# Contents

# 1 Fundamental

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## 1.1 Intervals

Intervals are basically subsets of $\mathbb{R}$ and are commonly used in solving inequalities or in finding domains. If there are two numbers $a, b \in \mathbb{R}$ such that $a < b$, there are four types of intervals:

- Open interval: $(a, b) = \{x : a < x < b\}$ i.e end points are not included. Symbols: () or ][
- Closed interval: $[a, b] = \{x : a \leq x \leq b\}$ i.e end points are also included. Symbol: [ ]
- Open-Closed interval: $(a, b) = \{x : a < x \leq b\}$. Symbols: ( ] or ] ]
- Closed-Open interval: $(a, b) = \{x : a \leq x < b\}$. Symbols: [ ) or [ [

**Infinite Intervals**

- $(a, \infty) = \{x : x > a\}$
- $[a, \infty) = \{x : x \geq a\}$
- $(-\infty, b) = \{x : x < b\}$
- $(\infty, b] = \{x : x \leq b\}$
- $(-\infty, \infty) = \{x : x \in \mathbb{R}\}$

## 1.2 Sets and Relations

### 1.2.1 Set

A collection of any kind of objects. The objects that make up a set are called *elements* or *members*. The statement '$a$ is an element of set $A$' can be written as $a \in A$ and set containing elements $a, b$ and $c$ is denoted by $\{a, b, c\}$. A *empty* or *null* set is denoted by $\varnothing$, which is the set that contains no elements.

**Union(join,sum):** The union of two sets $A$ and $B$, denoted by $A \cup B$, consists of those elements that belong to $A$ or to $B$:

$$A \cup B = \{x : (x \in A) \vee (x \in B)\}$$

For example, if $A$ is $\{1, 2, 3, 4\}$ and $B$ is $\{1, 4, 5, 6\}$ then $A \cup B$ is $\{1, 2, 3, 4, 5, 6\}$.
**Intersection(meet,product):** The intersection of two sets $A$ and $B$, denoted by $A \cap B$, consists of those elements that belong to both $A$ and $B$:

$$A \cap B = \{x : (x \in A) \wedge (x \in B)\}$$

For example, if $A$ is $\{1, 2, 3, 4, 5, 6\}$ and $B$ is $\{1, 4, 5, 6, 7, 8\}$ then $A \cap B$ is $\{1, 4, 5, 6\}$.
**Complement:** The complement of a set $A$, denoted by $A'$ or $A^{\complement}$, consists of all those elements that are not members of $A$:

$$A' = \{x : x \notin A\}$$

For example, in the domain of natural numbers, if $A$ is set of even numbers the its complement $A'$ is the set of odd numbers.
**Universal Set:** Relative to a particular domain, the universal set, denoted by $\mathbf{U}$, is set of all objects of that domain:

$$\mathbf{U} = \{x : x = x\}$$

### 1.2.2 Properties of sets

**Commutative law:**

- $(A \cup B) = B \cup A$
- $(A \cap B) = B \cap A$

**Associative law:**

- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C = A \cap (B \cap C)$

**Distributive law:**

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

**De Morgan's law:**

- $(A \cup B)' = A' \cap B'$
- $(A \cap B)' = A' \cup B'$

**Identity law:**

- $A \cup \mathbf{U} = A$
- $A \cup \varnothing = A$

**Complement law:**

- $A \cup A' = \mathbf{U}$
- $A \cap A' = \varnothing$
- $(A')' = A$

**Idempotent law:**

- $A \cap A = A$
- $A \cup A = A$

### 1.2.3 Results on number of elements in sets

If $A, B, C$ are finite sets and $\mathbf{U}$ be Universal finite set then:

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A - B) = n(A) - n(A \cap B)$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

### 1.2.4 Relations

An association between, or property of, two or more objects. Thus $(x = y)$ and ($a$ lies between $b$ and $c$) are relations, but ($N$ is a prime) is not.

### 1.2.5 Types of Relations

**Void Relation:** Let $A$ be a set. Then $\phi \subseteq A \times A$ and so it is a relation on $A$. This relation is also called empty relation on $A$.
**Universal Relation:**Let $A$ be a set. Then $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on $A$.
**Identity relation:** Let $A$ be a set. Then the relation $I_A = \{(a,a) : a \in A\}$is called the identity relation on $A$. In other words, a relation $I_A$ on $A$ is called the identity relation if every element of $A$ is related to itself only.
**Reflexive Relation:** A relation $R$ on a set $A$ is said to be reflexive if every element of $A$ is related to itself. Thus $R$ on set $A$ is not reflexive if there exits an element $a \in A$ such that $(a,a) \notin R$.
**Symmetric Relation:** A Relation $R$ on set $A$ is said to be symmetric if

$$(a,b) \in R \implies (b,a) \in R \, \forall \, a,b \in A$$

i.e

$$a \, R \, b \implies b \, R \, a \, \forall \, a,b \in A$$

**Transitive Relation:** A Relation $R$ on set $A$ is said to be transitive if $(a,b) \in R$ and $(b,c) \in R \implies (a,c) \in R \, \forall \, a,b,c \in A$.
**Equivalence Relation:** A Relation is said to be equivalence if it satisfies $reflexive$, *symmetric* and *transitive* relations.
**Equivalence Class:** If $R$ is an equivalence relation defined on set $A$ then the equivalence class of any element $x \in A$, denoted by $[x]$, is the set of elements to which $x$ is related by equivalence relation $R$:

$$[x] = \{y : x \, \mathbf{R} \, y\}$$

For example, if $R$ is the equivalence relation (the same height as), then the equivalence class of the element $x \in A$ consists of all elements of $A$ with same height as $x$.

# 2 Quadratic Equations

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## 2.1 Equation and Basic Results

An equation of the form

$$ax^2 + bx + c = 0$$

where $a \neq 0$ and $a, b, c \in \mathbb{R}$ is called a quadratic equation. The roots of quadratic equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quantity D ($D = b^2 - 4ac$) is known as discriminant of equation.

### 2.1.1 Results

1. Let $\alpha$ and $\beta$ be two roots of given quadratic equation. Then

- $\alpha + \beta = -\frac{b}{a}$
- $\alpha\beta = \frac{c}{a}$

2. A quadratic equation, whose roots are $\alpha$ and $\beta$ can be written as

$$(x - \alpha)(x - \beta) = 0$$

i.e

$$ax^2 + bx + c \equiv a(x - \alpha)(x - \beta)$$

3. If the quadratic equation is satisfied by more than two distinct numbers (real or complex), then it becomes an identity, i.e

$$a = b = c = 0$$

4. The quadratic equation has real and equal roots if and only if $D = 0$ i.e $b^2 - 4ac = 0$.
5. The quadratic equation has real and distinct roots if and only if $D > 0$ i.e $b^2 - 4ac > 0$.
6. The quadratic equation has complex roots with non-zero imaginary parts if and only if $D < 0$ i.e $b^2 - 4ac < 0$.
7. If $p + iq$ ($p, q \in \mathbb{R}$) is root of quadratic equation where $i = \sqrt{-1}$, the $p - iq$ is also root of quadratic equation. Provided $a, b, c \in \mathbb{R}$.
8. If $p + \sqrt{q}$ is an irrational root of quadratic equation, then $p - \sqrt{q}$ is also a root of equation provided that all coefficients are rational.

## 2.2 Conditions for Common Root(s)

Let $ax^2 + bx + c = 0$ and $dx^2 + ex + f = 0$ have a common root $\alpha$. Then $a\alpha^2 + b\alpha + c = 0$ and $d\alpha^2 + e\alpha + f = 0$.
Solving for $\alpha^2$ and $\alpha$ :

$$\frac{\alpha^2}{bf - ce} = \frac{\alpha}{dc - af} = \frac{1}{ae - bd} \implies$$

$$\alpha^2 = \frac{bf - ce}{ae - bd}$$

and

$$\alpha = \frac{dc - af}{ae - bd} \implies$$

$$(dc - af)^2 = (bf - ce)(ae - bd)$$

which is a required condition for the two equation to have a common root.

- Condition for both the roots to be common is $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$

## 2.3 Wavy Curve Method

The Method of intervals is used for solving inequalities of the form

$$f(x) = \frac{(x - a_1)^{n_1}(x - a_2)^{n_2}...(x - a_k)^{n_k}}{(x - b_1)^{m_1}(x - b_2)^{m_2}...(x - b_p)^{n_p}} > 0 (< 0, \leq 0, or \geq 0)$$

where $n_1, n_2, n_3...n_k, m_1, m_2, m_3..m_p \in \mathbb{N}$ and the numbers $a_1, a_2, ...a_k, b_1, b_2, ...b_p$ $\in \mathbb{R}$ such that $a_i \neq b_i$ where $i = 1, 2, 3, ...k$ and $j = 1, 2, 3, ...p$.
**Statements**

- All *zeros*[1] of the function $f(x)$ contained on left hand side of the inequality should be marked on the number line with black circles.
- All points of *discontinuities*[2] of the function $f(x)$ contained on left hand side of the inequality should be marked with white circles.
- Check the value of $f(x)$ for any real number greater than the right most marked number on the number line.

- From right to left, beginning above the number line, a wavy curve should be drawn which passes through all the marked points so that when passes through *singlepoint*[3], the curve intersects the number line, and when passing through a *double point*[4], the curve remains located on the one side of number line.
- The appropriate intervals are chosen in accordance with sign of inequality. Their union just represents the solution of inequality.

**Remarks**

- The points of discontinuity will never be included in answers.
- *Zeros*[1]:The point for which $f(x)$ vanishes(becomes zero) is called the *zeros* of function e.g. $x = a_i$
- *Discontinuities*[2]:The points $x = b_j$ are the points of the *discontinuity* of the function.
- *Single point*[3]:If the exponents of factor is *odd* then the point is called *single point*.
- *Double point*[4]: If the exponent of factor is *even* then the point is called *double point*.

## 2.4 Quadratic Expression

Let $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$ $a \neq 0$ It can be rewritten as

$$f(x) = a\{(x + \frac{b}{2a})^2 + \frac{D}{4a^2}\}$$

where $D = b^2 - 4ac$. Then $y - f(x)$ represents a parabola whose axis is parallel to the y axis, with vertex at $A(-\frac{b}{2a}, \frac{-D}{4a})$. Now depending upon the values of $a$ and $D$ the parabola will have different shapes: will add images later...

## 2.5 Interval in which the roots lie

Let $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Suppose $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 < \lambda_2$ Then the following results hold true:

- Both roots of $f(x) = 0$ will be greater than $\lambda$, if $D \geq 0$, $af(\lambda) > 0$ and $\lambda < -\frac{b}{2a}$.
- Both roots of $f(x) = 0$ will be less than $\lambda$, if $D \geq 0$, $af(\lambda) > 0$ and $\lambda > -\frac{b}{2a}$.
- If $\lambda$ lies between the roots of $f(x) = 0$ then $af(\lambda) < 0$.
- Both roots of $f(x) = 0$ will belong to interval $(\lambda_1, \lambda_2)$, if $D \geq 0$, $af(\lambda_1) > 0$,$af(\lambda_2) > 0$ and $\lambda_1 < -\frac{b}{2a} < \lambda_2$.
- Exactly one root of $f(x) = 0$ will lie in $(\lambda_1, \lambda_2)$ if $f(\lambda_1).f(\lambda_2) < 0$.
- Interval $(\lambda_1, \lambda_2)$ will be contained between the roots of $f(x) = 0$ if $af(\lambda_1) < 0, af(\lambda_2) < 0$.
- $\lambda$ will be the required root of $f(x) = 0$ if $f(\lambda) = 0$ and $f'(\lambda) = 0$.

# 3 Progression and Series

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## 3.1 Arithmetic Progression

If $a$ is the first term and $d$ the common difference, the $A.P.$ can be written as $a, a+d, a+2d, ....$ The $nth$ term $a_n$ is given by

$$a_n = a + (n-1)d$$

The sum of $S_n$ of the first terms of such an $A.P.$ is given by

$$S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(a + a_n)$$

If three terms are in $A.P.$, then the middle term is called the *Arimethic Mean* ($A.M.$) between the other two i.e $a, b, c$ are in $A.P.$ then

$$2b = a + c$$

is the $A.M$ of $a$ and $c$.
If $a_1, a_2, a_3...a_n$ are $n$ numbers, then the arithmetic mean of these numbers

$$A.M. = \frac{1}{n}(a_1 + a_2 + a_3 + ... + a_n)$$

The numbers $A_1, A_2, A_3...A_n$ are said to be $A.M.$'s between the numbers $a$ and $b$ if $a, A_1, A_2...A_n, b$ are in $A.P.$ then

$$A_n = a + \frac{n(b-a)}{n+1} = \frac{a+nb}{n+1}$$

## 3.2 Geometric Progression

If $a$ is the first term and $r$ the common ratio, then the $G.P.$ can be written as $a, ar, ar^2....$ The $nth$ term $a_n$ is given by

$$a_n = ar^{n-1}$$

The sum $S_n$ of first $n$ terms of the $G.P.$ is

$$S_n = \begin{cases} \frac{a(r^n-1)}{r-1}, r \neq 1 \\ na, r = 1 \end{cases}$$

If $-1 < r < 1$, then the sum of infinite $G.P.$ is $a + ar + ar^2 + ...$

$$= \frac{a}{1-r}$$

If three terms are in $G.P.$, then the middle term is called the geometric mean $G.M.$ between the two. So if $a, b, c$ are in $G.P.$ then $b = \sqrt{ac}$ is geometric mean of $a$ and $c$.
If $a_1, a_2...a_n$ are non-zero positive numbers, then their $G.M$ is given by

$$G = (a_1 a_2 a_3 ... a_n)^{1/n}$$

If $G_1, G_2...G_n$ are $n$ geometric means between $a$ and $b$, $a, G_1, G_2....G_n, b$ will be a $G.P.$

$$G_n = a(1 + \sqrt[n+1]{\frac{b}{a}})^n$$

## 3.3 Harmonic Progression

The sequence $a_1, a_2, a_3....a_n (a_i \neq 0)$ is said to be $H.P.$ if the sequence $\frac{1}{a_1}, \frac{1}{a_2}...\frac{1}{a_n}$ is an $A.P.$. The $nth$ term $a_n$ of the $H.P.$ is

$$a_n = \frac{1}{a + (n-1)d}$$

where $a = \frac{1}{a_i}$ and $d = \frac{1}{a_2} - \frac{1}{a_1}$.
**Harmonic Means:**
If $a$ and $b$ are two non-zero numbers, then the harmonic mean of $a$ and $b$ is a number $H$ such that the numbers $a, H, b$ are in $H.P.$ We have

$$\frac{1}{H} = \frac{1}{2}(\frac{1}{a} + \frac{1}{b})$$

There is no formula for sum of $n$ terms of an $H.P.$, if $a_1, a_2...a_n$ are $n$ non-zero numbers, then the harmonic mean $H$ of these numbers is given by

$$\frac{1}{H} = \frac{1}{n}(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n})$$

The $n$ number $H_1, H_2...H_n$ are said to be n-harmonic means between $a$ and $b$, if $a, H_1, H_2...H_n, b$ are in $H.P.$ if $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}...\frac{1}{H_n}, \frac{1}{b}$ are in $A.P.$

$$\frac{1}{H_n} = \frac{1}{a} + \frac{n(a-b)}{(n+1)ab}$$

## 3.4 Arithmetico-Geometric Progression

The sum $S_n$ of first $n$ terms of an $A.G.P.$ is obtained in the following way:

$$S_n = ab+(a+d)br+(a+2d)br^2+...+(a+(n-2)d)br^{n-2}+(a+(n-1)d)br^{n-1}$$

Multiplying both sides by $r$, so that

$$rS_n = abr+(a+d)br^2+(a+2d)br^3+...+(a+(n-2)d)br^{n-2}+(a+(n-1)d)br^{n-1}+(a+(n-1)d)br^n$$

Subtracting,we get

$$S_n = \frac{ab}{1-r} + \frac{dbr(1-r^{n-1})}{(1-r)^2} - \frac{(a+(n-1)d)br^n}{1-r}$$

If $-1 < r < 1$, the sum of infinite number of terms of progression is

$$\lim_{n\to\infty} S_n = S = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$$

## 3.5 Misc. Progressions

**Some Important Results:**

- $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$
- $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$
- $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4} = (1 + 2 + 3 + ... + n)^2$
- $1 + x + x^2 + x^3 + ... = (1 - x)^{-1}$ if $-1 < x < 1$
- $1 + 2x + 3x^2 + 4x^3 + ... = (1 - x)^{-2}$ if $-1 < x < 1$

**Method of Differences:**
Suppose $a_1, a_2, a_3...a_n$ is a sequence such that sequence $a_2 - a_1, a_3 - a_2...$ is either an $A.P.$ or a $G.P.$ The $nth$ term $a_n$ of this sequence is obtained as follows:

$$S_n = a_1 + a_2 + a_3 + ... + a_{n-2} + a_{n-1} + a_n$$

$$\implies a_n = a_1 + [(a_2 - a_1) + (a_3 - a_2) + ... + (a_n - a_{n-1})]$$

Since the term within the brackets are either in an $A.P.$ or in a $G.P.$, we can find the value of $a_n$ the $nth$ term. We can now find the sum of $n$ terms of sequence as

$$S = \sum_{k=1}^{n} a_k$$