Project Summary: Finite Element Analysis for Options Pricing (Focus on Call Options)

Option Contract: A contractual right but not an obligation to purchase (call option) or sell (put option) an asset (typically in bundles of 100 stocks) at a certain strike price by a certain maturity date. European option contracts only allow the contract to be exercised at the maturity date. American option contracts allow the contract to be exercised prior to the maturity date as long as the option is in-the-money or at-the-money.

In-the-money: For call options, stock price is above the agreed upon strike price. For put options, stock price is below the agreed upon strike price.

Out-of-the-money: For call options, stock price is below the agreed upon strike price. For put options, stock price is above the agreed upon strike price.

At-the-money: Stock price is equal to strike price.

At the expiration date, if the option is out-of-the-money, the contract expires worthless, if the contract is in-the-money, the contract will be automatically exercised. Buying and selling option contracts is allowed at any time the market is open.

Black-Scholes PDE: A PDE that is used to determine European options pricing. Works for both calls and puts.

Classical Black-Scholes PDE:

$$\frac{\partial V}{\partial t} - rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

Transformed Black-Scholes PDE:

$$\frac{\partial F}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}$$

The classical Black-Scholes PDE has top, bottom, and terminal boundary conditions. To convert the terminal condition to an initial condition, define $\tau = T - t$, and the PDE becomes:

$$-\frac{\partial V}{\partial \tau} - rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

Boundary conditions for call option contracts:

$V(0,\ \tau) = 0,\quad V(\infty, \tau) = S - Ke^{-r\,\tau},\quad V(S, 0) = \max\,(S - K, 0)$

Boundary conditions for put option contracts:

$V(0,\ \tau) = \ Ke^{-r\,\tau},\ \ V(\infty, \tau) = 0,\ \ V(S, 0) = \max\,(K - S, 0)$

The boundary conditions use the time value of money and assumes continuous compounding. The difference between the FE model of call option and a put option are the boundary conditions. Focus of this project is on call options.

Three models were created and compared, also with a theoretical solution:

1. Finite difference model using the classical Black-Scholes PDE and applying central differencing to the stock price dimension and Crank-Nicolson to the time dimension
2. Finite element model using the classical Black-Scholes PDE and applying 1D Galerkin Method of Weighted Residuals (GMWR) to the stock price dimension and Crank-Nicolson to the time dimension

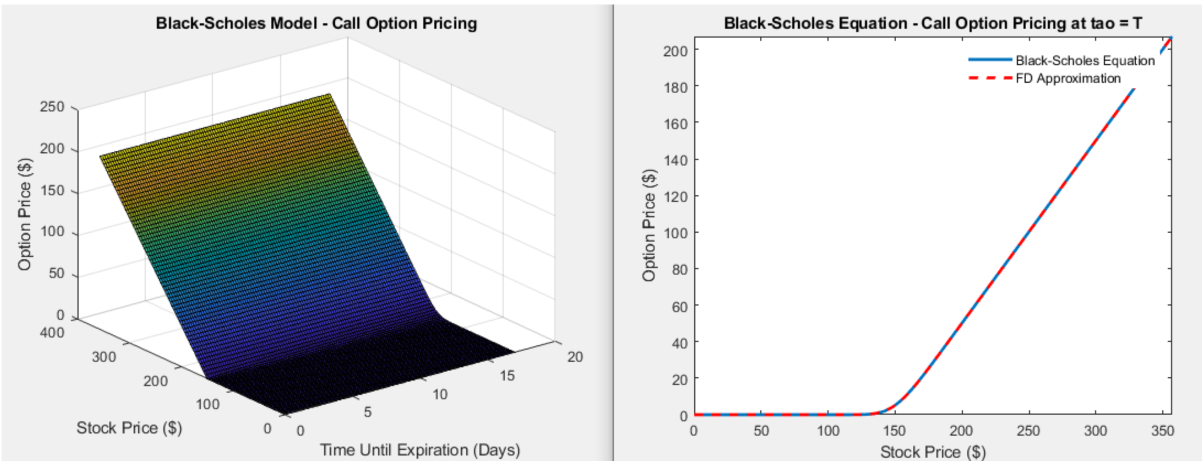
3. Finite element model using the transformed Black-Scholes PDE and applying 1D GMWR to the stock price dimension and Crank-Nicolson to the time dimension

Made a mistake on the second model and cannot find the mistake. It is wrong.

Theoretical solution: $Call\ Option\ Price = S * N(d_1) - Ke^{-r\tau} * N(d_2)$

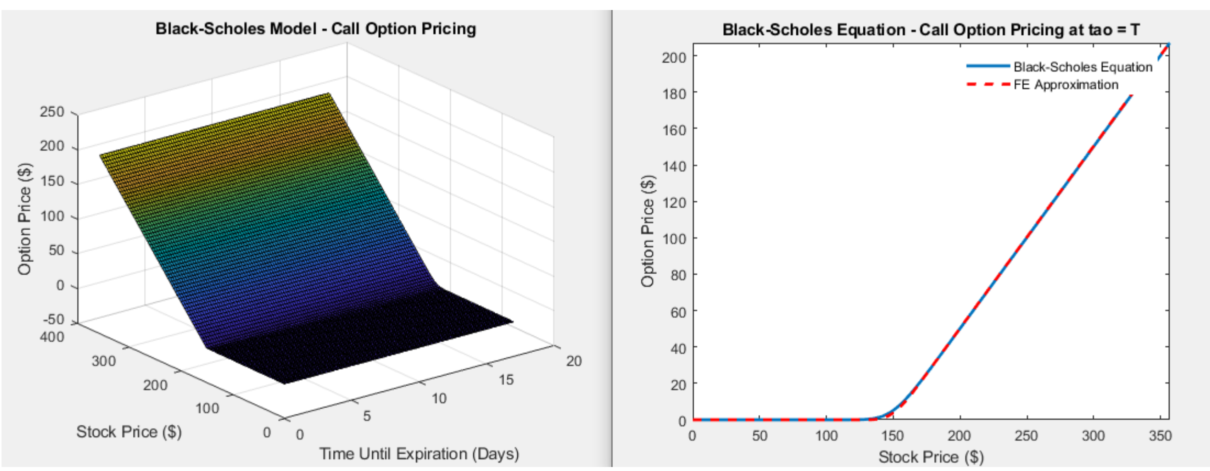
$$d_1 = \frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = \frac{\ln\left(\frac{S}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

Example mesh using model 1:




```
The current option price using Finite Difference Method is $28.7152
The current option price using Black-Scholes Equation is $28.7122>>
```

Example mesh using model 3:




```
The current option price using Finite Element Method is $28.7261
The current option price using Black-Scholes Equation is $28.7122>>
```

For some reason, the finite difference model is more accurate than the finite element model. Online sources seemed to have found the same trend (sanity check).