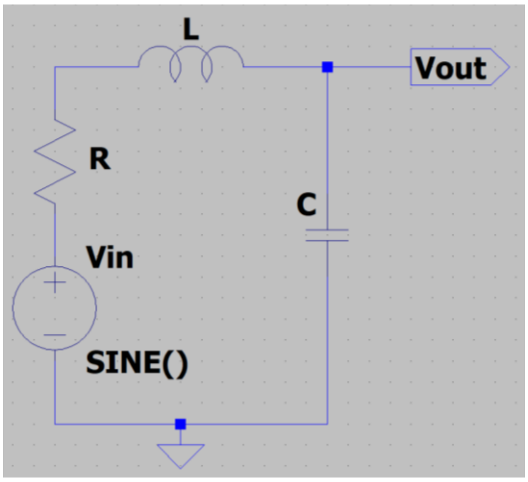
On network measurements with Analog Discovery 2

A low-pass passive LCR filter is shown in the circuit below.

- Derive the complex transfer function $\hat{\mathbf{F}}(\omega) = \frac{V_{out}}{V_{in}}$ in the frequency domain.
- Derive the magnitude of the transfer function $\left|\hat{\mathbf{F}}(\omega)\right|$.



Solution:

Total impedance: $Z = R + i\left(\omega L - \frac{1}{C\omega}\right)$

Current: $I = \frac{V_{in}}{Z}$, where the voltage must be written in the complex exponential form $\sim\exp\,(i\omega t)$

Output voltage: $V_{out} = I \times Z_C = \frac{V_{in}}{Z} \times Z_C$ (AC potential divider)

Transfer function in the frequency domain:

$$\hat{\mathbf{F}}(\omega) = \frac{V_{out}}{V_{in}} = \frac{Z_C}{Z} = \frac{-\frac{i}{C\omega}}{R+i\left(\omega L-\frac{1}{C\omega}\right)} = \frac{-i}{RC\omega+i(LC\omega^2-1)} = \frac{(1-LC\omega^2)-iRC\omega}{(RC\omega)^2+(LC\omega^2-1)^2}$$

$\left|\hat{\mathbf{F}}(\omega)\right| = \frac{1}{\sqrt{(RC\omega)^2+(LC\omega^2-1)^2}}$ is the transfer function magnitude

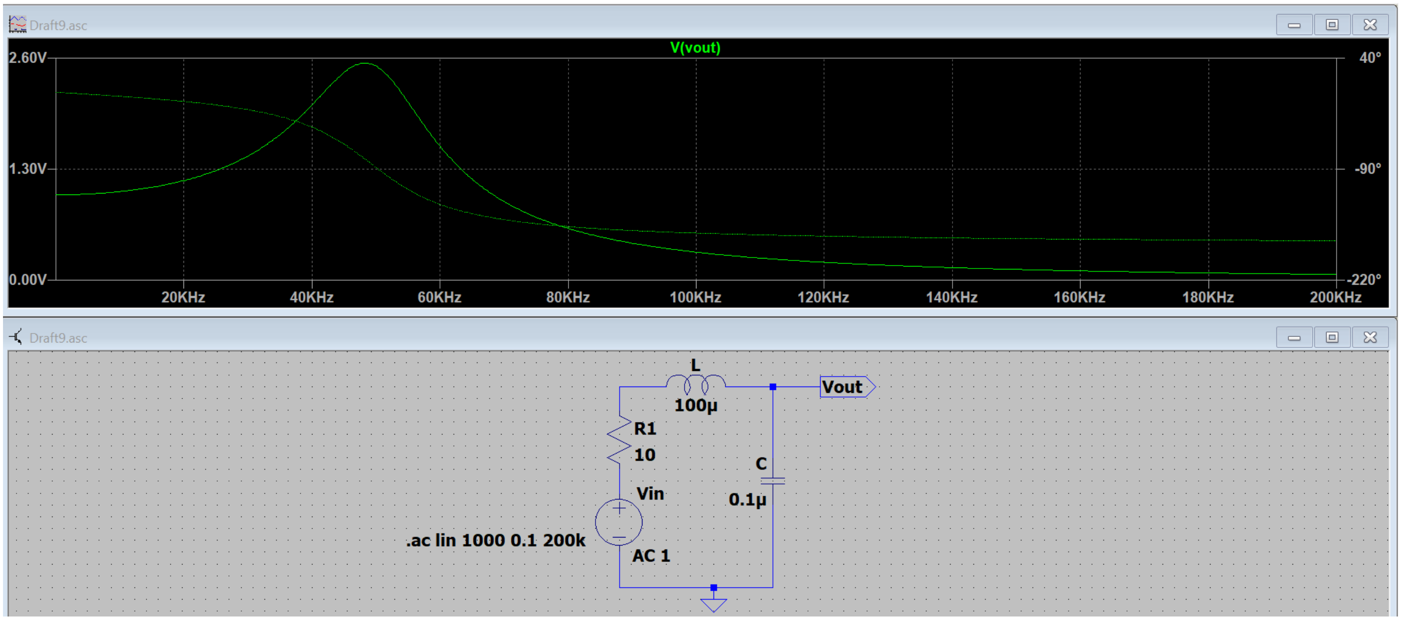
$\mathrm{Re}\left[\hat{\mathbf{F}}(\omega)\right] = \frac{(1-LC\omega^2)}{(RC\omega)^2+(LC\omega^2-1)^2}$ is the real part which can be positive or negative as a function of frequency, and it becomes zero at the resonance frequency $\omega_{res} = 1/\sqrt{LC}$

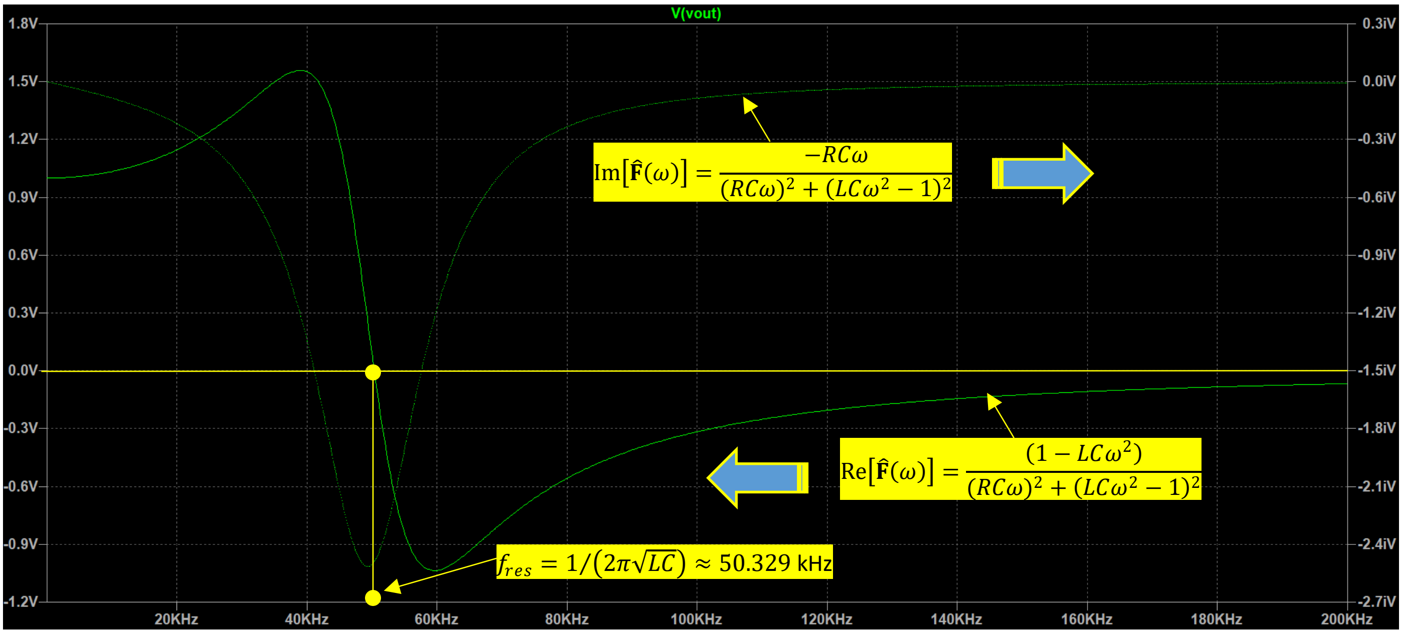
$\mathrm{Im}\left[\hat{\mathbf{F}}(\omega)\right] = \frac{-RC\omega}{(RC\omega)^2+(LC\omega^2-1)^2}$ is the imaginary part (only negative)

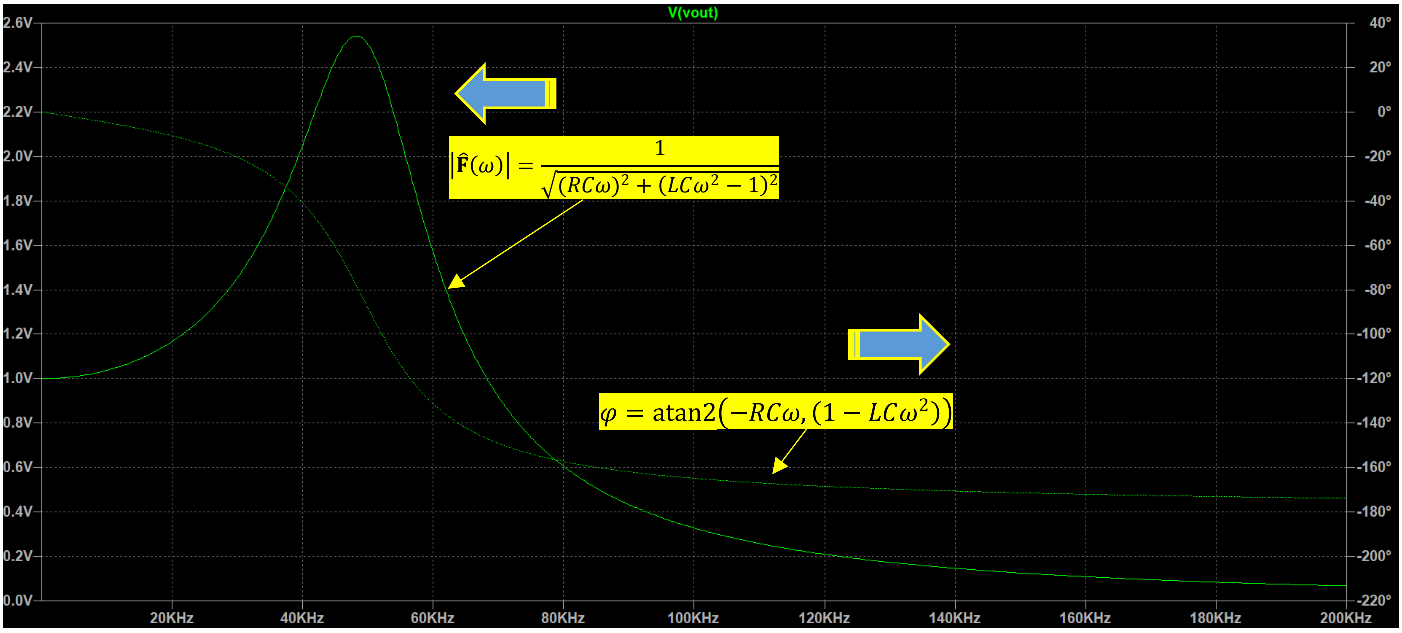
$\varphi = \mathrm{atan2}\left(-RC\omega, (1-LC\omega^2)\right)$ is the phase calculated using atan2 function which returns values ranging from $-\pi$ to π, as in electronic instruments

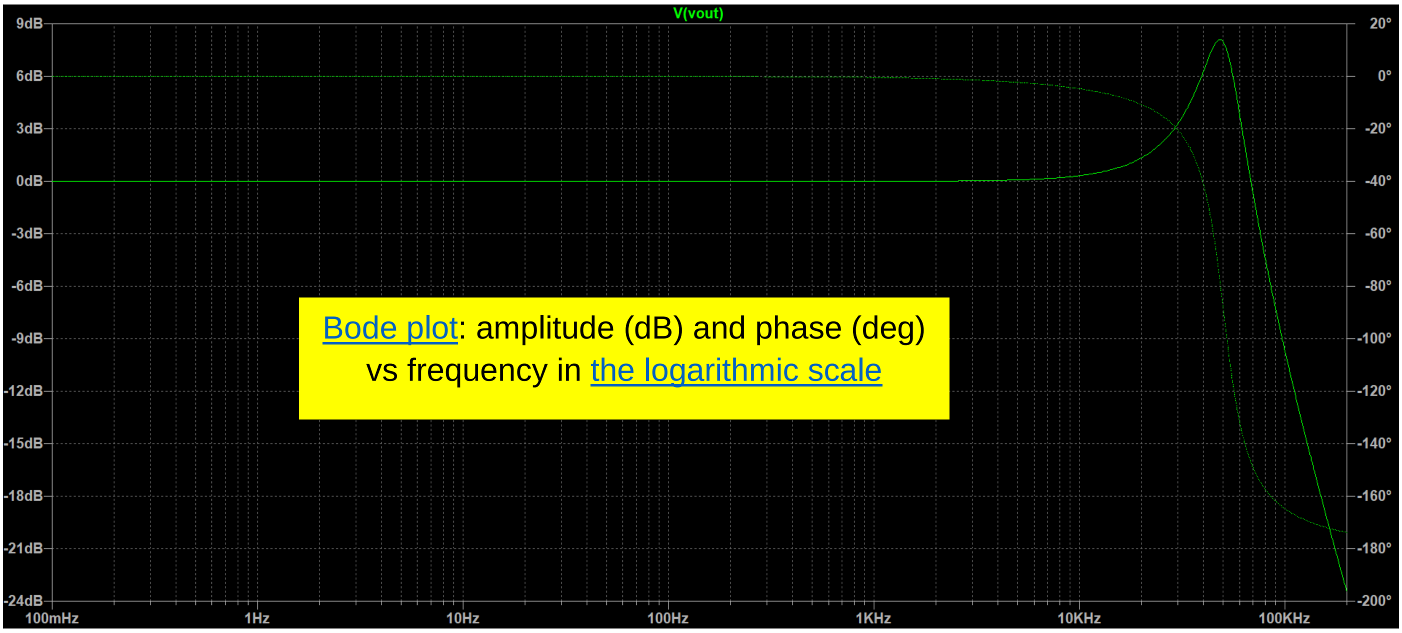
If the generator has an internal resistance, it should be considered part of the network under test. Additionally, any cascade connected to the network's output will alter the transfer function's properties unless it has a very high input impedance. These nuances must be considered when analysing circuits.

To develop an understanding of transfer functions, it would be useful to model a circuit using a simulator for some specific LCR values. For the circuit in LTspice shown below, we used the following parameters: a resistor of $R = 10\ \Omega$, a capacitor of $C = 0.1\ \mu\text{F}$, and an inductor of $L = 100\ \mu\text{H}$ with an internal resistance of $2.7\ \Omega$ (as in a real component). For a low resistance R, the circuit demonstrates a pronounced resonance at $\omega_{res} = 1/\sqrt{LC}$. The AC generator with a fixed amplitude of 1 V scans the frequency in a wide range that covers the resonance $f_{res} = 1/\left(2\pi\sqrt{LC}\right) \approx 50.329$ kHz calculated for the above values of L and C. Given that the generator's amplitude is constant at 1 V and its internal resistance is zero, the voltage across the capacitor V_{out} automatically represents the transfer function, obviating the need for division by 1. Below we show the transfer function in linear and logarithmic formats.

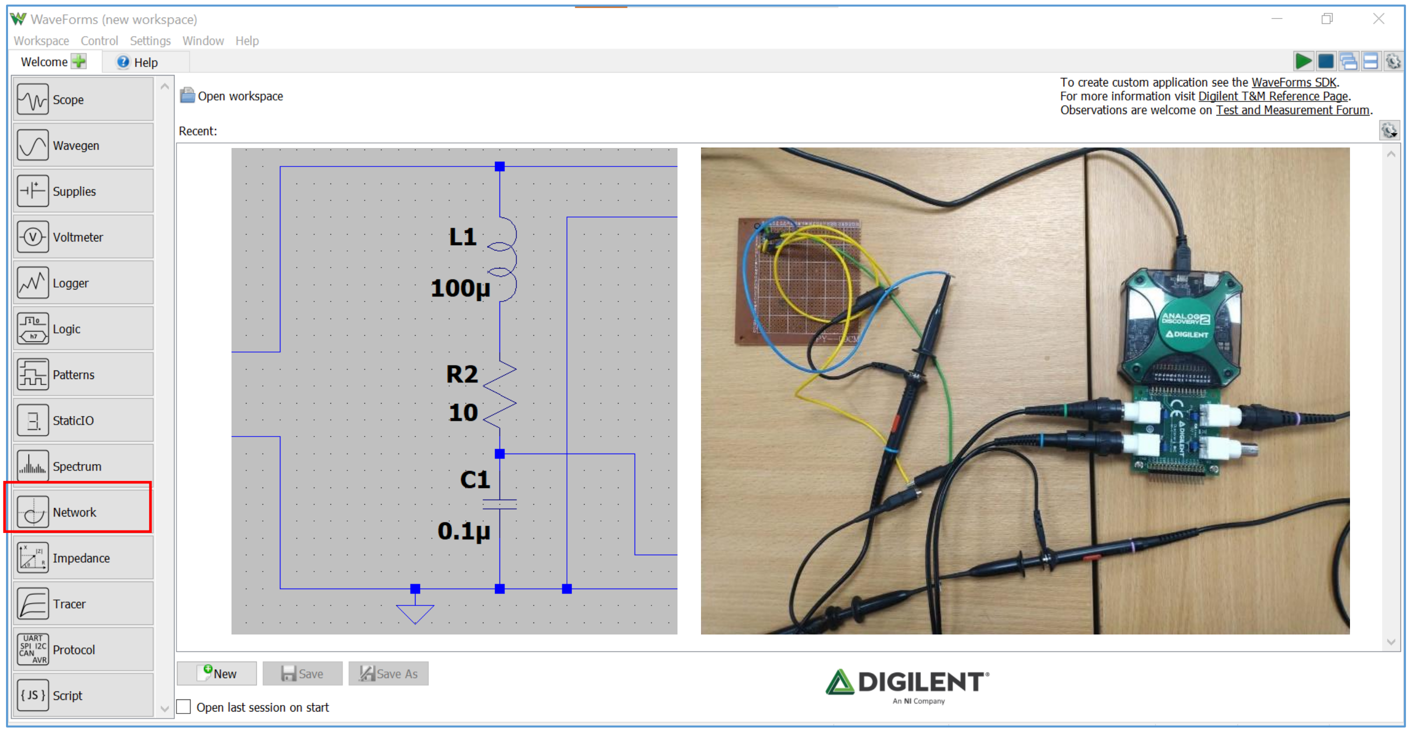



V(vout)
1.8V 1.5V 1.2V 0.9V 0.6V 0.3V 0.0V -0.3V -0.6V -0.9V -1.2V
0.3iV 0.0iV -0.3iV -0.6iV -0.9iV -1.2iV -1.5iV -1.8iV -2.1iV -2.4iV -2.7iV
20KHz 40KHz 60KHz 80KHz 100KHz 120KHz 140KHz 160KHz 180KHz 200KHz
$\mathrm{Im}[\hat{\mathbf{F}}(\omega)] = \dfrac{-RC\omega}{(RC\omega)^2 + (LC\omega^2 - 1)^2}$
$\mathrm{Re}[\hat{\mathbf{F}}(\omega)] = \dfrac{(1 - LC\omega^2)}{(RC\omega)^2 + (LC\omega^2 - 1)^2}$
$f_{res} = 1/\left(2\pi\sqrt{LC}\right) \approx 50.329$ kHz

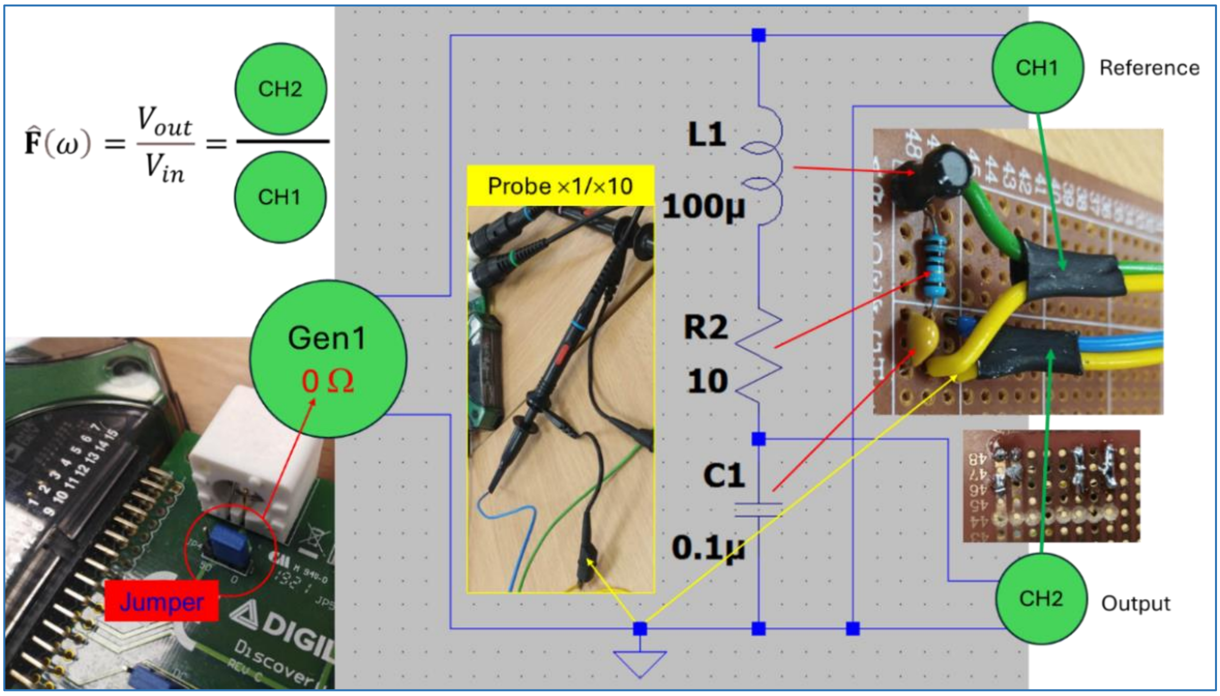

V(vout)
2.6V 2.4V 2.2V 2.0V 1.8V 1.6V 1.4V 1.2V 1.0V 0.8V 0.6V 0.4V 0.2V 0.0V
40° 20° 0° -20° -40° -60° -80° -100° -120° -140° -160° -180° -200° -220°
20KHz 40KHz 60KHz 80KHz 100KHz 120KHz 140KHz 160KHz 180KHz 200KHz
$|\hat{\mathbf{F}}(\omega)| = \dfrac{1}{\sqrt{(RC\omega)^2 + (LC\omega^2 - 1)^2}}$
$\varphi = \mathrm{atan2}\left(-RC\omega, (1 - LC\omega^2)\right)$


V(vout)
9dB 6dB 3dB 0dB -3dB -6dB -9dB -12dB -15dB -18dB -21dB -24dB
20° 0° -20° -40° -60° -80° -100° -120° -140° -160° -180° -200°
100mHz 1Hz 10Hz 100Hz 1KHz 10KHz 100KHz
Bode plot: amplitude (dB) and phase (deg)
vs frequency in the logarithmic scale

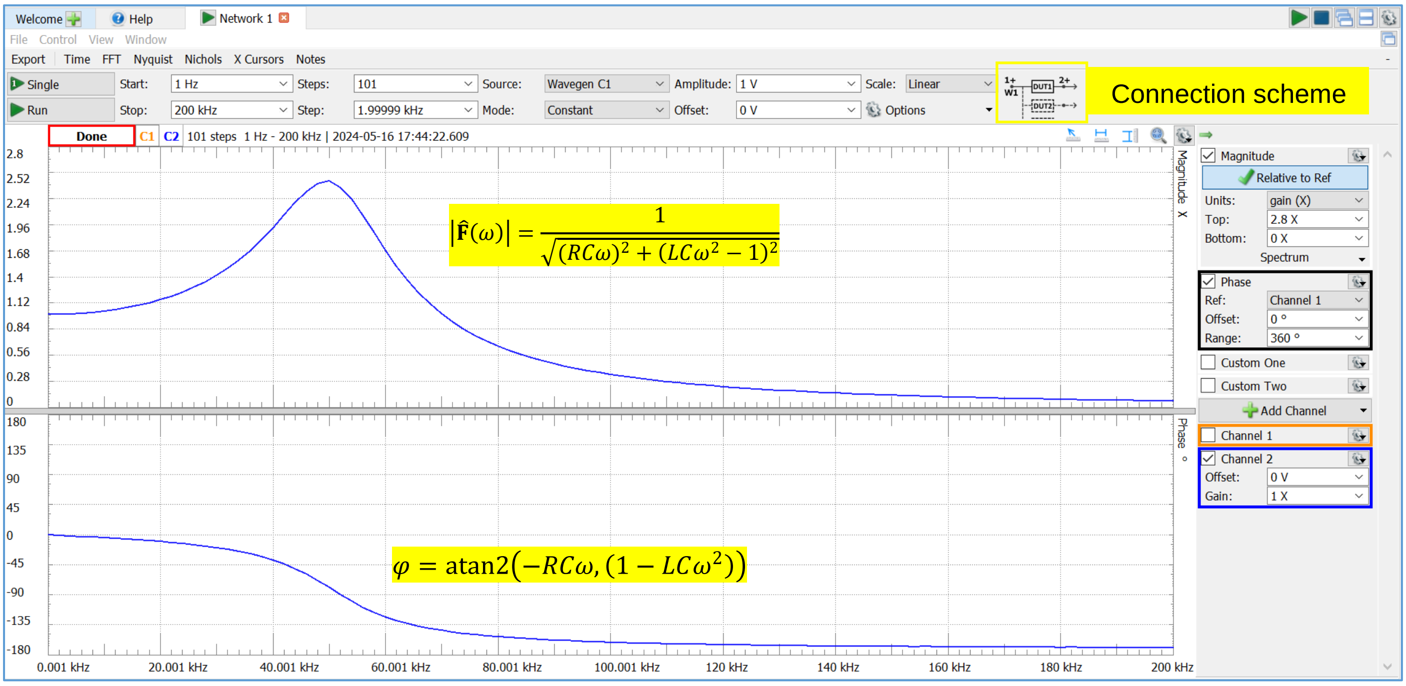
We reproduced the low-pass filter circuit for $R = 10\ \Omega$, $C = 0.1\ \mu\text{F}$, $L = 100\ \mu\text{H}$ purchased from Amazon. We measured its transfer function using Analog Discovery 2 multi-function instrument. The instrument can be configured to measure the transfer functions of two-port networks ("Network" option) up to 25 MHz, effectively functioning as a Vector Network Analyser (VNA). An on-board generator provides frequency scanning, while two synchronous oscilloscope channels deliver the reference signal (CH1) and output signal (CH2). The ratio of these signals is calculated in complex form, allowing the phase and amplitude of the output signal to be determined relative to the reference signal. Below we show screenshots of our network measurements using the device's graphical user interface.

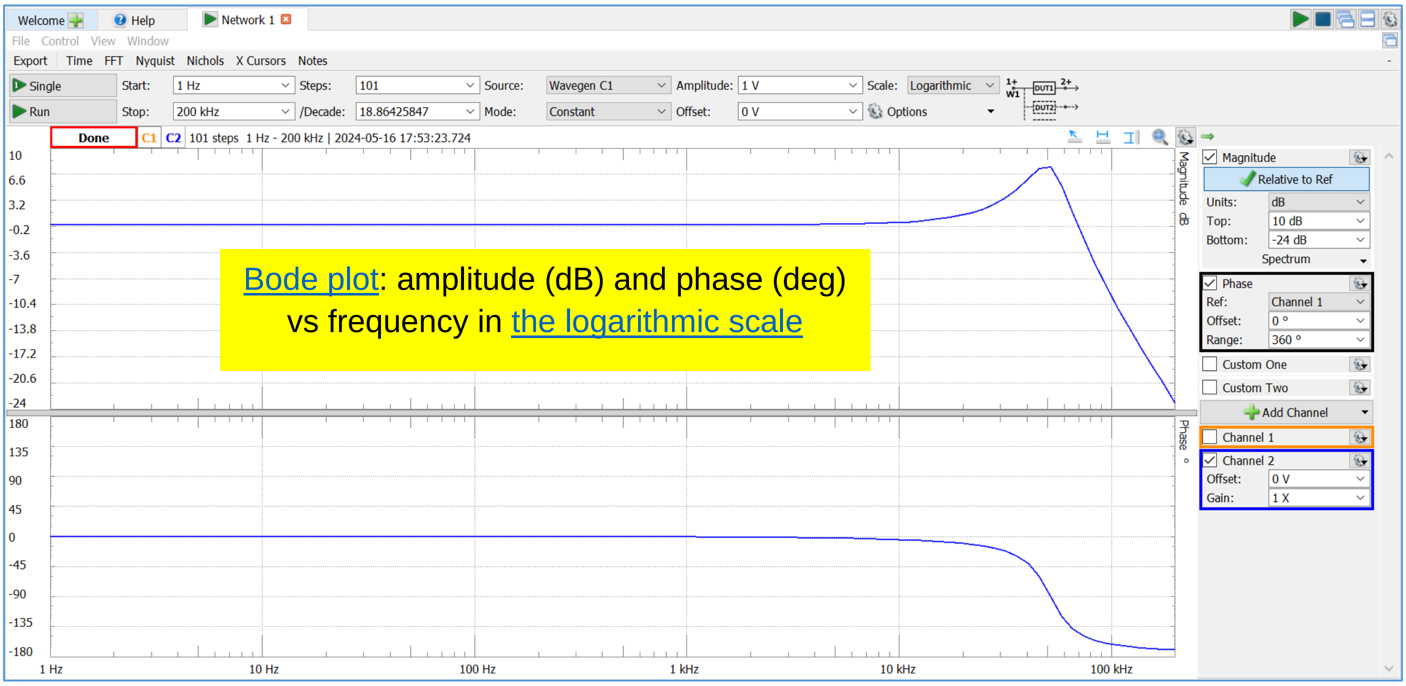


The oscilloscope input channels with $\times 1$ probes have a high input resistance ($\sim 1\ \text{M}\Omega$), preventing distortions in the circuit. A jumper on the signal board (see below) configures the generator with a negligible output resistance ($0\ \Omega$). In this case, the reference signal at CH1 will be independent of the frequency, which is a desirable condition for network measurements. In a practical circuit, if the voltage or current source has an internal resistance, it should be attributed to the network under test. However, in our measurement scheme, if the jumper is changed to $50\ \Omega$, we will obtain the same transfer function (the complex ratio CH2/CH1) as for the $0\ \Omega$ output. This occurs because the reference signal at CH1, which now depends on frequency due to the voltage drop at the source's internal resistance, is always taken after the jumper. Therefore, this $50\ \Omega$ will be effectively excluded, thus not affecting the transfer function. To avoid such uncertainties in network measurements with Analog Discovery 2, we suggest using the $0\ \Omega$ jumper. When implementing the circuit for network measurements, place all resistances used for the simulations after the jumper.



Network measurements with Analog Discovery 2: perfect agreement with the simulations in LTspice





In our lectures, we have explored how a complex-valued transfer function can be used to calculate the output signal of a linear network when subjected to a periodic input signal. According to the general theory of linear networks, we can calculate the Fourier series of the output $V_{out}(t)$ from the Fourier series of the input $V_{in}(t)$ using the following equations:

$$V_{in}(t) = A_0 + \sum_{k=1}^{\infty}[a_k \cos(\omega_k t) + b_k \sin(\omega_k t)]$$

$$V_{out}(t) = \mathrm{Re}\big[\hat{\mathbf{F}}(0)\big]\, A_0 + \sum_{k=1}^{\infty}\Big(\big(\, a_k\, \mathrm{Re}\big[\hat{\mathbf{F}}(\omega_k)\big] + b_k\, \mathrm{Im}\big[\hat{\mathbf{F}}(\omega_k)\big]\,\big)\cos(\omega_k t) + \big(\, b_k\, \mathrm{Re}\big[\hat{\mathbf{F}}(\omega_k)\big] - a_k\, \mathrm{Im}\big[\hat{\mathbf{F}}(\omega_k)\big]\,\big)\sin(\omega_k t)\Big)$$

Here, $\omega_k = k\omega$ are harmonic angular frequencies, ω is the fundamental angular frequency, V_{in} and V_{out} are signals of any nature defined for a specific network. For our low-pass filter network with $V_{out} = V_C$, we have:

$$\mathrm{Re}\big[\hat{\mathbf{F}}(0)\big] = 1$$

$$\mathrm{Re}\big[\hat{\mathbf{F}}(\omega_k)\big] = \frac{(1-LC\omega_k^2)}{(RC\omega_k)^2+\left(LC\omega_k^2-1\right)^2}$$

$$\mathrm{Im}\big[\hat{\mathbf{F}}(\omega)\big] = \frac{-RC\omega}{(RC\omega_k)^2+\left(LC\omega_k^2-1\right)^2}$$

These functions can be also measured with very good accuracy. Complex networks will always be either simulated or measured.