**Information and Coding Theory**

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# Homework 2-1

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1. **熵的上凸性：** 对 $\forall 0 \le \theta \le 1, H(\theta p_1 + (1-\theta)p_2) \ge \theta H(p_1) + (1-\theta)H(p_2)$

   证明：

$$
\begin{aligned}
&H(\theta p_1 + (1-\theta)p_2) - H(p_1) - (1-\theta)H(p_2) \\
&= -\theta \sum [\theta p_1 + (1-\theta)p_2] \log(\theta p_1 + (1-\theta)p_2) + \theta_1 \log p_1 + (1-\theta) \sum p_2 \log p_2 \\
&= \theta \sum p_1 \log \frac{p_1}{\theta p_1 + (1-\theta)p_2} + (1-\theta) \sum p_2 \log \frac{p_2}{\theta p_1 + (1-\theta)p_2} \\
&= -\theta \sum p_1 \log \frac{\theta p_1 + (1-\theta)p_2}{p_1} - (1-\theta) \sum p_2 \log \frac{\theta p_1 + (1-\theta)p_2}{p_2} \\
&\ge -(\theta \cdot \log 1 + (1-\theta) \cdot \log 1) = 0 \quad (Jenson\ Inequality)
\end{aligned}
$$

   证毕

2. 求证：当固定输入分布 $p(x)$ 给定时，$I(X;Y)$ 是条件概率 $p(y|x)$ 的严格下凸函数，即 $\alpha f(x_1) + \beta f(x_2) > f(\alpha x_1 + \beta x_2), \alpha + \beta = 1$

   证明：对于固定的信源，有 $p_1(x) = p_2(x) = p(x)$，且 $p(Y|X) = \alpha p_1(Y|X) + \beta p_2(Y|X)$

$$
\begin{aligned}
&\alpha I(X_1;Y_1) + \beta I(X_2;Y_2) - I(X|Y) \\
&= \alpha \sum_{x,y} p(x)p_1(y|x) \log \frac{p_1(y|x)}{p(x)} + \beta \sum_{x,y} p(x)p_2(y|x) \log \frac{p_2(y|x)}{p(x)} - \sum_{x,y} p(x)p(y|x) \log \frac{p(y|x)}{p(x)} \\
&= \alpha \sum_{x,y} p(x)p_1(y|x) \log \frac{p_1(y|x)p(x)}{p(y|x)p(x)} + \beta \sum_{x,y} p(x)p_2(y|x) \log \frac{p_2(y|x)p(x)}{p(y|x)p(x)} \\
&= \alpha \sum_{y} p_1(y) \log \frac{p_1(y)}{p(y)} + \beta \sum_{y} p_2(y) \log \frac{p_2(y)}{p(y)} \\
&\ge -(\alpha \log \sum_{y} p_1(y) \frac{p(y)}{p_1(y)} + \beta \log \sum_{y} p_2(y) \frac{p(y)}{p_2(y)}) = 0 \quad (Jenson\ Inequality)
\end{aligned}
$$

*3.* 已知 $XY$ 构成的联合概率空间为：$\begin{pmatrix} XY \\ P(XY) \end{pmatrix} = \begin{pmatrix} 00 & 01 & 10 & 11 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix}$，其中 $X, Y \in \{0, 1\}$，计算条件熵 $H(X|Y)$

由题可得：

$$P(y) = \sum_x P(Y|x)P(x) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$$

$$\begin{aligned} H(X|Y) &= -\sum_x \sum_y P(xy) \log P(x|y) = -\sum_x \sum_y P(xy) \log \frac{P(xy)}{P(y)} \\ &= -\frac{1}{8} \log \frac{1}{4} - \frac{3}{8} \log \frac{3}{4} - \frac{3}{8} \log \frac{3}{4} - \frac{1}{8} \log \frac{1}{4} \\ &= 0.244 \end{aligned}$$