An Algebraic Introduction to Mathematical Logic Chapter 4 Predicate Calculus Section 1 Algebra of Predicates Exercises

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The following is the construction of the Algebra of Predicates.

From the text, V is an infinite set who's elements are called individual variables. There is also a set of relation or predicate symbols $\mathscr{R}$, together with an arity function $ar : \mathscr{R} \to \mathbb{N}$. The set of generators used to construct the propositional algebra P is

$$\{(r, x_1, ...x_n) | r \in \mathscr{R}, x_i \in V, \text{ and } ar(r) = n\}$$

Let $\widetilde{P}(V, \mathscr{R})$ be the free algebra on the set above of type $\{\mathbf{F}, \Rightarrow, (\forall x) | x \in V\}$, where the arities are as usual, that is, $\mathbf{F}$ is nullary, and $\Rightarrow$ is binary. For each $x \in V$ we have a unary operator $(\forall x)$. A similar example can be found in the signature of a vector space over a feild k. Here, for each element of k, there is a unary operator which is scalar multiplication by that element.

Definition 1.1 Let $w \in \widetilde{P}(V, \mathscr{R})$ the set of *variables involved in* w, denoted by $V(w)$, is defined by

$$V(w) = \cap\{U | U \subseteq V, w \in \widetilde{P}(U, \mathscr{R})\}.$$

Problem 1 (Exercise 1.2 i))**.** *Show that* $V(\boldsymbol{F}) = \emptyset$.

Exercise 1.2 i). Since, $V(\mathbf{F}) = \cap\{U | U \subseteq V, \mathbf{F} \in \widetilde{P}(U, \mathscr{R})\}$ it suffices to show that $\emptyset$ is in this collection of sets such that $\emptyset \subseteq V$, and $\mathbf{F} \in \widetilde{P}(\emptyset, \mathscr{R})$, because then $V(F)$ which is the intersection of all such sets, must then be contained in the $\emptyset$. It is clearly true that $\emptyset \subseteq V$. Note by the construction of $\widetilde{P}(\emptyset, \mathscr{R})$ that T_0, the nullary operations for the type mentioned in the first paragraph, must be elements of $\widetilde{P}(\emptyset, \mathscr{R})$. That is, $\widetilde{P}(\emptyset, \mathscr{R}) = \cup F_n$ where $F_0 = T_0 \cup \emptyset$. Thus since $\mathbf{F} \in T_0$, $\mathbf{F} \in \widetilde{P}(\emptyset, \mathscr{R})$. It follows that $V(\mathbf{F}) = \emptyset$. $\square$

Problem 2 (Exercise 1.2 ii))**.** *Show that if* $r \in \mathscr{R}$, $ar(r) = n$, *and* $x_1, x_2, ..., x_n \in V$ *then* $V(r(x_1, x_2, ..., x_n)) = \{x_1, x_2, ..., x_n\}$.

Exercise 1.2 ii). Similarly, if we can show that $\{x_1, x_2, ..., x_n\}$ satisfies the properties that $\{x_1, x_2, ..., x_n\} \subseteq V$ and $r(x_1, x_2, ..., x_n) \in \widetilde{P}(\{x_1, x_2, ..., x_n\}, \mathscr{R})$, then we will have that $V(r(x_1, x_2, ..., x_n)) \subseteq \{x_1, x_2, ..., x_n\}$. By definition,

$\widetilde{P}(\{x_1, x_2, ..., x_n\}, \mathscr{R})$ has a generating set of things of the form $r(x_1, x_2, ..., x_n)$, and therefore must contain this element at it's nullary level. We initially supposed also that $x_1, x_2, ..., x_n \in V$ so $\{x_1, x_2, ..., x_n\} \subseteq V$ and both conditions are verified. Therefore it holds that $V(r(x_1, x_2, ..., x_n)) \subseteq \{x_1, x_2, ..., x_n\}$.

To show the reverse inclusion pick any U such that $U \subseteq V$ and $r(x_1, x_2, ..., x_n) \in \widetilde{P}(U, \mathscr{R})$. Looking at the form of $r(x_1, x_2, ..., x_n)$, we see that it must be a generator, and therefore, U must contain $\{x_1, x_2, ..., x_n\}$. Since U was chosen arbitrarily, the reverse inclusion holds and therefore $V(r(x_1, x_2, ..., x_n)) = \{x_1, x_2, ..., x_n\}$. $\square$

Problem 3 (Exercise 1.2 iii)). *Show that if $w_1, w_2 \in \widetilde{P}(V, \mathscr{R})$, then*

$$V(w_1 \Rightarrow w_2) = V(w_1) \cup V(w_2).$$

Exercise 1.2 iii). Let $\overline{U} = V(w_1) \cup V(w_2)$, then $w_1 \in \widetilde{P}(\overline{U}, \mathscr{R})$ and $w_2 \in \widetilde{P}(\overline{U}, \mathscr{R})$. Since $\Rightarrow$ is in the type for both of these free algebras, $w_1 \Rightarrow w_2 \in \widetilde{P}(\overline{U}, \mathscr{R})$. This implies that $V(w_1 \Rightarrow w_2) \subseteq U = (V(w_1) \cup V(w_2))$. This holds because we have verified that $\overline{U}$ is one of the sets which is in the intersection which $V(w_1 \Rightarrow w_2)$ is defined to be.

To show the reverse inclusion, that is that $V(w_1) \cup V(w_2) \subseteq V(w_1 \Rightarrow w_2)$, we need to show that $V(w_1) \subseteq V(w_1 \Rightarrow w_2)$ and that $V(w_2) \subseteq V(w_1 \Rightarrow w_2)$. This intuitively plausible pair of inclusions is verified if we can show without loss of generality that $w_1 \in \widetilde{p}(V(w_1 \Rightarrow w_2), \mathscr{R})$. Clearly, $w_1 \Rightarrow w_2 \in \widetilde{p}(V(w_1 \Rightarrow w_2), \mathscr{R})$, and since $\widetilde{p}(V(w_1 \Rightarrow w_2), \mathscr{R})$ is generated freely, then by construction, if $w_1 \Rightarrow w_2 \in F_n$, then if $w_2 \in F_k$ with $k < n$, $w_1 \in F_{n-k}$ and therefore, $w_1 \in \widetilde{p}(V(w_1 \Rightarrow w_2), \mathscr{R})$. The symmetry of the argument shows too that $w_2 \in \widetilde{p}(V(w_1 \Rightarrow w_2), \mathscr{R})$, and therefore, $V(w_1) \subseteq V(w_1 \Rightarrow w_2)$ and that $V(w_2) \subseteq V(w_1 \Rightarrow w_2)$. The reverse inclusion, $V(w_1) \cup V(w_2) \subseteq V(w_1 \Rightarrow w_2)$ follows and we obtain the final equality. $\square$