An Algebraic Introduction to Mathematical Logic
Chapter 1 Universal Algebra
Section 1 Introduction
Exercises

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Preliminary Definition of Operation: An n-ary operation on the set A is a function $t : A^n \to A$. The number n is called the arity of t.

Ex. 1.3 A 0-ary operation on a set A is a function from the set A^0 (whose only element is $\emptyset$)

Definition 1.4 A *type* $\mathscr{T}$ is a set T together with a function $ar : T \to \mathbb{N}$. We shall write, $\mathscr{T} = (T, ar)$, or, more simply, abuse notation and denote the type by T. It is also convenient to denote by T_n the set $\{t \in T | ar(t) = n\}$.

Definition 1.5 An *algebra* of type T, or a T-*algebra*, is a set A together with, for each $t \in T$, a function $t_A : A^{ar(t)} \to A$. The elements $t \in T_n$ are called n-ary T-algebra operations.

Definition 1.11 If A is a T-algebra, then a subset $B \subset A$ is called a T-*subalgebra* of A if it forms a T-algebra with operations the restrictions to B of those on A. That is to say, for each n-ary operation $t \in T_n$ on A, restricting the domain of t to just the set B, we have that $t|_B$ is an n-ary operation on B. For all $b_1, ...b_n \in B$, we have $t(b_1, ...b_n)|_B = t(b_1, ...b_n) \in B$

Problem 1 (1.12 a)**.** *A is a T-algebra. Show that $\emptyset$ is a subalgebra if and only if $T_0 = \emptyset$.*

Proof. The empty set is clearly contained in A. For any $n > 0$, $t_n : A^n \to A$ Since no element of A^n could possibly be contained in the empty set, the image of $\emptyset$ under T_n, for all operations of arity $n > 0$, must be the empty set. Thus we have shown for all $T_n|_{\emptyset}(\emptyset) = \emptyset$ where $n > 0$. It remains to show what happens in the case T_0.

If T_0 sends $\emptyset$ to $\emptyset$, then under all operations, T_n, the image of $\emptyset$ is itself. Thus if $T_0 = \emptyset$ then $\emptyset$ is a subalgebra.

Suppose $\emptyset$ constitutes a T-*subalgebra* for the T-algebra A. Then all operations of A send $\emptyset$ to itself. And therefore, the arity 0 operation T_0 sends $\emptyset$ to itself. □

Proposition: Any intersection of subalgebras is a subalgebra. Given any subset $X \subset A$, there is a unique smallest subalgebra containing X-namely, the subalgebra $\bigcap\{U|\ U \textit{ subalgebra of } A \textit{ and } X \subseteq U\}$. we call this the subalgebra generated by X and denote it $< X >_T$, or if there is no risk of confusion $< X >$.

Problem 2 (1.12 b)**.** *Show that for all T, every T-algebra has a unique smallest subalgebra.*

Solution (1.12 b). I claim that the unique smallest subalgebra of any $T\text{-}algebra$ A is the subalgebra generated by $\emptyset$, $< \emptyset >_T$. Let U be any subalgebra of A, $\emptyset \in U$. Therefore,

$< \emptyset >_T =$

$\bigcap\{U|\ U\ is\ a\ subalgebra\ of\ A\ and\ \emptyset \subseteq U\} = \bigcap\{U|\ U\ is\ a\ subalgebra\ of\ A\}$

All subalgebras U of A, appear in this intersection, therefore $< \emptyset >_T \subseteq U$ for all U. Uniqueness follows because $< \emptyset >_T$ is a subalgebra of A, and therefore if V is any other subalgebra with the property of being the smallest, it would have to be contained in $< \emptyset >_T$. So $V \subseteq < \emptyset >_T$ and $< \emptyset >_T \subseteq V$, so $V = < \emptyset >_T$ $\square$

Problem 3 (1.13 a)**.** *Groups may be regarded as the special case of T-algebras where $T = (\{*\}, ar)$ with $ar(*) = 2$, or of T'-algebras where $T' = (\{e, i, *\}, ar)$, $ar(e) = 0$, $ar(i) = 1$, and $ar(*) = 2$.*

Show that every T'-subalgebra of a group is a subgroup but not every non-empty T-subalgebra need be a group.

1.13 a. Let G be a T'-algebra and let H be any subalgebra. Then since H contains the empty set, $T'_n(\emptyset) = T'_n|_H(\emptyset) = e \in H$. So H has an identity. The fact that it is a subalgebra also means that it is closed under i and $*$.

Let g be an element of a group G such that g has infinite order. Letting H be the set of all products of g we see that g is a subalgebra. For any $m, n \geq 1$, we obtain $g^n * g^m = g^{m+n} \in H$. H is easily seen to not include the identity because otherwise $\exists k$ s.t. $g^k = e$. Therefore, H is not a subgroup. $\square$

Problem 4 (1.13 b)**.** *Show that if G is a finite group, then every non-empty T-subalgebra of G is itself a group.*

1.13 b. This is a consequence of Lagrange's Theorem. $\square$