An Algebraic Introduction to Mathematical Logic
Chapter 1 Universal Algebra
Section 2 Free Algebras
Exercises

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Problem 1 (2.6)**.** *T consists of one unary operation, and F is the free T-algebra on a one element set $X = \{x_0\}$. How many elements are there in F_n? How many elements are there in F?*

2.6 Solution. The set F is defined as the union of recursively defined F_n, for $n \in \mathbb{N}$. To begin, $F_0 = T_0 \cup X$. Since the type T has only one unary operation, $T_0 = \emptyset$ and $F_0 = \{x_0\}$. Suppose for all $k < n$ we have defined F_k. Define $F_n = \{(t, a_1, ..., a_k) | t \in T, ar(t) = k, a_i \in F_{r_i}, \sum_{i=1}^{k} r_i = n - 1\}$.

In the case where $n = 1$, $n - 1 = 0$, therefore all of the $r_i = 0$ so $a_i \in F_0$. Since $F_0 = x_0$ there is only one element each a_i can be. Therefore there is only one $a = x_0$. In this case $k = 1$, and we have a single arity 1 function $t \in T$. Therefore there is one element in F_1, namely, $F_1 = (t, x_0)$.

Generally, each F_n contains only one element $(t, (t, (t, ...(t, x_0)...)))$. The element obtained by n applications of the arity 1 operation written in prefix notation.

F has a countable infinity of elements. □

Problem 2 (2.7)**.** *If T is empty and X is any set, show that X is the free T-algebra on X.*

Solution 2.7. Trivially, X satisfies the requirements for being a T-algebra. In fact, any set is a T-algebra when T is empty. Let A be any other set and f a set map from X to A. Then the inclusion $\sigma : X \to X$ which takes the set X to itself with empty T-algebra structure admits a unique homomorphism from the trivial algebra X to the trivial algebra A. Since there are no operators in T, the function f will do perfectly. It trivially is seen to preserve the absent operators on the algebra X. Therefore, $f \circ i = f$ □

Problem 3 (2.8)**.** *T consists of a single binary operation, and F is the free T-algebraon a one element set X. How many elements are there in F?*

Solution 2.8. Again $F_0 = x_0$. When $k = 2$ then we may add elements to higher F_n. For n = 1, the sum of the r_i must be 0. Therefore, $F_1 = (t, x_0, x_0)$. The 2-tuples which sum to 1 are $(0, 1)$ and $(1, 0)$, Therefore we obtain, $F_2 = (t, x_0, (t, x_0, x_0)), (t, (t, x_0, x_0), x_0)$. Similarly we can compute F_n for any n. They with Bell's Numbers. Therefore there are countably infinitely many elements in F. □

Problem 4 (2.9)**.** *If T consists of one 0-ary operation and one 2-ary operation, and if $X = \emptyset$, then the free T-algebra F on X is countable.*

Solution 2.9. There is a recursive formula for the number of elements in any F_n Given by $|F_n| = |F_{n-1}| + \sum_{i=1}^{n-1} |F_i \times F_{n-1-i}|$. Since $|F_0| = 1$ the free T-algebra F on X is countable. □

Problem 5 (2.10)**.** *T is finite or countable, and contains at least one 0-ary operation and at least one operation t with $ar(t) > 0$. X is finite or countable. Prove that F is countable.*

Solution 2.10. The cardinality of F_n at each level is countable, and there are a countable infinity of levels, therefore, F, the union of all F_n is countable. □