1 Functions

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1.1 Functions. Domain of Definition

The independent variable x is defined by a set X of its values. If to each value of the independent variable $x \in X$ there corresponds one definite value of another variable y, then y is called the function of x with a domain of definition (or domain) X or, in functional notation, $y = y(x)$, or $y = f(x)$, or $y = \varphi(x)$, and so forth. The set of values of the function $y(x)$ is called the range of the given function.

1.2 Investigation of Functions

A function $f(x)$ defined on the set X is said to be non-decreasing on this set (respectively, increasing, non-increasing, decreasing), if for any numbers $x_1, x_2 \in X, x_1 < x_2$ the inequality $f(x_1) \leq f(x_2)$ (respectively, $f(x_1) < f(x_2), f(x_1) \geq f(x_2), f(x_1) > f(x_2)$) is satisfied.The function $f(x)$ is said to be monotonic on the set X if it possesses one of the four indicated properties. The function $f(x)$ is said to be bounded above (or below) on the set X if there exists a number $M(or\ m)$ such that $f(x) \leq M\ \forall\ x \in X$. The function $f(x)$ is said to be bounded on the set X if it is bounded above and below. The function $f(x)$ is called periodic if there exists a number $T > 0$ such that $f(x+T) = f(x)$ for all x belonging to the domain of definition of the function (together with any point x the point $x + T$ must belong to the domain of definition). The least number T possessing this property (if such a number exists) is called the period of the function $f(x)$. The function $f(x)$ takes on the maximum value at the point $x_o \in X$ if $f(x_o) \geq f(x)$ for all $x \in X$, and the minimum value if if $f(x_o) \leq f(x)$ for all $x \in X$. A function $f(x)$ defined on a set X which is symmetric $w.r.t$ origin of coordinates is called even if $f(-x) = f(x)$, and odd if $f(x) = -f(x)$.

1.3 Inverse of Function

Let the function $y = f(x)$ be defined on the set X and have a range Y. If for each $y \in Y$ there exists a single value of x such that $f(x) = y$, then this correspondence defines a certain function $x = g(y)$ called inverse $w.r.t$ given function $y = f(x)$. The sufficient condition for the existence of an inverse function is a strict monotony of the original function $y = f(x)$. If the function increases(decreases), then the inverse function also increases (decreases). The graph of the inverse function $x = g(y)$ coincides with that of the function $y = f(x)$ if the independent variable is marked off along the $y - axis$. If the independent variable is laid off along the x-axis,i. e. if the inverse function is written in the form $y = g(x)$, then the graph of the inverse function will be symmetric to that of the function $y = f(x)$ with respect to the bisector of the first and third quadrants.

2 Limits

2.1 Existence

Limit of function $f(x)$ is said to exist as $x \to a$ when,

$$\lim_{h \to 0^+} f(a - h) = \lim_{h \to 0^+} f(a + h)$$

equal to some finite value L.

2.2 Indeterminate forms

There are only seven indeterminate forms
$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, \infty^0, 0^0\ and\ 1^\infty$.

2.3 List of limits

Limits Operations
If $\lim_{x \to c} f(x) = L$

- $\lim_{x \to c}[f(x \pm a)] = L \pm a$
- $\lim_{x \to c} af(x) = aL$
- $\lim_{x \to c} \frac{1}{f(a)} = \frac{1}{L}$ for $L > 0$
- $\lim_{x \to c} f(x)^n = L^n$ for $n > 0$

Involving infinitesimal changes
If infinitesimal change h if denote by Δx. If $f(x)$ and $g(x)$ are differentiable at x.

- $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$
- $\lim_{h\to 0} \frac{fog(x+h)-fog(x)}{h} = f'[g(x)]g'(x)$
- $\lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} = f'(x)g(x) + f(x)g'(x)$
- $\lim_{h\to 0}(\frac{f(x+h)}{f(x)})^{\frac{1}{h}} = exp(\frac{f'(x)}{f(x)})$
- $\lim_{h\to 0}(\frac{f(e^h x)}{f(x)})^{\frac{1}{h}} = exp(\frac{xf'(x)}{f(x)})$

If $f(x)$ and $g(x)$are differentiable on an open interval containing c, except possibly c itself, and $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$.
Jean Bernoulli or L'Hopital's rule can be used:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

Inequalities
If $f(x) \leq g(x)$ for all x in interval that contains c, except possibly c itself, and the limit of $f(x)$ and $g(x)$ both exist at c, then $\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$
If $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$ and

$$f(x) \leq g(x) \leq h(x)$$

for all x in an open interval that contains c, except possibly c itself, $\lim_{x\to c} g(x) = L$. This is know as $Squeeze\ Theorem$.

2.4 Exponential Functions

Function of form $f(x)^{g(x)}$

- $\lim_{x\to +\infty}(\frac{x}{x+k})^x = e^{-k}$
- $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$
- $\lim_{x\to 0}(1+kx)^{\frac{m}{x}} = e^{mk}$
- $\lim_{x\to +\infty}(1+\frac{1}{x})^x = e$
- $\lim_{x\to +\infty}(1-\frac{1}{x})^x = \frac{1}{e}$
- $\lim_{x\to +\infty}(1+\frac{k}{x})^{mx} = e^{mk}$
- $\lim_{x\to 0}(1+a(e^{-x}-1)^{-\frac{1}{x}}) = e^a$

Sum products and Composites

- $\lim_{x\to 0}(\frac{a^x-1}{x}) = lna$
- $\lim_{x\to 0}(\frac{e^x-1}{x}) = 1$
- $\lim_{x\to 0}(\frac{e^{ax}-1}{x}) = a$

2.5 Logarithmic Functions

- $\lim_{x\to 1} \frac{lnx}{x-1} = 1$
- $\lim_{x\to 0} \frac{ln(x+1)}{x} = 1$
- $\lim_{x\to 0} \frac{-ln(1+a(e^{-x}-1)}{x} = a$

Some cases

- $\lim_{x\to 0^+} log_b x = -F(b)\infty$
- $\lim_{x\to \infty} log_b x = F(b)\infty$

where $F(x) = 2H(x-1)-1$ and $H(x)$ is Oliver Heaviside step function.

2.6 Trigonometric Functions

- $\lim_{x\to 0} \frac{sinax}{ax} = 1$ for $a \neq 0$
- $\lim_{x\to 0} \frac{sinax}{bx} = \frac{a}{b}$ for $b \neq 0$
- $\lim_{x\to \infty} xsin(\frac{1}{x}) = 1$
- $\lim_{x\to 0} \frac{tanax}{ax} = 1$ for $a \neq 0$
- $\lim_{x\to 0} \frac{tanax}{bx} = \frac{a}{b}$ for $b \neq 0$

2.7 Sums

- $\lim_{x\to \infty} \sum_{k=1}^{n} \frac{1}{k} = \infty$
- $\lim_{x\to \infty}(\sum_{k=1}^{n} \frac{1}{k} - logn) = \gamma$.This is Euler Mascheroni Constant.

2.8 Notable Special Limits

- $\lim_{x\to \infty} \frac{n}{\sqrt[n]{n!}} = e$
- $\lim_{x\to \infty} 2^n \sqrt{2-\sqrt{2+\sqrt{2+...+\sqrt{2}}}} = \pi$

2.9 Taylor Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \infty$$

$$ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + \infty$$

$$ln(1-x) = -(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ... + \infty)$$

$$ln(\frac{1+x}{1-x}) = 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + ...)$$

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... + \infty$$

$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... + \infty$$

$$tanx = x + \frac{x^3}{3} + \frac{2x^5}{15} + ... + \infty$$

$$secx = x + \frac{x^2}{2} + \frac{5x^4}{24} + ... + \infty$$

$$arcsinx/sin^{-1}x = x + \frac{x^3}{6} + \frac{3x^5}{40} + ... + \infty$$

$$arccosx/cos^{-1}x = \frac{\pi}{2} - (x + \frac{x^3}{6} + \frac{3x^5}{40} + ...)$$

$$arctanx/tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} + ... + \infty$$

3 Differentiation

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3.1 Elementary functions

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(a^x) = a^x lna$
- $\frac{d}{dx}(lnx) = \frac{1}{x}$
- $\frac{d}{dx}(log_a x) = \frac{1}{xlna}$
- $\frac{d}{dx}(sinx) = cosx$
- $\frac{d}{dx}(cosx) = -sinx$
- $\frac{d}{dx}(secx) = secxtanx$
- $\frac{d}{dx}(cosecx) = -cosecxcotx$
- $\frac{d}{dx}(tanx) = sec^2x$
- $\frac{d}{dx}(cotx) = -cosec^2x$

3.2 Basic Theorems

- $\frac{d}{dx}(f \pm g) = f'(x) \pm g'(x)$
- $\frac{d}{dx}(kf(x) = k\frac{d}{dx}(f(x))$
- $\frac{d}{dx}(f(x).g(x)) = f(x)g'(x) + g(x)f'(x)$
- $\frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$
- $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

3.3 Inverse Trigonometric Functions

- $\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(cot^{-1}x) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}(sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}(cosec^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$

3.4 Using Substitution

- $\sqrt{x^2+a^2} \implies x = atan\theta$
- $\sqrt{a^2-x^2} \implies x = asin\theta$
- $\sqrt{x^2-a^2} \implies x = asec\theta$
- $\sqrt{\frac{x+a}{a-x}} \implies x = acos\theta$

3.5 Parametric Differentiation

If $y = f(\theta)$ and $x = g(\theta)$ where θ is parameter then

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

3.6 Derivative of Determinant

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$ where $f, g, h, l, m, n, u, v, w$ are differentiable functions, then $F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$

4 Application of Derivatives

4.1 Equation of Tangent and Normal

Tangent at (x_1, y_1) is given by

$$(y - y_1) = f'(x_1)(x - x_1)$$

when $f'(x_1)$ is real and Normal at (x_1, y_1) is given by

$$(y - y_1) = \frac{-1}{f'(x_1)}(x - x_1)$$

when $f'(x_1)$ is non-zero and real.
Tangent from an external point
Given a point $\delta(a, b)$ which does not lie on the curve $y = f(x)$ then equation of possible tangents to the curve $y = f(x)$ passing through (a, b) can be found by solving for point of contact λ

$$f'(h) = \frac{f(h) - b}{h - a}$$

and equation of tangent

$$y - b = \frac{f(h) - b}{h - a}(x - a)$$

Length of tangent,normal,sub-tangent,sub-normal from point $\sigma(h, k)$ and slope m
Length of Tangent $= |k|\sqrt{1 + \frac{1}{m^2}}$
Length of Normal $= |k|\sqrt{1 + m^2}$
Length of Sub-Tangent $= |\frac{k}{m}|$
Length of Sub-Normal $= |km|$
Angle between the curves

$$tan\theta = |\frac{m_1 - m_2}{1 + m_1 m_2}|$$

4.2 Theorems

Rolle's Theorem
If a function f defined on $[a, b]$ and

- continuous on $[a, b]$
- derivable on (a, b)
- $f(a) = f(b)$

then there exists at least one real number c between a and b ($a < c < b$) such that $f'(c) = 0$.
Lagrange's Mean Value Theorem
If a function f defined on $[a, b]$ and

- continuous on $[a, b]$
- derivable on (a, b)

then there exists at least one real number c between a and b ($a < c < b$) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Cauchy's Mean Value theorem
If functions f and g defined on $[a, b]$ and

- continuous on $[a, b]$
- derivable on (a, b)
- $c \in (a, b)$ then

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

4.3 Maxima and Minima

If a function $y = f(x)$ is defined on interval X, then an interior point x_o of interval is called the point of *maximum* of function $f(x)$ [the point of *minimum* of function $f(x)$] if there exists a neighbourhood $U \in X$ of point x_o such that inequality $f(x) \leq f(x_o)[f(x) \geq f(x_o)]$ holds true within it.
A Necessary condition for the existence of an Extremum
At points of extremum the derivative $f'(x)$ is equal to zero or does not exist. The points at which $f'(x) = 0$ or does not exit are called *critical points*.

Sufficient conditions for the existence of an Extremum

1. Let the function $f(x)$ be continuous in some neighbourhood of point x_o

- If $f'(x) > 0$ at $x < x_o$ and $f'(x) < 0$ at $x > x_o$ (i.e if in moving from left to right through point x_o the derivative changes sign from plus to minus), then at point x_o the function reaches *maximum*.
- If $f'(x) < 0$ at $x < x_o$ and $f'(x) > 0$ at $x > x_o$ (i.e if in moving through the point x_o from left to right the derivative changes sign from minus to plus), then at point x_o the function reaches *minimum*.
- If the derivative does not change sign in moving through the point x_o, then there is no *extremum*.

2. Let the function $f(x)$ be twice differentiable (that is $f'(x_o) = 0$) at a critical point x_o. If $f''(x_o) < 0$ then at x_o the function has a *maximum*; if $f'(x_o) > 0$ then at x_o the function has *minimum* but if $f''(x_o) = 0$ then the question of existence of *extremum* at this point remains open.
3. Let $f(x_o) = f''(x_o) = ... = f^{n-1}(x_o) = 0$, but $f^n(x_o) \neq 0$. If n is even, then at $f^n(x_o) < 0$ there is a *maximum* at x_o, and at point $f^n(x_o) > 0$, a *minimum*.If n is odd then there is no *extremum* at point x_o.
4.Let the function $y = f(x)$ be represented parametrically:

$$x = \varphi(t),\ y = \psi(t)$$

where the functions $\varphi(t)$ and $\psi(t)$ have derivatives both of first and second orders within a certain interval of change of argument t, and $\varphi'(t) \neq 0$. Further, let, at $t = t_o$

$$\psi'(t) = 0$$

Then:

- If $\psi''(t_o) < 0$, the function $y = f(x)$ has a *maximum* at $x = x_o = \varphi(t_o)$
- If $\psi''(t_o) > 0$, the function $y = f(x)$ has a *minimum* at $x = x_o = \varphi(t_o)$
- If $\psi''(t_o) = 0$, the question of existence of *extremum* remains open.

5 Integration

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5.1 Standard Formula

- $\int(ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$
- $\int \frac{dx}{ax+b} = \frac{1}{a}ln(ax+b) + c$
- $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$
- $\int a^{px+q}dx = \frac{1}{p}\frac{a^{px+q}}{lna} + c$
- $\int sin(ax+b)dx = \frac{-1}{a}cos(ax+b) + c$
- $\int cos(ax+b)dx = \frac{1}{a}sin(ax+b) + c$
- $\int tan(ax+b)dx = \frac{1}{a}lnsec(ax+b) + c$
- $\int cot(ax+b)dx = \frac{1}{a}lnsin(ax+b) + c$
- $\int sec^2(ax+b)dx = \frac{1}{a}tan(ax+b) + c$
- $\int cosec^2(ax+b)dx = \frac{-1}{a}cot(ax+b) + c$
- $\int secxdx = ln(secx + tanx) + c$
- $\int cosexdx = ln(cosecx - cotx) + c$
- $\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}(\frac{x}{a}) + c$
- $\int \frac{dx}{a^2+x^2} = \frac{1}{a}tan^{-1}(\frac{x}{a}) + c$
- $\int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}sec^{-1}(\frac{x}{a}) + c$
- $\int \frac{dx}{\sqrt{x^2+a^2}} = ln[x + \sqrt{x^2+a^2}] + c$
- $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}ln|\frac{a+x}{a-x}| + c$
- $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}ln|\frac{x-a}{x+a}| + c$
- $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}sin^{-1}(\frac{x}{a}) + c$
- $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}ln(\frac{x+\sqrt{x^2+a^2}}{a}) + c$
- $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}ln(\frac{x+\sqrt{x^2-a^2}}{a}) + c$

5.2 Integration of types

1. $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\,dx$
$\implies$ Put $x + \frac{b}{2a} = t$

2. $\int \frac{px+q}{ax^2+bx+c}dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx,$
$\int (px+q)\sqrt{ax^2+bx+c}\,dx \implies$
Put $x + \frac{b}{2a} = t$ then split the integral.

3. $\int \frac{dx}{a+bsin^2x}, \int \frac{dx}{a+bcos^2x}, \int \frac{dx}{asin^2x+bsinxcosx+ccos^2x}$
$\implies$ Put $tanx = t$

4. $\int \frac{dx}{a+bsinx}, \int \frac{dx}{a+bcosx}, \int \frac{dx}{a+bsinx+ccosx}$
$\implies$ Put $tan(\frac{x}{2}) = t$

5. $\int \frac{dx}{(ax+b)\sqrt{px+q}}, \int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}} \implies$
Put $px + q = t^2$

6. $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}} \implies$ Put $ax + b = \frac{1}{t}$

7. $\int \frac{dx}{ax^2+b\sqrt{px^2+q}} \implies$ Put $x = \frac{1}{t}$

8. $\int \frac{x^2+1dx}{x^4+\lambda x^2+1}$ where λ is any constant $\implies$ Divide numerator and denominator by x^2 and Put $x \mp \frac{1}{x} = t$

5.3 Reduction Forms

- $\int sin^nxdx = \frac{-sin^{n-1}xcosx}{n} + \frac{n-1}{n}\int sin^{n-2}xdx$
- $\int cos^nxdx = \frac{cos^{n-1}xsinx}{n} + \frac{n-1}{n}\int cos^{n-2}xdx$
- $\int e^{ax}sinbxdx = \frac{e^{ax}}{a^2+b^2}(asinbx - bcosbx)$
- $\int e^{ax}cosbxdx = \frac{e^{ax}}{a^2+b^2}(acosbx + bsinbx)$
- $\int (lnx)^ndx = x(lnx)^n - n\int (lnx)^{n-1}dx$
- **For n>1**
- $\int tan^nxdx = \frac{tan^{n-1}x}{n-1} - \int tan^{n-2}xdx$
- $\int sec^nxdx = \frac{sec^{n-1}xsinx}{n-1} + \frac{n-2}{n-1}\int sec^{n-2}xdx$

5.4 Definite Integration

5.4.1 Properties

- $\int_a^b f(x)dx = \int_a^b f(t)dt$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- $\int_{-a}^a f(x)dx = \int_0^a (f(x) + f(-x))dx = \begin{cases} 2\int_0^a f(x)dx, f(-x) = f(x) \\ 0, f(-x) = f-(x) \end{cases}$
- $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
- $\int_0^a f(x)dx + \int_0^{f(a)} f^{-1}(x)dx = af(a)$

If $f(x)$ is a periodic function with period T

- $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx, n \in \mathbb{Z}$
- $\int_a^{a+nT} f(x)dx = n\int_0^T f(x)dx, n \in \mathbb{Z}, a \in \mathbb{R}$
- $\int_{mT}^{nT} f(x)dx = (n-m)\int_0^T f(x)dx, m, n \in \mathbb{Z}$
- $\int_{nT}^{a+nT} f(x)dx = \int_0^a f(x)dx, n \in \mathbb{Z}, a \in \mathbb{R}$
- $\int_{a+nT}^{b+nT} f(x)dx = \int_0^a f(x)dx, n \in \mathbb{Z}, a, b \in \mathbb{R}$

5.4.2 Inequalities

1. If $\Psi(x) \leq f(x) \leq \phi(x)$ for $a \leq x \leq b$,then

$$\int_a^b \Psi(x)dx \leq \int_a^b f(x)dx \leq \int_a^b \phi(x)dx$$

2. If $m \leq f(x) \leq M$ for $a \leq x \leq b$,then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

3. If $f(x) \geq 0$ on $[a, b]$,then

$$\int_a^b f(x)dx \geq 0$$

5.4.3 Leibniz Theorem

If $\varphi(x) = \int_{g(x)}^{h(x)} f(t)dt$, then

$$\frac{d}{dx}(\varphi(x)) = h'(x)f(h(x)) - g'(x)f(g(x))$$

6 Other Integrals

6.1 Wallis' Integral

$$\int_0^{\frac{\pi}{2}} sin^n x dx / \int_0^{\frac{\pi}{2}} cos^n x dx = \begin{cases} \frac{\pi}{2}\frac{(n-1)!}{n!}, \text{n is even} \\ \frac{(n-1)!}{n!}, \text{n is odd} \end{cases}$$

6.2 Pi Function

$$\Pi(n) = \int_0^{\infty} x^n e^{-x} dx$$

Properties:

- $\Pi(n+1) = (n+1)\Pi(n)$
- $\Pi(0) = 1 \implies \Pi(n) = n!$

6.3 Gamma Function

$$\Gamma(n) = \Pi(n-1) = \int_0^{\infty} x^{n-1} e^{-x} dx$$

Some Properties:

- $\Gamma(n+1) = n\Gamma(n)$
- $\Gamma(n) = (n-1)!,\ \Gamma(\frac{n}{2}) = \frac{2^{(1-n)}(n-1)!\sqrt{\pi}}{(\frac{n-1}{2})!}$

6.4 Gaussian Integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

6.4.1 Gaussian Integral Proof

Proof. Substitute $x^2 = u \implies 2xdx = du$

$$\int_{-\infty}^{\infty} e^{-x^2} dx \implies \frac{1}{2}\int_{-\infty}^{\infty} u^{-\frac{1}{2}} e^{-u} du$$

Using property $[\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx]$ for $[f(-x) = f(x)]$

$$= \int_0^{\infty} u^{-\frac{1}{2}} e^{-u} du$$

It is type of $\Gamma(n)$ for $n = \frac{1}{2}$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

Hence proved. □

7 Differential Equation

A relationship between between an independent variable x, a dependent variable y, and one or more of derivatives of y $w.r.t$ x.
A simple example of differential equation is

$$\frac{dy}{dx} = x$$

Order and Degree of DE
Order: The order of highest-order derivative in a differential equation.
Degree: The power to which the highest-order derivative is raised in a differential equation.
Solution
A solution of a differential equation is function that, when substituted for the dependent variable in equation, leads to an identity. Thus for above example $y = \frac{1}{2}x^2 + c$ is a *solution*.

7.1 DE of first order and first degree

7.1.1 Exact Equation

Equation of the form:

$$P(\frac{dy}{dx}) + Q = 0$$

are exact if left-hand side is differential coefficient of some function $f(x, y)$ $w.r.t$ x.. Integration gives the *solution* $f(x, y) = C$. An *exact* equation is one in which the total differential of function f is equal to zero.

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

Thus an equation $Ax + by = 0$ is exact if

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$

7.1.2 Variables Separable

In this case, the equation can be written in the form

$$f(x) + g(x)\frac{dy}{dx} = 0$$

Rearrangement gives

$$f(x)dx = -g(y)dy$$

Both sides then can be integrated.

7.1.3 Homogeneous Equations

These can be written in the form

$$\frac{dy}{dx} = f(\frac{y}{x})$$

The method of solution is to make substitution $y = vx$, which reduces the equation to one in v and x only. Resulting, the variables are separable.

7.1.4 Equations reducible to Homogeneous

Equation of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

can be handled by making substitution $x = X + h$ and $y = Y + k$ where h and k are constants.Then,

$$\frac{dy}{dx} = \frac{dY}{dX}$$

$$= \frac{a_1(X + h) + b_1(Y + k) + c_1}{a_2(X + h) + b_2(Y + k) + c_2}$$

If h and k are chosen to be the values of x and y, respectively, that satisfy the simultaneous equations

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

Then original equation becomes

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

which is homogeneous.
However if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c1}{c2}$ then h and k cannot be chosen as above.In this case, let $a_2 = ma_1$ and $u = a_1x + b_1y$ The equation becomes

$$\frac{du}{dx} - a_1 = b_1\frac{u + c_1}{mu + c_2}$$

and u and x can be separated.

7.1.5 Linear Equations

Equation of the form

$$\frac{dy}{dx} + Py = Q$$

where P and Q are the functions of x, or constants, are said to be linear in y and can be solved by multiplying integrating factor

$$e^{\int Pdx}$$

This makes left hand side of equation an exact differential:

$$e^{\int Pdx}(\frac{dy}{dx}) + e^{\int Pdx}(Py) = e^{\int Pdx}Q$$

$$\frac{d}{dx}[e^{\int Pdx}y] = e^{\int Pdx}Q$$

$$ye^{\int Pdx} = \int e^{\int Pdx}Qdx + c$$

7.2 Bernoulli's Differential Equation

A first order differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, n \in \mathbb{R}$$

7.2.1 Transformations

When $n = 0$ the differential equation is linear and $n = 1$, it is variable separable.For $n \neq 0, 1$ The substitution $u = y^{1-n}$ reduces Bernoulli equation to linear differential equation.

$$\frac{du}{dx} - (n-1)P(x)u = -(n-1)Q(x)$$

For example:
In case of $n = 2$, making substitution $u = y^{-1}$ in the differential equation

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2$$

produces the equation

$$\frac{du}{dx} - \frac{1}{u} = -x$$

which is a linear equation.