1 Functions

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1.1 Functions. Domain of Definition

The independent variable x is defined by a set X of its values. If to each value of the independent variable $x \in X$ there corresponds one definite value of another variable y, then y is called the function of x with a domain of definition (or domain) X or, in functional notation, $y = y(x)$, or $y = f(x)$, or $y = \varphi(x)$, and so forth. The set of values of the function $y(x)$ is called the range of the given function.

1.2 Investigation of Functions

A function $f(x)$ defined on the set X is said to be non-decreasing on this set (respectively, increasing, non-increasing, decreasing), if for any numbers $x_1, x_2 \in X, x_1 < x_2$ the inequality $f(x_1) \leq f(x_2)$ (respectively, $f(x_1) < f(x_2), f(x_1) \geq f(x_2), f(x_1) > f(x_2)$) is satisfied.The function $f(x)$ is said to be monotonic on the set X if it possesses one of the four indicated properties. The function $f(x)$ is said to be bounded above (or below) on the set X if there exists a number $M(or\ m)$ such that $f(x) \leq M\ \forall\ x \in X$. The function $f(x)$ is said to be bounded on the set X if it is bounded above and below. The function $f(x)$ is called periodic if there exists a number $T > 0$ such that $f(x+T) = f(x)$ for all x belonging to the domain of definition of the function (together with any point x the point $x + T$ must belong to the domain of definition). The least number T possessing this property (if such a number exists) is called the period of the function $f(x)$. The function $f(x)$ takes on the maximum value at the point $x_o \in X$ if $f(x_o) \geq f(x)$ for all $x \in X$, and the minimum value if if $f(x_o) \leq f(x)$ for all $x \in X$. A function $f(x)$ defined on a set X which is symmetric $w.r.t$ origin of coordinates is called even if $f(-x) = f(x)$, and odd if $f(x) = -f(x)$.

1.3 Inverse of Function

Let the function $y = f(x)$ be defined on the set X and have a range Y. If for each $y \in Y$ there exists a single value of x such that $f(x) = y$, then this correspondence defines a certain function $x = g(y)$ called inverse $w.r.t$ given function $y = f(x)$. The sufficient condition for the existence of an inverse function is a strict monotony of the original function $y = f(x)$. If the function increases(decreases), then the inverse function also increases (decreases). The graph of the inverse function $x = g(y)$ coincides with that of the function $y = f(x)$ if the independent variable is marked off along the $y - axis$. If the independent variable is laid off along the x-axis,i. e. if the inverse function is written in the form $y = g(x)$, then the graph of the inverse function will be symmetric to that of the function $y = f(x)$ with respect to the bisector of the first and third quadrants.

2 Limits

2.1 Existence

Limit of function $f(x)$ is said to exist as $x \to a$ when,

$$\lim_{h \to 0^+} f(a-h) = \lim_{h \to 0^+} f(a+h)$$

equal to some finite value L.

2.2 Indeterminate forms

There are only seven indeterminate forms
$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, \infty^0, 0^0\ and\ 1^\infty$.

2.3 List of limits

Limits Operations
If $\lim_{x \to c} f(x) = L$

- $\lim_{x \to c}[f(x \pm a)] = L \pm a$
- $\lim_{x \to c} af(x) = aL$
- $\lim_{x \to c} \frac{1}{f(a)} = \frac{1}{L}$ for $L > 0$
- $\lim_{x \to c} f(x)^n = L^n$ for $n > 0$

Involving infinitesimal changes
If infinitesimal change h if denote by Δx. If $f(x)$ and $g(x)$ are differentiable at x.

- $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$
- $\lim_{h\to 0} \frac{fog(x+h)-fog(x)}{h} = f'[g(x)]g'(x)$
- $\lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} = f'(x)g(x) + f(x)g'(x)$
- $\lim_{h\to 0}(\frac{f(x+h)}{f(x)})^{\frac{1}{h}} = exp(\frac{f'(x)}{f(x)})$
- $\lim_{h\to 0}(\frac{f(e^h x)}{f(x)})^{\frac{1}{h}} = exp(\frac{xf'(x)}{f(x)})$

If $f(x)$ and $g(x)$are differentiable on an open interval containing c, except possibly c itself, and $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$.
Jean Bernoulli or L'Hopital's rule can be used:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

Inequalities
If $f(x) \leq g(x)$ for all x in interval that contains c, except possibly c itself, and the limit of $f(x)$ and $g(x)$ both exist at c, then $\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$
If $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$ and

$$f(x) \leq g(x) \leq h(x)$$

for all x in an open interval that contains c, except possibly c itself, $\lim_{x\to c} g(x) = L$. This is know as $Squeeze\ Theorem$.

2.4 Exponential Functions

Function of form $f(x)^{g(x)}$

- $\lim_{x\to+\infty}(\frac{x}{x+k})^x = e^{-k}$
- $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$
- $\lim_{x\to 0}(1+kx)^{\frac{m}{x}} = e^{mk}$
- $\lim_{x\to+\infty}(1+\frac{1}{x})^x = e$
- $\lim_{x\to+\infty}(1-\frac{1}{x})^x = \frac{1}{e}$
- $\lim_{x\to+\infty}(1+\frac{k}{x})^{mx} = e^{mk}$
- $\lim_{x\to 0}(1+a(e^{-x}-1)^{-\frac{1}{x}}) = e^a$

Sum products and Composites

- $\lim_{x\to 0}(\frac{a^x-1}{x}) = lna$
- $\lim_{x\to 0}(\frac{e^x-1}{x}) = 1$
- $\lim_{x\to 0}(\frac{e^{ax}-1}{x}) = a$

2.5 Logarithmic Functions

- $\lim_{x\to 1} \frac{lnx}{x-1} = 1$
- $\lim_{x\to 0} \frac{ln(x+1)}{x} = 1$
- $\lim_{x\to 0} \frac{-ln(1+a(e^{-x}-1)}{x} = a$

Some cases

- $\lim_{x\to 0^+} log_b x = -F(b)\infty$
- $\lim_{x\to\infty} log_b x = F(b)\infty$

where $F(x) = 2H(x-1)-1$ and $H(x)$ is Oliver Heaviside step function.

2.6 Trigonometric Functions

- $\lim_{x\to 0} \frac{sinax}{ax} = 1$ for $a \neq 0$
- $\lim_{x\to 0} \frac{sinax}{bx} = \frac{a}{b}$ for $b \neq 0$
- $\lim_{x\to\infty} xsin(\frac{1}{x}) = 1$
- $\lim_{x\to 0} \frac{tanax}{ax} = 1$ for $a \neq 0$
- $\lim_{x\to 0} \frac{tanax}{bx} = \frac{a}{b}$ for $b \neq 0$

2.7 Sums

- $\lim_{x\to\infty} \sum_{k=1}^{n} \frac{1}{k} = \infty$
- $\lim_{x\to\infty}(\sum_{k=1}^{n} \frac{1}{k} - logn) = \gamma$.This is Euler Mascheroni Constant.

2.8 Notable Special Limits

- $\lim_{x\to\infty} \frac{n}{\sqrt[n]{n!}} = e$
- $\lim_{x\to\infty} 2^n \sqrt{2-\sqrt{2+\sqrt{2+...+\sqrt{2}}}} = \pi$

2.9 Taylor Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \infty$$

$$ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + \infty$$

$$ln(1-x) = -(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ... + \infty)$$

$$ln(\frac{1+x}{1-x}) = 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + ...)$$

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... + \infty$$

$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... + \infty$$

$$tanx = x + \frac{x^3}{3} + \frac{2x^5}{15} + ... + \infty$$

$$secx = x + \frac{x^2}{2} + \frac{5x^4}{24} + ... + \infty$$

$$arcsinx/sin^{-1}x = x + \frac{x^3}{6} + \frac{3x^5}{40} + ... + \infty$$

$$arccosx/cos^{-1}x = \frac{\pi}{2} - (x + \frac{x^3}{6} + \frac{3x^5}{40} + ...)$$

$$arctanx/tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} + ... + \infty$$

3 Differentiation

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3.1 Elementary functions

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(a^x) = a^x lna$
- $\frac{d}{dx}(lnx) = \frac{1}{x}$
- $\frac{d}{dx}(log_a x) = \frac{1}{xlna}$
- $\frac{d}{dx}(sinx) = cosx$
- $\frac{d}{dx}(cosx) = -sinx$
- $\frac{d}{dx}(secx) = secxtanx$
- $\frac{d}{dx}(cosecx) = -cosecxcotx$
- $\frac{d}{dx}(tanx) = sec^2 x$
- $\frac{d}{dx}(cotx) = -cosec^2 x$

3.2 Basic Theorems

- $\frac{d}{dx}(f \pm g) = f'(x) \pm g'(x)$
- $\frac{d}{dx}(kf(x) = k\frac{d}{dx}(f(x))$
- $\frac{d}{dx}(f(x).g(x)) = f(x)g'(x) + g(x)f'(x)$
- $\frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$
- $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

3.3 Inverse Trigonometric Functions

- $\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(cot^{-1}x) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}(sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}(cosec^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$

3.4 Using Substitution

- $\sqrt{x^2 + a^2} \implies x = atan\theta$
- $\sqrt{a^2 - x^2} \implies x = asin\theta$
- $\sqrt{x^2 - a^2} \implies x = asec\theta$
- $\sqrt{\frac{x+a}{a-x}} \implies x = acos\theta$

3.5 Parametric Differentiation

If $y = f(\theta)$ and $x = g(\theta)$ where θ is parameter then

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

3.6 Derivative of Determinant

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$ where $f, g, h, l, m, n, u, v, w$ are differentiable functions, then $F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$

4 Application of Derivatives

4.1 Equation of Tangent and Normal

Tangent at (x_1, y_1) is given by

$$(y - y_1) = f'(x_1)(x - x_1)$$

when $f'(x_1)$ is real and Normal at (x_1, y_1) is given by

$$(y - y_1) = \frac{-1}{f'(x_1)}(x - x_1)$$

when $f'(x_1)$ is non-zero and real.

Tangent from an external point

Given a point $\delta(a, b)$ which does not lie on the curve $y = f(x)$ then equation of possible tangents to the curve $y = f(x)$ passing through (a, b) can be found by solving for point of contact λ

$$f'(h) = \frac{f(h) - b}{h - a}$$

and equation of tangent

$$y - b = \frac{f(h) - b}{h - a}(x - a)$$

Length of tangent,normal,sub-tangent,sub-normal from point $\sigma(h, k)$ and slope m

Length of Tangent = $|k|\sqrt{1 + \frac{1}{m^2}}$

Length of Normal = $|k|\sqrt{1 + m^2}$

Length of Sub-Tangent = $|\frac{k}{m}|$

Length of Sub-Normal = $|km|$

Angle between the curves

$$tan\theta = |\frac{m_1 - m_2}{1 + m_1 m_2}|$$

4.2 Theorems

Rolle's Theorem

If a function f defined on $[a, b]$ and

- continuous on $[a, b]$
- derivable on (a, b)
- $f(a) = f(b)$

then there exists at least one real number c between a and b ($a < c < b$) such that $f'(c) = 0$.

Lagrange's Mean Value Theorem

If a function f defined on $[a, b]$ and

- continuous on $[a, b]$
- derivable on (a, b)

then there exists at least one real number c between a and b ($a < c < b$) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Cauchy's Mean Value theorem

If functions f and g defined on $[a, b]$ and

- continuous on $[a, b]$
- derivable on (a, b)
- $c \in (a, b)$ then

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

4.3 Maxima and Minima

If a function $y = f(x)$ is defined on interval X, then an interior point x_o of interval is called the point of *maximum* of function $f(x)$ [the point of *minimum* of function $f(x)$] if there exists a neighbourhood $U \in X$ of point x_o such that inequality $f(x) \leq f(x_o)[f(x) \geq f(x_o)]$ holds true within it.

A Necessary condition for the existence of an Extremum

At points of extremum the derivative $f'(x)$ is equal to zero or does not exist. The points at which $f'(x) = 0$ or does not exit are called *critical points*.

Sufficient conditions for the existence of an Extremum

1. Let the function $f(x)$ be continuous in some neighbourhood of point x_o

- If $f'(x) > 0$ at $x < x_o$ and $f'(x) < 0$ at $x > x_o$ (i.e if in moving from left to right through point x_o the derivative changes sign from plus to minus), then at point x_o the function reaches *maximum*.

- If $f'(x) < 0$ at $x < x_o$ and $f'(x) > 0$ at $x > x_o$ (i.e if in moving through the point x_o from left to right the derivative changes sign from minus to plus), then at point x_o the function reaches *minimum*.

- If the derivative does not change sign in moving through the point x_o, then there is no *extremum*.

2. Let the function $f(x)$ be twice differentiable (that is $f'(x_o) = 0$) at a critical point x_o. If $f''(x_o) < 0$ then at x_o the function has a *maximum*; if $f'(x_o) > 0$ then at x_o the function has *minimum* but if $f''(x_o) = 0$ then the question of existence of *extremum* at this point remains open.

3. Let $f(x_o) = f''(x_o) = ... = f^{n-1}(x_o) = 0$, but $f^n(x_o) \neq 0$. If n is even, then at $f^n(x_o) < 0$ there is a *maximum* at x_o, and at point $f^n(x_o) > 0$, a *minimum*.If n is odd then there is no *extremum* at point x_o.

4.Let the function $y = f(x)$ be represented parametrically:

$$x = \varphi(t),\ y = \psi(t)$$

where the functions $\varphi(t)$ and $\psi(t)$ have derivatives both of first and second orders within a certain interval of change of argument t, and $\varphi'(t) \neq 0$. Further, let, at $t = t_o$

$$\psi'(t) = 0$$

Then:

- If $\psi''(t_o) < 0$, the function $y = f(x)$ has a *maximum* at $x = x_o = \varphi(t_o)$

- If $\psi''(t_o) > 0$, the function $y = f(x)$ has a *minimum* at $x = x_o = \varphi(t_o)$

- If $\psi''(t_o) = 0$, the question of existence of *extremum* remains open.