1 Functions

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1.1 Functions. Domain of Definition

The independent variable x is defined by a set X of its values. If to each value of the independent variable $x \in X$ there corresponds one definite value of another variable y, then y is called the function of x with a domain of definition (or domain) X or, in functional notation, $y = y(x)$, or $y = f(x)$, or $y = \varphi(x)$, and so forth. The set of values of the function $y(x)$ is called the range of the given function.

1.2 Investigation of Functions

A function $f(x)$ defined on the set X is said to be non-decreasing on this set (respectively, increasing, non-increasing, decreasing), if for any numbers $x_1, x_2 \in X, x_1 < x_2$ the inequality $f(x_1) \leq f(x_2)$ (respectively, $f(x_1) < f(x_2), f(x_1) \geq f(x_2), f(x_1) > f(x_2)$) is satisfied.The function $f(x)$ is said to be monotonic on the set X if it possesses one of the four indicated properties. The function $f(x)$ is said to be bounded above (or below) on the set X if there exists a number $M(or\ m)$ such that $f(x) \leq M\ \forall\ x \in X$. The function $f(x)$ is said to be bounded on the set X if it is bounded above and below. The function $f(x)$ is called periodic if there exists a number $T > 0$ such that $f(x+T) = f(x)$ for all x belonging to the domain of definition of the function (together with any point x the point $x + T$ must belong to the domain of definition). The least number T possessing this property (if such a number exists) is called the period of the function $f(x)$. The function $f(x)$ takes on the maximum value at the point $x_o \in X$ if $f(x_o) \geq f(x)$ for all $x \in X$, and the minimum value if if $f(x_o) \leq f(x)$ for all $x \in X$. A function $f(x)$ defined on a set X which is symmetric $w.r.t$ origin of coordinates is called even if $f(-x) = f(x)$, and odd if $f(x) = -f(x)$.

1.3 Inverse of Function

Let the function $y = f(x)$ be defined on the set X and have a range Y. If for each $y \in Y$ there exists a single value of x such that $f(x) = y$, then this correspondence defines a certain function $x = g(y)$ called inverse $w.r.t$ given function $y = f(x)$. The sufficient condition for the existence of an inverse function is a strict monotony of the original function $y = f(x)$. If the function increases(decreases), then the inverse function also increases (decreases). The graph of the inverse function $x = g(y)$ coincides with that of the function $y = f(x)$ if the independent variable is marked off along the $y - axis$. If the independent variable is laid off along the x-axis,i. e. if the inverse function is written in the form $y = g(x)$, then the graph of the inverse function will be symmetric to that of the function $y = f(x)$ with respect to the bisector of the first and third quadrants.

2 Limits

2.1 Existence

Limit of function $f(x)$ is said to exist as $x \to a$ when,

$$\lim_{h \to 0^+} f(a - h) = \lim_{h \to 0^+} f(a + h)$$

equal to some finite value L.

2.2 Indeterminate forms

There are only seven indeterminate forms
$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, \infty^0, 0^0\ and\ 1^\infty$.

2.3 List of limits

Limits Operations
If $\lim_{x \to c} f(x) = L$

- $\lim_{x \to c}[f(x \pm a)] = L \pm a$
- $\lim_{x \to c} af(x) = aL$
- $\lim_{x \to c} \frac{1}{f(a)} = \frac{1}{L}$ for $L > 0$
- $\lim_{x \to c} f(x)^n = L^n$ for $n > 0$

Involving infinitesimal changes
If infinitesimal change h if denote by Δx. If $f(x)$ and $g(x)$ are differentiable at x.

- $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$
- $\lim_{h\to 0} \frac{fog(x+h)-fog(x)}{h} = f'[g(x)]g'(x)$
- $\lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} = f'(x)g(x) + f(x)g'(x)$
- $\lim_{h\to 0} (\frac{f(x+h)}{f(x)})^{\frac{1}{h}} = exp(\frac{f'(x)}{f(x)})$
- $\lim_{h\to 0} (\frac{f(e^h x)}{f(x)})^{\frac{1}{h}} = exp(\frac{xf'(x)}{f(x)})$

If $f(x)$ and $g(x)$are differentiable on an open interval containing c, except possibly c itself, and $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$.
Jean Bernoulli or L'Hopital's rule can be used:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

Inequalities
If $f(x) \leq g(x)$ for all x in interval that contains c, except possibly c itself, and the limit of $f(x)$ and $g(x)$ both exist at c, then $\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$
If $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$ and

$$f(x) \leq g(x) \leq h(x)$$

for all x in an open interval that contains c, except possibly c itself, $\lim_{x\to c} g(x) = L$. This is know as $Squeeze\ Theorem$.

2.4 Exponential Functions

Function of form $f(x)^{g(x)}$

- $\lim_{x\to+\infty} (\frac{x}{x+k})^x = e^{-k}$
- $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$
- $\lim_{x\to 0} (1+kx)^{\frac{m}{x}} = e^{mk}$
- $\lim_{x\to+\infty} (1+\frac{1}{x})^x = e$
- $\lim_{x\to+\infty} (1-\frac{1}{x})^x = \frac{1}{e}$
- $\lim_{x\to+\infty} (1+\frac{k}{x})^{mx} = e^{mk}$
- $\lim_{x\to 0} (1+a(e^{-x}-1)^{-\frac{1}{x}}) = e^a$

Sum products and Composites

- $\lim_{x\to 0} (\frac{a^x-1}{x}) = lna$
- $\lim_{x\to 0} (\frac{e^x-1}{x}) = 1$
- $\lim_{x\to 0} (\frac{e^{ax}-1}{x}) = a$

2.5 Logarithmic Functions

- $\lim_{x\to 1} \frac{lnx}{x-1} = 1$
- $\lim_{x\to 0} \frac{ln(x+1)}{x} = 1$
- $\lim_{x\to 0} \frac{-ln(1+a(e^{-x}-1)}{x} = a$

Some cases

- $\lim_{x\to 0^+} log_b x = -F(b)\infty$
- $\lim_{x\to\infty} log_b x = F(b)\infty$

where $F(x) = 2H(x-1)-1$ and $H(x)$ is Oliver Heaviside step function.

2.6 Trigonometric Functions

- $\lim_{x\to 0} \frac{sinax}{ax} = 1$ for $a \neq 0$
- $\lim_{x\to 0} \frac{sinax}{bx} = \frac{a}{b}$ for $b \neq 0$
- $\lim_{x\to\infty} xsin(\frac{1}{x}) = 1$
- $\lim_{x\to 0} \frac{tanax}{ax} = 1$ for $a \neq 0$
- $\lim_{x\to 0} \frac{tanax}{bx} = \frac{a}{b}$ for $b \neq 0$

2.7 Sums

- $\lim_{x\to\infty} \sum_{k=1}^{n} \frac{1}{k} = \infty$
- $\lim_{x\to\infty} (\sum_{k=1}^{n} \frac{1}{k} - logn) = \gamma$.This is Euler Mascheroni Constant.

2.8 Notable Special Limits

- $\lim_{x\to\infty} \frac{n}{\sqrt[n]{n!}} = e$
- $\lim_{x\to\infty} 2^n \sqrt{2-\sqrt{2+\sqrt{2+...+\sqrt{2}}}} = \pi$

2.9 Taylor Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \infty$$

$$ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + \infty$$

$$ln(1-x) = -(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ... + \infty)$$

$$ln(\frac{1+x}{1-x}) = 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + ...)$$

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... + \infty$$

$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... + \infty$$

$$tanx = x + \frac{x^3}{3} + \frac{2x^5}{15} + ... + \infty$$

$$secx = x + \frac{x^2}{2} + \frac{5x^4}{24} + ... + \infty$$

$$arcsinx/sin^{-1}x = x + \frac{x^3}{6} + \frac{3x^5}{40} + ... + \infty$$

$$arccosx/cos^{-1}x = \frac{\pi}{2} - (x + \frac{x^3}{6} + \frac{3x^5}{40} + ...)$$

$$arctanx/tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} + ... + \infty$$