An Algebraic Introduction to Mathematical Logic
Chapter 4 Predicate Calculus
Section 3 Proof in $Pred(V, \mathscr{R})$

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We will denote the logic called the First Order Predicate Calculus on $(V, \mathscr{R})$ by $Pred(V, \mathscr{R})$. Now we will have to define the semantics for the logic.

Definition 1. We define the axioms of $Pred(V, \mathscr{R})$ to be
$\mathscr{A}_1 = \{p \Rightarrow (q \Rightarrow p) | p, q, r \in P(V, \mathscr{R})\}$,
$\mathscr{A}_2 = \{p \Rightarrow (q \Rightarrow r) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) | p, q \in P(V, \mathscr{R})\}$,
$\mathscr{A}_3 = \{\sim\sim p \Rightarrow p | p \in P(V, \mathscr{R})\}$,
$\mathscr{A}_4 = \{(\forall x)(p \Rightarrow q) \Rightarrow (p \Rightarrow (\forall x)q) | p, q \in P(V, \mathscr{R}), x \notin var(p)\}$[i],
$\mathscr{A}_5 = \{(\forall x)p(x) \Rightarrow p(y) | p(x) \in P(V, \mathscr{R}), y \in V\}$.

Let (U, φ, ψ, v) be an interpretation.

Recall the following 3 conditions:
(a) If $r \in \mathscr{R}$ and $x_1, ..., x_n \in V$ then $v(r(x_1, ..., x_n)) = 1$ if $(\varphi x_1, ..., \varphi x_n) \in \psi r$, and 0 otherwise.
(b) v is an $\{F, \Rightarrow\}$-algebra homomorphism.
($\mathrm{c_k}$) Suppose $p = (\forall x)q(x)$ has depth k. Put $V' = V \cup \{t\}$ where $t \notin V$. If for every extension $\varphi' : V' \to U$ of φ and for every $v_{k-1} : P_{k-1}(V, \mathscr{R}) \to \mathbb{Z}_2$, such that $(\varphi', \psi, v'_{k-1})$ satisfy (a), (b) and ($\mathrm{c_i}$), for all $i < k$, we have $v'_{k-1}(q(t)) = 1$, then $v(p) = 1$, otherwise $v(p) = 0$.

Problem 1. Show that every axiom of $Pred(V, \mathscr{R})$ is a tautology.

Proof. For $\mathscr{A}_1$, $\mathscr{A}_2$, and $\mathscr{A}_3$ refer to the notes for Chapter 2 Section 3.

For $\mathscr{A}_4$, note by the definition that $(\forall x)(p \Rightarrow q)$ is $(\forall x)p \Rightarrow (\forall x)q$. Since $x \notin var(p)$, then no matter how t is interpreted, $v(p) = v((\forall x)p)$. We compute,

$$
\begin{aligned}
v(((\forall x)p \Rightarrow (\forall x)q) \Rightarrow (p \Rightarrow (\forall x)q)) = \\
1 + v((\forall x)p \Rightarrow (\forall x)q)(1 + v(p \Rightarrow (\forall x)q)) = \\
1 + (1 + v((\forall x)p)(1 + v((\forall x)q)))(1 + (1 + v(p)(1 + v((\forall x)q))))) = \\
1 + (1 + v(p)(1 + v((\forall x)q)))(1 + (1 + v(p)(1 + v((\forall x)q)))).
\end{aligned}
$$

[i] var(p) denotes all free variables of p, that is all unquantified variables.

Case 1: If $v(p) = 0$ we obtain

$$\begin{aligned}
1 + (1 + v(p)(1 + v((\forall x)q)))(1 + (1 + v(p)(1 + v((\forall x)q)))) = \\
1 + (1 + 0(1 + v((\forall x)q)))(1 + (1 + 0(1 + v((\forall x)q)))) = \\
1 + (1)(1 + (1)) = \\
1 + (1 + 1) = 1.
\end{aligned}$$

Case 2: If $v(p) = 1$ we obtain

$$\begin{aligned}
1 + (1 + v(p)(1 + v((\forall x)q)))(1 + (1 + v(p)(1 + v((\forall x)q)))) = \\
1 + (1 + (1 + v((\forall x)q)))(1 + (1 + (1 + v((\forall x)q)))).
\end{aligned}$$

Case 2a: If $v((\forall x)q) = 1$ then

$$\begin{aligned}
1 + (1 + (1 + v((\forall x)q)))(1 + (1 + (1 + v((\forall x)q)))) = \\
1 + (1 + (1 + 1))(1 + (1 + (1 + 1))) = \\
1 + (1 + (0))(1 + (1 + (0))) = \\
1 + (1)(1 + 1) = \\
1 + (1)(0) = 1
\end{aligned}$$

Case 2b: If $v((\forall x)q) = 0$ then

$$\begin{aligned}
1 + (1 + (1 + v((\forall x)q)))(1 + (1 + (1 + v((\forall x)q)))) = \\
1 + (1 + (1 + 0))(1 + (1 + (1 + 0))) = \\
1 + (1 + (1))(1 + (1 + (1))) = \\
1 + (1 + 1)(1 + (1 + 1)) = \\
1 + (0)(1) = 1
\end{aligned}$$

It follows that $\mathscr{A}_4$ is a tautology.

To show $\mathscr{A}_5$ is a tautology, let $t \notin V$, if y is interpreted in such a way that $\varphi y \in U$ makes $v'_{k-1}(p(y)) = 0$, then letting $\varphi'(t) = \varphi(y)$ in U it must also follow that $v'_{k-1}(p(t)) = 0$. Then by definition $v((\forall x)p(x)) = 0$. If y must be interpreted so that $v'_{k-1}(p(y)) = 1$, we have still the possibility that $v((\forall x)p(x)) = 0$ since φ may not be surjective on U. In particular, $\varphi t \in U$ may be such that $v'_{k-1}(p(t)) = 0$. Now since v is a $\{F, \Rightarrow\}$-algebra homomorphism, it follows that

$$v((\forall x)p(x) \Rightarrow p(y)) = 1 + v((\forall x)p(x))(1 + v(p(y))).$$

If $v(p(y)) = 0$ then $v((\forall x)p(x)) = 0$ and the expression values to 1. If $v(p(y)) = 1$, then the expression also values to 1 regardless of the value of $(\forall x)p(x)$. $\square$

The semantics of the logic $Pred(V, \mathscr{R})$ is similar to semantics for the logic $Prop(V)$, however, there is one additional rule, call the rule of Generalization, that must be implemented to handle quantification. Suppose we exhibit proof of $p(x)$ but the specifics of x are not used in the proof, that is x is general, then we may also deduce $(\forall x)p(x)$. The following is an inductive definition.

Definition 2. Let $A \subseteq P$, $p \in P$. A *proof of length* n of p from A is a sequence p_1,...,p_n of n elements of P such that $p_n = p$, the sequence p_1,...,p_{n-1} is a proof of length $n-1$ of p from A and

1. $p_n \in \mathscr{A} \cup A$, or
2. $p_i = p_j \Rightarrow p_n$, for some $i, j < n$, or
3. $p_n = (\forall x)w(x)$ and some subsequence p_{k_1},...,p_{k_r}, of p_1,...,p_{n-1} is a proof of length less than n of $w(x)$ from a subset A_0 of A such that $x \notin var(A_0)$.

Problem 2. Construct a proof in $Pred(V, \mathscr{R})$ of $(\forall x)(\forall y)p(x,y)$ from $\{(\forall y)(\forall x)p(x,y)\}$.

Proof. By $\mathscr{A}_5$,

$$p_1 = (\forall y)(\forall x)p(x,y) \Rightarrow (\forall x)p(x,y)$$

By assumption,

$$p_2 = (\forall y)(\forall x)p(x,y).$$

By modus ponens,

$$p_3 = (\forall x)p(x,y).$$

Then by $\mathscr{A}_5$ again,

$$p_4 = (\forall x)p(x,y) \Rightarrow p(x,y).$$

By modus ponens,

$$p_5 = p(x,y).$$

Since $y \notin var(\{(\forall y)(\forall x)p(x,y)\})$,[ii] we may use generalization

$$p_6 = (\forall y)p(x,y).$$

Since $x \notin var(\{(\forall y)(\forall x)p(x,y)\})$, we may use generalization

$$p_7 = (\forall x)(\forall y)p(x,y).$$

□

[ii]var only accounts for unbound variables