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# 1 Fundamental

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## 1.1 Intervals

Intervals are basically subsets of  $\mathbb{R}$  and are commonly used in solving inequalities or in finding domains. If there are two numbers  $a, b \in \mathbb{R}$  such that  $a < b$ , there are four types of intervals:

- Open interval:  $(a, b) = \{x : a < x < b\}$  i.e end points are not included. Symbols:  $()$  or  $] [$
- Closed interval:  $[a, b] = \{x : a \leq x \leq b\}$  i.e end points are also included. Symbol:  $[]$
- Open-Closed interval:  $(a, b) = \{x : a < x \leq b\}$ . Symbols:  $()]$  or  $] [$
- Closed-Open interval:  $(a, b) = \{x : a \leq x < b\}$ . Symbols:  $[])$  or  $] [$

### Infinite Intervals

- $(a, \infty) = \{x : x > a\}$
- $[a, \infty) = \{x : x \geq a\}$
- $(-\infty, b) = \{x : x < b\}$
- $(-\infty, b] = \{x : x \leq b\}$
- $(-\infty, \infty) = \{x : x \in \mathbb{R}\}$

## 1.2 Sets and Relations

### 1.2.1 Set

A collection of any kind of objects. The objects that make up a set are called *elements* or *members*. The statement ' $a$  is an element of set  $A$ ' can be written as  $a \in A$  and set containing elements  $a, b$  and  $c$  is denoted by  $\{a, b, c\}$ . A *empty* or *null* set is denoted by  $\emptyset$ , which is the set that contains no elements.

**Union(join,sum):** The union of two sets  $A$  and  $B$ , denoted by  $A \cup B$ , consists of those elements that belong to  $A$  or to  $B$ :

$$A \cup B = \{x : (x \in A) \vee (x \in B)\}$$

For example, if  $A$  is  $\{1, 2, 3, 4\}$  and  $B$  is  $\{1, 4, 5, 6\}$  then  $A \cup B$  is  $\{1, 2, 3, 4, 5, 6\}$ .

**Intersection(meet,product):** The intersection of two sets  $A$  and  $B$ , denoted by  $A \cap B$ , consists of those elements that belong to both  $A$  and  $B$ :

$$A \cap B = \{x : (x \in A) \wedge (x \in B)\}$$

For example, if  $A$  is  $\{1, 2, 3, 4, 5, 6\}$  and  $B$  is  $\{1, 4, 5, 6, 7, 8\}$  then  $A \cap B$  is  $\{1, 4, 5, 6\}$ .

**Complement:** The complement of a set  $A$ , denoted by  $A'$  or  $A^c$ , consists of all those elements that are not members of  $A$ :

$$A' = \{x : x \notin A\}$$

For example, in the domain of natural numbers, if  $A$  is set of even numbers the its complement  $A'$  is the set of odd numbers.

**Universal Set:** Relative to a particular domain, the universal set, denoted by  $\mathbf{U}$ , is set of all objects of that domain:

$$\mathbf{U} = \{x : x = x\}$$

### 1.2.2 Properties of sets

**Commutative law:**

- $(A \cup B) = B \cup A$
- $(A \cap B) = B \cap A$

**Associative law:**

- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C = A \cap (B \cap C)$

**Distributive law:**

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

**De Morgan's law:**

- $(A \cup B)' = A' \cap B'$
- $(A \cap B)' = A' \cup B'$

**Identity law:**

- $A \cup \mathbf{U} = A$
- $A \cup \emptyset = A$

**Complement law:**

- $A \cup A' = \mathbf{U}$
- $A \cap A' = \emptyset$
- $(A')' = A$

**Idempotent law:**

- $A \cap A = A$
- $A \cup A = A$

### 1.2.3 Results on number of elements in sets

If  $A, B, C$  are finite sets and  $\mathbf{U}$  be Universal finite set then:

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A - B) = n(A) - n(A \cap B)$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

### 1.2.4 Relations

An association between, or property of, two or more objects. Thus  $(x = y)$  and  $(a \text{ lies between } b \text{ and } c)$  are relations, but  $(N \text{ is a prime})$  is not.

### 1.2.5 Types of Relations

**Void Relation:** Let  $A$  be a set. Then  $\phi \subseteq A \times A$  and so it is a relation on  $A$ . This relation is also called empty relation on  $A$ .

**Universal Relation:** Let  $A$  be a set. Then  $A \times A \subseteq A \times A$  and so it is a relation on  $A$ . This relation is called the universal relation on  $A$ .

**Identity relation:** Let  $A$  be a set. Then the relation  $I_A = \{(a, a) : a \in A\}$  is called the identity relation on  $A$ . In other words, a relation  $I_A$  on  $A$  is called the identity relation if every element of  $A$  is related to itself only.

**Reflexive Relation:** A relation  $R$  on a set  $A$  is said to be reflexive if every element of  $A$  is related to itself. Thus  $R$  on set  $A$  is not reflexive if there exists an element  $a \in A$  such that  $(a, a) \notin R$ .

**Symmetric Relation:** A Relation  $R$  on set  $A$  is said to be symmetric if

$$(a, b) \in R \implies (b, a) \in R \forall a, b \in A$$

i.e

$$a R b \implies b R a \forall a, b \in A$$

**Transitive Relation:** A Relation  $R$  on set  $A$  is said to be transitive if  $(a, b) \in R$  and  $(b, c) \in R \implies (a, c) \in R \forall a, b, c \in A$ .

**Equivalence Relation:** A Relation is said to be equivalence if it satisfies *reflexive*, *symmetric* and *transitive* relations.

**Equivalence Class:** If  $R$  is an equivalence relation defined on set  $A$  then the equivalence class of any element  $x \in A$ , denoted by  $[x]$ , is the set of elements to which  $x$  is related by equivalence relation  $R$ :

$$[x] = \{y : x R y\}$$

For example, if  $R$  is the equivalence relation (the same height as), then the equivalence class of the element  $x \in A$  consists of all elements of  $A$  with same height as  $x$ .

## 2 Quadratic Equations

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## 2.1 Equation and Basic Results

An equation of the form

$$ax^2 + bx + c = 0$$

where  $a \neq 0$  and  $a, b, c \in \mathbb{R}$  is called a quadratic equation. The roots of quadratic equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quantity  $D$  ( $D = b^2 - 4ac$ ) is known as discriminant of equation.

### 2.1.1 Results

1. Let  $\alpha$  and  $\beta$  be two roots of given quadratic equation. Then

- $\alpha + \beta = -\frac{b}{a}$
- $\alpha\beta = \frac{c}{a}$

2. A quadratic equation, whose roots are  $\alpha$  and  $\beta$  can be written as

$$(x - \alpha)(x - \beta) = 0$$

i.e

$$ax^2 + bx + c \equiv a(x - \alpha)(x - \beta)$$

3. If the quadratic equation is satisfied by more than two distinct numbers (real or complex), then it becomes an identity, i.e

$$a = b = c = 0$$

4. The quadratic equation has real and equal roots if and only if  $D = 0$  i.e  $b^2 - 4ac = 0$ .

5. The quadratic equation has real and distinct roots if and only if  $D > 0$  i.e  $b^2 - 4ac > 0$ .

6. The quadratic equation has complex roots with non-zero imaginary parts if and only if  $D < 0$  i.e  $b^2 - 4ac < 0$ .

7. If  $p + iq$  ( $p, q \in \mathbb{R}$ ) is root of quadratic equation where  $i = \sqrt{-1}$ , the  $p - iq$  is also root of quadratic equation. Provided  $a, b, c \in \mathbb{R}$ .

8. If  $p + \sqrt{q}$  is an irrational root of quadratic equation, then  $p - \sqrt{q}$  is also a root of equation provided that all coefficients are rational.

## 2.2 Conditions for Common Root(s)

Let  $ax^2 + bx + c = 0$  and  $dx^2 + ex + f = 0$  have a common root  $\alpha$ . Then  $a\alpha^2 + b\alpha + c = 0$  and  $d\alpha^2 + e\alpha + f = 0$ .

Solving for  $\alpha^2$  and  $\alpha$  :

$$\frac{\alpha^2}{bf - ce} = \frac{\alpha}{dc - af} = \frac{1}{ae - bd} \implies$$
$$\alpha^2 = \frac{bf - ce}{ae - bd}$$

and

$$\alpha = \frac{dc - af}{ae - bd} \implies$$
$$(dc - af)^2 = (bf - ce)(ae - bd)$$

which is a required condition for the two equation to have a common root.

- Condition for both the roots to be common is  $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$

## 2.3 Wavy Curve Method

The Method of intervals is used for solving inequalities of the form

$$f(x) = \frac{(x - a_1)^{n_1}(x - a_2)^{n_2} \dots (x - a_k)^{n_k}}{(x - b_1)^{m_1}(x - b_2)^{m_2} \dots (x - b_p)^{n_p}} > 0 (< 0, \leq 0, \text{ or } \geq 0)$$

where  $n_1, n_2, n_3 \dots n_k, m_1, m_2, m_3 \dots m_p \in \mathbb{N}$  and the numbers  $a_1, a_2, \dots a_k, b_1, b_2, \dots b_p \in \mathbb{R}$  such that  $a_i \neq b_j$  where  $i = 1, 2, 3, \dots k$  and  $j = 1, 2, 3, \dots p$ .

**Statements**

- All *zeros*<sup>1</sup> of the function  $f(x)$  contained on left hand side of the inequality should be marked on the number line with black circles.
- All points of *discontinuities*<sup>2</sup> of the function  $f(x)$  contained on left hand side of the inequality should be marked with white circles.
- Check the value of  $f(x)$  for any real number greater than the right most marked number on the number line.

- From right to left, beginning above the number line, a wavy curve should be drawn which passes through all the marked points so that when passes through *singlepoint*<sup>3</sup>, the curve intersects the number line, and when passing through a *double point*<sup>4</sup>, the curve remains located on the one side of number line.
- The appropriate intervals are chosen in accordance with sign of inequality. Their union just represents the solution of inequality.

### Remarks

- The points of discontinuity will never be included in answers.
- *Zeros*<sup>1</sup>:The point for which  $f(x)$  vanishes(becomes zero) is called the *zeros* of function e.g.  $x = a_i$
- *Discontinuities*<sup>2</sup>:The points  $x = b_j$  are the points of the *discontinuity* of the function.
- *Single point*<sup>3</sup>:If the exponents of factor is *odd* then the point is called *single point*.
- *Double point*<sup>4</sup>: If the exponent of factor is *even* then the point is called *double point*.

## 2.4 Quadratic Expression

Let  $f(x) = ax^2 + bx + c$  where  $a, b, c \in \mathbb{R}$   $a \neq 0$  It can be rewritten as

$$f(x) = a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{D}{4a^2}\right\}$$

where  $D = b^2 - 4ac$ . Then  $y - f(x)$  represents a parabola whose axis is parallel to the y axis, with vertex at  $A(-\frac{b}{2a}, \frac{-D}{4a})$ . Now depending upon the values of  $a$  and  $D$  the parabola will have different shapes: will add images later...

## 2.5 Interval in which the roots lie

Let  $f(x) = ax^2 + bx + c$  where  $a, b, c \in \mathbb{R}$  and  $a \neq 0$ . Suppose  $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$  and  $\lambda_1 < \lambda_2$  Then the following results hold true:



- Both roots of  $f(x) = 0$  will be greater than  $\lambda$ , if  $D \geq 0$ ,  $af(\lambda) > 0$  and  $\lambda < -\frac{b}{2a}$ .
- Both roots of  $f(x) = 0$  will be less than  $\lambda$ , if  $D \geq 0$ ,  $af(\lambda) > 0$  and  $\lambda > -\frac{b}{2a}$ .
- If  $\lambda$  lies between the roots of  $f(x) = 0$  then  $af(\lambda) < 0$ .
- Both roots of  $f(x) = 0$  will belong to interval  $(\lambda_1, \lambda_2)$ , if  $D \geq 0$ ,  $af(\lambda_1) > 0, af(\lambda_2) > 0$  and  $\lambda_1 < -\frac{b}{2a} < \lambda_2$ .
- Exactly one root of  $f(x) = 0$  will lie in  $(\lambda_1, \lambda_2)$  if  $f(\lambda_1).f(\lambda_2) < 0$ .
- Interval  $(\lambda_1, \lambda_2)$  will be contained between the roots of  $f(x) = 0$  if  $af(\lambda_1) < 0, af(\lambda_2) < 0$ .
- $\lambda$  will be the required root of  $f(x) = 0$  if  $f(\lambda) = 0$  and  $f'(\lambda) = 0$ .

### 3 Progression and Series

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#### 3.1 Arithmetic Progression

If  $a$  is the first term and  $d$  the common difference, the *A.P.* can be written as  $a, a + d, a + 2d, \dots$ . The  $n$ th term  $a_n$  is given by

$$a_n = a + (n - 1)d$$

The sum of  $S_n$  of the first terms of such an *A.P.* is given by

$$S_n = \frac{n}{2}(2a + (n - 1)d) = \frac{n}{2}(a + a_n)$$

If three terms are in *A.P.*, then the middle term is called the *Arimethic Mean* (*A.M.*) between the other two i.e  $a, b, c$  are in *A.P.* then

$$2b = a + c$$

is the *A.M* of  $a$  and  $c$ .

If  $a_1, a_2, a_3 \dots a_n$  are  $n$  numbers, then the arithmetic mean of these numbers

$$A.M. = \frac{1}{n}(a_1 + a_2 + a_3 + \dots + a_n)$$

The numbers  $A_1, A_2, A_3 \dots A_n$  are said to be *A.M.*'s between the numbers  $a$  and  $b$  if  $a, A_1, A_2 \dots A_n, b$  are in *A.P.* then

$$A_n = a + \frac{n(b-a)}{n+1} = \frac{a+nb}{n+1}$$

### 3.2 Geometric Progression

If  $a$  is the first term and  $r$  the common ratio, then the *G.P.* can be written as  $a, ar, ar^2 \dots$ . The  $n$ th term  $a_n$  is given by

$$a_n = ar^{n-1}$$

The sum  $S_n$  of first  $n$  terms of the *G.P.* is

$$S_n = \begin{cases} \frac{a(r^n-1)}{r-1}, & r \neq 1 \\ na, & r = 1 \end{cases}$$

If  $-1 < r < 1$ , then the sum of infinite *G.P.* is  $a + ar + ar^2 + \dots$

$$= \frac{a}{1-r}$$

If three terms are in *G.P.*, then the middle term is called the geometric mean *G.M.* between the two. So if  $a, b, c$  are in *G.P.* then  $b = \sqrt{ac}$  is geometric mean of  $a$  and  $c$ .

If  $a_1, a_2 \dots a_n$  are non-zero positive numbers, then their *G.M* is given by

$$G = (a_1 a_2 a_3 \dots a_n)^{1/n}$$

If  $G_1, G_2 \dots G_n$  are  $n$  geometric means between  $a$  and  $b$ ,  $a, G_1, G_2 \dots G_n, b$  will be a *G.P.*

$$G_n = a \left(1 + \sqrt[n+1]{\frac{b}{a}}\right)^n$$

### 3.3 Harmonic Progression

The sequence  $a_1, a_2, a_3 \dots a_n (a_i \neq 0)$  is said to be *H.P.* if the sequence  $\frac{1}{a_1}, \frac{1}{a_2} \dots \frac{1}{a_n}$  is an *A.P.*. The  $n$ th term  $a_n$  of the *H.P.* is

$$a_n = \frac{1}{a + (n-1)d}$$

where  $a = \frac{1}{a_i}$  and  $d = \frac{1}{a_2} - \frac{1}{a_1}$ .

### Harmonic Means:

If  $a$  and  $b$  are two non-zero numbers, then the harmonic mean of  $a$  and  $b$  is a number  $H$  such that the numbers  $a, H, b$  are in  $H.P.$  We have

$$\frac{1}{H} = \frac{1}{2} \left( \frac{1}{a} + \frac{1}{b} \right)$$

There is no formula for sum of  $n$  terms of an  $H.P.$ , if  $a_1, a_2 \dots a_n$  are  $n$  non-zero numbers, then the harmonic mean  $H$  of these numbers is given by

$$\frac{1}{H} = \frac{1}{n} \left( \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right)$$

The  $n$  number  $H_1, H_2 \dots H_n$  are said to be  $n$ -harmonic means between  $a$  and  $b$ , if  $a, H_1, H_2 \dots H_n, b$  are in  $H.P.$  if  $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2} \dots \frac{1}{H_n}, \frac{1}{b}$  are in  $A.P.$

$$\frac{1}{H_n} = \frac{1}{a} + \frac{n(a-b)}{(n+1)ab}$$

## 3.4 Arithmetico-Geometric Progression

The sum  $S_n$  of first  $n$  terms of an  $A.G.P.$  is obtained in the following way:

$$S_n = ab + (a+d)br + (a+2d)br^2 + \dots + (a+(n-2)d)br^{n-2} + (a+(n-1)d)br^{n-1}$$

Multiplying both sides by  $r$ , so that

$$rS_n = abr + (a+d)br^2 + (a+2d)br^3 + \dots + (a+(n-2)d)br^{n-1} + (a+(n-1)d)br^n$$

Subtracting, we get

$$S_n = \frac{ab}{1-r} + \frac{dbr(1-r^{n-1})}{(1-r)^2} - \frac{(a+(n-1)d)br^n}{1-r}$$

If  $-1 < r < 1$ , the sum of infinite number of terms of progression is

$$\lim_{n \rightarrow \infty} S_n = S = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$$

### 3.5 Misc. Progressions

**Some Important Results:**

- $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$
- $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$
- $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4} = (1 + 2 + 3 + \dots + n)^2$
- $1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n+1)}{30}$
- $1 + x + x^2 + x^3 + \dots = (1 - x)^{-1}$  if  $-1 < x < 1$
- $1 + 2x + 3x^2 + 4x^3 + \dots = (1 - x)^{-2}$  if  $-1 < x < 1$

**Method of Differences:**

Suppose  $a_1, a_2, a_3 \dots a_n$  is a sequence such that sequence  $a_2 - a_1, a_3 - a_2 \dots$  is either an *A.P.* or a *G.P.* The  $n$ th term  $a_n$  of this sequence is obtained as follows:

$$S_n = a_1 + a_2 + a_3 + \dots + a_{n-2} + a_{n-1} + a_n$$
$$\implies a_n = a_1 + [(a_2 - a_1) + (a_3 - a_2) + \dots + (a_n - a_{n-1})]$$

Since the term within the brackets are either in an *A.P.* or in a *G.P.*, we can find the value of  $a_n$  the  $n$ th term. We can now find the sum of  $n$  terms of sequence as

$$S = \sum_{k=1}^n a_k$$

### 3.6 Inequalities

**A.M.  $\geq$  G.M.  $\geq$  H.M**

Let  $a_1, a_2 \dots a_n$  be  $n$  positive real numbers, then we define their arithmetic mean  $A$ , geometric mean  $G$  and harmonic mean  $H$  as

$$A = \frac{a_1 + a_2 + \dots + a_n}{n}$$

$$G = (a_1 a_2 a_3 \dots a_n)^{1/n}$$

$$H = \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right)}$$

## 4 Binomial Theorem

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### 4.1 Basic Results

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \dots + \binom{n}{r}a^{n-r}x^r + \dots + \binom{n}{n}x^n = \sum_{r=0}^n \binom{n}{r}x^r.a^{n-r}$$

$$(a-x)^n = \binom{n}{0} - \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 - \dots + (-1)^n \binom{n}{n}x^n.a^{n-r} = \sum_{r=0}^n (-1)^r \binom{n}{r}x^r$$

### 4.2 Properties of Binomial Coefficient

- $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$
- $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots = 2^{n-1}$
- $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$
- $\frac{\binom{n}{r}}{\binom{n}{r-1}} = \frac{n-r+1}{r}$
- $r\binom{n}{r} = n\binom{n-1}{r-1}$

### 4.3 Basic Properties of Binomial Expansion of $\text{pow}(a+x,n)$

- There are  $n+1$  terms in expansion.
- Binomial coefficients of the terms equidistant from beginning and end are equal.
- General term of the expansion is  $T_{r+1} = \binom{n}{r}a^{n-r}x^r$
- Binomial coefficient of middle term is greatest binomial coefficient occurring in the expansion.

## 4.4 Multinomial Expansion

In the expansion  $(x_1 + x_2 + \dots + x_n)^m$  where  $m, n \in \mathbb{N}$  and  $x_1, x_2, \dots, x_n$  are independent variables,

- Total number of terms in expansion  $\binom{m+n-1}{n-1}$
- Coefficient of  $x_1^{r_1} x_2^{r_2} \dots x_n^{r_n}$  (where  $\sum_{i=1}^n r_i = m, r_i \in \mathbb{N} \cup [0]$ ) is  $\frac{m!}{r_1! r_2! \dots r_n!}$
- Sum of all coefficient is obtained by putting all the variables  $x_i$  equal to 1 and is  $n^m$