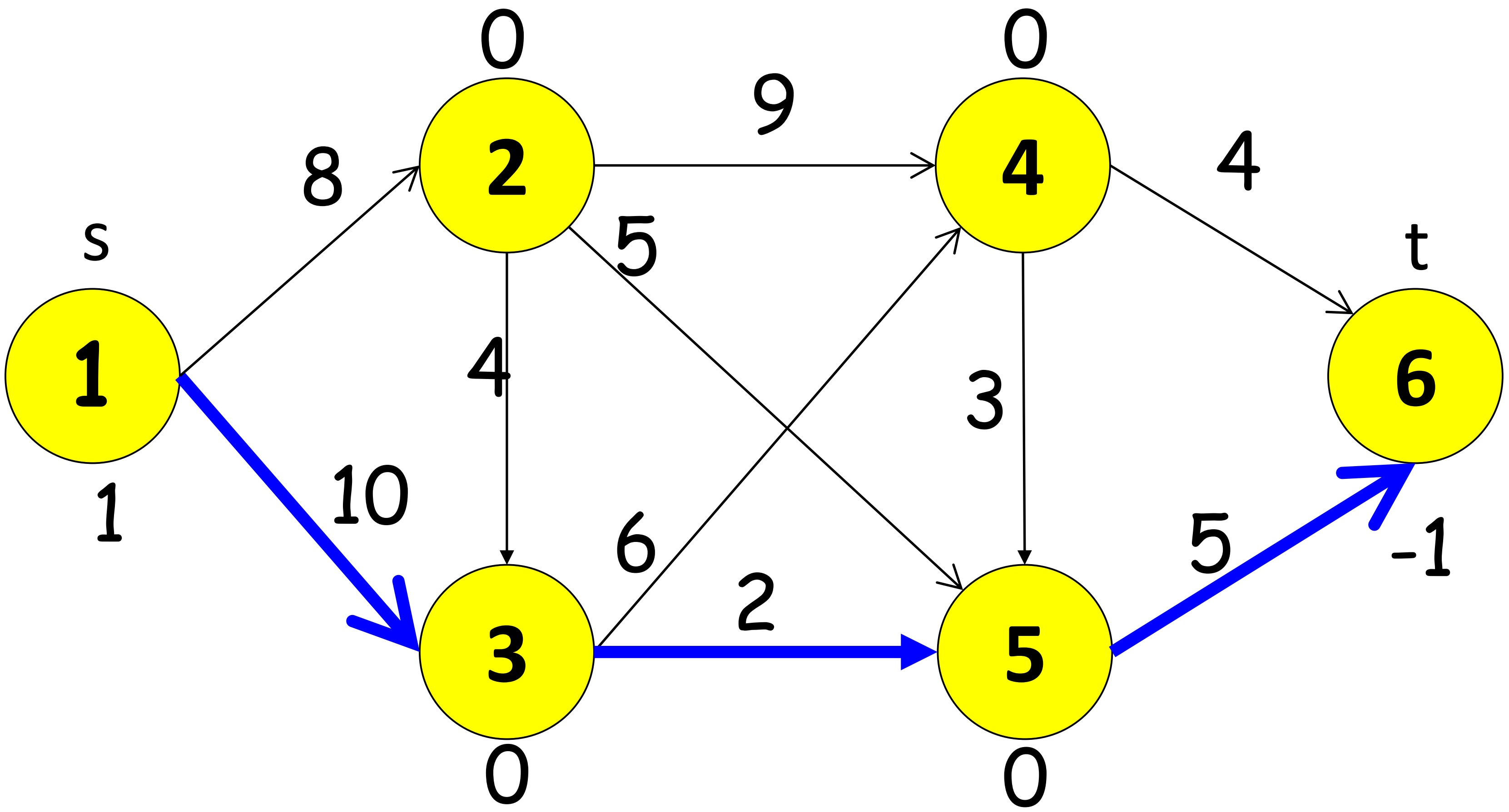
- every column relates to an arc
- every row relates to a node

"greedy" solution



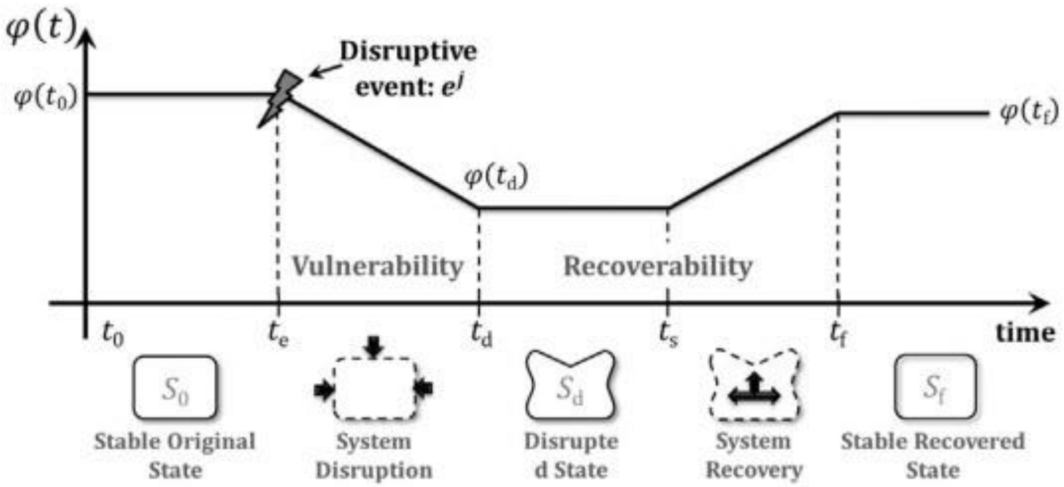
$x_{13} = x_{23} = x_{35} = x_{35} = 1$, $x_{ij} = 0$ for all other arcs.
Objective function value = 19.

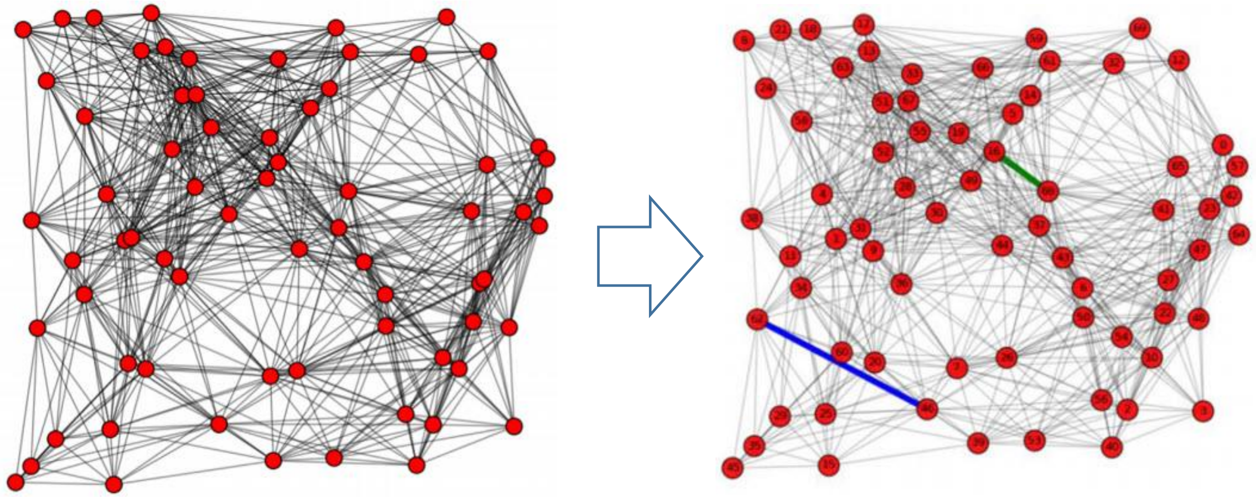
shortest path: optimal solution

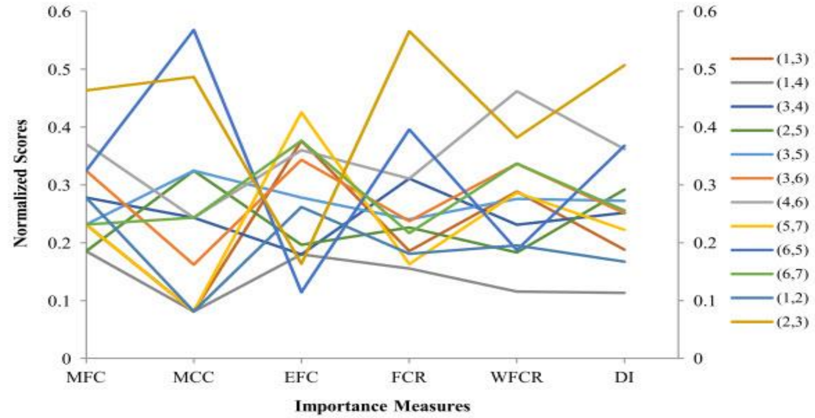


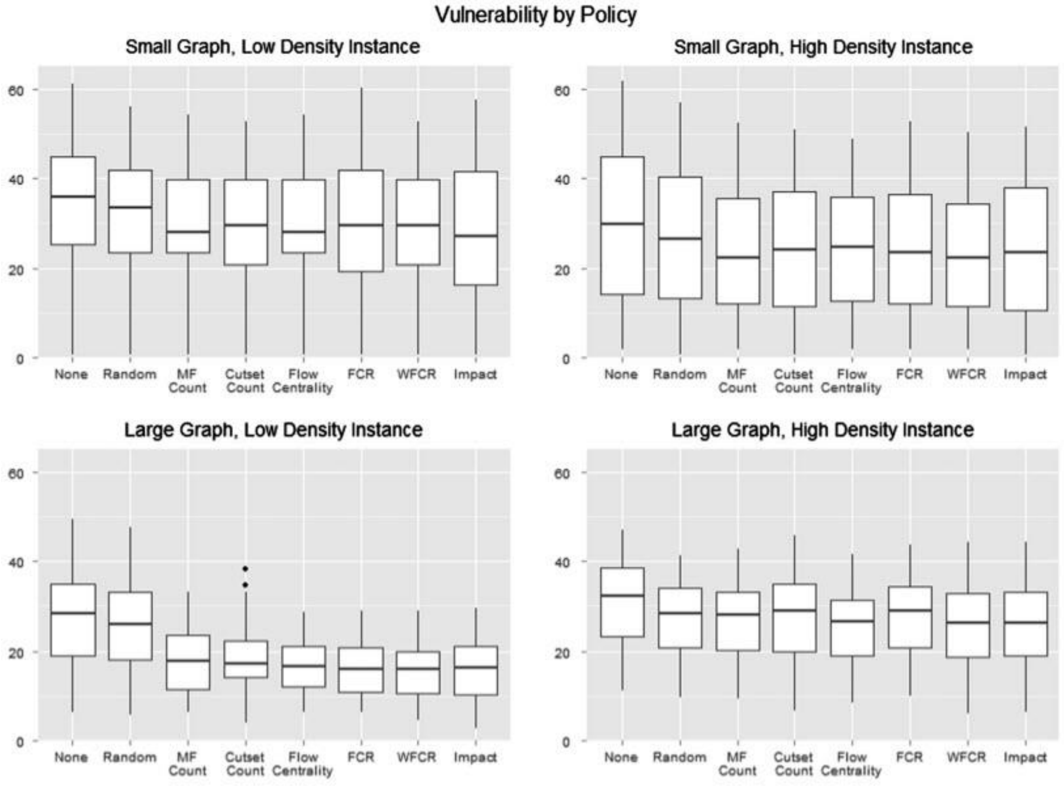
$x_{13} = x_{35} = x_{56} = 1$, $x_{ij} = 0$ for all other arcs.
Objective function value = 17.

Min-Cost Network Flow Problem Examples:
Maximum Flow Problem









Policy	SGLD		SGHD		LGLD		LGHD	
	Mean	Max	Mean	Max	Mean	Max	Mean	Max
None	34.6%	61.2%	30.0%	61.7%	27.6%	49.5%	31.1%	46.9%
Random	32.4%	55.9%	27.2%	56.8%	26.0%	47.6%	27.7%	**41.4%**
MF count	29.6%	54.3%	24.3%	52.4%	17.7%	33.2%	27.5%	42.7%
Cutset count	**28.9%**	**52.7%**	24.3%	51.1%	18.1%	38.1%	27.9%	45.8%
Flow centrality	29.6%	54.3%	24.8%	**48.7%**	16.8%	**28.6%**	**26.0%**	**41.4%**
FCR	31.3%	60.2%	24.8%	52.7%	15.9%	28.8%	28.4%	43.8%
WFCR	**28.9%**	**52.7%**	**23.6%**	50.4%	**15.6%**	28.9%	**26.0%**	44.4%
Impact	29.6%	57.6%	24.2%	51.4%	**15.6%**	29.6%	26.2%	44.4%

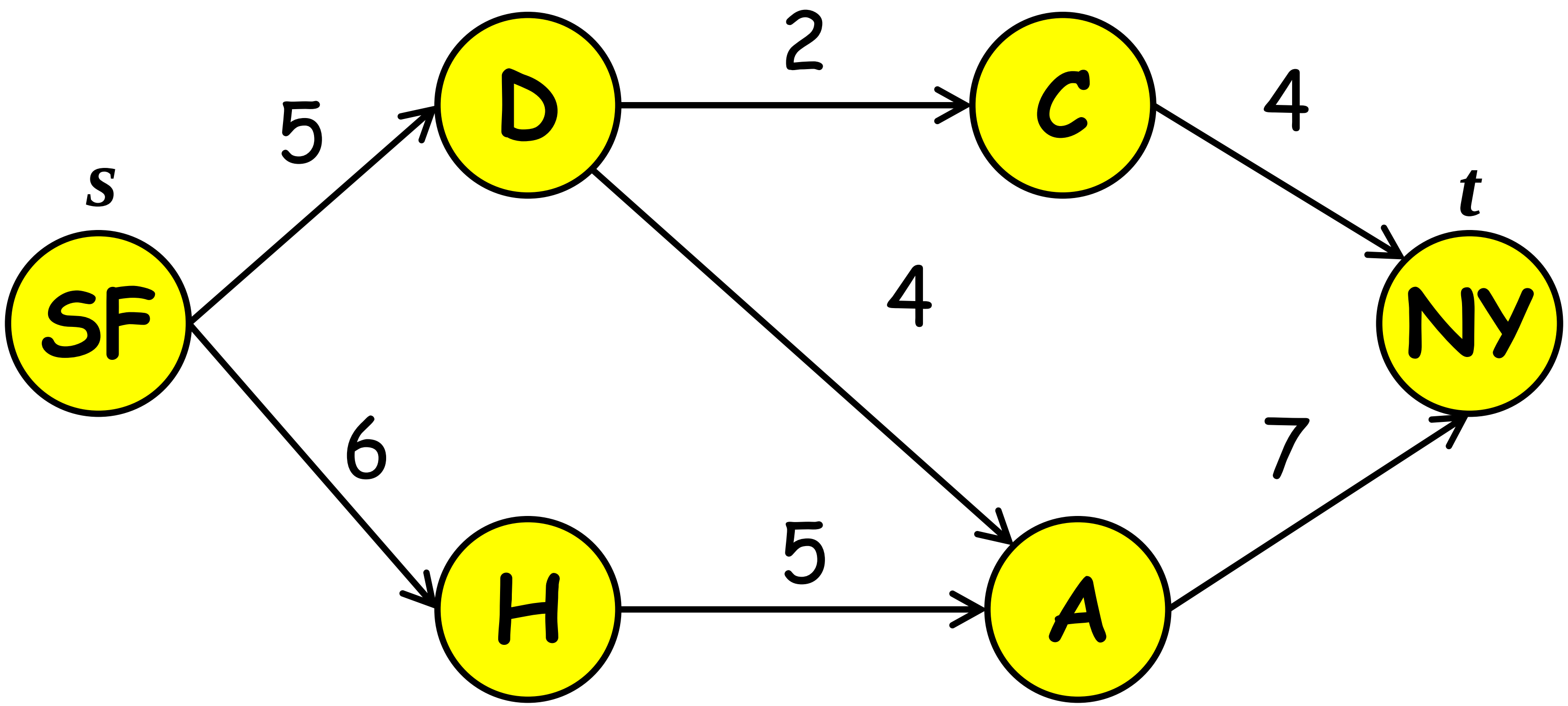
rerouting airline passengers example

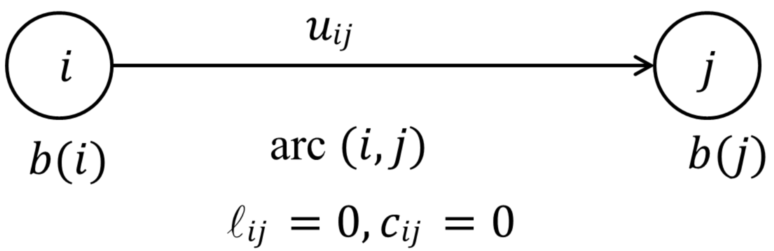
Due to a mechanical problem, Fly-By-Night Airlines had to cancel flight 162 - its only non-stop flight from San Francisco to New York.

Formulate a maximum flow problem to reroute as many passengers as possible from San Francisco to New York.

Flight	From	To	# of seats
160	San Francisco	Denver	5
115	San Francisco	Houston	6
153	Denver	Atlanta	4
102	Denver	Chicago	2
170	Houston	Atlanta	5
150	Atlanta	New York	7
180	Chicago	New York	4

network representation





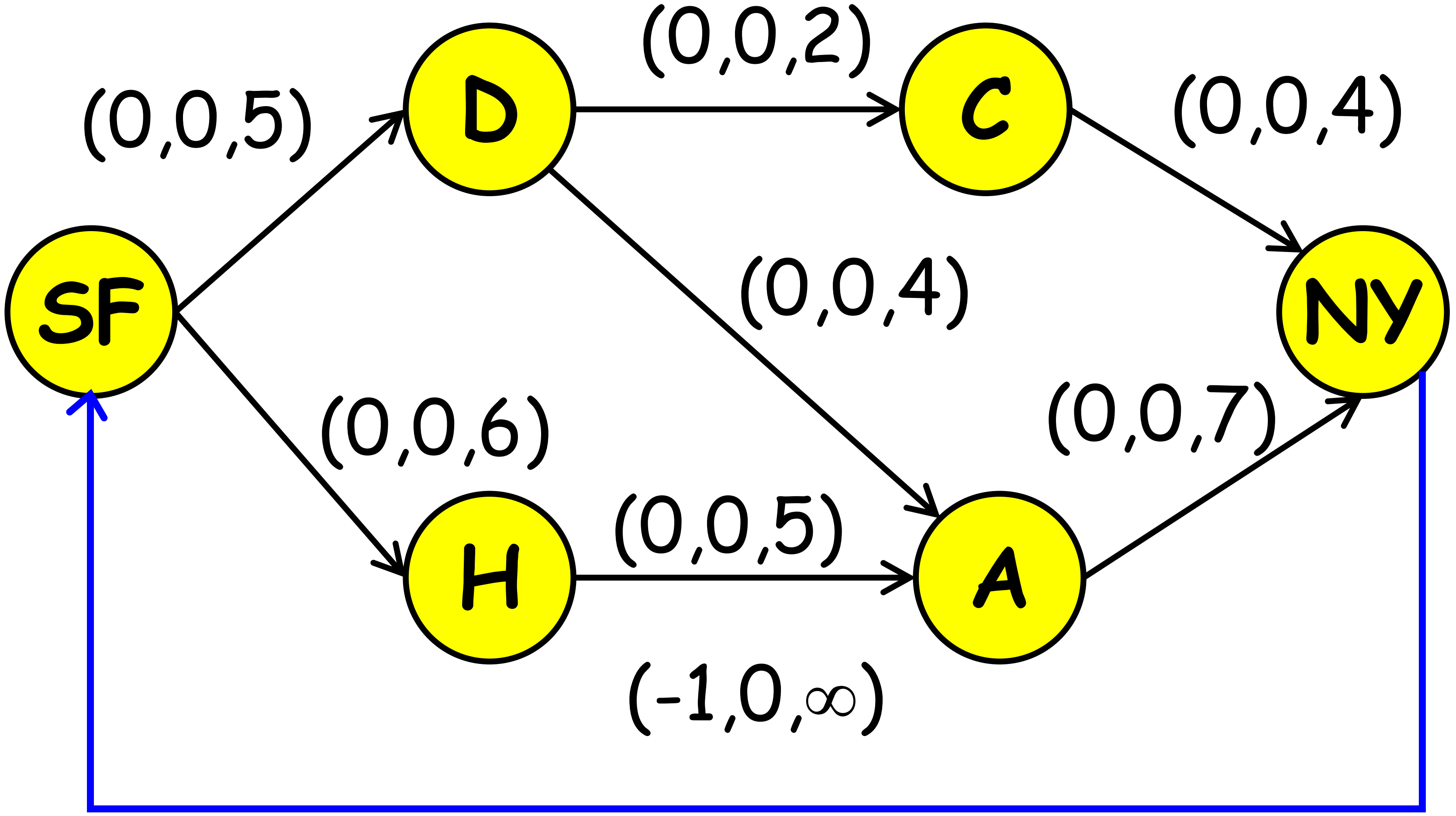
MCNFP formulation of maximum flow problem

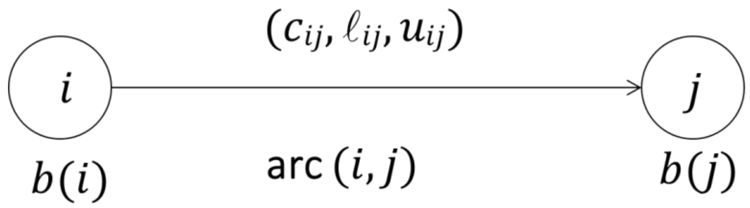
1. Let arc cost = 0 for all arcs
2. Add an arc from t to s

   Give this arc a cost of –1 and infinite capacity
3. All nodes are transshipment nodes

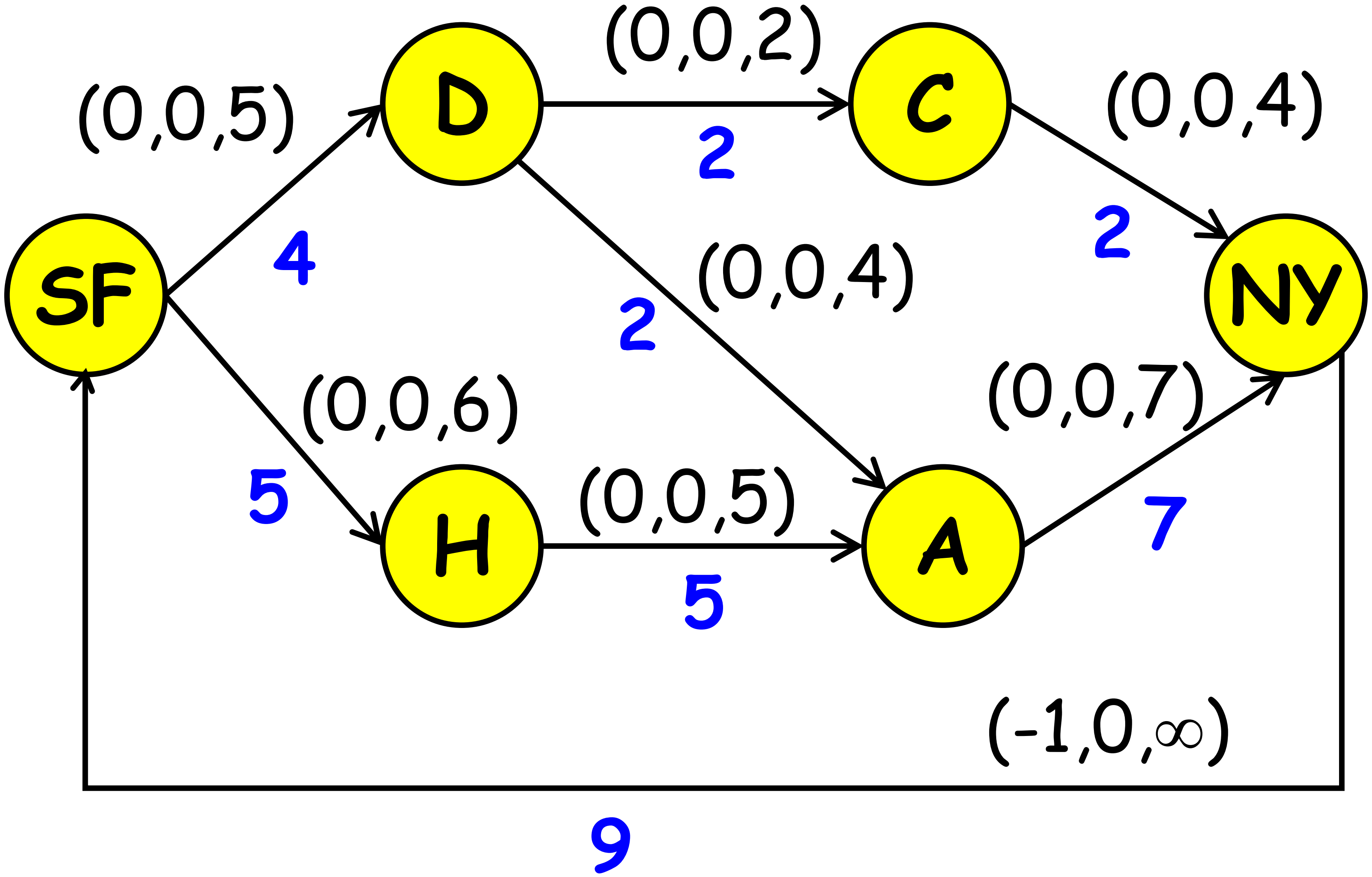
“Circulation Problem” formulation

formulation of maximum flow problem as MCNFP





optimal solution



Widgets International Inc.

Widgets International Inc. (***WII***) manufactures widgets in various locations around the country.

The cost of manufacturing (excluding raw material costs), for each widget along with the plant's minimum and maximum monthly production levels is given in the table.

Plant	Cost per widget	Production max	Production min
1	\$5	500	0
2	\$7	750	100
3	\$3	1000	500
4	\$4	250	250

Each widget requires 20 lbs of raw material.

There are two suppliers of the necessary raw material.

- The suppliers can provide as much raw material as ***WII*** wants.
- Long standing contracts stipulate that ***WII*** will purchase 14,000 lbs of raw material from each (at minimum).
- The suppliers charge different amounts per pound of raw material:

Supplier #1	$0.11/lbs
Supplier #2	$0.09/lbs

Shipping costs from suppliers to plants ($ per lbs):

	Plant 1	Plant 2	Plant 3	Plant 4
Supplier #1	$0.01	$0.02	$0.035	$0.04
Supplier #2	$0.04	$0.03	$0.02	$0.02

The widgets are sold in 4 cities: Norman; Ft. Collins; Austin; and San Jose. Transportation costs ($ per widget) between plants and cities are:

Plant	Norman	Ft. Collins	Austin	San Jose
1	$1	$1	$2	$0
2	$3	$6	$7	$3
3	$3	$1	$5	$3
4	$8	$2	$1	$4

WII knows it's market really well.

In particular, the selling price per widget, the maximum demand, and minimum demand at each city is known.

City	Selling price	Max Demand	Min Demand
Norman	$19	2,000	500
Ft. Collins	$15	400	100
Austin	$22	1,500	500
San Jose	$24	1,550	400

Note: The max demand implies that the max number of widgets that will be bought in the city (if delivered); the min demand is the min amount of demand that must be sent to the city (or the population will riot).

Can you formulate this as a MCNFP problem? If so, how? If not, why?