An Algebraic Introduction to Mathematical Logic
Chapter 4 Predicate Calculus
Section 1 Algebra of Predicates
Proof of Transitivity

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Definition 1.4 Let w_1, $w_2 \in \widetilde{P}(V, \mathscr{R})$ define $w_1 \approx w_2$ if

1. $d(w_1) = d(w2) = 0$ and $w_1 = w_2$. Generating the free proposition algebra on F_0 that is, all generators and **F**, that is F_0 are of depth 0. The use of $\Rightarrow$ does not increase the depth of quantification. Thus the equivalence relation on the proposition algebra contained in the predicate algebra $\widetilde{P}(V, \mathscr{R})$ is just equality.
2. $d(w_1) = d(w_2) > 0$, $w_1 = a_1 \Rightarrow b_1$, $w_2 = a_2 \Rightarrow b_2$, $a_1 \approx a_2$ and $b_1 \approx b_2$ or
3. $w_1 = (\forall x)(a)$, $w_2 = (\forall y)b$, and either
   (a) $x = y$ and $a \approx b$ (actually up to this point, this means iff they are equal)
   (b) There exists a $c = c(x)$ such that $c(x) \approx a$ and $c(y) \approx b$ and $y \notin V(c)$

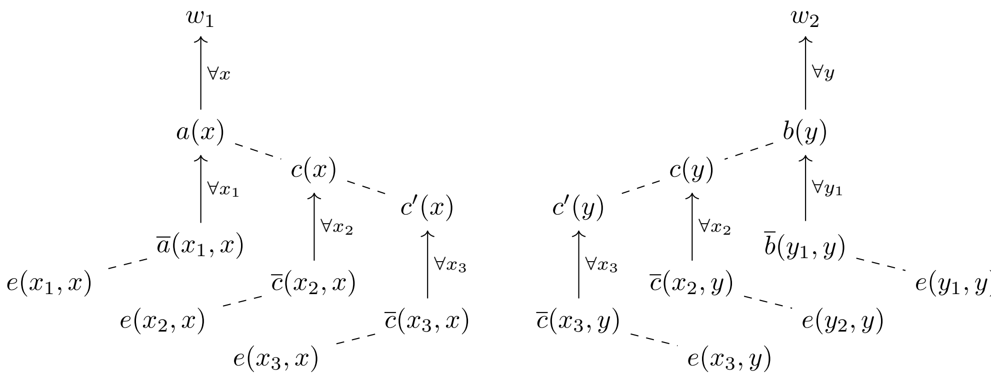
Problem 1 (Exercise 1.5 i))**.** *Given that $z \notin V(w_1) \cup V(w_2)$ show by induction over $d(w_1)$ that the element $c = c(x)$ in **Definition 1.4** 3(b) can always be chosen such that $z \notin V(c)$.*

Proof. The base case holds because the relation is equality for depth 0.

Assuming the notation of **Definition 1.4** 3(b), we have $c(x) \approx a$ and $c(y) \approx b$. Assume again that $c(x) \approx a$ because there exists some $e(x_1)$ such that $e(x_1) \approx \overline{a}(x_1)$ and $e(x_2) \approx \overline{c}(x_2)$ where $a(x) = \forall x_1 \overline{a}(x_1, x)$ and $c(x) = \forall x_2 \overline{c}(x_2, x)$ and $x_2 \notin V(e(x_1))$. Now pick any variable $z' \notin V(w_1) \cup V(w_2) \cup V(c(x))$.

By the induction hypothesis, $e(x_1)$ can be chosen so that it does not contain the variable z'. Thus we can substitute any mentions of the variable z in $e(x_1)$ with z' which we have ensured is not already present without disrupting the equivalence $c(x) \approx a$.

Suppose however, that the variable $x_2 = z$, then we may rechose x_2 which is quantified over when it appears in $c(x)$ and $c(y)$. That is, $c(x) = \forall x_2 c(x_2, x)$ so let $c'(x) = \forall x \overline{c}(x_3, x)$. The important thing is that c' still works for $w_1 \approx w_2$. By construction, it remains true that $a \approx c'(x)$ and $b \approx c'(y)$. See the figure below



Therefore, $z \notin V(e(x_3))$, morover, $z \neq x_3$. In order to show that $z \notin c'(x)$ all that remains is to show that $z \notin V(\overline{c}(x_3)) - x_3$. Notice that we have just reduced the problem by one degree of quantification. Since each expression has finite depth of quantification this process will terminate eventually when $e_n(x_n) \approx \overline{c}^{(n)}(x_n)$ is strict equality and therefore, if $z \notin V(e(x_3))$ then $z \notin V(\overline{c}^{(n)}(x_n))$. □

Problem 2 (Exercise 1.5 ii)). *If $u(x) \approx v(x)$ and $y \notin V(u(x)) \cup V(v(x))$, show by induction over $d(u(x))$ that $u(y) \approx v(y)$.*

Proof. The base case holds because the relation is equality for depth 0.

Suppose the result holds for depth of quantification less than n. Then let $d(u(x)) = d(v(x)) = n$. Let $u(x) = \forall x_1 u_1(x, x_1)$ and $v(x) = \forall x_2 v_1(x, x_2)$. Then there exists a $c(x, x_1)$ such that $u_1(x, x_1) \approx c(x, x_1)$ and $c(x, x_2) \approx v_1(x, x_2)$ such that $x_2 \notin V(c)$. By **Problem 1** we have that $y \notin V(c)$.

By the induction hypothesis, since the depth of u_1 v_1 and c are all $n-1$ we have that $u_1(y, x_1) \approx c(y, x_1)$ and $c(y, x_2) \approx v_1(y, x_2)$. Then by definition, since $u(y) = \forall x_1 u_1(y, x_1)$ and $v(y) = \forall x_2 v_1(y, x_2)$, we have proven $u(y) \approx v(y)$. □

Problem 3 (Exercise 1.5 iii)). *Show that the relation is transitive.*

Proof. Let $w_1 \approx w_2$ and $w_2 \approx w_3$. Assume the relation is transitive for lower depths of quantification. Since the relation is defined to be strict equivalence at depth 0, the base case holds.

The proof is given in the worst possible case: We begin by twice invoking **Definition 1.4** 3(b).

Let $w_1 = \forall x a_1(x)$, $w_2 = \forall y a_2(y)$, and $w_3 = \forall z a_3(z)$. We have that there exists $c_1(x)$ and $c_2(y)$, such that $y \notin c_1(x)$ and $z \notin c_2(y)$. Also, because of **Problem 1** we may require that $z \notin c_1(x)$ and $x \notin c_2(y)$. By part 3(b), of the definition $w_1 \approx w_2$ is defined to mean, $a_1(x) \approx c_1(x)$ and $c_1(y) \approx a_2(y)$.

Likewise, $w_2 \approx w_3$ means that $a_2(y) \approx c_2(y)$ and $c_2(z) \approx a_3(z)$.

By the induction hypothesis, since $c_1(y) \approx a_2(y)$ and $a_2(y) \approx c_2(y)$, we have $c_1(y) \approx c_2(y)$. Note also that neither $c_1(y)$ nor $c_2(y)$ contains either x or z.

By **Problem 2** we have that $c_1(x) \approx c_2(x)$. Then applying the induction hypothesis again, we obtain from $a_1(x) \approx c_1(x)$ and $c_1(x) \approx c_2(x)$, that $a_1(x) \approx c_2(x)$. Note also that since $c_2(z) \approx a_3(z)$, we have bridged the gap and it follows that $w_1 \approx w_3$.

Other cases follow from just **Problem 1** and the induction hypothesis alone.

$\square$