An Algebraic Introduction to Mathematical Logic
Chapter 2 Propositional Calculus
Section 4 Proof in The Propositional Calculus
Exercises

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For the propositional calculus on the set X, we take as axioms all elements of the subset $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3$ of $P(X)$, where

$\mathscr{A}_1 = \{p \Rightarrow (q \Rightarrow p) | p, q \in P(X)\}$
$\mathscr{A}_2 = \{(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) | p, q, r \in P(X)\}$ and
$\mathscr{A}_3 = \{\sim\sim p \Rightarrow p | p \in P(X)\}$.

As our one rule of inference, we take the rule known as modus ponens: from p and $p \Rightarrow q$, deduce q. We may now give a formal definition of a proof.

Definition 4.1 Let $q \in P(X)$ and let $A \subseteq P(X)$. In the propositional calculus on the set X, a *proof of q from the assumptions A* is a finite sequence p_1, p_2, ... p_n of elements $p_i \in P(X)$ such that $p_n = q$ and for each i, either $p_i \in \mathscr{A} \cup A$ or for some $j, k < i$ we have $p_k = (p_j \Rightarrow p_i)$.

Definition 4.2 Let $q \in P(X)$ and let $A \subseteq P(X)$. We say that q is a *deduction* from A, or q is *provable* from A, or that A *syntactically implies* q, if there exists a proof of q from A. We shall write this as $\vdash$, and we shall denote by $Ded(A)$ the set of all deduction from A.

Problem 1 (4.9)**.** *Show that $Ded(A)$ is the smallest subset D of $P(X)$ such that $\mathscr{A} \cup A \subseteq D$ and closed under modus ponens.*

4.9 Solution. Pick any $r \in Ded(A)$. We will show that it must be in any other subset D of $P(X)$ satisfying the above requirements. This is sufficient to show that $Ded(A) \subseteq D$.

Since $r \in Ded(A)$, there exists a proof of r. If the proof is of length 1, then $r \in \mathscr{A} \cup A$ and therefore in D, and we are finished. The result follows inductively, suppose it holds that $r \in D$ for all r with proofs of up to length $n-1$. Then either $p_n = r \in \mathscr{A} \cup A$ or for some $i, j < n$ $p_i = p_j \Rightarrow r$ in which case since p_i and p_j have proofs of length $n-1$ or less, p_j and p_j lie in D. By closure under modus ponens, $r \in D$.

Therefore, $Ded(A) \subseteq D$ for all such D and is thus the smallest, ordered by inclusions. $\square$

Problem 2 (4.10)**.** *Construct a proof in the propositional calculus of $p \Rightarrow r$ from the assumptions $A = \{p \Rightarrow q, q \Rightarrow r\}$.*

4.10. By assumption, $(q \Rightarrow r) \in Ded(A)$. Let $p_1 = (q \Rightarrow r)$. Axiom 1 of $\mathscr{A}$ says that $p_1 \Rightarrow (p \Rightarrow p_1) \in \mathscr{A} \cup A$ and is therefore in $Ded(A)$.

We have, $p_1 \in Ded(A)$ and $p_1 \Rightarrow (p \Rightarrow p_1) \in Ded(A)$. By modus ponens, $p \Rightarrow p_1 \in Ded(A)$. Applying Axiom 2 to $(p \Rightarrow p_1) = (p \Rightarrow (q \Rightarrow r))$, We obtain $((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$. Noting that we began with the assumption $(p \Rightarrow q)$, by modus ponens we obtain $(p \Rightarrow r)$. $\square$