An Algebraic Introduction to Mathematical Logic
Chapter 4 Predicate Calculus
Section 1 Algebra of Predicates
Proof of Transitivity

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Definition 1.4 Let w_1, $w_2 \in \widetilde{P}(V, \mathscr{R})$ define $w_1 \approx w_2$ if

1. $d(w_1) = d(w2) = 0$ and $w_1 = w_2$. Generating the free proposition algebra on F_0 that is, all generators and $\mathbf{F}$, that is F_0 are of depth 0. The use of $\Rightarrow$ does not increase the depth of quantification. Thus the equivalence relation on the proposition algebra contained in the predicate algebra $\widetilde{P}(V, \mathscr{R})$ is just equality.

2. $d(w_1) = d(w_2) > 0$, $w_1 = a_1 \Rightarrow b_1$, $w_2 = a_2 \Rightarrow b_2$, $a_1 \approx a_2$ and $b_1 \approx b_2$ or

3. $w_1 = (\forall x)(a)$, $w_2 = (\forall y)b$, and either

   (a) $x = y$ and $a \approx b$ (actually up to this point, this means iff they are equal)

   (b) There exists a $c = c(x)$ such that $c(x) \approx a$ and $c(y) \approx b$ and $y \notin V(c)$

Lemma 1. *Assume that $a(x) \approx c(x)$. Then any variable in $V(c(x))$ not present in $V(a(x))$ may be rechosen arbitrarily from the infinite set of variables not present in either. We show this by induction on the depth of quantification of $a(x)$.*

Proof. Let $d(a(x)) = 0$. Then $a(x) = c(x)$ and therefore $V(c(x)) \backslash V(a(x)) = \emptyset$. So the induction hypothesis holds in the base case.

Assume for all depths of quantification less than n the induction hypothesis holds. Let $d(a(x)) = d(c(x)) = n$ and $a(x) \approx c(x)$. Also let $a(x) = \forall x_1 a_1(x, x_1)$ and $c(x) = \forall x_2 c_1(x, x_2)$. Then by **Definition 1.4** $3(b)$, there exists a $e(x, x_1)$ not containing x_2 such that $e(x, x_1) \approx a_1(x, x_1)$ and $e(x, x_2) \approx c_1(x, x_2)$.

Since the depth $d(a_1(x, x_1)) = d(e(x, x_1)) = n - 1$ we have that any varible $z \in V(e(x, x_1))$ which is not in $a_1(x, x_1)$ can be rechosen arbitrarily. Since $e(x, x_2) \approx c_1(x, x_2)$, any variable $z \in V(c_1(x, x_2))$ not in $V(e(x, x_2))$ may be rechosen arbitrarily. I then claim that all variables of $V(c_1(x, x_2))$ not in $V(a_1(x, x_1))$.

Suppose there is a variable $z \notin V(a_1(x, x_1))$ but present in $V(c_1(x, x_2))$.

Case 1. If $z \notin V(e(x, x_2))$, we can use the congruence $e(x, x_2) \approx c_1(x, x_2)$ to eradicate z.

Case 2. If $z \in V(e(x, x_2))$, then we can use the congruence $a_1(x, x_1) \approx e(x, x_1)$ to rechose z in $e(x, x_1)$ if it appears there. Consequently, the only place that z could be in $e(x, x_2)$ is as in x_2, that is $z = x_2$. this violates the assumption that y NEQ.

It follows that any variable in $V(c_1(x, x_2))$ not present in $V(a_1(x, x_1))$ may be rechosen arbitrarily.

Since $a(x) = \forall x_1 a_1(x, x_1)$ and $c(x) = \forall x_2 c_1(x, x_2)$. The result follows: □

Problem 1 (Exercise 1.5 i)). *Given that $z \notin V(w_1) \cup V(w_2)$ show by induction over $d(w_1)$ that the element $c = c(x)$ in **Definition 1.4** 3(b) can always be chosen such that $z \notin V(c)$.*

Proof. By definition, there exists a $c(x)$ as in **Definition 1.4** 3(b) such that if $w_1 = \forall x a_1(x)$ and $w_2 = \forall y a_2(y)$, we have $a_1(x) \approx c(x)$, $y \notin V(c(x))$, and $a_2(y) \approx c(y)$.

By the lemma, since $a_1(x) \approx c(x)$, if $z \notin V(a(x))$, then if $z \in V(c(x))$, z can be rechosen arbitrarily. Since $z \notin V(w_2)$, $z \neq y$. □

Problem 2 (Exercise 1.5 ii)). *If $u(x) \approx v(x)$ and $y \notin V(u(x)) \cup V(v(x))$, show by induction over $d(u(x))$ that $u(y) \approx v(y)$.*

Proof. The base case holds because the relation is equality for depth 0.

Suppose the result holds for depth of quantification less than n. Then let $d(u(x)) = d(v(x)) = n$. Let $u(x) = \forall x_1 u_1(x, x_1)$ and $v(x) = \forall x_2 v_1(x, x_2)$. Then there exists a $c(x, x_1)$ such that $u_1(x, x_1) \approx c(x, x_1)$ and $c(x, x_2) \approx v_1(x, x_2)$ such that $x_2 \notin V(c)$. By **Problem 1** we have that $y \notin V(c)$.

By the induction hypothesis, since the depth of u_1 v_1 and c are all $n - 1$ we have that $u_1(y, x_1) \approx c(y, x_1)$ and $c(y, x_2) \approx v_1(y, x_2)$. Then by definition, since $u(y) = \forall x_1 u_1(y, x_1)$ and $v(y) = \forall x_2 v_1(y, x_2)$, we have proven $u(y) \approx v(y)$. □

Problem 3 (Exercise 1.5 iii)). *Show that the relation is transitive.*

Proof. Let $w_1 \approx w_2$ and $w_2 \approx w_3$. Assume the relation is transitive for lower depths of quantification. Since the relation is defined to be strict equivalence at depth 0, the base case holds.

The proof is given in the worst possible case: We begin by twice invoking **Definition 1.4** 3(b).

Let $w_1 = \forall x a_1(x)$, $w_2 = \forall y a_2(y)$, and $w_3 = \forall z a_3(z)$. We have that there exists $c_1(x)$ and $c_2(y)$, such that $y \notin c_1(x)$ and $z \notin c_2(y)$. Also, because of **Problem 1** we may require that $z \notin c_1(x)$ and $x \notin c_2(y)$. By part 3(b), of the definition $w_1 \approx w_2$ is defined to mean, $a_1(x) \approx c_1(x)$ and $c_1(y) \approx a_2(y)$.

Likewise, $w_2 \approx w_3$ means that $a_2(y) \approx c_2(y)$ and $c_2(z) \approx a_3(z)$.

By the induction hypothesis, since $c_1(y) \approx a_2(y)$ and $a_2(y) \approx c_2(y)$, we have $c_1(y) \approx c_2(y)$. Note also that neither $c_1(y)$ nor $c_2(y)$ contains either x or z.

By **Problem 2** we have that $c_1(x) \approx c_2(x)$. Then applying the induction hypothesis again, we obtain from $a_1(x) \approx c_1(x)$ and $c_1(x) \approx c_2(x)$, that $a_1(x) \approx c_2(x)$. Note also that since $c_2(z) \approx a_3(z)$, we have bridged the gap and it follows that $w_1 \approx w_3$.

Other cases follow from just **Problem 1** and the induction hypothesis alone.

$\square$