1 Fundamental

28 May 2023

1.1 Intervals

Intervals are basically subsets of $\mathbb{R}$ and are commonly used in solving inequalities or in finding domains. If there are two numbers $a, b \in \mathbb{R}$ such that $a < b$, there are four types of intervals:

- Open interval: $(a, b) = \{x : a < x < b\}$ i.e end points are not included. Symbols: () or][
- Closed interval: $[a, b] = \{x : a \leq x \leq b\}$ i.e end points are also included. Symbol: [ ]
- Open-Closed interval: $(a, b) = \{x : a < x \leq b\}$. Symbols: ( ] or]]
- Closed-Open interval: $(a, b) = \{x : a \leq x < b\}$. Symbols: [ ) or [[

Infinite Intervals

- $(a, \infty) = \{x : x > a\}$
- $[a, \infty) = \{x : x \geq a\}$
- $(-\infty, b) = \{x : x < b\}$
- $(\infty, b] = \{x : x \leq b\}$
- $(-\infty, \infty) = \{x : x \in \mathbb{R}\}$

1.2 Sets and Relations

1.2.1 Set

A collection of any kind of objects. The objects that make up a set are called *elements* or *members*. The statement 'a is an element of set A' can be written as $a \in A$ and set containing elements a, b and c is denoted by $\{a, b, c\}$. A *empty* or *null* set is denoted by $\varnothing$, which is the set that contains no elements.
Union(join,sum): The union of two sets A and B, denoted by $A \cup B$, consists of those elements that belong to A or to B:

$$A \cup B = \{x : (x \in A) \vee (x \in B)\}$$

For example, if A is $\{1, 2, 3, 4\}$ and B is $\{1, 4, 5, 6\}$ then $A \cup B$ is $\{1, 2, 3, 4, 5, 6\}$.
Intersection(meet,product): The intersection of two sets A and B, denoted by $A \cap B$, consists of those elements that belong to both A and B:

$$A \cap B = \{x : (x \in A) \wedge (x \in B)\}$$

For example, if A is $\{1, 2, 3, 4, 5, 6\}$ and B is $\{1, 4, 5, 6, 7, 8\}$ then $A \cap B$ is $\{1, 4, 5, 6\}$.
Complement: The complement of a set A, denoted by A' or $A^{\complement}$, consists of all those elements that are not members of A:

$$A' = \{x : x \notin A\}$$

For example, in the domain of natural numbers, if A is set of even numbers the its complement A' is the set of odd numbers.
Universal Set: Relative to a particular domain, the universal set, denoted by $\mathbf{U}$, is set of all objects of that domain:

$$\mathbf{U} = \{x : x = x\}$$

1.2.2 Properties of sets

Commutative law:

- $(A \cup B) = B \cup A$
- $(A \cap B) = B \cap A$

Associative law:

- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C = A \cap (B \cap C)$

Distributive law:

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

De Morgan's law:

- $(A \cup B)' = A' \cap B'$
- $(A \cap B)' = A' \cup B'$

Identity law:

- $A \cup \mathbf{U} = A$
- $A \cup \varnothing = A$

Complement law:

- $A \cup A' = \mathbf{U}$
- $A \cap A' = \varnothing$
- $(A')' = A$

Idempotent law:

- $A \cap A = A$
- $A \cup A = A$

1.2.3 Results on number of elements in sets

If A, B, C are finite sets and $\mathbf{U}$ be Universal finite set then:

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A - B) = n(A) - n(A \cap B)$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

1.2.4 Relations

An association between, or property of, two or more objects. Thus ($x = y$) and (a lies between b and c) are relations, but (N is a prime) is not.

1.2.5 Types of Relations

Void Relation: Let A be a set. Then $\phi \subseteq A \times A$ and so it is a relation on A. This relation is also called empty relation on A.
Universal Relation: Let A be a set. Then $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A.
Identity relation: Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$is called the identity relation on A. In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only.
Reflexive Relation: A relation R on a set A is said to be reflexive if every element of A is related to itself. Thus R on set A is not reflexive if there exits an element $a \in A$ such that $(a, a) \notin R$.
Symmetric Relation: A Relation R on set A is said to be symmetric if

$$(a, b) \in R \implies (b, a) \in R \,\forall\, a, b \in A$$

i.e

$$a\,R\,b \implies b\,R\,a \,\forall\, a, b \in A$$

Transitive Relation: A Relation R on set A is said to be transitive if $(a, b) \in R$ and $(b, c) \in R \implies (a, c) \in R \,\forall\, a, b, c \in A$.
Equivalence Relation: A Relation is said to be equivalence if it satisfies $reflexive, symmetric$ and $transitive$ relations.
Equivalence Class: If R is an equivalence relation defined on set A then the equivalence class of any element $x \in A$, denoted by $[x]$, is the set of elements to which x is related by equivalence relation R:

$$[x] = \{y : x\,\mathbf{R}\,y\}$$

For example, if R is the equivalence relation (the same height as), then the equivalence class of the element $x \in A$ consists of all elements of A with same height as x.

2 Quadratic Equations

29 May 2023

2.1 Equation and Basic Results

An equation of the form

$$ax^2 + bx + c = 0$$

where $a \neq 0$ and $a, b, c \in \mathbb{R}$ is called a quadratic equation. The roots of quadratic equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quantity D ($D = b^2 - 4ac$) is known as discriminant of equation.

2.1.1 Results

1. Let α and β be two roots of given quadratic equation. Then

- $\alpha + \beta = -\frac{b}{a}$
- $\alpha\beta = \frac{c}{a}$

2. A quadratic equation, whose roots are α and β can be written as

$$(x-\alpha)(x-\beta) = 0$$

i.e

$$ax^2 + bx + c \equiv a(x-\alpha)(x-\beta)$$

3. If the quadratic equation is satisfied by more than two distinct numbers (real or complex), then it becomes an identity, i.e

$$a = b = c = 0$$

4. The quadratic equation has real and equal roots if and only if $D = 0$ i.e $b^2 - 4ac = 0$.
5. The quadratic equation has real and distinct roots if and only if $D > 0$ i.e $b^2 - 4ac > 0$.
6. The quadratic equation has complex roots with non-zero imaginary parts if and only if $D < 0$ i.e $b^2 - 4ac < 0$.
7. If $p + iq$ $(p, q \in \mathbb{R})$ is root of quadratic equation where $i = \sqrt{-1}$, the $p - iq$ is also root of quadratic equation. Provided $a, b, c \in \mathbb{R}$.
8. If $p + \sqrt{q}$ is an irrational root of quadratic equation, then $p - \sqrt{q}$ is also a root of equation provided that all coefficients are rational.

2.2 Conditions for Common Root(s)

Let $ax^2 + bx + c = 0$ and $dx^2 + ex + f = 0$ have a common root α. Then $a\alpha^2 + b\alpha + c = 0$ and $d\alpha^2 + e\alpha + f = 0$.
Solving for α^2 and α :

$$\frac{\alpha^2}{bf - ce} = \frac{\alpha}{dc - af} = \frac{1}{ae - bd} \implies$$

$$\alpha^2 = \frac{bf - ce}{ae - bd}$$

and

$$\alpha = \frac{dc - af}{ae - bd} \implies$$

$$(dc - af)^2 = (bf - ce)(ae - bd)$$

which is a required condition for the two equation to have a common root.

- Condition for both the roots to be common is $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$

2.3 Wavy Curve Method

The Method of intervals is used for solving inequalities of the form

$$f(x) = \frac{(x-a_1)^{n_1}(x-a_2)^{n_2}...(x-a_k)^{n_k}}{(x-b_1)^{m_1}(x-b_2)^{m_2}...(x-b_p)^{n_p}} > 0 (< 0, \leq 0, or \geq 0)$$

where $n_1, n_2, n_3...n_k,\ m_1, m_2, m_3..m_p \in \mathbb{N}$ and the numbers $a_1, a_2, ...a_k, b_1, b_2, ...b_p \in \mathbb{R}$ such that $a_i \neq b_i$ where $i = 1, 2, 3, ...k$ and $j = 1, 2, 3, ...p$.

Statements

- All *zeros*[1] of the function $f(x)$ contained on left hand side of the inequality should be marked on the number line with black circles.
- All points of *discontinuities*[2] of the function $f(x)$ contained on left hand side of the inequality should be marked with white circles.
- Check the value of $f(x)$ for any real number greater than the right most marked number on the number line.
- From right to left, beginning above the number line, a wavy curve should be drawn which passes through all the marked points so that when passes through *single point*[3], the curve intersects the number line, and when passing through a *double point*[4], the curve remains located on the one side of number line.
- The appropriate intervals are chosen in accordance with sign of inequality. Their union just represents the solution of inequality.

Remarks

- The points of discontinuity will never be included in answers.
- *Zeros*[1]:The point for which $f(x)$ vanishes(becomes zero) is called the *zeros* of function e.g. $x = a_i$
- *Discontinuities*[2]:The points $x = b_j$ are the points of the *discontinuity* of the function.
- *Single point*[3]:If the exponents of factor is *odd* then the point is called *single point*.
- *Double point*[4]: If the exponent of factor is *even* then the point is called *double point*.