

# 1 Functions

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## 1.1 Functions. Domain of Definition

The independent variable  $x$  is defined by a set  $X$  of its values. If to each value of the independent variable  $x \in X$  there corresponds one definite value of another variable  $y$ , then  $y$  is called the function of  $x$  with a domain of definition (or domain)  $X$  or, in functional notation,  $y = y(x)$ , or  $y = f(x)$ , or  $y = \varphi(x)$ , and so forth. The set of values of the function  $y(x)$  is called the range of the given function.

## 1.2 Investigation of Functions

A function  $f(x)$  defined on the set  $X$  is said to be non-decreasing on this set (respectively, increasing, non-increasing, decreasing), if for any numbers  $x_1, x_2 \in X, x_1 < x_2$  the inequality  $f(x_1) \leq f(x_2)$  (respectively,  $f(x_1) < f(x_2), f(x_1) \geq f(x_2), f(x_1) > f(x_2)$ ) is satisfied. The function  $f(x)$  is said to be monotonic on the set  $X$  if it possesses one of the four indicated properties. The function  $f(x)$  is said to be bounded above (or below) on the set  $X$  if there exists a number  $M$  (or  $m$ ) such that  $f(x) \leq M \forall x \in X$ . The function  $f(x)$  is said to be bounded on the set  $X$  if it is bounded above and below. The function  $f(x)$  is called periodic if there exists a number  $T > 0$  such that  $f(x+T) = f(x)$  for all  $x$  belonging to the domain of definition of the function (together with any point  $x$  the point  $x+T$  must belong to the domain of definition). The least number  $T$  possessing this property (if such a number exists) is called the period of the function  $f(x)$ . The function  $f(x)$  takes on the maximum value at the point  $x_o \in X$  if  $f(x_o) \geq f(x)$  for all  $x \in X$ , and the minimum value if  $f(x_o) \leq f(x)$  for all  $x \in X$ . A function  $f(x)$  defined on a set  $X$  which is symmetric *w.r.t* origin of coordinates is called even if  $f(-x) = f(x)$ , and odd if  $f(x) = -f(x)$ .

## 1.3 Inverse of Function

Let the function  $y = f(x)$  be defined on the set  $X$  and have a range  $Y$ . If for each  $y \in Y$  there ex-

ists a single value of  $x$  such that  $f(x) = y$ , then this correspondence defines a certain function  $x = g(y)$  called inverse *w.r.t* given function  $y = f(x)$ . The sufficient condition for the existence of an inverse function is a strict monotony of the original function  $y = f(x)$ . If the function increases(decreases), then the inverse function also increases (decreases). The graph of the inverse function  $x = g(y)$  coincides with that of the function  $y = f(x)$  if the independent variable is marked off along the  $y$ -axis. If the independent variable is laid off along the  $x$ -axis, i. e. if the inverse function is written in the form  $y = g(x)$ , then the graph of the inverse function will be symmetric to that of the function  $y = f(x)$  with respect to the bisector of the first and third quadrants.

# 2 Limits

## 2.1 Existence

Limit of function  $f(x)$  is said to exist as  $x \rightarrow a$  when,

$$\lim_{h \rightarrow 0^+} f(a-h) = \lim_{h \rightarrow 0^+} f(a+h)$$

equal to some finite value  $L$ .

## 2.2 Indeterminate forms

There are only seven indeterminate forms  $\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, \infty^0, 0^0$  and  $1^\infty$ .

## 2.3 List of limits

### Limits Operations

If  $\lim_{x \rightarrow c} f(x) = L$

- $\lim_{x \rightarrow c} [f(x \pm a)] = L \pm a$
- $\lim_{x \rightarrow c} a f(x) = aL$
- $\lim_{x \rightarrow c} \frac{1}{f(x)} = \frac{1}{L}$  for  $L > 0$
- $\lim_{x \rightarrow c} f(x)^n = L^n$  for  $n > 0$

### Involving infinitesimal changes

If infinitesimal change  $h$  if denote by  $\Delta x$ . If  $f(x)$  and  $g(x)$  are differentiable at  $x$ .

- $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} = f'(x)$
- $\lim_{h \rightarrow 0} \frac{f \circ g(x+h)-f \circ g(x)}{h} = f'[g(x)]g'(x)$
- $\lim_{h \rightarrow 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} = f'(x)g(x) + f(x)g'(x)$
- $\lim_{h \rightarrow 0} \left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}} = \exp\left(\frac{f'(x)}{f(x)}\right)$
- $\lim_{h \rightarrow 0} \left(\frac{f(e^h x)}{f(x)}\right)^{\frac{1}{h}} = \exp\left(\frac{x f'(x)}{f(x)}\right)$

If  $f(x)$  and  $g(x)$  are differentiable on an open interval containing  $c$ , except possibly  $c$  itself, and  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$  or  $\pm\infty$ .

Jean Bernoulli or L'Hopital's rule can be used:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

### Inequalities

If  $f(x) \leq g(x)$  for all  $x$  in interval that contains  $c$ , except possibly  $c$  itself, and the limit of  $f(x)$  and  $g(x)$  both exist at  $c$ , then  $\lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} g(x)$ .  
If  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$  and

$$f(x) \leq g(x) \leq h(x)$$

for all  $x$  in an open interval that contains  $c$ , except possibly  $c$  itself,  $\lim_{x \rightarrow c} g(x) = L$ . This is known as *Squeeze Theorem*.

## 2.4 Exponential Functions

Function of form  $f(x)^{g(x)}$

- $\lim_{x \rightarrow +\infty} \left(\frac{x}{x+k}\right)^x = e^{-k}$
- $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$
- $\lim_{x \rightarrow 0} (1+kx)^{\frac{m}{x}} = e^{mk}$
- $\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x = e$
- $\lim_{x \rightarrow +\infty} \left(1 - \frac{1}{x}\right)^x = \frac{1}{e}$
- $\lim_{x \rightarrow +\infty} \left(1 + \frac{k}{x}\right)^{mx} = e^{mk}$
- $\lim_{x \rightarrow 0} (1 + a(e^{-x} - 1))^{-\frac{1}{x}} = e^a$

Sum products and Composites

- $\lim_{x \rightarrow 0} \left(\frac{a^x - 1}{x}\right) = \ln a$
- $\lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x}\right) = 1$
- $\lim_{x \rightarrow 0} \left(\frac{e^{ax} - 1}{x}\right) = a$

## 2.5 Logarithmic Functions

- $\lim_{x \rightarrow 1} \frac{\ln x}{x-1} = 1$
- $\lim_{x \rightarrow 0} \frac{\ln(x+1)}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{-\ln(1+a(e^{-x}-1))}{x} = a$

Some cases

- $\lim_{x \rightarrow 0^+} \log_b x = -F(b)\infty$
- $\lim_{x \rightarrow \infty} \log_b x = F(b)\infty$

where  $F(x) = 2H(x-1) - 1$  and  $H(x)$  is Oliver Heaviside step function.

## 2.6 Trigonometric Functions

- $\lim_{x \rightarrow 0} \frac{\sin ax}{ax} = 1$  for  $a \neq 0$
- $\lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \frac{a}{b}$  for  $b \neq 0$
- $\lim_{x \rightarrow \infty} x \sin\left(\frac{1}{x}\right) = 1$
- $\lim_{x \rightarrow 0} \frac{\tan ax}{ax} = 1$  for  $a \neq 0$
- $\lim_{x \rightarrow 0} \frac{\tan ax}{bx} = \frac{a}{b}$  for  $b \neq 0$

## 2.7 Sums

- $\lim_{x \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} = \infty$
- $\lim_{x \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n\right) = \gamma$ . This is Euler Mascheroni Constant.

## 2.8 Notable Special Limits

- $\lim_{x \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} = e$
- $\lim_{x \rightarrow \infty} 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \dots + \sqrt{2}}}} = \pi$

## 2.9 Taylor Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \infty$$

$$\ln(1-x) = -(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots + \infty)$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots\right)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \infty$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots + \infty$$

$$\sec x = x + \frac{x^2}{2} + \frac{5x^4}{24} + \dots + \infty$$

$$\arcsin x / \sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots + \infty$$

$$\arccos x / \cos^{-1} x = \frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots\right)$$

$$\arctan x / \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots + \infty$$

## 3 Differentiation

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### 3.1 Elementary functions

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(a^x) = a^x \ln a$
- $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$

- $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

### 3.2 Basic Theorems

- $\frac{d}{dx}(f \pm g) = f'(x) \pm g'(x)$
- $\frac{d}{dx}(kf(x)) = k \frac{d}{dx}(f(x))$
- $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + g(x)f'(x)$
- $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$
- $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

### 3.3 Inverse Trigonometric Functions

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

### 3.4 Using Substitution

- $\sqrt{x^2 + a^2} \implies x = a \tan \theta$
- $\sqrt{a^2 - x^2} \implies x = a \sin \theta$
- $\sqrt{x^2 - a^2} \implies x = a \sec \theta$
- $\sqrt{\frac{x+a}{a-x}} \implies x = a \cos \theta$

### 3.5 Parametric Differentiation

If  $y = f(\theta)$  and  $x = g(\theta)$  where  $\theta$  is parameter then

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

### 3.6 Derivative of Determinant

If  $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$  where  $f, g, h, l, m, n, u, v, w$  are differentiable functions, then  $F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$

## 4 Application of Derivatives

### 4.1 Equation of Tangent and Normal

Tangent at  $(x_1, y_1)$  is given by

$$(y - y_1) = f'(x_1)(x - x_1)$$

when  $f'(x_1)$  is real and Normal at  $(x_1, y_1)$  is given by

$$(y - y_1) = \frac{-1}{f'(x_1)}(x - x_1)$$

when  $f'(x_1)$  is non-zero and real.

#### Tangent from an external point

Given a point  $\delta(a, b)$  which does not lie on the curve  $y = f(x)$  then equation of possible tangents to the curve  $y = f(x)$  passing through  $(a, b)$  can be found by solving for point of contact  $\lambda$

$$f'(h) = \frac{f(h) - b}{h - a}$$

and equation of tangent

$$y - b = \frac{f(h) - b}{h - a}(x - a)$$

**Length of tangent, normal, sub-tangent, sub-normal** from point  $\sigma(h, k)$  and slope  $m$

$$\text{Length of Tangent} = |k| \sqrt{1 + \frac{1}{m^2}}$$

$$\text{Length of Normal} = |k| \sqrt{1 + m^2}$$

$$\text{Length of Sub-Tangent} = \left| \frac{k}{m} \right|$$

$$\text{Length of Sub-Normal} = |km|$$

**Angle between the curves**

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

### 4.2 Theorems

#### Rolle's Theorem

If a function  $f$  defined on  $[a, b]$  and

- continuous on  $[a, b]$
- derivable on  $(a, b)$
- $f(a) = f(b)$

then there exists at least one real number  $c$  between  $a$  and  $b$  ( $a < c < b$ ) such that  $f'(c) = 0$ .

#### Lagrange's Mean Value Theorem

If a function  $f$  defined on  $[a, b]$  and

- continuous on  $[a, b]$
- derivable on  $(a, b)$

then there exists at least one real number  $c$  between  $a$  and  $b$  ( $a < c < b$ ) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

#### Cauchy's Mean Value theorem

If functions  $f$  and  $g$  defined on  $[a, b]$  and

- continuous on  $[a, b]$
- derivable on  $(a, b)$
- $c \in (a, b)$  then

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

### 4.3 Maxima and Minima

If a function  $y = f(x)$  is defined on interval  $X$ , then an interior point  $x_o$  of interval is called the point of *maximum* of function  $f(x)$  [the point of *minimum* of function  $f(x)$ ] if there exists a neighbourhood  $U \in X$  of point  $x_o$  such that inequality  $f(x) \leq f(x_o)$  [ $f(x) \geq f(x_o)$ ] holds true within it.

#### A Necessary condition for the existence of an Extremum

At points of extremum the derivative  $f'(x)$  is equal to zero or does not exist. The points at which  $f'(x) = 0$  or does not exist are called *critical points*.

### Sufficient conditions for the existence of an Extremum

1. Let the function  $f(x)$  be continuous in some neighbourhood of point  $x_o$

- If  $f'(x) > 0$  at  $x < x_o$  and  $f'(x) < 0$  at  $x > x_o$  (i.e if in moving from left to right through point  $x_o$  the derivative changes sign from plus to minus), then at point  $x_o$  the function reaches *maximum*.
- If  $f'(x) < 0$  at  $x < x_o$  and  $f'(x) > 0$  at  $x > x_o$  (i.e if in moving through the point  $x_o$  from left to right the derivative changes sign from minus to plus), then at point  $x_o$  the function reaches *minimum*.
- If the derivative does not change sign in moving through the point  $x_o$ , then there is no *extremum*.

2. Let the function  $f(x)$  be twice differentiable (that is  $f'(x_o) = 0$ ) at a critical point  $x_o$ . If  $f''(x_o) < 0$  then at  $x_o$  the function has a *maximum*; if  $f''(x_o) > 0$  then at  $x_o$  the function has *minimum* but if  $f''(x_o) = 0$  then the question of existence of *extremum* at this point remains open.

3. Let  $f(x_o) = f''(x_o) = \dots = f^{n-1}(x_o) = 0$ , but  $f^n(x_o) \neq 0$ . If  $n$  is even, then at  $f^n(x_o) < 0$  there is a *maximum* at  $x_o$ , and at point  $f^n(x_o) > 0$ , a *minimum*. If  $n$  is odd then there is no *extremum* at point  $x_o$ .

4. Let the function  $y = f(x)$  be represented parametrically:

$$x = \varphi(t), y = \psi(t)$$

where the functions  $\varphi(t)$  and  $\psi(t)$  have derivatives both of first and second orders within a certain interval of change of argument  $t$ , and  $\varphi'(t) \neq 0$ . Further, let, at  $t = t_o$

$$\psi'(t) = 0$$

Then:

- If  $\psi''(t_o) < 0$ , the function  $y = f(x)$  has a *maximum* at  $x = x_o = \varphi(t_o)$
- If  $\psi''(t_o) > 0$ , the function  $y = f(x)$  has a *minimum* at  $x = x_o = \varphi(t_o)$

- If  $\psi''(t_o) = 0$ , the question of existence of *extremum* remains open.

## 5 Integration

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### 5.1 Standard Formula

- $\int (ax + b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$
- $\int \frac{dx}{ax+b} = \frac{1}{a} \ln(ax+b) + c$
- $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$
- $\int a^{px+q} dx = \frac{1}{p} \frac{a^{px+q}}{\ln a} + c$
- $\int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b) + c$
- $\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + c$
- $\int \tan(ax+b) dx = \frac{1}{a} \ln \sec(ax+b) + c$
- $\int \cot(ax+b) dx = \frac{1}{a} \ln \sin(ax+b) + c$
- $\int \sec^2(ax+b) dx = \frac{1}{a} \tan(ax+b) + c$
- $\int \operatorname{cosec}^2(ax+b) dx = -\frac{1}{a} \cot(ax+b) + c$
- $\int \sec x dx = \ln(\sec x + \tan x) + c$
- $\int \operatorname{cosec} x dx = \ln(\operatorname{cosec} x - \cot x) + c$
- $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln[x + \sqrt{x^2+a^2}] + c$
- $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + c$
- $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c$
- $\int \sqrt{a^2-x^2} dx = \frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c$
- $\int \sqrt{x^2+a^2} dx = \frac{x}{2} \sqrt{x^2+a^2} + \frac{a^2}{2} \ln \left( \frac{x+\sqrt{x^2+a^2}}{a} \right) + c$
- $\int \sqrt{x^2-a^2} dx = \frac{x}{2} \sqrt{x^2-a^2} - \frac{a^2}{2} \ln \left( \frac{x+\sqrt{x^2-a^2}}{a} \right) + c$

## 5.2 Integration of types

1.  $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c} dx$   
 $\Rightarrow$  Put  $x + \frac{b}{2a} = t$
2.  $\int \frac{px+q}{ax^2+bx+c} dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx,$   
 $\int (px+q)\sqrt{ax^2+bx+c} dx \Rightarrow$   
Put  $x + \frac{b}{2a} = t$  then split the integral.
3.  $\int \frac{dx}{a+b\sin^2 x}, \int \frac{dx}{a+b\cos^2 x}, \int \frac{dx}{a\sin^2 x+b\sin x\cos x+c\cos^2 x}$   
 $\Rightarrow$  Put  $\tan x = t$
4.  $\int \frac{dx}{a+b\sin x}, \int \frac{dx}{a+b\cos x}, \int \frac{dx}{a+b\sin x+c\cos x}$   
 $\Rightarrow$  Put  $\tan(\frac{x}{2}) = t$
5.  $\int \frac{dx}{(ax+b)\sqrt{px+q}}, \int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}} \Rightarrow$   
Put  $px+q = t^2$
6.  $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}} \Rightarrow$  Put  $ax+b = \frac{1}{t}$
7.  $\int \frac{dx}{ax^2+b\sqrt{px^2+q}} \Rightarrow$  Put  $x = \frac{1}{t}$
8.  $\int \frac{x^2+1dx}{x^4+\lambda x^2+1}$  where  $\lambda$  is any constant  $\Rightarrow$   
Divide numerator and denominator by  $x^2$  and Put  $x \mp \frac{1}{x} = t$

## 5.3 Reduction Forms

- $\int \sin^n x dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx$
- $\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx$
- $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$
- $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$
- $\int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx$
- For  $n > 1$
- $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$
- $\int \sec^n x dx = \frac{\sec^{n-1} x \sin x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx$

## 5.4 Definite Integration

### 5.4.1 Properties

- $\int_a^b f(x) dx = \int_a^b f(t) dt$
- $\int_a^b f(x) dx = - \int_b^a f(x) dx$
- $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$
- $\int_{-a}^a f(x) dx = \int_0^a (f(x) + f(-x)) dx = \begin{cases} 2 \int_0^a f(x) dx, & f(-x) = f(x) \\ 0, & f(-x) = -f(x) \end{cases}$
- $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$
- $\int_0^a f(x) dx + \int_0^{f(a)} f^{-1}(x) dx = af(a)$

If  $f(x)$  is a periodic function with period  $T$

- $\int_0^{nT} f(x) dx = n \int_0^T f(x) dx, n \in \mathbb{Z}$
- $\int_a^{a+nT} f(x) dx = n \int_0^T f(x) dx, n \in \mathbb{Z}, a \in \mathbb{R}$
- $\int_m^n f(x) dx = (n-m) \int_0^T f(x) dx, m, n \in \mathbb{Z}$
- $\int_n^{a+nT} f(x) dx = \int_0^a f(x) dx, n \in \mathbb{Z}, a \in \mathbb{R}$
- $\int_{a+nT}^{b+nT} f(x) dx = \int_0^a f(x) dx, n \in \mathbb{Z}, a, b \in \mathbb{R}$

### 5.4.2 Inequalities

1. If  $\Psi(x) \leq f(x) \leq \phi(x)$  for  $a \leq x \leq b$ , then

$$\int_a^b \Psi(x) dx \leq \int_a^b f(x) dx \leq \int_a^b \phi(x) dx$$

2. If  $m \leq f(x) \leq M$  for  $a \leq x \leq b$ , then

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

3. If  $f(x) \geq 0$  on  $[a, b]$ , then

$$\int_a^b f(x) dx \geq 0$$

### 5.4.3 Leibniz Theorem

If  $\varphi(x) = \int_{g(x)}^{h(x)} f(t)dt$ , then

$$\frac{d}{dx}(\varphi(x)) = h'(x)f(h(x)) - g'(x)f(g(x))$$

## 6 Other Integrals

### 6.1 Wallis' Integral

$$\int_0^{\frac{\pi}{2}} \sin^n x dx / \int_0^{\frac{\pi}{2}} \cos^n x dx = \begin{cases} \frac{\pi}{2} \frac{(n-1)!}{n!}, n \text{ is even} \\ \frac{(n-1)!}{n!}, n \text{ is odd} \end{cases}$$

### 6.2 Pi Function

$$\Pi(n) = \int_0^{\infty} x^n e^{-x} dx$$

Properties:

- $\Pi(n+1) = (n+1)\Pi(n)$
- $\Pi(0) = 1 \implies \Pi(n) = n!$

### 6.3 Gamma Function

$$\Gamma(n) = \Pi(n-1) = \int_0^{\infty} x^{n-1} e^{-x} dx$$

Some Properties:

- $\Gamma(n+1) = n\Gamma(n)$
- $\Gamma(n) = (n-1)!, \Gamma(\frac{n}{2}) = \frac{2^{(1-n)}(n-1)!\sqrt{\pi}}{(\frac{n-1}{2})!}$

### 6.4 Gaussian Integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

#### 6.4.1 Gaussian Integral Proof

*Proof.* Substitute  $x^2 = u \implies 2x dx = du$

$$\int_{-\infty}^{\infty} e^{-x^2} dx \implies \frac{1}{2} \int_{-\infty}^{\infty} u^{-\frac{1}{2}} e^{-u} du$$

Using property  $[\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx]$  for  $[f(-x) = f(x)]$

$$= \int_0^{\infty} u^{-\frac{1}{2}} e^{-u} du$$

It is type of  $\Gamma(n)$  for  $n = \frac{1}{2}$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

Hence proved.  $\square$

## 7 Differential Equation

A relationship between an independent variable  $x$ , a dependent variable  $y$ , and one or more of derivatives of  $y$  w.r.t  $x$ .

A simple example of differential equation is

$$\frac{dy}{dx} = x$$

### Order and Degree of DE

**Order:** The order of highest-order derivative in a differential equation.

**Degree:** The power to which the highest-order derivative is raised in a differential equation.

### Solution

A solution of a differential equation is function that, when substituted for the dependent variable in equation, leads to an identity. Thus for above example  $y = \frac{1}{2}x^2 + c$  is a *solution*.

### 7.1 DE of first order and first degree

#### 7.1.1 Exact Equation

Equation of the form:

$$P(\frac{dy}{dx}) + Q = 0$$

are exact if left-hand side is differential coefficient of some function  $f(x, y)$  w.r.t  $x$ . Integration gives the *solution*  $f(x, y) = C$ . An *exact* equation is one in which the total differential of function  $f$  is equal to zero.

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

Thus an equation  $Ax + by = 0$  is exact if

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$

### 7.1.2 Variables Separable

In this case, the equation can be written in the form

$$f(x) + g(x) \frac{dy}{dx} = 0$$

Rearrangement gives

$$f(x)dx = -g(y)dy$$

Both sides then can be integrated.

### 7.1.3 Homogeneous Equations

These can be written in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

The method of solution is to make substitution  $y = vx$ , which reduces the equation to one in  $v$  and  $x$  only. Resulting, the variables are separable.

### 7.1.4 Equations reducible to Homogeneous

Equation of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

can be handled by making substitution  $x = X + h$  and  $y = Y + k$  where  $h$  and  $k$  are constants. Then,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dY}{dX} \\ &= \frac{a_1(X + h) + b_1(Y + k) + c_1}{a_2(X + h) + b_2(Y + k) + c_2} \end{aligned}$$

If  $h$  and  $k$  are chosen to be the values of  $x$  and  $y$ , respectively, that satisfy the simultaneous equations

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

Then original equation becomes

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

which is homogeneous.

However if  $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$  then  $h$  and  $k$  cannot be chosen as above. In this case, let  $a_2 = ma_1$  and  $u = a_1x + b_1y$ . The equation becomes

$$\frac{du}{dx} - a_1 = b_1 \frac{u + c_1}{mu + c_2}$$

and  $u$  and  $x$  can be separated.

### 7.1.5 Linear Equations

Equation of the form

$$\frac{dy}{dx} + Py = Q$$

where  $P$  and  $Q$  are the functions of  $x$ , or constants, are said to be linear in  $y$  and can be solved by multiplying integrating factor

$$e^{\int P dx}$$

This makes left hand side of equation an exact differential:

$$e^{\int P dx} \left( \frac{dy}{dx} \right) + e^{\int P dx} (Py) = e^{\int P dx} Q$$

$$\frac{d}{dx} [e^{\int P dx} y] = e^{\int P dx} Q$$

$$ye^{\int P dx} = \int e^{\int P dx} Q dx + c$$