Introduction to elastic wave equation

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Motivation

- Elastic wave equation has been widely used to describe wave propagation in an elastic medium, such as seismic waves in Earth and ultrasonic waves in human body.
- **Seismic waves** are waves of energy that travel through the earth, and are a result of an earthquake, explosion, or a volcano.

Elastic wave equation

- The standard form for seismic elastic wave equation in homogeneous media is :

$$\rho \ddot{u} = (\lambda + 2\mu)\nabla\nabla . u - \mu\nabla \times \nabla \times u$$

ρ : is the density

u : is the displacement

λ, μ : Lame parameters

Equation of Motion

- We will depend on Newton's second law F=ma

$$\mathrm{m} : \mathrm{mass} = \rho dx_1 dx_2 dx_3$$

$$a : \mathrm{acceleration} = \frac{\partial^2 u}{\partial t^2}$$

- The total force from stress field:

$$F = F_i + F_i^{body}$$

$$F_i^{body} = f_i dx_1 dx_2 dx_3$$

$$F_i = \sum \frac{\partial \tau_{ij}}{\partial x_j} dx_1 dx_2 dx_3 = \partial_j \tau_{ij} dx_1 dx_2 dx_3$$

Equation of Motion

- Combining these information together we get the Momentum equation (Equation of Motion)

$$\rho \frac{\partial^2 u}{\partial t^2} = \partial_j \tau_{ij} + f_i$$

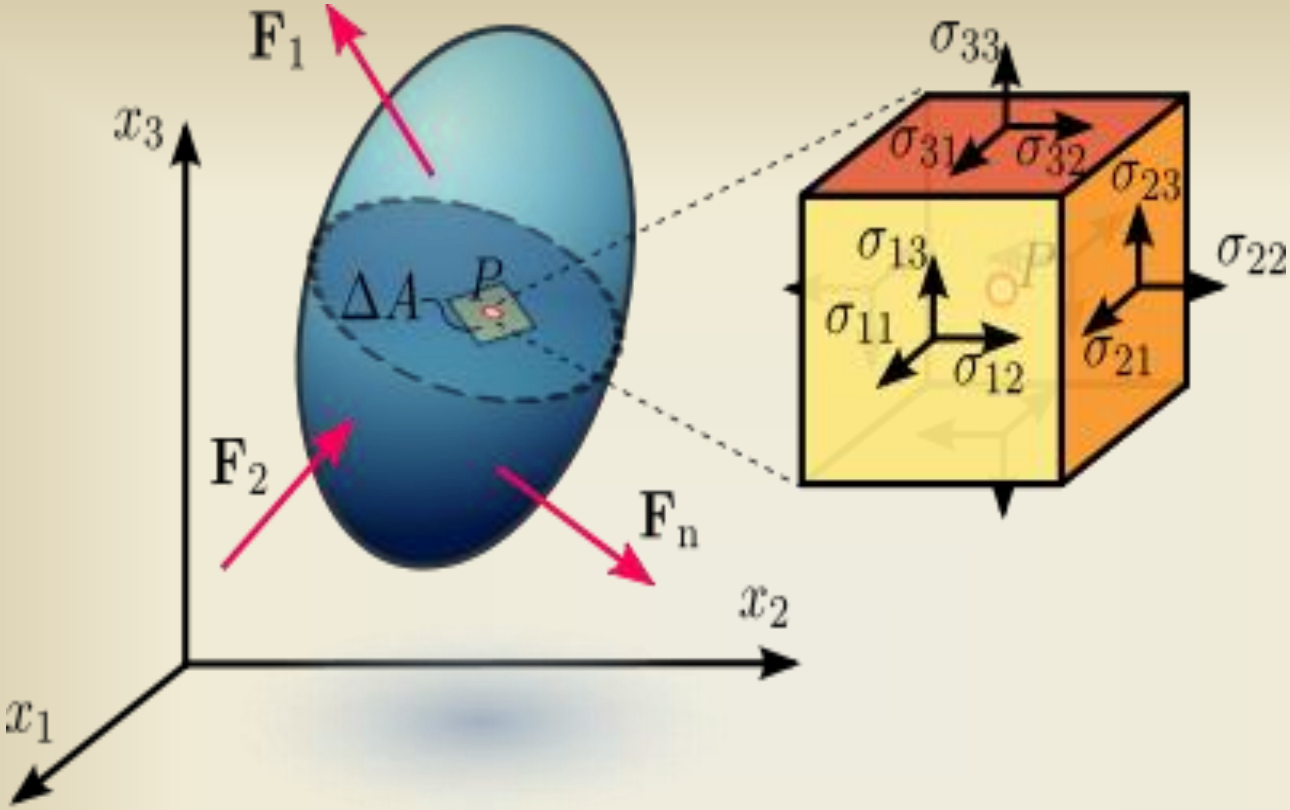
- where ρ is the density, u is the displacement, and τ is the stress tensor.

Definitions

- **Stress** : A measure of the internal forces acting within a deformable body.

  (The force acting on a solid to deform it)

The *stress at any point* in an object, assumed to behave as a continuum, is completely defined by nine component stresses: three orthogonal normal stresses and six orthogonal shear stresses.



$$\begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

This can be expressed as a second-order tensor known as the Cauchy stress tensor.

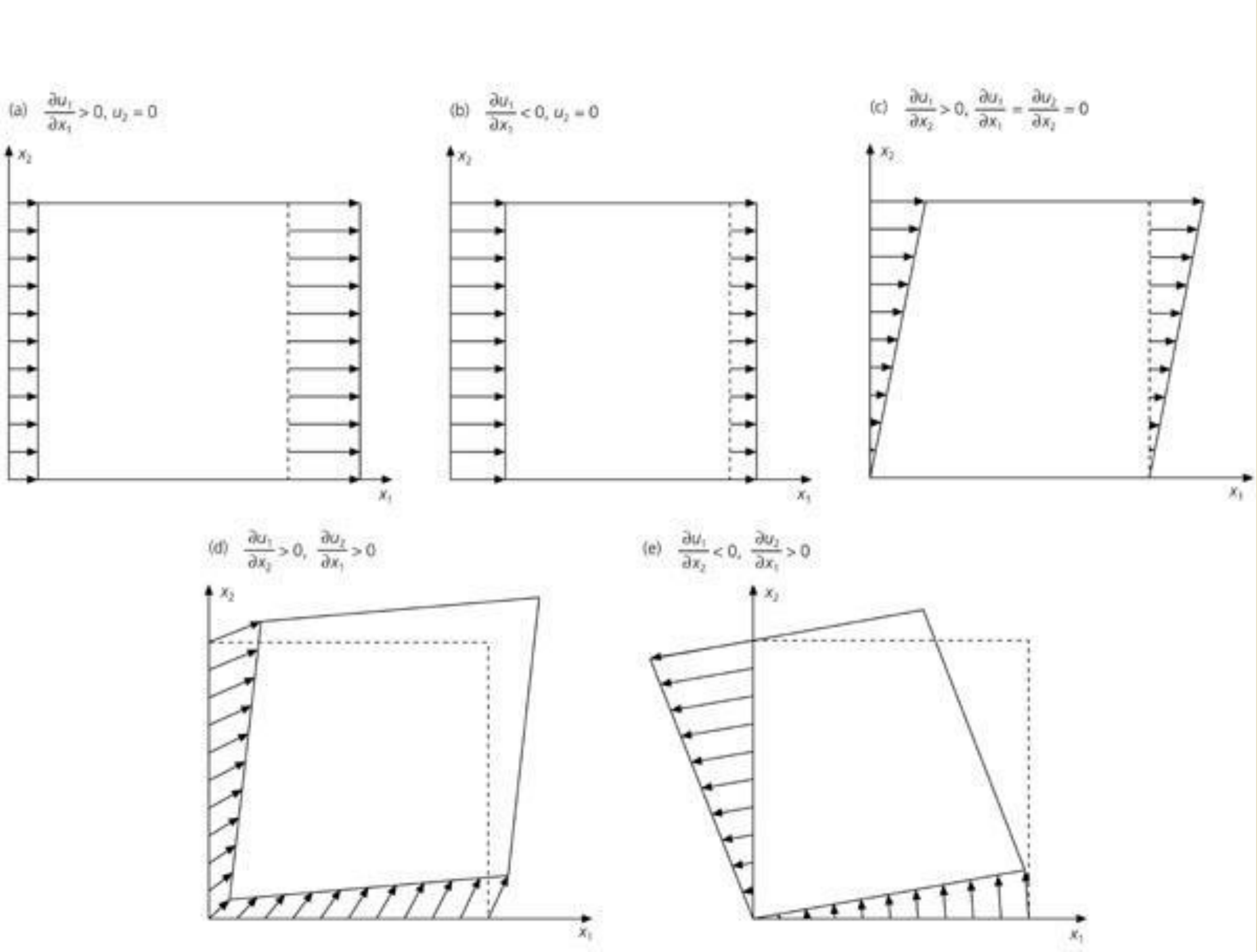
Definitions

- Strain : A local measure of relative change in the displacement field, that is , the spatial gradients in the displacement field. And it related to deformation, or change in shape, of a material rather than any change in position.

$$e_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$$

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

Some possible strains for two- dimensional element



The linear Stress-Strain Relationship

- Stress and Strain are linked in elastic media by Stress - Strain or constitutive relationship.
- The most general linear relationship between Stress and Strain is :

$$\tau_{ij} = C_{ijkl} e_{kl}$$

where,

$$C \rightarrow Stiffness\,(\text{or Elastic coefficient})$$

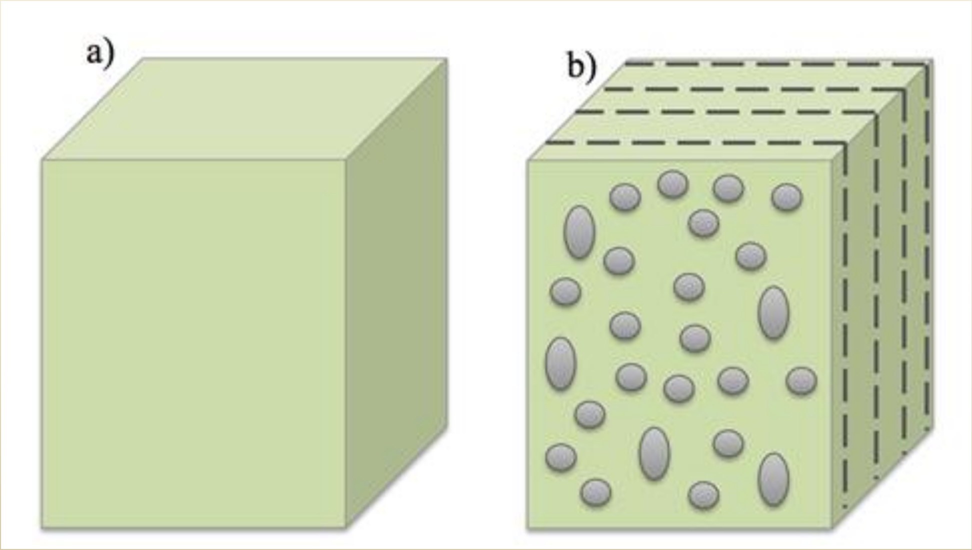
- C_{ijkl} is termed the elastic tensor.

The linear Stress-Strain Relationship

- The elastic tensor C_{ijkl}, is forth-order with 81 components ($1 \le_{i,j,k,l} \le 3$).
- Because of the symmetry of the stress and strain tensors and the thermodynamic considerations, only 21 of these components are independent.
- The 21 components are necessary to specify the stress-strain relationship for the most general form of an elastic solid.

The linear Stress-Strain Relationship

- The material is **isotropic** if the properties of the solid are the same in all directions.
- The material is **anisotropic** if the properties of the media vary with direction.



The linear Stress-Strain Relationship

- If we assume isotropy , the number of the independent parameters is reduced to two :

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})$$

where λ and μ are called the *Lame parameters*

$\delta_{ij} = 1 \text{ for } i = j \, , \delta_{ij} = 0 \text{ for } i \neq j$

μ : A measure of the resistance of the material to shearing

$$\mu = \frac{\tau_{xy}}{2e_{xy}}$$

λ : Has no simple physical explanation.

The linear Stress-Strain Relationship

- The stress-strain equation for an isotropic media :

$$\tau_{ij} = [\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})]e_{kl}$$

$$= \lambda\delta_{ij}e_{kk} + 2\mu e_{ij}$$

The linear Stress-Strain Relationship

- The linear isotropic stress-strain relationship

$$\tau_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} \qquad (1)$$

- The strain tensor is defined as :

$$e_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \qquad (2)$$

- Substituting for (2) in (1) we obtain :

$$\boxed{\tau_{ij} = \lambda\delta_{ij}\partial_k u_k + \mu(\partial_i u_j + \partial_j u_i) \qquad (3)}$$

The Seismic Wave Equation in Isotropic Media

- Substituting (3) in the homogeneous equation of motion :

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{t}^2} = \partial_j [\lambda \delta_{ij} \partial_k \boldsymbol{u}_k + \mu (\partial_i \boldsymbol{u}_j + \partial_j \boldsymbol{u}_i)\,]$$

$$= \partial_i \lambda \partial_k u_k + \lambda \partial_i \partial_k u_k + \partial_j \mu (\partial_i u_j + \partial_j u_i) + \mu \partial_j \partial_i u_j + \mu \partial_j \partial_j u_i$$

$$= \partial_i \lambda \partial_k u_k + \partial_j \mu (\partial_i u_j + \partial_j u_i) + \lambda \partial_i \partial_k u_k + \mu \partial_i \partial_j u_j + \mu \partial_j \partial_j u_i$$

The Seismic Wave Equation in Isotropic Media

Defining $\ddot{u} = \dfrac{\partial^2 u}{\partial t^2}$ we can write this in vector form as

$$\rho\ddot{u} = \nabla\lambda(\nabla . u) + \nabla\mu.[\nabla u + (\nabla u)^T] + (\lambda + \mu)\nabla\nabla . u + \mu\nabla^2 u$$

use the vector identity $\nabla^2 u = \nabla\nabla . u - \nabla \times \nabla \times u$

we obtain:

$$\rho\ddot{u} = \nabla\lambda(\nabla . u) + \nabla\mu.[\nabla u + (\nabla u)^T] + (\lambda + 2\mu)\nabla\nabla . u - \mu\nabla \times \nabla \times u$$

The Seismic Wave Equation in Isotropic Media

- This is one form of the seismic wave equation

$$\rho \ddot{u} = \nabla \lambda (\nabla . u) + \nabla \mu . [\nabla u + (\nabla u)^T] + (\lambda + 2\mu) \nabla \nabla . u - \mu \nabla \times \nabla \times u$$

- The first two terms on the (r.h.s) involve gradient in the *Lame parameters* and are non-zero whenever the material is inhomogeneous (i.e. : contains velocity gradient)
- Including these factors makes the equations very complicated and difficult to solve efficiently.

The Seismic Wave Equation in Isotropic Media

- If velocity is only a function of depth , then the material can be modeled as a series of homogeneous layers.
- Within each layer , there are no gradients in the *Lames parameters* and so these terms go to zero.
- The standard form for seismic wave equation in homogeneous media is :

$$\rho\ddot{u} = (\lambda + 2\mu)\nabla\nabla.u - \mu\nabla \times \nabla \times u$$

- Note : Here we neglected the gravity and velocity gradient terms and has assumed a linear , isotropic Earth model

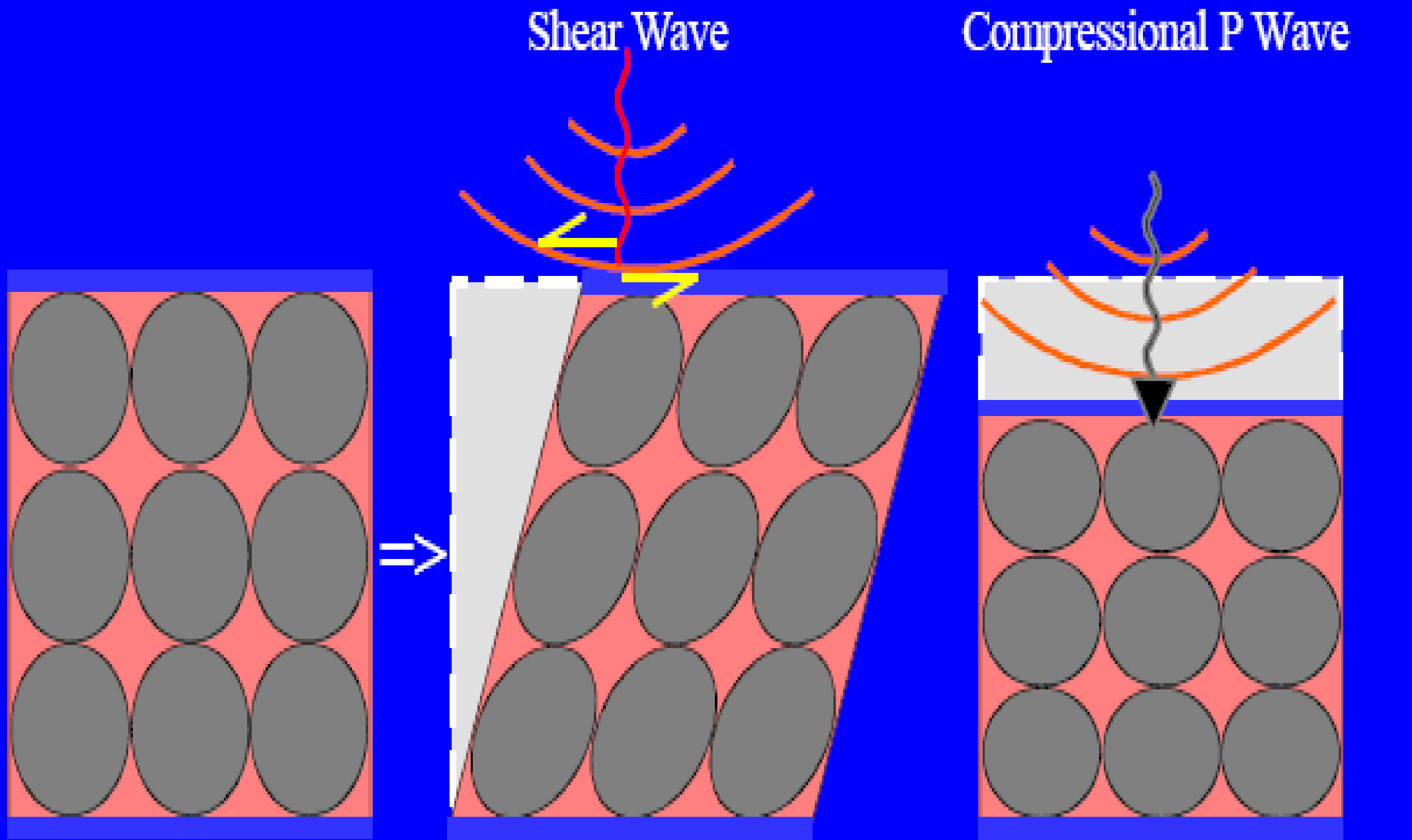
Seismic Wave Equation in homogeneous media

If ρ, λ and μ are constants, the wave equation is simplified as:

$$\ddot{u} = \alpha^2 \nabla\nabla . u - \beta^2 \nabla \times \nabla \times u$$

where the P-wave velocity $\alpha^2 = \frac{\lambda + 2\mu}{\rho}$

the S-wave velocity $\beta^2 = \frac{\mu}{\rho}$



Undeformed schematic rock volume of porous sandstone

Shear Velocity (Dipole Log)

$$Vs = \sqrt{\frac{\mu}{\rho}}$$

Compressional Velocity (Sonic Log)

$$Vp = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

Acoustic Wave Equation

- If the Lame parameter μ = 0 (i.e. No shearing) then we get :

$$\tfrac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \nabla^2\phi \quad \text{where } c^2 = \tfrac{\lambda}{\rho} \text{ the speed of propagation}$$

- In this case, the Elastic wave equation is reduced to an acoustic wave equation.

Short Summary

- We introduced definitions of Stress and Strain and the relationship between them.
- We depend on Newton's 2^{nd} law to get the equation of motion and from it we Derive the general form of Elastic wave equation .
- We simplify it to the standard form by modeling the material as series of homogeneous layers.
- We discussed two types of waves
  - P-waves(Compressional)
  - S-waves(Shear)
- Finally, if we assume no shearing then we reduced it to an acoustic wave equation .