

Contents

1	Functions	3
1.1	Functions. Domain of Definition	3
1.2	Investigation of Functions	3
1.3	Inverse of Function	3
2	Limits	4
2.1	Existence	4
2.2	Indeterminate forms	4
2.3	List of limits	4
2.4	Exponential Functions	5
2.5	Logarithmic Functions	6
2.6	Trigonometric Functions	6
2.7	Sums	7
2.8	Notable Special Limits	7
2.9	Taylor Series	7
3	Differentiation	8
3.1	Elementary functions	8
3.2	Basic Theorems	8
3.3	Inverse Trigonometric Functions	9
3.4	Using Substitution	9
3.5	Parametric Differentiation	9
3.6	Derivative of Determinant	9
4	Application of Derivatives	10
4.1	Equation of Tangent and Normal	10
4.2	Theorems	11
4.3	Maxima and Minima	11
5	Integration	13
5.1	Standard Formula	13
5.2	Integration of types	14
5.3	Reduction Forms	15
5.4	Definite Integration	16
5.4.1	Properties	16
5.4.2	Inequalities	16

5.4.3	Leibniz Theorem	17
6	Other Integrals	17
6.1	Wallis' Integral	17
6.2	Pi Function	17
6.3	Gamma Function	17
6.4	Gaussian Integral	18
6.4.1	Gaussian Integral Proof	18
7	Differential Equation	18
7.1	DE of first order and first degree	19
7.1.1	Exact Equation	19
7.1.2	Variables Separable	19
7.1.3	Homogeneous Equations	19
7.1.4	Equations reducible to Homogeneous	20
7.1.5	Linear Equations	20
7.2	Bernoulli's Differential Equation	21
7.2.1	Transformations	21

1 Functions

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1.1 Functions. Domain of Definition

The independent variable x is defined by a set X of its values. If to each value of the independent variable $x \in X$ there corresponds one definite value of another variable y , then y is called the function of x with a domain of definition (or domain) X or, in functional notation, $y = y(x)$, or $y = f(x)$, or $y = \varphi(x)$, and so forth. The set of values of the function $y(x)$ is called the range of the given function.

1.2 Investigation of Functions

A function $f(x)$ defined on the set X is said to be non-decreasing on this set (respectively, increasing, non-increasing, decreasing), if for any numbers $x_1, x_2 \in X, x_1 < x_2$ the inequality $f(x_1) \leq f(x_2)$ (respectively, $f(x_1) < f(x_2)$, $f(x_1) \geq f(x_2)$, $f(x_1) > f(x_2)$) is satisfied. The function $f(x)$ is said to be monotonic on the set X if it possesses one of the four indicated properties. The function $f(x)$ is said to be bounded above (or below) on the set X if there exists a number M (or m) such that $f(x) \leq M \forall x \in X$. The function $f(x)$ is said to be bounded on the set X if it is bounded above and below. The function $f(x)$ is called periodic if there exists a number $T > 0$ such that $f(x+T) = f(x)$ for all x belonging to the domain of definition of the function (together with any point x the point $x + T$ must belong to the domain of definition). The least number T possessing this property (if such a number exists) is called the period of the function $f(x)$. The function $f(x)$ takes on the maximum value at the point $x_o \in X$ if $f(x_o) \geq f(x)$ for all $x \in X$, and the minimum value if $f(x_o) \leq f(x)$ for all $x \in X$. A function $f(x)$ defined on a set X which is symmetric *w.r.t* origin of coordinates is called even if $f(-x) = f(x)$, and odd if $f(x) = -f(x)$.

1.3 Inverse of Function

Let the function $y = f(x)$ be defined on the set X and have a range Y . If for each $y \in Y$ there exists a single value of x such that $f(x) = y$, then this correspondence defines a certain function $x = g(y)$ called inverse *w.r.t*

given function $y = f(x)$. The sufficient condition for the existence of an inverse function is a strict monotony of the original function $y = f(x)$. If the function increases(decreases), then the inverse function also increases(decreases). The graph of the inverse function $x = g(y)$ coincides with that of the function $y = f(x)$ if the independent variable is marked off along the y -axis. If the independent variable is laid off along the x -axis, i. e. if the inverse function is written in the form $y = g(x)$, then the graph of the inverse function will be symmetric to that of the function $y = f(x)$ with respect to the bisector of the first and third quadrants.

2 Limits

2.1 Existence

Limit of function $f(x)$ is said to exist as $x \rightarrow a$ when,

$$\lim_{h \rightarrow 0^+} f(a - h) = \lim_{h \rightarrow 0^+} f(a + h)$$

equal to some finite value L .

2.2 Indeterminate forms

There are only seven indeterminate forms
 $\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, \infty^0, 0^0$ and 1^∞ .

2.3 List of limits

Limits Operations

If $\lim_{x \rightarrow c} f(x) = L$

- $\lim_{x \rightarrow c} [f(x \pm a)] = L \pm a$
- $\lim_{x \rightarrow c} af(x) = aL$
- $\lim_{x \rightarrow c} \frac{1}{f(a)} = \frac{1}{L}$ for $L > 0$
- $\lim_{x \rightarrow c} f(x)^n = L^n$ for $n > 0$

Involving infinitesimal changes

If infinitesimal change h is denoted by Δx . If $f(x)$ and $g(x)$ are differentiable at x .

- $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} = f'(x)$
- $\lim_{h \rightarrow 0} \frac{f(g(x+h))-f(g(x))}{h} = f'[g(x)]g'(x)$
- $\lim_{h \rightarrow 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} = f'(x)g(x) + f(x)g'(x)$
- $\lim_{h \rightarrow 0} \left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}} = \exp\left(\frac{f'(x)}{f(x)}\right)$
- $\lim_{h \rightarrow 0} \left(\frac{f(e^h x)}{f(x)}\right)^{\frac{1}{h}} = \exp\left(\frac{x f'(x)}{f(x)}\right)$

If $f(x)$ and $g(x)$ are differentiable on an open interval containing c , except possibly c itself, and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$ or $\pm\infty$.

Jean Bernoulli or L'Hopital's rule can be used:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

Inequalities

If $f(x) \leq g(x)$ for all x in interval that contains c , except possibly c itself, and the limit of $f(x)$ and $g(x)$ both exist at c , then $\lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} g(x)$

If $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$ and

$$f(x) \leq g(x) \leq h(x)$$

for all x in an open interval that contains c , except possibly c itself, $\lim_{x \rightarrow c} g(x) = L$. This is known as *Squeeze Theorem*.

2.4 Exponential Functions

Function of form $f(x)^{g(x)}$

- $\lim_{x \rightarrow +\infty} \left(\frac{x}{x+k}\right)^x = e^{-k}$
- $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$
- $\lim_{x \rightarrow 0} (1+kx)^{\frac{m}{x}} = e^{mk}$

- $\lim_{x \rightarrow +\infty} (1 + \frac{1}{x})^x = e$
- $\lim_{x \rightarrow +\infty} (1 - \frac{1}{x})^x = \frac{1}{e}$
- $\lim_{x \rightarrow +\infty} (1 + \frac{k}{x})^{mx} = e^{mk}$
- $\lim_{x \rightarrow 0} (1 + a(e^{-x} - 1))^{-\frac{1}{x}} = e^a$

Sum products and Composites

- $\lim_{x \rightarrow 0} (\frac{a^x - 1}{x}) = \ln a$
- $\lim_{x \rightarrow 0} (\frac{e^x - 1}{x}) = 1$
- $\lim_{x \rightarrow 0} (\frac{e^{ax} - 1}{x}) = a$

2.5 Logarithmic Functions

- $\lim_{x \rightarrow 1} \frac{\ln x}{x-1} = 1$
- $\lim_{x \rightarrow 0} \frac{\ln(x+1)}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{-\ln(1+a(e^{-x}-1))}{x} = a$

Some cases

- $\lim_{x \rightarrow 0^+} \log_b x = -F(b)\infty$
- $\lim_{x \rightarrow \infty} \log_b x = F(b)\infty$

where $F(x) = 2H(x-1) - 1$ and $H(x)$ is Oliver Heaviside step function.

2.6 Trigonometric Functions

- $\lim_{x \rightarrow 0} \frac{\sin ax}{ax} = 1$ for $a \neq 0$
- $\lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \frac{a}{b}$ for $b \neq 0$
- $\lim_{x \rightarrow \infty} x \sin(\frac{1}{x}) = 1$
- $\lim_{x \rightarrow 0} \frac{\tan ax}{ax} = 1$ for $a \neq 0$
- $\lim_{x \rightarrow 0} \frac{\tan ax}{bx} = \frac{a}{b}$ for $b \neq 0$

2.7 Sums

- $\lim_{x \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} = \infty$
- $\lim_{x \rightarrow \infty} (\sum_{k=1}^n \frac{1}{k} - \log n) = \gamma$. This is Euler Mascheroni Constant.

2.8 Notable Special Limits

- $\lim_{x \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} = e$
- $\lim_{x \rightarrow \infty} 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \dots + \sqrt{2}}}} = \pi$

2.9 Taylor Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \infty$$

$$\ln(1-x) = -(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots + \infty)$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots\right)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \infty$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots + \infty$$

$$\sec x = x + \frac{x^2}{2} + \frac{5x^4}{24} + \dots + \infty$$

$$\arcsin x / \sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots + \infty$$

$$\arccos x / \cos^{-1} x = \frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots\right)$$

$$\arctan x / \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots + \infty$$

3 Differentiation

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3.1 Elementary functions

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(a^x) = a^x \ln a$
- $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\csc x) = -\csc x \cot x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$

3.2 Basic Theorems

- $\frac{d}{dx}(f \pm g) = f'(x) \pm g'(x)$
- $\frac{d}{dx}(kf(x)) = k \frac{d}{dx}(f(x))$
- $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + g(x)f'(x)$
- $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$
- $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

3.3 Inverse Trigonometric Functions

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

3.4 Using Substitution

- $\sqrt{x^2 + a^2} \implies x = a \tan \theta$
- $\sqrt{a^2 - x^2} \implies x = a \sin \theta$
- $\sqrt{x^2 - a^2} \implies x = a \sec \theta$
- $\sqrt{\frac{x+a}{a-x}} \implies x = a \cos \theta$

3.5 Parametric Differentiation

If $y = f(\theta)$ and $x = g(\theta)$ where θ is parameter then

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

3.6 Derivative of Determinant

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$ where $f, g, h, l, m, n, u, v, w$ are differentiable

functions, then $F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$

4 Application of Derivatives

4.1 Equation of Tangent and Normal

Tangent at (x_1, y_1) is given by

$$(y - y_1) = f'(x_1)(x - x_1)$$

when $f'(x_1)$ is real and Normal at (x_1, y_1) is given by

$$(y - y_1) = \frac{-1}{f'(x_1)}(x - x_1)$$

when $f'(x_1)$ is non-zero and real.

Tangent from an external point

Given a point $\delta(a, b)$ which does not lie on the curve $y = f(x)$ then equation of possible tangents to the curve $y = f(x)$ passing through (a, b) can be found by solving for point of contact λ

$$f'(h) = \frac{f(h) - b}{h - a}$$

and equation of tangent

$$y - b = \frac{f(h) - b}{h - a}(x - a)$$

Length of tangent, normal, sub-tangent, sub-normal from point $\sigma(h, k)$ and slope m

$$\text{Length of Tangent} = |k| \sqrt{1 + \frac{1}{m^2}}$$

$$\text{Length of Normal} = |k| \sqrt{1 + m^2}$$

$$\text{Length of Sub-Tangent} = \left| \frac{k}{m} \right|$$

$$\text{Length of Sub-Normal} = |km|$$

Angle between the curves

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

4.2 Theorems

Rolle's Theorem

If a function f defined on $[a, b]$ and

- continuous on $[a, b]$
- derivable on (a, b)
- $f(a) = f(b)$

then there exists at least one real number c between a and b ($a < c < b$) such that $f'(c) = 0$.

Lagrange's Mean Value Theorem

If a function f defined on $[a, b]$ and

- continuous on $[a, b]$
- derivable on (a, b)

then there exists at least one real number c between a and b ($a < c < b$) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Cauchy's Mean Value theorem

If functions f and g defined on $[a, b]$ and

- continuous on $[a, b]$
- derivable on (a, b)
- $c \in (a, b)$ then

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

4.3 Maxima and Minima

If a function $y = f(x)$ is defined on interval X , then an interior point x_o of interval is called the point of *maximum* of function $f(x)$ [the point of *minimum* of function $f(x)$] if there exists a neighbourhood $U \in X$ of point x_o such that inequality $f(x) \leq f(x_o)$ [$f(x) \geq f(x_o)$] holds true within it.

A Necessary condition for the existence of an Extremum

At points of extremum the derivative $f'(x)$ is equal to zero or does not exist. The points at which $f'(x) = 0$ or does not exist are called *critical points*.

Sufficient conditions for the existence of an Extremum

1. Let the function $f(x)$ be continuous in some neighbourhood of point x_o
 - If $f'(x) > 0$ at $x < x_o$ and $f'(x) < 0$ at $x > x_o$ (i.e if in moving from left to right through point x_o the derivative changes sign from plus to minus), then at point x_o the function reaches *maximum*.
 - If $f'(x) < 0$ at $x < x_o$ and $f'(x) > 0$ at $x > x_o$ (i.e if in moving through the point x_o from left to right the derivative changes sign from minus to plus), then at point x_o the function reaches *minimum*.
 - If the derivative does not change sign in moving through the point x_o , then there is no *extremum*.
2. Let the function $f(x)$ be twice differentiable (that is $f'(x_o) = 0$) at a critical point x_o . If $f''(x_o) < 0$ then at x_o the function has a *maximum*; if $f''(x_o) > 0$ then at x_o the function has *minimum* but if $f''(x_o) = 0$ then the question of existence of *extremum* at this point remains open.
3. Let $f(x_o) = f''(x_o) = \dots = f^{n-1}(x_o) = 0$, but $f^n(x_o) \neq 0$. If n is even, then at $f^n(x_o) < 0$ there is a *maximum* at x_o , and at point $f^n(x_o) > 0$, a *minimum*. If n is odd then there is no *extremum* at point x_o .
4. Let the function $y = f(x)$ be represented parametrically:

$$x = \varphi(t), y = \psi(t)$$

where the functions $\varphi(t)$ and $\psi(t)$ have derivatives both of first and second orders within a certain interval of change of argument t , and $\varphi'(t) \neq 0$. Further, let, at $t = t_o$

$$\psi'(t) = 0$$

Then:

- If $\psi''(t_o) < 0$, the function $y = f(x)$ has a *maximum* at $x = x_o = \varphi(t_o)$
- If $\psi''(t_o) > 0$, the function $y = f(x)$ has a *minimum* at $x = x_o = \varphi(t_o)$
- If $\psi''(t_o) = 0$, the question of existence of *extremum* remains open.

5 Integration

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5.1 Standard Formula

- $\int (ax + b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$
- $\int \frac{dx}{ax+b} = \frac{1}{a} \ln(ax + b) + c$
- $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$
- $\int a^{px+q} dx = \frac{1}{p} \frac{a^{px+q}}{\ln a} + c$
- $\int \sin(ax + b) dx = -\frac{1}{a} \cos(ax + b) + c$
- $\int \cos(ax + b) dx = \frac{1}{a} \sin(ax + b) + c$
- $\int \tan(ax + b) dx = \frac{1}{a} \ln \sec(ax + b) + c$
- $\int \cot(ax + b) dx = \frac{1}{a} \ln \sin(ax + b) + c$
- $\int \sec^2(ax + b) dx = \frac{1}{a} \tan(ax + b) + c$
- $\int \csc^2(ax + b) dx = -\frac{1}{a} \cot(ax + b) + c$
- $\int \sec x dx = \ln(\sec x + \tan x) + c$
- $\int \csc x dx = \ln(\csc x - \cot x) + c$
- $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{dx}{|x|\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln[x + \sqrt{x^2 + a^2}] + c$
- $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + c$
- $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c$

- $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c$
- $\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right) + c$
- $\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) + c$

5.2 Integration of types

By Partial Fraction

A method of integrating rational functions that are fractions in which the denominator has a higher degree than the numerator. For example the integral

$$\int \frac{x+3}{x^2+3x+2} dx$$

can be put in the form

$$\frac{A}{x+2} + \frac{B}{x+1}$$

A and B can be found by putting this expression in the form

$$\frac{A(x+1) + B(x+2)}{x^2+3x+2}$$

Then

$$x+3 = (A+B)x + (A+2B)$$

Coefficient of like power are equated to give $A+B=1$ and $A+2B=3$ i.e $A=-1$ and $B=2$. Thus the partial fractions becomes

$$\int \frac{2}{x+1} dx - \int \frac{1}{x+2} dx$$

By Parts

A method of integration using the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

For example, it is possible to integrate $x \cos x$ using $x = u$ and $\cos x = \frac{dv}{dx}$ so that $\frac{du}{dx} = 1$ and $v = \sin x$. Then the formula gives

$$\int x \cos x dx = x \sin x - \int \sin x dx$$

$$= x \sin x + \cos x$$

Other types

$$1. \int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c} dx \\ \Rightarrow \text{Put } x + \frac{b}{2a} = t$$

$$2. \int \frac{px+q}{ax^2+bx+c} dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx, \\ \int (px+q)\sqrt{ax^2+bx+c} dx \Rightarrow \\ \text{Put } x + \frac{b}{2a} = t \text{ then split the integral.}$$

$$3. \int \frac{dx}{a+b\sin^2 x}, \int \frac{dx}{a+b\cos^2 x}, \int \frac{dx}{a\sin^2 x+b\sin x \cos x+c\cos^2 x} \\ \Rightarrow \text{Put } \tan x = t$$

$$4. \int \frac{dx}{a+b\sin x}, \int \frac{dx}{a+b\cos x}, \int \frac{dx}{a+b\sin x+c\cos x} \\ \Rightarrow \text{Put } \tan\left(\frac{x}{2}\right) = t$$

$$5. \int \frac{dx}{(ax+b)\sqrt{px+q}}, \int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}} \Rightarrow \\ \text{Put } px+q = t^2$$

$$6. \int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}} \Rightarrow \text{Put } ax+b = \frac{1}{t}$$

$$7. \int \frac{dx}{ax^2+b\sqrt{px^2+q}} \Rightarrow \text{Put } x = \frac{1}{t}$$

$$8. \int \frac{x^2+1dx}{x^4+\lambda x^2+1} \text{ where } \lambda \text{ is any constant} \Rightarrow \text{Divide numerator and de-} \\ \text{nominator by } x^2 \text{ and Put } x \mp \frac{1}{x} = t$$

5.3 Reduction Forms

- $\int \sin^n x dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx$
- $\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx$
- $\int e^{ax} \sin bxdx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$
- $\int e^{ax} \cos bxdx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$
- $\int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx$
- **For n>1**

- $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$
- $\int \sec^n x dx = \frac{\sec^{n-1} x \sin x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx$

5.4 Definite Integration

5.4.1 Properties

- $\int_a^b f(x) dx = \int_a^b f(t) dt$
- $\int_a^b f(x) dx = - \int_b^a f(x) dx$
- $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$
- $\int_{-a}^a f(x) dx = \int_0^a (f(x) + f(-x)) dx = \begin{cases} 2 \int_0^a f(x) dx, & f(-x) = f(x) \\ 0, & f(-x) = -f(x) \end{cases}$
- $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$
- $\int_0^a f(x) dx + \int_0^{f(a)} f^{-1}(x) dx = af(a)$

If $f(x)$ is a periodic function with period T

- $\int_0^{nT} f(x) dx = n \int_0^T f(x) dx, n \in \mathbb{Z}$
- $\int_a^{a+nT} f(x) dx = n \int_0^T f(x) dx, n \in \mathbb{Z}, a \in \mathbb{R}$
- $\int_{mT}^{nT} f(x) dx = (n-m) \int_0^T f(x) dx, m, n \in \mathbb{Z}$
- $\int_{nT}^{a+nT} f(x) dx = \int_0^a f(x) dx, n \in \mathbb{Z}, a \in \mathbb{R}$
- $\int_{a+nT}^{b+nT} f(x) dx = \int_a^b f(x) dx, n \in \mathbb{Z}, a, b \in \mathbb{R}$

5.4.2 Inequalities

1. If $\Psi(x) \leq f(x) \leq \phi(x)$ for $a \leq x \leq b$, then

$$\int_a^b \Psi(x) dx \leq \int_a^b f(x) dx \leq \int_a^b \phi(x) dx$$

2. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

3. If $f(x) \geq 0$ on $[a, b]$, then

$$\int_a^b f(x)dx \geq 0$$

5.4.3 Leibniz Theorem

If $\varphi(x) = \int_{g(x)}^{h(x)} f(t)dt$, then

$$\frac{d}{dx}(\varphi(x)) = h'(x)f(h(x)) - g'(x)f(g(x))$$

6 Other Integrals

6.1 Wallis' Integral

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx = \begin{cases} \frac{\pi}{2} \frac{(m-1)!!(n-1)!!}{(m+n)!!}, & \text{when } m \text{ and } n \text{ both are even.} \\ \frac{(m-1)!!(n-1)!!}{(m+n)!!}, & \text{else} \end{cases}$$

6.2 Pi Function

$$\Pi(n) = \int_0^\infty x^n e^{-x} dx$$

Properties:

- $\Pi(n+1) = (n+1)\Pi(n)$
- $\Pi(0) = 1 \implies \Pi(n) = n!$

6.3 Gamma Function

$$\Gamma(n) = \Pi(n-1) = \int_0^\infty x^{n-1} e^{-x} dx$$

Some Properties:

- $\Gamma(n+1) = n\Gamma(n)$
- $\Gamma(n) = (n-1)!, \Gamma(\frac{n}{2}) = \frac{2^{(1-n)}(n-1)!\sqrt{\pi}}{(\frac{n-1}{2})!}$

6.4 Gaussian Integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

6.4.1 Gaussian Integral Proof

Proof. Substitute $x^2 = u \implies 2x dx = du$

$$\int_{-\infty}^{\infty} e^{-x^2} dx \implies \frac{1}{2} \int_{-\infty}^{\infty} u^{-\frac{1}{2}} e^{-u} du$$

Using property $[\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx]$ for $[f(-x) = f(x)]$

$$= \int_0^{\infty} u^{-\frac{1}{2}} e^{-u} du$$

It is type of $\Gamma(n)$ for $n = \frac{1}{2}$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Hence proved. □

7 Differential Equation

A relationship between an independent variable x , a dependent variable y , and one or more of derivatives of y w.r.t x .

A simple example of differential equation is

$$\frac{dy}{dx} = x$$

Order and Degree of DE

Order: The order of highest-order derivative in a differential equation.

Degree: The power to which the highest-order derivative is raised in a differential equation.

Solution

A solution of a differential equation is function that, when substituted for the dependent variable in equation, leads to an identity. Thus for above example $y = \frac{1}{2}x^2 + c$ is a *solution*.

7.1 DE of first order and first degree

7.1.1 Exact Equation

Equation of the form:

$$P\left(\frac{dy}{dx}\right) + Q = 0$$

are exact if left-hand side is differential coefficient of some function $f(x, y)$ w.r.t x . Integration gives the *solution* $f(x, y) = C$. An *exact* equation is one in which the total differential of function f is equal to zero.

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

Thus an equation $Ax + by = 0$ is exact if

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$

7.1.2 Variables Separable

In this case, the equation can be written in the form

$$f(x) + g(x)\frac{dy}{dx} = 0$$

Rearrangement gives

$$f(x)dx = -g(y)dy$$

Both sides then can be integrated.

7.1.3 Homogeneous Equations

These can be written in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

The method of solution is to make substitution $y = vx$, which reduces the equation to one in v and x only. Resulting, the variables are separable.

7.1.4 Equations reducible to Homogeneous

Equation of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

can be handled by making substitution $x = X + h$ and $y = Y + k$ where h and k are constants. Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dY}{dX} \\ &= \frac{a_1(X + h) + b_1(Y + k) + c_1}{a_2(X + h) + b_2(Y + k) + c_2}\end{aligned}$$

If h and k are chosen to be the values of x and y , respectively, that satisfy the simultaneous equations

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

Then original equation becomes

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

which is homogeneous.

However if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ then h and k cannot be chosen as above. In this case, let $a_2 = ma_1$ and $u = a_1x + b_1y$. The equation becomes

$$\frac{du}{dx} - a_1 = b_1 \frac{u + c_1}{mu + c_2}$$

and u and x can be separated.

7.1.5 Linear Equations

Equation of the form

$$\frac{dy}{dx} + Py = Q$$

where P and Q are the functions of x , or constants, are said to be linear in y and can be solved by multiplying integrating factor

$$e^{\int P dx}$$

This makes left hand side of equation an exact differential:

$$e^{\int P dx} \left(\frac{dy}{dx} \right) + e^{\int P dx} (Py) = e^{\int P dx} Q$$

$$\frac{d}{dx} [e^{\int P dx} y] = e^{\int P dx} Q$$

$$ye^{\int P dx} = \int e^{\int P dx} Q dx + c$$

7.2 Bernoulli's Differential Equation

A first order differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, n \in \mathbb{R}$$

7.2.1 Transformations

When $n = 0$ the differential equation is linear and $n = 1$, it is variable separable. For $n \neq 0, 1$ The substitution $u = y^{1-n}$ reduces Bernoulli equation to linear differential equation.

$$\frac{du}{dx} - (n-1)P(x)u = -(n-1)Q(x)$$

For example:

In case of $n = 2$, making substitution $u = y^{-1}$ in the differential equation

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2$$

produces the equation

$$\frac{du}{dx} - \frac{1}{u} = -x$$

which is a linear equation.