Min-Cost Network Flow Problem Examples: Shortest Path Problem

Shortest Path Problem (MCNFP Formulation)

- Defined on a network with two special nodes: s and t
- A *path* from s to t is an alternating sequence of distinct nodes and distinct arcs starting at s and ending at t: $s, (s, n_1), n_1, (n_1, n_2), \dots, (n_i, n_j), n_j, (n_j, t), t$
  - There are *directed* and *undirected* paths
  - A networks is *connected* if an undirected path exists between any pairs of nodes
  - If $s \ = \ t$, the path is called a *cycle*
  - A connected network without cycles is called a *tree*

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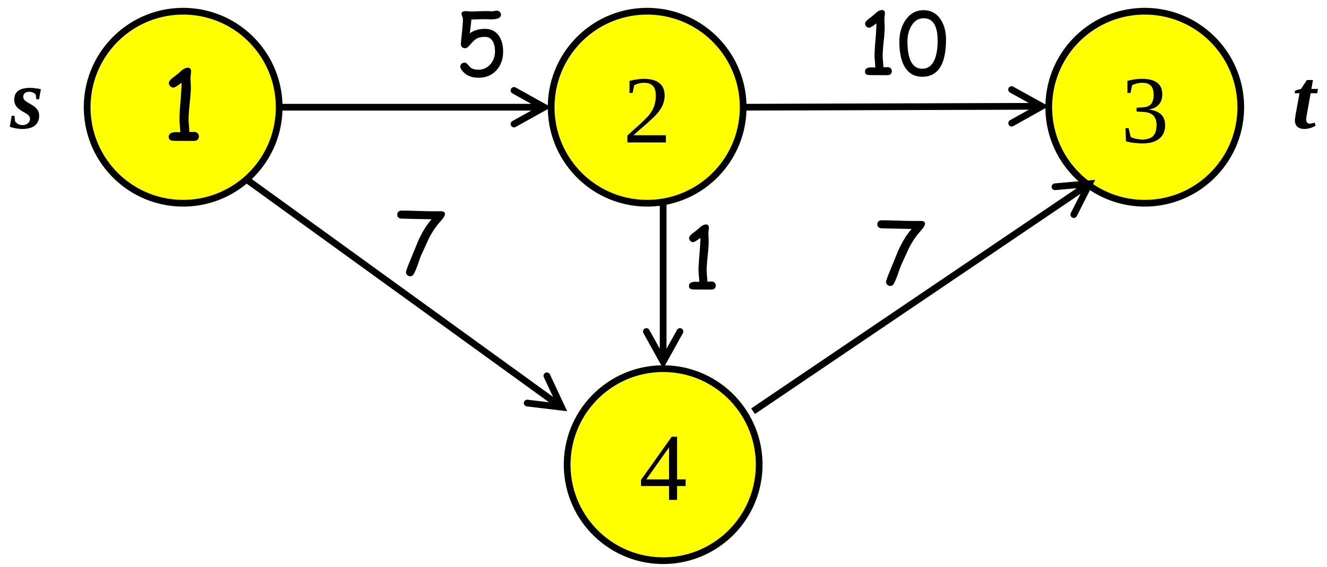
- Find a minimum-cost directed path from s to t

shortest path example



1,(1,2),2,(2,3),3 Length = 15

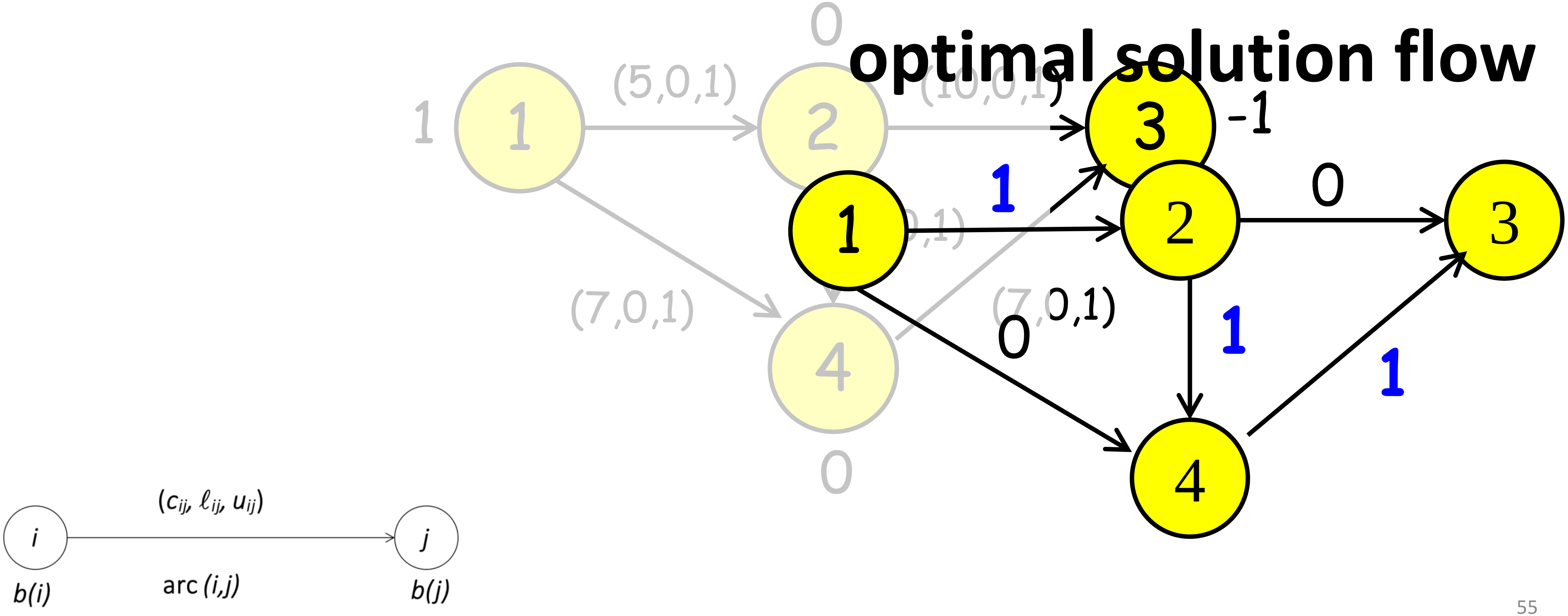
1,(1,2),2,(2,4),4,(4,3) *Length = 13*

1,(1,4),4,(4,3),3 Length = 14

MCNFP formulation of shortest path problem

- *Source* node s has a supply of 1
- *Terminal* node t has a demand of 1
- All other nodes are transshipment nodes
- Each arc has capacity 1
- Tracing the unit of flow from s to t gives a path from s to t

shortest path as MCNFP



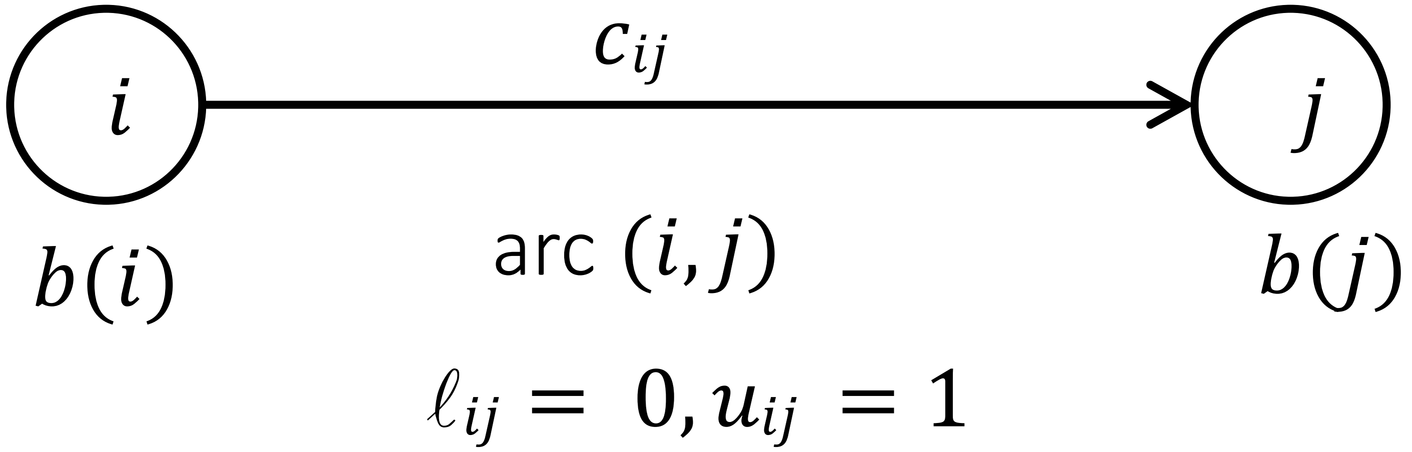
shortest path example

In a rural area of Texas, there are six farms connected by small roads. The distances in miles between the farms are given in the table.

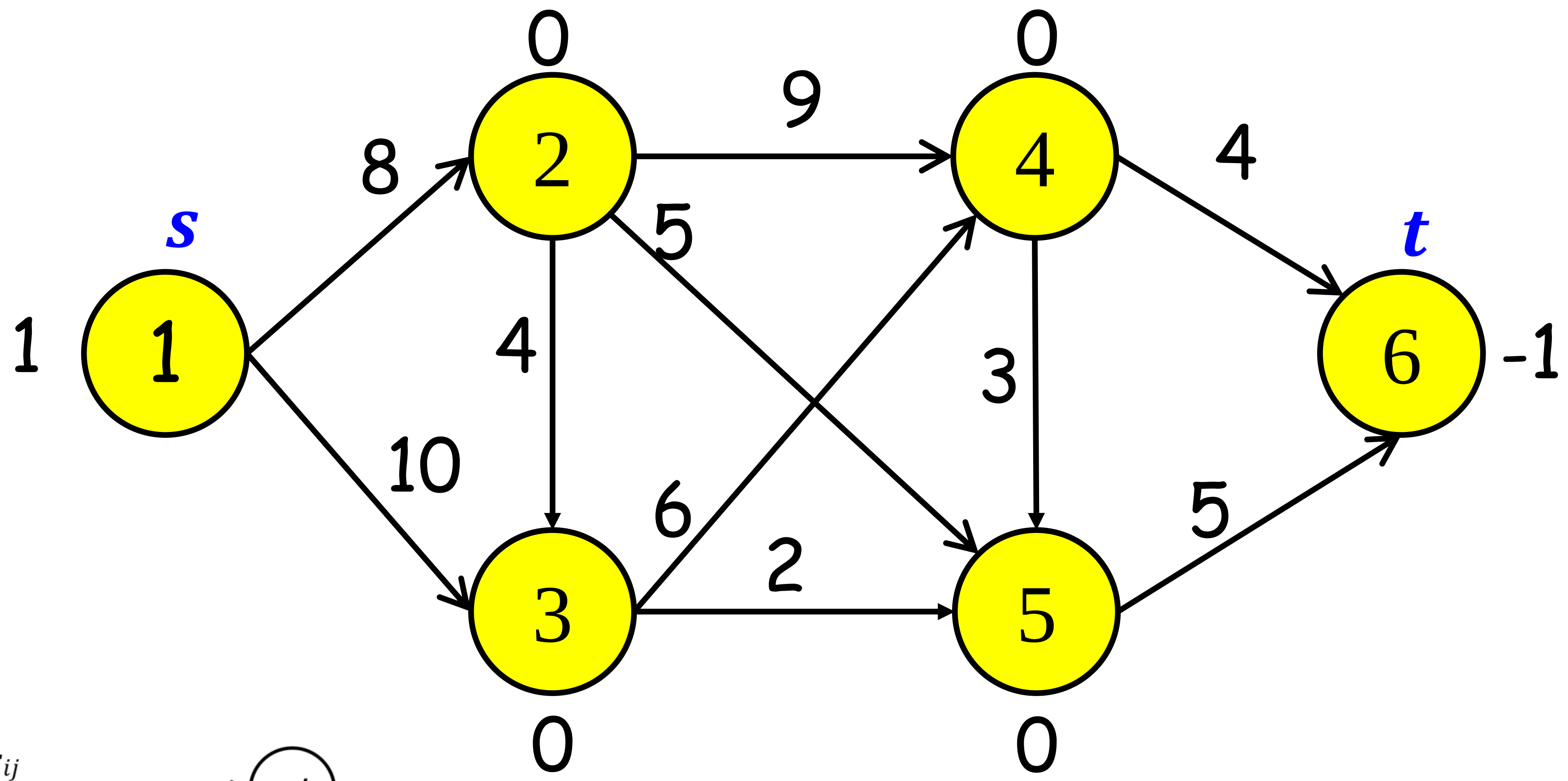
What is the minimum distance to get from Farm 1 to Farm 6?

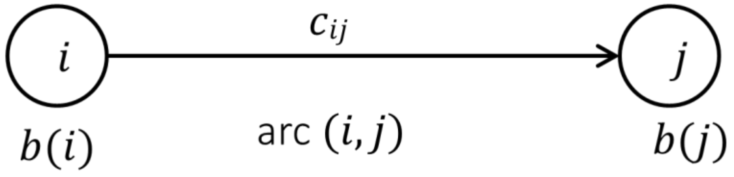
From Farm	To Farm	Distance
1	2	8
1	3	10
2	3	4
2	4	9
2	5	5
3	4	6
3	5	2
4	5	3
4	6	6
5	6	5

graphical network flow formulation



formulation as shortest path





$\ell_{ij} = 0, u_{ij} = 1$

matrix representation

node requirements:
+1 for node s;
-1 for node t

one variable per arc: denotes the "flow" on an edge

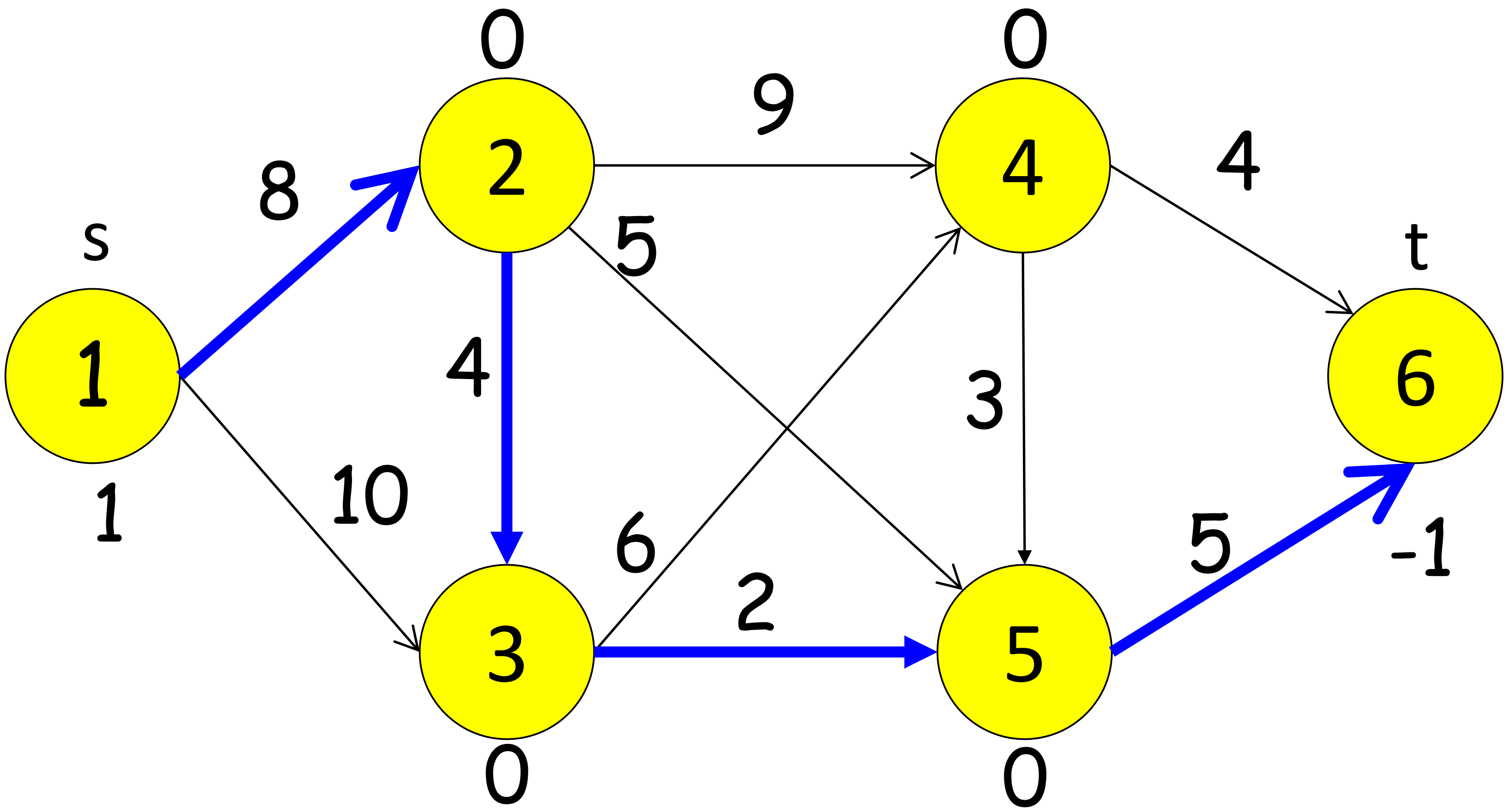
$$\lfloor x_{12} \quad x_{13} \quad x_{23} \quad x_{24} \quad x_{25} \quad x_{34} \quad x_{35} \quad x_{45} \quad x_{46} \quad x_{56} \rfloor$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

<u>node-arc incidence matrix</u> stipulates the "flow-balance constraints"

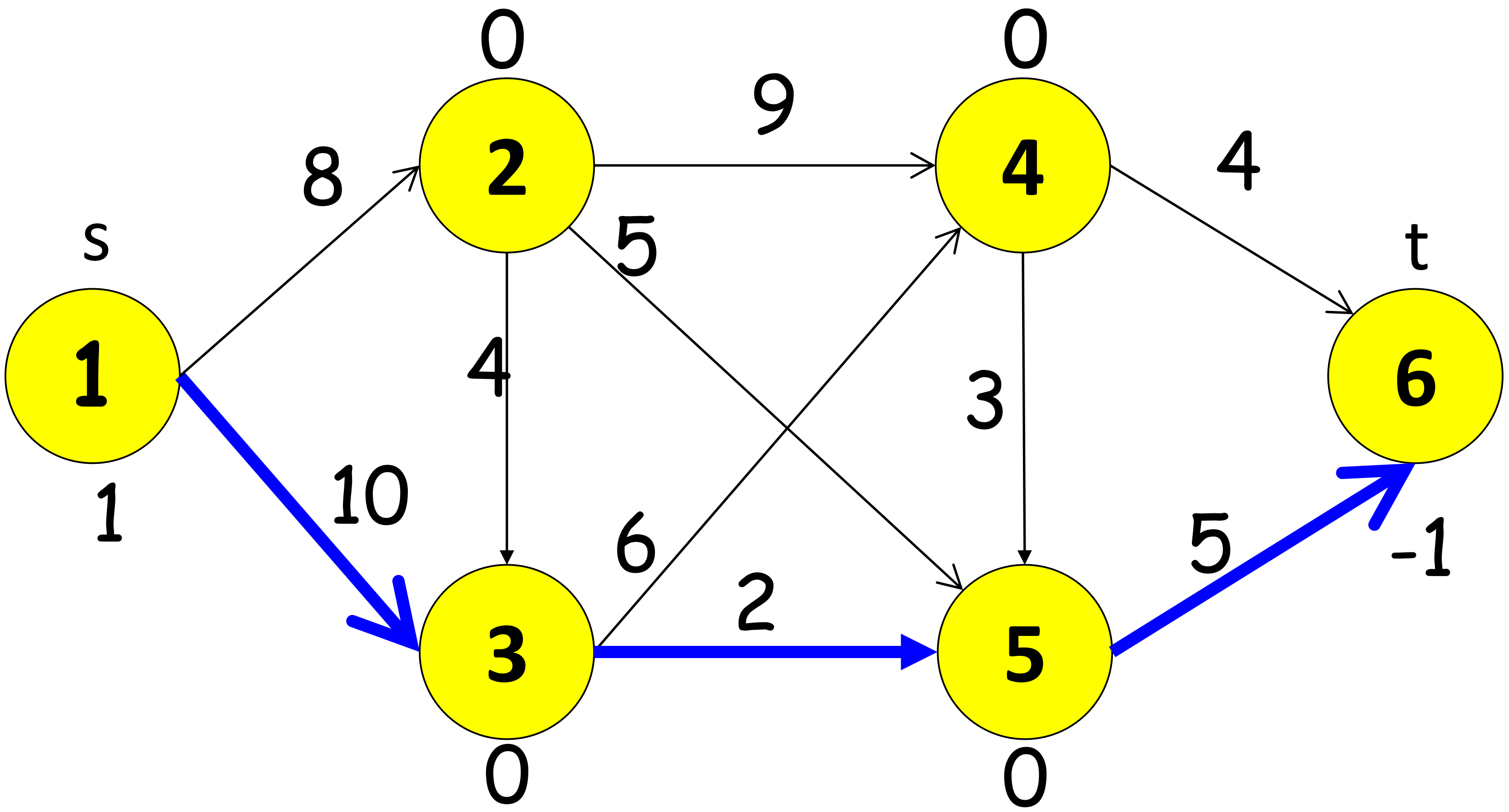
- every column relates to an arc
- every row relates to a node

“greedy” solution



$x_{13} = x_{23} = x_{35} = x_{35} = 1$, $x_{ij} = 0$ **for all other arcs.**
Objective function value = 19.

shortest path: optimal solution



$x_{13} = x_{35} = x_{56} = 1$, $x_{ij} = 0$ for all other arcs.
Objective function value = 17.