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A fast reduced-rank interpolation method for prestack seismic volumes that depend on four spatial dimensions

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ABSTRACT

Rank reduction strategies can be employed to attenuate noise and for prestack seismic data regularization. We present a fast version of Cadzow reduced-rank reconstruction method. Cadzow reconstruction is implemented by embedding 4D spatial data into a level-four block Toeplitz matrix. Rank reduction of this matrix via the Lanczos bidiagonalization algorithm is used to recover missing observations and to attenuate random noise. The computational cost of the Lanczos bidiagonalization is dominated by the cost of multiplying a level-four block Toeplitz matrix by a vector. This is efficiently implemented via the 4D fast Fourier transform. The proposed algorithm significantly decreases the computational cost of rank-reduction methods for multidimensional seismic data denoising and reconstruction. Synthetic and field prestack data examples are used to examine the effectiveness of the proposed method.

INTRODUCTION

During the last decade, many methods were proposed to reconstruct seismic data. These methods can be divided into three general categories. The first category includes methods based on transforms such as the Radon transform (Bardan, 1987; Darche, 1990; Kabir and Verschuur, 1995), the Fourier transform (Duijndam et al., 1999; Hindriks and Duijndam 2000; Liu and Sacchi, 2004; Xu et al., 2005; Zwartjes and Sacchi, 2007; Trad, 2009; Curry, 2010), and localized transforms (Herrmann and Hennenfent, 2008; Sacchi et al., 2009; Hennenfent et al., 2010; Naghizadeh and Sacchi, 2010; Wang et al., 2010). These methods are based on mathematical transforms and signal analysis principles. Transform methods do not require geologic assumptions about the subsurface and can reconstruct sparsely sampled seismic data (Liu and Sacchi, 2004; Trad, 2009) and irregularly sampled seismic data (Duijndam et al., 1999). A second category groups methods based on prediction filters (Spitz, 1991; Porsani, 1999; Gulunay, 2003; Naghizadeh and Sacchi, 2007, 2009) which use the predictability of linear events in the frequency-space domain to interpolate aliased data at high frequencies with filters derived from nonaliased low frequencies. The merit of these methods is that they can attain antialiasing reconstruction results. The third category of methods encompasses those techniques that use wave-equation principles (Ronen, 1987; Stolt, 2002; Fomel, 2003; Ramirez et al., 2006; Kaplan et al., 2010). These methods allow the use of subsurface information.

In recent years, a new class of reconstruction methods has been proposed. Rank-reduction methods for seismic volume reconstruction assume that the ideal data can be represented via a low-rank matrix or tensor (Trickett, 2008; Oropeza and Sacchi, 2011; Kreimer and Sacchi, 2011, 2012). Denoising and completion (regularization) is possible by iteratively finding a low-rank data structure that honors the original observations. Rank reduction seismic denoising and regularization methods were inspired by Cadzow filtering (Cadzow, 1988) or its equivalent formulation from the field of time series analysis called multichannel singular spectrum analysis (Broomhead and King, 1986; Ghil et al., 2002). It is important to notice that Cadzow filtering and the multichannel singular spectrum analysis reconstruction method lead to equivalent reconstruction algorithms. In addition, their only difference lies in the field of application where they were initially proposed. Cadzow Filtering was introduced to the seismic processing community for denoising and regularization by Trickett (2008), Trickett and Burroughs (2009), and Trickett et al. (2010). Similarly, the multichannel singular spectrum analysis reconstruction method was introduced by Sacchi (2009) and Oropeza and Sacchi (2011).

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Properly sampled multichannel data can be embedded into a low-rank Hankel or Toeplitz matrix (Trickett et al., 2010; Sacchi, 2009; Oropeza and Sacchi, 2011). Missing observations and noise increase the rank of the matrix. Therefore, seismic data reconstruction and noise attenuation can be posed as a rank reduction problem. Rank reduction requires the application of the singular value decomposition (SVD) to estimate a few dominant eigenvectors of the Toeplitz (or Hankel) matrix of the data. The computational cost of the SVD makes rank reduction methods prohibitively expensive for multidimensional seismic signal processing.

In this paper, we develop a rank reduction denoising and reconstruction scheme that is used to reconstruct prestack data that depends on four spatial dimensions. This is often called "5D interpolation" (Trad, 2009) because reconstruction algorithms operate on volumes that depend on four spatial dimensions and time or frequency. However, we stress that because the reconstruction is carried out in the frequency-space domain for individual temporal frequencies, the problem entails the reconstruction of a 4D volume and, therefore, we use the terminology "4D reconstruction." For this purpose, we first embed the 4D spatial data into a level-four Toeplitz matrix. The latter can be easily expanded into a multilevel circulant matrix (Oudin and Delmas, 2009). The multiplication of a multilevel circulant matrix and a vector can be implemented by the multidimensional fast Fourier transform (FFT). This line of thought leads to a fast algorithm for rank reduction that uses the Lanczos bidiagonalization (Golub and Van Loan, 1996; Trickett, 2003; Simon and Zha, 2000) and a fast level-four block Toeplitz matrix-vector multiplication implemented via 4D FFT.

Finally, we point out that Cadzow reconstruction method fails to interpolate regularly decimated volumes (Trickett et al., 2010). It is possible, however, to dealias Cadzow filtering by a strategy that uses low-frequency rank-reduced data to reconstruct high-frequency aliased spatial data (M. Naghizadeh, 2012, personal communication).

Main contribution of this article

Previous contributions in the area of seismic reconstruction via rank reduction methods are the articles by Trickett (2008) and Trickett et al. (2010). These articles addressed the multidimensional reconstruction problem using an algorithm with similarities to the one discussed in this paper. However, these contributions do not provide details about the algorithm that is used for rank reduction. Rank reduction algorithms that use Hankel matrices were also investigated by Oropeza and Sacchi (2010, 2011). These articles provide a comparison of the computational cost of rank reduction via the classical SVD approach and the randomized SVD approach proposed by Liberty et al. (2007). Oropeza and Sacchi (2011) showed that the randomized SVD algorithm provides an important speed up with respect to the classical implementation of rank reduction via the truncated SVD. However, neither the truncated SVD nor the randomized SVD algorithm profits from the special structure of the matrix used by rank reduction methods. This article, on the other hand, exploits the special structure of multilevel block Toeplitz matrices that arise in reduced-rank reconstruction methods and proposes an iterative algorithm based on Lanczos bidiagonalization that can perform efficient matrix times vector multiplications via multidimensional FFT. The latter allows for the acceleration of the rank-reduction algorithm needed by our reconstruction algorithm.

RANK REDUCTION OF A LEVEL-ONE TOEPLITZ MATRIX

For a 1D signal (Hansen and Jensen, 1998; Oropeza and Sacchi, 2011) reduced-rank denoising is implemented via the following three steps:

1) Form a Toeplitz (or Hankel) matrix from the original 1D data.
2) Find a low-rank approximation of the Toeplitz (or Hankel) matrix via, for instance, the truncated SVD.
3) Recover the output denoised signal from the low-rank approximation by averaging along its diagonal (or antidiagonals)

This procedure is also the main template for iterative reconstruction via rank reduction (Oropeza and Sacchi, 2011). Our exposition uses Toeplitz forms. However, it is important to notice that a similar algorithm can be developed using Hankel matrices. The development of denoising and reconstruction for N-dimensional signals involves forming multilevel block Toeplitz (or Hankel) matrices, followed by rank reduction and diagonal (or antidiagonal) block averaging (Trickett, 2002; Oropeza and Sacchi, 2011).

We commence our discussion by examining the 1D spatial case. In other words, we consider seismic data that depends on two variables t and x (time and space). These data are represented in the space-frequency domain $D(x,\omega)$ where ω represents the temporal frequency. A monochromatic frequency slice of $D(x,\omega)$, with discrete spatial variable $x_j = (j-1)\Delta x$, $j = 1\ldots N_1$ is denoted by the vector $\mathbf{D}(\omega) = [D_1(\omega), D_2(\omega), \cdots, D_{N_1}(\omega)]^T$. To avoid notational clutter, we drop the dependency on ω and, from now on, we understand that the subsequent analysis is carried out for all frequencies ω. With the latter in mind, the vector of spatial observations is denoted via $\mathbf{D} = [D_1, D_2, \cdots, D_{N_1}]^T$.

The level-one Toeplitz matrix $\mathbf{T}^{(1)}$ of $\mathbf{D}$ is given by

$$\mathbf{T}^{(1)} = \begin{pmatrix} D_{N_1-L_1+1} & D_{N_1-L_1} & \cdots & D_3 & D_2 & D_1 \\ D_{N_1-L_1+2} & D_{N_1-L_1+1} & \cdots & D_4 & D_3 & D_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ D_{N_1-1} & D_{N_1-2} & \cdots & D_{L_1+1} & D_{L_1} & D_{L_1-1} \\ D_{N_1} & D_{N_1-1} & \cdots & D_{L_1+2} & D_{L_1+1} & D_{L_1} \end{pmatrix}. \tag{1}$$

A good strategy is to choose L_1 such that the Toeplitz matrix is approximately square (Trickett, 2008). In other words, we choose $L_1 = \lfloor \frac{N_1}{2} \rfloor + 1$ where the symbol $\lfloor\rfloor$ indicates the integer part of its argument. Equation 1 is a level-one Toeplitz matrix of size $L_1 \times (N_1 - L_1 + 1)$.

It can be shown that if $\mathbf{D}$ is composed of K complex sinusoids (K linear events in $t-x$), the rank of $\mathbf{T}^{(1)}$ is K (Hua, 1992; Yang and Hua, 1996). Therefore, rank reduction via the truncated SVD can capture the ideal data that one would have measured in the absence of noise and/or missing samples. The cost associated with the computation of the truncated SVD can make rank-reduction methods unfeasible for multidimensional seismic data reconstruction.

Lanczos bidiagonalization

This article proposes rank reduction via the Lanczos bidiagonalization (O'Leary and Simmons, 1981; Golub and Van Loan, 1996; Simon and Zha, 2000). The Lanczos bidiagonalization method is

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used to decompose the $L_1 \times (N_1 - L_1 + 1)$ Toeplitz matrix $\mathbf{T}^{(1)}$ into the low-rank form

$$\mathbf{T}^{(1)} = \mathbf{U}\mathbf{B}\mathbf{Q}^H, \tag{2}$$

where $\mathbf{U}$ and $\mathbf{Q}$ are orthogonal matrices of size $(L_1 \times L_1)$ and $(N_1 - L_1 + 1) \times (N_1 - L_1 + 1)$, respectively. The matrix $\mathbf{B}$ is a $L_1 \times (N_1 - L_1 + 1)$, and is a bidiagonal matrix with elements given by

$$B_{i,i} = \alpha_i, \quad i = 1 \ldots L_1 \quad B_{i,i+1} = \beta_i, \quad i = 1 \ldots L_1 - 1. \tag{3}$$

Finally, the low-rank approximation of the matrix $\mathbf{T}^{(1)}$ can be found via the expression (Simon and Zha, 2000)

$$\tilde{\mathbf{T}}^{(1)} = \mathbf{U}_K \mathbf{B}_K \mathbf{Q}_K^H, \tag{4}$$

where $\mathbf{U}_K = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_K]$ and $\mathbf{Q}_K = [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_K]$ are the matrices composed of the first K columns of $\mathbf{U}$ and $\mathbf{Q}$, respectively. The elements of the bidiagonal matrix $\mathbf{B}_k$ are given by

$$B_{i,i} = \alpha_i, \quad i = 1 \ldots K \quad B_{i,i+1} = \beta_i, \quad i = 1 \ldots K - 1. \tag{5}$$

Below, we provide the selective reorthogonalized Lanczos bidiagonalization algorithm that we used to estimate the elements of the matrices $\mathbf{U}_K$, $\mathbf{Q}_K$, and $\mathbf{B}_K$ (O'Leary and Simmons, 1981; Sacchi, 1998):

$$\begin{aligned}
&\texttt{Set :} \quad \mathbf{z}_1 \neq \mathbf{0}, \quad \mathbf{q}_1 = \frac{\mathbf{z}_1}{\|\mathbf{z}_1\|} \\
&\qquad \mathbf{y}_1 = \underbrace{\mathbf{T}^{(1)}\mathbf{q}_1}_{(6a)}, \quad \alpha_1 = \|\mathbf{y}_1\|, \quad \mathbf{u}_1 = \frac{\mathbf{y}_1}{\alpha_1} \\
&\texttt{for} \quad i = 1 \quad \texttt{to}\ K - 1 \\
&\qquad \mathbf{z}_{i+1} = \underbrace{\mathbf{T}^{(1)H}\mathbf{u}_i}_{(6b)} - \alpha_i \mathbf{q}_i \\
&\qquad \mathbf{z}_{i+1} = \mathbf{z}_{i+1} - (\mathbf{z}_{i+1}^H \mathbf{q}_j)\mathbf{q}_j, \quad j = i, i-1, \ldots, i-s \\
&\qquad \beta_i = \|\mathbf{z}_{i+1}\| \\
&\qquad \mathbf{q}_{i+1} = \frac{\mathbf{z}_{i+1}}{\beta_i} \\
&\qquad \mathbf{y}_{i+1} = \underbrace{\mathbf{T}^{(1)}\mathbf{q}_{i+1}}_{(6a)} - \beta_i \mathbf{u}_i \\
&\qquad \mathbf{y}_{i+1} = \mathbf{y}_{i+1} - (\mathbf{y}_{i+1}^H \mathbf{u}_j)\mathbf{u}_j, \quad j = i, i-1, \ldots, i-s \\
&\qquad \alpha_{i+1} = \|\mathbf{y}_{i+1}\| \\
&\qquad \mathbf{u}_{i+1} = \frac{\mathbf{y}_{i+1}}{\alpha_{i+1}} \\
&\texttt{end}
\end{aligned} \tag{6}$$

The parameter s indicates the number of preceding $\mathbf{q}$ and $\mathbf{u}$ vectors that undergo reorthogonalization. The cost of the algorithm is about $O(L_1^2)$ operations per iteration when the matrix-vector multiplications $\mathbf{T}^{(1)}\mathbf{q}_{i+1}$ and $\mathbf{T}^{(1)H}\mathbf{u}_i$ are naively implemented. By exploiting the special structure of the level-one Toeplitz matrix $\mathbf{T}^{(1)}$, one can adopt the FFT algorithm to compute these products in $O(M_1 \log_2 M_1)$ operations with M_1 defined in next section.

In general, K iterations of Lanczos bidiagonalization will approximate the reduced-rank matrix that one would have obtained using the K-largest singular values of the SVD. However, there is no guarantee that this will happen; the convergence to the SVD solution depends on the distribution of the singular values. In our tests, however, we have found that rank reduction via K iterations of Lanczos bidiagonalization and by the truncated SVD leads to very similar results. A problem with Lanczos bidiagonalization is its numerical instability (Kahan and Parlett, 1976). In practice, the Lanczos vectors may loose orthogonality after a few iterations. Different strategies have been proposed to enforce orthogonality. One solution is to perform complete or selected reorthogonalization (O'Leary and Simmons, 1981) of each Lanczos vector with all preceding vectors. In our problem, we retrieve a small subspace composed of about $K = 5$–10 Lanczos vectors and our experience indicates that full or selected reorthogonalization is not needed.

Fast matrix-vector product for circulant matrices

A circulant matrix can be multiplied by a vector via the FFT (Strang, 1986; Sacchi and Porsani, 1999). In addition, a Toeplitz matrix can be embedded into a circulant matrix leading to an algorithm that permits to multiply a Toeplitz matrix times a vector via the FFT. We start by defining a level-one $M_1 \times M_1$ $\mathbf{C}^{(1)}$ circulant matrix

$$\mathbf{C}^{(1)} = \begin{pmatrix} C_1 & C_{M_1} & \cdots & C_3 & C_2 \\ C_2 & C_1 & \cdots & C_4 & C_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{M_1-1} & C_{M_1-2} & \cdots & C_1 & C_{M_1} \\ C_{M_1} & C_{M_1-1} & \cdots & C_2 & C_1 \end{pmatrix}. \tag{7}$$

It is well known that $\mathbf{C}^{(1)}$ can be diagonalized by the unitary Fourier matrix of the same size (Chan and Strang, 1987; Chan and Jin, 2007)

$$\mathbf{C}^{(1)} = \mathbf{F}^H \mathbf{\Lambda} \mathbf{F}, \tag{8}$$

where $\mathbf{F}$ and $\mathbf{F}^H$ denotes the unitary discrete Fourier and inverse discrete Fourier matrices, respectively. The elements of the $M_1 \times M_1$ operator $\mathbf{F}$ are given by

$$F_{jk} = \frac{1}{\sqrt{M_1}} e^{\frac{-i2\pi(j-1)(k-1)}{M_1}}, \qquad j, k = 1 \ldots M_1 - 1. \tag{9}$$

Using equation 8 and the particular structure of the DFT matrix $\mathbf{F}$, one can easily show that the diagonal matrix of eigenvalues $\mathbf{\Lambda}$ can be computed via

$$\mathbf{\Lambda} = diag(\mathbf{F}\mathbf{c}), \tag{10}$$

where the $M_1 \times 1$ vector $\mathbf{c}$ is the first column of $\mathbf{C}^{(1)}$. If M_1 is an integer power of two, the product between $\mathbf{C}^{(1)}$ and an $M_1 \times 1$ column vector $\mathbf{x}$ can be efficiently computed in $O(M_1 \log_2 M_1)$ operations via the fast Fourier transform (`fft`) and its inverse (`ifft`), respectively (Chan and Jin, 2007),

$$\mathbf{y} = \mathbf{C}^{(1)}\mathbf{x} \quad = \mathbf{F}^H diag(\mathbf{Fc})\mathbf{Fx} \quad = \mathbf{F}^H((\mathbf{Fc}) \circ (\mathbf{Fx})), \tag{11}$$

where, $\circ$ means Hadamard (componentwise) multiplication. It is understood that in equation 11 the discrete Fourier matrices $\mathbf{F}$ and $\mathbf{F}^H$ are replaced by FFTs: $\mathbf{F}^H((\mathbf{Fc}) \circ (\mathbf{Fx})) \equiv \mathtt{ifft}[\mathtt{fft}(\mathbf{c}) \circ \mathtt{fft}(\mathbf{x})]$. We stress that we do not save the complete matrix $\mathbf{C}^{(1)}$, we use only the first column of it, $\mathbf{c}$.

Fast matrix-vector product for Toeplitz matrices

At this point, it is clear that we have an efficient way of evaluating the product of a circulant matrix times an arbitrary vector. However, our reduced-rank filtering approach requires fast matrix-vector products for Toeplitz matrices. A Toeplitz matrix can be embedded into a circulant matrix. Consider, for instance, a signal of length $N_1 = 6$, $\mathbf{D} = [D_1, D_2, D_3, D_4, D_5, D_6]^T$, we choose $L_1 = 4$ and form the corresponding Toeplitz matrix of $\mathbf{D}$

$$\mathbf{T}^{(1)} = \begin{pmatrix} D_3 & D_2 & D_1 \\ D_4 & D_3 & D_2 \\ D_5 & D_4 & D_3 \\ D_6 & D_5 & D_4 \end{pmatrix}. \tag{12}$$

We now choose $M_1 = 8$, and form the new, zero-padded signal $\mathbf{D} = [D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8]^T$ where $D_7 = 0$ and $D_8 = 0$. The level-one circulant matrix (Elden and Sjostrom, 1996) is given by

$$\mathbf{C}^{(1)} = \begin{pmatrix} D_3 & D_2 & D_1 & D_8 & D_7 & D_6 & D_5 & D_4 \\ D_4 & D_3 & D_2 & D_1 & D_8 & D_7 & D_6 & D_5 \\ D_5 & D_4 & D_3 & D_2 & D_1 & D_8 & D_7 & D_6 \\ D_6 & D_5 & D_4 & D_3 & D_2 & D_1 & D_8 & D_7 \\ D_7 & D_6 & D_5 & D_4 & D_3 & D_2 & D_1 & D_8 \\ D_8 & D_7 & D_6 & D_5 & D_4 & D_3 & D_2 & D_1 \\ D_1 & D_8 & D_7 & D_6 & D_5 & D_4 & D_3 & D_2 \\ D_2 & D_1 & D_8 & D_7 & D_6 & D_5 & D_4 & D_3 \end{pmatrix} = \begin{pmatrix} \mathbf{T}^{(1)} & * \\ * & * \end{pmatrix}, \tag{13}$$

where $*$ represents the remaining elements of $\mathbf{C}^{(1)}$. The multiplication of $\mathbf{T}^{(1)}$ with an arbitrary column vector $\mathbf{x}$, for example, $\mathbf{x} = [x_1, x_2, x_3]^T$ is implemented by multiplying $\mathbf{C}^{(1)}$ with the augmented vector $\tilde{\mathbf{x}} = [x_1, x_2, x_3, 0, 0, 0, 0, 0]^T$. In other words,

$$\tilde{\mathbf{y}} = \mathbf{C}^{(1)}\tilde{\mathbf{x}} = \begin{pmatrix} \mathbf{T}^{(1)} & * \\ * & * \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{T}^{(1)}\mathbf{x} \\ * \end{pmatrix}. \tag{14}$$

Then, $\tilde{\mathbf{y}}$ can be calculated using equation 11,

$$\tilde{\mathbf{y}} = \mathbf{C}^{(1)}\tilde{\mathbf{x}} = \mathbf{F}^H((\mathbf{F}\tilde{\mathbf{c}}) \circ (\mathbf{F}\tilde{\mathbf{x}})) \equiv \mathtt{ifft}[\mathtt{fft}(\tilde{\mathbf{c}}) \circ \mathtt{fft}(\tilde{\mathbf{x}})], \tag{15}$$

where now $\tilde{\mathbf{c}} = [D_3, D_4, D_5, D_6, D_7, D_8, D_1, D_2]^T$ is the first column of the circulant $\mathbf{C}^{(1)}$. In the general case with parameters N_1, L_1 and M_1, the first column of $\mathbf{C}^{(1)}$ is given by

$$\tilde{\mathbf{c}} = [D_{N_1-L_1+1} \dots D_{M_1}, D_1, D_2, \dots D_{N_1-L_1}]^T. \tag{16}$$

Our algorithm chooses $M_1 = 2^J$, $J \in Z$. We point out that one could have also used FFT algorithms that do not require input signals of length 2^J. We finally compute the desired vector $\mathbf{y} = \mathbf{T}^{(1)}\mathbf{x}$ as the first L_1 top elements of vector $\tilde{\mathbf{y}}$. Note that the Lanczos algorithm in equation 6 also requires multiplications of the form $\mathbf{T}^{(1)H}\mathbf{u}$ (see expression 6b). Because the Hermitian transpose of a Toeplitz matrix is also a Toeplitz matrix, the algorithm outlined above is also valid to compute products of the form $\mathbf{T}^{(1)H}\mathbf{u}$.

RANK REDUCTION OF A LEVEL-FOUR TOEPLITZ MATRIX

We turn now our attention to the case where seismic data depends on four spatial dimensions. In other words, the observations are denoted by $D(\omega, x_1, x_2, x_3, x_4)$, where the spatial variables x_1, x_2, x_3, and x_4 can be used to identify x and y midpoints and x and y offsets, respectively. We assume that the spatial variables are regularly sampled and that each observation can be represented via D_{j_1,j_2,j_3,j_4} with $j_i = 1 \dots N_i$, $i = 1, 2, 3, 4$. Again, we have dropped the dependency on frequency and, it is understood that the analysis is carried out for all frequencies. We choose to follow the strategy developed for the 1D case, and define parameters $L_i = \lfloor \frac{N_i}{2} \rfloor + 1$, $i = 1, 2, 3, 4$. The 4D spatial hypercube can be embedded in a level-four block Toeplitz matrix as follows. We first embed the seismic data in a level-one Toeplitz matrix using all data components of, for instance, the first dimension of the tensor D_{j_1,j_2,j_3,j_4}. This generates the following Toeplitz matrices,

$$\mathbf{T}^{(1)}_{j_2,j_3,j_4} = \begin{pmatrix} D_{N_1-L_1+1,j_2,j_3,j_4} & \dots & D_{1,j_2,j_3,j_4} \\ \vdots & \dots & \vdots \\ D_{N_1,j_2,j_3,j_4} & \dots & D_{L_1,j_2,j_3,j_4} \end{pmatrix}, \quad j_i = 1 \dots N_i, \quad i = 2, 3, 4. \tag{17}$$

Matrices in 17 are of size $L_1 \times (N_1 - L_1 + 1)$. These matrices are embedded in a level-two Toeplitz matrix

$$\mathbf{T}^{(2)}_{j_3,j_4} = \begin{pmatrix} \mathbf{T}^{(1)}_{N_2-L_2+1,j_3,j_4} & \dots & \mathbf{T}^{(1)}_{1,j_3,j_4} \\ \vdots & \dots & \vdots \\ \mathbf{T}^{(1)}_{N_2,j_2,j_3,j_4} & \dots & \mathbf{T}^{(1)}_{L_2,j_3,j_4} \end{pmatrix}, \quad j_i = 1 \dots N_i, \quad i = 3, 4. \tag{18}$$

Matrices in 18 are of size $(L_1 L_2) \times (N_1 - L_1 + 1)(N_2 - L_2 + 1)$. These matrices are now embedded in a level-three Toeplitz matrix

$$\mathbf{T}^{(3)}_{j_4} = \begin{pmatrix} \mathbf{T}^{(2)}_{N_3-L_3+1,j_4} & \dots & \mathbf{T}^{(2)}_{1,j_4} \\ \dots & \dots & \vdots \\ \mathbf{T}^{(2)}_{N_3,j_4} & \dots & \mathbf{T}^{(2)}_{L_3,j_4} \end{pmatrix}, \qquad j_4 = 1 \dots N_4. \tag{19}$$

The latter are matrices of size $(L_1 L_2 L_3) \times (N_1 - L_1 + 1)(N_2 - L_2 + 1)(N_3 - L_3 + 1)$. Finally, matrices with the form given by equation 19 are embedded in the final level-four block Toeplitz matrix

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$$\mathbf{T}^{(4)} = \begin{pmatrix} \mathbf{T}^{(3)}_{N_4-L_4+1} & \dots & \mathbf{T}^{(3)}_1 \\ \dots & \dots & \vdots \\ \mathbf{T}^{(3)}_{N_4} & \dots & \mathbf{T}^{(3)}_{L_4} \end{pmatrix}. \tag{20}$$

The latter is the grand matrix of our problem. The size of the level-four block Toeplitz matrix $\mathbf{T}^{(4)}$ is $(L_1L_2L_3L_4) \times (N_1 - L_1 + 1)(N_2 - L_2 + 1)(N_3 - L_3 + 1)(N_4 - L_4 + 1)$. Recalling that we choose $L_i = \lfloor \frac{N_i}{2} \rfloor + 1$ to have blocks that are close to square matrices and considering the simple case where N_i is an odd integer, the size of $\mathbf{T}^{(4)}$ is $(L_1L_2L_3L_4) \times (L_1L_2L_3L_4)$. In this case, the cost of naively multiplying $\mathbf{T}^{(4)}$ times a vector is $O((L_1L_2, L_3L_4)^2)$ operations and, consequently, K iterations of the Lanczos bidiagonalization procedure yield a cost proportional to $O(K(L_1L_2L_3L_4)^2)$ operations. We stress that if one were to adopt the SVD to reduce the rank of $\mathbf{T}^{(4)}$, the overall cost of the SVD is $O((L_1L_2, L_3, L_4)^3)$ (Golub and Van Loan, 1996). Our motivation for using a Lanczos diagonalization that exploits the special structure of the level-four block Toeplitz matrix $\mathbf{T}^{(4)}$ is evident.

We could have provided the lengthy development needed to show that an algorithm similar to the one used to multiply a level-one Toeplitz matrix with a vector via the FFT can be also developed for level-four block Toeplitz forms. We prefer to refer the reader to contributions by Tyrtyshnikov (1996), Chan and Jin (2007), and Oudin and Delmas (2009) for the proof of the following recipe. The algorithm to multiply a level-four block Toeplitz by a vector entails appending each dimension of the original data by zeros, transforming the level-four Toeplitz form into a level-four circulant matrix and, finally, recognizing that the FFT can also be used to diagonalize block circulant matrices. The final step entails showing that a fast algorithm for the multiplication of a level-four Toeplitz matrix times a vector can be obtained using the 4D FFT and IFFT in a similar way expression 15 was used for the level-one Toeplitz matrix-vector product. To summarize, to multiply $\mathbf{T}^{(4)}$ or its hermitian transpose by a vector we implement the following steps:

- The first step of the algorithm requires to append zeros to each dimension of the data. The total number of zeros to append is $(M_1 - N_1)(M_2 - N_2)(M_3 - N_3)(M_4 - N_4)$. In other words, we extend the signal D_{j_1,j_2,j_3,j_4} with $j_i = 1 \dots N_i$ and form the new signal to D_{j_1,j_2,j_3,j_4}, $j_i = 1 \dots M_i$ where, $D_{j_1,j_2,j_3,j_4} = 0$ with indexes $j_i = N_i + 1, \dots, M_i$, $i = 1, 2, 3, 4$. Now, we recall equation 16 and we form a new hypercube with indexes analogous to the ones used for the 1D case

$$\tilde{c}_{j_1,j_2,j_3,j_4} = \begin{cases} D_{N_1-L_1+j_1,N_2-L_2+j_2,N_3-L_3+J_3,N_4-L_4+j_4} & j_i = 1 \dots M_i - N_i + L_i \\ D_{j_1-M_1+N_1-L_1,j_2-M_2+N_2-L_2,j_3-M_3+N_3-L_3,j_4-M_4+N_4-L_4} & j_i = M_i - N_i + L_i + 1 \dots M_i. \\ & i = 1,2,3,4 \end{cases} \tag{21}$$

- The product $\mathbf{T}^{(4)}\mathbf{x}$ needed by Lanczos bidiagonalzation is performed via the following elementwise multiplication of the 4D FFT of $\tilde{c}$ (equation 21) and the 4D FFT of $\tilde{\mathbf{x}}$, where $\tilde{\mathbf{x}}$ is the vector $\mathbf{x}$ reshaped into a hypercube of dimensions $M_1 \times M_2 \times M_3 \times M_4$

$$\tilde{\mathbf{y}} = \mathbf{F}^H_{4D}((\mathbf{F}_{4D}\tilde{\mathbf{c}}) \circ (\mathbf{F}_{4D}\tilde{\mathbf{x}})) \equiv \mathtt{ifft4}D[\mathtt{fft4}D(\tilde{\mathbf{c}}) \circ \mathtt{fft4}D(\tilde{\mathbf{x}})]. \tag{22}$$

- We reshape the hypercube of $\tilde{\mathbf{y}}$ of size $M_1 \times M_2 \times M_3 \times M_4$ into a vector of size $(M_1M_2M_3M_4) \times 1$ and we finally extract the first $L_1L_2L_3L_4$ elements of the vector $\tilde{\mathbf{y}}$. It is important to stress that the algorithm for multiplying $\mathbf{T}^{(4)}$ and a vector $\mathbf{x}$ does not require us to form the level-four block Toeplitz matrix or the associated level-four circulant matrix. Products of the form $\mathbf{T}^{(4)H}\mathbf{u}$ are implemented in a similar fashion.

The cost of the Lanczos algorithm is $O(KM\log_2 M)$ with $M = M_1M_2M_3M_4$, which is below the cost of the naive matrix times vector multiplication discussed at the beginning of this section.

SEISMIC DATA RECONSTRUCTION

Properly sampled multidimensional data can be embedded into a low-rank multilevel Hankel or multilevel Toeplitz matrix. Missing data and noise will increase the rank of this matrix. In this section, we describe the iterative algorithm discussed in Oropeza and Sacchi (2011).

We first form the multidimensional hypercube $D^{obs}_{j_1,j_2,j_3,j_3}$, $j_i = 1 \dots N_i$, $i = 1, 2, 3, 4$. The latter, for simplicity, is denoted via the multiway array or tensor $\mathcal{D}^{obs}$. The following iterative algorithm is used to retrieve the missing data

$$\mathcal{D}^k = \mathcal{D}^{obs} + (\mathcal{I} - \mathcal{S})\mathcal{F}\mathcal{D}^{k-1}, \qquad k = 1, \dots N_{iter}, \tag{23}$$

where $\mathcal{S}$ is the sampling operator, a multiarray of the same size of $\mathcal{D}$ that is equal to one for points with observations and zero for grid points with unrecorded observations. Similarly, $\mathcal{I}$ is a multiarray of the size of $\mathcal{S}$, but composed of ones. In other words, $\mathcal{I} - S$ is a reinsertion operator. The operator $\mathcal{F}$ indicates the concatenated application of: (1) Forming the data $\tilde{\mathbf{c}}$ (equation 21), (2) rank-reduction via the Lanczos bidiagonalization with matrix-vector multiplications via the 4D FFT, and (3) finally, block averaging along diagonals to recover the 4D data array. In general, we prefer to reinsert the weighted average $\alpha\mathcal{D}^{obs} + (1 - \alpha)\mathcal{S}\mathcal{F}\mathcal{D}^{k-1}$ to alleviate the influence of the noise contained in the original observations (Oropeza and Sacchi, 2011). Replacing $\mathcal{D}^{obs}$ in equation 23 by the aforementioned weighted average leads to the following iterative algorithm

$$\mathcal{D}^k = \alpha\mathcal{D}^{obs} + (\mathcal{I} - \alpha\mathcal{S})\mathcal{F}\mathcal{D}^{k-1}, k = 1, \dots N_{iter}, \tag{24}$$

the scalar $\alpha \leq 1$ is used for simultaneous data reconstruction and denoising. If $\alpha = 1$, the noisy observations are reinserted at each iteration.

EXAMPLES

Synthetic example

Our first example shows the performance of the proposed algorithm in comparison to the typical implementation of rank reduction using the truncated SVD. We compare the proposed Lanczos diagonalization algorithm with fast matrix-vector multiplication via 4D FFT and a similar algorithm where the full level-four Toeplitz matrix of the problem is rank reduced via the function `svds` provided by MATLAB. For this test, we construct a series of 4D spatial volumes of size $N \times N \times N \times N$, with $N = 5, 6, \dots, 13$ and test the computational time of the proposed method versus `svds`. In this example, we have timed the cost of one rank reduction for the desired rank $K = 5$ for one frequency slice. Figure 1 shows a computational time comparison of the two algorithms. These results were obtained using MATLAB on a desktop with an Intel Core 2 Duo processor of 2.66 Ghz. We confirm with this example that the Lanczos algorithm outperforms the SVD algorithm

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due to its ability to fully exploit the special structure of the level-four Toeplitz matrix. The cost of the reconstruction is dominated by the rank reduction stage of the algorithm. Therefore, Figure 1 can also be used to estimate the approximate cost of the algorithm per number of spatial grid points and per frequency by dividing the time by the total number of grid points (N^4). For example, for $N = 13$, the computation time is about two seconds per frequency slice. A typical seismic data set may have in the order of 2000 samples, and therefore in the order of 1000 frequency slices. For a typical number of iterations $N_{iter} = 10$, the cost of reconstructing the cube using the proposed algorithm is about $10 \times 1000 \times 2$ s (about five hours). The computation using the truncated SVD will be about 10 times the aforementioned time. Clearly, the proposed method is more efficient than the truncated SVD. However, the proposed method is not as efficient as methods based on Fourier synthesis that operate with the FFT. For instance, the same hypercube can be reconstructed in about half the time of the proposed method using our version of minimum weighted norm interpolation (Liu and Sacchi, 2004; Trad, 2009). We stress that code parallelization is required for industrial applications of 5D reconstruction via rank reduction.

The second example aims to examine the reconstruction accuracy of the proposed method. For this purpose, we prepare a synthetic model that consists of $8 \times 8 \times 8 \times 8$ traces and 512 time samples per trace. The data include three linear events ($S/N = \infty$). Figure 2 shows the reconstruction quality for different percentages of decimated data. The simulation was run for the proposed new rank-reduction algorithm and with SVD function provided by MATLAB (`svds`). We set the iteration number $N_{iter} = 10$ and $\alpha = 1$. The reconstruction error was measured via the following expression

$$Q = 10 \log\left(\frac{|\mathcal{D}^{true}|^2}{|\mathcal{D}^{true} - \mathcal{D}^{recon}|^2}\right), \tag{25}$$

where $\mathcal{D}^{true}$ and $\mathcal{D}^{recon}$ represent the true noise-free complete data and reconstructed data, respectively. The reconstruction quality obtained by the truncated SVD and the proposed Lanczos bidiagonalization algorithm are extremely similar. This gives us confidence in replacing the classical rank reduction via the truncated SVD by a faster Lanczos bidiagonalization strategy.

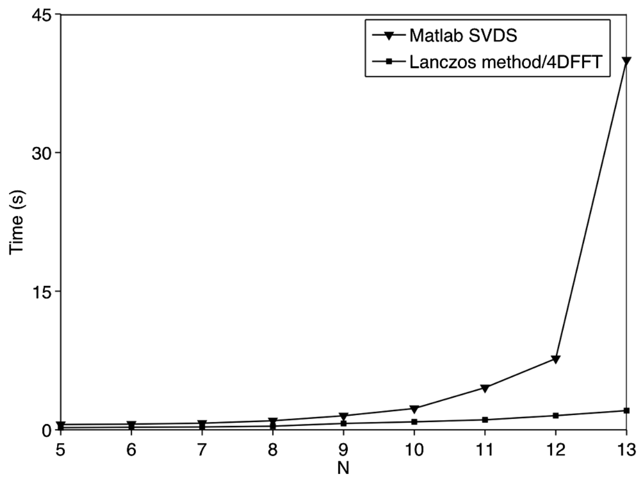

Figure 1. Comparison of two rank reduction algorithms. Computational time for rank reduction of a hypercube of size N^4 via Lanczos bidiagonalization with fast matrix-vector multiplications and the truncated SVD.

In our next examples, we test the reconstruction on a 4D spatial volume composed of $8 \times 8 \times 8 \times 8$ traces. Each seismogram contains 512 samples and the sampling interval is 2 msec. We have identified the first and second spatial coordinates of the data D_{j_1,j_2,j_3,j_4} with the CMP_x and CMP_y coordinates, respectively. Similarly, we have identified the third and fourth coordinates of the data with offsets (h_x, h_y) in the x- and y-directions, respectively. We removed 50% of the data via random decimation and show the reconstruction results in Figure 3 ($S/N = \infty$) and Figure 4 ($S/N = 1$). In both tests, the algorithm was run for a total of $N_{iter} = 10$ iterations and for temporal frequencies in the band 1–80 Hz. It is important to mention that in Figure 3, the noise-free case, we adopted $\alpha = 1$. In

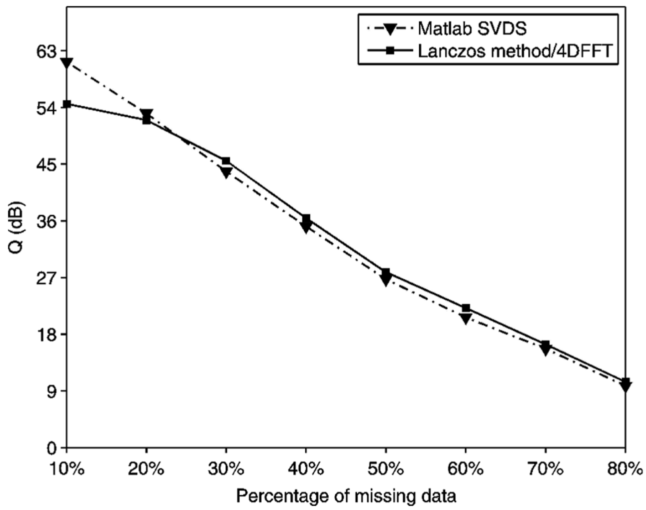

Figure 2. Reconstruction quality Q for data regularization via rank reduction using Lanczos bidiagonalization with fast matrix-vector multiplications and the truncated SVD.

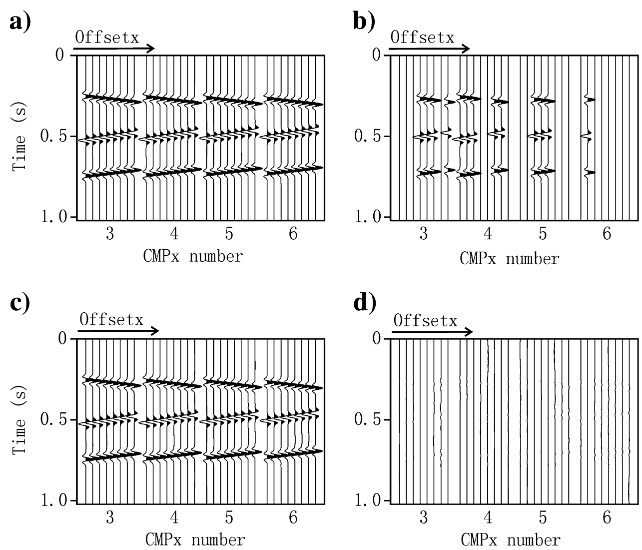

Figure 3. (a) Original data. (b) Decimated data with 50% of the traces missing. (c) Reconstructed data. (d) Difference between (a) and (c).

other words, total reinsertion of observed data was permitted in each iteration. For the noisy synthetics in Figure 4, we adopted $\alpha = 0.8$. In both examples, we portray a subset of the 5D data consisting of a data slice for fixed $\mathrm{CMP}_y = 5$ and $h_y = 6$ and CMP_x varying from three to six. Lastly, we mention that a rank $K = 5$ was used for both examples.

Western Canadian Sedimentary Basin field example

The following example involves testing the reconstruction performance of the proposed method on a real prestack 5D volume. Figure 5 represents the distribution of sources and receivers for a small patch of an orthogonal land survey (Cordsen et al., 2000). The seismic traces were assigned to a regular 4D midpoint offset grid of size $15 \times 15 \times 13 \times 13$. We selected a time window in the interval 900–1250 msec that corresponds to 351 samples. The binning process leads to a grid that contains 6650 traces distributed in $15 \times 15 \times 13 \times 13 = 38{,}025$ grid points. In other words, about 18% of the grid is occupied with observations. Midpoint and offset intervals and the number of grid points were carefully selected to maximize resolution without magnifying the sparsity of the grid. The following proof of concept parameters were chosen to test our algorithm

$$
\begin{aligned}
\mathrm{cmp}_x:\ n_1 &= 15, \quad [\min, \max] = [1700\ \mathrm{m}, 2300\ \mathrm{m}], \quad \text{increment} = 43\ \mathrm{m} \\
\mathrm{cmp}_y:\ n_2 &= 15, \quad [\min, \max] = [1300\ \mathrm{m}, 2700\ \mathrm{m}], \quad \text{increment} = 100\mathrm{m} \\
h_x:\ n_3 &= 13, \quad [\min, \max] = [-3400\ \mathrm{m}, 3400\ \mathrm{m}], \quad \text{increment} = 567\ \mathrm{m} \\
h_y:\ n_4 &= 13, \quad [\min, \max] = [-3000\ \mathrm{m}, 3000\ \mathrm{m}], \quad \text{increment} = 500\ \mathrm{m}.
\end{aligned}
$$

The maximum fold of the data is 26 and the minimum fold is 10. After reconstruction, the data will have a constant fold of $13 \times 13 = 169$ traces.

Figure 6a shows a subset of the data prior to reconstruction. Figure 6b and 6c illustrates the reconstruction via the proposed rank reduction method with $K = 8$. These figures correspond to a slice of

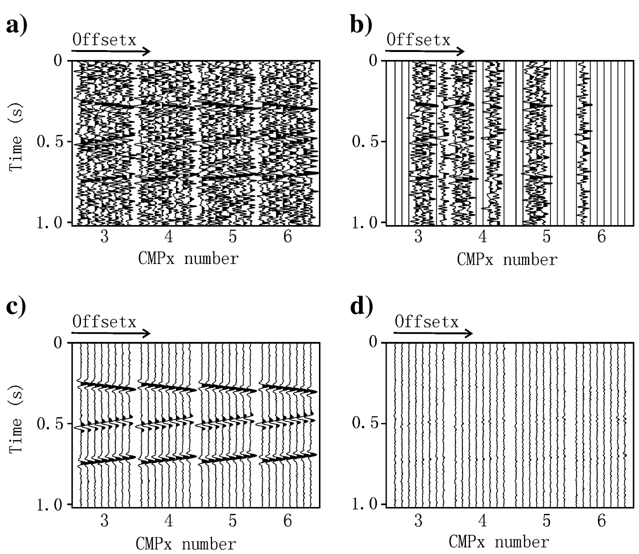

Figure 4. Denoising and reconstruction using equation 24 with $\alpha = 0.8$. (a) Data contaminated with noise. (b) Decimated data. (c) Reconstructed data. (d) Difference between Figure 3a and 4c.

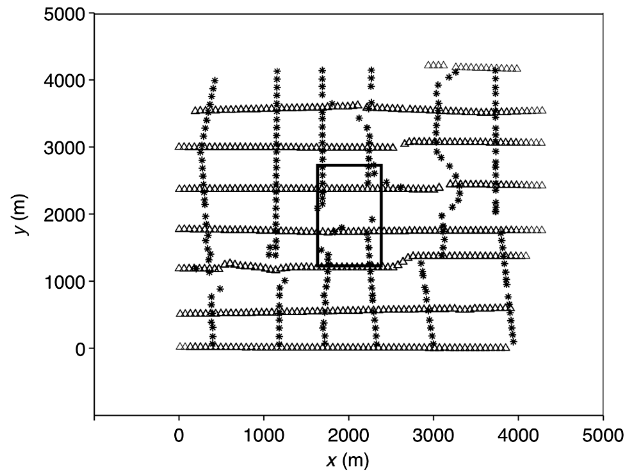

Figure 5. Field acquisition geometry. The asterisks represent shots and triangles represent receivers. The rectangle in the centre of the figure indicates the CMP coverage.

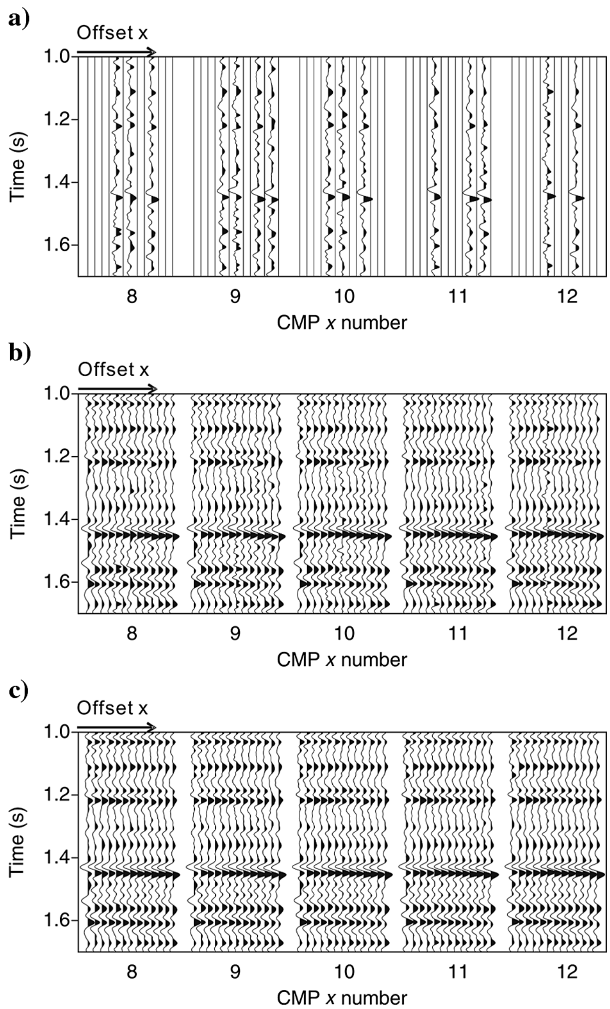

Figure 6. Subset of the data used to test the proposed rank reduction algorithm. (a) Data for fixed CMP_y bin 11 and h_y bin 11. (b) Reconstruction using $\alpha = 1$ and $K = 8$. (c) Reconstruction using $\alpha = 0.4$ and $K = 8$.

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the 5D volume that is obtained by fixing the CMP_y (bin 11, 2300 m) and h_y (bin 11, 2000 m). The results were computed for values of the parameter $\alpha = 1$ (Figure 6b) and $\alpha = 0.4$ (Figure 6c).

Figure 7a, 7b, and 7c shows the original and reconstructed data for a subset of the data that correspond to fixed CMP_y (bin 11, 2300 m) and h_y (bin 1, –3000 m), respectively. The results were computed for values of the parameter $\alpha = 1$ (Figure 7b) and $\alpha = 0.4$ (Figure 7c).

Finally, Figures 8 and 9 display stacks computed prior to reconstruction (normalized by fold) and after reconstruction. The stacks indicate that the algorithm has not introduced new signals in the reconstruction. We recognize that our test with a data set from the Western Canadian Sedimentary Basin is a simple example where most reconstruction/interpolation methods would succeed in reconstructing the missing data. More complex data examples might need values of $K > 8$.

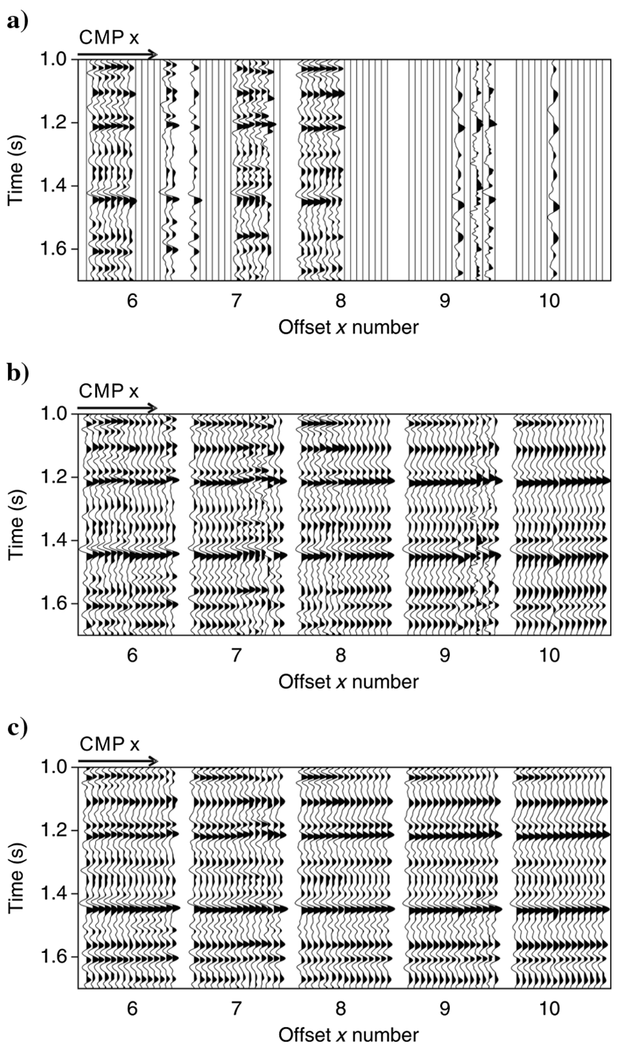

Figure 7. Subset of the data used to test the proposed rank reduction algorithm. (a) Partial view of the data for fixed CMP_y bin 11 and h_y bin 1. (b) Reconstruction using $\alpha = 1$ and $K = 8$. (c) Reconstruction using $\alpha = 0.4$ and $K = 8$.

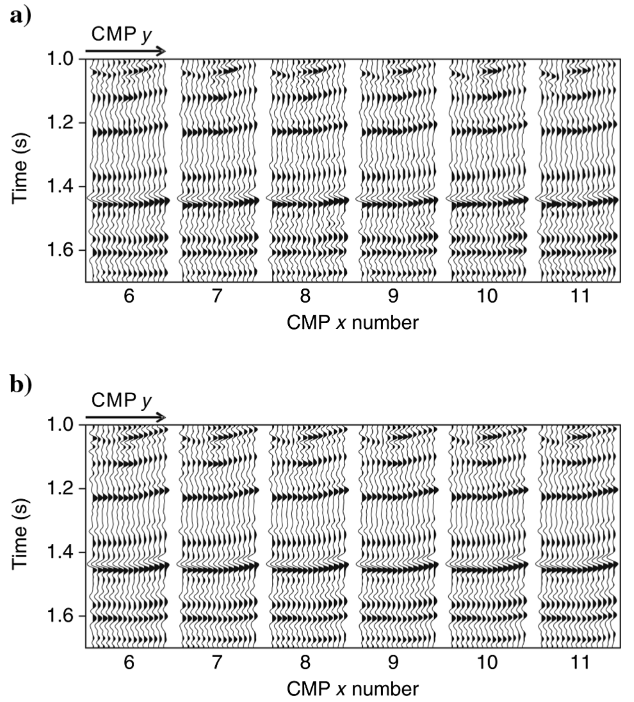

Figure 8. (a) Stack of the data prior to reconstruction. (b) Stack obtained with the reconstructed data with parameters $\alpha = 0.4$ and $K = 8$.

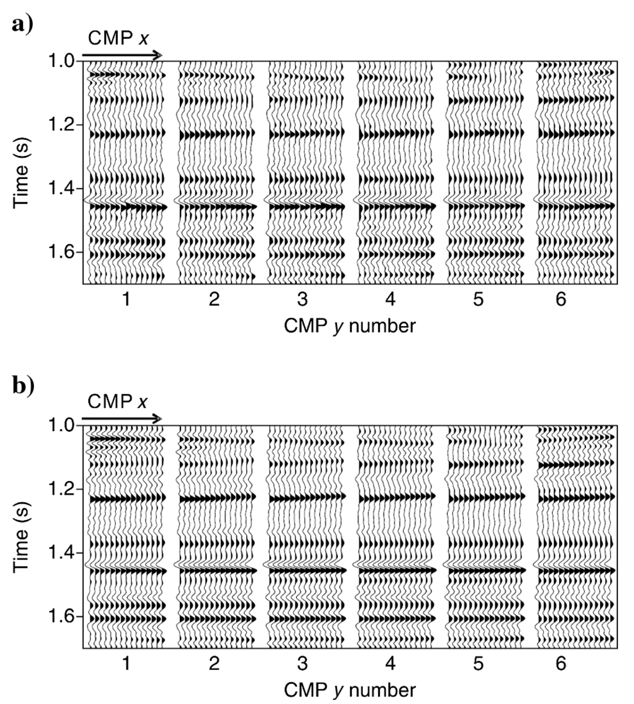

Figure 9. (a) Stack of the data prior to reconstruction. (b) Stack obtained with the reconstructed data with parameters $\alpha = 0.4$ and $K = 8$.

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CONCLUSIONS

In this paper, we have reformulated the Cadzow, or multichannel singular spectrum analysis, reconstruction and denoising method for volumes that depend on four spatial dimensions. A new algorithm is proposed where we combined Lanczos bidiagonalization and multilevel Toeplitz matrix-vector multiplications to reduce the rank of large matrices arising in reduced-rank denosing and reconstruction techniques. Synthetic and real data examples indicate that our new algorithm can greatly improve the computational efficiency and maintain the quality of the reconstruction at the level of the classical rank reduction implementation via the truncated SVD method.

The proposed algorithm requires some parameters to attain an optimal reconstruction. The number of iterations, the size of the subspace K, and the parameter α can be easily found by testing the algorithm with a patch of data. Once these parameters have been selected, the same parameters are adopted to process all the data patches that are contained in the full prestack volume. The parameter K is proportional to the number of dips in the data. When the data are composed by K linear events, the rank of the multilevel Toeplitz matrix of the ideal data is K. In general, the departure from the linear event assumption and lateral variations of amplitudes will preclude us from identifying K with the number of dips. Our tests indicates that the rank K needs to be about two to three times the number of dips in the data. The parameter α is used to control the reinsertion of the original data in the iterative reconstruction algorithm. We have found that $\alpha = 1$ works well for data with high signal-to-noise ratio. For our field data example, we have experimented with values in the range $\alpha = 0.4 - 0.6$ noticing minimal differences.

Finally, we reiterate that rank reduction reconstruction methods that use multilevel Toeplitz (or Hankel) matrices have been already reported in the geophysical literature. Fast algorithms for rank reduction, on the other hand, have not been described by previous contributions. In essence, this article provides the theoretical and practical framework for the development of fast reduced-rank denoising and reconstruction algorithms for industrial applications.

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