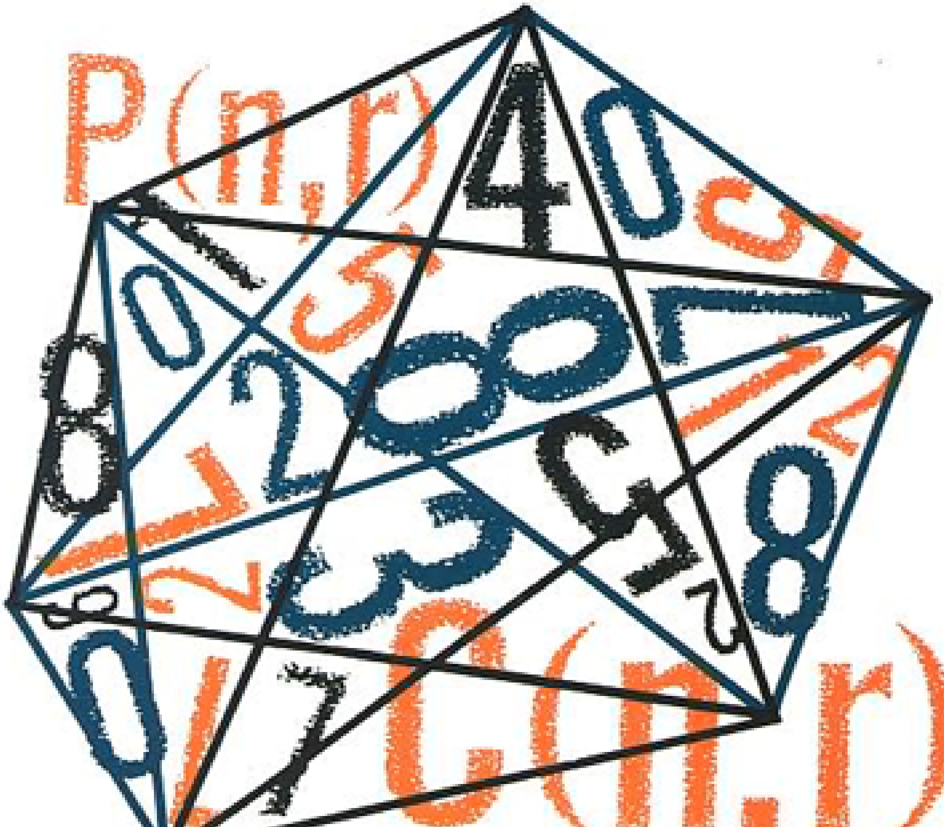
MATHEMATICS OF CHOICE

HOW TO COUNT WITHOUT COUNTING

IVAN NIVEN



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MATHEMATICS OF CHOICE
OR
HOW TO COUNT WITHOUT COUNTING

by

Ivan Niven

University of Oregon

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The best way to learn mathematics is to *do* mathematics, and each book includes problems, some of which may require considerable thought. The reader is urged to acquire the habit of reading with paper and pencil in hand; in this way mathematics will become increasingly meaningful to him.

For the authors and editors this is a new venture. They wish to acknowledge the generous help given them by the many high school teachers and students who assisted in the preparation of these monographs. The editors are interested in reactions to the books in tbis series and hope that readers will write to: Editorial Committee of the NML series, New York University, The Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, N. Y. 10012.

The Editors

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Contents

MATHEMATICS OF CHOICE

OR

HOW TO COUNT WITHOUT COUNTING

Preface

The subject of this book is often called "combinatorial analysis" or "combinatorics". The questions discussed are of the sort "In how many ways is it possible to . . . ? ", or variations on that theme. Permutations and combinations form a part of combinatorial analysis, a part with which the reader may be already acquainted. If so, he may be familiar with some of the material in the first three chapters.

The book is self-contained with the rudiments of algebra the only prerequisite. Summaries including all formulas are given at the ends of the chapters. Throughout the book there are many problems for the reader. In fact the entire monograph is in large measure a problem book with enough background information furnished for attacking the questions. A list of miscellaneous problems follows the final chapter. Solutions, or at least sketches of solutions, are given in the back of the book for questions of any depth, and numerical answers are given for the simpler problems.

Helpful suggestions were given by the members of the S. M. S. G. Monograph Panel, and also by Herbert S. Zuckerman. Max Bell used some of the material with his students, and forwarded their comments to me. The witty subtitle of the book was suggested by Mark Kac. For all this help I express my appreciation.

CHAPTER ONE

Introductory Questions

The purpose of this chapter is to present a few sample problems to illustrate the theme of the whole volume. A systematic development of the subject is started in the next chapter. While some of the sample questions introduced here can be solved with no theoretical background, the solution of others must be postponed until the necessary theory is developed.

The idea of this book is to examine certain aspects of the question "how many?". Such questions may be very simple; for example, "How many pages are there from page 14 to page 59, inclusive?" In some cases, the answer may be nothing more than a matter of common knowledge, as for example the number of days in October, or the number of yards in a mile. In other cases, the answer may require technical information, such as the number of chemical elements known at the present time, or the number of cubic centimeters of displacement in the engine of a certain automobile. But our concern is with questions that involve thought. They may also require some prior knowledge, which will be supplied if it is not common information. Some mathematical formulas are helpful, and these will be developed in due course. However, many problems require nothing more than a little ingenuity. We begin with such a question.

PROBLEM 1.1† In any calendar year how many Friday the thirteenths can there be? What is the smallest number possible?

† This occurs as Problem E1541 on p. 919 of the *American Mathematical Monthly*, November, 1962.

Like many other questions in this book, this problem is solved in the *Answers and Solutions* section at the end. Of course the reader is urged to try the question himself before turning to the solution provided. Problem 1.1 *can* be done by simply consulting a calendar, or rather a set of annual calendars giving all possible arrangements of the days of the year. The challenge is to solve the problem in an even simpler way by devising a system. It might be noted for example that years having 365 days can be separated into seven different types, one beginning on a Monday, one on a Tuesday, etc. Similarly there are seven different types of leap years, and so there are in all fourteen types of years for the purposes of this problem. Next a system can be devised for studying the number of Friday the thirteenths in any one of these types of years. However, we drop the analysis here, and leave the rest to the reader.

Problem 1.2 A manufacturer makes blocks for children, each block being a two inch cube whose faces are painted one of two colors, blue and red. Some blocks are all blue, some all red, and some have a mixture of blue and red faces. How many different kinds of blocks can the manufacturer make?

It is necessary to define what is meant by "different" blocks before the question has a precise meaning. We shall say that two blocks are the same if they can be put into matching positions so that corresponding faces have identical colors, that is, so that the bottom faces have the same color, the top faces the same color, the front faces the same color, etc. If two blocks are not the same in this sense, we say that they are different. For example, any two blocks with five blue faces and one red face are the same. But consider as another example two blocks with four red faces and two blue faces. Two such blocks may or may not be the same. If the two blue faces are adjacent on each block, then the blocks are the same. Or if the two blue faces are opposite on each block, they are the same. But if on one block the two blue faces are adjacent, whereas on the other block the two blue faces are opposite, then the blocks are different. See Figure 1.1.

This problem is also solved in the *Answers and Solutions* section, but again the reader is urged to solve it for himself, using the solution at the back of the book as a check against his work.

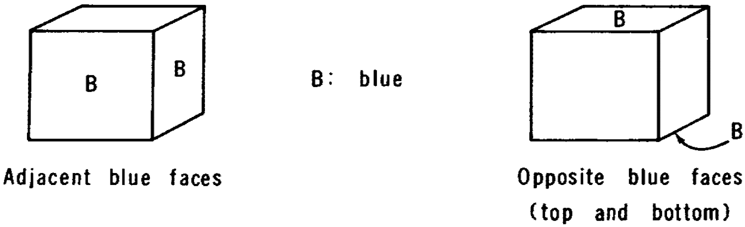


Figure 1.1

We turn now to three problems which are so much more difficult that the solutions are postponed until the needed theory is worked out.

PROBLEM 1.3 *A Path Problem.* A man works in a building located seven blocks east and eight blocks north of his home. (See Figure 1.2.) Thus in walking to work each day he goes fifteen blocks. All the streets in the rectangular pattern are available to him for walking. In how many different ways can he go from home to work, walking only fifteen blocks?

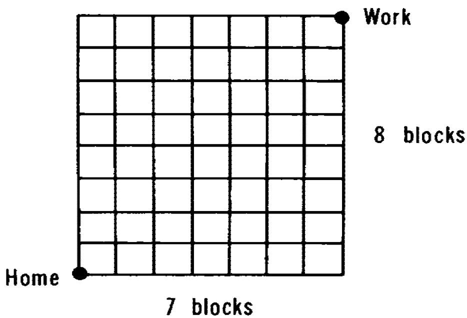


Figure 1.2

One obvious approach to this problem would be to draw diagrams of all possible paths, and then to count them. But there happen to be 6435 different paths, and so the direct approach is somewhat impractical. This problem is not very difficult if we look at it in the right way. The solution is given in Chapter 3.

We turn now to another problem whose solution must await some theoretical analysis.

PROBLEM 1.4 The governor of a state attended the centennial celebration of a famous publishing house. To express his appreciation, the publisher offered to present to the governor any selection of ten books chosen from the twenty best-sellers of the company. The governor was permitted to select ten different books from the twenty, or ten all alike (ten copies of one book), or any other combination he might prefer, provided only that the total was ten. (a) In how many ways could the governor make his selection? (b) If the governor had been requested to choose ten distinct books, in how many ways could he have made his selection?

Question (b) is easier than question (a), because (b) is a straightforward matter of choosing ten things from twenty. The number of different selections of ten things from twenty is denoted by the symbol $C(20, 10)$, and is easily evaluated as we shall see in the next chapter. The solution to part (a) of the question is given on page 59.

PROBLEM 1.5 In how many ways is it possible to change a dollar bill into coins? (Presume that the coins are in denominations 1, 5, 10, 25, and 50 cents, also known as cents, nickels, dimes, quarters and half dollars.)

This problem, like many others in this book, can be solved by simply enumerating all cases and counting them A more systematic scheme for solving it is given in Chapter 7.

We conclude this chapter by stating a basic principle about counting. It arises in such a simple question as finding the number of pages from page 14 to page 59 inclusive. The answer is 46, one more than the difference between the two integers† 14 and 59. In general, *the number of integers from* k *to* n *inclusive is* $n - k + 1$, *where* n *is presumed larger than* k, *i.e.* $n > k$

† Integers, sometimes called "whole numbers", are of three types: the positive integers or natural numbers 1, 2, 3, 4, $\cdots$, where the three dots "$\cdots$" stand for "and so on"; the negative integers -1, -2, -3, -4, $\cdots$; and 0 which is neither positive nor negative. The non-negative integers are 0, 1, 2, 3, 4, $\cdots$.

Problem Set 1

1. How many integers are there from 25 to 79 inclusive?

2. What is the 53rd integer in the sequence 86, 87, 88, $\cdots$?

3. The largest of 123 consecutive integers is 307. What is the smallest?

4. The smallest of r consecutive integers is n. What is the largest?

5. The largest of r consecutive integers is k. What is the smallest?

6. How many integers are there in the sequence $n,\ n+1,\ n+2,\ \cdots,$ $n+h$?

7. How many integers x satisfy the inequalities $12 < \sqrt{x} < 15$, that is $\sqrt{x}$ exceeds 12, but $\sqrt{x}$ is less than 15?

8. How many integers are there in the sequences

   (a) 60, 70, 80, $\cdots$, 540; (b) 15, 18, 21, $\cdots$, 144;

   (c) 17, 23, 29, 35, $\cdots$, 221?

9. How many integers between 1 and 2000 (a) are multiples of 11; (b) are multiples of 11 but not multiples of 3; (c) are multiples of 6 but not multiples of 4?

10. What is the smallest number of coins needed to pay in exact change any charge less than one dollar? (Coins are in the denominations 1, 5, 10, 25 and 50 cents.)

11. A man has 47 cents in change coming. Assuming that the cash register contains an adequately large supply of 1, 5, 10 and 25 cent coins, with how many different combinations of coins can the clerk give the man his change?

12. A man has six pairs of cuff links scrambled in a box. No two pairs are alike. How many cuff links does he have to draw out all at once (in the dark) in order to be certain to get a pair that match?

13. A man has twelve blue socks and twelve black socks scrambled in a drawer. How many socks does he have to draw out all at once (in the dark) to be certain to get a matching pair? (Any two blue socks, or any two black socks, constitute a pair.)

14. The measure in degrees of an angle of a regular polygon is an integer. How many sides can such a polygon have?

15. A man has a large supply of wooden regular tetrahedra, all the same size. (A regular tetrahedron is a solid figure bounded by four congruent equilateral triangles; see Figure 1.3.) If he paints each tri-

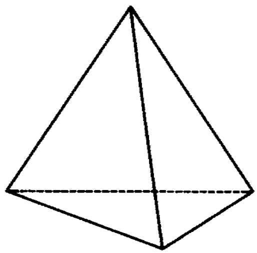


Figure 1.3

angular face in one of four colors, how many different painted tetrahedra can he make, allowing all possible combinations of colors? (Say that two blocks are different if they cannot be put into matching positions with identical colors on corresponding faces.)

16. How many paths are there from one corner of a cube to the opposite corner, each possible path being along three of the twelve edges of the cube?

17. At formal conferences of the United States Supreme Court each of the nine justices shakes hands with each of the others at the beginning of the session. How many handshakes initiate such a session?

CHAPTER TWO

Permutations and Combinations

This chapter and the next introduce some of the fundamental ideas of the subject of this book. The reader may recognize a number of these concepts from previous study. However, at several places in Chapters 2 and 3 the topics are discussed in more detail than is usually the case in elementary books on algebra. It will smooth the way for the reader in subsequent chapters if he fully understands these fundamental ideas. If he is able to answer the questions in the problem sets, he can be sure of his understanding of the subject. Much of the basic notation of combinatorics is set forth in these two chapters. Out of the variety of notation used throughout mathematical literature, we outline several of the standard forms, but subsequently stick to only one.

To introduce the subject we consider the following simple problem. A clothing store for men and boys has belts in five styles, and there are seven sizes available in each style. How many different kinds of belts does the store have?

The answer, 35, can be obtained by multiplying 5 by 7 because there are 7 belts in style number 1, 7 belts in style number 2, $\cdots$, 7 belts in style number 5, and so we have

$$7 + 7 + 7 + 7 + 7 = 5 \cdot 7 = 35.$$

This easy question illustrates a basic principle.

2.1 The Multiplication Principle

If a collection of things can be separated into m different types, and if each of these types can be separated into k different subtypes, then there are mk different types in all.

This principle can be extended beyond a classification according to two properties, such as styles and sizes of belts, to classifications according to three properties, four properties, and more. As an example consider the following question. A drugstore stocks toothpaste from seven different manufacturers. Each manufacturer puts out three sizes, each available in fluoridated form and plain. How many different kinds of toothpaste tubes does the store have? The answer, on the basis of the multiplication principle, is $7 \cdot 3 \cdot 2$ or 42, because of 7 manufacturers, 3 sizes, and 2 types as regards fluoridation.

The multiplication principle is applicable to many problems besides that of classifying objects. As an example, consider a man who decides to go to Europe by plane and to return by ship. If there are eight different airlines available to him, and nine different shipping companies, then he can make the round trip in $8 \cdot 9$ or 72 different ways.

Here is another simple example. At a big picnic the snack lunch consists of a sandwich (choice of four kinds), a beverage (choice of coffee, tea or milk) and an ice cream cup (choice of three flavors). In how many ways can a person make his selection? By the multiplication principle we see that the answer is $4 \cdot 3 \cdot 3$ or 36 ways.

Because of the various applications of the multiplication principle it is often formulated in terms of events: *If one event can occur in m ways, and a second event can occur independently of the first in k ways, then the two events can occur in mk ways.*

The word "independently" is essential here because the principle is not necessarily valid in situations where the second event is dependent on, or restricted by, the first. For example, a girl with seven skirts and five blouses might not have 35 skirt-blouse combinations because some of the colors or patterns might clash aesthetically; a certain red skirt might not go well with a certain orange blouse. However, the following example illustrates a standard kind of dependency of events wherein the principle can still be used.

PROBLEM 2.1 In how many different orders can the four letters A, B, C, D be written, no letter being repeated in any one arrangement?

This question can be answered by simply writing out all possible orders of the letters: $ABCD$, $ACBD$, $ABDC$, etc. But it is simpler, and in more complicated problems necessary, to devise a system to solve the problem. Consider the first letter in any arrangement. There are four choices for the letter in this position. For any given selection of the first letter, there are three possible choices for the second letter. If, for example, the first letter is B, then the second letter must be chosen from A, C or D. Similarly, after the first two letters of the foursome have been selected, the third letter can be chosen in two ways. And when we get to the fourth letter it can be chosen in only one way; that is, there is only one letter that can be used in the fourth place. Thus the multiplication principle gives the answer

$$4 \cdot 3 \cdot 2 \cdot 1 = 24.$$

The reader should verify this by listing all 24 cases. Here are those that begin with the letter A:

$$ABCD, \quad ABDC, \quad ACBD, \quad ACDB, \quad ADBC, \quad ADCB.$$

PROBLEM 2.2 In a certain (mythical) country the automobile license plates have letters, not numbers, as distinguishing marks. Precisely three letters are used, for example, BQJ, CCT and DWD. If the alphabet has 26 letters, how many different license plates can be made?

As the examples show, repetition of letters is allowed on a license plate. There being 26 choices for each of the three letters, the answer is

$$26 \cdot 26 \cdot 26 = 17576.$$

PROBLEM 2.3 What would be the answer to Problem 2.2 if the repetition of letters on a license plate were not allowed?

An argument similar to that used in the solution of Problem 2.1 can be made. There are 26 choices for the first letter, but only 25 for the second, and only 24 for the third. Thus the answer is

$$26 \cdot 25 \cdot 24 = 15600.$$

Problem Set 2

1. Of the arrangements in Problem 2.2, how many begin with the letter Q ?

2. Of the arrangements in Problem 2.3, how many begin with the letter Q ? How many end with the letter Q ?

3. Of the arrangements in Problem 2.2, how many end with a vowel (A, E, I, O, U)?

4. Of the arrangements in Problem 2.3, how many end with a vowel?

5. A room has six doors. In how many ways is it possible to enter by one door and leave by another?

6. A tire store carries eight different sizes of tires, each in both tube and tubeless variety, each with either nylon or rayon cord, and each with white sidewalls or plain black. How many different kinds of tires does the store have?

7. A mail order company offers 23 styles of ladies' slippers. If each style were available in twelve lengths, three widths and six colors, how many different kinds of ladies' slippers would the warehouse have to keep in stock?

8. How many of the integers (whole numbers) between 10,000 and 100,000 have no digits other than 6, 7, or 8 ? How many have no digits other than 6, 7, 8 or 0 ?

2.2 Factorials

In many situations it is useful to have a simple notation for products such as

$$4\cdot 3\cdot 2\cdot 1, \qquad 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1, \qquad 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1,$$

each of which is the product of a sequence of consecutive integers all the way down to one. Such products are called *factorials*. The standard mathematical notation uses what is ordinarily an exclamation point; thus

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040.$$

We read $4!$ as "four factorial", $6!$ as "six factorial", and $7!$ as "seven factorial". In general, for any positive integer n we define $n!$ (read this as "n factorial") as

$$n! = n(n-1)(n-2)(n-3) \cdots 1.$$

This is the product of all integers from n down to 1. Note that $1!$ is equal to 1.

Problem Set 3

1. Formulate as a product and then evaluate† each of $3!$, $5!$ and $8!$.

2. Evaluate the following: $12!/10!$; $2!$; $4! + 3!$; $(4 + 3)!$.

3. Evaluate $(n + 1)!$ in case $n = 4$.

4. Evaluate $n! + 1$ in case $n = 4$.

5. Evaluate $(n - 1)!$ in case $n = 4$.

6. Evaluate $(n - r)!$ in case $n = 10$ and $r = 8$.

7. Compute $(n - r)!$ in case $n = 12$ and $r = 6$.

8. Compute $\dfrac{n!}{(n-r)!}$ in case $n = 12$ and $r = 4$; also in the case $n = 10$ and $r = 6$.

† Whereas the reader is asked to evaluate or compute such numbers as 5! and 8!, he would not be expected to compute (say) 20!. If such a number were the answer to a question in this book, it would be left in precisely that form. Computational techniques are very important, but they are not stressed in this volume.

9. Compute $\dfrac{n!}{r!(n-r)!}$ in case $n = 10$ and $r = 6$.

10. Which of the following are true and which false?

(a) $8! = 8 \cdot 7!$ (b) $10!/9! = 9$ (c) $4! + 4! = 8!$

(d) $2! - 1! = 1!$ (e) $n! = n \cdot (n-1)!$

(f) $n! = (n^2 - n) \cdot (n-2)!$

2.3 Permutations

Permutations are ordered arrangements of objects. As examples of permutations, consider again Problems 2.1 and 2.3 from Section 2.1.

PROBLEM 2.1 In how many different orders can the four letters A, B, C, D be written, no letter being repeated in any one arrangement?

This is the same as asking how many permutations there are on four letters, taken four at a time. The number of such permutations is denoted by the symbol $P(4, 4)$.

PROBLEM 2.3 How many different license plates can be made if each plate has three letters and repetition of letters on a license plate is not allowed?

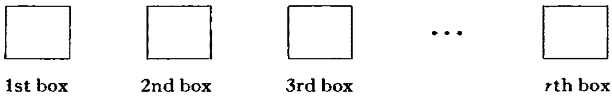
This is the same as asking how many permutations there are on twenty-six letters, taken three at a time. The number of such permutations is denoted by $P(26, 3)$. We have solved these problems in Section 2.1 and may now write the answers, in our new notation, as

$$P(4, 4) = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \quad \text{and} \quad P(26, 3) = 26 \cdot 25 \cdot 24 = 15600.$$

Each of these $P(26, 3) = 15600$ permutations of 26 objects taken 3 at a time is called a 3-permutation. In general, an r-permutation is an ordered arrangement of r objects, and $P(n, r)$ denotes the number of r-permutations of a set of n distinct objects. This is the same as saying that $P(n, r)$ is the number of permutations of n

things taken r at a time. Of course, it is presumed that r does not exceed n, that is $r \leqslant n$. Note that the n things or objects must be distinct, i.e. we must be able to tell them apart.

To derive a formula for $P(n, r)$ we conceive of r distinct boxes into which the n objects can be put:



Then $P(n, r)$ can be thought of as the number of ways of putting n distinct objects in the r boxes, *one object in each box.*

First consider the special case where the number of objects is the same as the number of boxes. For the first box we can select any one of the n objects. That done, there remain $n - 1$ objects from which to choose for the second box. Similarly, there remain $n - 2$ objects from which to choose for the third box. Continuing in this way we see that when we get to the last box there is only one object left, so we choose one out of one. By the multiplication principle we have

$$P(n, n) = n(n-1)(n-2) \cdots 1 \qquad \text{or} \qquad P(n, n) = n!\ .$$

For example,

$$P(7, 7) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7!\ ,$$
$$P(28, 28) = 28 \cdot 27 \cdot 26 \cdot \cdots \cdot 3 \cdot 2 \cdot 1 = 28!\ .$$

The argument just used to evaluate $P(n, n)$ can be applied just as well to $P(n, r)$. For instance, we note that

$$P(28, 5) = 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24.$$

We observe that in this product of 5 consecutive integers the difference between the largest and the smallest, that is, between 28 and 24, is 4. In general, $P(n, r)$ is the product of the r integers n, $n - 1$, $n - 2$, $\cdots$, $n - r + 1$:

$$P(n, r) = n(n-1)(n-2) \cdots (n-r+1).$$

To see that this is the product of r consecutive integers we recall that the number of integers from k to n inclusive is $n - k + 1$ (see p. 4), so the number of integers from $n - r + 1$ to n inclusive is

$$n - (n - r + 1) + 1 = r.$$

Note that, if $r = n$, this formula for $P(n, r)$ is in harmony with the earlier formula

$$P(n, n) = n(n - 1)(n - 2) \cdots 1 .$$

We now proceed to another formula for $P(n, r)$. As an example consider

$$P(10, 4) = 10 \cdot 9 \cdot 8 \cdot 7,$$

which can be written as a fraction involving factorials thus:

$$P(10, 4) = 10 \cdot 9 \cdot 8 \cdot 7 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{10!}{6!} .$$

The same procedure works in the general case $P(n, r)$:

$$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1)$$

$$= \frac{n(n - 1)(n - 2) \cdots (n - r + 1)(n - r)(n - r - 1) \cdots 1}{(n - r)(n - r - 1) \cdots 1},$$

$$P(n, r) = \frac{n!}{(n - r)!} . \tag{2.1}$$

Example. How many integers between 100 and 999 inclusive consist of distinct odd digits?

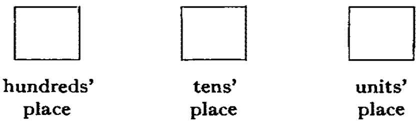
Solution. The odd digits are 1, 3, 5, 7, 9; the even digits are 0, 2, 4, 6, 8. An integer such as 723 is not to be counted because it contains the even digit 2; and an integer such as 373 is not to be counted because it does not have distinct digits. The question amounts to asking for the number of permutations of the five distinct digits 1, 3, 5, 7, 9, taken three at a time. The answer is

$$P(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5 \cdot 4 \cdot 3 = 60.$$

Formula (2.1) for $P(n, r)$ cannot be used to solve all problems about permutations, because not all such problems admit as solutions all ordered arrangements of n distinct objects, r at a time. A problem can sometimes be solved by direct use of the multiplication principle, as the following examples illustrate.

Example. How many integers between 100 and 999 have distinct digits?

Solution. This is not simply $P(10, 3)$, the number of permutations of all ten digits taken three at a time, because 086, for example, is not a number between 100 and 999. The digit 0 can be used in the units' place (as in 860), or in the tens' place (as in 806), but not in the hundreds' place. Consider three boxes to be filled by the digits of any of the integers under consideration:



There are nine choices for the digit in the hundreds' place, because 0 cannot be used. There are then nine choices for the digit in the tens' place, namely 0 together with the eight non-zero digits not used already. Similarly there are eight choices for the digit used in the units' place. Hence the answer is $9 \cdot 9 \cdot 8$ or 648.

Example. Of the 648 integers in the preceding problem, how many are odd numbers?

Solution. A number is odd if its units digit is odd, i.e. if the digit in the units' place is one of 1, 3, 5, 7, 9. So it is best to begin the argument by asking how many choices there are for the digit in the units' place; the answer is five. Next, turn to the hundreds' place; there are eight digits from which a selection can be made, namely all the non-zero digits except the one already used in the units' place. Finally there are eight choices for the digit in the tens' place, so the answer is $5 \cdot 8 \cdot 8$ or 320.

Some problems can be solved most readily by considering separate cases.

Example. How many of the first 1000 positive integers have distinct digits?

Solution. Setting aside the integer 1000, whose digits are not distinct, the others can be separated into three types:

Integers with one digit: $1, 2, 3, \cdots, 9$;

Integers with two digits: $10, 11, 12, \cdots, 99$;

Integers with three digits: $100, 101, 102, \cdots, 999$.

The number of three-digit integers with distinct digits is 648, as shown in a previous example. A similar argument shows that there are 81 two-digit integers and, of course, 9 one-digit integers that meet the specification of distinct digits. Hence the answer is

$$648 + 81 + 9 = 738.$$

The idea used here is called the *addition principle:* If the things to be counted are separated into cases, the total number is the *sum* of the numbers in the various cases.

2.4 Zero Factorial

An interesting phenomenon turns up if we use formula (2.1) for $P(n, r)$ in a case such as

$$P(7, 7) = \frac{7!}{(7-7)!} = \frac{7!}{0!}.$$

The notation $0!$, in words, "zero factorial", has so far not been defined. In mathematics we can define the meaning of the symbols in any way we please, provided of course that there is consistency. In the present case, since we had determined earlier that $P(7,7) = 7!$, consistency requires that

$$P(7, 7) = 7! = \frac{7!}{0!}.$$

Thus we should and do define zero factorial to be one:

$$0! = 1.$$

This may look strange but it is a useful definition. It is related to other combinatorial notation, not just to $P(n, r)$.

Problem Set 4

1. Evaluate $P(7, 3)$, $P(8, 4)$ and $P(20, 2)$.

2. Verify that $P(7, 3) = P(15, 2)$ and that $P(6, 3) = P(5, 5)$.

3. Prove that $P(n, 1) + P(m, 1) = P(n + m, 1)$ for all positive integers m and n.

4. Prove that $P(n, n) = P(n, n - 1)$ for all positive integers n.

5. How many fraternity names consisting of three different Greek letters can be formed? (There are twenty-four letters in the Greek alphabet.)

6. What would be the answer to the preceding question if repetitions of letters were allowed? What would it be if repetitions of letters were allowed and two-letter names were also included in the count?

7. How many integers between 1000 and 9999 inclusive have distinct digits? Of these how many are odd numbers?

8. From the digits 1, 2, 3, 4, 5, how many four-digit numbers with distinct digits can be constructed? How many of these are odd numbers?

9. From the digits 0, 1, 2, 3, 4, 5, 6, how many four-digit numbers with distinct digits can be constructed? How many of these are even numbers?

10. How many integers greater than 53000 have the following two properties: (a) the digits of the integer are distinct; (b) the digits 0 and 9 do not occur in the number?

11. In the preceding problem what would the answer be if condition (b) were changed to "the digits 8 and 9 do not occur in the integer"?

2.5 Combinations

Whereas a permutation is an ordered arrangement of objects, a combination is a selection made *without regard to order*. The notation $C(n, r)$ is used for the number of combinations of a certain special type, in parallel with the notation $P(n, r)$ for permutations. Thus $C(n, r)$ denotes the number of combinations, r at a time, that can be selected out of a total of n distinct objects.

Consider $C(5, 3)$ for example. Let the five objects be A, B, C, D, E. Then it can be observed that $C(5, 3) = 10$, because there are ten combinations of the objects taken three at a time:

$$\begin{matrix} A, B, C & A, B, D & A, B, E & A, C, D & A, C, E \\ A, D, E & B, C, D & B, C, E & B, D, E & C, D, E. \end{matrix} \tag{2.2}$$

Notice that each of these ten triples is simply a collection in which order does not matter. The triple C, D, E for example could have been written D, E, C or E, C, D, or in any other order; it counts as just one triple.

Given n distinct objects, $C(n, r)$ is the number of ways of choosing r objects from the total collection. Of course it is presumed that r does not exceed n, that is $r \leqslant n$. The meaning of $C(n, r)$ can also be stated in terms of a set of n elements. $C(n, r)$ is the number of subsets containing exactly r elements. For example, the listing (2.2) above gives all subsets of three elements selected from the set A, B, C, D, E.

Before deriving a general formula for $C(n, r)$, we compute the value of $C(26, 3)$ to illustrate the argument. $C(26, 3)$ can be thought of as the number of ways of choosing three letters out of a 26 letter alphabet. One such choice, for example, is the triple D, Q, X, taken without regard to order. This one combination D, Q, X corresponds to the six distinct permutations

$$DQX \quad DXQ \quad QDX \quad QXD \quad XDQ \quad XQD.$$

In fact, each of the $C(26, 3)$ combinations corresponds to $P(3, 3)$ or $3! = 6$ permutations. Hence there are six times as many permutations as there are combinations:

$$P(26, 3) = 6C(26, 3).$$

But we have already computed the value

$$P(26, 3) = 26 \cdot 25 \cdot 24 = 15600$$

in Problem 2.3. Hence we get

$$6C(26, 3) = 15600, \qquad \text{so that} \qquad C(26, 3) = 2600.$$

We now generalize this argument to get a relationship between $C(n, r)$ and $P(n, r)$ and then evaluate $C(n, r)$ by use of the formula (2.1) for $P(n, r)$. With n distinct objects, $C(n, r)$ counts the number of ways of choosing r of them without regard to order. Any one of these choices is simply a collection of r objects. Such a collection can be ordered in $r!$ different ways. Since to each combination there correspond $r!$ permutations, there are $r!$ times as many permutations as there are combinations:

$$P(n, r) = r!C(n, r) \qquad \text{or} \qquad C(n, r) = \frac{P(n, r)}{r!} .$$

But we know by formula (2.1) that $P(n, r)$ equals $n!/(n - r)!$, and hence we get the basic formula for $C(n, r)$,

$$(2.3) \qquad C(n, r) = \frac{n!}{r!(n - r)!} .$$

This is perhaps the most widely used formula in combinatorial analysis. The number $C(n, r)$ is often represented in other ways, for example

$$nCr, \quad {}^nCr, \quad C_r^n, \quad \text{and} \quad \binom{n}{r} .$$

The last of these is very common; it is to be read "n over r" or "the binomial coefficient n over r". Binomial coefficients occur in the expansion of a power of a sum of two terms, such as $(x + y)^8$; this is one of the topics of the next chapter.

There is one simple property of $C(n, r)$ that is almost obvious, namely

$$C(n, r) = C(n, n - r). \tag{2.4}$$

Let us take $n = 5$ and $r = 3$ as an illustration. Then the equation (2.4) becomes $C(5, 3) = C(5, 2)$ and can be verified as follows. Taking the five objects to be $A, B, C, D, E,$ we have seen that $C(5, 3) = 10,$ the ten triples having been written out in full detail in (2.2). Now when a triple, such as $A, C, D,$ is selected, there is a pair (in this case B, E) left unselected. So corresponding to each selected triple in (2.2) we can write a corresponding unselected pair (in parentheses):

$$\begin{array}{lll}
A, B, C(D, E) & A, B, D(C, E) & A, B, E(C, D) \\
A, C, D(B, E) & A, C, E(B, D) & A, D, E(B, C) \\
B, C, D(A, E) & B, C, E(A, D) & B, D, E(A, C) \\
C, D, E(A, B) & &
\end{array}$$

It follows that the number of ways of choosing three objects out of five is the same as the number of ways of choosing two objects out of five, so $C(5, 3) = C(5, 2) = 10.$

In general, corresponding to every selection of r things out of n there is a set of $n - r$ unselected things, the ones not in the selection. Hence the number of ways of choosing r things must be the same as the number of ways of choosing $n - r$ things, and so formula (2.4) is established.

If in formula (2.4) we replace r by 0 we get $C(n, 0) = C(n, n)$. Now $C(n, n)$ means the number of ways of choosing n things out of n, so $C(n, n) = 1$. But $C(n, 0)$ seems to have no meaning: "the number of ways of choosing no things out of n ". It is convenient to define $C(n, 0)$ to be 1. Notice that this harmonizes with formula (2.3) which gives, for $r = 0$,

$$C(n, 0) = \frac{n!}{0!(n - 0)!} = \frac{n!}{0!n!} = 1$$

because $0! = 1$. We also define $C(0, 0) = 1$.

It is convenient to extend the definition of $C(n, r)$ to *all* integers n and r, even negative integers, for then various formulas can be written without qualification or added explanation. If n is negative, if r is negative, or if $r > n$, $C(n, r)$ is defined to be zero. For example, $C(-10, 8)$, $C(5, -8)$, and $C(10, 12)$ are zero by definition. In other words

$$C(n, r) = 0 \quad \text{in case one or more of } n,\ r,\ n - r \text{ is negative,}$$

$$C(n, r) = \frac{n!}{r!(n - r)!} \quad \text{in all other cases.}$$

Problem Set 5

1. Evaluate $C(6, 2)$, $C(7, 4)$ and $C(9, 3)$.

2. Show that $C(6, 2) = C(6, 4)$ by pairing off the 2-subsets and the 4-subsets of the set A, B, C, D, E, F.

3. An examination consists of ten questions, of which a student is to answer eight and omit two. (a) In how many ways can a student make his selection? (b) If a student should answer two questions and omit eight, in how many ways can he make his selection?

4. A college has 720 students. In how many ways can a delegation of ten be chosen to represent the college? (Leave the answer in factorial form.)

5. Verify that $C(n, r) = C(n, n - r)$ by use of formula (2.3).

6. Twenty points lie in a plane, no three collinear, i.e. no three on a straight line. How many straight lines can be formed by joining pairs of points? How many triangles can be formed by joining triples of points?

7. In how many ways can ten persons be seated in a row so that a certain two of them are not next to each other?

8. Prove that the product of five consecutive positive integers is divisible by 5!, and more generally, that the product of r consecutive

integers is divisible by $r!$. Suggestion: Examine the formula for $C(n, r)$.

9. There are nine different books on a shelf; four are red and five are green. In how many different orders is it possible to arrange the books on the shelf if
   (a) there are no restrictions;
   (b) the red books must be together and the green books together;
   (c) the red books must be together whereas the green books may be, but need not be, together;
   (d) the colors must alternate, i.e. no two books of the same color may be adjacent?

10. A certain men's club has sixty members; thirty are business men and thirty are professors. In how many ways can a committee of eight be selected (a) if at least three must be business men and at least three professors; (b) the only condition is that at least one of the eight must be a business man? (Leave the answers in $C(n, r)$ symbols.)

11. In how many ways can a ballot be validly marked if a citizen is to choose one of three candidates for mayor, one of four for city councilman, and one of three for district attorney. A citizen is not required to vote for all three positions, but he is expected to vote for at least one.

12. If 20! were multiplied out, how many consecutive zeros would occur on the right hand end?

13. If 52! were multiplied out, how many consecutive zeros would occur on the right hand end?

14. Signals are made by running five colored flags up a mast. How many different signals can be made if there is an unlimited supply of flags of seven different colors?

15. In the preceding question what would be the answer if (a) adjacent flags in a signal must not be of the same color; (b) all five flags in a signal must be of different colors?

16. From the 26 letters of the alphabet, how many subsets of three letters are there such that no two of the three are consecutive letters of the alphabet?

17. In how many ways can all of n distinct objects be put in k distinct boxes, not more than one in each box, if there are more boxes than things?

2.6 Permutations of Things in a Circle

The permutations that we have considered so far are called *linear permutations* because they are permutations of things in a line or in a row. Permutations of things in a circle, or *circular permutations*, occur in such a problem as: In how many ways can five persons be seated at a round table?

First solution. If we label the persons $A, B, C, D, E,$ we see that the five linear permutations

$$ABCDE, \quad BCDEA, \quad CDEAB, \quad DEABC, \quad EABCD$$

are identical when thought of as circular permutations. This is so because two arrangements of people at a round table are considered to form the same circular permutation if one can be obtained from the other by rotating everybody around the circle by the same amount and in the same direction. This is the case, for example, if everybody moves one place to his right. Hence we can get at the number of circular permutations by relating them to the linear permutations: each circular permutation corresponds to five linear permutations, so there are only $\frac{1}{5}$ as many circular permutations as there are linear permutations. But there are 5! linear permutations of five objects, and hence the answer to the question is

$$\tfrac{1}{5}(5!) = \tfrac{1}{5}(5\cdot4\cdot3\cdot2\cdot1) = 4\cdot3\cdot2\cdot1 = 4!.$$

Second solution. Since a circular arrangement is unchanged if every object (or every person) is moved uniformly one place to the right, or uniformly two places to the right, etc., we can fix the place of one person and arrange the others with reference to him around the table. Putting A in a fixed place, we see that any one of four persons can be immediately to A's right, then any one of three remaining persons in the next place to the right, any one of two in the next place, and the remaining person in the final place; see Figure 2.1. Using the multiplication principle we get the answer $4\cdot3\cdot2\cdot1$.

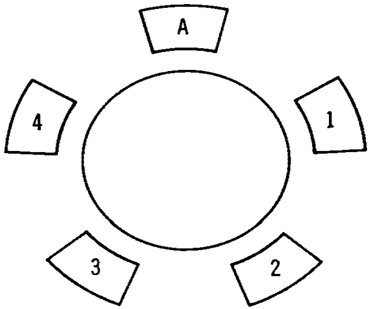


Figure 2.1

In general there are $(n - 1)!$ circular permutations of n distinct objects. To show this, we can argue as we did in the solutions just given for the special case $n = 5$. In particular, let us follow the second solution. We think of n persons, say $A, B, C, D, \cdots$, being seated at a round table. Since a uniform rotation of the persons does not alter an arrangement, we might as well put person A in one fixed place and then consider the number of ways of arranging all the others. In the chair to the right of A we can put any of the other $n - 1$ people. That done, we move to the next chair to the right into which we can place any one of the remaining $n - 2$ persons. Continuing in this counterclockwise fashion around the circle, we see that the multiplication principle gives the answer

$$(n - 1)(n - 2)(n - 3) \cdots 1 = (n - 1)!.$$

Problem Set 6

1. In how many ways is it possible to seat eight persons at a round table?

2. In the preceding question, what would be the answer if a certain two of the eight persons must not sit in adjacent seats?

3. In how many ways can four men and four ladies be seated at a round table, if no two men are to be in adjacent seats?

4. In the preceding question, suppose the persons are four married couples. What would be the answer to the question if no husband and wife, as well as no two men, are to be in adjacent seats?

5. How many different firing orders are theoretically possible in a six cylinder engine? (If the cylinders are numbered from 1 to 6, a firing order is a list, such as 1, 4, 2, 5, 3, 6, giving the rotational order in which the fuel is ignited in the cylinders.)

6. How many differently colored blocks of a fixed cubical shape can be made if six colors are available, and a block is to have a different color on each of its six faces? The definition of differently colored blocks is the same as in Problem 1.2 in Chapter 1.

7. How many different cubes with the six faces numbered from 1 to 6 can be made, if the sum of the numbers on each pair of opposite faces is 7?

2.7 Summary

The multiplication principle: If one event can occur in m ways, and a second event can occur independently of the first in k ways, then the two events can occur in mk ways.

Formula for n factorial:

$$n! = n(n-1)(n-2)\cdots 1 \quad \text{for positive integers } n\ ,$$

$$0! = 1.$$

The number of permutations (i.e. ordered arrangements) of n distinct objects, taken r at a time, is

$$P(n, r) = \frac{n!}{(n-r)!}.$$

The number of combinations (i.e. selections, without regard to order) of n distinct objects, taken r at a time, is

$$C(n, r) = \frac{n!}{r!(n-r)!}.$$

$C(n, r)$ can also be interpreted as the number of r-subsets (subsets containing r elements) of a set of n objects. A frequently used alternative notation for $C(n, r)$ is $\binom{n}{r}$. A basic property of $C(n, r)$ is

$$C(n, r) = C(n, n - r).$$

The symbol $C(n, r)$ was given a numerical value for all pairs of integers n and r, as follows:

$C(n, r) = 0$ in case one or more of n, r, $n - r$ is negative;

$C(n, r) = \dfrac{n!}{r!(n - r)!}$ in all other cases.

The number of circular permutations (i.e. arrangements in a circle) of n distinct objects is $(n - 1)!$.

The formulas for $P(n, r)$ and $C(n, r)$ apply only to special situations of ordered arrangements and unordered selections where the n objects are distinct and repetitions in the r-sets are not allowed. They are not universal formulas for permutations and combinations. However, in later chapters many problems are reduced to these special cases.

CHAPTER THREE

Combinations and Binomial Coefficients

There are other ways, besides those in the preceding chapter, of looking at $C(n, r)$, the number of combinations of n different things taken r at a time. Several of these possibilities are studied in this chapter. We begin by pointing out that we can easily solve the path problem which was listed as Problem 1.3 in Chapter 1. For convenience, we repeat the statement of the question.

3.1 A Path Problem

A man works in a building located seven blocks east and eight blocks north of his home. Thus in walking to work each day he goes fifteen blocks. All the streets in the rectangular pattern are available to him for walking. In how many different paths can he go from home to work, walking only fifteen blocks?

Let us denote by E the act of walking a block east, and by N the act of walking a block north, and let us interpret a string of E's and N's such as

$$EENNNENN$$

as meaning (reading from left to right) that a man walks two blocks east, then three blocks north, then one block east, and finally two blocks north. Then any path from home to work can be identified with an appropriate pattern of seven E's and eight N's in a row. For example, the path beginning with three blocks east, then two north, then four east, and finally six north is

$$EEENNEEEENNNNNN.$$

Thus to each path there corresponds a string of seven E's and eight N's properly interspersed in a row; and conversely, to any such string of E's and N's there corresponds exactly one path. We can therefore rephrase the problem as follows: In how many ways can seven E's and eight N's be written in a row?

□ □ □ □ □ □ □ □ □ □ □ □ □ □ □

If we think of fifteen boxes to be filled with seven E's and eight N's, we see that the answer to this question is just the number of ways we can choose seven boxes out of fifteen to fill with E's, and this number is

$$C(15, 7) = \frac{15!}{7!8!} = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = 6435.$$

It is also the same as the number of ways we can choose eight boxes out of fifteen to fill with N's, that is $C(15, 8)$. In Chapter 2 we saw that

$$C(n, r) = C(n, n - r), \quad \text{and so} \quad C(15, 7) = C(15, 8) = 6435.$$

3.2 Permutations of Things Not All Alike

We have just seen that $C(15, 7)$ can be interpreted as the number of permutations of fifteen things of which seven are alike and the other eight are alike. In general $C(n, r)$ can be interpreted as the number of permutations of n things of which r are alike and the other $n - r$ are alike. This idea can also be generalized from two batches of things, like E's and N's, to more batches. We begin with an example.

PROBLEM 3.1 How many different permutations are there of the letters of the word *Mississippi*, taken all at a time? In other words, in how many different orders is it possible to write the letters of the word *Mississippi*?

SOLUTION. There are eleven letters of which four are alike (the i's), another four alike (the s's), and another two alike (the p's). Consider eleven boxes for insertion of letters to give the various permutations. Choose four of these for the i's; there are $C(11, 4)$ ways of doing this. Then choose four of the remaining seven boxes for the s's; there are $C(7, 4)$ ways of doing this. Then from the remaining three choose two boxes for the p's; there are $C(3, 2)$ ways of doing this. The letter M will fill the remaining box. By the multiplication principle we get the answer

$$C(11, 4)\cdot C(7, 4)\cdot C(3, 2) = \frac{11!}{4!7!}\cdot\frac{7!}{4!3!}\cdot\frac{3!}{2!1!} = \frac{11!}{4!4!2!1!}.$$

Of course, if we choose the letters for the boxes in some other order, the calculation looks a little different, but the final answer is the same. For example, suppose we begin by choosing one box out of the eleven for the letter M, then four boxes for the s's, then two boxes for the p's, with the remaining four boxes for the i's; then the total number of different arrangements of the letters is

$$C(11, 1)\cdot C(10, 4)\cdot C(6, 2) = \frac{11!}{1!10!}\cdot\frac{10!}{4!6!}\cdot\frac{6!}{2!4!} = \frac{11!}{4!4!2!1!},$$

the same as before.

SECOND SOLUTION. An alternative argument of quite a different kind goes like this: Let x denote the number of permutations in our answer. If we were to replace the four i's by four letters different from each other and from the remaining letters of *Mississippi*, such as i, j, k, and l, we would obtain $x\cdot 4!$ permutations from the original x because each of the original permutations would give rise to $4!$. Similarly if the four s's were replaced by four distinct letters, again we would have $4!$ times as many permutations. And if the two p's were replaced by unlike letters, we would have $2!$ times as many permutations as before. But now we would have eleven letters, all *different*, and so 11! permutations. This gives the equation

$$x \cdot 4! \cdot 4! \cdot 2! = 11!, \qquad \text{so that} \qquad x = \frac{11!}{4!4!2!}.$$

More generally, if there are n things of which a are alike, another b are alike, another c are alike, and finally the remaining d are alike, we can find the number of permutations of the $n = a + b + c + d$ things taken all at a time by a similar argument: If x denotes the number of different permutations,

$$x \cdot a! \cdot b! \cdot c! \cdot d! = n!, \qquad \text{so that} \qquad x = \frac{n!}{a!b!c!d!}.$$

There need not be just four batches of like things. In general, if there are n things of which a are alike, another b are alike, another c are alike, etc., then the number of permutations of the n things taken all at a time is

$$(3.1) \qquad \frac{n!}{a!b!c! \cdots}, \qquad \text{where } n = a + b + c + \cdots.$$

Here the dots in the denominator stand for "and so on", that is, for as many additional factorial terms as may be necessary.

Problem Set 7

1. How many permutations are there of the letters, taken all at a time, of the words (a) assesses, (b) humuhumunukunukuapuaa (Hawaiian word for a species of fish).

2. Derive formula (3.1) in the case of four batches, $n = a + b + c + d$, by paralleling the first argument given for Problem 3.1.

3. In the path problem in Section 3.1, denote the north-south streets by $A, B, C, \cdots, H$ and the east-west streets by 1st, 2nd, $\cdots$, 9th. Presume that the man lives at the corner of 1st and A, and works at the corner of 9th and H. Given the information that all streets are available for walking with one exception, namely that E street from 5th

to 6th is not cut through, in how many different paths can the man walk from home to work, walking only fifteen blocks?

4. As a generalization of the path problem to three dimensions, consider a three-dimensional steel framework; how many different paths of length fifteen units are there from one intersection point in the framework to another that is located four units to the right, five units back, and six units up?

5. In how many different orders can the following 17 letters be written?

$$x\ x\ x\ x\ y\ y\ y\ y\ y\ z\ z\ z\ z\ z\ z\ w\ w$$

3.3 Pascal's Formula for $C(n, r)$

Consider the r-subsets, i.e. subsets consisting of r elements, of a set of n objects. The number of r-subsets is $C(n, r)$. Of the set of n objects, let us single out one and label it T. The r-subsets can be separated into two types:

(a) those that contain the object T;

(b) those not containing the object T.

Those that contain the object T are in number $C(n - 1, r - 1)$, because along with T in any r-subset there are $r - 1$ other objects selected from $n - 1$ objects. Those that do not contain the object T are in number $C(n - 1, r)$, because these r-subsets are selected from $n - 1$ objects, T being out. We have separated the entire collection of r-subsets into two types and then determined the number of each type, thus establishing Pascal's formula

$$C(n, r) = C(n - 1, r) + C(n - 1, r - 1). \tag{3.2}$$

There is a simple device that extends the use of such formulas. We observe that the reasoning leading to relation (3.2) does not depend on the precise number of objects n or r. The argument would have made equally good sense if we had begun with m objects from which k were to be selected to form k-subsets, and would have led to the equally meaningful formula

$$C(m, k) = C(m - 1, k) + C(m - 1, k - 1).$$

Similarly, had we begun with $n + 1$ objects and selected r of these, we would have derived the formula

$$(3.3) \qquad C(n+1, r) = C(n, r) + C(n, r-1).$$

There is really no need to rethink the whole process to get formula (3.3); it can be obtained from (3.2) by replacing n by $n + 1$; thus

$C(n, r)$ becomes $C(n+1, r)$;

$C(n-1, r)$ becomes $C(n+1-1, r)$ or $C(n, r)$;

$C(n-1, r-1)$ becomes $C(n+1-1, r-1)$ or $C(n, r-1)$;

and formula (3.2) becomes formula (3.3).

We can replace n by $n + 1$ in formula (3.2) and wind up with a valid formula because formula (3.2) holds *for any positive integers* n *and* r, *provided only that* $n \geqslant r$. So we can replace the symbols n and r by any other symbols subject only to the conditions that (i) the new symbols denote positive integers and (ii) the symbol replacing n stands for an integer at least as large as the integer denoted by the symbol replacing r. For example, in formula (3.2) we can replace n by $n + 1$, or $n + 2$, or $n + 3$. [We cannot replace n by $(n + 1)/2$, because of (i), nor by $r - 3$, because of condition (ii).]

In one sense such replacements give no new information. For example, formula (3.2) with $n = 20$ and $r = 6$ gives the information

$$C(20, 6) = C(19, 6) + C(19, 5).$$

Exactly the same equation comes from (3.3) with $n = 19$ and $r = 6$. However, if we add equations (3.2) and (3.3) we get

$$\begin{aligned} &C(n, r) + C(n+1, r) \\ &\qquad = C(n, r) + C(n, r-1) + C(n-1, r) + C(n-1, r-1); \end{aligned}$$

and, by subtracting $C(n, r)$ from both sides, we obtain the new formula

$$(3.4) \quad C(n+1, r) = C(n, r-1) + C(n-1, r) + C(n-1, r-1).$$

This illustrates the fact that we can get new formulas from simpler ones like (3.2) without making any arguments about the meaning of the symbols themselves, but just by the manipulation of the notation.

Problem Set 8

1. Calculate $C(6, 2)$, $C(5, 2)$ and $C(5, 1)$ and verify that the first of these is the sum of the other two.

2. Write $C(9, 4) + C(9, 3)$ as a single combination form $C(n, r)$.

3. Write $C(50, 10) - C(49, 9)$ as a single combination form $C(n, r)$.

4. What is the resulting equation if (a) we replace n by $n - 1$ in formula (3.2); (b) we replace n by $n - 1$ and r by $r - 1$ in (3.2)?

5. What are the resulting formulas, if, in

$$C(n, r) = \frac{n!}{r!(n-r)!},$$

(a) we replace n by $n - 1$; (b) we replace n by $n - 1$ and r by $r - 1$?

6. Using the results of the preceding question give a proof of formula (3.2) different from the one given in the text, by an argument involving factorials.

7. The proof in the text of formula (3.2) involved consideration of a special one, T, of the n things. Consider now two special ones, say S and T. The combinations can be divided into four classes: those that contain both S and T; those that contain S but not T; those that contain T but not S; those that contain neither S nor T. What formula results if we write $C(n, r)$ as a sum of the numbers of members of these four classes? Derive the formula thus obtained in another way, by using formula (3.2).

8. Apart from one exception, Pascal's formula (3.2) holds for all pairs of integers n and r, positive, negative or zero. What is this one exception?

3.4 The Binomial Expansion

Any sum of two unlike symbols, such as $x + y$, is called a binomial. The binomial expansion, or binomial theorem, is a formula for the powers of a binomial. If we compute the first few powers of $x + y$, we obtain

$$\begin{aligned}
(x+y)^1 &= x + y\\
(x+y)^2 &= x^2 + 2xy + y^2\\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3.\\
(x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,\\
(x+y)^5 &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5. \qquad (3.5)
\end{aligned}$$

Using equation (3.5) as a basis for discussion, we note that the right member has six terms: x^5, $5x^4y$, $10x^3y^2$, $10x^2y^3$, $5xy^4$ and y^5. What we want to do is explain the coefficients of these terms, 1, 5, 10, 10, 5, 1, by means of the theory of combinations.

First let us examine the results of multiplying several binomials. For example, to multiply $(a + b)$ by $(c + d)$ we apply the distributive law and obtain

$$(a+b)(c+d) = (a+b)c + (a+b)d = ac + bc + ad + bd.$$

Each of the terms in this sum is a product of two symbols, one taken from the first parenthesis of our original product and the other from the second. Notice that there are precisely $2 \cdot 2 = 4$ different ways of selecting one symbol from the first binomial and one symbol from the second.

We now examine the product of three binomials

$$(a+b)(c+d)(e+f)$$
$$= ace + acf + ade + adf + bce + bcf + bde + bdf$$

and observe that it consists of eight terms, each a product of three symbols selected, respectively, from the three binomials. Again we observe that $8 = 2 \cdot 2 \cdot 2$ is precisely the number of different ways that three symbols can be selected, one from each binomial. Similar results hold for the expanded product of four or more binomials. Let us consider the product

$$(a+b)(c+d)(e+f)(p+q)(r+s).$$

Its expansion which we will not write out in full, is a sum of $2^5 = 32$ terms. As sample terms we cite

$$adeqs \qquad \text{and} \qquad bceps.$$

Each term is a product of five symbols, one selected from each of the five original binomials.

Now in the light of these observations let us look at $(x + y)^5$ as the product

$$(x + y)(x + y)(x + y)(x + y)(x + y).$$

There are 32 ways of selecting five symbols, one from each parenthesis, but the resulting 32 expressions are not all distinct. For example, multiplication of the particular x's and y's shown here

$$(\overset{\downarrow}{x} + y)(x + \overset{\downarrow}{y})(x + \overset{\downarrow}{y})(\overset{\downarrow}{x} + y)(\overset{\downarrow}{x} + y)$$

with arrows directed at them results in the product

$$xyyxx = x^3y^2.$$

But x^3y^2 also arises if we select x's from the first three parentheses and y's from the remaining two. In fact, the expression x^3y^2 arises in the expansion of $(x + y)^5$ in exactly as many ways as three x's and two y's can be written in different orders:

$$xyyxx, \quad xxxyy, \quad yxxxy, \quad \text{etc.}$$

By the theory of Section 3.2 there are

$$C(5, 2) = \frac{5!}{2!3!} = 10$$

different arrangements of these symbols. This analysis explains the coefficient 10 of x^3y^2 in equation (3.5), the expansion of $(x + y)^5$. The other coefficients can be obtained in a similar way, so we have

$$\begin{aligned}(x + y)^5 = {} & C(5, 0)x^5 + C(5, 1)x^4y + C(5, 2)x^3y^2 \\ & + C(5, 3)x^2y^3 + C(5, 4)xy^4 + C(5, 5)y^5.\end{aligned}$$

This is not as concise as formula (3.5), but it suggests a general pattern. It suggests that the coefficient of x^3y^3 in the expansion of $(x + y)^6$ is $C(6, 3)$, the number of ways of writing three x's and three y's in a row; and that the coefficient of x^2y^4 in the same expansion is $C(6, 4)$, the number of ways of writing two x's and four y's in a row. (Of course $C(6, 4)$ is the same as $C(6, 2)$, but we shall make the combination symbol follow the number of y's rather than the number of x's. It could be done the other way around.)

Now let n be any positive integer. The expression $(x + y)^n$ is defined as

$$(x + y)(x + y)(x + y) \cdots (x + y) \qquad (n \text{ factors}).$$

In the expansion of this product $x^{n-j}y^j$ arises in as many ways as a batch of $n - j$ x's and a batch of j y's can be written in a row. Hence the coefficient of $x^{n-j}y^j$ is $C(n, j)$. Thus the binomial expansion can be written as

$$\begin{aligned}(x + y)^n = {} & C(n, 0)x^n + C(n, 1)x^{n-1}y + C(n, 2)x^{n-2}y^2 \\ & + C(n, 3)x^{n-3}y^3 + \cdots + C(n, j)x^{n-j}y^j \\ & + \cdots + C(n, n)y^n.\end{aligned}$$

As was remarked in Chapter 2, the notation $\binom{n}{j}$ is often used in place of $C(n, j)$, particularly in the binomial expansion. So in many books it looks like this:

$$\begin{aligned}(x + y)^n = {} & \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 \\ & + \cdots + \binom{n}{j}x^{n-j}y^j + \cdots + \binom{n}{n}y^n.\end{aligned}$$

The first and last terms can be written more simply as x^n and y^n, and this suggests yet another form in which the binomial expansion is often given:

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2\cdot 1} x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3\cdot 2\cdot 1} x^{n-3}y^3 + \frac{n(n-1)(n-2)(n-3)}{4\cdot 3\cdot 2\cdot 1} x^{n-4}y^4 + \cdots + y^n.$$

Problem Set 9

1. How many terms are there in the expansion of $(x + y)^5$? of $(x + y)^n$?

2. Write out the expansion of $(x + y)^6$ with the coefficients in the $C(n, r)$ form. Substitute 1 for x and 1 for y and so evaluate the sum

$$C(6,0) + C(6,1) + C(6,2) + C(6,3) + C(6,4) + C(6,5) + C(6,6).$$

3. Substitute 1 for x and -1 for y in the expansion of $(x + y)^6$ and so evaluate the sum

$$C(6,0) - C(6,1) + C(6,2) - C(6,3) + C(6,4) - C(6,5) + C(6,6).$$

4. What is the coefficient of u^3v^7 in $(u + v)^{10}$, expressed as a natural number?

5. Write out the full expansion of $(u + v)^7$ with the coefficients written as natural numbers.

6. Verify that the expansion of $(x + y)^8$ can be expressed in this way: the sum of all terms of the form

$$\frac{8!}{a!b!} x^a y^b,$$

where a and b range over all pairs of non-negative integers a and b such that $a + b = 8$.

7. Verify that $(x + y)^n$ is the sum of all terms of the form

$$\frac{n!}{a!b!}\,x^a y^b,$$

where a and b range over all possible pairs of non-negative integers such that $a + b = n$.

8. Without expanding the product

$$(a + b + c)(d + e + f)(p + q + r + s)(x + y + u + v + w)$$

answer the following questions: How many terms will there be? Which of the following are actual terms in the expansion? *adps, bdsw, bfpu, bfxw.*

3.5 The Multinomial Expansion

The idea of the preceding section carries over from binomials to sums of more than two elements. As an example consider the expression

$$(x + y + z + w)^{17};$$

this, by definition, is a product of seventeen identical factors $x + y + z + w$:

$$(x + y + z + w)(x + y + z + w) \cdots (x + y + z + w).$$

The expansion of this has a term, for example, of the form $x^4y^5z^6w^2$, because the sum of the exponents is $4 + 5 + 6 + 2 = 17$. This particular term occurs in the expansion as often as x can be chosen from four of the seventeen factors, y from five of the remaining thirteen factors, z from six of the remaining eight factors, and w then taken automatically from the other two factors. Paralleling the argument made in the case of the binomial expansion, we see that this is simply

$$C(17, 4)\cdot C(13, 5)\cdot C(8, 6)\cdot 1 = \frac{17!}{13!4!}\cdot\frac{13!}{8!5!}\cdot\frac{8!}{6!2!} = \frac{17!}{4!5!6!2!}.$$

The expansion of $(x + y + z + w)^{17}$ has been shown to contain the term

$$\frac{17!}{4!5!6!2!} x^4y^5z^6w^2.$$

This coefficient is very much like the numbers obtained in Section 3.2; and this is not surprising since all we are calculating here is the number of ways of ordering the following seventeen letters:

$$x\,x\,x\,x\,y\,y\,y\,y\,y\,z\,z\,z\,z\,z\,z\,w\,w.$$

More generally, we can say that the expansion of $(x + y + z + w)^{17}$ is the sum of all terms of the form

$$\frac{17!}{a!b!c!d!} x^ay^bz^cw^d,$$

where a, b, c, d range over all possible sets of non-negative integers satisfying $a + b + c + d = 17$. As a simple case we note the solution $a = 17$, $b = 0$, $c = 0$, $d = 0$, belonging to the term

$$\frac{17!}{17!0!0!0!} x^{17}y^0z^0w^0$$

or more simply x^{17} in the expansion. Other solutions of

$$a + b + c + d = 17$$

are, for example, $a = 4$, $b = 5$, $c = 6$, $d = 2$ and $a = 4$, $b = 5$, $c = 2$, $d = 6$ and belong to the terms

$$\frac{17!}{4!5!6!2!} x^4y^5z^6w^2 \quad \text{and} \quad \frac{17!}{4!5!2!6!} x^4y^5z^2w^6,$$

respectively.

Further generalization is apparent. For any positive integer n (in place of the special number 17) we see that the expansion of $(x + y + z + w)^n$ is the sum of all terms of the form

$$\frac{n!}{a!b!c!d!} x^ay^bz^cw^d,$$

where a, b, c, d range over all solutions of $a + b + c + d = n$ in non-negative integers.

There is no reason to restrict attention to a sum of four elements x, y, z, w. For any positive integer n, the multinomial expansion of

$$(x + y + z + w + \cdots)^n$$

is the sum of all terms of the form

$$\frac{n!}{a!b!c!d! \cdots} x^a y^b z^c w^d \cdots,$$

where $a, b, c, d, \cdots$ range over all solutions of

$$a + b + c + d + \cdots = n$$

in non-negative integers.

Problem Set 10

1. Write out the trinomial expansion of $(x + y + z)^4$ in full.

2. What is the coefficient of $x^2y^2z^2w^2u^2$ in the expansion of

$$(x + y + z + w + u)^{10}?$$

3. What is the coefficient of $xyzwuv$ in the expansion of

$$(x + y + z + w + u + v)^6?$$

4. What is the sum of all the coefficients in the expansion of $(x + y + z)^8$? of $(x + y + z + w)^{17}$?

5. What is the sum of all numbers of the form

$$\frac{12!}{a!b!c!},$$

where a, b, c range over all non-negative integers satisfying

$$a + b + c = 12?$$

3.6 Pascal's Triangle

The binomial coefficients in the expansion of $(x + y)^n$ form an interesting pattern if listed with increasing values of n. We begin with $(x + y)^0 = 1$ to give symmetry to the table:

1									from $(x + y)^0$
1	1								from $(x + y)^1$
1	2	1							from $(x + y)^2$
1	3	3	1						from $(x + y)^3$
1	4	6	4	1					etc.
1	5	10	10	5	1				
1	6	15	20	15	6	1			
1	7	21	35	35	21	7	1		
1	8	28	56	70	56	28	8	1	

This array, listed here as far as $n = 8$, is called Pascal's triangle.

The recursion relation $C(n, r) = C(n - 1, r) + C(n - 1, r - 1)$ of Section 3.3 reveals how this table can be made and extended without difficulty. For example, the three numbers 21, 35 and 56 that have been circled are the same as $C(7, 2)$, $C(7, 3)$ and $C(8, 3)$, the last of which is the sum of the first two by the recursion relation. Thus, any number $C(n, r)$ in Pascal's triangle is the sum of the number directly above, $C(n - 1, r)$, and the one to the left of that, $C(n - 1, r - 1)$. For example, if we wanted to extend the above table to the next row, namely the tenth row, we would write

$$1,\quad 1 + 8,\quad 8 + 28,\quad 28 + 56,\quad 56 + 70,\quad 70 + 56,\quad 56 + 28,\quad 28 + 8,\quad 8 + 1,\quad 1$$

or

$$1,\ 9,\ 36,\ 84,\ 126,\ 126,\ 84,\ 36,\ 9,\ 1\ .$$

These numbers are the coefficients in the expansion of $(x + y)^9$, and if we substitute $x = 1$ and $y = 1$ we get $(1 + 1)^9$ or 2^9.

Hence the sum of the elements in the tenth row of Pascal's triangle, $1 + 9 + 36 + 84 + \cdots$, is 2^9. In general, if we substitute $x = 1$ and $y = 1$ in $(x + y)^n$ we get 2^n, and so we can conclude that the sum of the elements in the $(n + 1)$st row of Pascal's triangle is

$$C(n, 0) + C(n, 1) + C(n, 2) + \cdots + C(n, n) = 2^n. \tag{3.6}$$

On the other hand, if we substitute $x = 1$ and $y = -1$ in $(x + y)^n$ we get 0^n or 0, and so we can conclude that

$$C(n, 0) - C(n, 1) + C(n, 2) - C(n, 3) + \cdots + (-1)^n C(n, n) = 0. \tag{3.7}$$

Problem Set 11

1. Extend Pascal's triangle to include $n = 9, 10, 11, 12, 13$.

2. Prove that the sum of the elements in the ninth row equals the sum of the elements of all previous rows, with 1 added.

3. Prove that in any row of Pascal's triangle the sum of the first, third, fifth, $\cdots$ elements equals the sum of the second, fourth, sixth, $\cdots$ elements.

3.7 The Number of Subsets of a Set

A man says to his son, "In cleaning up the attic I came across seven issues of an old magazine named *Colliers*. Look them over and take any you want. Whatever you don't want I will throw away." How many different selections are possible? Another way of stating this problem is: How many subsets are there of a set of seven things?

One way to solve the problem is to say that the son may select all seven, $C(7, 7)$, or six out of seven, $C(7, 6)$, or five out of seven, $C(7, 5)$, and so on. This gives the answer

$$C(7, 7) + C(7, 6) + C(7, 5) + C(7, 4) + C(7, 3) + C(7, 2) + C(7, 1) + C(7, 0).$$

By formula (3.6) on page 42 this is the same as 2^7.

Another way to solve the problem is to concentrate on the individual copies of the magazine rather than on sets of them. Let us denote the seven issues of the magazine by A, B, C, D, E, F, G. Then A may be taken or rejected (two possibilities); B may be taken or rejected (two possibilities); $\cdots$; G may be taken or rejected (two possibilities). Using the multiplication principle we have the answer

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7.$$

Thus a set of seven different things has 2^7 subsets, including the whole set of all seven, and the empty set or null set with no elements. If we disregard the whole set, the others are called *proper subsets*, and so a set of seven different things has $2^7 - 1$ proper subsets. In general, a set of n different things has 2^n subsets of which $2^n - 1$ are proper subsets. Among these subsets there are exactly $C(n, r)$ having r members.

Problem Set 12

1. How many different sums of money can be made up using one or more coins selected from a cent, a nickel, a dime, a quarter, a half dollar, and a silver dollar?

2. The members of a club are to vote "yes" or "no" on each of eight issues. In marking his ballot, a member has the option of abstaining on as many as seven of the issues, but he should not abstain in all eight cases. In how many ways can a ballot be marked?

3. A travel agency has ten different kinds of free folders. The agent tells a boy to take any he wants, but not more than one of a kind. Assuming that the boy takes at least one folder, how many selections are possible?

4. A biologist is studying patterns of male (M) and female (F) children in families. A family type is designated by a code; for example, FMM denotes a family of three children of which the oldest is a female and the other two males. Note that FMM, MFM, and MMF are different types. How many family types are there among families with at least one but not more than seven children?

3.8 Sums of Powers of Natural Numbers

As a by-product of the theory of combinations we can get formulas for the sum

$$1 + 2 + 3 + 4 + \cdots + n$$

of positive integers (natural numbers) from 1 to n, for the sum of their squares,

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2,$$

for the sum of their cubes, and so on. The idea is to use the recursion relation (3.2) for $C(n, r)$, which we rewrite in the form

$$C(n-1, r-1) = C(n, r) - C(n-1, r). \tag{3.8}$$

As an example to illustrate the method, let us write formula (3.8) in succession with $n = 9$, $n = 8$, $n = 7$, $\cdots$, $n = 3$, but with $r = 2$ in all cases:

$$\begin{aligned} C(8, 1) &= C(9, 2) - C(8, 2) \\ C(7, 1) &= C(8, 2) - C(7, 2) \\ C(6, 1) &= C(7, 2) - C(6, 2) \\ C(5, 1) &= C(6, 2) - C(5, 2) \\ C(4, 1) &= C(5, 2) - C(4, 2) \\ C(3, 1) &= C(4, 2) - C(3, 2) \\ C(2, 1) &= C(3, 2) - C(2, 2). \end{aligned} \tag{3.9}$$

If we add these equations there is much cancellation on the right, with the result

$$\begin{aligned} &C(2, 1) + C(3, 1) + C(4, 1) + C(5, 1) \\ &\qquad + C(6, 1) + C(7, 1) + C(8, 1) = C(9, 2) - C(2, 2), \end{aligned}$$

$$2 + 3 + 4 + 5 + 6 + 7 + 8 = \tfrac{1}{2}9 \cdot 8 - 1.$$

After adding 1 to both sides of this identity, we see that we have found the sum of the natural numbers from 1 to 8 by an indirect method, and this sum is $\frac{1}{2}(72)$ or 36.

To do this in general, replace n in formula (3.8) successively by $m + 1$, m, $m - 1$, $\cdots$, 4, 3, keeping r fixed as before, $r = 2$. This gives a chain of equations

$$
\begin{aligned}
C(m, 1) &= C(m + 1, 2) - C(m, 2) \\
C(m - 1, 1) &= C(m, 2) - C(m - 1, 2) \\
C(m - 2, 1) &= C(m - 1, 2) - C(m - 2, 2) \qquad (3.10)\\
&\cdots\cdots\cdots\cdots\cdots \\
C(3, 1) &= C(4, 2) - C(3, 2) \\
C(2, 1) &= C(3, 2) - C(2, 2).
\end{aligned}
$$

Adding these equations we note that again there is considerable cancellation on the right side, and the result is

$$
\begin{aligned}
C(2, 1) &+ C(3, 1) + \cdots + C(m-2, 1) \\
&+ C(m-1, 1) + C(m, 1) = C(m+1, 2) - C(2, 2),
\end{aligned}
$$

or

$$2 + 3 + \cdots + (m - 2) + (m - 1) + m = \tfrac{1}{2}(m + 1)m - 1,$$

so that

$$1 + 2 + 3 + \cdots + (m-2) + (m-1) + m = \tfrac{1}{2}m(m+1). \qquad (3.11)$$

It should be understood that if for example $m = 2$, the left side is to be interpreted simply as $1 + 2$.

To get a formula for the sum of the squares of the natural numbers, we write the analogues of (3.10) with all values of n and r raised by 1; that is, we write equation (3.8) with $r = 3$ and n replaced successively by $m + 2$, $m + 1$, m, $\cdots$, 5, 4 to get

$$
\begin{aligned}
C(m + 1, 2) &= C(m + 2, 3) - C(m + 1, 3) \\
C(m, 2) &= C(m + 1, 3) - C(m, 3) \\
C(m - 1, 2) &= C(m, 3) - C(m - 1, 3) \qquad (3.12)\\
&\cdots\cdots\cdots\cdots\cdots \\
C(4, 2) &= C(5, 3) - C(4, 3) \\
C(3, 2) &= C(4, 3) - C(3, 3).
\end{aligned}
$$

Adding these, and then substituting the values for the combination symbols, we have

$$C(3, 2) + C(4, 2) + \cdots + C(m - 1, 2) + C(m, 2) + C(m + 1, 2) = C(m + 2, 3) - C(3, 3),$$

$$\tfrac{1}{2}3 \cdot 2 + \tfrac{1}{2}4 \cdot 3 + \cdots + \tfrac{1}{2}m(m - 1) + \tfrac{1}{2}(m + 1)m = \tfrac{1}{6}(m + 2)(m + 1)m - 1.$$

By adding 1 to both sides, then multiplying both sides by 2, we bring this into the form

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (m - 1)m + m(m + 1) = \tfrac{1}{3}m(m + 1)(m + 2).$$

The products on the left side of this equation can be rewritten as

$$\begin{aligned} 1 \cdot 2 &= 1(1 + 1) = 1^2 + 1, \\ 2 \cdot 3 &= 2(2 + 1) = 2^2 + 2, \\ 3 \cdot 4 &= 3(3 + 1) = 3^2 + 3, \\ &\cdots\cdots\cdots\cdots \\ (m-2)(m-1) &= (m-2)[(m-2) + 1] = (m-2)^2 + (m-2), \\ (m-1)m &= (m-1)[(m-1) + 1] = (m-1)^2 + (m-1), \\ m(m + 1) &= m^2 + m. \end{aligned}$$

Substituting these we obtain

$$1^2 + 1 + 2^2 + 2 + 3^2 + 3 + \cdots + (m - 1)^2 + (m - 1) + m^2 + m = \tfrac{1}{3}m(m + 1)(m + 2)$$

or

$$[1^2 + 2^2 + \cdots + (m - 1)^2 + m^2] + [1 + 2 + \cdots + (m - 1) + m] = \tfrac{1}{3}m(m + 1)(m + 2).$$

We recognize the second expression in brackets as the sum of the first m natural numbers calculated in formula (3.11). Using formula (3.11) we see that

$$[1^2 + 2^2 + \cdots + (m-1)^2 + m^2] + \tfrac{1}{2}m(m+1) = \tfrac{1}{3}m(m+1)(m+2).$$

This can be rewritten as

$$[1^2 + 2^2 + \cdots + (m-1)^2 + m^2] = \tfrac{1}{3}m(m+1)(m+2) - \tfrac{1}{2}m(m+1).$$

The right member of this equation reduces to $\frac{1}{6}m(m+1)(2m+1)$, as can be readily calculated by simple algebra. Hence we have derived the formula for the sum of the squares of the first m natural numbers:

$$1^2 + 2^2 + 3^2 + \cdots + m^2 = \tfrac{1}{6}m(m+1)(2m+1).$$

Problem Set 13

1. Find the sum of the integers from 1 to 100 inclusive.

2. Do the preceding problem again by writing the sum of the numbers both forwards and backwards:†

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & \cdots\cdots\cdots\cdots & 97 & 98 & 99 & 100 \\ 100 & 99 & 98 & 97 & \cdots\cdots\cdots\cdots & 4 & 3 & 2 & 1 \end{array}$$

   Note that the addition of each pair of numbers in this array (i.e., $1 + 100$, $2 + 99$, $3 + 98$, $4 + 97$, etc.) always gives a total of 101. Hence the sum of the numbers from 1 to 100, taken twice, is the same as 101 added to itself 100 times. The rest is left to the reader.

3. Generalize the method outlined in the preceding problem to the integers from 1 to n inclusive, and thereby derive formula (3.11) in a different way. (Note that this device will not work on the sum of the squares of the numbers from 1 to n.)

† This method of adding the integers from 1 to 100 is said to have been used by a famous nineteenth century mathematician, C. F. Gauss, when he was a schoolboy. A teacher (so the story goes) assigned the problem to the class, hoping to keep them occupied for perhaps fifteen or twenty minutes, and was startled when the young Gauss gave the answer in a much shorter time.

4. Find the sum of the squares of the integers from 1 to 100 inclusive.

5. How many solutions in positive integers x and y are there of the equation $x + y = 100$? (In counting solutions in this and subsequent problems, treat such solutions as $x = 10$, $y = 90$ and $x = 90$, $y = 10$, as different. In other words, by a solution, we mean an ordered pair (x, y) that satisfies the equation.) How many solutions in non-negative integers?

6. How many solutions in positive integers are there of the equation $x + y = n$, where n is a fixed positive integer? How many solutions in non-negative integers?

7. How many ordered triples (x, y, z) of positive integers are solutions of $x + y + z = 100$? How many triples of non-negative integers?

8. Generalize Problem 7 to $x + y + z = n$.

9. How many terms are there in the expansion of $(x + y + z)^3$? of $(x + y + z)^4$? of $(x + y + z)^n$?

10. Extend the procedure used in equations (3.10), (3.11), (3.12) and so on, to get a formula for $1^3 + 2^3 + 3^3 + 4^3 + \cdots + m^3$.

3.9 Summary

Given n things which are alike in batches: a are alike; another b are alike; another c are alike, and so on; then the number of permutations of the n things taken all at a time is

$$\frac{n!}{a!b!c!\cdots}.$$

Pascal's formula for $C(n, r)$ is

$$C(n, r) = C(n - 1, r) + C(n - 1, r - 1).$$

The binomial expansion of $(x + y)^n$, for any positive integer n, is

$$(x + y)^n = C(n, 0)x^n + C(n, 1)x^{n-1}y + C(n, 2)x^{n-2}y^2 + \cdots + C(n, j)x^{n-j}y^j + \cdots + C(n, n)y^n.$$

An alternative notation for this expansion is

$$(x + y)^n = x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \binom{n}{3} x^{n-3}y^3$$
$$+ \cdots + \binom{n}{j} x^{n-j}y^n + \cdots + y^n.$$

The multinomial expansion of $(x + y + z + w + \cdots)^n$ is the sum of all terms of the form

$$\frac{n!}{a!b!c!d! \cdots} x^a y^b z^c w^d \cdots,$$

where $a, b, c, d, \cdots$ range over all solutions of

$$a + b + c + d + \cdots = n$$

in non-negative integers.

Pascal's triangle is given as far as $n = 8$ in Section 3.6. Also the following two relations were proved:

$$C(n, 0) + C(n, 1) + C(n, 2) + C(n, 3) + \cdots + C(n, n) = 2^n,$$
$$C(n, 0) - C(n, 1) + C(n, 2) - C(n, 3) + \cdots + (-1)^n C(n, n) = 0.$$

The number of subsets of a set of n different things is 2^n; the number of proper subsets is $2^n - 1$; the number of subsets having r elements is $C(n, r)$.

Methods for deriving the sum of kth powers of the first n natural numbers were sketched. In particular, the sum of the natural numbers from 1 to n (i.e., $k = 1$)and the sum of their squares (i.e. $k = 2$) were obtained:

$$1 + 2 + 3 + 4 + \cdots + n = \tfrac{1}{2}n(n + 1),$$
$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \tfrac{1}{6}n(n + 1)(2n + 1).$$

CHAPTER FOUR

Some Special Distributions

Many a problem in combinatorial analysis is solved by first reformulating it. This point was illustrated on page 28 where a path problem was reduced to an equivalent question involving combinations. In this chapter we look at some other problems which, when viewed from the proper perspective, also reduce to questions involving combinations.

4.1 Fibonacci Numbers

Consider the question: In how many ways can eight plus signs and five minus signs be lined up in a row so that no two minus signs are adjacent? An example of such an arrangement is:

$$+ + - + - + + + - + - + - .$$

The problem is easy to solve if we look at it this way; write the eight plus signs as an expression (4.1) with m's between, and also m's at the start and finish:

$$m + m + m + m + m + m + m + m + m . \tag{4.1}$$

Thus we have eight plus signs and nine m's. Now the five minus signs can be selected as any of the nine m's, and so the answer to the question is $C(9, 5)$.

In general, we can say that *the number of ways of writing k plus signs and r minus signs in a row so that no two minus signs are adjacent is* $C(k + 1, r)$. The reason for this is exactly the same as in the special case above where $k = 8$ and $r = 5$. In place of (4.1) we now have

$$m + m + m + m + \cdots + m + m, \tag{4.2}$$

namely k plus signs separating $k + 1$ symbols m. We convert r of the m's into minus signs and let the others disappear. Thus we select r out of the $k + 1$ symbols m, turn them into minus signs, and obtain the answer $C(k + 1, r)$.

If r exceeds $k + 1$ the notation $C(k + 1, r)$ has value zero. This is as it should be because, if $k = 8$ and $r = 12$ for example, there is no way to write eight plus signs and twelve minus signs in a row so that no two minus signs are adjacent.

Next we turn to another question. Consider a series of ten x's in a row,

$$x \ x \ x \ x \ x \ x \ x \ x \ x \ x \, .$$

Suppose that each x may be a plus sign or a minus sign, so that there are 2^{10} or 1024 cases in all. The question is: How many of the 1024 cases do not have two minus signs in adjacent positions? By considering successively the types,

10 plus signs,

9 plus signs and 1 minus sign,

8 plus signs and 2 minus signs,

7 plus signs and 3 minus signs, etc.,

and using the result previously developed, we get the answer

$$C(11, 0) + C(10, 1) + C(9, 2) + C(8, 3) + C(7, 4) + C(6, 5).$$

There are other terms, $C(5, 6)$, $C(4, 7)$, and so on, but each of these is zero. The calculation is simple:

$$1 + 10 + 36 + 56 + 35 + 6 = 144.$$

More generally, let us use n in place of 10. If there were n symbols x in a row, and if each x could be a plus sign or a minus sign, there would be 2^n cases in all. Of these, the number not having two minus signs in adjacent positions is

$$(4.3) \qquad C(n+1, 0) + C(n, 1) + C(n-1, 2) + C(n-2, 3) + \cdots,$$

where the sum continues until symbols of the sort $C(u, v)$ with $u < v$ are reached; such symbols denote zero by definition. Thus (4.3) denotes *the number of ways of writing n signs in a row, each being a plus sign or a minus sign, so that no two minus signs are adjacent*. This number is a function of n; we write it as $F(n)$, and then look at the question another way.

First consider those of the $F(n)$ sequences of signs that begin with a plus. They are $F(n-1)$ in number, because each of them is obtained by placing a plus sign in front of each of the $F(n-1)$ arrangements of $n-1$ signs.

Second consider those of the $F(n)$ sequences of signs that begin with a minus. The next sign in any such sequence must be plus, since adjacent minus signs are ruled out. Thus we are considering sequences of n signs beginning with $-\,+$. These are $F(n-2)$ in number, because each of them can be obtained by placing the pair $-\,+$ in front of each of the $F(n-2)$ arrangements of $n-2$ signs. Thus we conclude that

$$(4.4) \qquad F(n) = F(n-1) + F(n-2).$$

This formula is a recursion relation for the Fibonacci numbers, as the numbers $F(n)$ are called. It can be used to compute $F(n)$ whenever $F(n-1)$ and $F(n-2)$ are known. For $n = 3$, for example, it states that

$$F(3) = F(2) + F(1).$$

From the definition (4.3) of $F(n)$ we see that $F(1) = 2$ and

$F(2) = 3$, so that $F(3) = 5$. Similarly, by use of (4.4) we can compute

$$\begin{aligned}
F(4) &= F(3) + F(2) = 5 + 3 = 8,\\
F(5) &= F(4) + F(3) = 8 + 5 = 13,\\
F(6) &= F(5) + F(4) = 13 + 8 = 21,\\
F(7) &= F(6) + F(5) = 21 + 13 = 34, \text{ etc.}
\end{aligned}$$

The Fibonacci sequence† is usually written with additional terms at the start, for example $1, 2, 3, 5, 8, 13, \cdots$, or $1, 1, 2, 3, 5, 8, 13, \cdots$, or $0, 1, 1, 2, 3, 5, 8, 13, \cdots$. We shall use the first of these three versions. To do this we define $F(0) = 1$, and then use our results $F(1) = 2$, $F(2) = 3$, etc., each term being the sum of the preceding two.

Fibonacci numbers are usually *defined* by property (4.4), $F(n) = F(n - 1) + F(n - 2)$, whereas we approached them through a problem in arrangements. This approach gave us not only formula (4.4), but also the fact that $F(n)$ is expressible in the form (4.3), which can be visualized by means of Pascal's triangle:

$F(0)$	1	1							
$F(1)$	1	2	1						
$F(2)$	1	3	3	1					
$F(3)$	1	4	6	4	1				
$F(4)$	1	5	10	10	5	1			
$F(5)$	1	6	15	20	15	6	1		
$F(6)$	1	7	21	35	35	21	7	1	
$F(7)$	1	8	28	56	70	56	28	8	1

The sum of the elements along the diagonals are the values of $F(n)$. For example, consider $F(5)$. By (4.3) we see that

† For another approach to these sequences see, for example, C. D. Olds, *Continued Fractions*, Vol. 9 in this series, p. 80.

$$\begin{aligned} F(5) &= C(6,0) + C(5,1) + C(4,2) + C(3,3) \\ &= 1 + 5 + 6 + 1 = 13. \end{aligned}$$

Problem Set 14

1. Evaluate $F(11)$ by use of (4.4). Check the answer by use of (4.3).

2. For which integers n is $F(n)$ an even number, for which odd?

3. What does formula (4.4) become if n is replaced by $n + 1$?

4. Prove that $F(n+1) = 2F(n-1) + F(n-2)$.

5. In how many ways can ten A's and six B's be lined up in a row so that no two B's are adjacent?

6. In how many ways can ten A's, six B's and five C's be lined up in a row so that no two B's are adjacent?

7. How many permutations are there of the letters of the word *Mississippi*, taken all at a time, subject to the restriction that no two *i*'s are adjacent?

4.2 Linear Equations with Unit Coefficients

Consider the solutions of the equation $x + y + z + w = 12$ in positive integers x, y, z, w. (Recall that the positive integers are $1, 2, 3, 4, \cdots$.) In counting the solutions we shall say that, for example,

$$\begin{array}{llll} x = 9 & x = 1 & x = 1 & x = 1 \\ y = 1 & y = 9 & y = 1 & y = 1 \\ z = 1 & z = 1 & z = 9 & z = 1 \\ w = 1 & w = 1 & w = 1 & w = 9 \end{array}$$

are four different solutions because, as *ordered* quadruples of integers, they are distinct. In general, solutions are regarded as the same only if the values of x are equal, of y are equal, of z are equal, and of w are equal. (The situation where the four solutions just given are treated as though they were a single solution comes under the heading of "partitions of the number 12"; such questions will be considered in Chapter 6.) The number of solutions of the given equation is easily determined by viewing the problem in this way: If 12 units (denoted by 12 u's) separated by 11 spaces (denoted by 11 s's) are lined up,

$$u\,s\,u\,s\,u\,s\,u\,s\,u\,s\,u\,s\,u\,s\,u\,s\,u\,s\,u\,s\,u\,s\,u\,, \tag{4.5}$$

and if we choose any 3 of the s's and let the others disappear, for example,

$$u\quad u\,s\,u\quad u\quad u\quad u\,s\,u\quad u\quad u\quad u\quad u\,s\,u\,,$$

then the remaining s's separate the units into four batches. The number of units in these batches can be used as the values of x, y, z, w. In the example the values are $x = 2$, $y = 4$, $z = 5$, $w = 1$. Thus any selection of three of the s's in (4.5) gives a solution of $x + y + z + w = 12$ in positive integers, and any solution corresponds to such a selection. It follows that the number of solutions of this equation in positive integers is the same as the number of ways of choosing 3 things out of 11, which is

$$C(11, 3) = \frac{11\cdot 10\cdot 9}{3\cdot 2\cdot 1} = 165\,.$$

For a more general result let us replace 12 by m and $x + y + z + w$ by a sum of k variables, and ask for the number of solutions in positive integers of the equation

$$x_1 + x_2 + x_3 + \cdots + x_k = m\,. \tag{4.6}$$

In the special case just studied, $m = 12$ and $k = 4$, so the equation could have been written $x_1 + x_2 + x_3 + x_4 = 12$; using x_1, x_2, x_3, x_4 in place of x, y, z, w makes no difference in the number of solutions, of course.

The number of solutions of equation (4.6) *in positive integers* $x_1, x_2, \cdots, x_k$ *is*

$$C(m - 1, k - 1). \tag{4.7}$$

This can be seen by a direct generalization of the argument used in the special case $m = 12$, $k = 4$. We now have the symbol u repeated m times and the symbol s repeated $m - 1$ times, separating the u's:

$$u\ s\ u\ s\ u\ s\ u\ s \cdots s\ u\ s\ u\ .$$

We choose $k - 1$ of the symbols s (and let the others disappear) in order to separate the u's into k batches. Such a selection will give a unique solution of (4.6), namely x_1 is the number of u's in the first batch, x_2 the number in the second batch, and so on. Hence the number of solutions of (4.6) in positive integers is the same as the number of ways of selecting $k - 1$ of the symbols s out of $m - 1$, and that number is given in (4.7).

It may be noted that if k is larger than m, then equation (4.6) has no solutions in positive integers, and formula (4.7) is still valid since by our definition (see p. 21) the value of $C(m - 1, k - 1)$ is zero in this case.

Let us turn to the question of the number of solutions of an equation in non-negative integers, the difference here being that zero values for the variables are now allowed. To begin with a special case, we ask for the number of solutions of the equation

$$x + y + z + w = 12$$

in non-negative integers. Let us take 12 u's and 3 s's lined up in a row, for example:

$$u\ s\ u\ u\ u\ u\ u\ s\ u\ u\ s\ u\ u\ u\ u \tag{4.8}$$

If we look upon the s's as separators of the 12 u's into four batches, we see that the illustration (4.8) gives batches of 1, 5, 2, 4 corresponding to the solution $x = 1$, $y = 5$, $z = 2$, $w = 4$ of the equation $x + y + z + w = 12$ under discussion.

We claim that every arrangement of 12 u's and 3 s's gives a solution. To find the value of x, we locate the first s and count the number of u's to its left. If the arrangement begins with an s, $x = 0$ because no u's are to its left. To find the value of y, we count the number of u's between the first and second s. If the first and second s are adjacent, then $y = 0$. Similarly, the value of z

is the number of u's between the second and third s, and the value of w is the number of u's to the right of the third s.

Conversely, if we start with any solution, such as $x = 0$, $y = 1$, $z = 2$, $w = 9$ we can write 12 u's and 3 s's in a row

$$s\ u\ s\ u\ u\ s\ u\ u\ u\ u\ u\ u\ u\ u\ u$$

corresponding to this solution. The solution $x = 9$, $y = 0$, $z = 1$, $w = 2$ corresponds to the arrangement

$$u\ u\ u\ u\ u\ u\ u\ u\ u\ s\ s\ u\ s\ u\ u\ .$$

Thus the number of solutions of $x + y + z + w = 12$ in non-negative integers is the same as the number of ways of writing 12 u's and 3 s's in a row. This number is $C(15, 3)$ by the theory in Section 3.2.

We generalize this result to the equation

$$x_1 + x_2 + x_3 + \cdots + x_k = m\ , \tag{4.9}$$

in k variables whose sum must always be m, by an argument similar to that given in the special case. The number of solutions of equation (4.9) in non-negative integers is the same as the number of arrangements in a row of m u's and $k - 1$ s's, and there are

$$C(m + k - 1, k - 1)$$

such arrangements.

Let m and k be fixed positive integers. The number of solutions of equation (4.9) *in non-negative integers is*

$$C(m + k - 1, k - 1) \qquad \text{or} \qquad C(m + k - 1, m) \tag{4.10}$$

The second part of (4.10) follows from the first by virtue of the basic property $C(n, r) = C(n, n - r)$.

Problem Set 15†

1. How many solutions are there of $x + y + z + w = 50$ (a) in positive integers? (b) in non-negative integers?

2. Prove that the number of solutions of $x_1 + x_2 + x_3 + x_4 = 9$ in positive integers is the same as the number of solutions of

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 9$$

in positive integers.

3. Check that the two forms given in (4.10) are equal.

4. Prove that the number of solutions of equation (4.6) in positive integers equals the number of solutions in positive integers of the equation in $m - k + 1$ variables

$$x_1 + x_2 + x_3 + \cdots + x_{m-k+1} = m.$$

5. Prove that the number of solutions in non-negative integers is the same for the two equations

$$x_1 + x_2 + \cdots + x_6 = 8 \quad \text{and} \quad x_1 + x_2 + \cdots + x_9 = 5.$$

6. How many integers between 1 and 1,000,000 inclusive have sum of digits (a) equal to 6; (b) less than 6?

7. How many terms are there in the expansion of

$$\text{(a)} \quad (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)^{17};$$

$$\text{(b)} \quad (\alpha_1 + \alpha_2 + \cdots + \alpha_k)^t,$$

where k and t are positive integers?

†Answers to these and subsequent problems may be given in the $C(n, r)$ notation. For example, an answer such as $C(37, 5) - C(26, 4)$ is acceptable as it stands without further elaboration or simplification.

4.3 Combinations with Repetitions

The symbol $C(n, r)$ denotes the number of combinations of n *different* things, taken r at a time. Suppose now that there are several identical copies of each of the n things, as for example identical copies of the books in a bookstore. Thus we can ask the question: Given n distinct categories of things, each category containing an unlimited supply, how many different combinations of r things are there? (A combination may include several indistinguishable objects from the same category.) This question amounts to asking for the number of solutions of

$$x_1 + x_2 + x_3 + \cdots + x_n = r$$

in non-negative integers, because we can take x_1 of the first thing, x_2 of the second thing, $\cdots$, x_n of the n-th thing. Thus we can use formula (4.10) with k and m replaced by n and r to get the following conclusion.

The number of combinations, r at a time, of n different things each of which is available in unlimited supply is

$$C(n + r - 1, r). \tag{4.11}$$

For example, let us ask how many different sets of three coins can be formed, each coin being a cent, a nickel, a dime or a quarter. Here we have $r = 3$ and $n = 4$, and so the answer, by (4.11), is $C(6, 3)$ or 20. Going back to the analysis that led to (4.11) we note that the question amounts to asking for the number of solutions in non-negative integers of $x_1 + x_2 + x_3 + x_4 = 3$, where we interpret x_1 as the number of cents, x_2 as the number of nickels, x_3 as the number of dimes, and x_4 as the number of quarters used in forming a set of 3 coins.

As another example, consider Problem 1.4 of Chapter 1. Briefly, the problem is this: Consider twenty different books each of which is available in unlimited supply. How many selections of ten books can be made, (a) if repetitions are allowed, (b) if repetitions are not allowed, so that the ten books must all be different? Part (b) is just a matter of choosing ten things out of twenty, so the answer is $C(20, 10)$. Part (a) is a question of the number of combinations, ten at a time, of twenty different things each of which is available in multiple copies. So the answer to part (a) is given by formula (4.11) with $r = 10$ and $n = 20$, namely

$$C(29, 10) = \frac{29!}{10!19!}.$$

Problem Set 16

1. In the explanation of formula (4.11) in the text it is stated that each of the things is "available in unlimited supply". Actually the supply need not be unlimited. How many copies of each of the n things must there be?

2. How many different collections of six coins can be formed, if each coin may be a cent, a nickel, a dime, a quarter, a half dollar, or a silver dollar?

3. Poker chips come in three colors, red, white, and blue. How many different combinations of ten poker chips are there?

4. A toy store has marbles in five colors, all uniform in size. They are priced at 10 cents a dozen. How many different color combinations are available for ten cents?

5. Consider the integers with seven digits, namely the integers from 1,000,000 to 9,999,999 inclusive. Separate these into subsets as follows: put numbers into the same subset if and only if their digits as a collection are the same. For example, 8,122,333 and 3,213,283 are in the same subset. How many subsets are there?

4.4 Equations with Restricted Solutions

In Section 4.2 we considered the question of the number of solutions of such an equation as $x + y + z + w = 12$; first the solutions were restricted to the set of positive integers, then to the set of non-negative integers. To say that x is a positive integer is the same as saying that x is an integer satisfying $x > 0$, or that x is an integer satisfying $x \geqslant 1$. To say that x is a non-negative integer is the same as saying that x is an integer satisfying $x \geqslant 0$.

Now we raise the question: How many solutions in integers greater than 5 are there of the equation

$$x + y + z + w = 48 \, ? \tag{4.12}$$

For example, $x = 6$, $y = 10$, $z = 12$, $w = 20$ is a solution, and we want to count the number of such solutions. Since each variable must be greater than 5, the subtraction of 5 from each gives a new set of four positive numbers

$$r = x - 5, \quad s = y - 5, \quad t = z - 5, \quad u = w - 5 \tag{4.13}$$

whose sum is 28:

$$\begin{aligned} r + s + t + u &= x - 5 + y - 5 + z - 5 + w - 5 \\ &= x + y + z + w - 20 = 28, \end{aligned}$$

$$r + s + t + u = 28 \, . \tag{4.14}$$

Thus equations (4.14) and (4.12) are related by the substitutions, or transformation, (4.13). For example, the solution $x = 6$, $y = 10$, $z = 12$, $w = 20$ of equation (4.12) corresponds to the solution $r = 1$, $s = 5$, $t = 7$, $u = 15$ of equation (4.14). Here are other examples:

$$\begin{matrix} x = 10 & & r = 5 \\ y = 11 & \text{corresponds to} & s = 6 \\ z = 13 & & t = 8 \\ w = 14 & & u = 9\,; \end{matrix}$$

$$\begin{matrix} x = 25 & & r = 20 \\ y = 9 & \text{corresponds to} & s = 4 \\ z = 8 & & t = 3 \\ w = 6 & & u = 1. \end{matrix}$$

Now since each of x, y, z, w must be greater than 5, each of r, s, t, u must be greater than 0, that is to say, each of r, s, t, u must be a positive integer. Thus corresponding to each solution of equation (4.12) in integers greater than 5 there is a solution of equation (4.14) in positive integers, and to each solution of equation (4.14) in positive integers there corresponds a solution of equation (4.12) in integers greater than 5. For example, if we start with the solution

$$r = 15, \quad s = 1, \quad t = 8, \quad u = 4$$

of equation (4.14), we use the transformation (4.13) to get the solution

$$x = 20, \quad y = 6, \quad z = 13, \quad w = 9$$

of equation (4.12). This establishes a one-to-one correspondence between the solutions of equation (4.12) in integers exceeding 5, and the solutions of equation (4.14) in positive integers.

Hence we see that the number of solutions of equation (4.12) in integers greater than 5 is the same as the number of solutions of equation (4.14) in positive integers. Applying formula (4.7) of page 55 to equation (4.14), we see that this number is

$$C(28 - 1, 4 - 1) \qquad \text{or} \qquad C(27, 3).$$

The idea used here works just as well if each variable is subjected to a different condition. For example, let us consider the question: *How many solutions are there of the equation* $x + y + z + w = 48$ *in integers satisfying*

$$x > 5, \quad y > 6, \quad z > 7, \quad w > 8\,? \tag{4.15}$$

In this case we would subtract 5 from x, 6 from y, 7 from z, and 8 from w in any solution of the equation to get a new set of numbers (call them r, s, t, and u) each of which is a positive integer. Thus the transformation, this time, is

$$r = x - 5, \qquad s = y - 6, \qquad t = z - 7, \qquad u = w - 8,$$

or

$$x = r + 5, \qquad y = s + 6, \qquad z = t + 7, \qquad w = u + 8.$$

If we substitute these expressions in the equation $x+y+z+w=48$ we obtain the equation

$$r+5+s+6+t+7+u+8 = 48,$$

or

$$r+s+t+u = 22. \tag{4.16}$$

Any solution of $x+y+z+w = 48$ in integers subject to the restrictions (4.15) corresponds to a solution in positive integers of the equation (4.16). For example, the solution $x = 6$, $y = 10$, $z = 12$, $w = 20$ corresponds to the solution $r = 1$, $s = 4$, $t = 5$, $u = 12$ of the equation (4.16). There is a one-to-one correspondence between the solutions, so the number of solutions of

$$x+y+z+w = 48$$

in integers subject to the restrictions (4.15) equals the number of solutions of $r+s+t+u = 22$ in positive integers. By formula (4.7) of page 55 this number is

$$C(22-1, \quad 4-1) \qquad \text{or} \qquad C(21, 3).$$

Let us now formalize these ideas to get some general formulas. Let us replace 48 by m, and so consider the equation

$$x+y+z+w = m.$$

Furthermore, in place of the conditions (4.15) on x, y, z, and w, suppose we have the restrictions

$$x > c_1, \quad y > c_2, \quad z > c_3, \quad w > c_4, \tag{4.17}$$

where c_1, c_2, c_3, c_4 are some fixed integers. The transformation is now

$$r = x - c_1, \qquad s = y - c_2, \qquad t = z - c_3, \qquad u = w - c_4,$$

or

$$x = r + c_1, \qquad y = s + c_2, \qquad z = t + c_3, \qquad w = u + c_4.$$

If we substitute these in the equation $x + y + z + w = m$ we get the equation

$$r + c_1 + s + c_2 + t + c_3 + u + c_4 = m,$$

or

$$r + s + t + u = m - c_1 - c_2 - c_3 - c_4 . \tag{4.18}$$

Thus the number of solutions of the equation $x + y + z + w = m$ in integers subject to the restrictions (4.17) is the same as the number of solutions of (4.18) in positive integers r, s, t, u. By formula (4.7) of page 55 this number is

$$C(m-c_1-c_2-c_3-c_4-1, \quad 3) . \tag{4.19}$$

If we replace x, y, z, w, by x_1, x_2, x_3, x_4 respectively, the conclusion can be stated as follows: The number of solutions in integers x_1, x_2, x_3, x_4 of the equation

$$x_1 + x_2 + x_3 + x_4 = m$$

subject to the restrictions

$$x_1 > c_1, \quad x_2 > c_2, \quad x_3 > c_3, \quad x_4 > c_4$$

is given by the formula (4.19).

Suppose now that there are k variables $x_1, x_2, x_3, \cdots, x_k$ and k fixed integers $c_1, c_2, c_3, \cdots, c_k$. Then an immediate extension of the above theory gives the result:

The number of solutions of the equation

$$x_1 + x_2 + x_3 + \cdots + x_k = m \tag{4.20}$$

in integers satisfying the conditions

$$x_1 > c_1, \quad x_2 > c_2, \quad x_3 > c_3, \quad \cdots, \quad x_k > c_k \tag{4.21}$$

is

$$C(m - c_1 - c_2 - c_3 - \cdots - c_k - 1, \quad k - 1). \tag{4.22}$$

Note: Although in the examples leading up to this general result the integers c_1, c_2, etc. were taken to be *positive*, this proposition holds for any integers $c_1, c_2, c_3, \cdots, c_k$, positive, negative, or zero.

Problem Set 17

1. Write out in detail a one-to-one correspondence between the solutions of $x + y + z + w = 27$ in integers greater than 5, and the solutions of $r + s + t + u = 7$ in positive integers.

2. How many solutions are there of $x + y + z + w = 100$ in integers greater than 7?

3. Find the number of solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = 50$ in positive integers (a) with $x_5 > 12$; (b) with $x_5 > 12$ and $x_4 > 7$.

4. Find the number of solutions of $x + y + z + w = 1$ in integers greater than -4, i.e., in integers selected from $-3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots$.

5. Find the number of solutions of $x + y + z + w = 20$ in positive integers (a) with $x > 6$; (b) with $x > 6$ and $y > 6$; (c) with $x > 6$, $y > 6$, and $z > 6$.

6. Find the number of solutions of $x + y + z + w = 20$ in non-negative integers (a) with $x \geqslant 6$; (b) with $x \geqslant 6$ and $y \geqslant 6$.

7. Find a formula for the number of solutions of $x + y + z + w = m$ in non-negative integers satisfying (a) $x \geqslant c_1$; (b) $x \geqslant c_1$ and $y \geqslant c_2$.

8. How many integers between 1 and 1,000,000 inclusive have sum of digits 13?

4.5 Summary

The Fibonacci numbers $1, 2, 3, 5, 8, 13, 21, \cdots$, have the property that each member of the sequence (except the first two) is the sum of the preceding two numbers. This property is stated in the recursion formula

$$F(n) = F(n-1) + F(n-2).$$

Although in many books the Fibonacci numbers are defined in this way, we approached them by defining $F(n)$ as the number of ways oi writing n signs, each plus or minus, so that no two minus signs are adjacent; thus $F(1) = 2$, $F(2) = 3$, $F(3) = 5$, $F(5) = 8$, etc. It was established that, defining $F(0)$ to be 1, we get the entire sequence of Fibonacci numbers from the formula

$$F(n) = C(n+1, 0) + C(n, 1) + C(n-1, 2) + C(n-2, 3) + \cdots,$$

where the sum on the right terminates when zero terms (terms of the form $C(u, v)$ with $u < v$) begin to appear.

Let m and k be fixed positive integers. The number of solutions of

$$x_1 + x_2 + x_3 + \cdots + x_k = m$$

in positive integers is $C(m-1, k-1)$; in non-negative integers is $C(m+k-1, m)$. Furthermore, if $c_1, c_2, c_3, \cdots, c_k$ are fixed integers, then the number of solutions in integers satisfying the conditions

$$x_1 > c_1, \quad x_2 > c_2, \quad x_3 > c_3, \quad \cdots, \quad x_k > c_k$$

is

$$C(m - c_1 - c_2 - c_3 - \cdots - c_k - 1, \quad k - 1).$$

Two solutions are said to be the same if and only if the values of x_1 are identical, and the values of x_2 are identical, and the values of x_3 are identical, and so on. (Questions about the number of solutions of the given equation subject to even more restrictions on the solutions, such as for example solutions in integers from 1 to 7, are treated in the next chapter.)

The number of combinations, r at a time, of n different things each of which (like coins, books, or stamps) is available in an unlimited supply is $C(n+r-1, r)$. Here n and r denote positive integers, but n may be greater than, equal to, or less than r.

CHAPTER FIVE

The Inclusion-Exclusion Principle; Probability

In this chapter we prove a theorem of a very broad kind and then apply it to particular problems. The idea of *probability* is introduced towards the end of the chapter.

5.1 A General Result

It will be convenient to lead up to the inclusion-exclusion principle by a sequence of three problems listed in increasing order of difficulty. The first problem is not very difficult at all.

PROBLEM 5.1 How many integers between 1 and 6300 inclusive are not divisible by 5? Since precisely every fifth number is divisible by 5, we see that of the 6300 numbers under consideration, exactly 6300/5 or 1260 are divisible by 5. Hence the answer to the question is

$$6300 - 1260 = 5040.$$

PROBLEM 5.2 How many integers between 1 and 6300 inclusive are divisible by neither 5 nor 3? To answer this we could begin by paralleling the argument in Problem 5.1 and say that the number of integers under consideration that are divisible by 5 is 1260, and the number divisible by 3 is 6300/3 or 2100. But

$$6300 - 2100 - 1260$$

is not the correct answer to the problem, because too many integers have been subtracted from the 6300. Numbers like 15, 30, 45, $\cdots$ which are divisible by both 3 and 5 have been removed twice from the 6300 integers under consideration. So we see that we must add back the number of integers divisible by both 3 and 5, that is, divisible† by 15. There are 6300/15 or 420 of those. Thus we get the answer

$$6300 - 2100 - 1260 + 420 = 3360.$$

PROBLEM 5.3 How many integers between 1 and 6300 inclusive are divisible by none of 3, 5, 7? To solve this we can begin with an analogy to the previous argument and first remove from the 6300 integers those divisible by 3, in number 2100, those divisible by 5, in number 1260, and those divisible by 7, in number 900. Thus

$$6300 - 2100 - 1260 - 900$$

is a start toward the answer. However, numbers divisible by both 3 and 5 have been removed twice; likewise numbers divisible by both 3 and 7; likewise numbers divisible by both 5 and 7. Hence we add back the number of integers divisible by both 3 and 5, namely 6300/15 or 420, also the number divisible by both 3 and 7, namely 6300/21 or 300, and also the number divisible by both 5 and 7, namely 6300/35 or 180. We now have

$$6300 - 2100 - 1260 - 900 + 420 + 300 + 180$$

and are closer to the answer. But one final adjustment must be made because of integers divisible by 3, by 5, and by 7, e.g., 105, 210, 315

† A fuller discussion of such divisibility properties is given in Chapter 1 of I. Niven's *Numbers: Rational and Irrational* in this series.

and so on. Such integers are counted in the original 6300, are counted out in the 2100, 1260 and 900, and then counted back in the 420, 300, and 180. The net effect is that each such integer has been counted in once, out three times, and then back in three times. Hence the final adjustment is to count them out again, and so we subtract 6300/105 or 60. Thus the answer to Problem 5.3 is

$$6300-2100-1260-900+420+300+180-60 = 2880. \tag{5.1}$$

There are 2880 integers between 1 and 6300 inclusive that are divisible by none of 3, 5, 7.

The three problems just discussed can be answered by appeal to a general principle. Suppose that we have N objects. Suppose that some of these objects have property α, and some do not. Let $N(\alpha)$ denote the number having property α. Similarly, suppose that some of the objects have property β, and some do not. Let $N(\beta)$ denote the number having property β. If there are other properties γ, δ, $\cdots$, let $N(\gamma)$, $N(\delta)$, $\cdots$ denote the number of objects having property γ, the number having property δ, $\cdots$.

In the problems above the objects are the integers from 1 to 6300 inclusive, and so $N = 6300$. The properties $\alpha, \beta, \cdots$ are the divisibility properties; for example, an integer has the property γ if it is divisible by 7.

Continuing the general analysis, let $N(\alpha, \beta)$ denote the number of objects having both properties α and β. Let $N(\alpha, \beta, \gamma)$ denote the number having the three properties α, β and γ. In the same way $N(\alpha, \beta, \gamma, \delta)$ denotes the number of objects having the four properties α, β, γ and δ.

Now suppose we ask the question: How many of the N objects do not have property α? The answer, $N - N(\alpha)$, is obtained by a simple subtraction. This is analogous to Problem 5.1.

How many objects have neither the property α nor β? The answer is

$$N - N(\alpha) - N(\beta) + N(\alpha, \beta).$$

This is analogous to Problem 5.2.

How many of the objects have none of the three properties α, β, γ?

The answer is

$$(5.2)\qquad \begin{aligned} &N - N(\alpha) - N(\beta) - N(\gamma) \\ &\qquad + N(\alpha, \beta) + N(\alpha, \gamma) + N(\beta, \gamma) - N(\alpha, \beta, \gamma). \end{aligned}$$

Let us check this result, which is analogous to the answer in Problem 5.3.

First consider an object having none of the properties α, β, γ. Such an object is counted by the term N but by none of the other terms in (5.2). Hence such an object is counted in once.

Next consider an object that has exactly one of the three properties; say it has the property β. Such an object is counted by two terms in (5.2), namely N and $N(\beta)$; but since $N(\beta)$ is prefaced by a minus sign, such an object is not counted by (5.2).

Next consider an object that has exactly two of the properties, say β and γ. Such an object is counted by the terms N, $N(\beta)$, $N(\gamma)$ and $N(\beta, \gamma)$ in (5.2), and by no other terms. Hence, in effect, such an object is not counted at all by (5.2), because of the arrangement of plus and minus signs in that formula.

Finally consider an object that has all three properties α, β and γ. It is counted by every one of the eight terms in (5.2), but again this means that in effect it is not counted at all because of the arrangement of signs.

Summing up the argument, we see that formula (5.2) in effect counts those objects, and only those objects, having none of the properties α, β, γ.

Formula (5.2) can be extended to any number of properties. *The number of objects having none of the properties* α, β, γ, $\cdots$ *is*

$$(5.3)\qquad \begin{aligned} &N \\ &-N(\alpha) - N(\beta) - N(\gamma) - \cdots \\ &+N(\alpha, \beta) + N(\alpha, \gamma) + N(\beta, \gamma) + \cdots \\ &-N(\alpha, \beta, \gamma) - \cdots \\ &\cdots\cdots\cdots \end{aligned}$$

This is the inclusion-exclusion principle which gives the title to this chapter. To prove it, we shall show that an object having one or more of the properties α, β, γ, $\cdots$ is, in effect, not counted by (5.3). This argument will establish that the expression (5.3) counts precisely those objects having none of the properties, because such objects are counted by the term N but by no other term in (5.3).

Consider an object, say T, that has exactly j of the properties, where j is some positive integer. In formula (5.3), T is counted by the term N. In the second line,

$$-N(\alpha) - N(\beta) - N(\gamma) - \cdots ,$$

the object T is counted j times, or what is the same thing, $C(j, 1)$ times. In the third line

$$+N(\alpha, \beta) + N(\alpha, \gamma) + N(\beta, \gamma) + \cdots ,$$

the object T is counted $C(j, 2)$ times, because this is the number of terms with two of the j properties of T. Similarly the fourth line counts T exactly $C(j, 3)$ times, and so on. Because of the arrangement of plus and minus signs in (5.3), we see that T is counted in effect

$$1 - C(j, 1) + C(j, 2) - C(j, 3) + C(j, 4) - \cdots$$

times. The value of this expression is zero, by property (3.7) of Section 3.6. Thus we have established the general theorem.

Problem Set 18

1. Write out formula (5.3) in full for the case of four properties α, β, γ and δ.

2. How many terms are there in formula (5.3), presuming that there are r properties α, β, γ, $\cdots$?

3. How many integers from 1 to 33,000 inclusive are divisible by none of 3, 5, 11?

4. How many integers from 1 to 1,000,000 inclusive are neither perfect squares, perfect cubes, nor perfect fourth powers?

5. Using the same notation as in formula (5.3), with exactly five properties $\alpha, \beta, \gamma, \delta, \epsilon$ under consideration, write a formula for the number of objects having all three properties α, β and γ, but having neither of the properties δ, ϵ.

6. Presuming a set of objects, and four properties $\alpha, \beta, \gamma, \delta$ under consideration, write a formula for the number of objects having property β but none of the properties α, γ, δ.

5.2 Applications to Equations and to Combinations with Repetitions

PROBLEM 5.4 How many solutions are there of the equation

$$x_1 + x_2 + x_3 + x_4 = 20 \tag{5.4}$$

in positive integers with $x_1 \leqslant 6$, $x_2 \leqslant 7$, $x_3 \leqslant 8$, and $x_4 \leqslant 9$? In terms of the notation of the preceding section, let the "objects" be the solutions in positive integers of equation (5.4). For example, the set

$$x_1 = 2, \qquad x_2 = 8, \qquad x_3 = 9, \qquad x_4 = 1, \tag{5.5}$$

is one "object". Say that a solution has property α in case $x_1 > 6$, property β in case $x_2 > 7$, property γ in case $x_3 > 8$, property δ in case $x_4 > 9$. We want to find the number of solutions having none of the properties $\alpha, \beta, \gamma, \delta$, so we can use formula (5.3) of the preceding section.

From Chapter 4 (Section 4.5) we know that the number of solutions of (5.4) in positive integers is $C(19, 3)$. Since this is the total number N of "objects" under consideration, we set $N = C(19, 3)$. Next we want to find the value of $N(\alpha)$, the number of solutions of (5.4) in positive integers with $x_1 > 6$. Again by the results derived in Chapter 4, we see that

$$N(\alpha) = C(20-6-1, \; 4-1) = C(13, 3).$$

By a similar argument we conclude that

$$N(\beta) = C(12, 3), \qquad N(\gamma) = C(11, 3), \qquad N(\delta) = C(10, 3).$$

Next, $N(\alpha, \beta)$ denotes the number of solutions of (5.4) in positive integers satisfying both $x_1 > 6$ and $x_2 > 7$, so

$$N(\alpha, \beta) = C(20-6-7-1, \quad 4-1) = C(6, 3).$$

Parallel arguments show that

$$N(\alpha, \gamma) = C(5, 3), \qquad N(\alpha, \delta) = C(4, 3), \qquad N(\beta, \gamma) = C(4, 3),$$
$$N(\beta, \delta) = C(3, 3), \qquad N(\gamma, \delta) = C(2, 3) = 0.$$

All further terms in formula (5.3) are zero. For example, consider $N(\alpha, \beta, \gamma)$. This denotes the number of solutions of equation (5.4) in positive integers satisfying the conditions $x_1 > 6$, $x_2 > 7$, and $x_3 > 8$. There are no such solutions since $6 + 7 + 8 = 21$. Hence the solution of Problem 5.4 can be written

$$\begin{aligned} &C(19, 3) - C(13, 3) - C(12, 3) - C(11, 3) - C(10, 3) \\ &\qquad + C(6, 3) + C(5, 3) + C(4, 3) + C(4, 3) + C(3, 3) \\ &= 969 - 286 - 220 - 165 - 120 + 20 + 10 + 4 + 4 + 1 = 217. \end{aligned}$$

In many applications of the inclusion-exclusion principle (5.3) there is a symmetry about the properties $\alpha, \beta, \gamma, \cdots$ such that the following conditions hold:

$$\begin{aligned} &N(\alpha) = N(\beta) = N(\gamma) = \cdots, \\ &N(\alpha, \beta) = N(\alpha, \gamma) = N(\beta, \gamma) = \cdots, \\ &N(\alpha, \beta, \gamma) = \cdots, \\ &\cdot \ \cdot \ \cdot \ . \end{aligned}$$

In words, say that the properties $\alpha, \beta, \gamma, \cdots$ are symmetric if the number of objects having any one property equals the number of objects having any other single property, if the number of objects having two of the properties is the same no matter which two are considered, and likewise for three properties, four properties, and so on. Thus the properties are symmetric if the number of objects having a certain j properties (j being fixed) equals the number of objects having any other collection of j properties; furthermore, this must be true for $j = 1$, $j = 2$, $j = 3$ and so on as far as it makes sense to go.

Let the total number of properties $\alpha, \beta, \gamma, \cdots$ be r. If these are symmetric properties, then the formula (5.3) for the number of objects having none of the properties is

$$N - C(r, 1)N(\alpha) + C(r, 2)N(\alpha, \beta) - C(r, 3)N(\alpha, \beta, \gamma) + \cdots . \tag{5.6}$$

The reason for this is that the terms of (5.3) can be collected into batches with equal members, there being $C(r, 1)$ of the sort $N(\alpha)$, $C(r, 2)$ of the sort $N(\alpha, \beta)$, etc.

To illustrate a set of symmetric properties, consider the following question.

PROBLEM 5.5 How many solutions are there of the equation

$$x_1 + x_2 + x_3 + x_4 = 26 \tag{5.7}$$

in integers between 1 and 9 inclusive?

Again we can use the theory of the preceding section. The "objects" under consideration are all solutions in positive integers of the equation (5.7), so that $N = C(25, 3)$. A solution of the equation has property α in case $x_1 > 9$, property β in case $x_2 > 9$, property γ in case $x_3 > 9$, and property δ in case $x_4 > 9$. These four properties are completely symmetric in the sense that led from formula (5.3) to formula (5.6). Hence we need only compute

$$N(\alpha), \quad N(\alpha, \beta), \quad N(\alpha, \beta, \gamma) \quad \text{and} \quad N(\alpha, \beta, \gamma, \delta).$$

We find that

$$N(\alpha) = C(26 - 9 - 1,\ 4 - 1) = C(16, 3);$$
$$N(\alpha, \beta) = C(26 - 9 - 9 - 1,\ 4 - 1) = C(7, 3);$$
$$N(\alpha, \beta, \gamma) = C(26 - 9 - 9 - 9 - 1,\ 3) = C(-2, 3) = 0;$$
$$N(\alpha, \beta, \gamma, \delta) = 0.$$

Therefore, by formula (5.6), the answer is

$$C(25, 3) - C(4, 1)C(16, 3) + C(4, 2)C(7, 3)$$
$$= 2300 - 2240 + 210 = 270.$$

Let us now generalize from Problem 5.5 to this question: How many solutions does the equation

$$x_1 + x_2 + x_3 + \cdots + x_k = m \tag{5.8}$$

have in integers from 1 to c inclusive, where c is some fixed positive integer? Again we apply the theory of the preceding section, where the "objects" under study are now all solutions of (5.8) in positive integers. We say that a solution has property α if $x_1 > c$, property β if $x_2 > c$, property γ if $x_3 > c$, and so on. These properties are symmetric, and so we can use formula (5.6). By the theory of page 64, we see that

$$N = C(m - 1, k - 1)$$
$$N(\alpha) = C(m - c - 1, k - 1)$$
$$N(\alpha, \beta) = C(m - 2c - 1, k - 1)$$
$$N(\alpha, \beta, \gamma) = C(m - 3c - 1, k - 1),$$

etc.

Then formula (5.6) gives us the following result. *Let c be any fixed positive integer. The number of solutions of equation* (5.8) *in positive integers not exceeding c is*

$$\begin{aligned} C(m - 1, k - 1) &- C(k, 1)C(m - c - 1, k - 1) \\ &+ C(k, 2)C(m - 2c - 1, k - 1) \\ &- C(k, 3)C(m - 3c - 1, k - 1) \\ &+ C(k, 4)C(m - 4c - 1, k - 1) - \cdots \end{aligned} \tag{5.9}$$

where the series continues until zero terms arise.

The special case $m > kc$ is of some interest; for, in this case, equation (5.8) can have no solutions in integers not exceeding c.

For example, if $m = 25$, $k = 4$ and $c = 6$, then (5.8) becomes $x_1 + x_2 + x_3 + x_4 = 25$ and this has no solutions in integers from 1 to 6 because the maximum value of the sum $x_1 + x_2 + x_3 + x_4$, under our restriction, is 24. In this case, the value of expression (5.9) is zero, and we obtain the identity

$$C(24, 3) - C(4, 1)C(18, 3) + C(4, 2)C(12, 3) - C(4, 3)C(6, 3) = 0.$$

As another example of the inclusion-exclusion principle consider the following question which may be thought of as a problem in combinations with repetitions:

A bag of coins contains eight cents, seven nickels, four dimes, and three quarters. Assuming that the coins of any one denomination are identical (for example the eight cents are identical), in how many ways can a collection of six coins be made up from the whole bagful?

In making up a collection of coins, let x be the number of cents, y the number of nickels, z the number of dimes, and w the number of quarters. Then the problem amounts to asking for the number of solutions of the equation

$$x + y + z + w = 6$$

in non-negative integers satisfying the conditions

$$x \leqslant 8, \qquad y \leqslant 7, \qquad z \leqslant 4, \qquad w \leqslant 3.$$

Let N be the count of all solutions in non-negative integers; its value (see page 57) is

$$N = C(6 + 4 - 1, 4 - 1) = C(9, 3).$$

If we say that a solution has property α in case $x \geqslant 9$, property β in case $y \geqslant 8$, property γ in case $z \geqslant 5$, and property δ in case $w \geqslant 4$, then most of the terms in formula (5.3) on page 70 are zero. For example $N(\alpha) = 0$ because there are no solutions of $x + y + z + w = 6$ in non-negative integers with $x \geqslant 9$. In fact the only non-zero terms in this formula are N, $N(\gamma)$, and $N(\delta)$. Furthermore we can calculate that

$$N(\gamma) = C(6 + 4 - 5 - 1,\ 4 - 1) = C(4, 3);$$
$$N(\delta) = C(6 + 4 - 4 - 1,\ 4 - 1) = C(5, 3);$$
$$N - N(\gamma) - N(\delta) = C(9, 3) - C(4, 3) - C(5, 3) = 70.$$

Problem Set 19

1. Find the number of solutions of $x_1 + x_2 + x_3 + x_4 = 14$ in integers from 1 to 6 inclusive.

2. Find the number of solutions of $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 34$ in positive even integers not exceeding 10.

3. Find the number of solutions of $x_1 + x_2 + x_3 + x_4 = 20$ in integers satisfying $1 \leqslant x_1 \leqslant 6$, $1 \leqslant x_2 \leqslant 7$, $3 \leqslant x_3 \leqslant 9$, $4 \leqslant x_4 \leqslant 11$.

4. Find the number of solutions of $x_1 + x_2 + x_3 + x_4 = 1$ in integers between -3 and 3 inclusive.

5. A bag of coins contains eight cents, seven nickles, four dimes, and three quarters. Assuming that coins of any one denomination are identical, in how many ways can a collection of ten coins be made up from the bagful?

6. In the preceding problem, how many of the collections contain no quarters?

7. The equation $x_1 + x_2 + x_3 + x_4 = 12$ has exactly one solution in positive integers not exceeding 3, as a moment's reflection will show. Apply formula (5.9) to get an identity in the $C(n, r)$ symbols.

8. Find the number of solutions of the equation $y_1 + y_2 + y_3 + y_4 = 14$ in integers between 1 and 9 inclusive.

9. The numerical answer to the preceding problem is the same as in Problem 5.5. (The problems look somewhat similar, the difference being that the constants in the equations are 26 in one instance, 14 in the other.) Show that the equation in the preceding question can be obtained from that in Problem 5.5 by use of the substitution

$$x_1 = 10 - y_1, \quad x_2 = 10 - y_2, \quad x_3 = 10 - y_3, \quad x_4 = 10 - y_4.$$

10. Using the principle sketched in the preceding problem find a specific integer value for c, other than $c = 12$, so that the number of solutions of the two equations

$$x_1 + x_2 + x_3 + x_4 + x_5 = 12 \quad \text{and} \quad y_1 + y_2 + y_3 + y_4 + y_5 = c,$$

in positive integers from 1 to 6, is the same. Then use (5.9) to get an identity between two expressions in $C(n, r)$.

11. Let k, m, c_1, c_2 and c_3 be positive integers. Write a formula for the number of solutions in positive integers of

$$x_1 + x_2 + x_3 + x_4 + \cdots + x_k = m$$

subject to the restrictions $x_1 \leqslant c_1, \quad x_2 \leqslant c_2, \quad x_3 \leqslant c_3$.

12. Find the number of seven-digit positive integers such that the sum of the digits is 19.

5.3 Derangements

As another application of the general theorem of Section 5.1 we turn to quite a different question. Consider the permutations of the numbers $1, 2, 3, \cdots, n$, taken all at a time. Among these permutations there are some, called *derangements*, in which none of the n integers appears in its natural place, that is, 1 is not in its natural place (the first place), 2 is not in its natural place, $\cdots$, and n is not in its natural place. The number of derangements of n things will be denoted by $D(n)$.

As illustrations we note: $D(1) = 0$; $D(2) = 1$ because there is one derangement, 2, 1; $D(3) = 2$ because the derangements are 2, 3, 1 and 3, 1, 2; $D(4) = 9$ because the derangements are

2, 1, 4, 3	3, 1, 4, 2	4, 1, 2, 3
2, 3, 4, 1	3, 4, 1, 2	4, 3, 1, 2
2, 4, 1, 3	3, 4, 2, 1	4, 3, 2, 1.

We want to derive a formula for $D(n)$, valid for any positive integer n. This can be achieved with very little difficulty by use of the inclusion–exclusion principle. To make the idea concrete, let us begin by computing $D(7)$. Let N denote the number of permutations of 1, 2, 3, 4, 5, 6, 7 taken all at a time, so that $N = 7!$. Say that a permutation has property α if 1 is in its natural place, β if 2 is in its proper place, γ if 3 is in its proper place, δ if 4 is in its proper place, ϵ if 5 is in its proper place, ζ if 6 is in its proper place, η if 7 is in its proper place. For example, the permutation

$$7, 2, 6, 1, 5, 3, 4$$

has properties β and ϵ, but no others. A derangement is a permutation having none of the properties $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta$.

We compute $N(\alpha)$, the number of permutations of 1, 2, 3, 4, 5, 6, 7 such that 1 is in the first place (regardless of whether the others are in their natural positions or not) by putting 1 in first place and permuting the others. The result is $N(\alpha) = 6!$. Similarly, if 2 is held in the second place and the others permuted, we obtain $N(\beta) = 6!$. In fact, it does not matter which of the seven numbers we keep fixed in its natural position; the remaining six can be arranged in 6! ways. Therefore $N(\alpha) = N(\beta) = \cdots = N(\eta) = 6!$.

Next, we compute $N(\alpha, \beta)$ by holding 1 and 2 in the first and second places while the remaining five numbers are permuted. This leads to 5! different arrangements. Again, if any two of the numbers are held in their natural positions while the remaining five are permuted, we get 5! permutations so that

$$N(\alpha, \beta) = N(\alpha, \gamma) = \cdots = N(\beta, \gamma) = \cdots = N(\zeta, \eta) = 5!.$$

Similarly, holding three of the numbers fixed leads to

$$N(\alpha, \beta, \gamma) = \cdots = N(\epsilon, \zeta, \eta) = 4!,$$

holding four fixed leads to $N(\alpha, \beta, \gamma, \delta) = \cdots = 3!$, holding five fixed leads to $N(\alpha, \beta, \gamma, \delta, \epsilon) = \cdots = 2!$, holding six fixed leads to $N(\alpha, \beta, \gamma, \delta, \epsilon, \zeta) = 1!$, and holding all seven fixed leads to $N(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta) = 0! = 1$. Clearly, these are the symmetry conditions described in Section 5.2, and so we may use formula (5.6) to compute

$$\begin{aligned} D(7) = {} & 7! - C(7,1)\cdot 6! + C(7,2)\cdot 5! - C(7,3)\cdot 4! \\ & + C(7,4)\cdot 3! - C(7,5)\cdot 2! + C(7,6)\cdot 1! - C(7,7)\cdot 0!. \end{aligned}$$

We simplify this by expressing each $C(n, r)$ in terms of factorials; for example,

$$C(7,4)\cdot 3! = \frac{7!}{4!3!}\cdot 3! = \frac{7!}{4!}.$$

The result can be written

$$(5.10) \qquad D(7) = 7! - \frac{7!}{1!} + \frac{7!}{2!} - \frac{7!}{3!} + \frac{7!}{4!} - \frac{7!}{5!} + \frac{7!}{6!} - \frac{7!}{7!}$$

$$= 7!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}\right].$$

This entire argument generalizes directly to $D(n)$, the number of derangements of n things, and yields the following equations:

$$D(n) = n! - C(n, 1)(n-1)! + C(n, 2)(n-2)! - \cdots + (-1)^n C(n, n)0!$$

$$= n! - \frac{n!}{1!(n-1)!}(n-1)! + \frac{n!}{2!(n-2)!}(n-2)! - \cdots + (-1)^n \frac{n!}{n!0!} 0!$$

$$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!}.$$

Thus

$$(5.11) \qquad D(n) = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right].$$

There is another interpretation of $D(n)$, which we now explain in terms of the special case $D(7)$. Consider a fixed permutation of the integers from 1 to 7, for example

$$P_0: \quad 7, 2, 6, 1, 5, 3, 4.$$

Say that a permutation of the integers from 1 to 7 is *incompatible with* P_0 if it does not have 7 in the first place, nor 2 in the second place, nor 6 in the third place, nor 1 in the fourth place, nor 5 in the fifth place, nor 3 in the sixth place, nor 4 in the seventh place. For example, 1, 3, 5, 7, 2, 4, 6 is incompatible with P_0, whereas 1, 3, 5, 7, 2, 6, 4 is not. The question is: How many permutations are there that are incompatible with P_0?

If we think about the definition of compatibility for a moment and compare it with the definition of derangement, we notice that a derangement is just a permutation incompatible with the "natural" order 1, 2, 3, 4, 5, 6, 7. Since the number of permutations incompatible with some fixed ordering clearly does not depend on which fixed ordering ("natural" or otherwise) is given, we conclude that the number of permutations incompatible with P_0 is $D(7)$, the number of derangements.

We can also show this directly by applying the argument at the beginning of this section; we merely interpret

α as the property that a permutation has 7 in first position,

β as the property that a permutation has 2 in second position,

γ as the property that a permutation has 6 in third position,

. . . .

η as the property that a permutation has 4 in seventh position,

and rederive formula (5.10).

There is nothing special about the one permutation P_0 under discussion. In general, we can say that *if we take any fixed permutation of the integers from* 1 *to* 7, *the number of permutations incompatible with it is* $D(7)$. The derangements are simply all the permutations that are incompatible with the natural arrangement 1, 2, 3, 4, 5, 6, 7.

More generally, the following statements can be made. Say that two permutations $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ of the integers $1, 2, \cdots, n$ are *incompatible* if $a_1 \neq b_1$, $a_2 \neq b_2$, $\cdots$, and $a_n \neq b_n$. The number of permutations of the integers from 1 to n that are incompatible with any fixed permutation is $D(n)$, the number of derangements. Furthermore, derangements are simply permutations of the integers from 1 to n that are incompatible with the natural ordering $1, 2, 3, \cdots, n$.

Problem Set 20

1. Evaluate $D(5)$ and $D(6)$.

2. List all the permutations of 1, 2, 3, 4 that are incompatible with the particular permutation 4, 3, 2, 1.

3. Find the number of derangements of the integers from 1 to 10 inclusive, satisfying the condition that the set of elements in the first five places is (a) 1, 2, 3, 4, 5, in some order; (b) 6, 7, 8, 9, 10, in some order.

4. Find the number of permutations of 1, 2, 3, 4, 5, 6, 7 that do not have 1 in the first place, nor 4 in the fourth place, nor 7 in the seventh place.

5. How many permutations of the integers from 1 to 9 inclusive have exactly three of the numbers in their natural positions, and the other six not?

6. A simple code is made by permuting the letters of the alphabet, with every letter replaced by a different one. How many codes can be made in this way?

7. Prove that $D(n) - nD(n-1) = (-1)^n$ for $n \geqslant 2$.

5.4 Combinatorial Probability

Probability, an important branch of mathematics with an extensive literature, will be treated here in a very limited fashion. We shall restrict attention to a few questions closely related to the main subject of this book. Because of this restriction it will suffice to give a simple definition of probability which, although inadequate for a more sophisticated study of the subject, encompasses all the problems brought into our discussion.

To be specific, we shall confine attention to situations where we can presume what are called *equally likely cases*. For example, if a coin is tossed, we shall take it for granted that the two outcomes, heads and tails, are equally likely. If a die (plural "dice") is cast, we shall assume that the six outcomes, namely 1, 2, 3, 4, 5, or 6 coming up, are equally likely to occur. Or if a card is drawn at random from an ordinary deck, we shall assume that all 52 cards have an equal chance of being drawn, that it is just as likely that the three of hearts (say) will turn up as any other card.

The probability assigned to the three of hearts turning up is 1/52. In general, probability is defined as the ratio of the number of "favorable" cases to the total number of equally likely cases:

$$\frac{\text{Number of favorable cases}}{\text{Total number of equally likely cases}}.$$

Thus the probability of getting heads when a coin is tossed is 1/2; of a 4 coming up when a die is cast is 1/6; of an even number turning up when a die is cast is 3/6 or 1/2; of getting an ace in a single random draw from a deck of cards is 4/52 or 1/13.

An important condition imposed on this definition is that the cases entering into the calculation be equally likely cases. For example, consider the question: What is the probability of getting a total of 12 when two dice are thrown? We could argue *incorrectly* that the sum of the two numbers turning up on a pair of dice may be 2, or 3, or 4, $\cdots$, or 12, so the total number of cases is 11, and so the probability of throwing a 12 is 1/11. This answer is incorrect because these 11 cases are not equally likely. The chances of throwing a 12 are clearly not as great as those of throwing (say) 8, because for a 12, there must be a 6 turned up on each die, whereas for an 8 there may be a 4 on each die, or a 3 and a 5, or a 2 and a 6.

When two dice are thrown the proper number of equally likely outcomes can be found by thinking of the dice as two distinct independent objects, say one white die and one blue die. There are 6 possibilities for the white die and 6 for the blue die, and so by the multiplication principle of Chapter 2 there are 36 equally likely cases:

1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6, 1	6, 2	6, 3	6, 4	6, 5	6, 6.

Of these 36 cases there is only one, namely 6, 6, that gives a total of 12. Hence the probability of getting a total of 12 is 1/36.

Consider the question: What is the probability of a sum of 8 when two dice are thrown? From the above table of 36 cases we see that a sum of 8 arises in the 5 cases 2, 6 3, 5 4, 4 5, 3 6, 2. Hence the answer is 5/36.

Consider the question: What is the probability of getting one head and two tails when three coins are tossed? To get the total number of equally likely cases we conceive of the three coins as distinct, then use the multiplication principle of Chapter 2, and find that there are $2 \cdot 2 \cdot 2$ or eight cases, namely

$$\begin{array}{cccc} HHH & HTH & THH & TTH \\ HHT & HTT & THT & TTT \end{array}$$

where H stands for heads and T for tails. Thus the answer to the question is 3/8 since the favorable cases are HTT, THT and TTH.

PROBLEM 5.6 If ten coins fall to the floor, what is the probability that there are five heads and five tails?

SOLUTION: We conceive of the coins as distinct, a first coin, a second coin, and so on. There are 2^{10} outcomes because each coin can land in two possible ways, heads or tails. An outcome can be designated by a string of ten letters, each either an H (for heads) or a T (for tails); for example

$$T\ T\ H\ H\ T\ H\ H\ H\ T\ T \tag{5.12}$$

means that the first coin is tails, the second tails, the third heads, etc. The number of favorable cases, therefore, is the number of ways that five H's and five T's can be written in a row, and by the work of Chapter 3, this number is $C(10, 5)$. Hence the answer to the question is

$$\frac{C(10, 5)}{2^{10}} = \frac{63}{256}.$$

PROBLEM 5.7 What is the probability that six cards drawn at random from a standard deck of 52 cards will be red cards?

SOLUTION. The total number of cases is the number of ways of selecting six out of 52, and this is $C(52, 6)$. There being 26 red cards in a deck, the number of favorable cases is $C(26, 6)$, the number of ways of choosing six out of 26. Hence the answer is

$$\frac{C(26, 6)}{C(52, 6)}.$$

PROBLEM 5.8 What is the probability of getting a total of 13 when four dice are thrown?

SOLUTION. Since each die can come up in six ways the total number of cases is 6^4. Consider the dice as identifiable in some way, such as by color, so that we can refer to the first die, the second die, etc. If the number turning up on the first die is x_1, on the second die x_2, on the third die x_3, and on the fourth die x_4, then the number of favorable cases is the number of solutions of

$$x_1 + x_2 + x_3 + x_4 = 13$$

in positive integers from 1 to 6. By (5.9) this number of solutions is

$$C(12, 3) - C(4, 1)C(6, 3) = 220 - 80 = 140.$$

Hence the answer is $140/6^4$ or $35/324$.

PROBLEM 5.9 If a permutation of the integers $1, 2, 3, \cdots, n$ is taken at random, what is the probability that it is a derangement?

SOLUTION. The total number of permutations is $n!$ and the number of favorable cases is $D(n)$ as given in (5.11). Hence the probability is

$$\frac{D(n)}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!}. \tag{5.13}$$

This probability has some interesting aspects, some of which will be mentioned here. (Others will be elicited in the next problem set.) For $n = 1, 2, 3, 4, 5, 6, 7, 8$, the values of $D(n)/n!$ are

(5.14) 0, .5000, .3333, .3750, .3667, .3681, .3679, .3679,

to four decimal places of accuracy. Note that the four place approximation to the value of $D(n)/n!$ does not change from $n = 7$ to $n = 8$. It is interesting that this does not change beyond $n = 8$, so that .3679 is accurate for $D(n)/n!$ to four decimal places for all n from 7 onwards, $n = 7$, $n = 8$, $n = 9$, and so on. In other words, no matter how large n is, $D(n)/n!$ remains within .00005 of .3679.

As n increases without bound, the right side of (5.13) has more and more terms, and $D(n)/n!$ tends to the limiting value $1/e$ where e is a basic mathematical constant.

In computing the probability of the occurrence of an event it is sometimes more convenient to begin by computing the "complementary probability", namely the probability that the event will not occur. The probability of an event is defined as

$$p = \frac{\text{Number of favorable cases}}{\text{Total number of equally likely cases}},$$

so the complementary probability is defined as

$$q = \frac{\text{Number of unfavorable cases}}{\text{Total number of equally likely cases}}.$$

Since the number of favorable cases added to the number of unfavorable cases is the total number of cases, we see that

$$p + q = 1 \qquad \text{or} \qquad p = 1 - q.$$

Problem Set 21

1. Check the calculations giving the values in (5.14).

2. Find the probability of getting two tails if two coins are tossed.

3. What is the probability of getting a total of 7 when two dice are thrown?

4. Two dice, one red and one white, are tossed. What is the probability that the white die turns up a larger number than the red die?

5. If four dice are thrown, what is the probability that the four numbers turning up will be all different?

6. If seven dice are thrown, what is the probability that exactly three 6's will turn up?

7. Show that the terms of the binomial expansion of $(\frac{1}{6} + \frac{5}{6})^7$ are the probabilities that when seven dice are cast the number of 6's turning up will be respectively 0, 1, 2, 3, 4, 5, 6, 7.

8. What is the probability of getting a total of 15 when five dice are thrown?

9. When eight coins are tossed what is the probability of (a) exactly five heads; (b) at least five heads?

10. What is the probability that four cards dealt at random from an ordinary deck of 52 cards will contain one from each suit, that is to say, one heart, one spade, one club and one diamond?

11. Find the probability that when 13 cards are dealt from an ordinary deck of 52 cards (a) at least two are face cards; (b) exactly one ace is present; (c) at least one ace is present.

12. The letters of the alphabet are written in random order. What is the probability that x and y are adjacent?

13. If a five-digit integer is chosen at random, what is the probability that (a) the sum of the digits is 20; (b) the product of the digits is 20?

14. A teacher is going to separate ten boys into two teams of five each to play basketball by drawing five names out of a hat containing all ten names. As the drawing is about to start, one boy says to a good friend, "I hope we get on the same team." His friend replies, "Well, we have a fifty-fifty chance." Is he right, in the sense that the probability that the two boys will be on the same team is $\frac{1}{2}$?

15. A man took the eight spark plugs out of his auto to clean them. He intended to put each one back into the same cylinder it came from, but he got mixed up. Assuming that the plugs were put back in random fashion, what is the probability that at least one went back into the cylinder it came from? at least two?

16. A solitaire type of card game is played as follows: The player has two shuffled decks, each with the usual 52 cards. With the decks face down the player turns up a pair of cards, one from each deck.

If they are matching cards (for example, if both are the seven of spades) he has lost the game. If they are not matching cards he continues and turns up another pair of cards, one from each deck. Again he has lost if they are a matching pair. The player wins if he can turn up all 52 pairs, none matching. What is the probability of a win?

17. In the preceding problem suppose the game is played with two "decks" of 13 cards each, for example the spades from two decks of cards. What is the probability of a win in this case?

18. In Problem 16 suppose that a win is defined differently: The player wins if there is exactly one matching pair in the entire 52 pairs. What is the probability of a win?

5.5 Summary

Consider a collection of N different objects, some of which have property α, some property β, some property γ, and so on. Let $N(\alpha)$ be the number of objects having property α, $N(\beta)$ the number having property β, $\cdots$, $N(\alpha, \beta)$ the number having both properties α and β, $N(\alpha, \beta, \gamma)$ the number having all three properties α, β, and γ, and so on. Then the number of objects having none of the properties is

$$
\begin{aligned}
&N\\
&-N(\alpha) - N(\beta) - N(\gamma) - \cdots\\
&+N(\alpha, \beta) + N(\alpha, \gamma) + N(\beta, \gamma) + \cdots\\
&-N(\alpha, \beta, \gamma) - \cdots\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{aligned}
$$

Suppose that there are r properties under consideration. Also suppose that the number of objects having any one property is the same as the number of objects having any other single property, that the number of objects having two properties is the same no matter which two are considered, and likewise for three properties, four properties, and so on. Then the number of objects having none of the properties can be written in the simpler form

$$N - C(r, 1)N(\alpha) + C(r, 2)N(\alpha, \beta) - C(r, 3)N(\alpha, \beta, \gamma) + \cdots .$$

With the use of this inclusion-exclusion principle, as it is called, the discussion of the equation

$$x_1 + x_2 + x_3 + \cdots + x_k = m \tag{5.15}$$

is continued from the preceding chapter. We are now able to find the number of solutions of an equation of type (5.15) under the conditions that each variable is restricted to a specific set of consecutive integral values. Whereas detailed formulas were not developed in general, the following case was treated: Let c be any fixed positive integer; the number of solutions of equation (5.15) in positive integers from 1 to c inclusive is

$$\begin{aligned} C(m - 1, k - 1) - C(k, 1)C(m - c - 1, k - 1) \\ + C(k, 2)C(m - 2c - 1, k - 1) \\ - C(k, 3)C(m - 3c - 1, k - 1) \\ + C(k, 4)C(m - 4c - 1, k - 1) - \cdots , \end{aligned}$$

where the series continues until zero terms arise.

The discussion of combinations with repetitions is continued from the preceding chapter to a wider variety of cases, again by use of the inclusion-exclusion principle.

Define $D(n)$ as the number of derangements of $1, 2, 3, \cdots, n$, that is, the number of permutations with 1 not in the first place, 2 not in the second place, $\cdots$, and n not in the n-th place. It was established that

$$D(n) = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + (-1)^n \frac{1}{n!}\right].$$

Say that two permutations of n objects are *incompatible* if in the two arrangements, all pairs of objects in corresponding positions consist of two distinct objects. If P is a fixed permutation of the integers from 1 to n, the number of permutations incompatible with P is $D(n)$. Derangements are permutations incompatible with the natural ordering $1, 2, 3, \cdots, n$.

Probability was defined in simple combinatorial situations as the ratio of the number of favorable cases to the total number of equally likely cases. The meaning of "equally likely cases" was taken as intuitively clear in certain basic situations, and then extended to more complex ones by use of the multiplication principle of Chapter 3.

CHAPTER SIX

Partitions of an Integer

In this chapter we discuss partitions of an integer, or what is the same thing, partitions of a collection of identical objects. In case the objects are not identical, the problem comes under the heading "partitions of a set" and is discussed in Section 8.2.

The partitions of a positive integer are the ways of writing that integer as a sum of positive integers. The partitions of 5, for example, are

$$\begin{array}{llll} 5 & 4+1 & 3+1+1 & 2+1+1+1 \\ & 3+2 & 2+2+1 & 1+1+1+1+1 \end{array}$$

Since there are seven partitions of 5, we write $p(5) = 7$; in general, we let $p(n)$ denote the number of partitions of the positive integer n. In such a partition as $3 + 2$ above, the numbers 3 and 2 are called the summands. Thus 5 has one partition with one summand, two partitions with two summands, two partitions with three summands, one with four summands, and one with five summands.

Whereas 5 has two partitions with three summands, the equation $x_1 + x_2 + x_3 = 5$ has six solutions in positive integers; they are

$$(3, 1, 1),\ (1, 3, 1),\ (1, 1, 3),\ (2, 2, 1),\ (2, 1, 2),\ (1, 2, 2).$$

In counting the number of solutions of an equation, order is taken into account; but in counting the number of partitions, the order of the summands is irrelevant.

6.1 Graphs of Partitions

Let us look at the partitions of 6:

$$
\begin{array}{llll}
 & 1+1+1+1+1+1 & 4+1+1 & 5+1 \\
 & 2+1+1+1+1 & 3+2+1 & 4+2 \\
(6.1) & 3+1+1+1 & 2+2+2 & 3+3 \\
 & 2+2+1+1 & & 6
\end{array}
$$

There are eleven, so we write $p(6) = 11$. Also we see that the number of partitions of 6

$$
\begin{array}{ll}
 & \text{into 6 summands is 1,} \\
 & \text{into 5 summands is 1,} \\
 & \text{into 4 summands is 2,} \\
(6.2) & \text{into 3 summands is 3,} \\
 & \text{into 2 summands is 3,} \\
 & \text{into 1 summand is 1.}
\end{array}
$$

The notation $q_k(n)$ will denote the number of partitions of n with k or fewer summands. For $n = 6$, the listing (6.2) above shows that

$$
(6.3) \qquad \begin{array}{lll}
q_1(6) = 1 & q_3(6) = 7 & q_5(6) = 10 \\
q_2(6) = 4 & q_4(6) = 9 & q_6(6) = 11.
\end{array}
$$

Since the number 6 cannot be partitioned into more than six summands, we would expect that $q_6(6)$ would be the same as $p(6)$. Similarly $q_n(n)$ means the number of partitions of n having n or fewer summands, and so

$$(6.4) \qquad q_n(n) = p(n) .$$

Partitions can also be classified according to the size of the summands. The listing (6.1) shows that the number of partitions of 6

$$
(6.5)\qquad
\begin{array}{l}
\text{with 6 as the largest summand is 1,}\\
\text{with 5 as the largest summand is 1,}\\
\text{with 4 as the largest summand is 2,}\\
\text{with 3 as the largest summand is 3,}\\
\text{with 2 as the largest summand is 3,}\\
\text{with 1 as the largest summand is 1.}
\end{array}
$$

Note the resemblance of this list to the list (6.2). This is no coincidence, as we shall see. Furthermore, if we define $p_k(n)$ as the number of partitions of n with summands no larger than k, we find that

$$
(6.6)\qquad
\begin{array}{lll}
p_1(6) = 1 & p_3(6) = 7 & p_5(6) = 10\\
p_2(6) = 4 & p_4(6) = 9 & p_6(6) = 11\,.
\end{array}
$$

This list resembles (6.3). In general it is true that $p_k(n) = q_k(n)$. Let us look at some special cases to see why this is so.

The partitions of 6 into three summands are

$$
(6.7)\qquad 4 + 1 + 1, \qquad 3 + 2 + 1, \qquad 2 + 2 + 2.
$$

The partitions of 6 with 3 as the largest summand are

$$
(6.8)\qquad 3 + 1 + 1 + 1, \qquad 3 + 2 + 1, \qquad 3 + 3.
$$

To see why it is no accident that there are the same number (three) of partitions listed in (6.7) and (6.8), we use the so-called *graphs of partitions*. The graph of the partition $4 + 1 + 1$ is

$$
\begin{array}{llll}
\cdot & \cdot & \cdot & \cdot\\
\cdot & & & \\
\cdot & & &
\end{array}
$$

Similarly, the graphs of $3 + 2 + 1$ and $2 + 2 + 2$ are

$$
\begin{array}{lll}
\cdot & \cdot & \cdot\\
\cdot & \cdot & \\
\cdot & &
\end{array}
\qquad\qquad\qquad
\begin{array}{ll}
\cdot & \cdot\\
\cdot & \cdot\\
\cdot & \cdot
\end{array}
$$

Thus the graph of a partition of n with k summands simply consists of k rows of dots, one row for each summand; the row representing the largest summand appears at the top, that representing the next largest summand appears under it, and so on. There are as many rows as there are summands, and the number of dots in each row corresponds to the size of each summand. The total number of dots in the graph of a partition of n is n.

The *reverse* of a graph is obtained by interchanging the horizontal and vertical rows of dots; for example,

GRAPH	REVERSE OF THE GRAPH
. . . . . .	. . . . . .
$4+1+1$	$3+1+1+1$
. . . . . .	. . . . . .
$3+2+1$	$3+2+1$
. . . . . .	. . . . . .
$2+2+2$	$3+3$
. . . .	. . . .
$6+4+1+1$	$4+2+2+2+1+1$

The reverse of a graph of a partition of n is again a graph of a partition of n. If the original graph represents a partition with k summands (i.e. has k rows), then the reverse graph has k dots in its first (longest) row and hence represents a partition with maximum summand k. For example, the partition of 12 into 4 summands $6 + 4 + 1 + 1$ has the graph

$$\begin{array}{llllll} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & & & & & \\ \cdot & & & & & \end{array}$$

The reverse graph

$$\begin{array}{llll} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & & & \\ \cdot & & & \end{array}$$

represents the partition $4 + 2 + 2 + 2 + 1 + 1$ of 12, where 4 is the maximum summand. Thus the one-to-one correspondence between graphs and reverse graphs can be interpreted as a one-to-one correspondence between partitions of n with k summands and partitions of n with greatest summand k. It follows that the number of partitions of n into k summands is the same as the number of partitions of n with maximum summand k.

Moreover, since the number of partitions of n into 1, or 2, or 3, $\cdots$, or k summands is the same as the number of partitions of n with maximum summand 1, or 2, or 3, $\cdots$, or k, we may say:

> *The number of partitions of n into k or fewer summands equals the number of partitions of n having summands no larger than k; in symbols,*

$$q_k(n) = p_k(n) \, . \tag{6.9}$$

This result is illustrated for the special case $n = 6$ in the lists of equations (6.3) and (6.6).

Problem Set 22

1. Evaluate $p(1)$, $p(2)$, $p(3)$, $p(4)$ and $p(5)$.

2. Evaluate $p_1(n)$ and $q_1(n)$.

3. Evaluate $q_2(8)$, $q_2(9)$ and, in general, $q_2(n)$.

4. Find the value of $p_{99}(99) - p_{98}(99)$.

5. Evaluate $p_{67}(67) - p_{65}(67)$.

6. Prove that $p_n(n) = p_{n+1}(n)$ and, in general, that $p_k(n) = p_n(n)$ if $k > n$.

7. Prove that $p_n(n) = p_{n-1}(n) + 1$.

6.2 The Number of Partitions

The number of partitions of n has been denoted by $p(n)$, the number of partitions of n with k or fewer summands by $q_k(n)$, and the number of partitions of n with summands no larger than k by $p_k(n)$. The relations obtained so far are

$$(6.10) \qquad p_k(n) = q_k(n) \qquad \text{and} \qquad p(n) = q_n(n) = p_n(n) .$$

In order to calculate the numerical values of these partitions we establish one more result:

$$(6.11) \qquad p_k(n) = p_{k-1}(n) + p_k(n - k).$$

To prove this for integers n and k satisfying $1 < k < n$, we separate the $p_k(n)$ partitions of n with summands no larger than k into two types:

(a) those having k as a summand;

(b) those not having k as a summand.

We observe first that the partitions of type (b) are precisely the $p_{k-1}(n)$ partitions of n having summands no larger than $k - 1$. Next we note that, since the summand k occurs at least once in each partition of type (a), we can remove a summand k from each of these partitions. If we do so, the resulting partitions are precisely the partitions of $n - k$ into summands no larger than k, in number $p_k(n - k)$. Thus (6.11) is established for integers n and k such that $1 < k < n$.

As an illustration of this argument we take the case $n = 6$, $k = 4$, so that (6.11) becomes $p_4(6) = p_3(6) + p_4(2)$. All the partitions of the number 6 are listed in (6.1) of the preceding section. There are nine partitions of 6 having summands no larger than 4, so that $p_4(6) = 9$. These nine partitions are separated into type (a), those having 4 as a summand, and type (b), those not having 4 as a summand:

Type (a)	Type (b)
$4 + 1 + 1$	$1 + 1 + 1 + 1 + 1 + 1$
$4 + 2$	$2 + 1 + 1 + 1 + 1$
	$3 + 1 + 1 + 1$
	$2 + 2 + 1 + 1$
	$3 + 2 + 1$
	$2 + 2 + 2$
	$3 + 3$

The partitions of type (b) are all partitions of 6 having summands no larger than 3; the number of these is $p_3(6)$. When we remove a summand 4 from each partition of type (a) we get the partitions $1 + 1$ and 2. The number of these is $p_4(2)$ because

$$p_4(2) = p_2(2) = p(2) = 2.$$

Formula (6.11) is valid for positive integers k and n satisfying $1 < k < n$. To make a table of values for $p_k(n)$ we need a few additional observations. First, for $k = 1$ we note that

$$p_1(n) = 1 \quad \text{for all } n \geqslant 1, \tag{6.12}$$

because there is only one partition of n with summands no larger than 1. Next, there is no partition of n with a summand exceeding n, and so

$$p_k(n) = p_n(n) \quad \text{if } k \geqslant n. \tag{6.13}$$

In case $n = 1$ this gives

$$1 = p_1(1) = p_2(1) = p_3(1) = \cdots .$$

Also, there is exactly one partition of n having n as a summand, and hence

$$p_n(n) = 1 + p_{n-1}(n). \tag{6.14}$$

With these results it is a simple matter to make a table of values of $p_k(n)$. To begin, we can write 1's in the first horizontal row and the first vertical column because of formulas (6.12) and (6.13). Then the best way to proceed, perhaps, is to fill in the values of $p_2(n)$ for $n = 2, 3, 4, \cdots$, then $p_3(n)$ for $n = 2, 3, 4, \cdots$, then $p_4(n)$ for $n = 2, 3, 4, \cdots$, and so on, using formulas (6.11), (6.13) and (6.14).

TABLE OF VALUES OF $p_k(n)$

	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$
$n = 1$	1	1	1	1	1	1	1
$n = 2$	1	2	2	2	2	2	2
$n = 3$	1	2	3	3	3	3	3
$n = 4$	1	3	4	5	5	5	5
$n = 5$	1	3	5	6	7	7	7
$n = 6$	1	4	7	9	10	11	11
$n = 7$	1	4	8	11	13	14	15

Problem Set 23

1. Extend the table of values of $p_k(n)$ as far as $n = 12$ and $k = 12$.

2. Evaluate $q_7(5)$, $q_7(7)$ and $q_7(9)$.

3. Evaluate $p(7)$, $p(8)$, $p(9)$ and $p(10)$.

6.3 Summary

The number of partitions of a positive integer n, denoted by $p(n)$, is the number of ways of writing n as a sum of positive integers. In such a partition as $7 = 4 + 2 + 1$ there are three summands, 4, 2 and 1. The order of the summands does not matter, so that $7 = 2 + 1 + 4$ is the same partition. By $q_k(n)$ is meant the number of partitions of n having k or fewer summands; by $p_k(n)$ is meant the number of partitions of n with no summand greater than k. The following results were established:

$$p_k(n) = q_k(n),$$

$$p(n) = p_n(n) = p_{n+1}(n) = p_{n+2}(n) = p_{n+3}(n) = \cdots,$$

$$p_k(n) = p_{k-1}(n) + p_k(n - k) \quad \text{for } 1 < k < n.$$

A short table of partitions was developed by use of these results together with the following simple observations:

$$p_1(n) = 1 \qquad \text{and} \qquad p_n(n) = 1 + p_{n-1}(n).$$

CHAPTER SEVEN

Generating Polynomials

In this chapter we shall use polynomials to "generate" the solutions of a class of problems. For example, we shall solve Problem 1.5 of Chapter 1: In how many ways is it possible to make change for a dollar bill? The method introduced in this chapter is, in its level of sophistication, just one step above the enumeration of cases.

In order to find the number of ways of changing a dollar bill, we first examine the well-known technique of multiplying polynomials. We shall be concerned, in particular, with multiplying polynomials whose coefficients are 1. For example,

$$\begin{aligned}(1 + x + x^2 + x^4 + x^8)&(1 + x^3 + x^6 + x^9)\\ &= 1 + x + x^2 + x^3 + 2x^4 + x^5 + x^6 + 2x^7\\ &\quad + 2x^8 + x^9 + 2x^{10} + 2x^{11} + x^{13} + x^{14} + x^{17}.\end{aligned}$$

Now suppose we are interested in the terms of the product only up to x^9. Then we would neglect terms involving higher powers of x and we would write

$$\begin{aligned}(1 + x + x^2 + x^4 + x^8)&(1 + x^3 + x^6 + x^9)\\ &= 1 + x + x^2 + x^3 + 2x^4 + x^5 + x^6 + 2x^7 + 2x^8 + x^9 + \cdots .\end{aligned}$$

There would be no need to calculate powers of x beyond x^9 in the process of multiplication. To illustrate this point, let us find the expansion, up to x^7, of the product

$$(1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)(1 + x^8)(1 + x^7).$$

Working from the right-hand end we could write the multiplication process as follows:

$$
\begin{aligned}
&(1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)(1 + x^6)(1 + x^7) \\
&\qquad = (1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5) \\
&\qquad\qquad \cdot (1 + x^6 + x^7 + \cdots) \\
&\qquad = (1 + x)(1 + x^2)(1 + x^3)(1 + x^4) \\
&\qquad\qquad \cdot (1 + x^5 + x^6 + x^7 + \cdots) \\
&\qquad = (1 + x)(1 + x^2)(1 + x^3)(1 + x^4 + x^5 + x^6 + x^7 + \cdots) \\
&\qquad = (1 + x)(1 + x^2)(1 + x^3 + x^4 + x^5 + x^6 + 2x^7 + \cdots) \\
&\qquad = (1 + x)(1 + x^2 + x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + \cdots) \\
&\qquad = 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + \cdots .
\end{aligned} \tag{7.1}
$$

A considerable amount of work is saved since the full expansion includes terms up to x^{28}. This saving in labor can be achieved, of course, only if we are not concerned with terms beyond x^7. As we shall see, such a limitation will be acceptable in the problems of this chapter.

Problem Set 24

1. Expand the product $(1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 + x^{16})$ including terms up to x^{16}.

2. Expand the product

$$(1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)(1 + x^6)(1 + x^7)(1 + x^8)$$

including terms up to x^8.

3. Multiply out the product

$$(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7)(1 + x^2 + x^4 + x^6)(1 + x^3 + x^6) \cdot (1 + x^4)(1 + x^5)(1 + x^6)(1 + x^7)$$

including terms up to x^7.

7.1 Partitions and Products of Polynomials

Let us look at the term $5x^7$ in the expansion of the product (7.1) of binomials of the form $(1 + x^n)$, $n = 1, 2, \cdots, 7$. The coefficient 5 tells us, in effect, that x^7 turns up five times in the multiplication process. Tracking down these five cases we see that x^7 arises from the products

$$x^7, \quad x^6x, \quad x^5x^2, \quad x^4x^3, \quad x^4x^2x,$$

where the factors 1 have been omitted for simplicity. The exponents in these five cases correspond to the equations

$$7 = 7, \quad 7 = 6 + 1, \quad 7 = 5 + 2, \quad 7 = 4 + 3, \quad 7 = 4 + 2 + 1.$$

We observe that these five equations are precisely the partitions of the number 7 with *distinct* summands.

As a second example, consider all partitions of 6 with distinct summands,

$$6 = 6, \quad 6 = 5 + 1, \quad 6 = 4 + 2, \quad 6 = 3 + 2 + 1.$$

Here we have four equations, or four partitions, and this corresponds to the coefficient 4 in the term $4x^6$ of the expansion (7.1).

If we wanted to use polynomial products to find the number of partitions of 8 with distinct summands, the expansion (7.1) would be inadequate since it stops with $(1 + x^7)$. We would look at the coefficient of x^8 in the expansion of

$$(1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)(1 + x^6) \cdot (1 + x^7)(1 + x^8). \tag{7.2}$$

(See Problem 2 of Problem Set 24.)

Another point can be made. The coefficients of x, x^2, x^3, x^4, x^5, x^6, x^7 in the expansion (7.1) are respectively the numbers of partitions of 1, 2, 3, 4, 5, 6, 7 with distinct summands. Similarly, the coefficients of x, x^2, x^3, x^4, x^5, x^6, x^7, x^8 in the expansion of the product (7.2) are respectively the number of partitions of 1, 2, 3, 4, 5, 6, 7, 8 with distinct summands. It follows that the expansions of (7.1) and (7.2) are identical up to the term involving x^7, that is, up to $5x^7$.

Can we use polynomial multiplication to get at the ordinary partitions of a number, without the "distinct summands" restriction? We can, provided we choose the correct polynomials for multiplication. Consider the product

$$\begin{aligned}&(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7)\\ &\quad\cdot (1 + x^2 + x^4 + x^8)(1 + x^3 + x^6)\\ &\quad\cdot (1 + x^4)(1 + x^5)(1 + x^6)(1 + x^7).\end{aligned} \tag{7.3}$$

Let us look at the third, second and first factors in the forms

$$1 + x^3 + x^6 = 1 + x^8 + x^{3+3},$$

$$1 + x^2 + x^4 + x^6 = 1 + x^2 + x^{2+2} + x^{2+2+2},$$

$$\begin{aligned}1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 &= 1 + x^1 + x^{1+1} + x^{1+1+1}\\ &\quad + x^{1+1+1+1} + x^{1+1+1+1+1} + x^{1+1+1+1+1+1} + x^{1+1+1+1+1+1+1}.\end{aligned}$$

Viewing these factors in this way (and not altering the factors $1 + x^4$, $1 + x^5$, $1 + x^6$, $1 + x^7$) we see that the coefficient of x^7 in the entire product expansion can be thought of as the number of ways of writing 7 as a sum of numbers selected from one or more of the following batches, where at most one member may be taken from any one batch.

First batch	$1, \quad 1+1, \quad 1+1+1, \quad 1+1+1+1,$ $1+1+1+1+1, \quad 1+1+1+1+1+1,$ $1+1+1+1+1+1+1;$
Second batch:	$2, \quad 2+2, \quad 2+2+2;$
Third batch:	$3, \quad 3+3;$
Fourth batch:	$4;$
Fifth batch:	$5;$
Sixth batch:	$6;$
Seventh batch:	$7.$

But this is just an elaborate description of *the number of partitions of* 7.

Thus we see that the coefficients of x, x^2, x^3, x^4, x^5, x^6, x^7 in the expansion of the product (7.3) are simply the numbers of partitions of 1, 2, 3, 4, 5, 6, 7 respectively. In the notation of the preceding chapter, these coefficients are the numerical values of $p(1)$, $p(2)$, $p(3)$, $p(4)$, $p(5)$, $p(6)$, $p(7)$.

As another example, consider the product

$$(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9) \cdot (1 + x^3 + x^6 + x^9)(1 + x^5)(1 + x^7)(1 + x^9). \tag{7.4}$$

An argument similar to that used with the product (7.3) shows that the coefficients of x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9 are the numbers of partitions of 1, 2, 3, 4, 5, 6, 7, 8, 9, with odd summands only.

These examples suggest the following general principle. *Let* a, b, c, d, e *be unequal positive integers. Then the coefficient of* x^n *in the expansion of*

$$\begin{aligned}&(1 + x^a + x^{2a} + x^{3a} + \cdots)(1 + x^b + x^{2b} + x^{3b} + \cdots)\\ &\cdot(1 + x^c + x^{2c} + x^{3c} + \cdots)(1 + x^d + x^{2d} + x^{3d} + \cdots)\\ &\cdot(1 + x^e + x^{2e} + x^{3e} + \cdots)\end{aligned} \tag{7.5}$$

equals the number of partitions of n *with summands restricted to* a, b, c, d, e. *Each factor in* (7.5) *must include all exponents not exceeding* n.

To illustrate this last remark, consider the case $n = 34$ and $a = 6$; the first factor in (7.5) would be

$$1 + x^6 + x^{12} + x^{18} + x^{24} + x^{30}.$$

No harm would be done by the presence of higher powers such as x^{36}, x^{42}, and so on, but these are not necessary in case $n = 34$.

There is, of course, no reason to restrict the summands to five items a, b, c, d, e. The extension of formula (7.5) to more summands merely involves additional appropriate factors, and the contraction to fewer summands merely involves the removal of appropriate factors.

Question: What product can be used to give the number of partitions of 20 with summands 3, 4, 5, 6? *Answer:* The number of such partitions is the coefficient of x^{20} in the expansion of the product

$$\begin{aligned} &(1 + x^3 + x^6 + x^9 + x^{12} + x^{15} + x^{18}) \\ &\cdot (1 + x^4 + x^8 + x^{12} + x^{16} + x^{20}) \\ &\cdot (1 + x^5 + x^{10} + x^{15} + x^{20})(1 + x^6 + x^{12} + x^{18}). \end{aligned} \tag{7.6}$$

Finally, we note that there is another interpretation of the coefficient of x^{20} in this expansion. It is the number of solutions, in non-negative integers, of

$$3y + 4z + 5u + 6v = 20;$$

for, each such solution corresponds to a partition of 20 with summands 3, 4, 5, 6. For example, the solution $y = 1$, $z = 3$, $u = 1$, $v = 0$ corresponds to the partition $20 = 3 + 4 + 4 + 4 + 5$.

Problem Set 25

1. Give an interpretation of each of the following in terms of partitions:

(a) the coefficient of x^{12} in the expansion of

$$(1 + x^2 + x^4 + x^6 + x^8 + x^{10} + x^{12})(1 + x^4 + x^8 + x^{12})(1 + x^6 + x^{12}) \cdot (1 + x^8)(1 + x^{10})(1 + x^{12});$$

(b) the coefficient of x^9 in the expansion of

$$(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9)$$
$$\cdot (1 + x^2 + x^4 + x^6 + x^8)(1 + x^3 + x^6 + x^9);$$

(c) the coefficient of x^6 in the expansion of

$$(1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)(1 + x^6).$$

2. Calculate the indicated coefficients in the preceding question.

3. Write a polynomial product whose expansion can be used to find

(a) the number of partitions of 38 with summands restricted to 6, 7, 12, 20;

(b) the number of partitions of 15 with summands greater than 2;

(c) the number of partitions of 9 with distinct (i.e. unequal) summands.

Calculate the number of partitions in each case.

4. How many solutions are there in non-negative integers of the equation

$$2y + 3z + 5w + 7t = 18?$$

5. How many solutions in positive integers are there of the equation

$$3u + 5v + 7w + 9t = 40?$$

7.2 Change for a Dollar Bill

In the light of the general principle formulated in the preceding section, it is not difficult now to determine in how many ways it is possible to break a dollar bill into change. Since coins come in the denominations 1, 5, 10, 25 and 50 cents, our task is to find the number of partitions of 100 with summands restricted to 1, 5, 10, 25, 50. Thus we can apply the formulation (7.5) with

$$a = 1, \quad b = 5, \quad c = 10, \quad d = 25, \quad e = 50;$$

the answer to the question is the coefficient of x^{100} in the expansion of the product $P_1P_2P_3P_4P_5$, where the P's are the polynomials

$$\begin{aligned}
P_1 &= 1 + x + x^2 + x^3 + x^4 + \cdots + x^{99} + x^{100}, \\
P_2 &= 1 + x^5 + x^{10} + x^{15} + x^{20} + \cdots + x^{95} + x^{100}, \\
P_3 &= 1 + x^{10} + x^{20} + x^{30} + x^{40} + \cdots + x^{90} + x^{100}, \\
P_4 &= 1 + x^{25} + x^{50} + x^{75} + x^{100}, \\
P_5 &= 1 + x^{50} + x^{100}.
\end{aligned}$$

All calculations will be made up to x^{100}. We compute

$$P_4P_5 = 1 + x^{25} + 2x^{50} + 2x^{75} + 3x^{100} + \cdots,$$

$$\begin{aligned}
P_3P_4P_5 = {}& 1 + x^{10} + x^{20} + x^{25} + x^{30} + x^{35} + x^{40} + x^{45} + 3x^{50} \\
& + x^{55} + 3x^{60} + x^{85} + 3x^{70} + 3x^{75} \\
& + 3x^{80} + 3x^{85} + 3x^{90} + 3x^{95} + 6x^{100} + \cdots,
\end{aligned}$$

$$\begin{aligned}
P_2P_3P_4P_5 = {}& 1 + x^5 + 2x^{10} + 2x^{15} + 3x^{20} + 4x^{25} + 5x^{80} + 6x^{85} \\
& + 7x^{40} + 8x^{45} + 11x^{50} + 12x^{55} + 15x^{60} + 16x^{65} + 19x^{70} \\
& + 22x^{75} + 25x^{80} + 28x^{85} + 31x^{90} + 34x^{95} + 40x^{100} + \cdots .
\end{aligned}$$

The final multiplication need not be done in detail, since we are concerned only with the coefficient of x^{100}. We notice that each term in the polynomial product $P_2P_3P_4P_5$ enters exactly once in contributing to the coefficient of x^{100} in the product of P_1 and $P_2P_3P_4P_5$. It follows that this coefficient can be calculated simply by adding all the coefficients in $P_2P_3P_4P_5$ (including the constant term 1):

$$\begin{aligned}
1 + 1 + 2 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 11 + 12 \\
+ 15 + 16 + 19 + 22 + 25 + 28 + 31 + 34 + 40.
\end{aligned}$$

This sum is 292, and so there are 292 ways of changing a dollar bill.

Problem Set 26

1. Find the number of ways of changing a hundred dollar bill into bills of smaller denominations, namely 1, 5, 10, 20, 50 dollar bills.

2. In how many ways can the sum of 53 cents be made up in coins of denominations 1, 5, 10, 25 cents?

3. Find the number of solutions in non-negative integers of the equation

$$5y + 10z + 25w + 50t = 95.$$

4. Find the number of solutions in positive integers of the equation

$$5y + 10z + 25w + 50t = 155.$$

7.3 Summary

The incomplete multiplication of polynomials with unit coefficients —incomplete in the sense that the result is obtained only up to a certain power of the variable x —is used to determine the number of certain partitions and solutions of equations.

For five summands, the procedure is illustrated by the following general principle. Let a, b, c, d, e be unequal positive integers. Then the coefficient of x^n in the expansion of

$$(1 + x^a + x^{2a} + x^{3a} + \cdots)(1 + x^b + x^{2b} + x^{3b} + \cdots)$$
$$\cdot(1 + x^c + x^{2c} + x^{3c} + \cdots)(1 + x^d + x^{2d} + x^{3d} + \cdots)$$
$$\cdot(1 + x^e + x^{2e} + x^{3e} + \cdots)$$

equals the number of partitions of n with summands restricted to a, b, c, d, e. (Each of the five factors in parentheses in the product must include all exponents not exceeding n.) This coefficient is also the number of solutions in non-negative integers y, z, w, u, v of the equation

$$ay + bz + cw + du + ev = n.$$

This theory is used to determine, for example, the number of ways of making change for a dollar bill.

CHAPTER EIGHT

Distribution of Objects Not All Alike

Many problems of combinatorial analysis can be stated in terms of the number of ways of distributing *objects* in *boxes*. Some of these distribution problems were considered in earlier chapters. We now make a brief classification of the various types of questions.

First, the objects may be considered to be alike, and the boxes also indistinguishable from one another. These are partition problems. For example, the number of ways of distributing nine objects in four boxes is the same as the number of partitions of 9 into at most four summands. Problems of this sort were discussed in Chapters 6 and 7.

Next, the objects may be alike, but the boxes may be thought of as different. Under these conditions, the number of ways of distributing nine objects in four boxes equals the number of solutions of

$$x_1 + x_2 + x_3 + x_4 = 9$$

in non-negative integers. If no box is to be empty, solutions in positive integers are to be counted. Similarly, other restrictions on the number of elements in the boxes correspond to restrictions on the solutions of the equations. Questions of this sort were studied in Chapters 4 and 5.

In the present chapter we study the distribution of objects that are not all alike. The words "not all alike" admit two interpretations: (1) objects all different in the sense of no two alike; (2) a mixed collection of objects, some alike and some different, such as coins, for example. In the first section we discuss the case of objects all different, with boxes also different; in the second section, the case of objects all different with boxes alike; in the third section, objects some alike and some different, with boxes different.

8.1 Objects Different, Boxes Different

If m objects, no two alike, are to be distributed in k boxes, no two alike, the number of ways this can be done is k^m since there are k alternatives for the disposal of the first object, k alternatives for the disposal of the second, and so on.

But now suppose the additional requirement that there be no empty box is imposed; that is, we are to count only those distributions in which each box receives at least one object. Of course we must now have at least as many objects as boxes, $m > k$; otherwise no such distribution can be made. Let $f(m, k)$ denote the number of ways of putting m different objects into k different boxes, with no box empty. For example, $f(3, 2) = 6$. For convenience we define $f(m, k) = 0$ if $m < k$.

We derive a formula for $f(m, k)$ by using the inclusion-exclusion principle of Chapter 5. The method is illustrated by the computation of $f(m, 7)$. Consider the total number of arrangements, 7^m, of m different objects in seven different boxes. Say that such an arrangement has property α in case the first box is empty, property β in case the second box is empty, and similarly properties $\gamma, \delta, \epsilon, \zeta, \eta$ for the other five boxes respectively. To find the number of distributions with no box empty, we simply count the number of distributions having none of the properties α, β, γ, etc. We can apply formula (5.6) of page 74 because of the symmetry of the seven properties. Here $N = 7^m$ is the total number of distributions. By $N(\alpha)$ we mean the number of distributions with the first box empty, and so $N(\alpha) = 6^m$. Similarly, $N(\alpha, \beta)$ is the number of distributions with the first two boxes empty. But this is the same as the number of distributions into five boxes, and hence $N(\alpha, \beta) = 5^m$. Thus we can write

$$N = 7^m, \quad N(\alpha) = 6^m, \quad N(\alpha, \beta) = 5^m, \quad N(\alpha, \beta, \gamma) = 4^m,$$
$$N(\alpha, \beta, \gamma, \delta) = 3^m, \quad N(\alpha, \beta, \gamma, \delta, \epsilon) = 2^m$$
$$N(\alpha, \beta, \gamma, \delta, \epsilon, \zeta) = 1^m, \quad N(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta) = 0.$$

Applying formula (5.6) of page 74 with $r = 7$ we get

$$f(m, 7) = 7^m - C(7, 1)6^m + C(7, 2)5^m - C(7, 3)4^m + C(7, 4)3^m - C(7, 5)2^m + C(7, 6)1^m.$$

By a direct generalization of this with k in place of 7, we see that

$$f(m, k) = k^m - C(k, 1)(k - 1)^m + C(k, 2)(k - 2)^m - C(k, 3)(k - 3)^m + \cdots + (-1)^{k-1}C(k, k - 1)1^m. \tag{8.1}$$

If $m < k$ then $f(m, k) = 0$. In such cases formula (8.1) can be used to give identities about $C(n, r)$. For example, if $m = 6$ and $k = 7$, then (8.1) tells us that

$$7^6 - C(7, 1)6^6 + C(7, 2)5^6 - C(7, 3)4^6 + C(7, 4)3^6 - C(7, 5)2^6 + C(7, 6)1^6 = 0.$$

Problem Set 27

1. Find the number of distributions of five different objects in three different boxes, with no box empty.

2. Find the value of $f(5, 2)$.

3. It is stated in the text that $f(3, 2) = 6$. Verify this both by an actual count of the cases, and by use of formula (8.1).

4. In how many ways is it possible to distribute k distinct objects in k distinct boxes with no box empty? Answer this question in two ways, namely by direct consideration and by use of formula (8.1), and derive an identity.

5. Prove that if m is any positive integer less than 8,

$$8^m - C(8, 1)7^m + C(8, 2)6^m - C(8, 3)5^m + C(8, 4)4^m - C(8, 5)3^m + C(8, 6)2^m - C(8, 7) = 0.$$

8.2 Objects Different, Boxes Alike (Partitions of a Set)

If a set contains m elements, it is always presumed as part of the meaning of the word "set" that the elements are different from one another. Thus the number of ways that m different objects can be put into k like boxes is the same as the number of partitions of a set of m elements into k subsets. Note that nothing is said about the number of elements in the k subsets. However, in some problems it will be specified that the subsets are non-empty.

Let $G(m, k)$ denote the number of distributions of m different things into k like boxes, i.e., boxes that are not ordered, and cannot be distinguished in any way. To say it another way, $G(m, k)$ is the number of separations of m different objects into k or fewer batches; we include the words "or fewer" because one or more of the batches may be empty. For example, consider $G(3, 2)$. Denoting the three objects by A, B, and C, we see that there are four cases:

(8.2)

A, B, and C in one box, nothing in the other;

A in one box, B and C in the other;

B in one box, A and C in the other;

C in one box, A and B in the other.

Thus $G(3, 2) = 4$.

Now let $g(m, k)$ denote the number of distributions of m different objects in k like boxes, *with no box empty*. Thus $g(m, k)$ is the number of ways of separating m different objects into k non-empty batches, or the number of ways of separating a set of m things into k non-empty subsets. Looking at the cases listed as (8.2) we see that

$$g(3, 1) = 1 \quad \text{and} \quad g(3, 2) = 3.$$

In general, we can separate the $G(m, k)$ distributions into those where no box is empty, those where exactly one box is empty, those where exactly two boxes are empty, and so on, to get

$$
\begin{aligned}
(8.3)\qquad G(m, k) &= g(m, k) + g(m, k-1) + g(m, k-2) \\
&\quad + g(m, k-3) + \cdots + g(m, 1).
\end{aligned}
$$

Next we derive a formula for $g(m, k)$. There is a simple relationship between $g(m, k)$ and $f(m, k)$; it parallels the relationship between combinations and permutations in the elementary theory. To see this, consider any distribution counted by $g(m, k)$; since there are $k!$ ways of numbering the k boxes to change them from like to unlike boxes, each distribution gives rise to $k!$ distributions of the $f(m, k)$ type. It follows that

$$f(m, k) = g(m, k) \cdot k! \qquad \text{or} \qquad g(m, k) = f(m, k)/k!$$

In view of equation (8.1) of the preceding section, this last equation can be rewritten in the form

$$
\begin{aligned}
(8.4)\qquad g(m, k) &= \frac{1}{k!}[k^m - C(k, 1)(k-1)^m + C(k, 2)(k-2)^m \\
&\quad - \cdots + (-1)^{k-1}C(k, k-1)1^m].
\end{aligned}
$$

It is an easy matter to determine the value of $g(m, k)$ by (8.4), and then to determine $G(m, k)$ by (8.3). Also, since formula (8.4) gives the number of partitions of a set of m elements into k non-empty subsets, the total number of partitions of a set of m elements can be obtained by adding the values of $g(m, k)$ for all the appropriate values of k, namely $k = 1$, $k = 2$, $\cdots$, $k = m$. Thus the total number of partitions of a set of m elements is

$$g(m, 1) + g(m, 2) + g(m, 3) + \cdots + g(m, m),$$

where each term of this sum can be evaluated by use of (8.4).

Problem Set 28

1. In how many ways is it possible to separate the nine letters $a, b, c, d, e, f, g, h, i$ into three non-empty batches?

2. If we separate four distinct objects into four non-empty batches, it is clear that there is just one way to do it. Check that formula (8.4) gives this result.

3. In how many ways is it possible to separate m distinct objects into two non-empty batches?

4. In how many ways is it possible to factor the number 30,030 into three positive integer factors (a) if 1 is allowed as a factor, (b) if each factor must be greater than 1? (Order does not count: $30 \cdot 77 \cdot 13$ is the same factoring as $13 \cdot 30 \cdot 77$.)

5. Find the total number of ways of partitioning a set of five (distinct) elements.

6. Without using the formulas of the text, establish from the meaning of the symbolism that $g(m, m) = 1$. Hence prove the identity

$$\begin{aligned} m! = m^m - C(m, 1)(m-1)^m + C(m, 2)(m-2)^m \\ - C(m, 3)(m-3)^m + \cdots + (-1)^{m-1}C(m, m-1)(1)^m. \end{aligned}$$

7. Without using the formulas of the text, establish that

$$g(m, m-1) = C(m, 2).$$

Then use a similar kind of analysis to evaluate $g(m, m-2)$.

8.3 Objects Mixed, Boxes Different

Consider several different objects, each of which may be in more than one copy; for example, a collection of stamps, or of books. Let there be a copies of the first object, b copies of the second, c of the third, and so on. Let the total number of objects be n, so that

$$a + b + c + \cdots = n. \tag{8.5}$$

These objects are to be distributed into several unlike boxes as follows: α objects into the first box, β into the second box, γ into the third box, and so on. Each distribution into the boxes will use up all the objects, so that the sum of α, β, γ and so on is n:

$$\alpha + \beta + \gamma + \cdots = n. \tag{8.6}$$

We shall use the symbolism†

$$[a,\ b,\ c,\ \cdots \,\|\, \alpha, \beta, \gamma, \cdots] \tag{8.7}$$

to denote the number of distributions of the objects in the boxes, as specified. As an example, we consider the notation $[1,\ 2,\ 2 \,\|\, 2,\ 3]$, where $n = 5$. One way to visualize the five objects is to think of colored balls, say one red ball, two blue balls, and two white balls. The two blue balls are identical; the two white balls are identical. The problem is to find the number of ways of putting two of the balls into the first box, and three into the second box. It is not difficult to verify that there are five ways of doing this, that is,

$$[1,\ 2,\ 2 \,\|\, 2,\ 3] = 5.$$

We shall not derive any general formula for the number of distributions denoted by (8.7). The problem of this distribution number is rather difficult, and so we shall analyze some special cases only. First we examine two basic properties of the distribution number (8.7). One property is that the order of the terms on either side of the vertical separator is immaterial. For example,

$$[1,\ 2,\ 2 \,\|\, 2,\ 3] = [2,\ 1,\ 2 \,\|\, 2,\ 3] = [2,\ 2,\ 1 \,\|\, 3,\ 2].$$

The second, less obvious, property is that the two sides can be interchanged, as in the examples

$$[1,\ 2,\ 2 \,\|\, 2,\ 3] = [2,\ 3 \,\|\, 1,\ 2,\ 2],$$
$$[4,\ 5,\ 6 \,\|\, 2,\ 2,\ 3,\ 8] = [2,\ 2,\ 3,\ 8 \,\|\, 4,\ 5,\ 6]. \tag{8.8}$$

† Adapted from H. Rademacher and O. Toeplitz, *The Enjoyment of Mathematics*, Princeton, 1957, with permission.

We now establish the validity of the second statement, i.e., of equation (8.8). The notation $[4, 5, 6 \,\|\, 2, 2, 3, 8]$ means the number of ways of putting 4 red balls, 5 blue balls, and 6 white balls into 4 boxes, with 2 balls in the first box, 2 in the second box, 3 in the third box, and 8 in the fourth box. The balls are distinguishable only by color. In any distribution we shall denote by x_1 the number of red balls put into the first box, and similarly by x_2, x_3, x_4 the numbers of red balls put into the second, third, and fourth boxes respectively. In the same way, let y_1, y_2, y_3, y_4 denote the numbers of blue balls put into the boxes, and z_1, z_2, z_3, z_4 the number of white balls. Then the notation $[4, 5, 6 \,\|\, 2, 2, 3, 8]$ can be interpreted as the number of solutions in non-negative integers of the system of equations

$$
(8.9) \qquad
\begin{array}{lcl}
x_1 + x_2 + x_3 + x_4 & = & 4 \\
y_1 + y_2 + y_3 + y_4 & = & 5 \\
z_1 + z_2 + z_3 + z_4 & = & 6 \\
 & &
\end{array}
\qquad
\begin{array}{lcl}
x_1 + y_1 + z_1 & = & 2 \\
x_2 + y_2 + z_2 & = & 2 \\
x_3 + y_3 + z_3 & = & 3 \\
x_4 + y_4 + z_4 & = & 8
\end{array}
$$

Now let us consider the right hand side of equation (8.8). The symbolism $[2, 2, 3, 8 \,\|\, 4, 5, 6]$ denotes the number of ways of putting 2 green balls, 2 orange balls, 3 yellow balls, and 8 black balls into 3 boxes with 4 balls in the first box, 5 balls in the second box, and 6 balls in the third box. Let t_1, t_2, t_3 denote the numbers of green balls put in the first, second and third boxes respectively, in any distribution. Similarly, let u_1, u_2, u_3 denote the numbers of orange balls, v_1, v_2, v_3 the numbers of yellow balls, and w_1, w_2, w_3 the numbers of black balls, put in the first, second and third boxes respectively. Then the notation $[2, 2, 3, 8 \,\|\, 4, 5, 6]$ can be interpreted to mean the number of solutions in non-negative integers of the system of equations

$$
(8.10) \qquad
\begin{array}{lcl}
t_1 + t_2 + t_3 & = & 2 \\
u_1 + u_2 + u_3 & = & 2 \\
v_1 + v_2 + v_3 & = & 3 \\
w_1 + w_2 + w_3 & = & 8
\end{array}
\qquad
\begin{array}{lcl}
t_1 + u_1 + v_1 + w_1 & = & 4 \\
t_2 + u_2 + v_2 + w_2 & = & 5 \\
t_3 + u_3 + v_3 + w_3 & = & 6 \\
 & &
\end{array}
$$

To see that the system of equations (8.10) is the same as the system of equations (8.9), let

$$
\begin{array}{llll}
t_1 = x_1 & u_1 = x_2, & v_1 = x_3, & w_1 = x_4, \\
t_2 = y_1, & u_2 = y_2, & v_2 = y_3, & w_2 = y_4, \\
t_3 = z_1, & u_3 = z_2, & v_3 = z_3, & w_3 = z_4.
\end{array}
$$

This shows that (8.8) holds. A similar argument with more elaborate systems of equations can be used to prove that

$$(8.11)\qquad [a, b, c, \cdots \square\ \alpha, \beta, \gamma, \cdots] = [\alpha, \beta, \gamma, \cdots \square\ a, b, c, \cdots]$$

for any integers satisfying $a + b + c + \cdots = \alpha + \beta + \gamma + \cdots = n$. A result of this type in mathematics is called a *duality* principle.

As a special case of (8.11), consider the situation where $\alpha = 1$, $\beta = 1$, $\gamma = 1$ and so on:

$$(8.12)\qquad [a, b, c, \cdots \square\ 1, 1, 1, \cdots, 1] = [1, 1, 1, \cdots, 1 \square\ a, b, c, \cdots];$$

here each block of 1's has n terms, and $a + b + c + \cdots = n$. The notation on the left of (8.12) can be interpreted to mean the number of permutations of n things taken all at a time, where a of the things are alike, another b alike, another c alike, and so on. The number of such permutations, as given in the summary of Chapter 3, is

$$(8.13)\qquad \frac{n!}{a!b!c!\cdots}.$$

Now we can assert that the right member of (8.12) also has the value (8.13). Thus (8.13) is the number of ways of distributing n distinct things in boxes with a in the first box, b in the second box, c in the third box, and so on.

Problem Set 29

1. Find the numerical values of the following:

   (a) $[1, 1, 1, 1 \square\ 1, 1, 1, 1]$

(b) $[1, 1, 1, 1, 1, 1, 1, 1, 1 \,\|\, 4, 5]$

(c) $[1, 1, 1, 1, 1, 1, 1, 1, 1, 1 \,\|\, 1, 1, 1, 1, 1, 1, 4]$

(d) $[30, 10 \,\|\, 10, 10, 10, 10]$

(e) $[2, 1, 1 \,\|\, 2, 1, 1]$

(f) $[2, 2, 2 \,\|\, 2, 2, 2]$

(g) $[4, 4, 4 \,\|\, 6, 6]$

2. Express each of the following in the notation of previous chapters:

(a) $[1, 1, 1, \cdots, 1 \,\|\, 1, 1, 1, \cdots, 1]$, with n ones on each side of the separator;

(b) $[1, 1, 1, \cdots, 1 \,\|\, 1, 1, 1, \cdots, 1, n-r]$, with n ones to the left of the separator, and r ones to the right;

(c) $[1, 1, 1, \cdots, 1 \,\|\, r, n-r]$, with n ones to the left of the separator;

(d) $[a_1, a_2, a_3, \cdots, a_k \,\|\, r, n-r]$, where

$$a_1 + a_2 + a_3 + \cdots + a_k = n$$

and each of the a's is not less than r.

3. Evaluate $[2, 1, 1, 1, \cdots, 1 \,\|\, 2, 1, 1, 1, \cdots, 1]$, with k ones on each side of the separator.

8.4 Summary

The number of distributions of m different objects in k different boxes is k^m. If no box is to be empty, the notation $f(m, k)$ is used for the number of distributions, with $f(m, k) = 0$ in case $m < k$. It was shown that

$$f(m, k) = k^m - C(k, 1)(k-1)^m + C(k, 2)(k-2)^m - C(k, 3)(k-3)^m + \cdots + (-1)^{k-1}C(k, k-1)1^m.$$

If we have m different objects and k boxes that are indistinguishable from each other, the notation $G(m, k)$ is used for the total number of distributions of the objects in the boxes, and $g(m, k)$ for the number of distributions with no box empty. It is proved that

$$g(m, k) = f(m, k)/k!$$

and

$$G(m, k) = g(m, k) + g(m, k-1) + g(m, k-2) + \cdots + g(m, 1).$$

Another interpretation of $G(m, k)$ is the number of partitions of a set of m (distinct) elements into k (unordered) subsets; note that there is no restriction on the number of elements in the subsets. However, if the k subsets are required to be non-empty, the number of partitions of the set is $g(m, k)$. The total number of ways of partitioning a set of m (distinct) elements is $G(m, m)$. To evaluate this for any specific value of m, we use the result

$$G(m, m) = g(m, m) + g(m, m-1) + g(m, m-2) + \cdots + g(m, 1),$$

together with $g(m, k) = f(m, k)/k!$ and the formula above for $f(m, k)$.

Let $a, b, c, \cdots$ and $\alpha, \beta, \gamma, \cdots$ be two sets of positive integers having the same sum n:

$$a + b + c + \cdots = n, \qquad \alpha + \beta + \gamma + \cdots = n.$$

Let there be n objects of which a are alike, another b are alike, another c are alike, and so on. The number of distributions of these objects into boxes, with α objects in the first box, β objects in the second box, γ objects in the third box, etc., is denoted by $[a, b, c, \cdots \,[]\, \alpha, \beta, \gamma, \cdots]$. It was shown that

$$[a, b, c, \cdots \,[]\, \alpha, \beta, \gamma, \cdots] = [\alpha, \beta, \gamma, \cdots \,[]\, a, b, c, \cdots].$$

CHAPTER NINE

Configuration Problems

The questions discussed in this chapter are related to geometric patterns or configurations of one kind or another. We begin with a concept which is widely used throughout mathematics—the pigeonhole principle.

9.1 The Pigeonhole Principle

If eight pigeons fly into seven pigeonholes, at least one of the pigeon holes will contain two or more pigeons. More generally, if $n + 1$ pigeons are in n pigeon holes, at least one of the holes contains two or more pigeons.

This simple form of the pigeonhole principle can be generalized as follows: If $2n + 1$ pigeons are in n pigeonholes at least one of the holes contains three or more pigeons. Here is an even stronger statement that includes all the preceding assertions as special cases: If $kn + 1$ pigeons are in n pigeonholes, at least one of the holes contains $k + 1$ or more pigeons.

It is not difficult to prove this; for, if it were not so, then every hole would contain k or fewer pigeons. Thus there would be n holes with k or fewer pigeons in each hole, and so a total of at most nk pigeons could be accommodated. But this is a contradiction because there are $kn + 1$ pigeons in all. Hence we have established the result by an indirect proof.

Problem Set 30

1. Given the information that no human being has more than 300,000 hairs on his head, and that New York City, by a recent census, has a population of 7,781,984, observe that there are at least two persons in New York with the same number of hairs on their heads. What is the largest integer that can be used for n in the following assertion? There are n persons in New York with the same number of hairs on their heads.

2. Assume the information that at least one of a_1 and b_1 has a certain property P, and at least one of a_2 and b_2 has property P, and at least one of a_3 and b_3 has property P. Prove that at least two of a_1, a_2, a_3 or at least two of b_1, b_2, b_3 have property P.

3. Assume the same information as in the preceding question, and also that at least one of a_4 and b_4 has property P, and at least one of a_5 and b_5 has property P. Prove that at least three of a_1, a_2, a_3, a_4, a_5 or at least three of b_1, b_2, b_3, b_4, b_5 have property P.

4. Assume that at least one of a_1, b_1, c_1 has property Q, and likewise for a_2, b_2, c_2, and likewise for $a_3, b_3, c_3, \cdots$, and likewise for a_{10}, b_{10}, c_{10}. What is the largest integer that can be used for k to make the following assertion correct? At least k of $a_1, a_2, a_3, \cdots, a_{10}$, or at least k of $b_1, b_2, b_3, \cdots, b_{10}$, or at least k of $c_1, c_2, c_3, \cdots, c_{10}$, have property Q.

5. Assume that at least two of a_1, b_1, c_1 have property T, and likewise for $a_2, b_2, c_2, \cdots$, and likewise for a_5, b_5, c_5. What is the largest integer that can be used for r to make the following assertion correct? At least r of a_1, a_2, a_3, a_4, a_5, or at least r of b_1, b_2, b_3, b_4, b_5, or at least r of c_1, c_2, c_3, c_4, c_5 have property T.

9.2 Chromatic Triangles

Consider six points in a plane, no three of which are collinear (on a straight line). There are $C(6, 2)$ or fifteen line segments connecting the points. Let these fifteen segments be colored in any way by the use of two colors, say red and white; all the segments may be red, all may be white, or some may be red and the rest white. Say that any triangle connecting three of the points is *chromatic* if its sides have the same color.

We shall prove that no matter how the fifteen line segments are colored, it is always possible to find a chromatic triangle. It is a property of the number 6 that 6 is the smallest number of points in the plane (no three collinear) such that, no matter how each of the segments joining pairs of points is colored in one of two colors, it is always possible to find a chromatic triangle.

Proof that it is always possible to find a chromatic triangle: Take any one of the six points, say A, and consider the five segments AB, AC, AD, AE, AF emanating from A. (See Figure 9.1.) By the pigeonhole principle, at least three of these five segments must have the same color. There is no loss of generality in presuming that the three segments having the same color are AB, AC, and AD. (In Figure 9.1, where one color is indicated by dashed lines, the other by solid lines, it is actually AB, AC, and AE that have the same color. But since we can interchange the letters on the points B, C, D, E, and F —in the case illustrated we would interchange the labels on the points D and E —we can always fix it so that the segments AB, AC, and AD have the same color.)

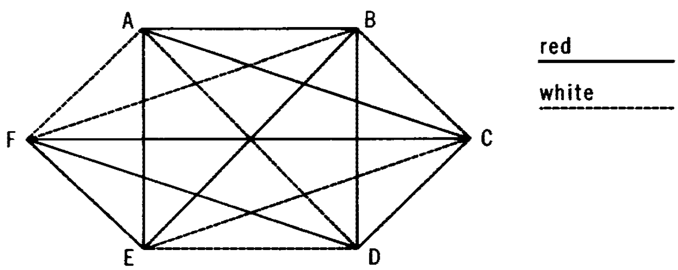


Figure 9.1

Next, there is no loss of generality in presuming that the three segments AB, AC, and AD are red. For if they were white, we would simply reverse the color of every one of the fifteen line segments with no effect on the existence of a chromatic triangle: any red chromatic triangle would become a white chromatic triangle, and vice versa; furthermore, no new chromatic triangles would be created in the process.

We now take the three red segments, AB, AC, and AD emanating from A and consider the triangle BCD formed by their endpoints. There are two possibilities: either all three sides of BCD are white, or it has at least one red side. If all three sides of BCD are white, then BCD is a chromatic triangle. On the other hand, if at least one side of triangle BCD is red, then this red side together with the appropriate two of the three red segments AB, AC, and AD, forms a red chromatic triangle. In full detail, if BC is red, then ABC is a chromatic triangle; if BD is red, then ABD is a chromatic triangle; if CD is red, then ACD is a chromatic triangle. This completes the proof.

We note two other ways of stating the same principle. Among any six persons, it is possible to find three who are mutually acquainted, or it is possible to find three no two of whom are acquainted. Among any six persons, it is possible to find three each of whom has shaken hands with the other two, or it is possible to find three no two of whom have shaken hands.

Problem Set 31

1. Prove that 6 is the smallest number of points having the chromatic triangle property; that is, exhibit 5 points in the plane, no 3 collinear, with each of the 10 line segments joining pairs of points colored in one of two colors, either red or white, in such a way that the configuration has no chromatic triangle. (Note that if 5 such points are exhibited, this implies that 6 is the smallest number.)

2. Consider 17 points in the plane, no 3 collinear, with each of the segments joining the points colored red, white, or blue. Prove that there is a chromatic triangle no matter what color pattern is present. (The reader might wish to solve the analogous problem for an integer

larger than 17. The number 17 is the smallest that can be used in this problem, in the sense that the proposition is not true for 16 or fewer points. However the proof that 17 is the smallest, given by R. E. Greenwood and A. M. Gleason in 1955, is beyond the scope of this book.)

Additional problems on chromatic triangles are included in the Miscellaneous Problems following Chapter 11.

9.3 Separations of the Plane

Consider n straight lines in a plane satisfying the following conditions: (1) each line is infinite in extent in both directions; (2) no two lines are parallel; (3) no three lines are concurrent, i.e. no three lines meet in a point. Into how many regions is the plane separated by the n lines? Let $f(n)$ denote the number of regions into which n such lines separate the plane; we find, by simple observation, that $f(1)=2$, $f(2)=4$, $f(3)=7$ (see Figure 9.2). The problem is to evaluate $f(n)$ in the general case.

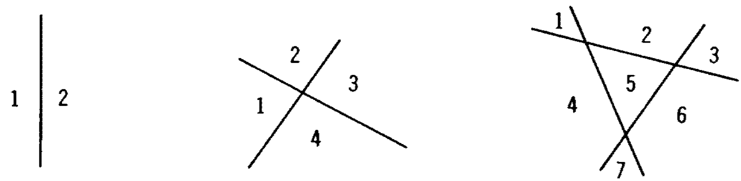


Figure 9.2

To solve this problem we employ a technique which has already been used in Section 3.8. It consists in finding expressions for the differences $f(k) - f(k-1)$ for $k = 2, 3, \cdots, n$ and adding them. Their sum is just $f(n) - f(1)$ since each intermediate term is subtracted and then added. Such a sum is often called a "telescoping" sum. To find the appropriate expressions in this case, consider $n - 1$ straight lines in the plane that separate it into $f(n - 1)$ regions. Now introduce the n-th line. Far out on the line—farther out than any intersection point—this n-th line is dividing a region in two. Then, if we move along the line, we observe that whenever this n-th line crosses one of the other lines, it splits another region in two. For example, let $n = 4$; if we move along the fourth line in Figure 9.3 from left to right, we see that it splits region "4", and then successively regions "5", "6", and "3" as it crosses the three other lines. Thus the fourth line creates four new regions. By the same reasoning we conclude that the n-th line creates n new regions, and we express this fact by the equation $f(n) = n + f(n-1)$, or

$$f(n) - f(n-1) = n. \tag{9.1}$$

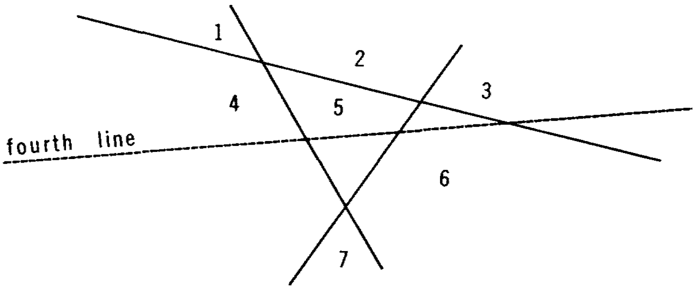


Figure 9.3

Now we apply the method of the "telescoping" sum; that is, we write equation (9.1) followed by its counterparts with n replaced successively by $n-1, n-2, \cdots, 3, 2$:

$$\begin{aligned} f(n) - f(n-1) &= n, \\ f(n-1) - f(n-2) &= n-1, \\ f(n-2) - f(n-3) &= n-2, \\ &\cdots\cdots\cdot \\ f(3) - f(2) &= 3, \\ f(2) - f(1) &= 2. \end{aligned}$$

When these equations are added, the sum of all the left members is simply $f(n) - f(1)$. Thus we have

$$f(n) - f(1) = 2 + 3 + \cdots + (n-2) + (n-1) + n.$$

The right side of this equation is the sum of the natural numbers from 2 to n. Now by Section 3.8 the sum of the natural numbers from 1 to n is $n(n+1)/2$, and so

$$f(n) - f(1) = \frac{n(n+1)}{2} - 1$$

Next we replace $f(1)$ by its known value, 2, and add 2 to both sides of our equation to obtain the final solution,

$$(9.2) \qquad f(n) = \frac{n(n+1)}{2} + 1 = \frac{n^2+n+2}{2}.$$

As another illustration of the "telescoping sum" method, consider the following

PROBLEM: Let there be $n + k$ lines in the plane satisfying these conditions: (1) k of the lines are parallel to each other; (2) there are no other cases of parallel lines; (3) no three of the $n + k$ lines are concurrent. Into how many regions is the plane separated by the $n + k$ lines?

SOLUTION: Let $G(n, k)$ denote the number of regions of separation. For example, $G(1, 2) = 6$ and $G(2, 2) = 10$. The argument leading to equation (9.1) can be modified to get a similar kind of equation in this case; that is, we study the effect of introducing the k-th parallel line, thus changing the number of regions of separation from $G(n, k-1)$ to $G(n, k)$. The k-th parallel line crosses n lines, and so creates $n + 1$ new regions. Thus

$$G(n, k) = n + 1 + G(n, k-1),$$

or

$$(9.3) \qquad G(n, k) - G(n, k-1) = n + 1.$$

Again we write this equation followed by its counterparts with k replaced successively by $k-1$, $k-2$, $\cdots$, 2, 1:

$$\begin{aligned}
G(n, k) - G(n, k-1) &= n+1, \\
G(n, k-1) - G(n, k-2) &= n+1, \\
G(n, k-2) - G(n, k-3) &= n+1, \\
\cdots\cdots\cdots\cdots& \\
G(n, 3) - G(n, 2) &= n+1, \\
G(n, 2) - G(n, 1) &= n+1, \\
G(n, 1) - G(n, 0) &= n+1.
\end{aligned}$$

Here we have k equations, each having the right member $n + 1$. When we add these equations, the sum of the right members is $k(n + 1)$, and the sum of the left members is $G(n, k) - G(n, 0)$, so

$$G(n, k) - G(n, 0) = k(n + 1).$$

The symbol $G(n, 0)$ denotes the number of regions created by n lines none of which are parallel and no three of which are concurrent; this number is the same as $f(n)$ in the preceding problem, and hence we can use equation (9.2) to get

$$G(n, k) - \frac{n^2 + n + 2}{2} = k(n + 1),$$

or

$$G(n, k) = \frac{n^2 + 2nk + n + 2k + 2}{2}.$$

Problem Set 32

1. Consider n straight lines in the plane, no two of which are parallel. However, three of the lines, and only three, are concurrent. Into how many regions is the plane separated?

2. A set of k parallel lines in the plane is intersected by another set of m parallel lines. Into how many regions is the plane separated?

3. In the preceding question, introduce another line, parallel to no previous line, and passing through none of the mk previous intersection points. Into how many regions is the plane separated?

4. Let there be $q + t$ straight lines in the plane satisfying the following conditions: no two lines are parallel; q of the lines pass through a certain point A; t of the lines pass through another point B; no line passes through both A and B. Into how many regions is the plane separated?

5. Let there be $k + q$ straight lines in the plane satisfying the following conditions: k of the lines are parallel to each other; there are no other cases of parallel lines; q of the lines, but none of the k parallel lines, pass through a certain point A. Into how many regions is the plane separated?

6. In addition to the $k + q$ lines in the preceding question, let there be introduced n more straight lines in the plane, such that there are no other cases of parallelism beyond the k parallel lines, and no further cases of concurrency beyond the q lines intersecting at the point A. Into how many regions is the plane separated?

9.4 Summary

The pigeonhole principle, in its simplest form, states that if $n + 1$ pigeons are in n holes, then at least one of the holes contains two or more pigeons. More generally, if $kn + 1$ pigeons are in n holes, then at least one of the holes contains $k + 1$ or more pigeons.

This principle is applied to prove that, given six points in the plane, no three collinear, if each of the 15 line segments joining pairs of points is colored with one of two colors, then there is a chromatic triangle present for every possible color pattern. By "chromatic triangle" is meant one whose three sides are of the same color.

The "telescoping sum" method is used to prove that a plane is separated into $\frac{1}{2}(n^2 + n + 2)$ regions by n straight lines satisfying the conditions (a) no two lines are parallel, and (b) no three lines are concurrent. A somewhat more general situation is handled by the same method and further generalizations are indicated in the problems.

CHAPTER TEN

Mathematical Induction

Consider the sums of the odd integers:

$$
\begin{aligned}
1 &= 1 \\
1 + 3 &= 4 \\
1 + 3 + 5 &= 9 \\
1 + 3 + 5 + 7 &= 16 \\
1 + 3 + 5 + 7 + 9 &= 25 \\
1 + 3 + 5 + 7 + 9 + 11 &= 36
\end{aligned}
\tag{10.1}
$$

A clear pattern emerges in the sums 1, 4, 9, 16, 25, 36; they are the squares of the natural numbers 1, 2, 3, 4, 5, 6. These equations suggest the general proposition that the sum of the first n odd positive integers is equal to n^2, or, stated in symbols,

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2. \tag{10.2}$$

Of course the verification of the first few cases in equations (10.1) does not in any way guarantee that the general formula (10.2) is correct for every positive integer n. Perhaps the easiest way to prove formula (10.2), which is valid for every positive integer n, is to use mathematical induction.

10.1 The Principle of Mathematical Induction

Let us use the symbol P_n to denote equation (10.2). To every positive integer n there corresponds an equation of the form (10.2); for example, the equations listed in (10.1) are of this form, and we designate them by P_1, P_2, P_3, P_4, P_5, P_6. Moreover, we can tell by actual calculation that all six statements of equality are true. The assertion "P_n is true for every positive integer n" actually comprises the infinitely many assertions "P_1 is true, P_2 is true, P_3 is true, $\cdots$". So far, we have seen only that $P_1, P_2, \cdots, P_6$ are true and we want to establish the truth of all (infinitely many) of these propositions. *The principle of mathematical induction states that we can establish the truth of any such infinite sequence of propositions if we can prove two results:*

(i) *that* P_1 *is true;*

(ii) *that* P_{k+1} *follows from* P_k *for every positive integer* k.

Other ways of stating (ii) are that "P_k implies P_{k+1}" and "P_{k+1} is implied by P_k".

The idea is that if we can prove (ii), that P_k implies P_{k+1}, then we can conclude

$$\begin{aligned} &P_1 \text{ implies } P_2,\\ &P_2 \text{ implies } P_3,\\ &P_3 \text{ implies } P_4,\\ &P_4 \text{ implies } P_5, \end{aligned}$$

and so on. So if we prove (i) we will have a start on this chain, and the truth of P_2, P_3, P_4, P_5, and so on, will follow by (ii) from the truth of P_1.

Let us return to the special case where the proposition P_n is the equation (10.2). There is no difficulty with (i), since P_1 is simply $1 = 1$. To prove (ii), we must show that

$$P_k\colon\quad 1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2$$

implies:

$$P_{k+1}\colon\quad 1 + 3 + 5 + 7 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2;$$

that is, we are to assume P_k and prove that P_{k+1} follows as a consequence. Assuming the proposition P_k, let us add $2k + 1$ to both sides of the equation:

$$\begin{aligned} 1 + 3 + 5 + 7 + \cdots + (2k - 1) + (2k + 1) &= k^2 + (2k + 1) \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2. \end{aligned}$$

Thus P_{k+1} follows from P_k, and we have proved the validity of equation (10.2) for all positive integers n.

As a second illustration, consider the sums of the cubes of the natural numbers:

$$\begin{aligned} &1^3 = 1, \\ &1^3 + 2^3 = 9, \\ &1^3 + 2^3 + 3^3 = 36, \\ &1^3 + 2^3 + 3^3 + 4^3 = 100, \\ &1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225, \\ &1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 = 441. \end{aligned} \tag{10.3}$$

The numbers 1, 9, 36, 100, 225, 441 on the right sides of these equations are all squares, namely the squares of 1, 3, 6, 10, 15, 21. If we look at Pascal's triangle on page 41 we note that these numbers are in the third vertical column, and so can be written, in terms of combination symbols, as

$$C(2, 2), \quad C(3, 2), \quad C(4, 2), \quad C(5, 2), \quad C(6, 2), \quad C(7, 2).$$

Might it be that $C(n + 1, 2)$ is the appropriate number whose square is the sum of the cubes of the natural numbers from 1 to n? Since

$$C(n + 1, 2) = \tfrac{1}{2}n(n + 1),$$

this conjecture may be expressed by

$$\begin{aligned} 1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 &= [C(n + 1, 2)]^2 \\ &= \tfrac{1}{4}n^2(n + 1)^2. \end{aligned} \tag{10.4}$$

Let us now regard (10.4) as the proposition P_n, or rather as the infinite collection of propositions, one for $n = 1$, a second for $n = 2$, a third for $n = 3$, and so on; then equations (10.3) are the propositions P_1, P_2, P_3, P_4, P_5, P_6. To prove P_n by mathematical induction, we must establish (i) that P_1 is true, and (ii) that P_k implies P_{k+1} for every positive integer k. Now P_1, the first of equations (10.3), simply states that $1^3 = 1$, and this is clearly true.

Before proving (ii), let us write out P_k and P_{k+1} in full, by replacing n by k, and then n by $k + 1$ in (10.4),

$$P_k:\quad 1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 = \tfrac{1}{4}k^2(k+1)^2;$$

$$P_{k+1}:\quad 1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 + (k+1)^3 = \tfrac{1}{4}(k+1)^2(k+2)^2.$$

We are to assume P_k and establish P_{k+1}. Adding $(k + 1)^3$ to both sides of the equation P_k, we obtain

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 + (k+1)^3 = \tfrac{1}{4}k^2(k+1)^2 + (k+1)^3.$$

The question is whether this is the same as P_{k+1}. Using basic algebra we shall see that it is P_{k+1}.

$$\begin{aligned}\tfrac{1}{4}k^2(k+1)^2 + (k+1)^3 &= (k+1)^2[\tfrac{1}{4}k^2 + (k+1)]\\ &= (k+1)^2\left[\frac{k^2+4k+4}{4}\right]\\ &= \tfrac{1}{4}(k+1)^2(k+2)^2,\end{aligned}$$

and so the proof of (10.4) is complete.

Problem Set 33

1. Prove that $1 + 2 + 3 + 4 + \cdots + n = \frac{1}{2}n(n + 1)$ by mathematical induction.

2. Prove that $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6$ by mathematical induction.

3. Let $K(n)$ denote the number of unordered pairs of integers selected from $1, 2, 3, \cdots, n$, subject to the restriction that no pair is consecutive. For example, $K(5)$ is the count of the pairs

$$1,3 \qquad 1,4 \qquad 1,5 \qquad 2,4 \qquad 2,5 \qquad 3,5$$

and so $K(5) = 6$. By such counting it can be determined that

$$K(3) = 1 \qquad K(4) = 3 \qquad K(5) = 6$$
$$K(6) = 10 \qquad K(7) = 15 \qquad K(8) = 21.$$

Make a conjecture about $K(n)$ from this information, and, if possible, prove your conjecture by mathematical induction.

4. Some of the following equations hold for all positive integers n. Try to establish each by mathematical induction.

(a) $1 + 4 + 7 + 10 + \cdots + (3n - 2) = n^2 + n - 1$;

(b) $1\cdot2 + 2\cdot3 + 3\cdot4 + \cdots + n(n + 1) = (n^3 + 3n^2 + 2n)/3$;

(c) $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = (4n^3 - n)/3$;

(d) $1\cdot3 + 3\cdot5 + 5\cdot7 + \cdots + (2n - 1)(2n + 1)$
$= (5n^3 + 10n - 6)/3$;

(e) $1\cdot2 + 3\cdot3 + 5\cdot4 + 7\cdot5 + \cdots + (2n - 1)(n + 1)$
$= (n^3 + 5n^2 - 4n + 2)/2$;

(f) $1\cdot1\cdot2 + 2\cdot2\cdot3 + 3\cdot3\cdot4 + \cdots + n\cdot n\cdot(n + 1)$
$= n(3n^3 + 10n^2 + 9n + 2)/12$.

10.2 Notation for Sums and Products

In writing such an equation as

$$1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n + 1)$$

there is some difficulty with the notation. Whereas with $n = 10$ there is no doubt as to what is meant by $1 + 2 + 3 + \cdots + n$, in the case $n = 2$ one must interpret $1 + 2 + 3 + \cdots + n$ as simply

$1 + 2$. There is a notation that avoids this confusion and at the same time has the virtue of greater compactness, namely

$$\sum_{j=1}^{n} j \quad \text{in place of} \quad 1 + 2 + 3 + \cdots + n.$$

This is read "sigma j, $j = 1$ to n", and means "sum the element j for all integer values from $j = 1$ to $j = n$". Here are some other illustrations:

$$\sum_{j=1}^{n} j^2 \quad \text{means} \quad 1^2 + 2^2 + 3^2 + \cdots + n^2;$$

$$\sum_{j=1}^{100} j(j + 3) \quad \text{means} \quad 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7 + \cdots + 100 \cdot 103;$$

$$\sum_{j=1}^{n} (j^3 + 1) \quad \text{means} \quad (1^3 + 1) + (2^3 + 1) + (3^3 + 1) + \cdots + (n^3 + 1);$$

$$\sum_{j=4}^{n-1} (j^3 + 1) \quad \text{means} \quad (4^3 + 1) + (5^3 + 1) + (6^3 + 1) + \cdots + ((n - 1)^3 + 1).$$

We note that constant factors can be moved to the left of the sigma:

$$(10.6) \qquad \sum_{j=1}^{n} 4j^2 = 4 \sum_{j=1}^{n} j^2, \qquad \sum_{j=1}^{n} 5(j^3 + 1) = 5 \sum_{j=1}^{n} (j^3 + 1);$$

the reason is that the constant factor multiplies each term in the sum and may therefore be written as a factor in front of the entire sum. Also we note that expressions consisting of several terms may be summed termwise; for example,

$$(10.7) \qquad \begin{aligned} \sum_{j=1}^{n} (j^2 + 3j) &= \sum_{j=1}^{n} j^2 + \sum_{j=1}^{n} 3j, \\ \sum_{j=1}^{n} (j^3 + 3j^2 - j) &= \sum_{j=1}^{n} j^3 + \sum_{j=1}^{n} 3j^2 - \sum_{j=1}^{n} j. \end{aligned}$$

This is a consequence of a mere re-grouping of terms.

The sum of the natural numbers from 1 to n and the sum of their squares were evaluated in Chapter 3, page 47. The sum (10.4) of the cubes of the natural numbers from 1 to n was found by mathematical induction in the preceding section. With the sigma notation, these sums can be written as follows:

$$\sum_{j=1}^{n} j = \tfrac{1}{2}n(n+1),$$

$$(10.8) \qquad \sum_{j=1}^{n} j^2 = \tfrac{1}{6}n(n+1)(2n+1),$$

$$\sum_{j=1}^{n} j^3 = \tfrac{1}{4}n^2(n+1)^2.$$

Note that the symbol j is a "dummy"; we could just as well write

$$\sum_{k=1}^{n} k = \tfrac{1}{2}n(n+1), \qquad \text{and} \qquad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

instead of the first two formulas in (10.8). Let us use sigma notation to evaluate the sum

$$1\cdot 1\cdot 2 + 2\cdot 2\cdot 3 + 3\cdot 3\cdot 4 + \cdots + n\cdot n\cdot (n+1).$$

We write it in the form

$$\sum_{j=1}^{n} j^2(j+1) \qquad \text{or} \qquad \sum_{j=1}^{n} (j^3 + j^2).$$

Using the property illustrated in (10.7) and formulas (10.8) we get

$$\begin{aligned} \sum_{j=1}^{n} (j^3 + j^2) &= \sum_{j=1}^{n} j^3 + \sum_{j=1}^{n} j^2 \\ &= \tfrac{1}{4}n^2(n+1)^2 + \tfrac{1}{6}n(n+1)(2n+1) \\ &= n(n+1)(n+2)(3n+1)/12, \end{aligned}$$

where we have omitted the simple algebra involved in arriving at the last formulation.

As another example, consider the sum

$$1 + 4 + 7 + 10 + \cdots + (3n - 2)$$

which, in sigma notation, can be written

$$\sum_{j=1}^{n} (3j - 2).$$

Again using (10.8) along with basic properties described above, we have

$$\sum_{j=1}^{n} (3j - 2) = \sum_{j=1}^{n} 3j - \sum_{j=1}^{n} 2 = 3\sum_{j=1}^{n} j - \sum_{j=1}^{n} 2$$
$$= \frac{3n(n+1)}{2} - 2n = \frac{n(3n-1)}{2}.$$

Sums whose terms are preceded by alternating plus and minus signs are written in sigma notation with the help of $(-1)^j$; for example,

$$\sum_{j=1}^{8} (-1)^j j^2 = -1^2 + 2^2 - 3^2 + 4^2 - 5^2 + 6^2 - 7^2 + 8^2,$$

$$\sum_{j=0}^{7} (-1)^j C(7,j) = C(7,0) - C(7,1) + C(7,2) - C(7,3)$$
$$+ C(7,4) - C(7,5) + C(7,6) - C(7,7).$$

As another illustration, consider the formula from Chapter 8 for the number of distributions of m distinct objects into k distinct boxes with no box empty:

$$f(m,k) = k^m - C(k,1)(k-1)^m + C(k,2)(k-2)^m$$
$$- C(k,3)(k-3)^m + \cdots + (-1)^{k-1}C(k,k-1)(1)^m.$$

This can be written in compact form as

$$f(m, k) = \sum_{j=0}^{k-1} (-1)^j C(k, j)(k - j)^m.$$

Also in Chapter 8 the number of partitions of a set of m (distinct) elements into k (non-distinct) subsets with no subset empty was denoted by $g(m, k)$, with the relation

$$g(m, k) = f(m, k)/k!\,.$$

Thus a compact formula for $g(m, k)$ would be

$$g(m, k) = \sum_{j=0}^{k-1} (-1)^j C(k, j)(k - j)^m/k!\,.$$

If in this formula we replace $C(k, j)$ by its factorial form, there is a cancellation of $k!$ and the result becomes

$$g(m, k) = \sum_{j=0}^{k-1} \frac{(-1)^j (k - j)^m}{j!(k - j)!}\,.$$

There is also a convenient shorthand notation for products; it uses the upper case Greek letter pi, $\prod$, in place of the upper case sigma. For example, $n!$ can be written

$$n! = \prod_{j=1}^{n} j.$$

Here are some other examples:

$$\prod_{j=1}^{n} (j^2 + 1) \quad \text{means} \quad (1^2 + 1)(2^2 + 1)(3^2 + 1)\cdots(n^2 + 1);$$

$$\prod_{j=1}^{n} (3j - 1) \quad \text{means} \quad 2\cdot 5\cdot 8\cdot 11\cdots(3n - 1);$$

$$\prod_{j=1}^{7} (1 + x^j) \quad \text{means} \quad (1 + x)(1 + x^2)(1 + x^3)(1 + x^4) \cdot (1 + x^5)(1 + x^6)(1 + x^7).$$

Problem Set 34

1. Express the following sums without the sigma notation:

$$\text{(a)} \sum_{j=1}^{5} j^2; \qquad \text{(b)} \sum_{j=1}^{4} (2j^2 - 1); \qquad \text{(c)} \sum_{k=3}^{6} (k^2 + 2).$$

2. Evaluate the following sums by use of the sigma notation and formulas (10.8):

   (a) $3 + 6 + 9 + 12 + \cdots + 3n;$

   (b) $2 + 5 + 8 + 11 + \cdots + (3n - 1);$

   (c) $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + 7 \cdot 9 + \cdots + (2n - 1)(2n + 1);$

   (d) $1 \cdot 2 + 3 \cdot 3 + 5 \cdot 4 + 7 \cdot 5 + \cdots + (2n - 1)(n + 1);$

   (e) $5 + 9 + 13 + 17 + 21 + \cdots + (4n + 1).$

3. Write formulas similar to formulas (10.8) for the sums

   (a) $1 + 2 + 3 + \cdots + (n - 1);$

   (b) $1 + 2 + 3 + \cdots + (n + 1);$

   (c) $1^2 + 2^2 + 3^2 + \cdots + (n + 1)^2.$

4. Find the numerical value of

$$\sum_{j=1}^{100} (-1)^j \cdot j.$$

5. Write the equation

$$C(n, 0) + C(n, 1) + C(n, 2) + C(n, 3) + \cdots + C(n, n) = 2^n$$

in sigma notation.

6. Identify

$$\sum_{j=0}^{n} (-1)^j C(n, j)$$

as a sum discussed earlier in this book, and so evaluate it.

7. Prove

$$\sum_{j=0}^{n} 2^j = 2^{n+1} - 1$$

by mathematical induction. Hence prove that, in Pascal's triangle (page 41), the sum of the elements in any row equals the sum of all elements in all preceding rows, with 1 added.

8. Evaluate the product $\prod_{j=2}^{6} (j+1)$.

9. Express the product $\prod_{j=1}^{n} (2j)$ in factorial notation.

10. Verify that

$$\prod_{j=1}^{n} (2j-1) = \frac{(2n)!}{2^n \cdot n!}.$$

11. Let $f(n)$ be defined by

$$f(n) = \sum_{j=1}^{n} j \cdot j!.$$

Verify that $f(1) = 1$, $f(2) = 5$, $f(3) = 23$, and find the numerical values of $f(4)$, $f(5)$, and $f(6)$. Then compare these with the numerical values of $2!$, $3!$, $4!$, $5!$, $6!$, and $7!$, conjecture a formula for $f(n)$, and try to prove it by mathematical induction.

10.3 Summary

The proof technique known as mathematical induction can be used to establish an infinite sequence of propositions P_1, P_2, P_3, $\cdots$, provided one can show

(i) that P_1 is true,

(ii) that P_{k+1} follows from P_k for every positive integer k.

The sigma notation for sums and the pi notation for products are explained.

CHAPTER ELEVEN

Interpretations of a Non-Associative Product

Consider the mathematical expression

$$2^{3^4}.$$

There appear to be two ways of interpreting this—one by starting with 2^3 and so interpreting the expression as 8^4; another by starting with 3^4 and so interpreting the expression as 2^{81}. These ways lead to different results because 8^4 is 4096 whereas 2^{81} is much larger. We can indicate these two interpretations by using parentheses; thus

$$(11.1) \qquad (2^3)^4 = 8^4, \quad 2^{(3^4)} = 2^{81}, \quad \text{and} \quad (2^3)^4 \neq 2^{(3^4)}.$$

Now in actual fact there is a convention or agreement in mathematics as to precisely how 2^{3^4} is to be interpreted, namely as the second form in (11.1),

$$2^{3^4} = 2^{(3^4)} = 2^{81}.$$

For the purposes of this chapter we disregard this convention. We look upon (11.1) as a demonstration that exponentiation is not associative, in contrast, for example, to addition and multiplication;

$$(2 + 3) + 4 = 2 + (3 + 4), \qquad (2\cdot 3)\cdot 4 = 2\cdot(3\cdot 4).$$

Ignoring the conventional meaning for such expressions as

$$2^{3^{4^5}} \quad \text{or} \quad a^{b^{c^d}}, \tag{11.2}$$

we ask how many interpretations there are when four numbers are stacked up this way in exponential fashion. More generally, how many interpretations are there when n numbers are stacked up in exponential fashion?

11.1 A Recursion Relation

To simplify the typography we shall write the second expression in (11.2) as though it were a "product" $abcd$. We presume that such "products" are non-associative, so that the 3-products $a(bc)$ and $(ab)c$ are different. All possible interpretations of a 4-product can be readily enumerated:

$$a((bc)d), \quad a(b(cd)), \quad (ab)(cd), \quad (a(bc))d, \quad ((ab)c)d. \tag{11.3}$$

Let us define $F(n)$ as the number of interpretations of a non-associative n-product; then the enumeration (11.3) shows that $F(4) = 5$. Also, we know that $F(3) = 2$, because of the two cases $a(bc)$ and $(ab)c$. There is only one interpretation of a 2-product ab, and likewise only one interpretation for a 1-product a, and hence we can write $F(2) = 1$, $F(1) = 1$.

The general problem of this chapter is the evaluation of $F(n)$, the number of interpretations of an n-product

$$x_1x_2x_3\cdots x_n \tag{11.4}$$

with no associative property. $F(n)$ can be thought of as the number of ways of putting parentheses on (11.4) to make it non-ambiguous.

To illustrate what we are about to do, let us take the special case $n = 6$. In putting parentheses on a 6-product, one possible first step is to split the product into two parts. This can be done in any of the following five ways:

$$
\text{(11.5)} \qquad
\begin{array}{ll}
\text{(a)} & x_1(x_2x_3x_4x_5x_6), \\
\text{(b)} & (x_1x_2)(x_3x_4x_5x_6), \\
\text{(c)} & (x_1x_2x_3)(x_4x_5x_6), \\
\text{(d)} & (x_1x_2x_3x_4)(x_5x_6), \\
\text{(e)} & (x_1x_2x_3x_4x_5)x_6.
\end{array}
$$

In how many ways can additional parentheses be installed? In expression (a), there are $F(5)$ interpretations of $x_2x_3x_4x_5x_6$. In expression (b), there are $F(4)$ interpretations of $x_3x_4x_5x_6$. In expression (c), there are $F(3)$ interpretations for each of $x_1x_2x_3$ and $x_4x_5x_6$, and so $F(3)\cdot F(3)$ interpretations in all for (c). Expressions (d) and (e) are similar to (b) and (a) respectively. Putting all this information together we see that

$$F(6) = F(5) + F(4) + F(3)F(3) + F(4) + F(5).$$

Since $F(1) = 1$ and $F(2) = 1$, this can be written in the more symmetric form

$$F(6) = F(1)F(5)+F(2)F(4)+F(3)F(3)+F(4)F(2)+F(5)F(1).$$

By a similar argument we can conclude that

$$F(7) = F(1)F(6) + F(2)F(5) + F(3)F(4) + F(4)F(3) + F(5)F(2) + F(6)F(1),$$

and, in general, that

$$
\text{(11.6)} \qquad F(n) = F(1)F(n-1) + F(2)F(n-2) + F(3)F(n-3) + \cdots + F(n-1)F(1).
$$

With the sigma notation for sums the recursion relation (11.6) can be written in the form

$$F(n) = \sum_{j=1}^{n-1} F(j)F(n-j)$$

and used to calculate successive values of $F(n)$ as n increases through the natural numbers. For example, if we take $F(1) = 1$ and $F(2) = 1$ as our starting values, we can find

$$\begin{aligned}
F(3) &= F(1)F(2) + F(2)F(1) = 1 + 1 = 2,\\
F(4) &= F(1)F(3) + F(2)F(2) + F(3)F(1) = 2 + 1 + 2 = 5,\\
F(5) &= F(1)F(4) + F(2)F(3) + F(3)F(2) + F(4)F(1)\\
&= 5 + 2 + 2 + 5 = 14,
\end{aligned}$$

and so on.

Problem Set 35

1. Find the number of interpretations of (i) a 6-product, (ii) a 7-product, (iii) an 8-product, in a non-associative system.

2. What is the conventional non-ambiguous meaning of $5^{4^{3^{2}}}$?

3. Enumerate the fourteen interpretations of a 5-product, analogous to the formulation (11.3) in the text.

11.2 The Development of an Explicit Formula

Consider any non-associative product such as

$$(((x_1x_2)(x_3x_4))x_5)(((x_6x_7)x_8)x_9). \tag{11.7}$$

The parentheses, which serve to indicate the arrangement of association of the elements, occur in pairs, with a left and a right parenthesis in each pair. To any product such as (11.7) we attach two numbers, denoted by n and k; n is the number of elements in the product [$n = 9$ in the example (11.7)], and k denotes the number of elements preceding the rightmost of the left parentheses. (In the example (11.7) the rightmost of the left parentheses is the one immediately preceding x_6, and hence $k = 5$.) As another example consider

$$x_1(x_2(((x_3(x_4x_5))x_6)x_7)), \tag{11.8}$$

where $n = 7$ and $k = 3$. In any such product the rightmost of the left parentheses is followed by two elements and the corresponding right parenthesis;† in (11.7) this pattern is (x_6x_7), and in (11.8) it is (x_4x_5).

Next we define a transformation which takes any product and transforms it into a product having one element less. The transformation removes the rightmost of the left parentheses, the element following it, and the corresponding right parenthesis. Thus (11.7) is transformed into

$$(((x_1x_2)(x_3x_4))x_5)((x_7x_8)x_9), \tag{11.9}$$

and (11.8) into

$$x_1(x_2(((x_3x_5)x_6)x_7)). \tag{11.10}$$

In (11.9) we see that $n = 8$ and $k = 5$, and in (11.10) $n = 6$ and $k = 2$.

In general, an n-product with k elements preceding the rightmost of the left parentheses is transformed into an $(n - 1)$-product because one element is removed. The transformed expression either has k elements preceding the rightmost of the left parentheses (this is the case whenever that parenthesis is adjacent to another left parenthesis), or the transformed product has fewer than k elements preceding it.

Let us denote by $F(n, k)$ the number of non-associative n-products with exactly k elements preceding the rightmost of the left parentheses. It turns out that the following relation holds:

$$\begin{aligned} F(n, k) &= F(n-1, k) + F(n-1, k-1) \\ &\quad + F(n-1, k-2) + \cdots + F(n-1, 0). \end{aligned} \tag{11.11}$$

We illustrate this in the case $n = 5$ and $k = 3$. There are five products corresponding to these specific values of n and k; that is to say, $F(5, 3) = 5$. We list these five products in the left column and the corresponding transformed products in the right column:

† By "such a product", we mean one which has been made unambiguous by the insertion of sufficiently many parentheses.

Product	Transformed product	n	k
$(x_1x_2)(x_3(x_4x_5))$	$(x_1x_2)(x_3x_5)$	4	2
$x_1(x_2(x_3(x_4x_5)))$	$x_1(x_2(x_3x_5))$	4	2
$x_1((x_2x_3)(x_4x_5))$	$x_1((x_2x_3)x_5)$	4	1
$(x_1(x_2x_3))(x_4x_5)$	$(x_1(x_2x_3))x_5$	4	1
$((x_1x_2)x_3)(x_4x_5)$	$((x_1x_2)x_3)x_5$	4	0

The first two transformed products are of the $F(4, 2)$ type, the next two of the $F(4, 1)$ type, and the last one of the $F(4, 0)$ type. In fact, these collections of types are complete, so that $F(4, 2) = 2$, $F(4, 1) = 2$, $F(4, 0) = 1$. Furthermore, there is no product of the $F(4, 3)$ type, and so $F(4, 3) = 0$. Thus by actual count we have verified the special case

$$F(5, 3) = F(4, 3) + F(4, 2) + F(4, 1) + F(4, 0)$$

of (11.11).

A proof of (11.11) is suggested by this example. First, if any product of the $F(n, k)$ type is transformed by the procedure described above, there results a product of one of the types listed on the right side of (11.11). Secondly, the transformation is reversible as follows: take any product of a type listed on the right side of (11.11); replace the $(k + 1)$-st element, say y, by two elements in parentheses, say (yz); this procedure gives a product of the type $F(n, k)$. So we have a one-to-one correspondence between the types of products listed on the two sides of equation (11.11), and the result is thereby established.

Now if, in formula (11.11), we replace k by $k - 1$ the result is

$$F(n, k - 1) = F(n - 1, k - 1) + F(n - 1, k - 2) + F(n - 1, k - 3) + \cdots + F(n - 1, 0).$$

Subtracting this from (11.11), we get

$$F(n, k) - F(n, k - 1) = F(n - 1, k)$$

or

$$F(n, k) = F(n, k - 1) + F(n - 1, k). \tag{11.12}$$

This formula somewhat resembles the result

$$C(n, r) = C(n - 1, r) + C(n - 1, r - 1).$$

There is a connection between the functions $F(n, k)$ and $C(n, r)$ which we now reveal by comparing brief numerical tables.

To develop a table of values of $F(n, k)$ we use (11.12) along with certain basic results. Because parentheses are not needed in the simple cases $n = 1$ and $n = 2$, let us confine attention to values of $n \geqslant 3$. For any value of n, the corresponding values of k are $0, 1, 2, \cdots, n - 1$. (It turns out to be convenient in the formulas to ignore the case $k = n$, for which there are no products.) The values of $F(n, 0)$ and $F(n, n - 1)$ can easily be determined from the definition of $F(n, k)$. First, $F(n, 0)$ means the number of n-products having no element preceding the rightmost of the left parentheses. There is one such product illustrated, in the case $n = 8$, by

$$((((((x_1x_2)x_3)x_4)x_5)x_6)x_7)x_8,$$

and so $F(n, 0) = 1$. Now the rightmost of the left parentheses is followed by at least two elements (since we do not enclose a single element in parentheses), and so there is no product of type $F(n, n - 1)$. Thus we have

$$F(n, 0) = 1, \quad F(n, n - 1) = 0 \quad \text{for all } n \geqslant 3. \tag{11.13}$$

With this information and the easily established result $F(3, 1) = 1$, it is now possible to use (11.12) to develop a table of values.

TABLE OF VALUES OF $F(n, k)$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
3	1	1	0							
4	1	2	2	0						
5	1	3	5	5	0					
6	1	4	9	14	14	0				
7	1	5	14	28	42	42	0			
8	1	6	20	48	90	132	132	0		
9	1	7	27	75	165	297	429	429	0	
10	1	8	35	110	275	572	1001	1430	1430	0
11	1	9	44	154	429	etc.				
12	1	10	54	208	637					
13	1	11	65	273	910					
14	1	12	77	350	1260					

This is to be compared with a table of values of $C(n, r)$, that is, with Pascal's triangle. In every row of Pascal's triangle, we list the differences of adjacent pairs of values, the left one subtracted from the right one, by writing these in parentheses between each pair; but we do not list negative differences (see the table on p. 148).

A comparison of these tables shows that the entries in the $F(n, k)$ table turn up as differences in the $C(n, r)$ table; for example,

$$F(7, 3) = C(8, 3) - C(8, 2),$$

$$F(8, 4) = C(10, 4) - C(10, 3),$$

$$F(9, 6) = C(13, 6) - C(13, 5),$$

$$F(12, 3) = C(13, 3) - C(13, 2).$$

These results suggest the general proposition

$$(11.14) \quad F(n, k) = C(n+k-2, k) - C(n+k-2, k-1).$$

This conjecture is correct, but of course it cannot be proved by an examination of a few special cases in the tables.

TABLE OF VALUES OF $C(n, r)$, DIFFERENCES IN PARENTHESES

$n \backslash r$	0	1	2	3	4	5	6	7
1	1(0)	1	0	0	0	0	0	0
2	1(1)	2	1	0	0	0	0	0
3	1(2)	3(0)	3	1	0	0	0	0
4	1(3)	4(2)	6	4	1	0	0	0
5	1(4)	5(5)	10(0)	10	5	1	0	0
6	1(5)	6(9)	15(5)	20	15	6	1	0
7	1(6)	7(14)	21(14)	35(0)	35	21	7	1
8	1(7)	8(20)	28(28)	56(14)	70	56	28	8
9	1(8)	9(27)	36(48)	84(42)	126(0)	126	84	36
10	1(9)	10(35)	45(75)	120(90)	210(42)	252	210	120
11	1(10)	11(44)	55(110)	165(165)	330(132)	462(0)	462	330
12	1(11)	12(54)	66(154)	220(275)	495(297)	792(132)	924	792
13	1(12)	13(65)	78(208)	286(429)	715(572)	1287(429)	1716(0)	1716

11.3 Proof of the Conjecture

Before proving the conjecture (11.14) we write it in a different form by making an algebraic calculation:

$$\begin{aligned} C(n+k-2,\ k) - C(n+k-2,\ k-1) \\ &= \frac{(n+k-2)!}{k!(n-2)!} - \frac{(n+k-2)!}{(k-1)!(n-1)!} \\ &= \frac{(n+k-2)!}{k!(n-1)!}[(n-1)-k] . \end{aligned}$$

Thus (11.14) can be written as

$$(11.15) \qquad F(n, k) = \frac{(n+k-2)!}{k!(n-1)!}(n-k-1) .$$

We note that this is true for $k = 0$, because

$$F(n, 0) = \frac{(n-2)!}{0!(n-1)!}(n-1) = 1$$

agrees with the value calculated before. Also we note that (11.15) gives the correct results $F(3, 1) = 1$ and $F(3, 2) = 0$.

We are now in a position to prove (11.15) by mathematical induction. It is necessary to argue in a slightly more sophisticated way than in the proofs by induction of the preceding chapter, because there are now two variables, n and k. However, we can reduce this problem to one in a single variable by the following device. Let P_m denote all cases of (11.15) with $n + k = m$. Since $n \geqslant 3$ we start with $m = 3$:

P_3 is equation (11.15) in the one case $n = 3,\ k = 0$;

P_4 is (11.15) in the cases $n = 4,\ k = 0$ and $n = 3,\ k = 1$;

P_5 is (11.15) in the cases $n = 5,\ k = 0$; $n = 4,\ k = 1$;

$n = 3,\ k = 2$.

Similarly P_6 would consist of 4 cases, P_7 of 5 cases, and so on. Our inductive proof of (11.15) will consist in proving (i) that P_3 is true, and (ii) that P_m implies P_{m+1}.

We have already checked that P_3 holds, so we turn to (ii). Assuming that P_m holds, we are to prove P_{m+1}, that is, equation (11.15), for any pair of integers n and k whose sum is $m + 1$. So in what follows we regard n and k as integers for which $n + k = m + 1$. Of course, in proving that P_m implies P_{m+1} we make use of P_m, and so we use (11.15) for $F(n, k - 1)$ and $F(n - 1, k)$ because

$n + k = m + 1$ implies $n + (k - 1) = m$ and $(n - 1) + k = m$.

Thus P_m includes the two statements

$$F(n, k-1) = \frac{(n+k-3)!}{(k-1)!(n-1)!}(n-k)$$

and

$$F(n-1, k) = \frac{(n+k-3)!}{k!(n-2)!}(n-k-2).$$

Making use of (11.12), we get

$$\begin{aligned}
F(n, k) &= F(n, k-1) + F(n-1, k) \\
&= \frac{(n+k-3)!}{(k-1)!(n-1)!}(n-k) + \frac{(n+k-3)!}{k!(n-2)!}(n-k-2) \\
&= \frac{(n+k-3)!}{k!(n-1)!}[k(n-k) + (n-1)(n-k-2)] \\
&= \frac{(n+k-3)!}{k!(n-1)!}(n+k-2)(n-k-1) \\
&= \frac{(n+k-2)!}{k!(n-1)!}(n-k-1),
\end{aligned}$$

and so (11.15) is established.

11.4 A Formula for $F(n)$

Our aim now is to answer the question posed in the introduction to this chapter: What is the number $F(n)$ of n-products in a non-associative system? By using the results of the intervening sections, we now derive a simple formula for $F(n)$.

First we observe that the total number of n-products consists of those with no element preceding the rightmost of the left parentheses, plus those with one element preceding it, plus those with two elements preceding it, $\cdots$, plus those with all but one preceding it, plus those with all preceding it. In symbols,

$$F(n) = F(n, 0) + F(n, 1) + \cdots + F(n, n - 2) + F(n, n - 1) + F(n, n). \tag{11.16}$$

In view of formula (11.13) and the fact that $F(n, n) = 0$, we may write (11.16) in the form

$$F(n) = \sum_{j=0}^{n-2} F(n, j). \tag{11.16'}$$

Next, we write formula (11.11), derived in Section 11.2, with n replaced by $n + 1$ and k replaced by $n - 1$. This yields

$$F(n + 1, n - 1) = F(n, n - 1) + F(n, n - 2) + \cdots + F(n, 0). \tag{11.17}$$

But $F(n, n - 1)$ is zero, so we may write (11.17) as

$$F(n + 1, n - 1) = \sum_{j=0}^{n-2} F(n, j),$$

and, comparing it with (11.16′), we see that

$$F(n) = F(n + 1, n - 1). \tag{11.18}$$

Finally, we apply formula (11.15), derived in Section 11.3, to the right member of (11.18); in other words, we replace n by $n + 1$ and k by $n - 1$ in (11.15) and obtain

$$F(n+1, n-1) = \frac{[(n+1) + (n-1) - 2]!}{(n - 1)!n!} [n+1 - (n-1) - 1]$$

$$= \frac{(2n - 2)!}{n!(n - 1)!}.$$

Substituting this into (11.18) gives the desired result

$$F(n) = \frac{(2n-2)!}{n!(n-1)!}$$

for the number of ways of meaningfully inserting parentheses into an expression of the form $x_1x_2x_3\cdots x_n$.

11.5 Summary

A mathematical "product" is non-associative if $a(bc) = (ab)c$ does not hold in all cases. The word "product" is in quotation marks because for the purposes of this chapter ab can represent the result of any binary operation on elements a and b yielding a non-associative system. One illustration of this arises from interpreting ab as the exponential form a^b.

In a non-associative system there are two interpretations of the 3-product abc, namely $a(bc)$ and $(ab)c$. The topic of this chapter is the number of interpretations, denoted by $F(n)$, of a non-associative n-product $x_1x_2x_3\cdots x_n$. First a recursion relation

$$F(n) = \sum_{j=1}^{n-1} F(j)F(n-j)$$

is established; then the explicit formula

$$F(n) = \frac{(2n-2)!}{n!(n-1)!}$$

is derived. We proved this result by separating $F(n)$ into parts, with $F(n, k)$ denoting the number of n-products having exactly k elements preceding the rightmost of the left parentheses. Properties of $F(n, k)$ developed in formulas (11.11) and (11.12) resulted in a table of values for this function. On comparing these values with differences in Pascal's triangle, it was easy to guess how $F(n, k)$ is related to the $C(m, j)$. This guess, conjecture (11.14), was proved by mathematical induction, and thus $F(n)$ was evaluated.

Miscellaneous Problems

1. A class is given a true-false test consisting of 12 questions. One of the students, rather unprepared, decides on the following strategy. He answers 3 of the questions about which he feels absolutely certain, and then handles the other 9 by tossing a coin to make his decision in each case. Assuming that the student answered those 3 correctly, establish that his probability of getting at least half the answers right is greater than 9/10.

2. How many terms of the sequence of natural numbers $1, 2, 3, 4, \cdots$ must be added to give a sum exceeding one million?

3. Consider the sequence $2, 22, 84, 212, \cdots$, whose terms are obtained by taking $j = 1,\ j = 2,\ j = 3,\ \cdots$ in the expression $4j^3 - 3j^2 + j$. Find a formula for the sum of the first n terms.

4. If 12 boys are separated at random into 3 teams of 4 each, what is the probability that 2 particular boys will be on different teams?

5. In the preceding question, what is the probability that 3 particular boys will be completely separated, one on each team?

6. Given a set of N objects of which $N(\alpha)$ have a certain property α, and $N(\alpha, \beta)$ have both properties α and β, etc., prove that

$$3N + N(\alpha, \beta) + N(\alpha, \gamma) + N(\beta, \gamma) \geqslant 2N(\alpha) + 2N(\beta) + 2N(\gamma).$$

7. Write a polynomial product so that the coefficient of x^{100} denotes the number of partitions of 100 into unequal positive odd integers.

8. In a certain mythical country, postage stamps come in the following denominations: 3 kinds of 1 cent stamps (the regular kind and two commemoratives), 3 kinds of 2 cent stamps, 2 kinds of 3 cent stamps, and one kind each of 4 cent, 5 cent, 10 cent and 20 cent stamps. Write a polynomial product so that the coefficient of x^{20} denotes the number of ways of getting 20 cents worth of stamps.

9. Prove that the Fibonacci numbers $F(0) = 1$, $F(1) = 2$, $F(2) = 3$, $F(3) = 5$, $F(5) = 8$, etc. have the property

$$F(n) = 2 + \sum_{j=0}^{n-2} F(j) \qquad \text{if } n > 2.$$

10. For any given positive integer n, prove that

$$\sum_{j+k=n+1} C(j, k) = 1 + \sum_{j+k<n} C(j, k),$$

where the sum on the left includes all terms $C(j, k)$ with non-negative integers j, k satisfying $j + k = n + 1$, and the sum on the right all such terms with $j + k < n$. (Suggestion: Use the result of the preceding problem.)

11. Of the 30! permutations of the integers $1, 2, 3, \cdots, 30$, how many have the property that multiples of 3 are not in adjacent places, that is, no two of the integers 3, 6, 9, 12, 15, 18, 21, 24, 27, 30 are adjacent?

12. Find the number of permutations of the 8 letters a, b, c, d, e, f, g, h, taken all at a time, subject to the condition that b does not immediately follow a, c does not immediately follow b, $\cdots$, and h does not immediately follow g.

13. A collection of 100 coins, 20 of which are cents, 20 nickels, 20 dimes, 20 quarters, and 20 fifty cent pieces, are to be put into 5 distinct boxes. In how many ways can this be done if no box is to be empty? (Presume that the 20 coins of each single denomination are indistinguishable.)

14. Find the number of permutations of the 8 letters $AABBCCDD$, taken all at a time, such that no two adjacent letters are alike.

15. How many permutations are there of the 9 letters $D, D, D, E, E, E, F, F, F,$ taken all at a time, subject to the restriction that no two D's are adjacent?

16. What would be the answer to the preceding question if the additional restriction were imposed that no two E's are adjacent?

17. What would be the answer to the preceding question if yet another restriction were imposed, namely that no two F's are adjacent?

18. Find the number of quintuples (x, y, z, u, v) of positive integers satisfying both equations

$$x + y + z + u = 30 \quad \text{and} \quad x + y + z + v = 27.$$

19. Of the solutions in positive integers of $x + y + z + w = 26$, how many have $x > y$?

20. Evaluate $[n, n, n, n, \| 2n, 2n]$, that is, the number of ways of dividing $4n$ objects, which are alike in batches of n, equally between two persons.

21. In how many ways is it possible to separate nj different objects into n batches with j objects in each batch?

22. Which would you expect to be larger, the number of partitions of 1000 into 3 positive even integers, or the number of partitions of 1000 into 3 positive odd integers? Give a proof of your conjecture.

23. Which would you expect to be larger, the number of partitions of 1000 into 4 positive even integers, or the number of partitions of 1000 into four positive odd integers? Give a proof of your conjecture.

24. Which would you expect to be larger, the number of partitions of 1000 into positive even integers, or the number of partitions of 1000 into positive odd integers? Give a proof of your conjecture. (This question differs from the preceding two questions in that the number of summands is now unrestricted.)

25. How many integers between 1 and 1,000,000 inclusive have the property that at least two consecutive digits are equal? (For example, 1007 has the property but 1017 does not.)

26. Find the number of permutations of the letters of the alphabet, taken all at a time, such that (i) no letter is in its natural place, and (ii) the letters A and B are adjacent.

27. Find the number of permutations of the 6 letters a, b, c, d, e, f, taken all at a time, subject to the condition that letters which are consecutive in the alphabet are not adjacent. (For example, a and b are not adjacent, b and c are not adjacent, etc.)

28. Prove that the number of people through all of history who have shaken hands (with other people) an odd number of times is even.

29. In any group of people, prove that there are two persons having the same number of acquaintances within the group. (Presume, of course, that if A is acquainted with B, then B is acquainted with A.)

30. Given n points in the plane, no 3 collinear, let each of the line segments joining pairs of points be colored one of two colors, say red and white. Then from each point there emanate $n - 1$ line segments, some red and some white. Prove that no matter what configuration of colors is used, there are two points out of which there emanate the same number of red segments, and hence also the same number of white segments.

31. Let there be $m + 1$ equally spaced parallel lines, intersected at right angles by $k + 1$ equally spaced parallel lines. Presuming $m \leqslant k$, what is the total number of squares in the network?

32. *The Tower of Hanoi Puzzle.* There are 8 circular discs placed over one of three vertical pegs. The discs are of 8 unequal radii, with the largest disc at the bottom of the pile on one peg, covered by successively smaller discs so that the smallest one is on top. The problem is to transfer the tower of discs from the peg on which they rest initially to one of the other two pegs. The rules are that the discs may be moved freely, one at a time, from peg to peg, except that no disc can ever be placed on top of a smaller disc. The question is whether it is possible under these rules to move the tower of discs from one peg to another, and if so, how many moves are needed to effect the transfer.

33. Given 6 points in the plane, no 3 collinear, let each of the line segments joining the points be colored one of two colors, say red or white. Prove that no matter what configuration of colors is used,

there are always at least *two* chromatic triangles present, that is, two triangles whose three sides have the same color. (The two triangles need not be of the same color; one may be a red chromatic triangle, and the other white.)

34. In the preceding problem prove that there need not be 3 chromatic triangles. That is, exhibit a configuration of colored segments with only 2 chromatic triangles.

35. Given 7 points in the plane, no 3 collinear, let each of the line segments joining the points be colored one of two colors, say red or white. Prove that no matter what configuration of colors is used, there are always at least three chromatic triangles present.

36. Consider 66 points in the plane, no 3 collinear, with each of the line segments joining these points colored one of 4 colors. Prove that for any arrangement of colors whatsoever there is always a chromatic triangle present, i.e. a triangle whose 3 sides have the same color. (The information that with 17 points and 3 colors there is a chromatic triangle might be useful.)

37. Consider 17 points in a plane, no three collinear, with each of the segments joining two points colored red, white or blue. Prove that there are at least two chromatic triangles in the configuration.

38. Consider 24 points in the plane, no 3 collinear, with each of the segments joining the points colored one of two colors, say red or white. Prove that no matter what distribution of colors is made, it is always possible to find 4 points such that the 6 line segments joining them are of the same color. (The reader might wish to solve this problem with a larger integer substituted for 24. The smallest number that can be used to replace 24 is 18, in the sense that the proposition is not true for 17 or fewer points. However the proof that 18 is the smallest, given by R. E. Greenwood and A. M. Gleason in 1955, is beyond the scope of this book.)

39. Given n points on the circumference of a circle, there are $C(n, 2)$ or $n(n-1)/2$ line segments joining pairs of points. Suppose the n points are spaced so that no 3 line segments have a common intersection point inside the circle. What is the total number $I(n)$ of intersection points inside (not on the circumference of) the circle? For example, $I(4) = 1$, $I(5) = 5$, $I(6) = 15$.

40. In the preceding problem, into how many regions is the interior of the circle divided by the line segments joining the n points? Let $R(n)$ be the number of regions; for example, $R(2) = 2$, $R(3) = 4$, $R(4) = 8$, $R(5) = 16$.

41. Given n equally spaced points on the circumference of a circle (the vertices of a regular n-gon); consider the

$$C(n, 3) = \tfrac{1}{6}n(n-1)(n-2)$$

triangles that can be formed by straight line segments linking the points. How many of these triangles are isosceles?

42. If n identical dice are thrown, how many possible outcomes are there? (Say that two outcomes are the same if they contain the same number of ones, the same number of twos, $\cdots$, and the same number of sixes.)

43. Consider n planes in 3-dimensional space satisfying the following conditions: no two are parallel; no two lines of intersection are parallel; no four intersect in a point. Into how many regions is space separated by the planes?

44. What is the probability that a randomly selected permutation of $1, 2, 3, \cdots, n$ has the "2" somewhere between the "1" and the "3"?

45. Find the number of permutations of $2n$ things which are alike in pairs (for example $AABBCCDDEE\cdots$) taken all at a time, such that no two adjacent things are alike.

46. Find the probability that a randomly selected permutation of $1, 2, 3, \cdots, n$ taken all at a time has exactly j of the numbers out of their natural positions.

47. How many permutations are there of the $n + k$ letters

$$AAA\cdots ABBB\cdots B,$$

of which n are A's and k are B's, subject to the condition that no three A's are adjacent?

48. Find the number of permutations of $1, 2, 3, \cdots, 2n$ taken all at a time, such that no odd number is in its natural position.

49. Suppose that the prime factorization of an integer n has exactly m factors, all distinct. How many factorings are there of n into k factors, where k is some integer $\leqslant m$, (i) if each factor must be greater than 1, (ii) if 1 is allowed as a factor? (Factorings that differ only in the order of the factors are not counted separately.)

50. Consider the integers $1, 2, 3, \cdots, n$. Let $K(n, j)$ denote the number of subsets of these n integers satisfying the conditions (i) each subset contains j integers, (ii) no subset contains a consecutive pair of integers. For example, $K(5, 3) = 1$ because the only subset of 1, 2, 3, 4, 5, satisfying the conditions is 1, 3, 5. By separating the subsets counted by $K(n, j)$ into two types, those that contain n and those that do not, obtain a recursion relation for $K(n, j)$. Then use this relation to construct a short table of values of $K(n, j)$, say up to $n = 100$ and $j = 10$. This table, when contrasted with Pascal's triangle, should suggest a conjecture about the value of $K(n, j)$. Find the proper conjecture and then prove it by mathematical induction.

51. Three persons, strangers to one another, enter a room in which there are 3 mutual acquaintances. Prove that among the 6 people there are at least 3 other triples each of which consists of 3 strangers or 3 mutual acquaintances. A more definite statement of the problem follows: Say that a set of 3 persons has property α in case they are pairwise strangers, property β in case they are pairwise acquainted. Consider six persons A, B, C, D, E, F, such that A, B, C have property α, and D, E, F have property β. Prove that the sum of the number of triples with property α and the number of triples with property β is at least 5.

52. A flight of stairs has 14 steps. A boy can go up the stairs one at a time, two at a time, or any combination of ones and twos. In how many ways can the boy go up the stairs?

53. How many integers between 1 and 1,000,000 inclusive have the property that no digit is smaller than a digit to its left? (For example 1468 has the property stated, but 1648 does not.)

Answers and Solutions

Answers are given for almost all problems. Solutions are given for many, although for the most part the "solution" is a mere sketch, in some instances nothing more than a suggestion or two. If the reader's answer to a problem is not the same as the one given here, he should allow for the possibility that the difference is merely one of form. For it should be kept in mind that most problems admit more than one method of solution and that two answers may be equal without looking the same.

Problem 1.1, page 1. 1, 2 or 3

We give here an analysis of years having 365 days; the analysis for 366 day years is similar. First, let us call Sunday a type 0 day, Monday a type 1 day, Tuesday type 2, $\cdots$, Saturday type 6. If January 13 is a type 0 day, then February 13 is a type 3 day since it is 31 or $3 + 28$ days later, March 13 is type 3, April 13 is type 6, May 13 is type 1, $\cdots$, December 13 is type 5. The entire list of types from January 13 to December 13 is

$$0, 3, 3, 6, 1, 4, 6, 2, 5, 0, 3, 5.$$

There are two Friday the thirteenths, since Friday is of type 5. The analysis thus far has been on the assumption that January 13 is a Sunday. The easiest way to proceed is to vary what is meant by type 0. For example, if we redefine type 0 to be Monday, then Friday is of type 4, and the list above shows that in such a year there is only one Friday the thirteenth. Thus the list reveals the answer to the problem by consideration of all seven interpretations of what is meant by type 0. For a 366 day year, the corresponding list is 0, 3, 4, 0, 2, 5, 0, 3, 6, 1, 4, 6.

Problem 1.2, page 2. 10

There is 1 kind of block with six blue faces; 1 kind with five blue faces; 2 kinds with four blue faces, because the two red faces may be opposite or adjacent to each other; 2 kinds with three blue faces, because there may be, or may not be, two blue faces opposite one another. The number of different kinds of blocks with two blue faces is the same as the number with four blue faces; with one blue face, the same as with five blue faces; with no blue face, the same as six blue faces.

Problem 1.3, page 3, is solved on page 27.

Problem 1.4, page 4, is solved on page 59.

Problem 1.5, page 4, is solved on page 106.

Problem Set 1, page 5

1. 55 **2.** 138 **3.** 185 **4.** $n + r - 1$ **5.** $k - r + 1$ **6.** $h + 1$

7. 80 integers from $x = 145$ to $x = 224$.

8. (a) 49 (b) 44 (c) 35

Argument for part (c). Subtract 11 from each integer: 6, 12, 18, 24, $\cdots$, 210. Divide each by 6: 1, 2, 3, 4, $\cdots$, 35. These operations have not changed the number of elements.

9. (a) 181: the integers 11, 22, 33, $\cdots$, 1991;
(b) 121: delete from the integers in (a) the following: 33, 66, 99, $\cdots$, 1980; these are 60 in number; hence $181 - 60$;
(c) 167: delete from 6, 12, 18, 24, $\cdots$, 1998 the integers 12, 24, 36, $\cdots$, 1992; thus $333 - 166$.

10. 9: 4 cents, 2 nickels, 1 dime, 1 quarter, 1 fifty-cent piece (alternatively, replace 2 nickels, 1 dime by 1 nickel, 2 dimes).

11. 39

Let a, b, c denote a cents, b nickels, c dimes. Then with no 25 cent piece the solutions in triples a, b, c are

47, 0, 0	42, 1, 0	37, 2, 0	37, 0, 1	32, 3, 0
32, 1, 1	27, 4, 0	27, 2, 1	27, 0, 2	22, 5, 0
22, 3, 1	22, 1, 2	17, 6, 0	17, 4, 1	17, 2, 2
17, 0, 3	12, 7, 0	12, 5, 1	12, 3, 2	12, 1, 3
7, 8, 0	7, 6, 1	7, 4, 2	7, 2, 3	7, 0, 4
2, 9, 0	2, 7, 1	2, 5, 2	2, 3, 3	2, 1, 4

With one 25 cent piece the solutions are

22, 0, 0	17, 1, 0	12, 2, 0	12, 0, 1	7, 3, 0
7, 1, 1	2, 4, 0	2, 2, 1	2, 0, 2	

12. 7

13. 3

14. 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360

A regular polygon with n sides has exterior angle $360/n$ degrees, and so an interior angle has size $360 - (360/n)$ degrees. Hence we choose all positive integers n such that $360/n$ is an integer, except $n = 1$ and $n = 2$.

15. 36

Let the colors be denoted by R, G, B and W, say for red, green, blue and white. There are 4 cases of solids painted one color: all R, all G, all B, all W. Solids painted two colors yield 18 cases: if the colors are R and G there are 3 cases because the number of R-faces may be 1, 2 or 3; similarly there are 3 cases for each of the other color combinations RB, RW, GB, GW, BW. There are 12 different kinds of solids painted three colors: if the colors are R, G, B, there are 3 cases—for example, one case with two R-faces, one G-face and one B-face. Solids painted four colors can be of 2 kinds: orient the tetrahedron so that the bottom is R and there is a G-face towards you; then the other two faces can be BW or WB.

16. 6

17. 36

Problem Set 2, page 10

1. 676 (or $26 \cdot 26$)
2. 600 (or $25 \cdot 24$)
3. 3380 (or $26 \cdot 26 \cdot 5$)
4. 3000 (or $5 \cdot 25 \cdot 24$)
5. 30 (or $6 \cdot 5$)
6. 64 (or $8 \cdot 2 \cdot 2 \cdot 2$)
7. 4968 (or $23 \cdot 12 \cdot 3 \cdot 6$)
8. 243 (or $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$); 768 (or $3 \cdot 4 \cdot 4 \cdot 4 \cdot 4$)

Problem Set 3, page 11

1. 6; 120; 40320
2. 132; 2; 30; 5040
3. 120
4. 25
5. 6
6. 2
7. 720
8. 11880; 151200
9. 210
10. (b) and (c) are false.

Problem Set 4, page 17

1. 210, 1680 and 380

3. $P(n, 1) = n$; $P(m, 1) = m$; $P(n + m, 1) = n + m$

4. $P(n, n) = n!$ and $P(n, n - 1) = n(n - 1)(n - 2) \cdots 2 = n!$

5. 12144 (or $24 \cdot 23 \cdot 22$)

6. 13824 (or $24 \cdot 24 \cdot 24$); 14400 (by adding $24 \cdot 24$ to 13824)

7. 4536 (or $9 \cdot 9 \cdot 8 \cdot 7$); 2240, since there are 5 choices for the units' digit (digit on the right end), 8 choices for the thousands' digit, 8 choices for the hundreds' digit, and 7 choices for the tens' digit.

8. 120 (or $5 \cdot 4 \cdot 3 \cdot 2$); 72 (or $4 \cdot 3 \cdot 2 \cdot 3$)

9. 720 (or $6\cdot 6\cdot 5\cdot 4$); 420

10. 103920

There are $P(8, 8) = 40320$ integers with 8 digits; $P(8, 7) = 40320$ integers with 7 digits; $P(8, 6) = 20160$ integers with 6 digits. Integers with 5 digits are separated into two types depending on the digit on the left end; if the left end digit is 5, there are $1\cdot 5\cdot 6\cdot 5\cdot 4 = 600$ possibilities because, taking the digits from left to right, there is one possibility for the first digit, 5 possibilities for the second digit (namely the digits 3, 4, 6, 7, 8 since integers with 0, 1 or 2 in this place would be less than 53000); if the first digit is 6, 7, or 8, there are $3\cdot 7\cdot 6\cdot 5\cdot 4 = 2520$ possibilities. The answer is obtained by adding these various results.

11. 90360

The solution of the previous problem can be used as a model, but must be modified because of the presence now of the digit 0. This digit cannot be used as the first or left end digit in an integer. The number of possibilities can be obtained by thinking about the number of possibilities for each digit position, from left to right. Thus,

8 digit numbers: $7\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 35280$;
7 digit numbers: $7\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2 = 35280$;
6 digit numbers: $7\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3 = 17640$;
5 digit numbers beginning with the digit 5: $1\cdot 4\cdot 6\cdot 5\cdot 4 = 480$;
5 digit numbers beginning with the digit 6 or 7: $2\cdot 7\cdot 6\cdot 5\cdot 4 = 1680$.

Problem Set 5, page 21

1. 15, 35 and 84

3. (a) $C(10, 2) = C(10, 8) = 45$; (b) 45

4. $C(720, 10)$ or $\dfrac{720!}{10!710!}$

6. $C(20, 2)$ or 190; $C(20, 3)$ or 1140

7. $10! - 2\cdot 9! = 8\cdot 9!$

The number of unrestricted arrangements is $10!$. The number of arrangements with two specific persons together is $2\cdot 9!$ because the two persons can be regarded as a unit, but in two ways.

8. Let n be the largest of the five integers, so that we must prove that $n(n-1)(n-2)(n-3)(n-4)$ is divisible by $5!$. Now we solve the problem by observing that $C(n, 5)$ is an integer given by the formula

$$C(n, 5) = \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}.$$

More generally, the formula for the integer $C(n, r)$ shows that the product of r consecutive integers is divisible by $r!$.

9. (a) 362880 or $9!$; (b) 5760 or $2(5!4!)$; (c) 17280 or $6!4!$; (d) 2880 or $5!4!$

In part (c) the red books may be treated as a unit, so there are 6 items to be permuted; this gives $6!$. But in any one of these arrangements the red books can be permuted in 4! ways. In part (d), the green books can be permuted in their allocated positions in 5! ways, the red books in 4! ways.

10. (a) $2C(30, 3)C(30, 5) + C(30, 4)C(30, 4)$

Add the results of three cases, namely 3 professors, 4 professors, or 5 professors. For example, the 3 professors case implies 5 business men, and so there are $C(30, 3)C(30, 5)$ possibilities.

(b) $C(60, 8) - C(30, 8)$

If none of the eight were a business man, the number of possibilities would be $C(30, 8)$. This is then subtracted from the total number of unrestricted possibilities.

11. 79 (or $4 \cdot 5 \cdot 4 - 1$)

12. 4

There are as many zeros as the number of occurrences of 10 as a factor. Now 5 occurs as a factor four times, namely in 5, 10, 15, 20; and 2 occurs as a factor many more times.

13. 12

The argument is similar to that in the preceding problem, with this difference: the terms 25 and 50 in the product have 5 as a factor twice.

14. 7^5, because there are 7 choices for each flag.

15. (a) $7 \cdot 6^4$; (b) $P(7, 5) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$

16. 2024

The total number of unrestricted subsets is $C(26, 3) = 2600$. From this number we subtract the number of cases with three consecutive letters, such as J, K, L. There are 24 of these. Then we subtract the number of cases with two, but not three consecutive letters; if the letters are A, B there are 23 cases; $B, C,$ 22 cases; $\cdots$; $X, Y,$ 22 cases; $Y, Z,$ 23 cases; so 552 in all. The answer is thus $2600 - 24 - 552$.

17. $k!/(k-n)!$

First choose n of the k boxes to receive one object each; this can be done in $C(k, n)$ ways. For each such choice, the things can be put in the boxes in $n!$ ways. Thus the answer is $C(k, n) \cdot n!$.

Problem Set 6, page 24

1. 5040 or $7!$

2. 3600

Start with the solution of the preceding problem and subtract the number of cases where the two persons, say A and B, are in adjacent seats. Taking A and B as a unit we see that there are 6! cases with A to the left of B, and 6! cases the other way about. Thus the answer is $7! - 6! - 6!$.

3. 144

The ladies can be seated in alternate seats in $3!$ ways. The answer is obtained by multiplying this by $4!$, the number of ways the men can be seated for any fixed arrangement of the ladies.

4. 12

Given any of the $3!$ seating arrangements of the ladies, the men can be seated in exactly two ways.

5. 120

The number of firing orders is simply the number of ways of arranging $1, 2, 3, 4, 5, 6$ in a circle.

6. 30

Let one of the colors be white. Then since one face must be white, let it be the bottom of the block. The top can be colored in any of the 5 remaining colors. That done, the vertical faces are to be colored with the 4 remaining colors. This now amounts to a problem in circular permutations, because the cubical block can now be rotated about a vertical axis through the center of the block without altering the colors of the top and bottom. Hence there are $3!$ ways of coloring the vertical faces, and this is multiplied by 5 to get the answer.

7. 2

Starting with a blank block, number two opposite faces 1 and 6, and place the block with the 6 on top. The four vertical faces are to be numbered 2, 3, 4, 5. Number the front face 2, and the back face 5. There remain two choices for the numbers 3 and 4.

Problem Set 7, page 30

1. (a) 168 or $8!/(5!2!)$ (b) $21!/(2!2!2!2!3!9!)$

2. Choose a out of n, then b out of $n - a$, then c out of $n - a - b$. This gives $C(n, a)C(n - a, b)C(n - a - b, c)$, which can be evaluated further to give the factorial answer.

3. 5035

We subtract from the total number 6435 of unrestricted paths the number that includes the block from 5th to 6th on E. From 1st and A to 5th and E there are $C(8, 4)$ paths. From 6th and E to 9th and H there are $C(6, 3)$ paths. Hence from 6435 we subtract

$$C(8, 4) \cdot C(6, 3).$$

4. $15!/(4!5!6!)$

Denoting by R, B, U the motions of distance one unit to the right, back, and up, we see that the problem is the same as finding the number of permutations, all at a time, of the fifteen letters

$$R\,R\,R\,R\,B\,B\,B\,B\,B\,U\,U\,U\,U\,U\,U$$

5. $17!/(4!5!6!2!)$

Problem Set 8, page 33

1. 15, 10, 5 **2.** $C(10, 4)$ **3.** $C(49, 10)$

4. (a) $C(n-1, r) = C(n-2, r) + C(n-2, r-1)$;
(b) $C(n-1, r-1) = C(n-2, r-1) + C(n-2, r-2)$

5. (a) $C(n-1, r) = \dfrac{(n-1)!}{r!(n-r-1)!}$

(b) $C(n-1, r-1) = \dfrac{(n-1)!}{(r-1)!(n-r)!}$

6. Adding the results of the preceding problem, we get

$$\begin{aligned} C(n-1, r) + C(n-1, r-1) &= \frac{(n-1)!}{r!(n-r-1)!} + \frac{(n-1)!}{(r-1)!(n-r)!} \\ &= \frac{(n-1)!(n-r) + (n-1)!r}{r!(n-r)!} = \frac{(n-1)!n}{r!(n-r)!} \\ &= \frac{n!}{r!(n-r)!} = C(n, r). \end{aligned}$$

7. $C(n, r) = C(n-2, r-2) + 2C(n-2, r-1) + C(n-2, r)$

8. $n = r = 0$

Problem Set 9, page 37

1. 6; $n + 1$

2. $(x+y)^6 = C(6, 0)x^6 + C(6, 1)x^5y + C(6, 2)x^4y^2 + C(6, 3)x^3y^3 + C(6, 4)x^2y^4 + C(6, 5)xy^5 + C(6, 6)y^6$;

with $x = y = 1$, this sum equals 2^6 or 64.

3. $(1-1)^6 = 0$

4. $C(10, 7) = 120$

5. $u^7 + 7u^6v + 21u^5v^2 + 35u^4v^3 + 35u^3v^4 + 21u^2v^5 + 7uv^6 + v^7$

8. 180 terms; $bdsw$ and $bfpu$ are actual terms.

Problem Set 10, page 40

1. $x^4 + y^4 + z^4 + 4x^3y + 4xy^3 + 4x^3z + 4xz^3 + 4y^3z + 4yz^3 + 6x^2y^2 + 6x^2z^2 + 6y^2z^2 + 12x^2yz + 12xy^2z + 12xyz^2$

2. $10!/(2!2!2!2!2!)$ 3. $6!$ 4. 3^8; 4^{17}

5. 3^{12} because the numbers are precisely the coefficients in the expansion of $(x + y + z)^{12}$.

Problem Set 11, page 42

1. 1, 9, 36, 84, 126, 126, 84, 36, 9, 1
   1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1
   1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, 1
   1, 12, 66, 220, 495, 792, 924, 792, 495, 220, 66, 12, 1
   1, 13, 78, 286, 715, 1287, 1716, 1716, 1287, 715, 286, 78, 13, 1

2. By formula (3.6) the sum of the elements of the ninth row is 2^8. Similarly the sums of the elements of preceding rows are 2^7, 2^6, etc. So it must be verified that

$$2^8 = 2^7 + 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 + 1.$$

By virtue of the identity

$$a^n - 1 = (a - 1)(a^{n-1} + a^{n-2} + \cdots + a + a^0),$$

we have $2^8 - 1 = (2 - 1)(2^7 + 2^6 + \cdots + 2^0)$ or

$$2^8 - 1 = 2^7 + 2^6 + \cdots + 2^0$$

which is equivalent to the equation we wanted to verify.

3. This can be deduced from equation (3.7) by transposing terms with minus signs.

Problem Set 12, page 43

1. 63 or $2^6 - 1$

2. 6560 or $3^8 - 1$, because on each issue a member has three choices: yes, no, or abstention.

3. 1023 or $2^{10} - 1$

4. 254, because there are 2^7 types for a family of 7 children, 2^6 for 6 children, and so on.

Problem Set 13, page 47

1. 5050 or $\frac{1}{2}(100)(101)$

2. Denoting the sum by s we have $2s = 101 + 101 + \cdots + 101$ with 100 summands. Hence $2s = 10100$ and $s = 5050$.

3. Denoting the sum by S we have

$$\begin{aligned} S &= 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\ S &= n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \\ 2S &= (n+1) + (n+1) + \cdots + (n+1) = n(n+1), \\ S &= \tfrac{1}{2}n(n+1) \end{aligned}$$

4. 338350 or $\frac{1}{6}(100)(101)(201)$

5. (a) 99 (b) 101 6. (a) $n - 1$ (b) $n + 1$.

7. 4851; 5151
With $x = 1$ the equation becomes $y + z = 99$ with 98 solutions in positive integers; with $x = 2$ we have $y + z = 98$ with 97 solutions in positive integers; $\cdots$; with $x = 98$ we have $y + z = 2$ with 1 solution. The total number of solutions in positive integers is

$$98 + 97 + \cdots + 2 + 1 = \tfrac{1}{2}(98)(99).$$

8. $\frac{1}{2}(n-2)(n-1)$; $\frac{1}{2}(n+1)(n+2)$

9. 10; 15; $\frac{1}{2}(n+1)(n+2)$
The number of terms in the expansion of $(x + y + z)^4$ is the number of solutions of $a + b + c = 4$ in non-negative integers.

10. Write equation (3.8) with $r = 4$ and n replaced by $m + 3$, $m + 2$, $m + 1$, $\cdots$, 6, 5, to get

$$\begin{aligned}
C(m+2, 3) &= C(m+3, 4) - C(m+2, 4)\\
C(m+1, 3) &= C(m+2, 4) - C(m+1, 4)\\
C(m, 3) &= C(m+1, 4) - C(m, 4)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
C(5, 3) &= C(6, 4) - C(5, 4)\\
C(4, 3) &= C(5, 4) - C(4, 4)
\end{aligned}$$

Adding these we get

$$C(4,3) + C(5,3) + \cdots + C(m,3) + C(m+1,3) + C(m+2,3)$$
$$= C(m+3,4) - C(4,4).$$

$$\tfrac{1}{6}(4)(3)(2) + \tfrac{1}{6}(5)(4)(3) + \cdots + \tfrac{1}{6}(m+2)(m+1)(m)$$
$$= \tfrac{1}{24}(m+3)(m+2)(m+1)(m) - 1.$$

Transpose the term -1 and multiply by 6 to get

$$(3)(2)(1) + (4)(3)(2) + (5)(4)(3) + \cdots + (m+2)(m+1)(m)$$
$$= \tfrac{1}{4}(m+3)(m+2)(m+1)(m).$$

The term $(m + 2)(m + 1)m$, for example, can be written as $m^3 + 3m^2 + 2m$, so the whole left side can be separated into three sums

$$(1^3 + 2^3 + 3^3 + \cdots + m^3) + 3(1^2 + 2^2 + 3^2 + \cdots + m^2)$$
$$+ 2(1 + 2 + 3 + \cdots + m).$$

Writing S for the sum of the cubes from 1^3 to m^3, and substituting the known formulas for the other sums, we get

$$S + \tfrac{1}{2}m(m+1)(2m+1) + m(m+1)$$
$$= \tfrac{1}{4}(m+3)(m+2)(m+1)m.$$

This reduces to $S = \frac{1}{4}m^2(m + 1)^2$ and so the answer is

$$1^3 + 2^3 + 3^3 + \cdots + m^3 = \tfrac{1}{4}m^2(m+1)^2.$$

Problem Set 14, page 54

1. $F(11) = 233$.

2. $F(n)$ is even if $n = 1, 4, 7, 10, 13, 16, \cdots$. In general $F(n)$ is even if n is of the form $3k + 1$, and $F(n)$ is odd in all other cases. This is an immediate consequence of Formula (4.4) and the fact that the sum of two integers is odd only if one of them is odd and the other even.

3. $F(n + 1) = F(n) + F(n - 1)$

4. Add the result of the preceding question to formula (4.4).

5. $C(11, 6)$

6. $C(15, 5)C(16, 6)$
Ignoring the B's momentarily we observe that the A's and C's can be arranged in order in any one of $C(15, 5)$ ways. Then there are 16 places between the A's and C's and at the ends where the B's may be inserted. Thus for each arrangement of A's and C's, there are $C(16, 6)$ ways of inserting the B's.

7. 7350
Ignoring the i's momentarily we note that the other letters can be arranged in order in $7!/(4!2!) = 105$ ways. Then there are 8 places between and at the ends of these letters where the i's may be inserted. Hence there are $C(8, 4)$ or 70 ways of inserting the i's. The answer is $70 \cdot 105$.

Problem Set 15, page 58

1. $C(49, 3)$; $C(53, 3)$

2. We note that $C(8, 3) = C(8, 5)$

4. It suffices to establish that $C(m - 1, k - 1) = C(m - 1, m - k)$, and this follows from formula (2.4).

5. Use formula (4.10).

6. (a) $C(11, 6)$

Ignoring the integer 1,000,000 the sum of whose digits is not 6, we interpret the integers from 1 to 999,999 as having six digits by allowing zero as a digit. For example the integer 8365 can be written 008365. If we write $x_1, \cdots, x_6$ for the six digits, we can interpret the problem as the number of solutions of $x_1 + x_2 + \cdots + x_6 = 6$ in non-negative integers.

(b) $C(10, 5) + C(9, 4) + C(8, 3) + C(7, 2) + C(6, 1) + 1$

7. (a) $C(21, 17)$

Each term of the expansion is of the form $\alpha_1^{x_1}\alpha_2^{x_2}\alpha_3^{x_3}\alpha_4^{x_4}\alpha_5^{x_5}$ (with an appropriate coefficient), where the sum of the exponents is 17; thus $x_1 + x_2 + x_3 + x_4 + x_5 = 17$. Hence the answer is the number of solutions of this equation in non-negative integers.

(b) $C(t + k - 1, t)$

Problem Set 16, page 60

1. r 2. $C(11, 6)$ 3. $C(12, 10)$ 4. $C(16, 12)$.

5. $C(16, 7) - 1$

The question amounts to asking for the number of combinations, seven at a time, of the ten digits 0, 1, 2, $\cdots$, 9, each of which may be repeated in the combination. The "-1" in the answer accounts for the case of seven zeros, to which there corresponds no integer.

Problem Set 17, page 65

1. One of the twenty parts of the answer is: 6, 8, 7, 6 corresponds to 1, 3, 2, 1.

2. $C(71, 3)$

3. (a) $C(37, 4)$ (b) $C(30, 4)$

4. $C(16, 3)$

5. (a) $C(13, 3)$ (b) $C(7, 3)$ (c) None

6. (a) $C(17, 3)$ (b) $C(11, 3)$

7. (a) $C(m - c_1 + 3, 3)$ (b) $C(m - c_1 - c_2 + 3, 3)$

8. $C(18, 5) - 6C(8, 5)$

Set aside the integer 1,000,000 the sum of whose digits is not 13. We interpret the integers from 1 to 999,999 as having six digits by allowing zero as a digit. If we write $x_1, \cdots, x_6$ for the six digits we can interpret the problem as the number of solutions of $x_1 + x_2 + \cdots + x_6 = 13$ in non-negative integers not exceeding nine. Ignoring the "not exceeding nine" limitation momentarily, we note that the equation has $C(18, 5)$ solutions in non-negative integers. Next, the number of solutions in non-negative integers with $x_1 > 9$ is seen to be $C(8, 5)$. This is subtracted from $C(18, 5)$, and analogous subtractions are made for the cases $x_2 > 9$, $x_3 > 9$, etc.

Problem Set 18, page 71

1. $$N - N(\alpha) - N(\beta) - N(\gamma) - N(\delta) + N(\alpha, \beta) + N(\alpha, \gamma) + N(\alpha, \delta) + N(\beta, \gamma) + N(\beta, \delta) + N(\gamma, \delta) - N(\alpha, \beta, \gamma) - N(\alpha, \beta, \delta) - N(\alpha, \gamma, \delta) - N(\beta, \gamma, \delta) + N(\alpha, \beta, \gamma, \delta)$$

2. 2^r, the total number of subsets of a set of r objects.

3. 16000

Let divisibility by 3, 5, 11 be denoted by α, β, γ, respectively. Then formula (5.3) gives

$$33000 - 11000 - 6600 - 3000 + 2200 + 1000 + 600 - 200.$$

4. 998910

The fourth powers are included among the squares, so they can be left out of the consideration. Say that an integer has property α if it is a perfect square, property β if a perfect cube. An integer has both properties α and β if it is a perfect sixth power. Thus we make the computation

$$N - N(\alpha) - N(\beta) + N(\alpha, \beta) = 1{,}000{,}000 - 1000 - 100 + 10.$$

5. $N(\alpha, \beta, \gamma) - N(\alpha, \beta, \gamma, \delta) - N(\alpha, \beta, \gamma, \epsilon) + N(\alpha, \beta, \gamma, \delta, \epsilon)$

6. $$N(\beta) - N(\beta, \alpha) - N(\beta, \gamma) - N(\beta, \delta) + N(\beta, \alpha, \gamma) + N(\beta, \alpha, \delta) + N(\beta, \gamma, \delta) - N(\beta, \alpha, \gamma, \delta)$$

Problem Set 19, page 77

1. $C(13, 3) - 4C(7, 3)$

2. $C(16, 5) - 6C(11, 5) + 15C(6, 5)$
The question amounts to asking for the number of solutions of $y_1 + y_2 + \cdots + y_6 = 17$ in positive integers not exceeding 5, because any even integer x_1 can be written as $2y_1$, where y_1 is again an integer.

3. $C(14, 3) - C(8, 3) - 2C(7, 3) - C(6, 3)$
Let N denote the number of solutions of the equation which satisfy the conditions $x_1 > 0$, $x_2 > 0$, $x_3 > 2$, $x_4 > 3$. Thus $N = C(14, 3)$ by the formula (4.22). If one of these solutions has $x_1 > 6$, say it has property α. Likewise let $x_2 > 7$, $x_3 > 9$ and $x_4 > 11$ correspond to properties β, γ and δ. Thus we want to find how many of the $C(14, 3)$ solutions have none of the properties $\alpha, \beta, \gamma, \delta$. By use of formula (4.22) we compute

$$N(\alpha) = C(8, 3), \quad N(\beta) = C(7, 3), \quad N(\gamma) = C(7, 3), \quad N(\delta) = C(6, 3).$$

All further terms in formula (5.3) are zero.

4. $C(16, 3) - 4C(9, 3)$

5. $C(13, 3) - C(4, 3) - C(5, 3) - C(8, 3) - C(9, 3) + C(4, 3)$
This amounts to finding the number of solutions of

$$x_1 + x_2 + x_3 + x_4 = 10$$

in non-negative integers subject to the restrictions $x_1 \leqslant 8$, $x_2 \leqslant 7$, $x_3 \leqslant 4$, $x_4 \leqslant 3$.

6. $C(12, 2) - C(3, 2) - C(4, 2) - C(7, 2)$
This amounts to finding the number of solutions of $x_1 + x_2 + x_3 = 10$ in integers satisfying the inequalities $0 \leqslant x_1 \leqslant 8$, $0 \leqslant x_2 \leqslant 7$, $0 \leqslant x_3 \leqslant 4$.

7. $C(11, 3) - 4C(8, 3) + 6C(5, 3) = 1$

8. $C(13, 3) - 4C(4, 3)$

10. $c = 23$; $C(11, 4) - 5C(5, 4) = C(22, 4) - 5C(16, 4) + 10C(10, 4) - 10C(4, 4)$
The substitution or transformation $x_j = 7 - y_j$ for $j = 1, 2, 3, 4, 5$ will solve the problem.

11. $C(m-1, k-1) - C(m-1-c_1, k-1) - C(m-1-c_2, k-1)$
$- C(m-1-c_3, k-1) + C(m-1-c_1-c_2, k-1)$
$+ C(m-1-c_1-c_3, k-1) + C(m-1-c_2-c_3, k-1)$
$- C(m-1-c_1-c_2-c_3, k-1)$

12. $C(24, 6) - C(15, 6) - 6C(14, 6)$

Denote the digits from left to right by $x_1, \cdots, x_7$. Then the answer is the number of solutions of $x_1 + \cdots + x_7 = 19$ in non-negative integers not exceeding 9, but with the additional restriction that x_1 is positive.

Problem Set 20, page 81

1. $D(5) = 44$; $D(6) = 265$

2. 1234 2134 3142 1243 2143 3214 1432 2413 3412

3. (a) 1936

The integers 1, 2, 3, 4, 5 can be put into the first five places in $D(5)$ ways, because there are $D(5)$ derangements of five things; the remaining integers from 6 to 10 can be put into the last five places in $D(5)$ ways, so the answer is $D(5) \cdot D(5)$.

(b) $(5!)^2 = 14400$

Any arrangement of 6, 7, 8, 9, 10 in the first five places is a derangement, so there are $5!$ possibilities; the same is true for the integers 1, 2, 3, 4, 5 in the last five places.

4. 3216

We use the inclusion-exclusion principle with three properties of the permutations: 1 in the first place; 4 in the fourth place; 7 in the seventh place. Thus the answer is $7! - 6! - 6! - 6! + 5! + 5! + 5! - 4!$.

5. 22260

There are $C(9, 3) = 84$ ways of choosing the three numbers which are to be in their natural positions; for each such choice, there are $D(6) = 265$ derangements of the other six numbers. The product of 84 and 265 is the answer.

6. $D(26)$ or $26!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{26!}\right]$

7. $$D(n) - nD(n-1) = n!\left[1 - \frac{1}{1!} + \cdots + \frac{(-1)^n}{n!}\right] - n!\left[1 - \frac{1}{1!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!}\right]$$

After the subtraction is performed, the only remaining term is

$$n!\left[\frac{(-1)^n}{n!}\right].$$

Problem Set 21, page 86

2. $\frac{1}{4}$ 3. $\frac{1}{6}$.

4. 5/12
There are 36 equally likely cases. Of these, 15 have a larger number on the white die.

5. 5/18
There are 6^4 equally likely cases. For convenience we suppose that the dice are of different colors, say white, red, blue and green. When the dice are thrown we can argue that any outcome on the white die will be satisfactory, so 6 possibilities; but whatever the outcome on the white die, we want a different outcome on the red die, so 5 possibilities; similarly there are 4 possibilities on the blue die, and 3 on the green die. So the number of favorable cases is $6 \cdot 5 \cdot 4 \cdot 3$. (Another way of calculating the number of favorable cases is to count the number of four-digit integers made up entirely with the digits 1, 2, 3, 4, 5, 6, and having distinct digits. Any such integer, say 3516, can be interpreted as meaning that the white die comes up "3", the red die "5", the blue die "1", and the green die "6".)

6. $7 \cdot 5^5/6^7$
There are 6^7 equally likely cases. To calculate the number of favorable cases, we count the number of seven digit numbers (one digit for each die) made up entirely of the digits 1, 2, 3, 4, 5, 6, and having exactly three sixes present. This is seen to be $1 \cdot 1 \cdot 1 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot C(7, 3)$.

8. $651/6^5$ or $[C(14, 4) - 5C(8, 4)]/6^5$
There are 6^5 equally likely cases. The number of favorable cases is the

same as the number of solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = 15$ in integers from 1 to 6.

9. (a) 7/32

There are 2^8 equally likely cases. Of these the number of favorable cases is $C(8, 5)$, because it amounts to the number of ways of choosing five out of eight coins.

(b) 93/256 or $[C(8, 5) + C(8, 6) + C(8, 7) + C(8, 8)]/2^8$.

10. $13^4/C(52, 4)$

The number of selections of four cards from a deck is $C(52, 4)$. The number of selections of four cards, one from each suit, is $[C(13, 1)]^4$, or 13^4.

11. (a) $1 - [C(40, 13) + 12C(40, 12)]/C(52, 13)$

Compute the complementary probability. The total number of equally likely cases is $C(52, 13)$. The number of selections of 13 cards with no face cards present is $C(40, 13)$; with exactly one face card present is $C(12, 1)C(40, 12)$.

(b) $C(4, 1)C(48, 12)/C(52, 13)$

(c) $1 - C(48, 13)/C(52, 13)$

This answer is arrived at by computing the complementary probability. (The answer by a direct method looks different.) The number of selections of 13 cards with no aces present is $C(48, 13)$.

12. 1/13

There are $26!$ orders in all. Of these, x and y are adjacent in $2 \cdot 25!$ cases.

13. (a) $[C(23, 4) - C(14, 4) - 4C(13, 4) + 4C(4, 4)]/90000$

There are 90000 five digit integers. The number of favorable cases is the number of solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ in non-negative integers not exceeding 9, but with the additional restriction that x_1 must be positive. The inclusion-exclusion principle along with formula (4.22) can then be used.

(b) 1/1800

There are 50 five-digit integers satisfying the conditions of the problem. Twenty of them have digits 5, 4, 1, 1, 1, and thirty of them have digits 5, 2, 2, 1, 1.

14. No, the probability is 4/9

There are $C(10, 5)/2$ equally likely cases, because this is the number of ways that 10 boys can be separated into two teams of 5. To compute the number of favorable cases, we set aside the two friends and choose three out of eight to accompany them on "favorable" teams; this gives $C(8, 3)$ favorable cases.

15. $1 - D(8)/8!$; $1 - [D(8) + 8D(7)]/8!$

The answers given arise from the complementary probability in each part. The number of equally likely cases is $8!$. The number of ways in which no spark plug can go back into its original cylinder is $D(8)$, the number of derangements of 8 things. Furthermore, the number of arrangements with exactly one plug in its original cylinder is $8D(7)$.

16. The probability of a win is the same as the probability that the arrangement of cards in one deck is compatible with that in the other. Since there are 52! possible arrangements and $D(52)$ derangements, the ratio of the number of favorable cases to the total number of equally likely cases is

$$\frac{D(52)}{52!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{52!} \qquad \text{(approximately .3679).}$$

17. $1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots - \frac{1}{13!}$

(To four decimal places, this answer is the same as that to Problem 16.)

18. The probability of a win is the probability that a shuffled deck will produce a total derangement except for one card. There are 52 ways of holding one card fixed and $D(51)$ derangements of the remaining 51 cards. Hence, the answer is

$$52\,\frac{D(51)}{52!} = \frac{D(51)}{51!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots - \frac{1}{51!}.$$

(This differs from the answer to Problem 16 by 1/52! which is an extremely small number.)

Problem Set 22, page 96

1. 1, 2, 3, 5, 7 **2.** $p_1(n) = 1, \quad q_1(n) = 1$

3. $q_2(8) = 5$, $q_2(9) = 5$, $q_2(n) = 1 + \frac{1}{2}n$ if n is even,
$q_2(n) = \frac{1}{2}(n+1)$ if n is odd.

4. 1 5. 2

6. The largest summand occurring among all the partitions of n is n itself, and hence there are no partitions counted by $p_{n+1}(n)$ not already counted by $p_n(n)$; similarly for $p_k(n)$ with $k > n$.

7. There is only one partition, namely n itself, that is counted by $p_n(n)$ but not by $p_{n-1}(n)$.

Problem Set 23, page 99

1. Partial solution:

$n \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12
8	1	5	10	15	18	20	21	22	22	22	22	22
9	1	5	12	18	23	26	28	29	30	30	30	30
10	1	6	14	23	30	35	38	40	41	42	42	42
11	1	6	16	27	37	44	49	52	54	55	56	56
12	1	7	19	34	47	58	65	70	73	75	76	77

2. 7, 15, 28 3. 15, 22, 30, 42

Problem Set 24, page 101

1. $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + \cdots$

2. $1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + \cdots$

3. $1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + \cdots$

Problem Set 25, page 105

1. (a) the number of partitions of 12 with even summands;
   (b) the number of partitions of 9 with summands not exceeding 3;
   (c) the number of partitions of 6 with distinct summands.

2. (a) 11 (b) 12 (c) 4

3. (a) $(1 + x^6 + x^{12} + x^{18} + x^{24} + x^{30} + x^{36}) \cdot (1 + x^7 + x^{14} + x^{21} + x^{28} + x^{35})(1 + x^{12} + x^{24} + x^{36})(1 + x^{20})$;

   (b) $(1 + x^3 + x^6 + x^9 + x^{12} + x^{15})(1 + x^4 + x^8 + x^{12}) \cdot (1 + x^5 + x^{10} + x^{15})(1 + x^6 + x^{12})(1 + x^7 + x^{14})(1 + x^8) \cdot (1 + x^9) \cdots (1 + x^{15})$;

   (c) $(1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)(1 + x^6)(1 + x^7) \cdot (1 + x^8)(1 + x^9)$; 5, 17, 8 partitions respectively.

4. 14
   Compute the coefficient of x^{18} in the expansion of

$$(1 + x^2 + x^4 + x^6 + \cdots + x^{18})(1 + x^3 + x^6 + \cdots + x^{18}) \cdot (1 + x^5 + x^{10} + x^{15})(1 + x^7 + x^{14}).$$

5. 3
   The answer is the number of solutions of

$$3U + 5V + 7W + 9T = 16$$

in non-negative integers, as can be seen by use of the transformation $u = 1 + U$, $v = 1 + V$, $w = 1 + W$, $t = 1 + T$. This is the coefficient of x^{16} in the expansion of

$$(1 + x^3 + x^6 + x^9 + x^{12} + x^{15})(1 + x^5 + x^{10} + x^{15}) \cdot (1 + x^7 + x^{14})(1 + x^9).$$

Problem Set 26, page 108

(In these solutions P_1, P_2, P_3, P_4, P_5 denote the polynomials given in the text on page 107.)

1. 343
Compute the coefficient of x^{100} in the expansion of $P_1P_2P_3QP_5$, where $Q = 1 + x^{20} + x^{40} + x^{60} + x^{80} + x^{100}$.

2. 49
Compute the coefficient of x^{53} in the expansion of $P_1P_2P_3P_4$. Each polynomial can be abbreviated to exclude powers higher than x^{53}.

3. 34
This is the coefficient of x^{95} in the expansion of $P_2P_3P_4P_5$.

4. 16
Apply the transformation $y = 1 + Y$, $z = 1 + Z$, $w = 1 + W$, $t = 1 + T$ to get the equation $5Y + 10Z + 25W + 50T = 65$ and then find the number of solutions of this equation in non-negative integers. This is the coefficient of x^{65} in the expansion of $P_2P_3P_4P_5$.

Problem Set 27, page 111

1. 150 or $3^5 - 3\cdot 2^5 + 3$

2. $f(5, 2) = 2^5 - 2 = 30$

4. $k! = k^k - C(k, 1)(k - 1)^k + C(k, 2)(k - 2)^k - C(k, 3)(k - 3)^k + \cdots + (-1)^{k-1}C(k, k - 1)$

5. The expression in the problem comes from formula (8.1) with $k = 8$; since there are no ways of distributing fewer than 8 objects into 8 boxes with no box empty, $f(m, 8) = 0$ for $m < 8$.

Problem Set 28, page 114

1. 3025 or $\dfrac{3^9 - 3\cdot 2^9 + 3}{3!}$

2. $\dfrac{4^4 - 4\cdot 3^4 + 6\cdot 2^4 - 4}{4!} = 1$

3. $2^{m-1} - 1$

4. (a) 122 (b) 90

The number 30,030 has six distinct prime factors, and we want to separate these into three sets. The notation for part (a) is $G(6, 3)$, for part (b) is $g(6, 3)$.

5. 52 7. $g(m, m - 2) = C(m, 3) + 3C(m, 4)$

Problem Set 29, page 117

1. (a) 24 (b) 126 or $C(9, 4)$ (c) 151200 or $P(10, 6)$
(d) 286 (e) 7 (f) 21 (g) 19

Solution of (d). This is $C(13, 3)$, the number of solutions of $x + y + z + w = 10$ in non-negative integers.

Solution of (g). This is $C(8, 2) - 3C(3, 2)$, the number of solutions of $x + y + z = 6$ in non-negative integers not exceeding 4.

2. (a) $n!$ (b) $P(n, r)$ (c) $C(n, r)$ (*d*) $C(r + k - 1, r)$

Solution of (d). This is the number of solutions of

$$x_1 + x_2 + \cdots + x_k = r$$

in non-negative integers.

3. $\frac{1}{4}(k^2 + 3k + 4) \cdot k!$

Denote the two like objects by A, A. The first box is to contain two objects. The number of distributions with at least one A in the first box is $(k + 1)!$. The number of distributions with the A's in boxes other than the first is $C(k, 2)C(k, 2) \cdot (k - 2)!$, because there are $C(k, 2)$ ways of choosing the two objects to go in the first box, $C(k, 2)$ ways of choosing the two boxes for the A's, and $(k - 2)!$ ways of distributing the other $k - 2$ objects.

Problem Set 30, page 121

4. $k = 4$

Suppose that at most three of $a_1, a_2, \cdots, a_{10}$, and at most three of $b_1, b_2, \cdots, b_{10}$, and at most three of $c_1, c_2, \cdots, c_{10}$ have property Q. Then by simple addition at most nine of the entire thirty items have property Q. This contradicts the given information.

5. $r = 4$

Problem Set 31, page 123

1. One way to handle this is to color AB, BC, CD, DE, EA blue, and all other line segments red.

2. Denote one of the seventeen points by A and denote the others by $B_1, B_2, \cdots, B_{16}$. Consider the sixteen line segments emanating from A, namely $AB_1, AB_2, \cdots, AB_{16}$. By the pigeonhole principle at least six of these segments are of one color, say blue. We may as well take these six blue segments to be $AB_1, AB_2, \cdots, AB_6$. Now if there is at least one blue segment among the fifteen segments linking B_1, B_2, B_3, B_4, B_5, B_6, then we have a blue chromatic triangle. (For example if the segment B_3B_5 is blue, then AB_3B_5 is a blue triangle.) On the other hand if there is no blue segment in this batch of fifteen, this means that all segments linking the points B_1, B_2, B_3, B_4, B_5, B_6 are red or white. In this case we apply the basic result proved in Section 9.2.

Problem Set 32, page 127

1. $\frac{1}{2}(n^2 + n)$
This result can be obtained from the answer $\frac{1}{2}(n^2 + n + 2)$ in the text for the number of regions created by n lines, no two parallel, no three concurrent. For if in that case we slide one of the lines across the plane in such a way that it becomes concurrent with two other lines, then one region is lost.

2. $(m + 1)(k + 1)$

3. $(m + 1)(k + 1) + m + k + 1$
The new line creates $m + k + 1$ new regions.

4. $qt + 2q + 2t - 1$
Denote the number of regions by $F(q, t)$. If we remove one of the q lines we see that $2 + t$ regions are lost, and so $F(q, t)$ exceeds $F(q - 1, t)$ by $2 + t$. Thus we have

$$
\begin{aligned}
F(q, t) - F(q - 1, t) &= 2 + t,\\
F(q - 1, t) - F(q - 2, t) &= 2 + t,\\
F(q - 2, t) - F(q - 3, t) &= 2 + t,\\
&\cdots\cdots\\
F(2, t) - F(1, t) &= 2 + t,\\
F(1, t) - F(0, t) &= 1 + t.
\end{aligned}
$$

(Note the slight difference in the last equation.) Adding these we get the answer by use of $F(0, t) = 2t$.

5. $kq + 2q + k$

Denote the number of regions by $H(q, k)$. If one of the k parallel lines is removed, the number of regions is reduced by $q + 1$. Hence we see that $H(q, k) - H(q, k - 1) = q + 1$. Forming a telescoping sum as in the preceding question, and using $H(q, 0) = 2q$, we get the answer.

6. $kq + 2q + k + nk + nq + n(n + 1)/2$

Denote the number of regions by $H(q, k, n)$ so that $H(q, k, 0)$ is the same as $H(q, k)$ of the preceding question. If one of the n lines is withdrawn, there is a decrease of $n + k + q$ regions. Thus,

$$H(q, k, n) - H(q, k, n - 1) = k + q + n,$$

and from this we can proceed as in the two preceding solutions.

Problem Set 33, page 132

3. A comparison with Pascal's triangle suggests the conjecture

$$K(n) = C(n - 1, 2) = \tfrac{1}{2}(n - 1)(n - 2).$$

This turns out to be correct, because $K(n + 1) = K(n) + n - 1$ by the following observation: $K(n + 1)$ counts not only the pairs counted by $K(n)$ but also the pairs

$$1,\ n + 1 \qquad 2,\ n + 1 \qquad 3,\ n + 1 \qquad \cdots \qquad n - 1,\ n + 1.$$

4. (a) False (b) True (c) True (d) False (e) False (f) True

Problem Set 34, page 138

2. (a) $\displaystyle\sum_{j=1}^{n} 3j = \frac{3n(n + 1)}{2}$

$$\text{(c)} \quad \sum_{j=1}^{n} (2j-1)(2j+1) = \sum_{j=1}^{n} (4j^2-1) = 4\sum_{j=1}^{n} j^2 - n$$

$$= \frac{2n(n+1)(2n+1)}{3} - n$$

$$= \frac{n(4n^2+6n-1)}{3}$$

3. (a) $\dfrac{n(n-1)}{2}$ (c) $\dfrac{(n+1)(n+2)(2n+3)}{6}$

4. 50 6. 0 8. 2520 9. $2^n n!$ 11. $f(n) = (n+1)! - 1$

Problem Set 35, page 143

1. (i) 42 (ii) 132 (iii) 429

2. 5^{262144}

Solutions of Miscellaneous Problems

1. Consider the complementary probability, namely the chances of getting fewer than half the answers correct. The number of equally likely possibilities is 2^9. The student fails to get at least three right out of nine in $C(9, 0) + C(9, 1) + C(9, 2)$ cases, the terms of this sum corresponding to none, one or two right. Noting that this sum is $1 + 9 + 36 = 46$, we see that the complementary probability is 46/512, which is less than 1/10.

2. 1414

The answer to the question is the smallest positive integer n such that $\frac{1}{2}(n^2 + n) > 1{,}000{,}000$. The corresponding equation

$$\tfrac{1}{2}(n^2 + n) = 1{,}000{,}000$$

has a positive root between 1413 and 1414.

3. $n^3(n+1)$

The problem is to evaluate

$$\sum_{j=1}^{n}(4j^3-3j^2+j) \qquad \text{or} \qquad 4\sum_{j=1}^{n}j^3-3\sum_{j=1}^{n}j^2+\sum_{j=1}^{n}j.$$

Formulas for these sums can be found in the Summary of Chapter 3 and the answer to Problem 10 of Set 13. Thus we get

$$n^2(n+1)^2-\tfrac{1}{2}n(n+1)(2n+1)+\tfrac{1}{2}n(n+1),$$

which reduces to the answer given.

4. 8/11

It makes no difference whether or not the teams are "identified", that is, given specific labels such as the red team, the blue team and the green team. Here is a solution using identified teams. The total number of ways of forming the teams is $C(12, 4)C(8, 4)$, by first choosing four boys for the red team and then four for the blue. Label the particular boys A and B; the number of ways of forming the teams with A on the red and B on the blue is $C(10, 3)C(7, 3)$. Consequently the answer is $6C(10, 3)C(7, 3)/[C(12, 4)C(8, 4)]$.

5. 16/55

Using the background of the preceding solution, with the boys labeled A, B, C, we note that there are $C(9, 3)C(6, 3)$ ways of forming the teams with A on the red, B on the blue and C on the green. Hence the answer is $6C(9, 3)C(6, 3)/[C(12, 4)C(8, 4)]$.

6. The number of objects having neither of the properties α, β is $N-N(\alpha)-N(\beta)+N(\alpha,\beta)$. This is not negative and so

$$N-N(\alpha)-N(\beta)+N(\alpha,\beta)\geqslant 0$$

or

$$N+N(\alpha,\beta)\geqslant N(\alpha)+N(\beta).$$

Similarly we have

$$N+N(\alpha,\gamma)\geqslant N(\alpha)+N(\gamma) \quad \text{and} \quad N+N(\beta,\gamma)\geqslant N(\beta)+N(\gamma),$$

and the result follows by addition of these inequalities.

7. $(1 + x)(1 + x^3)(1 + x^5)\cdots(1 + x^{99})$ or $\prod_{j=1}^{50} (1 + x^{2j-1})$

8. $\left\{\sum_{j=0}^{20} x^j\right\}^3 \left\{\sum_{j=0}^{10} x^{2j}\right\}^3 \left\{\sum_{j=0}^{6} x^{3j}\right\}^2 \cdot \sum_{j=0}^{5} x^{4j} \cdot \sum_{j=0}^{4} x^{5j} \cdot \sum_{j=0}^{2} x^{10j} \cdot \sum_{j=0}^{1} x^{20j}.$

9. Add the equations

$$\begin{aligned} F(n) - F(n-1) &= F(n-2), \\ F(n-1) - F(n-2) &= F(n-3), \\ F(n-2) - F(n-3) &= F(n-4), \\ &\cdots\cdots \\ F(2) - F(1) &= F(0). \end{aligned}$$

10. The relation $F(n) = C(n+1, 0) + C(n, 1) + C(n-1, 2) + \cdots$ from the Summary of Chapter 4 can be written as

$$F(n) = \sum_{j+k=n+1} C(j, k).$$

Similarly we see that

$$F(n-2) = \sum_{j+k=n-1} C(j, k), \qquad F(n-3) = \sum_{j+k=n-2} C(j, k),$$

$$\cdots, \qquad F(0) = \sum_{j+k=1} C(j, k).$$

Adding these to $1 = C(0, 0)$ we get

$$F(n-2) + F(n-1) + \cdots + F(0) + 1 = \sum_{j+k<n} C(j, k).$$

The conclusion then follows by use of the preceding problem.

11. $(20!)(21!)/11!$

Set aside the multiples of 3 momentarily; the other twenty integers can be permuted in 20! ways. Given any one of these permutations, there are 21 spaces between and at the ends of the integers. Choose 10 of these spaces to insert the multiples of 3; thus $C(21, 10)$ choices. But then the multiples of 3 can be inserted in 10! ways. So the answer is

$$(20!) \cdot C(21, 10) \cdot (10!).$$

12. 16687.

Say that a permutation has property α_1 in case a is followed immediately by b; property α_2 in case b is followed immediately by c; $\cdots$; property α_7 in case g is immediately followed by h. The problem is to find the number of permutations having none of these properties, and this can be done by use of the inclusion-exclusion principle. It can be seen that $N(\alpha_1) = N(\alpha_2) = \cdots = N(\alpha_7) = 7!$, $N(\alpha_1, \alpha_2) = 6!$, $N(\alpha_1, \alpha_2, \alpha_3) = 5!$, etc. Thus the answer is

$$8! - C(7, 1)\cdot 7! + C(7, 2)\cdot 6! - C(7, 3)\cdot 5! + C(7, 4)\cdot 4! \\ - C(7, 5)\cdot 3! + C(7, 6)\cdot 2! - C(7, 7)\cdot 1!$$

13. $\{C(24, 4)\}^5 - 5\{C(23, 3)\}^5 + 10\{C(22, 2)\}^5 - 10\{C(21, 1)\}^5 + 5\{C(20, 0)\}^5$

Ignore momentarily the condition that no box is to be empty. Label the boxes A, B, C, D, E. Then the number of distributions of the cents is $C(24, 4)$, the number of solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ in non-negative integers, where x_1 is interpreted as the number of cents in box A, etc. Hence the number of distributions of all the coins is $\{C(24, 4)\}^5$. Now introduce the condition that no box be empty and use the inclusion-exclusion principle. Say that a distribution has property α in case box A is empty, property β in case box B is empty, etc. Thus we see that

$$N = \{C(24, 4)\}^5, \quad N(\alpha) = \{C(23, 3)\}^5, \quad N(\alpha, \beta) = \{C(22, 2)\}^5,$$

and so on, which give the answer.

14. 864

First disregard the restriction that no two adjacent letters be alike. The total number of permutations is then

$$N = \frac{8!}{2!2!2!2!} = 2520.$$

Now apply the inclusion-exclusion principle, where a permutation has property α in case the A's are adjacent, property β in case the B's are adjacent, etc. It can be calculated that

$$N(\alpha) = \frac{7!}{2!2!2!} = 630, \quad N(\alpha, \beta) = \frac{6!}{2!2!} = 180,$$

$$N(\alpha, \beta, \gamma) = 60, \quad N(\alpha, \beta, \gamma, \delta) = 24.$$

Hence the answer is

$$N - 4N(\alpha) + 6N(\alpha, \beta) - 4N(\alpha, \beta, \gamma) + N(\alpha, \beta, \gamma, \delta) = 864.$$

15. 700

First permute the E's and F's to get $(6!)/(3!3!)$ or 20 permutations. Then in each of these 20 permutations insert the D's in any 3 of the 7 spaces between letters or at the ends. Thus the answer is $20C(7, 3)$.

16. 340

In the solution to the preceding problem separate the 20 permutations of the E's and F's into three types: 4 permutations having no E's adjacent; 12 permutations having exactly two E's adjacent; 4 permutations with all three E's adjacent. The D's can be inserted in $C(7, 3)$, $C(6, 2)$, and $C(5, 1)$ ways respectively for these three types. Hence the answer is $4C(7, 3) + 12C(6, 2) + 4C(5, 1)$.

17. 174

First consider all arrangements with DE on the left end, followed by the other seven letters. With the F's omitted these arrangements are six in number, namely

(1) $DEDDEE$	(2) $DEDEDE$	(3) $DEDEED$
(4) $DEEEDD$	(5) $DEEDED$	(6) $DEEDDE$

The F's can be inserted in the following number of ways in these six arrangements: $C(3, 1)$, $C(5, 3)$, $C(4, 2)$, $C(3, 3)$, $C(4, 2)$, $C(3, 1)$. This totals 29, and the answer is obtained by multiplying by 6 to allow for other pairs of letters on the left end, besides DE.

18. 2600

To any solution of $x + y + z + v = 27$ in positive integers, there corresponds a unique solution of the other equation since

$$u = 30 - x - y - z.$$

Hence the question amounts to asking for the number of solutions of

$$x + y + z + v = 27$$

in positive integers. The answer is $C(26, 3)$.

19. 1078

By symmetry the number of solutions with $x > y$ equals the number with $x < y$. The total number of solutions is $C(25, 3)$ or 2300. The number of solutions with $x = y$ is

$$\sum_{x=1}^{12} C(25 - 2x, 1) = \sum_{x=1}^{12} (25 - 2x) = 144,$$

because for a fixed value of x the equation $2x + z + w = 26$ or $z + w = 26 - 2x$ has $C(25 - 2x, 1)$ solutions in positive integers. Hence the answer is $(2300 - 144)/2$.

20. $(n + 1)(2n^2 + 4n + 3)/3$

The question amounts to asking for the number of selections of $2n$ objects from $4n$ objects, given that the $4n$ objects are identical in sets of n. (Thus there are just 4 different kinds of objects.) This is the same as asking for the number of solutions of $x_1 + x_2 + x_3 + x_4 = 2n$ in non-negative integers not exceeding n. By the work of Chapter 5 this is $C(2n + 3, 3) - 4C(n + 2, 3)$.

21. $(nj)!/\{(j!)^n n!\}$

A first batch of j objects can be chosen in $C(nj, j)$ ways, a second batch in $C(nj - j, j)$ ways, and so on. Since the order of the batches does not matter, the answer is

$$C(nj, j)C(nj - j, j)C(nj - 2j, j)\cdots C(3j, j)C(2j, j)/n!\,.$$

22. Partitions into positive even integers.
There are no partitions of 1000 into three positive odd integers.

23. Partitions into four positive odd integers.

Let $a + b + c + d$ be a partition of 1000 into four positive even integers, with $a \leqslant b \leqslant c \leqslant d$. Then

$$(a - 1) + (b - 1) + (c - 1) + (d + 3)$$

is a partition of 1000 into four positive odd integers. Moreover this gives a one-to-one correspondence between *all* the partitions into four even integers and *some* of the partitions into four odd integers. The correspondence gives only "some" of the partitions into odd integers because $d + 3$ exceeds $c - 1$ by at least 4, and so such a partition as $249 + 249 + 251 + 251$ is not present.

24. Partitions into positive odd integers.

Any partition into even summands can be transformed into one having odd summands by separating each even summand $2j$ into two parts, 1 and $2j - 1$. For example $200 + 300 + 500$ is transformed into $1 + 1 + 1 + 199 + 299 + 499$. Thus each partition with even summands is transformed into a unique partition with odd summands, but this procedure does not give *all* partitions with odd summands.

25. 402130

Count the integers *not* having the property. For example, there are 9^6 integers with 6 digits such that no two adjacent integers are equal. Thus the answer is $1{,}000{,}000 - (9^6 + 9^5 + 9^4 + 9^3 + 9^2 + 9)$.

26. $-24 + \sum_{j=0}^{22} (-1)^j \{C(24, j) + C(23, j) + 46C(22, j)\}(24 - j)!$.

First consider those permutations with B in first place and A in second place. Of these there are

$$D(24) \qquad \text{or} \qquad \sum_{j=0}^{24} (-1)^j C(24, j)(24 - j)!,$$

because these permutations are derangements. Next consider those with A in second place and B in third place. Of these there are, by an argument using the inclusion-exclusion principle as in the theory for derangements,

$$\sum_{j=0}^{23} (-1)^j C(23, j)(24 - j)!.$$

Finally with A and B in any other specified adjacent positions, there are

$$\sum_{j=0}^{22} (-1)^j C(22, j)(24 - j)!$$

possibilities.

27. 90

Apply the inclusion-exclusion principle to the 6! unrestricted partitions, taking α as the property that a and b are adjacent, β the property that b and c are adjacent, and so on.

28. Each time two people shake hands, the number of people who have shaken hands an odd number of times changes by 2, 0, or -2.

29. The possible number of acquaintances of each person in a group of n people is one of the n integers $0, 1, 2, \cdots, n-1$. If no two people have the same number of acquaintances, then all n integers are represented. But 0 and $n-1$ cannot occur simultaneously because it would mean one person is acquainted with everybody, another with nobody.

30. This is simply the preceding question in a different form.

31. $m(3mk - m^2 + 3k + 1)/6$

We count the number $s_1(k, m)$ of squares of side 1, then the number $s_2(k, m)$ of squares of side 2, etc. Then the total number of squares in the grid is

$$S(k, m) = \sum_{i=1}^{m} s_i(k, m).$$

Now $s_1(k, 1) = k, \quad s_1(k, m) = ms_1(k, 1) = mk;$ similarly,

$$s_2(k, m) = (m-1)s_2(k, 2) = (m-1)(k-1),$$

and so on. We evaluate

$$S(k, m) = mk + (m-1)(k-1) + (m-2)(k-2) + \cdots + 1(k-m+1).$$

Reversing the order of the terms in this sum, we write

$$\begin{aligned} S(k, m) &= k - m + 1 + 2(k-m+2) + 3(k-m+3) + \cdots + m(k-m+m) \\ &= (k-m)(1+2+3+\cdots+m) + (1^2+2^2+3^2+\cdots+m^2) \\ &= (k-m)\frac{m(m+1)}{2} + \frac{m(m+1)(2m+1)}{6}, \end{aligned}$$

and this is the number given above.

32. It is possible in 255 moves.

The problem can be stated for n discs, requiring say, $f(n)$ moves. A simple analysis reveals that $f(n) = 2f(n-1) + 1$.

33. Let the points be A, B, C, D, E, F. By the work of Section 9.2 we may presume that ABC (say) is a red chromatic triangle. If DEF is not chromatic it has a white side, say DE. If ADE is not chromatic then at least one of AD and AE is red. Likewise if BDE and CDE are not chromatic, at least one of BD and BE, and at least one of CD and CE, are red. So at least two of AD, BD, CD are red, or at least two of AE, BE, CE are red. In the first case consider the triangles ABD, ACD, BCD, and in the second case ABE, ACE, BCE.

34. Here is one pattern. Let the sides of ABC be red, and the sides of DEF red, and let all other segments be white.

35. By the solution to Problem 33, we can take any six of the points and get two chromatic triangles. Let A be one of the vertices of one of these triangles. Then apply the solution to Problem 33 to the six points other than A.

36. From any one of the points, say A, there are 65 emanating segments. Of these, at least 17 are of one color; say that AB_1, AB_2, $\cdots$, AB_{17} are red. If any line segment joining two of B_1, B_2, $\cdots$, B_{17} is red, there is a red chromatic triangle. Otherwise we can apply the 17 points, 3 colors result in Problem 2 of Set 31.

37. This problem can be solved by a slight sharpening of the argument given in the solution of Problem 2 of Set 31. Take A *not as any one* of the points, but as a point which is not the vertex of a chromatic triangle, and use the result of Problem 33 of this set.

38. From any one of the points there are 23 emanating segments of which at least 12 are of one color. Say that AB_1, AB_2, $\cdots$, AB_{12} are red segments. If among the 12 points B_1 to B_{12} there is a red chromatic triangle, then such a triangle together with the point A gives a solution. Otherwise consider the 11 segments B_1B_2, B_1B_3, $\cdots$, B_1B_{12}; at least 6 of these, say B_1B_2, B_1B_3, $\cdots$, B_1B_7, are of one color. If the color is red, the points B_2, B_3, B_4, B_5 give a solution. If the color is white, then consider the six points B_2, B_3, $\cdots$, B_7. There is a chromatic triangle, necessarily white, among these six points. This triangle, together with B_1, gives a solution.

39. $n(n-1)(n-2)(n-3)/24$

Any four of the points on the circle determine a unique intersection point, so $C(n, 4)$ is the answer.

40. $C(n, 4) + 1 + \frac{1}{2}n(n-1)$

Adding one point at a time, consider the increase $R(j) - R(j-1)$ in the number of regions as we pass from $j-1$ to j points. Let P be the j-th point, and Q any of the $j-1$ points. If the line PQ has k intersection points inside the circle, then the line PQ creates $k+1$ new regions. But all the lines PQ (with P fixed and Q any one of the $j-1$ points) create $C(j, 4) - C(j-1, 4)$ intersection points by the result of the preceding problem. Thus we have

$$R(j) - R(j-1) = j - 1 + C(j, 4) - C(j-1, 4),$$

and the problem can be solved by summing this from $j = 2$ to $j = n$.

41. $(n^2 - n)/2$ if n is odd and not a multiple of 3; $(n^2 - 2n)/2$ if n is even and not a multiple of 3; subtract $2n/3$ in case n is a multiple of 3.

42. $C(n+5, 5)$

This is the number of solutions of $x_1 + x_2 + \cdots + x_6 = n$ in non-negative integers, where x_1 is the number of the dice showing ones, x_2 the number showing twos, and so on.

43. $(n^3 + 5n + 6)/6$

We make use of the number of regions into which a plane is separated by n lines of which no two are parallel and no three concurrent. This was denoted by $f(n)$ in Section 9.2, and it was established that

$$f(n) = \tfrac{1}{2}(n^2 + n + 2).$$

Now in the present problem if we introduce one plane at a time, and if the j-th plane thereby causes an increase of $g(j) - g(j-1)$ in the number of regions of space, then it can be argued that

$$g(j) - g(j-1) = f(j-1).$$

The answer is obtained by summing this equation from $j = 2$ to $j = n$.

44. $1/3$

Consider any arrangement of the n integers; 1, 2, 3 occur somewhere, say in the i-th, the j-th and the k-th positions, respectively. Now hold all other integers fixed in their positions, but permute 1, 2, 3. There are 6 ways of placing 1, 2, 3 into the i-th, j-th and k-th positions; two of these ways are favorable, in the sense that 2 occurs between 1 and 3. Since this reasoning may be applied to every selection of positions i, j, k, we see that the probability is $\frac{2}{6}$ or $\frac{1}{3}$.

45. $$\sum_{j=0}^{n} \frac{(-1)^j C(n, j) \cdot (2n - j)!}{2^{n-j}}$$

Apply the inclusion-exclusion principle to the unrestricted permutations, taking the property α_1 to be A's adjacent, property α_2 to be B's adjacent, etc.

46. $C(n, j)D(j)/n!$

Choose $n - j$ integers to be in their natural positions, and then the others can be arranged in $D(j)$ ways, where $D(j)$ denotes derangements.

47. $$\sum_{j=0}^{n} C(k + 1, j)C(k + 1 - j, n - 2j)$$

First line up the B's, with $k + 1$ spaces between them and at the ends. Then consider the number of permutations having j pairs of A's in j of the spaces, and $n - 2j$ single A's in another $n - 2j$ spaces.

48. $$\sum_{j=0}^{n} (-1)^j C(n, j)(2n - j)!$$

Use the inclusion-exclusion principle applied to the total number $(2n)!$ of permutations. Say that a permutation has property α_1 in case 1 is in its natural position, α_2 in case 3 is in its natural position, and so on.

49. (i) $g(n, k)$ of Chapter 8;
(ii) $G(n, k)$ of Chapter 8.

50. The recursion relation is $K(n, j) = K(n - 2, j - 1) + K(n - 1, j)$. The proper conjecture is $K(n, j) = C(n - j + 1, j)$.

51. Consider the table:

	Column 1	Column 2	Column 3
Row 1:	AD	BD	CD
Row 2:	AE	BE	CE
Row 3:	AF	BF	CF

If a row contains two pairs of strangers, we get another triple with property α. If a row contains three pairs of strangers, we get three triples with property α. Two similar observations can be made about a column containing two pairs, or three pairs, of acquaintances. Next, if the table contains seven or more pairs of strangers, there must be one row with three pairs of strangers, and the problem is solved. A similar argument applies if the table contains seven or more pairs of acquaintances. If the table contains six pairs of strangers and three pairs of acquaintances, then either some row has three pairs of strangers or every row has two pairs of strangers. In either case the problem is solved. The rest of the argument is left to the reader.

52. 610

Say that the boy can go up n steps in $f(n)$ ways, so we are to determine $f(14)$. Now $f(n) = f(n-1) + f(n-2)$ because the boy can move to the n-th step directly from either of the two preceding steps. This recursion relation is the same as that for the Fibonacci numbers $F(n)$ of Section 4.1, but the beginning values here are $f(1) = 1$ and $f(2) = 2$; so $f(n) = F(n-1)$, and so $f(14) = F(13)$.

53. 4995 or $\sum_{j=1}^{9} (10-j)C(j+4, 4)$

Disregarding the integer 1,000,000 we note that all the others can be thought of as six-digit integers, with digits $x_1, x_2, \cdots, x_6$ from left to right, where zeros are allowed as digits. Define the differences $d_1, d_2, \cdots, d_5$ between adjacent digits by $d_1 = x_2 - x_1$, $d_2 = x_3 - x_2$, $\cdots$, $d_5 = x_6 - x_5$; the integers with the desired property have non-negative values for the differences d_1 to d_5. Furthermore, their digits are uniquely determined from each set of values $x_1, d_1, d_2, d_3, d_4, d_5$, and each such set satisfies the equation $d_1 + d_2 + d_3 + d_4 + d_5 = x_6 - x_1$. The number of solutions of this equation in non-negative integers is $C(x_6 - x_1 + 4, 4)$. Writing j for $x_6 - x_1$ we can assemble the answer given, keeping in mind that $0 \leqslant x_1 \leqslant x_6 \leqslant 9$.

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[3] H. J. Ryser, *Combinatorial Mathematics*, Carus Monograph No. 14, John Wiley, New York, 1963.

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General combinatorial problems are treated in [2], [3] and [4] whereas [1] deals with questions akin to Chapter 9 of this book. References [1], [2], [3] are advanced books; reference [4] is a reprinting of an older book, easy to read, but not very up-to-date.

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Symbols