An Algebraic Introduction to Mathematical Logic
Chapter 4 Predicate Calculus
Section 2 Interpretations

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Let $P(V, \mathscr{R})$ be the reduced first order algebra on $(V, \mathscr{R})$. We would like to create semantics for $P(V, \mathscr{R})$ in such a way that the statement that a certain collection of variables in V which we interpret in the context of some set U, satisfy some relation in $\mathscr{R}$ is true if and only if the relation is satisfied in that context.

In particular, if we are interested in some collection of mathematical objects U we can think of an element $x \in V$ as being the name of the object $u \in U$ by defining a function $\varphi : V \to U$ such that $\varphi(x) = u$. Similarly, if there exists a relation on the set U and we want to disucss whether or not variables like x satisfy the relation in $\mathscr{R}_U$, we define a function $\psi : \mathscr{R} \to \mathscr{R}_U$. By letting $\psi(r) = r_U$ we represent a relation r_U on U by a relation r on V.

Let $P_k(V, \mathscr{R})$ be the set of all elements p of $P(V, \mathscr{R})$ with $d(p) \leq k$.

Define the valuation $v : P \to \mathbb{Z}_2$ by the following properties:

(a) $v(r(x_1, .., x_n)) = 1$ if $\varphi(x_1), .., \varphi(x_n) \in \psi(r)$ and 0 otherwise.
(b) v is still a $\{\Rightarrow, \mathbf{F}\}$-algebra homomorphism.
($\mathrm{c_k}$) Suppose $p = (\forall x)q(x)$ has depth k. Put $V' = V \cup \{t\}$ where $t \notin V$. If for every extension $\varphi' : V' \to U$ of φ and for every $v_{k-1} : P_{k-1}(V, \mathscr{R}) \to \mathbb{Z}_2$, such that $(\varphi', \psi, v'_{k-1})$ satisfy (a), (b) and ($\mathrm{c_i}$), for all $i < k$, we have $v'_{k-1}(q(t)) = 1$, then $v(p) = 1$, otherwise $v(p) = 0$.

The idea is that $\forall x q(x)$ should be true if and only if for every possible interpretation t $q(t)$ is still true at depth $k - 1$, where t is a new variable and thus interpreting t differently in U changes only t.

Problem 1. Given (U, φ, ψ) there is exactly one function $v : P \to \mathbb{Z}_2$ satisfying (a), (b) and ($\mathrm{c_k}$) for all k.

Proof. Let w and v both be functions which satistfy (a), (b) and ($\mathrm{c_k}$) for all k. Let $p \in P$ at depth 0. By (a), $w(p) = v(p)$. Suppose $v = w$ up to depth $k - 1$. Let $p = (\forall x)q(x) \in P$ at depth k.

We have that $v(p) = 1$ only if for all possible interpretations of a new variable t, $v'_{k-1}(q(t)) = 1$ suppose there is an interpretation of t, $\varphi'(t)$ such that

$v'_{k-1}(q(t)) = 0$, then $w'_{k-1}(q(t)) = 0$ by the induction hypothesis $v'_{k-1} = w'_{k-1}$ since they have the same triple (U, φ', ψ), therefore, $v = w$. □

Definition 1. An *interpretation* of $P = P(V, \mathscr{R})$ in the domain U is a quadruple (U, φ, ψ, v) satisfy the conditions (a), (b) and ($\mathrm{c_k}$) for all k.

Problem 2. Let $w(u_1, .., u_n)$ be any tautology of $Prop(\{u_1, ..., u_n\})$. Let $p_1, ..., p_n \in P(V, \mathscr{R})$. Prove that $\vDash w(p_1, ..., p_n)$

Proof. Note that v is required to be a $\{\Rightarrow, \mathbf{F}\}$-algebra homomorphism. Thus $v(w(p_1, ..., p_n)) = w(v(p_1), ..., v(p_n))$ in $\mathbb{Z}_2$. However, regardless of the truth value of $v(p_1), ..., v(p_n)$ in $\mathbb{Z}_2$, $w(v(p_1), ..., v(p_n)) = 1$. □

Problem 3. Let $A \subseteq P(V, \mathscr{R})$ and $p(x) \in A$ for all $x \in V$. Does it follow that $A \vDash (\forall x)p(x)$? No, it does not follow.

Proof. Let $u \in U$ such that $u \notin range(\varphi) \subseteq U$.[i] Let (U, φ, ψ, v) be an interpretation such that $v(q) = 1$ for all $q \in A$. Let φ' be the extension of φ such that $\varphi'(t) = u$. Then we have no knowledge about the value of $v(p(t))$. For contrast, if we take $v(p(x))$, we know that $p(x) \in A$ thus $v(p(x)) = 1$. □

In particular, the statement "For any x in A, $v(p(x)) = 1$" describes quantification over the range of ϕ, a subset of U, while the expression "$v((\forall x)p(x)) = 1$" describes quantification over the entirety of U. The trick of requiring $t \notin V$ is exactly adding a basepoint of V to encode that ϕ is potentially a partial function. [ii]

[i] The text explicitly states "Of course, not every element $u \in U$ need have a name, while some elements u may well have more than one name."

[ii] The category of sets and partial functions is equivalent to the category of sets with specified base point and total functions.