An Algebraic Introduction to Mathematical Logic
Chapter 3 Propositional Calculus
Section 2 Soundness and Adequacy of Prop(X)
Exercises

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Theorem 2.1 (The Soundness Theorem) Let $A \subseteq P(X)$, $p \in P(X)$. *If* $A \vdash p$, *then* $A \models p$.

Corollary 2.2 (The Consistency Theorem) F *is not a theorem of* $Prop(x)$.

Problem 1 (Exercise 2.3)**.** *Show that* $Con(A)$ *is closed with respect to modus ponens (i.e. if* p, $p \Rightarrow q \in Con(A)$, *then* $q \in Con(A)$*). Use Exercise 4.9 of Chapter II to prove that* $Ded(A) \subseteq Con(A)$. *This is another way of stating the soundness theorem.*

Exercise 2.3. We show first that $Con(A)$ is closed under modus ponens. Let $A \models p$ and $A \models p \Rightarrow q$. Then for all valuations v such that $v(A) = 1$, we have $v(p) = 1$ and $v(p \Rightarrow q) = 1$.

$v(p \Rightarrow q) = v(1 + p(1 + q))$

Since we have that v is a homomorphism,

$v(1 + p(1 + q)) = 1 + v(p)(1 + v(q)) = 1 + 1 * (1 + v(q)) = 1 + (1 + v(q))$,

Since we are working in $\mathbb{Z}_2$

$v(q) + 1 = 0$ which implies $v(q) = 1$.

This completes the verification that $A \models q$, and hence A is closed under modus ponens.

By exercise 4.9 of Chapter II Section IV, we have that $Ded(A)$ is the smallest subset of $P(X)$ which is closed under modus ponens and contains all of the axioms. Since the axioms are tautologies, (valid for all valuations), we must have $Ded(A) \subseteq Con(A)$. □

Theorem 2.4 (The Deduction Theorem) Let $A \subseteq P(X)$, and let $p, q \in P(X)$. Then $A \vdash p \Rightarrow q$ *if and only if* $A \cup \{p\} \vdash q$.

Problem 2 (Exercise 2.6)**.** *Show that* $p \Rightarrow r \in Ded\{p \Rightarrow q, p \Rightarrow (q \Rightarrow r)\}$. *Hence show that if* $p \Rightarrow q$, $p \Rightarrow (q \Rightarrow r) \in Ded(A)$, *then* $p \Rightarrow r \in Ded(A)$, *and* *so prove the Deduction Theorem without giving an explicit construction for a proof in $Prop(X)$.*

Lemma The deduction of $(p \Rightarrow r)$ goes as follows:
$p \Rightarrow (q \Rightarrow r)$ Given
$(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$ Axiom 2
$p \Rightarrow q$ Given
$p \Rightarrow r$ Modus Ponens
End Lemma

Solution 2.6. Now we give the outlined proof of The Deduction Theorem. Suppose first that $A \vdash p \Rightarrow q$. Adding anything to our assumptions will not remove any deductions. Thus, $A \cup \{p\} \vdash p \Rightarrow q$ remains true. Modus ponens applied to any proof of $p \Rightarrow q$, along with p, results in a valid proof of q. Therefore, $A \cup \{p\} \vdash q$.

Suppose now $A \cup \{p\} \vdash q$. We want to show that $A \vdash p \Rightarrow q$. Let n be the length of proof of $p \Rightarrow q$. Suppose that it is true for all p_i with proofs of length $n-1$ or less that $A \cup \{p\} \vdash p_i$ implies that $A \vdash p \Rightarrow p_i$. If $q \notin A \cup \mathscr{A}$, (the alternative takes care of the base case of the induction), then the last step in a proof of q from $A \cup \{p\}$ would be modus ponens. Therefore, there exists some p_i deducible from $A \cup \{p\}$ which equals $p_j \Rightarrow q$ where p_j is also deducible from $A \cup \{p\}$, both of which have proofs of length strictly less than $n-1$.

By the inductive hypothesis, we have $A \vdash p \Rightarrow p_i$ and $A \vdash p \Rightarrow p_j$. Notice that $p \Rightarrow p_i = p \Rightarrow (p_j \Rightarrow q)$. Therefore, we have $p \Rightarrow (p_j \Rightarrow q) \in Ded(A)$ and $p \Rightarrow p_j \in Ded(A)$. By the lemma we have $p \Rightarrow q \in Ded(A)$, which completes the proof of The Deduction Theorem. □

Problem 3 (Exercise 2.7)**.** *Show that $\vdash p \Rightarrow \sim\sim p$ and construct a proof of $p \Rightarrow \sim\sim p$ in $Prop(X)$.*

Exercise 2.7. Recall that $\sim p = p \Rightarrow F$. The following is a proof of F from $\{p, \sim p\}$.
Given, p
$\sim p = p \Rightarrow F$
By modus ponens, F.

Therefore, $\{p, \sim p\} \vdash F$. By the Deduction Theorem, $\{p\} \vdash \sim p \Rightarrow F$. A second application of the Deduction Theorem gives $\emptyset \vdash p \Rightarrow (\sim p \Rightarrow F)$. By definition, we have $\vdash p \Rightarrow \sim\sim p$. □

Theorem 2.13 (The Adequacy Theorem) *Let $A \subseteq P(X)$, and $p \in P(X)$. If $A \models p$ in $Prop(X)$*, then *$A \vdash p$ in $Prop(X)$.*

Problem 4 (Exercise 2.14 (The Compactness Theorem))**.** *Show that if $A \models p$, then $A_0 \models p$ for some finite subset A_0 of A.*

Exercise 2.14. By the Adequacy Theorem, there $A \vdash p$ in $Prop(X)$. By Lemma 4.4 of chapter II section IV, $p \in Ded(A)$, implies that $p \in Ded(A_0)$ for some finite subset A_0 of A. Thus $A_0 \vdash p$. By the Soundness Theorem $A_0 \models p$. □