An Algebraic Introduction to Mathematical Logic
Chapter 4 Predicate Calculus
Section 1 Algebra of Predicates
Proof of Transitivity

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Definition 1.3 Let $w \in \widetilde{P}(V, \mathscr{R})$ the *depth of quantification* of w, denoted by $d(w)$, is defined by

1. $d(\mathbf{F}) = 0$, $d(r(x_1, x_2, ..., x_n)) = 0$ for every free generator of $\widetilde{P}(V, \mathscr{R})$. Note, since F_0 in the construction of $\widetilde{P}(V, \mathscr{R})$ is the union of $T_0 = \{\mathbf{F}\}$ and the generators which are of the form $r(x_1, x_2, ..., x_n)$ for $r \in \mathscr{R}$ and $x_1, x_2, ..., x_n \in V$, then we can say for all $f \in F_0$, $d(f) = 0$.
2. $d(w_1 \Rightarrow w_2) = max(d(w_1), d(w_2))$
3. $d((\forall x)(w)) = 1 + d(w)$ for $x \in V$

Our desired congruence relation on $\widetilde{P}(V, \mathscr{R})$ may now be defined.

Definition 1.4 Let w_1, $w_2 \in \widetilde{P}(V, \mathscr{R})$ define $w_1 \approx w_2$ if

1. $d(w_1) = d(w2) = 0$ and $w_1 = w_2$.
2. $d(w_1) = d(w_2) > 0$, $w_1 = a_1 \Rightarrow b_1$, $w_2 = a_2 \Rightarrow b_2$, $a_1 \approx a_2$ and $b_1 \approx b_2$ or
3. $w_1 = (\forall x)(a)$, $w_2 = (\forall y)b$, and either
   (a) $x = y$ and $a \approx b$
   (b) There exists a $c = c(x)$ such that $c(x) \approx a$ and $c(y) \approx b$ and $y \notin V(c)$

Problem 1 (Exercise 1.5 i))**.** Given that $z \notin V(w_1) \cup V(w_2)$ show by induction over $d(w_1)$ that the element $c = c(x)$ in **Definition 1.4** $3(b)$ can always be chosen such that $z \notin V(c)$.

Proof. Assuming the notation of **Definition 1.4** $3(b)$, we have $c(x) \approx a$ and $c(y) \approx b$.

The base case $d(w_1) = 1$ holds because the relation $c(x) \approx a$ is equality for depth 0 and therefore, $V(c(x)) = V(a)$. Since $w_1 = (\forall x)(a)$, we have that $z \notin V(a)$ and therefore, $z \notin V(c)$.

Now let $d(w_1) = n + 1$. The relations $c(x) \approx a$ and $c(y) \approx b$ are happening at depth n. The claim is that at every depth n the relations $c(x) \approx a$ and $c(y) \approx b$ can be replaced by depth $n - 1$ expressions. Finally, since all expressions are finite compositions of operations, this process terminates at depth zero where we invoke the base case.

The relations $c(x) \approx a$ and $c(y) \approx b$ at depth $n > 0$ hold because of any one of the following 3 possiblilities: **Definition 1.4** 2, **Definition 1.4** 3(a) or **Definition 1.4** 3(b). Note that $c(x) \approx a$ and $c(y) \approx b$ could each hold because of a different case potentially. However, their depths of quantification must match.
So without loss of generality we just show how the relation $c(x) \approx a$ can be reduced to a lower depth expression $\overline{c}(x) \approx \overline{a}$ while guaranteeing that $z \notin V(c) - V(\overline{c})$.

Cases

Definition 1.4 2

Assume now that $c(x) \approx a$ because of **Definition 1.4** 2. That is to say $a = a_1 \Rightarrow a_2$, $c = c_1 \Rightarrow c_2$, and therefore, $a_1 \approx c_1$ and $a_2 \approx c_2$. The definition of depth of a is $max(a_1, a_2)$ and since $a_1 \approx c_1$ and $a_2 \approx c_2$ implies that $d(a_1) = d(c_1)$ and $d(a_2) = d(c_2)$, $d(a) = d(max(a_1, a_2)) = d(max(c_1, c_2)) = d(c(x))$. We already knew in fact that $d(a) = d(c(x)$. Thus the relation $c(x) \approx a$ has been reduced to the two relations $a_1 \approx c_1$ and $a_2 \approx c_2$. One of these expressions is of depth equal to $c(x) \approx a$ and one is of depth less than or equal to $c(x) \approx a$. The relations $a_1 \approx c_1$ and $a_2 \approx c_2$ each hold because of any one of the 3 cases. Since (in prefix notation) an expression may only begin with at most finitely many $\Rightarrow$ eventually it will begin with a quantifier which brings us to cases 2 and 3 where we lower the depth of quantification in all cases.

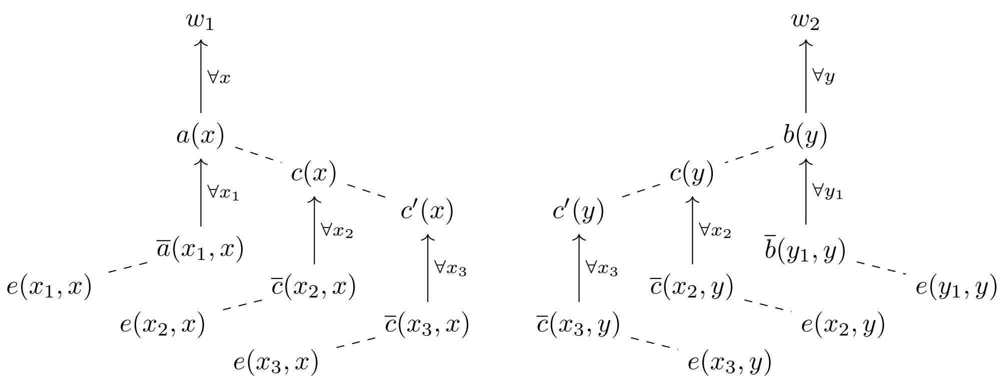
Definition 1.4 3 (a)

Assume now that $c(x) \approx a$ because of **Definition 1.4** 3(a). That is, $c(x) = \forall x_2 \overline{c}(x_2, x)$ and $a(x) = \forall x_2 \overline{a}(x_2, x)$. Then $\forall x_2 \overline{c}(x_2, x) \approx \forall x_2 \overline{a}(x_2, x)$ and $\overline{c}(x_2, x) \approx \overline{a}(x_2, x)$. The depth of the problem has just been reduced by one depth of quantification. Note that the relation $\overline{c}(x_2, x) \approx \overline{a}(x_2, x)$ could hold because of any one of the three cases.

Definition 1.4 3 (b)

Assume now that $c(x) \approx a$ because there exists some $e(x_1)$ such that $e(x_1) \approx \overline{a}(x_1)$ and $e(x_2) \approx \overline{c}(x_2)$ where $a(x) = \forall x_1 \overline{a}(x_1, x)$ and $c(x) = \forall x_2 \overline{c}(x_2, x)$ and $x_2 \notin V(e(x_1))$. Now pick any variable $z' \notin V(w_1) \cup V(w_2) \cup V(c(x))$.
By the induction hypothesis, $e(x_1)$ can be chosen so that it does not contain the variable z'. Thus we can substitute any mentions of the variable z in $e(x_1)$

with z' which we have ensured is not already present without disrupting the equivalence $c(x) \approx a$.
Suppose however, that the variable $x_2 = z$, then we may rechose x_2 which is quantified over when it appears in $c(x)$ and $c(y)$. That is, $c(x) = \forall x_2 c(x_2, x)$ so let $c'(x) = \forall x \overline{c}(x_3, x)$. The important thing is that c' still works in **Definition 1.4** 3(b) for $w_1 \approx w_2$. By construction, it remains true that $a \approx c'(x)$ and $b \approx c'(y)$ because $e(x_1, x) \approx \overline{a}(x_1, x)$ and $e(x_3, x) \approx \overline{c}(x_3, x)$. *In the figure below both of the relations* $c(x) \approx a$ *and* $c(y) \approx b$ *hold because of case 3 although this need not be the case*



Therefore, $z \notin V(e(x_3))$, moreover, $z \neq x_3$. In order to show that $z \notin c'(x)$ all that remains is to show that $z \notin V(\overline{c}(x_3)) - x_3$. Notice that we have just reduced the problem by one degree of quantification.
More precisely, $z \notin V(e(x_3, x))$ and $z \notin V(e(x_3, y))$. Also, $e(x_3, x) \approx \overline{c}(x_3, x)$ and $e(x_3, y) \approx \overline{c}(x_3, y)$. At the previous level, $a(x)$ played the role of $e(x_3, x)$ in the induction and $b(y)$ played the role of $e(x_3, y)$. Explicitly in the statement of the problem, since $x_3 \neq z$, we could let $\forall x_3 e(x_3, x)$ play the role of w_1 and depending on why $c(y) \approx b$ in case 3 let $\forall x_3 e(x_3, y)$ play the role of w_2 or if $c(y) \approx b$ holds because of case 2 let b play the role of w_2, c is then played by $\overline{c}(x_3, x)$.
Thus we have reduced the depth of quantification by 1. $\square$

Problem 2 (Exercise 1.5 ii))**.** If $u(x) \approx v(x)$ and $y \notin V(u(x)) \cup V(v(x))$, show by induction over $d(u(x))$ that $u(y) \approx v(y)$.

Proof. The base case holds because the relation is equality for depth 0.
Suppose the result holds for depth of quantification less than n. Then let $d(u(x)) = d(v(x)) = n$. Let $u(x) = \forall x_1 u_1(x, x_1)$ and $v(x) = \forall x_2 v_1(x, x_2)$. Then there exists a $c(x, x_1)$ such that $u_1(x, x_1) \approx c(x, x_1)$ and $c(x, x_2) \approx v_1(x, x_2)$ such that $x_2 \notin V(c)$. By **Problem 1** we have that $y \notin V(c)$.

By the induction hypothesis, since the depth of u_1 v_1 and c are all $n - 1$ we have that $u_1(y, x_1) \approx c(y, x_1)$ and $c(y, x_2) \approx v_1(y, x_2)$. Then by definition, since $u(y) = \forall x_1 u_1(y, x_1)$ and $v(y) = \forall x_2 v_1(y, x_2)$, we have proven $u(y) \approx v(y)$. $\square$

Problem 3 (Exercise 1.5 iii))**.** Show that the relation is transitive.

Proof. Let $w_1 \approx w_2$ and $w_2 \approx w_3$. Assume the relation is transitive for lower depths of quantification. Since the relation is defined to be strict equivalence at depth 0, the base case holds.

The proof is given in the worst possible case: We begin by twice invoking **Definition 1.4** $3(b)$.

Let $w_1 = \forall x a_1(x)$, $w_2 = \forall y a_2(y)$, and $w_3 = \forall z a_3(z)$. We have that there exists $c_1(x)$ and $c_2(y)$, such that $y \notin c_1(x)$ and $z \notin c_2(y)$. Also, because of **Problem 1** we may require that $z \notin c_1(x)$ and $x \notin c_2(y)$. By part $3(b)$, of the definition $w_1 \approx w_2$ is defined to mean, $a_1(x) \approx c_1(x)$ and $c_1(y) \approx a_2(y)$. Likewise, $w_2 \approx w_3$ means that $a_2(y) \approx c_2(y)$ and $c_2(z) \approx a_3(z)$.

By the induction hypothesis, since $c_1(y) \approx a_2(y)$ and $a_2(y) \approx c_2(y)$, we have $c_1(y) \approx c_2(y)$. Note also that neither $c_1(y)$ nor $c_2(y)$ contains either x or z.

By **Problem 2** we have that $c_1(x) \approx c_2(x)$. Then applying the induction hypothesis again, we obtain from $a_1(x) \approx c_1(x)$ and $c_1(x) \approx c_2(x)$, that $a_1(x) \approx c_2(x)$. Note also that since $c_2(z) \approx a_3(z)$, we have bridged the gap and it follows that $w_1 \approx w_3$.

Other cases follow from just **Problem 1** and the induction hypothesis alone. $\square$

Definition 1.6 The reduced first order algebra $P(V, \mathscr{R})$ on $(V, \mathscr{R})$ is the factor algebra of $\widetilde{P}(V, \mathscr{R})$ by the congruence relation $\approx$.

The elements of $P = P(V, \mathscr{R})$ are the congruence classes.

Proposition If $w \in \widetilde{P}$ and $[w]$ is the congruence class of w, then

$$(\forall x)[w] = [(\forall x)w],$$

and

$$[w_1] \Rightarrow [w_2] = [w_1 \Rightarrow w_2].$$

Proof. Let $\widetilde{w}_1$ and $\widetilde{w}_2$ be representatives of $[w]$. Thus $\widetilde{w}_1 \approx \widetilde{w}_2$. By **Definition 1.4** $3a)$ $(\forall x)\widetilde{w}_1 \approx (\forall x)\widetilde{w}_2$.

By definition **Definition 1.4** 2 the second result follows. $\square$

Definition 1.7 Let $w \in P$. We define the set $var(w)$ of *(free) variables of* w by putting $var(w) = var(\widetilde{w})$, where $\widetilde{w} \in \widetilde{P}$ is some representative of the congruence class w, and where $var(\widetilde{w})$ is defined inductively by

1. $var(\mathbf{F}) = \emptyset$,

2. $var(r(x_1, ..., x_n)) = \{x_1, ..., x_n\}$ for $r \in \mathscr{R}$, $x_1, ..., x_n \in V$,
3. $var(\widetilde{w}_1 \Rightarrow \widetilde{w}_2) = var(\widetilde{w_1}) \cup var(\widetilde{w_2}))$,
4. $var((\forall x)\widetilde{w}) = var(\widetilde{w}) - \{x\}$.

This captures the free or unbound variables in the predicate.

Definition 1.8 Let $A \subseteq P$. Put

$$var(A) = \bigcup_{p \in A} var(p).$$

Problem 4. Show that if $\widetilde{w}_1 \approx \widetilde{w}_2$, then $var(\widetilde{w}_1) = var(\widetilde{w}_2)$, and conclude that $var(w)$ is well defined for $w \in P$.

Proof. Suppose that $\widetilde{w}_1 \approx \widetilde{w}_2$. If the depth of quantification is 0 the equivalence relation is strict equality and the sets of variables are obviously equal.
Suppose the result holds for depths of quantification lower than n and that the $\widetilde{w}_1 \approx \widetilde{w}_2$ holds at depth n. We will only discuss the case where **Definition 1.4** $3a$) holds. In this case apply **Definition 1.7** 4. The result follows. $\square$