# An Algebraic Introduction to Mathematical Logic Chapter 4 Predicate Calculus Section 1 Algebra of Predicates Exercises

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The following is the construction of the Algebra of Predicates.

From the text, $V$ is an infinite set who's elements are called individual variables. There is also a set of relation or predicate symbols $\mathscr{R}$, together with an arity function $ar : \mathscr{R} \to \mathbb{N}$. The set of generators used to construct the propositional algebra $P$ is

$$\{(r, x_1, ...x_n) | r \in \mathscr{R}, x_i \in V, \text{ and } ar(r) = n\}$$

Let $\widetilde{P}(V, \mathscr{R})$ be the free algebra on the set above of type $\{\mathbf{F}, \Rightarrow, (\forall x) | x \in V\}$, where the arities are as usual, that is, $\mathbf{F}$ is nullary, and $\Rightarrow$ is binary. For each $x \in V$ we have a unary operator $(\forall x)$. A similar example can be found in the signature of a vector space over a feild $k$. Here, for each element of $k$, there is a unary operator which is scalar multiplication by that element.

**Definition 1.1** Let $w \in \widetilde{P}(V, \mathscr{R})$ the set of *variables involved in* $w$, denoted by $V(w)$, is defined by

$$V(w) = \cap\{U | U \subseteq V, w \in \widetilde{P}(U, \mathscr{R})\}.$$

**Problem 1** (Exercise 1.2 i))**.** *Show that* $V(\boldsymbol{F}) = \emptyset$*.*

*Exercise 1.2 i).* Since, $V(\mathbf{F}) = \cap\{U | U \subseteq V, \mathbf{F} \in \widetilde{P}(U, \mathscr{R})\}$ it suffices to show that $\emptyset$ is in this collection of sets such that $\emptyset \subseteq V$, and $\mathbf{F} \in \widetilde{P}(\emptyset, \mathscr{R})$, because then $V(F)$ which is the intersection of all such sets, must then be contained in the $\emptyset$. It is clearly true that $\emptyset \subseteq V$. Note by the construction of $\widetilde{P}(\emptyset, \mathscr{R})$ that $T_0$, the nullary operations for the type mentioned in the first paragraph, must be elements of $\widetilde{P}(\emptyset, \mathscr{R})$. That is, $\widetilde{P}(\emptyset, \mathscr{R}) = \cup F_n$ where $F_0 = T_0 \cup \emptyset$. Thus since $\mathbf{F} \in T_0$, $\mathbf{F} \in \widetilde{P}(\emptyset, \mathscr{R})$. It follows that $V(\mathbf{F}) = \emptyset$. □

**Problem 2** (Exercise 1.2 ii))**.** *Show that if* $r \in \mathscr{R}$*,* $ar(r) = n$*, and* $x_1, x_2, ..., x_n \in V$ *then* $V(r(x_1, x_2, ..., x_n)) = \{x_1, x_2, ...., x_n\}$*.*

*Exercise 1.2 ii).* Similarly, if we can show that $\{x_1, x_2, ...., x_n\}$ satisfies the properties that $\{x_1, x_2, ...., x_n\} \subseteq V$ and $r(x_1, x_2, ..., x_n) \in \widetilde{P}(\{x_1, x_2, ...., x_n\}, \mathscr{R})$, then we will have that $V(r(x_1, x_2, ..., x_n)) \subseteq \{x_1, x_2, ...., x_n\}$. By definition,

$\widetilde{P}(\{x_1, x_2, ...., x_n\}, \mathscr{R})$ has a generating set of things of the form $r(x_1, x_2, ..., x_n)$, and therefore must contain this element at it's nullary level. We initially supposed also that $x_1, x_2, ..., x_n \in V$ so $\{x_1, x_2, ...., x_n\} \subseteq V$ and both conditions are verified. Therefore it holds that $V(r(x_1, x_2, ..., x_n)) \subseteq \{x_1, x_2, ...., x_n\}$.

To show the reverse inclusion pick any $U$ such that $U \subseteq V$ and $r(x_1, x_2, ..., x_n) \in \widetilde{P}(U, \mathscr{R})$. Looking at the form of $r(x_1, x_2, ..., x_n)$, we see that it must be a generator, and therefore, $U$ must contain $\{x_1, x_2, ...., x_n\}$. Since $U$ was chosen arbitrarily, the reverse inclusion holds and therefore $V(r(x_1, x_2, ..., x_n)) = \{x_1, x_2, ...., x_n\}$. □

**Problem 3** (Exercise 1.2 iii)). *Show that if $w_1, w_2 \in \widetilde{P}(V, \mathscr{R})$, then*

$$V(w_1 \Rightarrow w_2) = V(w_1) \cup V(w_2).$$

*Exercise 1.2 iii).* □