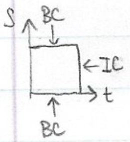
(1)

General Advection-Diffusion Partial Differential Equation.

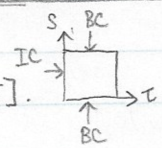
$\frac{\partial C}{\partial t} = D\nabla^2 C - \nabla\cdot vC + R$, $\nabla = \frac{\partial}{\partial x}$. C = call option price (concentration), S = stock price (displacement), r = risk-free rate, σ = implied volatility

$\frac{\partial C}{\partial t} + V\frac{\partial C}{\partial S} - D\frac{\partial^2 C}{\partial S^2} - R = 0$. → Convert to Black Schole's equation. $R = rC$, $D = -\frac{1}{2}\sigma^2 S^2$, $V = rS$.

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} - rC = 0. \to \boxed{-\frac{\partial C}{\partial \tau} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0}$$



We want IC at x-axis = 0. Set $\tau = T - t$ for $t[0,T]$.



$d\tau = -dt \Rightarrow \boxed{\frac{dt}{d\tau} = -1}$

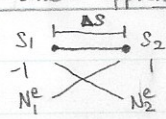
$\frac{\partial C}{\partial t} \times \frac{\partial t}{\partial \tau} = \frac{\partial C}{\partial \tau} = -\frac{\partial C}{\partial t}$.

Boundary Conditions {
European Call: BC → $C(0,\tau) = 0$, $C(\infty,\tau) = BIG = S - Ke^{-r\tau}$ → intrinsic + TVM. IC → $C(S,0) = \max(S-K, 0)$.
European Put: BC → $P(\infty,\tau) = 0$, $P(0,\tau) = BIG = Ke^{-r\tau} - S$ ↑ IC → $P(S,0) = \max(K-S, 0)$.

Try a 1D FEM solution with stock price and a Crank-Nicolson scheme for time steps.

$\tilde{C} = N_1^e C_1 + N_2^e C_2$ (1D Linear Approximation). → $N_i^e(S)$, $C_i(\tau)$, $\tilde{C}(S,\tau)$.

In global coordinates:



$N_1^e = \frac{S_2 - S}{S_2 - S_1} = \frac{S_2 - S}{\Delta S}$.

$N_2^e = \frac{S - S_1}{S_2 - S_1} = \frac{S - S_1}{\Delta S}$

$\frac{dN}{dS} = \frac{dN}{d\eta}\cdot\frac{d\eta}{dS}$: $\boxed{\frac{dN_1^e}{dS} = \left(\frac{-1}{2}\right)\left(\frac{2}{\Delta S}\right) = \frac{-1}{\Delta S}}$, $\boxed{\frac{dN_2^e}{dS} = \left(\frac{1}{2}\right)\left(\frac{2}{\Delta S}\right) = \frac{1}{\Delta S}}$

In local coordinates:

$N_1^e(\eta) = \frac{1}{2}(1-\eta)$

$N_2^e(\eta) = \frac{1}{2}(1+\eta)$ $\boxed{\partial S = \frac{\Delta S}{2}\partial \eta}$

$\int_{S_1}^{S_2} = \int_{\eta=-1}^{\eta=1}$

Transformation: $S(\eta) = N_1^e(\eta)S_1 + N_2^e(\eta)S_2$
$= \frac{1}{2}(1-\eta)S_1 + \frac{1}{2}(1+\eta)S_2$

$\boxed{\frac{\partial S}{\partial \eta} = \frac{S_2 - S_1}{2} = \frac{\Delta S}{2}}$

Apply MWR

$$\int_{S_1}^{S_2}\left(-\frac{\partial V}{\partial \tau} + \frac{\partial V}{\partial S}rS + \frac{1}{2}\sigma^2 S^2\left(\frac{\partial^2 V}{\partial S^2}\right) - rV\right)N_i\, dS = 0,$$

$N_j = \{N\}$, $N_i = \langle N \rangle$, j = row # (col vector), i = col # (row vector)

$$\int_{S_1}^{S_2}\left(\frac{-\partial(\Sigma N_j V_j)}{\partial \tau} + rS\frac{\partial(\Sigma N_j V_j)}{\partial S} + \frac{1}{2}\sigma^2 S^2\left(\frac{\partial^2(\Sigma N_j V_j)}{\partial S^2}\right) - r(\Sigma N_j V_j)\right)N_i\, dS = 0,$$

$\tilde{V}(\tau, S) = \Sigma V_j(\tau)\cdot N_j(S)$
$V_j(\tau) = \{V\} = V_j$, $V_i = \langle V \rangle$

Use V as general value, works for both C/P.
C=Call
P=Put

$$-\frac{\partial V_j}{\partial \tau}\int_{S_1}^{S_2} N_j N_i\, dS + \int_{S_1}^{S_2}\left(rS\frac{\partial N_j}{\partial S} + \frac{1}{2}\sigma^2 S^2\left(\frac{\partial^2(N_j)}{\partial S^2}\right) - rN_j\right)N_i\, dS\, V_j = 0$$

$S = \frac{1}{2}(1-\eta)S_1 + \frac{1}{2}(1+\eta)S_2$.
Where S_1, S_2 = constant, known.

✱

$$-\frac{\partial V_j}{\partial \tau}\overset{\text{D}}{\int_{S_1}^{S_2} N_j N_i\, dS} + V_j\left(\overset{\text{B}}{r\int_{S_1}^{S_2} S\frac{\partial N_j}{\partial S}N_i\, dS} + \overset{\text{C}}{\frac{1}{2}\sigma^2\int_{S_1}^{S_2} S^2\left(\frac{\partial^2 N_j}{\partial S^2}\right)N_i\, dS} - \overset{\text{D}}{r\int_{S_1}^{S_2} N_j N_i\, dS}\right) = 0$$

For C: IBP: $u = S^2 N_i$, $du = \left(2SN_i + S^2\frac{\partial N_i}{\partial S}\right)dS$, $dv = \frac{\partial^2 N_j}{\partial S^2}dS$, $v = \frac{\partial N_j}{\partial S}$

$$\text{C} = \frac{1}{2}\sigma^2\int_{S_1}^{S_2} S^2\left(\frac{\partial^2 N_j}{\partial S^2}\right)N_i\, dS = \frac{1}{2}\sigma^2\left(S^2 N_i\left(\frac{\partial N_j}{\partial S}\right)\Big|_{S_1}^{S_2} - \int_{S_1}^{S_2}\frac{\partial N_j}{\partial S}\left(2SN_i + S^2\frac{\partial N_i}{\partial S}\right)dS\right) = \boxed{\frac{1}{2}\sigma^2\left(S^2 N_i\left(\frac{\partial \tilde{N}_j}{\partial S}\right)\Big|_{S_1}^{S_2} - \int_{S_1}^{S_2} 2S\frac{\partial N_j}{\partial S}N_i\, dS - \int_{S_1}^{S_2} S^2\frac{\partial N_j}{\partial S}\frac{\partial N_i}{\partial S}dS\right)}$$

Sub C in ✱

$$-\frac{\partial V_j}{\partial \tau}\int_{S_1}^{S_2} N_j N_i\, dS + V_j\left(r\int_{S_1}^{S_2} S\frac{\partial N_j}{\partial S}N_i\, dS - \sigma^2\int_{S_1}^{S_2} S\frac{\partial N_j}{\partial S}N_i\, dS - \frac{1}{2}\sigma^2\int_{S_1}^{S_2} S^2\frac{\partial N_j}{\partial S}\frac{\partial N_i}{\partial S}dS - r\int_{S_1}^{S_2} N_j N_i\, dS\right) = -\frac{1}{2}\sigma^2 S^2 N_i\left(\frac{\partial \tilde{V}}{\partial S}\right)\Big|_{S_1}^{S_2}$$

✱²

$$\frac{\partial V_j}{\partial \tau}\overset{\text{D}}{\int_{S_1}^{S_2} N_j N_i\, dS} + V_j\left(\overset{\text{B}}{-r\int_{S_1}^{S_2} S\frac{\partial N_j}{\partial S}N_i\, dS} + \overset{\text{B}}{\sigma^2\int_{S_1}^{S_2} S\frac{\partial N_j}{\partial S}N_i\, dS} + \overset{\text{E}}{\frac{1}{2}\sigma^2\int_{S_1}^{S_2} S^2\frac{\partial N_j}{\partial S}\frac{\partial N_i}{\partial S}dS} + \overset{\text{D}}{r\int_{S_1}^{S_2} N_j N_i\, dS}\right) = \frac{1}{2}\sigma^2 S^2 N_i\left(\frac{\partial \tilde{V}}{\partial S}\right)\Big|_{S_1}^{S_2}$$

Look at RHS of ✱²: $\frac{1}{2}\sigma^2\left(S^2 N_i\left(\frac{\partial \tilde{V}}{\partial S}\right)\Big|_{S_2} - S^2 N_i\left(\frac{\partial \tilde{V}}{\partial S}\right)\Big|_{S_1}\right)$, $i = 1, 2$. $\left(\frac{\partial \tilde{V}}{\partial S}\right)$ = estimate of ΔV for every \$1 change in S.

When $i=1$: $\frac{1}{2}\sigma^2\left(\cancel{S_2^2 N_1(S_2)\left(\frac{\partial \tilde{V}}{\partial S}\right)\Big|_{S_2}} - S_1^2 N_1(S_1)\left(\frac{\partial \tilde{V}}{\partial S}\right)\Big|_{S_1}\right) = -\frac{1}{2}\sigma^2 S_1^2\left(\frac{\partial \tilde{V}}{\partial S}\right)\Big|_{S_1}$

When $i=2$: $\frac{1}{2}\sigma^2\left(S_2^2 N_2(S_2)\left(\frac{\partial \tilde{V}}{\partial S}\right)\Big|_{S_2} - \cancel{S_1^2 N_2(S_1)\left(\frac{\partial \tilde{V}}{\partial S}\right)\Big|_{S_1}}\right) = \frac{1}{2}\sigma^2 S_2^2\left(\frac{\partial \tilde{V}}{\partial S}\right)\Big|_{S_2}$

$$\text{RHS} = \frac{1}{2}\sigma^2\begin{bmatrix} -S_1^2\left(\frac{\partial \tilde{V}}{\partial S}\right)\Big|_{S_1} \\ S_2^2\left(\frac{\partial \tilde{V}}{\partial S}\right)\Big|_{S_2}\end{bmatrix}$$

②

$$\frac{\partial v_j}{\partial \tau}\underbrace{\int_{S_1}^{S_2} N_j N_i\, dS}_{D} + V_j\left((\sigma^2-r)\underbrace{\int_{S_1}^{S_2} S\frac{\partial N_j}{\partial S}N_i\,dS}_{B} + \frac12\sigma^2\underbrace{\int_{S_1}^{S_2} S^2\frac{\partial N_j}{\partial S}\frac{\partial N_i}{\partial S}dS}_{E} + r\underbrace{\int_{S_1}^{S_2}N_jN_i\,dS}_{D}\right) = \frac12\sigma^2S^2N_i\left(\frac{\partial\tilde V}{\partial S}\right)\Big|_{S_1}^{S_2}$$

From here on, the letters B, D, E only correspond to the integrals.

Convert to local coordinates:

(*) ③

$$\frac{\Delta S}{2}\left(\frac{\partial v_j}{\partial \tau}\underbrace{\int_{-1}^{1} N_j N_i\, d\eta}_{D} + V_j\left((\sigma^2-r)\underbrace{\int_{-1}^{1} S\frac{\partial N_j}{\partial S}N_i\,d\eta}_{B} + \frac12\sigma^2\underbrace{\int_{-1}^{1} S^2\frac{\partial N_j}{\partial S}\frac{\partial N_i}{\partial S}d\eta}_{E} + r\underbrace{\int_{-1}^{1}N_jN_i\,d\eta}_{D}\right)\right) = \frac12\sigma^2\begin{bmatrix}-S_1^2\left(\frac{\partial\tilde V}{\partial S}\right)\Big|_{S_1}\\ S_2^2\left(\frac{\partial\tilde V}{\partial S}\right)\Big|_{S_2}\end{bmatrix}$$

For B: $S=\frac12(1-\eta)S_1+\frac12(1+\eta)S_2$

$$\frac12 S_1\underbrace{\int_{-1}^{1}(1-\eta)\begin{bmatrix}\frac{\partial N_1^e}{\partial S}\\ \frac{\partial N_2^e}{\partial S}\end{bmatrix}\begin{bmatrix}N_1^e & N_2^e\end{bmatrix}d\eta}_{B1} + \frac12 S_2\underbrace{\int_{-1}^{1}(1+\eta)\begin{bmatrix}\frac{\partial N_1^e}{\partial S}\\ \frac{\partial N_2^e}{\partial S}\end{bmatrix}\begin{bmatrix}N_1^e & N_2^e\end{bmatrix}d\eta}_{B2}$$

$$=\frac12 S_1\int_{-1}^{1}(1-\eta)\begin{bmatrix}\frac{-1}{\Delta S}\left(\frac12(1-\eta)\right) & \frac{-1}{\Delta S}\left(\frac12(1+\eta)\right)\\ \frac{1}{\Delta S}\left(\frac12(1-\eta)\right) & \frac{1}{\Delta S}\left(\frac12(1+\eta)\right)\end{bmatrix}d\eta + \frac12 S_2\int_{-1}^{1}(1+\eta)\begin{bmatrix}\frac{-1}{\Delta S}\left(\frac12(1-\eta)\right) & \frac{-1}{\Delta S}\left(\frac12(1+\eta)\right)\\ \frac{1}{\Delta S}\left(\frac12(1-\eta)\right) & \frac{1}{\Delta S}\left(\frac12(1+\eta)\right)\end{bmatrix}d\eta$$

$$=\frac12 S_1\int_{-1}^{1}\begin{bmatrix}\frac{-1}{\Delta S}\left(\frac12(1-\eta)^2\right) & \frac{-1}{\Delta S}\left(\frac12(1+\eta)(1-\eta)\right)\\ \frac{1}{\Delta S}\left(\frac12(1-\eta)^2\right) & \frac{1}{\Delta S}\left(\frac12(1+\eta)(1-\eta)\right)\end{bmatrix}d\eta + \frac12 S_2\int_{-1}^{1}\begin{bmatrix}\frac{-1}{\Delta S}\left(\frac12(1+\eta)(1-\eta)\right) & \frac{-1}{\Delta S}\left(\frac12(1+\eta)^2\right)\\ \frac{1}{\Delta S}\left(\frac12(1+\eta)(1-\eta)\right) & \frac{1}{\Delta S}\left(\frac12(1+\eta)^2\right)\end{bmatrix}d\eta$$

$$=\frac{S_1}{4\Delta S}\begin{bmatrix}\frac{-(1-\eta)^3}{3}(-1) & -\left(\eta-\frac{\eta^3}{3}\right)\\ \frac{(1-\eta)^3}{3(-1)} & \left(\eta-\frac{\eta^3}{3}\right)\end{bmatrix}\Bigg|_{-1}^{1} + \frac{S_2}{4\Delta S}\begin{bmatrix}-\left(\eta-\frac{\eta^3}{3}\right) & -\frac{(1+\eta)^3}{3}\\ \left(\eta-\frac{\eta^3}{3}\right) & \frac{(1+\eta)^3}{3}\end{bmatrix}\Bigg|_{-1}^{1}$$

$$=\frac{S_1}{4\Delta S}\begin{bmatrix}0-\frac83 & -\frac23+\left(-1+\frac13\right)\\ 0+\frac83 & \frac23-\left(-1+\frac13\right)\end{bmatrix} + \frac{S_2}{4\Delta S}\begin{bmatrix}-\frac43 & -\frac83+0\\ \frac43 & \frac83-0\end{bmatrix}$$

$$=\frac{1}{4\Delta S}\left(S_1\begin{bmatrix}-\frac83 & -\frac43\\ \frac83 & \frac43\end{bmatrix} + S_2\begin{bmatrix}-\frac43 & -\frac83\\ \frac43 & \frac83\end{bmatrix}\right) = \boxed{\frac{1}{3\Delta S}\left(S_1\begin{bmatrix}-2 & -1\\ 2 & 1\end{bmatrix} + S_2\begin{bmatrix}-1 & -2\\ 1 & 2\end{bmatrix}\right)}$$

For D:

$$\int_{-1}^{1}\begin{bmatrix}N_1^eN_1^e & N_1^eN_2^e\\ N_2^eN_1^e & N_2^eN_2^e\end{bmatrix}d\eta = \int_{-1}^{1}\begin{bmatrix}\frac14(1-\eta)^2 & \frac14(1-\eta)(1+\eta)\\ \frac14(1-\eta)(1+\eta) & \frac14(1+\eta)^2\end{bmatrix}d\eta = \frac14\begin{bmatrix}\frac{(1-\eta)^3}{-3} & \left(\eta-\frac{\eta^3}{3}\right)\\ \left(\eta-\frac{\eta^3}{3}\right) & \frac{(1+\eta)^3}{3}\end{bmatrix}\Bigg|_{-1}^{1}$$

$$=\frac14\begin{bmatrix}0+\frac83 & \frac43\\ \frac43 & \frac83-0\end{bmatrix} = \boxed{\frac13\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}}$$

For E: $S^2=\frac14(1-\eta)^2S_1^2+\frac12(1-\eta)(1+\eta)S_1S_2+\frac14(1+\eta)^2S_2^2$

$$\frac14\left(S_1^2\underbrace{\int_{-1}^{1}(1-\eta)^2\begin{bmatrix}\frac{\partial N_1^e}{\partial S}\frac{\partial N_1^e}{\partial S} & \frac{\partial N_1^e}{\partial S}\frac{\partial N_2^e}{\partial S}\\ \frac{\partial N_2^e}{\partial S}\frac{\partial N_1^e}{\partial S} & \left(\frac{\partial N_2^e}{\partial S}\right)^2\end{bmatrix}d\eta}_{E1} + 2S_1S_2\underbrace{\int_{-1}^{1}(1-\eta)(1+\eta)\begin{bmatrix}\frac{1}{(\Delta S)^2} & -\left(\frac{1}{\Delta S}\right)^2\\ -\left(\frac{1}{\Delta S}\right)^2 & \left(\frac{1}{\Delta S}\right)^2\end{bmatrix}d\eta}_{E2} + S_2^2\underbrace{\int_{-1}^{1}(1+\eta)^2\begin{bmatrix}\left(\frac{1}{\Delta S}\right)^2 & -\left(\frac{1}{\Delta S}\right)^2\\ -\left(\frac{1}{\Delta S}\right)^2 & \left(\frac{1}{\Delta S}\right)^2\end{bmatrix}d\eta}_{E3}\right)$$

$$=\left(\frac{1}{2\Delta S}\right)^2\left(S_1^2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\frac{(1-\eta)^3}{-3}\Bigg|_{-1}^{1} + 2S_1S_2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\left(\eta-\frac{\eta^3}{3}\right)\Bigg|_{-1}^{1} + S_2^2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\frac{(1+\eta)^3}{3}\Bigg|_{-1}^{1}\right)$$

$$=\frac{1}{4\Delta S^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\left(S_1^2\left(\frac83\right)+2S_1S_2\left(\frac43\right)+S_2^2\left(\frac83\right)\right) = \boxed{\frac{(2S_1^2+2S_1S_2+2S_2^2)}{3\Delta S^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}}$$

$$\frac{\Delta S}{2}\left(\frac{\partial v_j}{\partial \tau}\left(\frac13\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right) + V_j\left((\sigma^2-r)\left(\frac{1}{3\Delta S}\left(S_1\begin{bmatrix}-2 & -1\\ 2 & 1\end{bmatrix}+S_2\begin{bmatrix}-1 & -2\\ 1 & 2\end{bmatrix}\right)\right)\right) + \frac12\sigma^2\left(\frac{(2S_1^2+2S_1S_2+2S_2^2)}{3\Delta S^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right) + \frac{r}{3}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)$$

$$=\frac12\sigma^2\begin{bmatrix}-S_1^2\left(\frac{\partial\tilde V}{\partial S}\right)\Big|_{S_1}\\ S_2^2\left(\frac{\partial\tilde V}{\partial S}\right)\Big|_{S_2}\end{bmatrix}$$

$$\left(\frac{\Delta S}{6}\right)\frac{\partial V_j}{\partial \tau}\begin{bmatrix}2&1\\1&2\end{bmatrix}+\frac{V_j\Delta S}{2}\left(\frac{\sigma^2-r}{3\Delta S}\left(S_1\begin{bmatrix}-2&-1\\2&1\end{bmatrix}+S_2\begin{bmatrix}-1&-2\\1&2\end{bmatrix}\right)+\frac{(2S_1^2+2S_1S_2+2S_2^2)\sigma^2}{6\Delta S^2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}+\frac{r}{3}\begin{bmatrix}2&1\\1&2\end{bmatrix}\right)=\frac{1}{2}\sigma^2\begin{bmatrix}-S_1^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_1}\\ S_2^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_2}\end{bmatrix}$$

$$\frac{\partial V_j}{\partial \tau}\begin{bmatrix}2&1\\1&2\end{bmatrix}=\left(\frac{6}{\Delta S}\right)\left(\frac{1}{2}\sigma^2\begin{bmatrix}-S_1^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_1}\\ S_2^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_2}\end{bmatrix}-\frac{V_j\Delta S}{2}\left(\frac{\sigma^2-r}{3\Delta S}\left(S_1\begin{bmatrix}-2&-1\\2&1\end{bmatrix}+S_2\begin{bmatrix}-1&-2\\1&2\end{bmatrix}\right)+\frac{(2S_1^2+2S_1S_2+2S_2^2)\sigma^2}{6\Delta S^2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}+\frac{r}{3}\begin{bmatrix}2&1\\1&2\end{bmatrix}\right)\right)$$

$$\frac{\partial V_j}{\partial \tau}=\frac{1}{4-1}\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\left(\frac{6}{\Delta S}\right)\left(\frac{1}{2}\sigma^2\begin{bmatrix}-S_1^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_1}\\ S_2^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_2}\end{bmatrix}-\frac{V_j\Delta S}{2}\left(\frac{\sigma^2-r}{3\Delta S}\left(S_1\begin{bmatrix}-2&-1\\2&1\end{bmatrix}+S_2\begin{bmatrix}-1&-2\\1&2\end{bmatrix}\right)+\frac{(2S_1^2+2S_1S_2+2S_2^2)\sigma^2}{6\Delta S^2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}+\frac{r}{3}\begin{bmatrix}2&1\\1&2\end{bmatrix}\right)\right)$$

$$\frac{\partial V_j}{\partial \tau}=\frac{1}{\Delta S}\begin{bmatrix}4&-2\\-2&4\end{bmatrix}\left(\frac{1}{2}\sigma^2\begin{bmatrix}-S_1^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_1}\\ S_2^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_2}\end{bmatrix}-\frac{V_j\Delta S}{2}\left(\frac{\sigma^2-r}{3\Delta S}\left(S_1\begin{bmatrix}-2&-1\\2&1\end{bmatrix}+S_2\begin{bmatrix}-1&-2\\1&2\end{bmatrix}\right)+\frac{(2S_1^2+2S_1S_2+2S_2^2)\sigma^2}{6\Delta S^2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}+\frac{r}{3}\begin{bmatrix}2&1\\1&2\end{bmatrix}\right)\right)$$

$$\frac{\partial V_j}{\partial \tau}=\frac{\sigma^2}{\Delta S}\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\begin{bmatrix}-S_1^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_1}\\ S_2^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_2}\end{bmatrix}-V_j\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\left(\frac{\sigma^2-r}{3\Delta S}\begin{bmatrix}-2S_1-S_2&-S_1-2S_2\\2S_1+S_2&S_1+2S_2\end{bmatrix}+\frac{(2S_1^2+2S_1S_2+2S_2^2)\sigma^2}{6\Delta S^2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}+\frac{r}{3}\begin{bmatrix}2&1\\1&2\end{bmatrix}\right)$$

$*^{4}$

$$\boxed{\frac{\partial V_j}{\partial \tau}=\frac{\sigma^2}{\Delta S}\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\begin{bmatrix}-S_1^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_1}\\ S_2^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_2}\end{bmatrix}-\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\left(\frac{\sigma^2-r}{3\Delta S}\begin{bmatrix}-(2S_1+S_2)&-(S_1+2S_2)\\(2S_1+S_2)&(S_1+2S_2)\end{bmatrix}+\frac{(2S_1^2+2S_1S_2+2S_2^2)\sigma^2}{6\Delta S^2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}+\frac{r}{3}\begin{bmatrix}2&1\\1&2\end{bmatrix}\right)\begin{bmatrix}V_1^e\\V_2^e\end{bmatrix}}$$

Apply theta differencing to $*^{4}$

$*^{f}$

$$\begin{bmatrix}\frac{V_1^{\tau+\Delta\tau}-V_1^{\tau}}{\Delta\tau}\\ \frac{V_2^{\tau+\Delta\tau}-V_2^{\tau}}{\Delta\tau}\end{bmatrix}^e=\theta\left\{\frac{\sigma^2}{\Delta S}\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\begin{bmatrix}-S_1^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_1}\\ S_2^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_2}\end{bmatrix}-\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\left(\frac{\sigma^2-r}{3\Delta S}\begin{bmatrix}-(2S_1+S_2)&-(S_1+2S_2)\\2S_1+S_2&S_1+2S_2\end{bmatrix}+\frac{(2S_1^2+2S_1S_2+2S_2^2)\sigma^2}{6\Delta S^2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}+\frac{r}{3}\begin{bmatrix}2&1\\1&2\end{bmatrix}\right)\begin{bmatrix}V_1^{\tau+\Delta\tau}\\V_2^{\tau+\Delta\tau}\end{bmatrix}^e\right.$$

$$\left.+(1-\theta)\left\{\underbrace{\frac{\sigma^2}{\Delta S}\begin{bmatrix}2&-1\\-1&2\end{bmatrix}}_{[\alpha]\text{ coeff.}}\underbrace{\begin{bmatrix}-S_1^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_1}\\ S_2^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_2}\end{bmatrix}}_{[\alpha]}-\underbrace{\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\left(\frac{\sigma^2-r}{3\Delta S}\begin{bmatrix}-(2S_1+S_2)&-(S_1+2S_2)\\2S_1+S_2&S_1+2S_2\end{bmatrix}+\frac{(2S_1^2+2S_1S_2+2S_2^2)\sigma^2}{6\Delta S^2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}+\frac{r}{3}\begin{bmatrix}2&1\\1&2\end{bmatrix}\right)}_{[\beta]}\begin{bmatrix}V_1^{\tau}\\V_2^{\tau}\end{bmatrix}^e\right\}\right\}$$

$\Delta S=S_2-S_1$

$S_2=S_1+\Delta S$

$$\frac{V_j^{\tau+\Delta\tau}-V_j^{\tau}}{\Delta\tau}=\theta\left(\frac{\sigma^2}{\Delta S}\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\begin{bmatrix}-S_1^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_1}\\ S_2^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_2}\end{bmatrix}^{\tau+\Delta\tau}-[\beta]V_j^{\tau+\Delta\tau}\right)+(1-\theta)\left(\frac{\sigma^2}{\Delta S}\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\begin{bmatrix}-S_1^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_1}\\ S_2^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_2}\end{bmatrix}^{\tau}-[\beta]V_j^{\tau}\right)$$

$$\left(\frac{1}{\Delta\tau}+\theta[\beta]\right)V_j^{\tau+\Delta\tau}=\left(\frac{1}{\Delta\tau}-(1-\theta)[\beta]\right)V_j^{\tau}+\frac{\sigma^2}{\Delta S}\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\left(\theta\begin{bmatrix}-S_1^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_1}\\ S_2^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_2}\end{bmatrix}^{\tau+\Delta\tau}+(1-\theta)\begin{bmatrix}-S_1^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_1}\\ S_2^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_2}\end{bmatrix}^{\tau}\right)$$

$$\begin{bmatrix}2&-1\\-1&2\end{bmatrix}^{-1}\left(\frac{1}{\Delta\tau}+\theta[\beta]\right)V_j^{\tau+\Delta\tau}=\begin{bmatrix}2&-1\\-1&2\end{bmatrix}^{-1}\left(\frac{1}{\Delta\tau}-(1-\theta)[\beta]\right)V_j^{\tau}+\frac{\sigma^2}{\Delta S}\cancel{\begin{bmatrix}2&-1\\-1&2\end{bmatrix}^{-1}\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\left(\theta\begin{bmatrix}-S_1^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_1}\\ S_2^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_2}\end{bmatrix}^{\tau+\Delta\tau}\right.}\left.+(1-\theta)\begin{bmatrix}-S_1^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_1}\\ S_2^2\left(\frac{\partial\tilde{V}}{\partial S}\right)\Big|_{S_2}\end{bmatrix}^{\tau}\right)$$

All free degrees of freedom cancel out and fixed degrees of freedom are boundary conditions
Can't manipulate the matrices any further because the last term can only be ignored when there is an identity matrix as its coefficient (scalar multiples are fine, they cancel out).

Assemble this into global matrices, solve for $V^{\tau+\Delta\tau}$, set $V^{\tau+\Delta\tau}=V^{\tau}$ and the new $V^{\tau+\Delta\tau}$ is unknown (next time step), iterate over and over. (Ignore the fixed dofs those are boundary conditions)

(4)

Black-Scholes Equation Using FDM. → Central Differencing + Crank-Nicolson Method.

$-\frac{\partial C}{\partial \tau} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0.$ → Use V for value instead of C for call option.

$$\frac{\partial V}{\partial \tau} = rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV$$

$$\frac{\partial V}{\partial S} = \frac{V_{i+1} - V_{i-1}}{2\Delta S}. \qquad \frac{\partial^2 V}{\partial S^2} = \frac{V_{i+1} - 2V_i + V_{i-1}}{\Delta S^2}$$

$$\frac{V_i^{\tau+\Delta\tau} - V_i^{\tau}}{\Delta \tau} = \theta\left[rS\left(\frac{V_{i+1}^{\tau+\Delta\tau} - V_{i-1}^{\tau+\Delta\tau}}{2\Delta S}\right) + \frac{1}{2}\sigma^2 S^2\left(\frac{V_{i+1}^{\tau+\Delta\tau} - 2V_i^{\tau+\Delta\tau} + V_{i-1}^{\tau+\Delta\tau}}{\Delta S^2}\right) - rV_i^{\tau+\Delta\tau}\right]$$

$$+(1-\theta)\left[rS\left(\frac{V_{i+1}^{\tau} - V_{i-1}^{\tau}}{2\Delta S}\right) + \frac{1}{2}\sigma^2 S^2\left(\frac{V_{i+1}^{\tau} - 2V_i^{\tau} + V_{i-1}^{\tau}}{\Delta S^2}\right) - rV_i^{\tau}\right]$$

In the form: $a_1 V_{i-1}^{\tau+\Delta\tau} + a_2 V_i^{\tau+\Delta\tau} + a_3 V_{i+1}^{\tau+\Delta\tau} = b_1 V_{i-1}^{\tau} + b_2 V_i^{\tau} + b_3 V_{i+1}^{\tau}$

$$\frac{V_i^{\tau+\Delta\tau}}{\Delta\tau} - \frac{V_i^{\tau}}{\Delta\tau} - \frac{\theta r S}{2\Delta S}\left(V_{i+1}^{\tau+\Delta\tau} - V_{i-1}^{\tau+\Delta\tau}\right) - \frac{\theta\sigma^2 S^2}{2\Delta S^2}\left(V_{i+1}^{\tau+\Delta\tau} - 2V_i^{\tau+\Delta\tau} + V_{i-1}^{\tau+\Delta\tau}\right) + \theta r V_i^{\tau+\Delta\tau} - \frac{(1-\theta) r S}{2\Delta S}\left(V_{i+1}^{\tau} - V_{i-1}^{\tau}\right) - \frac{(1-\theta)\sigma^2 S^2}{2\Delta S^2}\left(V_{i+1}^{\tau} - 2V_i^{\tau} + V_{i-1}^{\tau}\right)$$
$$+ (1-\theta) r V_i^{\tau} = 0$$

$$\left(\frac{\theta r S}{2\Delta S} - \frac{\theta\sigma^2 S^2}{2\Delta S^2}\right)V_{i-1}^{\tau+\Delta\tau} + \left(\frac{1}{\Delta\tau} + \frac{\theta\sigma^2 S^2}{\Delta S^2} + \theta r\right)V_i^{\tau+\Delta\tau} + \left(-\frac{\theta r S}{2\Delta S} - \frac{\theta\sigma^2 S^2}{2\Delta S^2}\right)V_{i+1}^{\tau+\Delta\tau}$$

$$= \left(-\frac{(1-\theta) r S}{2\Delta S} + \frac{(1-\theta)\sigma^2 S^2}{2\Delta S^2}\right)V_{i-1}^{\tau} + \left(\frac{1}{\Delta\tau} - \frac{(1-\theta)\sigma^2 S^2}{\Delta S^2} - (1-\theta) r\right)V_i^{\tau} + \left(\frac{(1-\theta) r S}{2\Delta S} + \frac{(1-\theta)\sigma^2 S^2}{2\Delta S^2}\right)V_{i+1}^{\tau}$$

Boundary Conditions for Call: $V(0)=0$, $V(5000)=5000-Ke^{-r\tau}$ → $V(0)=\hat{V}_1$, $V(5000)=\hat{V}_6$

Assume 5 elements with \$1000 steps. (Mesh centered)

Block 1: $V_1 = \hat{V}_1$, Block 6: $V_6 = \hat{V}_6$. Block 2: $a_1\hat{V}_1 + a_2 V_2^{\tau+\Delta\tau} + a_3 V_3^{\tau+\Delta\tau} = b_1\hat{V}_1 + b_2 V_2^{\tau} + b_3 V_3^{\tau}$.

Block 3: $a_1 V_2^{\tau+\Delta\tau} + a_2 V_3^{\tau+\Delta\tau} + a_3 V_4^{\tau+\Delta\tau} = b_1 V_2^{\tau} + b_2 V_3^{\tau} + b_3 V_4^{\tau}$ → block 4 similar.

Block 5: $a_1 V_4^{\tau+\Delta\tau} + a_2 V_5^{\tau+\Delta\tau} + a_3\hat{V}_6 = b_1 V_4^{\tau} + b_2 V_5^{\tau} + b_3\hat{V}_6$

$$\begin{bmatrix} a_2 & a_3 & 0 & 0 \\ a_1 & a_2 & a_3 & 0 \\ 0 & a_1 & a_2 & a_3 \\ 0 & 0 & a_1 & a_2 \end{bmatrix}\begin{bmatrix} V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix}^{\tau+\Delta\tau} = \begin{bmatrix} (b_1 - a_1)\hat{V}_1 + b_2 V_2^{\tau} + b_3 V_3^{\tau} \\ b_1 V_2^{\tau} + b_2 V_3^{\tau} + b_3 V_4^{\tau} \\ b_1 V_3^{\tau} + b_2 V_4^{\tau} + b_3 V_5^{\tau} \\ b_1 V_4^{\tau} + b_2 V_5^{\tau} + b_3\hat{V}_6^{\tau} - a_3\hat{V}_6^{\tau+\Delta\tau} \end{bmatrix}$$

start with initial conditions.

$$a_1 = \frac{\theta S}{2\Delta S}\left(r - \frac{\sigma^2 S}{\Delta S}\right), \quad a_2 = \left(\frac{1}{\Delta\tau} + \frac{\theta\sigma^2 S^2}{\Delta S^2} + \theta r\right), \quad a_3 = \frac{-\theta S}{2\Delta S}\left(r + \frac{\sigma^2 S}{\Delta S}\right)$$

$$b_1 = \frac{(1-\theta)S}{2\Delta S}\left(-r + \frac{\sigma^2 S}{\Delta S}\right), \quad b_2 = \left(\frac{1}{\Delta\tau} - (1-\theta)\left(\frac{\sigma^2 S^2}{\Delta S^2} + r\right)\right), \quad b_3 = \frac{(1-\theta)S}{2\Delta S}\left(r + \frac{\sigma^2 S}{\Delta S}\right)$$

S = Stock Price
V = Option Price
σ = implied volatility (constant)
K = Option Strike Price (constant)
r = risk-free interest rate (constant).

Transformation of Black-Scholes PDE to Heat Equation (1D Diffusion).

Source: quantpie youtube channel video titled ↑. (June 18, 2019).

Derivation uses Itô's lemma and random price action assumptions using Geometric Brownian Motion.

Start with the Black-Scholes PDE: $\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0} \rightarrow V(S,t)$.

T = time at option price maturity.

For Call Option Pricing

$t=0$: $S(0)$ = known, $V(0) = ?$; $t=T$: $S(T)$ = known, $V(T) = \max(S(T)-K, 0)$ → time

$S(t)$, $V(t)$, and we are solving for $V(t=0)$ with terminal BC's, so this is a backward looking equation.

Reverse the time to make the PDE a forward looking equation with initial BC's. $\boxed{\tau = T - t = \text{time until maturity}}$.

time to maturity

$\tau = T$: $\ddot{S}(T)$ = known, $\ddot{V}(T) = ?$; $\tau = 0$: $\ddot{S}(0)$ = known, $\ddot{V}(0) = \max(\ddot{S}(0)-K, 0)$. Denote variable as functions of τ, i.e. $\ddot{S}(\tau)$, $\ddot{V}(S,\tau)$.

Stock price follows geometric Brownian motion: $S(T) = e^{\ln S(t) + (r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}$

Ignore the stochastic term $\sigma(W_T - W_t)$ because the Black-Scholes PDE assumes full delta hedging of the portfolio (i.e. the random weighting variables W_T and W_t cancel out).

$\ddot{S}(0) = e^{\ln \ddot{S}(\tau) + (r - \frac{1}{2}\sigma^2)\tau}$, set $\boxed{x = \ln(\ddot{S}(0)) = \ln \ddot{S}(\tau) + (r - \frac{1}{2}\sigma^2)\tau}$

Assuming a single stock price, $F(S,\tau)$ and $\ddot{V}(S,\tau)$: Forward Value of Option: $\boxed{F(\tau) = \ddot{V}(\tau)e^{r\tau}}$ → Continuous compounding until maturity date.

Start Transformation: ① $\tau = T - t$, ② $x = \ln\ddot{S}(\tau) + (r - \frac{1}{2}\sigma^2)\tau$, ③ $F(\ddot{S},\tau) = \ddot{V}(\ddot{S},\tau)e^{r\tau} \Rightarrow F_\tau = \ddot{V}_\tau e^{r\tau}$

①: $\frac{\partial V}{\partial S} = \frac{\partial \ddot{V}}{\partial \ddot{S}}$, $\frac{\partial V}{\partial t} = \frac{\partial \ddot{V}}{\partial \tau} \times \frac{\partial \tau}{\partial t}$, $\frac{\partial \tau}{\partial t} = -1 \Rightarrow \partial\tau = -\partial t$. $\frac{\partial V}{\partial t} = -\frac{\partial \ddot{V}}{\partial \tau}$.

Sub into Black-Scholes PDE: $\boxed{-\frac{\partial \ddot{V}}{\partial \tau} + \frac{1}{2}\sigma^2 \ddot{S}^2 \frac{\partial^2 \ddot{V}}{\partial \ddot{S}^2} + r\ddot{S}\frac{\partial \ddot{V}}{\partial \ddot{S}} - r\ddot{V} = 0}$ $(*)^{1}$ → now moving forward in time.

②: $\ddot{V}(\ddot{S},\tau) \overset{\text{def}}{=} v(x,\tau)$. → Short form $\ddot{V}_\tau \overset{\text{def}}{=} v_\tau$.

$\frac{\partial \ddot{V}}{\partial \ddot{S}} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \ddot{S}}$, $\frac{\partial x}{\partial \ddot{S}} = \frac{1}{\ddot{S}}$. → $\frac{\partial \ddot{V}}{\partial \ddot{S}} = \frac{\partial v}{\partial x}\left(\frac{1}{\ddot{S}}\right)$. $\frac{\partial^2 \ddot{V}}{\partial \ddot{S}^2} = \frac{\partial}{\partial \ddot{S}}\left(\frac{\partial v}{\partial x}\left(\frac{1}{\ddot{S}}\right)\right) = \frac{\partial}{\partial \ddot{S}}\left(\frac{\partial v}{\partial x}\right)\left(\frac{1}{\ddot{S}}\right) - \frac{\partial v}{\partial x}\left(\frac{1}{\ddot{S}^2}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)\left(\frac{1}{\ddot{S}}\right)\left(\frac{\partial x}{\partial \ddot{S}}\right) - \frac{\partial v}{\partial x}\left(\frac{1}{\ddot{S}^2}\right)$

$= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)\left(\frac{1}{\ddot{S}^2}\right) - \frac{\partial v}{\partial x}\left(\frac{1}{\ddot{S}^2}\right) = \boxed{\left(\frac{1}{\ddot{S}^2}\right)\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right)}$

$\frac{\partial \ddot{V}}{\partial \tau} = \frac{\partial v}{\partial \tau} + \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \tau}$ → When τ changes, x changes (Total derivative). $\frac{\partial x}{\partial \tau} = r - \frac{1}{2}\sigma^2$

$\frac{\partial \ddot{V}}{\partial \tau} = \frac{\partial v}{\partial \tau} + \frac{\partial v}{\partial x}\left(r - \frac{1}{2}\sigma^2\right)$

Sub into $(*)^{1}$: $-\frac{\partial v}{\partial \tau} - \frac{\partial v}{\partial x}\left(\cancel{r} - \cancel{\tfrac{1}{2}}\sigma^2\right) + \frac{1}{2}\sigma^2\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + \cancel{r\frac{\partial v}{\partial x}} - rv = 0$.

$\boxed{-\frac{\partial v}{\partial \tau} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2} - rv = 0.}$ $(*)^{2}$

③: $\ddot{V}_\tau = F_\tau e^{-r\tau} = v_\tau$, $\frac{\partial v}{\partial x} = e^{-r\tau}\left(\frac{\partial F}{\partial x}\right)$, $\frac{\partial v}{\partial \tau} = e^{-r\tau}\left(\frac{\partial F}{\partial \tau}\right) - rFe^{-r\tau}$, $\frac{\partial^2 v}{\partial x^2} = e^{-r\tau}\left(\frac{\partial^2 F}{\partial x^2}\right)$.

$-e^{-r\tau}\left(\frac{\partial F}{\partial \tau}\right) + \cancel{rFe^{-r\tau}} + \frac{1}{2}\sigma^2 e^{-r\tau}\left(\frac{\partial^2 F}{\partial x^2}\right) - \cancel{rFe^{-r\tau}} = 0$

$-\frac{\partial F}{\partial \tau} + \frac{1}{2}\sigma^2\left(\frac{\partial^2 F}{\partial x^2}\right) = 0$

$\boxed{\frac{\partial F}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}}$ $(*)^{f}$ — Resembling a heat equation (diffusion equation).

Boundary Conditions: $\ddot{S}_\tau = e^{x - (r - \frac{1}{2}\sigma^2)\tau}$, $v_\tau = F_\tau e^{-r\tau}$. $\ddot{S}(0) = e^x$, $v(x,0) = v_0 = F_0 = F(x,0)$, $\ddot{V}_0 = v_0$.

$V(S,T) = \max(S(T) - K, 0)$

$\ddot{V}(\ddot{S},0) = \max(\ddot{S}(0) - K, 0)$

$v(x,0) = \max(e^x - K, 0)$

$F(x,0) = \max(e^x - K, 0)$. $F(\infty,\tau) = \left(e^{x - (r - \frac{1}{2}\sigma^2)\tau} - Ke^{-r\tau}\right)e^{r\tau}$, $F(0,\tau) = 0$.

6

Apply MWR to the Black-Scholes Equation (1D Diffusion).

$$\frac{\partial F}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2} = 0.$$

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}\right) N_i \, dx = 0. \qquad \tilde{F} = N_1^e F_1 + N_2^e F_2 \text{ (linear approximation).}$$

$$\int_{x_1}^{x_2} \left(\frac{\partial N_j F_j}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 N_j F_j}{\partial x^2}\right) N_i \, dx = 0. \qquad = \sum_{j=1}^{2} N_j^e F_j \qquad N_j = \{N\}, \; N_i = \langle N \rangle$$

(*) $$\underbrace{\int_{x_1}^{x_2} \frac{\partial F}{\partial \tau} N_i \, dx}_{A} - \frac{1}{2}\sigma^2 \underbrace{\int_{x_1}^{x_2} \frac{\partial^2 N_j}{\partial x^2} N_i \, dx}_{B} \, (F_j) = 0$$

For B: IBP: $u = N_i$, $du = \frac{dN_i}{dx} dx$, $dv = \frac{\partial^2 N_j}{\partial x^2} dx$, $v = \frac{\partial \tilde{N}_j}{\partial x}$

$$\int_{x_1}^{x_2} \frac{\partial^2 N_j}{\partial x^2} N_i \, dx = N_i \left(\frac{\partial \tilde{N}_j}{\partial x}\right)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x} dx$$

Sub B into (*)

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial \tau} N_i \, dx - \frac{1}{2}\sigma^2 \left(N_i \left(\frac{\partial \tilde{N}_j}{\partial x}\right)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x} dx \right) (F_j) = 0$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial \tau} N_i \, dx + \left(\frac{1}{2}\sigma^2 \int_{x_1}^{x_2} \frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x} dx\right)(F_j) = \frac{1}{2}\sigma^2 N_i \left(\frac{\partial \tilde{F}}{\partial x}\right)\Big|_{x_1}^{x_2}$$

(*)[2] $$\frac{\Delta x}{2}\left(\underbrace{\int_{-1}^{1} \frac{\partial F}{\partial \tau} N_i \, d\eta}_{A} + \left(\frac{1}{2}\sigma^2 \underbrace{\int_{-1}^{1} \frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x} d\eta}_{C}\right)(F_j)\right) = \frac{1}{2}\sigma^2 N_i \left(\frac{\partial \tilde{F}}{\partial x}\right)\Big|_{x_1}^{x_2}$$

Look at RHS $\quad \frac{1}{2}\sigma^2 \left(N_i \left(\frac{\partial \tilde{F}}{\partial x}\right)\Big|_{x_2} - N_i \left(\frac{\partial \tilde{F}}{\partial x}\right)\Big|_{x_1} \right)$

When $i=1$: $\frac{1}{2}\sigma^2 \left(\cancel{N_1(x_2) \left(\frac{\partial \tilde{F}}{\partial x}\right)\Big|_{x_2}} - N_1(\cancel{x_1}) \left(\frac{\partial \tilde{F}}{\partial x}\right)\Big|_{x_1} \right) = -\left(\frac{\partial \tilde{F}}{\partial x}\right)\Big|_{x_1}$

When $i=2$: $\frac{1}{2}\sigma^2 \left(N_2(x_2) \left(\frac{\partial \tilde{F}}{\partial x}\right)\Big|_{x_2} - \cancel{N_2(x_2) \left(\frac{\partial \tilde{F}}{\partial x}\right)\Big|_{x_1}} \right) = \left(\frac{\partial \tilde{F}}{\partial x}\right)\Big|_{x_2}$

$$\text{RHS} = \frac{1}{2}\sigma^2 \begin{bmatrix} -\left(\frac{\partial \tilde{F}}{\partial x}\right)\Big|_{x_1} \\ \left(\frac{\partial \tilde{F}}{\partial x}\right)\Big|_{x_2} \end{bmatrix}$$

For A: $$\int_{-1}^{1} \frac{\partial F}{\partial \tau} N_i \, d\eta = \frac{\partial F_j}{\partial \tau} \int_{-1}^{1} N_j N_i \, d\eta = \frac{\partial F_j}{\partial \tau}\left(\frac{1}{3}\right)\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

For C: $$\int_{-1}^{1} \frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x} d\eta = \left(\frac{1}{\Delta x}\right)^2 \int_{-1}^{1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\eta = \frac{2}{\Delta x^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\frac{\Delta x}{2}\left(\frac{\partial F_j}{\partial \tau}\left(\frac{1}{3}\right)\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \left(\frac{1}{2}\sigma^2\left(\frac{2}{\Delta x^2}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\{F\}\right) = \frac{1}{2}\sigma^2 \begin{bmatrix} -\left(\frac{\partial \tilde{F}}{\partial x}\right)\Big|_{x_1} \\ \left(\frac{\partial \tilde{F}}{\partial x}\right)\Big|_{x_2} \end{bmatrix}$$

$$\frac{\Delta x}{6}\frac{\partial F_j}{\partial \tau}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{\sigma^2}{2\Delta x}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\{F\} = \frac{1}{2}\sigma^2 \begin{bmatrix} -\frac{\partial \tilde{F}}{\partial x}\Big|_{x_1} \\ \frac{\partial \tilde{F}}{\partial x}\Big|_{x_2} \end{bmatrix}$$

$$\frac{\partial F_j}{\partial \tau} = \frac{6}{3\Delta x}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\left(\frac{1}{2}\sigma^2 \begin{bmatrix} -\frac{\partial \tilde{F}}{\partial x}\Big|_{x_1} \\ \frac{\partial \tilde{F}}{\partial x}\Big|_{x_2} \end{bmatrix} - \frac{\sigma^2}{2\Delta x}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\{F\}\right)$$

$$\frac{\partial F_j}{\partial \tau} = \frac{3\sigma^2}{3\Delta x}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -\frac{\partial \tilde{F}}{\partial x}\Big|_{x_1} \\ \frac{\partial \tilde{F}}{\partial x}\Big|_{x_2} \end{bmatrix} - \frac{3\sigma^2}{3\Delta x^2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\{F\} = \frac{\sigma^2}{\Delta x}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\overbrace{\begin{bmatrix} -\frac{\partial \tilde{F}}{\partial x}\Big|_{x_1} \\ \frac{\partial \tilde{F}}{\partial x}\Big|_{x_2} \end{bmatrix}}^{[\alpha]} - \overbrace{\frac{3\sigma^2}{\Delta x^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}^{[\beta]}\{F\}$$

Apply theta differencing

$$\frac{[F_j]^{\tau+\Delta\tau} - [F_j]^{\tau}}{\Delta\tau} = \theta\left\{\frac{\sigma^2}{\Delta x}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}[\alpha]^{\tau+\Delta\tau} - [\beta][F_j]^{\tau+\Delta\tau}\right\} + (1-\theta)\left\{\frac{\sigma^2}{\Delta x}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}[\alpha]^{\tau} - [\beta][F_j]^{\tau}\right\}$$

$$\left(\frac{1}{\Delta\tau} + \theta[\beta]\right)[F_j]^{\tau+\Delta\tau} = \left(\frac{1}{\Delta\tau} - (1-\theta)[\beta]\right)[F_j]^{\tau} + \frac{\sigma^2}{\Delta x}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\left(\theta\begin{bmatrix} -\frac{\partial \tilde{F}}{\partial x}\Big|_{x_1} \\ \frac{\partial \tilde{F}}{\partial x}\Big|_{x_2} \end{bmatrix}^{\tau+\Delta\tau} + (1-\theta)\begin{bmatrix} -\frac{\partial \tilde{F}}{\partial x}\Big|_{x_1} \\ \frac{\partial \tilde{F}}{\partial x}\Big|_{x_2} \end{bmatrix}^{\tau}\right)$$

$$\frac{\Delta x}{\sigma^2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1}\left(\frac{1}{\Delta\tau} + \theta[\beta]\right)[F_j]^{\tau+\Delta\tau} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1}\left(\frac{1}{\Delta\tau} - (1-\theta)[\beta]\right)[F_j]^{\tau}\frac{\Delta x}{\sigma^2} + \cancel{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}\cancel{\left(\theta\begin{bmatrix} -\frac{\partial \tilde{F}}{\partial x}\Big|_{x_1} \\ \frac{\partial \tilde{F}}{\partial x}\Big|_{x_2} \end{bmatrix}^{\tau+\Delta\tau} + (1-\theta)\begin{bmatrix} -\frac{\partial \tilde{F}}{\partial x}\Big|_{x_1} \\ \frac{\partial \tilde{F}}{\partial x}\Big|_{x_2} \end{bmatrix}^{\tau}\right)}$$

↳ Same logic as bottom of page 3. Note Δx is not constant, unlike ΔS on pages 1 to 3

Sanity Check with Assignment 3 – Chloride Concrete Problem.

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial \phi}{\partial x}\right) - \cancel{V\frac{\partial \phi}{\partial x}}, \quad V=0$$

$$\frac{\partial \phi}{\partial t} - D\frac{\partial \phi}{\partial x^2} = 0.$$

Apply GMWR

$$\int_{x_1}^{x_2}\left(\frac{\partial \phi}{\partial t} - D\frac{\partial^2 \phi}{\partial x^2}\right)N_i\,dx = 0 \qquad \tilde{\phi} = \phi_1 N_1 + \phi_2 N_2 = \sum_{j=1}^{2}\phi_j N_j$$

$$\int_{x_1}^{x_2} N_j N_i\,dx\,\frac{\partial \phi_j}{\partial t} - D\int_{x_1}^{x_2}\frac{\partial^2 N_j}{\partial x^2}N_i\,dx\,\phi_j = 0.$$

$$\int_{x_1}^{x_2} N_j N_i\,dx\,\frac{\partial \phi_j}{\partial t} - D\int_{x_1}^{x_2}\frac{\partial^2 N_j}{\partial x^2}N_i\,dx\,\phi_j\Big) = 0$$

$$\frac{\Delta x}{2}\left(\int_{-1}^{1} N_j N_i\,d\eta\,\frac{\partial \phi_j}{\partial t} + D\int_{-1}^{1}\frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x}\,d\eta\,\phi_j\right) = DN_i\left(\frac{\partial \tilde{N}_j}{\partial x}\right)\Big|_{x_1}^{x_2}$$

Apply Theta Differencing

$$\int_{-1}^{1} N_j N_i\,d\eta\left(\frac{\phi_j^{t+\Delta t} - \phi_j^{t}}{\Delta t}\right) + D\int_{-1}^{1}\frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x}\,d\eta\left(\theta\phi_j^{t+\Delta t} + (1-\theta)\phi_j^{t}\right) = \frac{2D}{\Delta x}N_i\left(\frac{\partial \tilde{\phi}}{\partial x}\right)\Big|_{x_1}^{x_2}$$

$$\left(\frac{\int_{-1}^{1} N_j N_i\,d\eta}{\Delta t} + \theta D\int_{-1}^{1}\frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x}\,d\eta\right)\phi_j^{t+\Delta t} = \frac{2D}{\Delta x}N_i\left(\frac{\partial \tilde{\phi}}{\partial x}\right)\Big|_{x_1}^{x_2} + \left(\frac{\int_{-1}^{1} N_j N_i\,d\eta}{\Delta t} - (1-\theta)D\int_{-1}^{1}\frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x}\,d\eta\right)\phi_j^{t}$$

$$\left(\frac{1}{3\Delta t}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} + \frac{2\theta D}{\Delta x^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right)\phi_j^{t+\Delta t} = \cancel{\frac{2D}{\Delta x}N_i\left(\frac{\partial \tilde{\phi}}{\partial x}\right)\Big|_{x_1}^{x_2}} + \left(\frac{1}{3\Delta t}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} - \frac{2(1-\theta)D}{\Delta x^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right)\phi_j^{t}$$

↑

Cancels out as seen in pages 6 and 3