Washkewicz College of Engineering

Path Integral Stochastic Optimal Control for Dynamic System: Swing-up Task

Mini-project for Stochastic System Class

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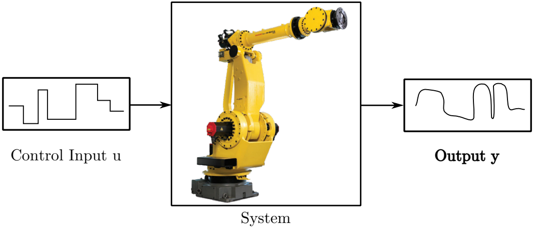
"Modeling is neither science nor mathematics; it is the craft that builds bridges between the two." (Morrison 2012)

Dynamics of Mechanical System

Input-Output of Robot Manipulator Dynamics.

- A requirement from an external signal (input)
- A process of the given information (system)
- A consequence of the process (output)

Dynamics of Mechanical System



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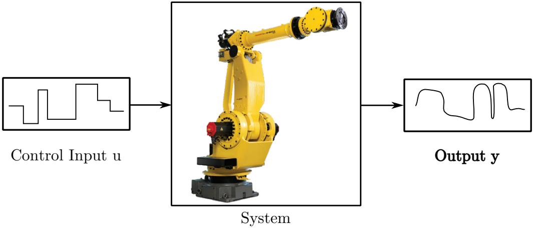

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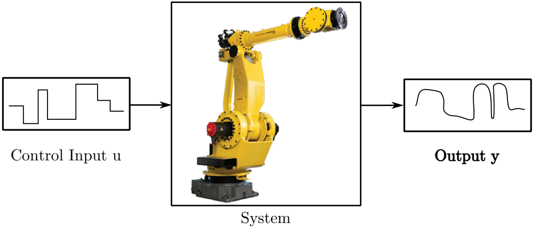

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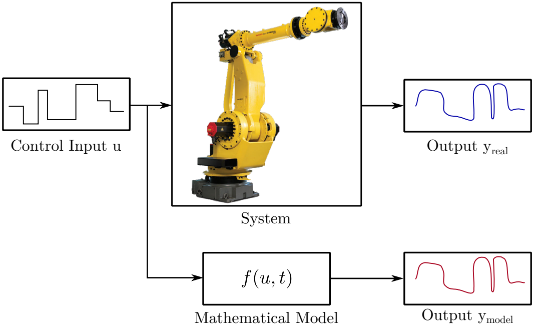

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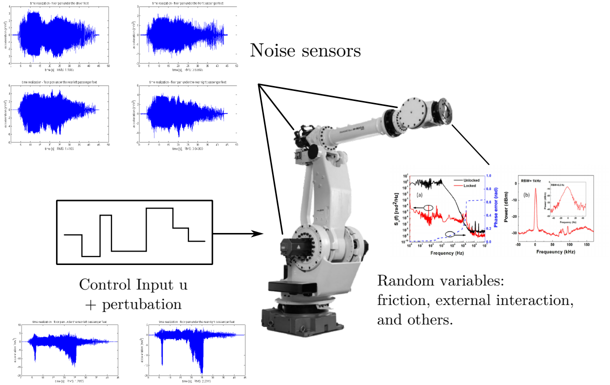


Stochastic Effects in Dynamics.

Deterministic vs. Stochastic System

Consider the continuous-time dynamic in the following:

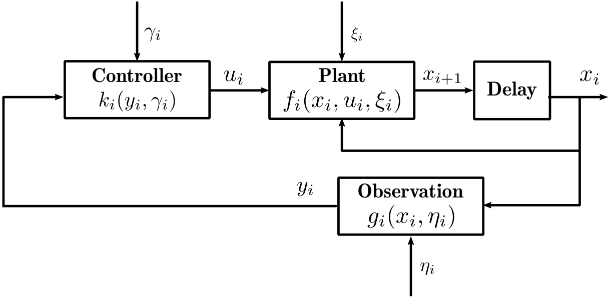
$$\dot{x} = f(t, x, u) \quad (1)$$

Digital implementations of control system and estimation requires a discrete-time dynamic system:

$$x_{k+1} = f\left(k, x_k, u_k\right), \quad t = 0, 1, \ldots, T \quad (2)$$

For a generic system, x_k is a *n*-dimensional state vector and u_k is a *m*-dimensional control vector at *k*.

General Stochastic Control System



Schematic diagram of general stochastic control system.

- η_k, γ_k, and ξ_k are *random noises*.
- $E(\gamma_i) = 0$ and $E(\gamma_i) = \delta_i^2 \quad 0 \leqslant i \leqslant N-1$
- $E(x_i \mid y_0, ...y_{i-1}) = \mu_i$

Stochastic Differential Equation

Particular stochastic differential equation using **Brownian motion**, also known as **Wiener Process**.

$$\mathrm{d}x = f[x(t), u(t), t]\,\mathrm{d}t + \mathrm{d}\xi \tag{3}$$

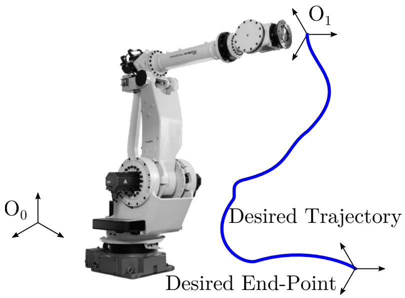
For instance, if the state variable in equation 3 is a single state, with means the value of v is a positive scalar, we have:

$$x_{t+1} = x_t + \xi, \quad \xi_t = \pm\sqrt{v} \tag{4}$$

Then, the conditional probability distribution of x at time t given initial condition is Gaussian and specified by its mean and variance.

$$\rho\left(x, t \mid x_0, 0\right) = \frac{1}{\sqrt{2\pi\nu t}} \exp\left(-\frac{\left(x - x_0\right)^2}{2\nu t}\right) \tag{5}$$

Stochastic Optimal Control



Optimization Goal.

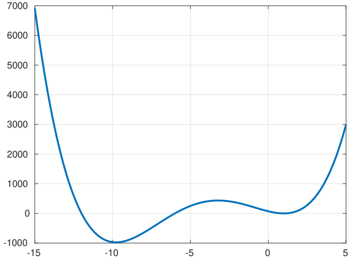
For instance, given a continuous-time dynamic:

$$\dot{x} = f(t, x, u)$$

where x is the states of the system. Find an optimal control information for the robot to follow a desired trajectory $\bar{x}$:

$$\begin{aligned} J(x,t) = \min_{u} \quad & (x - \bar{x})^2 \\ \text{s.t.} \quad & \dot{x} = f(x, u, t) \end{aligned}$$

Optimization in Math



Optimization Goal.

Given:

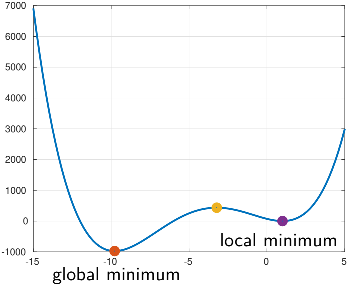
$$y = f(x) = (x+6)(x+12)(x-1)^2$$

Find:

$$J(x) = \min_x \quad y$$
$$\text{s.t.} \quad y = f(x)$$

- Critical point c: $\frac{\mathrm{d}f(x)}{\mathrm{d}x} = 0$
- $\frac{\mathrm{d}^2 f(c)}{\mathrm{d}x^2} > 0$ Local minimum
- $\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} < 0$ Local maximum

Optimization in Math



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Cost function for a stochastic Optimal control problem:

$$J(t,x) = \mathbb{E}_{\mathbb{Q}}\left[\phi\left(\mathrm{x}_T, T\right) + \int_{t_0}^{T} \mathrm{R}\left(\mathrm{x}_t, \mathrm{u}_t, t\right) \mathrm{d}t\right] \tag{6}$$

where the *end cost* $\phi\left(\mathrm{x}_T, T\right)$ gives the penalty to the last state configuration and integral term is the cost associate to the trajectories $x(0 \longmapsto T)$, also known as *optimal cost to go*.

Solution of *Hamilton-Jacobi-Bellman* (HJB):

$$-\partial_t J(t,x) = \min_u \left[\begin{matrix} R(t,x,u) + f(x,u,t)\partial_x J(x,t) \\ +\frac{1}{2}\nu(t,x,u)\partial_x^2 J(x,t)] \end{matrix}\right]$$

Path Integral Control

- The solution of functional integral solutions of second order elliptic and **parabolic partial differential equations** (PDEs) have been known for a long time.

- For stochastic methods are limited in scope because they are based on the **Wiener process**.

- A **path integral** solves a general class of linear second order partial differential equations with Dirichlet/Neumann B.C.

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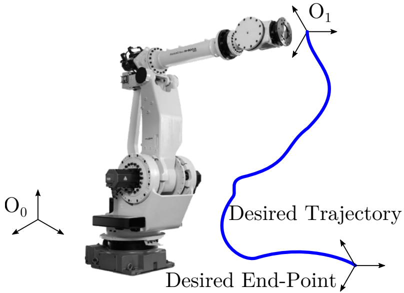


Optimization Goal.

- Probability distribution from the uncontrolled $\mathbb{P}$
- Probability distribution from the controlled $\mathbb{Q}$ dynamics
- Relative entropy $\mathbb{D}_{KL}\left(\mathbb{Q}^* \| \mathbb{Q}(\mathrm{u})\right)$

The optimization problem becomes:

$$\min_{\mathfrak{u}(\cdot)} \mathbb{D}_{KL}\left(\mathbb{Q}^* \| \mathbb{Q}(\mathbf{u})\right) \qquad (7)$$

Summary

Given stochastic discrete-time system:

$$\mathrm{d}x = \left[f\left(x_t, t\right) + G\left(x_t, t\right] u\left(x_t, t\right)\right) dt + B\left(x_t, t\right) \mathrm{d}w \tag{8}$$

Optimization Formulation:

$$\mathbf{u}^*(\cdot) = \underset{\mathfrak{u}(\cdot)}{\operatorname{argmin}}\, \mathbb{E}_{\mathbb{Q}}\left[\phi\left(\mathrm{x}_T, T\right) + \int_{t_0}^{T} q\left(\mathrm{x}_t, t\right) + \frac{1}{2}\mathbf{u}_t^{\mathrm{T}}\mathbf{R}\left(\mathrm{x}_t, t\right)\mathbf{u}_t \mathrm{d}t\right]$$

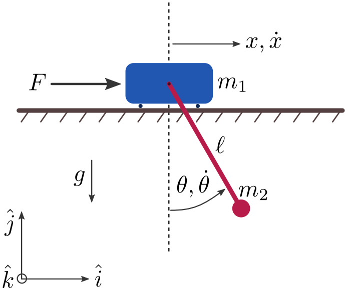
Optimal Control:

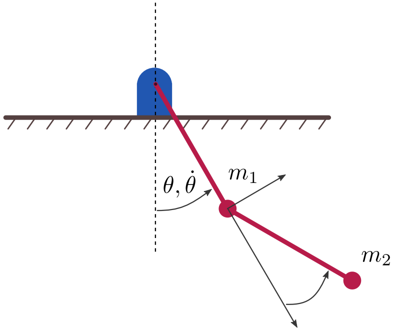
$$\mathrm{u}_j^* = \frac{1}{\Delta t}\mathrm{E}_{\mathbb{P}}\left[\frac{\exp\left(-\frac{1}{\lambda}S(\tau)\right)\mathcal{H}\left(\mathrm{x}_{t_j}, t_j\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{\lambda}S(\tau)\right)\right]}\right]^{-1} \mathbb{E}_{\mathbb{P}}\left[\frac{\exp\left(-\frac{1}{\lambda}S(\tau)\right)\mathcal{G}\left(\mathrm{x}_{t_j}, t_j\right)\epsilon_j\sqrt{\Delta t}}{\mathrm{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{\lambda}S(\tau)\right)\right]}\right]$$

Examples

Applied *Path Integral Stochastic Optimal Control* for the following:

- Inverted pendulum on a cart;
- Planar two-link robotic arm.

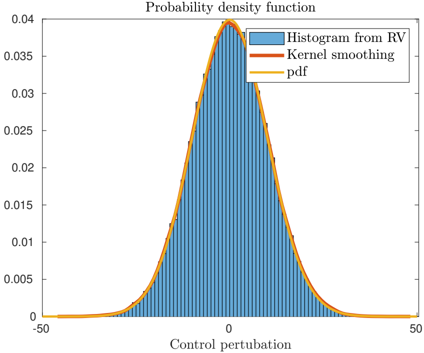




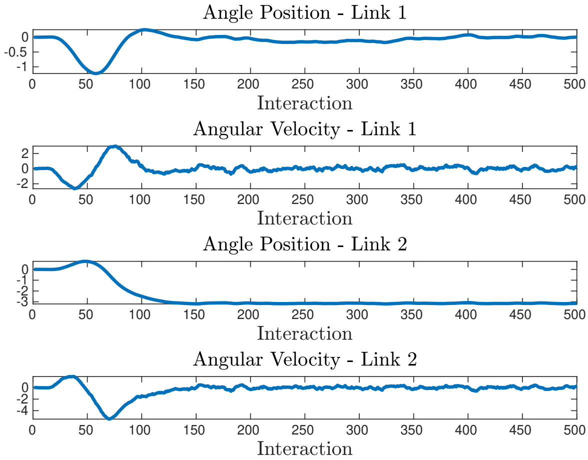
Objective function: Find an optimal control solution to drive the states into a vertical configuration then balance.

Both applications contain perturbation γ_i in the control input as:

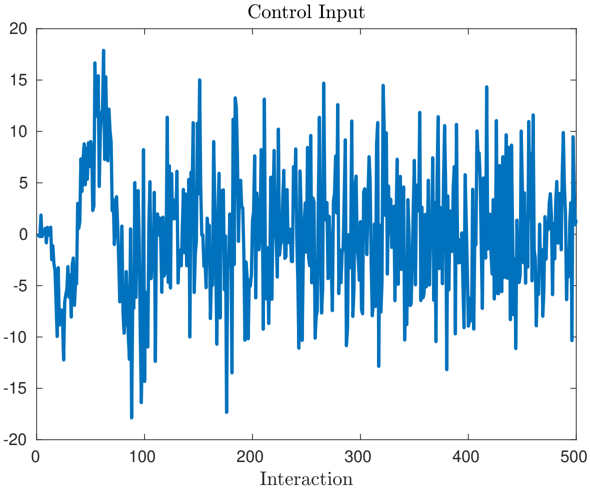
$$\begin{aligned} E(\gamma_i) &= 0 \\ E(\gamma_i) &= \delta_i^2 \quad 0 \leqslant i \leqslant N-1 \end{aligned} \tag{9}$$



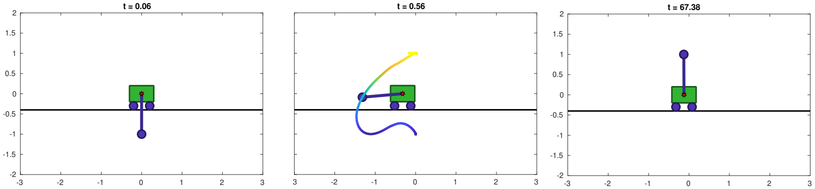
State response for Inverted pendulum on a cart model:



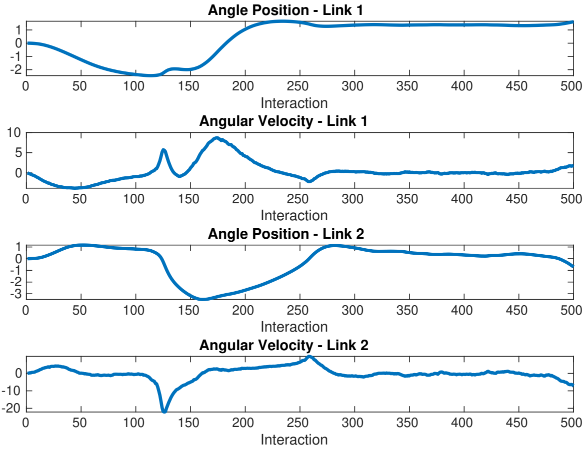
Control Input for Inverted pendulum on a cart model:



Optimal trajectory for the Inverted Pendulum System on a cart.



State response for Planar two-link robotic arm:



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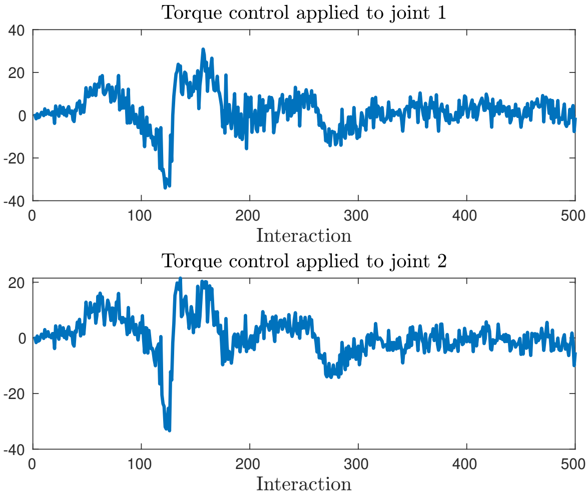

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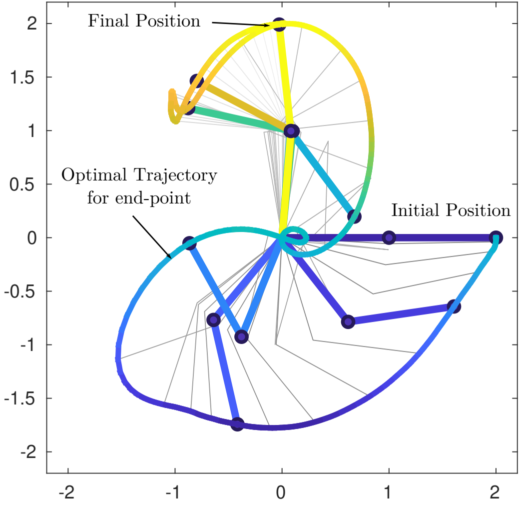

Thank you!