

# An Algebraic Introduction to Mathematical Logic

## Chapter 4 Predicate Calculus

### Section 1 Algebra of Predicates

### Exercises

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The following is the start of the construction of the Algebra of Predicates.

From the text,  $V$  is an infinite set whose elements are called individual variables. There is also a set of relation or predicate symbols  $\mathcal{R}$ , together with an arity function  $ar : \mathcal{R} \rightarrow \mathbb{N}$ . The set of generators used to construct the propositional algebra  $P$  is

$$\{(r, x_1, \dots, x_n) | r \in \mathcal{R}, x_i \in V, \text{ and } ar(r) = n\}$$

Let  $\tilde{P}(V, \mathcal{R})$  be the free algebra on the set above of type  $\{\mathbf{F}, \Rightarrow, (\forall x) | x \in V\}$ , where the arities are as usual, that is,  $\mathbf{F}$  is nullary, and  $\Rightarrow$  is binary. For each  $x \in V$  we have a unary operator  $(\forall x)$ . A similar example can be found in the signature of a vector space over a field  $k$ . Here, for each element of  $k$ , there is a unary operator which is scalar multiplication by that element.

**Definition 1.1** Let  $w \in \tilde{P}(V, \mathcal{R})$  the set of *variables involved in*  $w$ , denoted by  $V(w)$ , is defined by

$$V(w) = \cap \{U | U \subseteq V, w \in \tilde{P}(U, \mathcal{R})\}.$$

**Problem 1** (Exercise 1.2 i)). *Show that  $V(\mathbf{F}) = \emptyset$ .*

*Solution exercise 1.2 i).* Since,  $V(\mathbf{F}) = \cap \{U | U \subseteq V, \mathbf{F} \in \tilde{P}(U, \mathcal{R})\}$  it suffices to show that  $\emptyset$  is in this collection of sets such that  $\emptyset \subseteq V$ , and  $\mathbf{F} \in \tilde{P}(\emptyset, \mathcal{R})$ , because then  $V(\mathbf{F})$  which is the intersection of all such sets, must then be contained in  $\emptyset$ . It is clearly true that  $\emptyset \subseteq V$ . Note by the construction of  $\tilde{P}(\emptyset, \mathcal{R})$  that  $T_0$ , the nullary operations for the type mentioned in the first paragraph, must be elements of  $\tilde{P}(\emptyset, \mathcal{R})$ . That is,  $\tilde{P}(\emptyset, \mathcal{R}) = \cup F_n$  where  $F_0 = T_0 \cup \emptyset$ . Thus since  $\mathbf{F} \in T_0$ ,  $\mathbf{F} \in \tilde{P}(\emptyset, \mathcal{R})$ . It follows that  $V(\mathbf{F}) = \emptyset$ .  $\square$

**Problem 2** (Exercise 1.2 ii)). *Show that if  $r \in \mathcal{R}$ ,  $ar(r) = n$ , and  $x_1, x_2, \dots, x_n \in V$  then  $V(r(x_1, x_2, \dots, x_n)) = \{x_1, x_2, \dots, x_n\}$ .*

*Solution exercise 1.2 ii).* Similarly, if we can show that  $\{x_1, x_2, \dots, x_n\}$  satisfies the properties that  $\{x_1, x_2, \dots, x_n\} \subseteq V$  and  $r(x_1, x_2, \dots, x_n) \in \tilde{P}(\{x_1, x_2, \dots, x_n\}, \mathcal{R})$ , then we will have that  $V(r(x_1, x_2, \dots, x_n)) \subseteq \{x_1, x_2, \dots, x_n\}$ . By definition,

$\tilde{P}(\{x_1, x_2, \dots, x_n\}, \mathcal{R})$  has a generating set of things of the form  $r(x_1, x_2, \dots, x_n)$ , and therefore must contain this element at its nullary level. We initially supposed also that  $x_1, x_2, \dots, x_n \in V$  so  $\{x_1, x_2, \dots, x_n\} \subseteq V$  and both conditions are verified. Therefore it holds that  $V(r(x_1, x_2, \dots, x_n)) \subseteq \{x_1, x_2, \dots, x_n\}$ .

To show the reverse inclusion pick any  $U$  such that  $U \subseteq V$  and  $r(x_1, x_2, \dots, x_n) \in \tilde{P}(U, \mathcal{R})$ . Looking at the form of  $r(x_1, x_2, \dots, x_n)$ , we see that it must be a generator, and therefore,  $U$  must contain  $\{x_1, x_2, \dots, x_n\}$ . Since  $U$  was chosen arbitrarily, the reverse inclusion holds and therefore  $V(r(x_1, x_2, \dots, x_n)) = \{x_1, x_2, \dots, x_n\}$ .  $\square$

**Problem 3** (Exercise 1.2 iii). *Show that if  $w_1, w_2 \in \tilde{P}(V, \mathcal{R})$ , then*

$$V(w_1 \Rightarrow w_2) = V(w_1) \cup V(w_2).$$

*Solution exercise 1.2 iii.* Let  $\bar{U} = V(w_1) \cup V(w_2)$ , then  $w_1 \in \tilde{P}(\bar{U}, \mathcal{R})$  and  $w_2 \in \tilde{P}(\bar{U}, \mathcal{R})$ . Since  $\Rightarrow$  is in the type for both of these free algebras,  $w_1 \Rightarrow w_2 \in \tilde{P}(\bar{U}, \mathcal{R})$ . This implies that  $V(w_1 \Rightarrow w_2) \subseteq \bar{U} = (V(w_1) \cup V(w_2))$ . This holds because we have verified that  $\bar{U}$  is one of the sets which is in the intersection which  $V(w_1 \Rightarrow w_2)$  is defined to be.

To show the reverse inclusion, that is that  $V(w_1) \cup V(w_2) \subseteq V(w_1 \Rightarrow w_2)$ , we need to show that  $V(w_1) \subseteq V(w_1 \Rightarrow w_2)$  and that  $V(w_2) \subseteq V(w_1 \Rightarrow w_2)$ . This intuitively plausible pair of inclusions is verified if we can show without loss of generality that  $w_1 \in \tilde{P}(V(w_1 \Rightarrow w_2), \mathcal{R})$ . Clearly,  $w_1 \Rightarrow w_2 \in \tilde{P}(V(w_1 \Rightarrow w_2), \mathcal{R})$ , and since  $\tilde{P}(V(w_1 \Rightarrow w_2), \mathcal{R})$  is generated freely, then by construction, if  $w_1 \Rightarrow w_2 \in F_n$ , then if  $w_2 \in F_k$  with  $k < n$ ,  $w_1 \in F_{n-k}$  and therefore,  $w_1 \in \tilde{P}(V(w_1 \Rightarrow w_2), \mathcal{R})$ . The symmetry of the argument shows too that  $w_2 \in \tilde{P}(V(w_1 \Rightarrow w_2), \mathcal{R})$ , and therefore,  $V(w_1) \subseteq V(w_1 \Rightarrow w_2)$  and that  $V(w_2) \subseteq V(w_1 \Rightarrow w_2)$ . The reverse inclusion,  $V(w_1) \cup V(w_2) \subseteq V(w_1 \Rightarrow w_2)$  follows and we obtain the final equality.  $\square$

**Problem 4** (Exercise 1.2 iv). *Show that if  $x \in V$ , and  $w \in \tilde{P}(V, \mathcal{R})$ , then  $V((\forall x)(w)) = x \cup V(w)$*

*Solution exercise 1.2 iv.* To show  $V((\forall x)(w)) \subseteq x \cup V(w)$ , we need to show that  $(\forall x)(w) \in \tilde{P}(x \cup V(w), \mathcal{R})$ . Since  $x \in x \cup V(w)$ , the type of  $\tilde{P}(x \cup V(w), \mathcal{R})$  contains the necessary quantifier  $(\forall x)$ . Clearly,  $w \in \tilde{P}(x \cup V(w), \mathcal{R})$ . Since  $\tilde{P}(x \cup V(w), \mathcal{R})$  is freely generated and we have the necessary quantifier, then since for some  $n$   $w \in F_n$ , we then must have that  $(\forall x)(w) \in F_{n+1}$ . Therefore  $(\forall x)(w) \in \tilde{P}(x \cup V(w), \mathcal{R})$ , and the inclusion  $V((\forall x)(w)) \subseteq x \cup V(w)$  holds. The nice part about the free construction is that we need not actually define what the value of  $(\forall x)$  is at  $w$ .

To show the reverse inclusion, that is  $x \cup V(w) \subseteq V((\forall x)(w))$ , it is a consequence of part iii above, that it suffices to show that  $x \Rightarrow w \in \tilde{P}(V((\forall x)(w)), \mathcal{R})$ . Nonsensical as it sounds, but nonetheless by definition, the existence of the quantifier  $(\forall x)$ , only occurs when the underlying set contains  $x$ . Thus  $x \in \tilde{P}(V((\forall x)(w)), \mathcal{R})$ . Also  $w \in \tilde{P}(V((\forall x)(w)), \mathcal{R})$  because  $(\forall x)(w) \in \tilde{P}(V((\forall x)(w)), \mathcal{R})$ .

at  $F_{n+1}$ , so  $w \in F_n$  and therefore, in  $w \in \tilde{P}(V((\forall x)(w)), \mathcal{R})$ . Since  $\Rightarrow$  is present in the type of  $\tilde{P}(V((\forall x)(w)), \mathcal{R})$  we have that  $x \Rightarrow w \in \tilde{P}(V((\forall x)(w)), \mathcal{R})$  and therefore that  $x \cup V(w) \subseteq V((\forall x)(w))$ . The result follows.  $\square$

**Problem 5** (Exercise 1.2 v). *Show further that (i)-(iv) may be taken as the definition of the function  $V(w)$ .*

*Solution exercise 1.2 v).* Define a function  $V : \tilde{P}(V, \mathcal{R}) \rightarrow V$  by properties (i)-(iv). We verify that it is the same as defining the function  $V$  using **Definition 1.1** by examining what it does to each level of the recursive construction of  $\tilde{P}(V, \mathcal{R})$

The first level,  $F_0 = T_0 \cup \{(r, x_1, \dots, x_n) \mid r \in \mathcal{R}, x_i \in V, \text{ and } ar(r) = n\}$ , that is, the union of the nullary operation **F** and the generators. The proof of part (i) shows that the functions agree on **F**. The proof of part (ii) shows that the functions agree on the generators. Parts (iii) and (iv) show, that if the functions agree on all elements at level  $F_n$ , then the use of additional operators respectively,  $\Rightarrow$  and  $(\forall x)$ , from  $T_{k=1,2}$ , which then propel the elements of type  $F_n$  into type  $F_{n+1}$ , still agree on the next level. This completes an inductive argument that these ways of defining  $V$  are the same on all of  $\tilde{P}(V, \mathcal{R})$ .  $\square$

**Definition 1.3** Let  $w \in \tilde{P}(V, \mathcal{R})$  the *depth of quantification* of  $w$ , denoted by  $d(w)$ , is defined by

1.  $d(\mathbf{F}) = 0$ ,  $d(r(x_1, x_2, \dots, x_n)) = 0$  for every free generator of  $\tilde{P}(V, \mathcal{R})$ .  
Note, since  $F_0$  in the construction of  $\tilde{P}(V, \mathcal{R})$  is the union of  $T_0 = \{\mathbf{F}\}$  and the generators which are of the form  $r(x_1, x_2, \dots, x_n)$  for  $r \in \mathcal{R}$  and  $x_1, x_2, \dots, x_n \in V$ , then we can say for all  $f \in F_0$ ,  $d(f) = 0$ .
2.  $d(w_1 \Rightarrow w_2) = \max(d(w_1), d(w_2))$
3.  $d((\forall x)(w)) = 1 + d(w)$  for  $x \in V$

Our desired congruence relation on  $\tilde{P}(V, \mathcal{R})$  may now be defined.

**Definition 1.4** Let  $w_1, w_2 \in \tilde{P}(V, \mathcal{R})$  define  $w_1 \approx w_2$  if

1.  $d(w_1) = d(w_2) = 0$  and  $w_1 = w_2$ . Generating the free proposition algebra on  $F_0$  that is, all generators and **F**, that is  $F_0$  are of depth 0. By item 2 of the previous definition the use of  $\Rightarrow$  does not increase the depth of quantification. Thus the equivalence relation on the proposition algebra contained in the predicate algebra  $\tilde{P}(V, \mathcal{R})$  is just equality.
2.  $d(w_1) = d(w_2) > 0$ ,  $w_1 = a_1 \Rightarrow b_1$ ,  $w_2 = a_2 \Rightarrow b_2$ ,  $a_1 \approx a_2$  and  $b_1 \approx b_2$  or
3.  $w_1 = (\forall x)(a)$ ,  $w_2 = (\forall y)b$ , and either
  - (a)  $x = y$  and  $a \approx b$  (actually up to this point, this means iff they are equal)

- (b) There exists a  $c = c(x)$  such that  $c(x) \approx a$  and  $c(y) \sim b$  and  $y \notin V(c)$

The following is my interpretation of the paragraph directly after this definition in the text. The way that  $c$  is defined is as a word in the generators of the form  $r(x_1, x_2, \dots, x_n)$  as discussed. Then  $c$  can be thought of, (just like any word) as a function of the variables that are contained in its generators. So for example, if  $c = r_1(x_1, x_2) \Rightarrow r_2(x_1, x_2, x_3)$  then we can say that  $c = c(x_1, x_2, x_3)$ . Ignoring the dependence on  $x_2$  and  $x_3$  we could further say  $c = c(x_1)$ . We would like it to be true that  $(\forall x_1)c(x_1) \approx (\forall y_1)c(y_1)$