# An Algebraic Introduction to Mathematical Logic Chapter 4 Predicate Calculus Section 2 Interpretations

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Let $P(V, \mathscr{R})$ be the reduced first order algebra on $(V, \mathscr{R})$. We would like to create semantics for $P(V, \mathscr{R})$ in such a way that the statement that a certain collection of variables in $V$ which we interpret in the context of some set $U$, satisfy some relation in $\mathscr{R}$ is true if and only if the relation is satisfied in that context.

In particular, if we are interested in some collection of mathematical objects $U$ we can think of an element $x \in V$ as being the name of the object $u \in U$ by defining a function $\varphi : V \to U$ such that $\varphi(x) = u$. Similarly, if there exists a relation on the set $U$ and we want to disucss whether or not variables like $x$ satisfy the relation in $\mathscr{R}_U$, we define a function $\psi : \mathscr{R} \to \mathscr{R}_U$. By letting $\psi(r) = r_U$ we represent a relation $r_U$ on $U$ by a relation $r$ on $V$.

Let $P_k(V, \mathscr{R})$ be the set of all elements $p$ of $P(V, \mathscr{R})$ with $d(p) \leq k$.

Define the valuation $v : P \to \mathbb{Z}_2$ by the following properties:

(a) $v(r(x_1, .., x_n)) = 1$ if $\varphi(x_1), .., \varphi(x_n) \in \psi(r)$ and 0 otherwise.
(b) $v$ is still a $\{\Rightarrow, \mathbf{F}\}$-algebra homomorphism.
($\mathrm{c_k}$) Suppose $p = (\forall x)q(x)$ has depth $k$. Put $V' = V \cup \{t\}$ where $t \notin V$. If for every extension $\varphi' : V' \to U$ of $\varphi$ and for every $v_{k-1} : P_{k-1}(V, \mathscr{R}) \to \mathbb{Z}_2$, such that $(\varphi', \psi, v'_{k-1})$ satisfy (a), (b) and ($\mathrm{c_i}$), for all $i < k$, we have $v'_{k-1}(q(t)) = 1$, then $v(p) = 1$, otherwise $v(p) = 0$.

The idea is that $\forall x q(x)$ should be true if and only if for every possible interpretation $t$ $q(t)$ is still true at depth $k - 1$, where $t$ is a new variable and thus interpreting $t$ differently in $U$ changes only $t$.

**Problem 1.** Given $(U, \varphi, \psi)$ there is exactly one function $v : P \to \mathbb{Z}_2$ satisfying (a), (b) and ($\mathrm{c_k}$) for all $k$.

*Proof.* Let $w$ and $v$ both be functions which satistfy (a), (b) and ($\mathrm{c_k}$) for all $k$. Let $p \in P$ at depth $k = 0$. By (a), $w(p) = v(p)$. Suppose $v = w$ up to depth $k - 1$. Let $p \in P$ at depth $k = 0$. $\square$