

1 Functions

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1.1 Functions. Domain of Definition

The independent variable x is defined by a set X of its values. If to each value of the independent variable $x \in X$ there corresponds one definite value of another variable y , then y is called the function of x with a domain of definition (or domain) X or, in functional notation, $y = y(x)$, or $y = f(x)$, or $y = \varphi(x)$, and so forth. The set of values of the function $y(x)$ is called the range of the given function.

1.2 Investigation of Functions

A function $f(x)$ defined on the set X is said to be non-decreasing on this set (respectively, increasing, non-increasing, decreasing), if for any numbers $x_1, x_2 \in X, x_1 < x_2$ the inequality $f(x_1) \leq f(x_2)$ (respectively, $f(x_1) < f(x_2), f(x_1) \geq f(x_2), f(x_1) > f(x_2)$) is satisfied. The function $f(x)$ is said to be monotonic on the set X if it possesses one of the four indicated properties. The function $f(x)$ is said to be bounded above (or below) on the set X if there exists a number M (or m) such that $f(x) \leq M \forall x \in X$. The function $f(x)$ is said to be bounded on the set X if it is bounded above and below. The function $f(x)$ is called periodic if there exists a number $T > 0$ such that $f(x+T) = f(x)$ for all x belonging to the domain of definition of the function (together with any point x the point $x+T$ must belong to the domain of definition). The least number T possessing this property (if such a number exists) is called the period of the function $f(x)$. The function $f(x)$ takes on the maximum value at the point $x_o \in X$ if $f(x_o) \geq f(x)$ for all $x \in X$, and the minimum value if $f(x_o) \leq f(x)$ for all $x \in X$. A function $f(x)$ defined on a set X which is symmetric *w.r.t* origin of coordinates is called even if $f(-x) = f(x)$, and odd if $f(x) = -f(x)$.

1.3 Inverse of Function

Let the function $y = f(x)$ be defined on the set X and have a range Y . If for each $y \in Y$ there ex-

ists a single value of x such that $f(x) = y$, then this correspondence defines a certain function $x = g(y)$ called inverse *w.r.t* given function $y = f(x)$. The sufficient condition for the existence of an inverse function is a strict monotony of the original function $y = f(x)$. If the function increases(decreases), then the inverse function also increases (decreases). The graph of the inverse function $x = g(y)$ coincides with that of the function $y = f(x)$ if the independent variable is marked off along the y -axis. If the independent variable is laid off along the x -axis, i. e. if the inverse function is written in the form $y = g(x)$, then the graph of the inverse function will be symmetric to that of the function $y = f(x)$ with respect to the bisector of the first and third quadrants.

2 Limits

2.1 Existence

Limit of function $f(x)$ is said to exist as $x \rightarrow a$ when,

$$\lim_{h \rightarrow 0^+} f(a-h) = \lim_{h \rightarrow 0^+} f(a+h)$$

equal to some finite value L .

2.2 Indeterminate forms

There are only seven indeterminate forms $\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, \infty^0, 0^0$ and 1^∞ .

2.3 List of limits

Limits Operations

If $\lim_{x \rightarrow c} f(x) = L$

- $\lim_{x \rightarrow c} [f(x \pm a)] = L \pm a$
- $\lim_{x \rightarrow c} af(x) = aL$
- $\lim_{x \rightarrow c} \frac{1}{f(a)} = \frac{1}{L}$ for $L > 0$
- $\lim_{x \rightarrow c} f(x)^n = L^n$ for $n > 0$

Involving infinitesimal changes

If infinitesimal change h if denote by Δx . If $f(x)$ and $g(x)$ are differentiable at x .

- $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} = f'(x)$
- $\lim_{h \rightarrow 0} \frac{f \circ g(x+h)-f \circ g(x)}{h} = f'[g(x)]g'(x)$
- $\lim_{h \rightarrow 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} = f'(x)g(x) + f(x)g'(x)$
- $\lim_{h \rightarrow 0} \left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}} = \exp\left(\frac{f'(x)}{f(x)}\right)$
- $\lim_{h \rightarrow 0} \left(\frac{f(e^h x)}{f(x)}\right)^{\frac{1}{h}} = \exp\left(\frac{xf'(x)}{f(x)}\right)$

If $f(x)$ and $g(x)$ are differentiable on an open interval containing c , except possibly c itself, and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$ or $\pm\infty$.

Jean Bernoulli or L'Hopital's rule can be used:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

Inequalities

If $f(x) \leq g(x)$ for all x in interval that contains c , except possibly c itself, and the limit of $f(x)$ and $g(x)$ both exist at c , then $\lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} g(x)$.
If $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$ and

$$f(x) \leq g(x) \leq h(x)$$

for all x in an open interval that contains c , except possibly c itself, $\lim_{x \rightarrow c} g(x) = L$. This is known as *Squeeze Theorem*.

2.4 Exponential Functions

Function of form $f(x)^{g(x)}$

- $\lim_{x \rightarrow +\infty} \left(\frac{x}{x+k}\right)^x = e^{-k}$
- $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$
- $\lim_{x \rightarrow 0} (1+kx)^{\frac{m}{x}} = e^{mk}$
- $\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x = e$
- $\lim_{x \rightarrow +\infty} \left(1 - \frac{1}{x}\right)^x = \frac{1}{e}$
- $\lim_{x \rightarrow +\infty} \left(1 + \frac{k}{x}\right)^{mx} = e^{mk}$
- $\lim_{x \rightarrow 0} (1 + a(e^{-x} - 1))^{-\frac{1}{x}} = e^a$

Sum products and Composites

- $\lim_{x \rightarrow 0} \left(\frac{a^x - 1}{x}\right) = \ln a$
- $\lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x}\right) = 1$
- $\lim_{x \rightarrow 0} \left(\frac{e^{ax} - 1}{x}\right) = a$

2.5 Logarithmic Functions

- $\lim_{x \rightarrow 1} \frac{\ln x}{x-1} = 1$
- $\lim_{x \rightarrow 0} \frac{\ln(x+1)}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{-\ln(1+a(e^{-x}-1))}{x} = a$

Some cases

- $\lim_{x \rightarrow 0^+} \log_b x = -F(b)\infty$
- $\lim_{x \rightarrow \infty} \log_b x = F(b)\infty$

where $F(x) = 2H(x-1) - 1$ and $H(x)$ is Oliver Heaviside step function.

2.6 Trigonometric Functions

- $\lim_{x \rightarrow 0} \frac{\sin ax}{ax} = 1$ for $a \neq 0$
- $\lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \frac{a}{b}$ for $b \neq 0$
- $\lim_{x \rightarrow \infty} x \sin\left(\frac{1}{x}\right) = 1$
- $\lim_{x \rightarrow 0} \frac{\tan ax}{ax} = 1$ for $a \neq 0$
- $\lim_{x \rightarrow 0} \frac{\tan ax}{bx} = \frac{a}{b}$ for $b \neq 0$

2.7 Sums

- $\lim_{x \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} = \infty$
- $\lim_{x \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n\right) = \gamma$. This is Euler Mascheroni Constant.

2.8 Notable Special Limits

- $\lim_{x \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} = e$
- $\lim_{x \rightarrow \infty} 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \dots + \sqrt{2}}}} = \pi$

2.9 Taylor Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \infty$$

$$\ln(1-x) = -(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots + \infty)$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots\right)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \infty$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots + \infty$$

$$\sec x = x + \frac{x^2}{2} + \frac{5x^4}{24} + \dots + \infty$$

$$\arcsin x / \sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots + \infty$$

$$\arccos x / \cos^{-1} x = \frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots\right)$$

$$\arctan x / \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots + \infty$$

3 Differentiation

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3.1 Elementary functions

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(a^x) = a^x \ln a$
- $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$

- $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

3.2 Basic Theorems

- $\frac{d}{dx}(f \pm g) = f'(x) \pm g'(x)$
- $\frac{d}{dx}(kf(x)) = k \frac{d}{dx}(f(x))$
- $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + g(x)f'(x)$
- $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$
- $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

3.3 Inverse Trigonometric Functions

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

3.4 Using Substitution

- $\sqrt{x^2 + a^2} \implies x = a \tan \theta$
- $\sqrt{a^2 - x^2} \implies x = a \sin \theta$
- $\sqrt{x^2 - a^2} \implies x = a \sec \theta$
- $\sqrt{\frac{x+a}{a-x}} \implies x = a \cos \theta$

3.5 Parametric Differentiation

If $y = f(\theta)$ and $x = g(\theta)$ where θ is parameter then

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

3.6 Derivative of Determinant

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$ where $f, g, h, l, m, n, u, v, w$ are differentiable functions, then $F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$

4 Application of Derivatives

4.1 Equation of Tangent and Normal

Tangent at (x_1, y_1) is given by

$$(y - y_1) = f'(x_1)(x - x_1)$$

when $f'(x_1)$ is real and Normal at (x_1, y_1) is given by

$$(y - y_1) = \frac{-1}{f'(x_1)}(x - x_1)$$

when $f'(x_1)$ is non-zero and real.

Tangent from an external point

Given a point $\delta(a, b)$ which does not lie on the curve $y = f(x)$ then equation of possible tangents to the curve $y = f(x)$ passing through (a, b) can be found by solving for point of contact λ

$$f'(h) = \frac{f(h) - b}{h - a}$$

and equation of tangent

$$y - b = \frac{f(h) - b}{h - a}(x - a)$$

Length of tangent, normal, sub-tangent, sub-normal from point $\sigma(h, k)$ and slope m

$$\text{Length of Tangent} = |k| \sqrt{1 + \frac{1}{m^2}}$$

$$\text{Length of Normal} = |k| \sqrt{1 + m^2}$$

$$\text{Length of Sub-Tangent} = \left| \frac{k}{m} \right|$$

$$\text{Length of Sub-Normal} = |km|$$

Angle between the curves

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

4.2 Theorems

Rolle's Theorem

If a function f defined on $[a, b]$ and

- continuous on $[a, b]$
- derivable on (a, b)
- $f(a) = f(b)$

then there exists at least one real number c between a and b ($a < c < b$) such that $f'(c) = 0$.

Lagrange's Mean Value Theorem

If a function f defined on $[a, b]$ and

- continuous on $[a, b]$
- derivable on (a, b)

then there exists at least one real number c between a and b ($a < c < b$) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Cauchy's Mean Value theorem

If functions f and g defined on $[a, b]$ and

- continuous on $[a, b]$
- derivable on (a, b)
- $c \in (a, b)$ then

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

4.3 Maxima and Minima

If a function $y = f(x)$ is defined on interval X , then an interior point x_o of interval is called the point of *maximum* of function $f(x)$ [the point of *minimum* of function $f(x)$] if there exists a neighbourhood $U \in X$ of point x_o such that inequality $f(x) \leq f(x_o)$ [$f(x) \geq f(x_o)$] holds true within it.

A Necessary condition for the existence of an Extremum

At points of extremum the derivative $f'(x)$ is equal to zero or does not exist. The points at which $f'(x) = 0$ or does not exist are called *critical points*.

Sufficient conditions for the existence of an Extremum

1. Let the function $f(x)$ be continuous in some neighbourhood of point x_o

- If $f'(x) > 0$ at $x < x_o$ and $f'(x) < 0$ at $x > x_o$ (i.e if in moving from left to right through point x_o the derivative changes sign from plus to minus), then at point x_o the function reaches *maximum*.
- If $f'(x) < 0$ at $x < x_o$ and $f'(x) > 0$ at $x > x_o$ (i.e if in moving through the point x_o from left to right the derivative changes sign from minus to plus), then at point x_o the function reaches *minimum*.
- If the derivative does not change sign in moving through the point x_o , then there is no *extremum*.

2. Let the function $f(x)$ be twice differentiable (that is $f'(x_o) = 0$) at a critical point x_o . If $f''(x_o) < 0$ then at x_o the function has a *maximum*; if $f''(x_o) > 0$ then at x_o the function has *minimum* but if $f''(x_o) = 0$ then the question of existence of *extremum* at this point remains open.

3. Let $f(x_o) = f''(x_o) = \dots = f^{n-1}(x_o) = 0$, but $f^n(x_o) \neq 0$. If n is even, then at $f^n(x_o) < 0$ there is a *maximum* at x_o , and at point $f^n(x_o) > 0$, a *minimum*. If n is odd then there is no *extremum* at point x_o .

4. Let the function $y = f(x)$ be represented parametrically:

$$x = \varphi(t), \quad y = \psi(t)$$

where the functions $\varphi(t)$ and $\psi(t)$ have derivatives both of first and second orders within a certain interval of change of argument t , and $\varphi'(t) \neq 0$. Further, let, at $t = t_o$

$$\psi'(t) = 0$$

Then:

- If $\psi''(t_o) < 0$, the function $y = f(x)$ has a *maximum* at $x = x_o = \varphi(t_o)$
- If $\psi''(t_o) > 0$, the function $y = f(x)$ has a *minimum* at $x = x_o = \varphi(t_o)$

- If $\psi''(t_o) = 0$, the question of existence of *extremum* remains open.

5 Integration

5.1 Standard Formula

- $\int (ax + b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$
- $\int \frac{dx}{ax+b} = \frac{1}{a} \ln(ax+b) + c$
- $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$
- $\int a^{px+q} dx = \frac{1}{p} \frac{a^{px+q}}{\ln a} + c$
- $\int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b) + c$
- $\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + c$
- $\int \tan(ax+b) dx = \frac{1}{a} \ln \sec(ax+b) + c$
- $\int \cot(ax+b) dx = \frac{1}{a} \ln \sin(ax+b) + c$
- $\int \sec^2(ax+b) dx = \frac{1}{a} \tan(ax+b) + c$
- $\int \operatorname{cosec}^2(ax+b) dx = -\frac{1}{a} \cot(ax+b) + c$
- $\int \sec x dx = \ln(\sec x + \tan x) + c$
- $\int \operatorname{cosec} x dx = \ln(\operatorname{cosec} x - \cot x) + c$
- $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln|x + \sqrt{x^2+a^2}| + c$
- $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + c$
- $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c$
- $\int \sqrt{a^2-x^2} dx = \frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c$
- $\int \sqrt{x^2+a^2} dx = \frac{x}{2} \sqrt{x^2+a^2} + \frac{a^2}{2} \ln \left(\frac{x+\sqrt{x^2+a^2}}{a} \right) + c$
- $\int \sqrt{x^2-a^2} dx = \frac{x}{2} \sqrt{x^2-a^2} - \frac{a^2}{2} \ln \left(\frac{x+\sqrt{x^2-a^2}}{a} \right) + c$

5.2 Integration of types

1. $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c} dx$
 \Rightarrow Put $x + \frac{b}{2a} = t$

2. $\int \frac{px+q}{ax^2+bx+c} dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx,$
 $\int (px+q)\sqrt{ax^2+bx+c} dx \Rightarrow$
 Put $x + \frac{b}{2a} = t$ then split the integral.

3. $\int \frac{dx}{a+b\sin^2 x}, \int \frac{dx}{a+b\cos^2 x}, \int \frac{dx}{a\sin^2 x+b\sin x\cos x+c\cos^2 x}$
 \Rightarrow Put $\tan x = t$

4. $\int \frac{dx}{a+b\sin x}, \int \frac{dx}{a+b\cos x}, \int \frac{dx}{a+b\sin x+c\cos x}$
 \Rightarrow Put $\tan(\frac{x}{2}) = t$

5. $\int \frac{dx}{(ax+b)\sqrt{px+q}}, \int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}} \Rightarrow$
 Put $px+q = t^2$

6. $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}} \Rightarrow$ Put $ax+b = \frac{1}{t}$

7. $\int \frac{dx}{ax^2+b\sqrt{px^2+q}} \Rightarrow$ Put $x = \frac{1}{t}$

8. $\int \frac{x^2+1dx}{x^4+\lambda x^2+1}$ where λ is any constant \Rightarrow
 Divide numerator and denominator by x^2 and Put
 $x \mp \frac{1}{x} = t$

5.3 Reduction Forms

- $\int \sin^n x dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx$

- $\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx$

- $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$

- $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$

- $\int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx$

- For $n > 1$**

- $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$

- $\int \sec^n x dx = \frac{\sec^{n-1} x \sin x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx$

5.4 Definite Integration

5.4.1 Properties

- $\int_a^b f(x) dx = \int_a^b f(t) dt$

- $\int_a^b f(x) dx = - \int_b^a f(x) dx$

- $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$