Contents

1 Fundamental

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1.1 Intervals

Intervals are basically subsets of $\mathbb{R}$ and are commonly used in solving inequalities or in finding domains. If there are two numbers $a, b \in \mathbb{R}$ such that $a < b$, there are four types of intervals:

- Open interval: $(a, b) = \{x : a < x < b\}$ i.e end points are not included. Symbols: () or][
- Closed interval: $[a, b] = \{x : a \leq x \leq b\}$ i.e end points are also included. Symbol: [ ]
- Open-Closed interval: $(a, b) = \{x : a < x \leq b\}$. Symbols: ( ] or]]
- Closed-Open interval: $(a, b) = \{x : a \leq x < b\}$. Symbols: [ ) or [[

Infinite Intervals

- $(a, \infty) = \{x : x > a\}$
- $[a, \infty) = \{x : x \geq a\}$
- $(-\infty, b) = \{x : x < b\}$
- $(\infty, b] = \{x : x \leq b\}$
- $(-\infty, \infty) = \{x : x \in \mathbb{R}\}$

1.2 Sets and Relations

1.2.1 Set

A collection of any kind of objects. The objects that make up a set are called *elements* or *members*. The statement 'a is an element of set A' can be written as $a \in A$ and set containing elements a, b and c is denoted by $\{a, b, c\}$. A *empty* or *null* set is denoted by $\varnothing$, which is the set that contains no elements.

Union(join,sum): The union of two sets A and B, denoted by $A \cup B$, consists of those elements that belong to A or to B:

$$A \cup B = \{x : (x \in A) \vee (x \in B)\}$$

For example, if A is $\{1, 2, 3, 4\}$ and B is $\{1, 4, 5, 6\}$ then $A \cup B$ is $\{1, 2, 3, 4, 5, 6\}$.
Intersection(meet,product): The intersection of two sets A and B, denoted by $A \cap B$, consists of those elements that belong to both A and B:

$$A \cap B = \{x : (x \in A) \wedge (x \in B)\}$$

For example, if A is $\{1, 2, 3, 4, 5, 6\}$ and B is $\{1, 4, 5, 6, 7, 8\}$ then $A \cap B$ is $\{1, 4, 5, 6\}$.
Complement: The complement of a set A, denoted by A' or $A^{\complement}$, consists of all those elements that are not members of A:

$$A' = \{x : x \notin A\}$$

For example, in the domain of natural numbers, if A is set of even numbers the its complement A' is the set of odd numbers.
Universal Set: Relative to a particular domain, the universal set, denoted by $\mathbf{U}$, is set of all objects of that domain:

$$\mathbf{U} = \{x : x = x\}$$

1.2.2 Properties of sets

Commutative law:

- $(A \cup B) = B \cup A$
- $(A \cap B) = B \cap A$

Associative law:

- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C = A \cap (B \cap C)$

Distributive law:

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

De Morgan's law:

- $(A \cup B)' = A' \cap B'$
- $(A \cap B)' = A' \cup B'$

Identity law:

- $A \cup \mathbf{U} = A$
- $A \cup \varnothing = A$

Complement law:

- $A \cup A' = \mathbf{U}$
- $A \cap A' = \varnothing$
- $(A')' = A$

Idempotent law:

- $A \cap A = A$
- $A \cup A = A$

1.2.3 Results on number of elements in sets

If A, B, C are finite sets and $\mathbf{U}$ be Universal finite set then:

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A - B) = n(A) - n(A \cap B)$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

1.2.4 Relations

An association between, or property of, two or more objects. Thus $(x = y)$ and (a lies between b and c) are relations, but (N is a prime) is not.

1.2.5 Types of Relations

Void Relation: Let A be a set. Then $\phi \subseteq A \times A$ and so it is a relation on A. This relation is also called empty relation on A.
Universal Relation: Let A be a set. Then $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A.
Identity relation: Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$is called the identity relation on A. In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only.
Reflexive Relation: A relation R on a set A is said to be reflexive if every element of A is related to itself. Thus R on set A is not reflexive if there exits an element $a \in A$ such that $(a, a) \notin R$.
Symmetric Relation: A Relation R on set A is said to be symmetric if

$$(a, b) \in R \implies (b, a) \in R \,\forall\, a, b \in A$$

i.e

$$a\, R\, b \implies b\, R\, a \,\forall\, a, b \in A$$

Transitive Relation: A Relation R on set A is said to be transitive if $(a, b) \in R$ and $(b, c) \in R \implies (a, c) \in R \,\forall\, a, b, c \in A$.
Equivalence Relation: A Relation is said to be equivalence if it satisfies $reflexive$, *symmetric* and *transitive* relations.
Equivalence Class: If R is an equivalence relation defined on set A then the equivalence class of any element $x \in A$, denoted by $[x]$, is the set of elements to which x is related by equivalence relation R:

$$[x] = \{y : x\, \mathbf{R}\, y\}$$

For example, if R is the equivalence relation (the same height as), then the equivalence class of the element $x \in A$ consists of all elements of A with same height as x.

2 Quadratic Equations

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2.1 Equation and Basic Results

An equation of the form

$$ax^2 + bx + c = 0$$

where $a \neq 0$ and $a, b, c \in \mathbb{R}$ is called a quadratic equation. The roots of quadratic equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quantity D ($D = b^2 - 4ac$) is known as discriminant of equation.

2.1.1 Results

1. Let α and β be two roots of given quadratic equation. Then

- $\alpha + \beta = -\frac{b}{a}$
- $\alpha\beta = \frac{c}{a}$

2. A quadratic equation, whose roots are α and β can be written as

$$(x - \alpha)(x - \beta) = 0$$

i.e

$$ax^2 + bx + c \equiv a(x - \alpha)(x - \beta)$$

3. If the quadratic equation is satisfied by more than two distinct numbers (real or complex), then it becomes an identity, i.e

$$a = b = c = 0$$

4. The quadratic equation has real and equal roots if and only if $D = 0$ i.e $b^2 - 4ac = 0$.
5. The quadratic equation has real and distinct roots if and only if $D > 0$ i.e $b^2 - 4ac > 0$.
6. The quadratic equation has complex roots with non-zero imaginary parts if and only if $D < 0$ i.e $b^2 - 4ac < 0$.
7. If $p + iq$ ($p, q \in \mathbb{R}$) is root of quadratic equation where $i = \sqrt{-1}$, the $p - iq$ is also root of quadratic equation. Provided $a, b, c \in \mathbb{R}$.
8. If $p + \sqrt{q}$ is an irrational root of quadratic equation, then $p - \sqrt{q}$ is also a root of equation provided that all coefficients are rational.

2.2 Conditions for Common Root(s)

Let $ax^2 + bx + c = 0$ and $dx^2 + ex + f = 0$ have a common root α. Then $a\alpha^2 + b\alpha + c = 0$ and $d\alpha^2 + e\alpha + f = 0$.
Solving for α^2 and α :

$$\frac{\alpha^2}{bf - ce} = \frac{\alpha}{dc - af} = \frac{1}{ae - bd} \implies$$

$$\alpha^2 = \frac{bf - ce}{ae - bd}$$

and

$$\alpha = \frac{dc - af}{ae - bd} \implies$$

$$(dc - af)^2 = (bf - ce)(ae - bd)$$

which is a required condition for the two equation to have a common root.

- Condition for both the roots to be common is $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$

2.3 Wavy Curve Method

The Method of intervals is used for solving inequalities of the form

$$f(x) = \frac{(x - a_1)^{n_1}(x - a_2)^{n_2}...(x - a_k)^{n_k}}{(x - b_1)^{m_1}(x - b_2)^{m_2}...(x - b_p)^{n_p}} > 0(< 0, \leq 0, or \geq 0)$$

where $n_1, n_2, n_3...n_k$, $m_1, m_2, m_3..m_p \in \mathbb{N}$ and the numbers $a_1, a_2, ...a_k$, $b_1, b_2, ...b_p$ $\in \mathbb{R}$ such that $a_i \neq b_i$ where $i = 1, 2, 3, ...k$ and $j = 1, 2, 3, ...p$.
Statements

- All *zeros*[1] of the function $f(x)$ contained on left hand side of the inequality should be marked on the number line with black circles.
- All points of *discontinuities*[2] of the function $f(x)$ contained on left hand side of the inequality should be marked with white circles.
- Check the value of $f(x)$ for any real number greater than the right most marked number on the number line.

- From right to left, beginning above the number line, a wavy curve should be drawn which passes through all the marked points so that when passes through *singlepoint*[3], the curve intersects the number line, and when passing through a *double point*[4], the curve remains located on the one side of number line.
- The appropriate intervals are chosen in accordance with sign of inequality. Their union just represents the solution of inequality.

Remarks

- The points of discontinuity will never be included in answers.
- *Zeros*[1]:The point for which $f(x)$ vanishes(becomes zero) is called the *zeros* of function e.g. $x = a_i$
- *Discontinuities*[2]:The points $x = b_j$ are the points of the *discontinuity* of the function.
- *Single point*[3]:If the exponents of factor is *odd* then the point is called *single point*.
- *Double point*[4]: If the exponent of factor is *even* then the point is called *double point*.

2.4 Quadratic Expression

Let $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$ $a \neq 0$ It can be rewritten as

$$f(x) = a\{(x + \frac{b}{2a})^2 + \frac{D}{4a^2}\}$$

where $D = b^2 - 4ac$. Then $y - f(x)$ represents a parabola whose axis is parallel to the y axis, with vertex at $A(-\frac{b}{2a}, \frac{-D}{4a})$. Now depending upon the values of a and D the parabola will have different shapes: will add images later...

2.5 Interval in which the roots lie

Let $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Suppose $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 < \lambda_2$ Then the following results hold true:

- Both roots of $f(x) = 0$ will be greater than λ, if $D \geq 0$, $af(\lambda) > 0$ and $\lambda < -\frac{b}{2a}$.
- Both roots of $f(x) = 0$ will be less than λ, if $D \geq 0$, $af(\lambda) > 0$ and $\lambda > -\frac{b}{2a}$.
- If λ lies between the roots of $f(x) = 0$ then $af(\lambda) < 0$.
- Both roots of $f(x) = 0$ will belong to interval (λ_1, λ_2), if $D \geq 0$, $af(\lambda_1) > 0$,$af(\lambda_2) > 0$ and $\lambda_1 < -\frac{b}{2a} < \lambda_2$.
- Exactly one root of $f(x) = 0$ will lie in (λ_1, λ_2) if $f(\lambda_1).f(\lambda_2) < 0$.
- Interval (λ_1, λ_2) will be contained between the roots of $f(x) = 0$ if $af(\lambda_1) < 0, af(\lambda_2) < 0$.
- λ will be the required root of $f(x) = 0$ if $f(\lambda) = 0$ and $f'(\lambda) = 0$.

3 Progression and Series

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3.1 Arithmetic Progression

If a is the first term and d the common difference, the $A.P.$ can be written as $a, a+d, a+2d, ...$ The nth term a_n is given by

$$a_n = a + (n-1)d$$

The sum of S_n of the first terms of such an $A.P.$ is given by

$$S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(a + a_n)$$

If three terms are in $A.P.$, then the middle term is called the *Arimethic Mean* $(A.M.)$ between the other two i.e a, b, c are in $A.P.$ then

$$2b = a + c$$

is the $A.M$ of a and c.
If $a_1, a_2, a_3...a_n$ are n numbers, then the arithmetic mean of these numbers

$$A.M. = \frac{1}{n}(a_1 + a_2 + a_3 + ... + a_n)$$

The numbers $A_1, A_2, A_3...A_n$ are said to be $A.M.$'s between the numbers a and b if $a, A_1, A_2...A_n, b$ are in $A.P.$ then

$$A_n = a + \frac{n(b-a)}{n+1} = \frac{a+nb}{n+1}$$

3.2 Geometric Progression

If a is the first term and r the common ratio, then the $G.P.$ can be written as $a, ar, ar^2....$ The nth term a_n is given by

$$a_n = ar^{n-1}$$

The sum S_n of first n terms of the $G.P.$ is

$$S_n = \begin{cases} \frac{a(r^n-1)}{r-1}, r \neq 1 \\ na, r = 1 \end{cases}$$

If $-1 < r < 1$, then the sum of infinite $G.P.$ is $a + ar + ar^2 + ...$

$$= \frac{a}{1-r}$$

If three terms are in $G.P.$, then the middle term is called the geometric mean $G.M.$ between the two. So if a, b, c are in $G.P.$ then $b = \sqrt{ac}$ is geometric mean of a and c.
If $a_1, a_2...a_n$ are non-zero positive numbers, then their $G.M$ is given by

$$G = (a_1 a_2 a_3 ... a_n)^{1/n}$$

If $G_1, G_2...G_n$ are n geometric means between a and b, $a, G_1, G_2....G_n, b$ will be a $G.P.$

$$G_n = a(1 + \sqrt[n+1]{\frac{b}{a}})^n$$

3.3 Harmonic Progression

The sequence $a_1, a_2, a_3....a_n (a_i \neq 0)$ is said to be $H.P.$ if the sequence $\frac{1}{a_1}, \frac{1}{a_2} ... \frac{1}{a_n}$ is an $A.P.$. The nth term a_n of the $H.P.$ is

$$a_n = \frac{1}{a + (n-1)d}$$

where $a = \frac{1}{a_i}$ and $d = \frac{1}{a_2} - \frac{1}{a_1}$.
Harmonic Means:
If a and b are two non-zero numbers, then the harmonic mean of a and b is a number H such that the numbers a, H, b are in $H.P.$ We have

$$\frac{1}{H} = \frac{1}{2}(\frac{1}{a} + \frac{1}{b})$$

There is no formula for sum of n terms of an $H.P.$, if $a_1, a_2...a_n$ are n non-zero numbers, then the harmonic mean H of these numbers is given by

$$\frac{1}{H} = \frac{1}{n}(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n})$$

The n number $H_1, H_2...H_n$ are said to be n-harmonic means between a and b, if $a, H_1, H_2...H_n, b$ are in $H.P.$ if $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}...\frac{1}{H_n}, \frac{1}{b}$ are in $A.P.$

$$\frac{1}{H_n} = \frac{1}{a} + \frac{n(a-b)}{(n+1)ab}$$

3.4 Arithmetico-Geometric Progression

The sum S_n of first n terms of an $A.G.P.$ is obtained in the following way:

$S_n = ab + (a+d)br + (a+2d)br^2 + ... + (a+(n-2)d)br^{n-2} + (a+(n-1)d)br^{n-1}$

Multiplying both sides by r, so that

$rS_n = abr + (a+d)br^2 + (a+2d)br^3 + ... + (a+(n-2)d)br^{n-2} + (a+(n-1)d)br^{n-1} + (a+(n-1)d)br^n$

Subtracting,we get

$$S_n = \frac{ab}{1-r} + \frac{dbr(1-r^{n-1})}{(1-r)^2} - \frac{(a+(n-1)d)br^n}{1-r}$$

If $-1 < r < 1$, the sum of infinite number of terms of progression is

$$\lim_{n\to\infty} S_n = S = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$$

3.5 Misc. Progressions

Some Important Results:

- $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$
- $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$
- $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4} = (1 + 2 + 3 + ... + n)^2$
- $1 + x + x^2 + x^3 + ... = (1 - x)^{-1}$ if $-1 < x < 1$
- $1 + 2x + 3x^2 + 4x^3 + ... = (1 - x)^{-2}$ if $-1 < x < 1$

Method of Differences:
Suppose $a_1, a_2, a_3 ... a_n$ is a sequence such that sequence $a_2 - a_1, a_3 - a_2 ...$ is either an $A.P.$ or a $G.P.$ The nth term a_n of this sequence is obtained as follows:

$$S_n = a_1 + a_2 + a_3 + ... + a_{n-2} + a_{n-1} + a_n$$

$$\implies a_n = a_1 + [(a_2 - a_1) + (a_3 - a_2) + ... + (a_n - a_{n-1})]$$

Since the term within the brackets are either in an $A.P.$ or in a $G.P.$, we can find the value of a_n the nth term. We can now find the sum of n terms of sequence as

$$S = \sum_{k=1}^{n} a_k$$

3.6 Inequalities

A.M. $\geq$ G.M. $\geq$ H.M
Let $a_1, a_2 ... a_n$ be n positive real numbers, then we define their arithmetic mean A, geometric mean G and harmonic mean H as

$$A = \frac{a_1 + a_2 + ... + a_n}{n}$$

$$G = (a_1 a_2 a_3 ... a_n)^{1/n}$$

$$H = \frac{n}{(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n})}$$

4 Binomial Theorem

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4.1 Basic Results

$$(a+x)^n =^n C_0 a^n +^n C_1 a^{n-1} x +^n C_2 a^{n-2} x^2 + ... +^n C_r a^{n-r} x^r + ... +^n C_n x^n = \sum_{r=0}^{n} {}^n C_r x^r . a^{n-r}$$

$$(a-x)^n =^n C_0 -^n C_1 a^{n-1} x +^n C_2 a^{n-2} x^2 - ... + (-1)^{r n} C_r a^{n-r} x^r + ... + (-1)^{n n} C_n x^n . a^{n-r}$$