

# 1 Fundamental

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## 1.1 Intervals

Intervals are basically subsets of  $\mathbb{R}$  and are commonly used in solving inequalities or in finding domains. If there are two numbers  $a, b \in \mathbb{R}$  such that  $a < b$ , there are four types of intervals:

- Open interval:  $(a, b) = \{x : a < x < b\}$  i.e end points are not included. Symbols:  $()$  or  $]$
- Closed interval:  $[a, b] = \{x : a \leq x \leq b\}$  i.e end points are also included. Symbol:  $[]$
- Open-Closed interval:  $(a, b) = \{x : a < x \leq b\}$ . Symbols:  $()]$  or  $][]$
- Closed-Open interval:  $(a, b) = \{x : a \leq x < b\}$ . Symbols:  $[])$  or  $][]$

### Infinite Intervals

- $(a, \infty) = \{x : x > a\}$
- $[a, \infty) = \{x : x \geq a\}$
- $(-\infty, b) = \{x : x < b\}$
- $(-\infty, b] = \{x : x \leq b\}$
- $(-\infty, \infty) = \{x : x \in \mathbb{R}\}$

## 1.2 Sets and Relations

### 1.2.1 Set

A collection of any kind of objects. The objects that make up a set are called *elements* or *members*. The statement ' $a$  is an element of set  $A$ ' can be written as  $a \in A$  and set containing elements  $a, b$  and  $c$  is denoted by  $\{a, b, c\}$ . A *empty* or *null* set is denoted by  $\emptyset$ , which is the set that contains no elements.

**Union(join,sum):** The union of two sets  $A$  and  $B$ , denoted by  $A \cup B$ , consists of those elements that belong to  $A$  or to  $B$ :

$$A \cup B = \{x : (x \in A) \vee (x \in B)\}$$

For example, if  $A$  is  $\{1, 2, 3, 4\}$  and  $B$  is  $\{1, 4, 5, 6\}$  then  $A \cup B$  is  $\{1, 2, 3, 4, 5, 6\}$ .

**Intersection(meet,product):** The intersection of two sets  $A$  and  $B$ , denoted by  $A \cap B$ , consists of those elements that belong to both  $A$  and  $B$ :

$$A \cap B = \{x : (x \in A) \wedge (x \in B)\}$$

For example, if  $A$  is  $\{1, 2, 3, 4, 5, 6\}$  and  $B$  is  $\{1, 4, 5, 6, 7, 8\}$  then  $A \cap B$  is  $\{1, 4, 5, 6\}$ .

**Complement:** The complement of a set  $A$ , denoted by  $A'$  or  $A^c$ , consists of all those elements that are not members of  $A$ :

$$A' = \{x : x \notin A\}$$

For example, in the domain of natural numbers, if  $A$  is set of even numbers the its complement  $A'$  is the set of odd numbers.

**Universal Set:** Relative to a particular domain, the universal set, denoted by  $\mathbf{U}$ , is set of all objects of that domain:

$$\mathbf{U} = \{x : x = x\}$$

### 1.2.2 Properties of sets

**Commutative law:**

- $(A \cup B) = B \cup A$
- $(A \cap B) = B \cap A$

**Associative law:**

- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C = A \cap (B \cap C)$

**Distributive law:**

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

**De Morgan's law:**

- $(A \cup B)' = A' \cap B'$
- $(A \cap B)' = A' \cup B'$

**Identity law:**

- $A \cup \mathbf{U} = A$

- $A \cup \emptyset = A$

**Complement law:**

- $A \cup A' = \mathbf{U}$

- $A \cap A' = \emptyset$

- $(A')' = A$

**Idempotent law:**

- $A \cap A = A$

- $A \cup A = A$

### 1.2.3 Results on number of elements in sets

If  $A, B, C$  are finite sets and  $\mathbf{U}$  be Universal finite set then:

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

- $n(A - B) = n(A) - n(A \cap B)$

- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

### 1.2.4 Relations

An association between, or property of, two or more objects. Thus  $(x = y)$  and  $(a \text{ lies between } b \text{ and } c)$  are relations, but  $(N \text{ is a prime})$  is not.

### 1.2.5 Types of Relations

**Void Relation:** Let  $A$  be a set. Then  $\phi \subseteq A \times A$  and so it is a relation on  $A$ . This relation is also called empty relation on  $A$ .

**Universal Relation:** Let  $A$  be a set. Then  $A \times A \subseteq A \times A$  and so it is a relation on  $A$ . This relation is called the universal relation on  $A$ .

**Identity relation:** Let  $A$  be a set. Then the relation  $I_A = \{(a, a) : a \in A\}$  is called the identity relation on  $A$ . In other words, a relation  $I_A$  on  $A$  is called the identity relation if every element of  $A$  is related to itself only.

**Reflexive Relation:** A relation  $R$  on a set  $A$  is said to be reflexive if every element of  $A$  is related to

itself. Thus  $R$  on set  $A$  is not reflexive if there exists an element  $a \in A$  such that  $(a, a) \notin R$ .

**Symmetric Relation:** A Relation  $R$  on set  $A$  is said to be symmetric if

$$(a, b) \in R \implies (b, a) \in R \forall a, b \in A$$

i.e

$$a R b \implies b R a \forall a, b \in A$$

**Transitive Relation:** A Relation  $R$  on set  $A$  is said to be transitive if  $(a, b) \in R$  and  $(b, c) \in R \implies (a, c) \in R \forall a, b, c \in A$ .

**Equivalence Relation:** A Relation is said to be equivalence if it satisfies *reflexive*, *symmetric* and *transitive* relations.

**Equivalence Class:** If  $R$  is an equivalence relation defined on set  $A$  then the equivalence class of any element  $x \in A$ , denoted by  $[x]$ , is the set of elements to which  $x$  is related by equivalence relation  $R$ :

$$[x] = \{y : x R y\}$$

For example, if  $R$  is the equivalence relation (the same height as), then the equivalence class of the element  $x \in A$  consists of all elements of  $A$  with same height as  $x$ .

## 2 Quadratic Equations

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### 2.1 Equation and Basic Results

An equation of the form

$$ax^2 + bx + c = 0$$

where  $a \neq 0$  and  $a, b, c \in \mathbb{R}$  is called a quadratic equation. The roots of quadratic equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quantity  $D$  ( $D = b^2 - 4ac$ ) is known as discriminant of equation.

### 2.1.1 Results

1. Let  $\alpha$  and  $\beta$  be two roots of given quadratic equation. Then

- $\alpha + \beta = -\frac{b}{a}$
- $\alpha\beta = \frac{c}{a}$

2. A quadratic equation, whose roots are  $\alpha$  and  $\beta$  can be written as

$$(x - \alpha)(x - \beta) = 0$$

i.e

$$ax^2 + bx + c \equiv a(x - \alpha)(x - \beta)$$

3. If the quadratic equation is satisfied by more than two distinct numbers (real or complex), then it becomes an identity, i.e

$$a = b = c = 0$$

4. The quadratic equation has real and equal roots if and only if  $D = 0$  i.e  $b^2 - 4ac = 0$ .
5. The quadratic equation has real and distinct roots if and only if  $D > 0$  i.e  $b^2 - 4ac > 0$ .
6. The quadratic equation has complex roots with non-zero imaginary parts if and only if  $D < 0$  i.e  $b^2 - 4ac < 0$ .
7. If  $p + iq$  ( $p, q \in \mathbb{R}$ ) is root of quadratic equation where  $i = \sqrt{-1}$ , the  $p - iq$  is also root of quadratic equation. Provided  $a, b, c \in \mathbb{R}$ .
8. If  $p + \sqrt{q}$  is an irrational root of quadratic equation, then  $p - \sqrt{q}$  is also a root of equation provided that all coefficients are rational.

### 2.2 Conditions for Common Root(s)

Let  $ax^2 + bx + c = 0$  and  $dx^2 + ex + f = 0$  have a common root  $\alpha$ . Then  $a\alpha^2 + b\alpha + c = 0$  and  $d\alpha^2 + e\alpha + f = 0$ .

Solving for  $\alpha^2$  and  $\alpha$  :

$$\frac{\alpha^2}{bf - ce} = \frac{\alpha}{dc - af} = \frac{1}{ae - bd} \implies$$

$$\alpha^2 = \frac{bf - ce}{ae - bd}$$

and

$$\alpha = \frac{dc - af}{ae - bd} \implies$$

$$(dc - af)^2 = (bf - ce)(ae - bd)$$

which is a required condition for the two equation to have a common root.

- Condition for both the roots to be common is  $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$

### 2.3 Wavy Curve Method

The Method of intervals is used for solving inequalities of the form

$$f(x) = \frac{(x - a_1)^{n_1}(x - a_2)^{n_2} \dots (x - a_k)^{n_k}}{(x - b_1)^{m_1}(x - b_2)^{m_2} \dots (x - b_p)^{n_p}} > 0 (< 0, \leq 0, \text{ or } \geq 0)$$

where  $n_1, n_2, n_3 \dots n_k, m_1, m_2, m_3 \dots m_p \in \mathbb{N}$  and the numbers  $a_1, a_2, \dots a_k, b_1, b_2, \dots b_p \in \mathbb{R}$  such that  $a_i \neq b_i$  where  $i = 1, 2, 3, \dots k$  and  $j = 1, 2, 3, \dots p$ .

#### Statements

- All *zeros*<sup>1</sup> of the function  $f(x)$  contained on left hand side of the inequality should be marked on the number line with black circles.
- All points of *discontinuities*<sup>2</sup> of the function  $f(x)$  contained on left hand side of the inequality should be marked with white circles.
- Check the value of  $f(x)$  for any real number greater than the right most marked number on the number line.
- From right to left, beginning above the number line, a wavy curve should be drawn which passes through all the marked points so that when passes through *single point*<sup>3</sup>, the curve intersects the number line, and when passing through a *double point*<sup>4</sup>, the curve remains located on the one side of number line.
- The appropriate intervals are chosen in accordance with sign of inequality. Their union just represents the solution of inequality.

#### Remarks

- The points of discontinuity will never be included in answers.
- *Zeros*<sup>1</sup>: The point for which  $f(x)$  vanishes (becomes zero) is called the *zeros* of function e.g.  $x = a_i$
- *Discontinuities*<sup>2</sup>: The points  $x = b_j$  are the points of the *discontinuity* of the function.
- *Single point*<sup>3</sup>: If the exponents of factor is *odd* then the point is called *single point*.
- *Double point*<sup>4</sup>: If the exponent of factor is *even* then the point is called *double point*.
- Interval  $(\lambda_1, \lambda_2)$  will be contained between the roots of  $f(x) = 0$  if  $af(\lambda_1) < 0, af(\lambda_2) < 0$ .
- $\lambda$  will be the required root of  $f(x) = 0$  if  $f(\lambda) = 0$  and  $f'(\lambda) = 0$ .

## 2.4 Quadratic Expression

Let  $f(x) = ax^2 + bx + c$  where  $a, b, c \in \mathbb{R}$   $a \neq 0$  It can be rewritten as

$$f(x) = a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{D}{4a^2}\right\}$$

where  $D = b^2 - 4ac$ . Then  $y - f(x)$  represents a parabola whose axis is parallel to the y axis, with vertex at  $A\left(-\frac{b}{2a}, \frac{-D}{4a}\right)$ . Now depending upon the values of  $a$  and  $D$  the parabola will have different shapes: will add images later...

## 2.5 Interval in which the roots lie

Let  $f(x) = ax^2 + bx + c$  where  $a, b, c \in \mathbb{R}$  and  $a \neq 0$ . Suppose  $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$  and  $\lambda_1 < \lambda_2$  Then the following results hold true:

- Both roots of  $f(x) = 0$  will be greater than  $\lambda$ , if  $D \geq 0, af(\lambda) > 0$  and  $\lambda < -\frac{b}{2a}$ .
- Both roots of  $f(x) = 0$  will be less than  $\lambda$ , if  $D \geq 0, af(\lambda) > 0$  and  $\lambda > -\frac{b}{2a}$ .
- If  $\lambda$  lies between the roots of  $f(x) = 0$  then  $af(\lambda) < 0$ .
- Both roots of  $f(x) = 0$  will belong to interval  $(\lambda_1, \lambda_2)$ , if  $D \geq 0, af(\lambda_1) > 0, af(\lambda_2) > 0$  and  $\lambda_1 < -\frac{b}{2a} < \lambda_2$ .
- Exactly one root of  $f(x) = 0$  will lie in  $(\lambda_1, \lambda_2)$  if  $f(\lambda_1) \cdot f(\lambda_2) < 0$ .