An Algebraic Introduction to Mathematical Logic
Chapter 4 Predicate Calculus
Section 4 Properties of $Pred(V, \mathscr{R})$

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Some informal comments

Recall that the logic called the First Order Predicate Calculus is comprised of an infinite set of variables V, a set of relation symbols $\mathscr{R}$, which together yeild a set of generators of the form $r(x_1, ..., x_n)$, where $r \in \mathscr{R}$ and each $x_i \in V$ of a free $\{\mathbf{F}, \Rightarrow, (\forall x) | x \in V\}$-algebra, called $\widetilde{P}(V, \mathscr{R})$. We call the factor algebra by the equivalence relation detailed in Chapter 4 Section 1 $P(V, \mathscr{R})$. When we equip $P(V, \mathscr{R})$ with semantics and syntax constructed in the previous two sections and denote the logic $Pred(V, \mathscr{R})$.

Definition 1. A predicate p is called a sentence if $var(p) = \emptyset$, that is, p has no unbound variables. Otherwise we call p a fragment.

Note that this corresponds to our intuitive notion. Not that the following is a sentence: "For all deltas, there exists an epsilon." This is an honest to god sentence since both variables delta and epsilon are quantified over. The following is not strictly: "For all clocks, rabbit." We can fix this as follows: "For all clocks, a rabbit is late." The word "a" now plays the role of the the existencial quantifier.

What is the proper notion of morphism for $\{\mathbf{F}, \Rightarrow, (\forall x) | x \in V\}$-algebras? The reliance of the quantifiers on the set of variables V implies that the variables of the two logics must be the same.

It turns out that homomorphism is too strict to be of use. Let ϕ be a homomorphism between two such algebras, P_1, and P_2. Let $p(x), q(y) \in P_1$, then

$$\phi((\forall x)p(x) \Rightarrow q(x)) = \phi((\forall x)p(x)) \Rightarrow \phi(q(x)) = ((\forall x)\phi(p(x))) \Rightarrow \phi(q(x)).$$

By definition, ϕ may not interact with quantifiers, $\mathbf{F}$, or $\Rightarrow$. Thus the only parts of the predicate that it can interact with are variables (bound or unbound) and relation symbols.

On a high level, we would like morphisms to preserve the meaning of the predicate. Barnes and Mack have little to say on how relation symbols change.

I think there are probably some easy contradictions that one runs into if we allow homomorphisms to change the arity of a relation symbol.

1. Can $\{\mathbf{F}, \Rightarrow, (\forall x) | x \in V\}$-algebra homomorphisms alter relation symbols at all?

2. Must $\{\mathbf{F}, \Rightarrow, (\forall x) | x \in V\}$-algebra homomorphisms preserve the arity of the relation symbol?

Barnes and Mack have more to say regarding the behavior of homomorphisms with respect to variables. To be even close to interesting, homomorphisms must be able to interchange variables. Suppose ϕ sends x to y and fixes all relation symbols. Then if x is bound in a predicate p, we run into issues. Computing,

$$\phi((\forall x)p(x)) = (\forall x)\phi(p(x)) = (\forall x)p(y),$$

we see that the homomorphism sends a sentence to a fragment.

Barnes and Mack give the following definition which allows for greater flexibility.

Properties of $Pred(V, \mathscr{R})$

Definition 2. Let $P_1 = P(V_1, \mathscr{R}^{(1)})$ and $P_2 = P(V_2, \mathscr{R}^{(2)})$. A *semi-homomorphism* $(\alpha, \beta) : (P_1, V_1) \to (P_2, V_2)$ is a pair of maps $\alpha : P_1 \to P_2$, $\beta : V_1 \to V_2$ such that

1. $\beta(V_1)$ is infinite.

2. α is an $\{F, \Rightarrow\}$-algebra homomorphism, and

3. $\alpha((\forall x)p) = (\forall x')\alpha(p)$ where $x' = \beta(x)$.

Lemma 1. *$var((\forall x)p) = var(p)$ iff $x \notin var(p)$.*

Proof. Suppose that $x \in var(p)$ then $var((\forall x)p) \neq var(p)$ since $x \notin var((\forall x)p)$ but $x \in var(p)$. Suppose now that $x \notin var(p)$. Then $var(p) - \{x\} = var(p)$. Since $var((\forall x)p) = var(p) - \{x\}$ the result follows. □

Lemma 2. *Suppose that $x \neq y$, then $(\forall x)p = (\forall y)p$ iff $x, y \notin var(p)$.*

Proof. One direction follows immediately from lemma 1. The other is as follows which we prove using the contrapositive. Suppose $x, y \in var(p)$ then $var((\forall x)p) \neq var((\forall y)p)$ because $y \in var((\forall x)p)$ but $y \notin var((\forall y)p)$ and thus $(\forall x)p \neq (\forall y)p$. □

Lemma 3. *Let $(\alpha, \beta) : (P_1, V_1) \to (P_2, V_2)$ be a semi-homomorphism. Let $p \in P_1$ and suppose that $x \in V_1 - var(p)$. Then $\beta(x) \notin var(\alpha(p))$.*

Proof. Since $\beta(V_1)$ is infinite there exists a $y' \in \beta(V_1)$ such that $y' \notin \beta(var(p))$ and $y' \neq \beta(x)$. Since $y' \in \beta(V_1)$ there exists a $y \in V_1$ such that $\beta(y) = y'$. Additionally choose y' so that $y \in V_1 - var(p)$. It follows from lemma 2 that $(\forall x)p = (\forall y)p$. Then we may compute

$$(\forall \beta(x))p = \alpha((\forall x)p) = \alpha((\forall y)p) = (\forall \beta(y))p$$

From which we may conclude by lemma 3 that $\beta(x) \notin var(\alpha(p))$. □

Theorem 1. *(The substitution theorem) Let* $(\alpha, \beta) : (P_1, V_1) \to (P_2, V_2)$ *be a semi-homomorphism. Let* $A \subseteq P_1$[i], $p \in P_1$. *Then*

(a) If $A \vdash p$, *then* $\alpha(A) \vdash \alpha(p)$.

(b) If $A \models p$, *then* $\alpha(A) \models \alpha(p)$.

Proof. $\square$

[i] the text has just $A \subseteq P$ which must be a typo.