Bayesian Reinforcement Learning

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Bayesian Reinforcement Learning - what is it?

Bayesian RL is about capturing and dealing with uncertainty, where 'classic RL' does not. Research in Bayesian RL includes modelling the transition-function, or value-function, policy, reward function probabilistically.

Differences over 'classic RL':

- Resolves exploitation & exploration dilemma by planning in belief space.
- Computationally intractable in general, but approximations exist.
- Uses and chooses samples to learn from efficiently, suitable when sample cost is high, e.g. robot motion.

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Motivating Problem: Two armed bandit (1)

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- You have n tokens, which may be used in one of two slot machines.
- The i'th machine returns \$0 or \$1 based on a fixed yet unknown probability $p_i \in [0, 1]$
- Objective: maximise your winnings.

Motivating Problem: Two armed bandit (2)

- As you play, you record what you see, formatted as (#wins, #losses).
  Your current records shows:
  Arm 1: (1,2)
  Arm 2: (21,19)

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  Arm 1: (1,2)
  Arm 2: (21,19)
- Which machine would you play next if you have 1 token remaining?
- How about if you have 100 tokens remaining?

Motivating Problem: Two armed bandit (3)

'Classic' Reinforcement Learning mentality:
Two action classes (not mutually exclusive):

Exploit

Select action of greatest expected return given current belief on reward probabilities. i.e. select best action according to best guess of underlying MDP: MLE or MAP $\rightarrow$ select Arm #2!

Explore

Select random action to increase our certainty of underlying MDP.
This may lead to higher returns when exploiting in the future.

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Select random action to increase our certainty of underlying MDP.
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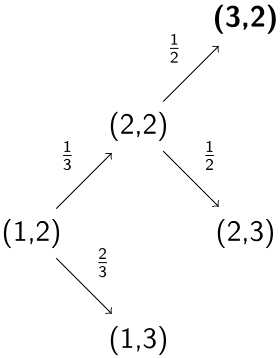
$\rightarrow$ Dilemma (?): how to choose between exploitation and exploration?
Seems like comparing apples and oranges...
Many heuristics exist, but is there a principled approach?

Motivating Problem: Two armed bandit (4)

Steps towards resolving 'exploitation' vs 'exploration': model future beliefs in Arm 1 (#wins, #losses):



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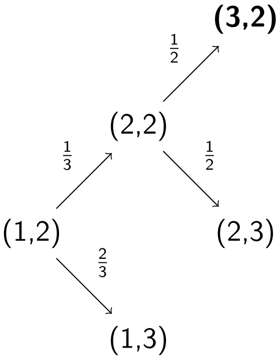

← higher expectation of rewards in this potential future!

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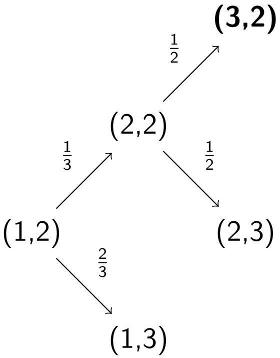

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we can **plan** in this space, and compute expected *additional* rewards gained from exploratory actions.

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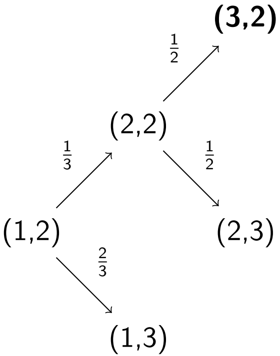

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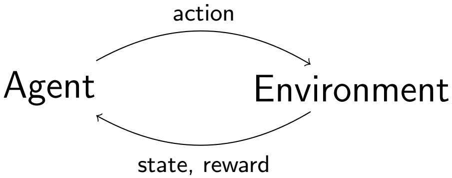
we can **plan** in this space, and compute expected *additional* rewards gained from exploratory actions.

Note: value of exploration depends on how much we can exploit that information gain later, i.e. # tokens remaining. Alternatively, with infinite tokens and discount rate γ on future rewards, effective horizon $\propto \frac{-1}{log(\gamma)}$

Planning in MDP Environments

Planning overview



- Environment: a *familiar* MDP (we can simulate interaction with the world accurately).
- Goal: compute a policy that maximises expected long-term discounted rewards over a horizon (episodic or continual).

Markov Decision Process

$\mathbf{S}$, set of states s

$\mathbf{A}$, set of action a

$\boldsymbol{\pi} : \mathbf{S} \rightarrow \mathbf{A}$, the policy, a mapping from state s to action a

$\mathbf{T}(\mathbf{s}, \mathbf{a}, \mathbf{s'}) = P(s'|s, a) \in [0, 1]$, transition probability, that state s' is reached by executing action a from state s

$\mathbf{R}(\mathbf{s}, \mathbf{a}, \mathbf{s'}) \in \mathbb{R}$, a reward distribution. An agent receives a reward drawn from this when taking action a from state s reaching state s'

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System dynamics

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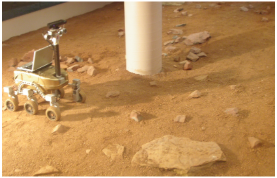
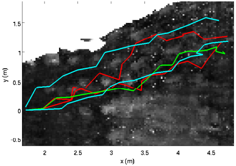
$\mathbf{R(s, a, s')} \in \mathbb{R}$, a reward distribution. An agent receives a reward drawn from this when taking action a from state s reaching state s'

Robot planning example

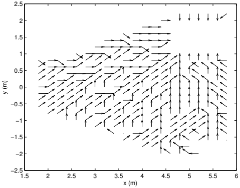
Goal: traverse to human-specified goal location 'safely'
States: physical space (x, y, yaw)
Action: move forward, left, spin anticlockwise, etc.
Rewards: 'dangerousness' of each (s, a) motion primitives



(a) Path Planning Scenario



(b) Rewards



(c) Policy

Rewards

A measure of *desirability* of the agent being in a particular state. Use to encode *what* we want the agent to achieve, not *how*.

example: Agent learns to play chess:
Don't reward agent for capturing opponent's queen, only for winning.
(don't want agent discovering novel ways to capture queens at expense of losing games!)

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(don't want agent discovering novel ways to capture queens at expense of losing games!)

Caveat: Reward *shaping*, the modification of reward function to give partial credit without affecting the optimal policy (much), can be important in practice.

Optimal Action-Value Function

Optimal action value: expectation of all future discounted rewards from taking action a from state s, assuming subsequent actions chosen by the optimal policy π^*. It can be re-expressed as a recursive relationship.

$$
\begin{aligned}
Q^*(s,a) &= E_{\pi^*}\left\{ \sum_{t=0}^{\infty} \gamma^t R(s_t, a_t) | s_0 = s, a_0 = a \right\} \\
&= E_{\pi^*}\left\{ R(s_0, a_0) + \gamma \sum_{t=0}^{\infty} \gamma^t R(s_{t+1}, a_{t+1}) | s_0 = s, a_0 = a \right\} \\
&= \bar{R}(s,a) + \gamma E_{\pi^*}\left\{ \max_{a'} Q^*(s_{t+1}, a') | s_0 = s, a_0 = a \right\} \\
&= \bar{R}(s,a) + \gamma \sum_{s'} T(s,a,s') [\max_{a'} Q^*(s', a')]
\end{aligned}
$$

Action-Value Optimisation

Need to satisfy the Bellman Optimality Equation:

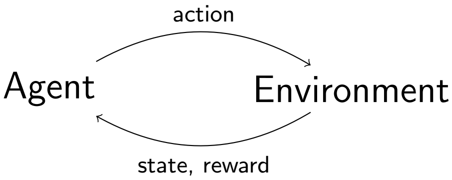
$$\begin{aligned} Q^*(s,a) &= \bar{R}(s,a) + \gamma \sum_{s'} T(s,a,s') \max_{a'} Q^*(s',a') \\ \pi^* &= \arg\max_{a} Q^*(s,a) \end{aligned}$$

An algorithm to compute $Q^*(s,a)$ is *value iteration*: for all $s \in S$ repeat until convergence:

$$Q_{t+1}(s,a) \leftarrow \bar{R}(s,a) + \gamma \sum_{s'} T(s,a,s') \max_{a'} Q_t(s',a')$$

Reinforcement Learning

Reinforcement learning overview



- Environment: an *unfamiliar* MDP ($T(s, a, s')$ and/or $\bar{R}(s, a)$ unknown) and possibly dynamic / changing.
- Consequence: agent cannot simulate interaction with world in advance, to predict future outcomes. Instead, the optimal policy is learned through sequential interaction and evaluative feedback.
- Goal: same as planning (compute a policy that maximises expected long-term discounted rewards over a horizon).

Q-learning

With **known** environmental models $\bar{R}(s,a)$ and $T(s,a,s')$, Q's computed iteratively using value iteration (e.g. planning): :

$$Q_{t+1}(s,a) \leftarrow \bar{R}(s,a) + \gamma \sum_{s'} T(s,a,s') \max_{a'} Q_t(s',a')$$

Q-learning

With **unknown** environmental models, Q's computed as *point estimates*:

on experience $\{s_t, a_t, r_t, s_{t+1}\}$:
$Q(s_t,a_t) \leftarrow Q(s_t,a_t) + \alpha_t(R_{t+1} + \gamma \max_{a'}(Q(s_{t+1},a')) - Q(s_t,a_t))$

if $\{s,a\}$ visited infinitely often, $\sum_t \alpha_t = \infty$, $\sum_t \alpha_t^2 < \infty$, then Q will converge to Q^* (independent of policy being followed!).

When to explore?: Heuristic approach to action selection

A couple heuristic examples agents use for action selection, to mostly exploit and sometimes and explore:

- ϵ-greedy:

$$\pi(a|s) = \begin{cases} (1-\epsilon), & \text{if } a = argmax_a Q_t(s,a) \\ \epsilon/|A|, & \text{if } a \neq argmax_a Q_t(s,a) \end{cases}$$

  e.g. $\epsilon = 5\%$

- Softmax: $\pi(a|s) = \frac{e^{Q_t(s,a)/\tau}}{\sum_i e^{Q_t(s,i)/\tau}}$
  i.e. biased towards more fruitful actions. τ is a crank for more frequent exploration.

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Note: often, heuristics are too inefficient for online learning! We wish to minimise wasteful exploration.

Bayesian Reinforcement Learning (Model-Based)

Brief Description

- Start with a prior over transition probabilities $T(s, a, s')$, maintain the posterior (update them) as evidence comes in.
- Now we can reason about more/less likely MDPs, instead of just possible MDPs or a single 'best guess' MDP.
- Can plan in the space of posteriors to:
  - evaluate the likelihood of any possible outcome of an action.
  - model how that outcome will change the posterior.

Motivation (1)

Resolves 'classic' RL dilemma:

- maximise immediate rewards (exploit), or
- maximise info gain (explore)?

Wrong question!

$\rightarrow$ Single objective: maximise expected rewards up to the horizon (as a weighted average over the possible futures).
(implicitly trades-off exploration with exploitation optimally)

Motivation (2)

More Pros:

- Prior information is easily used, can start planning straight away by running a full backup.
- Easy to explicit encoding of prior knowledge / domain assumptions.
- Easy to update belief if using conjugate priors, as we collect evidence

Cons:

- Computationally intractable except in special cases (bandits, short horizons)

Bayesian RL as a POMDP (1)

- Let $\theta_{sas'}$ denotes unknown MDP parameter $T(s, a, s') = P(s'|s, a) \in [0, 1]$
  Let $b(\theta)$ be the agent's prior belief over all unknown parameters $\theta_{sas'}$
- [Duff 2002]: Define hybrid state: $S_p = S$ (certain) $\times \theta_{sas'}$ (uncertain). Cast Bayesian RL as a Partially Observable Markov Decision Process (POMDP) $\mathscr{P} =< S_p, A_p, O_p, T_p, Z_p, R_p, \gamma, b_p^0 >$
- Use favourite POMDP solution technique. This provides a Bayes Optimal policy in our original state space.

Bayesian RL as a POMDP (2)

$$
\begin{aligned}
S_p &= S \times \theta, \text{ hybrid states of known } S \text{ and all unknown } \theta_{sas'} \\
A_p &= A, \text{ original action set (unchanged)} \\
O_p &= S: \text{ observation space}
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$$
\begin{aligned}
T_p(s, \theta_{sas'}, a, s', \theta'_{sas'}) &= P(s', \theta'_{sas'} | s, \theta_{sas'}, a) \\
&= P(\theta'_{sas'} | \theta_{sas'}) P(s' | s, \theta_{sas'}, a) \\
&= \delta(\theta'_{sas'} - \theta_{sas'}) \theta_{sas'} \text{ , assuming } \theta_{sas'} \text{ is stationary} \\
R_p(s, \theta_{sas'}, a, s', \theta'_{sas'}) &= R(s, a, s') \\
Z_p(s', \theta'_{sas'}, a, o) &= P(o | s', \theta'_{sas'}, a) = \delta(o - s') \text{ , as observation is } s'
\end{aligned}
$$

$T(.)$: transition probability (known), $R(.)$: reward distribution, $Z(.)$: observation function

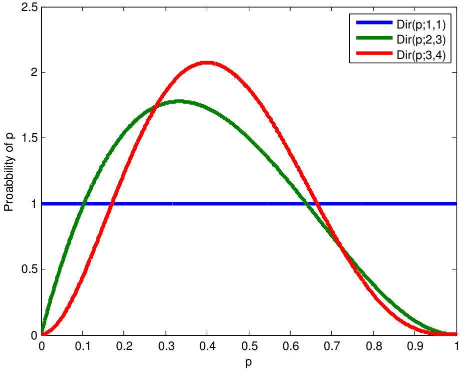
Bayesian Inference

let $b(\theta)$ be the agent's current (prior) belief over all unknown parameters $\theta_{sas'}$. For each $\{s, a, s'\}$ transition observed, the belief is updated accordingly:

$$\begin{aligned} b_{sas'}(\theta) &\propto b(\theta)P(s'|\theta_{sas'}, s, a) \\ &= b(\theta)\theta_{sas'} \\ (\textit{posterior}) &\propto (\textit{prior}) \times (\textit{likelihood}) \end{aligned}$$

Common Prior: Dirichlet Distribution

$Dir(\theta_{sa}; n_{sa}) = \frac{1}{B(n_{sa})} \prod_{s'} (\theta_{sas'})^{n_{sas'}-1}$
suitable for discrete state spaces
The Dirichlet distribution is a conjugate prior to a multinomial likelihood distribution (counts # a from s reached s'). Thus easy closed for Bayes updates.



Bayesian Inference: Discrete MDPs

$b_{sas'}(\theta) \propto b(\theta)\theta_{sas'}$

For discrete MDPs, we can define $\theta_{sa} = P(.|s,a)$, as a multinomial.

Choosing prior $b(\theta)$ form as a product of Dirichlets $\prod_{s,a} Dir(\theta_{sa}; n_{sa}) \propto \prod_{s,a} \prod_{s'} (\theta_{sas'})^{n_{sas'}-1}$, the posterior / updated-belief retains the same form:

$$\begin{aligned} b_{sas'}(\theta) &\propto (\prod_{\hat{s},\hat{a}} Dir(\theta_{\hat{s}\hat{a}}; n_{\hat{s}\hat{a}}))\theta_{sas'} \\ &\propto \prod_{\hat{s},\hat{a}} Dir(\theta_{\hat{s}\hat{a}}; n_{\hat{s}\hat{a}} + \delta_{\hat{s},\hat{a},\hat{s}'}(s,a,s')) \end{aligned} \tag{1}$$

(where n_{sa} is a vector of hyperparameters $n_{sas'}$, the $\#$ $\{s,a,s'\}$ transitions observed)

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(where n_{sa} is a vector of hyperparameters $n_{sas'}$, the $\#$ $\{s, a, s'\}$ transitions observed)
$\rightarrow$ So belief updated by incrementing corresponding $n_{sas'}$

Factoring structural priors

Can transition dynamics can be jointly expressed as a function of a smaller number of parameters?

Parameter *tying* is a special case of knowing $\theta_{sas'} = \theta_{\hat{s}\hat{a}\hat{s}'}$.

- realistic, real-life action outcomes from one state often generalise

- useful, speeds up convergence / less trials required $\rightarrow$ mitigates expensive hardware collisions etc.

Factoring structural priors: Example (1)

Taxi example: [Dietterich 1998]

- Goal: pick up passenger and drop at destination.
- States: 25 taxi location $\times$ 4 pickup locations $\times$ 4 dropoff destinations
- Actions: N, S, E, W, pickup, dropoff
- Rewards: $+20$ for successful delivery of passenger, -10 for illegal pickup or dropoff, -1 otherwise.

$\#\theta_{sa} = |S| \times |A| = 400 \times 6 = 2400.$

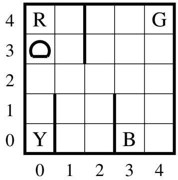

Figure: possible pickup, dropoff locations: R, Y, G, B

Factoring structural priors: Example (2)

We can factor θ_{sa}: We know *a priori* that navigation to pickup location is independent of dropoff-destination! Furthermore, navigation task is independent of purpose (pickup or dropoff).

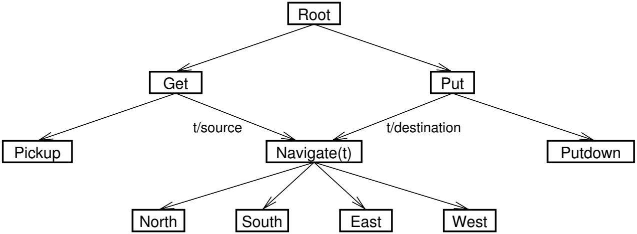

Figure: navigation as a subroutine

$\rightarrow$ $\#$ states required to learn navigation: $25 \times 4 = 100 < 400$.
Using a factored DBN model to generalises transitions for multiple states, we quarter the $\#$ of θ_{sa} to learn.

Value Optimisation

Classic RL Bellman equation:
$Q^*(s,a) = \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma \max_{a'} Q^*(s',a')]$

POMDP Bellman equation, in BRL context:

$Q^*(s,b,a) = \sum_{s'} P(s'|s,b,a)[R(s,a,s') + \gamma \max_{a'} Q^*(s', \mathbf{b_{sas'}}, a')]$

The Bayes-optimal policy is $\pi^*(s,b) = argmax_a Q^*(s,b,a)$, which maximises the predicted reward up to the horizon, over a weighted average of all the possible futures.

Big Picture

Task: solve:
$Q^*(s,b,a) = \sum_{s'} P(s'|s,b,a)[R(s,a,s') + \gamma \max_{a'} Q^*(s', b_{sas'}, a')]$

Challenge: Size of $s \times b_{sas'}$ space grows exponentially with number of $\theta_{sas'}$ parameters $\rightarrow$ Bayes-Optimal solution intractable.

Solutions: approximate $Q^*(s,b,a)$ via:

- discretisation
- exploration bonuses [BEB, Kolter 2009]
- myopic value of info [Bayesian Q-learning, Dearden 1999]
- sample beliefs [Bayesian Forward Search Sparse Sampling, Littman 2012]
- sample MDPs, update occasionally [Thompson Sampling, Strens 2000]

Algorithm: BEETLE (1)

[Poupart et al. 2006]

Exploits piecewise linear and convex property of POMDP value function [Sondik 1971].

- Sample a set of reachable (s, b) pairs by simulating a random policy.
- Uses Point Based Value Iteration (PBVI) [Pineau 2003] to approximate value iteration, by tracking value + derivative of sampled belief points. Proves α-functions (one per sampled belief) in Bayesian RL are a set of multivariate polynomials of $\theta_{sas'}$, and $V_s^*(\theta) = max_i poly_i(\theta)$.
- Scalable, has a closed form value representation under Bellman backups.

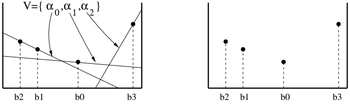

Figure 1: POMDP value function representation using PBVI (on the left) and a grid (on the right).

Algorithm: BEETLE (2)

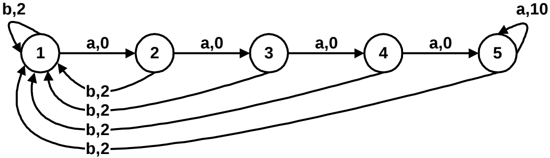

Figure 1. The "Chain" problem

Figure: [Strens 2002]

Table 1. Expected total reward for chain and handwashing problems. na-m indicates insufficient memory.

problem	|S|	|A|	free params	optimal (utopic)	discrete POMDP	exploit	Beetle	Beetle time (minutes) precomputation	optimization
chain_tied	5	2	1	3677	3661 ± 27	3642 ± 43	3650 ± 41	0.4	1.5
chain_semi	5	2	2	3677	3651 ± 32	3257 ± 124	3648 ± 41	1.3	1.3
chain_full	5	2	40	3677	na-m	3078 ± 49	1754 ± 42	14.8	18.0
handw_tied	9	2	4	1153	1149 ± 12	1133 ± 12	1146 ± 12	2.6	11.8
handw_semi	9	2	8	1153	990 ± 8	991 ± 31	1082 ± 17	3.4	52.3
handw_full	9	6	270	1083	na-m	297 ± 10	385 ± 10	125.3	8.3

Figure: [Poupart 2006]

PAC-MDP and Bayesian RL algorithms

Model Based Interval Estimation with Exploration Bonus MBIE-EB (PAC-MDP) and Bayesian Exploration Bonus BEB (Bayesian RL) will be compared. Both:

- count how many times each transition (s, a, s') has happened: $\alpha(s, a, s')$;
- use counts to produce an estimate of the underlying MDP;
- add exploration bonus to the reward for (s, a) pair if not visited enough;
- act greedily with respect to this modified MDP.

Bellman's optimality equations with exploration bonus

Let $\alpha_0(s,a) = \sum_{s'} \alpha(s,a,s')$ and $b = \{\alpha(s,a,s')\}$. Then
$P(s'|b,s,a) = \frac{\alpha(s,a,s')}{\alpha_0(s,a)}$
Attempts to maximize :

- BEB

$$\tilde{V}^*_H(b,s) = \max_a \left\{ R(s,a) + \frac{\beta}{1+\alpha_0(s,a)} \right.$$
$$\left. + \sum_{s'} P(s'|b,s,a)\tilde{V}^*_{H-1}(s') \right\}$$

- MBIE-EB

$$\tilde{V}^*_H(s) = \max_a \left\{ R(s,a) + \frac{\beta}{\sqrt{\alpha_0(s,a)}} \right.$$
$$\left. + \sum_{s'} P(s'|b,s,a)\tilde{V}^*_{H-1}(b,s') \right\}$$

Near Bayesian optimal

Approximate Bayes-Optimal:
If $\mathcal{A}_t$ denotes the policy followed by the algorithm at time t, then with probability greater than $1 - \delta$

$$V_t^{\mathcal{A}}(b_t, s_t) \geq V^*(b_t, s_t) - \epsilon$$

where $V^*(b, s)$ is the value function for a Bayes-optimal strategy.

Near Bayes Optimal:
With probability $\geq 1 - \delta$, an agent follows an approximate Bayes-optimal policy for all but a "small" number of steps, which is polynomial in quantities representing the system.

BEB near Bayesian optimality

Theorem (Kolter and Ng, 2009)

Let $\mathcal{A}_t$ denote the policy followed by the BEB algorithm (with $\beta = 2H^2$) at time t, and let s_t and b_t be the corresponding state and belief. Also suppose we stop updating the belief for a state-action pair when $\alpha_0(a, s) > 4H^3/\epsilon$. Then with probability at least $1 - \delta$,

$$V_H^{\mathcal{A}_t}(b_t, s_t) \geq V_H^*(b_t, s_t) - \epsilon$$

i..e, the algorithm is ϵ-close to the optimal Bayesian policy for all but

$$m = O\left(\frac{|S||A|H^6}{\epsilon^2} \log \frac{|S||A|}{\delta}\right)$$

time steps.

PAC-MDP

Theorem (Strehl, Li and Littman 2006)

Let $\mathcal{A}_t$ denote the policy followed by some algorithm. Also, let the algorithm satisfy the following properties, for some input ϵ:

- *acts greedily for every time step t;*
- *is optimistic ($V_t(s) \geq V_t^*(s) - \epsilon$)*
- *has bounded learning complexity (bounded number of action-value estimate updates and number of escape events)*
- *is accurate ($V_t(s) - V_{M_{Kt}}^{\pi_t}(s) \leq \epsilon$)*

Then, with probability greater than $1 - \delta$, for all but

$$\tilde{O}\left(\frac{|S|^2|A|H^6}{\epsilon^2}\right)$$

time steps, the algorithm follows an 4ϵ optimal policy.

Rate of Decay

Theorem (Kolter and Ng, 2009)

Let $\mathcal{A}_t$ denote the policy followed an algorithm using any (arbitrary complex) exploration bonus that is upper bounded by

$$\frac{\beta}{\alpha_0(s, a)^p}$$

for some constant β and $p > 1/2$. Then $\exists$ some MDP M and $\epsilon_0(\beta, p)$, s.t. with probability greater than $\delta_0 = 0.15$,

$$V_H^{\mathcal{A}_t}(s_t) < V_H^*(s_t) - \epsilon_0$$

will hold for an unbounded number of steps.

The proof uses the following inequality.

Lemma (Slud's inequality)

Let $X_1, ...X_n$ be i.i.d. Bernoulli random variables, with mean $\mu > 3/4$. Then

$$P\left(\mu - \frac{1}{n}\sum_{i=1}^{n} X_i > \epsilon\right) \geq 1 - \Phi\left(\frac{\epsilon\sqrt{n}}{\sqrt{\mu(1-\mu)}}\right)$$

Proof

The lower bound on the probability that the algorithm's estimate of the reward for playing a_1 plus the exploration bonus is pessimistic by at least β/n^p:

$$
\begin{aligned}
&P\left(3/4 - \frac{1}{n}\sum_{i=1}^{n} r_i - f(n) \geq \frac{\beta}{n^p}\right) \\
&\geq P\left(3/4 - \frac{1}{n}\sum_{i=1}^{n} r_i \geq \frac{2\beta}{n^p}\right) \\
&\geq 1 - \Phi\left(\frac{8\beta}{\sqrt{3}n^{p-1/2}}\right)
\end{aligned}
$$

Proof

Set

$$n \geq \left(\frac{8\beta}{\sqrt{3}}\right)^{\frac{2}{2p-1}}$$

and

$$\epsilon_0(\beta, p) = \beta / \left(\left(\frac{8\beta}{\sqrt{3}}\right)^{\frac{2p}{2p-1}}\right)$$

So at stage n with probability at least 0.15, action a_2 will be preferred over a_1 and the agent will stop exploring $\Rightarrow$ the algorithm will be more than ϵ suboptimal for an infinite number of steps, for any $\epsilon \geq \epsilon_0$.

Conclusions

- Both algorithms use the same intuition: in order to perform well, we want to explore enough that we learn an accurate model of the system;
- For PAC-MDP, exploration bonus cannot shrink at a rate faster that $\frac{1}{2}$ or they fail to be near optimal, and slow rate of decay results in more exploration;
- BEB reduces the amount of exploration needed, which allows us to achieve lower sample complexity and use greedier exploration method;
- a near Bayesian optimal policy is not near-optimal: the optimality is considered with respect to the Bayesian policy, rather than the optimal policy for some fixed MDP.