Textbook notes on Enumerative Combinatorics.

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The following are notes based on the book *Enumerative Combinatorics* by Bóna.

Chapter 1

Basic Methods

1.0.1 When we add

Theorem 1. ***Addition Priniciple*** *If A and B are two disjoint finite sets, then:*

$$|A \cup B| = |A| + |B|$$

Proof. Both sides of the above relation count the elements of the same set, the set $A \cup B$. The left-hand side does this directly, while the right-hand side counts the elements of A and B separately. In either case, each element is counted exactly once (as A and B are disjoint), so the two sides are indeed equal □

Observe that the previous theorem was about two disjoint finite sets.

Theorem 2. ***Generalized Addition Principle*** *Let $A_1, A_2, \ldots A_n$ be finite sets that are pairwise disjoint. Then*

$$|A_1 \cup A_2 \cup \ldots A_n| = |A_1| + |A_2| + \ldots |A_n|$$

Proof. Again, both sides count the elements of the same set, the set $A_1 \cup A_2 \cup \ldots A_n$, therefore they have to be equal. □

1.1 When We Subtract

For this section we will use the following definition

Definition 1. ***Difference of two sets*** *If A and B are two sets, then $A - B$ is the set consisting of the elements of A that are not elements of B*

Although the difference is defined for $B \not\subset A$, we will only consider cases on which $B \subseteq A$.

Theorem 3. ***Subtraction Priniciple*** *Let A be a finite set, and let $B \subseteq A$. Then $|A - B| = |A| - |B|$*

Proof. On a more easy way, let us prove the equivalent relation:

$$|A - B| + |B| = |A|$$

This relation holds true by the Addition Principle. Indeed, $A - B$ and B are disjoint set that their union is A. ☐

An important remark to denote here, is the fact that the hypothesis of $B \subseteq A$ is an important restriction.

The use of the Subtraction Principle is advisable in situations when it is easier to enumerate the elements of B ("bad guys") than the elements of $A - B$ (" good guys "). As a general rule of thumb, it is easy to compute $|A|$ and $|B|$ rather than compute $|A - B|$ directly.

When We Multiply

The Product Principle

Theorem 4. ***Product Principle*** *Let X and Y be two finite sets. Then the number of pairs (x, y) satisfying $x \in X$ and $y \in Y$ is $|X| \times |Y|$*

Proof. There are $|X|$ choices for the first element x of the pair (x, y), then regardless of what we choose for x, there are $|Y|$ choices for y. Each choice of x can be paired with each choice of y, so the statement is proved. ☐

A generalized version is the following:

Theorem 5. ***Generalized Product Principle*** *Let $X_1, x_2, \ldots X_k$ be inite sets. Then, the number of k-tuples $(x_1, \ldots, x_k)$ satisfying $x_i \in X_i$ is $|X_1| \times \ldots \times |X_k|$.*

Proof. We prove the statement by induction on k. For $k = 1$, there is nothing to prove, and for $k = 2$, the statement reduces to the Product Principle. Now let us assume that we know the statement for $k - 1$, and let us prove it for k. A k-tuple $(x_1, \ldots, x_k)$ satisfying $x_i \in X_i$ can be decomposed into an ordered pair $((x_1, \ldots, x_{k-1}), x_k)$, where we still have $x_i \in X_i$. The number of such k-1-tuples ia by our induction hypothesis, $|X_1| \times \ldots \times |X_{k-1}|$. The number of elements $x_k \in X_k$ is $|X_k|$. Therefore, by the Product Principle, the number of ordered pairs $((x_1, \ldots, x_{k-1}), x_k)$ satisfying the conditions is;

$$(|X_1| \times \ldots |X_{k-1}|) \times |X_k|$$

so this is also the number of k-tuples $(x_1, \ldots x_k)$ satisfying $x_i \in X_i$ ☐

An special case of the Theorem above, is when all $|X_i|$ have the same size. If A is a finite alphabet consisting of n letters, then a k-letter string over A is a sequence of k letters,each of which is an element of A.

Corollary 1. *The number of k-letters strings over an n-element alphabet A is n^k*

Proof. Apply the above theorem. ☐

Using Several Counting Principles
When Repetitions Are Not Allowed
Permutations

Definition 2. *Let n be a positive integer. Then the number:*

$$n \cdot (n-1) \dots 2 \cdot 1$$

is called n-factorial, and is denoted by $n!$

We define $0! = 1$. The set $\{1, 2, \dots, n\}$ will be denoted with: $[n]$

Theorem 6. *For any positive integers n, the number of ways to arange all elements of the set $[n]$ in a line is $n!$*

Proof. There are n ways to select the element that will be at the first place in our line. Then, regardless of this selection, there are $n-1$ ways to select the element that will be listed second, and so on .... □

an important definition is on place:

Definition 3. *A permutation of a finite set S is a list of the elements of S containing each element of S exactly once.*

Partial Lists Without Repetition

Theorem 7. *Let n and k be positive integers so that $n \geq k$. Then, the number of ways to make a k-element list from $[n]$ without repeating any elements is:*

$$n(n-1)(n-2)\dots(n-k+1)$$

Proof. There are n choices for the first element of the list, then $n-1$ choices for the second element of the list, and so on; finally there are $n-k+1$ choices for the las (kth) element of the list. The result then follows by the Product Principle. □

As a notation abbreviation, we define as:

$$(n)_k = n(n-1)(n-2)\dots(n-k+1)$$

When We Divide
The Division Principle

Definition 4. *Let S and T be finite sets, and let d be a fixed positive integer. We say that the function $f : T \to S$ is $d-$to-one*