An Algebraic Introduction to Mathematical Logic Chapter 4 Predicate Calculus Section 1 Algebra of Predicates Exercises

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The following is the start of the construction of the Algebra of Predicates.

From the text, V is an infinite set who's elements are called individual variables. There is also a set of relation or predicate symbols $\mathscr{R}$, together with an arity function $ar : \mathscr{R} \to \mathbb{N}$. The set of generators used to construct the propositional algebra P is

$$\{(r, x_1, ...x_n)|r \in \mathscr{R}, x_i \in V, \text{ and } ar(r) = n\}$$

Let $\widetilde{P}(V, \mathscr{R})$ be the free algebra on the set above of type $\{\mathbf{F}, \Rightarrow, (\forall x)|x \in V\}$, where the arities are as usual, that is, $\mathbf{F}$ is nullary, and $\Rightarrow$ is binary. For each $x \in V$ we have a unary operator $(\forall x)$. A similar example can be found in the signature of a vector space over a field k. Here, for each element of k, there is a unary operator which is scalar multiplication by that element.

Definition 1.1 Let $w \in \widetilde{P}(V, \mathscr{R})$ the set of *variables involved in* w, denoted by $V(w)$, is defined by

$$V(w) = \cap\{U|U \subseteq V, w \in \widetilde{P}(U, \mathscr{R})\}.$$

Problem 1 (Exercise 1.2 i))**.** *Show that* $V(\boldsymbol{F}) = \emptyset$.

Solution exercise 1.2 i). Since, $V(\mathbf{F}) = \cap\{U|U \subseteq V, \mathbf{F} \in \widetilde{P}(U, \mathscr{R})\}$ it suffices to show that $\emptyset$ is in this collection of sets such that $\emptyset \subseteq V$, and $\mathbf{F} \in \widetilde{P}(\emptyset, \mathscr{R})$, because then $V(F)$ which is the intersection of all such sets, must then be contained in $\emptyset$. It is clearly true that $\emptyset \subseteq V$. Note by the construction of $\widetilde{P}(\emptyset, \mathscr{R})$ that T_0, the nullary operations for the type mentioned in the first paragraph, must be elements of $\widetilde{P}(\emptyset, \mathscr{R})$. That is, $\widetilde{P}(\emptyset, \mathscr{R}) = \cup F_n$ where $F_0 = T_0 \cup \emptyset$. Thus since $\mathbf{F} \in T_0$, $\mathbf{F} \in \widetilde{P}(\emptyset, \mathscr{R})$. It follows that $V(\mathbf{F}) = \emptyset$. □

Problem 2 (Exercise 1.2 ii))**.** *Show that if* $r \in \mathscr{R}$, $ar(r) = n$, *and* $x_1, x_2, ..., x_n \in V$ *then* $V(r(x_1, x_2, ..., x_n)) = \{x_1, x_2, ..., x_n\}$.

Solution exercise 1.2 ii). Similarly, if we can show that $\{x_1, x_2, ..., x_n\}$ satisfies the properties that $\{x_1, x_2, ..., x_n\} \subseteq V$ and $r(x_1, x_2, ..., x_n) \in \widetilde{P}(\{x_1, x_2, ..., x_n\}, \mathscr{R})$, then we will have that $V(r(x_1, x_2, ..., x_n)) \subseteq \{x_1, x_2, ..., x_n\}$. By definition,

$\widetilde{P}(\{x_1, x_2, ..., x_n\}, \mathscr{R})$ has a generating set of things of the form $r(x_1, x_2, ..., x_n)$, and therefore must contain this element at it's nullary level. We initially supposed also that $x_1, x_2, ..., x_n \in V$ so $\{x_1, x_2, ..., x_n\} \subseteq V$ and both conditions are verified. Therefore it holds that $V(r(x_1, x_2, ..., x_n)) \subseteq \{x_1, x_2, ..., x_n\}$.

To show the reverse inclusion pick any U such that $U \subseteq V$ and $r(x_1, x_2, ..., x_n) \in \widetilde{P}(U, \mathscr{R})$. Looking at the form of $r(x_1, x_2, ..., x_n)$, we see that it must be a generator, and therefore, U must contain $\{x_1, x_2, ..., x_n\}$. Since U was chosen arbitrarily, the reverse inclusion holds and therefore $V(r(x_1, x_2, ..., x_n)) = \{x_1, x_2, ..., x_n\}$. $\square$

Problem 3 (Exercise 1.2 iii))**.** *Show that if $w_1, w_2 \in \widetilde{P}(V, \mathscr{R})$, then*

$$V(w_1 \Rightarrow w_2) = V(w_1) \cup V(w_2).$$

Solution exercise 1.2 iii). Let $\overline{U} = V(w_1) \cup V(w_2)$, then $w_1 \in \widetilde{P}(\overline{U}, \mathscr{R})$ and $w_2 \in \widetilde{P}(\overline{U}, \mathscr{R})$. Since $\Rightarrow$ is in the type for both of these free algebras, $w_1 \Rightarrow w_2 \in \widetilde{P}(\overline{U}, \mathscr{R})$. This implies that $V(w_1 \Rightarrow w_2) \subseteq U = (V(w_1) \cup V(w_2))$. This holds because we have verified that $\overline{U}$ is one of the sets which is in the intersection which $V(w_1 \Rightarrow w_2)$ is defined to be.

To show the reverse inclusion, that is that $V(w_1) \cup V(w_2) \subseteq V(w_1 \Rightarrow w_2)$, we need to show that $V(w_1) \subseteq V(w_1 \Rightarrow w_2)$ and that $V(w_2) \subseteq V(w_1 \Rightarrow w_2)$. This intuitively plausible pair of inclusions is verified if we can show without loss of generality that $w_1 \in \widetilde{p}(V(w_1 \Rightarrow w_2), \mathscr{R})$. Clearly, $w_1 \Rightarrow w_2 \in \widetilde{p}(V(w_1 \Rightarrow w_2), \mathscr{R})$, and since $\widetilde{p}(V(w_1 \Rightarrow w_2), \mathscr{R})$ is generated freely, then by construction, if $w_1 \Rightarrow w_2 \in F_n$, then if $w_2 \in F_k$ with $k < n$, $w_1 \in F_{n-k}$ and therefore, $w_1 \in \widetilde{p}(V(w_1 \Rightarrow w_2), \mathscr{R})$. The symmetry of the argument shows too that $w_2 \in \widetilde{p}(V(w_1 \Rightarrow w_2), \mathscr{R})$, and therefore, $V(w_1) \subseteq V(w_1 \Rightarrow w_2)$ and that $V(w_2) \subseteq V(w_1 \Rightarrow w_2)$. The reverse inclusion, $V(w_1) \cup V(w_2) \subseteq V(w_1 \Rightarrow w_2)$ follows and we obtain the final equality. $\square$

Problem 4 (Exercise 1.2 iv))**.** *Show that if $x \in V$, and $w \in \widetilde{P}(V, \mathscr{R})$, then $V((\forall x)(w)) = x \cup V(w)$*

Solution exercise 1.2 iv). To show $V((\forall x)(w)) \subseteq x \cup V(w)$, we need to show that $(\forall x)(w) \in \widetilde{P}(x \cup V(w), \mathscr{R})$. Since $x \in x \cup V(w)$, the type of $\widetilde{P}(x \cup V(w), \mathscr{R})$ contains the neccesary quantifier $(\forall x)$. Clearly, $w \in \widetilde{P}(x \cup V(w), \mathscr{R})$. Since $\widetilde{P}(x \cup V(w), \mathscr{R})$ is freely generated and we have the necesary quantifier, then since for some n $w \in F_n$, we then must have that $(\forall x)(w) \in F_{n+1}$. Therefore $(\forall x)(w) \in \widetilde{P}(x \cup V(w), \mathscr{R})$, and the inclusion $V((\forall x)(w)) \subseteq x \cup V(w)$ holds. The nice part about the free construction is that we need not actually define what the value of $(\forall x)$ is at w.

To show the reverse inclusion, that is $x \cup V(w) \subseteq V((\forall x)(w))$, it is a consequence of part *iii* above, that it suffices to show that $x \Rightarrow w \in \widetilde{P}(V((\forall x)(w)), \mathscr{R})$. Nonsensical as it sounds, but nonetheless by definition, the existence of the quantifier $(\forall x)$, only occurs when the underlying set contains x. Thus $x \in \widetilde{P}(V((\forall x)(w)), \mathscr{R})$. Also $w \in \widetilde{P}(V((\forall x)(w)), \mathscr{R})$ because $(\forall x)(w) \in \widetilde{P}(V((\forall x)(w)), \mathscr{R})$

at F_{n+1}, so $w \in F_n$ and therefore, in $w \in \widetilde{P}(V((\forall x)(w)), \mathscr{R})$. Since $\Rightarrow$ is present in the type of $\widetilde{P}(V((\forall x)(w)), \mathscr{R})$ we have that $x \Rightarrow w \in \widetilde{P}(V((\forall x)(w)), \mathscr{R})$ and therefore that $x \cup V(w) \subseteq V((\forall x)(w))$. The result follows. $\square$

Problem 5 (Exercise 1.2 v))**.** *Show further that (i)-(iv) may be taken as the definition of the function $V(w)$.*

Solution exercise 1.2 v). Define a function $V : \widetilde{P}(V, \mathscr{R}) \to V$ by properties (i)-(iv). We verify that it is the same as defining the function V using **Definition 1.1** by examining what it does to each level of the recursive construction of $\widetilde{P}(V, \mathscr{R})$

The first level, $F_0 = T_0 \cup \{(r, x_1, ...x_n) | r \in \mathscr{R}, x_i \in V, \text{ and } ar(r) = n\}$, that is, the union of the nullary operation $\mathbf{F}$ and the generators. The proof of part (i) shows that the functions agree on $\mathbf{F}$. The proof of part (ii) shows that the functions agree on the generators. Parts (iii) and (iv) show, that if the functions agree on all elements at level F_n, then the use of additional operators respectively, $\Rightarrow$ and $(\forall x)$, from $T_{k=1,2}$, which then propel the elements of type F_n into type F_{n+1}, still agree on the next level. This completes an inductive argument that these ways of defining V are the same on all of $\widetilde{P}(V, \mathscr{R})$. $\square$

Definition 1.3 Let $w \in \widetilde{P}(V, \mathscr{R})$ the *depth of quantification* of w, denoted by $d(w)$, is defined by

1. $d(\mathbf{F}) = 0$, $d(r(x_1, x_2, ..., x_n)) = 0$ for every free generator of $\widetilde{P}(V, \mathscr{R})$. Note, since F_0 in the construction of $\widetilde{P}(V, \mathscr{R})$ is the union of $T_0 = \{\mathbf{F}\}$ and the generators which are of the form $r(x_1, x_2, ..., x_n)$ for $r \in \mathscr{R}$ and $x_1, x_2, ..., x_n \in V$, then we can say for all $f \in F_0$, $d(f) = 0$.
2. $d(w_1 \Rightarrow w_2) = max(d(w_1), d(w_2))$
3. $d((\forall x)(w)) = 1 + d(w)$ for $x \in V$

Our desired congruence relation on $\widetilde{P}(V, \mathscr{R})$ may now be defined.

Definition 1.4 Let w_1, $w_2 \in \widetilde{P}(V, \mathscr{R})$ define $w_1 \approx w_2$ if

1. $d(w_1) = d(w2) = 0$ and $w_1 = w_2$. Generating the free proposition algebra on F_0 that is, all generators and $\mathbf{F}$, that is F_0 are of depth 0. By item 2 of the previous definition the use of $\Rightarrow$ does not increase the depth of quantification. Thus the equivalence relation on the proposition algebra contained in the predicate algebra $\widetilde{P}(V, \mathscr{R})$ is just equality.
2. $d(w_1) = d(w_2) > 0$, $w_1 = a_1 \Rightarrow b_1$, $w_2 = a_2 \Rightarrow b_2$, $a_1 \approx a_2$ and $b_1 \approx b_2$ or
3. $w_1 = (\forall x)(a)$, $w_2 = (\forall y)b$, and either
   (a) $x = y$ and $a \approx b$ (actually up to this point, this means iff they are equal)

(b) There exists a $c = c(x)$ such that $c(x) \approx a$ and $c(y) \approx b$ and $y \notin V(c)$

The following is my interpretation of the paragraph directly after this definition in the text. The way that c is defined is as a word in the generators of the form $r(x_1, x_2, ..., x_n)$ as discussed. Then c can be though of, (just like any word) as a function of the variables that are contained in its generators. So for example, if $c = r_1(x_1, x_2) \Rightarrow r_2(x_1, x_2, x_3)$ then we can say that $c = c(x_1, x_2, x_3)$. Ignoring the dependence on x_2 and x_3 we could further say $c = c(x_1)$. We would like it to be true that

$$(\forall x_1)c(x_1) \approx (\forall y_1)c(y_1) = (\forall y_1)(r_1(y_1, x_2) \Rightarrow r_2(y_1, x_2, x_3)).$$

For this reason to make this the case all that we need is to be able to say that $y_1 \notin V(c)$. The reason that the opposite condition $x_1 \notin V(c(y_1))$ holds is that by definition $V(c(y_1))$ has every instance of x_1 already replaced.

Problem 6 (Exercise 1.5 i))**.** *Given that $z \notin V(w_1) \cup V(w_2)$ show by induction over $d(w_1)$ that the element $c = c(x)$ in **Definition 1.4** 3(b) can always be chosen such that $z \notin V(c)$.*

Solution exercise 1.5 i). Implicit in the statement of the question is that $w_1 \approx w_2$ for the reason enumerated in **Definition 1.4** 3(b). That is, $w_1 = (\forall x)a$ and $w_2 = (\forall y)b$ and $\exists c = c(x)$ such that $c(x) \approx a$ and $c(y) \approx b$ (satisfying the condition $y \notin V(c)$).

Let us show now that in the case where $d(w_1) = 1$ the result holds. If $d(w_1) = 1$, then $d(a) = 0$. This implies that congruence with a must be strict equality. If $c(x) \approx a$, then indeed $c = c(x) = a$. This implies that $d(c) = 0$ we now have that $c(y) \approx b$, which implies that $c(y) = b$ and then $d(w_2) = 1$. (I realize that this is somewhat orthogonal to the proof of the base case but thinking about this is helping me get towards the proof.)

In fact we can use a double application of part 4 of **Exercise 1.2** to obtain that $z \notin V(w_1) \cup V(w_2)$ is equivalent to the statement

$$z \notin \{x\} \cup V(a) \cup \{y\} \cup V(b)$$

Clearly then $z \notin V(a)$ and since $c = c(x) = a$, and because V is a function, $z \notin V(c)$.

Now lets prove the inductive step. Assume true for all *depths of quantification* n or below that the result holds. Now for $w_1 \approx w_2$, we are given $z \notin V(w_1) \cup V(w_2)$, where $d(w_1) = n + 1$. Here we again omit a minor point, if the last operand in the words w_1 or w_2 is $\Rightarrow$, part 2 of **Definition 1.4** may be used to peel back the $\Rightarrow$. So we may assume that (as implicit in the statment of the problem) the condition of **Definition 1.4** 3(b) holds. That is to say that $w_1 = (\forall x)a$ and $w_2 = (\forall y)b$ and $\exists c = c(x)$ such that $c(x) \approx a$ and $c(y) \approx b$ (satisfying the condition $y \notin V(c)$). We wish to show that $z \notin c$. Note now that $d(a) = n$ and the result holds for a and b because if $z \notin V(w_1) \cup V(w_2)$, then

$z \notin \{x\} \cup V(a) \cup \{y\} \cup V(b)$ and therefore, $z \notin \{x\}$ or $V(a)$ or $\{y\}$ or $V(b)$. We may then apply **Definition 1.4** $3(b)$ to a and b playing the roles of w_1 and w_2 with the additional benifit that the newly existing c', does not contain z.

I will spell this out. Recall, $w_1 = (\forall x)a$ and $d(a) = n$. Similarly, $w_2 = (\forall y)b$. It follows from the above comments and (perhaps an application of part 2 of **Definition 1.4** that $a = (\forall x)\alpha$ and $b = (\forall y)\beta$, and by part 3 (b) of **Definition 1.4**, there exists a $c' \approx \alpha$ and $c'(y) \approx \beta$ such that $y \notin v(c')$ and by hypothesis that $z \notin v(c')$.

We have that by part 3 (a) of **Definition 1.4** that since $c' \approx \alpha$, $(\forall x)c' \approx (\forall x)\alpha = a$ and since $c'(y) \approx \beta$, $(\forall y)c'(y) \approx (\forall y)\beta = b$. If we define the c from before to be $(\forall x)c'$, we are finished. Using part 4 of **Exercise 1.2** and the fact that $z \notin \{x\}$, $z \notin (\forall y)c'(y)$ as desired. We have verified also that $(\forall x)c'(x) \approx a$ and $(\forall y)c'(y) \approx b$, and $y \notin V((\forall x)c'(x)) = \{x\} \cup V(c')$.

$\square$