

# 1 Functions

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## 1.1 Functions. Domain of Definition

The independent variable  $x$  is defined by a set  $X$  of its values. If to each value of the independent variable  $x \in X$  there corresponds one definite value of another variable  $y$ , then  $y$  is called the function of  $x$  with a domain of definition (or domain)  $X$  or, in functional notation,  $y = y(x)$ , or  $y = f(x)$ , or  $y = \varphi(x)$ , and so forth. The set of values of the function  $y(x)$  is called the range of the given function.

## 1.2 Investigation of Functions

A function  $f(x)$  defined on the set  $X$  is said to be non-decreasing on this set (respectively, increasing, non-increasing, decreasing), if for any numbers  $x_1, x_2 \in X, x_1 < x_2$  the inequality  $f(x_1) \leq f(x_2)$  (respectively,  $f(x_1) < f(x_2), f(x_1) \geq f(x_2), f(x_1) > f(x_2)$ ) is satisfied. The function  $f(x)$  is said to be monotonic on the set  $X$  if it possesses one of the four indicated properties. The function  $f(x)$  is said to be bounded above (or below) on the set  $X$  if there exists a number  $M$  (or  $m$ ) such that  $f(x) \leq M \forall x \in X$ . The function  $f(x)$  is said to be bounded on the set  $X$  if it is bounded above and below. The function  $f(x)$  is called periodic if there exists a number  $T > 0$  such that  $f(x+T) = f(x)$  for all  $x$  belonging to the domain of definition of the function (together with any point  $x$  the point  $x+T$  must belong to the domain of definition). The least number  $T$  possessing this property (if such a number exists) is called the period of the function  $f(x)$ . The function  $f(x)$  takes on the maximum value at the point  $x_o \in X$  if  $f(x_o) \geq f(x)$  for all  $x \in X$ , and the minimum value if  $f(x_o) \leq f(x)$  for all  $x \in X$ . A function  $f(x)$  defined on a set  $X$  which is symmetric *w.r.t* origin of coordinates is called even if  $f(-x) = f(x)$ , and odd if  $f(x) = -f(x)$ .

## 1.3 Inverse of Function

Let the function  $y = f(x)$  be defined on the set  $X$  and have a range  $Y$ . If for each  $y \in Y$  there ex-

ists a single value of  $x$  such that  $f(x) = y$ , then this correspondence defines a certain function  $x = g(y)$  called inverse *w.r.t* given function  $y = f(x)$ . The sufficient condition for the existence of an inverse function is a strict monotony of the original function  $y = f(x)$ . If the function increases(decreases), then the inverse function also increases (decreases). The graph of the inverse function  $x = g(y)$  coincides with that of the function  $y = f(x)$  if the independent variable is marked off along the  $y$ -axis. If the independent variable is laid off along the  $x$ -axis, i. e. if the inverse function is written in the form  $y = g(x)$ , then the graph of the inverse function will be symmetric to that of the function  $y = f(x)$  with respect to the bisector of the first and third quadrants.

# 2 Limits

## 2.1 Existence

Limit of function  $f(x)$  is said to exist as  $x \rightarrow a$  when,

$$\lim_{h \rightarrow 0^+} f(a-h) = \lim_{h \rightarrow 0^+} f(a+h)$$

equal to some finite value  $L$ .

## 2.2 Indeterminate forms

There are only seven indeterminate forms  $\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, \infty^0, 0^0$  and  $1^\infty$ .

## 2.3 List of limits

### Limits Operations

If  $\lim_{x \rightarrow c} f(x) = L$

- $\lim_{x \rightarrow c} [f(x \pm a)] = L \pm a$
- $\lim_{x \rightarrow c} a f(x) = aL$
- $\lim_{x \rightarrow c} \frac{1}{f(x)} = \frac{1}{L}$  for  $L \neq 0$
- $\lim_{x \rightarrow c} f(x)^n = L^n$  for  $n > 0$

### Involving infinitesimal changes

If infinitesimal change  $h$  if denote by  $\Delta x$ . If  $f(x)$  and  $g(x)$  are differentiable at  $x$ .

- $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} = f'(x)$
- $\lim_{h \rightarrow 0} \frac{f \circ g(x+h) - f \circ g(x)}{h} = f'[g(x)]g'(x)$
- $\lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = f'(x)g(x) + f(x)g'(x)$
- $\lim_{h \rightarrow 0} \left( \frac{f(x+h)}{f(x)} \right)^{\frac{1}{h}} = \exp\left(\frac{f'(x)}{f(x)}\right)$
- $\lim_{h \rightarrow 0} \left( \frac{f(e^h x)}{f(x)} \right)^{\frac{1}{h}} = \exp\left(\frac{x f'(x)}{f(x)}\right)$

If  $f(x)$  and  $g(x)$  are differentiable on an open interval containing  $c$ , except possibly  $c$  itself, and  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$  or  $\pm\infty$ .

Jean Bernoulli or L'Hopital's rule can be used:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

### Inequalities

If  $f(x) \leq g(x)$  for all  $x$  in interval that contains  $c$ , except possibly  $c$  itself, and the limit of  $f(c)$  and  $g(x)$  both exist at  $c$ , then