Structural Vector Autoregressions Identification & Short-Run Restrictions

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Structural and Reduced Form VARs

- Structural VAR:

$$B_0 y_t = B_1 y_{t-1} + \cdots + B_p y_{t-p} + w_t$$

- Reduced form VAR:

$$y_t = A_1 y_{t-1} + \cdots + A_p y_{t-p} + u_t$$

- $A_i = B_0^{-1} B_i$, for $i = 1 \ldots, p$.
- Relationship between reduced form residuals and structural shocks:

$$u_t = B_0^{-1} w_t$$

- More intuitively, B_0 transforms correlated reduced form residuals into uncorrelated structural shocks.

$$B_0 u_t = w_t$$

Identification Problem

- Identification problem: How to recover structural representation from an estimated reduced-form model.

- Knowledge of B_0 allows to recover:
  - Structural shocks: $B_0 u_t = w_t$
  - Structural parameters: $B_i = B_0 A_i$

- Intuition: The hard part is to understand what moves variables at time t. Once I know structure at time t (i.e. I have knowledge of B_0 or w_t), I can easily recover lagged structure, as variables in the past are predetermined (ie they do not depend on what happens today).

- The matrix B_0 that maps the reduced form into a structural model is not unique.

Identification Problem

- Recall that B_0 satisfies the following conditions:

$$\mathbb{E}[u_t u_t'] \equiv \Sigma_u = B_0^{-1} B_0^{-1'} \tag{1}$$

- Equation (1) describes a system of K^2 nonlinear equations...
- ... and B_0 has K^2 elements to compute.
- However, Σ_u is a symmetric matrix.
  - Σ_u has only $K(K+1)/2$ distinct elements.
  - Equation (1) has only $K(K+1)/2$ independent equations...
  - ... which can pin down only $K(K+1)/2$ elements of B_0.
- That is, Equation (1) describes an under–identified system.
- To obtain a unique solution, we fix $K(K-1)/2$ elements of B_0...
- ... to have a system of $K(K+1)/2$ equations in $K(K+1)/2$ unknown coefficients.

Identification problem: A Bivariate example

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} b_{11,0} & b_{12,0} \\ b_{21,0} & b_{22,0} \end{bmatrix}^{-1} \times \begin{bmatrix} b_{11,0} & b_{12,0} \\ b_{21,0} & b_{22,0} \end{bmatrix}^{-1'}$$

The system of equations is:

$$\frac{b_{22,0}^2 + b_{12,0}^2}{det(B_0)^2} = \sigma_{11} \quad (2)$$

$$-\frac{(b_{22,0}b_{21,0} + b_{12,0}b_{11,0})}{det(B_0)^2} = \sigma_{12} \quad (3)$$

$$-\frac{(b_{22,0}b_{21,0} + b_{12,0}b_{11,0})}{det(B_0)^2} = \sigma_{21} \quad (4)$$

$$\frac{b_{11,0}^2 + b_{21,0}^2}{det(B_0)^2} = \sigma_{22} \quad (5)$$

Since $\sigma_{12} = \sigma_{21}$, equations (2) and (3) are identical! Need to fix an element of B_0 to find unique solution for the system.

Conditions for Identification

- Previous discussion suggests that we need to impose $K(K-1)/2$ restrictions to find a unique solution for elements B_0.
- This is only a necessary condition for identification.
- That is, even if we impose $K(K-1)/2$ restrictions, we might fail to find a unique solution.
- Rubio-Ramirez, Waggoner, and Zha (2010) derive necessary and sufficient conditions for unique identification: counting condition + restrictions should follow this pattern:

$$q_k = K - k, \ \textit{for } 1 \leq k \leq K$$

  where q_k denote the number of restrictions imposed on equation k.
- Important: Number of restrictions need to be sequential and cannot be the same across equations.

Cholesky (Recursive) Identification

- Cholesky (or recursive) identification is a very popular identification strategy.
- Define P as a lower triangular $K \times K$ matrix such that:

$$\Sigma_u = PP'$$

- P os the lower-triangular Cholesky decomposition of Σ_u.
- Example for a 3-equation model:

$$P = \begin{bmatrix} p_{11} & 0 & 0 \\ p_{21} & p_{22} & 0 \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

- From $\Sigma_u = B_0^{-1} B_0^{-1'}$ it follows that $B_0^{-1'} = P$ is one possible solution to recover the structural representation.

Cholesky (Recursive) Identification

- In the previous slide we set B_0^{-1} to be the lower-triangular decomposition of Σ_u.
- Interpretation: B_0^{-1} characterizes the impact response of y_t to the structural shocks w_t: $y_t = B_0^{-1} w_t$.
- Alternatively, we can set B_0 to be the lower-triangular decomposition of Σ_u:

$$B_0 = \tilde{P} \equiv \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

- Interpretation: B_0 characterizes the contemporaneous relationship among the endogenous variables: $B_0 y_t = w_t$.
- Note that $\tilde{P}^{-1} = P$.

Cholesky (Recursive) Identification

- Applying a Cholesky decomposition is appropriate only if the recursive structure can be justified in economic terms!
- Cholesky imposes a particular causal chain through zero restrictions.
- The chain depends on how variables are ordered in the VAR model.
- 3-equation monetary VAR:

$$
\begin{aligned}
b_{11,0}\Delta x_t + b_{12,0}\pi_t + b_{13,0} i_t &= b_{11,1}\Delta x_{t-1} + b_{12,1}\pi_{t-1} + b_{13,1} i_{t-1} + w_{1t} \\
b_{21,0}\Delta x_t + b_{22,0}\pi_t + b_{23,0} i_t &= b_{21,1}\Delta x_{t-1} + b_{22,1}\pi_{t-1} + b_{23,1} i_{t-1} + w_{2t} \\
b_{31,0}\Delta x_t + b_{32,0}\pi_t + b_{33,0} i_t &= b_{31,1}\Delta x_{t-1} + b_{32,1}\pi_{t-1} + b_{33,1} i_{t-1} + w_{3t}
\end{aligned}
$$

Cholesky (Recursive) Identification

- Baseline variable ordering: $[\Delta x_t, \pi_t, i_t]$:

$$\begin{aligned} \Delta x_t &= \sigma_1 w_{1t} \\ \pi_t &= \psi_{21,0} \Delta x_t + \sigma_2 w_{2t} \\ i_t &= \psi_{31,0} \Delta x_t + \psi_{32,0} \pi_t + \sigma_3 w_{3t} \end{aligned}$$

- Equivalently:

$$\begin{aligned} \Delta x_t &= p_{11} w_{1t} \\ \pi_t &= p_{12} w_{1t} + p_{22} w_{2t} \\ i_t &= p_{31} w_{1t} + p_{32} w_{2t} + p_{33} w_{3t} \end{aligned}$$

- This Cholesky ordering has the following economic interpretation:
  - Real activity does not respond contemporaneously to inflation and the interest rate $\Leftrightarrow$ Δx_t responds only to the first shock (activity shock).
  - Inflation responds contemporaneously only to real activity $\Leftrightarrow$ π_t responds to the first two shocks (activity and price shocks).
  - Interest rate responds contemporaneously to activity and inflation $\Leftrightarrow$ i_t responds to all shocks.

Cholesky (Recursive) Identification

- An alternative ordering: $[i_t, \Delta x_t, \pi_t]$:

$$\begin{aligned} i_t &= \sigma_1 w_{1t} \\ \Delta x_t &= \psi_{21,0} i_t + \sigma_2 w_{2t} \\ \pi_t &= \psi_{31,0} i_t + \psi_{32,0} \Delta x_t + \sigma_3 w_{3t} \end{aligned}$$

- Changing ordering is simply a convenient way of imposing zeros for different entries of B_0 or B_0^{-1}.
- Economic justification for baseline ordering:
  - Monetary reaction function: monetary policy sets interest rate with reference to the current period's output and prices.
- Economic justification for alternative ordering:
  - Information friction: monetary policy reacts immediately only to variables that can observe without delay.

Criticisms to Baseline Ordering

- Model does not allow for contemporaneous feedback within quarter from monetary policy shocks to output and inflation.
  - Use of monthly data could attenuate this concern.
- Monetary policy could respond systematically to more variables that just output and inflation.
  - Add more variables!
- The monetary policy equation might have changed over time.
  - Estimate over subsamples with more stable conduct of monetary policy.
- The VAR model is linear and does not allow for the zero lower bound on the interest rate.
  - Estimate nonlinear model or use different variable to characterize monetary policy.
- Does not allow for policy experiments that change the systematic component (Lucas critique).

Sensitivity Analysis for Recursive Ordering

- Why thinking hard about the recursive ordering? Why not just trying different orderings?
- In a 3-equation VAR, six possible recursive orderings...
- In a K equation VAR, there are $K!$ orderings!
- We are confident about imposing zero restrictions, but we do not know where to impose them... not credible.
- Even if we can prove robustness across orderings, there is no reason for the model to be recursive.
- EXAMPLE:

$$\begin{bmatrix} p_t \\ q_t \end{bmatrix} = \begin{bmatrix} 1 & -0.5 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} w_t^d \\ w_t^s \end{bmatrix}$$