Problem Set #7 Properties of GARCH Models

In this problem set, we consider the properties of ARCH and GARCH models, and estimate a model for the daily return on the SP500 stock market index for the period 2009 to 2018. For the empirical analysis we consider if the properties of the data are different on the last trading day before a weekend and the first trading day after a weekend, respectively.

#7.1 SP500 Stock Market Index

In this exercise, we consider the daily return on the SP500 stock market index for the period January 2, 2009, to February 27, 2018. The data file `SP500Data_short.IN7` contains the daily returns calculated from closing prices, `Dlog(SP500)`, and well as a dummy variable for the first day after a closing day, `FirstTrade`, and for the last day before a closing day, `LastTrade`.

(1) Load the data into OxMetrics.
Draw graphs of the stock market return, the absolute return, and the squared return. Do the graphs indicate ARCH effects?

(2) Use PcGive to set up an autoregressive model for `Dlog(SP500)`.
Discuss if your preferred model is a reasonable characterization of a stock market.

(3) Test for ARCH effects in the residuals. What do you conclude?

(4) Select the package for `[GARCH Models using PcGive]`. Set up your preferred model for the mean and specify the conditional variance as a GARCH(1,1) model.
Estimate the model using maximum likelihood under the assumption of normality.

(5) Save the residuals, the scaled residuals, and the conditional variance from `[Store in Database]` in the `[Test]` menu.
Create the conditional standard deviation and the 95 percent confidence bands for the residuals using the Algebra Editor:

$$\mathsf{CondStdDev} = \mathsf{sqrt}(\mathsf{CondVar});$$
$$\mathsf{max} = 1.96 * \mathsf{CondStdDev};$$
$$\mathsf{min} = -1.96 * \mathsf{CondStdDev};$$

Plot the graphs of the residuals, the scaled residuals, the conditional variance, and the conditional standard deviation.

Additionally, plot the residuals and the 95 percent confidence bands. Look at the conditional standard deviation and compare with the residuals.

Interpret the model and the conditional variance.

(6) Make a 250 period out of sample forecast and interpret the outcome.

[Hint: Go to the `Test` menu and select `Forecast`. Set the `Number of forecasts` to 250 and set the `Number of pre-forecast observations to graph` to, say, 1,000.]

(7) Consider (in turn) the following extensions to the basic model:

(a) Insert the dummy variables in the conditional mean and the conditional variance.

(b) Allow for an asymmetric impact of shocks on the conditional variance σ_t^2 via the threshold specification.

(c) Allow for an asymmetric impact of shocks on the conditional variance σ_t^2 via the asymmetric specification.

(d) Allow for non-Gaussian errors by allowing z_t to have a Student's t distribution, $t(\nu)$, where ν is a parameter to be estimated.

(e) Allow for a risk-premium via the ARCH-in-Mean specification.

Which is your overall preferred model?

Try to combine several extensions and find a finally preferred model.

#7.2 The ARCH and GARCH Models

Consider the daily return $\{r_t\}_{t=1}^T$ and an ARCH(1) model given by:

$$r_t = \delta + \epsilon_t, \tag{7.1}$$

$$\epsilon_t = \sigma_t z_t, \qquad z_t|\mathcal{I}_{t-1} \overset{d}{=} N(0,1), \tag{7.2}$$

$$\sigma_t^2 = \varpi + \alpha\epsilon_{t-1}^2, \tag{7.3}$$

where $\varpi > 0$, $\alpha \geq 0$ and $\mathcal{I}_{t-1} = \{r_1, r_2, ..., r_{t-1}\}$ denotes the information set.

(1) Derive the conditional mean of r_t, $E(r_t|\mathcal{I}_{t-1})$.

Derive the conditional variance of r_t, $V(r_t^2|\mathcal{I}_{t-1})$.

State the distribution of r_t conditional on the information set $\mathcal{I}_{t-1}$.

(2) Derive the unconditional mean of r_t, $E(r_t)$.

Any random variable x can be decomposed into the sum of a conditional expectation given some information set $\mathcal{I}$ and a residual v which is uncorrelated with that information set:

$$x = E(x|\mathcal{I}) + v, \quad \text{where} \quad E(v|\mathcal{I}) = 0.$$

(3) Use the decomposition above for ϵ_t^2 conditional on the information set $\mathcal{I}_{t-1}$,

$$\epsilon_t^2 = E(\epsilon_t^2|\mathcal{I}_{t-1}) + v_t, \quad E(v_t|\mathcal{I}_{t-1}) = 0, \tag{7.4}$$

to show that the squared innovations ϵ_t^2 follow an AR(1) process with residual v_t.

(4) Assuming that the conditions for weak stationarity is fulfilled, derive the unconditional variance of the innovations ϵ_t, $E(\epsilon_t^2)$.
Explain what it means that the innovations ϵ_t in the weakly stationary ARCH(1) model are unconditionally homoskedastic and conditionally heteroskedastic?

Next, consider the following GARCH(1,1) model

$$
\begin{aligned}
r_t &= \delta + \epsilon_t \\
\epsilon_t &= \sigma_t z_t \\
\sigma_t^2 &= \varpi + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2,
\end{aligned}
$$

with $\varpi > 0$, $\alpha \geq 0$, $\beta \geq 0$, and z_t as given above. Assume that r_0 and σ_0^2 are given.

(5) Show that the process of squared residuals, ϵ_t^2, follows an ARMA(1,1) model.
Explain how this is a generalization of the ARCH case.

(6) Use the ARMA-representation to calculate the unconditional variance of ϵ_t, $\sigma^2 = E(\epsilon_t^2)$.