

Machine Learning

Blockkurs *Neuronale Netze und Deep Learning* vom 22.3.2021

Artificial Neural
Networks

Definition

Training (BackProp)

Activation Functions

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Definition 1 (Artificial Neural Network (NN, künstliches neuronales Netz))

A feed-forward **artificial neural network** with $L \geq 1$ layers of sizes s_1, \dots, s_L with $n =: s_0$ input variables $\mathbf{x} = (x_1, \dots, x_n)^T$ and $K := s_L$ output variables $\mathbf{t} = (t_1, \dots, t_K)^T$ is a function

$$\mathbf{t} = h_{\boldsymbol{\theta}}(\mathbf{x})$$

with parameters

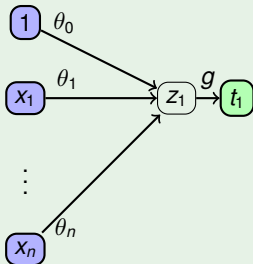
$$\boldsymbol{\theta} = (\boldsymbol{\Theta}^{(0)}, \dots, \boldsymbol{\Theta}^{(L-1)}) \quad , \text{ where } \boldsymbol{\Theta}^{(\ell)} \in \mathbb{R}^{s_{\ell+1} \times (s_{\ell}+1)} ,$$

defined by the following recursions

$$\begin{aligned} \mathbf{t} &= \mathbf{g}(\mathbf{z}^{(L)}) \in \mathbb{R}^K \\ \mathbf{z}^{(\ell)} &= \boldsymbol{\Theta}^{(\ell-1)} \mathbf{a}^{(\ell-1)} \in \mathbb{R}^{s_{\ell}} \quad (1 \leq \ell \leq L) \\ \mathbf{a}^{(\ell)} &= \begin{pmatrix} a_0^{(\ell)} \\ a_1^{(\ell)} \\ \vdots \\ a_{s_{\ell}}^{(\ell)} \end{pmatrix} = \begin{pmatrix} 1 \\ \sigma(z_1^{(\ell)}) \\ \vdots \\ \sigma(z_{s_{\ell}}^{(\ell)}) \end{pmatrix} \quad (1 \leq \ell < L) \\ \mathbf{a}^{(0)} &= \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} . \end{aligned} \tag{1}$$

Here, σ is called an **activation function** and we will call $\mathbf{g} : \mathbb{R}^K \rightarrow \mathbb{R}^K$ the **output activation function**.

Linear regression = NN with 1 layer and 1 output variable



$L = 1$ layers

$\theta = (\theta_0, \theta_1, \dots, \theta_n)$ parameters

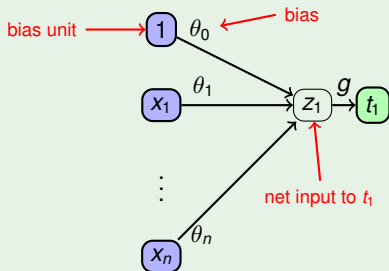
t_1 one output unit

x_1, \dots, x_n input units

output activation function $t_1 = g(z_1) = z_1$

$$t_1 = \theta_0 + \sum_{j=1}^n \theta_j x_j$$

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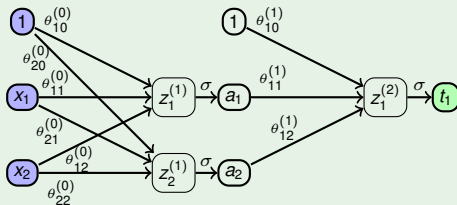
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Artificial Neural Networks

NN with 1 hidden layer



$L = 2$ layers, 1 hidden layer, $s_1 = 2$

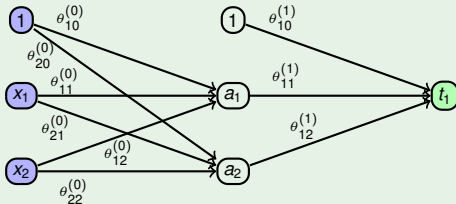
$\theta = (\Theta^{(0)}, \Theta^{(1)})$ parameters

x_1, x_2 input units, t_1 single output unit with output activation function σ

$$t_1 = \sigma \left(\theta_{10}^{(1)} + \theta_{11}^{(1)} a_1 + \theta_{12}^{(1)} a_2 \right) = \sigma \left(\theta_{10}^{(1)} + \theta_{11}^{(1)} \sigma(\theta_{10}^{(0)} + \theta_{11}^{(0)} x_1 + \theta_{12}^{(0)} x_2) + \theta_{12}^{(1)} \sigma(\theta_{20}^{(0)} + \theta_{21}^{(0)} x_1 + \theta_{22}^{(0)} x_2) \right)$$

Artificial Neural Networks

NN with 1 hidden layer



$L = 2$ layers, 1 hidden layer, $s_1 = 2$

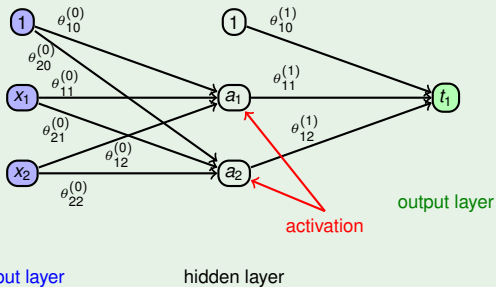
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Training of Artificial Neural Networks

Suppose a *single* training observation

$$\mathbf{y} = (y_1, \dots, y_K)^T$$

for training input

$$\mathbf{x} = (x_1, \dots, x_n)^T$$

is given.

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Consider some error function

$$D(\theta)$$

that typically depends on training data but that depends on the parameters θ only through the network output \mathbf{t} , e.g. a squared error function.

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$$\frac{\partial D}{\partial \theta_{ji}^{(\ell-1)}}$$

for ℓ , j and i (“gradient” in TensorFlow).

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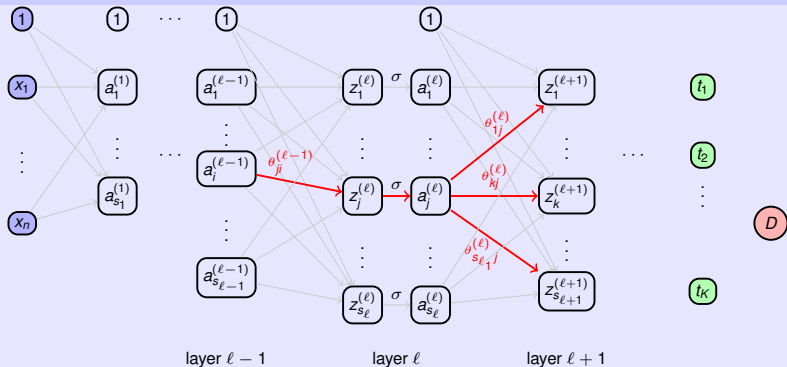
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$D(\boldsymbol{\theta})$ may be non-convex and have **local, non-global minima**.

NN Training: partial derivatives **backpropagate** through network



From

$$z_j^{(\ell)} = \sum_r \theta_{jr}^{(\ell-1)} a_r^{(\ell-1)}$$

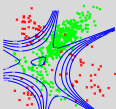
and using the multivariate chain rule that

we obtain using the univariate chain rule that

$$\frac{\partial D}{\partial \theta_{ji}^{(\ell-1)}} = \frac{\partial D}{\partial z_j^{(\ell)}} a_i^{(\ell-1)}$$

$$\frac{\partial D}{\partial z_j^{(\ell)}} = \left(\frac{\partial D}{\partial z_1^{(\ell+1)}}, \dots, \frac{\partial D}{\partial z_{s_{\ell+1}}^{(\ell+1)}} \right) \begin{pmatrix} \frac{\partial z_1^{(\ell+1)}}{\partial z_j^{(\ell)}} \\ \vdots \\ \frac{\partial z_{s_{\ell+1}}^{(\ell+1)}}{\partial z_j^{(\ell)}} \end{pmatrix}$$

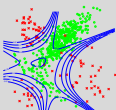
All derivatives can be computed efficiently in one right-to-left pass (“**backpropagation** algorithm”). TensorFlow does backprop automatically, for general models.



Artificial Neural Networks for Regression

Consider first a **single training instance** $\mathbf{x} \in \mathbb{R}^n$ with a single output $\mathbf{y} \in \mathbb{R}^K$.

(We simply average $D(\theta)$ over multiple training instances.)

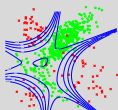


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(We simply average $D(\theta)$ over multiple training instances.)

- Choose the **identity function** as output activation function \mathbf{g} , i.e. $\mathbf{t} = \mathbf{z}^{(L)}$ (unbounded).
- Choose the **squared error function**

$$D(\theta) := \|\mathbf{t} - \mathbf{y}\|_2^2 = \sum_{k=1}^K (t_k - y_k)^2$$

Artificial Neural Networks for Classification

Again, consider first a single training instance. Here, let output $\mathbf{y} \in \mathbb{R}^K$ be one-hot encoded such that

$$\mathbf{y} = \mathbf{e}_c = c\text{-th unit vector}$$

if $c \in \{1, 2, \dots, K\}$ is the true class of the learning instance.

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- Choose the **softmax** function as output activation function \mathbf{g} .

$$\mathbf{t} = \mathbf{g}(z_1, \dots, z_K) = \frac{1}{\sum_{k=1}^K e^{z_k}} \begin{pmatrix} e^{z_1} \\ \vdots \\ e^{z_K} \end{pmatrix}$$

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Properties:

- $0 < t_k < 1$, $t_1 + \dots + t_K = 1$
- if $z_j \gg z_i$ for $i \neq j$, then $t_j \approx 1$ and $t_i \approx 0$ for $i \neq j$
- softmax is a generalization of the sigmoid function to more than 2 classes
- If $K = 2$ (binary classification), $t_1 = 1 - t_2$ is often not stored. The same neural network function can then be achieved by setting the output size to 1 and using the logistic sigmoid function as activation.

Artificial Neural Networks for Classification

Cross-entropy error function for multiclass problem

Consider the true class of the training instance to be a random variable $C \in \{1, \dots, K\}$ with observation c and the output vector \mathbf{t} of the net to be the parameters of a multinomial distribution for C . Then the likelihood is

$$L(\theta) =$$

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and we seek to minimize the negative log-likelihood

$$-\ln L(\boldsymbol{\theta}) = -\ln t_c = -\sum_{k=1}^K y_k \ln t_k$$

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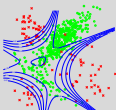
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$$-\ln L(\theta) = -\ln t_c = -\sum_{k=1}^K y_k \ln t_k$$

Define the **cross-entropy** error function

$$D(\theta) = -\sum_{k=1}^K y_k \ln t_k.$$



Artificial Neural Networks

Training set

Let

$$\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$$

be training **inputs** and

$$\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(m)}$$

be the corresponding training **outputs / labels**. Let

$$\mathbf{t}^{(i)} = h_{\Theta}(\mathbf{x}^{(i)}) \quad (i = 1..m)$$

be the outputs of the NN.

Overfitting

If the model has many parameters, the training may result in a model that fits the training data 'too well', therefore **does not generalize well** and performs poorly on **independent** test data. Remedies:

- ① Make model less complex, e.g. reduce number of parameters or change model class (e.g. lin. regresion over NN).
- ② **Regularize** the model: Penalize certain parameter values independent of the data.

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Error function with regularization for neural networks

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An common choice for the regularization term is the (scaled) L2 norm:

$$R(\theta) = \frac{1}{m} \sum_{\ell=0}^{L-1} \sum_{i=1}^{s_{\ell}} \sum_{j=1}^{s_{\ell+1}} \left(\theta_{ji}^{(\ell)} \right)^2.$$

Note that the bias terms $\theta_{j0}^{(\ell)}$ are left out (not penalized).

Artificial Neural Networks

Error function with regularization for regression

$$E(\theta) = \frac{1}{m} \left\{ \sum_{i=1}^m \sum_{k=1}^K \left(t_k^{(i)} - y_k^{(i)} \right)^2 + \lambda \sum_{\ell=0}^{L-1} \sum_{i=1}^{s_{\ell}} \sum_{j=1}^{s_{\ell+1}} \left(\theta_{ji}^{(\ell)} \right)^2 \right\} \quad (2)$$

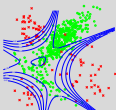
Artificial Neural Networks

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Error function with regularization for classification

$$E(\theta) = \frac{1}{m} \left\{ - \sum_{i=1}^m \sum_{k=1}^K y_k^{(i)} \ln t_k^{(i)} + \lambda \sum_{\ell=0}^{L-1} \sum_{i=1}^{s_{\ell}} \sum_{j=1}^{s_{\ell+1}} \left(\theta_{ji}^{(\ell)} \right)^2 \right\} \quad (3)$$



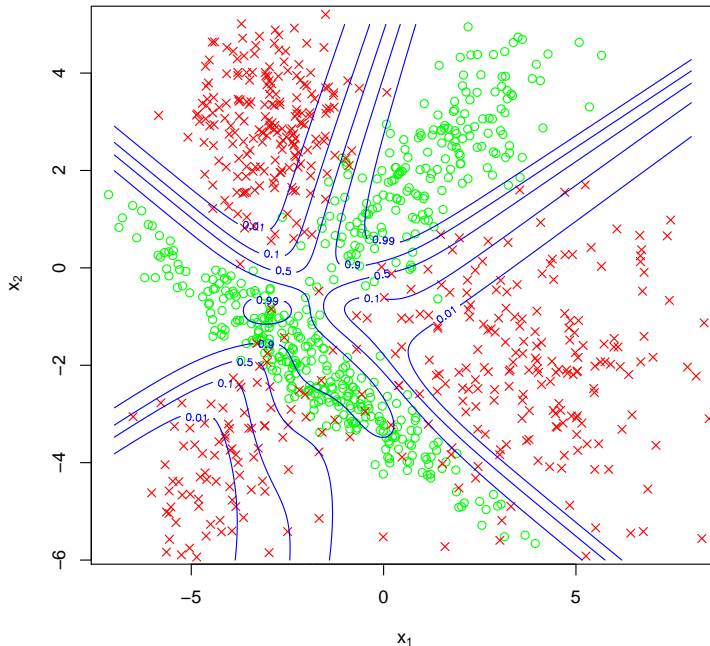
Artificial Neural Networks

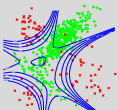
Definition

Training (BackProp)

Activation Functions

1 hidden layer of size 5, $\lambda = 0$ (no regularization)

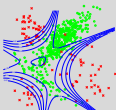




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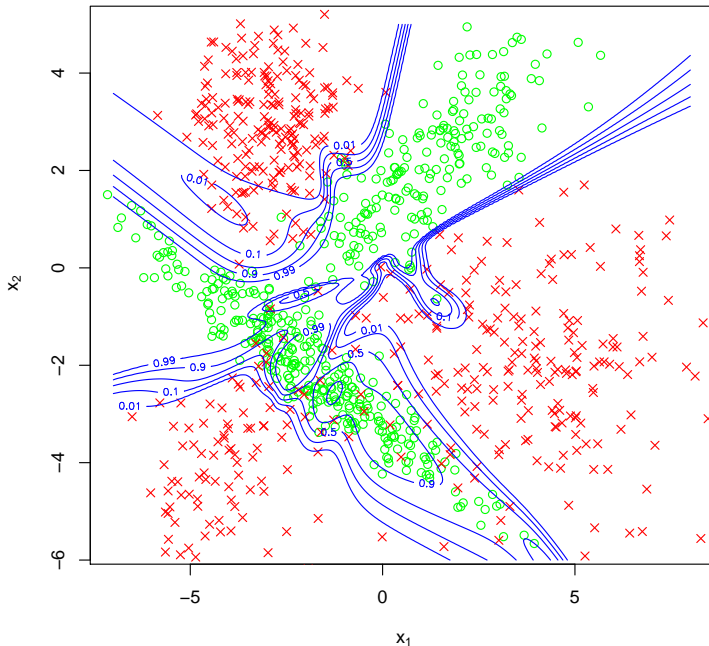


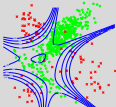
Artificial Neural Networks

Definition

Training (BackProp)

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2 hidden layers of size 20 each, $\lambda = 0$ (no regularization)

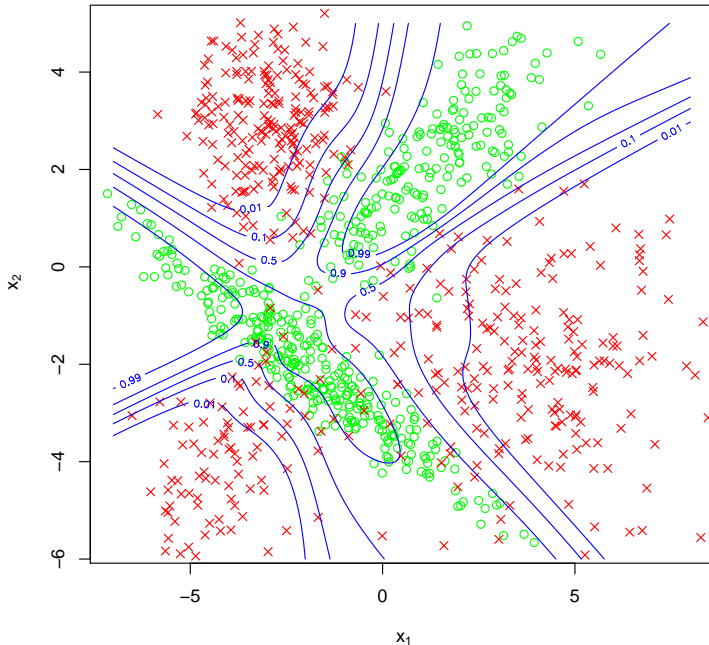


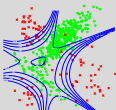
Artificial Neural Networks

Definition

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2 hidden layers of size 20 each, $\lambda = 0.1$ (with regularization)



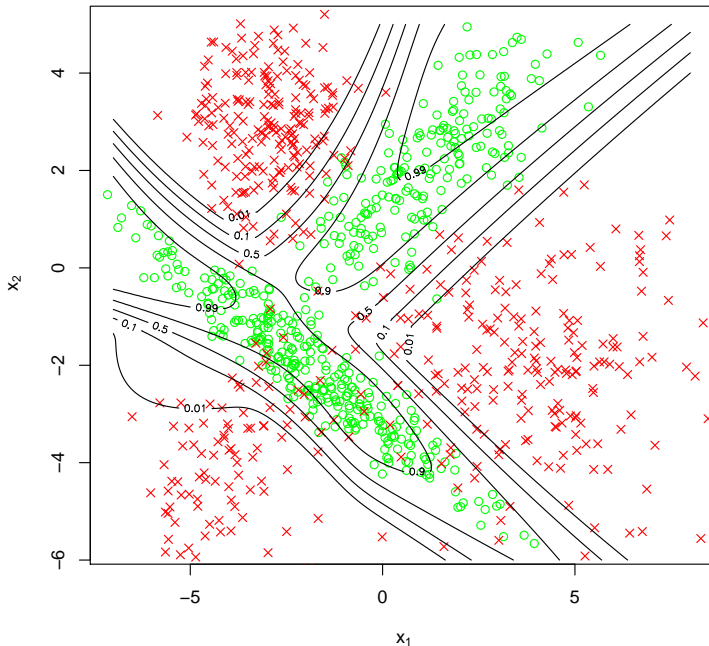
Artificial Neural Networks

Definition

Training (BackProp)

Activation Functions

Posterior probability of class (theoretical optimum if distr. was known)



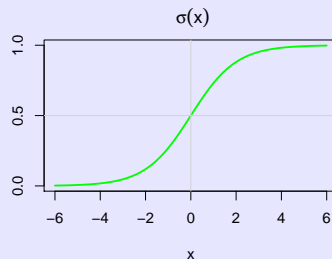


Activation Functions

Logistic sigmoid, `tf.nn.sigmoid`

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

$$\sigma' = \sigma(1 - \sigma)$$

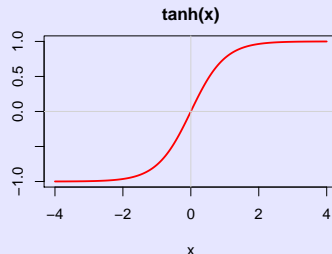


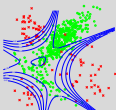
Tangens hyperbolicus

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$= 2\sigma(2x) - 1$$

$$\tanh'(x) = 1 - (\tanh(x))^2$$



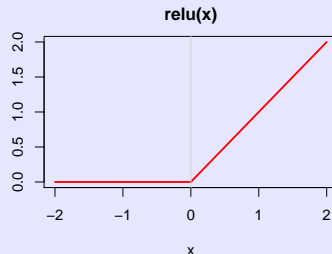


Rectified linear unit (ReLU), `tf.nn.relu`

$$\text{relu}(x) = x^+ = \max\{0, x\}$$

$$\text{relu}'(x) = \begin{cases} 1 & , \text{if } x > 0 \\ 0 & , \text{if } x < 0 \end{cases}$$

$$\text{relu}'(0) := 1 \quad (\text{arbitrary})$$



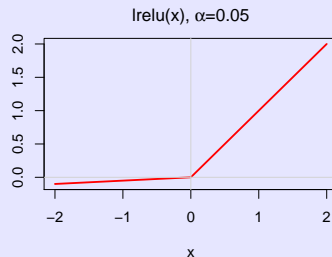
Leaky rectified linear unit (Leaky ReLU), `tf.nn.leaky_relu`

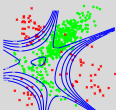
$$\text{lrelu}(x) = \max\{\alpha x, x\}$$

$$\text{lrelu}'(x) = \begin{cases} 1 & , \text{if } x > 0 \\ \alpha & , \text{if } x < 0 \end{cases}$$

$$\text{lrelu}'(0) := 1 \quad (\text{arbitrary})$$

$$\alpha = 0.01$$





Activation Functions

Vanishing gradients

- vanishing gradient problem:
 - during backprop in each layer, a derivative of the activation function is multiplied
 - for NNs with many layers (deep networks) this can result in gradients that are practically zero (“vanishing”), in particular for the sigmoid function
- ReLU
 - most frequently chosen
 - introduced in 2010

(Nair and Hinton, “Rectified linear units improve restricted Boltzmann machines”, *Proc. ICML*, 2010)

- The derivative is either 0 or 1. Can lead to sparse gradients.