Image Processing

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Homework 2

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1. 请计算如下两个向量与矩阵的卷积计算结果

(*1*) $[1\ 2\ 3\ 4\ 5\ 4\ 3\ 2\ 1] * [2\ 0\ -2]$

(*2*) $\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 3 & 2 & 0 & 4 \\ 1 & 0 & 3 & 2 & 3 \\ 0 & 4 & 1 & 0 & 5 \\ 2 & 3 & 2 & 1 & 4 \\ 3 & 1 & 0 & 4 & 2 \end{bmatrix}$

(*1*) 解：*"full" mode:* $[2\ 4\ 4\ 4\ 4\ 0\ -4\ -4\ -4\ -4\ -2]$

"same" mode: $[4\ 4\ 4\ 4\ 0\ -4\ -4\ -4\ -4]$

(*2*) 解：*"full" mode:* $\begin{bmatrix} -1 & -3 & -1 & 3 & -2 & 0 & 4 \\ -3 & -6 & -4 & 4 & -4 & 2 & 11 \\ -3 & -7 & -6 & 3 & -6 & 4 & 15 \\ -3 & -11 & -4 & 8 & -10 & 3 & 17 \\ -7 & -11 & 2 & 5 & -10 & 6 & 15 \\ -8 & -5 & 6 & -4 & -6 & 9 & 8 \\ -3 & -1 & 3 & -3 & -2 & 4 & 2 \end{bmatrix}$

"same" mode: $\begin{bmatrix} -6 & -4 & 4 & -4 & 2 \\ -7 & -6 & 3 & -6 & 4 \\ -11 & -4 & 8 & -10 & 3 \\ -11 & 2 & 5 & -10 & 6 \\ -5 & 6 & -4 & -6 & 9 \end{bmatrix}$

2. 卷积是用来描述线性移不变系统的。

(*1*) 请用数学语言定义描述定义什么是线性移不变系统？

(*2*) 请证明卷积具有交换性；

(*3*) 卷积是一种线性运算；

(*4*) 请证明任何函数与单位脉冲函数进行卷积时，结果为该函数本身。

(*5*) *(选作)* 说明卷积是线性移不变系统的数学描述

(注意：证明采用连续或者离散形式都可以)

解：(*1*) 对某一系统 $X(t) \xrightarrow{F} Y(t)$，即 $Y(t) = F\{X(t)\}$，若满足:(1)*Linear* $\forall a, b \in \mathbb{R}, aY(t_1) + bY(t_2) = F\{aX(t_1) + bX(t_2)\}$；(2)*Shift Invariance* $\forall T \in \mathbb{R}, Y(t-T) = F\{X(t-T)\}$，则称该系统 $X(t) \xrightarrow{F} Y(t)$ 为线性移不变系统。

(*2*) 设任意两个连续信号 $F(x), G(x)$，即证 $F(x) * G(x) = G(x) * F(x)$

卷积定义为 $(F(x) * G(x))(t) \triangleq \int_{-\infty}^{+\infty} F(x)G(t-x)dx$，

则

$$
\begin{aligned}
(G(x) * F(x))(t) &\triangleq \int_{-\infty}^{+\infty} G(x)F(t-x)dx \\
&= \int_{-\infty}^{+\infty} G(t-\tau)F(\tau)d(t-\tau) \quad (let\ \tau = t-x) \\
&= -\int_{+\infty}^{-\infty} G(t-\tau)F(\tau)d\tau \\
&= \int_{-\infty}^{+\infty} F(\tau)G(t-\tau)d\tau \\
&= (F(\tau) * G(\tau))(t)
\end{aligned}
$$

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(*3*) 设 $a, b \in \mathbb{R}$ 和一个额外的连续信号 $H(x)$

$$
\begin{aligned}
(aF(x) * bG(x))(t) &= \int_{-\infty}^{+\infty} aF(x)bG(t-x)dx \\
&= ab\int_{-\infty}^{+\infty} F(x)G(t-x)dx \\
&= ab(F(x) * G(x))(t)
\end{aligned}
$$

$$
\begin{aligned}
(F(x) * (G(x) + H(x)))(t) &= \int_{-\infty}^{+\infty} F(x)(G(t-x) + H(t-x))dx \\
&= \int_{-\infty}^{+\infty} F(x)G(t-x)dx + \int_{-\infty}^{+\infty} F(x)H(t-x)dx \\
&= (F(x) * G(x))(t) + (F(x) * H(x))(t)
\end{aligned}
$$

故卷积是一种线性运算

(*4*) 设单位脉冲信号为 $\delta(x)(\delta(0)=1)$

$$\begin{aligned}(F(x)*\delta(x))(t)=(\delta(x)*F(x))(t)&=\int_{-\infty}^{+\infty}\delta(x)F(t-x)dx\\&=\int_{-\infty}^{+\infty}\delta(x)F(t-0)dx\\&=F(t)\int_{-\infty}^{+\infty}\delta(x)dx\\&=F(t)\end{aligned}$$

(*5*) 卷积运算满足*:*(1)*Linear* $\forall a,b\in\mathbb{R},(aF(x)*bG(x))(t)ab(F(x)*G(x))(t)$; (2)*Shift Invariance* $\forall T\in\mathbb{R},(F(x)*G(x-t_0))(t)=(F(x)*G(x))(t-t_0)$，因此卷积为线性移不变系统。

3. 完成课本数字图像处理第二版 *116* 页，习题 *3.25*，即拉普拉斯算子具有理论上的旋转不变性。

解：定义二维空间上的拉普拉斯算子为：$\nabla^2 f\triangleq\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}$，设二维平面上一点 (x,y)，经过旋转角度 θ 后得到点 (x',y')，由极坐标易得：$x=x'\cos\theta-y'\sin\theta,y=x'\sin\theta+y'\cos\theta$，因此对点 (x',y')，其拉普拉斯算子为：

$$\begin{aligned}\nabla^2 f=\frac{\partial^2 f}{\partial x'^2}+\frac{\partial^2 f}{\partial y'^2}&=\frac{\partial}{\partial x'}(\frac{\partial f}{\partial x'})+\frac{\partial}{\partial y'}(\frac{\partial f}{\partial y'})\\&=\frac{\partial}{\partial x'}(\frac{\partial f}{\partial x}\frac{\partial x}{\partial x'}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial x'})+\frac{\partial}{\partial y'}(\frac{\partial f}{\partial x}\frac{\partial x}{\partial y'}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial y'})\\&=\frac{\partial}{\partial x'}(\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta)+\frac{\partial}{\partial y'}(-\frac{\partial f}{\partial x}\sin\theta+\frac{\partial f}{\partial y}\cos\theta)\\&=\frac{\partial}{\partial x}(\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta)\frac{\partial x}{\partial x'}+\frac{\partial}{\partial y}(\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta)\frac{\partial y}{\partial x'}+\\&\quad\frac{\partial}{\partial x}(-\frac{\partial f}{\partial x}\sin\theta+\frac{\partial f}{\partial y}\cos\theta)\frac{\partial x}{\partial y'}+\frac{\partial}{\partial y}(-\frac{\partial f}{\partial x}\sin\theta+\frac{\partial f}{\partial y}\cos\theta)\frac{\partial y}{\partial y'}\\&=\frac{\partial}{\partial x}(\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta)\cos\theta+\frac{\partial}{\partial y}(\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta)\sin\theta-\\&\quad\frac{\partial}{\partial x}(-\frac{\partial f}{\partial x}\sin\theta+\frac{\partial f}{\partial y}\cos\theta)\sin\theta+\frac{\partial}{\partial y}(-\frac{\partial f}{\partial x}\sin\theta+\frac{\partial f}{\partial y}\cos\theta)\cos\theta\\&=\frac{\partial^2 f}{\partial x^2}\cos^2\theta+2\frac{\partial^2 f}{\partial x\partial y}\sin\theta\cos\theta+\frac{\partial^2 f}{\partial y^2}\sin^2\theta+\frac{\partial^2 f}{\partial x^2}\sin^2\theta-\\&\quad 2\frac{\partial^2 f}{\partial x\partial y}\sin\theta\cos\theta+\frac{\partial^2 f}{\partial y^2}\cos^2\theta\\&=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\end{aligned}$$

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