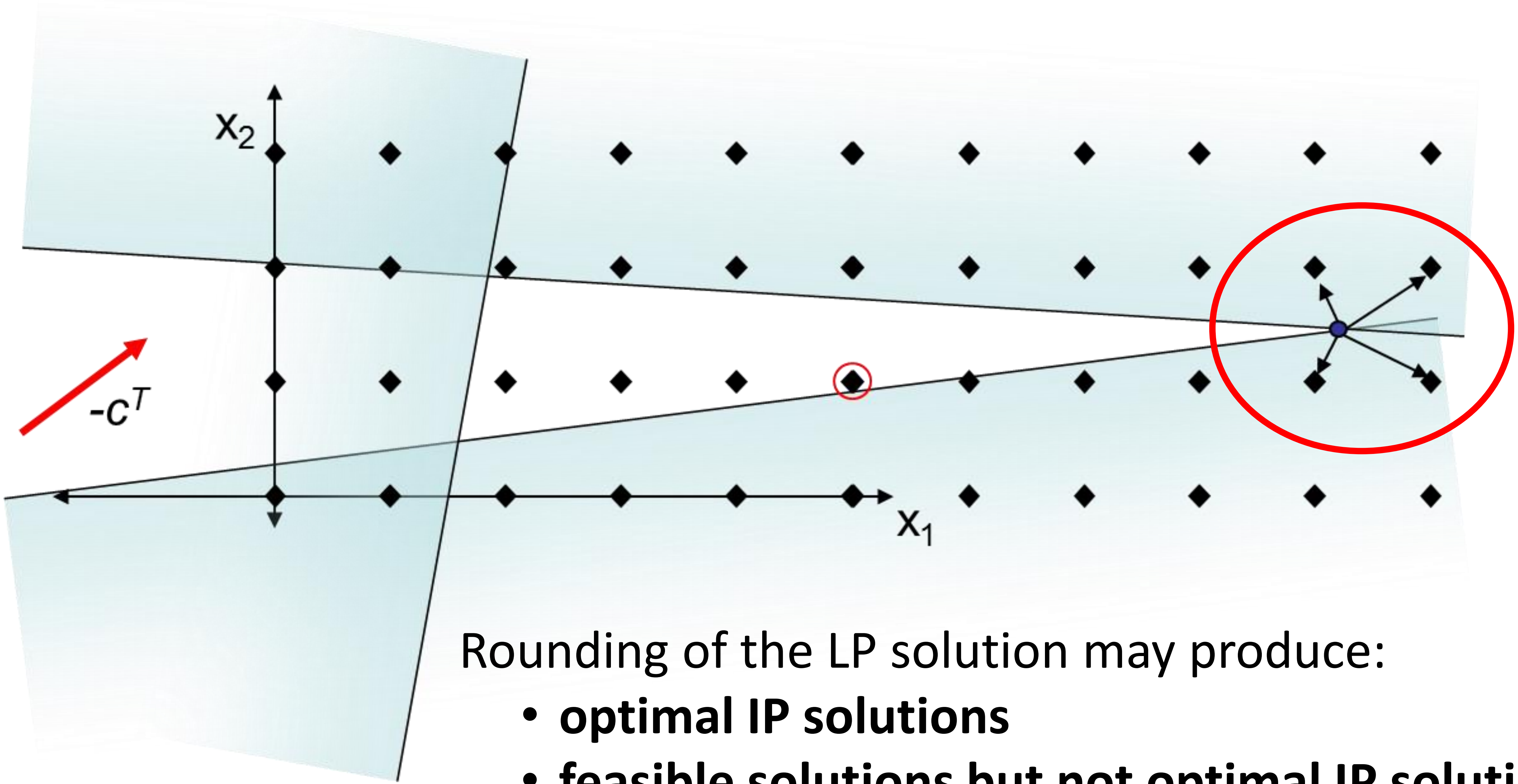
Integer programming: Modeling

Integer Programming

- Introduction and applications
- IP modeling formulations
- Fundamental algorithm for solving IP problems


Company
CAR POOL
PARKING



Rounding of the LP solution may produce:

- **optimal IP solutions**
- **feasible solutions but not optimal IP solutions**
- **infeasible IP solutions**

$$
\begin{aligned}
\text{maximize } & \mathbf{c}^T\mathbf{x} \\
A\mathbf{x} & \geq \mathbf{b} \\
x_j & \in \mathbb{Z}^+ \quad \forall j \in \mathcal{I} \\
x_j & \in \{0,1\} \quad \forall j \in \mathcal{B} \\
x_j & \in \mathbb{R}^+ \quad \forall j \in \mathcal{C}
\end{aligned}
$$

$$\text{maximize } \mathbf{c}^T\mathbf{x}$$

$$A\mathbf{x} \geq \mathbf{b}$$

[illegible]

$$x_j \in \mathbb{Z}^+ \quad \forall j \in \mathcal{I}$$

$$x_j \in \{0, 1\} \quad \forall j \in \mathcal{B}$$

$$x_j \in \mathbb{R}^+ \quad \forall j \in \mathcal{C}$$

applications of integer programming

Capital Budgeting

selection of a number of potential investments

Facility location

decide on facility location (e.g., fire station, emergency response center) to best cover an area

Warehouse Location

in distribution systems, decisions must be made about tradeoffs between transportation costs and fixed costs for opening distribution centers

Sequencing

many problems in sequencing and scheduling require the modeling of the order in which items appear in the sequence

Integer programming: Modeling

What can we model with IP that we cannot do with LP?

To model a variety of…

- Yes/No decisions
- Restrictions on number of options
- Contingent decisions
- Disjunctive constraints (Either/Or)
- Restricted Set or Range of Decision Values
- Fixed costs
- Piecewise linear costs

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Yes/No decisions

Suppose we are to determine whether or not to engage in the following:

1. to build a new plant,
2. to undertake an advertising campaign, or
3. to develop a new product

$$x_j = \begin{cases} 1, & \text{if the } j^{\text{th}} \text{ project is selected} \\ 0, & \text{otherwise} \end{cases}$$

Management says *at most one* project can be selected

$$\sum_{j=1}^{3} x_j = 1$$

restrictions on number of options

more generally, given a set T of n options

select at least k of n options $$\sum_{j \in T} x_j \geq k$$

select at most k of n options $$\sum_{j \in T} x_j \leq k$$

select exactly k of n options $$\sum_{j \in T} x_j = k$$

contingent decisions (x_1, x_2 binary)

Either x_1 or x_2	$x_1 + x_2 \geq 1$
If x_1, then x_2	$x_1 \leq x_2$
If x_1, then not x_2	$x_1 + x_2 \leq 1$

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Satisfy *at least one* of the following constraints:

$$x + y \leq 4$$
$$3x \; + \; 4y \leq 15$$

E.g.:

x = 1, y = 3 (Satisfies both)

x = 0, y = 4 (Satisfies 1, but not 2)

x = 5, y = 0 (Satisfies 2, but not 1)

x = 2, y = 3 (Satisfies neither)

Introduce a new binary variable $z \in \{0,1\}$ and re-write the constraints:

$$x + y \leq 4 \qquad\qquad x + y \leq 4 + Mz$$
$$3x + 4y \leq 15 \qquad\qquad 3x + 4y \leq 15 + M(1 - z)$$
$$z \in \{0,1\}$$

where M is a large, positive constant

$$x + y \leq 4 + Mz$$
$$3x + 4y \leq 15 + M(1 - z)$$
$$z \in \{0,1\}$$

Let $M = 1000$

To allow the solution: x = 5, y = 0 → set z = 1

$$5 + 0 < 4 + 1000 \quad ✓$$
$$15 + 0 = 15 + 1000\,(1 - 1) \quad ✓$$

$$x + y \leq 4 + Mz$$
$$3x + 4y \leq 15 + M(1 - z)$$
$$z \in \{0,1\}$$

Let $M = 1000$

To allow the solution: x = 0, y = 4; set z = 0:

$$0 + 4 = 4 + 1000(0)$$
$$0 + 16 \leq 15 + 1000\,(1 - 0)$$

Solutions with M = 1000

Solution		Conditions		OK?	Binary Variable	Formulation		Feasible?
x	y	$x + y$	$3x + 4y$		z	$4 + Mz$	$15 + M(1-z)$	
1	3	4	15	Yes	0	4	1015	Yes
1	3	4	15	Yes	1	1004	15	Yes
0	4	4	16	Yes	0	4	1015	Yes
0	4	4	16	Yes	1	1004	15	No
5	0	5	15	Yes	0	4	1015	No
5	0	5	15	Yes	1	1004	15	Yes
2	3	5	18	No	0	4	1015	No
2	3	5	18	No	1	1004	15	No

more generally, for any two constraints…

$$\mathbf{a}_1^T \mathbf{x} \leq b_1 + Mz$$

$$\mathbf{a}_2^T \mathbf{x} \leq b_2 + M(1 - z)$$

$$z \in \{0, 1\}$$

if $z = 1$, then here the second constraint is activated

to activate k out of m possible constraints…

$$\mathbf{a}_i^T \mathbf{x} \leq b_i + M(1 - z_i) \qquad \text{for } i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} z_i \geq k$$

$$z_i \in \{0, 1\} \qquad \text{for } i = 1, 2, \ldots, m$$

if $z_i = 1$, then the i^{th} constraint is activated

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restricted set or range of decision values

Decision variable can only take on certain discrete values: $x \in \{k_1, k_2, \ldots, k_m\}$

Create new constraints:

$$x = \sum_{i=1}^{m} y_i k_i$$

$$\sum_{i=1}^{m} y_i = 1$$

$$y_i \in \{0, 1\} \quad \text{for } i = 1, 2, \ldots, m$$

restricted set or range of decision values

Decision variable range is restricted as such: $x = 0 \quad \text{or} \quad x \geq k$

use binary variable $y = \begin{cases} 0, & \text{for} \quad x = 0 \\ 1, & \text{for} \quad x \geq k \end{cases}$

and the constraints:

$$x \leq My \qquad \text{(where } M \text{ is an upper bound on } x\text{)}$$

$$x \geq ky$$

$$y \in \{0, 1\}$$

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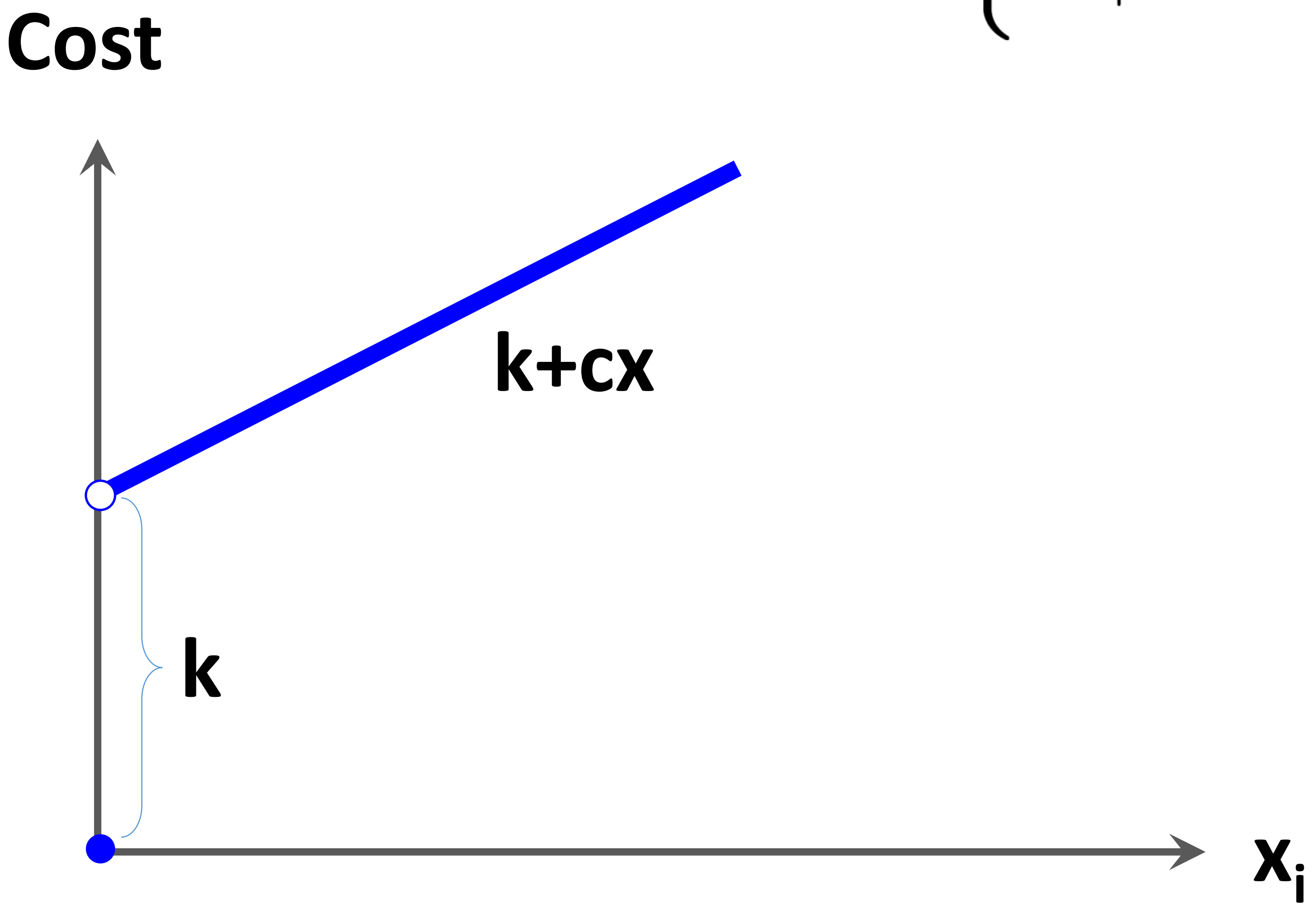
fixed cost

$$\text{Cost} = \begin{cases} 0 & \text{if } x = 0 \\ k + cx & \text{if } x > 0 \end{cases}$$



costs are function of DV (fixed cost)

$$\text{Cost} = \begin{cases} 0 & \text{if } x_i = 0 \\ k_i + c_i x_i & \text{if } x_i > 0 \end{cases}$$

use binary variable $y_i = \begin{cases} 0, & \text{for} \quad x_i = 0 \\ 1, & \text{for} \quad x_i > 0 \end{cases}$

Cost component: $k_i y_i + c_i x_i$

Constraints:

$$x_i \leq M y_i$$

$$y_i \in \{0, 1\}$$

To model a variety of…

- Yes/No decisions
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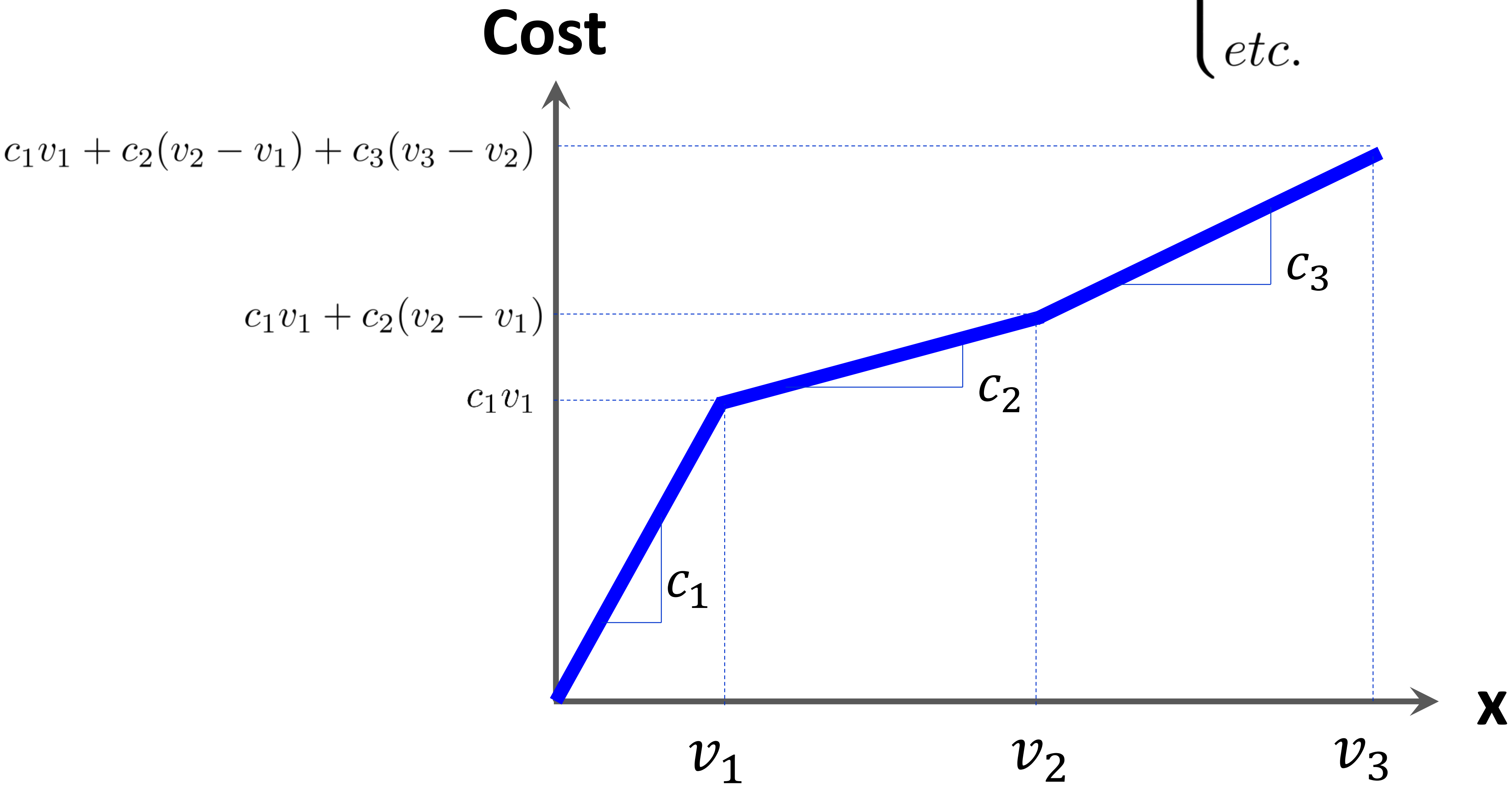
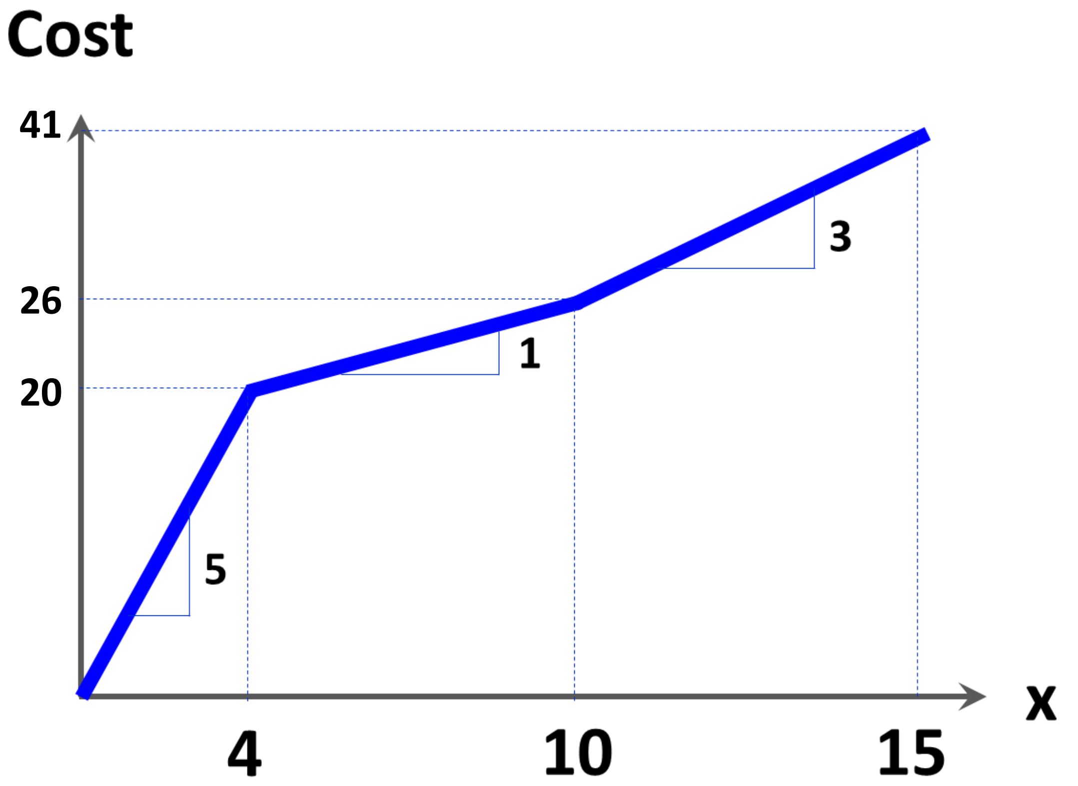
piecewise linear cost

$$\text{Cost} = \begin{cases} c_1 & \text{if } 0 \leq x < v_1 \\ c_2 & \text{if } v_1 \leq x < v_2 \\ c_3 & \text{if } v_2 \leq x < v_3 \\ etc. \end{cases}$$



costs are function of DV (piecewise linear)

$$\text{Cost} = \begin{cases} 5 & \text{if } 0 \le x < 4 \\ 1 & \text{if } 4 \le x < 10 \\ 3 & \text{if } 10 \le x < 15 \end{cases}$$



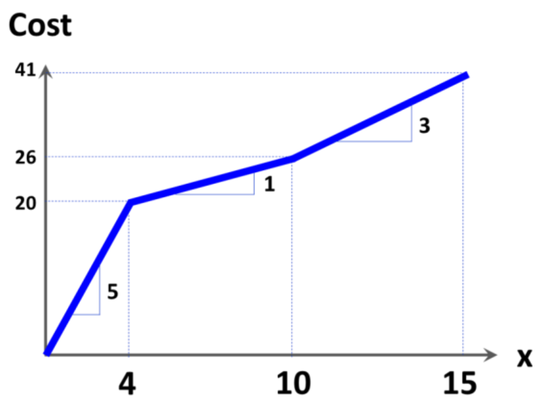
$$x = \delta_1 + \delta_2 + \delta_3$$

$$0 \le \delta_1 \le 4$$
$$0 \le \delta_2 \le 6$$
$$0 \le \delta_3 \le 5$$

$$\text{Cost} = 5\delta_1 + \delta_2 + 3\delta_3$$

We need: $\delta_1 = 4$ whenever $\delta_2 > 0$,
and $\delta_2 = 6$, whenever $\delta_3 > 0$

costs are function of DV (piecewise linear)



$$\text{Cost} = 5\delta_1 + \delta_2 + 3\delta_3$$

We need: $\delta_1 = 4$ whenever $\delta_2 > 0$,
and $\delta_2 = 6$, whenever $\delta_3 > 0$

use binary variables

$$y_1 = \begin{cases} 1, & \text{if} \quad \delta_1 = 4 \\ 0, & \text{otherwise} \end{cases} \qquad y_2 = \begin{cases} 1, & \text{if} \quad \delta_2 = 6 \\ 0, & \text{otherwise} \end{cases}$$

Constraints:

$$\cancel{\begin{aligned} 0 \le \delta_1 \le 4 \\ 0 \le \delta_2 \le 6 \\ 0 \le \delta_3 \le 5 \end{aligned}} \Rightarrow \begin{aligned} 4y_1 \le \delta_1 \le 4 \\ 6y_2 \le \delta_2 \le 6y_1 \\ 0 \le \delta_3 \le 5y_2 \\ y_1, y_2 \in \{0, 1\} \end{aligned}$$

costs are function of DV (piecewise linear)

$$\text{Cost} = 5\delta_1 + \delta_2 + 3\delta_3$$

Constraints

$$4y_1 \leq \delta_1 \leq 4$$
$$6y_2 \leq \delta_2 \leq 6y_1$$
$$0 \leq \delta_3 \leq 5y_2$$
$$y_1, y_2 \in \{0, 1\}$$

If $y_1 = 0$ $\Rightarrow$

$$0 \leq \delta_1 \leq 4$$
$$\boxed{6y_2 \leq \delta_2 \leq 0}$$
$$0 \leq \delta_3 \leq 5y_2$$

$\Rightarrow$ $y_2 = 0$ $\Rightarrow$

$$0 \leq \delta_1 \leq 4$$
$$0 \leq \delta_2 \leq 0$$
$$0 \leq \delta_3 \leq 0$$

$\Rightarrow$

$$\delta_2 = 0$$
$$\delta_3 = 0$$

$\Rightarrow$ **Cost $= 5\delta_1$**

costs are function of DV (piecewise linear)

$$\text{Cost} = 5\delta_1 + \delta_2 + 3\delta_3$$

Constraints

$$4y_1 \leq \delta_1 \leq 4$$

$$6y_2 \leq \delta_2 \leq 6y_1$$

$$0 \leq \delta_3 \leq 5y_2$$

$$y_1, y_2 \in \{0, 1\}$$

If $y_1 = 1$ and $y_2 = 0$

$$4 \leq \delta_1 \leq 4 \rightarrow \delta_1 = 4$$

$$0 \leq \delta_2 \leq 6$$

$$0 \leq \delta_3 \leq 0 \rightarrow \delta_3 = 0$$

$$\text{Cost} = 20 + \delta_2$$

costs are function of DV (piecewise linear)

$$\text{Cost} = 5\delta_1 + \delta_2 + 3\delta_3$$

Constraints

$$4y_1 \le \delta_1 \le 4$$
$$6y_2 \le \delta_2 \le 6y_1$$
$$0 \le \delta_3 \le 5y_2$$
$$y_1, y_2 \in \{0, 1\}$$

If $y_1 = 1$ and $y_2 = 1$

$$4 \le \delta_1 \le 4 \rightarrow \delta_1 = 4$$
$$6 \le \delta_2 \le 6 \rightarrow \delta_2 = 6$$
$$0 \le \delta_3 \le 5$$

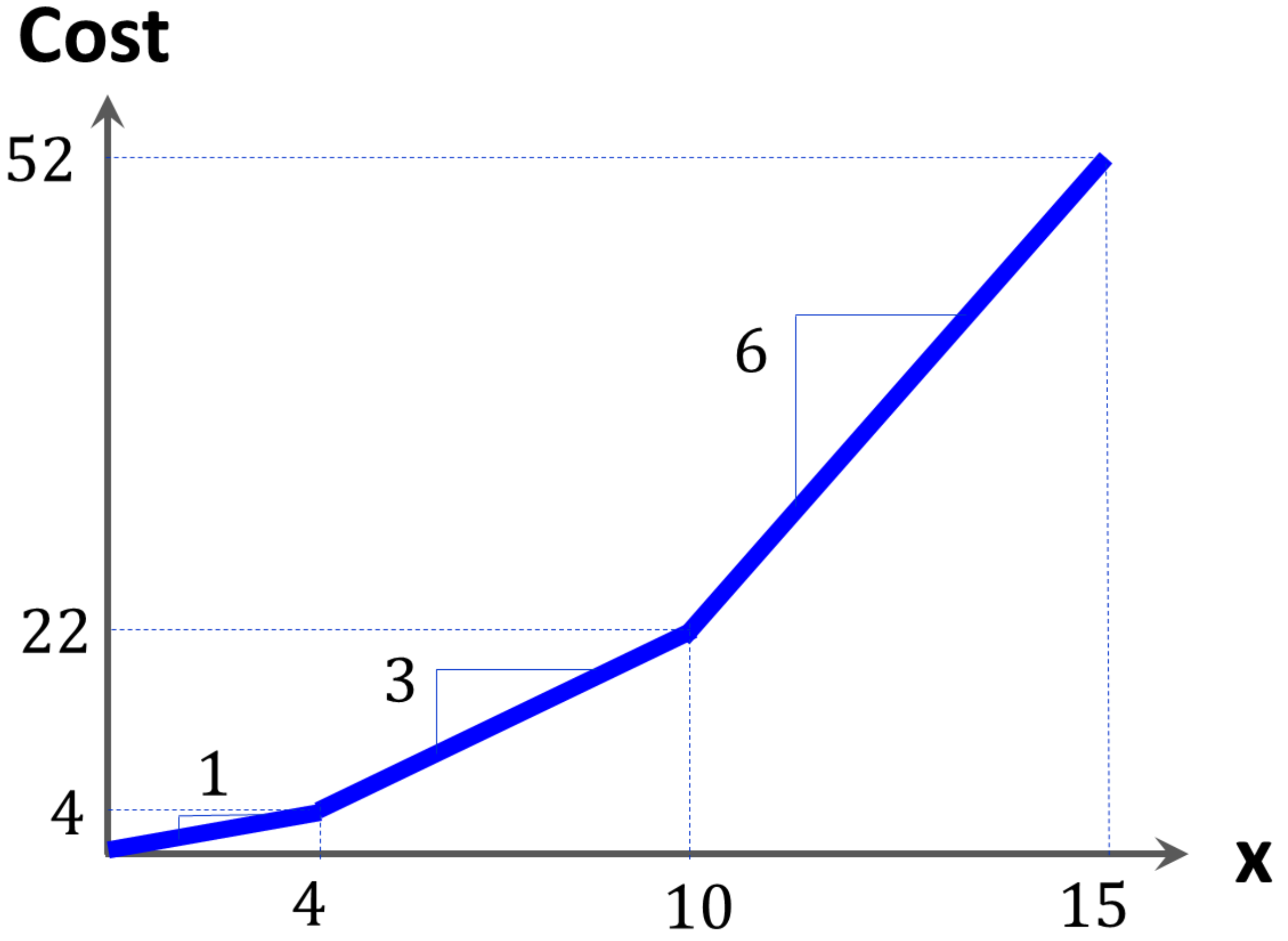
$$\text{Cost} = 26 + 3\delta_3$$

costs are function of DV (piecewise linear)

$$\text{Cost} = \begin{cases} 1 & \text{if } 0 \leq x < 4 \\ 3 & \text{if } 4 \leq x < 10 \\ 6 & \text{if } 10 \leq x < 15 \end{cases}$$

$$0 \leq \delta_1 \leq 4$$
$$0 \leq \delta_2 \leq 6$$
$$0 \leq \delta_3 \leq 5$$

$$\text{Cost} = \delta_1 + 3\delta_2 + 6\delta_3$$



For minimization problem
"diseconomy of scale"