Kurt Marti

Stochastic Optimization Methods

Applications in Engineering and Operations Research

Third Edition

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Preface

Optimization problems in practice depend mostly on several model parameters, noise factors, uncontrollable parameters, etc., which are not given fixed quantities at the planning stage. Typical examples from engineering and economics/operations research are: material parameters (e.g., elasticity moduli, yield stresses, allowable stresses, moment capacities, specific gravity), external loadings, friction coefficients, moments of inertia, length of links, mass of links, location of center of gravity of links, manufacturing errors, tolerances, noise terms, demand parameters, technological coefficients in input–output functions, cost factors, interest rates, exchange rates, etc. Due to the several types of stochastic uncertainties (physical uncertainty, economic uncertainty, statistical uncertainty, model uncertainty), these parameters must be modeled by random variables having a certain probability distribution. In most cases, at least certain moments of this distribution are known.

In order to cope with these uncertainties, a basic procedure in the engineering/economic practice is to replace first the unknown parameters by some chosen nominal values, e.g., estimates, guesses, of the parameters. Then, the resulting and mostly increasing deviation of the performance (output, behavior) of the structure/system from the prescribed performance (output, behavior), i.e., the "tracking error", is compensated by (online) input corrections. However, the online correction of a system/structure is often time-consuming and causes mostly increasing expenses (correction or recourse costs). Very large recourse costs may arise in case of damages or failures of the plant. This can be omitted to a large extent by taking into account already at the planning stage the possible consequences of the tracking errors and the known prior and sample information about the random data of the problem. Hence, instead of relying on ordinary deterministic parameter optimization methods—based on some nominal parameter values—and applying then just some correction actions, stochastic optimization methods should be applied: Incorporating stochastic parameter variations into the optimization process, expensive and increasing online correction expenses can be omitted or at least reduced to a large extent.

Consequently, for the computation of robust optimal decisions/designs, i.e., optimal decisions which are insensitive with respect to random parameter variations, appropriate deterministic substitute problems must be formulated first. Based on the decision theoretical principles, these substitute problems depend on probabilities of failure/success and/or on more general expected cost/loss terms. Since probabilities and expectations are defined by multiple integrals in general, the resulting often nonlinear and also nonconvex deterministic substitute problems can be solved by approximative methods only. Two basic types of deterministic substitute problems occur mostly in practice:

- Expected primary cost minimization subject to expected recourse (correction) cost constraints

  Minimization of the expected primary costs subject to expected recourse cost constraints (reliability constraints) and remaining deterministic constraints, e.g., box constraints. In case of piecewise constant cost functions, probabilistic objective functions and/or probabilistic constraints occur;
- Expected Total Cost Minimization Problems:

  Minimization of the expected total costs (costs of construction, design, recourse costs, etc.) subject to the remaining deterministic constraints.

The main analytical properties of the substitute problems have been examined in the first two editions of the book, where also appropriate deterministic and stochastic solution procedures can be found.

After an overview on basic methods for handling optimization problems with random data, in this present third edition transformation methods for optimization problems with random parameters into appropriate deterministic substitute problems are described for basic technical and economic problems. Hence, the aim of this present third edition is to provide now analytical and numerical tools—including their mathematical foundations—for the approximate computation of robust optimal decisions/designs/control as needed in concrete engineering/economic applications: stochastic Hamiltonian method for optimal control problems with random data, the H-minimal control and the Hamiltonian two-point boundary value problem, Stochastic optimal open-loop feedback control as an efficient method for the construction of optimal feedback controls in case of random parameters, adaptive optimal stochastic trajectory planning and control for dynamic control systems under stochastic uncertainty, optimal design of regulators in case of random parameters, optimal design of structures under random external load and with random model parameters.

Finally, a new tool is presented for the evaluation of the entropy of a probability distribution and the divergence of two probability distributions with respect to the use in an optimal decision problem with random parameters. Applications to statistics are given.

Realizing this monograph, the author was supported by several collaborators from the Institute for Mathematics and Computer Sciences. Especially, I would like to thank Ms Elisabeth Lößl for the excellent support in the LaTeX-typesetting of the manuscript. Without her very precise and careful work, the completion of this project could not have been realized. Moreover, I thank Ms Ina Stein for providing several figures. Last but not least, I am indebted to Springer-Verlag for inclusion

of the book into the Springer-program. I would like to thank especially the Senior Editor for Business/Economics of Springer-Verlag, Christian Rauscher, for his very long patience until the completion of this book.

Munich, Germany Kurt Marti
October 2014

Preface to the First Edition

Optimization problems in practice depend mostly on several model parameters, noise factors, uncontrollable parameters, etc., which are not given fixed quantities at the planning stage. Typical examples from engineering and economics/operations research are: material parameters (e.g., elasticity moduli, yield stresses, allowable stresses, moment capacities, specific gravity), external loadings, friction coefficients, moments of inertia, length of links, mass of links, location of center of gravity of links, manufacturing errors, tolerances, noise terms, demand parameters, technological coefficients in input–output functions, cost factors, etc. Due to the several types of stochastic uncertainties (physical uncertainty, economic uncertainty, statistical uncertainty, model uncertainty), these parameters must be modeled by random variables having a certain probability distribution. In most cases, at least certain moments of this distribution are known.

In order to cope with these uncertainties, a basic procedure in the engineering/economic practice is to replace first the unknown parameters by some chosen nominal values, e.g., estimates, guesses, of the parameters. Then, the resulting and mostly increasing deviation of the performance (output, behavior) of the structure/system from the prescribed performance (output, behavior), i.e., the "tracking error," is compensated by (online) input corrections. However, the online correction of a system/structure is often time-consuming and causes mostly increasing expenses (correction or recourse costs). Very large recourse costs may arise in case of damages or failures of the plant. This can be omitted to a large extent by taking into account already at the planning stage the possible consequences of the tracking errors and the known prior and sample information about the random data of the problem. Hence, instead of relying on ordinary deterministic parameter optimization methods—based on some nominal parameter values—and applying then just some correction actions, stochastic optimization methods should be applied: incorporating the stochastic parameter variations into the optimization process, expensive and increasing online correction expenses can be omitted or at least reduced to a large extent.

Consequently, for the computation of robust optimal decisions/designs, i.e., optimal decisions which are insensitive with respect to random parameter variations, appropriate deterministic substitute problems must be formulated first. Based on the decision theoretical principles, these substitute problems depend on probabilities of failure/success and/or on more general expected cost/loss terms. Since probabilities and expectations are defined by multiple integrals in general, the resulting often nonlinear and also nonconvex deterministic substitute problems can be solved by approximative methods only. Hence, the analytical properties of the substitute problems are examined, and appropriate deterministic and stochastic solution procedures are presented. The aim of this present book is therefore to provide analytical and numerical tools—including their mathematical foundations—for the approximate computation of robust optimal decisions/designs as needed in concrete engineering/economic applications.

Two basic types of deterministic substitute problems occur mostly in practice:

- Reliability-Based Optimization Problems:
  Minimization of the expected primary costs subject to expected recourse cost constraints (reliability constraints) and remaining deterministic constraints, e.g., box constraints. In case of piecewise constant cost functions, probabilistic objective functions and/or probabilistic constraints occur;
- Expected Total Cost Minimization Problems:
  Minimization of the expected total costs (costs of construction, design, recourse costs, etc.) subject to the remaining deterministic constraints.

Basic methods and tools for the construction of appropriate deterministic substitute problems of different complexity and for various practical applications are presented in Chap. 1 and part of Chap. 2 together with fundamental analytical properties of reliability-based optimization/expected cost minimization problems. Here, a main problem is the analytical treatment and numerical computation of the occurring expectations and probabilities depending on certain input variables.

For this purpose deterministic and stochastic approximation methods are provided which are based on Taylor expansion methods, regression and response surface methods (RSM), probability inequalities, differentiation formulas for probability and expectation value functions. Moreover, first order reliability methods (FORM), Laplace expansions of integrals, convex approximation/deterministic descent directions/efficient points, (hybrid) stochastic gradient methods, stochastic approximation procedures are considered. The mathematical properties of the approximative problems and iterative solution procedures are described.

After the presentation of the basic approximation techniques in Chap. 2, in Chap. 3 derivatives of probability and mean value functions are obtained by the following methods: transformation method, stochastic completion and transformation, orthogonal function series expansion. Chapter 4 shows how nonconvex deterministic substitute problems can be approximated by convex mean value minimization problems. Depending on the type of loss functions and the parameter distribution, feasible descent directions can be constructed at non-efficient points. Here, efficient points, determined often by linear/quadratic conditions involving first and second order moments of the parameters, are candidates for optimal solutions to be obtained with much larger effort. The numerical/iterative solution techniques presented in Chap. 5 are based on hybrid stochastic approximation, hence, methods using not only simple stochastic (sub)gradients, but also deterministic descent directions and/or more exact gradient estimators. More exact gradient estimators based on Response Surface Methods (RSM) are described in the first part of Chap. 5. The convergence in the mean square sense is considered, and results on the speed up of the convergence by using improved step directions at certain iteration points are given. Various extensions of hybrid stochastic approximation methods are treated in Chap. 6 on stochastic approximation methods with changing variance of the estimation error. Here, second order search directions, based, e.g., on the approximation of the Hessian, and asymptotic optimal scalar and matrix valued step sizes are constructed. Moreover, using different types of probabilistic convergence concepts, related convergence properties and convergence rates are given. Mathematical definitions and theorems needed here are given in the Appendix. Applications to the approximate computation of survival/failure probabilities for technical and economical systems/structures are given in the last Chap. 7. The reader of this book needs some basic knowledge in linear algebra, multivariate analysis and stochastics.

Realizing this monograph, the author was supported by several collaborators: First of all I would like to thank Ms Elisabeth Lößl from the Institute for Mathematics and Computer Sciences for the excellent LaTeXtypesetting of the whole manuscript. Without her very precise and careful work, this project could not have been realized. I also owe thanks to my former collaborators, Dr. Andreas Aurnhammer, for further LaTeXsupport and for providing English translation of some German parts of the original manuscript, and to Thomas Platzer for proof reading of some parts of the book. Last but not least, I am indebted to Springer-Verlag for inclusion of the book into the Springer-program. I would like to thank especially the Publishing Director Economics and Management Science of Springer-Verlag, Dr. Werner A. Müller, for his really very long patience until the completion of this book.

Munich, Germany Kurt Marti
August 2004

Preface to the Second Edition

The major change in the second edition of this book is the inclusion of a new, extended version of Chap. 7 on the approximate computation of probabilities of survival and failure of technical, economic structures and systems. Based on a representation of the state function of the structure/system by the minimum value of a certain optimization problem, approximations of the probability of survival/failure are obtained by means of polyhedral approximation of the so-called survival domain, certain probability inequalities and discretizations of the underlying probability distribution of the random model parameters. All the other chapters have been updated by including some more recent material and by correcting some typing errors.

The author thanks again Ms. Elisabeth Lößl for the excellent LaTeX-typesetting of all revisions and completions of the text. Moreover, the author thanks Dr. Müller, Vice President Business/Economics & Statistics, Springer-Verlag, for the possibility to publish a second edition of "Stochastic Optimization Methods."

Munich, Germany
March 2008

Kurt Marti

Outline of the 3rd Edition

A short overview on stochastic optimization methods including the derivation of appropriate deterministic substitute problems is given in *Chap. 1*.

Basic solution techniques for optimal control problems under stochastic uncertainty are presented in *Chap. 2*: optimal control problems as arising in different technical (mechanical, electrical, thermodynamic, chemical, etc.) plants and economic systems are modeled mathematically by a system of first order nonlinear differential equations for the plant state vector $z = z(t)$ involving, e.g., displacements, stresses, voltages, currents, pressures, concentration of chemicals, demands, etc. This system of differential equations depends on the vector $u(t)$ of input or control variables and a vector $a = a(\omega)$ of certain random model parameters. Moreover, also the vector z_0 of initial values of the plant state vector $z = z(t)$ at the initial time $t = t_0$ may be subject to random variations. While the actual realizations of the random parameters and initial values are not known at the planning stage, we may assume that the probability distribution or at least the occurring moments, such as expectations and variances, are known. Moreover, we suppose that the costs along the trajectory and the terminal costs G are convex functions with respect to the pair (u, z) of control and state variables u, z, the final state $z(t_f)$, respectively. The problem is then to determine an open-loop, closed-loop, or an intermediate open-loop feedback control law minimizing the expected total costs consisting of the sum of the costs along the trajectory and the terminal costs. For the computation of stochastic optimal open-loop controls at each starting time point t_b, the stochastic Hamilton function of the control problem is introduced first. Then, a H-minimal control can be determined by solving a finite-dimensional stochastic optimization problem for minimizing the conditional expectation of the stochastic Hamiltonian subject to the remaining deterministic control constraints at each time point t. Having a H-minimal control, the related Hamiltoninan two-point boundary value problem with random parameters is formulated for the computation of the stochastic optimal state and adjoint state trajectory. In case of a linear-quadratic control problem, the state and adjoint state trajectory can be determined analytically to a large extent. Inserting then these trajectories into the H-minimal control, stochastic optimal open-loop controls are found. For approximate solutions of the stochastic two-point boundary problem, cf. [112].

Stochastic optimal regulators or optimal feedback control under stochastic uncertainty can be obtained in general by approximative methods only. In *Chap. 3*, stochastic optimal feedback controls are determined by means of the fundamental *stochastic open-loop feedback method*. This very efficient approximation method is also the basis of *model predictive control* procedures. Using the methods described in Chap. 2, *stochastic optimal open-loop feedback controls* are constructed by computing next to stochastic optimal open-loop controls on the *remaining time intervals* $t_b \leq t \leq t_f$ with $t_0 \leq t_b \leq t_f$. Having a stochastic optimal open-loop feedback control on each remaining time interval $t_b \leq t \leq t_f$ with $t_0 \leq t_b \leq t_f$, a stochastic optimal open-loop feedback control law follows then immediately evaluating each of the stochastic optimal open-loop controls on $t_b \leq t \leq t_f$ at the corresponding initial time point $t = t_b$. The efficiency of this method has been proved already by applications to the stochastic optimization of regulators for robots.

Adaptive Optimal Stochastic Trajectory Planning and Control (*AOSTPC*) are considered in *Chap. 4*: in optimal control of dynamic systems, the standard procedure is to determine first off-line an optimal open-loop control, using some nominal or estimated values of the model parameters, and to correct then the resulting deviation of the actual trajectory or system performance from the prescribed trajectory (prescribed system performance) by on-line measurement and control actions. However, on-line measurement and control actions are very expensive and time-consuming. By adaptive optimal stochastic trajectory planning and control (AOSTPC), based on stochastic optimization methods, the available a priori and statistical information about the unknown model parameters is incorporating into the optimal control design. Consequently, the mean absolute deviation between the actual and prescribed trajectory can be reduced considerably, and robust controls are obtained. Using only some necessary stability conditions, by means of stochastic optimization methods also sufficient stability properties of the corresponding feedforward, feedback (PD-, PID-)controls, resp., are obtained. Moreover, analytical estimates are given for the reduction of the tracking error, hence, for the reduction of the on-line correction expenses by applying (AOSTPC).

Special methods for the optimal design of regulators are described in *Chap. 5*: the optimal design of regulators is often based on given, fixed nominal values of initial conditions, external loads, and other model parameters. However, due to the variations of material properties, measurement and observational errors, modeling errors, uncertainty of the working environment, the task to be executed, etc., the true external loadings, initial conditions, and model parameters are not known exactly in practice. Hence, a predetermined optimal regulator should be "*robust*," i.e., the regulator should guarantee satisfying results also in case of observational errors and variations of the different parameters occurring in the model. Robust regulators have been considered mainly for uncertainty models based on given fixed sets of parameters, especially certain multiple intervals containing the unknown, true parameter. However, since in many cases observational errors and parameter uncertainties can be described more adequately by means of stochastic models, in this chapter we suppose that the parameters involved in the regulator design problem are realizations of random variables having a known joint probability distribution. For the consideration of stochastic parameter variations within an optimal design process one has to introduce—as for any other optimization problem under stochastic uncertainty—an appropriate deterministic substitute problem. Based on the (optimal) reference trajectory and the related feedforward control, for the computation of stochastic optimal regulators, deterministic substitute control problems of the following type are considered: minimize the expected total costs arising from the deviation between the (stochastic optimal) reference trajectory and the effective trajectory of the dynamic system and the costs for the control correction subject to the following constraints: (i) dynamic equation of the stochastic system with the total control input being the sum of the feedforward control and the control correction, (ii) stochastic initial conditions, and (iii) additional conditions for the feedback law. The main problem is then the computation of the (conditional) expectation arising in the objective function. The occurring expectations are evaluated by Taylor expansions with respect to the stochastic parameter vector at its conditional mean. Applying *stochastic optimization methods*, the regulator optimization under stochastic uncertainty problem is converted into a deterministic optimal control problem involving the gain matrices as the unknown matrix parameters to be determined. This method has been validated already for the stochastic optimal design of regulators for industrial robots.

Stochastic optimization methods for the optimization of mechanical structures with random parameters are considered in *Chap. 6*: in the optimal design of mechanical structures, one has to minimize the weight, volume, or a more general cost function under yield, strength, or more general safety constraints. Moreover, for the design variables, e.g., structural dimensions, sizing variables, there are always some additional constraints, like box constraints. A basic problem is that the material parameters of the structure, manufacturing and modeling errors, as well as the external loads are not given, fixed quantities, but random variables having a certain probability distribution. In order to omit—as far as possible—very expensive observation and repair actions, or even severe damages, hence, in order to get robust designs with respect to random parameter variations, the always available information (e.g., certain moments) about the random variations of the unknown technical parameters should be taken into account already at the project definition stage. Hence, the original structural optimization problem with random parameters has to be replaced by an appropriate deterministic substitute problem. Starting from the linear yield, strength, or safety conditions and the linear equilibrium equation involving the member forces and bending moments, the problem can be described in the framework of a two-stage stochastic linear program (SLP) "with complete fixed recourse." For the evaluation of the violation of the yield, strength or safety condition, different types of linear and sublinear cost models are introduced by taking into account, e.g., the displacements of the elements of the structure caused by the member forces or bending moments, or by considering certain strength or safety "reserves" of the structure. The main properties of the resulting substitute problems are discussed, e.g., (i) the convexity of the problems and (ii) the "dual block angular" structure of the resulting (large-scale) deterministic linear program in case of discrete probability distributions. In this case, several special purpose LP-solvers are available. In case of a continuous parameter distribution, the resulting substitute problems can be solved approximatively by means of a discretization of the underlying probability distribution and using then the SLP-method as described above.

Structural Analysis and Optimal Structural Design under Stochastic Uncertainty using Quadratic Cost Functions are treated in *Chap. 7*: problems from plastic analysis and optimal plastic design are based on the convex, linear, or linearized yield/strength condition and the linear equilibrium equation for the stress (state) vector. In practice, one has to take into account stochastic variations of the vector $a = a(\omega)$ of model parameters (e.g., yield stresses, plastic capacities, external load factors, cost factors, etc.). Hence, in order to get robust optimal load factors x, robust optimal designs x, resp., the basic plastic analysis or optimal plastic design problem with random parameters has to be replaced by an appropriate deterministic substitute problem. As a basic tool in the analysis and optimal design of mechanical structures under uncertainty, a *state function* $s^* = s^*(a, x)$ of the underlying structure is introduced. The survival of the structure can be described then by the condition $s^* \leq 0$. Interpreting the state function s^* as a cost function, several relations s^* to other cost functions, especially quadratic cost functions, are derived. Bounds for the probability of survival p_s are obtained then by means of the Tschebyscheff inequality. In order to obtain robust optimal decisions x^*, i.e., maximum load factors, optimal designs insensitive with respect to variations of the model parameters $a = a(\omega)$, a direct approach is presented then based on the primary costs (weight, volume, costs of construction, costs for missing carrying capacity, etc.) and the recourse costs (e.g., costs for repair, compensation for weakness within the structure, damage, failure, etc.), where the abovementioned quadratic cost criterion is used. The minimum recourse costs can be determined then by solving an optimization problem having a quadratic objective function and linear constraints. For each vector $a = a(\omega)$ of model parameters and each design vector x, one obtains then an explicit representation of the "best" internal load distribution F^*. Moreover, also the expected recourse costs can be determined explicitly. The expected recourse function may be represented by means of a "generalized stiffness matrix." Hence, corresponding to an elastic approach, the expected recourse function can be interpreted here as a "generalized expected compliance function", which depends on a generalized "stiffness matrix." Based on the minimization of the expected primary costs subject to constraints for the expected recourse costs ("generalized compliance") or the minimization of the expected total primary and recourse costs, explicit finite dimensional parameter optimization problems are achieved for finding robust optimal design x^* or a maximal load factor x^*. The analytical properties of the resulting programming problem are discussed, and applications, such as limit load/shakedown analysis, are considered. Furthermore, based on the expected "compliance function," explicit upper and lower bounds for the probability p_s of survival can be derived using linearization methods.

Finally, in *Chap.* 8 the inference and decision strategies applied in stochastic optimization methods are considered in more detail:

A large number of optimization problems arising in engineering, control, and economics can be described by the minimization of a certain (cost) function $v = v(a, x)$ depending on a random parameter vector $a = a(\omega)$ and a decision vector $x \in D$ lying in a given set D of feasible decision, design, or control variables. Hence, in order to get *robust optimal decisions*, i.e., optimal decisions being most insensitive with respect to variations of the random parameter vector $a = a(\omega)$, the original optimization problem is replaced by the deterministic substitute problem which consists in the minimization of the expected objective function $\mathbf{E}v = \mathbf{E}v(a(\omega), x)$ subject to $x \in D$. Since the true probability distribution λ of $a = a(\omega)$ is not exactly known in practice, one has to replace λ by a certain estimate or guess β. Consequently, one has the following *inference and decision problem*:

- *inference/estimation step*
  Determine an estimation β of the true probability distribution λ of $a = a(\omega)$,
- *decision step*
  determine an optimal solution x^* of $\mathit{min} \int v(a(\omega), x)\beta(d\omega)$ s.t. $x \in D$.

Computing approximation, estimation β of λ, the criterion for judging an approximation β of λ should be based on its utility for the decision-making process, i.e., one should weight the approximation error according to its influence on decision errors, and the decision errors should be weighted in turn according to the loss caused by an incorrect decision.

Based on inferential ideas developed among others by Kerridge, Kullback, in this chapter generalized decision-oriented inaccuracy and divergence functions for probability distributions λ , β are developed, taking into account that the outcome β of the inferential stage is used in a subsequent (ultimate) decision-making problem modeled by the above-mentioned stochastic optimization problem. In addition, *stability properties* of the inference and decision process

$$\lambda \longrightarrow \beta \longrightarrow x \in D_\epsilon(\beta),$$

are studied, where $D_\epsilon(\beta)$ denotes the set of ϵ-optimal decisions with respect to probability distribution $P_{a(\cdot)} = \beta$ of the random parameter vector $a = a(\omega)$.

Contents

Chapter 1
Stochastic Optimization Methods

1.1 Introduction

Many concrete problems from engineering, economics, operations research, etc., can be formulated by an optimization problem of the type

$$\min f_0(a,x) \tag{1.1a}$$

s.t.

$$f_i(a,x) \le 0, i = 1,\ldots,m_f \tag{1.1b}$$

$$g_i(a,x) = 0, i = 1,\ldots,m_g \tag{1.1c}$$

$$x \in D_0. \tag{1.1d}$$

Here, the objective (goal) function $f_0 = f_0(a,x)$ and the constraint functions $f_i = f_i(a,x), i = 1,\ldots,m_f$ and $g_i = g_i(a,x), i = 1,\ldots,m_g$, defined on a joint subset of $\mathbb{R}^\nu \times \mathbb{R}^r$, depend on a decision, design, control or input vector $x = (x_1,x_2,\ldots,x_r)^T$ and a vector $a = (a_1,a_2,\ldots,a_\nu)^T$ of model parameters. Typical model parameters in technical applications, operations research and economics are material parameters, external load parameters, cost factors, technological parameters in input–output operators, demand factors. Furthermore, manufacturing and modeling errors, disturbances or noise factors, etc., may occur. Frequent decision, control or input variables are material, topological, geometrical and cross-sectional design variables in structural optimization [77], forces and moments in optimal control of dynamic systems and factors of production in operations research and economic design.

The objective function (1.1a) to be optimized describes the aim, the goal of the modeled optimal decision/design problem or the performance of a technical, economic system or process to be controlled optimally. Furthermore, the constraints

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(1.1b–d) represent the operating conditions guaranteeing a safe structure, a correct functioning of the underlying system, process, etc. Note that the constraint (1.1d) with a given, fixed convex subset $D_0 \subset \mathbb{R}^r$ summarizes all (deterministic) constraints being independent of unknown model parameters a, as e.g. box constraints:

$$x^L \leq x \leq x^U \tag{1.1d'}$$

with given bounds x^L, x^U.

Important concrete optimization problems, which may be formulated, at least approximatively, this way are problems from optimal design of mechanical structures and structural systems [7, 77, 147, 160], adaptive trajectory planning for robots [9, 11, 32, 93, 125, 148], adaptive control of dynamic system [150, 159], optimal design of economic systems, production planning, manufacturing [86, 129] and sequential decision processes [106] , etc.

In *optimal control*, cf. Chap. 2, the input vector $x := u(\cdot)$ is interpreted as a function, a *control or input function* $u = u(t), t_0 \leq t \leq t_f$, on a certain given time interval $[t_0, t_f]$. Moreover, see Chap. 2, the objective function $f_0 = f_0(a, u(\cdot))$ is defined by a certain integral over the time interval $[t_0, t_f]$. In addition, the constraint functions $f_j = f_j(a, u(\cdot))$ are defined by integrals over $[t_0, t_f]$, or $f_j = f_j(t, a, u(t))$ may be functions of time t and the control input $u(t)$ at time t.

A basic problem in practice is that the vector of model parameters $a = (a_1, \ldots, a_\nu)^T$ is not a given, fixed quantity. Model parameters are often unknown, only partly known and/or may vary randomly to some extent.

Several techniques have been developed in the recent years in order to cope with uncertainty with respect to model parameters a. A well known basic method, often used in engineering practice, is the following two-step procedure [11, 32, 125, 148, 150]:

I) *Parameter Estimation and Approximation*: Replace first the ν-vector a of the unknown or stochastic varying model parameters $a_1, \ldots, a_\nu$ by some estimated/chosen fixed vector a_0 of so-called *nominal* values $a_{0l}, l = 1, \ldots, \nu$. Apply then an optimal decision (control) $x^* = x^*(a_0)$ with respect to the resulting approximate optimization problem

$$\min f_0(a_0, x) \tag{1.2a}$$

s.t.

$$f_i(a_0, x) \leq 0, i = 1, \ldots, m_f \tag{1.2b}$$

$$g_i(a_0, x) = 0, i = 1, \ldots, m_g \tag{1.2c}$$

$$x \in D_0. \tag{1.2d}$$

Due to the deviation of the actual parameter vector a from the nominal vector a_0 of model parameters, deviations of the actual state, trajectory or performance of the system from the prescribed state, trajectory, goal values occur.

II) *Compensation or Correction*: The deviation of the actual state, trajectory or performance of the system from the prescribed values/functions is compensated then by online measurement and correction actions (decisions or controls). Consequently, in general, increasing measurement and correction expenses result in course of time.

Considerable improvements of this standard procedure can be obtained by taking into account already at the planning stage, i.e., offline, the mostly available a priori (e.g. the type of random variability) and sample information about the parameter vector a. Indeed, based e.g. on some structural insight, or by parameter identification methods, regression techniques, calibration methods, etc., in most cases information about the vector a of model parameters can be extracted. Repeating this information gathering procedure at some later time points $t_j > t_0$ (= initial time point), $j = 1, 2, \ldots,$ adaptive decision/control procedures occur [106].

Based on the inherent random nature of the parameter vector a, the observation or measurement mechanism, resp., or adopting a Bayesian approach concerning unknown parameter values [14], here we make the following basic assumption:

Stochastic (Probabilistic) Uncertainty: The unknown parameter vector a is a realization

$$a = a(\omega), \omega \in \Omega, \tag{1.3}$$

of a random ν-vector $a(\omega)$ on a certain probability space $(\Omega, \mathcal{A}_0, \mathcal{P})$, where the probability distribution $\mathcal{P}_{a(\cdot)}$ of $a(\omega)$ is known, or it is known that $\mathcal{P}_{a(\cdot)}$ lies within a given range W of probability measures on $\mathbb{R}^\nu$. Using a Bayesian approach, the probability distribution $\mathcal{P}_{a(\cdot)}$ of $a(\omega)$ may also describe the subjective or personal probability of the decision maker, the designer.

Hence, in order to take into account the stochastic variations of the parameter vector a, to incorporate the a priori and/or sample information about the unknown vector a, resp., the standard approach "insert a certain nominal parameter vector a_0, and correct then the resulting error" must be replaced by a more appropriate deterministic substitute problem for the basic optimization problem (1.1a–d) under stochastic uncertainty.

1.2 Deterministic Substitute Problems: Basic Formulation

The proper selection of a deterministic substitute problem is a decision theoretical task, see [90]. Hence, for (1.1a–d) we have first to consider the *outcome map*

$$e = e(a, x) := \Big(f_0(a, x), f_1(a, x), \ldots, f_{m_f}(a, x), g_1(a, x), \ldots, g_{m_g}(a, x)\Big)^T,$$
$$a \in \mathbb{R}^\nu, x \in \mathbb{R}^r (x \in D_0), \tag{1.4a}$$

and to evaluate then the outcomes $e \in \mathcal{E} \subset \mathbb{R}^{1+m_0}, m_0 := m_f + m_g$, by means of certain loss or cost functions

$$\gamma_i : \mathcal{E} \to \mathbb{R}, i = 0, 1, \ldots, m, \tag{1.4b}$$

with an integer $m \geq 0$. For the processing of the numerical outcomes $\gamma_i\big(e(a, x)\big), i = 0, 1, \ldots, m$, there are two basic concepts:

1.2.1 Minimum or Bounded Expected Costs

Consider the vector of (conditional) expected losses or costs

$$\mathbf{F}(x) = \begin{pmatrix} F_0(x) \\ F_1(x) \\ \vdots \\ F_m(x) \end{pmatrix} := \begin{pmatrix} E\gamma_0(e(a(\omega), x)) \\ E\gamma_1(e(a(\omega), x)) \\ \vdots \\ E\gamma_m(e(a(\omega), x)) \end{pmatrix}, x \in \mathbb{R}^r, \tag{1.5}$$

where the (conditional) expectation "E" is taken with respect to the time history $\mathcal{A} = \mathcal{A}_t, (\mathcal{A}_j) \subset \mathcal{A}_0$ up to a certain time point t or stage j. A short definition of expectations is given in Sect. 2.1, for more details, see e.g. [13, 47, 135].

Having different expected cost or performance functions $F_0, F_1, \ldots, F_m$ to be minimized or bounded, as a basic deterministic substitute problem for (1.1a–d) with a random parameter vector $a = a(\omega)$ we may consider the multi-objective expected cost minimization problem

$$\text{“}\min\text{”}\, \mathbf{F}(x) \tag{1.6a}$$

s.t.

$$x \in D_0. \tag{1.6b}$$

Obviously, a good compromise solution x^* of this vector optimization problem should have at least one of the following properties [27, 138]:

Definition 1.1 **a)** A vector $x^0 \in D_0$ is called a **functional-efficient** or **Pareto optimal** solution of the vector optimization problem (1.6a, b) if there is no $x \in D_0$ such that

$$F_i(x) \leq F_i(x^0), i = 0, 1, \ldots, m \tag{1.7a}$$

and

$$F_{i_0}(x) < F_{i_0}(x^0) \text{ for at least one } i_0, 0 \le i_0 \le m. \tag{1.7b}$$

b) A vector $x^0 \in D_0$ is called a **weak functional-efficient** or **weak Pareto optimal** solution of (1.6a, b) if there is no $x \in D_0$ such that

$$F_i(x) < F_i(x^0), i = 0, 1, \ldots, m \tag{1.8}$$

(Weak) Pareto optimal solutions of (1.6a, b) may be obtained now by means of scalarizations of the vector optimization problem (1.6a, b). Three main versions are stated in the following.

I) *Minimization of primary expected cost/loss under expected cost constraints*

$$\min F_0(x) \tag{1.9a}$$

s.t.

$$F_i(x) \le F_i^{\max}, i = 1, \ldots, m \tag{1.9b}$$

$$x \in D_0. \tag{1.9c}$$

Here, $F_0 = F_0(x)$ is assumed to describe the primary goal of the design/decision making problem, while $F_i = F_i(x), i = 1, \ldots, m$, describe secondary goals. Moreover, $F_i^{\max}, i = 1, \ldots, m$, denote given upper cost/loss bounds.

Remark 1.1 An optimal solution x^* of (1.9a–c) is a weak Pareto optimal solution of (1.6a, b).

II) *Minimization of the total weighted expected costs*

Selecting certain positive weight factors $c_0, c_1, \ldots, c_m$, the expected weighted total costs are defined by

$$\tilde{F}(x) := \sum_{i=0}^{m} c_i F_i(x) = Ef\Big(a(\omega), x\Big), \tag{1.10a}$$

where

$$f(a, x) := \sum_{i=0}^{m} c_i \gamma_i \Big(e(a, x)\Big). \tag{1.10b}$$

Consequently, minimizing the expected weighted total costs $\tilde{F} = \tilde{F}(x)$ subject to the remaining deterministic constraint (1.1d), the following deterministic substitute problem for (1.1a–d) occurs:

$$\min \sum_{i=0}^{m} c_i F_i(x) \tag{1.11a}$$

s.t.

$$x \in D_0. \tag{1.11b}$$

Remark 1.2 Let $c_i > 0, i = 1, 1, \ldots, m$, be any positive weight factors. Then an optimal solution x^* of (1.11a, b) is a Pareto optimal solution of (1.6a, b).

III) *Minimization of the maximum weighted expected costs*
Instead of adding weighted expected costs, we may consider the maximum of the weighted expected costs:

$$\tilde{F}(x) := \max_{0 \le i \le m} c_i F_i(x) = \max_{0 \le i \le m} c_i E\gamma_i\Big(e\big(a(\omega), x\big)\Big). \tag{1.12}$$

Here $c_0, c_1, \ldots, m$, are again positive weight factors. Thus, minimizing $\tilde{F} = \tilde{F}(x)$ we have the deterministic substitute problem

$$\min \max_{0 \le i \le m} c_i F_i(x) \tag{1.13a}$$

s.t.

$$x \in D_0. \tag{1.13b}$$

Remark 1.3 Let $c_i, i = 0, 1, \ldots, m$, be any positive weight factors. An optimal solution of x^* of (1.13a, b) is a weak Pareto optimal solution of (1.6a, b).

1.2.2 Minimum or Bounded Maximum Costs (Worst Case)

Instead of taking expectations, we may consider the worst case with respect to the cost variations caused by the random parameter vector $a = a(\omega)$. Hence, the random cost function

$$\omega \to \gamma_i\Big(e\big(a(\omega), x\big)\Big) \tag{1.14a}$$

is evaluated by means of

$$F_i^{\sup}(x) := \text{ ess sup } \gamma_i\Big(e\Big(a(\omega), x\Big)\Big), i = 0, 1, \ldots, m. \tag{1.14b}$$

Here, ess sup $(\ldots)$ denotes the (conditional) essential supremum with respect to the random vector $a = a(\omega)$, given information $\mathcal{A}$, i.e., the infimum of the supremum of (1.14a) on sets $A \in \mathcal{A}_0$ of (conditional) probability one, see e.g. [135].

Consequently, the vector function $\mathbf{F} = \mathbf{F}^{\sup}(x)$ is then defined by

$$\mathbf{F}^{\sup}(x) = \begin{pmatrix} F_0(x) \\ F_1(x) \\ \vdots \\ F_m(x) \end{pmatrix} := \begin{pmatrix} \text{ess sup } \gamma_0\Big(e\Big(a(\omega), x\Big)\Big) \\ \text{ess sup } \gamma_1\Big(e\Big(a(\omega), x\Big)\Big) \\ \vdots \\ \text{ess sup } \gamma_m\Big(e\Big(a(\omega), x\Big)\Big) \end{pmatrix}. \tag{1.15}$$

Working with the vector function $\mathbf{F} = \mathbf{F}^{\sup}(x)$, we have then the vector minimization problem

$$\text{“}\min\text{”}\, \mathbf{F}^{\sup}(x) \tag{1.16a}$$

s.t.

$$x \in D_0. \tag{1.16b}$$

By scalarization of (1.16a, b) we obtain then again deterministic substitute problems for (1.1a–d) related to the substitute problem (1.6a, b) introduced in Sect. 1.2.1.

More details for the selection and solution of appropriate deterministic substitute problems for (1.1a–d) are given in the next sections. Deterministic substitute problems for optimal control problems under stochastic uncertainty are considered in Chap. 2.

1.3 Optimal Decision/Design Problems with Random Parameters

In the optimal design of technical or economic structures/systems, in optimal decision problems arising in technical or economic systems, resp., two basic classes of criteria appear:

First there is a primary cost function

$$G_0 = G_0(a, x). \tag{1.17a}$$

Important examples are the total weight or volume of a mechanical structure, the costs of construction, design of a certain technical or economic structure/system, or the negative utility or reward in a general decision situation. Basic examples in optimal control, cf. Chap. 2, are the total run time, the total energy consumption of the process or a weighted mean of these two cost functions.

For the representation of the structural/system safety or failure, for the representation of the admissibility of the state, or for the formulation of the basic operating conditions of the underlying plant, structure/system certain **state, performance or response functions**

$$y_i = y_i(a, x),\ i = 1, \ldots, m_y, \tag{1.17b}$$

are chosen. In structural design these functions are also called "limit state functions" or "safety margins". Frequent examples are some displacement, stress, load (force and moment) components in structural design, or more general system output functions in engineering design. Furthermore, production functions and several cost functions are possible performance functions in production planning problems, optimal mix problems, transportation problems, allocation problems and other problems of economic decision.

In (1.17a, b), the design or input vector x denotes the r-vector of design or input variables, $x_1, x_2, \ldots, x_r$, as e.g. structural dimensions, sizing variables, such as cross-sectional areas, thickness in structural design, or factors of production, actions in economic decision problems. For the decision, design or input vector x one has mostly some basic deterministic constraints, e.g. nonnegativity constraints, box constraints, represented by

$$x \in D, \tag{1.17c}$$

where D is a given convex subset of $\mathbb{R}^r$. Moreover, a is the ν-vector of model parameters. In optimal structural/engineering design

$$a = \begin{pmatrix} p \\ R \end{pmatrix} \tag{1.17d}$$

is composed of the following two subvectors: R is the m-vector of the acting external loads or structural/system inputs, e.g. wave, wind loads, payload, etc. Moreover, p denotes the $(\nu - m)$-vector of the further model parameters, as e.g. material parameters, like strength parameters, yield/allowable stresses, elastic moduli, plastic capacities, etc., of the members of a mechanical structure, parameters of an electric circuit, such as resistances, inductances, capacitances, the manufacturing tolerances and weight or more general cost coefficients.

In linear programming, as e.g. in production planning problems,

$$a = (A, b, c) \tag{1.17e}$$

is composed of the $m \times r$ matrix A of technological coefficients, the demand m-vector b and the r-vector c of unit costs.

Based on the m_y-vector of state functions

$$y(a,x) := \left(y_1(a,x), y_2(a,x), \ldots, y_{m_y}(a,x)\right)^T, \tag{1.17f}$$

the admissible or safe states of the structure/system can be characterized by the condition

$$y(a,x) \in B, \tag{1.17g}$$

where B is a certain subset of $\mathbb{R}^{m_y}$; $B = B(a)$ may depend also on some model parameters.

In production planning problems, typical operating conditions are given, cf. (1.17e), by

$$y(a,x) := Ax - b \geq 0 \text{ or } y(a,x) = 0,\ x \geq 0. \tag{1.18a}$$

In mechanical structures/structural systems, the safety (survival) of the structure/system is described by the operating conditions

$$y_i(a,x) > 0 \ \text{ for all } i = 1, \ldots, m_y \tag{1.18b}$$

with state functions $y_i = y_i(a,x), i = 1, \ldots, m_y$, depending on certain response components of the structure/system, such as displacement, stress, force, moment components. Hence, a failure occurs if and only of the structure/system is in the i-th failure mode (failure domain)

$$y_i(a,x) \leq 0 \tag{1.18c}$$

for at least one index $i, 1 \leq i \leq m_y$.

Note. The number m_y of safety margins or limit state functions $y_i = y_i(a,x), i = 1, \ldots, m_y$, may be very large. For example, in optimal plastic design the limit state functions are determined by the extreme points of the admissible domain of the dual pair of static/kinematic LPs related to the equilibrium and linearized convex yield condition, see [99, 100].

Basic problems in optimal decision/design are:

I) *Primary (construction, planning, investment, etc.) cost minimization under operating or safety conditions*

$$\min G_0(a,x) \tag{1.19a}$$

s.t.

$$y(a,x) \in B \tag{1.19b}$$

$$x \in D. \tag{1.19c}$$

Obviously we have $B = (0, +\infty)^{m_y}$ in (1.18b) and $B = [0, +\infty)^{m_y}$ or $B = \{0\}$ in (1.18a).

II) *Failure or recourse cost minimization under primary cost constraints*

$$\text{“}\min\text{''}\gamma\Big(y(a,x)\Big) \tag{1.20a}$$

s.t.

$$G_0(a,x) \leq G^{\max} \tag{1.20b}$$

$$x \in D. \tag{1.20c}$$

In (1.20a) $\gamma = \gamma(y)$ is a scalar or vector valued cost/loss function evaluating violations of the operating conditions (1.19b). Depending on the application, these costs are called “failure” or “recourse” costs [70, 71, 97, 132, 146, 147]. As already discussed in Sect. 1.1, solving problems of the above type, a basic difficulty is the uncertainty about the true value of the vector a of model parameters or the (random) variability of a:

In practice, due to several types of uncertainties such as, see [163],

- physical uncertainty (variability of physical quantities, like material, loads, dimensions, etc.)
- economic uncertainty (trade, demand, costs, etc.)
- statistical uncertainty (e.g. estimation errors of parameters due to limited sample data)
- model uncertainty (model errors),

the ν-vector a of model parameters must be modeled by a random vector

$$a = a(\omega), \omega \in \Omega, \tag{1.21a}$$

on a certain probability space $(\Omega, \mathcal{A}_0, \mathcal{P})$ with sample space Ω having elements ω, see (1.3). For the mathematical representation of the corresponding (conditional) probability distribution $\mathcal{P}_{a(\cdot)} = \mathcal{P}^{\mathcal{A}}_{a(\cdot)}$ of the random vector $a = a(\omega)$ (given the time history or information $\mathcal{A} \subset \mathcal{A}_0$), two main distribution models are taken into account in practice:

i) Discrete probability distributions,
ii) Continuous probability distributions.

In the first case there is a finite or countably infinite number $l_0 \in \mathbb{N} \cup \{\infty\}$ of realizations or scenarios $a^l \in \mathbb{R}^\nu, l = 1, \ldots, l_0$,

$$\mathcal{P}\Big(a(\omega) = a^l\Big) = \alpha_l, l = 1, \ldots, l_0, \tag{1.21b}$$

taken with probabilities $\alpha_l, l = 1, \ldots, l_0$. In the second case, the probability that the realization $a(\omega) = a$ lies in a certain (measurable) subset $B \subset \mathbb{R}^\nu$ is described by the multiple integral

$$\mathcal{P}\Big(a(\omega) \in B\Big) = \int\limits_B \varphi(a)\, da \tag{1.21c}$$

with a certain probability density function $\varphi = \varphi(a) \geq 0, a \in \mathbb{R}^\nu, \displaystyle\int \varphi(a) da = 1$.

The properties of the probability distribution $\mathcal{P}_{a(\cdot)}$ may be described—fully or in part—by certain numerical characteristics, called parameters of $\mathcal{P}_{a(\cdot)}$. These distribution parameters $\theta = \theta_h$ are obtained by considering expectations

$$\theta_h := Eh\Big(a(\omega)\Big) \tag{1.22a}$$

of some (measurable) functions

$$(h \circ a)(\omega) := h\Big(a(\omega)\Big) \tag{1.22b}$$

composed of the random vector $a = a(\omega)$ with certain (measurable) mappings

$$h : \mathbb{R}^\nu \longrightarrow \mathbb{R}^{s_h}, s_h \geq 1. \tag{1.22c}$$

According to the type of the probability distribution $\mathcal{P}_{a(\cdot)}$ of $a = a(\omega)$, the expectation $Eh\Big(a(\omega)\Big)$ is defined, cf. [12, 13], by

$$Eh\Big(a(\omega)\Big) = \begin{cases} \displaystyle\sum_{l=1}^{l_0} h\Big(a^l\Big)\, \alpha_l, & \text{in the discrete case (1.21b)} \\ \displaystyle\int\limits_{\mathbb{R}^\nu} h(a)\varphi(a)\, da, & \text{in the continuous case (1.21c).} \end{cases} \tag{1.22d}$$

Further distribution parameters θ are functions

$$\theta = \Psi(\theta_{h_1}, \ldots, \theta_{h_s}) \tag{1.23}$$

of certain "h-moments" $\theta_{h_1}, \ldots, \theta_{h_s}$ of the type (1.22a). Important examples of the type (1.22a), (1.23), resp., are the expectation

$$\overline{a} = Ea(\omega) \text{ (for } h(a) := a, a \in \mathbb{R}^\nu) \tag{1.24a}$$

and the covariance matrix

$$Q := E\Big(a(\omega) - \overline{a}\Big)\Big(a(\omega) - \overline{a}\Big)^T = Ea(\omega)a(\omega)^T - \overline{a}\,\overline{a}^T \tag{1.24b}$$

of the random vector $a = a(\omega)$.

Due to the stochastic variability of the random vector $a(\cdot)$ of model parameters, and since the realization $a(\omega) = a$ is not available at the decision making stage, the optimal design problem (1.19a–c) or (1.20a–c) under stochastic uncertainty cannot be solved directly.

Hence, appropriate deterministic substitute problems must be chosen taking into account the randomness of $a = a(\omega)$, cf. Sect. 1.2.

1.4 Deterministic Substitute Problems in Optimal Decision/Design

According to Sect. 1.2, a basic deterministic substitute problem in optimal design under stochastic uncertainty is the minimization of the total expected costs including the expected costs of failure

$$\min c_G EG_0\Big(a(\omega), x\Big) + c_f p_f(x) \tag{1.25a}$$

s.t.

$$x \in D. \tag{1.25b}$$

Here,

$$p_f = p_f(x) := \mathcal{P}\Big(y\Big(a(\omega), x\Big) \notin B\Big) \tag{1.25c}$$

is the probability of failure or the probability that a safe function of the structure, the system is not guaranteed. Furthermore, c_G is a certain weight factor, and $c_f > 0$ describes the failure or recourse costs. In the present definition of expected failure costs, constant costs for each realization $a = a(\omega)$ of $a(\cdot)$ are assumed. Obviously, it is

$$p_f(x) = 1 - p_s(x) \tag{1.25d}$$

with the probability of safety or survival

$$p_s(x) := \mathcal{P}\left(y\Big(a(\omega), x\Big) \in B\right). \tag{1.25e}$$

In case (1.18b) we have

$$p_f(x) = \mathcal{P}\left(y_i\Big(a(\omega), x\Big) \le 0 \text{ for at least one index } i, 1 \le i \le m_y\right). \tag{1.25f}$$

The objective function (1.25a) may be interpreted as the Lagrangian (with given cost multiplier c_f) of the following reliability based optimization (RBO) problem, cf. [7, 92, 132, 147, 163]:

$$\min EG_0\Big(a(\omega), x\Big) \tag{1.26a}$$

s.t.

$$p_f(x) \le \alpha^{\max} \tag{1.26b}$$

$$x \in D, \tag{1.26c}$$

where $\alpha^{\max} > 0$ is a prescribed maximum failure probability, e.g. $\alpha^{\max} = 0.001$, cf. (1.19a–c).

The "dual" version of (1.26a–c) reads

$$\min p_f(x) \tag{1.27a}$$

s.t.

$$EG_0\Big(a(\omega), x\Big) \le G^{\max} \tag{1.27b}$$

$$x \in D \tag{1.27c}$$

with a maximal (upper) cost bound $G^{\max}$, see (1.20a–c).

1.4.1 Expected Cost or Loss Functions

Further substitute problems are obtained by considering more general expected failure or recourse cost functions

$$\Gamma(x) = E\gamma\left(y\Big(a(\omega), x\Big)\right) \tag{1.28a}$$

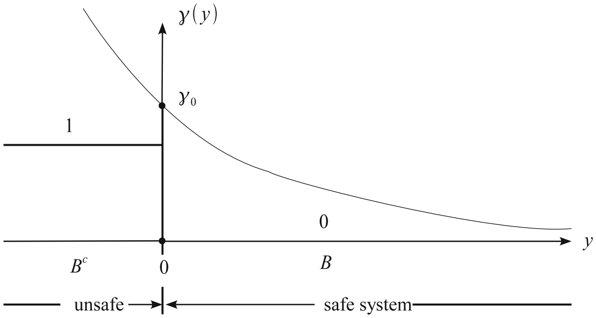

Fig. 1.1 Loss function γ

arising from structural systems weakness or failure, or because of false operation. Here,

$$y\Big(a(\omega), x\Big) := \Big(y_1\Big(a(\omega), x\Big), \ldots, y_{m_y}\Big(a(\omega), x\Big)\Big)^T \tag{1.28b}$$

is again the random vector of state or performance functions, and

$$\gamma : \mathbb{R}^{m_y} \to \mathbb{R}^{m_\gamma} \tag{1.28c}$$

is a scalar or vector valued cost or loss function (Fig. 1.1). In case $B = (0, +\infty)^{m_y}$ or $B = [0, +\infty)^{m_y}$ it is often assumed that $\gamma = \gamma(y)$ is a non increasing function, hence,

$$\gamma(y) \geq \gamma(z), \quad \text{if} \quad y \leq z, \tag{1.28d}$$

where inequalities between vectors are defined component-by-component.

Example 1.1 If $\gamma(y) = 1$ for $y \in B^c$ (complement of B) and $\gamma(y) = 0$ for $y \in B$, then $\Gamma(x) = p_f(x)$.

Example 1.2 Suppose that $\gamma = \gamma(y)$ is a nonnegative measurable scalar function on $\mathbb{R}^{m_y}$ such that

$$\gamma(y) \geq \gamma_0 > 0 \text{ for all } y \notin B \tag{1.29a}$$

with a constant $\gamma_0 > 0$. Then for the probability of failure we find the following upper bound

$$p_f(x) = \mathcal{P}\left(y\Big(a(\omega), x\Big) \notin B\right) \leq \frac{1}{\gamma_0} E\gamma\left(y\Big(a(\omega), x\Big)\right), \tag{1.29b}$$

where the right hand side of (1.29b) is obviously an expected cost function of type (1.28a–c). Hence, the condition (1.26b) can be guaranteed by the expected cost constraint

$$E\gamma\Big(y\big(a(\omega),x\big)\Big) \le \gamma_0\alpha^{\max}. \tag{1.29c}$$

Example 1.3 If the loss function $\gamma(y)$ is defined by a vector of individual loss functions γ_i for each state function $y_i = y_i(a,x), i = 1,\dots,m_y$, hence,

$$\gamma(y) = \Big(\gamma_1(y_1),\dots,\gamma_{m_y}(y_{m_y})\Big)^T, \tag{1.30a}$$

then

$$\Gamma(x) = \big(\Gamma_1(x),\dots,\Gamma_{m_y}(x)\big)^T,\ \Gamma_i(x) := E\gamma_i\left(y_i\Big(a(\omega),x\Big)\right), 1 \le i \le m_y, \tag{1.30b}$$

i.e., the m_y state functions $y_i, i = 1,\dots,m_y$, will be treated separately.

Working with the more general expected failure or recourse cost functions $\Gamma = \Gamma(x)$, instead of (1.25a–c), (1.26a–c) and (1.27a–c) we have the related substitute problems:

I) *Expected total cost minimization*

$$\min c_G EG_0\Big(a(\omega),x\Big) + c_f^T\Gamma(x), \tag{1.31a}$$

s.t.

$$x \in D \tag{1.31b}$$

II) *Expected primary cost minimization under expected failure or recourse cost constraints*

$$\min EG_0\Big(a(\omega),x\Big) \tag{1.32a}$$

s.t.

$$\Gamma(x) \le \Gamma^{\max} \tag{1.32b}$$

$$x \in D, \tag{1.32c}$$

III) *Expected failure or recourse cost minimization under expected primary cost constraints*

$$\text{“}\min\text{”}\,\Gamma(x) \tag{1.33a}$$

s.t.

$$EG_0\Big(a(\omega), x\Big) \le G^{\max} \tag{1.33b}$$

$$x \in D. \tag{1.33c}$$

Here, c_G, c_f are (vectorial) weight coefficients, $\Gamma^{\max}$ is the vector of upper loss bounds, and "min" indicates again that $\Gamma(x)$ may be a vector valued function.

1.5 Basic Properties of Deterministic Substitute Problems

As can be seen from the conversion of an optimization problem with random parameters into a deterministic substitute problem, cf. Sects. 2.1.1 and 2.1.2, a central role is played by expectation or mean value functions of the type

$$\Gamma(x) = E\gamma\left(y\Big(a(\omega), x\Big)\right), x \in D_0, \tag{1.34a}$$

or more general

$$\Gamma(x) = Eg\Big(a(\omega), x\Big), x \in D_0. \tag{1.34b}$$

Here, $a = a(\omega)$ is a random ν-vector, $y = y(a, x)$ is an m_y-vector valued function on a certain subset of $\mathbb{R}^\nu \times \mathbb{R}^r$, and $\gamma = \gamma(z)$ is a real valued function on a certain subset of $\mathbb{R}^{m_y}$.

Furthermore, $g = g(a, x)$ denotes a real valued function on a certain subset of $\mathbb{R}^\nu \times \mathbb{R}^r$. In the following we suppose that the expectation in (1.34a, b) exists and is finite for all input vectors x lying in an appropriate set $D_0 \subset \mathbb{R}^r$, cf. [16].

The following basic properties of the mean value functions Γ are needed in the following again and again.

Lemma 1.1 (Convexity) *Suppose that $x \to g\Big(a(\omega), x\Big)$ is convex a.s. (almost sure) on a fixed convex domain $D_0 \subset \mathbb{R}^r$. If $Eg\Big(a(\omega), x\Big)$ exists and is finite for each $x \in D_0$, then $\Gamma = \Gamma(x)$ is convex on D_0.*

Proof This property follows [70, 71, 90] directly from the linearity of the expectation operator.

If $g = g(a, x)$ is defined by $g(a, x) := \gamma\Big(y(a, x)\Big)$, see (1.34a), then the above theorem yields the following result:

Corollary 1.1 *Suppose that γ is convex and $E\gamma\left(y\Big(a(\omega), x\Big)\right)$ exists and is finite for each $x \in D_0$. a) If $x \to y\Big(a(\omega), x\Big)$ is linear a.s., then $\Gamma = \Gamma(x)$ is convex. b)*

If $x \to y\Big(a(\omega), x\Big)$ *is convex a.s., and* γ *is a convex, monotoneous nondecreasing function, then* $\Gamma = \Gamma(x)$ *is convex.*

It is well known [85] that a convex function is continuous on each open subset of its domain. A general sufficient condition for the continuity of Γ is given next.

Lemma 1.2 (Continuity) *Suppose that* $Eg\Big(a(\omega), x\Big)$ *exists and is finite for each* $x \in D_0$, *and assume that* $x \to g\Big(a(\omega), x\Big)$ *is continuous at* $x_0 \in D_0$ *a.s. If there is a function* $\psi = \psi\Big(a(\omega)\Big)$ *having finite expectation such that*

$$\left|g\Big(a(\omega), x\Big)\right| \leq \psi\Big(a(\omega)\Big) \text{ a.s. for all } x \in U(x_0) \cap D_0, \tag{1.35}$$

where $U(x_0)$ *is a neighborhood of* x_0, *then* $\Gamma = \Gamma(x)$ *is continuous at* x_0.

Proof The assertion can be shown by using Lebesgue's dominated convergence theorem, see e.g. [90].

For the consideration of the differentiability of $\Gamma = \Gamma(x)$, let D denote an open subset of the domain D_0 of Γ.

Lemma 1.3 (Differentiability) *Suppose that*

i) $Eg\Big(a(\omega), x\Big)$ *exists and is finite for each* $x \in D_0$,
ii) $x \to g\Big(a(\omega), x\Big)$ *is differentiable on the open subset* D *of* D_0 *a.s. and*
iii)

$$\left\|\nabla_x g\Big(a(\omega), x\Big)\right\| \leq \psi\Big(a(\omega)\Big), x \in D, \text{ a.s.}, \tag{1.36a}$$

where $\psi = \psi\Big(a(\omega)\Big)$ *is a function having finite expectation. Then the expectation of* $\nabla_x g\Big(a(\omega), x\Big)$ *exists and is finite,* $\Gamma = \Gamma(x)$ *is differentiable on* D *and*

$$\nabla \Gamma(x) = \nabla_x Eg\Big(a(\omega), x\Big) = E\nabla_x g\Big(a(\omega), x\Big), x \in D. \tag{1.36b}$$

Proof Considering the difference quotients $\dfrac{\Delta \Gamma}{\Delta x_k}, k = 1, \ldots, r$, of Γ at a fixed point $x_0 \in D$, the assertion follows by means of the mean value theorem, inequality (1.36a) and Lebesgue's dominated convergence theorem, cf. [70, 71, 90].

Example 1.4 In case (1.34a), under obvious differentiability assumptions concerning γ and y we have $\nabla_x g(a, x) = \nabla_x y(a, x)^T \nabla \gamma\Big(y(a, x)\Big)$, where $\nabla_x y(a, x)$

denotes the Jacobian of $y = y(a, x)$ with respect to a. Hence, if (1.36b) holds, then

$$\nabla \Gamma(x) = E\nabla_x y\Big(a(\omega), x\Big)^T \nabla\gamma\left(y\Big(a(\omega), x\Big)\right). \tag{1.36c}$$

1.6 Approximations of Deterministic Substitute Problems in Optimal Design/Decision

The main problem in solving the deterministic substitute problems defined above is that the arising probability and expected cost functions $p_f = p_f(x), \Gamma = \Gamma(x), x \in \mathbb{R}^r$, are defined by means of multiple integrals over a ν-dimensional space.

Thus, the substitute problems may be solved, in practice, only by some approximative analytical and numerical methods [44, 70, 90, 100]. In the following we consider possible approximations for substitute problems based on general expected recourse cost functions $\Gamma = \Gamma(x)$ according to (1.34a) having a real valued convex loss function $\gamma(z)$. Note that the probability of failure function $p_f = p_f(x)$ may be approximated from above, see (1.29a, b), by expected cost functions $\Gamma = \Gamma(x)$ having a nonnegative function $\gamma = \gamma(z)$ being bounded from below on the failure domain B^c. In the following several basic approximation methods are presented.

1.6.1 Approximation of the Loss Function

Suppose here that $\gamma = \gamma(y)$ is a continuously differentiable, convex loss function on $\mathbb{R}^{m_y}$. Let then denote

$$\overline{y}(x) := Ey\Big(a(\omega), x\Big) = \left(Ey_1\Big(a(\omega), x\Big), \ldots, Ey_{m_y}\Big(a(\omega), x\Big)\right)^T \tag{1.37}$$

the expectation of the vector $y = y\Big(a(\omega), x\Big)$ of state functions $y_i = y_i\Big(a(\omega), x\Big)$, $i = 1, \ldots, m_y$.

For an arbitrary continuously differentiable, convex loss function γ we have

$$\gamma\left(y\Big(a(\omega), x\Big)\right) \geq \gamma\Big(\overline{y}(x)\Big) + \nabla\gamma\Big(\overline{y}(x)\Big)^T\left(y\Big(a(\omega), x\Big) - \overline{y}(x)\right). \tag{1.38a}$$

Thus, taking expectations in (1.38a), we find Jensen's inequality

$$\Gamma(x) = E\gamma\left(y\Big(a(\omega), x\Big)\right) \geq \gamma\Big(\overline{y}(x)\Big) \tag{1.38b}$$

which holds for any convex function γ. Using the mean value theorem, we have

$$\gamma(y) = \gamma(\overline{y}) + \nabla\gamma(\hat{y})^T(y - \overline{y}), \tag{1.38c}$$

where $\hat{y}$ is a point on the line segment $\overline{y}y$ between $\overline{y}$ and y. By means of (1.38b, c) we get

$$0 \le \Gamma(x) - \gamma\Big(\overline{y}(x)\Big) \le E\left\|\nabla\gamma\left(\hat{y}\Big(a(\omega), x\Big)\right)\right\| \cdot \left\|y\Big(a(\omega), x\Big) - \overline{y}(x)\right\|. \tag{1.38d}$$

a) Bounded gradient

If the gradient $\nabla\gamma$ is bounded on the range of $y = y\Big(a(\omega), x\Big), x \in D$, i.e., if

$$\left\|\nabla\gamma\left(y\Big(a(\omega), x\Big)\right)\right\| \le \vartheta^{\max} \text{ a.s. for each } x \in D, \tag{1.39a}$$

with a constant $\vartheta^{\max} > 0$, then

$$0 \le \Gamma(x) - \gamma\Big(\overline{y}(x)\Big) \le \vartheta^{\max} E\left\|y\Big(a(\omega), x\Big) - \overline{y}(x)\right\|, x \in D. \tag{1.39b}$$

Since $t \to \sqrt{t}, t \ge 0$, is a concave function, we get

$$0 \le \Gamma(x) - \gamma\Big(\overline{y}(x)\Big) \le \vartheta^{\max}\sqrt{q(x)}\,, \tag{1.39c}$$

where

$$q(x) := E\left\|y\Big(a(\omega), x\Big) - \overline{y}(x)\right\|^2 = trQ(x) \tag{1.39d}$$

is the generalized variance, and

$$Q(x) := \operatorname{cov}\left(y\Big(a(\cdot), x\Big)\right) \tag{1.39e}$$

denotes the covariance matrix of the random vector $y = y\Big(a(\omega), x\Big)$. Consequently, the expected loss function $\Gamma(x)$ can be approximated from above by

$$\Gamma(x) \le \gamma\Big(\overline{y}(x)\Big) + \vartheta^{\max}\sqrt{q(x)} \text{ for } x \in D. \tag{1.39f}$$

b) Bounded eigenvalues of the Hessian

Considering second order expansions of γ, with a vector $\tilde{y} \in \bar{y}y$ we find

$$\gamma(y) - \gamma(\overline{y}) = \nabla\gamma(\overline{y})^T(y - \overline{y}) + \frac{1}{2}(y - \overline{y})^T\nabla^2\gamma(\tilde{y})(y - \overline{y}). \tag{1.40a}$$

Suppose that the eigenvalues λ of $\nabla^2\gamma(y)$ are bounded from below and above on the range of $y = y\Big(a(\omega), x\Big)$ for each $x \in D$, i.e.,

$$0 < \lambda^{\min} \leq \lambda\left(\nabla^2\gamma\left(y\Big(a(\omega), x\Big)\right)\right) \leq \lambda^{\max} < +\infty, \text{ a.s., } x \in D \tag{1.40b}$$

with constants $0 < \lambda^{\min} \leq \lambda^{\max}$. Taking expectations in (1.40a), we get

$$\gamma\Big(\overline{y}(x)\Big) + \frac{\lambda^{\min}}{2}q(x) \leq \Gamma(x) \leq \gamma\Big(\overline{y}(x)\Big) + \frac{\lambda^{\max}}{2}q(x), x \in D. \tag{1.40c}$$

Consequently, using (1.39f) or (1.40c), various approximations for the deterministic substitute problems (1.31a, b), (1.32a–c), (1.33a–c) may be obtained.

Based on the above approximations of expected cost functions, we state the following two approximates to (1.32a–c), (1.33a–c), resp., which are well known in *robust optimal design:*

i) *Expected primary cost minimization under approximate expected failure or recourse cost constraints*

$$\min EG_0\Big(a(\omega), x\Big) \tag{1.41a}$$

s.t.

$$\gamma\Big(\overline{y}(x)\Big) + c_0 q(x) \leq \Gamma^{\max} \tag{1.41b}$$

$$x \in D, \tag{1.41c}$$

where c_0 is a scale factor, cf. (1.39f) and (1.40c);

ii) *Approximate expected failure or recourse cost minimization under expected primary cost constraints*

$$\min \gamma\Big(\overline{y}(x)\Big) + c_0 q(x) \tag{1.42a}$$

s.t.

$$EG_0\Big(a(\omega), x\Big) \leq G^{\max} \tag{1.42b}$$

$$x \in D. \tag{1.42c}$$

Obviously, by means of (1.41a–c) or (1.42a–c) optimal designs x^* are achieved which

- yield a high mean performance of the structure/structural system
- are minimally sensitive or have a limited sensitivity with respect to random parameter variations (material, load, manufacturing, process, etc.) and
- cause only limited costs for design, construction, maintenance, etc.

1.6.2 Approximation of State (Performance) Functions

The numerical solution is simplified considerably if one can work with one single state function $y = y(a, x)$. Formally, this is possible by defining the function

$$y^{\min}(a, x) := \min_{1 \le i \le m_y} y_i(a, x). \tag{1.43a}$$

Indeed, according to (1.18b, c) the failure of the structure, the system can be represented by the condition

$$y^{\min}(a, x) \le 0. \tag{1.43b}$$

Thus, the weakness or failure of the technical or economic device can be evaluated numerically by the function

$$\Gamma(x) := E\gamma\Big(y^{\min}\big(a(\omega), x\big)\Big) \tag{1.43c}$$

with a nonincreasing loss function $\gamma : \mathbb{R} \to \mathbb{R}_+$, see Fig. 1.1.
However, the "min"-operator in (1.43a) yields a nonsmooth function $y^{\min} = y^{\min}(a, x)$ in general, and the computation of the mean and variance function

$$\overline{y^{\min}}(x) := Ey^{\min}\big(a(\omega), x\big) \tag{1.43d}$$

$$\sigma^2_{y^{\min}}(x) := V\Big(y^{\min}\big(a(\cdot), x\big)\Big) \tag{1.43e}$$

by means of Taylor expansion with respect to the model parameter vector a at $\overline{a} = Ea(\omega)$ is not possible directly, cf. Sect. 1.6.3.

According to the definition (1.43a), an upper bound for $\overline{y^{\min}}(x)$ is given by

$$\overline{y^{\min}}(x) \le \min_{1 \le i \le m_y} \overline{y}_i(x) = \min_{1 \le i \le m_y} Ey_i\big(a(\omega), x\big).$$

Further approximations of $y^{\min}(a, x)$ and its moments can be found by using the representation

$$\min(a, b) = \frac{1}{2}\big(a + b - |a - b|\big)$$

of the minimum of two numbers $a, b \in \mathbb{R}$. For example, for an even index m_y we have

$$y^{\min}(a, x) = \min_{i=1,3,\ldots,m_y-1} \min\Big(y_i(a, x), y_{i+1}(a, x)\Big)$$
$$= \min_{i=1,3,\ldots,m_y-1} \frac{1}{2}\Big(y_i(a, x) + y_{i+1}(a, x) - \big|y_i(a, x) - y_{i+1}(a, x)\big|\Big).$$

In many cases we may suppose that the state (performance) functions $y_i = y_i(a, x), i = 1, \ldots, m_y$, are bounded from below, hence,

$$y_i(a, x) > -A, i = 1, \ldots, m_y,$$

for all (a, x) under consideration with a positive constant $A > 0$. Thus, defining

$$\tilde{y}_i(a, x) := y_i(a, x) + A, i = 1, \ldots, m_y,$$

and therefore

$$\tilde{y}^{\min}(a, x) := \min_{1 \le i \le m_y} \tilde{y}_i(a, x) = y^{\min}(a, x) + A,$$

we have

$$y^{\min}(a, x) \le 0 \text{ if and only if } \tilde{y}^{\min}(a, x) \le A.$$

Hence, the survival/failure of the system or structure can be studied also by means of the positive function $\tilde{y}^{\min} = \tilde{y}^{\min}(a, x)$. Using now the theory of power or Hölder means [25], the minimum $\tilde{y}^{\min}(a, x)$ of positive functions can be represented also by the limit

$$\tilde{y}^{\min}(a, x) = \lim_{\lambda \to -\infty} \left(\frac{1}{m_y} \sum_{i=1}^{m_y} \tilde{y}_i(a, x)^{\lambda}\right)^{1/\lambda}$$

of the decreasing family of power means $M^{[\lambda]}(\tilde{y}) := \left(\frac{1}{m_y} \sum_{i=1}^{m_y} \tilde{y}_i^{\lambda}\right)^{1/\lambda}, \lambda < 0$. Consequently, for each fixed $p > 0$ we also have

$$\tilde{y}^{\min}(a, x)^p = \lim_{\lambda \to -\infty} \left(\frac{1}{m_y} \sum_{i=1}^{m_y} \tilde{y}_i(a, x)^{\lambda}\right)^{p/\lambda}.$$

Assuming that the expectation $EM^{[\lambda]}\Big(\tilde{y}\big(a(\omega)\big),x\Big)^p$ exists for an exponent $\lambda = \lambda_0 < 0$, by means of Lebesgue's bounded convergence theorem we get the moment representation

$$E\,\tilde{y}^{\min}\Big(a(\omega),x\Big)^p = \lim_{\lambda\to-\infty} E\left(\frac{1}{m_y}\sum_{i=1}^{m_y}\tilde{y}_i\Big(a(\omega),x\Big)^\lambda\right)^{p/\lambda}.$$

Since $t \to t^{p/\lambda}, t > 0$, is convex for each fixed $p > 0$ and $\lambda < 0$, by Jensen's inequality we have the lower moment bound

$$E\,\tilde{y}^{\min}\Big(a(\omega),x\Big)^p \geq \lim_{\lambda\to-\infty} \left(\frac{1}{m_y}\sum_{i=1}^{m_y}E\,\tilde{y}_i\Big(a(\omega),x\Big)^\lambda\right)^{p/\lambda}.$$

Hence, for the pth order moment of $\tilde{y}^{\min}\Big(a(\cdot),x\Big)$ we get the approximations

$$E\left(\frac{1}{m_y}\sum_{i=1}^{m_y}\tilde{y}_i\Big(a(\omega),x\Big)\right)^{p/\lambda} \geq \left(\frac{1}{m_y}\sum_{i=1}^{m_y}E\,\tilde{y}_i\Big(a(\omega),x\Big)^\lambda\right)^{p/\lambda}$$

for some $\lambda < 0$.

Using regression techniques, Response Surface Methods (RSM), etc., for given vector x, the function $a \to y^{\min}(a,x)$ can be approximated [17, 24, 66, 78, 144] by functions $\tilde{y} = \tilde{y}(a,x)$ being sufficiently smooth with respect to the parameter vector a.

In many important cases, for each $i = 1,\dots,m_y$, the state functions

$$(a,x) \longrightarrow y_i(a,x)$$

are bilinear functions. Thus, in this case $y^{\min} = y^{\min}(a,x)$ is a piecewise linear function with respect to a. Fitting a linear or quadratic Response Surface Model [22, 23, 115, 116]

$$\tilde{y}(a,x) := c(x) + q(x)^T(a-\overline{a}) + (a-\overline{a})^T Q(x)(a-\overline{a}) \tag{1.43f}$$

to $a \to y^{\min}(a,x)$, after the selection of appropriate reference points

$$a^{(j)} := \overline{a} + d_a^{(j)},\ j = 1,\dots,p, \tag{1.43g}$$

with "design" points $d_a^{(j)} \in \mathbb{R}^\nu, j = 1,\dots,p$, the unknown coefficients $c = c(x), q = q(x)$ and $Q = Q(x)$ are obtained by minimizing the mean square error

$$\rho(c,q,Q) := \sum_{j=1}^{p}\left(\tilde{y}(a^{(j)},x) - y^{\min}(a^{(j)},x)\right)^2 \tag{1.43h}$$

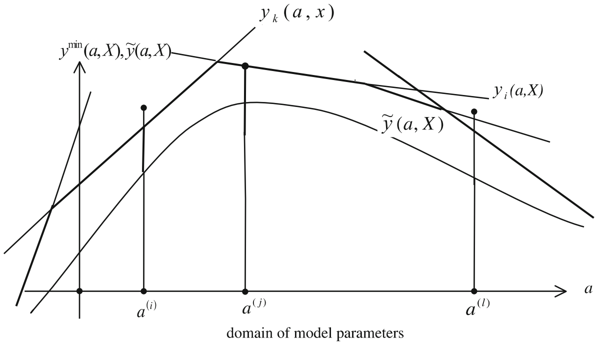

Fig. 1.2 Approximation $\tilde{y}(a, x)$ of $y^{\min}(a, x)$ for given x

with respect to (c, q, Q) (Fig. 1.2). Since the model (1.43f) depends linearly on the function parameters (c, q, Q), explicit formulas for the optimal coefficients

$$c^* = c^*(x), q^* = q^*(x), Q^* = Q^*(x), \tag{1.43i}$$

are obtained from this least squares estimation method, cf.[100].

Approximation of Expected Loss Functions

Corresponding to the approximation (1.43f) of $y^{\min} = y^{\min}(a, x)$, using again least squares techniques, a mean value function $\Gamma(x) = E\gamma\left(y\left(a(\omega), x\right)\right)$, cf. (1.28a), can be approximated at a given point $x_0 \in \mathbb{R}^\nu$ by a linear or quadratic Response Surface Function

$$\tilde{\Gamma}(x) := \beta_0 + \beta_I^T(x - x_0) + (x - x_0)^T B(x - x_0), \tag{1.43j}$$

with scalar, vector and matrix parameters β_0, β_I, B. In this case estimates $y^{(i)} = \hat{\Gamma}^{(i)}$ of $\Gamma(x)$ are needed at some reference points $x^{(i)} = x_0 + d^{(i)}, i = 1, \ldots, p$. Details are given in [100].

1.6.3 Taylor Expansion Methods

As can be seen above, cf. (1.34a, b), in the objective and/or in the constraints of substitute problems for optimization problems with random data mean value functions of the type

$$\Gamma(x) := Eg\Big(a(\omega), x\Big)$$

occur. Here, $g = g(a, x)$ is a real valued function on a subset of $\mathbb{R}^\nu \times \mathbb{R}^r$, and $a = a(\omega)$ is a ν-random vector.

(Complete) Expansion with Respect to *a*

Suppose that on its domain the function $g = g(a, x)$ has partial derivatives $\nabla_a^l g(a, x), l = 0, 1, \dots, l_g + 1$, up to order $l_g + 1$. Note that the gradient $\nabla_a g(a, x)$ contains the so–called *sensitivities* $\frac{\partial g}{\partial a_j}(a, x)$, $j = 1, \dots, \nu$, of g with respect to the parameter vector a at (a, x). In the same way, the higher order partial derivatives $\nabla_a^l g(a, x), l > 1$, represent the *higher order sensitivities* of g with respect to a at (a, x). Taylor expansion of $g = g(a, x)$ with respect to a at $\overline{a} := Ea(\omega)$ yields

$$g(a, x) = \sum_{l=0}^{l_g} \frac{1}{l!} \nabla_a^l g(\bar{a}, x) \cdot (a - \bar{a})^l + \frac{1}{(l_g + 1)!} \nabla_a^{lg+1} g(\hat{a}, x) \cdot (a - \bar{a})^{l_g+1} \tag{1.44a}$$

where $\hat{a} := \bar{a} + \vartheta(a - \bar{a}), 0 < \vartheta < 1$, and $(a - \bar{a})^l$ denotes the system of l-th order products

$$\prod_{j=1}^{\nu} (a_j - \bar{a}_j)^{l_j}$$

with $l_j \in \mathbb{N} \cup \{0\}, j = 1, \dots, \nu, l_1 + l_2 + \ldots + l_\nu = l$. If $g = g(a, x)$ is defined by

$$g(a, x) := \gamma\Big(y(a, x)\Big),$$

see (1.34a), then the partial derivatives $\nabla_a^l g$ of g up to the second order read:

$$\nabla_a g(a, x) = \Big(\nabla_a y(a, x)\Big)^T \nabla\gamma\Big(y(a, x)\Big) \tag{1.44b}$$

$$\begin{aligned} \nabla_a^2 g(a, x) = &\Big(\nabla_a y(a, x)\Big)^T \nabla^2\gamma\Big(y(a, x)\Big) \nabla_a y(a, x) \\ &+ \nabla\gamma\Big(y(a, x)\Big) \cdot \nabla_a^2 y(a, x), \end{aligned} \tag{1.44c}$$

where

$$(\nabla\gamma) \cdot \nabla_a^2 y := \left((\nabla\gamma)^T \frac{\partial^2 y}{\partial a_k \partial a_l} \right)_{k,l=1,\dots,\nu}. \tag{1.44d}$$

Taking expectations in (1.44a), $\Gamma(x)$ can be approximated, cf. Sect. 1.6.1, by

$$\tilde{\Gamma}(x) := g(\bar{a}, x) + \sum_{l=2}^{l_g} \nabla_a^l g(\bar{a}, x) \cdot E\Big(a(\omega) - \bar{a}\Big)^l, \tag{1.45a}$$

where $E\Big(a(\omega) - \bar{a}\Big)^l$ denotes the system of mixed lth central moments of the random vector $a(\omega) = \Big(a_1(\omega), \dots, a_\nu(\omega)\Big)^T$. Assuming that the domain of $g = g(a, x)$ is convex with respect to a, we get the error estimate

$$\begin{aligned}\Big|\Gamma(x) - \tilde{\Gamma}(x)\Big| \le \frac{1}{(l_g+1)!} E \sup_{0 \le \vartheta \le 1} \Big\| \nabla_a^{l_g+1} g\Big(\bar{a} + \vartheta\big(a(\omega) - \bar{a}\big), x\Big)\Big\| \\ \times \Big\| a(\omega) - \bar{a} \Big\|^{l_g+1}.\end{aligned} \tag{1.45b}$$

In many practical cases the random parameter ν-vector $a = a(\omega)$ has a convex, bounded support, and $\nabla_a^{l_g+1} g$ is continuous. Then the L_∞–norm

$$r(x) := \frac{1}{(l_g+1)!} \operatorname{ess\,sup} \Big\| \nabla_a^{l_g+1} g\Big(a(\omega), x\Big)\Big\| \tag{1.45c}$$

is finite for all x under consideration, and (1.45b, c) yield the error bound

$$\Big|\Gamma(x) - \tilde{\Gamma}(x)\Big| \le r(x) E \Big\| a(\omega) - \bar{a} \Big\|^{l_g+1}. \tag{1.45d}$$

Remark 1.1 The above described method can be extended to the case of vector valued loss functions $\gamma(z) = \Big(\gamma_1(z), \dots, \gamma_{m_\gamma}(z)\Big)^T$.

Inner (Partial) Expansions with Respect to *a*

In generalization of (1.34a), in many cases $\Gamma(x)$ is defined by

$$\Gamma(x) = E\gamma\left(a(\omega), y\Big(a(\omega), x\Big)\right), \tag{1.46a}$$

hence, the loss function $\gamma = \gamma(a, y)$ depends also explicitly on the parameter vector a. This may occur e.g. in case of randomly varying cost factors.

Linearizing now the vector function $y = y(a, x)$ with respect to a at $\bar{a}$, thus,

$$y(a,x) \approx y_{(1)}(a,x) := y(\bar{a},x) + \nabla_a y(\bar{a},x)(a-\bar{a}), \tag{1.46b}$$

the mean value function $\Gamma(x)$ is approximated by

$$\tilde{\Gamma}(x) := E\gamma\Big(a(\omega), y(\bar{a},x) + \nabla_a y(\bar{a},x)\big(a(\omega)-\bar{a}\big)\Big). \tag{1.46c}$$

This approximation is very advantageous in case that the cost function $\gamma = \gamma(a, y)$ is a *quadratic function in* y. In case of a cost function $\gamma = \gamma(a, y)$ being linear in the vector y, also *quadratic expansions of* $y = y(a, x)$ *with respect to* a many be taken into account.

Corresponding to (1.37), (1.39e), define

$$\overline{y}_{(1)}(x) := Ey_{(1)}\Big(a(\omega), x\Big) = y(\overline{a}, x) \tag{1.46d}$$

$$Q_{(1)}(x) := \mathrm{cov}\Big(y_{(1)}\Big(a(\cdot), x\Big)\Big) = \nabla_a y(\overline{a}, x)\,\mathrm{cov}\Big(a(\cdot)\Big)\nabla_a y(\overline{a}, x)^T. \tag{1.46e}$$

In case of convex loss functions γ, approximates of $\tilde{\Gamma}$ and the corresponding substitute problems based on $\tilde{\Gamma}$ may be obtained now by applying the methods described in Sect. 1.6.1. Explicit representations for $\tilde{\Gamma}$ are obtained in case of quadratic loss functions γ.

Error estimates can be derived easily for Lipschitz(L)-continuous or convex loss function γ. In case of a Lipschitz-continuous loss function $\gamma(a, \cdot)$ with Lipschitz constant $L = L(a) > 0$, e.g. for sublinear [90, 91] loss functions, using (1.46d) we have

$$\Big|\Gamma(x) - \tilde{\Gamma}(x)\Big| \le L_0 \cdot E\,\Big\| y\Big(a(\omega), x\Big) - y_{(1)}\Big(a(\omega), x\Big)\Big\|, \tag{1.46f}$$

provided that L_0 denotes a finite upper bound of the L-constants $L = L(a)$.

Applying the mean value theorem [37], under appropriate 2nd order differentiability assumptions, for the right hand side of (1.46f) we find the following stochastic version of the mean value theorem

$$\begin{aligned} &E\,\Big\| y\Big(a(\omega), x\Big) - y_{(1)}\Big(a(\omega), x\Big)\Big\| \\ &\le E\Big\|a(\omega) - \overline{a}\Big\|^2 \sup_{0\le\vartheta\le1} \Big\|\nabla_a^2 y\Big(\overline{a} + \vartheta\Big(a(\omega) - \overline{a}\Big), x\Big)\Big\|. \end{aligned} \tag{1.46g}$$

1.7 Approximation of Probabilities: Probability Inequalities

In reliability analysis of engineering/economic structures or systems, a main problem is the computation of probabilities

$$\mathcal{P}\left(\bigcup_{i=1}^{N} V_i\right) := \mathcal{P}\left(a(\omega) \in \bigcup_{i=1}^{N} V_i\right) \tag{1.47a}$$

or

$$\mathcal{P}\left(\bigcap_{j=1}^{N} S_j\right) := \mathcal{P}\left(a(\omega) \in \bigcap_{j=1}^{N} S_j\right) \tag{1.47b}$$

of unions and intersections of certain failure/survival domains (events) V_j, S_j, $j = 1, \ldots, N$. These domains (events) arise from the representation of the structure or system by a combination of certain series and/or parallel substructures/systems. Due to the high complexity of the basic physical relations, several approximation techniques are needed for the evaluation of (1.47a, b).

1.7.1 Bonferroni-Type Inequalities

In the following $V_1, V_2, \ldots, V_N$ denote arbitrary (Borel-)measurable subsets of the parameter space $\mathbb{R}^\nu$, and the abbreviation

$$\mathcal{P}(V) := \mathcal{P}\Big(a(\omega) \in V\Big) \tag{1.47c}$$

is used for any measurable subset V of $\mathbb{R}^\nu$.

Starting from the representation of the probability of a union of N events,

$$\mathcal{P}\left(\bigcup_{j=1}^{N} V_j\right) = \sum_{k=1}^{N} (-1)^{k-1} s_{k,N}, \tag{1.48a}$$

where

$$s_{k,N} := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le N} \mathcal{P}\left(\bigcap_{l=1}^{k} V_{i_l}\right), \tag{1.48b}$$

we obtain [48] the well known basic Bonferroni bounds

$$\mathcal{P}\left(\bigcup_{j=1}^{N} V_j\right) \leq \sum_{k=1}^{\rho}(-1)^{k-1} s_{k,N} \text{ for } \rho \geq 1, \rho \text{ odd} \tag{1.48c}$$

$$\mathcal{P}\left(\bigcup_{j=1}^{N} V_j\right) \geq \sum_{k=1}^{\rho}(-1)^{k-1} s_{k,N} \text{ for } \rho \geq 1, \rho \text{ even.} \tag{1.48d}$$

Besides (1.48c, d), a large amount of related bounds of different complexity are available, cf. [48, 166]. Important bounds of first and second degree are given below:

$$\max_{1 \leq j \leq N} q_j \leq \mathcal{P}\left(\bigcup_{j=1}^{N} V_j\right) \leq Q_1 \tag{1.49a}$$

$$Q_1 - Q_2 \leq \mathcal{P}\left(\bigcup_{j=1}^{N} V_j\right) \leq Q_1 - \max_{1 \leq l \leq N} \sum_{i \neq l} q_{il} \tag{1.49b}$$

$$\frac{Q_1^2}{Q_1 + 2Q_2} \leq \mathcal{P}\left(\bigcup_{j=1}^{N} V_j\right) \leq Q_1. \tag{1.49c}$$

The above quantities q_j, q_{ij}, Q_1, Q_2 are defined as follows:

$$Q_1 := \sum_{j=1}^{N} q_j \text{ with } q_j := \mathcal{P}(V_j) \tag{1.49d}$$

$$Q_2 := \sum_{j=2}^{N} \sum_{i=1}^{j-1} q_{ij} \text{ with } q_{ij} := \mathcal{P}(V_i \cap V_j). \tag{1.49e}$$

Moreover, defining

$$q := (q_1, \ldots, q_N),\ Q := (q_{ij})_{1 \leq i,j \leq N}, \tag{1.49f}$$

we have

$$\mathcal{P}\left(\bigcup_{j=1}^{N} V_j\right) \geq q^T Q^- q, \tag{1.49g}$$

where Q^- denotes the generalized inverse of Q, cf. [166].

1.7.2 Tschebyscheff-Type Inequalities

In many cases the survival or feasible domain (event) $S = \bigcap_{i=1}^{m} S_i$, is represented by a certain number m of inequality constraints of the type

$$y_{li} < (\le) y_i(a,x) < (\le) y_{ui}, i = 1, \ldots, m, \tag{1.50a}$$

as e.g. operating conditions, behavioral constraints. Hence, for a fixed input, design or control vector x, the event $S = S(x)$ is given by

$$S := \{a \in \mathbb{R}^{\nu} : y_{li} < (\le) y_i(a,x) < (\le) y_{ui}, i = 1, \ldots, m\}. \tag{1.50b}$$

Here,

$$y_i = y_i(a,x), i = 1, \ldots, m, \tag{1.50c}$$

are certain functions, e.g. response, output or performance functions of the structure, system, defined on (a subset of) $\mathbb{R}^{\nu} \times \mathbb{R}^{r}$.

Moreover, $y_{li} < y_{ui}, i = 1, \ldots, m$, are lower and upper bounds for the variables $y_i, i = 1, \ldots, m$. In case of one-sided constraints some bounds y_{li}, y_{ui} are infinite.

Two-Sided Constraints

If $y_{li} < y_{ui}, i = 1, \ldots, m$, are finite bounds, (1.50a) can be represented by

$$|y_i(a,x) - y_{ic}| < (\le) \rho_i, i = 1, \ldots, m, \tag{1.50d}$$

where the quantities $y_{ic}, \rho_i, i = 1, \ldots, m$, are defined by

$$y_{ic} := \frac{y_{li} + y_{ui}}{2}, \rho_i := \frac{y_{ui} - y_{li}}{2}. \tag{1.50e}$$

Consequently, for the probability $\mathcal{P}(S)$ of the event S, defined by (1.50b), we have

$$\mathcal{P}(S) = \mathcal{P}\left(\left|y_i\Big(a(\omega),x\Big) - y_{ic}\right| < (\le) \rho_i, i = 1, \ldots, m\right). \tag{1.50f}$$

Introducing the random variables

$$\tilde{y}_i\Big(a(\omega),x\Big) := \frac{y_i\Big(a(\omega),x\Big) - y_{ic}}{\rho_i}, i = 1, \ldots, m, \tag{1.51a}$$

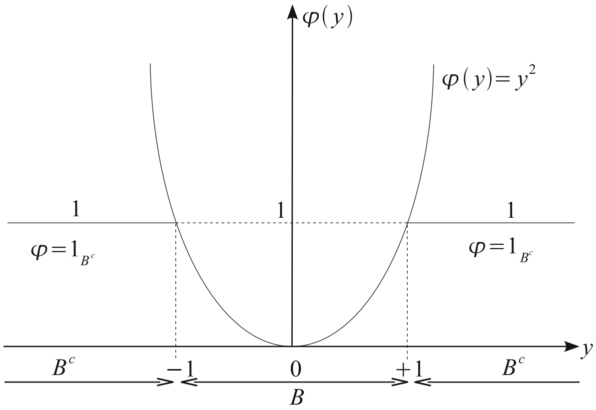

Fig. 1.3 Function $\varphi = \varphi(y)$

and the set

$$B := \{y \in \mathbb{R}^m : |y_i| < (\leq) 1, i = 1, \ldots, m\}, \tag{1.51b}$$

with $\tilde{y} = (\tilde{y}_i)_{1 \leq i \leq m}$, we get

$$\mathcal{P}(S) = \mathcal{P}\left(\tilde{y}\Big(a(\omega), x\Big) \in B\right). \tag{1.51c}$$

Considering any (measurable) function $\varphi : \mathbb{R}^m \to \mathbb{R}$ such that

$$i)\quad \varphi(y) \geq 0, y \in \mathbb{R}^m \tag{1.51d}$$

$$ii)\quad \varphi(y) \geq \varphi_0 > 0,\ \text{if } y \notin B, \tag{1.51e}$$

with a positive constant φ_0 (Fig. 1.3), we find the following result:

Theorem 1.1 *For any (measurable) function $\varphi : \mathbb{R}^m \to \mathbb{R}$ fulfilling conditions (1.51d, e), the following* **Tschebyscheff-type inequality** *holds:*

$$\mathcal{P}\left(y_{li} < (\leq) y_i\Big(a(\omega), x\Big) < (\leq) y_{ui}, i = 1, \ldots, m\right)$$
$$\geq 1 - \frac{1}{\varphi_0} E\varphi\left(\tilde{y}\Big(a(\omega), x\Big)\right), \tag{1.52}$$

provided that the expectation in (1.52) exists and is finite.

Proof If $\mathcal{P}_{\tilde{y}\left(a(\cdot),x\right)}$ denotes the probability distribution of the random m-vector $\tilde{y} = \tilde{y}\left(a(\omega), x\right)$, then

$$\begin{aligned}
E\varphi\left(\tilde{y}\left(a(\omega),x\right)\right) &= \int\limits_{y\in B} \varphi(y)\mathcal{P}_{\tilde{y}(a(\cdot),x)}(dy) + \int\limits_{y\in B^c} \varphi(y)\mathcal{P}_{\tilde{y}(a(\cdot),x)}(dy) \\
&\geq \int\limits_{y\in B^c} \varphi(y)\mathcal{P}_{\tilde{y}(a(\cdot),x)}(dy) \geq \varphi_0 \int\limits_{y\in B^c} \mathcal{P}_{\tilde{y}(a(\cdot),x)}(dy) \\
&= \varphi_0\mathcal{P}\left(\tilde{y}\left(a(\omega),x\right)\notin B\right) = \varphi_0\left(1-\mathcal{P}\left(\tilde{y}\left(a(\omega),x\right)\in B\right)\right),
\end{aligned}$$

which yields the assertion, cf. (1.51c).

Remark 1.2 Note that $\mathcal{P}(S) \geq \alpha_s$ with a given minimum reliability $\alpha_s \in (0,1]$ can be guaranteed by the expected cost constraint

$$E\varphi\left(\tilde{y}\left(a(\omega),x\right)\right) \leq (1-\alpha_s)\varphi_0.$$

Example 1.5 If $\varphi = 1_{B^c}$ is the indicator function of the complement B^c of B, then $\varphi_0 = 1$ and (1.52) holds with the equality sign.

Example 1.6 For a given positive definite $m \times m$ matrix C, define $\varphi(y) := y^TCy$, $y \in \mathbb{R}^m$. Then, cf. (1.51b, d, e),

$$\min_{y\notin B}\varphi(y) = \min_{1\leq i\leq m}\left\{\min_{y_i\geq 1} y^TCy,\ \min_{y_i\leq -1} y^TCy\right\}. \tag{1.53a}$$

Thus, the lower bound φ_0 follows by considering the convex optimization problems arising in the right hand side of (1.53a). Moreover, the expectation $E\varphi(\tilde{y})$ needed in (1.52) is given, see (1.51a), by

$$\begin{aligned}
E\varphi(\tilde{y}) &= E\tilde{y}^TC\tilde{y} = E\mathrm{tr}C\tilde{y}\tilde{y}^T, \\
&= \mathrm{tr}C(\mathrm{diag}\,\rho)^{-1}\left(\mathrm{cov}\ y\left(a(\cdot),x\right) + \left(\overline{y}(x)-y_c\right)\left(\overline{y}(x)-y_c\right)^T\right)(\mathrm{diag}\,\rho)^{-1},
\end{aligned} \tag{1.53b}$$

where "tr" denotes the trace of a matrix, diag ρ is the diagonal matrix diag $\rho := (\rho_i\delta_{ij})$, $y_c := (y_{ic})$, see (1.50e), and $\overline{y} = \overline{y}(x) := \left(Ey_i\left(a(\omega),x\right)\right)$. Since $\|Q\| \leq trQ \leq m\|Q\|$ for any positive definite $m \times m$ matrix Q, an upper bound of $E\varphi(\tilde{y})$

reads

$$E\varphi(\tilde{y}) \leq m\|C\| \, \|(\text{diag } \rho)^{-1}\|^2 \|\text{cov } y\Big(a(\cdot), x\Big) + \Big(\overline{y}(x) - y_c\Big)\Big(\overline{y}(x) - y_c\Big)^T \|. \tag{1.53c}$$

Example 1.7 Assuming in the above Example 1.6 that $C = \text{diag } (c_{ii})$ is a diagonal matrix with positive elements $c_{ii} > 0, i = 1, \ldots, m$, then

$$\min_{y \notin B} \varphi(y) = \min_{1 \leq i \leq m} c_{ii} > 0, \tag{1.53d}$$

and $E\varphi(\tilde{y})$ is given by

$$\begin{aligned} E\varphi(\tilde{y}) &= \sum_{i=1}^{m} c_{ii} \frac{E\Big(y_i\Big(a(\omega), x\Big) - y_{ic}\Big)^2}{\rho_i^2} \\ &= \sum_{i=1}^{m} c_{ii} \frac{\sigma_{y_i}^2\Big(a(\cdot), x\Big) + \Big(\overline{y}_i(x) - y_{ic}\Big)^2}{\rho_i^2}. \end{aligned} \tag{1.53e}$$

One-Sided Inequalities

Suppose that exactly one of the two bounds $y_{li} < y_{ui}$ is infinite for each $i = 1, \ldots, m$. Multiplying the corresponding constraints in (1.50a) by -1, the admissible domain $S = S(x)$, cf. (1.50b), can be represented always by

$$S(x) = \{a \in \mathbb{R}^\nu : \tilde{y}_i(a, x) < (\leq) \, 0, i = 1, \ldots, m\}, \tag{1.54a}$$

where $\tilde{y}_i := y_i - y_{ui}$, if $y_{li} = -\infty$, and $\tilde{y}_i := y_{li} - y_i$, if $y_{ui} = +\infty$. If we set $\tilde{y}(a, x) := \Big(\tilde{y}_i(a, x)\Big)$ and

$$\tilde{B} := \Big\{y \in \mathbb{R}^m : y_i < (\leq) \, 0, i = 1, \ldots, m\Big\}, \tag{1.54b}$$

then

$$\mathcal{P}\Big(S(x)\Big) = \mathcal{P}\left(\tilde{y}\Big(a(\omega), x\Big) \in \tilde{B}\right). \tag{1.54c}$$

Consider also in this case, cf. (1.51d, e), a function $\varphi : \mathbb{R}^m \to \mathbb{R}$ such that

$$i) \; \varphi(y) \geq 0, y \in \mathbb{R}^m \tag{1.55a}$$

$$ii) \varphi(y) \geq \varphi_0 > 0, \text{ if } y \notin \tilde{B}. \tag{1.55b}$$

Then, corresponding to Theorem 1.1, we have this result:

Theorem 1.2 (Markov-Type Inequality) *If $\varphi : \mathbb{R}^m \to \mathbb{R}$ is any (measurable) function fulfilling conditions (1.55a, b), then*

$$\mathcal{P}\left(\tilde{y}\Big(a(\omega), x\Big) < (\le)\ 0\right) \ge 1 - \frac{1}{\varphi_0} E\varphi\left(\tilde{y}\Big(a(\omega), x\Big)\right), \tag{1.56}$$

provided that the expectation in (1.56) exists and is finite.

Remark 1.3 Note that a related inequality was used already in Example 1.2.

Example 1.8 If $\varphi(y) := \sum_{i=1}^{m} w_i e^{\alpha_i y_i}$ with positive constants $w_i, \alpha_i, i = 1, \ldots, m$, then

$$\inf_{y \notin \tilde{B}} \varphi(y) = \min_{1 \le i \le m} w_i > 0 \tag{1.57a}$$

and

$$E\varphi\left(\tilde{y}\Big(a(\omega), x\Big)\right) = \sum_{i=1}^{m} w_i E e^{\alpha_i \tilde{y}_i \left(a(\omega), x\right)}, \tag{1.57b}$$

where the expectation in (1.57b) can be computed approximatively by Taylor expansion:

$$\begin{aligned} E e^{\alpha_i \tilde{y}_i} &= e^{\alpha_i \bar{\bar{y}}_i(x)} E e^{\alpha_i \left(\tilde{y}_i - \bar{\bar{y}}_i(x)\right)} \\ &\approx e^{\alpha_i \bar{\bar{y}}_i(x)} \left(1 + \frac{\alpha_i^2}{2} E\left(y_i\Big(a(\omega), x\Big) - \overline{y}_i(x)\right)^2\right) \\ &= e^{\alpha_i \bar{\bar{y}}_i(x)} \left(1 + \frac{\alpha_i^2}{2} \sigma^2_{y_i(a(\cdot), x)}\right). \end{aligned} \tag{1.57c}$$

Supposing that $y_i = y_i\Big(a(\omega), x\Big)$ is a normal distributed random variable, then

$$E e^{\alpha_i \tilde{y}_i} = e^{\alpha_i \bar{\bar{y}}_i(x)} e^{\frac{1}{2}\alpha_i^2 \sigma^2_{y_i(a(\cdot), x)}}. \tag{1.57d}$$

Example 1.9 Consider $\varphi(y) := (y - b)^T C (y - b)$, where, cf. Example 1.6, C is a positive definite $m \times m$ matrix and $b < 0$ a fixed m-vector. In this case we again have

$$\min_{y \notin \tilde{B}} \varphi(y) = \min_{1 \le i \le m} \min_{y_i \ge 0} \varphi(y)$$

and

$$E\varphi(\tilde{y}) = E\left(\tilde{y}\Big(a(\omega), x\Big) - b\right)^T C\left(\tilde{y}\Big(a(\omega), x\Big) - b\right)$$
$$= \mathrm{tr} CE\left(\tilde{y}\Big(a(\omega), x\Big) - b\right)\left(\tilde{y}\Big(a(\omega), x\Big) - b\right)^T.$$

Note that

$$\tilde{y}_i(a, x) - b_i = \begin{cases} y_i(a, x) - (y_{ui} + b_i), \text{ if } y_{li} = -\infty \\ y_{li} - b_i - y_i(a, x), \quad \text{ if } y_{ui} = +\infty, \end{cases}$$

where $y_{ui} + b_i < y_{ui}$ and $y_{li} < y_{li} - b_i$.

Remark 1.4 The one-sided case can also be reduced approximatively to the two-sided case by selecting a sufficiently large, but finite upper bound $\tilde{y}_{ui} \in \mathbb{R}$, lower bound $\tilde{y}_{li} \in \mathbb{R}$, resp., if $y_{ui} = +\infty$, $y_{li} = -\infty$.

Chapter 2
Optimal Control Under Stochastic Uncertainty

2.1 Stochastic Control Systems

Optimal control and regulator problems arise in many concrete applications (mechanical, electrical, thermodynamical, chemical, etc.) are modeled [5, 63, 136, 155] by dynamical control systems obtained from physical measurements and/or known physical laws. The basic control system (input–output system) is mathematically represented [73, 159] by a system of first order random differential equations:

$$\dot{z}(t) = g\Big(t, \omega, z(t), u(t)\Big), t_0 \le t \le t_f,\ \omega \in \Omega \tag{2.1a}$$

$$z(t_0) = z_0(\omega). \tag{2.1b}$$

Here, ω is the basic random element taking values in a probability space $(\Omega, \mathcal{A}, P)$, and describing the present random variations of model parameters or the influence of noise terms. The probability space $(\Omega, \mathcal{A}, P)$ consists of the sample space or set of elementary events Ω, the σ-algebra $\mathcal{A}$ of events and the probability measure P. The plant state vector $z = z(t, \omega)$ is an m-vector involving direct or indirect measurable/observable quantities like displacements, stresses, voltage, current, pressure, concentrations, etc., and their time derivatives (velocities), $z_0(\omega)$ is the random initial state. The plant control or control input $u(t)$ is a deterministic or stochastic n-vector denoting system inputs like external forces or moments, voltages, field current, thrust program, fuel consumption, production rate, etc. Furthermore, $\dot{z}$ denotes the derivative with respect to the time t. We assume that an input $u = u(t)$ is chosen such that $u(\cdot) \in U$, where U is a suitable linear space of input functions $u(\cdot) : [t_0, t_f] \to \mathbb{R}^n$ on the time interval $[t_0, t_f]$. Examples for U are subspaces of the space $PC_0^n[t_0, t_f]$ of piecewise continuous functions

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$u(\cdot) : [t_0, t_f] \to \mathbb{R}^n$ normed by the supremum norm

$$\left\| u(\cdot) \right\|_\infty = \sup \left\{ \| u(t) \| : t_0 \le t \le t_f \right\}.$$

Note that a function on a closed, bounded interval is called *piecewise continuous* if it is continuous up to at most a finite number of points, where the one-sided limits of the function exist. Other important examples for U are the Banach spaces of integrable, essentially bounded measurable or regulated [37] functions $L_p^n([t_0, t_f], \mathcal{B}^1, \lambda^1)$, $p \ge 1$, $L_\infty^n([t_0, t_f], \mathcal{B}^1, \lambda^1)$, $Reg([t_0, t_f]; \mathbb{R}^n)$, resp., on $[t_0, t_f]$. Here, $\big([t_0, t_f], \mathcal{B}^1, \lambda^1\big)$ denotes the measure space on $[t_0, t_f]$ with the σ-algebra $\mathcal{B}^1$ of Borel sets and the Lebesgue-measure λ^1 on $[t_0, t_f]$. Obviously, $PC_0^n[t_0, t_f] \subset L_\infty^n([t_0, t_f], \mathcal{B}^1, \lambda^1)$. If $u = u(t, \omega), t_0 \le t \le t_f$, is a random input function, then correspondingly we suppose that $u(\cdot, \omega) \in U$ a.s. (almost sure or with probability 1). Moreover, we suppose that the function $g = g(t, \omega, z, u)$ of the plant differential equation (2.1a) and its partial derivatives (Jacobians) $D_z g, D_u g$ with respect to z and u are at least measurable on the space $[t_0, t_f] \times \Omega \times \mathbb{R}^m \times \mathbb{R}^n$.

The possible trajectories of the plant, hence, absolutely continuous [118] m-vector functions, are contained in the linear space $Z = C_0^m[t_0, t_f]$ of continuous functions $z(\cdot) : [t_0, t_f] \to \mathbb{R}^m$ on $[t_0, t_f]$. The space Z contains the set $PC_1^m[t_0, t_f]$ of continuous, piecewise differentiable functions on the interval $[t_0, t_f]$. A function on a closed, bounded interval is called *piecewise differentiable* if the function is differentiable up to at most a finite number of points, where the function and its derivative have existing one-sided limits. The space Z is also normed by the supremum norm. $D(\subset U)$ denotes the convex set of admissible controls $u(\cdot)$, defined e.g. by some box constraints. Using the available information $\mathcal{A}_t$ up to a certain time t, the problem is then to find an optimal control function $u^* = u^*(t)$ being most insensitive with respect to random parameter variations. This can be obtained by minimizing the total (conditional) expected costs arising along the trajectory $z = z(t)$ and/or at the terminal state $z_f = z(t_f)$ subject to the plant differential equation (2.1a, b) and the required control and state constraints. Optimal controls being most insensitive with respect to random parameter variations are also called *robust* controls. Such controls can be obtained by stochastic optimization methods [100].

Since feedback (FB) control laws can be approximated very efficiently, cf. [2, 80, 136], by means of *open-loop feedback (OLF)* control laws, see Sect. 2.2, for practical purposes we may confine to the computation of deterministic stochastic optimal open-loop (OL) controls $u = u(\cdot; t_b), t_b \le t \le t_f$, on arbitrary *"remaining time intervals"* $[t_b, t_f]$ of $[t_0, t_f]$. Here, $u = u(\cdot; t_b)$ is stochastic optimal with respect to the information $\mathcal{A}_{t_b}$ at the "initial" time point t_b.

2.1.1 Random Differential and Integral Equations

In many technical applications the random variations is not caused by an additive white noise term, but by means of possibly time-dependent random parameters. Hence, in the following the dynamics of the control system is represented by random differential equation, i.e. a *system of ordinary differential equations (2.1a, b) with random parameters*. Furthermore, solutions of random differential equations are defined here in the parameter (point)-wise sense, cf. [10, 26].

In case of a *discrete* or *discretized* probability distribution of the random elements, model parameters, i.e., $\Omega = \{\omega_1, \omega_2, \dots, \omega_\varrho\}$, $P(\omega = \omega_j) = \alpha_j > 0$, $j = 1, \dots, \varrho$, $\sum_{j=1}^{\varrho} \alpha_j = 1$, we can *redefine* (2.1a, b) by

$$\dot{z}(t) = g\Big(t, z(t), u(t)\Big), t_0 \le t \le t_f, \tag{2.1c}$$

$$z(t_0) = z_0. \tag{2.1d}$$

with the vectors and vector functions

$$z(t) := \Big(z(t, \omega_j)\Big)_{j=1,\dots,\varrho}, z_0 := \Big(z_0(\omega_j)\Big)_{j=1,\dots,\varrho}$$

$$g(t, z, u) := \Big(g(t, \omega_j, z_{(j)}, u)\Big)_{j=1,\dots,\varrho}, z := (z_{(j)})_{j=1,\dots,\varrho} \in \mathbb{R}^{\varrho m}.$$

Hence, in this case (2.1c, d) represents again an ordinary system of first order differential equations for the ϱm unknown functions

$$z_{ij} = z_i(t, \omega_j), i = 1, \dots, m, j = 1, \dots, \varrho.$$

Results on the existence and uniqueness of the systems (2.1a, b) and (2.1c, d) and their dependence on the inputs can be found in [37].

Also in the general case we consider a *solution in the point-wise sense*. This means that for each random element $\omega \in \Omega$, (2.1a, b) is interpreted as a system of ordinary first order differential equations with the initial values $z_0 = z_0(\omega)$ and control input $u = u(t)$. Hence, we assume that to each deterministic control $u(\cdot) \in U$ and each random element $\omega \in \Omega$ there exists a unique solution

$$z(\cdot, \omega) = S\Big(\omega, u(\cdot)\Big) = S\Big(\omega, u(\cdot)\Big), \tag{2.2a}$$

$z(\cdot, \omega) \in C_0^m[t_0, t_f]$, of the integral equation

$$z(t) = z_0(\omega) + \int_{t_0}^{t} g\Big(s, \omega, z(s), u(s)\Big) ds, \; t_0 \le t \le t_f, \tag{2.2b}$$

such that $(t, \omega) \to S\Big(\omega, u(\cdot)\Big)(t)$ is measurable. This solution is also denoted by

$$z(t, \omega) = z_u(t, \omega) = z(t, \omega, u(\cdot)), \; t_0 \le t \le t_f. \tag{2.2c}$$

Obviously, the integral equation (2.2b) is the integral version of the initial value problem (2.1a, b): Indeed, if, for given $\omega \in \Omega$, $z = z(t, \omega)$ is a solution of (2.1a, b), i.e., $z(\cdot, \omega)$ is absolutely continuous, satisfies (2.1a) for almost all $t \in [t_0, t_f]$ and fulfills (2.1b), then $z = z(t, \omega)$ is also a solution of (2.2b). Conversely, if, for given $\omega \in \Omega$, $z = z(t, \omega)$ is a solution of (2.2b), such that the integral on the right hand side exists in the Lebesgue-sense for each $t \in [t_0, t_f]$, then this integral as a function of the upper bound t and therefore also the function $z = z(t, \omega)$ is absolutely continuous. Hence, by taking $t = t_0$ and by differentiation of (2.2b) with respect to t, cf. [118], we have that $z = z(t, \omega)$ is also a solution of (2.1a, b).

Parametric Representation of the Random Differential/Integral Equation

In the following we want to justify the above assumption that the initial value problem (2.1a, b), the equivalent integral equation (2.2b), resp., has a unique solution $z = z(t, \omega)$. For this purpose, let $\theta = \theta(t, \omega)$ be an r-dimensional stochastic process, as e.g. time-varying disturbances, random parameters, etc., of the system, such that the sample functions $\theta(\cdot, \omega)$ are continuous with probability one. Furthermore, let

$$\tilde{g} : [t_0, t_f] \times \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$$

be a continuous function having continuous Jacobians $D_\theta \tilde{q}, D_z \tilde{g}, D_u \tilde{g}$ with respect to θ, z, u. Now consider the case that the function g of the process differential equation (2.1a, b) is given by

$$g(t, \omega, z, u) := \tilde{g}\Big(t, \theta(t, \omega), z, u\Big),$$

$(t, \omega, z, u) \in [t_0, t_f] \times \Omega \times \mathbb{R}^m \times \mathbb{R}^n$. The spaces U, Z of possible inputs, trajectories, resp., of the plant are chosen as follows: $U := Reg([t_0, t_f]; \mathbb{R}^n)$ is the Banach space of all regulated functions $u(\cdot) : [t_0, t_f] \to \mathbb{R}^n$, normed by the supremum norm $\|\cdot\|_\infty$.

Furthermore, we set $Z := C_0^m[t_0, t_f]$ and $\Theta := C_0^r[t_0, t_f]$. Here, for an integer ν, $C_0^\nu[t_0, t_f]$ denotes the Banach space of all continuous functions of $[t_0, t_f]$ into $\mathbb{R}^\nu$ normed by the supremum norm $\| \cdot \|_\infty$. By our assumption we have $\theta(\cdot, \omega) \in \Theta$ a.s. (almost sure). Define

$$\Xi = \mathbb{R}^m \times \Theta \times U;$$

Ξ is the space of possible *initial values, time-varying model/environmental parameters and inputs* of the dynamic system. Hence, Ξ may be considered as the total space of inputs

$$\xi := \begin{pmatrix} z_0 \\ \theta(\cdot) \\ u(\cdot) \end{pmatrix}$$

into the plant, consisting of the random initial state z_0, the random input function $\theta = \theta(t, \omega)$ and the control function $u = u(t)$. Let now the mapping $\tau : \Xi \times Z \to Z$ related to the plant equation (2.1a, b) or (2.2b) be given by

$$\tau\Big(\xi, z(\cdot)\Big)(t) = z(t) - \left(z_0 + \int_{t_0}^{t} \tilde{g}\Big(s, \theta(s), z(s), u(s)\Big) ds \right), t_0 \le t \le t_f. \tag{2.2d}$$

Note that for each input vector $\xi \in \Xi$ and function $z(\cdot) \in Z$ the integrand in (2.2d) is piecewise continuous, bounded or at least essentially bounded on $[t_0, t_f]$. Hence, the integral in (2.2d) as a function of its upper limit t yields again a continuous function on the interval $[t_0, t_f]$, and therefore an element of Z. This shows that τ maps $\Xi \times Z$ into Z.
Obviously, the initial value problem (2.1a, b) or its integral form (2.2b) can be represented by the operator equation

$$\tau(\xi, z(\cdot)) = 0. \tag{2.2e}$$

Operators of the type (2.2d) are well studied, see e.g. [37, 85]: It is known that τ is continuously Fréchet (F)-differentiable [37, 85]. Note that the F-differential is a generalization of the derivatives (Jacobians) of mappings between finite-dimensional spaces to mappings between arbitrary normed spaces. Thus, the F-derivative $D\tau$ of τ at a certain point $\Big(\bar{\xi}, \bar{z}(\cdot)\Big)$ is given by

$$\begin{aligned}
&\left(D\tau\Big(\bar{\xi}, \bar{z}(\cdot)\Big) \cdot \Big(\xi, z(\cdot)\Big) \right)(t) \\
&= z(t) - \left(z_0 + \int_{t_0}^{t} D_z\tilde{g}\Big(s, \bar{\theta}(s), \bar{z}(s), \bar{u}(s)\Big) z(s) ds \right. \\
&+ \int_{t_0}^{t} D_\theta \tilde{g}\Big(s, \bar{\theta}(s), \bar{z}(s), \bar{u}(s)\Big) \theta(s) ds \\
&+ \left. \int_{t_0}^{t} D_u \tilde{g}\Big(s, \bar{\theta}(s), \bar{z}(s), \bar{u}(s)\Big) u(s) ds \right), t_0 \le t \le t_f,
\end{aligned} \tag{2.2f}$$

where $\bar{\xi} = \left(\bar{z}_0, \bar{\theta}(\cdot), \bar{u}(\cdot)\right)$ and $\xi = \left(z_0, \theta(\cdot), u(\cdot)\right)$. Especially, for the derivative of τ with respect to $z(\cdot)$ we find

$$\left(D_z\tau\left(\bar{\xi}, \bar{z}(\cdot)\right) \cdot z(\cdot)\right)(t) = z(t) - \int_{t_0}^{t} D_z\tilde{g}\left(s, \bar{\theta}(s), \bar{z}(s), \bar{u}(s)\right)z(s)ds, t_0 \le t \le t_f. \tag{2.2g}$$

The related equation

$$D_z\tau\left(\bar{\xi}, \bar{z}(\cdot)\right) \cdot z(\cdot) = y(\cdot), y(\cdot) \in Z, \tag{2.2h}$$

is a linear vectorial Volterra integral equation. By our assumptions this equation has a unique solution $z(\cdot) \in Z$. Note that the corresponding result for scalar Volterra equations, see e.g. [151], can be transferred to the present vectorial case. Therefore, $D_z\tau\left(\bar{\xi}, \bar{z}(\cdot)\right)$ is a linear, continuous one-to-one map from Z onto Z. Hence, its inverse $\left(D_z\tau\left(\bar{\xi}, \bar{z}(\cdot)\right)\right)^{-1}$ exists. Using the implicit function theorem [37, 85], we obtain now the following result:

Lemma 2.1 *For given $\bar{\xi} = \left(\bar{z}_0, \bar{\theta}(\cdot), \bar{u}(\cdot)\right)$, let $\left(\bar{\xi}, \bar{z}(\cdot)\right) \in \Xi \times Z$ be selected such that $\tau\left(\bar{\xi}, \bar{z}(\cdot)\right) = 0$, hence, $\bar{z}(\cdot) \in Z$ is supposed to be the solution of*

$$\dot{z}(t) = \tilde{g}\left(t, \bar{\theta}(t), z(t), \bar{u}(t)\right), t_0 \le t \le t_f, \tag{2.3a}$$

$$z(t_0) = \bar{z}_0 \tag{2.3b}$$

in the integral sense (2.2b). Then there is an open neighborhood of $\bar{\xi}$, denoted by $V^0(\bar{\xi})$, such that for each open connected neighborhood $V(\bar{\xi})$ of $\bar{\xi}$ contained in $V^0(\bar{\xi})$ there exists a unique continuous mapping $S : V(\bar{\xi}) \to Z$ such that a) $S(\bar{\xi}) = \bar{z}(\cdot)$; b) $\tau\left(\xi, S(\xi)\right) = 0$ for each $\xi \in V(\bar{\xi})$, i.e. $S(\xi) = S(\xi)(t), t_0 \le t \le t_f$, is the solution of

$$z(t) = z_0 + \int_{t_0}^{t} \tilde{g}\left(s, \theta(s), z(s), u(s)\right)ds, t_0 \le t \le t_f, \tag{2.3c}$$

where $\xi = \left(z_0, \theta(\cdot), u(\cdot)\right)$; c) S is continuously differentiable on $V(\bar{\xi})$, and it holds

$$D_uS(\xi) = -\left(D_z\tau\left(\xi, S(\xi)\right)\right)^{-1} D_u\tau\left(\xi, S(\xi)\right), \xi \in V(\bar{\xi}). \tag{2.3d}$$

An immediate consequence is given next:

Corollary 2.1 *The directional derivative* $\zeta(\cdot) = \zeta_{u,h}(\cdot) = D_u S(\xi)h(\cdot)(\in Z), h(\cdot) \in U$, *satisfies the integral equation*

$$\zeta(t) - \int_{t_0}^{t} D_z\tilde{g}\Big(s, \theta(s), S(\xi)(s), u(s)\Big)\zeta(s)ds$$

$$= \int_{t_0}^{t} D_u\tilde{g}\Big(s, \theta(s), S(\xi)(s), u(s)\Big)h(s)ds, \tag{2.3e}$$

where $t_0 \le t \le t_f$ *and* $\xi = \Big(z_0, \theta(\cdot), u(\cdot)\Big)$.

Remark 2.1 Taking the time derivative of Eq. (2.3e) shows that this integral equation is equivalent to the so-called *perturbation equation*, see e.g. [73].

For an arbitrary $h(\cdot) \in U$ the mappings

$$(t, \xi) \to S(\xi)(t),\ (t, \xi) \to \Big(D_u S(\xi)h(\cdot)\Big)(t), (t, \xi) \in [t_0, t_f] \times V(\bar{\xi}), \tag{2.3f}$$

are continuous and therefore also measurable.

The existence of a unique solution $\bar{z} = \bar{z}(t), t_0 \le t \le t_f$, of the reference differential equation (2.3a, b) can be guaranteed as follows, where *solution* is interpreted in the integral sense, i.e., $\bar{z} = \bar{z}(t), t_0 \le t \le t_f$, is absolutely continuous, satisfies equation (2.3a) almost everywhere in the time interval $[t_0, t_f]$ and the initial condition (2.3b), cf. [29] and [168].

Lemma 2.2 *Consider an arbitrary input vector* $\bar{\xi} = \Big(\bar{z}_0, \bar{\theta}(\cdot), \bar{u}(\cdot)\Big) \in \Xi$, *and define, see (2.3a, b), the function* $\tilde{g}_{\bar{\theta}(\cdot),\bar{u}(\cdot)} = \tilde{g}_{\bar{\theta}(\cdot),\bar{u}(\cdot)}\big(t, z\big) := \tilde{g}\Big(t, \bar{\theta}(t), z, \bar{u}(t)\Big)$. *Suppose that i)* $z \to \tilde{g}_{\bar{\theta}(\cdot),\bar{u}(\cdot)}\big(t, z\big)$ *is continuous for each time* $t \in [t_0, t_f]$, *ii)* $t \to \tilde{g}_{\bar{\theta}(\cdot),\bar{u}(\cdot)}\big(t, z\big)$ *is measurable for each vector z, iii) generalized Lipschitz-condition: For each closed sphere* $K \subset \mathbb{R}^n$ *there exists a nonnegative, integrable function* $L_S(\cdot)$ *on* $[t_0, t_f]$ *such that iii1)* $\|\tilde{g}_{\bar{\theta}(\cdot),\bar{u}(\cdot)}\big(t, 0\big)\| \le L_K(t)$, *and* $\|\tilde{g}_{\bar{\theta}(\cdot),\bar{u}(\cdot)}\big(t, z\big) - \tilde{g}_{\bar{\theta}(\cdot),\bar{u}(\cdot)}\big(t, w\big)\| \le L_K(t)\|z - w\|$ *on* $[t_0, t_f] \times K$. *Then, the initial value problem (2.3a, b) has a unique solution* $\bar{z} = \bar{z}(t; \bar{\xi})$.

Proof Proofs can be found in [29] and [168].

We observe that the controlled stochastic process $z = z(t, \omega)$ defined by the plant differential equation (2.1a, b) may be a non Markovian stochastic process, see [5, 63]. Moreover, note that the random input function $\theta = \theta(t, \omega)$ is not just an additive noise term, but may describe also a disturbance which is part of the nonlinear dynamics of the plant, random varying model parameters such as material, load or cost parameters, etc.

Stochastic Differential Equations

In some applications [5], instead of the system (2.1a, b) of ordinary differential equations with (time-varying) random parameters, a so–called stochastic differential equation [121] is taken into account:

$$dz(t,\omega) = \tilde{g}\Big(t, z(t,\omega), u(t)\Big)dt + \tilde{h}\Big(t, z(t,\omega), u(t)\Big)d\theta(t,\omega). \tag{2.4a}$$

Here, the "noise" term $\theta = \theta(t,\omega)$ is a certain stochastic process, as e.g. the Brownian motion, having continuous paths, and $\tilde{g} = \tilde{g}(t,z,u), \tilde{h} = \tilde{h}(t,z,u)$ are given, sufficiently smooth vector/matrix functions.

Corresponding to the integral equation (2.2a, b), the above stochastic differential equation is replaced by the stochastic integral equation

$$z(t,\omega) = z_0(\omega) + \int_{t_0}^{t} \tilde{g}\Big(s, z(s,\omega), u(s)\Big)ds + \int_{t_0}^{t} \tilde{h}\Big(s, z(s,\omega), u(s)\Big)d\theta(s,\omega). \tag{2.4b}$$

The meaning of this equation depends essentially on the definition (interpretation) of the "stochastic integral"

$$I\Big(\xi(\cdot,\cdot), z(\cdot,\cdot)\Big)(t) := \int_{t_0}^{t} \tilde{h}\Big(s, z(s,\omega), u(s)\Big)d\theta(s,\omega). \tag{2.4c}$$

Note that in case of closed loop and open-loop feedback controls, see the next Sect. 2.2, the control function $u = u(s,\omega)$ is random. If

$$\tilde{h}(s,z,u) = \tilde{h}\Big(s, u(s)\Big) \tag{2.5a}$$

with a deterministic control $u = u(t)$ and a matrix function $\tilde{h} = \tilde{h}(s,u)$, such that $s \to \tilde{h}\Big(s, u(s)\Big)$ is differentiable, by partial integration we get, cf. [121],

$$\begin{aligned} I\Big(\xi, z(\cdot)\Big)(t) = I\Big(\theta(\cdot), u(\cdot)\Big)(t) &= \tilde{h}\Big(t, u(t)\Big)\theta(t) - \tilde{h}\Big(t_0, u(t_0)\Big)\theta(t_0) \\ &\quad - \int_{t_0}^{t} \theta(s)\frac{d}{ds}\tilde{h}\Big(s, u(s)\Big)ds, \end{aligned} \tag{2.5b}$$

where $\theta = \theta(s)$ denotes a sample function of the stochastic process $\theta = \theta(t, \omega)$. Hence, in this case the operator $\tau = \tau\Big(\xi, z(\cdot)\Big)$, cf. (2.2d), is defined by

$$\tau\Big(\xi, z(\cdot)\Big)(t) := z(t) - \Bigg(z_0 + \int_{t_0}^{t} \tilde{g}\Big(s, z(s), u(s)\Big)ds + \tilde{h}\Big(t, u(t)\Big)\theta(t)$$
$$-\tilde{h}\Big(t_0, u(t_0)\Big)\theta(t_0) - \int_{t_0}^{t} \theta(s)\frac{d}{ds}\tilde{h}\Big(s, u(s)\Big)ds\Bigg). \tag{2.5c}$$

Obviously, for the consideration of the existence and differentiability of solutions $z(\cdot) = z(\cdot, \xi)$ of the operator equation $\tau\Big(\xi, z(\cdot)\Big) = 0$, the same procedure as in Sect. 2.1.1 may be applied.

2.1.2 Objective Function

The aim is to obtain optimal controls being **robust**, i.e., most insensitive with respect to stochastic variations of the model/environmental parameters and initial values of the process. Hence, incorporating stochastic parameter variations into the optimization process, for a *deterministic* control function $u = u(t), t_0 \le t \le t_f$, the objective function $F = F\Big(u(\cdot)\Big)$ of the controlled process $z = z(t, \omega, u(\cdot))$ is defined, cf. [100], by the conditional expectation of the total costs arising along the whole control process:

$$F\Big(u(\cdot)\Big) := Ef\Big(\omega, S\big(\omega, u(\cdot)\big), u(\cdot)\Big). \tag{2.6a}$$

Here, $E = E(\cdot|\mathcal{A}_{t_0})$, denotes the conditional expectation given the information $\mathcal{A}_{t_0}$ about the control process up to the considered starting time point t_0. Moreover, $f = f\Big(\omega, z(\cdot), u(\cdot)\Big)$ denote the stochastic total costs arising along the trajectory $z = z(t, \omega)$ and at the terminal point $z_f = z(t_f, \omega)$, cf. [5, 159]. Hence,

$$f\Big(\omega, z(\cdot), u(\cdot)\Big) := \int_{t_0}^{t_f} L\Big(t, \omega, z(t), u(t)\Big)dt + G\Big(t_f, \omega, z(t_f)\Big), \tag{2.6b}$$

$z(\cdot) \in Z, u(\cdot) \in U$. Here,

$$\begin{aligned} L &: [t_0, t_f] \times \Omega \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}, \\ G &: [t_0, t_f] \times \Omega \times \mathbb{R}^m \qquad\quad\; \to \mathbb{R} \end{aligned}$$

are given measurable cost functions. We suppose that $L(t,\omega,\cdot,\cdot)$ and $G(t,\omega,\cdot)$ are *convex functions* for each $(t,\omega)\in[t_0,t_f]\times\Omega$, having continuous partial derivatives $\nabla_z L(\cdot,\omega,\cdot,\cdot)$, $\nabla_u L(\cdot,\omega,\cdot,\cdot)$, $\nabla_z G(\cdot,\omega,\cdot)$. Note that in this case

$$\Big(z(\cdot),u(\cdot)\Big)\to\int_{t_0}^{t_f} L\Big(t,\omega,z(t),u(t)\Big)dt+G\Big(t_f,\omega,z(t_f)\Big) \tag{2.6c}$$

is a convex function on $Z\times U$ for each $\omega\in\Omega$. Moreover, assume that the expectation $F\Big(u(\cdot)\Big)$ exists and is finite for each admissible control $u(\cdot)\in D$.

In the case of random inputs $u=u(t,\omega), t_0\le t\le t_f, \omega\in\Omega$, with definition (2.6b), the objective function $F=F\Big(u(\cdot,\cdot)\Big)$ reads

$$F\Big(u(\cdot,\cdot)\Big):=Ef\bigg(\omega,S\Big(\omega,u(\cdot,\omega)\Big),u(\cdot,\omega)\bigg). \tag{2.6d}$$

Example 2.1 (Tracking Problems) If a trajectory $z_f=z_f(t,\omega)$, e.g., the trajectory of a moving target, known up to a certain stochastic uncertainty, must be followed or reached during the control process, then the cost function L along the trajectory can be defined by

$$L\Big(t,\omega,z(t),u\Big):=\Big\|\Gamma_z\Big(z(t)-z_f(t,\omega)\Big)\Big\|^2+\varphi(u). \tag{2.6e}$$

In (2.6e) Γ_z is a weight matrix, and $\varphi=\varphi(u)$ denotes the control costs, as e.g.

$$\varphi(u)=\|\Gamma_u u\|^2 \tag{2.6f}$$

with a further weight matrix Γ_u.

If a random target $z_f=z_f(\omega)$ has to be reached at the terminal time point t_f only, then the terminal cost function G may be defined e.g. by

$$G\Big(t_f,\omega,z(t_f)\Big):=\Big\|G_f\Big(z(t_f)-z_f(\omega)\Big)\Big\|^2 \tag{2.6g}$$

with a weight matrix G_f.

Example 2.2 (Active Structural Control, Control of Robots) In case of active structural control or for optimal regulator design of robots, cf. [98, 155], the total cost function f is given by defining the individual cost functions L and G as follows:

$$L(t,\omega,z,u):=z^TQ(t,\omega)z+u^TR(t,\omega)u \tag{2.6h}$$

$$G(t_f,\omega,z):=G(\omega,z). \tag{2.6i}$$

Here, $Q = Q(t, \omega)$ and $R = R(t, \omega)$, resp., are certain positive (semi)definite $m \times m, n \times n$ matrices which may depend also on (t, ω). Moreover, the terminal cost function G depends then on (ω, z). For example, in case of endpoint control, the cost function G is given by

$$G(\omega, z) = (z - z_f)^T G_f(\omega)(z - z_f) \tag{2.6j}$$

with a certain desired, possibly random terminal point $z_f = z_f(\omega)$ and a positive (semi)definite, possibly random weight matrix $G_f = G_f(\omega)$.

Optimal Control Under Stochastic Uncertainty

For finding *optimal controls being robust with respect to stochastic parameter variations* $u^{\star}(\cdot), u^{\star}(\cdot, \cdot)$, resp., in this chapter we are presenting now several methods for approximation of the following minimum expected total cost problem:

$$\min F\Big(u(\cdot)\Big) \text{ s.t. } u(\cdot) \in D, \tag{2.7}$$

$$\min F\Big(u(\cdot, \cdot)\Big) \text{ s.t. } u(\cdot, \omega) \in D \quad \text{a.s. (almost sure)},$$

$$u(t, \cdot)\ \mathcal{A}_t\text{-measurable}, \tag{$\widetilde{2.7}$}$$

where $\mathcal{A}_t \subset \mathcal{A}, t \geq t_0$, denotes the σ-algebra of events $A \in \mathcal{A}$ until time t.

Information Set $\mathcal{A}_t$ at Time t: In many cases, as e.g. for *PD*- and *PID* -controllers, see Chap. 4, the information σ-algebra $\mathcal{A}_t$ is given by $\mathcal{A}_t = \mathcal{A}(y(t, \cdot))$, where $y = y(t, \omega)$ denotes the $\bar{m}$-vector function of *state-measurements or -observations* at time t. Then, an $\mathcal{A}_t$-measurable control $u = u(t, \omega)$ has the representation, cf. [12],

$$u(t, \omega) = \eta(t, y(t, \omega)) \tag{2.8}$$

with a measurable function $\eta(t, \cdot) : \mathbb{R}^{\bar{m}} \to \mathbb{R}^m$.

Since parameter-insensitive optimal controls can be obtained by stochastic optimization methods incorporating random parameter variations into the optimization procedure, see [100], the aim is to determine *stochastic optimal controls*:

Definition 2.1 An optimal solution of the expected total cost minimization problem (2.7), ($\widetilde{2.7}$), resp., providing robust optimal controls, is called a **stochastic optimal control**.

Note. For controlled processes working on a time range $t_b \leq t \leq t_f$ with an intermediate starting time point t_b, the objective function $F = F(u(\cdot))$ is defined also by (2.6a), but with the conditional expectation operator $E = E(\cdot|\mathcal{A}_{t_b})$, where $\mathcal{A}_{t_b}$ denotes the information about the controlled process available up to time t_b.

Problem (2.7) is of course equivalent $\left(E = E(\cdot|\mathcal{A}_{t_0})\right)$ to the *optimal control problem under stochastic uncertainty*:

$$\min E\left(\int_{t_0}^{t_f} L\Big(t,\omega,z(t),u(t)\Big)dt + G\Big(t_f,\omega,z(t_f)\Big)\Big|\mathcal{A}_{t_0}\right) \tag{2.9a}$$

s.t.

$$\dot{z}(t) = g\Big(t,\omega,z(t),u(t)\Big), t_0 \le t \le t_f, \text{ a.s.} \tag{2.9b}$$

$$z(t_0,\omega) = z_0(\omega), \text{ a.s.} \tag{2.9c}$$

$$u(\cdot) \in D, \tag{2.9d}$$

cf. [89, 90].

Remark 2.2 Similar representations can be obtained also for the second type of stochastic control problem $(\widetilde{2.7})$.

Remark 2.3 (State Constraints) In addition to the plant differential equation (dynamic equation) (2.9b, c) and the control constraints (2.9d), we may still have some stochastic state constraints

$$h_I\Big(t,\omega,z(t,\omega)\Big) \le (=) 0 \text{ a.s.} \tag{2.9e}$$

as well as state constraints involving (conditional) expectations

$$Eh_{II}\Big(t,\omega,z(t,\omega)\Big) = E\left(h_{II}\Big(t,\omega,z(t,\omega)\Big)\Big|\mathcal{A}_{t_0}\right) \le (=)0. \tag{2.9f}$$

Here, $h_I = h_I(t,\omega,z), h_{II} = h_{II}(t,\omega,z)$ are given vector functions of (t,ω,z). By means of (penalty) cost functions, the random condition (2.9e) can be incorporated into the objective function (2.9a). As explained in Sect. 2.8, the expectations arising in the mean value constraints (2.9f) and in the objective function (2.9a) can be computed approximatively by means of Taylor expansion with respect to the vector $\vartheta = \vartheta(\omega) := (z_0(\omega), \theta(\omega)$ of random initial values and model parameters at the conditional mean $\overline{\vartheta} = \overline{\vartheta}^{(t_0)} := E\Big(\vartheta(\omega)|\mathcal{A}_{t_0}\Big)$. This yields then ordinary deterministic constraints for the extended deterministic trajectory (nominal state and sensitivity)

$$t \to \Big(z(t,\overline{\vartheta}), D_\vartheta z(t,\overline{\vartheta})\Big), t \ge t_0.$$

2.2 Control Laws

Control or guidance usually refers [5, 73, 85] to direct influence on a dynamic system to achieve desired performance. In optimal control of dynamic systems mostly the following types of *control laws* or *control policies* are considered:

I) *Open-Loop Control (OL)*

Here, the control function $u = u(t)$ is a **deterministic** function depending only on the (a priori) information $\mathcal{I}_{t_0}$ about the system, the model parameters, resp., available at the starting time point t_0. Hence, for the optimal selection of optimal (OL) controls

$$u(t) = u\Big(t; t_0, \mathcal{I}_{t_0}\Big), t \geq t_0, \tag{2.10a}$$

we get optimal control problems of type (2.7).

II) *Closed-Loop Control (CL) or Feedback Control*

In this case the control function $u = u(t)$ is a **stochastic** function

$$u = u(t, \omega) = u(t, \mathcal{I}_t), t \geq t_0 \tag{2.10b}$$

depending on time t and the total information $\mathcal{I}_t$ about the system available up to time t. Especially $\mathcal{I}_t$ may contain information about the state $z(t) = z(t, \omega)$ up to time t. Optimal (CL) or feedback controls are obtained by solving problems of type $(\widetilde{2.7})$.

Remark 2.4 Information Set $\mathcal{A}_t$ at Time t: Often the information $\mathcal{I}_t$ available up to time t is described by the *information set* or $\sigma-$algebra $\mathcal{A}_t \subset \mathcal{A}$ of events A occurred up to time t. In the important case $\mathcal{A}_t = \mathcal{A}(y(t, \cdot))$, where $y = y(t, \omega)$ denotes the $\bar{m}$-vector function of *state-measurements or -observations* at time t, then an $\mathcal{A}_t-$ measurable control $u = u(t, \omega)$, see problem $(\widetilde{2.7})$, has the representation, cf. [12],

$$u(t, \omega) = \eta_t(y(t, \omega)) \tag{2.10c}$$

with a measurable function $\eta_t : \mathbb{R}^{\bar{m}} \to \mathbb{R}^m$. Important examples of this type are the *PD*- and *PID* -controllers, see Chap. 4.

III) *Open-Loop Feedback (OLF) Control/Stochastic Open-Loop Feedback (SOLF) Control*

Due to their large complexity, in general, optimal feedback control laws can be determined approximatively only. A very efficient approximation procedure for optimal feedback controls, being functions of the information $\mathcal{I}_t$, is the approximation by means of optimal open-loop controls. In this combination of (OL) and (CL) control, at each intermediate time point $t_b := t, t_0 \leq t \leq t_f$, given the information $\mathcal{I}_t$ up to time t, first the open-loop

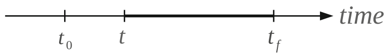

Fig. 2.1 Remaining time interval for intermediate time points t

control function for the remaining time interval $t \le s \le t_f$, see Fig. 2.1, is computed, hence,

$$u_{[t,t_f]}(s) = u\Big(s; t, \mathcal{I}_t\Big), s \ge t. \tag{2.10d}$$

Then, an approximate feedback control policy, originally proposed by Dreyfus (1964), cf. [40], can be defined as follows:

Definition 2.2 The hybrid control law, defined by

$$\varphi(t, \mathcal{I}_t) := u\Big(t; t, \mathcal{I}_t\Big), t \ge t_0, \tag{2.10e}$$

is called *open-loop feedback (OLF) control law*.

Thus, the OL control $u_{[t,t_f]}(s), s \ge t$, for the remaining time interval $[t, t_f]$ is used only at time $s = t$, see also [2, 40–42, 51, 80, 167]. Optimal (OLF) controls are obtained therefore by solving again control problems of the type (2.7) at each intermediate starting time point $t_b := t, t \in [t_0, t_f]$.

A major issue in optimal control is the **robustness**, cf. [43], i.e., the insensitivity of an optimal control with respect to parameter variations. In case of random parameter variations robust optimal controls can be obtained by means of stochastic optimization methods, cf. [100], incorporating the probability distribution, i.e., the random characteristics, of the random parameter variation into the optimization process, cf. Definition 2.1.

Thus, constructing stochastic optimal open-loop feedback controls, hence, optimal open-loop feedback control laws being insensitive as far as possible with respect to random parameter variations, means that besides the optimality of the control policy also its insensitivity with respect to stochastic parameter variations should be guaranteed. Hence, in the following sections we also develop a **stochastic version** of the optimal open-loop feedback control method, cf. [102–105]. A short overview on this novel stochastic optimal open-loop feedback control concept is given below:

At each intermediate time point $t_b = t \in [t_0, t_f]$, based on the given process observation $\mathcal{I}_t$, e.g. the observed state $z_t = z(t)$ at $t_b = t$, a stochastic optimal open-loop control $u^* = u^*(s) = u^*(s; t, \mathcal{I}_t), t \le s \le t_f$, is determined first on the remaining time interval $[t, t_f]$, see Fig. 2.1, by stochastic optimization methods, cf. [100].

Having a stochastic optimal open-loop control $u^* = u^*(s; t, \mathcal{I}_t), t \le s \le t_f$, on each remaining time interval $[t, t_f]$ with an arbitrary starting time point

t, $t_0 \le t \le t_f$, a *stochastic optimal open-loop feedback (SOLF) control law* is then defined—corresponding to Definition 2.2—as follows:

Definition 2.3 The hybrid control law, defined by

$$\varphi^*(t, \mathcal{I}_t) := u^*\Big(t; t, \mathcal{I}_t\Big), t \ge t_0. \tag{2.10f}$$

is called the *stochastic optimal open-loop feedback (SOLF) control law*.

Thus, at time $t_b = t$ just the "first" control value $u^*(t) = u^*(t; t, \mathcal{I}_t)$ of $u^* = u^*(\cdot; t, \mathcal{I}_t)$ is used only.

For finding stochastic optimal open-loop controls, on the remaining time intervals $t_b \le t \le t_f$ with $t_0 \le t_b \le t_f$, the stochastic Hamilton function of the control problem is introduced. Then, the class of H-minimal controls, cf. [73], can be determined in case of stochastic uncertainty by solving a finite-dimensional stochastic optimization problem for minimizing the conditional expectation of the stochastic Hamiltonian subject to the remaining deterministic control constraints at each time point t. Having a H-minimal control, the related two-point boundary value problem with random parameters will be formulated for the computation of a stochastic optimal state- and costate-trajectory. In the important case of a linear-quadratic structure of the underlying control problem, the state and costate trajectory can be **determined analytically** to a large extent. Inserting then these trajectories into the H-minimal control, stochastic optimal open-loop controls are found on an arbitrary remaining time interval. According to Definition 2.2, these controls yield then immediately a stochastic optimal open-loop feedback control law. Moreover, the obtained controls can be realized in **real-time**, which is already shown for applications in optimal control of industrial robots, cf. [139].

III.1) *Nonlinear Model Predictive Control (NMPC)/Stochastic Nonlinear Model Predictive Control (SNMPC)*

Optimal open-loop feedback (OLF) control is the basic tool in *Nonlinear Model Predictive Control (NMPC)*. Corresponding to the approximation technique for feedback controls described above, (NMPC) is a method to solve complicated feedback control problems by means of stepwise computations of open-loop controls. Hence, in (NMPC), see [1, 45, 49, 136] optimal open-loop controls

$$u = u_{[t,t+T_p]}(s), t \le s \le t + T_p, \tag{2.10g}$$

cf. (2.10c), are determined first on the time interval $[t, t + T_p]$ with a certain so-called *prediction time horizon* $T_p > 0$. In sampled-data MPC, cf. [45], optimal open-loop controls $u = u_{[t_i, t_i + T_p]}$, are determined at certain sampling instants $t_i, i = 0, 1, \ldots$, using the information $\mathcal{A}_{t_i}$ about the control process and its neighborhood up to time $t_i, i = 0, 1, \ldots$, see also [98]. The optimal open-loop control at stage "i" is applied then,

$$u = u_{[t_i, t_i + T_p]}(t), t_i \leq t \leq t_{i+1}, \tag{2.10h}$$

until the next sampling instant t_{i+1}. This method is closely related to the *Adaptive Optimal Stochastic Trajectory Planning and Control (AOSTPC) procedure* described in [95, 98].

Corresponding to the extension of (OLF) control to (SOLF) control, (NMPC) can be extended to *Stochastic Nonlinear Model Predictive Control (SNMPC)*. For control policies of this type, robust (NMPC) with respect to stochastic variations of model parameters and initial values are determined in the following way:

- Use the a posteriori distribution $P(d\omega|\mathcal{A}_t)$ of the basic random element $\omega \in \Omega$, given the process information $\mathcal{A}_t$ up to time t, and
- apply stochastic optimization methods to incorporate random parameter variations into the optimal (NMPC) control design.

2.3 Convex Approximation by Inner Linearization

We observe first that (2.7), ($\widetilde{2.7}$), resp., is in general a non convex optimization problem, cf. [85]. Since for convex (deterministic) optimization problems there is a well established theory, we approximate the original problems (2.7), ($\widetilde{2.7}$) by a sequence of suitable convex problems (Fig. 2.2). In the following we describe first a single step of this procedure. Due to the consideration in Sect. 2.2, we may concentrate here to problem (2.7) or (2.9a–d) for deterministic controls $u(\cdot)$, as needed in the computation of optimal (OL), (OLF), (SOLF) as well as (NMP), (SNMP) controls being most important for practical problems.

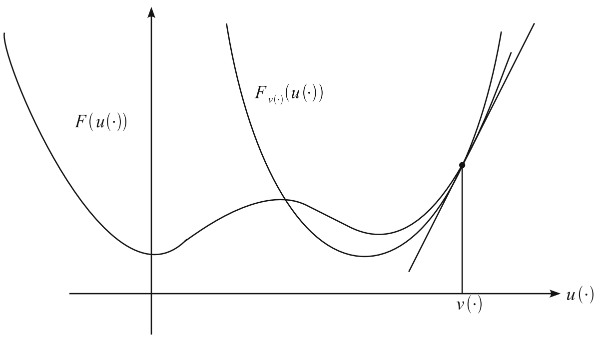

Fig. 2.2 Convex approximation

Let $v(\cdot) \in D$ be an arbitrary, but fixed admissible initial or reference control and assume, see Lemma 2.1, for the input–output map $z(\cdot) = S(\omega, u(\cdot))$:

Assumption 2.1 *$S(\omega, \cdot)$ is F-differentiable at $v(\cdot)$ for each $\omega \in \Omega$.*

Denote by $DS\Big(\omega, v(\cdot)\Big)$ the F-derivative of $S(\omega, \cdot)$ at $v(\cdot)$, and replace now the cost function $F = F\Big(u(\cdot)\Big)$ by

$$F_{v(\cdot)}\Big(u(\cdot)\Big) := Ef\Big(\omega, S\big(\omega, v(\cdot)\big) + DS\big(\omega, v(\cdot)\big)\big(u(\cdot) - v(\cdot)\big), u(\cdot)\Big) \tag{2.11}$$

where $u(\cdot) \in U$. Assume that $F_{v(\cdot)}\Big(u(\cdot)\Big) \in \mathbb{R}$ for all pairs $\Big(u(\cdot), v(\cdot)\Big) \in D \times D$.

Then, replace the optimization problem (2.7), see [89], by

$$\min F_{v(\cdot)}\Big(u(\cdot)\Big) \text{ s.t. } u(\cdot) \in D. \tag*{$(2.7)_{v(\cdot)}$}$$

Lemma 2.3 *$(2.7)_{v(\cdot)}$ is a convex optimization problem.*

Proof According to Sect. 1.1, function (2.6c) is convex. The assertion follows now from the linearity of the F-differential of $S(\omega, \cdot)$.

Remark 2.5 Note that the approximate $F_{v(\cdot)}$ of F is obtained from (2.6a, b) by means of linearization of the input–output map $S = S\Big(\omega, u(\cdot)\Big)$ with respect to the control $u(\cdot)$ at $v(\cdot)$, hence, by **inner linearization** of the control problem with respect to the control $u(\cdot)$ at $v(\cdot)$.

Remark 2.6 (Linear Input–Output Map) In case that $S = S(\omega, u(\cdot)) := S(\omega)u(\cdot)$ is linear with respect to the control $u(\cdot)$, then $DS(\omega, v(\cdot) = S(\omega)$ and we have $F_{v(\cdot)}\Big(u(\cdot)\Big) = F(u(\cdot))$. In this case the problems (2.7) and $(2.7)_{v(\cdot)}$ coincide for each input vector $v(\cdot)$.

For a real-valued convex function $\phi : X \to \mathbb{R}$ on a linear space X the directional derivative $\phi'_+(x; y)$ exists, see e.g. [60], at each point $x \in X$ and in each direction $y \in X$. According to Lemma 2.3 the objective function $F_{v(\cdot)}$ of the approximate problem $(2.7)_{v(\cdot)}$ is convex. Using the theorem of the *monotone convergence*, [12], for all $u(\cdot), v(\cdot) \in D$ and $h(\cdot) \in U$ the directional derivative of $F_{v(\cdot)}$ is given, see [90], Satz 1.4, by

$$\begin{aligned} F'_{v(\cdot)+}\Big(u(\cdot); h(\cdot)\Big) = Ef'_+\Big(\omega, S\Big(\omega, v(\cdot)\Big) + DS\Big(\omega, v(\cdot)\Big)\Big(u(\cdot) - v(\cdot)\Big), \\ u(\cdot); DS\Big(\omega, v(\cdot)\Big)h(\cdot), h(\cdot)\Big). \end{aligned} \tag{2.12a}$$

In the special case $u(\cdot) = v(\cdot)$ we get

$$F'_{v(\cdot)+}\Big(v(\cdot); h(\cdot)\Big) = Ef'_{+}\Big(\omega, S\Big(\omega, v(\cdot)\Big),$$
$$v(\cdot); DS\Big(\omega, v(\cdot)\Big)h(\cdot), h(\cdot)\Big). \tag{2.12b}$$

A solution $\bar{u}(\cdot) \in D$ of the convex problem $(2.7)_{v(\cdot)}$ is then characterized cf. [94], by

$$F'_{v(\cdot)+}\Big(\bar{u}(\cdot); u(\cdot) - \bar{u}(\cdot)\Big) \geq 0 \text{ for all } u(\cdot) \in D. \tag{2.13}$$

Definition 2.4 For each $v(\cdot) \in D$, let $M\Big(v(\cdot)\Big)$ be the set of solutions of problem $(2.7)_{v(\cdot)}$, i.e., $M\Big(v(\cdot)\Big)$
$:= \Big\{u^0(\cdot) \in D : F'_{v(\cdot)+}\Big(u^0(\cdot); u(\cdot) - u^0(\cdot)\Big) \geq 0, u(\cdot) \in D\Big\}$.

Note. If the input–output operator $S = S(\omega, u(\cdot)) := S(\omega)u(\cdot)$ is linear, then $M\Big(v(\cdot)\Big) = M$ for each input $v(\cdot)$, where M denotes the set of solutions of problem (2.7).

In the following we suppose that optimal solutions of $(2.7)_{v(\cdot)}$ exist for each $v(\cdot)$:

Assumption 2.2 $M\Big(v(\cdot)\Big) \neq \emptyset$ *for each* $v(\cdot) \in D$.

A first relation between our original problem (2.7) and the family of its approximates $(2.7)_{v(\cdot)}$, $v(\cdot) \in D$, is shown in the following.

Theorem 2.1 *Suppose that the directional derivative* $F'_{+} = F'_{+}\Big(v(\cdot); h(\cdot)\Big)$ *exists and*

$$F'_{+}\Big(v(\cdot); h(\cdot)\Big) = F'_{v(\cdot)+}\Big(v(\cdot); h(\cdot)\Big) \tag{2.14}$$

for each $v(\cdot) \in D$ *and* $h(\cdot) \in D - D$. *Then:*

I) If $\bar{u}(\cdot)$ *is an optimal control, then* $\bar{u}(\cdot) \in M\Big(\bar{u}(\cdot)\Big)$*, i.e.* $\bar{u}(\cdot)$ *is a solution of* $(2.7)_{\bar{u}(\cdot)}$.

II) If (2.7) is convex, then $\bar{u}(\cdot)$ *is an optimal control if and only if* $\bar{u}(\cdot) \in M\Big(\bar{u}(\cdot)\Big)$.

Proof Because of the convexity of the approximate control problem $(2.7)_{v(\cdot)}$, the condition $v(\cdot) \in M\Big(v(\cdot)\Big)$ holds, cf. (2.13), if and only if $F'_{v(\cdot)+}\Big(v(\cdot); u(\cdot) - v(\cdot)\Big) \geq 0$ for all $u(\cdot) \in D$. Because of (2.14), this is equivalent with $F'_{+}\Big(v(\cdot); u(\cdot) - v(\cdot)\Big) \geq 0$ for all $u(\cdot) \in D$. However, since the admissible control domain D is convex, for a an optimal solution $v(\cdot) := \bar{u}(\cdot)$ of (2.7) this condition is necessary, and necessary as also sufficient in case that (2.7) is convex.

Assuming that $f = f(\omega, \cdot, \cdot)$ is F-differentiable for each ω, by means of (2.12b) and the chain rule we have

$$\begin{aligned}
F'_{v(\cdot)+}\Big(v(\cdot); h(\cdot)\Big) &= Ef'_{+}\left(\omega, S\Big(\omega, v(\cdot)\Big), v(\cdot); DS\Big(\omega, v(\cdot)\Big)h(\cdot), h(\cdot)\right) \\
&= E \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left(f\Big(\omega, S\big(\omega, v(\cdot) + \varepsilon h(\cdot)\big), v(\cdot) + \varepsilon h(\cdot)\Big)\right. \\
&\qquad \left. - f\Big(\omega, S\big(\omega, v(\cdot)\big), v(\cdot)\Big)\right). \qquad (2.15)
\end{aligned}$$

Note. Because of the properties of the operator $S = S(\omega, u(\cdot))$, the above equations holds also for arbitrary convex functions f such that the expectations under consideration exist, see [90].

Due to the definition (2.6a) of the objective function $F = F(u(\cdot))$ for condition (2.14) the following criterion holds:

Lemma 2.4 *a) Condition (2.14) in Theorem 2.1 holds if and only if the expectation operator "E" and the limit process "*$\lim\limits_{\varepsilon \downarrow 0}$*" in (2.15) may be interchanged. b) This interchangeability holds e.g. if* $\sup\Big\{\big\|DS\big(\omega, v(\cdot) + \varepsilon h(\cdot)\big)\big\| : 0 \le \varepsilon \le 1\Big\}$ *is bounded with probability one, and the convex function* $f(\omega, \cdot, \cdot)$ *satisfies a Lipschitz condition*

$$\begin{aligned}
&\left|f\Big(\omega, z(\cdot), u(\cdot)\Big) - f\Big(\omega, \bar{z}(\cdot), \bar{u}(\cdot)\Big)\right| \\
&\le \gamma(\omega)\left\|\Big(z(\cdot), u(\cdot)\Big) - \Big(\bar{z}(\cdot), \bar{u}(\cdot)\Big)\right\|_{Z \times U}
\end{aligned}$$

on a set $Q \subset Z \times U$ *containing all vectors* $\Big(S\big(\omega, v(\cdot) + \varepsilon h(\cdot)\big), v(\cdot) + \varepsilon h(\cdot)\Big), 0 \le \varepsilon \le 1$*, where* $E\gamma(\omega) < +\infty$*, and* $\|\cdot\|_{Z \times U}$ *denotes the norm on* $Z \times U$*.*

Proof The first assertion a) follows immediately from (2.15) and the definition of the objective function F. Assertion b) can be obtained by means of the generalized mean value theorem for vector functions and Lebesgue's bounded convergence theorem.

Remark 2.7 Further conditions are given in [90].

A second relation between our original problem (2.7) and the family of its approximates $(2.7)_{v(\cdot)}$, $v(\cdot) \in D$, is shown next.

Lemma 2.5 *a) If* $\bar{u}(\cdot) \notin M\Big(\bar{u}(\cdot)\Big)$ *for a control* $\bar{u}(\cdot) \in D$*, then*

$$F_{\bar{u}(\cdot)}\Big(u(\cdot)\Big) < F_{\bar{u}(\cdot)}\Big(\bar{u}(\cdot)\Big) = F\Big(\bar{u}(\cdot)\Big) \textit{ for each } u(\cdot) \in M\Big(\bar{u}(\cdot)\Big)$$

b) Let the controls $u(\cdot), v(\cdot) \in D$ be related such that

$$F_{u(\cdot)}\Big(v(\cdot)\Big) < F_{u(\cdot)}\Big(u(\cdot)\Big).$$

If (2.14) holds for the pair $\Big(u(\cdot), h(\cdot)\Big), h(\cdot) = v(\cdot) - u(\cdot)$, then $h(\cdot)$ is an admissible direction of decrease for F at $u(\cdot)$, i.e. we have $F\Big(u(\cdot) + \varepsilon h(\cdot)\Big) < F\Big(u(\cdot)\Big)$ and $u(\cdot) + \varepsilon h(\cdot) \in D$ on a suitable interval $0 < \varepsilon < \bar{\varepsilon}$.

Proof According to Definition 2.4, for $u(\cdot) \in M\Big(\bar{u}(\cdot)\Big)$ we have $F_{\bar{u}(\cdot)}(u(\cdot)) \leq F_{\bar{u}(\cdot)}(v(\cdot))$ for all $v(\cdot) \in D$ and therefore also $F_{\bar{u}(\cdot)}(u(\cdot)) \leq F_{\bar{u}(\cdot)}(\bar{u}(\cdot))$. a) In case $F_{\bar{u}(\cdot)}(u(\cdot)) = F_{\bar{u}(\cdot)}(\bar{u}(\cdot))$ we get $F_{\bar{u}(\cdot)}(\bar{u}(\cdot)) \leq F_{\bar{u}(\cdot)}(v(\cdot))$ for all $v(\cdot) \in D$, hence, $\bar{u}(\cdot) \in M\Big(\bar{u}(\cdot)\Big)$. Since this is in contradiction to the assumption, it follows $F_{\bar{u}(\cdot)}(u(\cdot)) < F_{\bar{u}(\cdot)}(\bar{u}(\cdot))$. b) If controls $u(\cdot), v(\cdot) \in D$ are related $F_{u(\cdot)}\Big(v(\cdot)\Big) < F_{u(\cdot)}\Big(u(\cdot)\Big)$, then due to the convexity of $F_{u(\cdot)}$ we have $F'_{u(\cdot)+}\Big(u(\cdot); v(\cdot) - u(\cdot)\Big) \leq F_{u(\cdot)}(v(\cdot)) - F_{u(\cdot)}(u(\cdot)) < 0$. With (2.14) we then get $F'_{+}\Big(u(\cdot); v(\cdot) - u(\cdot)\Big) = F'_{u(\cdot)+}\Big(u(\cdot); v(\cdot) - u(\cdot)\Big) < 0$. This yields now that $h(\cdot) := v(\cdot) - u(\cdot)$ is a feasible descent direction for F at $u(\cdot)$.

If $\bar{u}(\cdot) \in M\Big(\bar{u}(\cdot)\Big)$, then the convex approximate $F_{\bar{u}}$ of F at $\bar{u}$ cannot be decreased further on D. Thus, the above results suggest the following definition:

Definition 2.5 A control $\bar{u}(\cdot) \in D$ such that $\bar{u}(\cdot) \in M\Big(\bar{u}(\cdot)\Big)$ is called a **stationary control** of the optimal control problem (2.7).

Under the rather weak assumptions in Theorem 2.1 an optimal control is also stationary, and in the case of a convex problem (2.7) the two concepts coincide. Hence, stationary controls are candidates for optimal controls. As an appropriate substitute/approximate for an optimal control we may determine therefore stationary controls. For this purpose algorithms of the following *conditional gradient-type* may be applied:

Algorithm 2.1 *I) Choose $u^1 \in D$, put $j = 1$*

II) If $u^j(\cdot) \in M\Big(u^j(\cdot)\Big)$, then $u^j(\cdot)$ is stationary and the algorithm stops; otherwise find a control $v^j(\cdot) \in M\Big(u^j(\cdot)\Big)$

III) Set $u^{j+1}(\cdot) = v^j(\cdot)$ and go to ii), putting $j \to j + 1$.

Algorithm 2.2 *I) Choose $u^1(\cdot) \in D$, put $j = 1$*

II) If $u^j(\cdot) \in M\Big(u^j(\cdot)\Big)$, then $u^j(\cdot)$ is stationary and the algorithm stops; otherwise find a $v^{(j)}(\cdot) \in M\Big(u^j(\cdot)\Big)$, define $h^j(\cdot) := v^j(\cdot) - u^j(\cdot)$

III) Calculate $\bar{u}(\cdot) \in m\Big(u^j(\cdot), h^j(\cdot)\Big)$*, set* $u^{j+1}(\cdot) := \bar{u}(\cdot)$ *and go to ii), putting* $j \to j+1$.

Here, based on line search, $m\Big(u(\cdot), h(\cdot)\Big)$ is defined by

$$m\Big(u(\cdot), h(\cdot)\Big) = \Bigg\{ u(\cdot) + \varepsilon^* h(\cdot) : F\Big(u(\cdot) + \varepsilon^* h(\cdot)\Big)$$
$$= \min_{0 \le \varepsilon \le 1} F\Big(u(\cdot) + \varepsilon h(\cdot)\Big) \text{ for } \varepsilon^* \in [0,1] \Bigg\}, u(\cdot), h(\cdot) \in U.$$

Concerning *Algorithm 2.1* we have the following result.

Theorem 2.2 *Let the set valued mapping* $u(\cdot) \to M\Big(u(\cdot)\Big)$ *be closed at each* $\bar{u}(\cdot) \in D\Bigg($*i.e. the relations* $u^j(\cdot) \to \bar{u}(\cdot), v^j(\cdot) \in M\Big(u^j(\cdot)\Big), j = 1, 2 \ldots,$ *and* $v^j(\cdot) \to \bar{v}(\cdot)$ *imply that also* $\bar{v}(\cdot) \in M\Big(\bar{u}(\cdot)\Big)\Bigg)$*. If a sequence* $u^1(\cdot), u^2(\cdot), \ldots$ *of controls generated by Algorithm 2.1 converges to an element* $\bar{u}(\cdot) \in D$*, then* $\bar{u}(\cdot)$ *is a stationary control.*

A sufficient condition for the closedness of the algorithmic map $u(\cdot) \to M\Big(u(\cdot)\Big)$ is given next:

Lemma 2.6 *Let* D *be a closed set of admissible controls, and let*

$$\Big(u(\cdot), v(\cdot)\Big) \to F'_{u(\cdot)+}\Big(v(\cdot); w(\cdot) - v(\cdot)\Big)$$

be continuous on $D \times D$ *for each* $w(\cdot) \in D$*. Then* $u(\cdot) \to M\Big(u(\cdot)\Big)$ *is closed at each element of* D.

While the convergence assumption for a sequence $u^1(\cdot), u^2(\cdot), \ldots$ generated by *Algorithm 2.1* is rather strong, only the existence of accumulation points of $u^j(\cdot), j = 1, 2, \ldots,$ has to be required in *Algorithm 2.2*.

2.4 Computation of Directional Derivatives

Suppose here again that $U := L^n_\infty([t_0, t_f], \mathcal{B}^1, \lambda^1)$ is the Banach space of all essentially bounded measurable functions $u(\cdot) : [t_0, t_f] \to \mathbb{R}^n$, normed by the essential supremum norm. According to Definitions 2.4, 2.5 of a stationary control and characterization (2.13) of an optimal solution of $(2.7)_{v(\cdot)}$, we first have to determined the directional derivative $F'_{v(\cdot)+}$. Based on the justification in Sect. 2.1, we assume again that the solution $z(t,\omega) = S\Big(\omega, u(\cdot)\Big)(t)$ of (2.2b) is measurable in $(t,\omega) \in [t_0, t_f] \times \Omega$ for each $u(\cdot) \in D$, and $u(\cdot) \to S\Big(\omega, u(\cdot)\Big)$ is continuously differentiable on D for each $\omega \in \Omega$. Furthermore, we suppose that the F-differential $\zeta(t) = \zeta(t,\omega) = \Big(D_u S\Big(\omega, u(\cdot)\Big)h(\cdot)\Big)(t), h(\cdot) \in U$, is measurable and essentially bounded in (t,ω), and is given according to (2.3a–f) by the linear integral equation

$$\zeta(t) - \int_{t_0}^{t} A\Big(t,\omega,u(\cdot)\Big)\zeta(s)ds = \int_{t_0}^{t} B\Big(t,\omega,u(\cdot)\Big)h(s)ds, \tag{2.16a}$$

$$t_0 \le t \le t_f,$$

with the Jacobians

$$A\Big(t,\omega,u(\cdot)\Big) := D_z g\Big(t,\omega,z_u(t,\omega),u(t)\Big) \tag{2.16b}$$

$$B\Big(t,\omega,u(\cdot)\Big) := D_u g\Big(t,\omega,z_u(t,\omega),u(t)\Big) \tag{2.16c}$$

and $z_u = z_u(t,\omega)$ defined, cf. (2.3f), by

$$z_u(t,\omega) := S\Big(\omega,u(\cdot)\Big)(t), \tag{2.16d}$$

$\Big(t,\omega,u(\cdot)\Big) \in [t_0,t_f] \times \Omega \times U$. Here, the random element "ω" is also used, cf. Sect. 2.1.1, to denote the realization

$$\omega := \begin{pmatrix} z_0 \\ \theta(\cdot) \end{pmatrix}$$

of the random inputs $z_0 = z_0(\omega), \theta(\cdot) = \theta(\cdot,\omega)$.

Remark 2.8 Due to the measurability of the functions $z_u = z_u(t,\omega)$ and $u = u(t)$ on $[t_0,t_f] \times \Omega$, $[t_0,t_f]$, resp., and the assumptions on the function g and its Jacobians $D_z g, D_u g$, see Sect. 2.1, also the matrix-valued functions $(t,\omega) \to A(t,\omega,u(\cdot))$, $(t,\omega) \to B(t,\omega,u(\cdot))$ are measurable and essentially bounded on $[t_0,t_f] \times \Omega$. Equation (2.16a) is again a vectorial Volterra integral equation, and the existence of a unique measurable solution $\zeta(t) = \zeta(t,\omega)$ can be shown as for the Volterra integral equation (2.2g, h).

The differential form of (2.16a) is then the *linear perturbation equation*

$$\dot{\zeta}(t) = A\Big(t,\omega,u(\cdot)\Big)\zeta(t) + B\Big(t,\omega,u(\cdot)\Big)h(t), t_0 \le t \le t_f, \omega \in \Omega \tag{2.16e}$$

$$\zeta(t_0) = 0. \tag{2.16f}$$

The solution $\zeta = \zeta(t, \omega)$ of (2.16a), (2.16e, f), resp., is also denoted, cf. (2.3f), by

$$\zeta(t, \omega) = \zeta_{u,h}(t, \omega) := \left(D_u S\Big(\omega, u(\cdot)\Big) h(\cdot) \right)(t), h(\cdot) \in U. \tag{2.16g}$$

This means that the approximate $(2.7)_{v(\cdot)}$ of (2.7) has the following explicit form:

$$\min E\left(\int_{t_0}^{t_f} L\Big(t, \omega, z_v(t, \omega) + \zeta(t, \omega), u(t)\Big) dt \right. \tag{2.17a}$$

$$\left. + G\Big(t_f, \omega, z_v(t_f, \omega) + \zeta(t_f, \omega)\Big) \right)$$

s.t.

$$\dot{\zeta}(t, \omega) = A\Big(t, \omega, v(\cdot)\Big)\zeta(t, \omega) + B\Big(t, \omega, v(\cdot)\Big)\Big(u(t) - v(t)\Big) \text{a.s.} \tag{2.17b}$$

$$\zeta(t_0, \omega) = 0 \text{ a.s.} \tag{2.17c}$$

$$u(\cdot) \in D. \tag{2.17d}$$

With the convexity assumptions in Sect. 2.1.2, Lemma 2.3 yields that (2.17a–d) is a convex stochastic control problem, with a *linear* plant differential equation.

For the subsequent analysis of the stochastic control problem we need now a representation of the directional derivative $F'_{v(\cdot)+}\Big(u(\cdot); h(\cdot)\Big)$ by a scalar product

$$F'_{v(\cdot)+}\Big(u(\cdot); h(\cdot)\Big) = \int_{t_0}^{t_f} q(t)^T h(t) dt$$

with a certain deterministic vector function $q = q(t)$. From representation (2.12a) of the directional derivative $F'_{v(\cdot)+}$ of the convex approximate $F_{v(\cdot)}$ of F, definition (2.6b) of $f = f\Big(\omega, z(\cdot), u(\cdot)\Big)$ by an integral over $[t_0, t_f]$ and [90], Satz 1.4, with (2.16g) we obtain

$$F'_{v(\cdot)+}\Big(u(\cdot); h(\cdot)\Big) = E\left(\int_{t_0}^{t_f} \left(\nabla_z L\Big(t, \omega, z_v(t, \omega) + \zeta_{v,u-v}(t, \omega), u(t)\Big)^T \zeta_{v,h}(t, \omega) \right. \right.$$

$$\left. + \nabla_u L\Big(t, \omega, z_v(t, \omega) + \zeta_{v,u-v}(t, \omega), u(t)\Big)^T h(t) \right)$$

$$\left. + \nabla_z G\Big(t_f, \omega, z_v(t_f, \omega) + \zeta_{v,u-v}(t_f, \omega)\Big)^T \zeta_{v,h}(t_f, \omega) \right). \tag{2.18}$$

Defining the gradients

$$a\Big(t,\omega,v(\cdot),u(\cdot)\Big) := \nabla_z L\Big(t,\omega,z_v(t,\omega)+\zeta_{v,u-v}(t,\omega),u(t)\Big) \tag{2.19a}$$

$$b\Big(t,\omega,v(\cdot),u(\cdot)\Big) := \nabla_u L\Big(t,\omega,z_v(t,\omega)+\zeta_{v,u-v}(t,\omega),u(t)\Big) \tag{2.19b}$$

$$c\Big(t_f,\omega,v(\cdot),u(\cdot)\Big) := \nabla_z G\Big(t_f,\omega,z_v(t_f,\omega)+\zeta_{v,u-v}(t_f,\omega)\Big), \tag{2.19c}$$

measurable with respect to (t,ω), the directional derivative $F'_{v(\cdot)+}$ can be represented by

$$F'_{v(\cdot)+}\Big(u(\cdot);h(\cdot)\Big) = E\left(\int_{t_0}^{t_f}\left(a\Big(t,\omega,v(\cdot),u(\cdot)\Big)^T\zeta_{v,h}(t,\omega)\right.\right.$$
$$\left.\left.+b\Big(t,\omega,v(\cdot),u(\cdot)\Big)^T h(t)\right)dt + c\Big(t_f,\omega,v(\cdot),u(\cdot)\Big)^T\zeta_{v,h}(t_f,\omega)\right). \tag{2.20a}$$

According to (2.16a), for $\zeta_{v,h} = \zeta_{v,h}(t,\omega)$ we have

$$\zeta_{v,h}(t_f,\omega) = \int_{t_0}^{t_f}\left(A\Big(t,\omega,v(\cdot)\Big)\zeta_{v,h}(t,\omega) + B\Big(t,\omega,v(\cdot)\Big)h(t)\right)dt. \tag{2.20b}$$

Putting (2.20b) into (2.20a), we find

$$F'_{v(\cdot)+}\Big(u(\cdot);h(\cdot)\Big) = E\left(\int_{t_0}^{t_f}\tilde{a}\Big(t,\omega,v(\cdot),u(\cdot)\Big)^T\zeta_{v,h}(t,\omega)dt\right.$$
$$\left.+\int_{t_0}^{t_f}\tilde{b}\Big(t,\omega,v(\cdot),u(\cdot)\Big)^T h(t)dt\right), \tag{2.20c}$$

where

$$\tilde{a}\Big(t,\omega,v(\cdot),u(\cdot)\Big) := a\Big(t,\omega,v(\cdot),u(\cdot)\Big) + A\Big(t,\omega,v(\cdot)\Big)^T c\Big(t_f,\omega,v(\cdot),u(\cdot)\Big) \tag{2.20d}$$

$$\tilde{b}\Big(t,\omega,v(\cdot),u(\cdot)\Big) := b\Big(t,\omega,v(\cdot),u(\cdot)\Big) + B\Big(t,\omega,v(\cdot)\Big)^T c\Big(t_f,\omega,v(\cdot),u(\cdot)\Big). \tag{2.20e}$$

Remark 2.9 According to Remark 2.8 also the functions $(t,\omega) \to \tilde{a}\Big(t,\omega,v(\cdot),u(\cdot)\Big)$, $(t,\omega) \to \tilde{b}\Big(t,\omega,v(\cdot),u(\cdot)\Big)$ are measurable on $[t_0,t_f] \times \Omega$.

In order to transform the first integral in (2.20c) into the form of the second integral in (2.20c), we introduce the m-vector function

$$\lambda = \lambda_{v,u}(t,\omega)$$

defined by the following integral equation depending on the random parameter ω:

$$\lambda(t) - A\Big(t,\omega,v(\cdot)\Big)^T \int_t^{t_f} \lambda(s)ds = \tilde{a}\Big(t,\omega,v(\cdot),u(\cdot)\Big). \tag{2.21}$$

Under the present assumptions, this *Volterra integral equation* has [90] a unique measurable solution $(t,\omega) \to \lambda_{v,u}(t,\omega)$, see also Remark 2.8. By means of (2.21) we obtain

$$\begin{aligned}
&\int_{t_0}^{t_f} \tilde{a}\Big(t,\omega,v(\cdot),u(\cdot)\Big)^T \zeta_{v,h}(t,\omega)dt \\
&= \int_{t_0}^{t_f} \left(\lambda(t) - A\Big(t,\omega,v(\cdot)\Big)^T \int_t^{t_f} \lambda(s)ds\right)^T \zeta_{v,h}(t,\omega)dt \\
&= \int_{t_0}^{t_f} \lambda(t)^T \zeta_{v,h}(t,\omega)dt \\
&\qquad - \int_{t_0}^{t_f} dt \int_{t_0}^{t_f} dsJ(s,t)\lambda(s)^T A\Big(t,\omega,v(\cdot)\Big)\zeta_{v,h}(t,\omega) \\
&= \int_{t_0}^{t_f} \lambda(s)^T \zeta_{v,h}(s,\omega)ds - \int_{t_0}^{t_f} ds \int_{t_0}^{t_f} dtJ(s,t)\lambda(s)^T A\Big(t,\omega,v(\cdot)\Big)\zeta_{v,h}(t,\omega) \\
&= \int_{t_0}^{t_f} \lambda(s)^T \left(\zeta_{v,h}(s,\omega) - \int_{t_0}^{s} A\Big(t,\omega,v(\cdot)\Big)\zeta_{v,h}(t,\omega)dt\right)ds,
\end{aligned} \tag{2.22a}$$

where $J = J(s,t)$ is defined by

$$J(s,t) := \begin{cases} 0, & t_0 \le s \le t \\ 1, & t < s \le t_f. \end{cases}$$

Using now again the perturbation equation (2.16a, b), from (2.22a) we get

$$\begin{aligned}
&\int_{t_0}^{t_f} \tilde{a}\Big(t,\omega,v(\cdot),u(\cdot)\Big)^T \zeta_{v,h}(t,\omega)dt = \int_{t_0}^{t_f} \lambda(s)^T \Bigg(\int_{t_0}^{s} B\Big(t,\omega,v(\cdot)\Big)h(t)dt\Bigg)ds \\
&= \int_{t_0}^{t_f} ds\lambda(s)^T \int_{t_0}^{t_f} dtJ(s,t)B\Big(t,\omega,v(\cdot)\Big)h(t) \\
&= \int_{t_0}^{t_f} dt \int_{t_0}^{t_f} dsJ(s,t)\lambda(s)^T B\Big(t,\omega,v(\cdot)\Big)h(t) \\
&= \int_{t_0}^{t_f} \Big(\int_{t}^{t_f} \lambda(s)ds\Big)^T B\Big(t,\omega,v(\cdot)\Big)h(t)dt = \int_{t_0}^{t_f} \Bigg(B\Big(t,\omega,v(\cdot)\Big)^T \int_{t}^{t_f} \lambda(s)ds\Bigg)^T h(t)dt.
\end{aligned} \tag{2.22b}$$

Inserting (2.22b) into (2.20c), we have

$$\begin{aligned}
F'_{v(\cdot)+}\Big(u(\cdot);h(\cdot)\Big) = E\Bigg(\int_{t_0}^{t_f} \Bigg(B\Big(t,\omega,v(\cdot)\Big)^T \int_{t}^{t_f} \lambda(s)ds \\
+\tilde{b}\Big(t,\omega,v(\cdot),u(\cdot)\Big)\Bigg)^T h(t)dt\Bigg).
\end{aligned} \tag{2.23}$$

By means of (2.20d), the integral equation (2.21) may be written by

$$\lambda(t) - A\Big(t,\omega,v(\cdot)\Big)^T \Bigg(c\Big(t_f,\omega,v(\cdot),u(\cdot)\Big) + \int_{t}^{t_f} \lambda(s)ds\Bigg) = a\Big(t,\omega,v(\cdot),u(\cdot)\Big). \tag{2.24}$$

According to (2.24), defining the m-vector function

$$y = y_{v,u}(t,\omega) := c\Big(t_f,\omega,v(\cdot),u(\cdot)\Big) + \int_{t}^{t_f} \lambda_{v,u}(s,\omega)ds, \tag{2.25a}$$

we get

$$\lambda(t) - A\Big(t, \omega, v(\cdot)\Big)^T y_{v,u}(t, \omega) = a\Big(t, \omega, v(\cdot), u(\cdot)\Big). \tag{2.25b}$$

Replacing in (2.25b) the variable t by s and integrating then the equation (2.25b) over the time interval $[t, t_f]$, yields

$$\int_t^{t_f} \lambda(s)ds = \int_t^{t_f} \left(A\Big(s, \omega, v(\cdot)\Big)^T y_{v,u}(s, \omega) + a\Big(s, \omega, v(\cdot), u(\cdot)\Big)\right) ds. \tag{2.25c}$$

Finally, using again (2.25a), from (2.25c) we get

$$\begin{aligned} y_{v,u}(t, \omega) &= c\Big(t_f, \omega, v(\cdot), u(\cdot)\Big) \\ &\quad + \int_t^{t_f} \left(A\Big(s, \omega, v(\cdot)\Big)^T y_{v,u}(s, \omega) + a\Big(s, \omega, v(\cdot), u(\cdot)\Big)\right) ds. \end{aligned} \tag{2.25d}$$

Obviously, the differential form of the Volterra integral equation (2.25d) for $y = y_{v,u}(t, \omega)$ reads:

$$\dot{y}(t) = -A\Big(t, \omega, v(\cdot)\Big)^T y(t) - a\Big(t, \omega, v(\cdot), u(\cdot)\Big), t_0 \le t \le t_f, \tag{2.26a}$$

$$y(t_f) = c\Big(t_f, \omega, v(\cdot), u(\cdot)\Big). \tag{2.26b}$$

System (2.25d), (2.26a, b), resp., is called the *adjoint integral, differential equation* related to the perturbation equation (2.16a, b).

By means of (2.25a), from (2.20e) and (2.23) we obtain now

$$\begin{aligned} F'_{v(\cdot)+}\Big(u(\cdot); h(\cdot)\Big) &= E \int_{t_0}^{t_f} \left(B\Big(t, \omega, v(\cdot)\Big)^T \int_t^{t_f} \lambda(s)ds + b\Big(t, \omega, v(\cdot), u(\cdot)\Big) \right. \\ &\qquad \left. + B\Big(t, \omega, v(\cdot)\Big)^T c\Big(t_f, \omega, v(\cdot), u(\cdot)\Big) \right)^T h(t)dt \\ &= E \int_{t_0}^{t_f} \left(B\Big(t, \omega, v(\cdot)\Big)^T y_{v,u}(t, \omega) + b\Big(t, \omega, v(\cdot), u(\cdot)\Big) \right)^T h(t)dt. \end{aligned}$$

Summarizing the above transformations, we have the following result:

Theorem 2.3 *Let* $(\omega, t) \to y_{v,u}(t, \omega)$ *be the unique measurable solution of the adjoint integral, differential equation (2.25d), (2.26a, b), respectively. Then,*

$$F'_{v(\cdot)+}\Big(u(\cdot); h(\cdot)\Big) = E \int_{t_0}^{t_f} \Bigg(B\Big(t, \omega, v(\cdot)\Big)^T y_{v,u}(t, \omega)$$

$$+ b\Big(t, \omega, v(\cdot), u(\cdot)\Big)\Bigg)^T h(t)dt. \tag{2.27}$$

Note that $F'_{v(\cdot)+}\Big(u(\cdot); \cdot\Big)$ is also the Gâteaux-differential of $F_{v(\cdot)}$ at $u(\cdot)$.

For a further discussion of formula (2.27) for $F'_{v(\cdot)+}\Big(u(\cdot); h(\cdot)\Big)$, in generalization of the *Hamiltonian* of a deterministic control problem, see e.g. [73], we introduce now the **stochastic Hamiltonian** related to the partly linearized control problem (2.17a–d) based on a reference control $v(\cdot)$:

$$H_{v(\cdot)}(t, \omega, \zeta, y, u) := L\Big(t, \omega, z_v(t, \omega) + \zeta, u\Big)$$

$$+ y^T \Bigg(A\Big(t, \omega, v(\cdot)\Big)\zeta + B\Big(t, \omega, v(\cdot)\Big)\Big(u - v(t)\Big)\Bigg), \tag{2.28a}$$

$(t, \omega, z, y, u) \in [t_0, t_f] \times \Omega \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$. Using $H_{v(\cdot)}$, for $F'_{v(\cdot)+}$ we find the representation

$$F'_{v(\cdot)+}\Big(u(\cdot); h(\cdot)\Big) = \int_{t_0}^{t_f} E\nabla_u H_{v(\cdot)}\Bigg(t, \omega, \zeta_{v,u-v}(t, \omega),$$

$$y_{v,u}(t, \omega), u(t)\Bigg)^T h(t)dt, \tag{2.28b}$$

where $\zeta_{v,u-v} = \zeta_{v,u-v}(t, \omega)$ is the solution of the perturbation differential, integral equation (2.17b, c), (2.20b), resp., and $y_{v,u} = y_{v,u}(t, \omega)$ denotes the solution of the adjoint differential, integral equation (2.26a, b), (2.25d).

Let $u^0(\cdot) \in U$ denote a given initial control. By means of (2.28a, b), the necessary and sufficient condition for a control $u^1(\cdot)$ to be an element of the set $M\Big(u^0(\cdot)\Big)$, i.e.

a solution of the approximate convex problem $(2.7)_{u^0(\cdot)}$, reads, see Definition 2.4 and (2.13):

$$\int_{t_0}^{t_f} E\nabla_u H_{u^0(\cdot)}\Big(t,\omega,\zeta_{u^0,u^1-u^0}(t,\omega), y_{u^0,u^1}(t,\omega),$$
$$u^1(t)\Big)^T \Big(u(t)-u^1(t)\Big)dt \geq 0, u(\cdot) \in D. \tag{2.29}$$

Introducing, for given controls $u^0(\cdot), u^1(\cdot)$, the convex mean value function $\overline{H}_{u^0(\cdot),u^1(\cdot)} = \overline{H}_{u^0(\cdot),u^1(\cdot)}(u(\cdot))$ defined by

$$\overline{H}_{u^0(\cdot),u^1(\cdot)}\Big(u(\cdot)\Big) := \int_{t_0}^{t_f} EH_{u^0(\cdot)}\Big(t,\omega,\zeta_{u^0,u^1-u^0}(t,\omega),$$
$$y_{u^0,u^1}(t,\omega), u(t)\Big)dt, \tag{2.30a}$$

corresponding to the representation (2.12a) and (2.18) of the directional derivative of F, it is seen that the left hand side of (2.29) is the directional derivative of the function $\overline{H}_{u^0(\cdot),u^1(\cdot)} = \overline{H}_{u^0(\cdot),u^1(\cdot)}(u(\cdot))$ at $u^1(\cdot)$ with increment $h(\cdot) := u(\cdot) - u^1(\cdot)$. Consequently, (2.29) is equivalent with the condition:

$$\overline{H}'_{u^0(\cdot),u^1(\cdot)+}\Big(u^1(\cdot)); \big(u(\cdot)-u^1(\cdot)\Big) \geq 0, u(\cdot) \in D, \tag{2.30b}$$

Due to the equivalence of the conditions (2.29) and (2.30b), for a control $u^1(\cdot) \in M\Big(u^0(\cdot)\Big)$, i.e. a solution of $(2.7)_{u^0(\cdot)}$ we have the following characterization:

Theorem 2.4 *Let $u^0(\cdot) \in D$ be a given initial control. A control $u^{(1)}(\cdot) \in U$ is a solution of $(2.7)_{u^0(\cdot)}$, i.e., $u^1(\cdot) \in M\Big(u^0(\cdot)\Big)$, if and only if $u^1(\cdot)$ is an optimal solution of the convex stochastic optimization problem*

$$\min \overline{H}_{u^0(\cdot),u^1(\cdot)}\Big(u(\cdot)\Big) \text{ s.t. } u(\cdot) \in D, \tag{2.31a}$$

In the following we study therefore the convex optimization problem (2.31a), where we replace next to the yet unknown functions $\zeta = \zeta_{u^0,u^1-u^0}(t,\omega), y = y_{u^0,u^1}(t,\omega)$ by arbitrary stochastic functions $\zeta = \zeta(t,\omega), y = y(t,\omega)$. Hence, we consider the mean value function $\overline{H}_{u^0(\cdot),\zeta(\cdot,\cdot),y(\cdot,\cdot)} = \overline{H}_{u^0(\cdot),\zeta(\cdot,\cdot),y(\cdot,\cdot)}(u(\cdot))$ defined, see (2.30a), by

$$\overline{H}_{u^0(\cdot),\zeta(\cdot,\cdot),y(\cdot,\cdot)}\Big(u(\cdot)\Big) = \int_{t_0}^{t_f} EH_{u^0(\cdot)}\Big(t,\omega,\zeta(t,\omega),y(t,\omega),u(t)\Big)dt. \tag{2.31b}$$

In practice, the admissible domain D is often given by

$$D = \Big\{u(\cdot) \in U : u(t) \in D_t, t_0 \le t \le t_f\Big\}, \tag{2.32}$$

where $D_t \subset \mathbb{R}^n$ is a given convex subset of $\mathbb{R}^n$ for each time t, $t_0 \le t \le t_f$. Since $\overline{H}_{u^0(\cdot),\zeta(\cdot,\cdot),y(\cdot,\cdot)}\Big(u(\cdot)\Big)$ has an integral form, the minimum value of $\overline{H}_{u^0(\cdot),\zeta(\cdot,\cdot),y(\cdot,\cdot)}\Big(u(\cdot)\Big)$ on D can be obtained—in case (2.32)—by solving the finite-dimensional stochastic optimization problem

$$\begin{aligned} \min EH_{u^0(\cdot)}\Big(t, \omega, \zeta(t,\omega),& \\ & y(t,\omega), u\Big) \text{ s.t. } u \in D_t \end{aligned} \qquad (P)^t_{u^0(\cdot),\zeta,y}$$

for each $t \in [t_0, t_f]$. Let denote then

$$\widetilde{u^*} = \widetilde{u^*}_{u^0(\cdot)}\Big(t, \zeta(t,\cdot), y(t,\cdot)\Big), t_0 \le t \le t_f, \tag{2.33a}$$

a solution of $(P)^t_{u^0(\cdot),\zeta,y}$ for each $t_0 \le t \le t_f$. Obviously, if

$$\widetilde{u^*}_{u^0(\cdot)}\Big(\cdot, \zeta(\cdot,\cdot), y(\cdot,\cdot)\Big) \in U \left(\text{ and therefore } \widetilde{u^*}_{u^0(\cdot)}\Big(\cdot, \zeta(\cdot,\cdot), y(\cdot,\cdot)\Big) \in D\right), \tag{2.33b}$$

then

$$\widetilde{u^*}_{u^0(\cdot)}\Big(\cdot, \zeta(\cdot,\cdot), y(\cdot,\cdot)\Big) \in \underset{u(\cdot)\in D}{\operatorname{argmin}}\ \overline{H}_{u^0(\cdot),\zeta(\cdot,\cdot),y(\cdot,\cdot)}\Big(u(\cdot)\Big). \tag{2.33c}$$

Because of Theorem 2.4, problems $(P)^t_{u^0(\cdot),\zeta,y}$ and (2.33a–c), we introduce, cf. [73], the following definition:

Definition 2.6 Let $u^0(\cdot) \in D$ be a given initial control. For measurable functions $\zeta = \zeta(t,\omega)$, $y = y(t,\omega)$ on $(\Omega, \mathcal{A}, P)$, let denote

$$\widetilde{u^*} = \widetilde{u^*}_{u^0(\cdot)}\Big(t, \zeta(t,\cdot), y(t,\cdot)\Big), t_0 \le t \le t_f,$$

a solution of $(P)^t_{u^0(\cdot),\zeta,y}$, $t_0 \le t \le t_f$. The function $\widetilde{u^*} = \widetilde{u^*}_{u^0(\cdot)}\Big(t, \zeta(t,\cdot), y(t,\cdot)\Big)$ is called a $\mathbf{H}_{\mathbf{u^0}(\cdot)}$**-minimal control** of (2.17a–d). The stochastic Hamiltonian $H_{u^0(\cdot)}$ is called **regular, strictly regular**, resp., if a $H_{u^0(\cdot)}$-minimal control exists, and is determined uniquely.

Remark 2.10 Obviously, by means of Definition 2.6, for an optimal solution $u^1(\cdot)$ of $(2.7)_{u^0(\cdot)}$ we have then the "*model control law*" $\widetilde{u^*} = \widetilde{u^*}_{u^0(\cdot)}\Big(t, \zeta(t,\cdot), y(t,\cdot)\Big)$ depending on still unknown state, costate functions $\zeta(t,\cdot)$, $y(\cdot)$, respectively.

2.5 Canonical (Hamiltonian) System of Differential Equations/Two-Point Boundary Value Problem

For a given initial control $u^0(\cdot) \in D$, let $\widetilde{u^*} = \widetilde{u^*}_{u^0(\cdot)}\Big(t, \zeta(\cdot), \eta(\cdot)\Big), t_0 \le t \le t_f$, denote a $H_{u^0(\cdot)}$-minimal control of (2.17a–d) according to Definition 2.6. Due to (2.17b, c) and (2.26a, b) we consider, cf. [73], the following so-called canonical or Hamiltonian system of differential equations, hence, a two-point boundary value problem, with random parameters for the vector functions $(\zeta, y) = \Big(\zeta(t,\omega), y(t,\omega)\Big), t_0 \le t \le t_f, \omega \in \Omega$:

$$\begin{aligned}\dot{\zeta}(t,\omega) &= A\Big(t,\omega,u^0(\cdot)\Big)\zeta(t,\omega)\\ &\quad + B\Big(t,\omega,u^0(\cdot)\Big)\bigg(\widetilde{u^*}_{u^0(\cdot)}\Big(t,\zeta(t,\cdot),y(t,\cdot)\Big) - u^0(t)\bigg),\\ &\quad t_0 \le t \le t_f, \qquad (2.34a)\\ \zeta(t_0,\omega) &= 0 \text{ a.s.} \qquad (2.34b)\\ \dot{y}(t,\omega) &= -A\Big(t,\omega,u^0(\cdot)\Big)^T y(t,\omega)\\ &\quad -\nabla_z L\bigg(t,\omega,z_{u^0}(t,\omega)+\zeta(t,\omega),\widetilde{u^*}_{u^0(\cdot)}\Big(t,\zeta(t,\cdot),y(t,\cdot)\Big)\bigg),\\ &\quad t_0 \le t \le t_f, \qquad (2.34c)\\ y(t_f,\omega) &= \nabla_z G\Big(t_f,\omega,z_{u^0}(t_f,\omega)+\zeta(t_f,\omega)\Big). \qquad (2.34d)\end{aligned}$$

Remark 2.11 Note that the (deterministic) control law $\widetilde{u^*} = \widetilde{u^*}_{u^0(\cdot)}\Big(t, \zeta(t,\cdot), y(t,\cdot)\Big)$ depends on the whole random variable $\Big(\zeta(t,\omega), y(t,\omega)\Big), \omega \in \Omega$, or the occurring moments. In case of a discrete parameter distribution $\Omega = \{\omega_1, \ldots, \omega_\varrho\}$, see Sect. 2.1, the control law $\widetilde{u^*}_{u^0(\cdot)}$ depends,

$$\widetilde{u^*}_{u^0(\cdot)} = \widetilde{u^*}_{u^0(\cdot)}\Big(t, \zeta(t,\omega_1), \ldots, \zeta(t,\omega_\varrho), y(t,\omega_1), \ldots, y(t,\omega_\varrho)\Big),$$

on the 2ϱ unknown functions $\zeta(t,\omega_i), y(t,\omega_i), i = 1, \ldots, \varrho$.

Suppose now that $(\zeta^1, y^1) = \Big(\zeta^1(t,\omega), y^1(t,\omega)\Big), t_0 \le t \le t_f,\ \omega \in (\Omega, \mathcal{A}, P)$, is the unique measurable solution of the canonical stochastic system (2.34a–d), and define

$$u^1(t) := \widetilde{u^*}_{u^0(\cdot)}\Big(t, \zeta^1(t,\cdot), y^1(t,\cdot)\Big), t_0 \le t \le t_f. \qquad (2.35)$$

System (2.34a–d) takes then the following form:

$$\dot{\zeta}^1(t,\omega) = A\Big(t,\omega,u^0(\cdot)\Big)\zeta^1(t,\omega) + B\Big(t,\omega,u^0(\cdot)\Big)\Big(u^1(t) - u^0(t)\Big),$$
$$t_0 \le t \le t_f, \tag{2.34a'}$$
$$\zeta^1(t_0,\omega) = 0 \text{ a.s.} \tag{2.34b'}$$
$$\dot{y}^1(t,\omega) = -A\Big(t,\omega,u^0(\cdot)\Big)^T y(t,\omega)$$
$$-\nabla_z L\Big(t,\omega,z_{u^0}(t,\omega) + \zeta^1(t,\omega), u^1(t)\Big), t_0 \le t \le t_f, \tag{2.34c'}$$
$$y^1(t_f,\omega) = \nabla_z G\Big(t_f,\omega,z_{u^0}(t_f,\omega) + \zeta^1(t_f,\omega)\Big). \tag{2.34d'}$$

Assuming that

$$u^1(\cdot) \in U, \tag{2.36a}$$

due to the definition of a $H_{u^0(\cdot)}$-minimal control we also have

$$u^1(\cdot) \in D. \tag{2.36b}$$

According to the notation introduced in (2.16a–g), (2.25a–d)/(2.26a, b), resp., and the above assumed uniqueness of the solution (ζ^1, y^1) of (2.34a–d) we have

$$\zeta^1(t,\omega) = \zeta_{u^0,u^1-u^0}(t,\omega) \tag{2.37a}$$
$$y^1(t,\omega) = y_{u^0,u^1}(t,\omega) \tag{2.37b}$$

with the control $u^1(\cdot)$ given by (2.35).

Due to the above construction, we know that $u^1(t)$ solves $(P)^t_{u^0(\cdot),\zeta,\eta}$ for

$$\zeta = \zeta(t,\omega) := \zeta^1(t,\omega) = \zeta_{u^0,u^1-u^0}(t,\omega)$$
$$\eta = \eta(t,\omega) := y^1(t,\omega) = y_{u^0,u^1}(t,\omega)$$

for each $t_0 \le t \le t_f$. Hence, control $u^1(\cdot)$, given by (2.35), is a solution of (2.31a).

Summarizing the above construction, from Theorem 2.4 we obtain this result:

Theorem 2.5 *Suppose that D is given by (2.32), $M\Big(u^0(\cdot)\Big) \neq \emptyset$, and $(P)^t_{u^0(\cdot),\zeta,y}$ has an optimal solution for each $t, t_0 \le t \le t_f$, and measurable functions $\zeta(t,\cdot), y(t,\cdot)$. Moreover, suppose that the canonical system (2.34a–d) has a unique measurable solution $\Big(\zeta^1(t,\omega), y^1(t,\omega)\Big), t_0 \le t \le t_f, \omega \in \Omega$, such that $u^1(\cdot) \in U$, where $u^1(\cdot)$ is defined by (2.35). Then $u^1(\cdot)$ is a solution of $(2.7)_{u^0(\cdot)}$.*

2.6 Stationary Controls

Suppose here that condition (2.14) holds for all controls $v(\cdot)$ under consideration, cf. Lemma 2.4. Having a method for the construction of improved approximative controls $u^1(\cdot) \in M\Big(u^0(\cdot)\Big)$ related to an initial control $u^0(\cdot) \in D$, we consider now the construction of stationary controls of the control problem (2.7), i.e. elements $\bar{u}(\cdot) \in D$ such that $\bar{u}(\cdot) \in M\Big(\bar{u}(\cdot)\Big)$, see Definition 2.5.

Starting again with formula (2.27), by means of (2.14), for an element $v(\cdot) \in D$ we have

$$F'_{+}\Big(v(\cdot); h(\cdot)\Big) = F'_{v(\cdot)+}\Big(v(\cdot); h(\cdot)\Big)$$

$$= \int_{t_0}^{t_f} E\left(B\Big(t,\omega,v(\cdot)\Big)^T y_v(t,\omega) + b\Big(t,\omega,v(\cdot),v(\cdot)\Big)\right)^T h(t)dt, \quad (2.38a)$$

where

$$y_v(t,\omega) := y_{v,v}(t,\omega) \quad (2.38b)$$

fulfills, cf. (2.26a, b), the adjoint differential equation

$$\dot{y}(t,\omega) = -A\Big(t,\omega,v(\cdot)\Big)^T y(t,\omega) - \nabla_z L\Big(t,\omega,z_v(t,\omega),v(t)\Big) \quad (2.39a)$$

$$y(t_f,\omega) = \nabla_z G\Big(t_f,\omega,z_v(t_f,\omega)\Big). \quad (2.39b)$$

Moreover, cf. (2.16b, c),

$$A\Big(t,\omega,v(\cdot)\Big) = D_z g\Big(t,\omega,z_v(t,\omega),v(t)\Big) \quad (2.39c)$$

$$B\Big(t,\omega,v(\cdot)\Big) = D_u g\Big(t,\omega,z_v(t,\omega),v(t)\Big) \quad (2.39d)$$

and, see (2.19b),

$$b\Big(t,\omega,v(\cdot),v(\cdot)\Big) = \nabla_u L\Big(t,\omega,z_v(t,\omega),v(t)\Big), \quad (2.39e)$$

where $z_v = z_v(t,\omega)$ solves the dynamic equation

$$\dot{z}(t,\omega) = g\Big(t,\omega,z(t,\omega),v(t)\Big), t_0 \le t \le t_f, \quad (2.39f)$$

$$z(t_0,\omega) = z_0(\omega). \quad (2.39g)$$

Using now the stochastic Hamiltonian, cf. (2.28a),

$$H(t,\omega,z,y,u) := L(t,\omega,z,u) + y^T g(t,\omega,z,u) \tag{2.40a}$$

related to the basic control problem (2.9a–d), from (2.38a, b), (2.39a–g) we get the representation, cf. (2.28a, b),

$$F'_+\Big(v(\cdot);h(\cdot)\Big) = \int_{t_0}^{t_f} E\nabla_u H\Big(t,\omega,z_v(t,\omega),y_v(t,\omega),v(t)\Big)^T h(t)dt. \tag{2.40b}$$

According to condition (2.13), a stationary control of (2.7), hence, an element $\bar{u}(\cdot) \in D$ such that $\bar{u}(\cdot)$ is an optimal solution of $(2.7)_{\bar{u}(\cdot)}$ is characterized, see (2.14), by

$$F'_+\Big(\bar{u}(\cdot);u(\cdot)-\bar{u}(\cdot)\Big) \geq 0 \text{ for all } u(\cdot) \in D.$$

Thus, for stationary controls $\bar{u}(\cdot) \in D$ of problem (2.7) we have the characterization

$$\int_{t_0}^{t_f} E\nabla_u H\Big(t,\omega,z_{\bar{u}}(t,\omega),y_{\bar{u}}(t,\omega),\bar{u}(t)\Big)^T \Big(u(t)-\bar{u}(t)\Big)dt \geq 0, u(\cdot) \in D. \tag{2.41}$$

Comparing (2.29) and (2.41), corresponding to (2.30a, b), for given $w(\cdot) \in D$ we introduce here the function

$$\overline{H}_w\Big(u(\cdot)\Big) := \int_{t_0}^{t_f} EH\Big(t,\omega,z_w(t,\omega),y_w(t,\omega),u(t)\Big)dt, \tag{2.42a}$$

and we consider the optimization problem

$$\min \overline{H}_w\Big(u(\cdot)\Big) \text{ s.t. } u(\cdot) \in D. \tag{2.42b}$$

Remark 2.12 Because of the assumptions in Sect. 2.1.2, problem (2.42b) is (strictly) convex, provided that the process differential equation (2.1a) is affine-linear with respect to u, hence,

$$\dot{z}(t,\omega) = g(t,\omega,z,u) = \hat{g}(t,\omega,z) + \hat{B}(t,\omega,z)u \tag{2.43}$$

with a given vector-, matrix-valued function $\hat{g} = \hat{g}(t,\omega,z)$, $\hat{B} = \hat{B}(t,\omega,z)$.

If differentiation and integration/expectation in (2.42a) may be interchanged, which is assumed in the following, then (2.41) is a necessary condition for

$$\bar{u}(\cdot) \in \underset{u(\cdot)\in D}{\operatorname{argmin}}\ \overline{H}_{\bar{u}}\Big(u(\cdot)\Big), \tag{2.44}$$

cf. (2.30a, b), (2.31a, b). Corresponding to Theorem 2.4, here we have this result:

Theorem 2.6 (Optimality Condition for Stationary Controls) *Suppose that a control $\bar{u}(\cdot) \in D$ fulfills (2.44). Then $\bar{u}(\cdot)$ is a stationary control of (2.7).*

2.7 Canonical (Hamiltonian) System of Differential

Assume now again that the feasible domain D is given by (2.32). In order to solve the optimization problem (2.42a, b), corresponding to $(P)^t_{u^0(\cdot),\zeta,\eta}$, here we consider the finite-dimensional optimization problem

$$\min EH\Big(t, \omega, \zeta(\omega), \eta(\omega), u\Big) \text{ s.t. } u \in D_t \qquad (P)^t_{\zeta,\eta}$$

for each $t, t_0 \le t \le t_f$. Furthermore, we use again the following definition, cf. Definition 2.6:

Definition 2.7 For measurable functions $\zeta(\cdot), \eta(\cdot)$ on $(\Omega, \mathcal{A}, P)$, let denote

$$\widetilde{u^*} = \widetilde{u^*}\Big(t, \zeta(\cdot), \eta(\cdot)\Big), t_0 \le t \le t_f,$$

a solution of $(P)^t_{\zeta,\eta}$. The function $\widetilde{u^*} = \widetilde{u^*}\Big(t, \zeta(\cdot), \eta(\cdot)\Big), t_0 \le t \le t_f$, is called a **H-minimal control** of (2.9a–d). The stochastic Hamiltonian H is called **regular, strictly regular**, resp., if a H-minimal control exists, exists and is determined uniquely.

For a given H-minimal control $u^* = u^*\Big(t, \zeta(\cdot), \eta(\cdot)\Big)$ we consider now, see (2.34a–d), the following canonical (Hamiltonian) two-point boundary value problem with random parameters:

$$\dot{z}(t,\omega) = g\Bigg(t, \omega, z(t,\omega), u^*\Big(t, z(t,\cdot), y(t,\cdot)\Big)\Bigg), t_0 \le t \le t_f \tag{2.45a}$$

$$z(t_0,\omega) = z_0(\omega) \tag{2.45b}$$

$$\dot{y}(t,\omega) = -D_z g\left(t,\omega,z(t,\omega),u^*\left(t,z(t,\cdot),y(t,\cdot)\right)\right)^T y(t,\omega)$$
$$-\nabla_z L\left(t,\omega,z(t,\omega),u^*\left(t,z(t,\cdot),y(t,\cdot)\right)\right), t_0 \le t \le t_f \quad (2.45c)$$
$$y(t_f,\omega) = \nabla_z G\left(t_f,\omega,z(t_f,\omega)\right). \quad (2.45d)$$

Remark 2.13 In case of a discrete distribution $\Omega = \{\omega_1,\ldots,\omega_\varrho\}$, $P(\omega=\omega_j)$, $j = 1,\ldots,\varrho$, corresponding to Sect. 3.1, for the H-minimal control we have

$$\widetilde{u^*} = \widetilde{u^*}\left(t,z(t,\omega_1),\ldots,z(t,\omega_\varrho),y(t,\omega_1),\ldots,y(t,\omega_\varrho)\right).$$

Thus, (2.45a–d) is then an ordinary two-point boundary value problem for the 2ϱ unknown functions $z = z(t,\omega_j)$, $y = y(t,\omega_j)$, $j = 1,\ldots,\varrho$.

Let denote $(\bar{z},\bar{y}) = \left(\bar{z}(t,\omega),\bar{y}(t,\omega)\right)$, $t_0 \le t \le t_f$, $\omega \in (\Omega,\mathcal{A},P)$, the unique measurable solution of (2.45a–d) and define:

$$\bar{u}(t) := \widetilde{u^*}\left(t,\bar{z}(t,\cdot),\bar{y}(t,\cdot)\right), t_0 \le t \le t_f. \quad (2.46)$$

Due to (2.16e) and (2.38b), (2.39a, b) we have

$$\bar{z}(t,\omega) = z_{\bar{u}}(t,\omega), t_0 \le t \le t_f, \omega \in (\Omega,\mathcal{A},P) \quad (2.47a)$$
$$\bar{y}(t,\omega) = y_{\bar{u}}(t,\omega), t_0 \le t \le t_f, \omega \in (\Omega,\mathcal{A},P), \quad (2.47b)$$

hence,

$$\bar{u}(t) = \widetilde{u^*}\left(t,z_{\bar{u}}(t,\cdot),y_{\bar{u}}(t,\cdot)\right), t_0 \le t \le t_f. \quad (2.47c)$$

Assuming that

$$\bar{u}(\cdot) \in U (\text{and therefore } \bar{u}(\cdot) \in D), \quad (2.48)$$

we get this result:

Theorem 2.7 *Suppose that the Hamiltonian system (2.45a–d) has a unique measurable solution $(\bar{z},\bar{y}) = \left(\bar{z}(t,\omega),\bar{y}(t,\omega)\right)$, and define $\bar{u}(\cdot)$ by (2.46) with a H-minimal control $\widetilde{u^*} = \widetilde{u^*}(t,\zeta,\eta)$. If $\bar{u}(\cdot) \in U$, then $\bar{u}(\cdot)$ is a stationary control.*

Proof According to the construction of $(\bar{z}, \bar{y}, \bar{u})$, the control $\bar{u}(\cdot) \in D$ minimizes $\overline{H}_{\bar{u}}\Big(u(\cdot)\Big)$ on D. Hence,

$$\bar{u}(\cdot) \in \operatorname*{argmin}_{u(\cdot) \in D} \overline{H}_{\bar{u}}\Big(u(\cdot)\Big).$$

Theorem 2.6 yields then that $\bar{u}(\cdot)$ is a stationary control.

2.8 Computation of Expectations by Means of Taylor Expansions

Corresponding to the assumptions in Sect. 2.1, based on a parametric representation of the random differential equation with a finite dimensional random parameter vector $\theta = \theta(\omega)$, we suppose that

$$g(t, w, z, u) = \tilde{g}(t, \theta, z, u) \tag{2.49a}$$

$$z_0(\omega) = \tilde{z}_0(\theta) \tag{2.49b}$$

$$L(t, \omega, z, u) = \tilde{L}(t, \theta, z, u) \tag{2.49c}$$

$$G(t, \omega, z) = \tilde{G}(t, \theta, z). \tag{2.49d}$$

Here,

$$\theta = \theta(\omega), \omega \in (\Omega, \mathcal{A}, P), \tag{2.49e}$$

denotes the time-independent r-vector of random model parameters and random initial values, and $\tilde{g}, \tilde{z}_0, \tilde{L}, \tilde{G}$ are sufficiently smooth functions of the variables indicated in (2.49a–d). For simplification of notation we omit symbol "~" and write

$$g(t, w, z, u) := g\Big(t, \theta(\omega), z, u\Big) \tag{2.49a'}$$

$$z_0(\omega) := z_0\Big(\theta(\omega)\Big) \tag{2.49b'}$$

$$L(t, \omega, z, u) := L\Big(t, \theta(\omega), z, u\Big) \tag{2.49c'}$$

$$G(t, \omega, z, u) := G\Big(t, \theta(\omega), z\Big). \tag{2.49d'}$$

Since the approximate problem (2.17a–d), obtained by the above described *inner linearization*, has the same basic structure as the original problem (2.9a–d), it is sufficient to describe the procedure for problem (2.9a–d). Again, for simplification, the conditional expectation $E(\ldots|\mathcal{A}_{t_0})$ given the information $\mathcal{A}_{t_0}$ up to the considered starting time t_0 is denoted by "E". Thus, let denote

$$\overline{\theta} = \overline{\theta}^{t_0} := E\theta(\omega) = E\Big(\theta(\omega)|\mathcal{A}_{t_0}\Big) \tag{2.50a}$$

the conditional expectation of the random vector $\theta(\omega)$ given the information $\mathcal{A}_{t_0}$ at time point t_0. Taking into account the properties of the solution

$$z = z(t,\theta) = S\Big(z_0(\theta), \theta, u(\cdot)\Big)(t), t \geq t_0, \tag{2.50b}$$

of the dynamic equation (2.3a–d), see Lemma 2.1, the expectations arising in the objective function (2.9a) can be computed approximatively by means of Taylor expansion with respect to θ at $\overline{\theta}$.

2.8.1 Complete Taylor Expansion

Considering first the costs L along the trajectory we obtain, cf. [100],

$$\begin{aligned} L\Big(t,\theta,z(t,\theta),u(t)\Big) &= L\Big(t,\overline{\theta},z(t,\overline{\theta}),u(t)\Big) \\ &+ \Big(\nabla_\theta L\Big(t,\overline{\theta},z(t,\overline{\theta}),u(t)\Big) + D_\theta z(t,\overline{\theta})^T \nabla_z L\Big(t,\overline{\theta},z(t,\overline{\theta}),u(t)\Big)\Big)^T (\theta-\overline{\theta}) \\ &+ \frac{1}{2}(\theta-\overline{\theta})^T Q_L\Big(t,\overline{\theta},z(t,\overline{\theta}),D_\theta z(t,\overline{\theta}),u(t)\Big)(\theta-\overline{\theta}) + \ldots . \end{aligned} \tag{2.51a}$$

Retaining only 1st order derivatives of $z = z(t,\theta)$ with respect to θ, the approximative Hessian Q_L of $\theta \to L\Big(t,\theta,z(t,\theta),u\Big)$ at $\theta = \overline{\theta}$ is given by

$$\begin{aligned} Q_L\Big(t,\overline{\theta},z(t,\overline{\theta}),D_\theta z(t,\overline{\theta}),u(t)\Big) &:= \nabla_\theta^2 L\Big(t,\overline{\theta},z(t,\overline{\theta}),u(t)\Big) \\ &+ D_\theta z(t,\overline{\theta})^T \nabla_{\theta z}^2 L\Big(t,\overline{\theta},z(t,\overline{\theta}),u(t)\Big) + \nabla_{\theta z}^2 L\Big(t,\overline{\theta},z(t,\overline{\theta}),u(t)\Big)^T D_\theta z(t,\overline{\theta}) \\ &+ D_\theta z(t,\overline{\theta})^T \nabla_z^2 L\Big(t,\overline{\theta},z(t,\overline{\theta}),u(t)\Big) D_\theta z(t,\overline{\theta}). \end{aligned} \tag{2.51b}$$

Here, $\nabla_\theta L, \nabla_z L$ denotes the gradient of L with respect to θ, z, resp., $D_\theta z$ is the Jacobian of $z = z(t,\theta)$ with respect to θ, and $\nabla_\theta^2 L, \nabla_z^2 L$, resp., denotes the Hessian of L with respect to θ, z. Moreover, $\nabla_{\theta z}^2 L$ is the $r \times m$ matrix of partial derivatives of L with respect to θ_i and z_k, in this order.

Taking expectations in (2.51a), from (2.51b) we obtain the expansion

$$
\begin{aligned}
EL\Big(t,\theta(\omega),z\big(t,\theta(\omega)\big),u(t)\Big) &= L\big(t,\overline{\theta},z(t,\overline{\theta}),u(t)\big)\\
&+\frac{1}{2}E\big(\theta(\omega)-\overline{\theta}\big)^T Q_L\big(t,\overline{\theta},z(t,\overline{\theta}),D_\theta z(t,\overline{\theta}),u(t)\big)\big(\theta(\omega)-\overline{\theta}\big)+\dots\\
= L\big(t,\overline{\theta},z(t,\overline{\theta}),u(t)\big) &+ \frac{1}{2}trQ_L\big(t,\overline{\theta},z(t,\overline{\theta}),D_\theta z(t,\overline{\theta}),u(t)\big)\mathrm{cov}\big(\theta(\cdot)\big)+\dots .
\end{aligned}
\tag{2.52}
$$

For the terminal costs G, corresponding to the above expansion we find

$$
\begin{aligned}
G\big(t_f,\theta,z(t_f,\theta)\big) &= G\big(t_f,\overline{\theta},z(t_f,\overline{\theta})\big)\\
&+\Big(\nabla_\theta G\big(t_f,\overline{\theta},z(t_f,\overline{\theta})\big)+D_\theta z(t_f,\overline{\theta})^T\nabla_z G\big(t_f,\overline{\theta},z(t_f,\overline{\theta})\big)\Big)^T(\theta-\overline{\theta})\\
&+\frac{1}{2}(\theta-\overline{\theta})^T Q_G\big(t_f,\overline{\theta},z(t_f,\overline{\theta}),D_\theta z(t_f,\overline{\theta})\big)(\theta-\overline{\theta})+\dots,
\end{aligned}
\tag{2.53a}
$$

where Q_G is defined in the same way as Q_L, see (2.51a). Taking expectations with respect to $\theta(\omega)$, we get

$$
\begin{aligned}
EG\Big(t_f,\theta(\omega),z\big(t_f,\theta(\omega)\big)\Big) &= G\big(t_f,\overline{\theta},z(t_f,\overline{\theta})\big)\\
&+\frac{1}{2}trQ_G\big(t_f,\overline{\theta},z(t_f,\overline{\theta}),D_\theta z(t_f,\overline{\theta})\big)\mathrm{cov}\big(\theta(\cdot)\big)+\dots .
\end{aligned}
\tag{2.53b}
$$

Note. Corresponding to Definition 2.1 and (2.50a), for the mean and covariance matrix of the random parameter vector $\theta=\theta(\omega)$ we have

$$
\begin{aligned}
\overline{\theta} = \overline{\theta}^{(t_0)} &:= E\big(\theta(\omega)|\mathcal{A}_{t_0}\big)\\
\mathrm{cov}\big(\theta(\cdot)\big) = \mathrm{cov}^{(t_0)}\big(\theta(\cdot)\big) &:= E\Big(\big(\theta(\omega)-\overline{\theta}^{(t_0)}\big)\big(\theta(\omega)-\overline{\theta}^{(t_0)}\big)^T\Big|\mathcal{A}_{t_0}\Big).
\end{aligned}
$$

2.8.2 Inner or Partial Taylor Expansion

Instead of a complete expansion of L, G with respect to θ, appropriate approximations of the expected costs EL, EG, resp., may by obtained by the inner 1st order approximation of the trajectory, hence,

$$L\Big(t,\theta,z(t,\theta),u(t)\Big) \approx L\Big(t,\theta,z(t,\overline{\theta}) + D_\theta z(t,\overline{\theta})(\theta-\overline{\theta}),u(t)\Big). \tag{2.54a}$$

Taking expectations in (2.54a), for the expected cost function we get the approximation

$$\begin{aligned} &EL\Big(t,\theta,z(t,\theta),u(t)\Big) \\ &\approx EL\Big(t,\theta(\omega),z(t,\overline{\theta}) + D_\theta z(t,\overline{\theta})(\theta(\omega)-\overline{\theta}),u(t)\Big). \end{aligned} \tag{2.54b}$$

In many important cases, as e.g. for cost functions L being quadratic with respect to the state variable z, the above expectation can be computed analytically. Moreover, if the cost function L is convex with respect to z, then the expected cost function EL is convex with respect to both, the state vector $z(t,\overline{\theta})$ and the Jacobian matrix of sensitivities $D_\theta z(t,\overline{\theta})$ evaluated at the mean parameter vector $\overline{\theta}$.

Having the approximate representations (2.52), (2.53b), (2.54b), resp., of the expectations occurring in the objective function (2.9a), we still have to compute the trajectory $t \to z(t,\overline{\theta}), t \geq t_0$, related to the mean parameter vector $\theta = \overline{\theta}$ and the sensitivities $t \to \frac{\partial z}{\partial \theta_i}(t,\overline{\theta}), i = 1,\ldots,r, t \geq t_0$, of the state $z = z(t,\theta)$ with respect to the parameters $\theta_i, i = 1,\ldots,r$, at $\theta = \overline{\theta}$. According to (2.3a, b) or (2.9b, c), for $z = z(t,\overline{\theta})$ we have the system of differential equations

$$\dot{z}(t,\overline{\theta}) = g\Big(t,\overline{\theta},z(t,\overline{\theta}),u(t)\Big),\ t \geq t_0, \tag{2.55a}$$

$$z(t_0,\overline{\theta}) = z_0(\overline{\theta}). \tag{2.55b}$$

Moreover, assuming that differentiation with respect to $\theta_i, i = 1,\ldots,r$, and integration with respect to time t can be interchanged, see Lemma 2.1, from (2.3c) we obtain the following system of linear perturbation differential equation for the Jacobian $D_\theta z(t,\overline{\theta}) = \left(\frac{\partial z}{\partial \theta_1}(t,\overline{\theta}), \frac{\partial z}{\partial \theta_2}(t,\overline{\theta}),\ldots,\frac{\partial z}{\partial \theta_r}(t,\overline{\theta})\right), t \geq t_0$:

$$\begin{aligned} \frac{d}{dt}\Big(D_\theta z(t,\overline{\theta})\Big) &= D_z g\Big(t,\overline{\theta},z(t,\overline{\theta}),u(t)\Big)D_\theta z(t,\overline{\theta}) \\ &\quad + D_\theta g\Big(t,\overline{\theta},z(t,\overline{\theta}),u(t)\Big), t \geq t_0, \end{aligned} \tag{2.56a}$$

$$D_\theta z(t_0,\overline{\theta}) = D_\theta z_0(\overline{\theta}). \tag{2.56b}$$

Note. Equation (2.56a, b) is closely related to the perturbation equation (2.16a, b) for representing the derivative $D_u z$ of z with respect to the control u. Moreover, the matrix differential equation (2.56a) can be decomposed into the following r

differential equations for the columns $\frac{\partial z}{\partial \theta_j}(t, \overline{\theta})$, $j = 1, \ldots, r$:

$$\frac{d}{dt}\Big(\frac{\partial z}{\partial \theta_j}(t, \overline{\theta})\Big) = \frac{\partial g}{\partial \theta_j}\Big(t, \overline{\theta}, z(t, \overline{\theta}), u(t)\Big)\frac{\partial z}{\partial \theta_j}(t, \overline{\theta})$$
$$+\frac{\partial g}{\partial \theta_j}\Big(t, \overline{\theta}, z(t, \overline{\theta}), u(t)\Big), t \geq t_0, j = 1, \ldots, r. \quad (2.56c)$$

Denoting by

$$\tilde{L} = \tilde{L}\Big(t, \theta, z(t, \overline{\theta}), D_\theta z(t, \overline{\theta}), u(t)\Big), \quad (2.57a)$$

$$\tilde{G} = \tilde{G}\Big(t_f, \theta, z(t_f, \overline{\theta}), D_\theta z(t_f, \overline{\theta})\Big), \quad (2.57b)$$

the approximation of the cost functions L, G by complete, partial Taylor expansion, for the optimal control problem under stochastic uncertainty (2.9a–d) we obtain now the following approximation:

Theorem 2.8 *Suppose that differentiation with respect to the parameters $\theta_i, i = 1, \ldots, r$, and integration with respect to time t can be interchanged in (2.3c). Retaining only 1st order derivatives of $z = z(t, \theta)$ with respect to θ, the optimal control problem under stochastic uncertainty (2.9a–d) can be approximated by the ordinary deterministic control problem:*

$$\min \int_{t_0}^{t_f} E\tilde{L}\Big(t, \theta(\omega), z(t, \overline{\theta}), D_\theta z(t, \overline{\theta}), u(t)\Big)dt$$
$$+E\tilde{G}\Big(t_f, \theta(\omega), z(t_f, \overline{\theta}, D_\theta z(t, \overline{\theta})\Big) \quad (2.58a)$$

subject to

$$\dot{z}(t, \overline{\theta}) = g\Big(t, \overline{\theta}, z(t, \overline{\theta}), u(t)\Big), t \geq t_0, \quad (2.58b)$$

$$z(t_0, \overline{\theta}) = z_0(\overline{\theta}) \quad (2.58c)$$

$$\frac{d}{dt}\Big(D_\theta z(t, \overline{\theta})\Big) = D_z g\Big(t, \overline{\theta}, z(t, \overline{\theta}), u(t)\Big)D_\theta z(t, \overline{\theta})$$
$$+D_\theta g\Big(t, \overline{\theta}, z(t, \overline{\theta}), u(t)\Big), t \geq t_0, \quad (2.58d)$$

$$D_\theta z(t_0, \overline{\theta}) = D_\theta z_0(\overline{\theta}) \quad (2.58e)$$

$$u(\cdot) \in D. \quad (2.58f)$$

Remark 2.14 Obviously, the trajectory of above deterministic substitute control problem (2.58a–f) of the original optimal control problem under stochastic uncertainty (2.9a–d) can be represented by the $m(r+1)$-vector function:

$$t \to \boldsymbol{\xi}(t) := \begin{pmatrix} z(t, \overline{\theta}) \\ \frac{\partial z}{\partial \theta_1}(t, \overline{\theta}) \\ \vdots \\ \frac{\partial z}{\partial \theta_r}(t, \overline{\theta}) \end{pmatrix}, t_0 \le t \le t_f. \tag{2.59}$$

Remark 2.15 Constraints of the expectation type (2.9f), i.e.,

$$Eh_{II}\left(t, \theta(\omega), z\left(t, \theta(\omega)\right)\right) \le (=) 0,$$

can be evaluated as in (2.52) and (2.53b). This yields then deterministic constraints for the unknown functions $t \to z(t, \overline{\theta})$ and $t \to D_\theta z(t, \overline{\theta}), t \ge t_0$.

Remark 2.16 The expectations $EH_{u^\circ(\cdot)}, EH$ arising in $(P)^t_{u^\circ(\cdot),\zeta,\eta}, (P)^t_{\zeta,\eta}$, resp., can be determined approximatively as described above.

Chapter 3
Stochastic Optimal Open-Loop Feedback Control

3.1 Dynamic Structural Systems Under Stochastic Uncertainty

3.1.1 Stochastic Optimal Structural Control: Active Control

In order to omit structural damages and therefore high compensation (recourse) costs, active control techniques are used in structural engineering. The structures usually are stationary, safe and stable without considerable external dynamic disturbances. Thus, in case of heavy dynamic external loads, such as earthquakes, wind turbulences, water waves, etc., which cause large vibrations with possible damages, additional control elements can be installed in order to counteract applied dynamic loads, see [18, 154, 155].

The structural dynamics is modeled mathematically by means of a linear system of second order differential equations for the m-vector $q = q(t)$ of displacements. The system of differential equations involves random dynamic parameters, random initial values, the random dynamic load vector and a control force vector depending on an input control function $u = u(t)$. Robust, i.e. parameter-insensitive optimal feedback controls u^* are determined in order to cope with the stochastic uncertainty involved in the dynamic parameters, the initial values and the applied loadings. In practice, the design of controls is directed often to reduce the mean square response (displacements and their time derivatives) of the system to a desired level within a reasonable span of time.

The performance of the resulting structural control problem under stochastic uncertainty is evaluated therefore by means of a convex quadratic cost function $L = L(t, z, u)$ of the state vector $z = z(t)$ and the control input vector $u = u(t)$. While the actual time path of the random external load is not known at the planning stage, we may assume that the probability distribution or at least the moments under consideration of the applied load and other random parameters are known. The problem is then to determine a robust, i.e. parameter-insensitive (open-loop)

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feedback control law by minimization of the expected total costs, hence, a stochastic optimal control law.

As mentioned above, in active control of dynamic structures, cf. [18, 117, 154–157, 171], the behavior of the m-vector $q = q(t)$ of displacements with respect to time t is described by a system of second order linear differential equations for $q(t)$ having a right hand side being the sum of the stochastic applied load process and the control force depending on a control n-vector function $u(t)$:

$$M\ddot{q} + D\dot{q} + Kq(t) = f\Big(t, \omega, u(t)\Big), t_0 \leq t \leq t_f \,. \tag{3.1a}$$

Hence, the force vector $f = f\Big(t, \omega, u(t)\Big)$ on the right hand side of the dynamic equation (3.1a) is given by the sum

$$f(t, \omega, u) = f_0(t, \omega) + f_a(t, \omega, u) \tag{3.1b}$$

of the applied load $f_0 = f_0(t, \omega)$ being a vector-valued stochastic process describing e.g. external loads or excitation of the structure caused by earthquakes, wind turbulences, water waves, etc., and the actuator or control force vector $f_a = f_a(t, \omega, u)$ depending on an input or control n-vector function $u = u(t), t_0 \leq t \leq t_f$. Here, ω denotes the random element, lying in a certain probability space $(\Omega, \mathcal{A}, \mathcal{P})$, used to represent random variations. Furthermore, M, D, K, resp., denotes the $m \times m$ mass, damping and stiffness matrix. In many cases the actuator or control force f_a is linear, i.e.

$$f_a = \Gamma_u u \tag{3.1c}$$

with a certain $m \times n$ matrix Γ_u.

By introducing appropriate matrices, the linear system of second order differential equations (3.1a, b) can be represented by a system of first order differential equations as follows:

$$\dot{z} = g(t, \omega, z(t\omega), u) := Az(t, \omega) + Bu + b(t, \omega) \tag{3.2a}$$

with

$$A := \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix}, \qquad B := \begin{pmatrix} 0 \\ M^{-1}\Gamma_u \end{pmatrix}, \tag{3.2b}$$

$$b(t, \omega) := \begin{pmatrix} 0 \\ M^{-1} f_0(t, \omega). \end{pmatrix} \tag{3.2c}$$

Moreover, $z = z(t)$ is the $2m$-state vector defined by

$$z = \begin{pmatrix} q \\ \dot{q} \end{pmatrix} \tag{3.2d}$$

fulfilling a certain initial condition

$$z(t_0) = \begin{pmatrix} q(t_0) \\ \dot{q}(t_0) \end{pmatrix} := \begin{pmatrix} q_0 \\ \dot{q}_0 \end{pmatrix} \tag{3.2e}$$

with given or stochastic initial values $q_0 = q_0(\omega), \dot{q}_0 = \dot{q}_0(\omega)$.

3.1.2 Stochastic Optimal Design of Regulators

In the optimal design of regulators for dynamic systems, see also Chap. 4, the (m-)vector $q = q(t)$ of *tracking errors* is described by a system of $2nd$ order linear differential equations:

$$M(t)\ddot{q} + D(t)\dot{q} + K(t)q(t) = -Y(t)\Delta p_D(\omega) + \Delta u(t,\omega), t_0 \le t \le t_f. \tag{3.3}$$

Here, $M(t), D(t), K(t), Y(t)$ denote certain time-dependent Jacobians arising from the linearization of the dynamic equation around the stochastic optimal reference trajectory and the conditional expectation $\overline{p_D}$ of the vector of dynamic parameters $p_D(\omega)$. The deviation between the vector of dynamic parameters $p_D(\omega)$ and its conditional expectation $\overline{p_D}$ is denoted by $\Delta p_D(\omega) := p_D(\omega) - \overline{p_D}$. Furthermore, $\Delta u(t)$ denotes the correction of the feedforward control $u^0 = u^0(t)$.

By introducing appropriate matrices, system (2.1a, b) can be represented by the $1st$ order system of linear differential equations:

$$\dot{z} = A(t)z(t,\omega) + B\Delta u + b(t,\omega) \tag{3.4a}$$

with

$$A(t) := \begin{pmatrix} 0 & I \\ -M(t)^{-1}K & -M(t)^{-1}D(t) \end{pmatrix}, \qquad B := \begin{pmatrix} 0 \\ M(t)^{-1} \end{pmatrix}, \tag{3.4b}$$

$$b(t,\omega) := \begin{pmatrix} 0 \\ -M(t)^{-1}Y(t)\Delta p_D(\omega) \end{pmatrix}. \tag{3.4c}$$

Again, the ($2m$-) state vector $z = z(t)$ is defined by

$$z = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}. \tag{3.4d}$$

3.1.3 Robust (Optimal) Open-Loop Feedback Control

According to the description in Sect. 2.2, a feedback control is defined, cf. (2.10b), by

$$u(t) := \varphi(t, \mathcal{I}_t), t \geq t_0, \tag{3.5a}$$

where $\mathcal{I}_t$ denotes again the total information about the control system up to time t and $\varphi(\cdot, \cdot)$ designates the feedback control law. If the state $z_t := z(t)$ is available at each time point t, the control input n-vector function $u = u(t)$, $\Delta u = \Delta u(t)$, resp., can be generated by means of a PD-controller, hence,

$$u(t)(\Delta u(t)) := \varphi\Big(t, z(t)\Big), t \geq t_0, \tag{3.5b}$$

with a feedback control law $\varphi = \varphi(t, q, \dot{q}) = \varphi(t, z(t))$. Efficient approximate feedback control laws are constructed here by using the concept of **open-loop feedback control**. Open-loop feedback control is the main tool in *model predictive control*, cf. [1, 101, 136], which is very often used to solve optimal control problems in practice. The idea of *open-loop feedback control* is to construct a feedback control law quasi *argument-wise*, see cf. [2, 80].

A major issue in optimal control is the **robustness**, cf. [43], i.e. the insensitivity of the optimal control with respect to parameter variations. In case of random parameter variations, robust optimal controls can be obtained by means of stochastic optimization methods, cf. [100]. Thus, we introduce the following concept of an *stochastic optimal (open-loop) feedback control*.

Definition 3.1 In case of stochastic parameter variations, robust, hence, parameter-insensitive optimal (open-loop) feedback controls obtained by stochastic optimization methods are also called **stochastic optimal (open-loop) feedback controls**.

3.1.4 Stochastic Optimal Open-Loop Feedback Control

Finding a stochastic optimal open-loop feedback control, hence, an optimal (open-loop) feedback control law, see Sect. 2.2, being insensitive as far as possible with respect to random parameter variations, means that besides optimality of the control law also its insensitivity with respect to stochastic parameter variations should be guaranteed. Hence, in the following sections we develop now a stochastic version of the (optimal) open-loop feedback control method, cf. [102–105]. A short overview on this novel stochastic optimal open-loop feedback control concept is given below:

At each intermediate time point $t_b \in [t_0, t_f]$, based on the information $\mathcal{I}_{t_b}$ available at time t_b, a stochastic optimal open-loop control $u^* = u^*\big(t; t_b, \mathcal{I}_{t_b}\big), t_b \leq$

Fig. 3.1 Remaining time interval

$t \leq t_f$, is determined first on the remaining time interval $[t_b, t_f]$, see Fig. 3.1, by stochastic optimization methods, cf. [100] .

Having a stochastic optimal open-loop control $u^* = u^*\big(t; t_b, \mathcal{I}_{t_b}\big), t_b \leq t \leq t_f$, on each remaining time interval $[t_b, t_f]$ with an arbitrary starting time t_b, $t_0 \leq t_b \leq t_f$, a stochastic optimal open-loop feedback control law is then defined, see Definition 2.3, as follows:

Definition 3.2

$$\varphi^* = \varphi\big(t_b, \mathcal{I}_{t_b}\big) := u^*(t_b) = u^*\big(t_b; t_b, \mathcal{I}_{t_b}\big), t_0 \leq t_b \leq t_f. \tag{3.5c}$$

Hence, at time $t = t_b$ just the "first" control value $u^*(t_b) = u^*\big(t_b; t_b, \mathcal{I}_{t_b}\big)$ of $u^*\big(\cdot; t_b, \mathcal{I}_{t_b}\big)$ is used only. For each other argument $\big(t, \mathcal{I}_t\big)$ the same construction is applied.

For finding stochastic optimal open-loop controls, based on the methods developed in Chap. 2, on the remaining time intervals $t_b \leq t \leq t_f$ with $t_0 \leq t_b \leq t_f$, the stochastic Hamilton function of the control problem is introduced. Then, the class of $H-$ minimal controls, cf. Definitions 2.6 and 2.7, can be determined in case of stochastic uncertainty by solving a finite-dimensional stochastic optimization problem for minimizing the conditional expectation of the stochastic Hamiltonian subject to the remaining deterministic control constraints at each time point t. Having a $H-$ minimal control, the related two-point boundary value problem with random parameters can be formulated for the computation of a stochastic optimal state- and costate-trajectory. Due to the linear-quadratic structure of the underlying control problem, the state and costate trajectory can be **determined analytically** to a large extent. Inserting then these trajectories into the H-minimal control, stochastic optimal open-loop controls are found on an arbitrary remaining time interval. According to Definition 3.2, these controls yield then immediately a stochastic optimal open-loop feedback control law. Moreover, the obtained controls can be realized in **real-time**, which is already shown for applications in optimal control of industrial robots, cf. [139].

Summarizing, we get *optimal (open-loop) feedback controls under stochastic uncertainty* minimizing the effects of external influences on system behavior, subject to the constraints of not having a complete representation of the system, cf. [43]. Hence, robust or stochastic optimal active controls are obtained by **new** techniques from *Stochastic Optimization*, see [100]. Of course, the construction can be applied also to *PD*- and *PID*-controllers.

3.2 Expected Total Cost Function

The performance function F for active structural control systems is defined , cf. [95, 98, 101], by the conditional expectation of the total costs being the sum of costs L along the trajectory, arising from the displacements $z = z(t, \omega)$ and the control input $u = u(t, \omega)$, and possible terminal costs G arising at the final state z_f. Hence, on the remaining time interval $t_b \leq t \leq t_f$ we have the following conditional expectation of the total cost function with respect to the information $\mathfrak{A}_{t_b}$ available up to time t_b:

$$F := \mathbb{E}\left(\int_{t_b}^{t_f} L\Big(t, \omega, z(t, \omega), u(t, \omega)\Big)dt + G(t_f, \omega, z(t_f, \omega)) \Big| \mathfrak{A}_{t_b}\right). \tag{3.6a}$$

Supposing quadratic costs along the trajectory, the function L is given by

$$L(t, \omega, z, u) := \frac{1}{2}z^T Q(t, \omega)z + \frac{1}{2}u^T R(t, \omega)u \tag{3.6b}$$

with positive (semi) definite $2m \times 2m, n \times n$, resp., matrix functions $Q = Q(t, \omega), R = R(t, \omega)$. In the simplest case the weight matrices Q, R are fixed. A special selection for Q reads

$$Q = \begin{pmatrix} Q_q & 0 \\ 0 & Q_{\dot{q}} \end{pmatrix} \tag{3.6c}$$

with positive (semi) definite weight matrices $Q_q, Q_{\dot{q}}$, resp., for $q, \dot{q}$. Furthermore, $G = G(t_f, \omega, z(t_f, \omega))$ describes possible terminal costs. In case of endpoint control G is defined by

$$G(t_f, \omega, z(t_f, \omega)) := \frac{1}{2}(z(t_f, \omega) - z_f(\omega))^T G_f(z(t_f, \omega) - z_f(\omega)), \tag{3.6d}$$

where $G_f = G_f(\omega)$ is a positive (semi) definite, possible random weight matrix, and $z_f = z_f(\omega)$ denotes the (possible random) final state.

Remark 3.1 Instead of $\frac{1}{2}u^T Ru$, in the following we also use a more general convex control cost function $C = C(u)$.

3.3 Open-Loop Control Problem on the Remaining Time Interval $[t_b, t_f]$

In the following we suppose next to that the $2m \times 2m$ matrix A and the $2m \times n$ matrix B are given, fixed matrices.

Having the differential equation with random coefficients derived above, describing the behavior of the dynamic mechanical structure/system under stochastic uncertainty, and the costs arising from displacements and at the terminal state, on a given remaining time interval $[t_b, t_f]$ a stochastic optimal open-loop control $u^* = u^*\big(t; t_b, \mathcal{I}_{t_b}\big), t_b \le t \le t_f$, is a solution of the following optimal control problem under stochastic uncertainty:

$$\min \quad \mathbb{E}\left(\int_{t_b}^{t_f} \frac{1}{2}\big(z(t,\omega)^T Q z(t,\omega) + u(t)^T R u(t)\big)\ dt + G(t_f, \omega, z(t_f, \omega))\Big|\mathfrak{A}_{t_b}\right) \tag{3.7a}$$

$$\text{s.t.} \quad \dot{z}(t,\omega) = Az(t,\omega) + Bu(t) + b(t,\omega), \ \text{a.s.}, \qquad t_b \le t \le t_f \tag{3.7b}$$

$$z(t_b,\omega) = \overline{z_b}^{(b)} \ \ \text{(estimated state at time } t_b) \tag{3.7c}$$

$$u(t) \in D_t, \qquad t_b \le t \le t_f\,. \tag{3.7d}$$

An important property of (3.7a–d) is stated next:

Lemma 3.1 *If the terminal cost function $G = G(t_f, \omega, z)$ is convex in z, and the feasible domain D_t is convex for each time point t, $t_0 \le t \le t_f$, then the stochastic optimal control problem (3.7a–d) is a convex optimization problem.*

3.4 The Stochastic Hamiltonian of (3.7a–d)

According to (2.28a), (2.40a), see also [101], the stochastic Hamiltonian H related to the stochastic optimal control problem (3.7a–d) reads:

$$\begin{aligned} H(t,\omega,z,y,u) &:= L(t,\omega,z,u) + y^T g(t,\omega,z,u) \\ &= \frac{1}{2} z^T Q z + C(u) + y^T\left(Az + Bu + b(t,\omega)\right)\,. \end{aligned} \tag{3.8a}$$

3.4.1 Expected Hamiltonian (with Respect to the Time Interval $[t_b, t_f]$ and Information $\mathfrak{A}_{t_b}$)

For the definition of a $H-$minimal control the conditional expectation of the stochastic Hamiltonian is needed:

$$\begin{aligned}\overline{H}^{(b)} &:= \mathbb{E}\left(H(t,\omega,z,y,u)\middle|\mathfrak{A}_{t_b}\right) = \mathbb{E}\left(\frac{1}{2}z^T Qz + y^T\left(Az + b(t,\omega)\right)\middle|\mathfrak{A}_{t_b}\right)\\ &\qquad + C(u) + \mathbb{E}\left(y^T Bu\middle|\mathfrak{A}_{t_b}\right)\\ &= C(u) + \mathbb{E}\left(B^T y(t,\omega)\middle|\mathfrak{A}_{t_b}\right)^T u + \dots = C(u) + h(t)^T u + \dots\end{aligned} \tag{3.8b}$$

with

$$h(t) = h\big(t; t_b, \mathcal{I}_{t_b}\big) := \mathbb{E}\left(B(\omega)^T y(t,\omega)\middle|\mathfrak{A}_{t_b}\right), \quad t \geq t_b\,. \tag{3.8c}$$

3.4.2 H-Minimal Control on $[t_b, t_f]$

In order to formulate the two-point boundary value problem for a stochastic optimal open-loop control $u^* = u^*\big(t; t_b, \mathcal{I}_{t_b}\big), t_b \leq t \leq t_f$, we need first an H-minimal control

$$\widetilde{u^*} = \widetilde{u^*}\Big(t, z(t,\cdot), y(t,\cdot); t_b, \mathcal{I}_{t_b}\Big), t_b \leq t \leq t_f,$$

defined, see Definitions 2.6 and 2.7 and cf. also [101], for $t_b \leq t \leq t_f$ as a solution of the following convex stochastic optimization problem, cf. [100]:

$$\min \mathbb{E}\left(H(t,\omega,z(t,\omega),y(t,\omega),u)\middle|\mathfrak{A}_{t_b}\right) \tag{3.9a}$$

s.t.

$$u \in D_t, \tag{3.9b}$$

where $z = z(t,\omega)$, $y = y(t,\omega)$ are certain trajectories.

According to (3.9a, b) the H-minimal control

$$\widetilde{u^*} = \widetilde{u^*}\Big(t, z(t,\cdot), y(t,\cdot); t_b, \mathcal{I}_{t_b}\Big) = \widetilde{u^*}\big(t, h\big(\cdot; t_b, \mathcal{I}_{t_b}\big)\big) \tag{3.10a}$$

is defined by

$$\widetilde{u^*}(t, h(\cdot; t_b, \mathcal{I}_{t_b})) := \underset{u \in D_t}{\operatorname{argmin}}\, C(u) + h(t; t_b, \mathcal{I}_{t_b})^T u \qquad \text{for } t \geq t_b\,. \tag{3.10b}$$

Strictly Convex Cost Function, No Control Constraints

For strictly convex, differentiable cost functions $C = C(u)$, as e.g. $C(u) = \frac{1}{2}u^T Ru$ with positive definite matrix R, the necessary and sufficient condition for $\widetilde{u^*}$ reads in case $D_t = \mathbb{R}^n$:

$$\nabla C(u) + h(t; t_b, \mathcal{I}_{t_b}) = 0\,. \tag{3.11a}$$

If $u \mapsto \nabla C(u)$ is a 1-1-operator, then the solution of (3.11a) reads

$$u = v(h(t; t_b, \mathcal{I}_{t_b})) := \nabla C^{-1}(-h(t; t_b, \mathcal{I}_{t_b}))\,. \tag{3.11b}$$

With (3.8c) and (3.10b) we then have

$$\widetilde{u^*}(t, h) = \widetilde{u^*}(h(t; t_b, \mathcal{I}_{t_b}))) := \nabla C^{-1}\left(-\mathbb{E}\left(B(\omega)^T y(t,\omega) \middle| \mathfrak{A}_{t_b}\right)\right)\,. \tag{3.11c}$$

3.5 Canonical (Hamiltonian) System

We suppose here that a H-minimal control $\widetilde{u^*} = \widetilde{u^*}(t, z(t,\cdot), y(t,\cdot); t_b, \mathcal{I}_{t_b})$, $t_b \leq t \leq t_f$, i.e., a solution $\widetilde{u^*} = \widetilde{u^*}(t, h) = v(h(t)))$ of the stochastic optimization problem (3.9a, b) is available. Moreover, the conditional expectation $\mathbb{E}\left(\xi \middle| \mathfrak{A}_{t_b}\right)$ of a random variable ξ is also denoted by $\overline{\xi}^{(b)}$, cf. (3.8b). According to (2.46), Theorem 2.7, a stochastic optimal open-loop control $u^* = u^*(t; t_b, \mathcal{I}_{t_b})$, $t_b \leq t \leq t_f$,

$$u^*(t; t_b, \mathcal{I}_{t_b}) = \widetilde{u^*}\Big(t, z^*(t,\cdot), y^*(t,\cdot); t_b, \mathcal{I}_{t_b}\Big), t_b \leq t \leq t_f\,, \tag{3.12}$$

of the stochastic optimal control problem (3.7a–d), can be obtained, see also [101], by solving the following stochastic two-point boundary value problem related to (3.7a–d):

Theorem 3.1 *If $z^* = z^*(t,\omega)$, $y^* = y^*(t,\omega)$, $t_0 \leq t \leq t_f$, is a solution of*

$$\dot{z}(t,\omega) = Az(t,\omega) + B\nabla C^{-1}\left(-\overline{B(\omega)^T y(t,\omega)}^{(b)}\right) + b(t,\omega), \quad t_b \leq t \leq t_f \tag{3.13a}$$

$$z(t_b, \omega) = \overline{z_b}^{(b)} \tag{3.13b}$$

$$\dot{y}(t,\omega) = -A^T y(t,\omega) - Qz(t,\omega) \tag{3.13c}$$

$$y(t_f,\omega) = \nabla G(t_f,\omega,z(t_f,\omega)), \tag{3.13d}$$

then the function $u^* = u^*(t;t_b,\mathcal{I}_{t_b}), t_b \le t \le t_f$, *defined by (3.12) is a stochastic optimal open-loop control for the remaining time interval* $t_b \le t \le t_f$.

3.6 Minimum Energy Control

In this case we have $Q = 0$, i.e., there are no costs for the displacements $z = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}$. In this case the solution of (3.13c, d) reads

$$y(t,\omega) = e^{A^T(t_f-t)}\nabla_z G(t_f,\omega,z(t_f,\omega)), \qquad t_b \le t \le t_f\,. \tag{3.14a}$$

This yields for fixed Matrix B

$$\widetilde{u^*}(t,h(t)) = v(h(t)) = \nabla C^{-1}\left(-B^T e^{A^T(t_f-t)}\overline{\nabla_z G(t_f,\omega,z(t_f,\omega))}^{(b)}\right), \tag{3.14b}$$

$$t_b \le t \le t_f.$$

Having (3.14a, b), for the state trajectory $z = z(t,\omega)$ we get, see (3.13a, b), the following system of ordinary differential equations

$$\begin{aligned}\dot{z}(t,\omega) &= Az(t,\omega) + B\nabla C^{-1}\big(-B^T e^{A^T(t_f-t)}\overline{\nabla_z G(t_f,\omega,z(t_f,\omega))}^{(b)}\big)\\ &\quad + b(t,\omega), \qquad t_b \le t \le t_f,\end{aligned} \tag{3.15a}$$

$$z(t_b,\omega) = \overline{z_b}^{(b)}. \tag{3.15b}$$

The solution of system (3.15a, b) reads

$$\begin{aligned} z(t,\omega) = e^{A(t-t_b)}\overline{z_b}^{(b)} + \int_{t_b}^{t} e^{A(t-s)}\Bigg(& b(s,\omega) \\ & + B\nabla C^{-1}\left(-B^T e^{A^T(t_f-s)}\overline{\nabla_z G(t_f,\omega,z(t_f,\omega))}^{(b)}\right)\Bigg)\,ds, \end{aligned}$$

$$t_b \le t \le t_f\,. \tag{3.16}$$

For the final state $z = z(t_f, \omega)$ we get the relation:

$$z(t_f, \omega) = e^{A(t_f - t_b)}\overline{z_b}^{(b)} + \int_{t_b}^{t_f} e^{A(t_f - s)}\Bigg(b(s, \omega) + B\nabla C^{-1}\left(-B^T e^{A^T(t_f - s)}\overline{\nabla_z G(t_f, \omega, z(t_f, \omega))}^{(b)}\right)\Bigg)\, ds\,. \tag{3.17}$$

3.6.1 Endpoint Control

In the case of endpoint control, the terminal cost function is given by the following definition (3.18a), where $z_f = z_f(\omega)$ denotes the desired—possible random—final state:

$$G(t_f, \omega, z(t_f, \omega)) := \frac{1}{2}\|z(t_f, \omega) - z_f(\omega)\|^2\,. \tag{3.18a}$$

Hence,

$$\nabla G(t_f, \omega, z(t_f, \omega)) = z(t_f, \omega) - z_f(\omega) \tag{3.18b}$$

and therefore

$$\begin{aligned} \overline{\nabla G(t_f, \omega, z(t_f, \omega))}^{(b)} &= \overline{z(t_f, \omega)}^{(b)} - \overline{z_f}^{(b)} \\ &= \mathbb{E}\left(z(t_f, \omega)\middle|\mathfrak{A}_{t_b}\right) - \mathbb{E}\left(z_f\middle|\mathfrak{A}_{t_b}\right)\,. \end{aligned} \tag{3.18c}$$

Thus

$$z(t_f, \omega) = e^{A(t_f - t_b)}\overline{z_b}^{(b)} + \int_{t_b}^{t_f} e^{A(t_f - s)}\Bigg(b(s, \omega) + B\nabla C^{-1}\left(-B^T e^{A^T(t_f - s)}\left(\overline{z(t_f, \omega)}^{(b)} - \overline{z_f}^{(b)}\right)\right)\Bigg)\, ds\,. \tag{3.19a}$$

Taking expectations $\mathbb{E}(\ldots|\mathfrak{A}_{t_b})$ in (3.19a), we get the following condition for $\overline{z(t_f,\omega)}^{(b)}$:

$$\overline{z(t_f,\omega)}^{(b)} = e^{A(t_f-t_b)}\overline{z_b}^{(b)} + \int\limits_{t_b}^{t_f} e^{A(t_f-s)}\overline{b(s,\omega)}^{(b)}\, ds$$
$$+ \int\limits_{t_b}^{t_f} e^{A(t_f-s)} B \nabla C^{-1} \left(-B^T e^{A^T(t_f-s)}\left(\overline{z(t_f,\omega)}^{(b)} - \overline{z_f}^{(b)}\right)\right)\, ds\,. \tag{3.19b}$$

Quadratic Control Costs

Here, the control cost function $C = C(u)$ reads

$$C(u) = \frac{1}{2}u^T R u\,, \tag{3.20a}$$

hence,

$$\nabla C = Ru \tag{3.20b}$$

and therefore

$$\nabla C^{-1}(w) = R^{-1}w\,. \tag{3.20c}$$

Consequently, (3.19b) reads

$$\overline{z(t_f,\omega)}^{(b)} = e^{A(t_f-t_b)}\overline{z_b}^{(b)} + \int\limits_{t_b}^{t_f} e^{A(t_f-s)}\overline{b(s,\omega)}^{(b)}\, ds$$
$$- \int\limits_{t_b}^{t_f} e^{A(t_f-s)} BR^{-1}B^T e^{A^T(t_f-s)}\, ds\overline{z(t_f,\omega)}^{(b)}$$
$$+ \int\limits_{t_b}^{t_f} e^{A(t_f-s)} BR^{-1}B^T e^{A^T(t_f-s)}\, ds\overline{z_f(\omega)}^{(b)}\,. \tag{3.21}$$

Define now

$$U := \int_{t_b}^{t_f} e^{A(t_f-s)} B R^{-1} B^T e^{A^T(t_f-s)} \, ds \,. \tag{3.22}$$

Lemma 3.2 *$I + U$ is regular.*

Proof Due to the previous considerations, U is a positive semidefinite $2m \times 2m$ matrix. Hence, U has only nonnegative eigenvalues.

Assuming that the matrix $I + U$ is singular, there is a $2m$-vector $w \neq 0$ such that

$$(I + U)\, w = 0.$$

However, this yields

$$U w = -I w = -w = (-1) w,$$

which means that $\lambda = -1$ is an eigenvalue of U. Since this contradicts to the above mentioned property of U, the matrix $I + U$ must be regular.

From (3.21) we get

$$(I + U)\, \overline{z(t_f, \omega)}^{(b)} = e^{A(t_f - t_b)} \overline{z_b}^{(b)} + \int_{t_b}^{t_b} e^{A(t_f - s)} \overline{b(s, \omega)}^{(b)} \, ds + U \overline{z_f}^{(b)}, \tag{3.23a}$$

hence,

$$\overline{z(t_f, \omega)}^{(b)} = (I + U)^{-1} e^{A(t_f - t_b)} z_b + (I + U)^{-1} \int_{t_b}^{t_b} e^{A(t_f - s)} \overline{b(s, \omega)}^{(b)} \, ds$$
$$+ (I + U)^{-1} U \overline{z_f}^{(b)} \,. \tag{3.23b}$$

Now, (3.23b) and (3.18b) yield

$$\begin{aligned} \overline{\nabla_z G(t_f, \omega, z(t_f, \omega))} &= \overline{z(t_f, \omega) - z_f}^{(b)} = \overline{z(t_f, \omega)}^{(b)} - \overline{z_f}^{(b)} \\ &= (I + U)^{-1} e^{A(t_f - t_b)} \overline{z_b}^{(b)} \\ &\quad + (I + U)^{-1} \int_{t_b}^{t_f} e^{A(t_f - s)} \overline{b(s, \omega)}^{(b)} \, ds \\ &\quad + \left((I + U)^{-1} U - I \right) \overline{z_f}^{(b)} \,. \end{aligned} \tag{3.24}$$

Thus, a stochastic optimal open-loop control $u^* = u^*(t; t_b, \mathcal{I}_{t_b})$, $t_b \le t \le t_f$, on $[t_b, t_f]$ is given by, cf. (3.11b),

$$u^*(t; t_b, \mathcal{I}_{t_b}) = -R^{-1} B^T e^{A^T(t_f - t)} \Bigg((I + U)^{-1} e^{A(t_f - t_b)} \overline{z_b}^{(b)} + (I + U)^{-1} \int\limits_{t_b}^{t_f} e^{A(t_f - s)} \overline{b(s, \omega)}^{(b)}\, ds + \Big((I + U)^{-1} U - I \Big) \overline{z_f}^{(b)} \Bigg), \qquad t_b \le t \le t_f. \tag{3.25}$$

Finally, the stochastic optimal open-loop feedback control law $\varphi = \varphi(t, \mathcal{I}_t)$ is then given by

$$\begin{aligned} \varphi(t_b, \mathcal{I}_{t_b}) &:= u^*(t_b; t_b, \mathcal{I}_{t_b}) \\ &= -R^{-1} B^T e^{A^T(t_f - t_b)} (I + U)^{-1} e^{A(t_f - t_b)} \overline{z_b}^{(b)} \\ &\quad -R^{-1} B^T e^{A^T(t_f - t_b)} (I + U)^{-1} \int\limits_{t_b}^{t_f} e^{A(t_f - s)} \overline{b(s, \omega)}^{(b)}\, ds \\ &\quad -R^{-1} B^T e^{A^T(t_f - t_b)} \big((I + U)^{-1} U - I \big) \overline{z_f}^{(b)} \end{aligned} \tag{3.26}$$

with $\mathcal{I}_{t_b} := \Big(\overline{z_b}^{(b)} := \overline{z(t_b)}^{(b)}, \overline{b(\cdot, \omega)}^{(b)}, \overline{z_f}^{(b)} \Big)$.

Replacing $t_b \to t$, we find this result:

Theorem 3.2 *The stochastic optimal open-loop feedback control law $\varphi = \varphi(t, \mathcal{I}_t)$ is given by*

$$\begin{aligned} \varphi(t, \mathcal{I}_t) = &\underbrace{-R^{-1} B^T e^{A^T(t_f - t)} (I + U)^{-1} e^{A(t_f - t)}}_{\Psi_0(t)} \overline{z(t)}^{(t)} \\ &\underbrace{-R^{-1} B^T e^{A^T(t_f - t)} (I + U)^{-1} \int\limits_{t}^{t_f} e^{A(t_f - s)} \overline{b(s, \omega)}^{(t)}\, ds}_{\Psi_1(t, \overline{b(\cdot, \omega)}^{(t)})} \\ &\underbrace{-R^{-1} B^T e^{A^T(t_f - t)} \big((I + U)^{-1} U - I \big)}_{\Psi_2(t)} \overline{z_f}^{(t)}, \end{aligned} \tag{3.27a}$$

hence,

$$\varphi(t, \mathcal{I}_t) \;=\; \Psi_0(t)\overline{z(t)}^{(t)} + \Psi_1(t, \overline{b(\cdot,\omega)}^{(t)}) + \Psi_2(t)\overline{z_f}^{(t)}\,,$$
$$\mathcal{I}_t := \left(\overline{z(t)}^{(t)}, \overline{b(\cdot,\omega)}^{(t)}, \overline{z_f}^{(t)}\right). \tag{3.27b}$$

Remark 3.2 Note that the stochastic optimal open-loop feedback law $\overline{z(t)}^{(t)} \mapsto \varphi(t, \mathcal{I}_t)$ is not linear in general, but affine-linear.

3.6.2 Endpoint Control with Different Cost Functions

In this section we consider more general terminal cost functions G. Hence, suppose

$$G(t_f, \omega, z(t_f, \omega)) := \kappa(z(t_f, \omega) - z_f(\omega))\,, \tag{3.28a}$$
$$\nabla G(t_f, \omega, z(t_f, \omega)) = \nabla\kappa(z(t_f, \omega) - z_f(\omega))\,. \tag{3.28b}$$

Consequently,

$$\widetilde{u^*}(t, h(t)) = v^*(h(t)) = \nabla C^{-1}\left(B^T e^{A^T(t_f - t)}\overline{\nabla\kappa(z(t_f, \omega) - z_f(\omega))}^{(b)}\right) \tag{3.29a}$$

and therefore, see (3.17)

$$\begin{aligned} z(t_f, \omega) &= e^{A(t_f - t_b)} z_b + \int_{t_b}^{t_f} e^{A(t_f - s)} b(s, \omega)\, ds \\ &\quad + \int_{t_b}^{t_f} e^{A(t_f - s)} B \nabla C^{-1}\left(-B^T e^{A^T(t_f - s)}\overline{\nabla\kappa(z(t_f, \omega) - z_f(\omega))}^{(b)}\right) ds\,, \\ &\quad t_b \le t \le t_f. \end{aligned} \tag{3.29b}$$

Special Case:
Now a special terminal cost function is considered in more detail:

$$\kappa(z - z_f) := \sum_{i=1}^{2m} (z_i - z_{f\,i})^4 \tag{3.30a}$$
$$\nabla\kappa(z - z_f) = 4\left((z_1 - z_{f\,1})^3, \ldots, (z_{2m} - z_{f\,2m})^3\right)^T\,. \tag{3.30b}$$

Here,

$$\begin{aligned}\overline{\nabla\kappa(z-z_f)}^{(b)} &= 4\left(\mathbb{E}\left((z_1-z_{f\,1})^3|\mathfrak{A}_{t_b}\right),\dots,\mathbb{E}\left((z_{2m}-z_{f\,2m})^3|\mathfrak{A}_{t_b}\right)\right)^T\\ &= 4\left(m_3^{(b)}(z_1(t_f,\cdot);z_{f\,1}(\cdot)),\dots,m_3^{(b)}(z_{2m}(t_f,\cdot);z_{f\,2m}(\cdot))\right)^T\\ &=: 4m_3^{(b)}(z(t_f,\cdot);z_f(\cdot))\,. \end{aligned} \tag{3.31}$$

Thus,

$$\begin{aligned} z(t_f,\omega) = e^{A(t_f-t_b)}z_b + \int_{t_b}^{t_f} e^{A(t_f-s)}b(s,\omega)\,ds\\ + \underbrace{\int_{t_b}^{t_f} e^{A(t_f-s)}B\nabla C^{-1}\left(-B^Te^{A^T(t_f-s)}4m_3^{(b)}(z(t_f,\cdot);z_f(\cdot))\right)\,ds}_{J\left(m_3^{(b)}(z(t_f,\cdot);z_f(\cdot))\right)}\,. \end{aligned} \tag{3.32}$$

Equation (3.32) yields then

$$\begin{aligned} &\left(z(t_f,\omega)-z_f(\omega)\right)^3\Big|_{c\text{-}by\text{-}c}\\ &= \left(e^{A(t_f-t_b)}z_b - z_f + \int_{t_b}^{t_f} e^{A(t_f-s)}b(s,\omega)\,ds + J\left(m_3^{(b)}(z(t_f,\cdot);z_f(\cdot))\right)\right)^3\Bigg|_{c\text{-}by\text{-}c}, \end{aligned} \tag{3.33a}$$

where *"c-by-c"* means *"component-by-component"*. Taking expectations in (3.33a), we get the following relation for the moment vector $m_3^{(b)}$:

$$m_3^{(b)}(z(t_f,\cdot);z_f(\cdot)) = \Psi\left(m_3^{(b)}(z(t_f,\cdot);z_f(\cdot))\right). \tag{3.33b}$$

Remark 3.3

$$\begin{aligned} &\mathbb{E}\left(\left(z(t_f,\omega)-z_f(\omega)\right)^3|\mathfrak{A}_{t_b}\right)\Big|_{c-by-c}\\ &= \mathbb{E}^{(b)}\left(z(t_f,\omega)-\overline{z}^{(b)}(t_f)+\overline{z}^{(b)}(t_f)-z_f(\omega)\right)^3 \end{aligned}$$

$$= \mathbb{E}^{(b)}\Bigg(\left(z(t_f,\omega) - \overline{z}^{(b)}(t_f)\right)^3 + 3\left(z(t_f,\omega) - \overline{z}^{(b)}(t_f)\right)^2 \left(\overline{z}^{(b)}(t_f) - z_f(\omega)\right)$$

$$+3\left(z(t_f,\omega) - \overline{z}^{(b)}(t_f)\right)\left(\overline{z}^{(b)}(t_f) - z_f(\omega)\right)^2 + \left(\overline{z}^{(b)}(t_f) - z_f(\omega)\right)^3 \Bigg). \quad (3.33c)$$

Assuming that $z(t_f,\omega)$ and $z_f(\omega)$ are stochastic independent, then

$$\mathbb{E}\left((z(t_f,\omega) - z_f(\omega))^3 \middle| \mathfrak{A}_{t_b}\right)$$
$$= m_3^{(b)}(z(t_f,\cdot)) + 3\sigma^{2(b)}(z(t_f,\cdot))(\overline{z}^{(b)}(t_f) - \overline{z_f}^{(b)}) + \overline{\left(\overline{z}^{(b)}(t_f) - z_f(\omega)\right)^3}^{(b)}, \quad (3.33d)$$

where $\sigma^{2(b)}(z(t_f,\cdot)$ denotes the conditional variance of the state reached at the final time point t_f, given the information at time t_b.

3.6.3 Weighted Quadratic Terminal Costs

With a certain (possibly random) weight matrix $\Gamma = \Gamma(\omega)$, we consider the following terminal cost function:

$$G(t_f,\omega,z(t_f,\omega)) := \frac{1}{2}\left\|\Gamma(\omega)\left(z(t_f,\omega) - z_f(\omega)\right)\right\|^2. \quad (3.34a)$$

This yields

$$\nabla G(t_f,\omega,z(t_f,\omega)) = \Gamma(\omega)^T\Gamma(\omega)(z(t_f,\omega) - z_f(\omega)), \quad (3.34b)$$

and from (3.14a) we get

$$\begin{aligned} y(t,\omega) &= e^{A^T(t_f-t)}\nabla_z G(t_f,\omega,z(t_f,\omega)) \\ &= e^{A^T(t_f-t)}\Gamma(\omega)^T\Gamma(\omega)(z(t_f,\omega) - z_f(\omega)), \end{aligned} \quad (3.35a)$$

hence,

$$\begin{aligned} \overline{y}^{(b)}(t) &= e^{A^T(t_f-t)}\overline{\Gamma(\omega)^T\left(\Gamma(\omega)(z(t_f,\omega) - \Gamma(\omega)z_f(\omega))\right)}^{(b)} \\ &= e^{A^T(t_f-t)}\left(\overline{\Gamma(\omega)^T\Gamma(\omega)z(t_f,\omega)}^{(b)} - \overline{\Gamma(\omega)^T\Gamma(\omega)z_f(\omega)}^{(b)}\right). \end{aligned} \quad (3.35b)$$

Thus, for the H-minimal control we find

$$
\begin{aligned}
\widetilde{u^*}(t,h) =& v(h(t)) \\
=& \nabla C^{-1}\left(-B^T \overline{y}^{(b)}(t)\right) \\
=& \nabla C^{-1}\Bigg(- B^T e^{A^T(t_f-t)}\Big(\overline{\Gamma(\omega)^T\Gamma(\omega)z(t_f,\omega)}^{(b)} \\
& - \overline{\Gamma(\omega)^T\Gamma(\omega)z_f(\omega)}^{(b)}\Big)\Bigg).
\end{aligned}
\tag{3.36}
$$

We obtain therefore, see (3.16),

$$
\begin{aligned}
z(t,\omega) = e^{A(t-t_b)}z_b + \int_{t_b}^{t} e^{A(t-s)}\Bigg(b(s,\omega) \\
+ B\nabla C^{-1}\Bigg(- B^T e^{A^T(t_f-s)}\Big(\overline{\Gamma(\omega)^T\Gamma(\omega)z(t_f,\omega)}^{(b)} \\
-\overline{\Gamma(\omega)^T\Gamma(\omega)z_f(\omega)}^{(b)}\Big)\Bigg)\Bigg)\, ds.
\end{aligned}
\tag{3.37a}
$$

Quadratic Control Costs

Assume that the control costs and its gradient are given by

$$
C(u) = \frac{1}{2}u^T Ru,\ \nabla C(u) = Ru\,.
\tag{3.37b}
$$

Here, (3.37a) yields

$$
\begin{aligned}
z(t_f,\omega) = e^{A(t_f-t_b)}z_b + \int_{t_b}^{t_f} e^{A(t_f-s)}\Bigg(b(s,\omega) \\
-BR^{-1}B^T e^{A^T(t_f-s)}\left(\overline{\Gamma(\omega)^T\Gamma(\omega)z(t_f,\omega)}^{(b)} - \overline{\Gamma(\omega)^T\Gamma(\omega)z_f(\omega)}^{(b)}\right)\Bigg)\, ds\,.
\end{aligned}
\tag{3.37c}
$$

Multiplying with $\Gamma(\omega)^T\Gamma(\omega)$ and taking expectations, from (3.37c) we get

$$
\begin{aligned}
\overline{\Gamma^T\Gamma z(t_f,\omega)}^{(b)} =& \overline{\Gamma^T\Gamma}^{(b)} e^{A(t_f-t_b)}\overline{z_b}^{(b)} + \int_{t_b}^{t_f} \overline{\Gamma^T\Gamma e^{A(t_f-s)}b(s,\omega)}^{(b)}\, ds \\
&-\overline{\Gamma^T\Gamma}^{(b)} \int_{t_b}^{t_f} e^{A(t_f-s)}BR^{-1}B^T e^{A^T(t_f-s)}\, ds \\
&\times\left(\overline{\Gamma(\omega)^T\Gamma(\omega)z(t_f,\omega)}^{(b)} - \overline{\Gamma(\omega)^T\Gamma(\omega)z_f(\omega)}^{(b)}\right). \qquad (3.38a)
\end{aligned}
$$

According to a former lemma, we define the matrix

$$
U = \int_{t_b}^{t_f} e^{A(t_f-s)}BR^{-1}B^T e^{A^T(t_f-s)}\, ds\,.
$$

From (3.38a) we get then

$$
\begin{aligned}
&\left(I + \overline{\Gamma^T\Gamma}^{(b)}U\right)\overline{\Gamma^T\Gamma z(t_f,\omega)}^{(b)} \\
=&\overline{\Gamma^T\Gamma}^{(b)} e^{A(t_f-t_b)}\overline{z_b}^{(b)} + \int_{t_b}^{t_f} \overline{\Gamma^T\Gamma e^{A(t_f-s)}b(s,\omega)}^{(b)}\, ds \\
&+\overline{\Gamma^T\Gamma}^{(b)} U\,\overline{\Gamma(\omega)^T\Gamma(\omega)z_f(\omega)}^{(b)}. \qquad (3.38b)
\end{aligned}
$$

Lemma 3.3 *$I + \overline{\Gamma^T\Gamma}^{(b)}U$ is regular.*

Proof First notice that not only U, but also $\overline{\Gamma^T\Gamma}^{(b)}$ is positive semidefinite:

$$
v^T\overline{\Gamma^T\Gamma}^{(b)}v = \overline{v^T\Gamma^T\Gamma v} = \overline{(\Gamma v)^T\,\Gamma v}^{(b)} = \overline{\|\Gamma v\|_2^2}^{(b)} \geq 0\,.
$$

Then their product $\overline{\Gamma^T\Gamma}^{(b)}U$ is positive semidefinite as well. This follows immediately from [122] as $\Gamma(\omega)^T\Gamma(\omega)$ is symmetric.

Since the matrix $I + \overline{\Gamma^T \Gamma}^{(b)} U$ is regular, we get cf. (3.23a, b),

$$\begin{aligned}\overline{\Gamma^T \Gamma z(t_f,\omega)}^{(b)} =& \left(I + \overline{\Gamma^T \Gamma}^{(b)} U\right)^{-1} \overline{\Gamma^T \Gamma}^{(b)} e^{A(t_f-t_b)} \overline{z_b}^{(b)} \\ &+ \left(I + \overline{\Gamma^T \Gamma}^{(b)} U\right)^{-1} \int_{t_b}^{t_f} \overline{\Gamma^T \Gamma e^{A(t_f-s)} b(s,\omega)}^{(b)} \, ds \\ &+ \left(I + \overline{\Gamma^T \Gamma}^{(b)} U\right)^{-1} \overline{\Gamma^T \Gamma}^{(b)} U \overline{\Gamma z_f(\omega)}^{(b)}. \end{aligned} \tag{3.38c}$$

Putting (3.38c) into (3.36), corresponding to (3.25) we get the stochastic optimal open-loop control

$$\begin{aligned} u^*(t; t_b, \mathcal{I}_{t_b}) =& - R^{-1} B^T e^{A^T(t_f-t)} \left(\overline{\Gamma(\omega)^T \Gamma(\omega) z(t_f,\omega)}^{(b)} \right. \\ & \left. - \overline{\Gamma(\omega)^T \Gamma(\omega) z_f(\omega)}^{(b)} \right) \\ =& \ldots, t_b \le t \le t_f , \end{aligned} \tag{3.39}$$

which yields then the related stochastic optima open-loop feedback control $\varphi = \varphi(t, \mathcal{I}_t)$ law corresponding to Theorem 3.2.

3.7 Nonzero Costs for Displacements

Suppose here that $Q \neq 0$. According to (3.13a–d), for the adjoint trajectory $y = y(t,\omega)$ we have the system of differential equations

$$\begin{aligned} \dot{y}(t,\omega) =& - A^T y(t,\omega) - Qz(t,\omega) \\ y(t_f,\omega) =& \nabla G(t_f, \omega, z(t_f,\omega)) , \end{aligned}$$

which has the following solution for given $z(t,\omega)$ and $\nabla G(t_f, \omega, z(t_f,\omega))$:

$$y(t,\omega) = \int_t^{t_f} e^{A^T(s-t)} Qz(s,\omega) \, ds + e^{A^T(t_f-t)} \nabla G(t_f, \omega, z(t_f,\omega)). \tag{3.40}$$

Indeed, we get

$$\begin{aligned} y(t_f,\omega) =& 0 + I \nabla_z G(t_f, \omega, z(t_f,\omega)) = \nabla_z G(t_f, \omega, z(t_f,\omega)) \\ \dot{y}(t,\omega) =& - e^{A^T \cdot 0} Qz(t,\omega) \end{aligned}$$

$$
\begin{aligned}
&- \int_t^{t_f} A^T e^{A^T(s-t)} Qz(s,\omega)\, ds - A^T e^{A^T(t_f-t)} \nabla G(t_f,\omega,z(t_f,\omega)) \\
=& - e^{A^T \cdot 0} Qz(t,\omega) \\
&- A^T \left(\int_t^{t_f} e^{A^T(s-t)} Qz(s,\omega)\, ds + e^{A^T(t_f-t)} \nabla G(t_f,\omega,z(t_f,\omega)) \right) \\
=& - A^T y(t,\omega) - Qz(t,\omega)\,.
\end{aligned}
$$

From (3.40) we then get

$$
\begin{aligned}
\overline{y}^{(b)}(t) =& \mathbb{E}^{(b)}(y(t,\omega)) = \mathbb{E}\left(y(t,\omega) | \mathfrak{A}_{t_b} \right) \\
=& \int_t^{t_f} e^{A^T(s-t)} Q \overline{z}^{(b)}(s)\, ds + e^{A^T(t_f-t)} \overline{\nabla G(t_f,\omega,z(t_f,\omega))}^{(b)}.
\end{aligned} \tag{3.41}
$$

The unknown function $\overline{z(t)}^{(b)}$, and the vector $z(t_f,\omega)$ in this equation are both given, based on $\overline{y(t)}^{(b)}$, by the initial value problem, see (3.13a, b),

$$
\dot{z}(t,\omega) = Az(t,\omega) + B \nabla C^{-1} \left(-B^T \overline{y}^{(b)}(t) \right) + b(t,\omega) \tag{3.42a}
$$

$$
z(t_b,\omega) = z_b\,. \tag{3.42b}
$$

Taking expectations, considering the state vector at the final time point t_f, resp., yields the expressions:

$$
\overline{z}^{(b)}(t) = e^{A(t-t_b)} \overline{z_b}^{(b)} + \int_{t_b}^{t} e^{A(t-s)} \left(\overline{b(s)}^{(b)} + B \nabla C^{-1} \left(-B^T \overline{y}^{(b)}(s) \right) \right) ds, \tag{3.43a}
$$

$$
z(t_f,\omega) = e^{A(t_f-t_b)} z_b + \int_{t_b}^{t_f} e^{A(t_f-s)} \left(b(s,\omega) + B \nabla C^{-1} \left(-B^T \overline{y}^{(b)}(s) \right) \right) ds. \tag{3.43b}
$$

3.7.1 Quadratic Control and Terminal Costs

Corresponding to (3.18a, b) and (3.20a, b), suppose

$$\nabla G(t_f, \omega, z(t_f, \omega)) = z(t_f, \omega) - z_f(\omega),$$
$$\nabla C^{-1}(w) = R^{-1} w.$$

According to (3.12) and (3.11c), in the present case the stochastic optimal open-loop control is given by

$$u^*(t; t_b, \mathcal{I}_{t_b}) = \widetilde{u^*}(t, h(t)) = R^{-1}\left(-\mathbb{E}\left(B^T y(t,\omega) \big| \mathfrak{A}_{t_b}\right)\right) = -R^{-1} B^T \overline{y(t)}^{(b)}. \tag{3.44a}$$

Hence, we need the function $\overline{y}^{(b)} = \overline{y(t)}^{(b)}$. From (3.41) and (3.18a, b) we have

$$\overline{y(t)}^{(b)} = e^{A^T(t_f - t)}\left(\overline{z(t_f)}^{(b)} - \overline{z_f}^{(b)}\right) + \int_t^{t_f} e^{A^T(s-t)} Q \overline{z(s)}^{(b)}\, ds. \tag{3.44b}$$

Inserting (3.43a, b) into (3.44b), we have

$$\begin{aligned}\overline{y(t)}^{(b)} =& e^{A^T(t_f - t)}\Bigg(e^{A(t_f - t_b)}\overline{z_b}^{(b)} - \overline{z_f}^{(b)} \\ &+ \int_{t_b}^{t_f} e^{A(t_f - s)}\left(\overline{b(s)}^{(b)} - BR^{-1}B^T\overline{y(s)}^{(b)}\right) ds\Bigg) \\ &+ \int_t^{t_f} e^{A^T(s-t)} Q \Bigg(e^{A(s-t_b)}\overline{z_b}^{(b)} \\ &+ \int_{t_b}^{s} e^{A(s-\tau)}\left(\overline{b(\tau)}^{(b)} - BR^{-1}B^T\overline{y(\tau)}^{(b)}\right) d\tau\Bigg) ds.\end{aligned} \tag{3.44c}$$

In the following we develop a condition that guarantees the existence and uniqueness of a solution $\overline{y}^b = \overline{y(t)}^{(b)}$ of Eq. (3.44c):

Theorem 3.3 *In the space of continuous functions, the above Eq. (3.44c) has a unique solution if*

$$c_B < \frac{1}{c_A \sqrt{c_{R^{-1}}(t_f - t_0)\left(1 + \frac{(t_f - t_0)c_Q}{2}\right)}}. \tag{3.45}$$

Here,

$$c_A := \sup_{t_b \le t \le s \le t_f} \|e^{A(t-s)}\|_F \quad c_B := \|B\|_F \quad c_{R^{-1}} := \|R^{-1}\|_F \quad c_Q := \|Q\|_F ,$$

and the index F denotes the Frobenius-Norm.

Proof The proof of the existence and uniqueness of such an solution is based on the Banach fixed point theorem. For applying this theorem, we consider the Banach space

$$\mathcal{X} = \left\{ f : [t_b; t_f] \to \mathbb{R}^{2m} \ : \ f \text{ continuous} \right\} \tag{3.46a}$$

equipped with the supremum norm

$$\|f\|_L := \sup_{t_b \le t \le t_f} \|f(t)\|_2 , \tag{3.46b}$$

where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^{2m}$.

Now we study the operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ defined by

$$\begin{aligned}(\mathcal{T}f)(t) = & e^{A^T(t_f - t)} \Bigg(e^{A(t_f - t_b)} \overline{z_b}^{(b)} - \overline{z_f}^{(b)} \\ & + \int_{t_b}^{t_f} e^{A(t_f - s)} \left(\overline{b(s)}^{(b)} - BR^{-1}B^T f(s) \right) ds \Bigg) \\ & + \int_{t}^{t_f} e^{A^T(s-t)} Q \Bigg(e^{A(s-t_b)} \overline{z_b}^{(b)} \\ & + \int_{t_b}^{s} e^{A(s-\tau)} \left(\overline{b(\tau)}^{(b)} - BR^{-1}B^T f(\tau) \right) d\tau \Bigg) ds .\end{aligned} \tag{3.47}$$

The norm of the difference $\mathcal{T}f - \mathcal{T}g$ of the images of two different elements $f, g \in \mathcal{X}$ with respect to $\mathcal{T}$ may be estimated as follows:

$$\begin{aligned}& \|\mathcal{T}f - \mathcal{T}g\| \\ = & \sup_{t_b \le t \le t_f} \Bigg\{ \Bigg\| e^{A^T(t_f - t)} \int_{t_b}^{t_f} e^{A(t_f - s)} BR^{-1}B^T (g(s) - f(s))\, ds \\ + & \int_{t}^{t_f} e^{A^T(s-t)} Q \int_{t_b}^{s} e^{A(s-\tau)} BR^{-1}B^T (g(\tau) - f(\tau))\, d\tau\, ds \Bigg\|_2 \Bigg\} .\end{aligned} \tag{3.48a}$$

Note that the Frobenius norm is submultiplicative and compatible with the Euclidian norm. Using these properties, we get

$$\begin{aligned}
&\|\mathcal{T}f - \mathcal{T}g\| \\
&\leq \sup_{t_b \leq t \leq t_f} \left\{ c_A \int_{t_b}^{t_f} c_A c_B c_{R^{-1}} c_B \| f(s) - g(s) \|_2 \, ds \right. \\
&\quad \left. + c_A c_Q \int_t^{t_f} \int_{t_b}^{s} c_A c_B c_{R^{-1}} c_B \| f(\tau) - g(\tau) \|_2 \, d\tau \, ds \right\} \\
&\leq \sup_{t_b \leq t \leq t_f} \left\{ c_A \int_{t_b}^{t_f} c_A c_B c_{R^{-1}} c_B \sup_{t_b \leq t \leq t_f} \| f(s) - g(s) \|_2 \, ds \right. \\
&\quad \left. + c_A c_Q \int_t^{t_f} \int_{t_b}^{s} c_A c_B c_{R^{-1}} c_B \sup_{t_b \leq t \leq t_f} \| f(\tau) - g(\tau) \|_2 \, d\tau \, ds \right\} \\
&= \| f - g \| c_A^2 c_B^2 c_{R^{-1}} \sup_{t_b \leq t \leq t_f} \left\{ (t_f - t_b) + \frac{c_Q}{2} \left((t_f - t_b)^2 - (t - t_b)^2 \right) \right\} \\
&\leq \| f - g \| c_A^2 c_B^2 c_{R^{-1}} (t_f - t_b)(1 + \frac{c_Q}{2} (t_f - t_b)) \, .
\end{aligned} \tag{3.48b}$$

Thus, $\mathcal{T}$ is a contraction if

$$c_B^2 < \frac{1}{c_A^2 c_{R^{-1}} (t_f - t_b) \left(1 + \frac{c_Q}{2} (t_f - t_b) \right)} \tag{3.48c}$$

and therefore

$$c_B < \frac{1}{c_A \sqrt{c_{R^{-1}} (t_f - t_b) \left(1 + \frac{c_Q}{2} (t_f - t_b) \right)}} \, . \tag{3.48d}$$

In order to get a condition that is independent of t_b, we take the worst case $t_b = t_0$, hence,

$$c_B < \frac{1}{c_A \sqrt{c_{R^{-1}} (t_f - t_0) \left(1 + \frac{(t_f - t_0) c_Q}{2} \right)}} \, . \tag{3.48e}$$

Remark 3.4 Condition (3.48e) holds if the matrix Γ in (3.1c) has a sufficiently small Frobenius norm. Indeed, according to (3.2b) we have

$$B = \begin{pmatrix} 0 \\ M^{-1}\Gamma \end{pmatrix}$$

and therefore

$$c_B = \|B\|_F = \|M^{-1}\Gamma\|_F \leq \|M^{-1}\|_F \cdot \|\Gamma\|_F \,.$$

Having $\overline{y}^{(b)}(t)$, according to (3.44a) a stochastic optimal open-loop control $u^*(t) = u^*(t; t_b, \mathcal{I}_{t_b})$, $t_b \leq t \leq t_f$, reads:

$$u^*(t; t_b, \mathcal{I}_{t_b}) = -R^{-1} B^T \overline{y(t)}^{(b)}. \tag{3.49a}$$

Moreover,

$$\varphi(t_b, \mathcal{I}_{t_b}) := u^*(t_b), t_0 \leq t_b \leq t_f \,, \tag{3.49b}$$

is then a stochastic optimal open-loop feedback control law.

Remark 3.5 Putting $Q = 0$ in (3.40), we again obtain the stochastic optimal open-loop feedback control law (3.26) in Sect. 3.6.1.

3.8 Stochastic Weight Matrix $Q = Q(t, \omega)$

In the following we consider the case that, cf.(2.6h, i), the weight matrix for the evaluation of the displacements $z = z(t, \omega)$ is stochastic and may depend also on time t, $t_0 \leq t \leq t_f$. In order to take into account especially the size of the additive disturbance term $b = b(t, \omega)$, cf.(3.7b), in the following we consider the stochastic weight matrix

$$Q(t, \omega) := \|b(t, \omega)\|^2 Q, \tag{3.50a}$$

where Q is again a positive (semi)definite $2m \times 2m$ matrix, and $\| \cdot \|$ denotes the Euclidian norm.
According to (3.13c, d), for the adjoint variable $y = y(t, \omega)$ we then have the system of differential equations

$$\begin{aligned} \dot{y}(t, \omega) =& - A^T y(t, \omega) - \beta(t, \omega) Q z(t, \omega) \\ y(t_f, \omega) =& \nabla G(t_f, \omega, z(t_f, \omega)), \end{aligned}$$

where the stochastic function $\beta = \eta(t, \omega)$ is defined by

$$\beta(t, \omega) := \|bt(t, \omega)|^2. \tag{3.50b}$$

Assuming that we have also the weighted terminal costs,

$$G(t_f, \omega, z(t_f, \omega)) := \frac{1}{2}\beta(t_f, \omega)\|z(t_f, \omega) - z_f(\omega)\|^2, \tag{3.50c}$$

for the adjoint variable $y = y(t, \omega)$, we have the boundary value problem

$$\dot{y}(t, \omega) = -A^T y(t, \omega) - \beta(t, \omega) Q z(t, \omega) \tag{3.51a}$$

$$\begin{aligned} y(t_f, \omega) =& \beta(t_f, \omega) \left(z(t_f, \omega) - z_f(\omega)\right) \\ =& \beta(t_f, \omega) z(t_f, \omega) - \beta(t_f, \omega) z_f(\omega) \end{aligned} \tag{3.51b}$$

Corresponding to (3.40), from (3.51a, b) we then get the solution

$$\begin{aligned} y(t, \omega) =& \int_t^{t_f} e^{A^T(s-t)} Q \beta(s, \omega) z(s, \omega)\, ds \\ &+ e^{A^T(t_f - t)} \left(\beta(t_f, \omega) z(t_f, \omega) - \beta(t_f, \omega) z_f(\omega)\right), \quad t_b \le t \le t_f \end{aligned} \tag{3.52a}$$

Taking conditional expectations of (3.52a) with respect to $\mathfrak{A}_{t_b}$, corresponding to (3.41) we obtain

$$\begin{aligned} \overline{y}^{(b)}(t) =& e^{A^T(t_f - t)} \left(\overline{\eta(t_f, \cdot) z(t_f, \cdot)}^{(b)} - \overline{\beta(t_f, \cdot) z_f(\cdot)}^{(b)}\right) \\ &+ \int_t^{t_f} e^{A^T(s-t)} Q \overline{\beta(s, \cdot) z(s, \cdot)}^{(b)}\, ds, \quad t \ge t_b \end{aligned} \tag{3.52b}$$

Since the matrices A, B are assumed to be fixed, see (3.7b), from (3.8c) and (3.52b) we get

$$h(t) = \mathbb{E}\left(B^T y(t, \omega) | \mathfrak{A}_{t_b}\right) = B^T \overline{y}^{(b)}(t), \quad t \ge t_b \tag{3.53a}$$

Consequently, corresponding to (3.11c) and (3.12), with (3.53a) the optimal open-loop control $u^* = u^*(t)$ is given then by

$$u^*(t; \mathcal{I}_{t_b}) = R^{-1}\left(-h(t)\right). \tag{3.53b}$$

Moreover, the weighted conditional mean trajectory

$$t \to \overline{\beta(t,\cdot)z(t,\cdot)}^{(b)} = \mathbb{E}\left(\beta(t,\omega)z(t,\omega)|\mathfrak{A}_{t_b}\right), \ t \geq t_b, \tag{3.54a}$$

is determined in the present open-loop feedback approach as follows. We first remember that the optimal trajectory is defined by the initial value problem (3.13a, b). Approximating the weighted conditional mean trajectory (3.54a) by

$$t \to \mathbb{E}\left(\beta(t_b,\omega)z(t,\omega)|\mathfrak{A}_{t_b}\right), \ t_b \leq t \leq t_f, \tag{3.54b}$$

we multiply (3.13a, b) by $\beta(t_b, \omega)$. Thus, the trajectory $t \to \beta(t_b, \omega)z(t, \omega)$, $t \geq t_b$, is the solution of the initial value problem

$$\frac{d}{dt}\beta(t_b,\omega)z(t,\omega) = A\beta(t_b,\omega)z(t,\omega) - BR^{-1}B\beta(t_b,\omega)\overline{y}^{(b)}(t) + \beta(t_b,\omega)b(t,\omega) \tag{3.55a}$$

$$\beta(t_b,\omega)z(t_b,\omega) = \beta(t_b,\omega)z_b\,. \tag{3.55b}$$

Taking conditional expectations of (3.55a, b) with respect to $\mathfrak{A}_{t_b}$, for the approximate weighted conditional mean trajectory (3.54b) we obtain the initial value problem

$$\frac{d}{dt}\overline{\beta(t_b,\cdot)z(t,\cdot)}^{(b)} = A\overline{\beta(t_b,\cdot)z(t,\cdot)}^{(b)} - BR^{-1}B\overline{\beta}^{(b)}(t_b)\overline{y}^{(b)}(t) + \overline{\beta(t_b,\cdot)b(t,\cdot)}^{(b)} \tag{3.56a}$$

$$\overline{\beta(t_b,\cdot)z(t_b,\cdot)}^{(b)} = \overline{\beta(t_b,\cdot)z_b(\cdot)}^{(b)}, \tag{3.56b}$$

where $\overline{\beta}^{(b)}(t) := \mathbb{E}\left(\beta(t,\omega)|\mathfrak{A}_{t_b}\right)$, $t \geq t_b$. Consequently, the approximate weighted conditional mean trajectory (3.54b) can be represented, cf. (3.43a, b), by

$$\begin{aligned}\overline{\beta(t_b,\cdot)z(t,\cdot)}^{(b)} =& e^{A(t-t_b)}\overline{\beta(t_b,\cdot)z_b}^{(b)} \\ &+ \int_{t_b}^{t} e^{A(t-s)}\left(\overline{\beta(t_b,\cdot)b(s,\cdot)}^{(b)} - BR^{-1}B^T\overline{\beta}^{(b)}(t_b)\overline{y}^{(b)}(s)\right)\,ds, \\ & t_b \leq t \leq t_f\,. \end{aligned} \tag{3.57}$$

Obviously, a corresponding approximate representation for

$$t \to \overline{\beta(t_f,\cdot)z(t_f,\cdot)}^{(b)}$$

can be obtained, cf. (3.43b), by using (3.57) for $t = t_f$.

Inserting now (3.57) into (3.52b), corresponding to (3.44c) we find the following approximate fixed point condition for the conditional mean adjoint trajectory $t \mapsto \overline{y}^{(b)}(t),\ t_b \le t \le t_f$, needed in the representation (3.53b) of the stochastic optimal open-loop control $u^* = u^*(t),\ t_b \le t \le t_f$:

$$\begin{aligned}
\overline{y}^{(b)}(t) =& e^{A^T(t_f-t)} \left(\overline{\beta(t_f,\cdot)z(t_f,\cdot)}^{(b)} - \overline{\beta(t_f,\cdot)z_f(\cdot)}^{(b)} \right) \\
& + \int_{t_b}^{t_f} e^{A^T(s-t)} Q \overline{\beta(s,\cdot)z(s,\cdot)}^{(b)}\, ds \\
\approx & e^{A^T(t_f-t)} \Bigg(e^{A(t_f-t_b)} \overline{\beta(t_b,\cdot)z_b(\cdot)}^{(b)} - \overline{\beta(t_f,\cdot)z_f(\cdot)}^{(b)} \\
& + \int_{t_b}^{t_f} e^{A(t_f-s)} \Big(\overline{\beta(t_b,\cdot)b(s,\cdot)}^{(b)} \\
& - BR^{-1}B^T \overline{\eta}^{(b)}(t_b) \overline{y}^{(b)}(s) \Big)\, ds \Bigg) \\
& + \int_{t}^{t_f} e^{A^T(s-t)} Q \Bigg(e^{A(s-t_b)} \overline{\beta(t_b,\cdot)z_b(\cdot)}^{(b)} \\
& + \int_{t_b}^{s} e^{A(s-\tau)} \Big(\overline{\beta(t_b,\cdot)b(\tau,\cdot)}^{(b)} \\
& - BR^{-1}B^T \overline{\beta}^{(b)}(t_b) \overline{y}^{(b)}(\tau) \Big)\, d\tau \Bigg)\, ds
\end{aligned} \tag{3.58}$$

Corresponding to Theorem 3.3 we can also develop a condition that guarantees the existence and uniqueness of a solution $\overline{y}^b = \overline{y}^{(b)}(t)$ of Eq. (3.58):

Theorem 3.4 *In the space of continuous functions, Eq. (3.58) has a unique solution if*

$$c_B < \frac{1}{c_A \sqrt{c_{R^{-1}} \overline{\eta}^{(b)}(t_b)(t_f - t_0) \left(1 + \frac{(t_f-t_0)c_Q}{2}\right)}}. \tag{3.59}$$

Here again,

$$c_A := \sup_{t_b \leq t \leq s \leq t_f} \|e^{A(t-s)}\|_F \quad c_B := \|B\|_F \quad c_{R^{-1}} := \|R^{-1}\|_F \quad c_Q := \|Q\|_F,$$

and the index F denotes the Frobenius-Norm.

According to (3.53a, b), the stochastic optimal open-loop control $u^*(t)$, $t_b \leq t \leq t_f$, can be obtained as follows:

Theorem 3.5 *With a solution $\overline{y}^{(b)}(t)$ of the fixed point condition (3.58), the stochastic optimal open-loop control $u^*(t)$, $t_b \leq t \leq t_f$, reads :*

$$u^*(t; \mathcal{I}_{t_b}) = -R^{-1} B^T \overline{y}^{(b)}(t). \tag{3.60a}$$

Moreover,

$$\varphi(t_b, \mathcal{I}_{t_b}) := u^*(t_b; \mathcal{I}_{t_b}), t_0 \leq t_b \leq t_f\ , \tag{3.60b}$$

is then the stochastic optimal open-loop feedback control law.

3.9 Uniformly Bounded Sets of Controls $D_t, t_0 \leq t \leq t_f$

The above shown Theorem 3.4 guaranteeing the existence of a solution of the fixed point condition (3.58) can be generalized considerably if we suppose that the sets D_t of feasible controls $u = u(t)$ are uniformly bounded with respect to the time $t, t_0 \leq t \leq t_f$. Hence, in the following we suppose again:

- time-independent and deterministic matrices of coefficients, hence

$$A(t, \omega) = A \qquad\qquad B(t, \omega) = B \tag{3.61a}$$

- quadratic cost functions,

$$\begin{aligned} C(u) &= \frac{1}{2} u^T R u \qquad Q(z) = \frac{1}{2} z^T Q z \\ G(t_f, \omega, z(t_f, \omega)) &= \frac{1}{2} \left\| z(t_f, \omega) - z_f(\omega) \right\|_2^2 \end{aligned} \tag{3.61b}$$

- uniformly bounded sets of feasible controls, hence, we assume that there exists a constant $C_D \geq 0$ such that

$$\|u\|_2 \leq c_D \quad \text{ for all } u \in \bigcup_{t \in T} D_t. \tag{3.61c}$$

According to the above assumed deterministic coefficient matrices A, b, the H-minimal control depends only on the conditional expectation of the adjoint trajectory, hence,

$$\widetilde{u^*}(t) = \widetilde{u^*}(t, \overline{y(t)}^{(b)}). \tag{3.62}$$

Thus, the integral form of the related 2-point boundary value problem reads:

$$z(\omega, t) = z_b + \int_{t_b}^{t} \left(Az(\omega, s) + b(s, \omega) + B\widetilde{u^*}(s, \overline{y(s)}^{(b)}) \right) ds \tag{3.63a}$$

$$y(\omega, t) = \big(z(t_f, \omega) - z_f(\omega)\big) + \int_{t}^{t_f} \big(A^T y(\omega, s) + Qz(s, \omega) \big) ds. \tag{3.63b}$$

Consequently, for the conditional expectations of the trajectories we get

$$\overline{z(t)}^{(b)} = \overline{z_b}^{(b)} + \int_{t_b}^{t} \left(A\overline{z(s)}^{(b)} + \overline{b(s)}^{(b)} + B\widetilde{u^*}(s, \overline{y(s)}^{(b)}) \right) ds \tag{3.64a}$$

$$\overline{y(t)}^{(b)} = \left(\overline{z(t_f)}^{(b)} - \overline{z_f}^{(b)} \right) + \int_{t}^{t_f} \left(A^T \overline{y(s)}^{(b)} + Q\overline{z(s)}^{(b)} \right) ds. \tag{3.64b}$$

Using the matrix exponential function with respect to A, we have

$$\overline{z(t)}^{(b)} = e^{A(t-t_b)}\overline{z_b}^{(b)} + \int_{t_b}^{t} e^{A(t-s)} \left(\overline{b(s)}^{(b)} + B\widetilde{u^*}(s, \overline{y(s)}^{(b)}) \right) ds \tag{3.65a}$$

$$\overline{y(t)}^{(b)} = e^{A^T(t_f-t)} \left(\overline{z(t_f)}^{(b)} - \overline{z_f}^{(b)} \right) + \int_{t}^{t_f} e^{A^T(s-t)} Q\overline{z(s)}^{(b)} ds. \tag{3.65b}$$

Putting (3.65a) into (3.65b), for $\overline{y(t)}^{(b)}$ we get then the following fixed point condition

$$\begin{aligned} \overline{y(t)}^{(b)} &= e^{A^T(t_f-t)} \left(\overline{z(t_f)}^{(b)} - \overline{z_f}^{(b)} \right) \\ &+ \int_{t}^{t_f} e^{A^T(s-t)} Q \left(e^{A(s-t_b)}\overline{z_b}^{(b)} + \int_{t_b}^{s} e^{A(s-\tau)} \left(\overline{b(\tau)}^{(b)} + B\widetilde{u^*}(\tau, \overline{y(\tau)}^{(b)}) \right) d\tau \right) ds \end{aligned} \tag{3.66}$$

For the consideration of the existence of a solution of the above fixed point equation (3.66) we need several auxiliary tools. According to the assumption (3.61c), next we have the following lemma:

Lemma 3.4 *There exist constants $c_z, c_G > 0$ such that*

$$\|\overline{z(f(\cdot), t)}^{(b)}\| \leq c_z \qquad \text{and} \qquad \|\overline{\nabla_z G(z(f(\cdot), t)}^{(b)}\| \leq c_G \tag{3.67a}$$

for each time $t, t_b \leq t \leq t_f$, and all $f \in \mathcal{C}(T; \mathbb{R}^m)$, where

$$\overline{z(f(\cdot), t)}^{(b)} := e^{A(t-t_b)} \overline{z_b}^{(b)} + \int_{t_b}^{t} e^{A(t-s)} \left(\overline{b(s)}^{(b)} + B\widetilde{u^*}(f(s), s) \right) ds \tag{3.67b}$$

Proof *With $c_A := e^{\|A\|_F (t_f - t_b)}, c_B := \|B\|_F$ and $c_{\overline{b}^{(b)}} := \|\overline{b(\cdot)}^{(b)}\|_\infty$ the following inequalities hold:*

$$\|\overline{z(f(\cdot), t)}^{(b)}\|_2 \leq c_A \left(\|\overline{z_b}^{(b)}\|_2 + \left(c_{\overline{b}^{(b)}} + c_B c_D \right) (t_f - t_b) \right) \leq c_z \tag{3.68a}$$

$$\|\overline{\nabla_z G(z(f(\cdot), t)}^{(b)}\|_2 = \|\overline{z(f(\cdot), t_f)}^{(b)} - \overline{z_f}^{(b)}\| \leq c_z + \|\overline{z_f}^{(b)}\|_2 \leq c_G, \tag{3.68b}$$

where c_z, c_G are arbitrary upper bounds of the corresponding left quantities. □

In the next lemma the operator defined by the right hand side of (3.66) is studied:

Lemma 3.5 *Let denote again $\mathcal{X} := \mathcal{C}(T; \mathbb{R}^m)$ the space of continuous functions f on T equipped with the supremum norm. If $\tilde{\mathcal{T}} : \mathcal{X} \to \mathcal{X}$ denotes the operator defined by*

$$\begin{aligned}(\tilde{\mathcal{T}} f)(t) &:= e^{A^T(t_f - t)} \left(\overline{z(t_f)}^{(b)} - \overline{z_f}^{(b)} \right) \\ &+ \int_{t}^{t_f} e^{A^T(s-t)} Q \left(e^{A(s-t_b)} \overline{z_b}^{(b)} + \int_{t_b}^{s} e^{A(s-\tau)} \left(\overline{b(\tau)}^{(b)} + B\widetilde{u^*}(\tau, f(\tau)) \right) d\tau \right) ds,\end{aligned} \tag{3.69}$$

then the image of $\tilde{\mathcal{T}}$ is relative compact.

Proof *Let $c_Q := \|Q\|_F$. We have to show that $\tilde{\mathcal{T}}(\mathcal{X})$ is bounded and equicontinuous.*

- *$\tilde{\mathcal{T}}(\mathcal{X})$ is bounded:*

$$
\begin{aligned}
&\Bigg\| e^{A^T(t_f-t)}\left(\overline{z(t_f)}^{(b)} - \overline{z_f}^{(b)}\right) \\
&+ \int_t^{t_f} e^{A^T(s-t)} Q \left(e^{A(s-t_b)}\overline{z_b}^{(b)} + \int_{t_b}^{s} e^{A(s-\tau)}\left(\overline{b(\tau)}^{(b)} \right.\right. \\
&\left.\left. + \ B\widetilde{u^*}(\tau, f(\tau))\right) \, d\tau \right) \, ds \Bigg\| \\
\leq & c_A c_G + c_A c_Q \left(c_A \|\overline{z_b}^{(b)}\|_2 (t_f - t_b) + (c_A c_{\overline{b}^{(b)}} + c_B c_D)\frac{t_f^2 - t_b^2}{2} \right)
\end{aligned}
\tag{3.70}
$$

- *$\tilde{\mathcal{T}}(\mathcal{X})$ is equicontinuous:*
We have to show that for each $\epsilon > 0$ there exists a $\delta > 0$ such that, independent of the mapped function f, the following inequality holds:

$$
|t - s| < \delta \quad \Rightarrow \quad \|\tilde{\mathcal{T}} f(t) - \tilde{\mathcal{T}} f(s)\|_2 \leq \epsilon.
$$

Defining the function

$$
\varrho(t) = e^{A(t-t_b)} z_b + \int_{t_b}^{t} e^{A(t-\mu)} \overline{b(\mu)}^{(b)} \, d\mu, \tag{3.71}
$$

the following inequalities hold:

$$
\begin{aligned}
&\left\| \tilde{\mathcal{T}} f(t) - \tilde{\mathcal{T}} f(s) \right\|_2 \\
=&\Bigg\| e^{A^T(t_f-t)}\left(\overline{z(t_f)}^{(b)} - \overline{z_f}^{(b)}\right) - e^{A^T(t_f-s)}\left(\overline{z(t_f)}^{(b)} - \overline{z_f}^{(b)}\right) \\
&\qquad + \int_t^{t_f} e^{A^T(\tau-t)} Q \left(e^{A(\tau-t_b)} z_b \right. \\
&\qquad \left. + \int_{t_b}^{\tau} e^{A(\tau-\mu)} \left(\overline{b(\mu)}^{(b)} + B\widetilde{u^*}(\mu, f(\mu))\right) \, d\mu \right) \, d\tau
\end{aligned}
$$

$$
\begin{aligned}
&- \int_s^{t_f} e^{A^T(\tau-s)} Q \left(e^{A(\tau-t_b)} z_b \right. \\
&\left. + \int_{t_b}^{\tau} e^{A(\tau-\mu)} \left(\overline{b(\mu)}^{(b)} + B\widetilde{u^*}(\mu, f(\mu)) \right) \, d\mu \right) \, d\tau \Bigg\|
\end{aligned}
\tag{3.72a}
$$

From Eq. (3.72a) we get then

$$
\begin{aligned}
&\left\| \tilde{\mathcal{T}} f(t) - \tilde{\mathcal{T}} f(s) \right\|_2 \\
=&\Bigg\| \left(e^{A^T(t_f-t)} - e^{A^T(t_f-s)} \right) \left(\overline{z(t_f)}^{(b)} - \overline{z_f}^{(b)} \right) \\
&+ \int_t^{t_f} e^{A^T(\tau-t)} Q \left(\varrho(\tau) + \int_{t_b}^{\tau} B\widetilde{u^*}(\mu, f(\mu)) \, d\mu \right) d\tau \\
&- \int_s^{t_f} e^{A^T(\tau-s)} Q \left(\varrho(\tau) + \int_{t_b}^{\tau} B\widetilde{u^*}(\mu, f(\mu)) \, d\mu \right) d\tau \Bigg\|_2 \\
\le& \left\| e^{A^T(t_f-t)} - e^{A^T(t_f-s)} \right\|_2 c_G \\
&+ \Bigg\| \left(e^{A^T(-t)} - e^{A^T(-s)} \right) \int_t^{t_f} e^{A^T\tau} Q \left(\varrho(\tau) + \int_{t_b}^{\tau} B\widetilde{u^*}(\mu, f(\mu)) \, d\mu \right) d\tau \\
&- e^{A^T(-s)} \int_s^{t} e^{A^T\tau} Q \left(\varrho(\tau) + \int_{t_b}^{\tau} B\widetilde{u^*}(\mu, f(\mu)) \, d\mu \right) d\tau \Bigg\|_2 \\
\le& \left\| e^{A^T(t_f-t)} - e^{A^T(t_f-s)} \right\|_2 c_G \\
&+ \left\| e^{A^T(-t)} - e^{A^T(-s)} \right\| e^{\|A\|_F t_f} c_Q (c_\varrho + c_B c_D (t_f - t_b))(t_f - t_b) \\
&+ c_A c_Q (c_\varrho + c_B c_D (t_f - t_b)) |t - s|.
\end{aligned}
\tag{3.72b}
$$

Obviously, the final expression in (3.72b) is independent of $f(\cdot)$. Hence, due to the continuity of the matrix exponential function and the function $\varrho(\cdot)$, the assertion follows. $\square$

From the above Lemma 3.5 we obtain now this result:

Theorem 3.6 *The fixed point equation (3.66) has a continuous, bounded solution.*

Proof Define again $\mathcal{X} := \mathcal{C}(T; \mathbb{R}^m)$ and consider the set $\mathcal{M} \subset \mathcal{X}$

$$\mathcal{M} := \left\{ f(\cdot) \in \mathcal{X} \mid \sup_{t \in T} \| f(t) \|_2 \leq C \right\}, \tag{3.73a}$$

where

$$C := c_A c_G + c_A c_Q \left(c_A \| z_b \|_2 (t_f - t_b) + (c_A c_{\overline{b}^{(b)}} + c_B c_D) \frac{t_f^2 - t_b^2}{2} \right). \tag{3.73b}$$

Moreover, let $\mathcal{T}$ denote the restriction of $\tilde{\mathcal{T}}$ to $\mathcal{M}$, hence,

$$\mathcal{T} : \mathcal{M} \to \mathcal{M}, \quad f \mapsto \tilde{\mathcal{T}} f. \tag{3.74}$$

Obviously, the operator $\mathcal{T}$ is continuous and, according to Lemma 3.5, the image of $\mathcal{T}$ is relative compact. Moreover, the set $\mathcal{M}$ is closed and convex. Hence, according to the fixed point theorem of Schauder, $\mathcal{T}$ has a fixed point in $\mathcal{M}$. □

3.10 Approximate Solution of the Two-Point Boundary Value Problem (BVP)

According to the previous sections, the remaining problem is then to solve the fixed point equation (3.44c) or (3.58). In the first case, the corresponding equation reads:

$$\begin{aligned}
\overline{y}^{(b)}(t) =& e^{A^T (t_f - t)} G_f \Bigg(e^{A(t_f - t_b)} \overline{z_b}^{(b)} - \overline{z_f}^{(b)} \\
&+ \int_{t_b}^{t_f} e^{A(t_f - s)} \left(\overline{b(s)}^{(b)} - BR^{-1} B^T \overline{y}^{(b)}(s) \right) ds \Bigg) \\
&+ \int_{t}^{t_f} e^{A^T (s - t)} Q \Bigg(e^{A(s - t_b)} \overline{z_b}^{(b)} \\
&+ \int_{t_b}^{s} e^{A(s - \tau)} \left(\overline{b(\tau)}^{(b)} - BR^{-1} B^T \overline{y}^{(b)}(\tau) \right) d\tau \Bigg) ds .
\end{aligned} \tag{3.75}$$

Based on the given stochastic open-loop feedback control approach, the fixed point equation (3.75) can be solved by the following approximation method:

Step I

According to Eqs. (3.12) and (3.44a) of the stochastic optimal $OLFC$, for each remaining time interval $[t_b, t_f]$ the value of the stochastic optimal open-loop control $u^* = u^*(t; t_b, \mathcal{I}_{t_b}), t \leq t_b$, is needed at the left time point t_b only.

Thus, putting first $t = t_b$ in (3.75), we get:

$$
\begin{aligned}
\overline{y}^{(b)}(t_b) =& e^{A^T(t_f - t_b)} G_f e^{A(t_f - t_b)} \overline{z_b}^{(b)} - e^{A^T(t_f - t_b)} G_f \overline{z_f}^{(b)} \\
&+ e^{A^T(t_f - t_b)} G_f \int_{t_b}^{t_f} e^{A(t_f - s)} \overline{b(s)}^{(b)} \, ds \\
&- e^{A^T(t_f - t_b)} G_f \int_{t_b}^{t_f} e^{A(t_f - s)} B R^{-1} B^T \overline{y}^{(b)}(s) \, ds \\
&+ \int_{t_b}^{t_f} e^{A^T(s - t_b)} Q e^{A(s - t_b)} \, ds \overline{z_b}^{(b)} \\
&+ \int_{t_b}^{t_f} e^{A^T(s - t_b)} Q \Big(\int_{t_b}^{s} e^{A(s - \tau)} \overline{b(\tau)}^{(b)} \, d\tau \Big) ds \\
&- \int_{t_b}^{t_f} e^{A^T(s - t_b)} Q \Big(\int_{t_b}^{s} e^{A(s - \tau)} B R^{-1} B^T \overline{y}^{(b)}(\tau) \, d\tau \Big) ds .
\end{aligned}
\tag{3.76a}
$$

Step II

Due to the representation (3.44a) of the stochastic optimal open-loop control u^* and the stochastic OLF construction principle (2.10d, e), the value of the conditional mean adjoint variable $\overline{y}^{(b)}(t)$ is needed at the left boundary point $t = t_b$ only. Consequently, $\overline{y}^b = \overline{y}^{(b)}(s)$ is approximated on $[t_b, t_f]$ by the constant function

$$
\overline{y}^{(b)}(s) \approx \overline{y}^{(b)}(t_b), t_b \leq s \leq t_f . \tag{3.76b}
$$

In addition, the related matrix exponential function $s \to e^{A(t_f - s)}$ is approximated on $[t_b, t_f]$ in the same way.

This approach is justified especially if one works with a *receding time horizon* or *moving time horizon*

$$t_f := t_b + \Delta$$

with a short *prediction time horizon* Δ.

$$\begin{aligned}
\overline{y}^{(b)}(t_b) \approx & e^{A^T(t_f-t_b)} G_f e^{A(t_f-t_b)} \overline{z_b}^{(b)} - e^{A^T(t_f-t_b)} G_f \overline{z_f}^{(b)} \\
& + e^{A^T(t_f-t_b)} G_f \int_{t_b}^{t_f} e^{A(t_f-s)} \overline{b(s)}^{(b)}\, ds \\
& -(t_f-t_b) e^{A^T(t_f-t_b)} G_f e^{A(t_f-t_b)} BR^{-1}B^T \overline{y}^{(b)}(t_b) \\
& + \int_{t_b}^{t_f} e^{A^T(s-t_b)} Q e^{A(s-t_b)} ds \overline{z_b}^{(b)} \\
& + \int_{t_b}^{t_f} e^{A^T(s-t_b)} Q \Big(\int_{t_b}^{s} e^{A(s-\tau)} \overline{b(\tau)}^{(b)}\, d\tau \Big) ds \\
& - \int_{t_b}^{t_f} (s-t_b) e^{A^T(s-t_b)} Q e^{A(s-t_b)}\, ds\, BR^{-1}B^T \overline{y}^{(b)}(t_b).
\end{aligned} \tag{3.76c}$$

Step III

Rearranging terms, (3.76c) yields a system of linear equations for $\overline{y}^{(b)}(t_b)$:

$$\begin{aligned}
\overline{y}^{(b)}(t_b) \approx & A_0((t_b, t_f, G_f, Q) \overline{z_b}^{(b)} - e^{A^T(t_f-t_b)} G_f \overline{z_f}^{(b)} \\
& + A_1(t_b, t_f, G_f, Q) \cdot \overline{b}^{(b)}_{[t_b,t_f]}(\cdot) \\
& - A_{23}(t_b, t_f, G_f, Q) BR^{-1}B^T \overline{y}^{(b)}(t_b),
\end{aligned} \tag{3.76d}$$

where the matrices, linear operator and function, resp., $A_0, A_1, A_{23}, \overline{b}^{(b)}_{[t_b,t_f]}$ can be easily read from relation (3.76c). Consequently, (3.76d) yields

$$\begin{aligned}
\Big(I + A_{23}(t_b, t_f, G_f, Q) BR^{-1}B^T \Big) \overline{y}^{(b)}(t_b) \approx & A_0((t_b, t_f, G_f, Q) \overline{z_b}^{(b)} \\
& - e^{A^T(t_f-t_b)} G_f \overline{z_f}^{(b)} + A_1(t_b, t_f, G_f, Q) \cdot \overline{b}^{(b)}_{[t_b,t_f]}(\cdot).
\end{aligned} \tag{3.76e}$$

For the matrix occurring in (3.76e) we have this result:

Lemma 3.6 *The matrix $I + A_{23}(t_b, t_f, G_f, Q)BR^{-1}B^T$ is regular.*

Proof According to (3.76c, d) we have

$$A_{23}(t_b, t_f, G_f, Q) = (t_f - t_b)e^{A^T(t_f - t_b)}G_f e^{A(t_f - t_b)} + \int_{t_b}^{t_f} (s - t_b)e^{A^T(s-t_b)}Qe^{A(s-t_b)}\,ds\,.$$

Hence, $A_{23} = A_{23}(t_b, t_f, G_f, Q)$ is a positive definite matrix. Moreover, $U := BR^{-1}B^T$ is at least positive semidefinite. Consider now the equation $(I + A_{23}U)w = 0$. We get $A_{23}Uw = -w$ and therefore $Uw = -A_{23}^{-1}w$, hence, $(U + A_{23}^{-1})w = 0$. However, since the matrix $U + A_{23}^{-1}$ is positive definite, we have $w = 0$, which proves now the assertion.

The above lemma and (3.76e) yields now

$$\overline{y}^{(b)}(t_b) \approx \Big(I + A_{23}(t_b, t_f, G_f, Q)BR^{-1}B^T\Big)^{-1}\Big(A_0((t_b, t_f, G_f, Q)\overline{z_b}^{(b)} - e^{A^T(t_f - t_b)}G_f\overline{z_f}^{(b)} + A_1(t_b, t_f, G_f, Q)\cdot\overline{b}^{(b)}_{[t_b,t_f]}(\cdot)\Big). \tag{3.76f}$$

According to (3.75) and (2.10d, e), the stochastic optimal open-loop feedback control $\varphi^* = \varphi^*(t_b, \mathcal{I}_{t_b})$ at $t = t_b$ is obtained as follows:

Theorem 3.7 *With the approximative solution $\overline{y}^{(b)}(t_b)$ of the fixed point condition (3.75) at $t = t_b$, represented by (3.76f), the stochastic optimal open-loop feedback control law at $t = t_b$ is given by*

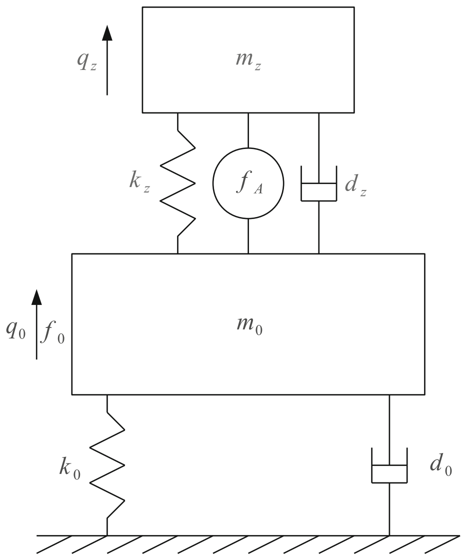
$$\varphi^*(t_b, \mathcal{I}_{t_b}) := -R^{-1}B^T\overline{y}^{(b)}(t_b). \tag{3.76g}$$

Moreover, the whole approximate stochastic optimal open-loop feedback control law $\varphi^* = \varphi^*(t, \mathcal{I}_t)$ is obtained from (3.76g) by replacing $t_b \to t$ for arbitrary $t, t_0 \leq t \leq t_f$.

3.11 Example

We consider the structure according to Fig. 3.2, see [18], where we want to control the supplementary active system while minimizing the expected total costs for the control and the terminal costs.

Fig. 3.2 Principle of active structural control



The behavior of the vector of displacements $q(t,\omega)$ can be described by a system of differential equations of second order:

$$M\begin{pmatrix}\ddot{q}_0(t,\omega)\\ \ddot{q}_z(t,\omega)\end{pmatrix}+D\begin{pmatrix}\dot{q}_0(t,\omega,t\\ \dot{q}_z(t,\omega)\end{pmatrix}+K\begin{pmatrix}q_0(t,\omega)\\ q_z(t,\omega)\end{pmatrix}=f_0(t,\omega)+f_a(t) \tag{3.77}$$

with

$$M=\begin{pmatrix}m_0 & 0\\ 0 & m_z\end{pmatrix} \quad \text{mass matrix} \tag{3.78a}$$

$$D=\begin{pmatrix}d_0+d_z & -d_z\\ -d_z & d_z\end{pmatrix} \quad \text{damping matrix} \tag{3.78b}$$

$$K=\begin{pmatrix}k_0+k_z & -k_z\\ -k_z & k_z\end{pmatrix} \quad \text{stiffness matrix} \tag{3.78c}$$

$$f_a(t)=\begin{pmatrix}-1\\ +1\end{pmatrix}u(t) \quad \text{actuator force} \tag{3.78d}$$

$$f_0(t,\omega)=\begin{pmatrix}f_{01}(t,\omega)\\ 0\end{pmatrix} \quad \text{applied load .} \tag{3.78e}$$

Here we have $n = 1$, i.e. $u(\cdot) \in \mathcal{C}(T,\mathbb{R})$, and the weight matrix R becomes a positive real number.

To represent the equation of motion (3.77) as a first order differential equation we set

$$z(t,\omega) := (q(t,\omega), \dot{q}(t,\omega))^T = \begin{pmatrix} q_0(t,\omega) \\ q_z(t,\omega) \\ \dot{q}_0(t,\omega) \\ \dot{q}_z(t,\omega) \end{pmatrix}.$$

This yields the dynamical equation

$$\begin{aligned} \dot{z}(t,\omega) &= \begin{pmatrix} \mathbf{0} & I_2 \\ -M^{-1}K & -M^{-1}D \end{pmatrix} z(t,\omega) + \begin{pmatrix} \mathbf{0} \\ M^{-1} f_a(s) \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ M^{-1} f_0(s,\omega) \end{pmatrix} = \\ &= \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_0+k_z}{m_0} & \frac{k_z}{m_0} & -\frac{d_0+d_z}{m_0} & \frac{d_z}{m_0} \\ \frac{k_z}{m_z} & -\frac{k_z}{m_z} & \frac{d_z}{m_z} & -\frac{d_z}{m_z} \end{pmatrix}}_{:=A} z(t,\omega) + \underbrace{\begin{pmatrix} 0 \\ 0 \\ -\frac{1}{m_0} \\ \frac{1}{m_z} \end{pmatrix}}_{:=B} u(s) + \underbrace{\begin{pmatrix} 0 \\ 0 \\ \frac{f_0(s,\omega)}{m_0} \\ 0 \end{pmatrix}}_{:=b(t,\omega)}, \end{aligned} \tag{3.79}$$

where I_p denotes the $p \times p$ identity matrix. Furthermore, we have the optimal control problem under stochastic uncertainty:

$$\min \quad F(u(\cdot)) := \mathbb{E}\frac{1}{2}\left(\int_{t_b}^{t_f} R\,(u(s))^2 \; ds + z(t_f,\omega)^T G z(t_f,\omega) \Big|\, \mathfrak{A}_{t_b} \right) \tag{3.80a}$$

$$s.t. \quad z(t,\omega) = z_b + \int_{t_b}^{t} \big(Az(s,\omega) + Bu(s) + b(s,\omega)\big) ds \tag{3.80b}$$

$$u(\cdot) \in \mathcal{C}(T,\mathbb{R}). \tag{3.80c}$$

Note that this problem is of the “Minimum-Energy Control”-type, if we apply no extra costs for the displacements, i.e. $Q \equiv 0$.

The two-point-boundary problem to be solved reads then, cf. (3.13a–d),

$$\dot{z}(t,\omega) = Az(t,\omega) - \frac{1}{R} BB^T \overline{y(t,\cdot)}^{(b)} + b(\omega,t) \tag{3.81a}$$

$$\dot{y}(t,\omega) = -A^T y(t,\omega) \tag{3.81b}$$

$$z(t_b,\omega) = z_b \tag{3.81c}$$

$$y(t_f,\omega) = Gz(t_f,\omega). \tag{3.81d}$$

Hence, the solution of (3.81a)–(3.81d), i.e. the optimal trajectories, reads, cf. (3.14a), (3.37a),

$$y(t,\omega) = e^{A^T(t_f-t)} G z(t_f,\omega) \tag{3.82a}$$

$$z(t,\omega) = e^{A(t-t_b)} z_b + \int_{t_b}^{t} e^{A(t-s)} \Bigg(b(s,\omega) - \frac{1}{R} BB^T e^{A^T(t_f-s)} G \overline{z(t_f,\omega)}^{(b)} \Bigg)\, ds. \tag{3.82b}$$

Finally, we get the optimal control, see (3.38c) and (3.39):

$$u^*(t) = -\frac{1}{R} B^T e^{A^T(t_f-t)} \left(I_4 + GU\right)^{-1} G e^{At_f} \left(e^{-At_b} z_b + \int_{t_b}^{t_f} e^{-As} \overline{b(s,\omega)}^{(b)}\, ds \right) \tag{3.83}$$

with

$$U = \frac{1}{R} \int_{t_b}^{t_f} e^{A(t_f-s)} BB^T e^{A^T(t_f-s)}\, ds. \tag{3.84}$$

Chapter 4
Adaptive Optimal Stochastic Trajectory Planning and Control (AOSTPC)

4.1 Introduction

An industrial, service or field robot is modeled mathematically by its **dynamic equation**, being a system of second order differential equations for the robot or configuration coordinates $q = (q_1, \ldots, q_n)'$ (rotation angles in case of revolute links, length of translations in case of prismatic links), and the **kinematic equation**, relating the space $\{q\}$ of robot coordinates to the work space $\{x\}$ of the robot. Thereby one meets [11, 126, 140, 148, 150, 161] several model parameters, such as length of links, $l_i(m)$, location of center of gravity of links, $l_{ci}(m)$, mass of links, $m_i(kg)$, payload (N), moments of inertia about centroid, $I_i(kgm^2)$, (Coulomb-)friction coefficients, $R_{ij0}(N)$, etc. Let p_D, p_K denote the vector of model parameters contained in the dynamic, kinematic equation, respectively. A further vector p_C of model parameters occurs in the formulation of several constraints, especially initial and terminal conditions, control and state constraints of the robot, as e.g. maximum, minimum torques or forces in the links, bounds for the position, maximum joint, path velocities. Moreover, certain parameters p_J, e.g. cost factors, may occur also in the objective (performance, goal) functional J.

Due to stochastic variations of the material, manufacturing errors, measurement (identification) errors, stochastic variations of the workspace environment, as e.g. stochastic uncertainty of the payload, randomly changing obstacles, errors in the selection of appropriate bounds for the moments, forces, resp., in the links, for the position and path velocities, errors in the selection of random cost factors, modeling errors, disturbances, etc., the total vector

$$p = \begin{pmatrix} p_D \\ p_K \\ p_C \\ p_J \end{pmatrix} \tag{4.1a}$$

K. Marti, *Stochastic Optimization Methods*, DOI 10.1007/978-3-662-46214-0_4

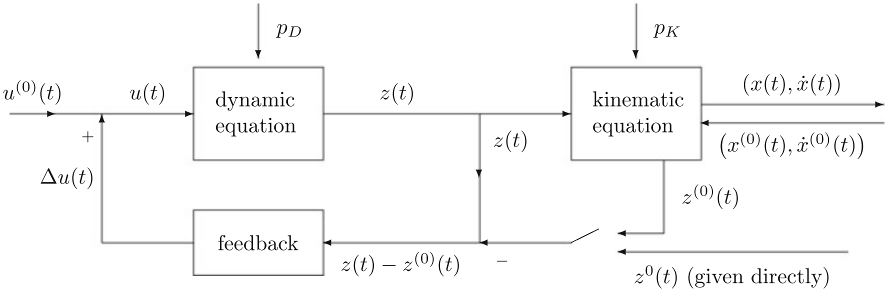


Fig. 4.1 Control of dynamic systems (robots). Here, $z(t) := \big(q(t), \dot{q}(t)\big)$, $z^{(0)}(t) := \big(q^{(0)}(t), \dot{q}^{(0)}(t)\big)$

of model parameters is not a given fixed quantity. The vector p must be represented therefore by a random vector

$$p = p(\omega), \quad \omega \in (\Omega, \mathcal{A}, P) \tag{4.1b}$$

on a certain probability space $(\Omega, \mathcal{A}, P)$, see [8, 58, 123, 148, 169].

Having to control a robotic or more general dynamical system, the control law $u = u(t)$, is represented usually by the sum

$$u(t) := u^{(0)}(t) + \Delta u(t), t_0 \leq t \leq t_f, \tag{4.2}$$

of a feedforward control (open-loop-control) $u_0(t), t_0 \leq t \leq t_f$, and an on-line local control correction (feedback control) $\Delta u(t)$ (Fig. 4.1).

In actual engineering practice [68, 125, 127, 170], the feedforward control $u^{(0)}(t)$ is determined off-line based on a certain reference trajectory $q^{(0)}(t), t_0 \leq t \leq t_f$, in configuration space, where the unknown parameter vector p is replaced by a certain vector $p^{(0)}$ of nominal parameter values, as e.g. the expectation $p^{(0)} := \overline{p} = Ep(\omega)$. The increasing deviation of the actual position and velocity of the robot from the prescribed values, caused by the deviation of the actual parameter values $p(\omega)$ from the chosen nominal values $p^{(0)}$, must be compensated by on-line control corrections $\Delta u(t), t > t_0$. This requires usually extensive on-line state observations (measurements) and feedback control actions.

In order to determine a more reliable reference path $q = q(t), t_0 \leq t \leq t_f$, in configuration space, being robust with respect to stochastic parameter variations, the a priori information (e.g. certain moments or parameters of the probability distribution of $p(\cdot)$) about the random variations of the vector $p(\omega)$ of model parameters of the robot and its working environment is taken into account already at the planning phase. Thus, instead of solving a deterministic trajectory planning problem with a fixed nominal parameter vector $p^{(0)}$, here, an optimal velocity profile $\beta^{(0)}, s_0 \leq s \leq s_f$, and—in case of point-to-point control problems—also an optimal geometric path $q_e^{(0)}(s), s_0 \le s \le s_f$, in configuration space is determined by using a stochastic optimization approach [93, 109–111, 130]. By means of $\beta^{(0)}(s)$ and $q_e^{(0)}(s), s_0 \le s \le s_f$, we then find a more reliable, robust reference trajectory $q^{(0)}(t), t_0 \le t \le t_f^{(0)}$, in configuration space. Applying now the so-called "inverse dynamics approach" [4, 11, 61], more reliable, robust open-loop controls $u^{(0)}(t), t_0 \le t \le t_f^{(0)}$, are obtained. Moreover, by linearization of the dynamic equation of the robot in a neighborhood of $\left(u^{(0)}(t), q^{(0)}(t), E\left(p_M(\omega)|\mathcal{A}_{t_0}\right)\right), t \ge t_0$, where $\mathcal{A}_{t_0}$ denotes the σ-algebra of informations up to the initial time point t_0, a control correction $\Delta u^{(0)}(t), t \ge t_0$, is obtained which is related to the so-called feedback linearization of a system [11, 61, 131, 150].

At later moments (main correction time points) t_j,

$$t_0 < t_1 < t_2 < \ldots < t_{j-1} < t_j < \ldots, \tag{4.3}$$

further information on the parameters of the control system and its environment are available, e.g., by process observation, identification, calibration procedures etc. Improvements $q^{(j)}(t), u^{(j)}(t), \Delta u^{(j)}(t), t \ge t_j, j = 1, 2, \ldots$, of the preceding reference trajectory $q^{(j-1)}(t)$, open loop control $u^{(j-1)}(t)$, and local control correction (feedback control) $\Delta u^{(j-1)}(t)$ can be determined by *replanning*, i.e., by optimal stochastic trajectory planning (OSTP) for the remaining time interval $t \ge t_j, j = 1, 2, \ldots$, and by using the information $\mathcal{A}_{t_j}$ on the robot and its working environment available up to the time point $t_j > t_0, j = 1, 2, \ldots$, see [62, 141, 142].

4.2 Optimal Trajectory Planning for Robots

According to [11, 126, 148], the dynamic equation for a robot is given by the following system of second order differential equations

$$M\left(p_D, q(t)\right)\ddot{q}(t) + h\left(p_D, q(t), \dot{q}(t)\right) = u(t), t \ge t_0, \tag{4.4a}$$

for the n-vector $q = q(t)$ of the robot or configuration coordinates $q_1, q_2, \ldots, q_n$. Here, $M = M(p_D, q)$ denotes the $n \times n$ inertia (or mass) matrix, and the vector function $h = h(p_D, q, \dot{q})$ is given by

$$h(p_D, q, \dot{q}) := C(p_D, q, \dot{q})\dot{q} + F_R(p_D, q, \dot{q}) + G(p_D, q), \tag{4.4b}$$

where $C(p_D, q, \dot{q}) = C(p_D, q)\dot{q}$, and $C(p_D, q) = \left(C_{ijk}(p_D, q)\right)_{1 \le i,j,k \le n}$ is the tensor of Coriolis and centrifugal terms, $F_R = F_R(p_D, q, \dot{q})$ denotes the vector of frictional forces and $G = G(p_D, q)$ is the vector of gravitational forces. Moreover, $u = u(t)$ is the vector of controls, i.e., the vector of torques/forces in the joints of the robot. Standard representations of the friction term F_R are given [11, 68, 148] by

$$F_R(p_D, q, \dot{q}) := R_v(p_D, q)\dot{q}, \tag{4.4c}$$

$$F_R(p_D, q, \dot{q}) := R(p_D, q)sgn(\dot{q}), \tag{4.4d}$$

where $sgn(\dot{q}) := \Big(sgn(\dot{q}_1), \dots, sgn(\dot{q}_n)\Big)^T$. In the first case (4.4c), $R_v = R_v(p_D, q)$ is the viscous friction matrix, and in the Coulomb approach (4.4d), $R = R(p_D, q) = \Big(R_i(p, q)\delta_{ij}\Big)$ is a diagonal matrix.

Remark 4.1 (Inverse Dynamics) Reading the dynamic equation (4.4a) from the left to the right hand side, hence, by inverse dynamics [4, 11, 61], the control function $u = u(t)$ may be described in terms of the trajectory $q = q(t)$ in configuration space.

The relationship between the so-called configuration space $\{q\}$ of robot coordinates $q = (q_1, \dots, q_n)'$ and the work space $\{x\}$ of world coordinates (position and orientation of the end-effector) $x = (x_1, \dots, x_n)'$ is represented by the kinematic equation

$$x = T(p_K, q). \tag{4.5}$$

As mentioned already in the introduction, p_D, p_K, denote the vectors of dynamic, kinematic parameters arising in the dynamic and kinematic equation (4.4a–d), (4.5).

Remark 4.2 (Linear Parameterization of Robots) Note that the parameterization of a robot can be chosen, cf. [4, 11, 61], so that the dynamic and kinematic equation depend linearly on the parameter vectors p_D, p_K.

The objective of optimal trajectory planning is to determine [19, 20, 68, 127, 149] a control function $u = u(t), t \geq t_0$, so that the cost functional

$$J\Big(u(\cdot)\Big) := \int_{t_0}^{t_f} L\Big(t, p_J, q(t), \dot{q}(t), u(t)\Big)\, dt + \phi\Big(t_f, p_J, q(t_f), \dot{q}(t_f)\Big) \tag{4.6}$$

is minimized, where the terminal time t_f may be given explicitly or implicitly, as e.g. in minimum-time problems. Standard examples are, see e.g. [126]: a) $\phi = 0, L = 1$ (minimum time), b) $\phi = 0, L =$ sum of potential, translatory and rotational energy of the robot (minimum energy), c) $\phi = 0, L = \sum_{i=1}^{n}\Big(\dot{q}_i(t)u_i(t)\Big)^2$ (minimum fuel consumption), d) $\phi = 0, L = \sum_{i=1}^{n}\Big(u_i(t)\Big)^2$ (minimum force and moment). Furthermore, an optimal control function $u^* = u^*(t)$ and the related optimal trajectory $q^* = q^*(t), t \geq t_0$, in configuration space must satisfy the dynamic equation (4.4a–d) and the following constraints [19, 20, 28]:

i) The initial conditions

$$q(t_0) = q_0(\omega), \dot{q}(t_0) = \dot{q}_0(\omega) \tag{4.7a}$$

Note that by means of the kinematic equation (4.5), the initial state $(q_0(\omega), \dot{q}_0(\omega))$ in configuration space can be represented by the initial state $\Big(x_0(\omega), \dot{x_0}(\omega)\Big)$ in work space.

ii) The terminal conditions

$$\psi\Big(t_f, p, q(t_f), \dot{q}(t_f)\Big) = 0, \tag{4.7b}$$

e.g.

$$q(t_f) = q_f(\omega), \dot{q}(t_f) = \dot{q}_f(\omega). \tag{4.7c}$$

Again, by means of (4.5), $(q_f, \dot{q}_f)$ may be described in terms of the final state $(x_f, \dot{x}_f)$ in work space.

Note that more general boundary conditions of this type may occur at some intermediate time points $t_0 < \tau_1 < \tau_2 < \ldots < \tau_r < t_f$.

iii) Control constraints

$$u^{\min}(t, p) \leq u(t) \leq u^{\max}(t, p), t_0 \leq t \leq t_f \tag{4.8a}$$

$$g_I\Big(t, p, q(t), \dot{q}(t), u(t)\Big) \leq 0, t_0 \leq t \leq t_f \tag{4.8b}$$

$$g_{II}\Big(t, p, q(t), \dot{q}(t), u(t)\Big) = 0, t_0 \leq t \leq t_f. \tag{4.8c}$$

iv) State constraints

$$S_I\Big(t, p, q(t), \dot{q}(t)\Big) \leq 0, t_0 \leq t \leq t_f \tag{4.9a}$$

$$S_{II}\Big(t, p, q(t), \dot{q}(t)\Big) = 0, t_0 \leq t \leq t_f. \tag{4.9b}$$

Using the kinematic equation (4.5), different types of obstacles in the work space can be described by (time-invariant) state constraints of the type (4.9a, b). In robotics [127] often the following state constraints are used:

$$q_{\min}(p_C) \leq q(t) \leq q_{\max}(p_C), t_0 \leq t \leq t_f \tag{4.9c}$$

$$\dot{q}_{\min}(p_C) \leq \dot{q}(t) \leq \dot{q}_{\max}(p_C), t_0 \leq t \leq t_f, \tag{4.9d}$$

with certain vectors $q_{\min}, q_{\max}, \dot{q}_{\min}, \dot{q}_{\max}$ of (random) bounds.

A special constraint of the type (4.9b) occurs if the trajectory in work space

$$x(t) := T\Big(p_K, q(t)\Big) \tag{4.10}$$

should follow as precise as possible a geometric path in work space

$$x_e = x_e(p_x, s), s_0 \le s \le s_f \tag{4.11}$$

being known up to a certain random parameter vector $p_x = p_x(\omega)$, which then is added to the total vector p of model parameters, cf. (4.4a, b).

Remark 4.3 In the following we suppose that the functions M, h, L, ϕ and T arising in (4.4a–d), (4.5), (4.6) as well as the functions $\psi, g_I, g_{II}, S_I, S_{II}$ arising in the constraints (4.7b–4.9b) are sufficiently smooth.

4.3 Problem Transformation

Since the terminal time t_f may be given explicitly or implicitly, the trajectory $q(\cdot)$ in configuration space may have a varying domain $[t_0, t_f]$. Hence, in order to work with a given fixed domain of the unknown functions, the reference trajectory $q = q(t), t \ge t_0$, in configuration space is represented, cf. [68], by

$$q(t) := q_e\Big(s(t)\Big), t \ge t_0. \tag{4.12a}$$

Here,

$$s = s(t), t_0 \le t \le t_f, \tag{4.12b}$$

is a strictly monotonous increasing transformation from the possibly varying time domain $[t_0, t_f]$ into a given fixed parameter interval $[s_0, s_f]$. For example, $s \in [s_0, s_f]$ may be the path parameter of a given path in work space, cf. (4.11). Moreover,

$$q_e = q_e(s), \ s_0 \le s \le s_f, \tag{4.12c}$$

denotes the so-called geometric path in configuration space.

Remark 4.4 In many more complicated industrial robot tasks such as grinding, welding, driving around difficult obstacles, complex assembly, etc., the geometric path $q_e(\cdot)$ in configuration space is predetermined off-line [21, 62, 64] by a separate path planning procedure for $q_e = q_e(s), s_0 \le s \le s_f$, only. Hence, the trajectory planning/replanning is reduced then to the computation/adaptation of the transformation $s = s(t)$ along a given fixed path $q_e(\cdot) = q_e^{(0)}(\cdot)$.

Assuming that the transformation $s = s(t)$ is differentiable on $[t_0, t_f]$ with the exception of at most a finite number of points, we introduce now the so-called velocity profile $\beta = \beta(s), s_0 \leq s \leq s_f$, along the geometric path $q_e(\cdot)$ in configuration space by

$$\beta(s) := \dot{s}^2\Big(t(s)\Big) = \left(\frac{ds}{dt}\right)^2 \Big(t(s)\Big), \tag{4.13}$$

where $t = t(s), s_0 \leq s \leq s_f$, is the inverse of $s = s(t), t_0 \leq t \leq t_f$. Thus, we have that

$$dt = \frac{1}{\sqrt{\beta(s)}}\, ds, \tag{4.14a}$$

and the time $t \geq t_0$ can be represented by the integral

$$t = t(s) := t_0 + \int_{s_0}^{s} \frac{d\sigma}{\sqrt{\beta(\sigma)}}. \tag{4.14b}$$

By using the integral transformation $\sigma := s_0 + (s - s_0)\rho, 0 \leq \rho \leq 1, t = t(s)$ may be represented also by

$$t(s) = t_0 + (s - s_0) \int_0^1 \frac{d\rho}{\sqrt{\beta\Big(s_0 + (s - s_0)\rho\Big)}}, s \geq s_0. \tag{4.15a}$$

By numerical quadrature, i.e., by applying a certain numerical integration formula of order ν and having weights $a_0, a_1, a_2, \ldots, a_\nu$ to the integral in (4.15a), the time function $t = t(s)$ can be represented approximatively (with an $\varepsilon_0 > 0$) by

$$\tilde{t}(s) := t_0 + (s - s_0) \sum_{k=0}^{\nu} \frac{a_k}{\sqrt{\beta\left(s_0 + \varepsilon_0 + (s - s_0 - 2\varepsilon_0)\frac{k}{\nu}\right)}}, s \geq s_0. \tag{4.15b}$$

In case of Simpson's rule ($\nu = 2$) we have that

$$\tilde{t}(s) := t_0 + \frac{s - s_0}{6} \left(\frac{1}{\sqrt{\beta(s_0 + \varepsilon_0)}} + \frac{4}{\sqrt{\beta\left(\frac{s + s_0}{2}\right)}} + \frac{1}{\sqrt{\beta(s - \varepsilon_0)}} \right). \tag{4.15c}$$

As long as the basic mechanical equations, the cost and constraint functions do **not** depend explicitly on time t, the transformation of the robot control problem from the time onto the s-parameter domain causes no difficulties. In the more general case one has to use the time representation (4.14b), (4.15a) or its approximates (4.15b, c).

Obviously, the terminal time t_f is given, cf. (4.14b), (4.15a), by

$$t_f = t(s_f) = t_0 + \int_{s_0}^{s_f} \frac{d\sigma}{\sqrt{\beta(\sigma)}} \tag{4.16}$$

$$= t_0 + (s_f - s_0) \int_0^1 \frac{d\rho}{\sqrt{\beta\Big(s_0 + (s_f - s_0)\rho\Big)}}.$$

4.3.1 Transformation of the Dynamic Equation

Because of (4.12a, b), we find

$$\dot{q}(t) = q_e'(s)\dot{s} \ \left(\dot{s} := \frac{ds}{dt}, q_e'(s) := \frac{dq_e}{ds}\right) \tag{4.17a}$$

$$\ddot{q}(t) = q_e'(s)\ddot{s} + q_e''(s)\dot{s}^2. \tag{4.17b}$$

Moreover, according to (4.13) we have that

$$\dot{s}^2 = \beta(s), \dot{s} = \sqrt{\beta(s)}, \tag{4.17c}$$

and the differentiation of (4.17c) with respect to time t yields

$$\ddot{s} = \frac{1}{2}\beta'(s). \tag{4.17d}$$

Hence, (4.17a–d) yields the following representation

$$\dot{q}(t) = q_e'(s)\sqrt{\beta(s)} \tag{4.18a}$$

$$\ddot{q}(t) = q_e'(s)\frac{1}{2}\beta'(s) + q_e''(s)\beta(s) \tag{4.18b}$$

of $\dot{q}(t), \ddot{q}(t)$ in terms of the new unknown functions $q_e(\cdot), \beta(\cdot)$.

Inserting now (4.18a, b) into the dynamic equation (4.4a), we find the equivalent relation

$$u_e\left(p_D, s; q_e(\cdot), \beta(\cdot)\right) = u(t) \text{ with } s = s(t), t = t(s), \tag{4.19a}$$

where the function u_e is defined by

$$u_e\Big(p_D, s; q_e(\cdot), \beta(\cdot)\Big) := M\Big(p_D, q_e(s)\Big)\left(\frac{1}{2}q_e'(s)\beta'(s) + q_e''(s)\beta(s)\right) \tag{4.19b}$$
$$+h\left(p_D, q_e(s), q_e'(s)\sqrt{\beta(s)}\right).$$

The initial and terminal conditions (4.7a–c) are transformed, see (4.12a, b) and (4.18a), as follows

$$q_e(s_0) = \mathbf{q_0}(\omega),\ \ q_e'(s_0)\sqrt{\beta(s_o)} = \dot{\mathbf{q}}_\mathbf{0}(\omega) \tag{4.20a}$$

$$\psi\Big(t(s_f), p, q_e(s_f), q_e'(s_f)\sqrt{\beta(s_f)}\Big) = 0 \tag{4.20b}$$

or

$$q_e(s_f) = \mathbf{q}_f(\omega),\ \ q_e'(s_f)\sqrt{\beta(s_f)} = \dot{\mathbf{q}}_\mathbf{f}(\omega). \tag{4.20c}$$

Remark 4.5 In most cases we have the robot resting at time $t = t_0$ and $t = t_f$, i.e. $\dot{q}(t_0) = \dot{q}(t_f) = 0$, hence,

$$\beta(s_0) = \beta(s_f) = 0. \tag{4.20d}$$

4.3.2 Transformation of the Control Constraints

Using (4.12a, b), the control constraints (4.8a–c) read in s-form as follows:

$$u^{\min}\Big(t(s), p_C\Big) \le u_e\Big(p_D, s; q_e(\cdot), \beta(\cdot)\Big) \le u^{\max}\Big(t(s), p_C\Big), s_0 \le s \le s_f \tag{4.21a}$$

$$g_I\left(t(s), p_C, q_e(s), q_e'(s)\sqrt{\beta(s)}, u_e\Big(p_D, s; q_e(\cdot), \beta(\cdot)\Big)\right) \le 0, s_0 \le s \le s_f \tag{4.21b}$$

$$g_{II}\left(t(s), p_C, q_e(s), q_e'(s)\sqrt{\beta(s)}, u_e\Big(p_D, s; q_e(\cdot), \beta(\cdot)\Big)\right) = 0, s_0 \le s \le s_f, \tag{4.21c}$$

where $t = t(s) = t\Big(s; \beta(\cdot)\Big)$ or its approximation $t = \tilde{t}(s) = \tilde{t}\Big(s; \beta(\cdot)\Big)$ is defined by (4.14b), (4.15a–c).

Remark 4.6 I) In the important case

$$u^{\min}(t, p_C) := u^{\min}\Big(p_C, q(t), \dot{q}(t)\Big), u^{\max}(t, p_C) := u^{\max}\Big(p_C, q(t), \dot{q}(t)\Big) \tag{4.22a}$$

that the bounds for $u = u(t)$ depend on the system state $\Big(q(t), \dot{q}(t)\Big)$ in configuration space, condition (4.21a) is reduced to

$$u^{\min}\Big(p_C, q_e(s), q_e'(s)\sqrt{\beta(s)}\Big) \leq u_e\Big(p_D, s; q_e(\cdot), \beta(\cdot)\Big) \qquad (4.22b)$$
$$\leq u^{\max}\Big(p_C, q_e(s), q_e'(s)\sqrt{\beta(s)}\Big), s_0 \leq s \leq s_f.$$

II) If the bounds for $u(t)$ in (4.22a) do not depend on the velocity $\dot{q}(t)$ in configuration space, and the geometric path $q_e(s) = q_e(s), s_0 \leq s \leq s_f$, in configuration space is known in advance, then the bounds

$$u^{\min}\Big(p_C, q_e(s)\Big) = \tilde{u}^{\min}(p_C, s) \qquad (4.22c)$$
$$u^{\max}\Big(p_C, q_e(s)\Big) = \tilde{u}^{\max}(p_C, s), s_0 \leq s \leq s_f,$$

depend on (p_C, s) only.

Bounds of the type (4.22c) for the control function $u(t)$ may be taken into account as an approximation of the more general bounds in (4.21a).

4.3.3 Transformation of the State Constraints

Applying the transformations (4.12a, b), (4.17a) and (4.14b) to the state constraints (4.9a, b), we find the following s-form of the state constraints:

$$S_I\Big(t(s), p_C, q_e(s), q_e'(s)\sqrt{\beta(s)}\Big) \leq 0, s_0 \leq s \leq s_f \qquad (4.23a)$$

$$S_{II}\Big(t(s), p_C, q_e(s), q_e'(s)\sqrt{\beta(s)}\Big) = 0, s_0 \leq s \leq s_f. \qquad (4.23b)$$

Obviously, the s-form of the special state constraints (4.9c, d) read

$$q^{\min}(p_C) \leq q_e(s) \leq q^{\max}(p_C), s_0 \leq s \leq s_f, \qquad (4.23c)$$

$$\dot{q}^{\min}(p_C) \leq q_e'(s)\sqrt{\beta(s)} \leq \dot{q}^{\max}(p_C), s_0 \leq s \leq s_f. \qquad (4.23d)$$

In the case that the end-effector of the robot has to follow a given path (4.11) in work space, Eq. (4.23b) reads

$$T\Big(p_K, q_e(s)\Big) - x_e(p_x, s) = 0, s_0 \leq s \leq s_f, \qquad (4.23e)$$

with the parameter vector p_x describing possible uncertainties in the selection of the path to be followed by the roboter in work space.

4.3.4 Transformation of the Objective Function

Applying the integral transformation $t = t(s), dt = \frac{ds}{\sqrt{\beta(s)}}$ to the integral in the representation (4.6) of the objective function $J = J\Big(u(\cdot)\Big)$, and transforming also the terminal costs, we find the following s-form of the objective function:

$$J\Big(u(\cdot)\Big) = \int_{s_0}^{s_f} L\left(t(s),\ p_J, q_e(s), q_e'(s)\sqrt{\beta(s)}, u_e\Big(p_D, s; q_e(\cdot), \beta(\cdot)\Big)\right) \frac{ds}{\sqrt{\beta(s)}}$$

$$+\ \phi\left(t(s_f), p_J, q_e(s_f), q_e'(s_f)\sqrt{\beta(s_f)}\right). \tag{4.24a}$$

Note that $\beta(s_f) = 0$ holds in many practical situations.

For the class of time-minimum problems we have that

$$J\Big(u(\cdot)\Big) := t_f - t_0 = \int_{t_0}^{t_f} dt = \int_{s_0}^{s_f} \frac{ds}{\sqrt{\beta(s)}}. \tag{4.24b}$$

Optimal Deterministic Trajectory Planning (OSTP) By means of the $t - s$-transformation onto the fixed s-parameter domain $[s_0, s_f]$, the optimal control problem (4.4a–d),(4.6)–(4.11) is transformed into a variational problem for finding, see (4.12a–c) and (4.13), an optimal velocity profile $\beta(s)$ and an optimal geometric path $q_e(s), s_0 \le s \le s_f$. In the determinsitstic case, i.e. if the parameter vector p is assumed to be known, then for the numerical solution of the resulting *optimal deterministic trajectory planning problem* several efficient solution techniques are available, cf. [19, 20, 28, 68, 93, 111, 149].

4.4 OSTP: Optimal Stochastic Trajectory Planning

In the following we suppose that the initial and terminal conditions (4.20d) hold, i.e.

$$\beta_0 = \beta(s_0) = \beta_f = \beta(s_f) = 0 \text{ or } \dot{q}(t_0) = \dot{q}(t_f) = 0.$$

Based on the $(t - s)$-transformation described in Sect. 4.2, and relying on the inverse dynamics approach, the *robot control problem* (4.6), (4.7a–c), (4.8a–c),(4.9a–c) can be represented now by a *variational problem* for $\Big(q_e(\cdot), \beta(\cdot)\Big), \beta(\cdot)$, resp., given in the following. Having $\Big(q_e(\cdot), \beta(\cdot)\Big), \beta(\cdot)$, resp., a reference trajectory and a feedforward control can then be constructed.

A) *Time-invariant case* (autonomous systems)
If the objective function and the constraint functions do not depend explicitly on time t, then the optimal control problem takes the following equivalent s-forms:

$$\min \int_{s_0}^{s_f} L^J\Big(p_J, q_e(s), q_e'(s), q_e''(s), \beta(s), \beta'(s)\Big)\, ds + \phi^J\Big(p_J, q_e(s_f)\Big) \tag{4.25a}$$

s.t.

$$f_I^u\Big(p, q_e(s), q_e'(s), q_e''(s), \beta(s), \beta'(s)\Big) \le 0, s_0 \le s \le s_f \tag{4.25b}$$

$$f_{II}^u\Big(p, q_e(s), q_e'(s), q_e''(s), \beta(s), \beta'(s)\Big) = 0, s_0 \le s \le s_f \tag{4.25c}$$

$$f_I^S\Big(p, q_e(s), q_e'(s), \beta(s)\Big) \le 0, s_0 \le s \le s_f \tag{4.25d}$$

$$f_{II}^S\Big(p, q_e(s), q_e'(s), \beta(s)\Big) = 0, s_0 \le s \le s_f \tag{4.25e}$$

$$\beta(s) \ge 0, s_0 \le s \le s_f \tag{4.25f}$$

$$q_e(s_0) = \mathbf{q}_0(\omega), q_e'(s_0)\sqrt{\beta(s_0)} = \dot{\mathbf{q}}_\mathbf{0}(\omega) \tag{4.25g}$$

$$q_e(s_f) = \mathbf{q}_f(\omega), \beta(s_f) = \beta_f. \tag{4.25h}$$

Under condition (4.20d), a more general version of the terminal condition (4.25h) reads, cf. (4.20b),

$$\psi\Big(p, q_e(s_f)\Big) = 0,\ \beta(s_f) = \beta_f := 0. \tag{4.25h'}$$

Here,

$$L^J = L^J\Big(p_J, q_e, q_e', q_e'', \beta, \beta'\Big), \phi^J = \phi^J(p_J, q_e) \tag{4.26a}$$

$$f_I^u = f_I^u(p, q_e, q_e', q_e'', \beta, \beta'), f_{II}^u = f_{II}^u(p, q_e, q_e', q_e'', \beta, \beta') \tag{4.26b}$$

$$f_I^S = f_I^S(p, q_e, q_e', \beta), f_{II}^S = f_{II}^S(p, q_e, q_e', \beta) \tag{4.26c}$$

are the functions representing the s-form of the objective function (4.24a), the constraint functions in the control constraints (4.21a–c), and in the state constraints (4.23a–e), respectively. Define then f^u and f^S by

$$f^u := \begin{pmatrix} f_I^u \\ f_{II}^u \end{pmatrix},\ f^S := \begin{pmatrix} f_I^S \\ f_{II}^S \end{pmatrix}. \tag{4.26d}$$

B) *Time-varying case* (non autonomous systems)
If the time t occurs explicitly in the objective and/or in some of the constraints of the robot control problem, then, using (4.14a, b), (4.15a–c), we have that $t = t\Big(s; t_0, s_0, \beta(\cdot)\Big)$, and the functions (4.26a–d) and ψ may depend then also on $\Big(s, t_0, s_0, \beta(\cdot)\Big), \Big(s_f, t_0, s_0, \beta(\cdot)\Big)$, resp., hence,

$$
\begin{aligned}
L^J &= L^J\Big(s, t_0, s_0, \beta(\cdot), p_J, q_e, q_e', q_e'', \beta, \beta'\Big)\\
\phi^J &= \phi^J\Big(s_f, t_0, s_0, \beta(\cdot), p_J, q_e\Big)\\
f^u &= f^u\Big(s, t_0, s_0, \beta(\cdot), p, q_e, q_e', q_e'', \beta, \beta'\Big)\\
f^S &= f^S\Big(s, t_0, s_0, \beta(\cdot), p, q_e, q_e', \beta\Big)\\
\psi &= \psi\Big(s_f, t_0, s_0, \beta(\cdot), p, q_e\Big).
\end{aligned}
$$

In order to get a reliable optimal geometric path $q_e^* = q_e^*(s)$ in configuration space and a reliable optimal velocity profile $\beta^* = \beta^*(s), s_0 \le s \le s_f$, being robust with respect to random parameter variations of $p = p(\omega)$, the variational problem (4.25a–h) under stochastic uncertainty must be replaced by an appropriate *deterministic substitute problem* which is defined according to the following principles [71, 76, 84, 90, 111], cf. also [70, 71, 90, 107, 108].

Assume first that the a priori information about the robot and its environment up to time t_0 is described by means of a σ-algebra $\mathcal{A}_{t_0}$, and let then

$$P_{p(\cdot)}^{(0)} = P_{p(\cdot)}\Big(\cdot|\mathcal{A}_{t_0}\Big) \tag{4.27}$$

denote the a priori distribution of the random vector $p = p(\omega)$ given $\mathcal{A}_{t_0}$.

Depending on the decision theoretical point of view, different approaches are possible, e.g. reliability-based substitute problems, belonging essentially to one of the following two basic classes of substitute problems:

I) Risk(recourse)-constrained minimum expected cost problems
II) Expected total cost-minimum problems.

Substitute problems are constructed by selecting certain scalar or vectorial loss or cost functions

$$\gamma_I^u, \gamma_{II}^u, \gamma_I^S, \gamma_{II}^S, \gamma^\psi, \ldots \tag{4.28a}$$

evaluating the violation of the random constraints (4.25b, c), (4.25d, e), (4.25h'), respectively.

In the following all expectations are conditional expectations with respect to the a priori distribution $P^{(0)}_{p(\cdot)}$ of the random parameter vector $p(\omega)$. Moreover, the following compositions are introduced:

$$f^u_\gamma := \begin{pmatrix} \gamma^u_I \circ f^u_I \\ \gamma^u_{II} \circ f^u_{II} \end{pmatrix}, \ f^S_\gamma := \begin{pmatrix} \gamma^S_I \circ f^S_I \\ \gamma^S_{II} \circ f^S_{II} \end{pmatrix} \tag{4.28b}$$

$$\psi_\gamma := \gamma^\psi \circ \psi. \tag{4.28c}$$

Now the two basic types of substitute problems are described.

I) *Risk(recourse)-based minimum expected cost problems*
Minimizing the expected (primal) costs $E\Big(J(u(\cdot))|\mathcal{A}_{t_0}\Big)$, and demanding that the risk, i.e. the expected (recourse) costs arising from the violation of the constraints of the variational problem (4.25a–h) do not exceed given upper bounds, in the **time-invariant case** we find the following substitute problem:

$$\min \int_{s_0}^{s_f} E\left(L^J\Big(p_J, q_e(s), q_e'(s), q_e''(s), \beta(s), \beta'(s)\Big)|\mathcal{A}_{t_0}\right) ds \tag{4.29a}$$

$$+E\left(\phi^J\Big(p_J, q_e(s_f)\Big)|\mathcal{A}_{t_0}\right)$$

s.t.

$$E\left(f^u_\gamma\Big(p, q_e(s), q_e'(s), q_e''(s), \beta(s), \beta'(s)\Big)|\mathcal{A}_{t_0}\right) \le \Gamma^u,$$

$$s_0 \le s \le s_f \tag{4.29b}$$

$$E\left(f^S_\gamma\Big(p, q_e(s), q_e'(s), \beta(s)\Big)|\mathcal{A}_{t_0}\right) \le \Gamma^S, s_0 \le s \le s_f \tag{4.29c}$$

$$\beta(s) \ge 0, s_0 \le s \le s_f \tag{4.29d}$$

$$q_e(s_0) = \overline{q}_0, \ q_e'(s_0)\sqrt{\beta(s_0)} = \overline{\dot{q}}_0 \tag{4.29e}$$

$$q_e(s_f) = \overline{q}_f \ (\text{if } \phi^J = 0), \ \beta(s_f) = \beta_f, \tag{4.29f}$$

and the more general terminal condition (4.25h') is replaced by

$$\beta(s_f) = \beta_f := 0, E\left(\psi_\gamma\Big(p, q_e(s_f)\Big)|\mathcal{A}_{t_0}\right) \le \Gamma_\psi. \tag{4.29f'}$$

Here,

$$\Gamma^u = \Gamma^u(s), \Gamma^S = \Gamma^S(s), \Gamma_\psi = \Gamma_\psi(s) \tag{4.29g}$$

denote scalar or vectorial upper risk bounds which may depend on the path parameter $s \in [s_0, s_f]$. Furthermore, the initial, terminal values $\overline{q}_0, \dot{\overline{q}}_0, \overline{q}_f$ in (4.29e, f) are determined according to one of the following relations:

a)

$$\overline{q}_0 := \hat{q}(t_0), \dot{\overline{q}}_0 := \hat{\dot{q}}(t_0), \overline{q}_f := \hat{q}(t_f), \tag{4.29h}$$

where $\left(\hat{q}(t), \hat{\dot{q}}(t)\right)$ denotes an estimate, observation, etc., of the state in configuration space at time t;

b)

$$\overline{q}_0 := E\left(q_0(\omega)|\mathcal{A}_{t_0}\right), \dot{\overline{q}}_0 := E\left(\dot{q}_0(\omega)|\mathcal{A}_{t_0}\right),$$

$$\overline{q}_f = \overline{q_f}^{(0)} := E\left(q_f(\omega)|\mathcal{A}_{t_0}\right), \tag{4.29i}$$

where $q_0(\omega), \dot{q}_0(\omega)$ is a random initial position, and $q_f(\omega)$ is a random terminal position.

Having corresponding information about initial and terminal values $x_0, \dot{x}_0, x_f$ in work space, related equations for $q_0, \dot{q}_0, q_f$ may be obtained by means of the kinematic equation (4.5).

Remark 4.7 Average Constraints

Taking the average of the pointwise constraints (4.29c, c) with respect to the path parameter $s, s_0 \leq s \leq s_f$, we get the simplified integrated constraints

$$\int_{s_0}^{s_f} E\left(f_\gamma^u\left(p, q_e, (s), q_e'(s), q_e''(s), \beta(s), \beta'(s)\right)|\mathcal{A}_{t_0}\right)\, ds \leq \tilde{\Gamma}^u \tag{4.29b'}$$

$$\int_{s_0}^{s_f} E\left(f_\gamma^S\left(p, q_e(s), q_e'(s), \beta(s)\right)|\mathcal{A}_{t_0}\right)\, ds \leq \tilde{\Gamma}^S \tag{4.29c'}$$

Remark 4.8 Generalized Area of Admissible Motion

In generalization of the admissible area of motion [68, 93, 125] for path planning problems with a prescribed geometrical path $q_e(\cdot) = \overline{q}_e(\cdot)$ in configuration space, for point-to-point problems the constraints (4.29c–i) define for each path point $s, s_0 \leq s \leq s_f$, a generalized admissible area of motion for the vector

$$\chi(s) := \left(q_e(s), q_e'(s), q_e''(s), \beta(s), \beta'(s)\right), \; s_0 \leq s \leq s_f, \tag{4.29j}$$

including information about the magnitude $\Big(\beta(s), \beta'(s)\Big)$ of the motion as well as information about the direction $\Big(q_e(s), q_e'(s), q_e''(s)\Big)$ of the motion.

Remark 4.9 Problems with Chance Constraints

Substitute problems having chance constraints are obtained if the loss functions γ^u, γ^S for evaluating the violation of the inequality constraints in (4.25a–h, h') are 0–1-functions, cf. [93].

To give a characteristic example, we demand that the control, state constraints (4.21a), (4.23c), (4.23d), resp., have to be fulfilled at least with probability $\alpha_u, \alpha_q, \alpha_{\dot{q}}$, hence,

$$P\left(u^{\min}(p_C) \le u_e\Big(p_D, s; q_e(\cdot), \beta(\cdot)\Big) \le u^{\max}(p_C)|\mathcal{A}_{t_0}\right) \ge \alpha_u,$$
$$s_0 \le s \le s_f, \tag{4.30a}$$

$$P\left(q^{\min}(p_C) \le q_e(s) \le q^{\max}(p_C)|\mathcal{A}_{t_0}\right) \ge \alpha_q, s_0 \le s \le s_f, \tag{4.30b}$$

$$P\left(\dot{q}^{\min}(p_C) \le q_e'(s)\sqrt{\beta(s)} \le \dot{q}^{\max}(p_C)|\mathcal{A}_{t_0}\right) \ge \alpha_{\dot{q}}, s_0 \le s \le s_f. \tag{4.30c}$$

Sufficient conditions for the chance constraints (4.30a–c) can be obtained by applying certain probability inequalities, see [93]. Defining

$$u^c(p_C) := \frac{u^{\max}(p_C) + u^{\min}(p_C)}{2}, \rho_u(p_C) := \frac{u^{\max}(p_C) - u^{\min}(p_C)}{2}, \tag{4.30d}$$

then a sufficient conditions for (4.30a) reads, cf. [93],

$$E\left(trB\rho_u(p_C)_d^{-1}\Big(u_e - u^c(p_C)\Big)\Big(u_e - u^c(p_C)\Big)^T \rho_u(p_C)_d^{-1}|\mathcal{A}_{t_0}\right)$$
$$\le 1 - \alpha_u, s_0 \le s \le s_f, \tag{4.30e}$$

where $u_e = u_e\Big(p_D, s; q_e(\cdot), \beta(\cdot)\Big)$ and $\rho_u(p_C)_d$ denotes the diagonal matrix containing the elements of $\rho_u(p_C)$ on its diagonal. Moreover, B denotes a positive definite matrix such that $z^T Bz \ge 1$ for all vectors z such that $\|z\|_\infty \ge 1$. Taking e.g. $B = I$, (4.30e) reads

$$E\left(\|\rho_u(p_C)_d^{-1}\Big(u_e\Big(p_D, s; q_e(\cdot), \beta(\cdot)\Big) - u^c(p_C)\Big)\|^2|\mathcal{A}_{t_0}\right) \le 1 - \alpha_u,$$
$$s_0 \le s \le s_f. \tag{4.30f}$$

Obviously, similar sufficient conditions may be derived for (4.30b, c).

We observe that the above class of risk-based minimum expected cost problems for the computation of $\left(q_e(\cdot), \beta(\cdot)\right), \beta(\cdot)$, resp., is represented completely by the following set of

$$\textit{initial parameters } \zeta_0 : \; t_0, s_0, \overline{q}_0, \dot{\overline{q}}_0, P^{(0)}_{p(\cdot)} \text{ or } \nu_0 \tag{4.31a}$$

and

$$\textit{terminal parameters } \zeta_f : \; t_f, s_f, \beta_f, \overline{q}_f. \tag{4.31b}$$

In case of problems with a given geometric path $q_e = q_e(s)$ in configuration space, the values q_0, q_f may be deleted. Moreover, approximating the expectations in (4.29a–f, f') by means of Taylor expansions with respect to the parameter vector p at the conditional mean

$$\overline{p}^{(0)} := E\left(p(\omega)|\mathcal{A}_{t_0}\right), \tag{4.31c}$$

the a priori distribution $P^{(0)}_{p(\cdot)}$ may be replaced by a certain vector

$$\nu_0 := \left(E\left(\prod_{k=1}^{r} p_l(\omega)|\mathcal{A}_{t_0}\right)_{(l_1,\ldots,l_r)\in\Lambda}\right) \tag{4.31d}$$

of a priori moments of $p(\omega)$ with respect to $\mathcal{A}_{t_0}$.
Here, Λ denotes a certain finite set of multiple indices $(l_1, \ldots, l_\tau), r \leq 1$.
Of course, in the *time-variant case* the functions $L^J, \phi^J, f^u, f^S, \psi$ as described in item *B)* have to be used. Thus, t_0, t_f occur then explicitly in the parameter list (4.31a, b).

II) *Expected total cost-minimum problem*
Here, the total costs arising from violations of the constraints in the variational problem (4.29a–f, f') are added to the (primary) costs arising along the trajectory, to the terminal costs, respectively. Of course, corresponding weight factors may be included in the cost functions (4.28a). Taking expectations with respect to $\mathcal{A}_{t_0}$, in the *time-invariant case* the following substitute problem is obtained:

$$\min \int_{s_0}^{s_f} E\left(L^J_\gamma\left(p, q_e(s), q_e'(s), q_e''(s), \beta(s), \beta'(s)\right)|\mathcal{A}_{t_0}\right)\, ds$$

$$+E\left(\phi^J_\gamma\left(p, q_e(s_f)\right)|\mathcal{A}_{t_0}\right) \tag{4.32a}$$

s.t.

$$\beta(s) \geq 0, s_0 \leq s \leq s_f \tag{4.32b}$$

$$q_e(s_0) = \overline{q}_0, q_e'(s_0)\sqrt{\beta(s_0)} = \bar{\dot{q}}_0 \tag{4.32c}$$

$$q_e(s_f) = \overline{q}_f(\text{ if } \phi_\gamma^J = 0), \beta(s_f) = \beta_f, \tag{4.32d}$$

where $L_\gamma^J, \phi_\gamma^J$ are defined by

$$L_\gamma^J := L^J + \nu^{uT} f_\gamma^u + \nu^{ST} f_\gamma^s \tag{4.32e}$$

$$\phi_\gamma^J := \phi^J \; or \; \phi_\gamma^J := \phi^J + \nu_\psi^T \psi_\gamma, \tag{4.32f}$$

and $\nu_I^u, \nu_{II}^u, \nu_I^S, \nu_{II}^S, \nu_\psi \geq 0$ are certain nonnegative (vectorial) scale factors which may depend on the path parameter s.

We observe that also in this case the initial/terminal parameters characterizing the second class of substitute problems (4.32a–f) are given again by (4.31a, b).

In the *time-varying case* the present substitute problems of *class II* reads:

$$\min \int\limits_{s_0}^{s_f} E\left(L_\gamma^J\Big(s, t_0, s_0, \beta(\cdot), p_J, q_e(s), q_e'(s, q_e''(s), \beta(s), \beta'(s)\Big)|\mathcal{A}_{t_0}\right)\, ds$$
$$+E\left(\phi_\gamma^J\Big(s_f, t_0, s_0, \beta(\cdot), p_J, q_e(s_f)\Big)|\mathcal{A}_{t_0}\right) \tag{4.33a}$$

s.t.

$$\beta(s) \geq 0, s_0 \leq s \leq s_f \tag{4.33b}$$

$$q_e(s_0) = q_0, q_e'(s_0)\sqrt{\beta(s_0)} = \dot{q}_0 \tag{4.33c}$$

$$q_e(s_f) = q_f \;(\text{ if } \phi_\gamma^J = 0), \beta(s_f) = \beta_f. \tag{4.33d}$$

Remark 4.10 Mixtures of (I), (II)

Several mixtures of the classes (I) and (II) of substitute problems are possible.

4.4.1 Computational Aspects

The following techniques are available for solving substitutes problems of type (I), (II):

a) *Reduction to a finite dimensional parameter optimization problem*
Here, the unknown functions $\left(q_e(\cdot), \beta(\cdot)\right)$ or $\beta(\cdot)$ are approximated by a linear combination

$$q_e(s) := \sum_{l=1}^{l_q} \hat{q}_l B_l^q(s), s_0 \le s \le s_f \tag{4.34a}$$

$$\beta(s) := \sum_{l=1}^{l_\beta} \hat{\beta}_l B_l^\beta(s), s_0 \le s \le s_f, \tag{4.34b}$$

where $B_l^q = B_l^q(s), B_l^\beta = B_l^\beta(s), s_0 \le s \le s_f, l = 1, \ldots, l_q(l_\beta)$, are given basis functions, e.g. B-splines, and $\hat{q}_l, \hat{\beta}_l, l = 1, \ldots, l_q(l_\beta)$, are vectorial, scalar coefficients. Putting (4.34a, b) into (4.29a–f, f'), (4.32a–f), resp., a semiinfinite optimization problem is obtained. If the inequalities involving explicitly the path parameter $s, s_0 \le s \le s_f$, are required for a finite number N of parameter values $s_1, s_2, \ldots, s_N$ only, then this problem is reduced finally to a finite dimensional parameter optimization problem which can be solved now numerically by standard mathematical programming routines or search techniques. Of course, a major problem is the approximative computation of the conditional expectations which is done essentially by means of Taylor expansion with respect to the parameter vector p at $\overline{p}^{(0)}$. Consequently, several conditional moments have to be determined (on-line, for stage $j \ge 1$). For details, see [109–111, 130] and the program package "OSTP" [8].

b) *Variational techniques*
Using methods from calculus of variations, necessary and—in some cases—also sufficient conditions in terms of certain differential equations may be derived for the optimal solutions $\left(q_e^{(0)}, \beta^{(0)}\right), \beta^{(0)}$, resp., of the variational problems (4.29a–f, f'), (4.32a–f). For more details, see [130].

c) *Linearization methods*
Here, we assume that we already have an approximative optimal solution $\left(\overline{q}_e(s), \overline{\beta}(s)\right), s_0 \le s \le s_f$, of the substitute problem (4.29a–f, f') or (4.32a–f) under consideration. For example, an approximative optimal solution $\left(\overline{q}_e(\cdot), \overline{\beta}(\cdot)\right)$ can be obtained by starting from the deterministic substitute problem obtained by replacing the random parameter vector $p(\omega)$ just by its conditional mean $\overline{p}^{(0)} := E\left(p(\omega)|\mathcal{A}_{t_0}\right)$.

Given an approximate optimal solution $\left(\overline{q}_e(\cdot), \overline{\beta}(\cdot)\right)$ of substitute problem (I) or (II), the unknown optimal solution $\left(q_e^{(0)}(\cdot), \beta^{(0)}(\cdot)\right)$ to be determined is represented by

$$q_e^{(0)}(s) := \overline{q}_e(s) + \Delta q_e(s), \;\; s_0 \leq s \leq s_f \tag{4.35a}$$

$$\beta^{(0)}(s) := \overline{\beta}(s) + \Delta\beta(s), s_0 \leq s \leq s_f, \tag{4.35b}$$

where $\left(\Delta q_e(s), \Delta\beta(s)\right), s_0 \leq s \leq s_f$, are certain (small) correction terms. In the following we assume that the changes $\Delta q_e(s), \Delta\beta(s)$ and their first and resp. second order derivatives $\Delta q_e'(s), \Delta q_e''(s), \Delta\beta'(s)$ are small.

We observe that the function arising in the constraints and in the objective of (4.29a–f, f'), (4.32a–f), resp., are of the following type:

$$\begin{aligned}&\overline{g}^{(0)}\left(q_e(s), q_e'(s), q_e''(s), \beta(s), \beta'(s)\right) \\ &:= E\left(g\left(p(\omega), q_e(s), q_e'(s), \;\; q_e''(s), \;\beta(s), \beta'(s)\right)|\mathcal{A}_{t_0}\right),\end{aligned} \tag{4.36a}$$

$$\overline{\phi}^{(0)}\left(q_e(s_f)\right) := E\left(\phi\left(p(\omega), q_e(s_f)\right)|\mathcal{A}_{t_0}\right) \tag{4.36b}$$

and

$$\overline{F}^{(0)}\left(q_e(\cdot), \beta(\cdot)\right) := \int_{s_0}^{s_f} \overline{g}^{(0)}\left(q_e(s), q_e'(s), q_e''(s), \beta(s), \beta'(s)\right) ds. \tag{4.36c}$$

with certain functions g, ϕ. Moreover, if for simplification the pointwise (cost-) constraints (4.29c, c) are averaged with respect to the path parameter $s, s_0 \leq s_f$, then also constraint functions of the type (4.36c) arise, see (4.29b', c').

By means of first order Taylor expansion of g, ϕ with respect to $\left(\Delta q_e(s), \Delta q_e'(s), \Delta q_e''(s), \Delta\beta(s), \Delta\beta'(s)\right)$ at $\left(\overline{q}_e(s), \overline{q}_e'(s), \overline{q}_e''(s), \overline{\beta}(s), \overline{\beta}'(s)\right)$, $s_0 \leq s \leq s_f$, we find then the following approximations

$$\begin{aligned}&\overline{g}^{(0)}\left(q_e(s), q_e'(s), q_e''(s), \beta(s), \beta'(s)\right) \approx \overline{g}^{(0)}\left(\overline{q}_e(s), \overline{q}_e'(s), \overline{q}_e''(s), \overline{\beta}(s), \overline{\beta}'(s)\right) \\ &+\overline{A}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s)^T \Delta q_e(s) + \overline{B}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s)^T \Delta q_e'(s) + \overline{C}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s)^T \Delta q_e''(s) \\ &+\overline{D}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s)\Delta\beta(s) + \overline{E}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s)\Delta\beta'(s)\end{aligned} \tag{4.37a}$$

and

$$\overline{\phi}^{(0)}\left(q_e(s_f)\right) \approx \overline{\phi}^{(0)}\left(\overline{q}_e(s_f)\right) + \overline{a}^{(0)}_{\phi,\overline{q}_e}(s_f)^T \Delta q_e(s_f), \tag{4.37b}$$

where the expected sensitivities of g, ϕ with respect to q, q', q'', β and β' are given by

$$\overline{A}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s) := E\left(\nabla_q g\Big(p(\omega), \overline{q}_e(s), \overline{q}'_e(s), \overline{q}''_e(s), \overline{\beta}(s), \overline{\beta}'(s)\Big)|\mathcal{A}_{t_0}\right) \tag{4.37c}$$

$$\overline{B}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s) := E\left(\nabla_{q'} g\Big(p(\omega), \overline{q}_e(s), \overline{q}'_e(s), \overline{q}''_e(s), \overline{\beta}(s), \overline{\beta}'(s)\Big)|\mathcal{A}_{t_0}\right) \tag{4.37d}$$

$$\overline{C}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s) := E\left(\nabla_{q''} g\Big(p(\omega), \overline{q}_e(s), \overline{q}'_e(s), \overline{q}''_e(s), \overline{\beta}(s), \overline{\beta}'(s)\Big)|\mathcal{A}_{t_0}\right) \tag{4.37e}$$

$$\overline{D}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s) := E\left(\frac{\partial g}{\partial \beta}\Big(p(\omega), \overline{q}_e(s), \overline{q}'_e(s), \overline{q}''_e(s), \overline{\beta}(s), \overline{\beta}'(s)\Big)|\mathcal{A}_{t_0}\right) \tag{4.37f}$$

$$\overline{E}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s) := E\left(\frac{\partial g}{\partial \beta'}\Big(p(\omega), \overline{q}_e(s), \overline{q}'_e(s), \overline{q}''_e(s), \overline{\beta}(s), \overline{\beta}'(s)\Big)|\mathcal{A}_{t_0}\right) \tag{4.37g}$$

$$\overline{a}^{(0)}_{\phi,\overline{q}_e}(s_f) := E\left(\nabla_q \phi\Big(p(\omega), \overline{q}_e(s)\Big)|\mathcal{A}_{t_0}\right), s_0 \le s \le s_f. \tag{4.37h}$$

As mentioned before, cf. (4.31c, d), the expected values $\overline{g}^{(0)}, \overline{\phi}^{(0)}$ in (4.37a, b) and the expected sensitivities defined by (4.37c–h) can be computed approximatively by means of Taylor expansion with respect to p at $\overline{p}^{(0)} = E\Big(p(\omega)|\mathcal{A}_{t_0}\Big)$.

According to (4.37a), and using partial integration, for the total costs $\overline{F}^{(0)}$ along the path we get the following approximation:

$$\overline{F}^{(0)} \approx \int_{s_0}^{s_f} \overline{g}^{(0)}\Big(\overline{q}_e(s), \overline{q}'_e(s), \overline{q}''_e(s), \overline{\beta}(s), \overline{\beta}'(s)\Big)\, ds \tag{4.38a}$$

$$+ \int_{s_0}^{s_f} \overline{G}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s)^T \Delta q_e(s)\, ds + \int_{s_0}^{s_f} \overline{H}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s) \Delta\beta(s)\, ds$$

$$+ \Big(\overline{B}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s_f) - \overline{C}^{(0)\prime}_{g,\overline{q}_e,\overline{\beta}}(s_f)\Big)^T \Delta q_e(s_f)$$

$$+ \Big(-\overline{B}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s_0) + \overline{C}^{(0)\prime}_{g,\overline{q}_e,\overline{\beta}}(s_0)\Big)^T \Delta q_e(s_0)$$

$$+ \overline{C}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s_f)^T \Delta q'_e(s_f) - \overline{C}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s_0)^T \Delta q'_e(s_0)$$

$$+ \overline{E}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s_f) \Delta\beta(s_f) - \overline{E}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s_0) \Delta\beta(s_0),$$

where

$$\overline{G}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s) := \overline{A}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s) - \overline{B}^{(0)\prime}_{g,\overline{q}_e,\overline{\beta}}(s) + \overline{C}^{(0)\prime\prime}_{g,\overline{q}_e,\overline{\beta}}(s) \tag{4.38b}$$

$$\overline{H}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s) := \overline{D}^{(0)}_{g,\overline{q}_e,\overline{\beta}}(s) - \overline{E}^{(0)\prime}_{g,\overline{q}_e,\overline{\beta}}(s). \tag{4.38c}$$

Conditions (4.29e–f, f'), (4.32c, d), resp., yield the following initial and terminal conditions for the changes $\Delta q_e(s)$, $\Delta\beta(s)$:

$$\Delta\beta(s_0) = 0, \Delta q_e(s_0) = \overline{q}_0 - \overline{q}_e(s_0) \tag{4.38d}$$

$$\Delta\beta(s_f) = 0, \Delta q_e(s_f) = \overline{q}_f - \overline{q}_e(s_f), \text{ if } \phi^J = 0. \tag{4.38e}$$

Moreover, if $\bar{\dot{q}}_0 \neq 0$ (as in later correction stages, cf. Sect. 4.5), according to (4.29e) or (4.32c), condition $\Delta\beta(s_0) = 0$ must be replaced by the more general one

$$\left(\overline{q}_e'(s_0) + \Delta q_e'(s_0)\right)\sqrt{\overline{\beta}(s_0) + \Delta\beta(s_0)} = \bar{\dot{q}}_0 \tag{4.38f}$$

which can be approximated by

$$\sqrt{\overline{\beta}(s_0)}\Delta q_e'(s_0) + \frac{1}{2}\frac{\Delta\beta(s_0)}{\sqrt{\overline{\beta}(s_0)}}\overline{q}_e'(s_0) \approx \bar{\dot{q}}_0 - \sqrt{\overline{\beta}(s_0)}\overline{q}_e'(s_0). \tag{4.38f'}$$

Applying the above described linearization (4.37a–h) to (4.29a–e) or to the constraints (4.29c, c) only, problem (4.29a–f, f') is approximated by a linear variational problem or a variational problem having linear constraints for the changes $\Delta q_e(\cdot)$, $\Delta\beta(\cdot)$. On the other hand, using linearizations of the type (4.37a–h) in the variational problem (4.32a–f), in the average constraints (4.29b', c'), resp., an optimization problem for $\Delta q_e(\cdot)$, $\Delta\beta(\cdot)$ is obtained which is linear, which has linear constraints, respectively.

d) *Separated computation of* $q_e(\cdot)$ *and* $\beta(\cdot)$
In order to reduce the computational complexity, the given trajectory planning problem is often split up [62] into the following two separated problems for $q_e(\cdot)$ and $\beta(\cdot)$:

  i) Optimal path planning: find the shortest collision-free geometric path $q_e^{(0)} = q_e^{(0)}(s), s_0 \leq s \leq s_f$, in configuration space from a given initial point q_0 to a prescribed terminal point q_f. Alternatively, with a given initial velocity profile $\beta(\cdot) = \overline{\beta}(\cdot)$, see (4.35b), the substitute problem (4.29a–f, f'), (4.32a–f), resp., may be solved for an approximate geometric path $q_e(\cdot) = q_e^{(0)}(\cdot)$ only.
  ii) Velocity planning: Determine then an optimal velocity profile $\beta^{(0)} = \beta^{(0)}(s), s_0 \leq s \leq s_f$, along the predetermined path $q_e^{(0)}(\cdot)$.

Having a certain collection $\left\{q_{e,\lambda}(\cdot)\ :\ \lambda\ \in\ \Lambda\right\}$ of admissible paths in configuration space, a variant of the above procedure (i), (ii) is to determine—in an inner optimization loop—the optimal velocity profile $\beta_\lambda(\cdot)$ with respect to a given path $q_{e,\lambda}(\cdot)$, and to optimize then the parameter λ in an outer optimization loop, see [68].

4.4.2 Optimal Reference Trajectory, Optimal Feedforward Control

Having, at least approximatively, the optimal geometric path $q_e^{(0)} = q_e^{(0)}(s)$ and the optimal velocity profile $\beta^{(0)} = \beta^{(0)}(s), s_0 \leq s \leq s_f$, i.e. the optimal solution $\left(q_e^{(0)}, \beta^{(0)}\right) = \left(q_e^{(0)}(s), \beta^{(0)}(s)\right), s_0 \leq s \leq s_f$, of one of the stochastic path planning problems (4.29a–f, f'), (4.32a–f), (4.33a–d), resp., then, according to (4.12a, b), (4.13), the optimal reference trajectory in configuration space $q^{(0)} = q^{(0)}(t), t \geq t_0$, is defined by

$$q^{(0)}(t) := q_e^{(0)}\left(s^{(0)}(t)\right), t \geq t_0. \tag{4.39a}$$

Here, the optimal $(t \leftrightarrow s)$-transformation $s^{(0)} = s^{(0)}(t), t \geq t_0$, is determined by the initial value problem

$$\dot{s}(t) = \sqrt{\beta^{(0)}(s)},\ t \geq t_0, s(t_0) := s_0. \tag{4.39b}$$

By means of the kinematic equation (4.5), the corresponding reference trajectory $x^{(0)} = x^{(0)}(t), t \geq t_0$, in workspace may be defined by

$$x^{(0)}(t) := E\left(T\left(p_K(\omega), q^{(0)}(t)\right)|\mathcal{A}_{t_0}\right) = T\left(\overline{p}_K^{(0)}, q^{(0)}(t)\right), t \geq t_0, \tag{4.39c}$$

where

$$\overline{p}_K^{(0)} := E\left(p_K(\omega)|\mathcal{A}_{t_0}\right). \tag{4.39d}$$

Based on the *inverse dynamics approach*, see Remark 4.1, the optimal reference trajectory $q^{(0)} = q^{(0)}(t), t \geq t_0$, is inserted now into the left hand side of the dynamic equation (4.4a). This yields next to the random optimal control function

$$\begin{aligned} v^{(0)}\left(t, p_D(\omega)\right) := {} & M\left(p_D(\omega), q^{(0)}(t)\right)\ddot{q}^{(0)}(t) \\ & + h\left(p_D(\omega), q^{(0)}(t), \dot{q}^{(0)}(t)\right), t \geq t_0. \end{aligned} \tag{4.40}$$

Starting at the initial state $(\overline{q}_0, \dot{\overline{q}}_0) := \left(q^{(0)}(t_0), \dot{q}^{(0)}(t_0)\right)$, this control obviously keeps the robot exactly on the optimal trajectory $q^{(0)}(t), t \geq t_0$, provided that $p_D(\omega)$ is the true vector of dynamic parameters.

An optimal feedforward control law $u^{(0)} = u^{(0)}(t), t \geq t_0$, related to the optimal reference trajectory $q^{(0)} = q^{(0)}(t), t \geq t_0$, can be obtained then by applying a certain averaging or estimating operator $\Psi = \Psi\left(\cdot|\mathcal{A}_{t_0}\right)$ to (4.40), hence,

$$u^{(0)} := \Psi\left(v^{(0)}\left(t, p_D(\cdot)\right)|\mathcal{A}_{t_0}\right), t \geq t_0. \tag{4.41}$$

If $\Psi(\cdot|\mathcal{A}_{t_0})$ is the conditional expectation, then we find the optimal feedforward control law

$$\begin{aligned} u^{(0)} &:= E\left(M\left(p_D(\omega), q^{(0)}(t)\right)\ddot{q}^{(0)}(t) + h\left(p_D(\omega), q^{(0)}(t), \dot{q}^{(0)}(t)\right)|\mathcal{A}_{t_0}\right), \\ &= M\left(\overline{p}_D^{(0)}, q^{(0)}(t)\right)\ddot{q}^{(0)}(t) + h\left(\overline{p}_D^{(0)}, q^{(0)}(t), \dot{q}^{(0)}(t)\right), t \geq t_0, \end{aligned} \tag{4.42a}$$

where $\overline{p}_D^{(0)}$ denotes the conditional mean of $p_D(\omega)$ defined by (4.31c), and the second equation in formula (4.42a) holds since the dynamic equation of a robot depends linearly on the parameter vector p_D, see Remark 4.2.

Inserting into the dynamic equation (4.4a), instead of the conditional mean $\overline{p}_D^{(0)}$ of $p_D(\omega)$ given $\mathcal{A}_{t_0}$, another estimator $p_D^{(0)}$ of the true parameter vector p_D or a certain realization $p_D^{(0)}$ of $p_D(\omega)$ at the time instant t_0, then we obtain the optimal feedforward control law

$$u^{(0)}(t) := M\left(p_D^{(0)}, q^{(0)}(t)\right)\ddot{q}^{(0)}(t) + h\left(p_D^{(0)}, q^{(0)}(t), \dot{q}^{(0)}(t)\right), t \geq t_0. \tag{4.42b}$$

4.5 AOSTP: Adaptive Optimal Stochastic Trajectory Planning

As already mentioned in the introduction, by means of direct or indirect measurements, observations of the robot and its environment, as e.g. by observations of the state $(x, \dot{x})$, $(q, \dot{q})$, resp., of the mechanical system in work or configuration space, further information about the unknown parameter vector $p = p(\omega)$ is available at each moment $t > t_0$. Let denote, cf. Sects. 4.1, 4.4,

$$\mathcal{A}_t(\subset \mathcal{A}), t \geq t_0, \tag{4.43a}$$

the σ-algebra of all information about the random parameter vector $p = p(\omega)$ up to time t. Hence, $(\mathcal{A}_t)$ is an increasing family of σ-algebras. Note that the flow of information in this control process can be described also by means of the stochastic process

$$p_t(\omega) := E\Big(p(\omega)|\mathcal{A}_t\Big), t \geq t_0, \tag{4.43b}$$

see [12].

By parameter identification [65, 143] or robot calibration techniques [15, 145] we may then determine the conditional distribution

$$P_{p(\cdot)}^{(t)} = P_{p(\cdot)|\mathcal{A}_t} \tag{4.43c}$$

of $p(\omega)$ given $\mathcal{A}_t$. Alternatively, we may determine the vector of conditional moments

$$\nu^{(t)} := \left(E\Big(\prod_{k=1}^{r} p_{l_k}(\omega)|\mathcal{A}_t\Big)\right)_{(l_1,\ldots,l_r)\in\Lambda} \tag{4.43d}$$

arising in the approximate computation of conditional expectations in (OSTP) with respect to $\mathcal{A}_t$, cf. (4.31c, d).

The increase of information about the unknown parameter vector $p(\omega)$ from one moment t to the next $t + dt$ may be rather low, and the determination of $P_{p(\cdot)}^{(t)}$ or $\nu^{(t)}$ at each time point t may be very expensive, though identification methods in real-time exist [143]. Hence, as already mentioned briefly in Sect. 4.1, the conditional distribution $P_{p(\cdot)}^{(t)}$ or the vector of conditional moments $\nu^{(t)}$ is determined/updated at discrete moments (t_j):

$$t_0 < t_1 < t_2 < \ldots < t_j < t_{j+1} < \ldots . \tag{4.44a}$$

The optimal functions $q_e^{(0)}(s), \beta^{(0)}(s), s_0 \leq s \leq s_f$, based on the a priori information $\mathcal{A}_{t_0}$, loose in course of time more or less their qualification to provide a satisfactory pair of guiding functions $\left(q^{(0)}(t), u^{(0)}(t)\right), t \geq t_0$.

However, having at the main correction time points $t_j, j = 1, 2, \ldots$, the updated information σ-algebras $\mathcal{A}_{t_j}$ and then the a posteriori probability distributions $P_{p(\cdot)}^{(t_j)}$ or the updated conditional moments $\nu^{(t_j)}$ of $p(\omega), j = 1, 2, \ldots$, the pair of guiding functions $\left(q^{(0)}(t), u^{(0)}(t)\right), t \geq t_0$, is replaced by a sequence of renewed pairs $\left(q^{(j)}(t), u^{(j)}(t)\right), t \geq t_j, j = 1, 2, \ldots$, of guiding functions determined by replanning, i.e. by repeated (OSTP) for the remaining time intervals $[t_j, t_f^{(j)}]$ and by using the new information given by $\mathcal{A}_{t_j}$. Since replanning at a later main correction time point $t_j, j \geq 1$, hence on-line, may be very time consuming, in order to maintain the real-time capability of the method, we may start the replanning

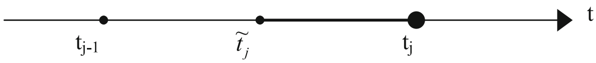


Fig. 4.2 Start of replanning

procedure for an update of the guiding functions at time t_j already at some earlier time $\tilde{t}_j$ with $t_{j-1} < \tilde{t}_j < t_j$, $j \geq 1$. Of course, in this case

$$\mathcal{A}_{t_j} := \mathcal{A}_{\tilde{t}_j} \tag{4.44b}$$

is defined to contain only the information about the control process up to time $\tilde{t}_j$ in which replanning for time t_j starts (Fig. 4.2).

The resulting substitute problem at a stage $j \geq 1$ follows from the corresponding substitute problem for the previous stage $j-1$ just by updating $\zeta_{j-1} \to \zeta_j, \zeta_f^{(j-1)} \to \zeta_f^{(j)}$, the initial and terminal parameters, see (4.31a, b). The renewed

$$\text{initial parameters } \zeta_j : \ t_j, s_j, \overline{q}_j, \overline{\dot{q}}_j, P_{p(\cdot)}^{(j)} \text{ or } \nu_j \tag{4.45a}$$

for the j-th stage, $j \geq 1$, are determined recursively as follows:

$$s_j := s^{(j-1)}(t_j) \quad (1-1-\text{transformation } s = s(t)) \tag{4.45b}$$

$$\overline{q}_j := \hat{q}(t_j), \overline{q}_j := q^{(j-1)}(t_j) \text{ or } \overline{q}_j := E\Big(q(t_j)|\mathcal{A}_{t_j}\Big) \tag{4.45c}$$

$$\overline{\dot{q}}_j := \hat{\dot{q}}(t_j), \overline{q}_j := \dot{q}^{(j-1)}(t_j) \text{ or } \overline{\dot{q}}_j := E\Big(\dot{q}(t_j)|\mathcal{A}_{t_j}\Big) \tag{4.45d}$$

$$(\text{observation or estimate of } q(t_j), \dot{q}(t_j))$$

$$P_{p(\cdot)}^{(j)} := P_{p(\cdot)}^{(t_j)} = P_{p(\cdot)|\mathcal{A}_{t_j}} \tag{4.45e}$$

$$\nu_j := \nu^{(t_j)}. \tag{4.45f}$$

The renewed

$$\text{terminal parameters } \zeta_f^{(j)} : \ t_f^{(j)}, s_f, \overline{q}_f^{(j)}, \beta_f \tag{4.46a}$$

for the j-th stage, $j \geq 1$, are defined by

$$s_f \text{ (given)} \tag{4.46b}$$

$$\overline{q}_f^{(j)} := \hat{q}(t_f) \text{ or } \overline{q}_f^{(j)} := E\Big(q_f(\omega)|\mathcal{A}_{t_j}\Big) \tag{4.46c}$$

$$\beta_f = 0 \tag{4.46d}$$

$$s^{(j)}\left(t_f^{(j)}\right) = s_f. \tag{4.46e}$$

As already mentioned above, the (OSTP) for the j-th stage, $j \geq 1$, is obtained from the substitute problems (4.29a–f, f'), (4.32a–f), (4.33a–d), resp., formulated for the 0-th stage, $j = 0$, just by substituting

$$\zeta_0 \to \zeta_j \text{ and } \zeta_f \to \zeta_f^{(j)}. \tag{4.47}$$

Let then denote

$$\left(q_e^{(j)}, \beta^{(j)}\right) = \left(q_e^{(j)}(s), \beta^{(j)}(s)\right), s_j \leq s \leq s_f, \tag{4.48}$$

the corresponding pair of optimal solutions of the resulting substitute problem for the j-th stage, $j \geq 1$.

The pair of guiding functions $\left(q^{(j)}(t), u^{(j)}(t)\right), t \geq t_j$, for the j-th stage, $j \geq 1$, is then defined as described in Sect. 4.4.2 for the 0-th stage. Hence, for the j-th stage, the reference trajectory in configuration space $q^{(j)}(t), t \geq t_j$, reads, cf. (4.39a),

$$q^{(j)}(t) := q_e^{(j)}\left(s^{(j)}(t)\right), t \geq t_j, \tag{4.49a}$$

where the transformation $s^{(j)} : \left[t_j, t_f^{(j)}\right] \to [s_j, s_f]$ is defined by the initial value problem

$$\dot{s}(t) = \sqrt{\beta^{(j)}(s)},\ t \geq t_j, s(t_j) = s_j. \tag{4.49b}$$

The terminal time $t_f^{(j)}$ for the j-th stage is defined by the equation

$$s^{(j)}\left(t_f^{(j)}\right) = s_f. \tag{4.49c}$$

Moreover, again by the inverse dynamics approach, the feedforward control $u^{(j)} = u^{(j)}(t), t \geq t_j$, for the j-th stage is defined, see (4.40), (4.41), (4.42a, b), by

$$u^{(j)}(t) := \Psi\left(v^{(j)}\Big(t, p_D(\omega)\Big)|\mathcal{A}_{t_j}\right), \tag{4.50a}$$

where

$$v^{(j)}(t, p_D) := M\Big(p_D, q^{(j)}(t)\Big)\ddot{q}^{(j)}(t) + h\Big(p_D, q^{(j)}(t), \dot{q}^{(j)}(t)\Big), t \geq t_j. \tag{4.50b}$$

Using the conditional expectation $\Psi(\cdot|\mathcal{A}_{t_j}) := E(\cdot|\mathcal{A}_{t_j})$, we find the feedforward control

$$u^{(j)}(t) := M\left(\overline{p}_D^{(j)}, q^{(j)}(t)\right)\ddot{q}^{(j)}(t) + h\left(\overline{p}_D^{(j)}, q^{(j)}, \dot{q}^{(j)}(t)\right), t \geq t_j, \tag{4.50c}$$

where, cf. (4.31c),

$$\overline{p}_D^{(j)} := E\left(p_D(\omega)|\mathcal{A}_{t_j}\right). \tag{4.50d}$$

Corresponding to (4.39c, d), the reference trajectory in work space $x^{(j)} = x^{(j)}(t), t \geq t_j$, for the remaining time interval $t_j \leq t \leq t_f^{(j)}$, is defined by

$$x^{(j)}(t) := E\left(T\left(p_K(\omega), q^{(j)}(t)\right)|\mathcal{A}_{t_j}\right) = T\left(\overline{p}_K^{(j)}, q^{(j)}(t)\right), t_j \leq t \leq t_f^{(j)}, \tag{4.51a}$$

where

$$\overline{p}_K^{(j)} := E\left(p_K(\omega)|\mathcal{A}_{t_j}\right). \tag{4.51b}$$

4.5.1 (OSTP)-Transformation

The variational problems (OSTP) at the different stages $j = 0, 1, 2\ldots$ are determined uniquely by the set of initial and terminal parameters $(\zeta_j, \zeta_f^{(j)})$, cf. (4.45a–f), (4.46a–e). Thus, these problems can be transformed to a reference problem depending on $\left(\zeta_j, \zeta_f^{(j)}\right)$ and having a certain fixed reference s-interval.

Theorem 4.5.1 *Let $[\tilde{s}_0, \tilde{s}_f], \tilde{s}_0 < \tilde{s}_f := s_f$, be a given, fixed reference s-interval, and consider for a certain stage j, $j = 0, 1, \ldots$, the transformation*

$$\tilde{s} = \tilde{s}(s) := \tilde{s}_0 + \frac{\tilde{s}_f - \tilde{s}_0}{s_f - s_j}(s - s_j), s_j \leq s \leq s_f, \tag{4.52a}$$

from $[s_j, s_f]$ onto $[\tilde{s}_0, s_f]$ having the inverse

$$s = s(\tilde{s}) = s_j + \frac{s_f - s_j}{\tilde{s}_f - \tilde{s}_0}(\tilde{s} - \tilde{s}_0), \tilde{s}_0 \leq \tilde{s} \leq \tilde{s}_f. \tag{4.52b}$$

Represent then the geometric path in work space $q_e = q_e(s)$ and the velocity profile $\beta = \beta(s), s_j \leq s \leq s_f$, for the j-th stage by

$$q_e(s) := \tilde{q}_e\left(\tilde{s}(s)\right), s_j \leq s \leq s_f \tag{4.53a}$$

$$\beta(s) := \tilde{\beta}\left(\tilde{s}(s)\right), s_j \leq s \leq s_f \tag{4.53b}$$

where $\tilde{q}_e = \tilde{q}_e(\tilde{s}), \tilde{\beta} = \tilde{\beta}(\tilde{s}), \tilde{s}_0 \leq \tilde{s} \leq \tilde{s}_f$, denote the corresponding functions on $[\tilde{s}_0, \tilde{s}_f]$. Then the (OSTP) for the j-th stage is transformed into a reference variational problem (stated in the following) for $(\tilde{q}_e, \tilde{\beta})$ depending on the parameters

$$(\zeta, \zeta_f) = (\zeta_j, \zeta_f^{(j)}) \in Z \times Z_f \tag{4.54}$$

and having the fixed reference s-interval $[\tilde{s}_0, \tilde{s}_f]$. Moreover, the optimal solution $\left(q_e^{(j)}, \beta^{(j)}\right) = \left(q_e^{(j)}(s), \beta^{(j)}(s)\right), s_j \le s \le s_f$, may be represented by the optimal adaptive law

$$q_e^{(j)}(s) = \tilde{q}_e^*\left(\tilde{s}(s); \zeta_j, \zeta_f^{(j)}\right), s_j \le s \le s_f, \tag{4.55a}$$

$$\beta^{(j)}(s) = \tilde{\beta}^*\left(\tilde{s}(s); \zeta_j, \zeta_f^{(j)}\right), s_j \le s \le s_f, \tag{4.55b}$$

where

$$\tilde{q}_e^* = \tilde{q}_e^*(\tilde{s}; \zeta, \zeta_f), \tilde{\beta}^* = \tilde{\beta}^*(\tilde{s}; \zeta, \zeta_f), \ \tilde{s}_0 \le \tilde{s} \le \tilde{s}_f, \tag{4.55c}$$

denotes the optimal solution of the above mentioned reference variational problem.

Proof According to (4.53a, b) and (4.52a, b), the derivatives of the functions $q_e(s), \beta(s), s_j \le s \le s_f$, are given by

$$q_e'(s) = \tilde{q}_e'\left(\tilde{s}(s)\right)\frac{\tilde{s}_f - \tilde{s}_0}{s_f - s_j}, \ s_j \le s \le s_f, \tag{4.56a}$$

$$q_e''(s) = \tilde{q}_e''\left(\tilde{s}(s)\right)\left(\frac{\tilde{s}_f - \tilde{s}_0}{s_f - s_j}\right)^2, s_j \le s \le s_f, \tag{4.56b}$$

$$\beta'(s) = \tilde{\beta}'\left(\tilde{s}(s)\right)\frac{\tilde{s}_f - \tilde{s}_0}{s_f - s_j}, \ s_j \le s \le s_f. \tag{4.56c}$$

Now putting the transformation (4.52a, b) and the representation (4.53a, b), (4.56a–c) of $q_e(x), \beta(s), s_j \le s \le s_f$, and their derivatives into one of the substitute problems (4.29a–f, f'), (4.32a–f) or their time-variant versions, the chosen substitute problem is transformed into a corresponding reference variational problem (stated in the following Sect. 4.5.1) having the fixed reference interval $[\tilde{s}_0, \tilde{s}_f]$ and depending on the parameter vectors $\zeta_j, \zeta_f^{(j)}$. Moreover, according to (4.53a, b), the optimal solution $\left(q_e^{(j)}, \beta^{(j)}\right)$ of the substitute problem for the j-th stage may be represented then by (4.55a–c).

Remark 4.11 Based on the above theorem, the stage-independent functions $\tilde{q}_e^*, \tilde{\beta}^*$ can be determined now off-line by using an appropriate numerical procedure.

4.5.2 The Reference Variational Problem

After the (OSTP)-transformation described in Sect. 4.5.1, in the *time-invariant case* for the problems of type (4.29a–f, f') we find

$$\min \int_{\tilde{s}_0}^{\tilde{s}_f} E\left(L^J\left(p_J, \tilde{q}_e(\tilde{s}), \tilde{q}_e'(\tilde{s})\frac{\tilde{s}_f - \tilde{s}_0}{s_f - s_j}, \tilde{q}_e''(\tilde{s})\left(\frac{\tilde{s}_f - \tilde{s}_0}{s_f - s_j}\right)^2, \tilde{\beta}(\tilde{s}), \tilde{\beta}'(\tilde{s})\frac{\tilde{s}_f - \tilde{s}_0}{s_f - s_j}\right)|\mathcal{A}_{t_j}\right) \frac{s_f - s_j}{\tilde{s}_f - \tilde{s}_0}\, d\tilde{s} + E\left(\phi^J\left(p_J, \tilde{q}_e(\tilde{s_f})\right)|\mathcal{A}_{t_j}\right) \tag{4.57a}$$

s.t.

$$E\left(f_\gamma\left(p, \tilde{q}_e(\tilde{s}), \tilde{q}_e'(\tilde{s})\frac{\tilde{s}_f - \tilde{s}_0}{s_f - s_j}, \tilde{q}_e''(\tilde{s})\left(\frac{\tilde{s}_f - \tilde{s}_0}{s_f - s_j}\right)^2, \tilde{\beta}(\tilde{s}), \tilde{\beta}'(\tilde{s})\frac{\tilde{s}_f - \tilde{s}_0}{s_f - s_j}\right)|\mathcal{A}_{t_j}\right) \le \Gamma_f, \tag{4.57b}$$

$$\tilde{s}_0 \le \tilde{s} \le \tilde{s}_f \tag{4.57c}$$

$$\tilde{\beta}(\tilde{s}) \ge 0, \tilde{s}_0 \le \tilde{s} \le \tilde{s}_f \tag{4.57d}$$

$$\tilde{q}_e(\tilde{s}_0) = \overline{q}_j,\ \tilde{q}_e'(\tilde{s}_0)\frac{\tilde{s}_f - \tilde{s}_0}{s_f - s_j}\sqrt{\tilde{\beta}(\tilde{s}_0)} = \dot{\overline{q}}_j \tag{4.57e}$$

$$\tilde{q}_e(\tilde{s}_f) = \overline{q}_f^{(j)} \text{ (if } \phi^J = 0),\ \tilde{\beta}(\tilde{s}_f) = 0 \tag{4.57f}$$

$$\tilde{\beta}(\tilde{s}_f) = 0, E\left(\psi\left(p, \tilde{q}_e(\tilde{s}_f)\right)|\mathcal{A}_{t_j}\right) \le \Gamma_\psi, \tag{4.57e'}$$

where f_γ, Γ_f are defined by

$$f_\gamma := \begin{pmatrix} f_\gamma^u \\ f_\gamma^S \end{pmatrix},\ \Gamma_f := \begin{pmatrix} \Gamma_f^u \\ \Gamma_f^S \end{pmatrix}. \tag{4.57g}$$

Moreover, for the problem type (4.32a–f) we get

$$\min \int_{\tilde{s}_0}^{\tilde{s}_f} E\left(L_\gamma^J\left(p, \tilde{q}_e(\tilde{s}), \tilde{q}_e'(\tilde{s})\frac{\tilde{s}_f - \tilde{s}_0}{s_f - s_j}, \tilde{q}_e''(\tilde{s})\left(\frac{\tilde{s}_f - \tilde{s}_0}{s_f - s_j}\right)^2, \tilde{\beta}(\tilde{s}), \tilde{\beta}'(\tilde{s})\frac{\tilde{s}_f - \tilde{s}_0}{s_f - s_j}\right)|\mathcal{A}_{t_j}\right) \frac{s_f - s_j}{\tilde{s}_f - \tilde{s}_0}\, d\tilde{s} + E\left(\phi_\gamma^J\left(p, \tilde{q}_e(\tilde{s}_f)\right)|\mathcal{A}_{t_j}\right) \tag{4.58a}$$

s.t.

$$\tilde{\beta}(\tilde{s}) \geq 0, \tilde{s}_0 \leq \tilde{s} \leq \tilde{s}_f \tag{4.58b}$$

$$\tilde{q}_e(\tilde{s}_0) = \overline{q}_j, \tilde{q}'_e(\tilde{s}_0)\frac{\tilde{s}_f - \tilde{s}_0}{s_f - s_j}\sqrt{\tilde{\beta}(\tilde{s}_0)} = \bar{\dot{q}}_j \tag{4.58c}$$

$$\tilde{q}_e(\tilde{s}_f) = \overline{q}_f^{(j)} \text{ (if } \phi_\gamma^J = 0\text{)}, \ \tilde{\beta}(\tilde{s}_f) = 0. \tag{4.58d}$$

For the consideration of the *time-variant case* we note first that by using the transformation (4.52a, b) and (4.53b) the time $t \geq t_j$ can be represented, cf. (4.14a, b) and (4.15a), also by

$$t = t\Big(\tilde{s}, t_j, s_j, \tilde{\beta}(\cdot)\Big) := t_j + \frac{s_f - s_j}{\tilde{s}_f - \tilde{s}_0}\int_{\tilde{s}_0}^{\tilde{s}} \frac{d\tilde{\sigma}}{\sqrt{\tilde{\beta}(\tilde{\sigma})}}. \tag{4.59a}$$

Hence, if the variational problems (4.57a–f) and (4.58a–d) for the j-th stage depend explicitly on time $t \geq t_j$, then, corresponding to Sect. 4.4, item *B)*, for the constituting functions $L^J, \phi^J, L_\gamma^J, \phi_\gamma^J$ of the variational problems we have that

$$L^J = L^J\Big(\tilde{s}, t_j, s_j, \tilde{\beta}(\cdot), p_J, q_e, q'_e, q''_e, \beta, \beta'\Big), \quad \tilde{s}_0 \leq \tilde{s} \leq \tilde{s}_f \tag{4.59b}$$

$$\phi^J = \phi^J\Big(\tilde{s}_f, t_j, s_j, \tilde{\beta}(\cdot), p_J, q_e\Big) \tag{4.59c}$$

$$f_\gamma = f_\gamma\Big(\tilde{s}, t_j, s_j, \tilde{\beta}(\cdot), p, q_e, q'_e, q''_e, \beta, \beta'\Big), \quad \tilde{s}_0 \leq \tilde{s} \leq \tilde{s}_f \tag{4.59d}$$

$$L_\gamma^J = L_\gamma^J\Big(\tilde{s}, t_j, s_j, \tilde{\beta}(\cdot), p, q_e, q''_e, \beta, \beta'\Big), \quad \tilde{s}_0 \leq \tilde{s} \leq \tilde{s}_f \tag{4.59e}$$

$$\phi_\gamma^J = \phi_\gamma^J\Big(\tilde{s}_f, t_j, s_j, \tilde{\beta}(\cdot), p, q_e\Big). \tag{4.59f}$$

Transformation of the Initial State Values

Suppose here that $\phi^J \neq 0, \phi_\gamma^J \neq 0$, resp., and the terminal state condition (4.57f, e'), (4.58d), resp., is reduced to

$$\tilde{\beta}(\tilde{s}_f) = 0. \tag{4.60a}$$

Representing then the unknown functions $\tilde{\beta}(\cdot), \tilde{q}_e(\cdot)$ on $[\tilde{s}_0, \tilde{s}_f]$ by

$$\tilde{\beta}(\tilde{s}) := \beta_j \tilde{\beta}_a(\tilde{s}), \quad \tilde{s}_0 \leq \tilde{s} \leq \tilde{s}_f \tag{4.60b}$$

$$\tilde{q}_e(\tilde{s}) := \overline{q}_{jd}\tilde{q}_{ea}(\tilde{s}), \quad \tilde{s}_0 \leq \tilde{s} \leq \tilde{s}_f, \tag{4.60c}$$

where $\overline{q}_{jd}$ denotes the diagonal matrix with the components of $\overline{q}_j$ on its main diagonal, then in terms of the new unknowns $\left(\tilde{\beta}_a(\cdot), \tilde{q}_{ea}(\cdot)\right)$ on $[\tilde{s}_0, \tilde{s}_f]$ we have the nonnegativity and fixed initial/terminal conditions

$$\tilde{\beta}_a(\tilde{s}) \geq 0, \quad \tilde{s}_0 \leq \tilde{s} \leq \tilde{s}_f \tag{4.61a}$$

$$\tilde{\beta}_a(\tilde{s}_0) = 1, \;\; \tilde{q}_{ea}(\tilde{s}_0) = \mathbf{1} \tag{4.61b}$$

$$\tilde{\beta}_a(\tilde{s}_f) = 0, \tag{4.61c}$$

where $\mathbf{1} := (1, \ldots, 1)'$.

4.5.3 Numerical Solutions of (OSTP) in Real-Time

With the exception of field robots (e.g. Mars rover) and service robots [62], becoming increasingly important, the standard industrial robots move very fast. Hence, for industrial robots the optimal solution $\left(q_e^{(j)}(s), \beta^{(j)}(s)\right), \beta^{(j)}(s)$, resp., $s_j \leq s \leq s_f$, generating the renewed pair of guiding functions $\left(q^{(j)}(t), u^{(j)}(t)\right), t \geq t_j$, on each stage $j = 1, 2, \ldots$ should be provided in *real-time*. This means that the optimal solutions $\left(q_e^{(j)}, \beta^{(j)}\right), \beta^{(j)}$, resp., must be prepared off-line as far as possible such that only relatively simple numerical operations are left on-line.

Numerical methods capable to generate approximate optimal solutions in real-time are based mostly on discretization techniques, neural network (NN) approximation [8, 111, 114, 133], linearization techniques (sensitivity analysis) [159].

Discretization Techniques

Partitioning the space $Z \times Z_f$ of initial/terminal parameters (ζ, ζ_f) into a certain (small) number l_0 of subdomains

$$Z \times Z_f = \bigcup_{l=1}^{l_0} Z^l \times Z_f^l, \tag{4.62a}$$

and selecting then a reference parameter vector

$$(\zeta^l, \zeta_f^l) \in Z^l \times Z_f^l, l = 1, \ldots, l_0, \tag{4.62b}$$

in each subdomain $Z^l \times Z_f^l$, the optimal adaptive law (4.55c) can be approximated, cf. [150], by

$$\left.\begin{array}{l}\hat{\tilde{q}}_e^*(\tilde{s};\zeta,\zeta_f) := \tilde{q}^*(\tilde{s};\zeta^l,\zeta_f^l), \tilde{s}_0 \leq \tilde{s} \leq \tilde{s_f} \\ \hat{\tilde{\beta}}^*(\tilde{s};\zeta,\zeta_f) := \tilde{\beta}^*(\tilde{s};\zeta^l,\zeta_f^l), \tilde{s}_0 \leq \tilde{s} \leq \tilde{s}_f\end{array}\right\} \text{ for } (\zeta,\zeta_f) \in Z^l \times Z_f^l. \tag{4.62c}$$

NN-Approximation

For the determination of the optimal adaptive law (4.55a–c) in real-time, according to (4.34a, b), the reference variational problem (4.57a–f) or (4.58a–d) is reduced first to a finite dimensional parameter optimization problem by

i) representing the unknown functions $\tilde{q}_e = \tilde{q}_e(\tilde{s}), \tilde{\beta} = \tilde{\beta}(\tilde{s}), \tilde{s}_0 \leq \tilde{s} \leq \tilde{s}_f$, as linear combinations

$$\tilde{q}_e(\tilde{s}) := \sum_{l=1}^{l_q} \hat{q}_e B_l^q(\tilde{s}), \tilde{s}_0 \leq \tilde{s} \leq \tilde{s}_f, \tag{4.63a}$$

$$\tilde{\beta}(\tilde{s}) := \sum_{l=1}^{l_\beta} \hat{\beta}_l B_l^\beta(\tilde{s}), \tilde{s}_0 \leq \tilde{s} \leq \tilde{s}_f, \tag{4.63b}$$

of certain basis functions, e.g. cubic B-splines, $B_l^q = B_l^q(\tilde{s}), B_l^\beta = B_l^\beta(\tilde{s}), \tilde{s}_0 \leq \tilde{s} \leq \tilde{s}_f, l = 1,\ldots,l_q(l_\beta)$, with unknown vectorial (scalar) coefficients $\hat{q}_l, \hat{\beta}_l, l = 1,\ldots,l_q(l_\beta)$, and

ii) demanding the inequalities in (4.57b, c), (4.58b), resp., only for a finite set of $\tilde{s}$-parameter values $\tilde{s}_0 < \tilde{s}_1 < \ldots < \tilde{s}_k < \tilde{s}_{k+1} < \ldots < \tilde{s}_\kappa = \tilde{s}_f$.

By means of the above described procedure (i), (ii), the optimal coefficients

$$\hat{q}_l^* = \hat{q}_l^*(\zeta,\zeta_f), l = 1,\ldots,l_q \tag{4.63c}$$

$$\hat{\beta}_l^* = \hat{\beta}_l^*(\zeta,\zeta_f), l = 1,\ldots,l_\beta \tag{4.63d}$$

become functions of the initial/terminal parameters ζ,ζ_f, cf. (4.55c). Now, for the numerical realization of the optimal parameter functions (4.63c, d), a Neural Network (NN) is employed generating an approximative representation

$$\hat{q}_e^*(\zeta,\zeta_f) \approx \hat{q}_e^{NN}(\zeta,\zeta_f;w_q), l = 1,\ldots,l_q \tag{4.64a}$$

$$\hat{\beta}_l^*(\zeta,\zeta_f) \approx \hat{\beta}_l^{NN}(\zeta,\zeta_f;w_\beta), l = 1,\ldots,l_\beta, \tag{4.64b}$$

where the vectors w_q, w_β of NN-weights are determined optimally

$$w_q = w_q^* \text{ (data)}, \; w_\beta = w_\beta^* \text{ (data)} \tag{4.64c}$$

in an off-line training procedure [8, 111, 133]. Here, the model (4.64a, b) is fitted in the LSQ-sense to given data

$$\begin{pmatrix} \zeta^\tau, \zeta_f^\tau \\ \hat{q}_l^{*\tau}, l = 1, \ldots, l_q \end{pmatrix}, \begin{pmatrix} \zeta^\tau, \zeta_f^\tau \\ \hat{\beta}_l^{*\tau}, l = 1, \ldots, l_\beta \end{pmatrix}, \; \tau = 1, \ldots, \tau_0, \tag{4.64d}$$

where

$$(\zeta^\tau, \zeta_f^\tau), \tau = 1, \ldots, \tau_0, \tag{4.64e}$$

is a certain collection of initial/terminal parameter vectors, and

$$\hat{q}_l^{*\tau} := \hat{q}_l^{*\tau}(\zeta^\tau, \zeta_f^\tau), l = 1, \ldots, l_q, \tau = 1, \ldots, \tau_0 \tag{4.64f}$$

$$\hat{\beta}_l^{*\tau} := \hat{\beta}_l^{*\tau}(\zeta^\tau, \zeta_f^\tau), l = 1, \ldots, l_\beta, \tau = 1, \ldots, \tau_0 \tag{4.64g}$$

are the related optimal coefficients in (4.63a, b) which are determined off-line by an appropriate parameter optimization procedure.

Having the vectors w_q^*, w_β^* of optimal NN-weights, by means of (4.64a–c), for given actual initial/terminal parameters $(\zeta, \zeta_f) = (\zeta_j, \zeta_f^{(j)})$ at stage $j \geq 0$, the NN yields then the optimal parameters

$$\hat{q}_l^*(\zeta_j, \zeta_f^{(j)}), \hat{\beta}_l^*(\zeta_j, \zeta_f^{(j)}), l = 1, \ldots, l_q(l_\beta)$$

in real-time; consequently, by means of (4.63a, b), also the optimal functions $\tilde{q}_e^*(\tilde{s}), \tilde{\beta}^*(\tilde{s}), \tilde{s}_0 \leq \tilde{s} \leq \tilde{s}_f$, are then available very fast. For more details, see [8, 111].

Linearization Methods

I) *Linearization of the optimal feedforward control law*
Expanding the optimal control laws (4.55c) with respect to the initial/terminal parameter vector $\hat{\zeta} = (\zeta, \zeta_f)$ at its value $\hat{\zeta}_0 = (\zeta_0, \zeta_f^{(0)})$ for stage $j = 0$,

approximatively we have that

$$\tilde{q}_e^*(\tilde{s},\zeta,\zeta_f) = \tilde{q}_e^*(\tilde{s};\zeta_0,\zeta_f^{(0)}) + \frac{\partial \tilde{q}_e^*}{\partial \hat{\zeta}}(\tilde{s};\zeta_0,\zeta_f^{(0)})(\zeta-\zeta_0,\zeta_f-\zeta_f^{(0)}) + \ldots \tag{4.65a}$$

$$\tilde{\beta}^*(\tilde{s};\zeta,\zeta_f) = \tilde{\beta}^*(\tilde{s};\zeta_0,\zeta_f^{(0)}) + \frac{\partial \tilde{\beta}^*}{\partial \hat{\zeta}}(\tilde{s};\zeta_0,\zeta_f^{(0)})(\zeta-\zeta_0,\zeta_f-\zeta_f^{(0)}) + \ldots, \tag{4.65b}$$

where the optimal starting functions $\tilde{q}_e^*(\tilde{s};\zeta_0,\zeta_f^{(0)}), \tilde{\beta}^*(\tilde{s};\zeta_0,\zeta_f^{(0)})$ and the derivatives $\frac{\partial \tilde{q}_e^*}{\partial \hat{\zeta}}(\tilde{s};\zeta_0,\zeta_f^{(0)}), \frac{\partial \tilde{\beta}^*}{\partial \hat{\zeta}}(\tilde{s};\zeta_0,\zeta_f^{(0)}), \ldots$ can be determined—on a certain grid of $[\tilde{s}_0,\tilde{s}_f]$—off-line by using sensitivity analysis [159]. The actual values of $\tilde{q}_e^*, \tilde{\beta}^*$ at later stages can be obtained then very rapidly by means of simple matrix operations. If necessary, the derivatives can be updated later on by a numerical procedure running in parallel to the control process.

II) *Sequential linearization of the (AOSTP) process*
Given the optimal guiding functions $q_e^{(j)} = q_e^{(j)}(s), \beta^{(j)} = \beta^{(j)}(s), s_j \le s \le s_f$ for the j-th stage (Fig. 4.3), corresponding to the representation (4.35a, b), the optimal guiding functions $q_e^{(j+1)}(s), \beta^{(j+1)}(s), s_{j+1} \le s \le s_f$, are represented by

$$q_e^{(j+1)}(s) := q_e^{(j)}(s) + \Delta q_e(s), \quad s_{j+1} \le s \le s_f, \tag{4.66a}$$

$$\beta^{(j+1)}(s) := \beta^{(j)}(s) + \Delta\beta(s), \quad s_{j+1} \le s \le s_f, \tag{4.66b}$$

where $s_j < s_{j+1} < s_f$, and $\big(\Delta q_e(s), \Delta\beta(s)\big), s_{j+1} \le s \le s_f$, are certain (small) changes of the j-th stage optimal guiding functions $\big(q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\big)$.

Obviously, the linearization technique described in Sect. 4.4.1c can be applied now also to the approximate computation of the optimal changes

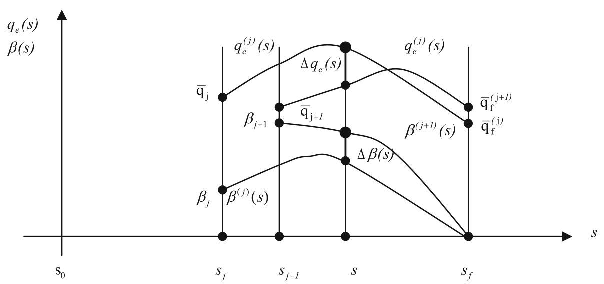


Fig. 4.3 Sequential linearization of (AOSTP)

$\Delta q_e(s), \Delta\beta(s), s_{j+1} \le s \le s_f$, if the following replacements are made in the formulas (4.35a, b), (4.36a–c), (4.37a–h) and (4.38a–f):

$$\left.\begin{array}{ll}
s_0 \le s \le s_f & \to s_{j+1} \le s \le s_f \\
\overline{q}_e, \overline{\beta} & \to q_e^{(j)}, \beta^{(j)} \\
\mathcal{A}_{t_0} & \to \mathcal{A}_{t_{j+1}} \\
\overline{g}^{(0)}, \overline{\phi}^{(0)}, \overline{F}^{(0)} & \to \overline{g}^{(j+1)}, \overline{\phi}^{(j+1)}, \overline{F}^{(j+1)} \\
\overline{A}^{(0)}_{g,\overline{q}_e,\overline{\beta}}, \ldots, \overline{H}^{(0)}_{g,\overline{q}_e,\overline{\beta}} & \to \overline{A}^{(j+1)}_{g,q_e^{(j)},\beta^{(j)}}, \ldots, \overline{H}^{(j+1)}_{g,q_e^{(j)},\beta^{(j)}} \\
\Delta\beta(s_0) = 0 & \to \Delta\beta(s_{j+1}) = \beta_{j+1} - \beta^{(j)}(s_{j+1}) \\
\Delta q_e(s_0) = q_0 - \overline{q}_e(s_0) & \to \Delta q_e(s_{j+1}) = \overline{q}_{j+1} - q_e^{(j)}(s_{j+1}) \\
\Delta\beta(s_f) = 0 & \to \Delta\beta(s_f) = 0 \\
\Delta q_e(s_f) = q_f^{(0)} - \overline{q}_e(s_f) & \to \Delta q_e(s_f) = \overline{q}_f^{(j+1)} - \overline{q}_f^{(j)}, \text{ if } \phi^J = 0,
\end{array}\right\} \tag{4.66c}$$

where $\overline{X}^{(j+1)}$ denotes the conditional expectation of a random variable X with respect to $\mathcal{A}_{t_{j+1}}$, cf. (4.50d), (4.51b). Furthermore, (4.37f) yields

$$\left(q_e^{(j)'}(s_{j+1}) + \Delta q_e'(s_{j+1})\right) \cdot \sqrt{\beta_{j+1}} = \bar{\dot{q}}_{j+1} \tag{4.66d}$$

which can be approximated, cf. (4.38f'), by

$$\begin{aligned}\sqrt{\beta^{(j)}(s_{j+1})}\Delta q_e'(s_{j+1}) + \frac{1}{2}\frac{\Delta\beta(s_{j+1})}{\sqrt{\beta^{(j)}(s_{j+1})}} q_e^{(j)'}(s_{j+1}) \\ \approx \bar{\dot{q}}_{j+1} - \sqrt{\beta^{(j)}(s_{j+1})} q_e^{(j)'}(s_{j+1}).\end{aligned} \tag{4.66d'}$$

Depending on the chosen substitute problem, by this linearization method we obtain then a variational problem, an optimization problem, resp., for the changes $\left(\Delta q_e(s), \Delta\beta(s)\right), s_{j+1} \le s \le s_f$, having a linear objective function and/or linear constraints.

To give a typical example, we consider now (AOSTP) on the $(j+1)$th stage with substitute problem (4.32a–f). Hence, the functions g, ϕ in (4.39a–c), (4.37a–h), and (4.38a–f) are given by

$$g := L_\gamma^J, \phi := \phi_\gamma^J.$$

Applying the linearization techniques developed in Sect. 4.4.1c now to (4.32a–f), according to (4.38a–c), (4.37b), for the correction terms $\Delta q_e(s), \Delta\beta(s)$,

$s_{j+1} \leq s \leq s_f$, we find the following linear optimization problem:

$$\min \int_{s_{j+1}}^{s_f} \overline{G}^{(j+1)}_{g,q_e^{(j)},\beta^{(j)}}(s)^T \Delta q_e(s)ds + \int_{s_{j+1}}^{s_f} \overline{H}^{(j+1)}_{g,q_e^{(j)},\beta^{(j)}}(s)\Delta(s)ds \tag{4.67a}$$

$$+R_j^T \Delta q_e(s_f) + S_j^T \Delta q_e'(s_f) + T_j \Delta\beta(s_{j+1})$$

s.t.

$$\Delta q_e(s_{j+1}) = \overline{q}_{j+1} - q_e^{(j)}(s_{j+1}) \tag{4.67b}$$

$$\Delta q_e(s_f) = \overline{q}_f^{(j+1)} - \overline{q}_f^{(j)}, \text{ if } \phi^J = 0 \tag{4.67c}$$

$$\Delta\beta(s_f) = 0$$

$$\Delta\beta(s) \geq -\beta^{(j)}(s), s_{j+1} \leq s \leq s_f, \tag{4.67d}$$

where

$$R_j := \overline{B}^{(j+1)}_{g,q_e^{(j)},\beta^{(j)}}(s_f) - \overline{C}^{(j+1)}_{g,q_e^{(j)},\beta^{(j)}}(s_f) + \overline{a}^{(j+1)}_{\phi,q_e^{(j)}}(s_f) \tag{4.67e}$$

$$S_j := \overline{C}^{(j+1)}_{g,q_e^{(j)},\beta^{(j)}}(s_f) \tag{4.67f}$$

$$T_j := \frac{1}{2}\overline{C}^{(j+1)}_{g,q_e^{(j)},\beta^{(j)}}(s_{j+1})^T \frac{q_e^{(j)'}(s_{j+1})}{\beta^{(j)}(s_{j+1})} - \overline{E}^{(j+1)}_{g,q_e^{(j)},\beta^{(j)}}(s_{j+1}). \tag{4.67g}$$

The linear optimization problem (4.67a–g) can be solved now by the methods developed e.g. in [52], where for the correction terms $\Delta q_e(s), \Delta\beta(s), s_{j+1} \leq s \leq s_f$, some box constraints or norm bounds have to be added to (4.67a–g). In case of $\Delta\beta(\cdot)$ we may replace, e.g., (4.67d) by the condition

$$-\beta^{(j)}(s) \leq \Delta\beta(s) \leq \Delta\beta^{\max}, s_{j+1} \leq s \leq s_f, \tag{4.67d'}$$

with some upper bound $\Delta\beta^{\max}$.

It is easy to see that (4.67a–g) can be split up into two separated linear optimization problems for $\Delta q_e(\cdot), \Delta\beta(\cdot)$, respectively. Hence, according to the simple structure of the objective function (4.67a), we observe that

$$\text{sign}\left(\overline{H}^{(j+1)}_{g,q_e^{(j)},\beta^{(j)}}(s)\right), s_{j+1} \leq s \leq s_f, \text{ and sign}(T_j)$$

indicates the points s in the interval $[s_{j+1}, s_f]$ with $\Delta\beta(s) < 0$ or $\Delta\beta(s) > 0$, hence, the points $s, s_{j+1} \leq s \leq s_f$, where the velocity profile should be decreased/increased. Moreover, using (4.67d'), the optimal correction $\Delta\beta(s)$ is equal to the lower/upper bound in (4.67d') depending on the above mentioned signs.

Obviously, the correction vectors $\Delta q_e(s),\ s_{j+1} \le s \le s_f$, for the geometric path in configuration space can be determined in the same way. Similar results are obtained also if we use L_2-norm bounds for the correction terms.

If the pointwise constraints (4.29b, c) in (4.29a–f, f') are averaged with respect to $s, s_{j+1} \le s \le s_f$, then functions of the type (4.36c) arise, cf. (4.29b', c'), which can be linearized again by the same techniques as discussed above. In this case, linear constraints are obtained for $\Delta\beta(s), \Delta q_e(s), s_{j+1} \le s \le s_f$, with constraint functions of the type (4.38a–c), cf. also (4.67a).

Combination of Discretization and Linearization

Obviously, the methods described briefly in Sect. 4.5.3 can be combined in the following way:

First, by means of discretization (Finite Element Methods), an approximate optimal control law $(\tilde{q}_e^*, \beta^*)$ is searched in a class of finitely generated functions of the type (4.63a, b). Corresponding to (4.65a, b), by means of Taylor expansion here the optimal coefficients $\hat{q}_l^*, \hat{\beta}_l^*, l = 1, \ldots, l_q(l_\beta)$, in the corresponding linear combination of type (4.63a, b) are represented, cf. (4.63c, d), by

$$\hat{q}_l^*(\zeta, \zeta_f) = \hat{q}_l^*\left(\zeta_0, \zeta_f^{(0)}\right) + \frac{\partial \hat{q}_l^*}{\partial \hat{\zeta}}(\zeta_0, \zeta_f^{(0)})\left(\zeta - \zeta_0, \zeta_f - \zeta_f^{(0)}\right) + \ldots, \tag{4.68a}$$

$$\hat{\beta}_l^*(\zeta, \zeta_f) = \hat{\beta}_l^*\left(\zeta_0, \zeta_f^{(0)}\right) + \frac{\partial \hat{\beta}_l^*}{\partial \hat{\zeta}}\left(\zeta_0, \zeta_f^{(0)}\right)\left(\zeta - \zeta_0, \zeta_f - \zeta_f^{(0)}\right) + \ldots, \tag{4.68b}$$

$l = 1, \ldots, l_q(l_\beta)$. Here, the derivatives

$$\frac{\partial \hat{q}_l^*}{\partial \hat{\zeta}}\left(\zeta_0, \zeta_f^{(0)}\right), \frac{\partial \hat{\beta}_l^*}{\partial \hat{\zeta}}\left(\zeta_0, \zeta_f^{(0)}\right), \ldots, l = 1, \ldots, l_q(l_\beta), \tag{4.68c}$$

can be determined again by sensitivity analysis [159] of a finite dimensional parameter-dependent optimization problem which may be much simpler than the sensitivity analysis of the parameter-dependent variational problem (4.57a–f) or (4.58a–d).

Stating the necessary (and under additional conditions also sufficient) Kuhn–Tucker conditions for the optimal coefficients $\hat{q}_l^*, \hat{\beta}_l^*, l = 1, \ldots, l_q(l_\beta)$, formulas for the derivatives (4.68c) may be obtained by partial differentiation with respect to the complete vector $z = (\zeta, \zeta_f)$ of initial/terminal parameters.

4.6 Online Control Corrections: PD-Controller

We now consider the control of the robot at the j-th stage, i.e., for time $t \geq t_j$, see [4, 6, 11, 20, 58, 61]. In practice we have random variations of the vector p of the model parameters of the robot and its environment, moreover, there are possible deviations of the true initial state $(q_j, \dot{q}_j) := \big(q(t_j), \dot{q}(t_j)\big)$ in configuration space from the corresponding initial values $(\overline{q}_j, \overline{\dot{q}}_j) = \Big(\overline{q}_j, q_e^{(j)'}(s_j)\sqrt{\beta_j}\Big)$ of the (OSTP) at stage j. Thus, the actual trajectory

$$q(t) = q\Big(t, p_D, q_j, \dot{q}_j, u(\cdot)\Big), t \geq t_j \tag{4.69a}$$

in configuration space of the robot will deviate more or less from the optimal reference trajectory

$$q^{(j)}(t) = q_e^{(j)}\Big(s^{(j)}(t)\Big) = q\left(t, \overline{p}_D^{(j)}, \overline{q}_j, \overline{\dot{q}}_j, u^{(j)}(\cdot)\right), \tag{4.69b}$$

see (4.44a, b), (4.45a–f), (4.49a, b) and (4.50c). In the following we assume that the state $\big(q(t), \dot{q}(t)\big)$ in configuration space may be observed for $t > t_j$. Now, in order to define an appropriate control correction (feedback control law), see (4.2) and Fig. 4.1,

$$\Delta u^{(j)}(t) = u(t) - u^{(j)}(t) := \varphi^{(j)}\left(t, \Delta z^{(j)}(t)\right), \ t \geq t_j, \tag{4.70a}$$

for the compensation of the tracking error

$$\Delta z^{(j)}(t) := z(t) - z^{(j)}(t), z(t) := \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix}, z^{(j)}(t) := \begin{pmatrix} q^{(j)}(t) \\ \dot{q}^{(j)}(t) \end{pmatrix}, \tag{4.70b}$$

where $\varphi^{(j)} = \varphi^{(j)}(t, \Delta q, \Delta\dot{q})$ is such a function that

$$\varphi^{(j)}(t, 0, 0) = 0 \text{ for all } t \geq t_j, \tag{4.70c}$$

the trajectories $q(t)$ and $q^{(j)}(t), t \geq t_j$, are embedded into a one-parameter family of trajectories $q = q(t, \epsilon), t \geq t_j, 0 \leq \epsilon \leq 1$, in configuration space which are defined as follows:

Consider first the following initial data for stage j:

$$q_j(\epsilon) := \overline{q}_j + \epsilon\Delta q_j, \Delta q_j := q_j - \overline{q}_j \tag{4.71a}$$

$$\dot{q}_j(\epsilon) := \overline{\dot{q}}_j + \epsilon\Delta\dot{q}_j, \Delta\dot{q}_j := \dot{q}_j - \overline{\dot{q}}_j \tag{4.71b}$$

$$p_D(\epsilon) := \overline{p}_D^{(j)} + \epsilon\Delta p_D, \Delta p_D := p_D - \overline{p}_D^{(j)}, 0 \leq \epsilon \leq 1. \tag{4.71c}$$

Moreover, define the control input $u(t), t \geq t_j$, by (4.70a), hence,

$$\begin{aligned} u(t) &= u^{(j)}(t) + \Delta u^{(j)}(t) \\ &= u^{(j)}(t) + \varphi^{(j)}\left(t, q(t) - q^{(j)}(t), \dot{q}(t) - \dot{q}^{(j)}(t)\right), \ t \geq t_j. \end{aligned} \tag{4.71d}$$

Let then denote

$$q(t,\epsilon) = q\left(t, p_D(\epsilon), q_j(\epsilon), \dot{q}_j(\epsilon), u(\cdot)\right), \ 0 \leq \epsilon \leq 1, \ t \geq t_j, \tag{4.72}$$

the solution of the following initial value problem consisting of the dynamic equation (4.4a) with the initial values, the vector of dynamic parameters and the total control input $u(t)$ given by (4.71a–d):

$$F\left(p_D(\epsilon), q(t,\epsilon), \dot{q}(t,\epsilon), \ddot{q}(t,\epsilon)\right) = u(t,\epsilon), 0 \leq \epsilon \leq 1, t \geq t_j, \tag{4.73a}$$

where

$$q(t_j,\epsilon) \ = \ q_j(\epsilon), \dot{q}(t_j,\epsilon) = \dot{q}_j(\epsilon), \tag{4.73b}$$

$$u(t,\epsilon) := u^{(j)}(t) + \varphi^{(j)}\Big(t, q(t,\epsilon) - q^{(j)}(t), \dot{q}(t,\epsilon) - \dot{q}^{(j)}(t)\Big), \tag{4.73c}$$

and $F = F(p_D, q, \dot{q}, \ddot{q})$ is defined, cf. (2.6a), by

$$F(p_D, q, \dot{q}, \ddot{q}) := M(p_D, q)\ddot{q} + h(p_D, q, \dot{q}). \tag{4.73d}$$

In the following we suppose [76] that the initial value problem (4.73a–d) has a unique solution $q = q(t,\epsilon), \ t \geq t_j$, for each parameter value $\epsilon, 0 \leq \epsilon \leq 1$ (Fig. 4.4).

4.6.1 Basic Properties of the Embedding $q(t,\epsilon)$

$\epsilon = \epsilon_0 := 0$

Because of condition (4.70c) of the feedback control law $\varphi^{(j)}$ to be determined, and due to the unique solvability assumption of the initial value problem (4.73a–d) at the j-th stage, for $\epsilon = 0$ we have that

$$q(t,0) = q^{(j)}(t), t \geq t_j. \tag{4.74a}$$

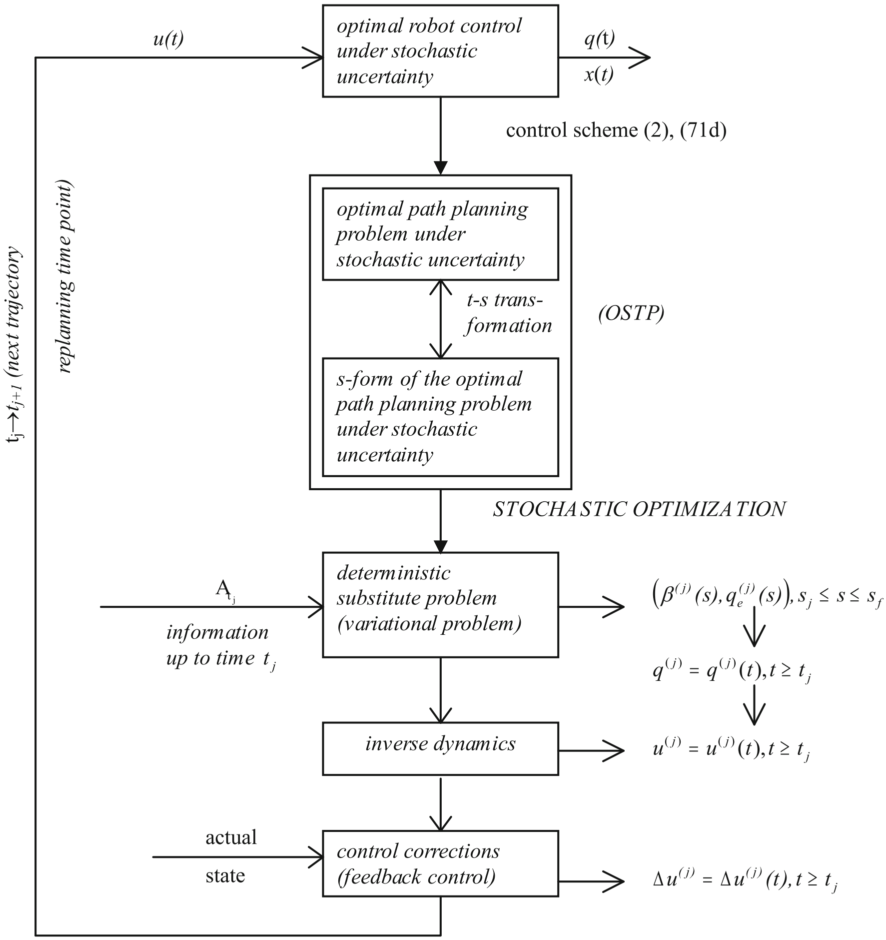


Fig. 4.4 Adaptive optimal stochastic trajectore planning and control (AOSTPC)

$\epsilon = \epsilon_1 := 1$

According to (4.69a), (4.70a–c) and (4.71a–d),

$$q(t,1) = q(t) = q\Big(t, p_D, q_j, \dot{q}_j, u(\cdot)\Big), t \ge t_j, \tag{4.74b}$$

is the actual trajectory in configuration space under the total control input $u(t) = u^{(j)}(t) + \Delta u^{(j)}(t), t \ge t_j$, given by (4.71d).

Taylor-Expansion with Respect to ϵ

Let $\Delta\epsilon = \epsilon_1 - \epsilon_0 = 1$, and suppose that the following property known from parameter-dependent differential equations, cf. [76], holds:

Assumption 4.1 *The solution $q = q(t, \epsilon), t \geq t_j, 0 \leq \epsilon \leq 1$, of the initial value problem (4.71a–d) has continuous derivatives with respect to ϵ up to order $\nu > 1$ for all $t_j \leq t \leq t_j + \Delta t_j, 0 \leq \epsilon \leq 1$, with a certain $\Delta t_j > 0$.*

Note that $(t, \epsilon) \rightarrow q(t, \epsilon), t \geq t_j, 0 \leq \epsilon \leq 1$, can be interpreted as a homotopy from the reference trajectory $q^{(j)}(t)$ to the actual trajectory $q(t), t \geq t_j$, cf. [134].

Based on the above assumption and (4.74a, b), by Taylor expansion with respect to ϵ at $\epsilon = \epsilon_0 = 0$, the actual trajectory of the robot can be represented by

$$\begin{aligned} q(t) &= q\Big(t, p_D, q_j, \dot{q}_j, u(\cdot)\Big) = q(t, 1) = q(t, \epsilon_0 + \Delta\epsilon) \\ &= q(t, 0) + \Delta q(t) = q^{(j)}(t) + \Delta q(t), \end{aligned} \tag{4.75a}$$

where the expansion of the tracking error $\Delta q(t), t \geq t_j$, is given by

$$\begin{aligned} \Delta q(t) &= \sum_{l=1}^{\nu-1} \frac{1}{l!} d^l q(t) (\Delta\epsilon)^l + \frac{1}{\nu!} \frac{\partial^\nu q}{\partial \epsilon^\nu}(t, \vartheta)(\Delta\epsilon)^\nu \\ &= \sum_{l=1}^{\nu-1} \frac{1}{l!} d^l q(t) + \frac{1}{\nu!} \frac{\partial^\nu q}{\partial \epsilon^\nu}(t, \vartheta), t \geq t_j. \end{aligned} \tag{4.75b}$$

Here, $\vartheta = \vartheta(t, \nu), 0 < \vartheta < 1$, and

$$d^l q(t) := \frac{\partial^l q}{\partial \epsilon^l}(t, 0), t \geq t_j, l = 1, 2, \ldots, \tag{4.75c}$$

denote the l-th order differentials of $q = q(t, \epsilon)$ with respect to ϵ at $\epsilon = \epsilon_0 = 0$. Obviously, differential equations for the differentials $d^l q(t), l = 1, 2, \ldots$, may be obtained, cf. [76], by successive differentiation of the initial value problem (4.73a–d) with respect to ϵ at $\epsilon_0 = 0$.

Furthermore, based on the Taylor expansion of the tracking error $\Delta q(t), t \geq t_j$, using some stability requirements, the tensorial coefficients $D_z^l \varphi^{(j)}(t, 0), l = 1, 2, \ldots$, of the Taylor expansion

$$\varphi^{(j)}(t, \Delta z) = \sum_{l=1}^{\infty} \mathbf{D}_z^l \varphi^{(j)}(t, 0) \cdot (\Delta z)^l \tag{4.75d}$$

of the feedback control law $\varphi^{(j)} = \varphi^{(j)}(t, \Delta z)$ can be determined at the same time.

4.6.2 The 1st Order Differential dq

Next to we have to introduce some definitions. Corresponding to (4.70b) and (4.72) we put

$$z(t,\epsilon) := \begin{pmatrix} q(t,\epsilon) \\ \dot{q}(t,\epsilon) \end{pmatrix}, t \geq t_j, 0 \leq \epsilon \leq 1; \tag{4.76a}$$

then, we define the following Jacobians of the function F given by (4.73d):

$$K(p_D, q, \dot{q}, \ddot{q}) := F_q(p_D, q, \dot{q}, \ddot{q}) = \mathbf{D}_q F(p_D, q, \dot{q}, \ddot{q}) \tag{4.76b}$$

$$D(p_D, q, \dot{q}) := F_{\dot{q}}(p_D, q, \dot{q}, \ddot{q}) = h_{\dot{q}}(p_D, q, \dot{q}). \tag{4.76c}$$

Moreover, it is

$$M(p_D, q) = F_{\ddot{q}}(p_D, q, \dot{q}, \ddot{q}), \tag{4.76d}$$

and due to the linear parameterization property of robots, see Remark 4.2, F may be represented by

$$F(p_D, q, \dot{q}, \ddot{q}) = Y(q, \dot{q}, \ddot{q}) p_D \tag{4.76e}$$

with a certain matrix function $Y = Y(q, \dot{q}, \ddot{q})$.

By differentiation of (4.73a–d) with respect to ϵ, for the partial derivative $\frac{\partial q}{\partial \epsilon}(t,\epsilon)$ of $q = q(t,\epsilon)$ with respect to ϵ we find, cf. (4.70b), the following linear initial value problem **(error differential equation)**

$$\begin{aligned}
&Y\Big(q(t,\epsilon), \dot{q}(t,\epsilon), \ddot{q}(t,\epsilon)\Big)\Delta p_D + K\Big(p_D(\epsilon), q(t,\epsilon), \dot{q}(t,\epsilon), \ddot{q}(t,\epsilon)\Big)\frac{\partial q}{\partial \epsilon}(t,\epsilon) \\
&+D\Big(p_D(\epsilon), q(t,\epsilon), \dot{q}(t,\epsilon)\Big)\frac{d}{dt}\frac{\partial q}{\partial \epsilon}(t,\epsilon) + M\Big(p_D(\epsilon), q(t,\epsilon)\Big)\frac{d^2}{dt^2}\frac{\partial q}{\partial \epsilon}(t,\epsilon) \\
&= \frac{\partial u}{\partial \epsilon}(t,\epsilon) = \frac{\partial \varphi^{(j)}}{\partial z}\Big(t, \Delta z^{(j)}(t)\Big)\frac{\partial z}{\partial \epsilon}(t,\epsilon)
\end{aligned} \tag{4.77a}$$

with the initial values, see (4.71a, b),

$$\frac{\partial q}{\partial \epsilon}(t_j,\epsilon) = \Delta q_j, \frac{d}{dt}\frac{\partial q}{\partial \epsilon}(t_j,\epsilon) = \Delta\dot{q}_j. \tag{4.77b}$$

Putting now $\epsilon = \epsilon_0 = 0$, because of (4.70a, b) and (4.74a), system (4.77a, b) yields then this system of 2nd order differential equations for the 1st order differential $dq(t) = \frac{\partial q}{\partial \epsilon}(t, 0)$:

$$
\begin{aligned}
&Y^{(j)}(t)\Delta p_D + K^{(j)}(t)dq(t) + D^{(j)}(t)\dot{d}q(t) + M^{(j)}(t)\ddot{d}q(t)\\
&\quad = du(t) = \varphi_z^{(j)}(t,0)dz(t) = \varphi_q^{(j)}(t,0)dq(t) + \varphi_{\dot{q}}^{(j)}(t,0)\dot{d}q(t), t \geq t_j,
\end{aligned}
\tag{4.78a}
$$

with the initial values

$$
dq(t_j) = \Delta q_j, \dot{d}q(t_j) = \dot{\Delta} q_j. \tag{4.78b}
$$

Here,

$$
du(t) := \frac{\partial u}{\partial \epsilon}(t, 0), \tag{4.78c}
$$

$$
dz(t) := \begin{pmatrix} dq(t) \\ \dot{d}q(t), \end{pmatrix}, \dot{d}q := \frac{d}{dt}dq, \ddot{d}q := \frac{d^2}{dt^2}dq, \tag{4.78d}
$$

and the matrices $Y^{(j)}(t)$, $K^{(j)}(t)$, $D^{(j)}(t)$ and $M^{(j)}(t)$ are defined, cf. (4.76b–e), by

$$
Y^{(j)}(t) := Y\left(q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\right) \tag{4.78e}
$$

$$
K^{(j)}(t) := K\left(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\right) \tag{4.78f}
$$

$$
D^{(j)} := D\left(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t)\right), M^{(j)}(t) := M\left(\overline{p}_D^{(j)}, q^{(j)}(t)\right). \tag{4.78g}
$$

Local (PD-)control corrections $du = du(t)$ stabilizing system (4.78a, b) can now be obtained by the following definition of the Jacobian of $\varphi^{(j)}(t, z)$ with respect to z at $z = 0$:

$$
\begin{aligned}
\varphi_z^{(j)}(t,0) &:= F_z\left(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\right) - M^{(j)}(t)(K_p, K_d)\\
&= \left(K^{(j)}(t) - M^{(j)}(t)K_p, D^{(j)}(t) - M^{(j)}(t)K_d\right),
\end{aligned}
\tag{4.79}
$$

where $K_p = (\gamma_{pk}\delta_{k\nu})$, $K_d = (\gamma_{dk}\delta_{k\nu})$ are positive definite diagonal matrices with positive diagonal elements $\gamma_{pk}, \gamma_{dk} > 0, k = 1, \ldots, n$.

Inserting (4.79) into (4.78a), and assuming that $M^{(j)} = M^{(j)}(t)$ is regular [11] for $t \geq t_j$, we find the following linear system of 2nd order differential equations for $dq = dq(t)$:

$$
\ddot{d}q(t) + K_d\dot{d}q(t) + K_p dq(t) = -M^{(j)}(t)^{-1}Y^{(j)}(t)\Delta p_D, t \geq t_j, \tag{4.80a}
$$

$$
dq(t_j) = \Delta q_j, \dot{d}q(t_j) = \dot{\Delta} q_j. \tag{4.80b}
$$

Considering the right hand side of (4.80a), according to (4.78e), (4.76e) and (4.73d) we have that

$$\begin{aligned} Y^{(j)}(t)\Delta p_D &= Y\left(q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\right)\Delta p_D = F\left(\Delta p_D, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\right) \\ &= M\left(\Delta p_D, q^{(j)}(t)\right)\ddot{q}^{(j)}(t) + h\left(\Delta p_D, q^{(j)}(t), \dot{q}^{(j)}(t)\right). \end{aligned} \tag{4.81a}$$

Using then definition (4.49a, b) of $q^{(j)}(t)$ and the representation (4.18a, b) of $\dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)$, we get

$$\begin{aligned} &Y^{(j)}(t)\Delta p_D \\ &= M\left(\Delta p_D, q_e^{(j)}\left(s^{(j)}(t)\right)\right)\left(q_e^{(j)\prime}\left(s^{(j)}(t)\right)\frac{1}{2}\beta^{(j)\prime}\left(s^{(j)}(t)\right)\right. \\ &\left.+q_e^{(j)\prime\prime}\left(s^{(j)}(t)\right)\beta^{(j)}\left(s^{(j)}(t)\right)\right) \\ &+h\left(\Delta p_D, q_e^{(j)}\left(s^{(j)}(t)\right), q_e^{(j)\prime}\left(s^{(j)}(t)\right)\sqrt{\beta^{(j)}\left(s^{(j)}(t)\right)}\right). \end{aligned} \tag{4.81b}$$

From (4.19b) we now obtain the following important representations, where we suppose that the feedforward control $u^{(j)}(t), t \geq t_j$, is given by (4.50c, d).

Lemma 4.1 *The following representations hold:*

a)

$$Y^{(j)}(t)\Delta p_D = u_e\left(\Delta p_D, s^{(j)}(t); q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\right), t \geq t_j; \tag{4.82a}$$

b)

$$u^{(j)}(t) = u_e\left(\overline{p}_D^{(j)}, s^{(j)}(t); q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\right), t \geq t_j; \tag{4.82b}$$

c)

$$u^{(j)}(t) + Y^{(j)}(t)\Delta p_D = u_e\left(p_D, s^{(j)}(t); q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\right), t \geq t_j. \tag{4.82c}$$

Proof The first equation follows from (4.81b) and (4.19b). Equations (4.50c), (4.18a, b) and (4.19b) yield (4.82b). Finally, (4.82c) follows from (4.82a, b) and the linear parameterization of robots, cf. Remark 4.2.

Remark 4.12 Note that according to the transformation (4.19a) of the dynamic equation onto the s-domain, for the control input $u(t)$ we have the representation

$$\begin{aligned} u(t) &= u_e\left(p_D, s; q_e(\cdot), \beta(\cdot)\right) \\ &= u_e\left(\overline{p}_D^{(j)}, s; q_e(\cdot), \beta(\cdot)\right) + u_e\left(\Delta p_D, s; q_e(\cdot), \beta(\cdot)\right) \end{aligned} \tag{4.82d}$$

with $s = s(t)$.

Using (4.78d), it is easy to see that (4.80a, b) can be described also by the 1st order initial value problem

$$\dot{d}z(t) = Adz(t) + \begin{pmatrix} 0 \\ \psi^{(j,1)}(t) \end{pmatrix}, t \geq t_j \tag{4.83a}$$

$$dz(t_j) = \Delta z_j = \begin{pmatrix} q_j - \overline{q}_j \\ \dot{q}_j - \dot{\overline{q}}_j \end{pmatrix}, \tag{4.83b}$$

where A is the stability or Hurwitz matrix

$$A := \begin{pmatrix} 0 & I \\ -K_p & -K_d \end{pmatrix}, \tag{4.83c}$$

and $\psi^{(j,1)}(t)$ is defined, cf. (4.82a), by

$$\begin{aligned} \psi^{(j,1)}(t) &:= -M^{(j)}(t)^{-1} Y^{(j)}(t) \Delta p_D \\ &= -M^{(j)}(t)^{-1} u_e \left(\Delta p_D, s^{(j)}(t); q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\right). \end{aligned} \tag{4.83d}$$

Consequently, for the first order expansion term $dz(t)$ of the deviation $\Delta z^{(j)}(t)$ between the actual state $z(t) = \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix}$ and the prescribed state $z^{(j)}(t) = \begin{pmatrix} q^{(j)}(t) \\ \dot{q}^{(j)}(t) \end{pmatrix}, t \geq t_j$, we have the representation [32, 76]

$$dz(t) = dz^{(j)}(t) = e^{A(t-t_j)} \Delta z_j + \int_{t_j}^{t} e^{A(t-\tau)} \begin{pmatrix} 0 \\ \psi^{(j,1)}(\tau) \end{pmatrix} d\tau. \tag{4.84a}$$

Because of $E\left(\Delta p_D(\omega) | \mathcal{A}_{t_j}\right) = 0$, we have that

$$E\left(\psi^{(j,1)}(t) | \mathcal{A}_{t_j}\right) = 0, \tag{4.84b}$$

$$E\left(dz(t) | \mathcal{A}_{t_j}\right) = e^{A(t-t_j)} E\left(\Delta z_j | \mathcal{A}_{t_j}\right), \ t \geq t_j, \tag{4.84c}$$

where , see (4.83a, b),

$$E\left(\Delta z_j | \mathcal{A}_{t_j}\right) = E\left(z(t_j) | \mathcal{A}_{t_j}\right) - \overline{z}_j. \tag{4.84d}$$

It is easy to see that the diagonal elements $\gamma_{dk}, \gamma_{pk} > 0, k = 1, \ldots, n$, of the positive definite diagonal matrices K_d, K_p, rep., can be chosen so that the fundamental matrix $\Phi(t, \tau) = e^{A(t-\tau)}, t \geq \tau$, is exponentially stable, i.e.

$$\|\Phi(t, \tau)\| \leq a_0 e^{-\lambda_0(t-\tau)}, t \geq \tau, \tag{4.85a}$$

with positive constants a_0, λ_0. A sufficient condition for (4.85a) reads

$$\gamma_{dk}, \gamma_{pk} > 0, k = 1, \ldots, n, \text{ and } \gamma_{dk} > 2 \text{ in case of double eigenvalues of } A. \tag{4.85b}$$

Define the generalized variance var $(Z|\mathcal{A}_{t_j})$ of a random vector $Z = Z(\omega)$ given $\mathcal{A}_{t_j}$ by $\mathrm{var}(Z|\mathcal{A}_{t_j}) := E\Big(\|Z - E(Z|\mathcal{A}_{t_j})\|^2|\mathcal{A}_{t_j}\Big)$, and let $\sigma_Z^{(j)} := \sqrt{\mathrm{var}\ (Z|\mathcal{A}_{t_j})}$. Then, for the behavior of the 1st order error term $dz(t), t \geq t_j$, we have the following result:

Theorem 4.6.1 *Suppose that the diagonal matrices K_d, K_p are selected such that (4.85a) holds. Moreover, apply the local (i.e. first order) control correction (PD-controller)*

$$du(t) := \varphi_z^{(j)}(t, 0) dz(t), \tag{4.86a}$$

where $\varphi_z^{(j)}(t, 0)$ is defined by (4.79). Then, the following relations hold:

a) Asymptotic local stability in the mean:

$$E\Big(dz(t)|\mathcal{A}_{t_j}\Big) \to 0, t \to \infty; \tag{4.86b}$$

b) Mean absolute 1st order tracking error:

$$\begin{aligned} E\Big(\|dz\||\mathcal{A}_{t_j}\Big) \leq{}& a_0 e^{-\lambda_0(t-t_j)} \sqrt{\sigma_{z(t_j)}^{(j)2} + \|E\Big(z(t_j)|\mathcal{A}_{t_j}\Big) - \bar{z}_j\|^2} \\ &+ a_0 \int_{t_j}^{t} e^{-\lambda_0(t-\tau)} \sqrt{E\left(\|\psi^{(j,1)}(\tau)\|^2|\mathcal{A}_{t_j}\right)}\, d\tau,\ t \geq t_j, \end{aligned} \tag{4.86c}$$

where

$$E\left(\|\psi^{(j,1)}(t)\|^2|\mathcal{A}_{t_j}\right) \leq \|M^{(j)}(t)^{-1}\|^2 \sigma_{u_e}^{(j)2}\Big(s^{(j)}(t)\Big), \tag{4.86d}$$

$$\sigma_{u_e}^{(j)^2}\Big(s^{(j)}(t)\Big) \leq \|Y^{(j)}(t)\|^2\ var\ \Big(p_D(\cdot)|\mathcal{A}_{t_j}\Big) \tag{4.86e}$$

with

$$\sigma_{u_e}^{(j)}(s) := \sqrt{var\left(u_e\left(p_D(\cdot), s; q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\right)|\mathcal{A}_{t_j}\right)}, \; s_j \leq s \leq s_f. \tag{4.86f}$$

Proof Follows from (4.84a), (4.82a–d) and the fact that by Jensen's inequality $E\sqrt{X(\omega)} \leq \sqrt{EX(\omega)}$ for a nonnegative random variable $X = X(\omega)$.

Note that $\sigma_{u_e}^{(j)2}\left(s^{(j)}(t)\right)$ can be interpreted as the risk of the feedforward control $u^{(j)}(t), \; t \geq t_j$. Using (4.69b), (4.78g), (4.86d, e) and then changing variables $\tau \to s$ in the integral in (4.86c), we obtain the following result:

Theorem 4.6.2 *Let denote $t^{(j)} = t^{(j)}(s), s \geq s_j$, the inverse of the parameter transformation $s^{(j)} = s^{(j)}(t), t \geq t_j$. Under the assumptions of Theorem 4.6.1, the following inequality holds for $t_j \leq t \leq t_f^{(j)}$:*

$$E\left(\|dz(t)\||\mathcal{A}_{t_j}\right) \leq a_0 e^{-\lambda_0(t-t_j)}\sqrt{\sigma_{z(t_j)}^{(j)2} + \|E\left(z(t_j)|\mathcal{A}_{t_j}\right) - \bar{z}_j\|^2}$$
$$+ \int_{s_j}^{s^{(j)}(t)} \frac{a_0 e^{-\lambda_0\left(t-t^{(j)}(s)\right)}\|M\left(\overline{p}_D^{(j)}, q_e^{(j)}(s)\right)^{-1}\|}{\sqrt{\beta^{(j)}(s)}}\, \sigma_{u_e}^{(j)}(s)\, ds. \tag{4.87a}$$

The minimality or boundedness of the right hand side of (4.87a), hence, the robustness [43] of the present control scheme, is shown next:

Corollary 4.6.1 *The meaning of the above inequality (4.87a) follows from the following important minimality/boundedness properties depending on the chosen substitute problem in (OSTP) for the trajectory planning problem under stochastic uncertainty:*

i) The error contribution of the initial value $\bar{z}_j$ takes a minimum for $\bar{z}_j := E\left(z(t_j)|\mathcal{A}_{t_j}\right)$, cf. (4.44a, b).

ii) The factor λ_0 can be increased by an appropriate selection of the matrices K_p, K_d;

iii)

$$\underline{c}_M \leq \|M\left(p_D^{(j)}, q_e^{(j)}(s)\right)^{-1}\| \leq \overline{c}_M, \; s_j \leq s \leq s_f, \tag{4.87b}$$

with positive constants $\underline{c}_M, \overline{c}_M > 0$. This follows from the fact that the mass matrix is always positive definite [11].

iv)

$$\int_{s_j}^{s^{(j)}(t)} \frac{ds}{\sqrt{\beta^{(j)}(s)}} \le \int_{s_j}^{s_f} \frac{ds}{\sqrt{\beta^{(j)}(s)}} = t_f^{(j)} - t_j, \tag{4.87c}$$

where according to (OSTP), for minimum-time and related substitute problems, the right hand side is a minimum.

v) Depending on the chosen substitute problem in (OSTP), the generalized variance $\sigma_{u_e}^{(j)}(s),\ s_f \le s \le s_f$*, is bounded pointwise by an appropriate upper risk level, or* $\sigma_{u_e}^{(j)}(\cdot)$ *minimized in a certain weighted mean sense.*

For the minimality or boundedness of the generalized variance $\sigma_{u_e}^{(j)^2}(s), s_j \le s \le s_f$, mentioned above, we give the following examples:

Working with the probabilistic control constraints (4.30a) and assuming that the vectors u^c and ρ_u are fixed, see (4.30d), according to (4.30f) we find that (4.30a) can be guaranteed by

$$\sigma_{u_e}^{(j)^2}(s) + \|\overline{u}_e^{(j)}(s) - u^c\|^2 \le (1-\alpha_u) \min_{1\le k\le n} \rho_{uk}^2,\ s_j \le s \le s_f, \tag{4.87d}$$

where $\overline{u}_e^{(j)}(s) := u_e\Big(\overline{p}_D^{(j)}, s; q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\Big)$. Hence, with (4.87d) we have then the condition

$$\sigma_{u_e}^{(j)^2}(s) \le (1-\alpha_u) \min_{1\le k\le n} \rho_{u_e}^2,\ s_j \le s \le s_f, \tag{4.87d'}$$

cf. (4.87a). Under special distribution assumptions for $p_D(\omega)$ more exact explicit deterministic conditions for (4.30a) may be derived, see Remark 4.2.

If minimum force and moment should be achieved along the trajectory, hence, if $\phi = 0$ and $L = \|u(t)\|^2$, see (4.6), then, according to substitute problem (4.29a–f, f') we have the following minimality property:

$$\int_{s_j}^{s_f} \Big(\sigma_{u_e}^{(j)^2}(s) + \big\|\overline{u}_e^{(j)}(s)\big\|^2\Big) \frac{ds}{\sqrt{\beta^{(j)}(s)}} = \min_{q_e(\cdot),\beta(\cdot)} E\Big(\int_{t_j}^{t_f} \|u(t)\|^2\, dt|\mathcal{A}_{t_j}\Big). \tag{4.87e}$$

Mean/variance condition for u_e: Condition (4.29c) in substitute problem (4.29a–f, f') may read in case of fixed bounds $u^{\min}, u^{\max}$ for the control $u(t)$ as follows:

$$u^{\min} \le u_e\Big(\overline{p}_D^{(j)}, s; q_e(\cdot), \beta(\cdot)\Big) \le u^{\max},\ s_j \le s \le s_f \tag{4.87f}$$

$$\sigma_{u_e}^{(j)}(s) \le \sigma_{u_e}^{\max},\ s_f \le s \le s_f \tag{4.87g}$$

with a given upper bound $\sigma_{u_e}^{\max}$, cf. (4.87d').

According to the above Theorem 4.6.2, further stability results, especially the convergence

$$E\Big(\|dz(t)\||\mathcal{A}_{t_j}\Big) \to 0 \text{ for } j \to \infty, t \to \infty \tag{4.88a}$$

of the mean absolute first order tracking error can be obtained if, by using a suitable update law [4, 6, 11, 32] for the parameter estimates, hence, for the a posteriori distribution $P(\cdot|\mathcal{A}_{t_j})$, we have that, see (4.86f),

$$\text{var}\,\Big(p_D(\cdot)|\mathcal{A}_{t_j}\Big) = \text{ var}\,\Big(p_D(\cdot)|\mathcal{A}_{t_j}\Big) \to 0 \text{ for } j \to \infty. \tag{4.88b}$$

4.6.3 The 2nd Order Differential d^2q

In order to derive a representation of the second order differential d^2q, Eq. (4.77a) for $\frac{\partial q}{\partial \epsilon}(t,\epsilon)$ is represented as follows:

$$F_{p_D}\Delta p_D + F_z\frac{\partial z}{\partial \epsilon} + F_{\ddot{q}}\frac{\partial \ddot{q}}{\partial \epsilon} = \frac{\partial u}{\partial \epsilon} = \varphi_z^{(j)}\frac{\partial z}{\partial \epsilon}, \tag{4.89a}$$

where $F = F(p_D, z, \ddot{q}), z = \binom{q}{\dot{q}}$, is given by (4.73d), see also (4.76e), and therefore

$$F_{p_D} = F_{p_D}(q,\dot{q},\ddot{q}) = Y(q,\dot{q},\ddot{q}), F_{\ddot{q}} = F_{\ddot{q}}(p_D, q) = M(p_D, q) \tag{4.89b}$$

$$F_z = F_z(p_D, q,\dot{q},\ddot{q}) = \big(F_q, F_{\dot{q}}\big) = \Big(K(p_D, q,\dot{q},\ddot{q}), D(p_D, q,\dot{q})\Big). \tag{4.89c}$$

Moreover, we have that

$$\varphi^{(j)} = \varphi^{(j)}\Big(t, z - z^{(j)}(t)\Big),\ z = \begin{pmatrix} q(t,\epsilon) \\ \dot{q}(t,\epsilon) \end{pmatrix},\ p_D = p_D(\epsilon). \tag{4.89d}$$

By differentiation of (4.89a) with respect to ϵ, we obtain

$$\begin{aligned} &2F_{p_D z}\cdot\left(\Delta p_D, \frac{\partial z}{\partial \epsilon}\right) + 2F_{p_D\ddot{q}}\cdot\left(\Delta p_D, \frac{\partial \ddot{q}}{\partial \epsilon}\right) + F_{zz}\cdot\left(\frac{\partial z}{\partial \epsilon}, \frac{\partial z}{\partial \epsilon}\right) \\ &+2F_{z\ddot{q}}\cdot\left(\frac{\partial z}{\partial \epsilon}, \frac{\partial \ddot{q}}{\partial \epsilon}\right) + F_z\frac{\partial^2 z}{\partial \epsilon^2} + F_{\ddot{q}}\frac{\partial^2 \ddot{q}}{\partial \epsilon^2} \\ &\qquad\qquad = \varphi_{zz}^{(j)}\cdot\left(\frac{\partial z}{\partial \epsilon}, \frac{\partial z}{\partial \epsilon}\right) + \varphi_z^{(j)}\frac{\partial^2 z}{\partial \epsilon^2}, \end{aligned} \tag{4.90a}$$

with the 2nd order partial derivatives

$$F_{p_D z} = F_{p_D z}(z, \ddot{q}), F_{p_D \ddot{q}} = F_{p_D \ddot{q}}(q) = M_{p_D}(q) \tag{4.90b}$$

$$F_{zz} = F_{zz}(p_D, z, \ddot{q}), F_{z\ddot{q}} = F_{z\ddot{q}}(p_D, z) = \Big(M_q(p_D, q), 0\Big). \tag{4.90c}$$

Moreover, differentiation of (4.77b) with respect to ϵ yields the initial values

$$\frac{\partial^2 q}{\partial \epsilon^2}(t_j, \epsilon) = 0, \frac{d}{dt}\frac{\partial^2 q}{\partial \epsilon^2}(t_j, \epsilon) = \frac{\partial^2 \dot{q}}{\partial \epsilon^2}(t_j, \epsilon) = 0 \tag{4.90d}$$

for $\frac{\partial^2 z}{\partial \epsilon^2} = \frac{\partial^2 z}{\partial \epsilon^2}(t, \epsilon) = \left(\frac{\partial^2 q}{\partial \epsilon^2}(t, \epsilon), \frac{\partial^2 \dot{q}}{\partial \epsilon^2}(t, \epsilon)\right), t \geq t_j, 0 \leq \epsilon \leq 1.$

Putting now $\epsilon = 0$, from (4.90a) we obtain the following differential equation for the 2nd order differential $d^2 q(t) = \frac{\partial^2 q}{\partial \epsilon^2}(t, 0)$ of $q = q(t, \epsilon)$:

$$\begin{aligned}
&K^{(j)}(t) d^2 q(t) + D^{(j)}(t) \frac{d}{dt} d^2 q(t) + M^{(j)}(t) \frac{d^2}{dt^2} d^2 q(t) \\
&+ \left(F_{zz}\Big(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\Big) - \varphi_{zz}^{(j)}(t, 0)\right) \cdot \Big(dz(t), dz(t)\Big) \\
&- \varphi_z^{(j)}(t, 0) d^2 z(t) = -2 F_{p_D z}^{(j)}(t) \cdot \Big(\Delta p_D, dz(t)\Big) \\
&+ 2 F_{p_D \ddot{q}}^{(j)}(t) \cdot \Big(\Delta p_D, \ddot{d} q(t)\Big) + 2 F_{z\ddot{q}}^{(j)}(t) \cdot \Big(dz(t), \ddot{d} q(t)\Big).
\end{aligned} \tag{4.91a}$$

Here, we set

$$d^2 z(t) := \begin{pmatrix} d^2 q(t) \\ \frac{d}{dt} d^2 q(t) \end{pmatrix}, \tag{4.91b}$$

and the vectorial Hessians $F_{p_D z}^{(j)}(t), F_{p_D \ddot{q}}^{(j)}(t), F_{z\ddot{q}}^{(j)}(t)$ follow from (4.90b) by inserting there the argument $(p_D, q, \dot{q}, \ddot{q}) := \Big(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\Big)$. Furthermore, (4.90d) yields the following initial condition for $d^2 q(t)$

$$d^2 q(t_j) = 0, \frac{d}{dt} d^2 q(t_j) = 0. \tag{4.91c}$$

According to (4.91a) we define now, cf. (4.79), the second order derivative of $\varphi^{(j)}$ with respect to z at $\Delta z = 0$ by

$$\varphi_{zz}^{(j)}(t, 0) := F_{zz}\left(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\right), \ t \geq t_j. \tag{4.92}$$

Using then definition (4.79) of the Jacobian of $\varphi^{(j)}$ with respect to z at $\Delta z = 0$, for $d^2q = d^2q(t), t \geq t_j$, we find the following initial value problem

$$\frac{d^2}{dt^2}d^2q(t) + K_d\frac{d}{dt}d^2q(t) + K_p d^2q(t) \tag{4.93a}$$

$$= -M^{(j)}(t)^{-1}\widetilde{\mathbf{D}^2F}^{(j)}(t)\cdot\Big(\Delta p_D, dz(t), \ddot{d}q(t)\Big)^2, t \geq t_j,$$

$$d^2q(t_j) = 0, \frac{d}{dt}d^2q(t_j) = 0, \tag{4.93b}$$

where the sub-Hessian

$$\widetilde{\mathbf{D}^2F}^{(j)}(t) := \widetilde{\mathbf{D}^2F}\Big(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\Big), \tag{4.93c}$$

of F results from the Hessian of F by replacing the diagonal block F_{zz} by zero. Of course, we have that, cf. (4.91a),

$$\widetilde{\nabla^2F}^{(j)}\cdot\Big(\Delta_{p_D}, dz(t), \ddot{d}q(t)\Big)^2 = 2\Big(F^{(j)}_{p_Dz}(t)\cdot\Big(\Delta p_D, dz(t)\Big)$$

$$+F^{(j)}_{p_D\ddot{q}}(t)\cdot\Big(\Delta p_D, \ddot{d}q(t)\Big) + F^{(j)}_{z\ddot{q}}(t)\cdot\Big(\ddot{d}q(t), dz(t)\Big)\Big). \tag{4.93d}$$

Comparing now the initial value problems (4.80a, b) and (4.93a, b) for the first and second order differential of $q = q(t,\epsilon)$, we recognize that the linear 2nd order differential equations have—up to the right hand side—exactly the same form.

According to (4.83a–d) and (4.84a) we know that the 1st order expansion terms

$$\Big(dz(t), \ddot{d}q(t)\Big) = \Big(dq(t), \dot{d}q(t), \ddot{d}q(t)\Big),\ t \geq t_j,$$

in the tracking error depend linearly on the error term

$$\Delta\theta^{(j)} = \Delta\theta^{(j)}(t) := \begin{pmatrix} e^{A(t-t_j)}\Delta z_j \\ \Delta p_D \end{pmatrix}\left(\to \begin{pmatrix} 0 \\ \Delta p_D \end{pmatrix}, t \to \infty\right) \tag{4.94}$$

corresponding to the variations/disturbances of the initial values $(q_j, \dot{q}_j)$ and dynamic parameters p_D. Consequently, we have this observation:

Lemma 4.2 *The right hand side*

$$\psi^{(j,2)}(t) := -M^{(j)}(t)^{-1}\widetilde{\mathbf{D}^2F}^{(j)}(t)\cdot\Big(\Delta p_D, dz(t), \ddot{d}q(t)\Big)^2, t \geq t_j, \tag{4.95}$$

of the error differential equation (4.93a) for $d^2q(t)$ is quadratic in the error term $\Delta\theta^{(j)}(\cdot)$.

According to (4.93a–c), the second order expansion term

$$d^2z(t) = \left(d^2q(t), \frac{d}{dt}d^2q(t)\right), t \geq t_j,$$

of the Taylor expansion of the tracking error can be represented again by the solution of the system of linear differential equations (4.83a–c), where now $\psi^{(j,1)}(t)$ is replaced by $\psi^{(j,2)}(t)$ defined by (4.95), and the initial values are given by $d^2z(t_j) = 0$. Thus, applying again solution formula (4.84a), we find

$$d^2z(t) = \int_{t_j}^{t} e^{A(t-\tau)} \begin{pmatrix} 0 \\ \psi^{(j,2)}(\tau) \end{pmatrix} d\tau. \tag{4.96}$$

From (4.94)–(4.96) and Lemma 4.2 we get now the following result:

Theorem 4.6.3 *The second order tracking error expansion terms $\left(d^2z(t), \frac{d^2}{dt^2}d^2q(t)\right) = \left(d^2q(t), \frac{d}{dt}d^2q(t), \frac{d^2}{dt^2}d^2q(t)\right), t \geq t_j$, depend*

i) quadratically on the 1st order error terms $\left(\Delta p_D, dz(t), d\ddot{q}(t)\right)$ and
ii) quadratically on the error term $\Delta\theta^{(j)}(\cdot)$ corresponding to the variations/ disturbances of the initial values and dynamics parameters.

Because of (4.96), the stability properties of the second order tracking error expansion term $d^2q(t), t \geq t_j$, are determined again by the matrix exponential function $\Phi(t,\tau) = e^{A(t-\tau)}$ and the remainder $\psi^{(j,2)}(t)$ given by (4.95).

According to Theorem 4.6.2 and Corollary 4.6.1 we know that the disturbance term $\psi^{(j,1)}(t)$ of (4.80a), (4.83a) is reduced directly by control-constraints (for u_e) present in (OSTP). Concerning the next disturbance term $\psi^{(j,2)}(t)$ of (4.93a), by (4.95) we note first that a reduction of the 1st order error terms $\left(\Delta p_D, dz(\cdot), d\ddot{q}(\cdot)\right)$ yields a reduction of $\psi^{(j,2)}$ and by (4.96) also a reduction of $d^2z(t), \frac{d^2}{dt^2}d^2q(t), t \geq t_j$. Comparing then definitions (4.83d) and (4.95) of the disturbances $\psi^{(j,1)}, \psi^{(j,2)}$, we observe that, corresponding to $\psi^{(j,1)}$, certain terms in $\psi^{(j,2)}$ depend only on the reference trajectory $q^{(j)}(t), t \geq t_j$, of stage j. Hence, this observation yields the following result.

Theorem 4.6.4 *The disturbance $\psi^{(j,2)}$ of (4.93a), and consequently also the second order tracking error expansion terms $d^2q(t), \frac{d}{dt}d^2q(t), \frac{d^2}{dt^2}d^2q(t),\ t \geq t_j$, can be diminished by*

i) reducing the 1st order error terms $\left(\Delta p_D, dz(\cdot), \ddot{q}(\cdot)\right)$, and by
ii) taking into (OSTP) additional conditions for the unknown functions $q_e(s), \beta(s), s_j \leq s \leq s_f$, guaranteeing that (the norm of) the sub-Hessian $\widetilde{\mathbf{D}^2 F}^{(j)}(t),\ t \geq t_j$, fulfills a certain minimality or boundedness condition.

Proof Follows from definition (4.95) of $\psi^{(j,2)}$ and representation (4.96) of the second order tracking error expansion term d^2z.

4.6.4 Third and Higher Order Differentials

By further differentiation of Eqs. (4.90a, d) with respect to ϵ and by putting $\epsilon = 0$, also the 3rd and higher order differentials $d^lq(t),\ t \geq t_j, l \geq 3$, can be obtained. We observe that the basic structure of the differential equations for the differentials $d^lq, l \geq 1$, remains the same. Hence, by induction for the differentials $d^lz, l \geq 1$, we have the following representation:

Theorem 4.6.5 *Defining the tensorial coefficients of the Taylor expansion (4.75d) for the feedback control law $\varphi^{(j)} = \varphi^{(j)}(t, \Delta z)$ by*

$$D_z^l\varphi^{(j)}(t,0) := D_z^l F\left(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\right),\ t \geq t_j, l = 1,2,\ldots, \tag{4.97}$$

the differentials $d^lz(t) = \left(d^lq(t), \frac{d}{dt}d^lq(t)\right),\ t \geq t_j$, may be represented by the systems of linear differential equations

$$\frac{d}{dt}d^lz(t) = Ad^lz(t) + \begin{pmatrix} 0 \\ \psi^{(j,l)}(t) \end{pmatrix},\ t \geq t_j, l = 1,2,\ldots, \tag{4.98a}$$

with the same system matrix A and the disturbance terms $\psi^{(j,l)}(t),\ t \geq t_j$, given by

$$\psi^{(j,l)}(t) = -M^{(j)}(t)^{-1}\pi\Big(\widetilde{\mathbf{D}^\lambda F}^{(j)}(t), 2 \leq \lambda \leq l; \Delta p_D, d^jz(t),$$
$$\frac{d^2}{dt^2}d^kq(t), 1 \leq j,k \leq l-1\Big),\ l \geq 2, \tag{4.98b}$$

where π is a polynomial in the variables Δp_D and $d^j z(t), \frac{d^2}{dt^2} d^k q(t), 1 \le j,k \le l-1$, having coefficients from the sub-operators $\widetilde{\mathbf{D}^\lambda} F^{(j)}(t)$ of $\mathbf{D}^\lambda F^{(j)}(t)$ containing mixed partial derivatives of F with respect to $\Delta p_D, z, \ddot{q}$ of order $\lambda = 2, 3, \ldots, l-1$ at $(p_D, q, \dot{q}, \ddot{q}) = \left(\overline{p_D}^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\right)$ such that the disturbance $\psi^{(j,l)}$ is a polynomial of order l with respect to the error term $\Delta\theta^{(j)}(\cdot)$.

According to (4.73d), (4.4a–d) and Remark 4.2 we know that the vector function $F = F(p_D, q, \dot{q}, \ddot{q})$ is linear in p_D, linear in $\ddot{q}$ and quadratic in $\dot{q}$ (supposing case 4.4c)) and analytical with respect to q. Hence, working with a polynomial approximation with respect to q, we may assume that

$$\mathbf{D}^l F^{(j)}(t) = \mathbf{D}^l F\left(\overline{p_D}^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\right) \approx 0,\ t \ge t_j, l \ge l_0 \tag{4.99}$$

for some index l_0.

According to the expansion (4.75d) of the feedback control law $\varphi^{(j)} = \varphi^{(j)}(t, \Delta z)$, definition (4.97) of the corresponding coefficients and Theorem 4.6.4 we have now this robustness [43] result:

Theorem 4.6.6 *The Taylor expansion (4.75d) of the feedback control law $\varphi^{(j)} = \varphi^{(j)}(t, \Delta z)$ stops after a finite number ($\le l_0$) of terms. Besides the conditions for u_e contained automatically in (OSTP) via the control constraints, the mean absolute tracking error $E\left(\|\Delta z(t)\| \,|\mathcal{A}_{t_j}\right)$ can be diminished further be including additional conditions for the functions $\left(q_e(s), \beta(s)\right)$, $s_j \le s \le s_f$, in (OSTP) which guarantee a minimality or boundedness condition for (the norm of) the sub-operators of mixed partial derivatives $\widetilde{\mathbf{D}^2} F^{(j)}(t)$, $t \ge t_j, \lambda = 2, 3, \ldots, l_0$.*

4.7 Online Control Corrections: PID Controllers

Corresponding to Sect. 4.5, Eq. (4.70a–c), at stage j we consider here control corrections, hence, feedback control laws, of the type

$$\Delta u^{(j)}(t) := u(t) - u^{(j)}(t) = \varphi^{(j)}\big(t, \Delta z^{(j)}(t)\big), t \ge t_j, \tag{4.100a}$$

where

$$\Delta z^{(j)}(t) := z(t) - z^{(j)}(t), t \ge t_j,$$

is the tracking error related now to the state vector

$$z(t) := \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{pmatrix} = \begin{pmatrix} q(t) \\ \int\limits_{t_j}^{t} q(s)ds \\ \dot{q}(t) \end{pmatrix}, t \geq t_j \tag{4.100b}$$

$$z^{(j)}(t) := \begin{pmatrix} z_1^{(j)}(t) \\ z_2^{(j)}(t) \\ z_3^{(j)}(t) \end{pmatrix} = \begin{pmatrix} q^{(j)}(t) \\ \int\limits_{t_j}^{t} q^{(j)}(s)ds \\ \dot{q}^{(j)}(t) \end{pmatrix}, t \geq t_j. \tag{4.100c}$$

Furthermore,

$$\begin{aligned} \varphi^{(j)} &= \varphi^{(j)}\big(t, \Delta z(t)\big) \\ &= \varphi^{(j)}\big(t, \Delta z_1(t), \Delta z_2(t), \Delta z_3(t)\big), t \geq t_j, \end{aligned} \tag{4.100d}$$

is a feedback control law such that

$$\varphi^{(j)}(t, 0, 0, 0) = 0, t \geq t_j.,\tag{4.100e}$$

Of course, we have

$$\Delta z(t) = \Delta z^{(j)}(t) = \begin{pmatrix} q(t) - q^{(j)}(t) \\ \int\limits_{t_j}^{t} \big(q(s) - q^{(j)}(s)\big)ds \\ \dot{q}(t) - \dot{q}^{(j)}(t) \end{pmatrix}, t \geq t_j. \tag{4.101}$$

Corresponding to Sect. 4.5, the trajectories $q = q(t)$ and $q^{(j)} = q^{(j)}(t), t \geq t_j$, are embedded into a one-parameter family of trajectories $q = q(t, \varepsilon), t \geq t_j, 0 \leq \varepsilon \leq 1$, in configuration space which is defined as follows.

At stage j, here we have the following **initial data**:

$$q_j(\varepsilon) := \overline{q}_j + \varepsilon \Delta q_j, \Delta q_j := q_j - \overline{q}_j \tag{4.102a}$$

$$\dot{q}_j(\varepsilon) := \overline{\dot{q}}_j + \varepsilon \Delta \dot{q}_j, \Delta \dot{q}_j := \dot{q}_j - \overline{\dot{q}}_j \tag{4.102b}$$

$$p_D(\varepsilon) := \overline{p}_D^{(j)} + \varepsilon \Delta p_D, \Delta p_D := p_D - \overline{p}_D^{(j)}, \tag{4.102c}$$

$0 \leq \varepsilon \leq 1$. Moreover, the control input $u = u(t), t \geq t_j$, is defined by

$$\begin{aligned} u(t) &= u^{(j)}(t) + \Delta u^{(j)}(t) \\ &= u^{(j)}(t) + \varphi^{(j)}\Big(t, q(t) - q^{(j)}(t), \int_{t_j}^{t} \big(q(s) - q^{(j)}(s)\big)ds, \\ &\qquad \dot{q}(t) - \dot{q}^{(j)}(t)\Big), t \geq t_j. \end{aligned} \tag{4.102d}$$

Let denote now

$$q(t,\varepsilon) = q\big(t, p_D(\varepsilon), q_j(\varepsilon), \dot{q}_j(\varepsilon), u(\cdot)\big), t \geq t_j, 0 \leq \varepsilon \leq 1, \tag{4.103}$$

the solution of following initial value problem based on the dynamic equation (4.4a) having the initial values and total control input $u(t)$ given by (4.102a–d):

$$F\big(p_D(\varepsilon), q(t,\varepsilon), \dot{q}(t,\varepsilon), \ddot{q}(t,\varepsilon)\big) = u(t,\varepsilon), t \geq t_j, 0 \leq \varepsilon \leq 1, \tag{4.104a}$$

where

$$q(t_j,\varepsilon) := q_j(\varepsilon), \dot{q}(t_j,\varepsilon) = \dot{q}_j(\varepsilon) \tag{4.104b}$$

$$\begin{aligned} u(t,\varepsilon) &:= u^{(j)}(t) + \varphi^{(j)}\Big(t, q(t,\varepsilon) - q^{(j)}(t), \int_{t_j}^{t} \big(q(s,\varepsilon) - q^{(j)}(s)\big)ds, \\ &\qquad \dot{q}(t,\varepsilon) - \dot{q}^{(j)}(t)\Big), \end{aligned} \tag{4.104c}$$

and the vector function $F = F(p_D, q, \dot{q}, \ddot{q})$ is again defined by

$$F(p_D, q, \dot{q}, \ddot{q}) = M(p_D, q)\ddot{q} + h(p_D, q, \dot{q}), \tag{4.104d}$$

cf. (4.104a).

In the following we assume that problem (4.104a–d) has a unique solution $q = q(t,\varepsilon), t \geq t_j$, for each parameter $\varepsilon, 0 \leq \varepsilon \leq 1$.

4.7.1 Basic Properties of the Embedding $q(t, \varepsilon)$

According to (4.102a–c), for $\varepsilon = 0$ system (4.104a–d) reads

$$F\left(\overline{p}_D^{(j)}, q(t,0), \dot{q}(t,0), \ddot{q}(t,0)\right)$$
$$= u^{(j)}(t) + \varphi^{(j)}\Big(t, q(t,0) - q^{(j)}(t), \int_{t_j}^{t} \left(q(s,0) - q^{(j)}(s)\right) ds,$$
$$\dot{q}(t,0) - \dot{q}^{(j)}(t)\Big), t \geq t_j, \tag{4.105a}$$

where

$$q(t_j, 0) = \overline{q}_j, \dot{q}(t_j, 0) = \overline{\dot{q}}_j. \tag{4.105b}$$

Since, due to (OSTP),

$$q^{(j)}(t_j) = \overline{q}_j, \dot{q}^{(j)}(t_j) = \overline{\dot{q}_j}$$

and

$$F\left(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t) = u^{(j)}(t), t \geq t_j,\right.$$

according to the unique solvability assumption for (4.104a–d) and condition (4.100e) we have

$$q(t,0) = q^{(j)}(t), t \geq t_j. \tag{4.106}$$

For $\varepsilon = 1$, from (4.102a–c) and (4.104a–d) we obtain the system

$$F\left(p_D, q(t,1), \dot{q}(t,1), \ddot{q}(t,1)\right)$$
$$= u^{(j)}(t) + \varphi^{(j)}\Big(t, q(t,1) - q^{(j)}(t), \int_{t_j}^{t} \left(q(s,1) - q^{(j)}(s)\right) ds, \dot{q}(t,1) - \dot{q}^{(j)}(t)\Big),$$
$$t \geq t_1 \tag{4.107a}$$

with

$$q(t_j, 1) = q_j, \dot{q}(t_j, 1) = \dot{q}_j. \tag{4.107b}$$

However, since the control input is defined by (4.102d), again due to the unique solvability property of (4.104a–d), (4.107a, b) yields

$$q(t,1) = q(t), t \geq t_j, \tag{4.108}$$

where $q = q(t)$ denotes the **actual trajectory.**

Remark 4.13 The integro-differential equation (4.104a–d) can be easily converted into an ordinary initial value problem. Indeed, using the state variables

$$z(t,\varepsilon) = \begin{pmatrix} q(t;\varepsilon) \\ q_I(t,\varepsilon) \\ \dot{q}(t,\varepsilon) \end{pmatrix}, t \geq t_j,$$

where $q_I = q_I(t,\varepsilon), t \geq t_j$, is defined, see (4.100b), by

$$q_I(t,\varepsilon) := \int_{t_j}^{t} q(s,\varepsilon)ds, \tag{4.109}$$

problem (4.104a–d) can be represented by the equivalent 2nd order initial value problem:

$$\begin{aligned} &F\big(p_D(\varepsilon), q(t,\varepsilon), \dot{q}(t,\varepsilon), \ddot{q}(t,\varepsilon)\big) \\ &\quad = u^{(j)}(t) + \varphi^{(j)}\big(t, q(t,\varepsilon) - q^{(j)}(t), q_I(t,\varepsilon) - q_I^{(j)}(t), \\ &\qquad\qquad \dot{q}(t,\varepsilon) - \dot{q}^{(j)}(t)\big) \end{aligned} \tag{4.110a}$$

$$\dot{q}_I(t,\varepsilon) := q(t,\varepsilon) \tag{4.110b}$$

with

$$q(t_j,\varepsilon) = q_j(\varepsilon) \tag{4.110c}$$

$$\dot{q}(t_j,\varepsilon) = \dot{q}_j(\varepsilon) \tag{4.110d}$$

$$q_I(t_j,\varepsilon) = 0. \tag{4.110e}$$

and

$$q_I^{(j)}(t) := \int_{t_j}^{t} q^{(j)}(s)ds. \tag{4.111}$$

4.7.2 Taylor Expansion with Respect to ε

Based on representation (4.110a–e) of problem (4.104a–d), we may again assume, cf. Assumption 1.1, that the solution $q = q(t,\varepsilon), t \geq t_j, 0 \leq \varepsilon \leq 1$, has continuous derivatives with respect to ε up to a certain order $\nu \geq 1$ for all $t \in [t_j, t_j + \Delta t_j], 0 \leq \varepsilon \leq 1$, with a certain $\Delta t_j > 0$.

Corresponding to (4.75a–c), the actual trajectory of the robot can be represented then, see (4.108), (4.106), (4.103), by

$$
\begin{aligned}
q(t) &= q\big(t, p_D, q_j, \dot{q}_j, u(\cdot)\big) = q(t,1) = q(t, \varepsilon_0 + \Delta\varepsilon) \\
&= q(t,\varepsilon_0) + \Delta q(t) = q^{(j)}(t) + \Delta q(t),
\end{aligned} \tag{4.112a}
$$

with $\varepsilon_0 = 0, \Delta\varepsilon = 1$. Moreover, the expansion on the tracking error $\Delta q = \Delta q(t), t \geq t_j$, is given by

$$
\begin{aligned}
\Delta q(t) &= \sum_{l=1}^{\nu-1} \frac{1}{l!} d^l q(t)(\Delta\varepsilon)^l + \frac{1}{\nu!}\frac{\partial^\nu}{\partial\varepsilon^\nu} q(t,\vartheta)(\Delta\varepsilon)^\nu \\
&= \sum_{l=1}^{\nu-1} \frac{1}{l!} d^l q(t) + \frac{1}{\nu!}\frac{\partial^\nu q}{\partial\varepsilon^\nu}(t,\vartheta), t \geq t_j.
\end{aligned} \tag{4.112b}
$$

Here, $\vartheta = \vartheta(t,\nu), 0 < \vartheta < 1$, and

$$
d^l q(t) := \frac{\partial^l q}{\partial\varepsilon^l}(t,0), t \geq t_j, l = 1,2,\ldots, \tag{4.112c}
$$

denote the l-th order differentials of $q = q(t,\varepsilon)$ with respect to ε at $\varepsilon = \varepsilon_0 = 0$. Differential equations for the differentials $d^l q(t), l = 1,2,\ldots$, may be obtained by successive differentiation of the initial value problem (4.104a–d) with respect to ε at $\varepsilon = 0$.

4.7.3 The 1st Order Differential dq

Corresponding to Sect. 4.6.2, we consider now the partial derivative in Eq. (4.110a–e) with respect to ε. Let

$$
K(p_D, q, \dot{q}, \ddot{q}) := F_q(p_D, q, \dot{q}, \ddot{q}) \tag{4.113a}
$$

$$
D(p_D, q, \dot{q}) := F_{\dot{q}}(p_D, q, \dot{q}, \ddot{q}) = h_{\dot{q}}(p_D, q, \dot{q}) \tag{4.113b}
$$

$$
Y(q, \dot{q}, \ddot{q}) := F_{p_D}(p_D, q, \dot{q}, \ddot{q}) \tag{4.113c}
$$

$$
M(p_D, q) := F_{\ddot{q}}(p_D, q, \dot{q}, \ddot{q}) \tag{4.113d}
$$

denote again the Jacobians of the vector function $F = F(p_D, q, \dot{q}, \ddot{q})$ with respect to $q, \dot{q}, \ddot{q}$ and p_D. According to the linear parametrization property of robots we have, see (4.76e)

$$
F(p_D, q, \dot{q}, \ddot{q}) = Y(q, \dot{q}, \ddot{q}) p_D. \tag{4.113e}
$$

Taking the partial derivative with respect to ε, from (4.110a–e) we obtain the following equations, cf. (4.77a, b),

$$
\begin{aligned}
& Y\big(q(t,\varepsilon),\dot q(t,\varepsilon),\ddot q(t,\varepsilon)\big)\Delta p_D + K\big(p_D(\varepsilon),\dot q(t,\varepsilon),\ddot q(t,\varepsilon)\big)\frac{\partial q}{\partial\varepsilon}(t,\varepsilon)\\
&+ D\big(p_D(\varepsilon),q(t,\varepsilon),\dot q(t,\varepsilon),\ddot q(t,\varepsilon)\big)\frac{d}{dt}\frac{\partial q}{\partial\varepsilon}(t,\varepsilon) + M\big(p_D(\varepsilon),q(t,\varepsilon)\big)\frac{d^2}{dt^2}\frac{\partial q}{\partial\varepsilon}(t,\varepsilon)\\
&= \varphi_q^{(j)}\big(t,q(t,\varepsilon)-q^{(j)}(t),q_I(t,\varepsilon)-q_I^{(j)}(t),\dot q(t,\varepsilon)-\dot q^{(j)}(t)\big)\frac{\partial q}{\partial\varepsilon}(t,\varepsilon)\\
&+ \varphi_{q_I}^{(j)}\big(t,q(t,\varepsilon)-q^{(j)}(t),q_I(t,\varepsilon)-q_I^{(j)}(t),\dot q(t,\varepsilon)-\dot q^{(j)}(t)\big)\frac{\partial q_I}{\partial\varepsilon}(t,\varepsilon)\\
&+ \varphi_{\dot q}^{(j)}\big(t,q(t,\varepsilon)-q^{(j)}(t),q_I(t,\varepsilon)-q_I^{(j)}(t),\dot q(t,\varepsilon)-\dot q^{(j)}(t)\big)\frac{d}{dt}\frac{\partial q}{\partial\varepsilon}(t,\varepsilon).
\end{aligned} \tag{4.114a}
$$

$$\frac{d}{dt}\frac{\partial q_I}{\partial\varepsilon}(t,\varepsilon) = \frac{\partial q}{\partial\varepsilon}(t,\varepsilon) \tag{4.114b}$$

$$\frac{\partial q}{\partial\varepsilon}(t_j,\varepsilon) = \Delta q_j \tag{4.114c}$$

$$\frac{d}{dt}\frac{\partial q}{\partial\varepsilon}(t_j,\varepsilon) = \dot\Delta q_j \tag{4.114d}$$

$$\frac{\partial q_I}{\partial\varepsilon}(t,\varepsilon) = 0. \tag{4.114e}$$

Putting now $\varepsilon = \varepsilon_0 = 0$, due to (4.106), (4.109), 4.112c) we obtain

$$
\begin{aligned}
& Y^{(j)}(t)\Delta p_D + K^{(j)}(t)dq(t) + D^{(j)}(t)\dot d q(t) + M^{(j)}(t)\ddot d q(t)\\
&= \varphi_q^{(j)}(t,0,0,0)dq(t) + \varphi_{q_I}^{(j)}(t,0,0,0)\int_{t_j}^{t} dq(s)ds + \varphi_{\dot q}^{(j)}(t,0,0,0)\dot d q(t),\ t\ge t_j
\end{aligned} \tag{4.115a}
$$

$$dq(t_j) = \Delta q_j \tag{4.115b}$$

$$\dot d q(t_j) = \dot\Delta q_j, \tag{4.115c}$$

where

$$\dot d q := \frac{d}{dt}dq, \quad \ddot d q := \frac{d^2}{dt^2}dq. \tag{4.116a}$$

From (4.114b, e) we obtain

$$\frac{\partial q_I}{\partial \varepsilon}(t,0) = \int_{t_j}^{t} dq(s)ds, \tag{4.116b}$$

which is already used in (4.115a). Moreover, the matrices $Y^{(j)}$, $K^{(j)}$, $D^{(j)}$ and $M^{(j)}$ are defined as in Eqs. (4.78e–g) of Sect. 4.6.2, hence,

$$Y^{(j)}(t) := Y\big(q^{(j)}(t), \dot{q}^{(j)}(t)\ddot{q}^{(j)}(t)\big) \tag{4.117a}$$

$$K^{(j)}(t) := K\big(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\big) \tag{4.117b}$$

$$D^{(j)}(t) := D\big(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t)\big) \tag{4.117c}$$

$$M^{(j)}(t) := M\big(\overline{p}_D^{(j)}, q^{(j)}(t)\big), t \geq t_j, \tag{4.117d}$$

see also (4.113a–d).

Multiplying now (4.115a) with the inverse $M^{(j)}(t)^{-1}$ of $M^{(j)}(t)$ and rearranging terms, we get

$$\begin{aligned} \ddot{d}q(t) &+ M^{(j)}(t)^{-1}\big(D^{(j)}(t) - \varphi_{\dot{q}}^{(j)}(t,0,0,0)\big)\dot{d}q(t) \\ &+ M^{(j)}(t)^{-1}\big(K^{(j)}(t) - \varphi_{q}^{(j)}(t,0,0,0)\big)dq(t) \\ &- M^{(j)}(t)^{-1}\varphi_{q_I}^{(j)}(t,0,0,0)\int_{t_j}^{t} dq(s)ds = -M^{(j)}(t)^{-1}Y^{(j)}\Delta p_D, t \geq t_j \end{aligned} \tag{4.118}$$

with the initial conditions (4.115b, c).

For given matrices K_d, K_p, K_i to be selected later on, the unknown Jacobians $\varphi_{\dot{q}}^{(j)}, \varphi_{q}^{(j)}$ and $\varphi_{q_I}^{(j)}$ are defined now by the equations

$$M^{(j)}(t)^{-1}\big(D^{(j)}(t) - \varphi_{\dot{q}}^{(j)}(t,0,0,0)\big) = K_d \tag{4.119a}$$

$$M^{(j)}(t)^{-1}\big(K^{(j)}(t) - \varphi_{q}^{(j)}(t,0,0,0)\big) = K_p \tag{4.119b}$$

$$M^{(j)}(t)^{-1}\big(-\varphi_{q_I}^{(j)}(t,0,0,0)\big) = K_i. \tag{4.119c}$$

Thus, we have

$$\varphi_{q}^{(j)}(t,0,0,0) = K^{(j)}(t) - M^{(j)}(t)K_p \tag{4.120a}$$

$$\varphi_{q_I}^{(j)}(t,0,0,0) = \qquad -M^{(j)}(t)K_i \tag{4.120b}$$

$$\varphi_{\dot{q}}^{(j)}(t,0,0,0) = D^{(j)}(t) - M^{(j)}(t)K_d. \tag{4.120c}$$

Putting (4.119a–c) into system (4.118), we find

$$\ddot{d}q(t) + K_d \dot{d}q(t) + K_p dq(t) + K_i \int_{t_j}^{t} dq(s)ds = \psi^{(j,1)}(t), t \geq t_j, \tag{4.121a}$$

with

$$dq(t_j) = \Delta q_j \tag{4.121b}$$

$$\dot{d}q(t_j) = \dot{\Delta} q_j, \tag{4.121c}$$

where $\psi^{(j,i1}(t)$ is given, cf. (4.83d), by

$$\psi^{(j,1)}(t) := -M^{(j)}(t)^{-1} Y^{(j)}(t) \Delta p_D. \tag{4.121d}$$

If K_p, K_d and K_i are fixed matrices, then by differentiation of the integro-differential equation (4.121a) with respect to time t we obtain the following 3rd order system of linear differential equations

$$\dddot{d}q(t) + K_d \ddot{d}q(t) + K_p \dot{d}q(t) + K_i dq(t) = \dot{\psi}^{(j,1)}(t), t \geq t_j, \tag{4.122a}$$

for the 1st order tracking error term $dq = dq(t), t = t_j$. Moreover, the initial conditions read, see (4.121b, c),

$$dq(t_j) = \Delta q_j \tag{4.122b}$$

$$\dot{d}q(t_j) = \dot{\Delta} q_j \tag{4.122c}$$

$$\ddot{d}q(t_j) = \ddot{\Delta} q_j := \ddot{q}_j - \ddot{q}^{(j)}, \tag{4.122d}$$

where $\ddot{q}_j := \ddot{q}(t_j)$, see (4.102a–c).

Corresponding to (4.83a–c), the system (4.121a) of 3rd order linear differential equations can be converted easily into the following system of 1st order differential equations

$$\dot{d}z(t) = Adz(t) + \begin{pmatrix} 0 \\ 0 \\ \dot{\psi}^{(j,1)}(t) \end{pmatrix}, t \geq t_j, \tag{4.123a}$$

where

$$A := \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -K_i & -K_p & -K_d \end{pmatrix}, \tag{4.123b}$$

$$dz(t_j) := \begin{pmatrix} \Delta q_j \\ \dot{\Delta} q_j \\ \ddot{\Delta} q_j \end{pmatrix} = \Delta z_j \tag{4.123c}$$

and

$$dz(t) := \begin{pmatrix} dq(t) \\ \dot{d}q(t) \\ \ddot{d}q(t) \end{pmatrix}. \tag{4.123d}$$

With the fundamental matrix $\Phi(t,\tau) := e^{A(t-\tau)}, t \geq \tau$, the solution of (4.123a–d) reads

$$dz(t) = e^{A(t-t_j)}\Delta z_j + \int_{t_j}^{t} e^{A(t-\tau)} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi}^{(j,1)}(\tau) \end{pmatrix} d\tau, t \geq t_j, \tag{4.124a}$$

where, see (4.121d),

$$\dot{\psi}^{(j,1)}(t) = -\frac{d}{dt}\Big(M^{(j)}(t)^{-1}Y^{(j)}(t)\Big)\Delta p_D. \tag{4.124b}$$

Because of

$$\Delta p_D = p_D(\omega) - \overline{p}_D^{(j)} = p_D(\omega) - E\Big(p_D(\omega)|\mathcal{A}_{t_j}\Big),$$

see (4.102c), for the conditional mean 1st order error term $E\Big(dz(t)|\mathcal{A}_{t_j}\Big)$ from (4.124a, b) we get

$$E\Big(dz(t)|\mathcal{A}_{t_j}\Big) = e^{A(t-t_j)}E(\Delta z_j|\mathcal{A}), t \geq t_j. \tag{4.124c}$$

Obviously, the properties of the 1st order error terms $dz(t), E(dz(t)|\mathcal{A}_{t_j})$, resp., $t \geq t_j$, or the stability properties of the 1st order system (4.123a–d) depend on the eigenvalues of the matrix A.

Diagonalmatrices K_p, K_d, K_i

Supposing here, cf. Sect. 4.5, that K_p, K_d, K_i are diagonal matrices

$$K_p = (\gamma_{pk}\delta_{k\kappa}), K_d = (\gamma_{dk}\delta_{k\kappa}), K_i = (\gamma_{ik}\delta_{k\kappa}) \tag{4.125}$$

with diagonal elements $\gamma_{pk}, \gamma_{dk}, \gamma_{ik}$, resp., $k = 1, \ldots, n$, system (4.122a) is divided into the separated ordinary differential equations

$$\dddot{d}q_k(t) + \gamma_{dk}\ddot{d}q_k(t) + \gamma_{pk}\dot{d}q_k(t) + \gamma_{ik}dq_k(t) = \dot{\psi}_k^{(j,1)}(t), t \geq t_j, \tag{4.126a}$$

$k = 1, 2, \ldots, n$. The related homogeneous differential equations read

$$\dddot{d}q_k + \gamma_{dk}\ddot{d}q_k + \gamma_{pk}\dot{d}q_k + \gamma_{ik}dq_k = 0, \tag{4.126b}$$

$k = 1, \ldots, n$, which have the characteristic equations

$$\lambda^3 + \gamma_{dk}\lambda^2 + \gamma_{pk}\lambda + \gamma_{ik} =: p_k(\lambda) = 0, k = 1, \ldots, n, \tag{4.126c}$$

with the polynomials $p_k = p_k(\lambda), k = 1, \ldots, n$, of degree $= 3$.

A system described by the homogeneous differential equation (4.126b) is called **uniformly (asymptotic) stable** if

$$\lim_{t \to \infty} dq_k(t) = 0 \tag{4.127a}$$

for arbitrary initial values (4.122b–d). It is well known that property (4.127a) holds if

$$Re(\lambda_{kl}) < 0, l = 1, 2, 3, \tag{4.127b}$$

where $Re(\lambda_{kl})$ denotes the real part of the zeros $\lambda_{k1}, \lambda_{k2}, \lambda_{k3}$ of the characteristic equation (4.126c). According to the **Hurwitz criterion**, a necessary and sufficient condition for (4.127b) is the set of inequalities

$$\det(\gamma_{dk}) > 0 \tag{4.128a}$$

$$\det\begin{pmatrix} \gamma_{dk} & 1 \\ \gamma_{ik} & \gamma_{pk} \end{pmatrix} = \gamma_{dk}\gamma_{pk} - \gamma_{ik} > 0 \tag{4.128b}$$

$$\det\begin{pmatrix} \gamma_{dk} & 1 & 0 \\ \gamma_{ik} & \gamma_{pk} & \gamma_{dk} \\ 0 & 0 & \gamma_{ik} \end{pmatrix} = \gamma_{ik}(\gamma_{dk}\gamma_{pk} - \gamma_{ik}) > 0. \tag{4.128c}$$

Note that

$$H_{3k} := \begin{pmatrix} \gamma_{dk} & 1 & 0 \\ \gamma_{ik} & \gamma_{pk} & \gamma_{dk} \\ 0 & 0 & \gamma_{ik} \end{pmatrix} \tag{4.128d}$$

is the so-called **Hurwitz-matrix** of (4.126b).

Obviously, from (4.128a–c) we obtain now this result:

Theorem 4.7.1 *The system represented by the homogeneous 3rd order linear differential equation (4.126b) is uniformly (asymptotic) stable if the (feedback) coefficients $\gamma_{pk}, \gamma_{dk}, \gamma_{ik}$ are selected such that*

$$\gamma_{pk} > 0, \gamma_{dk} > 0, \gamma_{ik} > 0 \tag{4.129a}$$

$$\gamma_{dk}\gamma_{pk} > \gamma_{ik}. \tag{4.129b}$$

Mean Absolute 1st Order Tracking Error

Because of $E\Big(\Delta p_D(\omega)|\mathcal{A}_{t_j}\Big) = 0$ and the representation (4.124b) of $\dot{\psi}^{(j,1)} = \dot{\psi}^{(j,1)}(t)$, corresponding to (4.84b) we have

$$E\Big(\dot{\psi}^{(j,1)}(t)|\mathcal{A}_{t_j}\Big) = 0, t \geq t_j. \tag{4.130a}$$

Hence, (4.124a) yields

$$E\Big(dz(t)|\mathcal{A}_{t_j}\Big) = e^{A(t-t_j)} E\Big(\Delta z_j|\mathcal{A}_{t_j}\Big), \tag{4.130b}$$

where

$$E\Big(\Delta z_j|\mathcal{A}_{t_j}\Big) = E\Big(z(t_j)|\mathcal{A}_{t_j}\Big) - \bar{z}_j \tag{4.130c}$$

with, cf. (4.105a), (4.122d),

$$\bar{z}_j = \begin{pmatrix} \overline{q}_j \\ \dot{\overline{q}}_j \\ \ddot{q}^{(j)} \end{pmatrix}, z(t_j) = \begin{pmatrix} q_j \\ \dot{q}_j \\ \ddot{q}_j \end{pmatrix}. \tag{4.130d}$$

The matrices K_p, K_i, K_d in the definition (4.119a–c) or (4.120a–c) of the Jacobians $\varphi_q^{(j)}(t, 0, 0, 0), \varphi_{qI}^{(j)}(t, 0, 0, 0), \varphi_{\dot{q}}^{(j)}(t, 0, 0, 0)$ of the feedback control law

$\Delta u^{(j)}(t) = \varphi^{(j)}(t, \Delta q, \Delta q_I, \dot{\Delta q})$ can be chosen now, see Theorem 4.7.1, such that the fundamental matrix $\Phi(t,\tau) = e^{A(t-\tau)}, t \geq \tau$, is exponentially stable, hence,

$$\|\Phi(t,\tau)\| \leq a_0 e^{-\lambda_0(t-\tau)}, t \geq \tau, \tag{4.131}$$

with constants $a_0 > 0, \lambda_0 > 0$, see also (4.85a).

Considering the Euclidean norm $\|dz(t)\|$ of the 1st order error term $dz(t)$, from (4.124a, b) and with (4.131) we obtain

$$\begin{aligned}\|dz(t)\| &\leq \|e^{A(t-t_j)}\Delta z_j\| + \left\| \int_{t_j}^{t} e^{A(t-\tau)} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi}^{(j,1)}(\tau) \end{pmatrix} d\tau \right\| \\ &\leq a_0 e^{-\lambda_0(t-t_j)}\|\Delta z_j\| + \int_{t_j}^{t} \left\| e^{A(t-\tau)} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi}^{(j,1)}(\tau) \end{pmatrix} \right\| d\tau \\ &\leq a_0 e^{-\lambda_0(t-t_j)}\|\Delta z_j\| + \int_{t_j}^{t} a_0 e^{-\lambda_0(t-\tau)} \|\dot{\psi}^{(j,1)}(\tau)\| d\tau. \end{aligned} \tag{4.132}$$

Taking the conditional expectation in (4.132), we find

$$\begin{aligned} E\left(\|dz(t)\| \Big| \mathcal{A}_{t_j}\right) \leq a_0 e^{-\lambda_0(t-t_j)} E\left(\|\Delta z_j\| \Big| \mathcal{A}_{t_j}\right) \\ + a_0 \int_{t_j}^{t} e^{-\lambda_0(t-\tau)} E\left(\|\dot{\psi}^{(j,1)}(\tau)\| \Big| \mathcal{A}_{t_j}\right) d\tau. \end{aligned} \tag{4.133}$$

Applying Jensen's inequality

$$\sqrt{EX(\omega)} \geq E\sqrt{X(\omega)},$$

where $X = X(\omega)$ is a nonnegative random variable, with (4.124b) we get, cf. (4.86d, e),

$$\begin{aligned} E\left(\|\dot{\psi}^{(j,1)}(\tau)\| \Big| \mathcal{A}_{t_j}\right) &= E\left(\sqrt{\|\dot{\psi}^{(j,1)}(\tau)\|^2} \Big| \mathcal{A}_{t_j}\right) \\ &\leq \sqrt{E\left(\|\dot{\psi}^{(j,1)}(\tau)\|^2 \Big| \mathcal{A}_{t_j}\right)} \qquad (4.134a) \\ &\leq \left\| \frac{d\left(M^{(j)}(t)^{-1} Y^{(j)}(t)\right)}{dt}(\tau) \right\| \sqrt{\operatorname{var}\left(p_D(\cdot) \Big| \mathcal{A}_{t_j}\right)}, \end{aligned}$$

where

$$\operatorname{var}\Big(p_D(\cdot)\big|\mathcal{A}_{t_j}\Big) := E\Big(\|\Delta p_D(\omega)\|^2\big|\mathcal{A}_{t_j}\Big). \tag{4.134b}$$

In the following we study the inhomogeneous term $\dot{\psi}^{(j,1)}(t)$ of the 3rd order linear differential equation (4.122a) in more detail. According to (4.121d) we have

$$\begin{aligned}\dot{\psi}^{(j,1)}(t) &= -\frac{d}{dt}\Big(M^{(j)}(t)^{-1}Y^{(j)}(t)\Big)\Delta p_D\\ &= -\left(\frac{d}{dt}M^{(j)}(t)^{-1}\right)Y^{(j)}(t)\Delta p_D\\ &\qquad -M^{(j)}(t)^{-1}\left(\frac{d}{dt}Y^{(j)}(t)\right)\Delta p_D.\end{aligned} \tag{4.135a}$$

and therefore

$$\begin{aligned}\|\dot{\psi}^{(j,1)}(t)\| \le &\left\|\frac{d}{dt}M^{(j)}(t)^{-1}\right\| \cdot \|Y^{(j)}(t)\Delta p_D\|\\ &+\|M^{(j)}(t)^{-1}\| \cdot \left\|\big(\dot{Y}^{(j)}(t)\big)\,\Delta p_D\right\|.\end{aligned} \tag{4.135b}$$

Now, according to (4.19a, b), (4.82a–c) it holds

$$u^{(j)}(t;\Delta p_D) := Y^{(j)}(t)\Delta p_D = u_e\Big(\Delta p_D, s^{(j)}(t); q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\Big), t \ge t_j. \tag{4.136a}$$

Thus, we find

$$\begin{aligned}\dot{u}^{(j)}(t;\Delta p_D) &= \big(\dot{Y}^{(j)}(t)\big)\,\Delta p_D\\ &= \frac{d}{dt}u_e\Big(\Delta p_D, s^{(j)}(t); q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\Big)\\ &= \frac{\partial u_e}{\partial s}\Big(\Delta p_D, s^{(j)}(t); q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\Big)\cdot\frac{d}{dt}s^{(j)}(t)\\ &= \frac{\partial u_e}{\partial s}\Big(\Delta p_D, s^{(j)}(t); q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\Big)\cdot\sqrt{\beta^{(j)}\Big(s^{(j)}(t)\Big)},\end{aligned} \tag{4.136b}$$

see (4.17c), (4.49b). Note that

$$u^{(j)}(t;\Delta p_D), \dot{u}^{(j)}(t;\Delta p_D)$$

are linear with respect to $\Delta p_D = p_D - E\Big(p_D(\omega)|\mathcal{A}_{t_j}\Big)$.

From (4.135a, b) and (4.136a, b) we get

$$
\begin{aligned}
E\left(\|\dot{\psi}^{(j,1)}(t)\|\Big|\mathcal{A}_{t_j}\right) &\le \|\tfrac{d}{dt}M^{(j)}(t)^{-1}\|\; E\left(\|Y^{(j)}(t)\Delta p_D\|\Big|\mathcal{A}_{t_j}\right)\\
&\quad +\|M^{(j)}(t)^{-1}\|E\left(\left\|\left(\dot{Y}^{(j)}(t)\right)\Delta p_D\right\|\Big|\mathcal{A}_{t_j}\right)\\
&= \left\|\frac{d\left(M^{(j)}(t)^{-1}\right)}{dt}\right\|\cdot E\left(\|u(j)(t;\Delta p_D)\|\Big|\mathcal{A}_{t_j}\right)\\
&\quad +\|M^{(j)}(t)^{-1}\|\cdot E\left(\|\dot{u}^{(j)}(t;\Delta p_D\|\Big|\mathcal{A}_{t_j}\right)\\
&= \left\|\frac{d\left(M^{(j)}(t)^{-1}\right)}{dt}\right\|\cdot E\left(\left\|u_e\left(\Delta p_D, s^{(j)}(t); q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\right)\right\|\Big|\mathcal{A}_{t_j}\right)\\
&\quad +\|M^{(j)}(t)^{-1}\|\cdot E\left(\left\|\tfrac{\partial u_e}{\partial s}\left(\Delta p_D, s^{(j)}(t); q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\right)\sqrt{\beta^{(j)}\left(s^{(j)}(t)\right)}\right\|\Bigg|\mathcal{A}_{t_j}\right).
\end{aligned}
\tag{4.137a}
$$

Using again Jensen's inequality, from (4.137a) we obtain

$$
\begin{aligned}
&E\left(\|\dot{\psi}^{(j,1)}(t)\|\Big|\mathcal{A}_{t_j}\right)\\
&\le \left\|\frac{d\left(M^{(j)}(t)^{-1}\right)}{dt}\right\|\sqrt{\operatorname{var}\left(u_e\left(p_D(\cdot), s^{(j)}(t); q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\right)\Big|\mathcal{A}_{t_j}\right)}\\
&\quad +\|M^{(j)}(t)^{-1}\|\sqrt{\operatorname{var}\left(\tfrac{\partial u_e}{\partial s}\left(p_D(\cdot), s^{(j)}(t); q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\right)\Big|\mathcal{A}_{t_j}\right)\beta^{(j)}\left(s^{(j)}(t)\right)},
\end{aligned}
\tag{4.137b}
$$

where

$$
\begin{aligned}
&\operatorname{var}\left(u_e\left(p_D(\cdot), s^{(j)}(t); q_0^{(j)}(\cdot), \beta^{(j)}(\cdot)\right)\Big|\mathcal{A}_{t_j}\right)\\
&:= E\left(\left\|u_e\left(\Delta p_D(\omega), s^{(j)}(t); q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\right)\right\|^2\Big|\mathcal{A}_{t_j}\right),
\end{aligned}
\tag{4.137c}
$$

$$
\begin{aligned}
&\operatorname{var}\left(\tfrac{\partial u_e}{\partial s}\left(p_D(\cdot), s^{(j)}(t); q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\right)\sqrt{\beta^{(j)}\left(s^{(j)}(t)\right)}\Bigg|\mathcal{A}_{t_j}\right)\\
&= E\left(\left\|\tfrac{\partial u_e}{\partial s}\left(\Delta p_D(\omega), s^{(j)}(t); q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\right)\right\|^2\Big|\mathcal{A}_{t_j}\right)\beta^{(j)}\left(s^{(j)}(t)\right).
\end{aligned}
\tag{4.137d}
$$

According to the representation (4.124a, b) of the first order tracking error form $dz = dz(t), t \geq t_j$, the behavior of $dz(t)$ is determined mainly by the system matrix A and the "inhomogeneous term" $\dot{\psi}^{(j,1)}(t), t \geq t_j$. Obviously, this term plays the same role as the expression $\psi^{(j,1)}(t), t \geq t_j$, in the representation (4.84a) for the first order error term in case of PD-controllers.

However, in the present case of PID-control the error estimates (4.137a–d) show that for a satisfactory behavior of the first order error term $dz = dz(t), t \geq t_j$, besides the control constraints (4.8a–c), (4.21a–c), (4.30a–f) for

$$u(t) = u_e\Big(p_D, s; q_e(.), \beta(.)\Big) \text{ with } s = s(t), \tag{4.138a}$$

here, also corresponding constraints for the input rate, i.e. the time derivative of the control

$$\dot{u}(t) = \frac{\partial u_e}{\partial s}\Big(p_D, s; q_e(.), \beta(.)\Big)\sqrt{\beta(s)} \text{ with } s = s(t) \tag{4.138b}$$

are needed!

The above results can be summarized by the following theorem:

Theorem 4.7.2 *Suppose that the matrices K_p, K_i, K_d are selected such that the fundamental matrix $\Phi(t,\tau) = e^{A(t-\tau)}, t \geq \tau$, is exponentially stable (cf. Theorem 4.7.1). Then, based on the definition (4.120a–c) of the linear approximation of the PID-controller, the following properties hold:*

a) ***Asymptotic local stability in the mean***

$$E\big(dz(t)\big|\mathcal{A}_{t_j}\big) \to 0, t \to \infty \tag{4.139a}$$

b) ***Mean absolute 1st order tracking error***

$$\begin{aligned} E\big(\|dz(t)\|\big|\mathcal{A}_{t_j}\big) &\leq a_0 e^{-\lambda_0(t-t_j)} E\big(\|\Delta z_j\|\big|\mathcal{A}_{t_j}\big) \\ &+ a_0 \int_{t_j}^{t} e^{-\lambda_0(t-\tau)} \left\| \frac{d\Big(M^{(j)}(t)^{-1}\Big)}{dt}(\tau) \right\| \sigma_{u_e}^{(j)}\Big(s^{(j)}(\tau)\Big) d\tau \\ &+ a_0 \int_{t_j}^{t} e^{-\lambda_0(t-\tau)} \left\| M^{(j)}(\tau)^{-1} \right\| \sigma_{\frac{\partial u_e}{\partial s}}^{(j)}\Big(s^{(j)}(\tau)\Big)\sqrt{\beta^{(j)}\big(s^{(j)}(\tau)\big)}\, d\tau, t \geq t_j \end{aligned} \tag{4.139b}$$

with

$$\sigma_{u_e}^{(j)}(s) := \sqrt{var\Big(u_e\big(p_D(.), s; q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\big)\Big|\mathcal{A}_{t_j}\Big)}, \tag{4.139c}$$

$$\sigma_{\frac{\partial u_e}{\partial s}}^{(j)}(s) := \sqrt{var\Big(\frac{\partial u_e}{\partial s}\big(p_D(\cdot), s; q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\big)\Big|\mathcal{A}_{t_j}\Big)}, \tag{4.139d}$$

$s \geq s_j$. *Moreover,*

$$E\big(\|dz(t)\|\big|\mathcal{A}_{t_j}\big) \leq a_0 e^{-\lambda_0(t-t_j)} E\big(\|\Delta z_j\|\big|\mathcal{A}_{t_j}\big) + a_0\Bigg(\int_{t_j}^{t} e^{-\lambda_0(t-\tau)}\left\|\frac{d\Big(M^{(j)}(t)^{-1}Y^{(j)}(t)\Big)}{dt}(\tau)\right\| d\tau\Bigg)\sigma_{p_D}^{(j)}, t \geq t_j, \tag{4.139e}$$

where

$$\sigma_{p_D}^{(j)} := \sqrt{var\big(p_D(.)\big|\mathcal{A}_{t_j}\big)}. \tag{4.139f}$$

Using the t-s-transformation $s = s^{(j)}(\tau), \tau \geq t_j$, the time-integrals in (4.139b) can be represented also in the following form:

$$\int_{t_j}^{t} e^{-\lambda_0(t-\tau)}\left\|\frac{d\Big(M^{(j)}(t)^{-1}\Big)}{dt}(\tau)\right\| \sigma_{u_e}^{(j)}\Big(s^{(j)}(\tau)\Big)d\tau = \int_{s_j}^{s^{(j)}(t)} e^{-\lambda_0\big(t-t^{(j)}(s)\big)}\left\|\frac{d\Big(M^{(j)}(t)^{-1}\Big)}{dt}\big(t^{(j)}(s)\big)\right\| \frac{\sigma_{u_e}^{(j)}(s)}{\sqrt{\beta^{(j)}(s)}}ds, \tag{4.140a}$$

$$\int_{t_j}^{t} e^{-\lambda_0(t-\tau)}\left\|M^{(j)}(\tau)^{-1}\right\|\sigma_{\frac{\partial u_e}{\partial s}}^{(j)}\Big(s^{(j)}(\tau)\Big)\sqrt{\beta^{(j)}\big(s^{(j)}(\tau)\big)}d\tau = \int_{s_j}^{s^{(j)}(t)} e^{-\lambda_0\big(t-t^{(j)}(s)\big)}\left\|M^{(j)}\big(t^{(j)}(s)\big)^{-1}\right\|\sigma_{\frac{\partial u_e}{\partial s}}^{(j)}(s)ds, \tag{4.140b}$$

where $\tau = t^{(j)}(s), s \geq s_j$, denotes the inverse of $s = s^{(j)}(\tau), \tau \geq t_j$.

Minimality or Boundedness Properties

Several terms in the above estimates of the 1st order tracking error $dz = dz(t), t \geq t_j$, are influenced by means of (OSTP) as shown in the following:

i) **Optimal velocity profile $\beta^{(j)}$**
For minimum-time and related substitute problems the total runtime

$$\int_{s_j}^{s^{(j)}(t)} \frac{1}{\sqrt{\beta^{(j)}(s)}} ds \leq \int_{s_j}^{s_f} \frac{1}{\sqrt{\beta^{(j)}(s)}} ds = t_f^{(j)} - t_j \tag{4.141}$$

is minimized by (OSTP).

ii) **Properties of the coefficient λ_0**
According to Theorem 4.7.1 and 4.7.2 the matrices K_p, K_i, K_d can be selected such that real parts $\mathrm{Re}\left(\lambda_{kl}\right), k = 1, \ldots, n, l = 1, 2, 3$, of the eigenvalues $\lambda_{kl}, k = 1, \ldots, n, l = 1, 2, 3$, of the matrix A, cf. (4.123b), are negative, see (4.127b), (4.128a–d). Then, the decisive coefficient $\lambda_0 > 0$ in the norm estimate (4.131) of the fundamental matrix $\Phi(t, \tau), t \geq \tau$, can be selected such that

$$0 < \lambda_0 < - \max_{\substack{1 \leq k \leq n \\ l=1,2,3}} \mathrm{Re}(\lambda_{kl}). \tag{4.142}$$

iii) **Chance constraints for the input**
With certain lower and upper vector bounds $u^{\min} \leq u^{\max}$ one usually has the input or control constraints, cf. (2.10a),

$$u^{\min} \leq u(t) \leq u^{\max}, t \geq t_j.$$

In the following we suppose that the bounds $u^{\min}, u^{\max}$ are given deterministic vectors. After the transformation

$$s = s^{(j)}(t), t \geq t_j,$$

from the time domain $[t_j, t_f^{(j)}]$ to the s-domain $[s_j, s_f]$, see (4.49b), due to (4.21a) we have the stochastic constraint for $\big(q_e(\cdot), \beta(\cdot)\big)$:

$$u^{\min} \leq u_e\big(p_D(\omega), s; q_e(\cdot), \beta(\cdot)\big) \leq u^{\max}, s_j \leq s \leq s_f.$$

Demanding that the above constraint holds at least with the probability α_u, we get, cf. 4.30a, the probabilistic constraint

$$P\left(u^{\min} \leq u_e\big(p_D(\omega), s; q_e(\cdot), \beta(\cdot)\big) \leq u^{\max} \middle| \mathcal{A}_{t_j}\right) \geq \alpha_u, s_j \leq s \leq s_f. \tag{4.143a}$$

Defining again, see (4.30d),

$$u^c := \frac{u^{\min} + u^{\max}}{2}, \ \rho_u := \frac{u^{\max} - u^{\min}}{2},$$

by means of Tschebyscheff-type inequalities, the chance constraint (4.143a) can be guaranteed, see (4.30e, f), (4.87d), by the condition

$$E\Big(\big\|u_e\big(p_D(\omega), s; q_e(.), \beta(.)\big) - u^c\big\|^2 \Big| \mathcal{A}_{t_j}\Big) \le (1-\alpha_u) \min_{1\le k\le n} \rho_{u_k}^2, \ s_j \le s \le s_f. \tag{4.143b}$$

According to the definition (4.139c) of $\sigma_{u_e}^{(j)}(s)$, inequality (4.143b) is equivalent to

$$\sigma_{u_e}^{(j)}(s)^2 + \big\|\overline{u}_e^{(j)}(s) - u^c\big\|^2 \le (1-\alpha_u) \min_{1\le k\le n} \rho_{u_k}^2, \ s_j \le s \le s_f, \tag{4.144a}$$

where

$$\begin{aligned}\overline{u}_e^{(j)} :=& E\Big(u_e\big(p_D(\omega), s; q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\big) \Big| \mathcal{A}_{t_j}\Big)\\ =& u_e\Big(\overline{p}_D^{(j)}, s; q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\Big).\end{aligned} \tag{4.144b}$$

Hence, the sufficient condition (4.144a) for the reliability constraint (4.143a) yields the variance constraint

$$\sigma_{u_e}^{(j)}(s)^2 \le (1-\alpha_u) \min_{1\le k\le n} \rho_{u_k}^2, \ s_j \le s \le s_f. \tag{4.144c}$$

iv) **Minimum force and moment**
According to the different performance functions $J\big(u(.)\big)$ mentioned after the definition 4.6, in case of minimum expected force and moment we have, using transformation formula 4.19a,

$$\begin{aligned}E\Big(J\big(u(\cdot)\big)\Big|\mathcal{A}_{t_j}\Big) =& E\Bigg(\int_{t_j}^{t_f^{(j)}} \big\|u(t)\big\|^2 dt \Big| \mathcal{A}_{t_j}\Bigg)\\ =& \int_{s_j}^{s_f} E\Big(\big\|u_e\big(p_D(\omega), s; q_e(\cdot), \beta(\cdot)\big\|^2 \Big| \mathcal{A}_{t_j}\Big) \frac{ds}{\sqrt{\beta(s)}}\\ =& \int_{s_j}^{s_f} \Bigg(E\Big(\big\|u_e\big(p_D(\omega), s; q_e(\cdot), \beta(\cdot) - u_e\big(\overline{p}_D^{(j)}, s; q_e(\cdot), \beta(\cdot)\big\|^2 \Big| \mathcal{A}_{t_j}\Big)\end{aligned}$$

$$+\left\|u_e\left(\overline{p}_D^{(j)}, s; q_e(\cdot), \beta(\cdot)\right)\right\|^2\Bigg)\frac{ds}{\sqrt{\beta(s)}}$$

$$= \int_{s_j}^{s_f} \left(\sigma_{u_e}^2(s) + \left\|u_e\left(\overline{p}_D^{(j)}, s; q_e(\cdot), \beta(\cdot)\right\|^2\right)\frac{ds}{\sqrt{\beta(s)}}, \tag{4.145a}$$

where

$$\sigma_{u_e}^2(s) := E\left(\left\|u_e\left(p_D(\omega), s; q_e(\cdot), \beta(\cdot) - u_e\left(\overline{p}_D^{(j)}, s; q_e(\cdot), \beta(\cdot)\right)\right\|^2 \middle| \mathcal{A}_{t_j}\right)$$

$$= E\left(\left\|u_e\left(\Delta p_D(\omega), s; q_e(\cdot), \beta(\cdot)\right\|^2 \middle| \mathcal{A}_{t_j}\right). \tag{4.145b}$$

Hence, (OSTP) yields the following **minimum property**:

$$\int_{s_j}^{s_f} \left(\sigma_{u_e}^{(j)}(s)^2 + \left\|\overline{u}_e^{(j)}(s)\right\|^2\right)\frac{ds}{\sqrt{\beta^{(j)}(s)}} = \min_{\substack{q_e(\cdot),\beta(\cdot)\\ s.t.\ (4.8a\text{–c}),\\ (4.9a\text{–d})}} E\left(\int_{t_j}^{t_f^{(j)}} \|u(t)\|^2 dt \middle| \mathcal{A}_{t_j}\right), \tag{4.146}$$

where $\overline{u}_e^{(j)}(s)$ is defined by (4.144b).

v) **Decreasing stochastic uncertainty**
According to (4.132), (4.133) and (4.134a, b) we have

$$E\left(\|dz(t)\|\mathcal{A}_{t_j}\right) \le \alpha_0 e^{-\lambda_0(t-t_j)} E\left(\|\Delta z_j\| \middle| \mathcal{A}_{t_j}\right)$$
$$+\left(a_o \int_{t_j}^{t} e^{-\lambda_0(t-\tau)} \left\| \frac{d\left(M^{(j)}(t)^{-1} Y^{(j)}(t)\right)}{dt}(\tau)\right\| d\tau\right)\sqrt{\operatorname{var}\left(p_D(.)\middle|\mathcal{A}_{t_j}\right)}. \tag{4.147}$$

Thus, the mean absolute 1st order tracking error can be decreased further by removing step by step the uncertainty about the vector $p_D = p_D(\omega)$ of dynamic parameter. This is done in practice by a parameter identification procedure running parallel to control process of the robot.

vi) **Chance constraints for the input rate**
According to the representation (4.138b) of the input or control rate we have

$$\dot{u}(t) = \frac{\partial u_e}{\partial s}\left(p_D, s; q_e(\cdot), \beta(\cdot)\right)\sqrt{\beta(s)}, \quad s = s(t).$$

From the input rate condition

$$\dot{u}^{\min} \leq \dot{u}(t) \leq \dot{u}^{\max}, t \geq t_j, \tag{4.148a}$$

with given, fixed vector bounds $\dot{u}^{\min} \leq \dot{u}^{\max}$, for the input rate $\frac{\partial u_e}{\partial s}$ with respect to the path parameter $s \geq s_j$ we obtain the constraint

$$\frac{\dot{u}^{\min}}{\sqrt{\beta(s)}} \leq \frac{\partial u_e}{\partial s}\big(p_D(\omega), s; q_e(\cdot), \beta(\cdot)\big) \leq \frac{\dot{u}^{\max}}{\sqrt{\beta(s)}}, s_j \leq s \leq s_f. \tag{4.148b}$$

If we require that the input rate condition (4.148a) holds at least with probability $\alpha_{\dot{u}}$, then corresponding to (4.143a) we get the chance constraint

$$P\left(\frac{\dot{u}^{\min}}{\sqrt{\beta(s)}} \leq \frac{\partial u_e}{\partial s}\big(p_D(\omega), s; q_e(\cdot), \beta(\cdot)\big) \leq \frac{\dot{u}^{\max}}{\sqrt{\beta(s)}} \middle| \mathcal{A}_{t_j}\right) \geq \alpha_{\dot{u}}. \tag{4.149}$$

In the same way as in (iii), condition (4.149) can be guaranteed, cf. (4.139d), by

$$\sigma^{(j)}_{\frac{\partial u}{\partial s}}(s)^2 + \left\| \overline{\frac{\partial u_e}{\partial s}}^{(j)}(s) - \frac{\dot{u}^c}{\sqrt{\beta^{(j)}(s)}} \right\|^2 \leq \big(1-\alpha_{\dot{u}}\big)\frac{1}{\beta^{(j)}(s)} \min_{1 \leq k \leq n} \rho^2_{\dot{u}_k}, \ s_j \leq s \leq s_f, \tag{4.150a}$$

where

$$\dot{u}^c := \frac{\dot{u}^{\min} + \dot{u}^{\max}}{2}, \ \rho_{\dot{u}} := \frac{\dot{u}^{\max} - \dot{u}^{\min}}{2}, \tag{4.150b}$$

$$\overline{\frac{\partial u_e}{\partial s}}^{(j)} := \frac{\partial u_e}{\partial s}\Big(\overline{p}_D^{(j)}, s; q_e^{(j)}(\cdot), \beta^{(j)}(\cdot)\Big), s \geq s_j. \tag{4.150c}$$

Hence, corresponding to (4.144c), here we get the following variance constraint:

$$\sigma^{(j)}_{\frac{\partial u}{\partial s}}(s)^2 \leq \big(1 - \alpha_{\dot{u}}\big)\frac{1}{\beta^{(j)}(s)} \min_{1 \leq k \leq n} \rho^2_{\dot{u}_k}, \ s_j \leq s \leq s_f. \tag{4.150d}$$

vii) **Minimum force and moment rate**
Corresponding to (4.145a) we may consider the integral

$$E\Big(J\big(\dot{u}(\cdot)\big|\mathcal{A}_{t_j}\Big) := E\left(\int_{t_j}^{t_f} \|\dot{u}(t)\|^2 dt \middle| \mathcal{A}_{t_j}\right). \tag{4.151a}$$

Again with (4.138b) we find

$$E\Big(J\big(\dot{u}(\cdot)\big|\mathcal{A}_{t_j}\Big) = E\left(\int_{t_j}^{t_f} \left\| \frac{\partial u_e}{\partial s}\big(p_D(\omega), s; q_e(\cdot), \beta(\cdot)\big)\sqrt{\beta(s)}\right\|^2 \frac{ds}{\sqrt{\beta(s)}}\middle|\mathcal{A}_{t_j}\right)$$

$$= \int_{s_j}^{s_f} E\left(\left\| \frac{\partial u_e}{\partial s}\big(p_D(\omega), s; q_e(\cdot), \beta(\cdot)\big)\right\|^2 \middle|\mathcal{A}_{t_j}\right)\sqrt{\beta(s)}ds$$

$$= \int_{s_j}^{s_f} \left(\mathrm{var}\Big(\frac{\partial u_e}{\partial s}\big(p_D(\omega), s; q_e(\cdot), \beta(\cdot)\big)\Big|\mathcal{A}_{t_j}\Big)\right.$$

$$\left. + \left\| \frac{\partial u_e}{\partial s}\big(\overline{p}_D^{(j)}, s; q_e(\cdot), \beta(\cdot)\big)\right\|^2\right)\sqrt{\beta(s)}ds. \tag{4.151b}$$

If we consider

$$E\Big(\tilde{J}\big(\dot{u}(\cdot)\big)\Big|\mathcal{A}_{t_j}\Big) := E\left(\int_{t_j}^{t_f} \|\dot{u}(t)\|dt\middle|\mathcal{A}_{t_j}\right), \tag{4.152a}$$

then we get

$$E\Big(\tilde{J}\big(\dot{u}(\cdot)\big)\Big|\mathcal{A}_{t_j}\Big) = \int_{s_j}^{s_f} E\left(\left\| \frac{\partial u_e}{\partial s}\big(p_D(\omega), s; q_e(\cdot), \beta(\cdot)\big)\right\|\middle|\mathcal{A}_{t_j}\right)ds. \tag{4.152b}$$

Chapter 5
Optimal Design of Regulators

The optimal design of regulators is often based on the use of given, fixed nominal values of initial conditions, pay loads and other model parameters. However, due to

- variations of the material properties,
- measurement errors (e.g. in case of parameter identification)
- modeling errors (complexity of real systems)
- uncertainty on the working environment, the task to be executed, etc.,

the true payload, initial conditions, and model parameters, like sizing parameters, mass values, gravity centers, moments of inertia, friction, tolerances, adjustment setting error, etc., are not known exactly in practice. Hence, a predetermined (optimal) regulator should be "*robust*", i.e., the controller should guarantee satisfying results also in case of variations of the initial conditions, load parameters and further model parameters.

Robust controls have been considered up to now mainly for uncertainty models based on given fixed sets of parameters, e.g. a certain multiple intervals, assumed to contain the unknown, true parameter. In this case one requires then often that the controlled system fulfills certain properties, as e.g. certain stability properties for all parameter vectors in the given parameter domain. If the required property can be described by a scalar criterion, then the controller design is based on a minimax criterion, such as the H^{∞}-criterion.

Since in many cases parameter uncertainty can be modeled more adequately by means of stochastic parameter models, in the following we suppose that the parameters involved in the regulator design problem are realizations of a random vector having a known or at least partly known joint probability distribution. The determination of an optimal controller under uncertainty with respect to model parameters, working neighborhood, modeling assumptions, etc., is a *decision theoretical problem*. Criteria of the "*holds for all parameter vectors in a given set*" and the *minmax*-criterion are very pessimistic and often too strong. Indeed, in many cases the available a priori and empirical information on the dynamic

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system and its working neighborhood allows a more adequate, flexible description of the uncertainty situation by means of stochastic approaches. Thus, it is often more appropriate to model unknown and varying initial values, external loads and other model parameters as well as modeling errors, e.g. incomplete representation of the dynamic system, by means of realizations of a random vector, a random function with a given or at least partly known probability distribution. Consequently, the optimal design of robust regulators is based on an optimization problem under stochastic uncertainty.

Stochastic Optimal Design of Regulators

For the consideration of stochastic parameter variations as well as other uncertainties within the optimal design process of a regulator one has to introduce—as for any other optimization problem under (stochastic) uncertainty—an appropriate **deterministic substitute problem**. In the present case of stochastic optimal design of a regulator, hence, a map from the state or observation space into the space of control corrections, one has a **control problem under stochastic uncertainty**. For the solution of the occurring deterministic substitute problems the methods of *stochastic optimization*, cf. [100], are available.

As shown in the preceding sections, the optimization of a regulator presupposes an optimal reference trajectory $q^R(t)$ in the configuration space and a corresponding feedforward control $u^R(t)$. As shown in preceding sections, the guiding functions $\big(q^R(t), u^R(t)\big)$ can be determined also by stochastic optimization methods.

For the computation of stochastic optimal regulators, deterministic substitute control problems of the following type are considered:

Minimize the expected total costs composed of (i) the costs arising from the deviation $\Delta z(t)$ between the (stochastic optimal) reference trajectory and the effective trajectory of the dynamic system and (ii) the costs for the regulation $\Delta u(t)$.

Subject to the following constraints:

- dynamic equation of the robot or a more general dynamic stochastic system with the total control input $u(t) = u^R(t) + \Delta u(t)$ being the sum of the feedforward control $u^R(t)$ and the control correction $\Delta u(t) = \varphi(t, \Delta z(t))$
- stochastic initial conditions $q(t_0) = q_0(\omega)$, $\dot{q}(t_0) = \dot{q}_0(\omega)$ for the state of the system at the starting time point t_0,
- conditions for the feedback law $\varphi = \varphi(t, z(t))$, such as $\varphi(t, 0) = 0$ (if the effective state is equal to the state prescribed by the reference trajectory, then no control correction is needed).

Here, as often in practice, we use quadratic cost functions. The above described deterministic substitute problem can be interpreted again as a control problem for the unknown feedback control law. A basic problem is then the computation of the (conditional) expectation arising in the objective function. Since the expectations are defined here by multiple integrals, the expectations can be determined in general only approximatively. In the following approximations based on Taylor expansions with respect to the stochastic parameter vector $a(\omega)$ at their conditional expectations $\bar{a}^{t_0} = Ea(\omega)|\mathcal{A}_{t_0})$.

5.1 Tracking Error

According to Sects. 4.6 and 4.7, at each stage $j = 0, 1, 2, \ldots$ the control input $u(t)$ is defined by

$$\begin{aligned} u(t) &:= \text{feedforward control plus control correction} \\ &= u^{(j)}(t) + \Delta u^{(j)}(t),\ t \geq t_j \end{aligned} \tag{5.1a}$$

where

$$\begin{aligned} \Delta u^{(j)}(t) &:= \varphi^{(j)}\left(t, \Delta z^{(j)}(t)\right) \\ &= \varphi^{(j)}\left(t, z(t) - z^{(j)}(t)\right) \\ &= \varphi^{(j)}\left(t, q(t) - q^{(j)}(t), \dot{q}(t) - \dot{q}^{(j)}(t)\right) \end{aligned} \tag{5.1b}$$

with a *feedback*-function or *feedback control law*

$$\varphi^{(j)} = \varphi^{(j)}(t, \Delta z). \tag{5.1c}$$

If $z(t) = z^{(j)}(t)$, hence, $\Delta z^{(j)}(t) = 0$, at the time point $t \geq t_j$ no control correction is needed. Thus, without further, global information one often requires that

$$\varphi^{(j)}(t, 0) = 0 \text{ for all } t \geq t_j. \tag{5.1d}$$

Inserting the total control input $u(t),\ t \geq t_j$, into the dynamic equation (4.4a), then the **effective further course** of the trajectory $q(t),\ t \geq t_j$, will be described by the system of ordinary differential equations (ODE):

$$\begin{aligned} F\left(p_D, q(t), \dot{q}(t), \ddot{q}(t)\right) &= u^{(j)}(t) + \Delta u^{(j)}(t) \qquad (5.2a) \\ &= u^{(j)}(t) + \varphi^{(j)}\left(t, q(t) - q^{(j)}(t), \dot{q}(t) - \dot{q}^{(j)}(t)\right),\ t \geq t_j \\ q(t_j) &= q_j,\ \dot{q}(t_j) = \dot{q}_j, \qquad (5.2b) \end{aligned}$$

where $z_j := (q_j, \dot{q}_j)$ denotes the **effective state** in the time point t_j.

Because of

$$\Delta q(t) = \Delta q^{(j)}(t) := q(t) - q^{(j)}(t),\ t \geq t_j \tag{5.3a}$$

$$\dot{\Delta q}(t) = \dot{\Delta q}^{(j)}(t) := \dot{q}(t) - \dot{q}^{(j)}(t),\ t \geq t_j \tag{5.3b}$$

$$\ddot{\Delta q}(t) = \ddot{\Delta q}^{(j)}(t) := \ddot{q}(t) - \ddot{q}^{(j)}(t),\ t \geq t_j \tag{5.3c}$$

and

$$\Delta q_j := q(t_j) - q^{(j)}(t_j) = q_j - \overline{q_j} \tag{5.3d}$$

$$\dot{\Delta q}_j := \dot{q}(t_j) - \dot{q}^{(j)}(t_j) = \dot{q}_j - \overline{\dot{q}_j} \tag{5.3e}$$

$$\Delta p_D := p_D - \overline{p}_D^{(j)}, \tag{5.3f}$$

system (5.2a, b) can be rewritten by the system of differential equations for the **tracking error**:

$$\begin{aligned} &F\Big(\overline{p}_D^{(j)} + \Delta p_D, q^{(j)}(t) + \Delta q(t), \dot{q}^{(j)}(t) \\ &\quad + \dot{\Delta q}(t), \ddot{q}^{(j)}(t) + \ddot{\Delta q}(t)\Big) \\ &= u^{(j)}(t) + \varphi^{(j)}\left(t, \Delta q(t), \dot{\Delta q}(t)\right),\ t \geq t_j \end{aligned} \tag{5.4a}$$

$$\Delta q(t_j) = \Delta q_j,\ \dot{\Delta q}(t_j) = \dot{\Delta q}_j. \tag{5.4b}$$

The solution of system (5.4a, b) is a function

$$\Delta q(t) = \Delta q\left(t, \Delta p_D, \Delta q_j, \dot{\Delta q}_j\right) = \Delta q^{(j)}\left(t, \Delta p_D, \Delta q_j, \dot{\Delta q}_j\right),\ t \geq t_j. \tag{5.4c}$$

5.1.1 Optimal PD-Regulator

For simplification, we first consider the optimal design of a PD-regulator. Of course, the **aim of the control correction** $\Delta u^{(j)}(t),\ t \geq t_j$, at the j-th stage reads:

$$\begin{aligned} &\text{i) } z(t) = \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} \longrightarrow z^{(j)}(t) = \begin{pmatrix} q^{(j)}(t) \\ \dot{q}^{(j)}(t) \end{pmatrix},\ t > t_j,\ t \to \infty \\ &\text{or} \\ &\text{ii) } z(t) \approx z^{(j)}(t) \text{ on } [t_j, t_f^{(j)}]. \end{aligned} \tag{5.5a}$$

In general, the control corrections $\Delta u^{(j)}(t),\ t \geq t_j$, cause considerable expenses for measurements and control. Hence, choosing a feedback-function $\varphi^{(j)} = \varphi^{(j)}(t, \Delta z)$, one has also to take into account certain **control costs**

$$\gamma = \gamma\left(t, \varphi(t, \cdot, \cdot), \Delta u(t)\right), \tag{5.5b}$$

depending on the whole feedback function $\varphi(t,\cdot,\cdot)$ and/or on the individual control inputs $\Delta u(t)$.

For given model parameters and initial conditions, the quality or performance of a feedback-function $\varphi^{(j)} = \varphi^{(j)}(t, \Delta z)$ can be evaluated by the functional

$$J = J^{(j)}\left(\varphi^{(j)}, p_D, z_j\right) \tag{5.6a}$$

$$= \int_{t_j}^{t_f^{(j)}} \Big(\underbrace{c(t, \underbrace{\Delta q(t), \dot{\Delta q}(t)}_{=\Delta z(t)})}_{\substack{\text{costs of the}\\ \text{tracking error}}} + \underbrace{\gamma\left(t, \varphi(t,\cdot,\cdot), \Delta u(t)\right)}_{\text{regulator costs}} \Big)\, dt$$

with a cost function

$$c = c\left(t, \Delta q, \dot{\Delta q}\right) = c(t, \Delta z) \tag{5.6b}$$

for evaluating the tracking error Δz, as e.g.

$$c(t, \Delta z) = \Delta q(t)^T C_q \Delta q(t) + \dot{\Delta q}(t) C_{\dot q} \dot{\Delta q}(t) \tag{5.6c}$$

involving positive (semi-) definite weight matrices $C_q = C_q(t), C_{\dot q} = C_{\dot q}(t)$. Moreover, the regulator costs can be defined by

$$\gamma = \gamma(t, \varphi(t,\cdot,\cdot), \Delta u(t)) := \Delta u^T C_u \Delta u \tag{5.6d}$$

with a positive (semi-) definite matrix $C_u = C_u(t)$.

Minimizing the conditional expected total costs EJ, for the selection of an optimal feedback-function $\varphi = \varphi(t, \Delta z)$, according to (5.3a)–(5.6d) at the j-th stage we have the following **regulator optimization problem**:

$$\min \mathrm{E}\left(\int_{t_j}^{t_f^{(j)}} \left(c\big(t, \Delta q(t), \dot{\Delta q}(t)\big) + \gamma\big(t, \varphi(t,\cdot,\cdot), \Delta u(t)\big)\right) dt \,\Big|\, \mathcal{A}_{t_j} \right) \tag{5.7a}$$

subject to

$$F\left(p_D(\omega), q^{(j)}(t) + \Delta q(t), \dot q^{(j)}(t) + \dot{\Delta q}(t), \ddot q^{(j)}(t) + \ddot{\Delta q}(t)\right)$$
$$= u^{(j)}(t) + \varphi\left(t, \Delta q(t), \dot{\Delta q}(t)\right),\ t \ge t_j,\ \text{a.s. (almost sure)} \tag{5.7b}$$
$$\Delta q(t_j) = \Delta q_j(\omega),\ \dot{\Delta q}(t_j) = \dot{\Delta q}_j(\omega),\ \text{a.s.}, \tag{5.7c}$$

where $\Delta u(t)$ is defined by (5.1b–d).

Obviously, (5.7a–c) is a stochastic optimization problem, or an optimal control problem under stochastic uncertainty. Transforming the second order dynamic equation (5.7b) into a first order system of differential equations, the above regulator optimization problem can also be represented, cf. Sect. 3.1.2, in the following form:

$$\min \mathrm{E}\left(\int_{t_j}^{t_f^{(j)}} \Big(c\big(t, \Delta z(t)\big) + \gamma\big(t, \varphi(t,\cdot,\cdot), \Delta u(t)\big)\Big) dt \,\Big|\, \mathcal{A}_{t_j} \right) \tag{5.8a}$$

subject to

$$\dot{\Delta z} = f_0\left(p_D(\omega), z^{(j)}(t) + \Delta z(t)\right) - \dot{z}^{(j)}(t) \tag{5.8b}$$

$$+B\left(p_D(\omega), z^{(j)}(t) + \Delta z(t)\right)\left(u^{(j)}(t) + \varphi(t, \Delta z(t))\right), \ t \geq t_j, \text{ a.s.}$$

$$\Delta z(t_j) = \Delta z_j(\omega), \text{ a.s..} \tag{5.8c}$$

Remark 5.1 Note that state and control constraints can also be considered within the stochastic optimal regulator optimization problem (5.7a–c).

5.2 Parametric Regulator Models

5.2.1 Linear Regulator

Lineare regulators are defined by

$$\underbrace{\varphi(t, \Delta z)}_{\text{n comp.}} := G(t)\Delta z = \big(G_p(t), G_d(t)\big)\begin{pmatrix} \Delta q \\ \dot{\Delta q} \end{pmatrix} = G_p(t)\Delta q + G_d(t)\dot{\Delta q} \tag{5.9a}$$

with **unknown** matrix functions $G_p = G_p(t), \ G_d = G_d(t)$.

Nonlinear (Polynomial) Regulator

Considering the m-th order Taylor expansion of the feedback function $\varphi = \varphi(t, \Delta z$ with respect to Δz, we obtain the representation

$$\underbrace{\varphi(t, \Delta z)}_{\text{n comp.}} := \sum_{l=1}^{m} \mathcal{G}_l(t) \cdot (\Delta z)^l = \sum_{l=1}^{m} \underbrace{\mathcal{G}_l(t)}_{\substack{\text{multilin. form} \\ \text{l-th order}}} \cdot \underbrace{(\Delta z, \Delta z, \ldots, \Delta z)}_{l}. \tag{5.9b}$$

This equation can also be represented by

$$\varphi(t, \Delta z) = G(t)h(\Delta z) \tag{5.9c}$$

involving a matrix function $G = G(t)$ with elements to be determined, and a known vector function $h = h(\Delta z)$ containing the weighted different products of order 1 up to m of the components of Δz.

5.2.2 *Explicit Representation of Polynomial Regulators*

With the unit vectors $e_1, \ldots, e_{2n}$ of $\mathbb{R}^{2n}$ we put

$$\Delta z = \sum_{k=1}^{2n} \Delta z_k e_k, \tag{5.10a}$$

where

$$\Delta z_k = \Delta q_k, \quad k = 1, \ldots, n, \tag{5.10b}$$

$$\Delta z_{n+l} = \Delta\dot{q}_l, \quad l = 1, \ldots, n. \tag{5.10c}$$

Thereby the summands in (5.9b) can be represented by

$$\mathcal{G}_l(t) \cdot \underbrace{(\Delta z, \Delta z, \ldots, \Delta z)}_{l} = \mathcal{G}_l(t) \cdot \left(\sum_{k_1=1}^{2n} \Delta z_{k_1} e_{k_1}, \ldots, \sum_{k_l=1}^{2n} \Delta z_{k_l} e_{k_l} \right)$$

$$= \sum_{k_1,k_2,\ldots,k_l=1}^{2n} \mathcal{G}_l(t) \cdot (e_{k_1}, \ldots, e_{k_l}) \Delta z_{k_1} \cdot \ldots \cdot \Delta z_{k_l} \tag{5.11a}$$

Defining the n-vector functions $g_{k_1\ldots k_l} = g_{k_1\ldots k_l}(t)$ by

$$g_{k_1\ldots k_l}(t) := \mathcal{G}_l(t) \cdot (e_{k_1}, \ldots, e_{k_l}), \tag{5.11b}$$

for (5.11a) we obtain also the representation

$$\mathcal{G}_l(t) \cdot (\Delta z)^l = \sum_{k_1,k_2,\ldots,k_l=1}^{2n} g_{k_1\ldots k_l}(t) \Delta z_{k_1} \cdot \ldots \cdot \Delta z_{k_l}. \tag{5.11c}$$

Obviously, (5.11c) depends linearly on the functions $g_{k_1 \dots k_l}(t)$. According to (5.9c) and (5.11c), the matrix function $G = G(t)$ has the following columns:

$$G(t) = (\underbrace{\underbrace{\underbrace{g_1(t), \dots, g_{2n}(t)}_{\text{linear regulator}}, g_{11}(t), \dots, g_{2n2n}(t)}_{\text{quadratic regulator}}, \dots, g_{1\dots1}(t), \dots, g_{2n\dots2n}(t)}_{\text{polynomial regulator m-th order}}). \tag{5.12}$$

5.2.3 *Remarks to the Linear Regulator*

The representation of the feedback function $\varphi(t, \Delta z)$ discussed in the previous Sect. 5.2.2 is still illustrated for the case of linear regulators. With (5.9a), (5.11b) and (5.12) we get:

$$\begin{aligned}
\underbrace{\varphi(t, \Delta z)}_{\text{n-vector}} &\overset{(5.9a)}{=} G_p(t)\Delta q + G_d(t)\Delta\dot{q} \\
&= G_p(t)\left(\sum_{k=1}^{n} \Delta q_k e_k\right) + G_d(t)\left(\sum_{l=1}^{n} \Delta\dot{q}_l e_l\right) \\
&= \sum_{k=1}^{n} \left(G_p(t)e_k\right)\Delta q_k + \sum_{l=1}^{n} \left(G_d(t)e_l\right)\Delta\dot{q}_l \\
&\overset{(5.11b)}{=} \sum_{k=1}^{n} g_{p_k}(t)\Delta q_k + \sum_{l=1}^{n} g_{d_l}(t)\Delta\dot{q}_l \\
&\overset{(5.12)}{=} G(t)\begin{pmatrix} \Delta q(t) \\ \Delta\dot{q}(t) \end{pmatrix}
\end{aligned}$$

with $\quad G(t) := \left(g_{p_1}(t), \dots, g_{p_n}(t), g_{d_1}(t), \dots, g_{d_n}(t)\right).$

Remark to the Regulator Costs

Consider once more (5.9c):

$$\varphi(t, \Delta z) = \underbrace{G(t)}_{\text{unknown}} \underbrace{h(\Delta z)}_{\text{given}}.$$

Thus, the cost function γ can be defined by

$$\gamma(t, \varphi(t, \cdot, \cdot), \Delta u(t)) =: \tilde{\gamma}(t, G(t)) \tag{5.13}$$

with an appropriate function $\tilde{\gamma}$. Consequently, the optimal regulator problem (5.7a–c) under stochastic uncertainty can also be represented by:

$$\min \mathrm{E} \left(\int_{t_j}^{t_f^{(j)}} \left(c\left(t, \Delta q(t), \Delta\dot{q}(t)\right) + \tilde{\gamma}\left(t, G(t)\right)\right) dt \,\Big|\, \mathcal{A}_{t_j} \right) \tag{5.14a}$$

subject to

$$F\left(p_D(\omega), q^{(j)}(t) + \Delta q(t), \dot{q}^{(j)}(t) + \Delta\dot{q}(t), \ddot{q}^{(j)}(t) + \Delta\ddot{q}(t)\right) \tag{5.14b}$$

$$= u^{(j)}(t) + G(t)h(\Delta q, \Delta\dot{q}(t)),\ t \geq t_j,\ \text{a.s. (almost sure)}$$

$$\Delta q(t_j) = \Delta q_j(\omega),\ \Delta\dot{q}(t_j) = \Delta\dot{q}_j(\omega),\ \text{f.s.} \tag{5.14c}$$

5.3 Computation of the Expected Total Costs of the Optimal Regulator Design

Working with the expected total costs, in this section we have to determine now the conditional expectation of the functional (5.6a):

$$\mathrm{E}\left(J^{(j)}|\mathcal{A}_{t_j}\right) = \int_{t_j}^{t_f^{(j)}} \left(\mathrm{E}\left(c(t, \Delta q(t), \Delta\dot{q}(t)) + \gamma(t, \varphi(t,\cdot,\cdot), \Delta u(t))\right)|\mathcal{A}_{t_j}\right) dt. \tag{5.15}$$

The system of differential equations (5.4a, b) yields the trajectory

$$\Delta q(t) = \Delta q(t, \Delta p_D, \Delta q_j, \Delta\dot{q}_j), \quad t_j \leq t \leq t_f^{(j)}, \tag{5.16a}$$

$$\Delta\dot{q}(t) = \Delta\dot{q}(t, \Delta p_D, \Delta q_j, \Delta\dot{q}_j), \quad t_j \leq t \leq t_f^{(j)}, \tag{5.16b}$$

where

$$\Delta p_D = \Delta p_D(\omega) \tag{5.16c}$$

$$\Delta q_j = \Delta q_j(\omega) \tag{5.16d}$$

$$\Delta\dot{q}_j = \Delta\dot{q}_j(\omega). \tag{5.16e}$$

According to (5.15), we consider first the computation of the expectations:

$$\mathrm{E}\Bigg(c\Big(t, \Delta q\big(t, \Delta p_D(\omega), \Delta q_j(\omega), \dot{\Delta q}_j(\omega)\big), \dot{\Delta q}\big(t, \Delta p_D(\omega), \Delta q_j(\omega),$$

$$\dot{\Delta q}_j(\omega)\big)\Big) \Big| \mathcal{A}_{t_j}\Bigg)$$

and

$$\mathrm{E}\Bigg(\gamma\Big(t, \varphi(t,\cdot,\cdot), \Delta u(t,\omega)\Big) \Big| \mathcal{A}_{t_j}\Bigg) \tag{5.17a}$$

with

$$\Delta u(t,\omega) = \varphi(t, \Delta z(t, \Delta p_D(\omega), \Delta z_j(\omega))) \tag{5.17b}$$

$$\Delta z(t) = \begin{pmatrix} \Delta q(t) \\ \dot{\Delta q}(t) \end{pmatrix} \tag{5.17c}$$

$$\Delta z_j = \begin{pmatrix} \Delta q_j \\ \dot{\Delta q}_j \end{pmatrix}. \tag{5.17d}$$

5.3.1 Computation of Conditional Expectations by Taylor Expansion

For abbreviation, we introduce the random parameter vector with conditional expectation

$$a := \begin{pmatrix} \Delta p_D(\omega) \\ \Delta q_j(\omega) \\ \dot{\Delta q}_j(\omega) \end{pmatrix}, \quad \overline{a} = \overline{a}^{(j)} := \begin{pmatrix} \overline{\Delta p_D}^{(j)} \\ \overline{\Delta q_j}^{(j)} \\ \overline{\dot{\Delta q}_j}^{(j)} \end{pmatrix} = \begin{pmatrix} \mathrm{E}(\Delta p_D(\omega)|\mathcal{A}_{t_j}) \\ \mathrm{E}(\Delta q_j(\omega)|\mathcal{A}_{t_j}) \\ \mathrm{E}(\dot{\Delta q}_j(\omega)|\mathcal{A}_{t_j}) \end{pmatrix}. \tag{5.18a}$$

Here, we may readily assume that the following property holds:

$$\textit{The random vectors } p_D(\omega) \textit{ and}$$

$$\Delta z_j(\omega) = \begin{pmatrix} \Delta q_j(\omega) \\ \dot{\Delta q}_j(\omega) \end{pmatrix} \textit{ are stochastically independent.} \tag{5.18b}$$

Then, the first conditional expectation (5.17a) can be represented by

$$E(c(t, \Delta z(t, a(\omega)))|\mathcal{A}_{t_j}) = E(\tilde{c}(t, a(\omega))|\mathcal{A}_{t_j}) \tag{5.19a}$$

where

$$\tilde{c}(t, a) := c(t, \Delta z(t, a)). \tag{5.19b}$$

For the computation of the expectation $E(\tilde{c}(t, a(\omega))|\mathcal{A}_{t_j})$ we consider the Taylor expansion of $a \to \tilde{c}(t, a)$ at $a = \overline{a}$ with

$$\overline{a} := E(a(\omega)|\mathcal{A}_{t_j}) = \left(E(a_1(\omega)|\mathcal{A}_{t_j}), \ldots, E(a_\nu(\omega)|\mathcal{A}_{t_j})\right)^T,$$

see Sect. 1.6.3. Hence,

$$\begin{aligned} \tilde{c}(t, a) &= \tilde{c}(t, \overline{a}) + \nabla_a \tilde{c}(t, \overline{a})^T (a - \overline{a}) + \frac{1}{2}(a - \overline{a})^T \nabla_a^2 \tilde{c}(t, \overline{a})(a - \overline{a}) \\ &\quad + \frac{1}{3!}\nabla_a^3 \tilde{c}(t, \overline{a}) \cdot (a - \overline{a})^3 + \ldots, \end{aligned} \tag{5.19c}$$

and therefore

$$\begin{aligned} E\left(\tilde{c}(t, a(\omega))|\mathcal{A}_{t_j}\right) &= \tilde{c}(t, \overline{a}) + \frac{1}{2}E\left((a - \overline{a})^T \nabla_a^2 \tilde{c}(t, \overline{a})(a - \overline{a})|\mathcal{A}_{t_j}\right) \\ &\quad + \frac{1}{3!}E\left(\nabla_a^3 \tilde{c}(t, \overline{a}) \cdot (a(\omega) - \overline{a})^3|\mathcal{A}_{t_j}\right) + \ldots . \end{aligned} \tag{5.19d}$$

Here, (5.19d) is obtained from (5.19c) due to the following equation:

$$E\left(\nabla_a \tilde{c}(t, \overline{a})^T (a(\omega) - \overline{a})|\mathcal{A}_{t_j}\right) = \nabla_a \tilde{c}(t, \overline{a})^T \underbrace{\left(\underbrace{E\left(a(\omega)|\mathcal{A}_{t_j}\right)}_{=\overline{a}^{(j)}} - \underbrace{\overline{a}}_{=\overline{a}^{(j)}}\right)}_{=0} = 0.$$

In practice, mostly Taylor expansions up to second or eventually third order are taken into account. Thus, in the following we consider Taylor expansions of $a \to \tilde{c}(t, a)$ up to third order. Consequently, for the expansion (5.19d) up to the 3rd order we need the derivatives $\nabla_a^2 \tilde{c}(t, \overline{a})$, $\nabla_a^3 \tilde{c}(t, \overline{a})$ at $\overline{a}$.

Computation of $\nabla_a \tilde{c}(t, a)$

According to (5.19b) we have

$$\tilde{c}(t, a) = c(t, \Delta z(t, a)).$$

Differentiation with respect to the components of a yields

$$\frac{\partial \tilde{c}}{\partial a_j}(t,a) = \sum_{k=1}^{2n} \frac{\partial c}{\partial \Delta z_k}(t, \Delta z(t,a)) \frac{\partial \Delta z_k}{\partial a_j}(t,a), \quad j = 1, \dots, \nu. \tag{5.20}$$

Writing this in vector-matrix form, we get, cf. Sect. 1.6.3,

$$\nabla_a \tilde{c}(t,a) = \begin{pmatrix} \vdots \\ \frac{\partial \tilde{c}}{\partial a_j} \\ \vdots \end{pmatrix} = \underbrace{\begin{pmatrix} \\ \frac{\partial \Delta z_1}{\partial a_j} & \dots & \frac{\partial \Delta z_{2n}}{\partial a_j} \\ \\ \end{pmatrix}}_{(\nabla_a \Delta z)^T} \underbrace{\begin{pmatrix} \vdots \\ \frac{\partial c}{\partial \Delta z_k} \\ \vdots \end{pmatrix}}_{\nabla_{\Delta z} c},$$

hence,

$$\nabla_a \tilde{c}(t,a) = \big(\nabla_a \Delta z(t,a)\big)^T \nabla_{\Delta z} c\big(t, \Delta z(t,a)\big). \tag{5.21}$$

Higher derivatives with respect to a follow by further differentiation of (5.21). However, in the following we consider *quadratic cost functions* which simplifies the computation of the expected cost function (5.15).

5.3.2 *Quadratic Cost Functions*

For the regulator optimization problem under stochastic uncertainty (5.7a–c) we consider now the following class of *quadratic cost functions*:

(i) Let the cost function $c = c(t, \Delta q, \Delta\dot{q})$ for the evaluation of the deviation $\Delta z = \begin{pmatrix} \Delta q \\ \Delta\dot{q} \end{pmatrix}$ between the effective trajectory $z = z(t) = \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix}$ and the optimal reference trajectory $z^{(j)} = z^{(j)}(t) = \begin{pmatrix} q^{(j)}(t) \\ \dot{q}^{(j)}(t) \end{pmatrix}$, $t \geq t_j$, be given by (5.6c), d.h.

$$c(t, \Delta q, \Delta\dot{q}) := \Delta q^T C_q \Delta q + \Delta\dot{q}^T C_{\dot{q}} \Delta\dot{q}, \ t \geq t_j,$$

with symmetric, positive (semi-)definite matrices $C_q = C_q(t), \quad C_{\dot{q}} = C_{\dot{q}}(t), \ t \geq t_j$.

(ii) Suppose that the cost function for the evaluation of the regulator expenses is given by (5.6d), hence,

$$\gamma(t, \varphi(t, \cdot, \cdot), \Delta u) := \Delta u^T C_u \Delta u,\ t \geq t_j,$$

with a symmetric, positive (semi-)definite matrix $C_u = C_u(t),\ t \geq t_j$.

5.4 Approximation of the Stochastic Regulator Optimization Problem

Under the cost assumptions $(i), (ii)$ in Sect. 5.3.2 and because of (5.1b), thus

$$\Delta u := \varphi(t, \Delta q(t), \dot{\Delta q}(t)),$$

from (5.7a–c) we obtain now (at stage j) the following optimal regulator problem under stochastic uncertainty:

$$\min \mathrm{E}\Bigg(\int_{t_j}^{t_f^{(j)}} \Big(\Delta q(t)^T C_q \Delta q(t) + \dot{\Delta q}(t) C_{\dot{q}} \dot{\Delta q}(t) + \varphi\big(t, \Delta q(t), \dot{\Delta q}(t)\big)^T C_u \varphi\big(t, \Delta q(t), \dot{\Delta q}(t)\big) \Big) dt \,\Big|\, \mathcal{A}_{t_j} \Bigg) \tag{5.22a}$$

s.t.

$$F\left(\overline{p}_D^{(j)} + \Delta p_D(\omega), q^{(j)}(t) + \Delta q(t), \dot{q}^{(j)}(t) + \dot{\Delta q}(t), \ddot{q}^{(j)}(t) + \ddot{\Delta q}(t)\right)$$

$$= u^{(j)}(t) + \varphi\left(t, \Delta q(t), \dot{\Delta q}(t)\right) \text{ f.s. }, t \geq t_j, \tag{5.22b}$$

$$\Delta q(t_j) = \Delta q_j(\omega), \dot{\Delta q}(t_j) = \dot{\Delta q}_j(\omega) \text{ f.s.} \tag{5.22c}$$

As already mentioned in Sect. 5.3, (5.16a, b), for the tracking error we obtain from (5.22b, c) the representation:

$$\Delta q(t) = \Delta q\left(t, \Delta p_D, \Delta q_j, \dot{\Delta q}_j\right),\ t \geq t_j$$

$$\dot{\Delta q}(t) = \dot{\Delta q}\left(t, \Delta p_D, \Delta q_j, \dot{\Delta q}_j\right),\ t \geq t_j,$$

where $\Delta p_D = \Delta p_D(\omega)$, $\Delta q_j = \Delta q_j(\omega)$, $\Delta\dot{q}_j = \Delta\dot{q}_j(\omega)$. Under appropriate regularity assumptions on the functions F, φ as well as on $q^{(j)}$ and $u^{(j)}$, the solution

$$\Delta z = \begin{pmatrix} \Delta q \\ \Delta\dot{q} \end{pmatrix} = \begin{pmatrix} \Delta q(t, \Delta p_D, \Delta q_j, \Delta\dot{q}_j) \\ \Delta\dot{q}(t, \Delta p_D, \Delta q_j, \Delta\dot{q}_j) \end{pmatrix}, \ t \geq t_j,$$

of (5.22b, c) has continuous partial derivatives—of a prescribed order—with respect to the parameter deviations $\Delta p_D, \Delta q_j, \Delta\dot{q}_j$. By Taylor expansion of Δz with respect to the vector of deviations $\Delta a = (\Delta p_D^T, \Delta q_j^T, \Delta\dot{q}_j^T)^T$ at $\Delta a = 0$ we find

$$\Delta q(t) = \Delta q(t, \mathbf{0}) + \frac{\partial \Delta q}{\partial \Delta p_D}(t, \mathbf{0})\Delta p_D + \frac{\partial \Delta q}{\partial \Delta q_j}(t, \mathbf{0})\Delta q_j + \frac{\partial \Delta q}{\partial \Delta\dot{q}_j}(t, \mathbf{0})\Delta\dot{q}_j + \ldots \tag{5.23a}$$

$$\Delta\dot{q}(t) = \Delta\dot{q}(t, \mathbf{0}) + \frac{\partial \Delta\dot{q}}{\partial \Delta p_D}(t, \mathbf{0}\Delta p_D + \frac{\partial \Delta\dot{q}}{\partial \Delta q_j}(t, \mathbf{0})\Delta q_j + \frac{\partial \Delta\dot{q}}{\partial \Delta\dot{q}_j}(t, 0)\Delta\dot{q}_j + \ldots \tag{5.23b}$$

with $\mathbf{0} := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and the Jacobians

$$\frac{\partial \Delta q}{\partial \Delta p_D}(t, \Delta p_D, \Delta q_j, \Delta\dot{q}_j) := \begin{pmatrix} \frac{\partial \Delta q_1}{\partial \Delta p_{D_1}} & \frac{\partial \Delta q_1}{\partial \Delta p_{D_2}} & \cdots & \frac{\partial \Delta q_1}{\partial \Delta p_{D_\nu}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial \Delta q_n}{\partial \Delta p_{D_1}} & \frac{\partial \Delta q_n}{\partial \Delta p_{D_2}} & \cdots & \frac{\partial \Delta q_n}{\partial \Delta p_{D_\nu}} \end{pmatrix}(t, \Delta p_D, \Delta q_j, \Delta\dot{q}_j),$$

etc. For the choice of parameters $\Delta a = 0$ or $\Delta p_D = 0, \Delta q_j = 0, \Delta\dot{q}_j = 0$, according to (5.22b, c), the function $\Delta z = \Delta z(t, 0) = \begin{pmatrix} \Delta q(t, 0) \\ \Delta\dot{q}(t, 0) \end{pmatrix}$ is the solution of the system of differential equations:

$$F\left(\overline{p}_D^{(j)}, q^{(j)}(t) + \Delta q(t), \dot{q}^{(j)}(t) + \Delta\dot{q}(t), \ddot{q}^{(j)}(t) + \Delta\ddot{q}(t)\right)$$
$$= u^{(j)}(t) + \varphi\left(t, \Delta q(t), \Delta\dot{q}(t)\right), \ t \geq t_j, \tag{5.24a}$$

$$\Delta q(t_j) = 0, \ \Delta\dot{q}(t) = 0. \tag{5.24b}$$

Because of (4.50c), i.e.,

$$F\left(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}, \ddot{q}^{(j)}(t)\right) = u^{(j)}(t), \ t \geq t_j,$$

and (5.1d), thus,

$$\varphi(t, 0, 0) = 0,$$

system (5.24a, b) has the unique solution

$$\Delta z(t, 0) = \begin{pmatrix} \Delta q(t, 0) \\ \Delta \dot{q}(t, 0) \end{pmatrix} = 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ t \geq t_j.$$

Consequently, the expansion (5.23a, b) reads

$$\Delta q(t) = \Delta q(t, \Delta p_D, \Delta q_j, \Delta \dot{q}_j)$$
$$= \frac{\partial \Delta q}{\partial \Delta p_D}(t, 0)\Delta p_D + \frac{\partial \Delta q}{\partial \Delta q_j}(t, 0)\Delta q_j + \frac{\partial \Delta q}{\partial \Delta \dot{q}_j}(t, 0)\Delta \dot{q}_j + \ldots \tag{5.25a}$$
$$\Delta \dot{q}(t) = \Delta \dot{q}(t, \Delta p_D, \Delta q_j, \Delta \dot{q}_j)$$
$$= \frac{\partial \Delta \dot{q}}{\partial \Delta p_D}(t, 0)\Delta p_D + \frac{\partial \Delta \dot{q}}{\partial \Delta q_j}(t, 0)\Delta q_j + \frac{\partial \Delta \dot{q}}{\partial \Delta \dot{q}_j}(t, 0)\Delta \dot{q}_j + \ldots \tag{5.25b}$$

The Jacobians $\frac{\partial \Delta q}{\partial \Delta p_D}, \frac{\partial \Delta q}{\partial \Delta q_j}, \ldots$ arising in (5.25a, b) can be obtained by partial differentiation of the system of differential equations (5.22b, c) with respect to $\Delta p_D, \Delta q_j, \Delta \dot{q}_j$, cf. Sect. 4.6.2.

5.4.1 Approximation of the Expected Costs: Expansions of 1st Order

According to (5.22a), we have to determine the following conditional expectation:

$$\begin{aligned} \mathrm{E}\left(f^{(j)} | \mathcal{A}_j\right) := \quad & \mathrm{E}\Big(\Delta q(t)^T C_q \Delta q(t) + \Delta \dot{q}(t)^T C_{\dot{q}} \Delta \dot{q}(t) \\ & + \varphi\big(t, \Delta q(t), \Delta \dot{q}(t)\big)^T C_u \varphi\big(t, \Delta q(t), \Delta \dot{q}(t)\big) \Big| \mathcal{A}_{t_j}\Big). \end{aligned} \tag{5.26}$$

For the computation of the expectation in (5.26) also the feedback function $\varphi = \varphi(t, \Delta q, \Delta \dot{q})$ is approximated by means of Taylor expansion at $\Delta q = 0, \Delta \dot{q} = 0$. Because of (5.1d), hence, $\varphi(t, 0, 0) = 0$, we get

$$\varphi(t, \Delta q, \Delta \dot{q}) = \frac{\partial \varphi}{\partial \Delta q}(t, 0, 0)\Delta q + \frac{\partial \varphi}{\partial \Delta \dot{q}}(t, 0, 0)\Delta \dot{q} + \ldots \tag{5.27a}$$

with unknown Jacobians

$$\frac{\partial \varphi}{\partial \Delta q}(t,0,0),\ \frac{\partial \varphi}{\partial \dot{\Delta} q}(t,0,0),\ldots \tag{5.27b}$$

to be determined. Using in the Taylor expansion (5.27a) of the feedback function $\varphi = \varphi(t,\cdot,\cdot)$ only the linear terms, we find, c.f. (5.26)

$$\mathrm{E}\left(f^{(j)}|\mathcal{A}_{t_j}\right) \approx \mathrm{E}\left(\tilde{f}^{(j)}|\mathcal{A}_{t_j}\right), \tag{5.28a}$$

where

$$\begin{aligned}
&\mathrm{E}\left(\tilde{f}^{(j)}|\mathcal{A}_j\right)\\
&:= \mathrm{E}\Bigg(\Delta q(t)^T C_q \Delta q(t) + \dot{\Delta} q(t)^T C_{\dot{q}} \dot{\Delta} q(t) + \Big(\frac{\partial \varphi}{\partial \Delta q}(t,0,0)\Delta q(t)\\
&+\frac{\partial \varphi}{\partial \dot{\Delta} q}(t,0,0)\dot{\Delta} q(t)\Big)^T C_u\Big(\frac{\partial \varphi}{\partial \Delta q}(t,0,0)\Delta q(t) + \frac{\partial \varphi}{\partial \dot{\Delta} q}(t,0,0)\dot{\Delta} q(t)\Big)\Big|\mathcal{A}_{t_j}\Bigg)\\
&= \mathrm{E}\Bigg(\Delta q(t)^T C_q \Delta q(t) + \dot{\Delta} q(t)^T C_{\dot{q}} \dot{\Delta} q(t) + \Big(\Delta q(t)^T \frac{\partial \varphi}{\partial \Delta q}(t,0,0)^T C_u\\
&+\dot{\Delta} q(t)^T \frac{\partial \varphi}{\partial \dot{\Delta} q}(t,0,0)^T C_u\Big)\Big(\frac{\partial \varphi}{\partial \Delta q}(t,0,0)\Delta q(t) + \frac{\partial \varphi}{\partial \dot{\Delta} q}(t,0,0)\dot{\Delta} q(t)\Big)\Big|\mathcal{A}_{t_j}\Bigg).
\end{aligned} \tag{5.28b}$$

This yields

$$\begin{aligned}
\mathrm{E}\left(\tilde{f}^{(j)}|\mathcal{A}_j\right) &:= \mathrm{E}\Bigg(\Delta q(t)^T C_q \Delta q(t) + \dot{\Delta} q(t)^T C_{\dot{q}} \dot{\Delta} q(t)\\
&+ \Delta q(t)^T \frac{\partial \varphi}{\partial \Delta q}(t,0,0)^T C_u \frac{\partial \varphi}{\partial \Delta q}(t,0,0)\Delta q(t)\\
&+\dot{\Delta} q(t)^T \frac{\partial \varphi}{\partial \dot{\Delta} q}(t,0,0)^T C_u \frac{\partial \varphi}{\partial \dot{\Delta} q}(t,0,0)\dot{\Delta} q(t)\\
&+\Delta q(t)^T \frac{\partial \varphi}{\partial \Delta q}(t,0,0)^T C_u \frac{\partial \varphi}{\partial \dot{\Delta} q}(t,0,0)\dot{\Delta} q(t)\\
&+\dot{\Delta} q(t)^T \frac{\partial \varphi}{\partial \dot{\Delta} q}(t,0,0)^T C_u \frac{\partial \varphi}{\partial \Delta q}(t,0,0)\Delta q(t)\Big|\mathcal{A}_{t_j}\Bigg)
\end{aligned} \tag{5.28c}$$

and therefore

$$\begin{aligned}
\mathrm{E}\left(\tilde{f}^{(j)}|\mathcal{A}_j\right) = \mathrm{E}\Bigg(& \Delta q(t)^T\left(C_q + \frac{\partial\varphi}{\partial\Delta q}(t,0,0)^T C_u \frac{\partial\varphi}{\partial\Delta q}(t,0,0)\right)\Delta q(t) \\
& + \dot{\Delta q}(t)^T\left(C_{\dot q} + \frac{\partial\varphi}{\partial\dot{\Delta q}}(t,0,0)^T C_u \frac{\partial\varphi}{\partial\dot{\Delta q}}(t,0,0)\right)\dot{\Delta q}(t) \\
& + 2\Delta q(t)^T \frac{\partial\varphi}{\partial\Delta q}(t,0,0)^T C_u \frac{\partial\varphi}{\partial\dot{\Delta q}}(t,0,0)\dot{\Delta q}(t)\Big|\mathcal{A}_{t_j}\Bigg).
\end{aligned} \tag{5.28d}$$

Putting

$$Q_q = Q_q(t,\varphi) := C_q + \frac{\partial\varphi}{\partial\Delta q}(t,0,0)^T C_u \frac{\partial\varphi}{\partial\Delta q}(t,0,0) \tag{5.29a}$$

$$Q_{\dot q} = Q_{\dot q}(t,\varphi) := C_{\dot q} + \frac{\partial\varphi}{\partial\dot{\Delta q}}(t,0,0)^T C_u \frac{\partial\varphi}{\partial\dot{\Delta q}}(t,0,0) \tag{5.29b}$$

$$Q_u = \quad Q_u(t,\varphi) := \frac{\partial\varphi}{\partial\Delta q}(t,0,0)^T C_u \frac{\partial\varphi}{\partial\dot{\Delta q}}(t,0,0), \tag{5.29c}$$

we obtain

$$\begin{aligned}
\tilde{f}^{(j)} :=\ & \Delta q(t)^T\left(C_q + \frac{\partial\varphi}{\partial\Delta q}(t,0,0)^T C_u \frac{\partial\varphi}{\partial\Delta q}(t,0,0)\right)\Delta q(t) \\
& + \dot{\Delta q}(t)^T\left(C_{\dot q} + \frac{\partial\varphi}{\partial\dot{\Delta q}}(t,0,0)^T C_u \frac{\partial\varphi}{\partial\dot{\Delta q}}(t,0,0)\right)\dot{\Delta q}(t) \\
& + 2\Delta q(t)^T \frac{\partial\varphi}{\partial\Delta q}(t,0,0)^T C_u \frac{\partial\varphi}{\partial\dot{\Delta q}}(t,0,0)\dot{\Delta q}(t) \\
=\ & \left(\Delta q(t)^T, \dot{\Delta q}(t)^T\right)\begin{pmatrix} Q_q & Q_u \\ Q_u^T & Q_{\dot q}\end{pmatrix}\begin{pmatrix}\Delta q(t) \\ \dot{\Delta q}(t)\end{pmatrix} \\
=\ & \Delta z(t)^T Q(t,\varphi)\Delta z(t).
\end{aligned} \tag{5.30a}$$

Here,

$$\Delta z(t) = \begin{pmatrix}\Delta q(t) \\ \dot{\Delta q}(t)\end{pmatrix}, \tag{5.30b}$$

$$\begin{aligned}
Q(t,\varphi) &:= \begin{pmatrix} Q_q & Q_u \\ Q_u^T & Q_{\dot q}\end{pmatrix} \\
&= \begin{pmatrix} C_q + \frac{\partial\varphi}{\partial\Delta q}(t,0,0)^T C_u \frac{\partial\varphi}{\partial\Delta q}(t,0,0) & \frac{\partial\varphi}{\partial\Delta q}(t,0,0)^T C_u \frac{\partial\varphi}{\partial\dot{\Delta q}}(t,0,0) \\ \frac{\partial\varphi}{\partial\dot{\Delta q}}(t,0,0)^T C_u \frac{\partial\varphi}{\partial\Delta q}(t,0,0) & C_{\dot q} + \frac{\partial\varphi}{\partial\dot{\Delta q}}(t,0,0)^T C_u \frac{\partial\varphi}{\partial\dot{\Delta q}}(t,0,0)\end{pmatrix}.
\end{aligned} \tag{5.30c}$$

Obviously, $Q_q, Q_{\dot{q}}$ and also $Q = Q(t, \varphi)$ are symmetric matrices. Since $C_q, C_{\dot{q}}, C_u$ are positive (semi-)definite matrices, according to the definition (5.28b) of the approximate total cost function we get

$$\tilde{f}^{(j)}(t, \Delta z) > (\geq)\ 0, \quad \text{for all } \Delta z \neq 0.$$

Thus, we have the following result on the approximate total cost function $\tilde{f}^{(j)} = \tilde{f}^{(j)}(t, \Delta z)$:

Lemma 5.1 *The quadratic approximation* $\tilde{f}^{(j)} = \tilde{f}^{(j)}(t, \Delta z)$ *of* $f^{(j)} = f^{(j)}(t, \Delta z)$ *is a positive semidefinite quadratic form in* $\Delta z = \begin{pmatrix} \Delta q \\ \Delta \dot{q} \end{pmatrix}$. *If* $C_q = C_q(t), C_{\dot{q}} = C_{\dot{q}}(t)$ *are positive definite weight matrices, then also the matrix* $Q = Q(t, \varphi)$ *is positive definite, and* $\tilde{f}^{(j)}$ *is then a positive definite quadratic form in* Δz.

Proof The first part of the assertion is clear. Suppose now that $C_q, C_{\dot{q}}$ are positive definite weight matrices. Then, according to the definition of $\tilde{f}^{(j)} = \tilde{f}^{(j)}(t, \Delta z)$ by (5.28b, c) and since C_u is positive (semi-)definite, we have $\tilde{f}^{(j)}(t, \Delta z) = 0$ if and only if $\Delta z) = 0$.

For the approximate computation of the expectation $\mathrm{E}\left(\tilde{f}^{(j)} | \mathcal{A}_{t_j}\right)$ we use now only the linear terms of (5.25a, b), hence

$$\begin{aligned}
\Delta z(t) &= \begin{pmatrix} \Delta q(t) \\ \Delta \dot{q}(t) \end{pmatrix} \\
&\cong \begin{pmatrix} \frac{\partial \Delta q}{\partial \Delta p_D}(t, 0) \\ \frac{\partial \Delta \dot{q}}{\partial \Delta p_D}(t, 0) \end{pmatrix} \Delta p_D + \begin{pmatrix} \frac{\partial \Delta q}{\partial \Delta q_j}(t, 0) \\ \frac{\partial \Delta \dot{q}}{\partial \Delta q_j}(t, 0) \end{pmatrix} \Delta q_j + \begin{pmatrix} \frac{\partial \Delta q}{\partial \Delta \dot{q}_j}(t, 0) \\ \frac{\partial \Delta \dot{q}}{\partial \Delta \dot{q}_j}(t, 0) \end{pmatrix} \Delta \dot{q}_j \\
&= \frac{\partial \Delta z}{\partial \Delta p_D}(t, 0) \Delta p_D + \frac{\partial \Delta z}{\partial \Delta q_j}(t, 0) \Delta q_j + \frac{\partial \Delta z}{\partial \Delta \dot{q}_j}(t, 0) \Delta \dot{q}_j .
\end{aligned} \tag{5.31}$$

Inserting (5.31) into (5.30a) yields

$$\begin{aligned}
\tilde{f}^{(j)} &= \Delta z^T Q(t, \varphi) \Delta z \\
&= \left(\Delta p_D^T \left(\frac{\partial \Delta z}{\partial \Delta p_D}(t, 0) \right)^T + \Delta q_j^T \left(\frac{\partial \Delta z}{\partial \Delta q_j}(t, 0) \right)^T + \Delta \dot{q}_j^T \left(\frac{\partial \Delta z}{\partial \Delta \dot{q}_j}(t, 0) \right)^T \right) \\
&\times Q \left(\frac{\partial \Delta z}{\partial \Delta p_D}(t, 0) \Delta p_D + \frac{\partial \Delta z}{\partial \Delta q_j}(t, 0) \Delta q_j + \frac{\partial \Delta z}{\partial \Delta \dot{q}_j}(t, 0) \Delta \dot{q}_j \right) \\
&= \Delta p_D^T \left(\frac{\partial \Delta z}{\partial \Delta p_D}(t, 0) \right)^T Q \frac{\partial \Delta z}{\partial \Delta p_D}(t, 0) \Delta p_D
\end{aligned}$$

$$
\begin{aligned}
&+ \ \Delta q_j^T \left(\frac{\partial \Delta z}{\partial \Delta q_j}(t,0) \right)^T Q \frac{\partial \Delta z}{\partial \Delta q_j}(t,0) \Delta q_j \\
&+ \ \Delta \dot{q}_j^T \left(\frac{\partial \Delta z}{\partial \Delta \dot{q}_j}(t,0) \right)^T Q \frac{\partial \Delta z}{\partial \Delta \dot{q}_j}(t,0) \Delta \dot{q}_j \\
&+ \ 2\Delta q_j^T \left(\frac{\partial \Delta z}{\partial \Delta q_j}(t,0) \right)^T Q \frac{\partial \Delta z}{\partial \Delta p_D}(t,0) \Delta p_D \\
&+ \ 2\Delta \dot{q}_j^T \left(\frac{\partial \Delta z}{\partial \Delta \dot{q}_j}(t,0) \right)^T Q \frac{\partial \Delta z}{\partial \Delta p_D}(t,0) \Delta p_D \\
&+ \ 2\Delta \dot{q}_j^T \left(\frac{\partial \Delta z}{\partial \Delta \dot{q}_j}(t,0) \right)^T Q \frac{\partial \Delta z}{\partial \Delta q_j}(t,0) \Delta q_j .
\end{aligned}
\tag{5.32}
$$

Readily we may assume that the random vectors

$$
\begin{aligned}
&\Delta p_D(\omega), \ \ \Delta q_j(\omega) \quad \text{as well as} \\
&\Delta p_D(\omega), \ \ \Delta \dot{q}_j(\omega)
\end{aligned}
$$

are stochastically independent. Because of

$$
\mathrm{E}\Big(\Delta p_D(\omega) \Big| \mathcal{A}_{t_j} \Big) = 0,
$$

(5.32) yields

$$
\begin{aligned}
\mathrm{E}\Big(\tilde{f}^{(j)} \Big| \mathcal{A}_{t_j} \Big) = \ & \mathrm{E}\Big(\Delta p_D(\omega)^T P_{\Delta p_D} \Delta p_D(\omega) \Big| \mathcal{A}_{t_j} \Big) \\
& + \mathrm{E}\Big(\Delta q_j(\omega)^T P_{\Delta q_j} \Delta q_j(\omega) \Big| \mathcal{A}_{t_j} \Big) + \mathrm{E}\Big(\Delta \dot{q}_j(\omega)^T P_{\Delta \dot{q}_j} \Delta \dot{q}_j(\omega) \Big| \mathcal{A}_{t_j} \Big) \\
& + 2\mathrm{E}\Big(\Delta \dot{q}_j(\omega)^T P_{\Delta \dot{q}_j, \Delta q_j} \Delta q_j(\omega) \Big| \mathcal{A}_{t_j} \Big)
\end{aligned}
\tag{5.33a}
$$

with the matrices

$$
P_{\Delta p_D} := \left(\frac{\partial \Delta z}{\partial \Delta p_D}(t,0) \right)^T Q(t,\varphi) \left(\frac{\partial \Delta z}{\partial \Delta p_D}(t,0) \right), \tag{5.33b}
$$

$$
P_{\Delta q_j} := \left(\frac{\partial \Delta z}{\partial \Delta q_j}(t,0) \right)^T Q(t,\varphi) \left(\frac{\partial \Delta z}{\partial \Delta q_j}(t,0) \right), \tag{5.33c}
$$

$$
P_{\Delta \dot{q}_j} := \left(\frac{\partial \Delta z}{\partial \Delta \dot{q}_j}(t,0) \right)^T Q(t,\varphi) \left(\frac{\partial \Delta z}{\partial \Delta \dot{q}_j}(t,0) \right), \tag{5.33d}
$$

$$P_{\dot{\Delta q}_j,\Delta q_j} := \left(\frac{\partial \Delta z}{\partial \dot{\Delta q}_j}(t,0)\right)^T Q(t,\varphi)\left(\frac{\partial \Delta z}{\partial \Delta q_j}(t,0)\right). \tag{5.33e}$$

The P-matrices defined by (5.33b–e) are deterministic and depend on the feedback law $\varphi = \varphi(t, \Delta z)$.

Die conditional expectations in (5.33a) can be determined as follows: Let $P = (p_{kl})$ denote one of the P-matrices, where the first three matrices $P_{\Delta p_D}, P_{\Delta q_j}, P_{\dot{\Delta q}_j}$ are symmetric. Now, for $P = P_{\Delta p_D}$ we get

$$\begin{aligned}
\mathrm{E}\Big(\Delta p_D(\omega)^T P_{\Delta p_D} \Delta p_D(\omega)\Big|\mathcal{A}_{t_j}\Big) &= \mathrm{E}\Big(\sum_{k,l=1}^{\nu} p_{kl}\Delta p_{Dk}(\omega)\Delta p_{Dl}(\omega)\Big|\mathcal{A}_{t_j}\Big)\\
&= \sum_{k,l=1}^{\nu} p_{kl}\mathrm{E}\Big(\Delta p_{Dk}(\omega)\Delta p_{Dl}(\omega)\Big|\mathcal{A}_{t_j}\Big)\\
&= \sum_{k,l=1}^{\nu} p_{kl}\,\mathrm{cov}^{(j)}\big(p_{Dk}, p_{Dl}\big).
\end{aligned} \tag{5.34a}$$

Here, the last equation follows, cf. (5.16c), from the following equations:

$$\begin{aligned}
\mathrm{cov}^{(j)}\big(p_{Dk}, p_{Dl}\big) &:= \mathrm{E}\Big(\big(p_{Dk}(\omega) - \overline{p_{Dk}}^{(j)}\big)\big(p_{Dl}(\omega) - \overline{p_{Dl}}^{(j)}\big)\Big|\mathcal{A}_{t_j}\Big)\\
&= \mathrm{E}\Big(\Delta p_{Dk}(\omega)\Delta p_{Dl}(\omega)\Big|\mathcal{A}_{t_j}\Big),\ k,l = 1,\dots,\nu.
\end{aligned} \tag{5.34b}$$

Because of the symmetry of the covariance matrix

$$\begin{aligned}
\mathrm{cov}^{(j)}\big(p_D(\cdot)\big) &:= \Big(\mathrm{cov}^{(j)}\big(p_{Dk}(\cdot), p_{Dl}(\cdot)\big)\Big)_{k,l=1,\dots,\nu}\\
&= \mathrm{E}\Big(\big(p_D(\omega) - \overline{p_D}^{(j)}\big)\big(p_D(\omega) - \overline{p_D}^{(j)}\big)^T\Big|\mathcal{A}_{t_j}\Big)
\end{aligned} \tag{5.34c}$$

of $p_D = p_D(\omega)$, from (5.34a) we obtain

$$\begin{aligned}
&\mathrm{E}\ \Big(\Delta p_D(\omega)^T P_{\Delta p_D} \Delta p_D(\omega)\Big|\mathcal{A}_{t_j}\Big) = \sum_{k,l=1}^{\nu} p_{kl}\,\mathrm{cov}^{(j)}\big(p_{Dk}, p_{Dl}\big)\\
&= \sum_{k,l=1}^{\nu} p_{kl}\,\mathrm{cov}^{(j)}\big(p_{Dl}, p_{Dk}\big)\\
&= \sum_{k=1}^{\nu}\Big(\sum_{l=1}^{\nu} p_{kl}\,\mathrm{cov}^{(j)}\big(p_{Dl}, p_{Dk}\big)\Big)
\end{aligned}$$

$$= \sum_{k=1}^{\nu} \Big(\text{ k-th row of } P \Big) \cdot \Big(\text{ k-th column of } \operatorname{cov}^{(j)}\big(p_D(\cdot)\big) \Big)$$

$$= \operatorname{tr} P_{\Delta p_D} \operatorname{cov}^{(j)}\big(p_D(\cdot)\big). \tag{5.35a}$$

In the same way we obtain

$$E\Big(\Delta q_j(\omega)^T P_{\Delta q_j} \Delta q_j(\omega) \Big| \mathcal{A}_{t_j}\Big) = \operatorname{tr} P_{\Delta q_j} \operatorname{cov}^{(j)}\big(\Delta q_j(\cdot)\big). \tag{5.35b}$$

$$E\Big(\Delta \dot{q}_j(\omega)^T P_{\Delta \dot{q}_j} \Delta \dot{q}_j(\omega) \Big| \mathcal{A}_{t_j}\Big) = \operatorname{tr} P_{\Delta \dot{q}_j} \operatorname{cov}^{(j)}\big(\Delta \dot{q}_j(\cdot)\big). \tag{5.35c}$$

For the last expectation in (5.33a) with $P := P_{\Delta \dot{q}_j, \Delta q_j}$ we find

$$E\ \Big(\Delta \dot{q}_j(\omega)^T P_{\Delta \dot{q}_j, \Delta q_j} \Delta q_j(\omega) \Big| \mathcal{A}_{t_j}\Big) = E\Big(\sum_{k,l=1}^{\nu} p_{kl} \Delta \dot{q}_{jk}(\omega) \Delta q_{jl}(\omega) \Big| \mathcal{A}_{t_j}\Big)$$

$$= \sum_{k,l=1}^{\nu} p_{kl} \operatorname{cov}^{(j)}\big(\Delta \dot{q}_{jk}(\cdot), \Delta q_{jl}(\cdot)\big)$$

$$= \sum_{k,l=1}^{\nu} p_{kl} \operatorname{cov}^{(j)}\big(\Delta q_{jl}(\cdot), \Delta \dot{q}_{jk}(\cdot)\big)$$

$$= \sum_{k=1}^{\nu} \Big(\sum_{l=1}^{\nu} p_{kl} \operatorname{cov}^{(j)}\big(\Delta q_{jl}(\cdot), \Delta \dot{q}_{jk}(\cdot)\big)\Big)$$

$$= \sum_{k=1}^{\nu} \Big(\text{ k-th row of } P \Big) \cdot \Big(\text{ k-th column of } \operatorname{cov}^{(j)}\big(\Delta q_j(\cdot), \Delta \dot{q}_j(\cdot)\big)\Big), \tag{5.35d}$$

where

$$\operatorname{cov}^{(j)}\big(\Delta q_j(\cdot), \Delta \dot{q}_j(\cdot)\big) := E\Big(\big(\Delta q_j(\omega) - \overline{\Delta q_j}^{(j)}\big)\big(\Delta \dot{q}_j(\omega) - \overline{\Delta \dot{q}_j}^{(j)}\big) \Big| \mathcal{A}_{t_j}\Big) \tag{5.35e}$$

According to (5.35d) we have

$$E\Big(\Delta \dot{q}_j(\omega)^T P_{\Delta \dot{q}_j, \Delta q_j} \Delta q_j(\omega) \Big| \mathcal{A}_{t_j}\Big) = \operatorname{tr} P_{\Delta \dot{q}_j, \Delta q_j} \operatorname{cov}^{(j)}\big(\Delta q_j(\cdot), \Delta \dot{q}_j(\cdot)\big) \tag{5.35f}$$

Inserting (5.35a–c) and (5.35f) into (5.33a), we finally get

$$
\begin{aligned}
\mathrm{E}\Big(\tilde{f}^{(j)}\Big|\mathcal{A}_{t_j}\Big) = \ &\mathrm{tr}\ P_{\Delta p_D}\ \mathrm{cov}^{(j)}\big(p_D(\cdot)\big) + \ \mathrm{tr}\ P_{\Delta q_j}\ \mathrm{cov}^{(j)}\big(\Delta q_j(\cdot)\big)\\
&+ \mathrm{tr}\ P_{\dot{\Delta q}_j}\ \mathrm{cov}^{(j)}\big(\dot{\Delta q}_j(\cdot)\big)\\
&+ \mathrm{tr}\ P_{\dot{\Delta q}_j,\Delta q_j}\ \mathrm{cov}^{(j)}\big(\Delta q_j(\cdot),\dot{\Delta q}_j(\cdot)\big).
\end{aligned} \tag{5.36}
$$

5.5 Computation of the Derivatives of the Tracking Error

As can be seen from (5.33a–e) and (5.36), for the approximate computation of the conditional expectation of the objective function (5.22a) we need the Jacobians or *"sensitivities"*:

$$
\frac{\partial \Delta q}{\partial \Delta p_D}(t,0), \frac{\partial \dot{\Delta q}}{\partial \Delta p_D}(t,0), \frac{\partial \Delta q}{\partial \Delta q_j}(t,0), \frac{\partial \dot{\Delta q}}{\partial \Delta q_j}(t,0), \frac{\partial \Delta q}{\partial \dot{\Delta q}_j}(t,0), \frac{\partial \dot{\Delta q}}{\partial \dot{\Delta q}_j}(t,0),
$$

of the functions

$$
\begin{aligned}
\Delta q &= \Delta q(t, \Delta p_D, \Delta q, \dot{\Delta q}),\ t \geq t_j\\
\dot{\Delta q} &= \dot{\Delta q}(t, \Delta p_D, \Delta q, \dot{\Delta q}),\ t \geq t_j.
\end{aligned}
$$

As already mentioned in Sect. 5.4, conditions in form of initial value problems can be obtained by differentiation of the dynamic equation with respect to the parameter deviations $\Delta p_D, \Delta q, \dot{\Delta q}$. According to (5.22b, c), for the actual trajectory $\Delta z(t) = \begin{pmatrix} \Delta q(t) \\ \dot{\Delta q}(t) \end{pmatrix}$, $t \geq t_j$, we have the second order initial value problem

$$
F\Big(\overline{p}_D^{(j)} + \Delta p_D, q^{(j)}(t) + \Delta q(t), \dot{q}^{(j)}(t) + \dot{\Delta q}(t), \ddot{q}^{(j)}(t) + \ddot{\Delta q}(t)\Big)
$$

$$
= u^{(j)}(t) + \varphi\left(t, \Delta q(t), \dot{\Delta q}(t)\right),\ t \geq t_j, \tag{5.37a}
$$

$$
\Delta q(t_j) = \Delta q_j, \tag{5.37b}
$$

$$
\dot{\Delta q}(t_j) = \dot{\Delta q}_j. \tag{5.37c}
$$

The needed derivatives follow from (5.37a–c) by implicit differentiation with respect to $\Delta p_D, \Delta q_j, \dot{\Delta q}_j$. Here, the following equations for the functional matrices have to be taken into account:

$$\frac{\partial \Delta q}{\partial \Delta p_D} = \begin{pmatrix} \left(\mathrm{grad}_{\Delta p_D} \Delta q_1\right)^T \\ \vdots \\ \left(\mathrm{grad}_{\Delta p_D} \Delta q_n\right)^T \end{pmatrix} = \left(\frac{\partial \Delta q}{\partial \Delta p_{D1}}, \frac{\partial \Delta q}{\partial \Delta p_{D2}}, \ldots, \frac{\partial \Delta q}{\partial \Delta p_{D\nu}}\right) \tag{5.38a}$$

$$\frac{\partial \Delta q}{\partial \Delta q_j} = \begin{pmatrix} \left(\mathrm{grad}_{\Delta q_j} \Delta q_1\right)^T \\ \vdots \\ \left(\mathrm{grad}_{\Delta q_j} \Delta q_n\right)^T \end{pmatrix} = \left(\frac{\partial \Delta q}{\partial \Delta q_{j1}}, \frac{\partial \Delta q}{\partial \Delta q_{j2}}, \ldots, \frac{\partial \Delta q}{\partial \Delta q_{jn}}\right) \tag{5.38b}$$

$$\frac{\partial \Delta q}{\partial \Delta \dot{q}_j} = \begin{pmatrix} \left(\mathrm{grad}_{\Delta \dot{q}_j} \Delta q_1\right)^T \\ \vdots \\ \left(\mathrm{grad}_{\Delta \dot{q}_j} \Delta q_n\right)^T \end{pmatrix} = \left(\frac{\partial \Delta q}{\partial \Delta \dot{q}_{j1}}, \frac{\partial \Delta q}{\partial \Delta \dot{q}_{j2}}, \ldots, \frac{\partial \Delta q}{\partial \Delta \dot{q}_{jn}}\right) \tag{5.38c}$$

Because of

$$\frac{\partial \Delta \dot{q}}{\partial \Delta p_D} = \frac{\partial}{\partial \Delta p_D}\left(\frac{\partial \Delta q}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \Delta q}{\partial \Delta p_D}\right) \tag{5.38d}$$

$$\frac{\partial \Delta \dot{q}}{\partial \Delta q_j} = \frac{\partial}{\partial \Delta q_j}\left(\frac{\partial \Delta q}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \Delta q}{\partial \Delta q_j}\right) \tag{5.38e}$$

$$\frac{\partial \Delta \dot{q}}{\partial \Delta \dot{q}_j} = \frac{\partial}{\partial \Delta \dot{q}_j}\left(\frac{\partial \Delta q}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \Delta q}{\partial \Delta \dot{q}_j}\right), \tag{5.38f}$$

the derivatives $\frac{\partial \Delta \dot{q}}{\partial \Delta p_D}, \frac{\partial \Delta \dot{q}}{\partial \Delta q_j}, \frac{\partial \Delta \dot{q}}{\partial \Delta \dot{q}_j}$ follow, as shown below, together with derivatives defined by (5.38a–c).

5.5.1 Derivatives with Respect to Dynamic Parameters at Stage j

According to (5.38a) and (5.38d), for each vector function

$$t \mapsto \frac{\partial \Delta q}{\partial \Delta p_{Di}}(t, \Delta p_D, \Delta q_j, \Delta \dot{q}_j),\ t \geq t_j,\ i = 1, \ldots, \nu,$$

we obtain a system of differential equations by differentiation of the systems (5.37a–c) with respect to Δp_{Di}, $i = 1, \ldots, \nu$. From (5.37a) next to we find

$$\frac{\partial F}{\partial p_D}\frac{\partial \Delta p_D}{\partial \Delta p_{Di}} + \frac{\partial F}{\partial q}\frac{\partial \Delta q}{\partial \Delta p_{Di}} + \frac{\partial F}{\partial \dot{q}}\frac{\partial \Delta \dot{q}}{\partial \Delta p_{Di}} + \frac{\partial F}{\partial \ddot{q}}\frac{\partial \Delta \ddot{q}}{\partial \Delta p_{Di}}$$
$$= \frac{\partial \varphi}{\partial \Delta q}\frac{\partial \Delta q}{\partial \Delta p_{Di}} + \frac{\partial \varphi}{\partial \Delta \dot{q}}\frac{\partial \Delta \dot{q}}{\partial \Delta p_{Di}},\ i = 1, \ldots, \nu, \tag{5.39a}$$

where $\frac{\partial \Delta p_D}{\partial \Delta p_{Di}} = e_i$ (= i-th unit vector). Differentiation of the initial conditions (5.37b, c) with respect to p_{Di} yields

$$\frac{\partial \Delta q}{\partial \Delta p_{Di}}(t_j, \Delta p_D, \Delta q_j, \Delta \dot{q}_j) = 0,\ i = 1, \ldots, \nu, \tag{5.39b}$$

$$\frac{\partial \Delta \dot{q}}{\partial \Delta p_{Di}}(t_j, \Delta p_D, \Delta q_j, \Delta \dot{q}_j) = 0,\ i = 1, \ldots, \nu, \tag{5.39c}$$

since the initial values $\Delta q_j, \Delta \dot{q}_j$ are independent of the dynamic parameters p_D, Δp_D, respectively.

Then, the initial value problems for the sensitivities

$$t \mapsto \frac{\partial \Delta q}{\partial \Delta p_{Di}}(t, 0),\ t \mapsto \frac{\partial \Delta \dot{q}}{\partial \Delta p_{Di}}(t, 0) = \frac{d}{dt}\frac{\partial \Delta q}{\partial \Delta p_{Di}}(t, 0)$$

follow by insertion of $\Delta a = \begin{pmatrix} \Delta p_D \\ \Delta q_j \\ \Delta \dot{q}_j \end{pmatrix} = 0$ into (5.39a–c). Defining still the expressions

$$\left.\frac{\partial F}{\partial p_D}\right|_{(j)} := \frac{\partial F}{\partial p_D}\left(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\right),\ t \geq t_j, \tag{5.40a}$$

$$\left.\frac{\partial F}{\partial q}\right|_{(j)} := \frac{\partial F}{\partial q}\left(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\right),\ t \geq t_j, \tag{5.40b}$$

$$\left.\frac{\partial F}{\partial \dot{q}}\right|_{(j)} := \frac{\partial F}{\partial \dot{q}}\left(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\right),\ t \geq t_j, \tag{5.40c}$$

$$\left.\frac{\partial F}{\partial \ddot{q}}\right|_{(j)} := \frac{\partial F}{\partial \ddot{q}}\left(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\right),\ t \geq t_j, \tag{5.40d}$$

for $i = 1, \ldots, \nu$, the initial value problems read:

$$\frac{\partial F}{\partial p_{Di}}\Big|_{(j)} + \frac{\partial F}{\partial q}\Big|_{(j)} \frac{\partial \Delta q}{\partial \Delta p_{Di}}(t,0) + \frac{\partial F}{\partial \dot{q}}\Big|_{(j)} \frac{\partial \Delta \dot{q}}{\partial \Delta p_{Di}}(t,0) + \frac{\partial F}{\partial \ddot{q}}\Big|_{(j)} \frac{\partial \Delta \ddot{q}}{\partial \Delta p_{Di}}(t,0)$$
$$= \frac{\partial \varphi}{\partial \Delta q}(t,0) \frac{\partial \Delta q}{\partial \Delta p_{Di}}(t,0) + \frac{\partial \varphi}{\partial \Delta \dot{q}}(t,0) \frac{\partial \Delta \dot{q}}{\partial \Delta p_{Di}}(t,0),\ t \geq t_j \tag{5.41a}$$
$$\frac{\partial \Delta q}{\partial \Delta p_{Di}}(t_j,0) = 0 \tag{5.41b}$$
$$\frac{\partial \Delta \dot{q}}{\partial \Delta p_{Di}}(t_j,0) = 0. \tag{5.41c}$$

Subsuming the above ν systems, we get a second order system of differential equations for the functional matrix $t \mapsto \frac{\partial \Delta q}{\partial \Delta p_D}(t,0)$.

$$\frac{\partial F}{\partial p_{D}}\Big|_{(j)} + \frac{\partial F}{\partial q}\Big|_{(j)} \frac{\partial \Delta q}{\partial \Delta p_{D}}(t,0) + \frac{\partial F}{\partial \dot{q}}\Big|_{(j)} \frac{\partial \Delta \dot{q}}{\partial \Delta p_{D}}(t,0) + \frac{\partial F}{\partial \ddot{q}}\Big|_{(j)} \frac{\partial \Delta \ddot{q}}{\partial \Delta p_{D}}(t,0),$$
$$= \frac{\partial \varphi}{\partial \Delta q}(t,0) \frac{\partial \Delta q}{\partial \Delta p_{D}}(t,0) + \frac{\partial \varphi}{\partial \Delta \dot{q}}(t,0) \frac{\partial \Delta \dot{q}}{\partial \Delta p_{D}}(t,0),\ t \geq t_j \tag{5.42a}$$
$$\frac{\partial \Delta q}{\partial \Delta p_{D}}(t_j,0) = 0 \tag{5.42b}$$
$$\frac{\partial \Delta \dot{q}}{\partial \Delta p_{D}}(t_j,0) = 0. \tag{5.42c}$$

Using the definition

$$\eta(t) := \frac{\partial \Delta q}{\partial \Delta p_{Di}}(t,0),\ t \geq t_j, \tag{5.43}$$

with $i = 1, \ldots, \nu$, the separated systems (5.41a–c) can be represented by:

$$\left(\frac{\partial F}{\partial q}\Big|_{(j)} - \frac{\partial \varphi}{\partial \Delta q}(t,0)\right)\eta(t) + \left(\frac{\partial F}{\partial \dot{q}}\Big|_{(j)} - \frac{\partial \varphi}{\partial \Delta \dot{q}}(t,0)\right)\dot{\eta}(t) + \frac{\partial F}{\partial \ddot{q}}\Big|_{(j)}\ddot{\eta}(t)$$
$$= -\frac{\partial F}{\partial p_{Di}}\Big|_{(j)}, t \geq t_j \tag{5.44a}$$
$$\eta(t_j) = 0 \tag{5.44b}$$
$$\dot{\eta}(t_j) = 0. \tag{5.44c}$$

Obviously, this property holds:

Lemma 5.2 *The sensitivities* $\eta(t) = \frac{\partial \Delta q}{\partial \Delta p_{Di}}(t, 0)$, $t \geq t_j$, *with respect to the deviations of the dynamic parameters* Δp_{Di}, $i = 1, \dots, \nu$, *are determined by linear initial value problems of second order.*

The following abbreviations are often used in control theory:

$$M^{(j)}(t) := \frac{\partial F}{\partial \ddot{q}}\Big|_{(j)} = \frac{\partial F}{\partial \ddot{q}}\left(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\right), \ t \geq t_j, \tag{5.45a}$$

$$D^{(j)}(t) := \frac{\partial F}{\partial \dot{q}}\Big|_{(j)} = \frac{\partial F}{\partial \dot{q}}\left(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\right), \ t \geq t_j, \tag{5.45b}$$

$$K^{(j)}(t) := \frac{\partial F}{\partial q}\Big|_{(j)} = \frac{\partial F}{\partial q}\left(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\right), \ t \geq t_j, \tag{5.45c}$$

$$Y^{(j)}(t) := \frac{\partial F}{\partial p_D}\Big|_{(j)} = \frac{\partial F}{\partial p_D}\left(\overline{p}_D^{(j)}, q^{(j)}(t), \dot{q}^{(j)}(t), \ddot{q}^{(j)}(t)\right), \ t \geq t_j. \tag{5.45d}$$

Putting

$$Y^{(j)}(t) := \left(y_1^{(j)}, y_2^{(j)}, \dots, y_\nu^{(j)}\right) \tag{5.45e}$$

with the columns

$$y_i^{(j)}(t) := \frac{\partial F}{\partial p_{Di}}\Big|_{(j)} = \text{ i-th column of } Y^{(j)}, \tag{5.45f}$$

then (5.44a–c) can be represented also by

$$\left(K^{(j)}(t) - \frac{\partial \varphi}{\partial \Delta q}(t, 0)\right)\eta(t) + \left(D^{(j)}(t) - \frac{\partial \varphi}{\partial \Delta \dot{q}}(t, 0)\right)\dot{\eta}(t) + M^{(j)}(t)\ddot{\eta}(t)$$

$$= -y_i^{(j)}(t), \quad t \geq t_j \tag{5.46a}$$

$$\eta(t_j) = 0 \tag{5.46b}$$

$$\dot{\eta}(t_j) = 0. \tag{5.46c}$$

In the present case the matrix $M^{(j)}(t)$ is regular. Thus, the above system can be represented also by the following initial value problem of 2nd order for $\eta = \eta(t)$:

$$\ddot{\eta}(t) + (M^{(j)}(t))^{-1}\left(D^{(j)}(t) - \tfrac{\partial \varphi}{\partial \Delta \dot{q}}(t, 0)\right)\dot{\eta}(t)$$

$$+ (M^{(j)}(t))^{-1}\left(K^{(j)}(t) - \tfrac{\partial \varphi}{\partial \Delta q}(t, 0)\right)\eta(t)$$

$$= -(M^{(j)}(t))^{-1} y_i^{(j)}(t), \ t \geq t_j \tag{5.47a}$$

$$\eta(t_j) = 0 \tag{5.47b}$$

$$\dot{\eta}(t_j) = 0. \tag{5.47c}$$

for each $i = 1, \ldots, \nu$.

5.5.2 Derivatives with Respect to the Initial Values at Stage j

Differentiation of the 2nd order system (5.37a–c) with respect to the initial values Δq_{jk}, $k = 1, \ldots, n$, at stage j-ten yields, cf. (5.39a–c),

$$\frac{\partial F}{\partial q}\frac{\partial \Delta q}{\partial \Delta q_{jk}} + \frac{\partial F}{\partial \dot{q}}\frac{\partial \dot{\Delta q}}{\partial \Delta q_{jk}} + \frac{\partial F}{\partial \ddot{q}}\frac{\partial \ddot{\Delta q}}{\partial \Delta q_{jk}} = \frac{\partial \varphi}{\partial \Delta q}\frac{\partial \Delta q}{\partial \Delta q_{jk}}$$
$$+\frac{\partial \varphi}{\partial \dot{\Delta q}}\frac{\partial \dot{\Delta q}}{\partial \Delta q_{jk}}, \quad k = 1, \ldots, n, \tag{5.48a}$$

$$\frac{\partial \Delta q}{\partial \Delta q_{jk}}(t_j) = e_k \ (= k\text{-th unit vector}) \tag{5.48b}$$

$$\frac{\partial \dot{\Delta q}}{\partial \Delta q_{jk}}(t_j) = 0. \tag{5.48c}$$

Inserting again $\Delta a = \begin{pmatrix} \Delta p_D \\ \Delta q_j \\ \dot{\Delta q}_j \end{pmatrix} = 0$ in (5.48a–c), for

$$\zeta(t) := \frac{\partial \Delta q}{\partial \Delta q_{jk}}(t, 0), t \geq t_j, \tag{5.49}$$

with a fixed index k, $1 \leq k \leq n$, cf. (5.43), we get the following linear initial value problem of 2nd order:

$$\left(K^{(j)}(t) - \frac{\partial \varphi}{\partial \Delta q}(t, 0)\right)\zeta(t) + \left(D^{(j)}(t) - \frac{\partial \varphi}{\partial \dot{\Delta q}}(t, 0)\right)\dot{\zeta}(t)$$
$$+M^{(j)}(t)\ddot{\zeta}(t) = 0, \qquad t \geq t_j \tag{5.50a}$$

$$\zeta(t_j) = e_k \tag{5.50b}$$

$$\dot{\zeta}(t_j) = 0. \tag{5.50c}$$

In the same way, for the partial derivative

$$\xi(t) := \frac{\partial \Delta q}{\partial \dot{\Delta q}_{jk}}(t, 0), \ t \geq t_j, \tag{5.51}$$

of Δq with respect to the deviation of the initial velocities $\dot{\Delta q}_{jk}$, $k = 1, \ldots, n$, at stage j we have the linear 2nd order initial value problem

$$\left(K^{(j)}(t) - \frac{\partial \varphi}{\partial \Delta q}(t,0)\right)\xi(t) + \left(D^{(j)}(t) - \frac{\partial \varphi}{\partial \dot{\Delta q}}(t,0)\right)\dot{\xi}(t)$$

$$+M^{(j)}(t)\ddot{\xi}(t) = 0, \qquad t \geq t_j \tag{5.52a}$$

$$\xi(t_j) = 0 \tag{5.52b}$$

$$\dot{\xi}(t_j) = e_k, \tag{5.52c}$$

where e_k denotes the k-th unit vector.
Corresponding to (5.47a–c), system (5.50a–c) and (5.52a–c) can be represented also as follows:

$$\ddot{\zeta}(t) + (M^{(j)}(t))^{-1}\left(D^{(j)}(t) - \tfrac{\partial \varphi}{\partial \dot{\Delta q}}(t,0)\right)\dot{\zeta}(t)$$

$$+(M^{(j)}(t))^{-1}\left(K^{(j)}(t) - \tfrac{\partial \varphi}{\partial \Delta q}(t,0)\right)\zeta(t) = 0,\ t \geq t_j \tag{5.53a}$$

$$\xi(t_j) = e_k \tag{5.53b}$$

$$\dot{\zeta}(t_j) = 0, \tag{5.53c}$$

$$\ddot{\xi}(t) + (M^{(j)}(t))^{-1}\left(D^{(j)}(t) - \tfrac{\partial \varphi}{\partial \dot{\Delta q}}(t,0)\right)\dot{\xi}(t)$$

$$+(M^{(j)}(t))^{-1}\left(K^{(j)}(t) - \tfrac{\partial \varphi}{\partial \Delta q}(t,0)\right)\xi(t) = 0,\ t \geq t_j \tag{5.54a}$$

$$\xi(t_j) = 0 \tag{5.54b}$$

$$\dot{\xi}(t_j) = e_k \tag{5.54c}$$

for each fixed $k = 1, \ldots, n$.

Definition 5.1 The 2nd order linear initial value problems (5.47a–c), (5.50a–c), (5.52a–c) are also called **perturbation equations**.

The coefficients of the perturbation equations (5.47a–c),(5.50a–c) and (5.52a–c) are defined often by the approach:

$$(M^{(j)}(t))^{-1}\left(K^{(j)}(t) - \frac{\partial \varphi}{\partial \Delta q}(t,0)\right) = K_p \tag{5.55a}$$

$$(M^{(j)}(t))^{-1}\left(D^{(j)}(t) - \frac{\partial \varphi}{\partial \dot{\Delta q}}(t,0)\right) = K_d \tag{5.55b}$$

with certain fixed matrices (often diagonal matrices) still to be determined

$$K_p = \left(\gamma_{pkl}\right), \ K_d = (\gamma_{dkl}). \tag{5.55c}$$

For the functional matrices $\frac{\partial \varphi}{\partial \Delta q}, \frac{\partial \varphi}{\partial \Delta \dot{q}}$ to be determined, the approach (5.55a–c) yields the following representation:

$$\frac{\partial \varphi}{\partial \Delta q}(t,0) = K^{(j)}(t) - M^{(j)}(t)K_p, \tag{5.56a}$$

$$\frac{\partial \varphi}{\partial \Delta \dot{q}}(t,0) = D^{(j)}(t) - M^{(j)}(t)K_d. \tag{5.56b}$$

Using only the linear terms $\frac{\partial \varphi}{\partial \Delta q}(t,0), \frac{\partial \varphi}{\partial \Delta \dot{q}}(t,0)$ of the Taylor expansion (5.27a, b) of the feedback function $\varphi = \varphi(t, \Delta q, \Delta \dot{q})$, and using the definition (5.55a–c), then the optimal regulator problem (5.22a–c) is reduced to the optimal selection of the regulator parameters in the matrices K_p, K_d.

Defining the Jacobians $\frac{\partial \varphi}{\partial \Delta q}(t,0), \frac{\partial \varphi}{\partial \Delta \dot{q}}(t,0)$ by (5.56a, b), the above described 2nd order linear initial value problems read as follows:

With an index $i = 1 \dots \nu$, for $\eta(t) = \frac{\partial \Delta q}{\partial \Delta q_{Di}}(t,0)$, $t \geq t_j$, according to (5.47a–c) we have

$$\ddot{\eta}(t) + K_d\dot{\eta}(t) + K_p\eta(t) = -(M^{(j)}(t))^{-1}y_i^{(j)}(t), \ \ t \geq t_j \tag{5.57a}$$

$$\eta(t_j) = 0 \tag{5.57b}$$

$$\dot{\eta}(t_j) = 0. \tag{5.57c}$$

For $\zeta(t) = \frac{\partial \Delta q}{\partial \Delta q_{jk}}(t,0)$, $t \geq t_j$, with an index $k = 1, \dots, n$ we get, cf. (5.53a–c),

$$\ddot{\zeta}(t) + K_d\dot{\zeta}(t) + K_p\zeta(t) = 0, \ \ t \geq t_j \tag{5.58a}$$

$$\zeta(t_j) = e_k \tag{5.58b}$$

$$\dot{\zeta}(t_j) = 0. \tag{5.58c}$$

Finally, for $\xi(t) := \frac{\partial \Delta q}{\partial \Delta \dot{q}_{jk}}(t,0)$, $t \geq t_j$, with an index $k = 1, \dots, n$ we have, see (5.54a–c),

$$\ddot{\xi}(t) + K_d\dot{\xi}(t) + K_p\xi(t) = 0, \ \ t \geq t_j \tag{5.59a}$$

$$\xi(t_j) = 0 \tag{5.59b}$$

$$\dot{\xi}(t_j) = e_k. \tag{5.59c}$$

5.5.3 Solution of the Perturbation Equation

For given fixed indices $i = 1, \ldots, \nu,\ k = 1, \ldots, n$, we put

$$\psi^{(j)}(t) := \left\{ \begin{array}{ll} -(M^{(j)}(t))^{-1} y_i^{(j)}(t),\ t \geq t_j, & \text{for (5.57a–c)} \\ 0, & \text{for (5.58a–c)} \\ 0, & \text{for (5.59a–c)} \end{array} \right\},\ t \geq t_j, \tag{5.60a}$$

and

$$dq_j := \begin{cases} 0, & \text{for (5.57a–c)} \\ e_k, & \text{for (5.58a–c)} \\ 0, & \text{for (5.59a–c),} \end{cases} \tag{5.60b}$$

$$\dot{d}q_j := \begin{cases} 0, & \text{for (5.57a–c)} \\ 0, & \text{for (5.58a–c)} \\ e_k, & \text{for (5.59a–c).} \end{cases} \tag{5.60c}$$

Then the 2nd order linear initial value problems (5.57a–c), (5.58a–c), (5.59a–c), resp., for the computation of the sensitivities

$$dq(t) := \left\{ \begin{array}{ll} \eta(t), & \text{for (5.57a–c)} \\ \zeta(t), & \text{for (5.58a–c)} \\ \xi(t), & \text{for (5.59a–c)} \end{array} \right\},\ t \geq t_j \tag{5.60d}$$

can be represented uniformly as follows:

$$\ddot{d}q(t) + K_d \dot{d}q(t) + K_p dq(t) = \psi^{(j)}(t),\ \ t \geq t_j \tag{5.61a}$$

$$dq(t_j) = dq_j \tag{5.61b}$$

$$\dot{d}q(t_j) = \dot{d}q_j. \tag{5.61c}$$

Defining now

$$dz(t) := \begin{pmatrix} dq(t) \\ \dot{d}q(t) \end{pmatrix},\ t \geq t_j, \tag{5.62}$$

the 2nd order linear initial value problem (5.61a–c) can be transformed into an equivalent linear 1st order initial value problem, see (1.2a, b),(1.3):

$$\dot{d}z(t) = A\, dz(t) + \begin{pmatrix} 0 \\ \psi^{(j)}(t) \end{pmatrix},\ t \geq t_j \tag{5.63a}$$

$$dz(t_j) = dz_j. \tag{5.63b}$$

Here, the matrix A and the vector dz_j of initial values are defined as follows:

$$A := \begin{pmatrix} 0 & I \\ -K_p & -K_d \end{pmatrix} \tag{5.63c}$$

$$dz_j := \begin{pmatrix} dq_j \\ \dot{d}q_j \end{pmatrix}. \tag{5.63d}$$

For the associated *homogeneous linear* 1st order system of differential equations

$$\dot{d}z(t) = A\,dz(t),\ t \geq t_j \tag{5.64a}$$

$$dz(t_j) = dz_j. \tag{5.64b}$$

we have this stability-criterion:

Lemma 5.3 (Routh–Hurwitz Criterion) *The homogeneous linear system (5.64a, b) is **asymptotic stable** for each initial state dz_j, if the real parts of all eigenvalues of A are negative. In this case we especially have*

$$dz(t) \to 0,\ t \to +\infty \tag{5.65}$$

for arbitrary dz_j.

According to Lemma 5.3, Definition (5.60d) of $dq = dq(t)$, $t \geq t_j$, and the Definitions (5.43), (5.49) and (5.51) of $\eta = \eta(t)$, $\zeta = \zeta(t)$, $\xi = \xi(t)$, resp., for the sensitivities we have the following asymptotic properties:

Corollary 5.5.1 *Suppose that the derivatives (5.43), (5.49) and (5.51) are defined for all $t \geq t_j$. If the matrix A has only eigenvalues with negative real parts, then*

$$\lim_{t\to+\infty} \left(\frac{\partial \Delta q}{\partial \Delta p_D}(t,0)\right)_{\substack{homog.\\ part\ of\\ solution}} = 0 \tag{5.66a}$$

$$\lim_{t\to+\infty} \frac{\partial \Delta q}{\partial \Delta q_j}(t,0) = 0 \tag{5.66b}$$

$$\lim_{t\to+\infty} \frac{\partial \Delta q}{\partial \Delta \dot{q}_j}(t,0) = 0. \tag{5.66c}$$

Proof Since the systems of differential equations for $\frac{\partial \Delta q}{\partial \Delta q_j}(t,0)$ and $\frac{\partial \Delta q}{\partial \Delta \dot{q}_j}(t,0)$ are homogeneous, the assertion follows immediately from Lemma 5.3.

By means of the matrix exponential function e^{At} of A, the solution $dz = dz(t),\ t \geq t_j$, of (5.63a–d) can be represented explicitly as follows:

$$dz(t) = e^{A(t-t_j)}dz_j + \int_{t_j}^{t} e^{A(t-\tau)} \begin{pmatrix} 0 \\ \psi^{(j)}(\tau) \end{pmatrix} d\tau,\ t \geq t_j. \tag{5.67}$$

Consequently, because of (5.60a–c), the sensitivities read:

$$\frac{\partial \Delta q}{\partial \Delta p_{Di}}(t,0) = -\int_{t_j}^{t} e^{A(t-\tau)} \begin{pmatrix} 0 \\ (M^{(j)}(\tau))^{-1} y_i^{(j)}(\tau) \end{pmatrix} d\tau,\ t \geq t_j, \tag{5.68a}$$

$$\frac{\partial \Delta q}{\partial \Delta q_{jk}}(t,0) = e^{A(t-t_j)} \begin{pmatrix} e_k \\ 0 \end{pmatrix},\ t \geq t_j, \tag{5.68b}$$

$$\frac{\partial \Delta q}{\partial \Delta \dot{q}_{jk}}(t,0) = e^{A(t-t_j)} \begin{pmatrix} 0 \\ e_k \end{pmatrix},\ t \geq t_j. \tag{5.68c}$$

Selection of the Matrices K_p, K_d

An explicit solution of (5.61a–c) and therefore also of the 2nd order linear systems of differential equations (5.57a–c),(5.58a–c) and (5.59a–c) as well as an explicit solution of the 1st order system (5.63a–d) can be obtained under the assumption that K_p and K_d are diagonal matrices:

$$K_p = (\gamma_{pl}\, \delta_{kl}),\ K_d = (\gamma_{dl}\, \delta_{kl}). \tag{5.69}$$

Hence, $\gamma_{pl}, \gamma_{dl},\ l = 1, \ldots, n$, resp., denote the diagonal elements of K_p, K_d. Under this assumption the system (5.61a–c) is decomposed into n separated equations for the components $dq_k = dq_k(t),\ t \geq t_j,\ k = 1, \ldots, n$ of $dq = dq(t),\ t \geq t_j$:

$$d\ddot{q}_k(t) + \gamma_{dk} d\dot{q}_k(t) + \gamma_{pd} dq_k(t) = \psi_k^{(j)}(t), t \geq t_j \tag{5.70a}$$

$$dq_k(t_j) = dq_{jk} \tag{5.70b}$$

$$d\dot{q}_k(t_j) = d\dot{q}_{jk} \tag{5.70c}$$

with $k = 1, \ldots, n$. Obviously, (5.70a–c) are ordinary linear differential equations of 2nd order with constant coefficients. Thus, up to possible integrals, the solution can be given explicitly.

For this purpose we consider first the characteristic polynomial

$$p_k(\lambda) := \lambda^2 + \gamma_{dk}\lambda + \gamma_{pk} \tag{5.71}$$

The zeros of $p_k = p_k(\lambda)$ read

$$\begin{aligned}\lambda_{k1,2} &:= -\frac{\gamma_{dk}}{2} \pm \sqrt{\left(\frac{\gamma_{pk}}{2}\right)^2 - \gamma_{pk}} \\ &= \frac{1}{2}\left(-\gamma_{dk} \pm \sqrt{\gamma_{dk}^2 - 4\gamma_{pk}}\right),\ k = 1, \ldots, n.\end{aligned} \tag{5.72}$$

As shown in the following, the roots $\lambda_{k1,2}$, $k = 1, \ldots, n$, are just the eigenvalues of the system matrix A, provided that K_d, K_p are defined by (5.69):

If λ is an eigenvalue of A and $w = \begin{pmatrix} u \\ v \end{pmatrix}$ an associated eigenvector, then (5.63c)

$$\lambda \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & I \\ -K_p & -K_d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -K_p u - K_d v \end{pmatrix}. \tag{5.73}$$

Thus, $\lambda u = v$, and the second equality in (5.73) yields

$$\lambda^2 u = \lambda(\lambda u) = \lambda v = -K_p u - K_d v = -K_p u - K_d \lambda u \tag{5.74a}$$

and therefore

$$(\lambda^2 I + \lambda K_d + K_p)u = 0. \tag{5.74b}$$

Because of $v = \lambda u$ each eigenvector $w = \begin{pmatrix} u \\ v \end{pmatrix}$ of A related to the eigenvalue λ of A has the form

$$w = \begin{pmatrix} u \\ \lambda u \end{pmatrix}. \tag{5.75a}$$

However, this means that

$$w \neq 0 \quad \Leftrightarrow u \neq 0. \tag{5.75b}$$

Hence, Eq. (5.74b) holds if and only if

$$\det(\lambda^2 I + \lambda K_d + K_p) = 0. \tag{5.76a}$$

An eigenvalue λ of A is therefore a root of the polynomial

$$q(\lambda) = \det(\lambda^2 I + \lambda K_d + K_p) \tag{5.76b}$$

with degree $2n$. If the matrices K_p and K_d have the diagonal form (5.69), then also $\lambda^2 I + \lambda K_d + K_p$ is a diagonal matrix. Thus, we have

$$q(\lambda) = \prod_{k=1}^{n} (\lambda^2 + \gamma_{dk}\lambda + \gamma_{pk}). \tag{5.76c}$$

Obviously, the eigenvalues λ of A are the quantities $\lambda_{k1,2}, k = 1, \ldots, n$, defined by (5.72).

According to Lemma 5.3 we consider now the real parts of the eigenvalues $\lambda_{k1,2},\ k = 1, \ldots, n$, of A, where we assume that

$$\gamma_{pk} > 0,\ \gamma_{dk} > 0,\ k = 1, \ldots, n. \tag{5.77}$$

Case 1:

$$\gamma_{dk}^2 > 4\gamma_{pk} \tag{5.78a}$$

Here, the eigenvalues $\lambda_{k1,2}$ are real valued, and because of

$$\sqrt{\gamma_{dk}^2 - 4\gamma_{pk}} < \gamma_{dk}$$

we have

$$\lambda_{k1} = \frac{1}{2}\left(-\gamma_{dk} + \sqrt{\gamma_{dk}^2 - 4\gamma_{pk}}\right) < 0, \tag{5.78b}$$

$$\lambda_{k2} = \frac{1}{2}\left(-\gamma_{dk} - \sqrt{\gamma_{dk}^2 - 4\gamma_{pk}}\right) < 0. \tag{5.78c}$$

Thus, in this case, both eigenvalues take negative real values.
Case 2:

$$\gamma_{dk}^2 = 4\gamma_{pk} \tag{5.79a}$$

In this case

$$\lambda_{k1,2} = -\frac{\gamma_{dk}}{2} \tag{5.79b}$$

is (at least) a double negative eigenvalue of A.
Case 3:

$$\gamma_{dk}^2 < 4\gamma_{pk} \tag{5.80a}$$

Here, we have

$$\begin{aligned}\lambda_{k1,2} &= \frac{1}{2}\left(-\gamma_{dk} \pm \sqrt{(-1)(4\gamma_{pk} - \gamma_{dk}^2)}\right)\\ &= \frac{1}{2}\left(-\gamma_{dk} \pm i\sqrt{4\gamma_{pk} - \gamma_{dk}^2}\right)\\ &= \frac{1}{2}\left(-\gamma_{dk} \pm i\omega_k\right)\end{aligned} \tag{5.80b}$$

with

$$\omega_k := \sqrt{4\gamma_{pk} - \gamma_{dk}^2}. \tag{5.80c}$$

In the present case $\lambda_{k1,2}$ are two different complex eigenvalues of A with negative real part $-\frac{1}{2}\gamma_{dk}$.

Consequently, we have this result:

Lemma 5.4 *If K_p, K_d are diagonal matrices having positive diagonal elements γ_{pk}, γ_{dk}, $k = 1, \ldots, n$,, then all eigenvalues $\lambda_{k1,2}$, $k = 1, \ldots, n$, of A have negative real parts.*

According to the Routh–Hurwitz criterion, under the assumption (5.77) of positive diagonal elements, system of differential equations (5.64a, b) is asymptotic stable.

With respect to the eigenvalues $\lambda_{k1,2}$ described in **Cases 1–3** we have the following contributions to the fundamental system of the differential equation (5.70a, b):

Case 1:

$$e^{\lambda_{k1}(t-t_j)},\ e^{\lambda_{k2}(t-t_j)}, t \geq t_j, \tag{5.81a}$$

with the negative real eigenvalues $\lambda_{k1,2}$, see (5.78b, c);

Case 2:

$$e^{\lambda_{k1}(t-t_j)},\ (t-t_j)e^{\lambda_{k1}(t-t_j)}, t \geq t_j, \tag{5.81b}$$

with $\lambda_{k1} = -\frac{1}{2}\gamma_{dk}$, see (5.79b);

Case 3:

$$e^{-\frac{1}{2}\gamma_{dk}(t-t_j)}\cos\omega_k(t-t_j),\ e^{-\frac{1}{2}\gamma_{dk}(t-t_j)}\sin\omega_k(t-t_j), t \geq t_j, \tag{5.81c}$$

with ω_k given by (5.80c).

Consequently, the solution of system (5.61a–c), (5.63a, b), resp., can be given explicitly—eventually up to certain integrals occurring in the computation of a particular solution of the inhomogeneous linear system of differential equations.

5.6 Computation of the Objective Function

For the optimal regulator problem under stochastic uncertainty (5.22a–c), we have to determine approximatively the conditional expectation

$$\begin{aligned} \mathrm{E}\Big(J^{(j)}\Big|\mathcal{A}_{t_j}\Big) &= \int\limits_{t_j}^{t_f^{(j)}} \mathrm{E}\Big(f^{(j)}(t)\Big|\mathcal{A}_{t_j}\Big)dt \\ &\approx \int\limits_{t_j}^{t_f^{(j)}} \mathrm{E}\Big(\tilde{f}^{(j)}(t)\Big|\mathcal{A}_{t_j}\Big)dt, \end{aligned} \tag{5.82a}$$

see (5.6a–d), (5.15), (5.26) and (5.28a). According to (5.36) we have

$$\begin{aligned} \mathrm{E}\Big(\tilde{f}^{(j)}\Big|\mathcal{A}_{t_j}\Big) = & \ \mathrm{tr}\ P_{\Delta p_D}\ \mathrm{cov}^{(j)}\big(p_D(\cdot)\big) + \ \mathrm{tr}\ P_{\Delta q_j}\ \mathrm{cov}^{(j)}\big(\Delta q_j(\cdot)\big) \\ & + \ \mathrm{tr}\ P_{\dot{\Delta} q_j}\ \mathrm{cov}^{(j)}\big(\dot{\Delta} q_j(\cdot)\big) \\ & + \ \mathrm{tr}\ P_{\dot{\Delta} q_j, \Delta q_j}\ \mathrm{cov}^{(j)}\big(\Delta q_j(\cdot), \dot{\Delta} q_j(\cdot)\big). \end{aligned} \tag{5.82b}$$

Here, the P-matrices are defined, see (5.33b–e), by

$$P_{\Delta p_D} := \left(\frac{\partial \Delta z}{\partial \Delta p_D}(t,0)\right)^T Q(t,\varphi)\left(\frac{\partial \Delta z}{\partial \Delta p_D}(t,0)\right), \tag{5.83a}$$

$$P_{\Delta q_j} := \left(\frac{\partial \Delta z}{\partial \Delta q_j}(t,0)\right)^T Q(t,\varphi)\left(\frac{\partial \Delta z}{\partial \Delta q_j}(t,0)\right), \tag{5.83b}$$

$$P_{\dot{\Delta} q_j} := \left(\frac{\partial \Delta z}{\partial \dot{\Delta} q_j}(t,0)\right)^T Q(t,\varphi)\left(\frac{\partial \Delta z}{\partial \dot{\Delta} q_j}(t,0)\right), \tag{5.83c}$$

$$P_{\dot{\Delta} q_j, \Delta q_j} := \left(\frac{\partial \Delta z}{\partial \dot{\Delta} q_j}(t,0)\right)^T Q(t,\varphi)\left(\frac{\partial \Delta z}{\partial \Delta q_j}(t,0)\right), \tag{5.83d}$$

where, see (5.30c),

$$Q(t,\varphi) = \begin{pmatrix} C_q + \frac{\partial\varphi}{\partial\Delta q}(t,0)^T C_u \frac{\partial\varphi}{\partial\Delta q}(t,0) & \frac{\partial\varphi}{\partial\Delta q}(t,0)^T C_u \frac{\partial\varphi}{\partial\Delta\dot{q}}(t,0) \\ \frac{\partial\varphi}{\partial\Delta\dot{q}}(t,0)^T C_u \frac{\partial\varphi}{\partial\Delta q}(t,0) & C_{\dot{q}} + \frac{\partial\varphi}{\partial\Delta\dot{q}}(t,0)^T C_u \frac{\partial\varphi}{\partial\Delta\dot{q}}(t,0) \end{pmatrix} \tag{5.84}$$

with, see (5.56a, b) and (5.45a–c),

$$\frac{\partial\varphi}{\partial\Delta q}(t,0) = K^{(j)}(t) - M^{(j)}(t)K_p \tag{5.85a}$$

$$\frac{\partial\varphi}{\partial\Delta\dot{q}}(t,0) = D^{(j)}(t) - M^{(j)}(t)K_d. \tag{5.85b}$$

By (5.31) and (5.38a–f) we further have

$$\begin{aligned} \frac{\partial\Delta z}{\partial\Delta p_D}(t,0) &= \begin{pmatrix} \frac{\partial\Delta q}{\partial\Delta p_D}(t,0) \\ \frac{\partial\Delta\dot{q}}{\partial\Delta p_D}(t,0) \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial\Delta q}{\partial\Delta p_{D_1}}(t,0) & \frac{\partial\Delta q}{\partial\Delta p_{D_2}}(t,0) & \dots & \frac{\partial\Delta q}{\partial\Delta p_{D_\nu}}(t,0) \\ \frac{\partial\Delta\dot{q}}{\partial\Delta p_{D_1}}(t,0) & \frac{\partial\Delta\dot{q}}{\partial\Delta p_{D_2}}(t,0) & \dots & \frac{\partial\Delta\dot{q}}{\partial\Delta p_{D_\nu}}(t,0) \end{pmatrix}, \end{aligned} \tag{5.86a}$$

$$\begin{aligned} \frac{\partial\Delta z}{\partial\Delta q_j}(t,0) &= \begin{pmatrix} \frac{\partial\Delta q}{\partial\Delta q_j}(t,0) \\ \frac{\partial\Delta\dot{q}}{\partial\Delta q_j}(t,0) \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial\Delta q}{\partial\Delta q_{j1}}(t,0) & \frac{\partial\Delta q}{\partial\Delta q_{j2}}(t,0) & \dots & \frac{\partial\Delta q}{\partial\Delta q_{jn}}(t,0) \\ \frac{\partial\Delta\dot{q}}{\partial\Delta q_{j1}}(t,0) & \frac{\partial\Delta\dot{q}}{\partial\Delta q_{j2}}(t,0) & \dots & \frac{\partial\Delta\dot{q}}{\partial\Delta q_{jn}}(t,0) \end{pmatrix}, \end{aligned} \tag{5.86b}$$

$$\begin{aligned} \frac{\partial\Delta z}{\partial\Delta\dot{q}_j}(t,0) &= \begin{pmatrix} \frac{\partial\Delta q}{\partial\Delta\dot{q}_j}(t,0) \\ \frac{\partial\Delta\dot{q}}{\partial\Delta\dot{q}_j}(t,0) \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial\Delta q}{\partial\Delta\dot{q}_{j1}}(t,0) & \frac{\partial\Delta q}{\partial\Delta\dot{q}_{j2}}(t,0) & \dots & \frac{\partial\Delta q}{\partial\Delta\dot{q}_{jn}}(t,0) \\ \frac{\partial\Delta\dot{q}}{\partial\Delta\dot{q}_{j1}}(t,0) & \frac{\partial\Delta\dot{q}}{\partial\Delta\dot{q}_{j2}}(t,0) & \dots & \frac{\partial\Delta\dot{q}}{\partial\Delta\dot{q}_{jn}}(t,0) \end{pmatrix}. \end{aligned} \tag{5.86c}$$

Using (5.85a, b), according to (5.42a–c) and (5.45a–d), the functional matrix

$$t \to \frac{\partial\Delta q}{\partial\Delta p_D}(t,0),\ t \geq t_j,$$

satisfies the 2nd order linear initial value problem

$$\frac{\partial \Delta \ddot{q}}{\partial \Delta p_D}(t,0) + K_d \frac{\partial \Delta \dot{q}}{\partial \Delta p_D}(t,0) + K_p \frac{\partial \Delta q}{\partial \Delta p_D}(t,0)$$

$$= -\left(M^{(j)}(t)\right)^{-1} Y^{(j)}(t),\ t \geq t_j \tag{5.87a}$$

$$\frac{\partial \Delta q}{\partial \Delta p_D}(t_j,0) = 0 \tag{5.87b}$$

$$\frac{\partial \Delta \dot{q}}{\partial \Delta p_D}(t_j,0) = 0. \tag{5.87c}$$

Moreover, using (5.85a, b), by (5.48a–c), (5.49) and (5.50a–c) the functional matrix

$$t \rightarrow \frac{\partial \Delta q}{\partial \Delta q_j}(t,0),\ t \geq t_j,$$

satisfies the

$$\frac{\partial \Delta \ddot{q}}{\partial \Delta q_j}(t,0) + K_d \frac{\partial \Delta \dot{q}}{\partial \Delta q_j}(t,0) + K_p \frac{\partial \Delta q}{\partial \Delta q_j}(t,0) = 0,\ t \geq t_j \tag{5.88a}$$

$$\frac{\partial \Delta q}{\partial \Delta q_j}(t_j,0) = I \tag{5.88b}$$

$$\frac{\partial \Delta \dot{q}}{\partial \Delta q_j}(t_j,0) = 0. \tag{5.88c}$$

Finally, using (5.85a, b), according to (5.51) and (5.52a–c) the functional matrix

$$t \rightarrow \frac{\partial \Delta q}{\partial \Delta \dot{q}_j}(t,0),\ t \geq t_j,$$

is the solution of the 2nd order linear initial value problem

$$\frac{\partial \Delta \ddot{q}}{\partial \Delta \dot{q}_j}(t,0) + K_d \frac{\partial \Delta \dot{q}}{\partial \Delta \dot{q}_j}(t,0) + K_p \frac{\partial \Delta q}{\partial \Delta \dot{q}_j}(t,0) = 0,\ t \geq t_j \tag{5.89a}$$

$$\frac{\partial \Delta q}{\partial \Delta \dot{q}_j}(t_j,0) = 0 \tag{5.89b}$$

$$\frac{\partial \Delta \dot{q}}{\partial \Delta \dot{q}_j}(t_j,0) = I. \tag{5.89c}$$

5.7 Optimal PID-Regulator

For simplification of the notation, in the following we consider only the 0-th stage j := 0. Moreover, we suppose again that the state $z(t) = (q(t), \dot{q}(t))$ is available exactly. Corresponding to Sect. 5.1, the tracking error, i.e., the deviation of the actual position $q = q(t)$ and velocity $\dot{q} = \dot{q}(t))$ from the reference position $q^{(0)} = q^{(0)}(t)$ and reference velocity $\dot{q}^{(0)} = \dot{q}^{(0)}(t)), t_0 \le t \le t_f$ is defined by

$$\Delta q(t) := q(t) - q^R(t), \; t_0 \le t \le t_f, \tag{5.90a}$$

$$\Delta \dot{q}(t) := \dot{q}(t) - \dot{q}^R(t), \; t_0 \le t \le t_f. \tag{5.90b}$$

Hence, for the difference between the actual state $z = z(t)$ and the reference state $z^{(0)} = z^{(0)}(t)$ we have

$$\Delta z(t) := z(t) - z^{(0)}(t) = \begin{pmatrix} q(t) - q^{(0)}(t) \\ \dot{q}(t) - \dot{q}^{(0)}(t) \end{pmatrix}, \; t_0 \le t \le t_f. \tag{5.91}$$

As in Sect. 5.1, the actual control input $u = u(t), t_0 \le t \le t_f$, is defined by

$$u(t) := u^{(0)}(t) + \Delta u(t), \; t_0 \le t \le t_f. \tag{5.92}$$

Here $u^{(0)} = u^{(0)}(t), t_0 \le t \le t_f$, denotes the feedforward control based on the reference trajectory $z^{(0)}(t) = z^{(0)}(t), t_0 \le t \le t_f$, obtained e.g. by *inverse dynamics*, cf. [4, 11, 61], and $\Delta u = \Delta u(t), t_0 \le t \le t_f$, is the optimal control correction to be determined.

Generalizing the PD-regulator, depending on the state $z(t) = (q(t), \dot{q}(t))$, for the PID-regulator, cf. Sect. 4.7, also the integrated position

$$q_I(t) := \int_{t_0}^{t} q(\tau)\, d\tau, \; t_0 \le t \le t_f, \tag{5.93}$$

is taken into account. Hence, the PID-regulator depends also on the *integrative position error*

$$\begin{aligned} \Delta q_I(t) &:= \int_{t_0}^{t} \Delta q(\tau)\, d\tau = \int_{t_0}^{t} \Big(q(\tau) - q^R(\tau) \Big)\, d\tau \\ &= \int_{t_0}^{t} q(\tau)\, d\tau - \int_{t_0}^{t} q^R(\tau)\, d\tau = q_I(t) - q_I^R(t). \end{aligned} \tag{5.94}$$

Thus, working with the PID-regulator, the control correction is defined by

$$\Delta u(t) := \varphi(t, \Delta q(t), \Delta q_I(t), \Delta \dot{q}(t)),\ t_0 \le t \le t_f. \tag{5.95}$$

Without further global information on the stochastic parameters also in the present case for the feedback law φ we suppose

$$\varphi(t, 0, 0, 0) = 0,\ t_0 \le t \le t_f. \tag{5.96}$$

Corresponding to the optimization problem (5.7a–c) for PD-regulators described in Sect. 5.1.1, for PID-regulators $\varphi^* = \varphi^*(t, \Delta q, \Delta q_I, \Delta \dot{q})$ we have the following optimization problem:

Definition 5.2 A stochastic optimal feedback law $\varphi^* = \varphi^*(t, \Delta q(t), \Delta q_I(t), \Delta \dot{q}(t))$ is an optimal solution of

$$\min \mathrm{E}\left(\int_{t_0}^{t_f} \Big(c\big(t, \Delta q(t), \Delta q_I(t), \Delta \dot{q}(t)\big) + \gamma\big(t, \varphi(t, \Delta q(t), \Delta q_I(t), \Delta \dot{q}(t))\big) \Big) dt \,\Big|\, \mathcal{A}_{t_0} \right) \tag{5.97a}$$

subject to

$$\begin{aligned} &F(p_D(\omega), q(t), \dot{q}(t), \ddot{q}(t)) \\ &= u^R(t) + \varphi(t, \Delta q(t), \Delta q_I(t), \Delta \dot{q}(t)), t_0 \le t \le t_f,\ a.s. \end{aligned} \tag{5.97b}$$

$$q(t_0, \omega) = q_0(\omega), \tag{5.97c}$$

$$\dot{q}(t_0, \omega) = \dot{q}_0(\omega), \tag{5.97d}$$

$$\varphi(t, \mathbf{0}) = 0. \tag{5.97e}$$

Here,

$$F(p_D(\omega), q(t), \dot{q}(t), \ddot{q}(t)) := M\Big(p_D, q(t)\Big)\ddot{q}(t) + h\Big(p_D, q(t), \dot{q}(t)\Big) \tag{5.98}$$

is again the left hand side of the dynamic equation as in (4.4a).

Corresponding to Sect. 5.1.1, the term $c\big(t, \Delta q(t), \Delta q_I(t), \Delta \dot{q}(t)\big)$ in the objective function (5.97a) describes the costs resulting from the deviation of the actual trajectory from the reference trajectory , and the costs for the control corrections are represented by $\gamma\big(t, \varphi(t, \Delta q(t), \Delta q_I(t), \Delta \dot{q}(t))\big)$. Moreover, cf. (5.3f),

$$\Delta p_D(\omega) := p_D(\omega) - \overline{p_D} \tag{5.99}$$

denotes again the deviation of the effective dynamic parameter vector $p_D = p_D(\omega)$ from its nominal value or expectation $\overline{p_D}$. In the present section, $\overline{p_D} :=$

$E(p_D(\omega)|\mathcal{A}_{t_0})$ denotes the (conditional) expectation of the random vector $p_D = p_D(\omega)$.

Having the stochastic optimal feedback control $\varphi^* = \varphi^*(t, \Delta q, \Delta q_I, \Delta\dot{q})$, the effective trajectory $q = q(t),\ t \geq t_0$, of the stochastic optimally controlled dynamic system is then given by the solution of the initial value problem (5.97b–d). Using the definitions, cf. (4.45c, d), (5.3d, e),

$$\Delta q_0 := q(t_0) - q^R(t_0) = q_0 - E(q_0|\mathcal{A}_{t_0}), \tag{5.100a}$$

$$\Delta\dot{q}_0 := \dot{q}(t_0) - \dot{q}^R(t_0) = \dot{q}_0 - E(\dot{q}_0|\mathcal{A}_{t_0}), \tag{5.100b}$$

the constraints of the regulator optimization problem (5.97a–f) can be represented also by

$$F(\overline{p_D} + \Delta p_D(\omega), q^R(t) + \Delta q(t), \dot{q}^R(t) + \Delta\dot{q}(t), \ddot{q}^R(t) + \Delta\ddot{q}(t))$$
$$= u^R(t) + \varphi(t, \Delta q(t), \Delta q_I(t), \Delta\dot{q}(t)), t_0 \leq t \leq t_f,\ a.s, \tag{5.101a}$$

$$\Delta q(t_0, \omega) = \Delta q_0(\omega), \tag{5.101b}$$

$$\Delta\dot{q}(t_0, \omega) = \Delta\dot{q}_0(\omega), \tag{5.101c}$$

$$\varphi(t, \mathbf{0}) = 0. \tag{5.101d}$$

Remark 5.2 For given control law $\varphi = \varphi(t, \Delta q(t), \Delta q_I(t), \Delta\dot{q}(t))$, the solution of the initial value problem (5.101a–d) yields position error function

$$\Delta q := \Delta q(t, a(\omega)), t_0 \leq t \leq t_f, \tag{5.102a}$$

depending on the random parameter vector

$$a = a(\omega) := \begin{pmatrix} \Delta p_D(\omega) \\ \Delta q_0(\omega) \\ \Delta\dot{q}_0(\omega) \end{pmatrix}. \tag{5.102b}$$

5.7.1 Quadratic Cost Functions

Corresponding to Sect. 5.3.2, here we use again *quadratic cost functions*. Hence, with symmetric, positive (semi-)definite matrices $C_q, C_{q_I}, C_{\dot{q}}$ the costs c arising from the tracking error are defined by

$$c(t, \Delta q(t), \Delta q_I(t), \Delta\dot{q}(t))$$
$$:= \Delta q(t)^T C_q \Delta q(t) + \Delta q_I(t)^T C_{q_I} \Delta q_I(t) + \Delta\dot{q}(t)^T C_{\dot{q}} \Delta\dot{q}(t). \tag{5.103a}$$

Moreover, with a positive (semi-)definite matrix C_u the regulator costs γ are defined by

$$\begin{aligned}&\gamma\big(t, \varphi(t, \Delta q(t), \Delta q_I(t), \Delta\dot{q}(t))\\&:= \varphi\big(t, \Delta q(t), \Delta q_I(t), \Delta\dot{q}(t)\big)^T C_u \varphi\big(t, \Delta q(t), \Delta q_I(t), \Delta\dot{q}(t)\big).\end{aligned} \tag{5.103b}$$

Corresponding to Sect. 5.4.1 the total cost function is defined by

$$\begin{aligned}f(t) := {}& \Delta q(t)^T C_q \Delta q(t) + \Delta q_I(t)^T C_{q_I} \Delta q_I(t)\\&+ \Delta\dot{q}(t)^T C_{\dot{q}} \Delta\dot{q}(t) + \Delta u(t)^T C_u \Delta u(t).\end{aligned} \tag{5.104}$$

Thus, for the objective function of the regulator optimization problem (5.97a–e) we obtain

$$\begin{aligned}&\mathrm{E}\left(\int_{t_0}^{t_f} f(t)\, dt \,\middle|\, \mathcal{A}_{t_0}\right) = \int_{t_0}^{t_f} \mathrm{E}\Big(f(t)\Big|\mathcal{A}_{t_0}\Big)\, dt\\
&= \int_{t_0}^{t_f} \mathrm{E}\Big(\big(\Delta q(t)\big)^T C_q \Delta q(t) + \big(\Delta q_I(t)\big)^T C_{q_I} \Delta q_I(t) + \big(\Delta\dot{q}(t)\big)^T C_{\dot{q}} \Delta\dot{q}(t)\\
&\quad + \big(\varphi\big(t, \Delta q(t), \Delta q_I(t), \Delta\dot{q}(t)\big)\big)^T C_u \varphi\big(t, \Delta q(t), \Delta q_I(t), \Delta\dot{q}(t)\big)\Big|\mathcal{A}_{t_0}\Big) dt.\end{aligned} \tag{5.105}$$

Computation of the Expectation by Taylor Expansion

Corresponding to Sect. 5.4, here we use also the Taylor expansion method to compute approximatively the arising conditional expectations. Thus, by Taylor expansion of the feedback function φ at $\Delta q = 0$, $\Delta q_I = 0$, $\Delta\dot{q} = 0$, we get

$$\begin{aligned}\varphi\big(t, \Delta q(t), \Delta q_I(t), \Delta\dot{q}(t)\big) = {}& \varphi\big(t, \mathbf{0}\big) + D_{\Delta q}\varphi(t, \mathbf{0})\Delta q(t)\\&+ D_{\Delta q_I}\varphi(t, \mathbf{0})\Delta q_I(t) + D_{\Delta\dot{q}}\varphi(t, \mathbf{0})\Delta\dot{q}(t) + \cdots .\end{aligned} \tag{5.106}$$

Here,

$$D_{\Delta q}\varphi = D_{\Delta q}\varphi(t, \mathbf{0}) = \frac{\partial\varphi}{\partial\Delta q}(t, \mathbf{0})$$

denotes the Jacobian, hence, the matrix of partial derivatives of φ with respect to Δq at $(t, \mathbf{0})$.

Taking into account only the linear terms in (5.106), because of $\varphi(t, \mathbf{0}) = 0$ we get the approximation

$$E\left(f \middle| \mathcal{A}_{t_0}\right) \approx E\left(\tilde{f} \middle| \mathcal{A}_{t_0}\right), \tag{5.107}$$

where

$$\begin{aligned}
E\left(\tilde{f}|\mathcal{A}_0\right) := E\Bigg(&\Delta q(t)^T C_q \Delta q(t) + \Delta q_I(t)^T C_{q_I} \Delta q_I(t) + \Delta\dot{q}(t)^T C_{\dot{q}} \Delta\dot{q}(t) \\
&+ \Big(D_{\Delta q}\varphi(t,\mathbf{0})\Delta q(t) + D_{\Delta q_I}\varphi(t,\mathbf{0})\Delta q_I(t) + D_{\Delta\dot{q}}\varphi(t,\mathbf{0})\Delta\dot{q}(t)\Big)^T \\
&\quad \times C_u\Big(D_{\Delta q}\varphi(t,\mathbf{0})\Delta q(t) + D_{\Delta q_I}\varphi(t,\mathbf{0})\Delta q_I(t) + D_{\Delta\dot{q}}\varphi(t,\mathbf{0})\Delta\dot{q}(t)\Big)^T \Bigg| \mathcal{A}_{t_0}\Bigg) \\
= E\Bigg(&\Delta q(t)^T C_q \Delta q(t) + \Delta q_I(t)^T C_{q_I} \Delta q_I(t) + \Delta\dot{q}(t)^T C_{\dot{q}} \Delta\dot{q}(t) \\
&+ \Big(\Delta q(t)^T D_{\Delta q}\varphi(t,\mathbf{0})^T C_u + \Delta q_I(t)^T D_{\Delta q_I}\varphi(t,\mathbf{0})^T C_u \\
&+ \Delta\dot{q}(t)^T D_{\Delta\dot{q}}\varphi(t,\mathbf{0})^T C_u\Big) \\
&\times \Big(D_{\Delta q}\varphi(t,\mathbf{0})\Delta q(t) + D_{\Delta q_I}\varphi(t,\mathbf{0})\Delta q_I(t) + D_{\Delta\dot{q}}\varphi(t,\mathbf{0})\Delta\dot{q}(t)\Big) \Bigg| \mathcal{A}_{t_0}\Bigg) \\
= E\Bigg(&\Delta q(t)^T C_q \Delta q(t) + \Delta q_I(t)^T C_{q_I} \Delta q_I(t) + \Delta\dot{q}(t)^T C_{\dot{q}} \Delta\dot{q}(t) \\
&+ \Delta q(t)^T D_{\Delta q}\varphi(t,\mathbf{0})^T C_u D_{\Delta q}\varphi(t,\mathbf{0})\Delta q(t) \\
&+ \Delta q(t)^T D_{\Delta q}\varphi(t,\mathbf{0})^T C_u D_{\Delta q_I}\varphi(t,\mathbf{0})\Delta q_I(t) \\
&+ \Delta q(t)^T D_{\Delta q}\varphi(t,\mathbf{0})^T C_u D_{\Delta\dot{q}}\varphi(t,\mathbf{0})\Delta\dot{q}(t) \\
&+ \Delta q_I(t)^T D_{\Delta q_I}\varphi(t,\mathbf{0})^T C_u D_{\Delta q}\varphi(t,\mathbf{0})\Delta q(t) \\
&+ \Delta q_I(t)^T D_{\Delta q_I}\varphi(t,\mathbf{0})^T C_u D_{\Delta q_I}\varphi(t,\mathbf{0})\Delta q_I(t) \\
&+ \Delta q_I(t)^T D_{\Delta q_I}\varphi(t,\mathbf{0})^T C_u D_{\Delta\dot{q}}\varphi(t,\mathbf{0})\Delta\dot{q}(t) \\
&+ \Delta\dot{q}(t)^T D_{\Delta\dot{q}}\varphi(t,\mathbf{0})^T C_u D_{\Delta q}\varphi(t,\mathbf{0})\Delta q(t) \\
&+ \Delta\dot{q}(t)^T D_{\Delta\dot{q}}\varphi(t,\mathbf{0})^T C_u D_{\Delta q_I}\varphi(t,\mathbf{0})\Delta q_I(t) \\
&+ \Delta\dot{q}(t)^T D_{\Delta\dot{q}}\varphi(t,\mathbf{0})^T C_u D_{\Delta\dot{q}}\varphi(t,\mathbf{0})\Delta\dot{q}(t) \Bigg| \mathcal{A}_{t_0}\Bigg)
\end{aligned}$$

$$
\begin{aligned}
=&\mathrm{E}\Bigg(\Delta q(t)^T\left(C_q + D_{\Delta q}\varphi(t,\mathbf{0})^T C_u D_{\Delta q}\varphi(t,\mathbf{0})\right)\Delta q(t)\\
&+ \Delta q_I(t)^T\left(C_{q_I} + D_{\Delta q_I}\varphi(t,\mathbf{0})^T C_u D_{\Delta q_I}\varphi(t,\mathbf{0})\right)\Delta q_I(t)\\
&+ \Delta \dot{q}(t)^T\left(C_{\dot{q}} + D_{\Delta \dot{q}}\varphi(t,\mathbf{0})^T C_u D_{\Delta \dot{q}}\varphi(t,\mathbf{0})\right)\Delta \dot{q}(t)\\
&+ 2\Delta q(t)^T D_{\Delta q}\varphi(t,\mathbf{0})^T C_u D_{\Delta q_I}\varphi(t,\mathbf{0})\Delta q_I(t)\\
&+ 2\Delta q(t)^T D_{\Delta q}\varphi(t,\mathbf{0})^T C_u D_{\Delta \dot{q}}\varphi(t,\mathbf{0})\Delta \dot{q}(t)\\
&+ 2\Delta q_I(t)^T D_{\Delta q_I}\varphi(t,\mathbf{0})^T C_u D_{\Delta \dot{q}}\varphi(t,\mathbf{0})\Delta \dot{q}(t)\Bigg|\mathcal{A}_{t_0}\Bigg).
\end{aligned}
\tag{5.108}
$$

In order to compute the above expectation, we introduce the following definition:

$$
Q_q := C_q + D_{\Delta q}\varphi(t,\mathbf{0})^T C_u D_{\Delta q}\varphi(t,\mathbf{0}), \tag{5.109a}
$$

$$
Q_{q_I} := C_{q_I} + D_{\Delta q_I}\varphi(t,\mathbf{0})^T C_u D_{\Delta q_I}\varphi(t,\mathbf{0}), \tag{5.109b}
$$

$$
Q_{\dot{q}} := C_{\dot{q}} + D_{\Delta \dot{q}}\varphi(t,\mathbf{0})^T C_u D_{\Delta \dot{q}}\varphi(t,\mathbf{0}), \tag{5.109c}
$$

$$
Q_{qq_I} := D_{\Delta q}\varphi(t,\mathbf{0})^T C_u D_{\Delta q_I}\varphi(t,\mathbf{0}), \tag{5.109d}
$$

$$
Q_{q\dot{q}} := D_{\Delta q}\varphi(t,\mathbf{0})^T C_u D_{\Delta \dot{q}}\varphi(t,\mathbf{0}), \tag{5.109e}
$$

$$
Q_{q_I\dot{q}} := D_{\Delta q_I}\varphi(t,\mathbf{0})^T C_u D_{\Delta \dot{q}}\varphi(t,\mathbf{0}). \tag{5.109f}
$$

Next to this yields for $\tilde{f}$ the representation

$$
\begin{aligned}
\tilde{f} :=& \Delta q(t)^T\left(C_q + D_{\Delta q}\varphi(t,\mathbf{0})^T C_u D_{\Delta q}\varphi(t,\mathbf{0})\right)\Delta q(t)\\
&+ \Delta q_I(t)^T\left(C_{q_I} + D_{\Delta q_I}\varphi(t,\mathbf{0})^T C_u D_{\Delta q_I}\varphi(t,\mathbf{0})\right)\Delta q_I(t)\\
&+ \Delta \dot{q}(t)^T\left(C_{\dot{q}} + D_{\Delta \dot{q}}\varphi(t,\mathbf{0})^T C_u D_{\Delta \dot{q}}\varphi(t,\mathbf{0})\right)\Delta \dot{q}(t)\\
&+ 2\Delta q(t)^T D_{\Delta q}\varphi(t,\mathbf{0})^T C_u D_{\Delta q_I}\varphi(t,\mathbf{0})\Delta q_I(t)\\
&+ 2\Delta q(t)^T D_{\Delta q}\varphi(t,\mathbf{0})^T C_u D_{\Delta \dot{q}}\varphi(t,\mathbf{0})\Delta \dot{q}(t)\\
&+ 2\Delta q_I(t)^T D_{\Delta q_I}\varphi(t,\mathbf{0})^T C_u D_{\Delta \dot{q}}\varphi(t,\mathbf{0})\Delta \dot{q}(t)\\
=& (\Delta q(t)^T, \Delta q_I(t)^T, \Delta \dot{q}(t)^T)\begin{pmatrix} Q_q & Q_{qq_I} & Q_{q\dot{q}} \\ Q_{qq_I}^T & Q_{q_I} & Q_{q_I\dot{q}} \\ Q_{q\dot{q}}^T & Q_{q_I\dot{q}}^T & Q_{\dot{q}} \end{pmatrix}\begin{pmatrix} \Delta q(t) \\ \Delta q_I(t) \\ \Delta \dot{q}(t) \end{pmatrix}\\
=& \Delta z_I(t)^T Q(t,\varphi)\Delta z_I(t),
\end{aligned}
\tag{5.110}
$$

where the extended state vector Δz_I and the matrix Q are defined by

$$\Delta z_I(t) := \begin{pmatrix} \Delta q(t) \\ \Delta q_I(t) \\ \Delta \dot{q}(t) \end{pmatrix}, \tag{5.111a}$$

$$Q(t,\varphi) := \begin{pmatrix} Q_q & Q_{qq_I} & Q_{q\dot{q}} \\ Q_{qq_I}^T & Q_{q_I} & Q_{q_I\dot{q}} \\ Q_{q\dot{q}}^T & Q_{q_I\dot{q}}^T & Q_{\dot{q}} \end{pmatrix}. \tag{5.111b}$$

Since $C_q, C_{q_I}, C_{\dot{q}}$ and C_u are positive (semi-)definite matrices, we have $\tilde{f} \geq 0$.

In order to determine the expectation of $\tilde{f}$, in addition to the Taylor approximation of the feedback law $\varphi = \varphi\big(t, \Delta q(t), \Delta q_I(t), \Delta \dot{q}(t)\big)$, we need also the Taylor expansion of Δz_I with respect to the parameter vector a at

$$a = (\Delta p_D(\omega)^T, \Delta q_0(\omega)^T, \Delta \dot{q}_0(\omega)^T)^T = \mathbf{0}. \tag{5.112}$$

Hence, we get

$$\begin{aligned}\Delta q(t) =& \Delta q(t,\mathbf{0}) + D_{\Delta p_D}\Delta q(t,\mathbf{0})\Delta p_D(\omega) + D_{\Delta q_0}\Delta q(t,\mathbf{0})\Delta q_0(\omega) \\ &+ D_{\Delta \dot{q}_0}\Delta q(t,\mathbf{0})\Delta \dot{q}_0(\omega) + \ldots, \end{aligned} \tag{5.113a}$$

$$\begin{aligned}\Delta \dot{q}(t) =& \Delta \dot{q}(t,\mathbf{0}) + D_{\Delta p_D}\Delta \dot{q}(t,\mathbf{0})\Delta p_D(\omega) \\ &+ D_{\Delta q_0}\Delta \dot{q}(t,\mathbf{0})\Delta q_0(\omega) + D_{\Delta \dot{q}_0}\Delta \dot{q}(t,\mathbf{0})\Delta \dot{q}_0(\omega) + \ldots, \end{aligned} \tag{5.113b}$$

as well as

$$\begin{aligned}\Delta q_I(t) =& \int_{t_0}^{t} \Delta q(\tau)d\tau = \int_{t_0}^{t} \Big(\Delta q(\tau,\mathbf{0}) + D_{\Delta p_D}\Delta q(\tau,\mathbf{0})\Delta p_D(\omega) \\ &+ D_{\Delta q_0}\Delta q(\tau,\mathbf{0})\Delta q_0(\omega) + D_{\Delta \dot{q}_0}\Delta q(\tau,\mathbf{0})\Delta \dot{q}_0(\omega) + \ldots\Big)d\tau. \end{aligned} \tag{5.113c}$$

Using only the linear terms, and taking into account the conditions (5.101a–d)

$$\Delta q(t,\mathbf{0}) = 0,\ t \geq t_0, \tag{5.114a}$$

$$\Delta \dot{q}(t,\mathbf{0}) = 0,\ t \geq t_0, \tag{5.114b}$$

then (5.113a, b) can be approximated by

$$\Delta q(t) \approx D_{\Delta p_D}\Delta q(t,\mathbf{0})\Delta p_D(\omega) + D_{\Delta q_0}\Delta q(t,\mathbf{0})\Delta q_0(\omega) + D_{\Delta \dot{q}_0}\Delta q(t,\mathbf{0})\Delta \dot{q}_0(\omega), \tag{5.115a}$$

$$\Delta\dot{q}(t) \approx D_{\Delta p_D}\Delta\dot{q}(t,\mathbf{0})\Delta p_D(\omega) + D_{\Delta q_0}\Delta\dot{q}(t,\mathbf{0})\Delta q_0(\omega) + D_{\Delta\dot{q}_0}\Delta\dot{q}(t,\mathbf{0})\Delta\dot{q}_0(\omega). \tag{5.115b}$$

For (5.113c) this yields then

$$\begin{aligned}
\Delta q_I(t) &= \int_{t_0}^{t} \Delta q(\tau)\, d\tau \\
&\approx \int_{t_0}^{t} \Big(D_{\Delta p_D}\Delta q(\tau,\mathbf{0})\Delta p_D(\omega) \\
&\quad + D_{\Delta q_0}\Delta q(\tau,\mathbf{0})\Delta q_0(\omega) + D_{\Delta\dot{q}_0}\Delta q(\tau,\mathbf{0})\Delta\dot{q}_0(\omega)\Big)\, d\tau \\
&= \int_{t_0}^{t} D_{\Delta p_D}\Delta q(\tau,\mathbf{0})\Delta p_D(\omega)\, d\tau + \int_{t_0}^{t} D_{\Delta q_0}\Delta q(\tau,\mathbf{0})\Delta q_0(\omega)\, d\tau \\
&\quad + \int_{t_0}^{t} D_{\Delta\dot{q}_0}\Delta q(\tau,\mathbf{0})\Delta\dot{q}_0(\omega)\, d\tau \\
&= \left(\frac{\partial}{\partial \Delta p_D} \underbrace{\int_{t_0}^{t} \Delta q(\tau,p)\, d\tau}_{=\Delta q_I(t,p)}\right)\Bigg|_{p=\mathbf{0}} \Delta p_D(\omega) + \left(\frac{\partial}{\partial \Delta q_0} \underbrace{\int_{t_0}^{t} \Delta q(\tau,p)\, d\tau}_{=\Delta q_I(t,p)}\right)\Bigg|_{p=\mathbf{0}} \Delta q_0(\omega) \\
&\quad + \left(\frac{\partial}{\partial \Delta \dot{q}_0} \underbrace{\int_{t_0}^{t} \Delta q(\tau,p)\, d\tau}_{=\Delta q_I(t,p)}\right)\Bigg|_{p=\mathbf{0}} \Delta \dot{q}_0(\omega) \\
&= D_{\Delta p_D}\Delta q_I(t,\mathbf{0})\Delta p_D(\omega) + D_{\Delta q_0}\Delta q_I(t,\mathbf{0})\Delta q_0(\omega) + D_{\Delta\dot{q}_0}\Delta q_I(t,\mathbf{0})\Delta\dot{q}_0(\omega).
\end{aligned} \tag{5.115c}$$

Consequently, we find

$$\begin{aligned}
\Delta z_I(t) &= \begin{pmatrix} \Delta q(t) \\ \Delta q_I(t) \\ \Delta \dot{q}(t) \end{pmatrix} \\
&\approx \begin{pmatrix} D_{\Delta p_D}\Delta q(t,\mathbf{0})\Delta p_D(\omega) + D_{\Delta q_0}\Delta q(t,\mathbf{0})\Delta q_0(\omega) + D_{\Delta\dot{q}_0}\Delta q(t,\mathbf{0})\Delta\dot{q}_0(\omega) \\ D_{\Delta p_D}\Delta q_I(t,\mathbf{0})\Delta p_D(\omega) + D_{\Delta q_0}\Delta q_I(t,\mathbf{0})\Delta q_0(\omega) + D_{\Delta\dot{q}_0}\Delta q_I(t,\mathbf{0})\Delta\dot{q}_0(\omega) \\ D_{\Delta p_D}\Delta\dot{q}(t,\mathbf{0})\Delta p_D(\omega) + D_{\Delta q_0}\Delta\dot{q}(t,\mathbf{0})\Delta q_0(\omega) + D_{\Delta\dot{q}_0}\Delta\dot{q}(t,\mathbf{0})\Delta\dot{q}_0(\omega) \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} D_{\Delta p_D} \Delta q(t, \mathbf{0}) \\ D_{\Delta p_D} \Delta q_I(t, \mathbf{0}) \\ D_{\Delta p_D} \Delta \dot{q}(t, \mathbf{0}) \end{pmatrix} \Delta p_D(\omega) + \begin{pmatrix} D_{\Delta q_0} \Delta q(t, \mathbf{0}) \\ D_{\Delta q_0} \Delta q_I(t, \mathbf{0}) \\ D_{\Delta q_0} \Delta \dot{q}(t, \mathbf{0}) \end{pmatrix} \Delta q_0(\omega) \\
&\quad + \begin{pmatrix} D_{\Delta \dot{q}_0} \Delta q(t, \mathbf{0}) \\ D_{\Delta \dot{q}_0} \Delta q_I(t, \mathbf{0}) \\ D_{\Delta \dot{q}_0} \Delta \dot{q}(t, \mathbf{0}) \end{pmatrix} \Delta \dot{q}_0(\omega) \\
&= D_{\Delta p_D} \Delta z_I(t, \mathbf{0}) \Delta p_D(\omega) + D_{\Delta q_0} \Delta z_I(t, \mathbf{0}) \Delta q_0(\omega) + D_{\Delta \dot{q}_0} \Delta z_I(t, \mathbf{0}) \Delta \dot{q}_0(\omega).
\end{aligned} \tag{5.116}
$$

Inserting this into equation (5.110), we get the approximation:

$$
\begin{aligned}
\tilde{f} =& \Delta z_I(t)^T Q \Delta z_I(t) \\
\approx& \Big(\Delta p_D(\omega)^T \Big(D_{\Delta p_D} \Delta z_I(t, \mathbf{0}) \Big)^T + \Delta q_0(\omega)^T \Big(D_{\Delta q_0} \Delta z_I(t, \mathbf{0}) \Big)^T \\
& + \Delta \dot{q}_0(\omega)^T \Big(D_{\Delta \dot{q}_0} \Delta z_I(t, \mathbf{0}) \Big)^T \Big) \\
& \times Q \, \Big(D_{\Delta p_D} \Delta z_I(t, \mathbf{0}) \Delta p_D(\omega) + D_{\Delta q_0} \Delta z_I(t, \mathbf{0}) \Delta q_0(\omega) \\
& + D_{\Delta \dot{q}_0} \Delta z_I(t, \mathbf{0}) \Delta \dot{q}_0(\omega) \Big) \\
=& \Delta p_D(\omega)^T \Big(D_{\Delta p_D} \Delta z_I(t, \mathbf{0}) \Big)^T Q \; D_{\Delta p_D} \Delta z_I(t, \mathbf{0}) \Delta p_D(\omega) \\
& + \Delta q_0(\omega)^T \Big(D_{\Delta q_0} \Delta z_I(t, \mathbf{0}) \Big)^T Q \; D_{\Delta q_0} \Delta z_I(t, \mathbf{0}) \Delta q_0(\omega) \\
& + \Delta \dot{q}_0(\omega)^T \Big(D_{\Delta \dot{q}_0} \Delta z_I(t, \mathbf{0}) \Big)^T Q \; D_{\Delta \dot{q}_0} \Delta z_I(t, \mathbf{0}) \Delta \dot{q}_0(\omega) \\
& + 2 \Delta q_0(\omega)^T \Big(D_{\Delta \dot{q}_0} \Delta z_I(t, \mathbf{0}) \Big)^T Q \; D_{\Delta \dot{q}_0} \Delta z_I(t, \mathbf{0}) \Delta p_D(\omega) \\
& + 2 \Delta \dot{q}_0(\omega)^T \Big(D_{\Delta \dot{q}_0} \Delta z_I(t, \mathbf{0}) \Big)^T Q \; D_{\Delta q_0} \Delta z_I(t, \mathbf{0}) \Delta q_0(\omega) \\
& + 2 \Delta p_D(\omega)^T \Big(D_{\Delta \dot{q}_0} \Delta z_I(t, \mathbf{0}) \Big)^T Q \; D_{\Delta \dot{q}_0} \Delta z_I(t, \mathbf{0}) \Delta \dot{q}_0(\omega).
\end{aligned} \tag{5.117}
$$

Assumption 1 *Without constraints we may suppose that the random vectors* $\Delta q_0(\omega), \Delta p_D(\omega)$ *as well as* $\Delta p_D(\omega), \Delta \dot{q}_0(\omega)$ *are stochastically independent.*

Since

$$
\mathrm{E}\Big(\Delta p_D(\omega) | \mathcal{A}_{t_0} \Big) = \overline{\Delta p_D} = 0, \tag{5.118}
$$

cf. (5.99), we have

$$\mathrm{E}\left(\Delta q_0(\omega)^T \left(D_{\Delta\dot{q}_0}\Delta z_I(t,\mathbf{0})\right)^T Q\, D_{\Delta\dot{q}_0}\Delta z_I(t,\mathbf{0})\Delta p_D\right) = 0, \tag{5.119a}$$

$$\mathrm{E}\left(\Delta p_D(\omega)^T \left(D_{\Delta\dot{q}_0}\Delta z_I(t,\mathbf{0})\right)^T Q\, D_{\Delta\dot{q}_0}\Delta z_I(t,\mathbf{0})\Delta\dot{q}_0(\omega)\right) = 0. \tag{5.119b}$$

Defining the deterministic matrices,

$$P_{\Delta p_D} := \left(D_{\Delta p_D}\Delta z_I(t,\mathbf{0})\right)^T Q\, D_{\Delta p_D}\Delta z_I(t,\mathbf{0}), \tag{5.120a}$$

$$P_{\Delta q_0} := \left(D_{\Delta q_0}\Delta z_I(t,\mathbf{0})\right)^T Q\, D_{\Delta q_0}\Delta z_I(t,\mathbf{0}), \tag{5.120b}$$

$$P_{\Delta\dot{q}_0} := \left(D_{\Delta\dot{q}_0}\Delta z_I(t,\mathbf{0})\right)^T Q\, D_{\Delta\dot{q}_0}\Delta z_I(t,\mathbf{0}), \tag{5.120c}$$

$$P_{\Delta\dot{q}_0,\Delta q_0} := \left(D_{\Delta\dot{q}_0}\Delta z_I(t,\mathbf{0})\right)^T Q\, D_{\Delta q_0}\Delta z_I(t,\mathbf{0}), \tag{5.120d}$$

for the conditional expectation of $\tilde{f}$ we obtain the approximation

$$\mathrm{E}\left(\tilde{f}\,\middle|\,\mathcal{A}_{t_0}\right) \approx \mathrm{E}\left(\hat{g}\,\middle|\,\mathcal{A}_{t_0}\right), \tag{5.121a}$$

where

$$\begin{aligned}\mathrm{E}\left(\hat{f}\,\middle|\,\mathcal{A}_{t_0}\right) :=\ & \mathrm{E}\left(\Delta p_D(\omega)^T P_{\Delta p_D}\Delta p_D(\omega)\,\middle|\,\mathcal{A}_{t_0}\right) + \mathrm{E}\left(\Delta q_0(\omega)^T P_{\Delta q_0}\Delta q_0(\omega)\,\middle|\,\mathcal{A}_{t_0}\right)\\ & + \mathrm{E}\left(\Delta\dot{q}_0(\omega)^T P_{\Delta\dot{q}_0}\Delta\dot{q}_0(\omega)\,\middle|\,\mathcal{A}_{t_0}\right) + \mathrm{E}\left(\Delta\dot{q}_0(\omega)^T P_{\Delta\dot{q}_0,\Delta q_0}\Delta q_0(\omega)\,\middle|\,\mathcal{A}_{t_0}\right).\end{aligned} \tag{5.121b}$$

Thus, we find

$$\begin{aligned}\mathrm{E}\left(\Delta p_D(\omega)^T P_{\Delta p_D}\Delta p_D(\omega)\,\middle|\,\mathcal{A}_{t_0}\right) =\ & \mathrm{E}\left(\sum_{k,l=1}^{\nu}[P_{\Delta p_D}]_{kl}\Delta p_{D_k}(\omega)\Delta p_{D_l}(\omega)\,\middle|\,\mathcal{A}_{t_0}\right)\\ &= \sum_{k,l=1}^{\nu}[P_{\Delta p_D}]_{kl}\mathrm{E}\left(\Delta p_{D_k}(\omega)\Delta p_{D_l}(\omega)\,\middle|\,\mathcal{A}_{t_0}\right)\\ &= \sum_{k,l=1}^{\nu}[P_{\Delta p_D}]_{kl}\mathrm{E}\left(\left(\Delta p_{D_k}(\omega) - \underbrace{\overline{\Delta p_{D_k}}}_{=0}\right)\left(\Delta p_{D_l}(\omega) - \underbrace{\overline{\Delta p_{D_l}}}_{=0}\right)\,\middle|\,\mathcal{A}_{t_0}\right)\end{aligned}$$

$$= \sum_{k,l=1}^{\nu} [P_{\Delta p_D}]_{kl} \mathrm{E}\Big(\big(p_{D_k}(\omega) - \overline{p_{D_k}}\big)\big(p_{D_l}(\omega) - \overline{p_{D_l}}\big) \Big| \mathcal{A}_{t_0} \Big)$$

$$= \sum_{k,l=1}^{\nu} [P_{\Delta p_D}]_{kl} \operatorname{cov}\big(p_{D_k}, p_{D_l}\big). \tag{5.122a}$$

With the covariance matrix

$$\operatorname{cov}\big(p_D(\cdot)\big) := \Big(\operatorname{cov}\big(p_{D_k}(\cdot), p_{D_l}(\cdot)\big)\Big)_{k,l=1,\ldots,\nu}$$

$$= \mathrm{E}\Big(\big(p_D(\omega) - \overline{p_D}\big)\big(p_D(\omega) - \overline{p_D}\big)^T \Big| \mathcal{A}_{t_0} \Big) \tag{5.122b}$$

of $p_D = p_D(\omega)$, from (5.122a) we obtain the representation

$$\mathrm{E}\Big(\Delta p_D(\omega)^T P_{\Delta p_D} \Delta p_D(\omega) \Big| \mathcal{A}_{t_0} \Big) = \sum_{k,l=1}^{\nu} [P_{\Delta p_D}]_{kl} \operatorname{cov}\big(p_{D_k}, p_{D_l}\big)$$

$$= \sum_{k,l=1}^{\nu} [P_{\Delta p_D}]_{kl} \operatorname{cov}\big(p_{D_l}, p_{D_k}\big)$$

$$= \sum_{k=1}^{\nu} \Big(\sum_{l=1}^{\nu} [P_{\Delta p_D}]_{kl} \operatorname{cov}\big(p_{D_l}, p_{D_k}\big) \Big)$$

$$= \sum_{k=1}^{\nu} \Big(\text{k-th row of } P_{\Delta p_D} \Big) \cdot \Big(\text{k-th column of } \operatorname{cov}\big(p_D(\cdot)\big) \Big)$$

$$= \operatorname{tr} P_{\Delta p_D} \operatorname{cov}\big(p_D(\cdot)\big). \tag{5.123a}$$

Here, "tr" denotes the trace of a quadratic matrix.

In the same way we have, cf. (5.100a, b),

$$\mathrm{E}\Big(\Delta q_0(\omega)^T P_{\Delta q_0} \Delta q_0(\omega) \Big| \mathcal{A}_{t_0} \Big) = \operatorname{tr} P_{\Delta q_0} \operatorname{cov}\big(\Delta q_0(\cdot)\big), \tag{5.123b}$$

$$\mathrm{E}\Big(\Delta \dot{q}_0(\omega)^T P_{\Delta \dot{q}_0} \Delta \dot{q}_0(\omega) \Big| \mathcal{A}_{t_0} \Big) = \operatorname{tr} P_{\Delta \dot{q}_0} \operatorname{cov}\big(\Delta \dot{q}_0(\cdot)\big). \tag{5.123c}$$

According to (5.100a, b), see also (5.3d, e), (5.118), we have

$$\mathrm{E}\Big(\Delta q_0(\omega) \Big| \mathcal{A}_{t_0} \Big) = 0, \quad \mathrm{E}\Big(\Delta \dot{q}_0(\omega) \Big| \mathcal{A}_{t_0} \Big) = 0. \tag{5.124}$$

Hence, for the fourth term in (5.121b) we get

$$
\begin{aligned}
E\Big(\Delta\dot{q}_0(\omega)^T P_{\Delta\dot{q}_0,\Delta q_0}\Delta q_0(\omega)\Big|\mathcal{A}_{t_0}\Big) &= E\Big(\sum_{k,l=1}^{\nu}[P_{\Delta\dot{q}_0,\Delta q_0}]_{kl}\Delta\dot{q}_{0_k}(\omega)\Delta q_{0_l}(\omega)\Big|\mathcal{A}_{t_0}\Big) \\
&= \sum_{k,l=1}^{\nu}[P_{\Delta\dot{q}_0,\Delta q_0}]_{kl}\,\mathrm{cov}\big(\Delta\dot{q}_{0_k}(\cdot),\Delta q_{0_l}(\cdot)\big) \\
&= \sum_{k,l=1}^{\nu}[P_{\Delta\dot{q}_0,\Delta q_0}]_{kl}\,\mathrm{cov}\big(\Delta q_{0_l}(\cdot),\Delta\dot{q}_{0_k}(\cdot)\big) \\
&= \sum_{k=1}^{\nu}\Big(\sum_{l=1}^{\nu}[P_{\Delta\dot{q}_0,\Delta q_0}]_{kl}\,\mathrm{cov}\big(\Delta q_{0_l}(\cdot),\Delta\dot{q}_{0_k}(\cdot)\big)\Big) \\
&= \sum_{k=1}^{\nu}\Big(\text{ k-th row of } P_{\Delta\dot{q}_0,\Delta q_0}\Big)\cdot \\
&\quad \Big(\text{ k-th column of } \mathrm{cov}\big(\Delta q_0(\cdot),\Delta\dot{q}_0(\cdot)\big)\Big) \\
&= \mathrm{tr}\ P_{\Delta\dot{q}_0,\Delta q_0}\,\mathrm{cov}\big(\Delta q_0(\cdot),\Delta\dot{q}_0(\cdot)\big).
\end{aligned}
\tag{5.125}
$$

Inserting now (5.123a–c) and (5.125) into (5.121b), for the objective function (5.97a) we find the approximation

$$
\begin{aligned}
E\Big(\tilde{f}\Big|\mathcal{A}_{t_0}\Big) \approx{}& \mathrm{tr}\ P_{\Delta p_D}\,\mathrm{cov}\big(p_D(\cdot)\big) + \mathrm{tr}\ P_{\Delta q_0}\,\mathrm{cov}\big(\Delta q_0(\cdot)\big) \\
&+ \mathrm{tr}\ P_{\Delta\dot{q}_0}\,\mathrm{cov}\big(\Delta\dot{q}_0(\cdot)\big) \\
&+ \mathrm{tr}\ P_{\Delta\dot{q}_0,\Delta q_0}\,\mathrm{cov}\big(\Delta q_0(\cdot),\Delta\dot{q}_0(\cdot)\big).
\end{aligned}
\tag{5.126}
$$

Calculation of the Sensitivities

Because of (5.126), (5.120a–d), resp., we need the derivatives of $\Delta z_I(t,p)$ with respect to $\Delta p_D, \Delta q_0$ and $\Delta\dot{q}_0$. The calculation of the derivatives is based on the initial value problem (5.97b–e), where, see 5.99, (5.100a, b), (5.118) and (5.124),

$$\Delta p_D(\omega) := p_D(\omega) - E\Big(p_D(\omega)|\mathcal{A}_{t_0}\Big), \tag{5.127a}$$

$$\Delta q_0(\omega) := q_0(\omega) - E\Big(q_0(\omega)|\mathcal{A}_{t_0}\Big), \tag{5.127b}$$

$$\Delta\dot{q}_0(\omega) := \dot{q}_0(\omega) - E\Big(\dot{q}_0(\omega)|\mathcal{A}_{t_0}\Big). \tag{5.127c}$$

Due to Eqs. (5.127a–c), the constraints of the regulator optimization problem (5.97a–f) will be replaced now by the equivalent ones

$$\min \int_{t_0}^{t_f} \Big(\operatorname{tr} P_{\Delta p_D} \operatorname{cov}\big(p_D(\cdot)\big) + \operatorname{tr} P_{\Delta q_0} \operatorname{cov}\big(\Delta q_0(\cdot)\big) + \operatorname{tr} P_{\Delta \dot{q}_0} \operatorname{cov}\big(\Delta \dot{q}_0(\cdot)\big)$$

$$+ \operatorname{tr} P_{\Delta \dot{q}_0, \Delta q_0} \operatorname{cov}\big(\Delta q_0(\cdot), \Delta \dot{q}_0(\cdot)\big)\Big)\, dt F(\overline{p}_D + \Delta p_D(\omega), q^R(t) + \Delta q(t), \dot{q}^R(t)$$

$$+\Delta \dot{q}(t), \ddot{q}^R(t) + \Delta \ddot{q}(t))$$

$$= F(p_D, q(t), \dot{q}(t), \ddot{q}(t)) = u^R(t) + \varphi(t, \Delta q(t), \Delta q_I(t), \Delta \dot{q}(t)),\ a.s. \tag{5.128a}$$

$$\Delta q(t_0, \omega) = \Delta q_0(\omega), \tag{5.128b}$$

$$\Delta \dot{q}(t_0, \omega) = \Delta \dot{q}_0(\omega), \tag{5.128c}$$

$$\varphi(t_0, \mathbf{0}) = 0. \tag{5.128d}$$

By partial differentiation of the equation (5.128a) with respect to the above mentioned parameter vectors we obtain three systems of differential equations for the sensitivities or partial derivatives of $\Delta z_I = \Delta z_I(t, p)$ with respect to the total parameter vector p needed in in (5.117), (5.120a–d),(5.126).

For the representation of these sensitivities we introduce the following definition:

Definition 5.3 The Jacobians of the vector function F in (5.128a) taken at $\big(\overline{p_D}, q^R(t), \dot{q}^R(t), \ddot{q}^R(t)\big)$ are denoted by

$$K^R(t) := D_q F(\overline{p_D}, q^R(t), \dot{q}^R(t), \ddot{q}^R(t)), \tag{5.129a}$$

$$D^R(t) := D_{\dot{q}} F(\overline{p_D}, q^R(t), \dot{q}^R(t), \ddot{q}^R(t)), \tag{5.129b}$$

$$Y^R(t) := D_{p_D} F(\overline{p_D}, q^R(t), \dot{q}^R(t), \ddot{q}^R(t)), \tag{5.129c}$$

$$M^R(t) := M(\overline{p_D}, q^R(t)) = D_{\ddot{q}} F(\overline{p_D}, q^R(t), \dot{q}^R(t), \ddot{q}^R(t)). \tag{5.129d}$$

Partial Derivative with Respect to Δp_D

By means of differentiation of (5.128a) with respect to the parameter sub vector Δp_D, using the property $\Delta q(t) = \Delta q(t, a) = \Delta q(t, \Delta p_D, \Delta q_0, \Delta \dot{q}_0)$, we obtain

$F = F(\overline{p}_D + \Delta p_D, q^R(t) + \Delta q(t), \dot{q}^R(t) + \Delta\dot{q}(t), \ddot{q}^R(t) + \Delta\ddot{q}(t))$:

$$
\begin{aligned}
\frac{\partial F}{\partial p_D}\frac{\partial \Delta p_D}{\partial \Delta p_D} + \frac{\partial F}{\partial q}\frac{\partial \Delta q(t,a)}{\partial \Delta p_D} + \frac{\partial F}{\partial \dot{q}}\frac{\partial \Delta\dot{q}(t,a)}{\partial \Delta p_D} + \frac{\partial F}{\partial \ddot{q}}\frac{\partial \Delta\ddot{q}(t,a)}{\partial \Delta p_D} &= \underbrace{\frac{\partial}{\partial p_D}u^R(t)}_{=0} \\
+\frac{\partial \varphi}{\partial \Delta q}(t, \Delta z_I)\frac{\partial \Delta q(t,a)}{\partial \Delta p_D} + \frac{\partial \varphi}{\partial \Delta q_I}(t, \Delta z_I)\frac{\partial \Delta q_I(t,a)}{\partial \Delta p_D} & \\
+\frac{\partial \varphi}{\partial \Delta \dot{q}}(t, \Delta z_I)\frac{\partial \Delta\dot{q}(t,a)}{\partial \Delta p_D}. & \quad (5.130\text{a})
\end{aligned}
$$

Moreover, by differentiation of Eqs. (5.128b, c) for the initial values with respect to Δp_D we still find

$$
\frac{\partial \Delta q}{\partial \Delta p_D}(t_0, a) = 0, \tag{5.130b}
$$

$$
\frac{\partial \Delta \dot{q}}{\partial \Delta p_D}(t_0, a) = 0. \tag{5.130c}
$$

Here, we have

$$
\frac{\partial \Delta q_I}{\partial \Delta p_D}(t,a) = \frac{\partial}{\partial \Delta p_D}\int_{t_0}^{t} \Delta q(\tau, a)\, d\tau = \int_{t_0}^{t} \frac{\partial \Delta q}{\partial \Delta p_D}(\tau, a)\, d\tau. \tag{5.131}
$$

Evaluating the initial value problem (5.130a–c) at $a = (\Delta p_D, \Delta q_0, \Delta\dot{q}_0) = 0$ yields, using the definitions (5.129a–d),

$$
\begin{aligned}
Y^R(t) + K^R(t)\frac{\partial \Delta q(t,\mathbf{0})}{\partial \Delta p_D} + D^R(t)\frac{\partial \Delta\dot{q}(t,\mathbf{0})}{\partial \Delta p_D} + M^R(t)\frac{\partial \Delta\ddot{q}(t,\mathbf{0})}{\partial \Delta p_D} & \\
= \frac{\partial \varphi}{\partial \Delta q}(t,\mathbf{0})\frac{\partial \Delta q(t,\mathbf{0})}{\partial \Delta p_D} + \frac{\partial \varphi}{\partial \Delta q_I}(t,\mathbf{0})\frac{\partial \Delta q_I(t,\mathbf{0})}{\partial \Delta p_D} & \\
+\frac{\partial \varphi}{\partial \Delta\dot{q}}(t,\mathbf{0})\frac{\partial \Delta\dot{q}(t,\mathbf{0})}{\partial \Delta p_D}, & \quad (5.132\text{a})
\end{aligned}
$$

$$
\frac{\partial \Delta q}{\partial \Delta p_D}(t_0, \mathbf{0}) = 0, \tag{5.132b}
$$

$$
\frac{\partial \Delta \dot{q}}{\partial \Delta p_D}(t_0, \mathbf{0}) = 0. \tag{5.132c}
$$

Defining the function

$$
\eta(t) := \frac{\partial \Delta q}{\partial \Delta p_D}(t, \mathbf{0}),\ t_0 \le t \le t_f, \tag{5.133}
$$

the initial value problem (5.132a–c) can be represented also by

$$
\begin{aligned}
&Y^R(t) + K^R(t)\eta(t) + D^R(t)\dot{\eta}(t)+M^R(t)\ddot{\eta}(t)\\
&\qquad = \frac{\partial\varphi}{\partial\Delta q}(t,\mathbf{0})\eta(t)+\frac{\partial\varphi}{\partial\Delta q_I}(t,\mathbf{0})\int\limits_{t_0}^{t}\eta(\tau)\,d\tau+\frac{\partial\varphi}{\partial\Delta\dot{q}}(t,\mathbf{0})\dot{\eta}(t),
\end{aligned}
\tag{5.134a}
$$

$$\eta(t_0) = 0, \tag{5.134b}$$

$$\dot{\eta}(t_0) = 0. \tag{5.134c}$$

Since the matrix $M = M(t)$ is regular, see e.g. [11], Eq. (5.134a) can be transformed into

$$
\begin{aligned}
&\ddot{\eta}(t) + M^R(t)^{-1}\big(D^R(t) - \frac{\partial\varphi}{\partial\Delta\dot{q}}(t,\mathbf{0})\big)\dot{\eta}(t) + M^R(t)^{-1}\big(K^R(t) - \frac{\partial\varphi}{\partial\Delta q}(t,\mathbf{0})\big)\eta(t)\\
&\quad +M^R(t)^{-1}\big(-\frac{\partial\varphi}{\partial\Delta q_I}(t,\mathbf{0})\big)\int\limits_{t_0}^{t}\eta(s)\,ds = -M^R(t)^{-1}Y(t).
\end{aligned}
\tag{5.135}
$$

Hence, for the unknown Jacobians of φ we use the following representation:

Definition 5.4 With given fixed, positive definite matrices K_p, K_i, K_d, define

$$K_d =: M^R(t)^{-1}\big(D^R(t) - \frac{\partial\varphi}{\partial\Delta\dot{q}}(t,\mathbf{0})\big), \tag{5.136a}$$

$$K_p =: M^R(t)^{-1}\big(K^R(t) - \frac{\partial\varphi}{\partial\Delta q}(t,\mathbf{0})\big), \tag{5.136b}$$

$$K_i =: M^R(t)^{-1}\big(-\frac{\partial\varphi}{\partial\Delta q_I}(t,\mathbf{0})\big). \tag{5.136c}$$

Remark 5.3 In many cases the matrices K_p, K_i, K_d in (5.136a–c) are supposed to be diagonal matrices.

With the definitions (5.136a–c), the initial value problem (5.134a–c) can be represented now as follows:

$$\ddot{\eta}(t) + K_d\dot{\eta}(t) + K_p\eta(t) + K_i\int\limits_{t_0}^{t}\eta(\tau)\,d\tau = -M^R(t)^{-1}Y^R(t),\ t \ge t_0, \tag{5.137a}$$

$$\eta(t_0) = 0, \tag{5.137b}$$

$$\dot{\eta}(t_0) = 0, \tag{5.137c}$$

Partial Derivative with Respect to Δq_0

Following the above procedure, we can also (i) determine the partial derivative of Δz_I with respect to the deviation Δq_0 in the initial position q_0, and (ii) evaluate then the derivatives at $a = (\Delta p_D, \Delta q_0, \dot{\Delta q}_0) = 0$.

This way, we get the next initial value problem:

$$Y^R(t)\frac{\partial \Delta p_D}{\partial \Delta q_0} + K^R(t)\frac{\partial \Delta q(t, \mathbf{0})}{\partial \Delta q_0} + D^R(t)\frac{\partial \Delta \dot{q}(t, \mathbf{0})}{\partial \Delta q_0} + M^R(t)\frac{\partial \Delta \ddot{q}(t, \mathbf{0})}{\partial \Delta q_0}$$

$$= \frac{\partial \varphi}{\partial \Delta q}(t, \mathbf{0})\frac{\partial \Delta q(t, \mathbf{0})}{\partial \Delta q_0} + \frac{\partial \varphi}{\partial \Delta q_I}(t, \mathbf{0})\frac{\partial \Delta q_I(t, \mathbf{0})}{\partial \Delta q_0}$$

$$+ \frac{\partial \varphi}{\partial \Delta \dot{q}}(t, \mathbf{0})\frac{\partial \Delta \dot{q}(t, \mathbf{0})}{\partial \Delta q_0} \tag{5.138a}$$

$$\frac{\partial \Delta q}{\partial \Delta q_0}(t_0, \mathbf{0}) = I, \tag{5.138b}$$

$$\frac{\partial \Delta \dot{q}}{\partial \Delta q_0}(t_0, \mathbf{0}) = 0, \tag{5.138c}$$

where I denotes the unit matrix .

Defining then

$$\xi(t) := \frac{\partial \Delta q}{\partial \Delta q_0}(t, \mathbf{0}),\ t \geq t_0, \tag{5.139}$$

and having $\frac{\partial \Delta p_D}{\partial \Delta q_0} = 0$, the differential equations (5.138a) reads

$$\ddot{\xi}(t) + M(t)^{-1}\big(D(t) - \frac{\partial \varphi}{\partial \Delta \dot{q}}(t, \mathbf{0})\big)\dot{\xi}(t) + M(t)^{-1}\big(K(t) - \frac{\partial \varphi}{\partial \Delta q}(t, \mathbf{0})\big)\xi(t)$$

$$+M(t)^{-1}\big(- \frac{\partial \varphi}{\partial \Delta q_I}(t, \mathbf{0})\big)\int_{t_0}^{t} \xi(\tau)\, d\tau = 0. \tag{5.140}$$

Using again definitions (5.136a–c), for $\xi = \xi(t)$ we get the initial value problem:

$$\ddot{\xi}(t) + K_d\dot{\xi}(t) + K_p\xi(t) + K_i\int_{t_0}^{t} \xi(\tau)\, d\tau = 0,\ t \geq t_0, \tag{5.141a}$$

$$\xi(t_0) = I, \tag{5.141b}$$

$$\dot{\xi}(t_0) = 0. \tag{5.141c}$$

Partial Derivative with Respect to $\Delta\dot{q}_0$

Finally, we consider the partial derivative with respect to the deviations $\Delta\dot{q}_0$ in the initial velocities $\dot{q}_0$. With a corresponding differentiation and evaluation at $a = (\Delta p_D, \Delta q_0, \dot{\Delta q}_0) = 0$, we get the in initial value problem

$$Y(t)\frac{\partial \Delta p_D}{\partial \Delta\dot{q}_0} + K(t)\frac{\partial \Delta q(t,\mathbf{0})}{\partial \Delta\dot{q}_0} + D(t)\frac{\partial \Delta\dot{q}(t,\mathbf{0})}{\partial \Delta\dot{q}_0} + M(t)\frac{\partial \Delta\ddot{q}(t,\mathbf{0})}{\partial \Delta\dot{q}_0}$$
$$= \frac{\partial\varphi}{\partial \Delta q}(t,\mathbf{0})\frac{\partial \Delta q(t,\mathbf{0})}{\partial \Delta\dot{q}_0} + \frac{\partial\varphi}{\partial \Delta q_I}(t,\mathbf{0})\frac{\partial \Delta q_I(t,\mathbf{0})}{\partial \Delta\dot{q}_0}$$
$$+\frac{\partial\varphi}{\partial \Delta\dot{q}}(t,\mathbf{0})\frac{\partial \Delta\dot{q}(t,\mathbf{0})}{\partial \Delta\dot{q}_0}, \tag{5.142a}$$

$$\frac{\partial \Delta q}{\partial \Delta\dot{q}_0}(t_0,\mathbf{0}) = 0, \tag{5.142b}$$

$$\frac{\partial \Delta\dot{q}}{\partial \Delta\dot{q}_0}(t_0,\mathbf{0}) = I. \tag{5.142c}$$

Also here I denotes the unit matrix and it holds $\frac{\partial \Delta p_D}{\partial \Delta\dot{q}_0} = 0$.

Defining here

$$\zeta(t) := \frac{\partial \Delta q}{\partial \Delta\dot{q}_0}(t,0),\ t \geq t_0, \tag{5.143}$$

for (5.142a) we find the representation

$$\ddot{\zeta}(t) + M^R(t)^{-1}\big(D^R(t) - \frac{\partial\varphi}{\partial \Delta\dot{q}}(t,\mathbf{0})\big)\dot{\zeta}(t) + M^R(t)^{-1}\big(K^R(t) - \frac{\partial\varphi}{\partial \Delta q}(t,\mathbf{0})\big)\zeta(t)$$
$$+M^R(t)^{-1}\big(-\frac{\partial\varphi}{\partial \Delta q_I}(t,\mathbf{0})\big)\int_{t_0}^{t}\zeta(\tau)\,d\tau = 0. \tag{5.144}$$

Taking into account (5.136a–c), one finally gets the initial value problem

$$\ddot{\zeta}(t) + K_d\dot{\zeta}(t) + K_p\zeta(t) + K_i\int_{t_0}^{t}\zeta(\tau)\,d\tau = 0,\ t \geq t_0, \tag{5.145a}$$

$$\zeta(t_0) = 0, \tag{5.145b}$$

$$\dot{\zeta}(t_0) = I. \tag{5.145c}$$

5.7.2 The Approximate Regulator Optimization Problem

Based on the above approximations, the regulator optimization problem under stochastic uncertainty can be represented by

$$\min \int_{t_0}^{t_f} \Big(\operatorname{tr} P_{\Delta p_D} \operatorname{cov}\big(p_D(\cdot)\big) + \operatorname{tr} P_{\Delta q_0} \operatorname{cov}\big(\Delta q_0(\cdot)\big) + \operatorname{tr} P_{\Delta \dot{q}_0} \operatorname{cov}\big(\Delta \dot{q}_0(\cdot)\big) + \operatorname{tr} P_{\Delta \dot{q}_0, \Delta q_0} \operatorname{cov}\big(\Delta q_0(\cdot), \Delta \dot{q}_0(\cdot)\big) \Big)\, dt \tag{5.146a}$$

subject to the integro-differential system

$$\ddot{\eta}(t) + K_d \dot{\eta}(t) + K_p \eta(t) + K_i \int_{t_0}^{t} \eta(\tau)\, d\tau = -M(t)^{-1} Y(t), t \geq t_0, \tag{5.146b}$$

$$\eta(t_0) = 0, \tag{5.146c}$$

$$\dot{\eta}(t_0) = 0, \tag{5.146d}$$

$$\ddot{\xi}(t) + K_d \dot{\xi}(t) + K_p \xi(t) + K_i \int_{t_0}^{t} \xi(\tau)\, d\tau = 0,\ t \geq t_0, \tag{5.146e}$$

$$\xi(t_0) = I, \tag{5.146f}$$

$$\dot{\xi}(t_0) = 0. \tag{5.146g}$$

$$\ddot{\zeta}(t) + K_d \dot{\zeta}(t) + K_p \zeta(t) + K_i \int_{t_0}^{t} \zeta(\tau)\, d\tau = 0,\ t \geq t_0, \tag{5.146h}$$

$$\zeta(t_0) = 0, \tag{5.146i}$$

$$\dot{\zeta}(t_0) = I. \tag{5.146j}$$

Introducing appropriate vector and vector functions, the above three systems of differential equations (5.146b–j) can be represented uniformly. Hence, we set:

$$\psi(t) := \left. \begin{cases} -M^R(t)^{-1} Y^R(t),\ t \geq t_0, & \text{for (5.146b–d)} \\ 0, & \text{for (5.146e–g)} \\ 0, & \text{for (5.146h–j)} \end{cases} \right\},\ t \geq t_0, \tag{5.147a}$$

and

$$dq_0 := \begin{cases} 0, & \text{for (5.146b–d)} \\ I, & \text{for (5.146e–g)} \\ 0, & \text{for (5.146h–j),} \end{cases} \tag{5.147b}$$

$$d\dot{q}_0 := \begin{cases} 0, & \text{for (5.146b–d)} \\ 0, & \text{for (5.146e–g)} \\ I, & \text{for (5.146h–j).} \end{cases} \tag{5.147c}$$

The three systems of differential equations (5.146b–d), (5.146e–g), (5.146h–j), resp., for the computation of the partial derivatives (sensitivities)

$$dq(t) := \left\{ \begin{matrix} \eta(t), & \text{for (5.146b–d)} \\ \zeta(t), & \text{for (5.146e–g)} \\ \xi(t), & \text{for (5.146h–j)} \end{matrix} \right\}, \ t \geq t_0 \tag{5.147d}$$

can be represented then by:

$$d\ddot{q}(t) + K_d d\dot{q}(t) + K_p dq(t) + K_i \int_{t_0}^{t} dq(\tau)\, d\tau = \psi(t), \ t \geq t_0 \tag{5.148a}$$

$$dq(t_0) = dq_0 \tag{5.148b}$$

$$d\dot{q}(t_0) = d\dot{q}_0. \tag{5.148c}$$

Defining

$$dz_I(t) := \begin{pmatrix} dq(t) \\ \int_{t_0}^{t} dq(\tau)\, d\tau \\ d\dot{q}(t) \end{pmatrix}, \ t \geq t_0, \tag{5.149}$$

the integro-differential equations (5.148a–c) can be transformed into linear systems of first order of the following type:

$$d\dot{z}_I(t) = A\, dz_I(t) + \begin{pmatrix} 0 \\ 0 \\ \psi(t) \end{pmatrix}, \ t \geq t_0, \tag{5.150a}$$

$$dz_I(t_0) = dz_{I_0}. \tag{5.150b}$$

Here, the matrix A and the vector of initial conditions dz_{I_0} is defined by:

$$A := \begin{pmatrix} 0 & 0 & I \\ I & 0 & 0 \\ -K_p & -K_i & -K_d \end{pmatrix}, \tag{5.150c}$$

$$dz_{I_0} := \begin{pmatrix} dq_0 \\ 0 \\ d\dot{q}_0 \end{pmatrix}. \tag{5.150d}$$

Chapter 6
Expected Total Cost Minimum Design of Plane Frames

6.1 Introduction

Yield stresses, allowable stresses, moment capacities (plastic moments with respect to compression, tension and rotation), applied loadings, cost factors, manufacturing errors, etc., are not given fixed quantities in structural analysis and optimal design problems, but must be modeled as random variables with a certain joint probability distribution. Problems from plastic analysis and optimal plastic design are based on the convex yield (feasibility) criterion and the linear equilibrium equation for the stress (state) vector.

After the formulation of the basic mechanical conditions including the relevant material strength parameters and load components as well as the involved design variables (as e.g. sizing variables) for plane frames, several approximations are considered: (i) approximation of the convex yield (feasible) domain by means of convex polyhedrons (piecewise linearization of the yield domain); (ii) treatment of symmetric and non symmetric yield stresses with respect to compression and tension; (iii) approximation of the yield condition by using given reference capacities.

As a result, for the survival of a plane frame a certain system of necessary and/or sufficient linear equalities and inequalities is obtained. Evaluating the recourse costs, i.e., the costs for violations of the survival condition by means of piecewise linear convex cost functions, a linear program is obtained for the minimization of the total costs including weight-, volume- or more general initial (construction) costs. Appropriate cost factors are derived. Considering then the expected total costs from construction as well as from possible structural damages or failures, a stochastic linear optimization problem (SLP) is obtained. Finally, discretization of the probability distribution of the random parameter vector yields a (large scale) linear program (LP) having a special structure. For LP's of this type numerous, very efficient LP-solvers are available—also for the solution of very large scale problems.

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6.2 Stochastic Linear Programming Techniques

6.2.1 Limit (Collapse) Load Analysis of Structures as a Linear Programming Problem

Assuming that the material behaves as an elastic-perfectly plastic material [59, 79, 119] a conservative estimate of the collapse load factor λ_T is based [30, 31, 36, 39, 54, 55, 77, 120, 164] on the following linear program:

$$\text{maximize } \lambda \tag{6.1a}$$

s.t.

$$F^L \leq F \leq F^U \tag{6.1b}$$

$$CF = \lambda R_0. \tag{6.1c}$$

Here, (6.1c) is the equilibrium equation of a statically indeterminate loaded structure involving an $m \times n$ matrix $C = (c_{ij}), m < n$, of given coefficients $c_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$, depending on the undeformed geometry of the structure having n_0 members (elements); we suppose that rank $C = m$. Furthermore, R_0 is an external load m-vector, and F denotes the n-vector of internal forces and bending-moments in the relevant points (sections, nodes or elements) of lower and upper bounds F^L, F^U.

For a plane or spatial truss [83, 158] we have that $n = n_0$, the matrix C contains the direction cosines of the members, and F involves only the normal (axial) forces moreover,

$$F_j^L := \sigma_{yj}^L A_j, F_j^U := \sigma_{yj}^U A_j, j = 1, \ldots, n(= n_0), \tag{6.2}$$

where A_j is the (given) cross-sectional area, and $\sigma_{yj}^L, \sigma_{yj}^U$, respectively, denotes the yield stress in compression (negative values) and tension (positive values) of the j-th member of the truss. In case of a plane frame, F is composed of subvectors [158],

$$F^{(k)} = \begin{pmatrix} F_1^{(k)} \\ F_2^{(k)} \\ F_3^{(k)} \end{pmatrix} = \begin{pmatrix} t_k \\ m_k^+ \\ m_k^- \end{pmatrix}, \tag{6.3a}$$

where $F_1^{(k)} = t_k$ denotes the normal (axial force, and $F_2^{(k)} = m_k^+, K_3^{(k)} = m_k^-$ are the bending-moments at the positive, negative end of the k-th member. In this case

F^L, F^U contain—for each member k—the subvectors

$$F^{(k)L} = \begin{pmatrix} \sigma_{yk}^L A_k \\ -M_{kpl} \\ -M_{kpl} \end{pmatrix}, F^{(k)U} = \begin{pmatrix} \sigma_{yk}^U A_k \\ M_{kpl} \\ M_{kpl} \end{pmatrix}, \tag{6.3b}$$

respectively, where $M_{kpl}, k = 1, \ldots, n_0$, denotes the plastic moments (moment capacities) [59, 119] given by

$$M_{kpl} = \sigma_{yk}^U W_{kpl}, \tag{6.3c}$$

and $W_{kpl} = W_{kpl}(A_k)$ is the plastic section modulus of the cross-section of the k-th member (beam) with respect to the local z-axis.

For a *spatial frame* [83, 158], corresponding to the k-th member (beam), F contains the subvector

$$F^{(k)} := (t_k, m_{kT}, m_{k\bar{y}}^+ m_{k\bar{z}}^+, m_{k\bar{y}}^- m_{k\bar{z}}^-)', \tag{6.4a}$$

where t_k is the normal (axial) force, m_{kT} the twisting moment, and $m_{k\bar{y}}^-, m_{k\bar{z}}^+,$ $m_{k\bar{y}}^-, m_{k\bar{z}}^-$ denote four bending moments with respect to the local $\bar{y}, z$-axis at the positive, negative end of the beam, respectively. Finally, the bounds F^L, F^U for F are given by

$$F^{(k)L} = \left(\sigma_{yk}^L A_k, -M_{kpl}^{\bar{p}}, -M_{kpl}^{\bar{y}}, -M_{kpl}^{\bar{z}}, -M_{kpl}^{\bar{y}}, -M_{kpl}^{z}\right)' \tag{6.4b}$$

$$F^{(k)U} = \left(\sigma_{yk}^U A_k, M_{kpl}^{\bar{p}}, M_{kpl}^{\bar{y}}, M_{kpl}^{\bar{z}}, M_{kpl}^{\bar{y}}, M_{kpl}^{\bar{z}}\right)', \tag{6.4c}$$

where [59, 119]

$$M_{kpl}^{\bar{p}} := \tau_{yk} W_{kpl}^{\bar{p}}, M_{kpl}^{\bar{y}} := \sigma_{yk}^U W_{kpl}^{\bar{y}}, M_{kpl}^{\bar{z}} := \sigma_{yk}^U W_{kpl}^{\bar{z}}, \tag{6.4d}$$

are the plastic moments of the cross-section of the k-th element with respect to the local twisting axis, the local $\bar{y}$-, $\bar{z}$-axis, respectively. In (6.4d), $W_{kpl}^{\bar{p}} = W_{kpl}^{\bar{p}}(x)$ and $W_{kpl}^{\bar{y}} = W_{kpl}^{\bar{y}}(X), W_{kpl}^{\bar{z}} = W_{kpl}^{\bar{z}}(X)$, respectively, denote the polar, axial modulus of the cross-sectional area of the k-th beam and τ_{yk} denotes the yield stress with respect to torsion; we suppose that $\tau_{yk} = \sigma_{yk}^U$.

Remark 6.1 Possible plastic hinges [59, 82, 119] are taken into account by inserting appropriate eccentricities $e_{kl} > 0, e_{kr} > 0, k = 1, \ldots, n_0$, with $e_{kl}, e_{kr} \ll L_k$, where L_k is the length of the k-th beam.

Remark 6.2 Working with more general yield polygons, see e.g. [164], the stress condition (6.1b) is replaced by the more general system of inequalities

$$H(F_d^U)^{-1} F \le h. \tag{6.5a}$$

Here, (H, h) is a given $\nu \times (n+1)$ matrix, and $F_d^U := (F_j^U \delta_{ij})$ denotes the $n \times n$ diagonal matrix of principal axial and bending plastic capacities

$$F_j^U := \sigma_{yk_j}^U A_{kj}, F_j^U := \sigma_{yk_j}^U W_{k_j pl}^{\kappa j}, \tag{6.5b}$$

where $kj, \kappa j$ are indices as arising in (6.3b)–(6.4d). The more general case (6.5a) can be treated by similar methods as the case (6.1b) which is considered here.

Plastic and Elastic Design of Structures

In the plastic design of trusses and frames [77, 87, 92, 97, 128, 152] having n_0 members, the n-vectors F^L, F^U of lower and upper bounds

$$F^L = F^L(\sigma_y^L, \sigma_y^U, x), F^U = F^U(\sigma_y^L, \sigma_y^U, x), \tag{6.6}$$

for the n-vector F of internal member forces and bending moments $F_j, j = 1, \ldots, n$, are determined [54, 77] by the yield stresses, i.e. compressive limiting stresses (negative values) $\sigma_y^L = \left(\sigma_{y1}^L, \ldots, \sigma_{yn_0}^L\right)^T$, the tensile yield stresses $\sigma_y^U = \left(\sigma_{y1}^U, \ldots, \sigma_{yn_0}^U\right)^T$, and the r-vector

$$x = (x_1, x_2, \ldots, x_r)^T \tag{6.7}$$

of design variables of the structure. In case of trusses we have that, cf. (6.2),

$$\begin{aligned} F^L &= \sigma_{yd}^L A(x) = A(x)_d \sigma_y^L, \\ F^U &= \sigma_{yd}^U A(x) = A(x)_d \sigma_y^U, \end{aligned} \tag{6.8}$$

where $n = n_0$, and $\sigma_{yd}^L, \sigma_{yd}^U$ denote the $n \times n$ diagonal matrices having the diagonal elements $\sigma_{yj}^L, \sigma_{yj}^U$, respectively, $j = 1, \ldots, n$, moreover,

$$A(x) = \left[A_1(x), \ldots, A_n(x)\right]^T \tag{6.9}$$

is the n-vector of cross-sectional area $A_j = A_j(x), j = 1, \ldots, n$, depending on the r-vector x of design variables $x_\kappa, \kappa = 1, \ldots, r$, and $A(x)_d$ denotes the $n \times n$ diagonal matrix having the diagonal elements $A_j = A_j(x), 1 < j < n$.

Corresponding to (6.1c), here the equilibrium equation reads

$$CF = R_u, \tag{6.10}$$

where R_u describes [77] the ultimate load [representing constant external loads or self-weight expressed in linear terms of $A(x)$].

The *plastic design* of structures can be represented then, cf. [7], by the optimization problem

$$\min G(x), \tag{6.11a}$$

s.t.

$$F^L(\sigma_y^L, \sigma_y^U, x) \le F \le F^U(\sigma_y^L, \sigma_y^U, x) \tag{6.11b}$$

$$CF = R_u, \tag{6.11c}$$

where $G = G(x)$ is a certain objective function, e.g. the volume or weight of the structure.

Remark 6.3 As mentioned in Remark 6.2, working with more general yield polygons, (6.11b) is replaced by the condition

$$H[F^U(\sigma_y^U, x)_d]^{-1} F \le h. \tag{6.11d}$$

For the *elastic design* we must replace the yield stresses σ_y^L, σ_y^U by the allowable stresses σ_a^L, σ_a^U and instead of ultimate loads we consider service loads R_s. Hence, instead of (6.11a–d) we have the related program

$$\min G(x), \tag{6.12a}$$

s.t.

$$F^L(\sigma_a^L, \sigma_a^U, x) \le F \le F^U(\sigma_a^L, \sigma_a^U, x), \tag{6.12b}$$

$$CF = R_s, \tag{6.12c}$$

$$x^L \le x \le x^U, \tag{6.12d}$$

where x^L, x^U still denote lower and upper bounds for x.

6.2.2 Plane Frames

For each bar $i = 1, \ldots, B$ of a plane frame with member load vector $F_i = (t_i, m_i^+, m_i^-)^T$ we consider [153, 165] the load at the negative, positive end

$$F_i^- := (t_i, m_i^-)^T, F_i^+ := (t_i, m_i^+)^T, \tag{6.13}$$

respectively.

Furthermore, for each bar/beam with rigid joints we have several plastic capacities: The plastic capacity N_{ipl}^{L} of the bar with respect to axial compression, hence, the maximum axial force under compression is given by

$$N_{ipl}^{L} = |\sigma_{yi}^{L}| \cdot A_i, \tag{6.14a}$$

where $\sigma_{yi}^{L} < 0$ denotes the (negative) yield stress with respect to compression and A_i is the cross sectional area of the ith element. Correspondingly, the plastic capacity with respect to (axial) tension reads:

$$N_{ipl}^{U} = \sigma_{yi}^{U} \cdot A_i, \tag{6.14b}$$

where $\sigma_{yi}^{U} > 0$ is the yield stress with respect to tension. Besides the plastic capacities with respect to the normal force, we have the moment capacity

$$M_{ipl} = \sigma_{yi}^{U} \cdot W_{ipl} \tag{6.14c}$$

with respect to the bending moments at the ends of the bar i.

Remark 6.1 Note that all plastic capacities have nonnegative values.

Using the plastic capacities (6.14a–c), the load vectors F_i^{+}, F_i^{+} given by (6.13) can be replaced by dimensionless quantities

$$F_i^{L-} := \left(\frac{t_i}{N_{ipl}^{L}}, \frac{m_i^{-}}{M_{ipl}}\right)^T, F_i^{U-} := \left(\frac{t_i}{N_{ipl}^{U}}, \frac{m_i^{-}}{M_{ipl}}\right)^T \tag{6.15a,b}$$

$$F_i^{L+} := \left(\frac{t_i}{N_{ipl}^{L}}, \frac{m_i^{+}}{M_{ipl}}\right)^T, F_i^{U+} := \left(\frac{t_i}{N_{ipl}^{U}}, \frac{m_i^{+}}{M_{ipl}}\right)^T \tag{6.15c,d}$$

for the negative, positive end, resp., of the ith bar.

Remark 6.2 (Symmetric Yield Stresses Under Compression and Tension) In the important special case that the absolute values of the yield stresses under compression (< 0) and tension (> 0) are equal, hence,

$$\sigma_{yi}^{L} = -\sigma_{yi}^{U} \tag{6.16a}$$

$$N_{ipl}^{L} = N_{ipl}^{U} =: N_{ipl}. \tag{6.16b}$$

The limit between the elastic and plastic state of the elements is described by the feasibility or yield condition: At the negative end we have the condition

$$F_i^{L-} \in K_i, F_i^{U-} \in K_i \tag{6.17a,b}$$

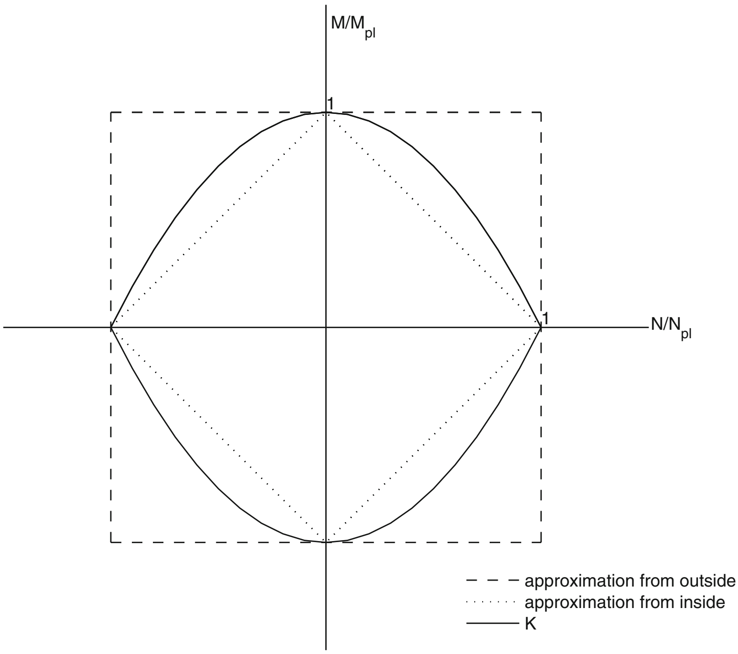

Fig. 6.1 Domain $K_{0,\text{sym}}$ with possible approximations

and at the positive end the condition reads

$$F_i^{L+} \in K_i, F_i^{U+} \in K_i. \tag{6.17c,d}$$

Here, K_i, $K_i \in \mathbb{R}^2$, denotes the feasible domain of bar "i" having the following properties:

- K_i is a closed, convex subset of $\mathbb{R}^2$
- the origin 0 of $\mathbb{R}^2$ is an interior point of K_i
- the interior K_i of K_i represents the elastic states
- at the boundary ∂K_i yielding of the material starts.

Considering e.g. bars with rectangular cross sectional areas and symmetric yield stresses, cf. Remark 6.2, K_i is given by $K_i = K_{0,sym}$, where [72, 74]

$$K_{0,\text{sym}} = \{(x, y)^T : x^2 + |y| \leq 1\} \tag{6.18}$$

where $x = \frac{N}{N_{pl}}$ and $y = \frac{M}{M_{pl}}$, (see Fig. 6.1).

In case (6.18) and supposing symmetric yield stresses, the yield condition (17a–d) reads

$$\left(\frac{t_i}{N_{ipl}}\right)^2 + \left|\frac{m_i^-}{M_{ipl}}\right| \le 1, \tag{6.19a}$$

$$\left(\frac{t_i}{N_{ipl}}\right)^2 + \left|\frac{m_i^+}{M_{ipl}}\right| \le 1. \tag{6.19b}$$

Remark 6.3 Because of the connection between the normal force t_i and the bending moments, (6.19a, b) are also called "M–N-interaction".

If the M–N-interaction is not taken into account, $K_{0,\text{sym}}$ is approximated from outside, see Fig. 6.1, by

$$K^u_{0,\text{sym}} := \{(x,y)^T : |x|, |y| \le 1\}. \tag{6.20}$$

Hence, (4.17a–d) are replaced, cf. (6.19a, b), by the simpler conditions

$$|t_i| \le N_{ipl} \tag{6.21a}$$

$$|m_i^-| \le M_{ipl} \tag{6.21b}$$

$$|m_i^+| \le M_{ipl}. \tag{6.21c}$$

Since the symmetry condition (6.16a) does not hold in general, some modifications of the basic conditions (6.19a, b) are needed. In the non symmetric case $K_{0,xm}$ must be replaced by the intersection

$$K_0 = K_0^U \cap K_0^L \tag{6.22}$$

of two convex sets K_0^U and K_0^L. For a rectangular cross-sectional area we have

$$K_0^U = \{(x,y)^T : \ x \le \sqrt{1-|y|}, \ |y| \le 1\} \tag{6.23a}$$

for tension and

$$K_0^L = \{(x,y)^T : -x \le \sqrt{1-|y|}, \ |y| \le 1\} \tag{6.23b}$$

compression, where again $x = \frac{N}{N_{pl}}$ and $y = \frac{M}{M_{pl}}$ (see Fig. 6.2).

In case of tension, see (6.17b, d), from (6.23a) we obtain then the feasibility condition

$$\frac{t_i}{N_{ipl}^U} \le \sqrt{1 - \left|\frac{m_i^-}{M_{ipl}}\right|} \tag{6.24a}$$

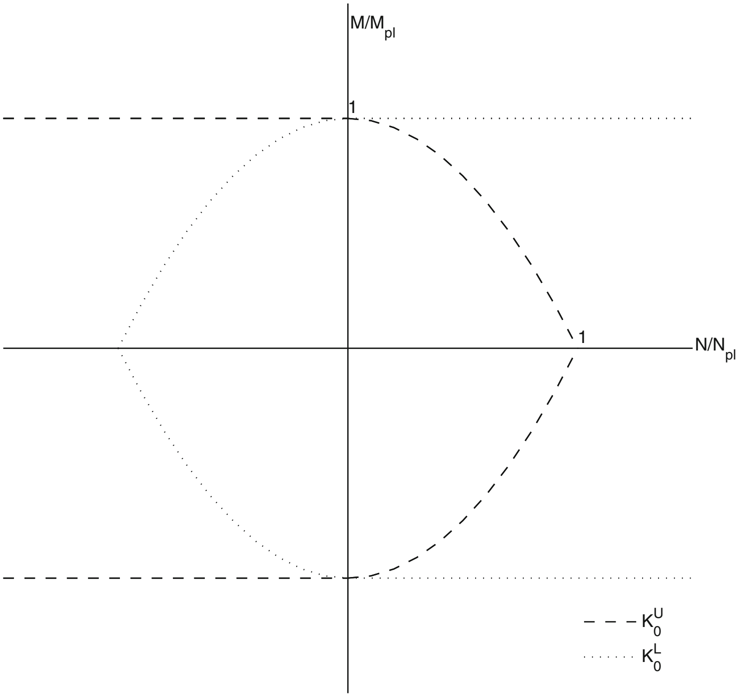

Fig. 6.2 Feasible domain as intersection of K_0^U and K_0^L

$$\frac{t_i}{N_{ipl}^U} \leq \sqrt{1 - \left|\frac{m_i^+}{M_{ipl}}\right|} \tag{6.24b}$$

$$\left|\frac{m_i^-}{M_{ipl}}\right| \leq 1 \tag{6.24c}$$

$$\left|\frac{m_i^+}{M_{ipl}}\right| \leq 1. \tag{6.24d}$$

For compression, with (4.17a, c) and (6.23b) we get the feasibility condition

$$-\frac{t_i}{N_{ipl}^L} \leq \sqrt{1 - \left|\frac{m_i^-}{M_{ipl}}\right|} \tag{6.24e}$$

$$-\frac{t_i}{N_{ipl}^L} \leq \sqrt{1 - \left|\frac{m_i^+}{M_{ipl}}\right|} \tag{6.24f}$$

$$\left|\frac{m_i^-}{M_{ipl}}\right| \leq 1 \tag{6.24g}$$

$$\left|\frac{m_i^+}{M_{ipl}}\right| \leq 1. \tag{6.24h}$$

From (6.24a), (6.24e) we get

$$-N_{ipl}^L\sqrt{1-\left|\frac{m_i^-}{M_{ipl}}\right|} \leq t_i \leq N_{ipl}^U\sqrt{1-\left|\frac{m_i^-}{M_{ipl}}\right|}. \tag{6.25a}$$

and (6.24b), (6.24f) yield

$$-N_{ipl}^L\sqrt{1-\left|\frac{m_i^+}{M_{ipl}}\right|} \leq t_i \leq N_{ipl}^U\sqrt{1-\left|\frac{m_i^+}{M_{ipl}}\right|}. \tag{6.25b}$$

Furthermore, (6.24c), (6.24g) and (6.24d), (6.24h) yield

$$|m_i^-| \leq M_{ipl} \tag{6.25c}$$

$$|m_i^+| \leq M_{ipl}. \tag{6.25d}$$

For computational purposes, piecewise linearizations are applied [99] to the nonlinear conditions (6.25a, b). A basic approximation of K_0^L and K_0^U is given by

$$K_0^{Uu} := \{(x,y)^T : x \leq 1, |y| \leq 1\} =]\infty, 1] \times [-1, 1] \tag{6.26a}$$

and

$$K_0^{Lu} := \{(x,y)^T : x \geq -1, |y| \leq 1\} = [-1, \infty] \times [-1, 1] \tag{6.26b}$$

with $x = \frac{N}{N_{pl}}$ and $y = \frac{M}{M_{pl}}$ (see Fig.6.3).

Since in this approximation the M–N-interaction is not taken into account, condition (6.17a–d) is reduced to

$$-N_{ipl}^L \leq t_i \leq N_{ipl}^U \tag{6.27a}$$

$$|m_i^-| \leq M_{ipl} \tag{6.27b}$$

$$|m_i^+| \leq M_{ipl}. \tag{6.27c}$$

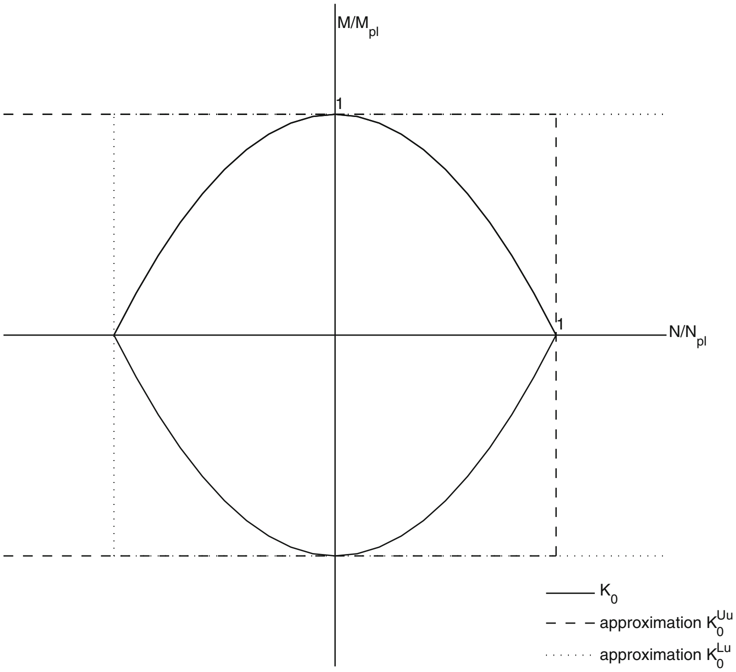

Fig. 6.3 Approximation of K_0 by K_0^{Lu} and K_0^{Uu}

6.2.3 Yield Condition in Case of M–N-Interaction

Symmetric Yield Stresses

Consider first the case

$$\sigma_{yi}^{U} = -\sigma_{yi}^{L} =: \sigma_{yi}, \; i = 1, \ldots, B. \tag{6.28a}$$

Then,

$$N_{ipl}^{L} := |\sigma_{yi}^{L}| A_i = \sigma_{yi}^{U} A_i =: N_{ipl}^{U}, \tag{6.28b}$$

hence,

$$N_{ipl} := N_{ipl}^{L} = N_{ipl}^{U} = \sigma_{yi} A_i \,. \tag{6.28c}$$

Moreover,

$$M_{ipl} = \sigma_{yi}^{U} W_{ipl} = \sigma_{yi} W_{ipl} = \sigma_{yi} A_i \, \bar{y}_{ic}, \; i = 1, \ldots, B, \tag{6.28d}$$

where $\overline{y}_{ic}$ denotes the arithmetic mean of the centroids of the two half areas of the cross-sectional area A_i of bar i.

Depending on the geometric from of the cross-sectional areal (rectangle, circle, etc.), for the element load vectors

$$F_i = \begin{pmatrix} t_i \\ m_i^+ \\ m_i^- \end{pmatrix}, i = 1, \dots, B, \tag{6.29}$$

of the bars we have the yield condition:

$$\left| \frac{t_i}{N_{ipl}} \right|^{\alpha} + \left| \frac{m_i^-}{M_{i\,pl}} \right| \leq 1 \quad \text{(negative end)} \tag{6.30a}$$

$$\left| \frac{t_i}{N_{ipl}} \right|^{\alpha} + \left| \frac{m_i^+}{M_{i\,pl}} \right| \leq 1 \quad \text{(positive end).} \tag{6.30b}$$

Here, $\alpha > 1$ is a constant depending on the type of the cross-sectional area of the ith bar. Defining the convex set

$$K_0^{\alpha} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : |x|^{\alpha} + |y| \leq 1 \right\}, \tag{6.31}$$

for (6.30a, b) we have also the representation

$$\begin{pmatrix} \frac{t_i}{N_{ipl}} \\ \frac{m_i^-}{M_{ipl}} \end{pmatrix} \in K_0^{\alpha} \quad \text{(negative end)} \tag{6.32a}$$

$$\begin{pmatrix} \frac{t_i}{N_{ipl}} \\ \frac{m_i^+}{M_{ipl}} \end{pmatrix} \in K_0^{\alpha} \quad \text{(positive end).} \tag{6.32b}$$

Piecewise Linearization of K_0^{α}

Due to the symmetry of K_0^{α} with respect to the transformation

$$x \to -x, y \to -y,$$

K_0^{α} is piecewise linearized as follows:

Starting from a boundary point of K_0^{α}, hence,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \partial K_0^{\alpha} \quad \text{with} \quad u_1 \geq 0,\ u_2 \geq 0, \tag{6.33a}$$

we consider the gradient of the boundary curve

$$f(x, y) := |x|^\alpha + |y| - 1 = 0$$

of K_0^α in the four points

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} -u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}. \tag{6.33b}$$

We have

$$\nabla f \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \alpha u_1^{\alpha-1} \\ 1 \end{pmatrix} \tag{6.34a}$$

$$\nabla f \begin{pmatrix} -u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\alpha(-(-u_1))^{\alpha-1} \\ 1 \end{pmatrix} = \begin{pmatrix} -\alpha u_1^{\alpha-1} \\ 1 \end{pmatrix} \tag{6.34b}$$

$$\nabla f \begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix} = \begin{pmatrix} -\alpha(-(-u_1))^{\alpha-1} \\ -1 \end{pmatrix} = \begin{pmatrix} -\alpha u_1^{\alpha-1} \\ -1 \end{pmatrix} \tag{6.34c}$$

$$\nabla f \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix} = \begin{pmatrix} \alpha u_1^{\alpha-1} \\ -1 \end{pmatrix}, \tag{6.34d}$$

where

$$\nabla f \begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix} = -\nabla f \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{6.35a}$$

$$\nabla f \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix} = -\nabla f \begin{pmatrix} -u_1 \\ u_2 \end{pmatrix}. \tag{6.35b}$$

Furthermore, in the two special points

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

of ∂K_0^α we have, cf. (6.34a), (6.34d), resp., the gradients

$$\nabla f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.36a}$$

$$\nabla f \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{6.36b}$$

Though $f(x, y) = |x|^\alpha + |y| - 1$ is not differentiable at

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

we define

$$\nabla f \begin{pmatrix} 1 \\ 0 \end{pmatrix} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{6.36c}$$

$$\nabla f \begin{pmatrix} -1 \\ 0 \end{pmatrix} := \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{6.36d}$$

Using a boundary point $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ of K_0^α with $u_1, u_2 > 0$, the feasible domain K_0^α can be approximated from outside by the convex polyhedron defined as follows:

From the gradients (6.36a–d) we obtain next to the already known conditions (no M–N-interaction):

$$\nabla f \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} x \\ y - 1 \end{pmatrix} \leq 0$$

$$\nabla f \begin{pmatrix} 0 \\ -1 \end{pmatrix}^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}^T \begin{pmatrix} x \\ y + 1 \end{pmatrix} \leq 0$$

$$\nabla f \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} x - 1 \\ y \end{pmatrix} \leq 0$$

$$\nabla f \begin{pmatrix} -1 \\ 0 \end{pmatrix}^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}^T \begin{pmatrix} x + 1 \\ y \end{pmatrix} \leq 0.$$

This yields

$$y - 1 \leq 0, -1(y + 1) \leq 0 \ \text{ and } \ x - 1 \leq 0, -1(x + 1) \leq 0,$$

hence,

$$|x| \leq 1 \tag{6.37a}$$

$$|y| \leq 1. \tag{6.37b}$$

Moreover, with the gradients (6.34a–d), cf. (6.35a, b), we get the additional conditions

$$\nabla f \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right)$$
$$= \begin{pmatrix} \alpha u_1^{\alpha-1} \\ 1 \end{pmatrix}^T \begin{pmatrix} x - u_1 \\ y - u_2 \end{pmatrix} \le 0 \qquad \text{(1st quadrant)} \tag{6.38a}$$
$$\nabla f \begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix}^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix} \right)$$
$$= -\begin{pmatrix} \alpha u_1^{\alpha-1} \\ 1 \end{pmatrix}^T \begin{pmatrix} x + u_1 \\ y + u_2 \end{pmatrix} \le 0 \qquad \text{(3rd quadrant)} \tag{6.38b}$$
$$\nabla f \begin{pmatrix} -u_1 \\ u_2 \end{pmatrix}^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -u_1 \\ u_2 \end{pmatrix} \right)$$
$$= \begin{pmatrix} -\alpha u_1^{\alpha-1} \\ 1 \end{pmatrix}^T \begin{pmatrix} x + u_1 \\ y - u_2 \end{pmatrix} \le 0 \qquad \text{(2nd quadrant)} \tag{6.38c}$$
$$\nabla f \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix} \right)$$
$$= -\begin{pmatrix} -\alpha u_1^{\alpha-1} \\ 1 \end{pmatrix}^T \begin{pmatrix} x - u_1 \\ y + u_2 \end{pmatrix} \le 0 \qquad \text{(4th quadrant)} \tag{6.38d}$$

This means

$$\alpha u_1^{\alpha-1} x - \alpha u_1^{\alpha} + y - u_2 = \alpha u_1^{\alpha-1} x + y - \left(\alpha u_1^{\alpha} + u_2\right) \le 0 \tag{6.39a}$$
$$-\left(\alpha u_1^{\alpha-1} x + \alpha u_1^{\alpha} + y + u_2\right) = -\left(\alpha u_1^{\alpha-1} x + y + \left(\alpha u_1^{\alpha} + u_2\right) -\right) \le 0 \tag{6.39b}$$
$$-\alpha u_1^{\alpha-1} x - \alpha u_1^{\alpha} + y - u_2 = -\alpha - u_1^{\alpha-1} x + y - \left(\alpha u_1^{\alpha} + u_2\right) \le 0 \tag{6.39c}$$
$$-\alpha u_1^{\alpha-1} x - \alpha u_1^{\alpha} - y - u_2 = \alpha - u_1^{\alpha-1} x - y - \left(\alpha u_1^{\alpha} + u_2\right) \le 0. \tag{6.39d}$$

With

$$\alpha u_1^{\alpha} + u_2 = \nabla f \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^T \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} =: \beta(u_1, u_2) \tag{6.40}$$

we get the equivalent constraints

$$\alpha u_1^{\alpha-1} x + y - \beta(u_1, u_2) \le 0$$
$$\alpha u_1^{\alpha-1} x + y + \beta(u_1, u_2) \ge 0$$

$$-\alpha u_1^{\alpha-1} x + y - \beta(u_1, u_2) \leq 0$$
$$\alpha u_1^{\alpha-1} x - y - \beta(u_1, u_2) \leq 0.$$

This yields the double inequalities

$$|\alpha u_1^{\alpha-1} x + y| \leq \beta(u_1, u_2) \tag{6.41a}$$
$$|\alpha u_1^{\alpha-1} x - y| \leq \beta(u_1, u_2). \tag{6.41b}$$

Thus, a point $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \partial K_0^\alpha,\ u_1 > 0, u_2 > 0$, generates therefore the inequalities

$$-1 \leq x \leq 1 \tag{6.42a}$$
$$-1 \leq y \leq 1 \tag{6.42b}$$
$$-\beta(u_1, u_2) \leq \alpha u_1^{\alpha-1} x + y \leq \beta(u_1, u_2) \tag{6.42c}$$
$$-\beta(u_1, u_2) \leq \alpha u_1^{\alpha-1} x - y \leq \beta(u_1, u_2). \tag{6.42d}$$

Obviously, each further point $\hat{u} \in \partial K_0^\alpha$ with $\hat{u}_1 > 0, \hat{u}_2 > 0$ yields additional inequalities of the type (6.42c, d).

Condition (6.42a–d) can be represented in the following vectorial form:

$$-\begin{pmatrix} 1 \\ 1 \end{pmatrix} \leq I \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{6.43a}$$

$$-\beta(u_1, u_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leq H(u_1, u_2) \begin{pmatrix} x \\ y \end{pmatrix} \leq \beta(u_1, u_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{6.43b}$$

with the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ H(u_1, u_2) = \begin{pmatrix} \alpha u_1^{\alpha-1} & 1 \\ \alpha u_1^{\alpha-1} & -1 \end{pmatrix}. \tag{6.44}$$

Choosing a further boundary point $\hat{u}$ of K_0^α with $\hat{u}_1 > 0, \hat{u}_2 > 0$, we get additional conditions of the type (6.43b).

Using (6.42a–d), for the original yield condition (6.32a, b) we get then the approximative feasibility condition:

i) Negative end of the bar

$$-N_{ipl} \leq t_i \leq N_{ipl} \tag{6.45a}$$
$$-M_{ipl} \leq m_i^- \leq M_{ipl} \tag{6.45b}$$

$$-\beta(u_1,u_2) \le \alpha u_1^{\alpha-1}\frac{t_i}{N_{ipl}} + \frac{m_i^-}{M_{ipl}} \le \beta(u_1,u_2) \tag{6.45c}$$

$$-\beta(u_1,u_2) \le \alpha u_1^{\alpha-1}\frac{t_i}{N_{ipl}} - \frac{m_i^-}{M_{ipl}} \le \beta(u_1,u_2). \tag{6.45d}$$

ii) Positive end of the bar

$$-N_{ipl} \le t_i \le N_{ipl} \tag{6.45e}$$

$$-M_{ipl} \le m_i^+ \le M_{ipl} \tag{6.45f}$$

$$-\beta(u_1,u_2) \le \alpha u_1^{\alpha-1}\frac{t_i}{N_{ipl}} + \frac{m_i^+}{M_{ipl}} \le \beta(u_1,u_2) \tag{6.45g}$$

$$-\beta(u_1,u_2) \le \alpha u_1^{\alpha-1}\frac{t_i}{N_{ipl}} - \frac{m_i^+}{M_{ipl}} \le \beta(u_1,u_2). \tag{6.45h}$$

Defining

$$\Gamma^{(i)}\Big(a(\omega),x\Big) := \begin{pmatrix} \frac{1}{N_{ipl}} & 0 & 0 \\ 0 & \frac{1}{M_{ipl}} & 0 \\ 0 & 0 & \frac{1}{M_{ipl}} \end{pmatrix}, \; F_i = \begin{pmatrix} t_i \\ m_i^+ \\ m_i^- \end{pmatrix}, \tag{6.46}$$

conditions (6.45a–h) can be represented also by

$$-\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \le \Gamma^{(i)}\Big(a(\omega),x\Big)F_i \le \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{6.47a}$$

$$-\beta(u_1,u_2)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \le \begin{pmatrix} \alpha u_1^{\alpha-1} & 1 & 0 \\ \alpha u_1^{\alpha-1} & -1 & 0 \\ \alpha u_1^{\alpha-1} & 0 & 1 \\ \alpha u_1^{\alpha-1} & 0 & -1 \end{pmatrix} \Gamma_i(a(\omega),x)F_i \le \beta(u_1,u_2)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{6.47b}$$

Multiplying (6.45a, c, d, g, h) with N_{ipl}, due to

$$\frac{N_{ipl}}{M_{ipl}} = \frac{\sigma_{yi}A_i}{\sigma_{yi}W_{ipl}} = \frac{\sigma_{yi}A_i}{\sigma_{yi}A_i\,\bar{y}_{ic}} = \frac{1}{\bar{y}_{ic}}, \tag{6.48}$$

for (6.45a, c, d, g, h) we also have

$$-\beta(u_1,u_2)N_{ipl} \le \alpha u_1^{\alpha-1}t_i + \frac{m_i^-}{\bar{y}_{ic}} \le \beta(u_1,u_2)N_{ipl} \tag{6.49a}$$

$$-\beta(u_1,u_2)N_{ipl} \le \alpha u_1^{\alpha-1}t_i - \frac{m_i^-}{\bar{y}_{ic}} \le \beta(u_1,u_2)N_{ipl} \tag{6.49b}$$

$$-\beta(u_1,u_2)N_{ipl} \le \alpha u_1^{\alpha-1}t_i + \frac{m_i^+}{\bar{y}_{ic}} \le \beta(u_1,u_2)N_{ipl} \tag{6.49c}$$

$$-\beta(u_1,u_2)N_{ipl} \le \alpha u_1^{\alpha-1}t_i - \frac{m_i^+}{\bar{y}_{ic}} \le \beta(u_1,u_2)N_{ipl}. \tag{6.49d}$$

6.2.4 Approximation of the Yield Condition by Using Reference Capacities

According to (6.31), (6.32a, b) for each bar $i = 1, \dots, B$ we have the condition

$$\begin{pmatrix} \frac{t}{N_{pl}} \\ \frac{m}{M_{pl}} \end{pmatrix} \in K_0^\alpha = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : |x|^\alpha + |y| \le 1 \right\}$$

with $(t,m) = (t_i, m_i^\pm),\ (N_{pl}, M_{pl}) = (N_{ipl}, M_{ipl})$.

Selecting fixed reference capacities

$$N_{i0} > 0, M_{i0} > 0, i = 1, \dots, B,$$

related to the plastic capacities N_{ipl}, M_{ipl}, we get

$$\left|\frac{t_i}{N_{ipl}}\right|^\alpha + \left|\frac{m_i^\pm}{M_{ipl}}\right| = \left|\frac{t_i}{N_{i0}} \cdot \frac{1}{\frac{N_{ipl}}{N_{i0}}}\right|^\alpha + \left|\frac{m_i^\pm}{M_{i0}} \cdot \frac{1}{\frac{M_{ipl}}{M_{i0}}}\right|.$$

Putting

$$\rho_i = \rho_i(a(\omega), x) := \min\left\{\frac{N_{ipl}}{N_{i0}}, \frac{M_{ipl}}{M_{i0}}\right\}, \tag{6.50}$$

we have

$$\frac{\rho_i}{\frac{N_{ipl}}{N_{i0}}} \le 1, \frac{\rho_i}{\frac{M_{ipl}}{M_{i0}}} \le 1$$

and therefore

$$\left|\frac{t_i}{N_{ipl}}\right|^\alpha + \left|\frac{m_i^\pm}{M_{ipl}}\right| = \left|\frac{t_i}{\rho_i N_{i0}}\right|^\alpha \cdot \left|\frac{\rho_i}{\frac{N_{ipl}}{N_{i0}}}\right|^\alpha + \left|\frac{m_i^\pm}{\rho_i M_{i0}}\right| \cdot \left|\frac{\rho_i}{\frac{M_{ipl}}{M_{i0}}}\right| \le \left|\frac{t_i}{\rho_i N_{i0}}\right|^\alpha + \left|\frac{m_i^\pm}{\rho_i M_{i0}}\right|. \tag{6.51}$$

Thus, the yield condition (6.30a, b) or (6.32a, b) is guaranteed by

$$\left|\frac{t_i}{\rho_i N_{i0}}\right|^{\alpha} + \left|\frac{m_i^{\pm}}{\rho_i M_{i0}}\right| \le 1$$

or

$$\begin{pmatrix} \frac{t_i}{\rho_i N_{i0}} \\ \frac{m_i^{\pm}}{\rho_i M_{i0}} \end{pmatrix} \in K_0^{\alpha}. \tag{6.52}$$

Applying the piecewise linearization described in Sect. 2.1 to condition (6.52), we obtain, cf. (6.45a–h), the approximation stated below. Of course, conditions (6.45a, b, e, f) are not influenced by this procedure. Thus, we find

$$-N_{ipl} \le t_i \le N_{ipl} \tag{6.53a}$$

$$-M_{ipl} \le m_i^- \le M_{ipl} \tag{6.53b}$$

$$-M_{ipl} \le m_i^+ \le M_{ipl} \tag{6.53c}$$

$$-\beta(u_1,u_2) \le \alpha u_1^{\alpha-1}\frac{t_i}{\rho_i N_{i0}} + \frac{m_i^-}{\rho_i M_{i0}} \le \beta(u_1,u_2) \tag{6.53d}$$

$$-\beta(u_1,u_2) \le \alpha u_1^{\alpha-1}\frac{t_i}{\rho_i N_{i0}} - \frac{m_i^-}{\rho_i M_{i0}} \le \beta(u_1,u_2) \tag{6.53e}$$

$$-\beta(u_1,u_2) \le \alpha u_1^{\alpha-1}\frac{t_i}{\rho_i N_{i0}} + \frac{m_i^+}{\rho_i M_{i0}} \le \beta(u_1,u_2) \tag{6.53f}$$

$$-\beta(u_1,u_2) \le \alpha u_1^{\alpha-1}\frac{t_i}{\rho_i N_{i0}} - \frac{m_i^+}{\rho_i M_{i0}} \le \beta(u_1,u_2). \tag{6.53g}$$

Remark 6.4 Multiplying with ρ_i we get quotients $\frac{t_i}{N_{i0}}, \frac{m_i^{\pm}}{M_{i0}}$ with fixed denominators. Hence, multiplying (6.53d–g) with ρ_i, we get the equivalent system

$$-N_{ipl} \le t_i \le N_{ipl} \tag{6.54a}$$

$$-M_{ipl} \le m_i^- \le M_{ipl} \tag{6.54b}$$

$$-M_{ipl} \le m_i^+ \le M_{ipl} \tag{6.54c}$$

$$-\beta(u_1,u_2)\rho_i \le \alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} + \frac{m_i^-}{M_{i0}} \le \beta(u_1,u_2)\rho_i \tag{6.54d}$$

$$-\beta(u_1,u_2)\rho_i \le \alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} - \frac{m_i^-}{M_{i0}} \le \beta(u_1,u_2)\rho_i \tag{6.54e}$$

$$-\beta(u_1,u_2)\rho_i \le \alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} + \frac{m_i^+}{M_{i0}} \le \beta(u_1,u_2)\rho_i \tag{6.54f}$$

$$-\beta(u_1,u_2)\rho_i \le \alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} - \frac{m_i^+}{M_{i0}} \le \beta(u_1,u_2)\rho_i\,. \tag{6.54g}$$

Obviously, (6.54a–g) can be represented also in the following form:

$$|t_i| \le N_{ipl} \tag{6.55a}$$

$$|m_i^-| \le M_{ipl} \tag{6.55b}$$

$$\left|m_i^+\right| \le M_{ipl} \tag{6.55c}$$

$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} + \frac{m_i^-}{M_{i0}}\right| \le \beta(u_1,u_2)\rho_i \tag{6.55d}$$

$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} - \frac{m_i^-}{M_{i0}}\right| \le \beta(u_1,u_2)\rho_i \tag{6.55e}$$

$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} + \frac{m_i^+}{M_{i0}}\right| \le \beta(u_1,u_2)\rho_i \tag{6.55f}$$

$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} - \frac{m_i^+}{M_{i0}}\right| \le \beta(u_1,u_2)\rho_i\,. \tag{6.55g}$$

By means of definition (6.50) of ρ_i, system (6.55a–g) reads

$$|t_i| \le N_{ipl} \tag{6.56a}$$

$$|m_i^-| \le M_{ipl} \tag{6.56b}$$

$$\left|m_i^+\right| \le M_{ipl} \tag{6.56c}$$

$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} + \frac{m_i^-}{M_{i0}}\right| \le \beta(u_1,u_2)\frac{N_{ipl}}{N_{i0}} \tag{6.56d}$$

$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} + \frac{m_i^-}{M_{i0}}\right| \le \beta(u_1,u_2)\frac{M_{ipl}}{M_{i0}} \tag{6.56e}$$

$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} - \frac{m_i^-}{M_{i0}}\right| \le \beta(u_1,u_2)\frac{N_{ipl}}{N_{i0}} \tag{6.56f}$$

$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} - \frac{m_i^-}{M_{i0}}\right| \le \beta(u_1,u_2)\frac{M_{ipl}}{M_{i0}} \tag{6.56g}$$

$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} + \frac{m_i^+}{M_{i0}}\right| \le \beta(u_1,u_2)\frac{N_{ipl}}{N_{i0}} \tag{6.56h}$$

$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} + \frac{m_i^+}{M_{i0}}\right| \le \beta(u_1,u_2)\frac{M_{ipl}}{M_{i0}} \tag{6.56i}$$

$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} - \frac{m_i^+}{M_{i0}}\right| \le \beta(u_1, u_2)\frac{N_{ipl}}{N_{i0}} \tag{6.56j}$$

$$\left|\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}} - \frac{m_i^+}{M_{i0}}\right| \le \beta(u_1, u_2)\frac{M_{ipl}}{M_{i0}} \tag{6.56k}$$

Corresponding to Remark 6.4, the variables

$$t_i, m_i^+, m_i^-, A_i \text{ or } x$$

enters linearly. Increasing the accuracy of approximation by taking a further point $\hat{u} = (\hat{u}_1, \hat{u}_2)$ with the related points $(\hat{u}_1, -\hat{u}_2), (-\hat{u}_1, \hat{u}_2), (-\hat{u}_1, -\hat{u}_2)$, we obtain further inequalities of the type (6.55d–g), (6.56d–g) respectively.

6.2.5 Asymmetric Yield Stresses

Symmetric yield stresses with respect to compression and tension, cf. (6.28a), do not hold for all materials. Thus, we still have to consider the case

$$\sigma_{yi}^L \ne -\sigma_{yi}^U \text{ for some } i \in \{1, \ldots, B\}. \tag{6.57}$$

In case (6.57) we have different plastic capacities $N_{ipl}^L \ne N_{ipl}^U$ with respect to compression and tension. Thus must be taken into account in the yield condition (6.30a, b), hence,

$$\left|\frac{t_i}{N_{ipl}}\right|^\alpha + \left|\frac{m_i^\pm}{M_{ipl}}\right| \le 1.$$

Corresponding to (6.20), (6.23a, b) for $\alpha = 2$, the feasible domain K_0^α is represented by the intersection of convex sets $K_0^{\alpha L}, K_0^{\alpha U}$ defined in the same way as K_0^{2L}, K_0^{2U} in (6.23a, b).

Corresponding to Sect. 6.2.4 we select then a vector

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \partial K_0^\alpha \text{ with } u_1 > 0, u_2 > 0, \tag{6.58a}$$

hence,

$$u_1^\alpha + u_2 = 1,\ u_1 > 0, u_2 > 0, \tag{6.58b}$$

see (6.33a). More accurate approximations follow by choosing additional points $\hat{u} = (\hat{u}_1, \hat{u}_2)$.

Set $K_0^{\alpha U}$ is then approximated, cf. (6.39a–d), by

$$x \le 1 \tag{6.59a}$$

$$-1 \le y \le 1 \tag{6.59b}$$

$$\alpha u_1^{\alpha-1} x + y \le \beta(u_1, u_2) \tag{6.59c}$$

$$\alpha u_1^{\alpha-1} x - y \le \beta(u_1, u_2) \tag{6.59d}$$

where, cf. (6.40),

$$\beta(u_1, u_2) = \alpha u_1^{\alpha} + u_2. \tag{6.59e}$$

Consider now the points of $K_0^{\alpha U}$ such that

$$\begin{pmatrix} x \\ y \end{pmatrix}, x \le 0, -1 \le y \le 1. \tag{6.60}$$

Because of

$$\beta(u_1, u_2) = \alpha u_1^{\alpha} + u_2 > u_1^{\alpha} + u_2 = 1 \tag{6.61}$$

and $\alpha > 1$, for the points of $K_0^{\alpha U}$ with (6.60) we have

$$\alpha u_1^{\alpha-1} x + y \le y \le 1 < \beta(u_1, u_2) \tag{6.62a}$$

$$\alpha u_1^{\alpha-1} x - y \le -y \le 1 < \beta(u_1, u_2). \tag{6.62b}$$

Obviously, the piecewise linearization (6.59a–d) of $K_0^{\alpha U}$ effects only the case $x > 0$.

Corresponding to (6.59a–e) the set $K_0^{\alpha L}$ is piecewise linearized by means of the conditions

$$-1 \le x \tag{6.63a}$$

$$-1 \le y \le 1 \tag{6.63b}$$

$$-\alpha u_1^{\alpha-1} x + y \le \beta(u_1, u_2) \tag{6.63c}$$

$$-\alpha u_1^{\alpha-1} x - y \le \beta(u_1, u_2). \tag{6.63d}$$

Considering here the points

$$\begin{pmatrix} x \\ y \end{pmatrix}, x \ge 0, -1 \le y \le 1, \tag{6.64}$$

of $K_0^{\alpha L}$, then corresponding to (6.60) we have

$$-\alpha u_1^{\alpha-1} x + y \leq y \leq 1 < \beta(u_1, u_2) \tag{6.65a}$$

$$-\alpha u_1^{\alpha-1} x - y \leq -y \leq 1 < \beta(u_1, u_2). \tag{6.65b}$$

Hence, the piecewise linearization (6.63a–d) of $K_0^{\alpha L}$ effects the case $x < 0$ only.

Formulation of the Yield Condition in the Non Symmetric Case

$$\sigma_{yi}^L \neq -\sigma_{yi}^U .$$

Since the plastic capacities $N_{ipl}^U = \sigma_{yi}^U A_i$ and $N_{ipl}^L = |\sigma_{yi}^L| A_i$ with respect to tension and compression are different, we have to consider the cases $t_i > 0$ (tension) and $t_i < 0$ (compression) differently.

Because of $K_0^\alpha = K_0^{\alpha L} \cap K_0^{\alpha U}$, we obtain the following conditions: From (6.59a–d), hence, for $K_0^{\alpha U}$ we get

$$\frac{t_i}{N_{ipl}^U} \leq 1 \tag{6.66a}$$

$$\left| \frac{m_i^\pm}{M_{ipl}} \right| \leq 1 \tag{6.66b}$$

$$\alpha u_1^{\alpha-1} \frac{t_i}{N_{ipl}^U} + \frac{m_i^\pm}{M_{ipl}} \leq \beta(u_1, u_2) \tag{6.66c}$$

$$\alpha u_1^{\alpha-1} \frac{t_i}{N_{ipl}^U} - \frac{m_i^\pm}{M_{ipl}} \leq \beta(u_1, u_2). \tag{6.66d}$$

Moreover, from (6.63a–d), hence, for $K_0^{\alpha L}$ we get

$$-1 \leq \tfrac{t_i}{N_{ipl}^L} \tag{6.67a}$$

$$\left| \tfrac{m_i^\pm}{M_{ipl}} \right| \leq 1 \tag{6.67b}$$

$$-\alpha u_1^{\alpha-1} \tfrac{t_i}{N_{ipl}^L} + \tfrac{m_i^\pm}{M_{ipl}} \leq \beta(u_1, u_2) \tag{6.67c}$$

$$-\alpha u_1^{\alpha-1} \tfrac{t_i}{N_{ipl}^L} - \tfrac{m_i^\pm}{M_{ipl}} \leq \beta(u_1, u_2). \tag{6.67d}$$

Summarizing the above conditions, we have

$$-N_{ipl}^{L} \le t_i \le N_{ipl}^{U} \tag{6.68a}$$

$$-M_{ipl} \le m_i^{-} \le M_{ipl} \tag{6.68b}$$

$$-M_{ipl} \le m_i^{+} \le M_{ipl} \tag{6.68c}$$

$$\alpha u_1^{\alpha-1}\frac{t_i}{N_{ipl}^{U}} + \frac{m_i^{+}}{M_{ipl}} \le \beta(u_1, u_2) \tag{6.68d}$$

$$\alpha u_1^{\alpha-1}\frac{t_i}{N_{ipl}^{U}} - \frac{m_i^{+}}{M_{ipl}} \le \beta(u_1, u_2) \tag{6.68e}$$

$$-\alpha u_1^{\alpha-1}\frac{t_i}{N_{ipl}^{L}} + \frac{m_i^{+}}{M_{ipl}} \le \beta(u_1, u_2) \tag{6.68f}$$

$$-\alpha u_1^{\alpha-1}\frac{t_i}{N_{ipl}^{L}} - \frac{m_i^{+}}{M_{ipl}} \le \beta(u_1, u_2) \tag{6.68g}$$

$$\alpha u_1^{\alpha-1}\frac{t_i}{N_{ipl}^{U}} + \frac{m_i^{-}}{M_{ipl}} \le \beta(u_1, u_2) \tag{6.68h}$$

$$\alpha u_1^{\alpha-1}\frac{t_i}{N_{ipl}^{U}} - \frac{m_i^{-}}{M_{ipl}} \le \beta(u_1, u_2) \tag{6.68i}$$

$$-\alpha u_1^{\alpha-1}\frac{t_i}{N_{ipl}^{L}} + \frac{m_i^{-}}{M_{ipl}} \le \beta(u_1, u_2) \tag{6.68j}$$

$$-\alpha u_1^{\alpha-1}\frac{t_i}{N_{ipl}^{L}} - \frac{m_i^{-}}{M_{ipl}} \le \beta(u_1, u_2). \tag{6.68k}$$

Corresponding to (6.49a–d), the inequalities (6.68d–k) may be multiplied by N_{ipl}^{U}, N_{ipl}^{L}, resp., in order to get denominators independent of A_i, x, respectively.

Two-sided constraints are obtained by introducing lower bounds in (6.68d–k)

$$-\mu\beta(u_1, u_2) \tag{6.69}$$

with a sufficiently large $\mu > 0$.

Use of Reference Capacities

Corresponding to Sect. 6.2.4 we introduce here reference capacities

$$N_{i0}^{L}, N_{i0}^{U}, M_{i0} > 0, \; i = 1, \ldots, B,$$

for the plastic capacities $N_{ipl}^L, N_{ipl}^U, M_{ipl}$. Consider then the relative capacities ρ_i^L, ρ_i^U, cf. (6.50),

$$\rho_i^L := \min\left\{\frac{N_{ipl}^L}{N_{i0}^L}, \frac{M_{ipl}}{M_{i0}}\right\}, \tag{6.70a}$$

$$\rho_i^U := \min\left\{\frac{N_{ipl}^U}{N_{i0}^U}, \frac{M_{ipl}}{M_{i0}}\right\}. \tag{6.70b}$$

Based again on (6.52) and the representation $K_0^\alpha = K_0^{\alpha L} \cap K_0^{\alpha U}$, corresponding to (6.53d–g) we get

$$\alpha u_1^{\alpha-1}\frac{t_i}{\rho_i^U N_{i0}^U} + \frac{m_i^{\pm}}{\rho_i^U M_{i0}} \le \beta(u_1,u_2) \tag{6.71a}$$

$$\alpha u_1^{\alpha-1}\frac{t_i}{\rho_i^U N_{i0}^U} - \frac{m_i^{\pm}}{\rho_i^U M_{i0}} \le \beta(u_1,u_2) \tag{6.71b}$$

$$-\alpha u_1^{\alpha-1}\frac{t_i}{\rho_i^L N_{i0}^L} + \frac{m_i^{\pm}}{\rho_i^L M_{i0}} \le \beta(u_1,u_2) \tag{6.71c}$$

$$-\alpha u_1^{\alpha-1}\frac{t_i}{\rho_i^L N_{i0}^L} - \frac{m_i^{\pm}}{\rho_i^L M_{i0}} \le \beta(u_1,u_2). \tag{6.71d}$$

Multiplying (6.71a–d) with ρ_i^U, ρ_i^L, resp., corresponding to (6.54a–d) we find the conditions

$$-N_{ipl}^L \le t_i \le N_{ipl}^U \tag{6.72a}$$

$$-M_{ipl} \le m_i^- \le M_{ipl} \tag{6.72b}$$

$$-M_{ipl} \le m_i^+ \le M_{ipl} \tag{6.72c}$$

$$\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}^U} + \frac{m_i^+}{M_{i0}} \le \rho_i^U\beta(u_1,u_2) \tag{6.72d}$$

$$\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}^U} - \frac{m_i^+}{M_{i0}} \le \rho_i^U\beta(u_1,u_2) \tag{6.72e}$$

$$-\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}^L} + \frac{m_i^+}{M_{i0}} \le \rho_i^L\beta(u_1,u_2) \tag{6.72f}$$

$$-\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}^L} - \frac{m_i^+}{M_{i0}} \le \rho_i^L\beta(u_1,u_2) \tag{6.72g}$$

$$\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}^U} + \frac{m_i^-}{M_{i0}} \le \rho_i^U\beta(u_1,u_2) \tag{6.72h}$$

$$\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}^U}-\frac{m_i^-}{M_{i0}}\le \rho_i^U\beta(u_1,u_2) \tag{6.72i}$$

$$-\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}^L}+\frac{m_i^-}{M_{i0}}\le \rho_i^L\beta(u_1,u_2) \tag{6.72j}$$

$$-\alpha u_1^{\alpha-1}\frac{t_i}{N_{i0}^L}-\frac{m_i^-}{M_{i0}}\le \rho_i^L\beta(u_1,u_2). \tag{6.72k}$$

According to the definition (6.70a, b) of ρ_i^L, ρ_i^U each of the above conditions can be portioned into two equivalent inequalities with the right hand sides

$$\frac{N_{ipl}^U}{N_{i0}^U}\beta(u_1,u_2),\ \frac{M_{ipl}}{M_{i0}}\beta(u_1,u_2) \tag{6.73a}$$

$$\frac{N_{ipl}^L}{N_{i0}^L}\beta(u_1,u_2),\ \frac{M_{ipl}}{M_{i0}}\beta(u_1,u_2). \tag{6.73b}$$

Stochastic Optimization

Due to (6.1c), (6.10), (6.11c), (6.12c), the $3B$-vector

$$F=(F_1^T,\dots,F_B^T)^T \tag{6.74a}$$

of all interior loads fulfills the equilibrium condition

$$CF=R \tag{6.74b}$$

with the external load vector R and the equilibrium matrix C.

In the following we collect all random model parameters [100], such as external load factors, material strength parameters, cost factors, etc., into the random ν-vector

$$a=a(\omega),\ \omega\in(\Omega,\mathcal{A},\mathcal{P}). \tag{6.75a}$$

Thus, since in some cases the vector R of external loads depend also on the design r-vector x, we get

$$R=R\left(a(\omega),x\right). \tag{6.75b}$$

Of course, the plastic capacities depend also on the vectors x and $a(\omega)$, hence,

$$N_{ipl}^{\Gamma}=N_{ipl}^{\Gamma}\left(a(\omega),x\right),\ \Gamma=L,U \tag{6.75c}$$

$$M_{ipl}=M_{ipl}\left(a(\omega),x\right). \tag{6.75d}$$

We assume that the probability distribution and/or the needed moments of the random parameter vector $a = a(\omega)$ are known [7, 100].

The remaining deterministic constraints for the design vector x are represented by

$$x \in \mathcal{D} \tag{6.76}$$

with a certain convex subset D of $\mathbb{R}^r$.

6.2.6 Violation of the Yield Condition

Consider in the following an interior load distribution F fulfilling the equilibrium condition (6.75b).

According to the analysis given in Sect. 6.2.2, after piecewise linearization, the yield condition for the ith bar can be represented by an inequality of the type

$$H\Gamma^{(i)}\Big(a(\omega), x\Big) \leq h^{(i)}\Big(a(\omega), x\Big), i = 1, \ldots, B, \tag{6.77}$$

with matrices $H, \Gamma^{(i)} = \Gamma^{(i)}\Big(a(\omega), x\Big)$ and a vector $h^{(i)} = h^{(i)}\Big(a(\omega), x\Big)$ described in Sect. 6.2

In order to take into account violations of condition (6.77), we consider the equalities

$$H\Gamma^{(i)}F_i + z_i = h^{(i)}, i = 1, \ldots, B. \tag{6.78}$$

If

$$z_i \geq 0, \text{ for all } i = 1, \ldots, B, \tag{6.79a}$$

then (6.77) holds, and the yield condition is then fulfilled too, or holds with a prescribed accuracy.

However, in case

$$z_i \not\geq 0 \text{ for some bars } i \in \{1, \ldots, B\}, \tag{6.79b}$$

the survival condition is violated at some points of the structure. Hence, structural failures may occur. The resulting costs Q of failure, damage and reconstructure of the frame is a function of the vectors $z_i, i = 1, \ldots, B$, defined by (6.78). Thus, we have

$$Q = Q(z) = Q(z_1, \ldots, z_B), \tag{6.80a}$$

where

$$z := (z_1^T, z_2^T, \dots, z_B^T)^T, \tag{6.80b}$$

$$z_i := h^{(i)} - H\Gamma^{(i)} F_i,\ i = 1, \dots, B. \tag{6.80c}$$

6.2.7 Cost Function

Due to the survival condition (6.79a), we may consider cost functions Q such that

$$Q(z) = 0,\ \text{if } z \geq 0, \tag{6.81}$$

hence, no (recourse) costs arise if the yield condition (6.77) holds.

In many cases the recourse or failure costs of the structure are defined by the sum

$$Q(z) = \sum_{i=1}^{B} Q_i(z_i), \tag{6.82}$$

of the element failure costs $Q_i = Q_i(z_i), i = 1, \dots, B$.

Using the representation

$$z_i = y_i^+ - y_i^-, \quad y_i^+, y_i^- \geq 0, \tag{6.83a}$$

the member cost functions $Q_i = Q_i(z_i)$ are often defined [70, 100] by considering the linear function

$$q_i^{-T} y_i^- + q_i^{+T} y_i^+ \tag{6.83b}$$

with certain vectors q_i^+, q_i^- of cost coefficients for the evaluation of the condition $z_i \geq 0, z_i \ngeq 0$, respectively.

The cost function $Q_i = Q_i(z_i)$ is then defined by the minimization problem

$$\min\ q_i^{-T} y_i^- + q_i^{+T} y_i^+ \tag{6.84a}$$

$$\text{s.t. } y_i^+ - y_i^- = z_i \tag{6.84b}$$

$$y_i^-, y_i^+ \geq 0. \tag{6.84c}$$

If $z_i := (z_{i1}, \dots, z_{i\mu})^T, y_i^\pm := (y_{i1}^\pm, \dots, y_{i\mu}^\pm)^T, i = 1, \dots, B$ then (6.84a–c) can also be represented by

$$\min \sum_{l=1}^{\mu} (q_{il}^- y_{il}^- + q_{il}^+ y_{il}^+) \tag{6.85a}$$

$$\text{s.t. } y_{il}^{+} - y_{il}^{-} = z_{il}, \quad l = 1, \dots, \mu \tag{6.85b}$$

$$y_{il}^{-}, y_{il}^{+} \geq 0, \quad l = 1, \dots, \mu. \tag{6.85c}$$

Obviously, (6.85a–c) can be decomposed into μ separated minimization problems

$$\min q_{il}^{-} y_{il}^{-} + q_{il}^{+} y_{il}^{+} \tag{6.86a}$$

$$\text{s.t. } y_{il}^{+} - y_{il}^{-} = z_{il} \tag{6.86b}$$

$$y_{il}^{-}, y_{il}^{+} \geq 0, \tag{6.86c}$$

for the pairs of variables (y_{il}^{-}, y_{il}^{+}), $l = 1, \dots, \mu$.

Under the condition

$$q_{il}^{-} + q_{il}^{+} \geq 0, l = 1, \dots, \mu, \tag{6.87}$$

the following result holds:

Lemma 6.1 *Suppose that (6.87) holds. Then the minimum value function $Q_{il} = Q_{il}(z_{il})$ of (6.86a–c) is a piecewise linear, convex function given by*

$$Q_{il}(z_{il}) := \max\{q_{il}^{+} z_{il}, -q_{il}^{-} z_{il}\}. \tag{6.88}$$

Hence, the member cost functions $Q_i = Q_i(z_i)$ reads

$$\begin{aligned} Q_i(z_i) &= \sum_{l=1}^{\mu} Q_{il}(z_{il}) \\ &= \sum_{l=1}^{\mu} \max\{q_{il}^{+} z_{il}, -q_{il}^{-} z_{il}\}, \end{aligned} \tag{6.89a}$$

and the total cost function $Q = Q(z)$ is given by

$$\begin{aligned} Q(z) &= \sum_{i=1}^{B} Q_i(z_i) \\ &= \sum_{i=1}^{B} \sum_{l=1}^{\mu} \max\{q_{il}^{+} z_{il}, -q_{il}^{-} z_{il}\}. \end{aligned} \tag{6.89b}$$

Choice of the Cost Factors

Under elastic conditions the change $\Delta\sigma$ of the total stress σ and the change ΔL of the element length L are related by

$$\Delta L = \frac{L}{E}\Delta\sigma, \tag{6.90a}$$

where E denotes the modulus of elasticity.

Assuming that the neutral fiber is equal to the axis of symmetry of the element, for the total stress $\Delta\sigma$ in the upper ("+") lower ("–") fibre of the boundary we have

$$\Delta\sigma = \frac{\Delta t}{A} \pm \frac{\Delta m}{W}, \tag{6.90b}$$

where W denotes the axial modulus of the cross-sectional area of the element (beam).

Representing the change ΔV of volume of an element by

$$\Delta V = A \cdot \Delta L, \tag{6.91a}$$

then

$$\begin{aligned}\Delta V &= A \cdot \Delta L = A \cdot \frac{L}{E}\Delta\sigma = A \cdot \frac{L}{E}\left(\frac{\Delta t}{A} \pm \frac{\Delta m}{W}\right)\\ &= \frac{L}{E}\Delta t \pm \frac{L}{E}\frac{A}{W}\Delta m = \frac{L}{E}\Delta t \pm \frac{L}{E}\frac{1}{\frac{W}{A}}\Delta m = \frac{L}{E}\Delta t \pm \frac{L}{E} \cdot \frac{1}{\bar{y}_c}\Delta m,\end{aligned} \tag{6.91b}$$

where $\bar{y}_c$ is the cross-sectional parameter as defined in (6.28d).

Consequently, due to (6.90a, b), for the evaluation of violations Δt of the axial force constraint we may use a cost factor of the type

$$\gamma_K := \frac{L}{E}, \tag{6.92a}$$

and an appropriate cost factor for the evaluation of violations of moment constraints reads

$$\gamma_M = \frac{L}{E} \cdot \frac{1}{\bar{y}_c}. \tag{6.92b}$$

Total Costs

Denoting by

$$G_0 = G_0\Big(a(\omega), x\Big) \tag{6.93a}$$

the primary costs, such as material costs, costs of construction, the total costs including failure or recourse costs are given by

$$G = G_0\Big(a(\omega), x\Big) + Q\Big(z\big(a(\omega), x, F(\omega)\big)\Big). \tag{6.93b}$$

Hence, the total costs $G = G\Big(a(\omega), x, F(\omega)\Big)$ depend on the vector $x = (x_1, \dots, x_r)^T$ of design variables, the random vector $a(\omega) = \Big(a_1(\omega), \dots, a_\nu(\omega)\Big)^T$ of model parameters and the random vector $F = F(\omega)$ of all internal loadings.

Minimizing the expected total costs, we get the following stochastic optimization problem of recourse type [90]

$$\min\ E\Big(G_0\big(a(\omega), x\big) + Q\big(z\left(a(\omega), x, F(\omega)\right)\big)\Big) \tag{6.94a}$$

$$\text{s.t. }\ H\Gamma^{(i)}\Big(a(\omega), x\Big)F_i(\omega) + z_i(\omega) = h^{(i)}\Big(a(\omega), x\Big) \quad \text{a.s.,} \quad i = 1, \dots, B \tag{6.94b}$$

$$CF(\omega) = R\Big(a(\omega), x\Big) \quad \text{a.s.} \tag{6.94c}$$

$$x \in D. \tag{6.94d}$$

Using representation (6.82), (6.84a–c) of the recourse or failure cost function $Q = Q(z)$, problem (6.94a–d) takes also the following equivalent form

$$\min\ E\left(G_0\Big(a(\omega), x\Big) + \sum_{i=1}^{B}\left(q_i^-(\omega)^T y_i^-(\omega) + q_i^+(\omega)^T y_i^+(\omega)\right)\right) \tag{6.95a}$$

$$\text{s.t. }\ H\Gamma^{(i)}\Big(a(\omega), x\Big)F_i(\omega) + y_i^+(\omega) - y_i^-(\omega) = h^{(i)}\Big(a(\omega), x\Big) \quad \text{a.s.,} \quad i = 1, \dots, B \tag{6.95b}$$

$$CF(\omega) = R\left(a(\omega), x\right) \quad \text{a.s.} \tag{6.95c}$$

$$x \in D,\ y_i^+(\omega), y_i^-(\omega) \ge 0 \quad \text{a.s.,} \quad i = 1, \dots, B. \tag{6.95d}$$

Remark 6.5 Stochastic optimization problems of the type (6.95a–d) are called "two-stage stochastic programs" or "stochastic problems with recourse".

In many cases the primary cost function $G_0 = G_0\Big(a(\omega), x\Big)$ represents the volume or weight of the structural, hence,

$$G_0\Big(a(\omega), x\Big) := \sum_{i=1}^{B} \gamma_i(\omega) V_i(x) = \sum_{i=1}^{B} \gamma_i(\omega) L_i A_i(x), \tag{6.96}$$

with certain (random) weight factors $\gamma_i = \gamma_i(\omega)$.

In case

$$A_i(x) := u_i x_i \tag{6.97}$$

with fixed sizing parameters $u_i, i = 1, \ldots, B$, we get

$$G_0\left(a(\omega), x\right) = \sum_{i=1}^{B} \gamma_i(\omega) L_i u_i(x) = \sum_{i=1}^{B} \gamma_i(\omega) L_i h_i x_i = c\left(a(\omega)\right)^T x, \tag{6.98a}$$

where

$$c\left(a(\omega)\right) := \left(\gamma_1(\omega) L_1 u_1, \ldots, \gamma_B(\omega) L_B u_B\right)^T. \tag{6.98b}$$

Thus, in case (6.97), $G_0 = G_0\Big(a(\omega), x\Big)$ is a linear function of x.

Discretization Methods

The expectation in the objective function of the stochastic optimization problem (6.95a–d) must be computed numerically. One of the main methods is based on the discretization of the probability distribution $P_{a(\cdot)}$ of the random parameter ν-vector $a = a(\omega)$, cf. [96], hence,

$$P_{a(\cdot)} \approx \mu := \sum_{k=1}^{s} \alpha_k \epsilon_{a^{(k)}} \tag{6.99a}$$

with

$$\alpha_k \geq 0, \ k = 1, \ldots, s, \ \sum_{k=1}^{s} \alpha_k = 1. \tag{6.99b}$$

Corresponding to the realizations $a^{(k)}, k = 1, \ldots, s$, of the discrete approximate (6.99a, b), we have the realizations $y_i^{-(k)}, y_i^{+(k)}$ and $F^{(k)}, F_i^{(k)}, k = 1, \ldots, s$, of the random vectors $y_i^-(\omega), y_i^+(\omega), i = 1, \ldots, B$, and $F(\omega)$. then

$$E\left(G_0\Big(a(\omega), x\Big) + \sum_{i=1}^{B}\left(q_i^{-T} y_i^-(\omega) + q_i^{+T} y_i^+(\omega)\right)\right) \tag{6.100a}$$
$$\approx \sum_{k=1}^{s} \alpha_k \left(G_0(a^{(k)}, x) + \sum_{i=1}^{B}(q_i^{-T} y_i^{-(k)} + q_i^{+T} y_i^{+(k)}\right).$$

Furthermore, the equilibrium equation (6.94c) is approximated by

$$CF^{(k)} = R(a^{(k)}, x),\ k = 1, \ldots, s, \tag{6.100b}$$

where $F^{(k)} := \left(F_1^{(k)^T}, \ldots, F_B^{(k)^T}\right)^T$, and we have, cf. (6.95d), the nonnegativity constraints

$$y_i^{+(k)},\ y_i^{-(k)} \geq 0,\ k = 1, \ldots, s,\ i = 1, \ldots, B. \tag{6.100c}$$

Thus, (SOP) (6.95a–d) is reduced to the parameter optimization problem

$$\min\ \overline{G}_0(a^{(k)}, x) + \sum_{i=1}^{B} \alpha_k \left(q_i^{-T} y_i^{-(k)} + q_i^{+T} y_i^{+(k)}\right) \tag{6.101a}$$

$$\text{s.t.}\ \ H\Gamma^{(i)}(a^{(k)}, x)F_i^{(k)} + y_i^{+(k)} - y_i^{-(k)} = h^{(i)}(a^{(k)}, x),$$
$$i = 1, \ldots, B,\ k = 1, \ldots, s \tag{6.101b}$$
$$CF^{(k)} = R(a^{(k)}, x),\ k = 1, \ldots, s \tag{6.101c}$$
$$x \in D,\ y_i^{+(k)},\ y_i^{-(k)} \geq 0,\ k = 1, \ldots, s,\ i = 1, \ldots, B. \tag{6.101d}$$

Complete Recourse

According to Sect. 6.2.6, the evaluation of the violation of the yield condition (6.77) is based on Eqs. (6.78), hence

$$H\Gamma^{(i)} F_i + z_i = h^{(i)}, i = 1, \ldots, B.$$

In generalization of the so-called "simple recourse" case (6.84a–c), in the "complete recourse" case the deviation

$$z_i = h^{(i)} - H\Gamma^{(i)} F_i$$

is evaluated by means of the minimum value $Q_i = Q_i(z_i)$ of the linear program, cf. (6.85a–c)

$$\min \; q^{(i)T} y^{(i)} \tag{6.102a}$$

$$\text{bzgl. } M^{(i)} y^{(i)} = z_i \tag{6.102b}$$

$$y^{(i)} \geq 0. \tag{6.102c}$$

Here, $q^{(i)}$ is a given cost vector and $M^{(i)}$ denotes the so-called recourse matrix [69, 70].

We assume that the linear equation (6.102b) has a solution $y^{(i)} \geq 0$ for each vector z_i. This property is called "complete recourse".

In the present case the stochastic optimization problem (6.94a–c) reads

$$\min E\left(G_0\Big(a(\omega), x\Big) + \sum_{i=1}^{B} q^{(i)T} y^{(i)}(\omega)\right) \tag{6.103a}$$

$$\text{s.t. } H\Gamma^{(i)}\Big(a(\omega), x\Big) F_i(\omega) + M^{(i)} y^{(i)}(\omega) = h^{(i)}\Big(a(\omega), x\Big) \text{ a.s.,} \quad i = 1, \dots, B \tag{6.103b}$$

$$CF(\omega) = R\,(a(\omega), x) \quad \text{a.s.} \tag{6.103c}$$

$$x \in D,\; y^{(i)}(\omega) \geq 0 \quad \text{a.s.,} \quad i = 1, \dots, B. \tag{6.103d}$$

As described in Sect. 6.2.7, problem (6.103a–d) can be solved numerically by means of discretization methods and application of linear/nonlinear programming techniques.

Numerical Implementation

A thorough consideration and numerical implementation of the linear optimization problems (6.95a–d) and (6.103a–d) can be found in [172], see also [160]. Especially, by certain linear transformations, a considerable reduction of the number of variables has been obtained. As a result, the problem has been reduced to a linear program with fixed recourse having the following *dual block angular structure*:

$$\min E(c^T x + q^T y) = c^T x + \sum_{r=1}^{R} p_r q^T y^r \tag{6.104a}$$

$$\begin{array}{lccl} Tx + Wy^1 & & & = h^1 \\ \vdots & & \ddots & \vdots \\ Tx + & & Wy^R & = h^R \end{array} \tag{6.104b}$$

$$x \in D,\; y^r \geq 0,\; r = 1, \dots, R. \tag{6.104c}$$

Here, the vectors and matrices $c, q, T, W, h^1, \ldots, h^R$ contain the data of the (discretized) stochastic structural optimization problem, and the vectors $x, y^1, \ldots, y^R$ involve the variables and slack variables of the given problem. Finally, $p^1, \ldots, p^R$ denote the probabilities of the R realizations of the (discretized) stochastic optimal design problem.

Chapter 7
Stochastic Structural Optimization with Quadratic Loss Functions

7.1 Introduction

Problems from plastic analysis and optimal plastic design are based on the convex, linear or linearized yield/strength condition and the linear equilibrium equation for the stress (state) vector. In practice one has to take into account stochastic variations of several model parameters. Hence, in order to get robust optimal decisions, the structural optimization problem with random parameters must be replaced by an appropriate deterministic substitute problem. A direct approach is proposed based on the primary costs (weight, volume, costs of construction, costs for missing carrying capacity, etc.) and the recourse costs (e.g. costs for repair, compensation for weakness within the structure, damage, failure, etc.). Based on the mechanical survival conditions of plasticity theory, a quadratic error/loss criterion is developed. The minimum recourse costs can be determined then by solving an optimization problem having a quadratic objective function and linear constraints. For each vector $a(\cdot)$ of model parameters and each design vector x, one obtains then an explicit representation of the "best" internal load distribution F^*. Moreover, also the expected recourse costs can be determined explicitly. It turns out that this function plays the role of a generalized expected *compliance function* involving a *generalized stiffness matrix*. For the solution of the resulting deterministic substitute problems, i.e., the minimization of the expected primary cost (e.g. volume, weight) subject to expected recourse cost constraints or the minimization of the expected total primary and recourse costs, explicit finite dimensional parameter optimization problems are obtained. Furthermore, based on the quadratic cost approach, lower and upper bounds for the probability of survival can be derived.

In optimal plastic design of mechanical structure [38] one has to minimize a weight, volume or more general cost function c, while in limit load analysis [74] of plastic mechanical structures the problem is to maximize the load factor μ—in both cases—subject to the survival or safety conditions, consisting of the equilibrium equation and the so-called yield (feasibility) condition of the structure.

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Thus, the objective function G_0 to be minimized is defined by

$$G_0(x) = \sum_{i=1}^{B} \gamma_{i0}(\omega) L_i A_i(x) \tag{7.1a}$$

in the case of optimal plastic design, and by

$$G_0 = G_0(a, x) := -\mu \tag{7.1b}$$

in the second case of limit load analysis.

Here, $x = (x_1, x_2, \ldots, x_r)^T, x := (x_1) = (\mu)$ is the decision vector, hence, the r-vector x of design variables $x_1, \ldots, x_r$, such as sizing variables, in the first case and the load factor $x_1 = \mu$ in the second case. For the decision vector x one has mostly simple feasibility conditions represented by $x \in D$, where $D \subset \mathbb{R}^r$ is a given closed convex set such as $D = \mathbb{R}_+$ in the second case. Moreover, $a = a(\omega)$ is the ν-vector of all random model parameters arising in the underlying mechanical model, such as weight or cost factors $\gamma_{i0} = \gamma_{i0}(\omega)$, yield stresses in compression and tension $\sigma_{yi}^L = \sigma_{yi}^L(\omega), \sigma_{yi}^U = \sigma_{yi}^U(\omega), i = 1, \ldots, B$, load factors contained in the external loading $P = P\Big(a(\omega), x\Big)$, etc. Furthermore, in the general cost function defined by (7.1a), $A_i = A_i(x), i = 1, \ldots, B$, denote the cross-sectional areas of the bars having length $L_i, i = 1, \ldots, B$.

As already mentioned above, the optimization of the function $G_0 = G_0(a, x)$ is done under the safety or survival conditions of plasticity theory which can be described [97, 158] for plane frames as follows:

I) *Equilibrium condition*

After taking into account the boundary conditions, the equilibrium between the m-vector of external loads $P = P\Big(a(\omega), x\Big)$ and the $3B$-vector of internal loads $F = (F_1^T, F_2^T, \ldots, F_B^T)^T$ can be described by

$$CF = P\Big(a(\omega), x\Big) \tag{7.2}$$

where C is the $m \times 3B$ equilibrium matrix having rank $C = m$.

II) *Yield condition (feasibility condition)*

If no interactions between normal (axial) forces t_i and bending moments m_i^l, m_i^r, resp., at the left, right end of the oriented i-th bar of the structure are taken into account, then the feasibility condition for the generalized forces of the bar

$$F_i = \begin{pmatrix} t_i \\ m_i^l \\ m_i^r \end{pmatrix}, i = 1, \ldots, B, \tag{7.3}$$

reads

$$\tilde{F}_i^L\Big(a(\omega), x\Big) \le F_i \le \tilde{F}_i^U\Big(a(\omega, x\Big), i = 1, \ldots, B, \tag{7.4a}$$

where the bounds $\tilde{F}_i^L, \tilde{F}_i^U$ containing the plastic capacities with respect to axial forces and moments are given by

$$\tilde{F}_i^L\Big(a(\omega), x\Big) := \begin{pmatrix} -N_{ipl}^L\Big(a(\omega), x\Big) \\ -M_{ipl}\Big(a(\omega), x\Big) \\ -M_{ipl}\Big(a(\omega), x\Big) \end{pmatrix} = \begin{pmatrix} \sigma_{yi}^L\Big(a(\omega)\Big) A_i(x) \\ -\sigma_{yi}^U\Big(a(\omega)\Big) W_{ipl}(x) \\ -\sigma_{yi}^U\Big(a(\omega)\Big) W_{ipl}(x) \end{pmatrix} \tag{7.4b}$$

$$\tilde{F}_i^U\Big(a(\omega), x\Big) = \begin{pmatrix} N_{ipl}^U\Big(a(\omega), x\Big) \\ M_{ipl}\Big(a(\omega), x\Big) \\ M_{ipl}\Big(a(\omega), x\Big) \end{pmatrix} = \begin{pmatrix} \sigma_{yi}^U\Big(a(\omega)\Big) A_i(x) \\ \sigma_{yi}^U\Big(a(\omega)\Big) W_{ipl}(x) \\ \sigma_{yi}^U\Big(a(\omega)\Big) W_{ipl}(x) \end{pmatrix}, \tag{7.4c}$$

Here,

$$W_{ipl} = A_i \bar{y}_{ic}. \tag{7.4d}$$

denotes the plastic section modulus with the arithmetic mean

$$\overline{y}_{ic} = \frac{1}{2}(y_{i1} + y_{i2}) \tag{7.4e}$$

of the centroids y_{i1}, y_{i2} of the two half areas of the cross-sectional areas A_i of the bars $i = 1, \ldots, B$.

Taking into account also interactions between normal forces t_i and moments m_i^l, m_i^r, besides (7.4a) we have additional feasibility conditions of the type

$$-h_l \eta_i^L\Big(a(\omega), x\Big) \le H_l^{(i)} F_i \le h_l \eta_i^U\Big(a(\omega), x\Big), \tag{7.4f}$$

where $\Big(H_l^{(i)}(N_{i0}, M_{i0}), h_l\Big), l = 4, \ldots, l_0 + 3$, are given row vectors depending on the yield domains of the bars, and η_i^L, η_i^U are defined by

$$\eta_i^L\Big(a(\omega), x\Big) = \min \left\{ \frac{N_{ipl}^L\Big(a(\omega), x\Big)}{N_{i0}}, \frac{M_{ipl}\Big(a(\omega), x\Big)}{M_{i0}} \right\} \tag{7.4g}$$

$$\eta_i^U\Big(a(\omega), x\Big) = \min \left\{ \frac{N_{ipl}^U\Big(a(\omega), x\Big)}{N_{io}}, \frac{M_{ipl}\Big(a(\omega), x\Big)}{M_{i0}} \right\} \tag{7.4h}$$

with certain chosen reference values $N_{i0}, M_{i0}, i = 1, \ldots, B$, for the plastic capacities.

According to (7.4a, f), the feasibility condition for the vector F of interior loads (member forces and moments) can be represented uniformly by the conditions

$$F_{il}^L\Big(a(\omega),x\Big) \le H_l^{(i)} F_i \le F_{il}^U\Big(a(\omega),x\Big), i = 1,\ldots,B, l = 1,2,\ldots,l_0+3, \tag{7.5a}$$

where the row 3-vectors $H_l^{(i)}$ and the bounds $F_{il}^L, F_{il}^U, i = 1,\ldots,B, l = 1,\ldots,l_0+3$, are defined by (7.4a–c) and (7.4f–h). Let $e_1 =: H_1^T, e_2 =: H_2^T, e_3 =: H_3^T$ denote the unit vectors of $\mathbb{R}^3$.

Defining the $(l_0+3)\times 3$ matrix $H^{(i)}$ by

$$H^{(i)} := \begin{pmatrix} e_1^T \\ e_2^T \\ e_3^T \\ H_4(N_{i0}, M_{i0}) \\ \vdots \\ H_{l_0+3}(N_{i0}, M_{i0}) \end{pmatrix} \tag{7.5b}$$

and the (l_0+3)-vectors $F_i^L = F_i^L\Big(a(\omega),x\Big), F_i^U = F_i^U\Big(a(\omega),x\Big)$ by

$$F_i^L := \begin{pmatrix} \tilde{F}_i^L \\ -h_1\eta_i^L \\ \vdots \\ -h_{l_0}\eta_i^L \end{pmatrix}, F_i^U := \begin{pmatrix} \tilde{F}_i^U \\ h_1\eta_i^U \\ \vdots \\ h_{l_0}\eta_i^U \end{pmatrix}, \tag{7.5c}$$

the feasibility condition can also be represented by

$$F_i^L\Big(a(\omega),x\Big) \le H^{(i)} F_i \le F_i^U\Big(a(\omega),x\Big), i = 1,\ldots,B. \tag{7.6}$$

7.2 State and Cost Functions

Defining the quantities

$$F_{il}^c = F_{il}^c\Big(a(\omega),x\Big) := \frac{F_{il}^L + F_{il}^U}{2} \tag{7.7a}$$

$$\varrho_{il} = \varrho_{il}\Big(a(\omega),x\Big) := \frac{F_{il}^U - F_{il}^L}{2}, \tag{7.7b}$$

$i = 1, \ldots, B, l = 1, \ldots, l_0 + 3$, the feasibility condition (7.5a) or (7.6) can be described by

$$|z_{il}| \le 1, i = 1, \ldots, B, l = 1, \ldots, l_0 + 3, \tag{7.8a}$$

with the quotients

$$z_{il} = z_{il}\big(F_i; a(\omega), x\big) = \frac{H_l^{(i)} F_i - F_{il}^c}{\varrho_{il}}, i = 1, \ldots, B, l = 1, \ldots, l_0 + 3. \tag{7.8b}$$

The quotient $z_{il}, i = 1, \ldots, B, l = 1, \ldots, l_0 + 3$, denotes the relative deviation of the load component $H_l^{(i)} F_i$ from its "ideal" value F_{il}^c with respect to the radius ϱ_{il} of the feasible interval $[F_{il}^L, F_{il}^U]$. According to (7.8a, b), the absolute values $|z_{il}|$ of the quotients z_{il} should not exceed the value 1. The absolute value $|z_{il}|$ of the quotient z_{il} denotes the percentage of use of the available plastic capacity by the corresponding load component. Obviously, $|z_{il}| = 1, |z_{il}| > 1$, resp., means maximal use, overcharge of the available resources.

Consider now the $(l_0 + 3)$-vectors

$$z_i := (z_{i1}, z_{i2}, \ldots, z_{il_0+3})^T = \left(\frac{H_1^{(i)} F_i - F_{i1}^c}{\varrho_{i1}}, \frac{H_2^{(i)} F_i - F_{i2}^c}{\varrho_{i2}}, \cdots, \right.$$
$$\left. \frac{H_{l_0+3}^{(i)} F_i - F_{il_0+3}^c}{\varrho_{il_0+3}} \right)^T. \tag{7.8c}$$

With

$$\varrho_i := \begin{pmatrix} \varrho_{i1} \\ \varrho_{i2} \\ \vdots \\ \varrho_{il_0+3} \end{pmatrix}, \varrho_{id} := \begin{pmatrix} \varrho_{i1} & 0 & \ldots & 0 \\ 0 & \varrho_{i2} & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & \varrho_{il_0+3} \end{pmatrix}, F_i^c := \begin{pmatrix} F_{i1}^c \\ F_{i2}^c \\ \vdots \\ F_{il_0+3}^c \end{pmatrix} \tag{7.8d}$$

we get

$$z_i = \varrho_{id}^{-1} (H^{(i)} F_i - F_i^c). \tag{7.8e}$$

Using (7.4b–d), we find

$$F_i^c = \left(A_i \frac{\sigma_{yi}^L + \sigma_{yi}^U}{2}, 0, 0, h_4 \frac{\eta_i^U - \eta_i^L}{2}, \ldots, h_{l_0+3} \frac{\eta_i^U - \eta_i^L}{2} \right)^T \tag{7.8f}$$

$$\varrho_i = \left(A_i \frac{\sigma_{yi}^U - \sigma_{yi}^L}{2}, A_i \sigma_{yi}^U \bar{y}_{ic}, A_i \sigma_{yi}^U \bar{y}_{ic}, h_4 \frac{\eta_i^U + \eta_i^L}{2}, \ldots, h_{l_0+3} \frac{\eta_i^U + \eta_i^L}{2} \right)^T. \tag{7.8g}$$

The vector z_i can be represented then, cf. (7.3), by

$$z_i = \left(\frac{t_i - A_i \frac{\sigma_{yi}^L + \sigma_{yi}^U}{2}}{A_i \frac{\sigma_{yi}^U - \sigma_{yi}^L}{2}}, \frac{m_i^l}{A_i \sigma_{yi}^U \bar{y}_{ic}}, \frac{m_i^r}{A_i \sigma_{yi}^U \bar{y}_{ic}}, \frac{H_4^{(i)} F_i - h_4 \frac{\eta_i^U - \eta_i^L}{2}}{h_4 \frac{\eta_i^U + \eta_i^L}{2}}, \ldots, \right.$$

$$\left. \frac{H_{l_0+3}^{(i)} F_i - h_{l_0+3} \frac{\eta_i^U - \eta_i^L}{2}}{h_{l_0+3} \frac{\eta_i^U + \eta_i^L}{2}} \right)^T . \tag{7.9a}$$

In case of symmetry $\sigma_{yi}^L = -\sigma_{yi}^U$ we get

$$z_i = \left(\frac{t_i}{A_i \sigma_{y_i}^U}, \frac{m_i^l}{A_i \sigma_{y_i}^U \bar{y}_{ic}}, \frac{m_i^r}{A_i \sigma_{y_i}^U \bar{y}_{ic}}, \frac{H_4^{(i)} F_i,}{h_4 \eta_i^U}, \ldots, \frac{H_{l_0+3}^{(i)} F_i}{h_{l_0+3} \eta_i^U} \right)^T . \tag{7.9b}$$

According to the methods introduced in [97, 99, 100], the fulfillment of the survival condition for elastoplastic frame structures, hence, the equilibrium condition (7.2) and the feasibility condition (7.6) or (7.8a, b), can be described by means of the *state function* $s^* = s^*\Big(a(\omega), x\Big)$ defined, in the present case, by

$$s^* = s^*\Big(a(\omega), x\Big) := \min \Big\{ s : \Big| z_{il}\big(F_i; a(\omega), x\big) \Big| - 1 \le s, i = 1, \ldots, B,$$

$$l = 1, 2, \ldots, l_0 + 3, CF = P\Big(a(\omega), x\Big) \Big\} . \tag{7.10}$$

Hence, the state function s^* is the minimum value function of the linear program (LP)

$$\min s \tag{7.11a}$$

s.t.

$$\Big| z_{il}\Big(F_i; a(\omega), x\Big) \Big| - 1 \le s, \quad i = 1, \ldots, B, \quad l = 1, \ldots, l_0 + 3 \tag{7.11b}$$

$$CF = P\Big(a(\omega), x\Big). \tag{7.11c}$$

Since the objective function s is bounded from below and a feasible solution (s, F) always exists, LP (7.11a–c) has an optimal solution $(s^*, F^*) = \Big(s^*\Big(a(\omega), x\Big), F^*\Big(a(\omega), x\Big) \Big)$ for each configuration $\Big(a(\omega), x\Big)$ of the structure.

Consequently, for the survival of the structure we have the following criterion, cf. [99]:

Theorem 7.1 *The elastoplastic frame structure having configuration* (a, x) *carries the exterior load* $P = P(a, x)$ *safely if and only if*

$$s^*(a, x) \leq 0. \tag{7.12}$$

Obviously, the constraint (7.11b) in the LP (7.11a–c) can also be represented by

$$\left\| z\Big(F; a(\omega), x\Big) \right\|_\infty - 1 \leq s, \tag{7.13a}$$

where $z = z\Big(F; a(\omega), x\Big)$ denotes the $B(l_0 + 3)$-vector

$$z\Big(F; a(\omega), x\Big) := \left(z_1\Big(F; a(\omega), x\Big)^T, \ldots, z_B\Big(F; a(\omega), x\Big)^T \right)^T, \tag{7.13b}$$

and $\|z\|_\infty$ is the maximum norm

$$\|z\|_\infty := \max_{\substack{1 \leq i \leq B \\ 1 \leq l \leq l_0 + 3}} |z_{il}|. \tag{7.13c}$$

If we put

$$\hat{s} = 1 + s \text{ or } s = \hat{s} - 1, \tag{7.14}$$

from (7.10) we obtain

$$s^*(a, x) = \hat{s}^*(a, x) - 1, \tag{7.15a}$$

where the transformed state function $\hat{s}^* = \hat{s}^*(a, x)$ reads

$$\hat{s}^*(a, x) := \min \left\{ \Big\| z(F; a, x) \Big\|_\infty : CF = P(a, x) \right\}. \tag{7.15b}$$

Remark 7.1 According to (7.15a, b) and (7.12), the safety or survival condition of the plane frame with plastic material can be represented also by

$$\hat{s}^*(a, x) \leq 1.$$

The state function $\hat{s}^* = \hat{s}^*(a, x)$ describes the maximum percentage of use of the available plastic capacity within the plane frame for the best internal load distribution with respect to the configuration (a, x).

Obviously, $\hat{s}^* = \hat{s}^*(a, x)$ is the minimum value function of the LP

$$\min_{CF=P(a,x)} \left\| z(F; a, x) \right\|_\infty . \tag{7.16}$$

The following inequalities for norms or power/Hölder means $\|z\|$ in $\mathbb{R}^{B(l_0+3)}$ are well known [25, 57]:

$$\begin{aligned} \frac{1}{B(l_0+3)} \|z\|_\infty &\le \frac{1}{B(l_0+3)} \|z\|_1 \\ &\le \frac{1}{\sqrt{B(l_0+3)}} \|z\|_2 \le \|z\|_\infty \le \|z\|_2, \end{aligned} \tag{7.17a}$$

where

$$\|z\|_1 := \sum_{i=1}^{B} \sum_{l=1}^{l_0+3} |z_{il}|, \ \|z\|_2 := \sqrt{\sum_{i=1}^{B} \sum_{l=1}^{l_0+3} z_{il}^2}. \tag{7.17b}$$

Using (7.17a), we find

$$\frac{1}{B(l_0+3)} \hat{s}^*(a, x) \le \frac{1}{\sqrt{B(l_0+3)}} \hat{s}_2^*(a, x) \le \hat{s}^*(a, x) \le \hat{s}_2^*(a, x), \tag{7.18a}$$

where $\hat{s}_2^* = \hat{s}_2^*(a, x)$ is the modified state function function defined by

$$\hat{s}_2^*(a, x) := \min \left\{ \left\| z(F; a, x) \right\|_2 : CF = P(a, x) \right\}. \tag{7.18b}$$

Obviously, we have

$$\hat{s}_2^*(a, x) = \sqrt{G_1^*(a, x)}, \tag{7.18c}$$

where $G_1^*(a, x)$ is the minimum value function of the **quadratic program**

$$\min_{CF=P\left(a(\omega), x\right)} \sum_{i=1}^{B} \sum_{l=1}^{l_0+3} z_{il}(F_i; a, x)^2. \tag{7.19}$$

7.2.1 Cost Functions

The inequalities in (7.18a) show that for structural analysis and optimal design purposes we may work also with the state function $\hat{s}_2^* = \hat{s}_2^*(a, x)$ which can be defined easily by means of the quadratic program (7.19).

According to the definition (7.8b) and the corresponding technical interpretation of the quotients z_{il}, the transformed state function $\hat{s}^* = \hat{s}^*(a,x)$ represents—for the best internal load distribution—the maximum percentage of use of the plastic capacities *relative* to the available plastic capacities in the members (bars) of the plane frame with configuration (a,x). While the definition (7.15b) of $\hat{s}^*$ is based on the absolute value function

$$c_1(z_{il}) = |z_{il}|, \tag{7.20a}$$

in definition (7.18b) of $\hat{s}_2^*$ occur quadratic functions

$$c_2(z_{il}) = z_{il}^2, \quad i = 1,\dots,B, \quad l = 1,\dots,l_0+3. \tag{7.20b}$$

Obviously,

$$c_p(z_{il}) = |z_{il}|^p \text{ with } p = 1,2 \text{ (or also } p \geq 1)$$

are possible convex functions measuring the **costs** resulting from the position z_{il} of a load component $\tilde{H}_l^{(i)} F_i$ relative to the corresponding safety interval (plastic capacity) $[\tilde{F}_{il}^L, \tilde{F}_{il}^U]$ (Fig. 7.1).

If different weights are used in the objective function (7.19), then for the bars we obtain, cf. (7.8c), the cost functions

$$q_i(z_i) = \|W_{i0} z_i\|^2, \tag{7.20c}$$

with $(l_0+3)\times(l_0+3)$ weight matrices W_{i0}, $i = 1,\dots,B$.

The total weighted quadratic costs resulting from a load distribution F acting on the plastic plane frame having configuration (a,x) are given, cf. (7.18c), (7.19), (7.20c), by

$$G_1 := \sum_{i=1}^{B} \|W_{i0} z_i\|^2 = \sum_{i=1}^{B} z_i^T W_{i0}^T W_{i0} z_i. \tag{7.21a}$$

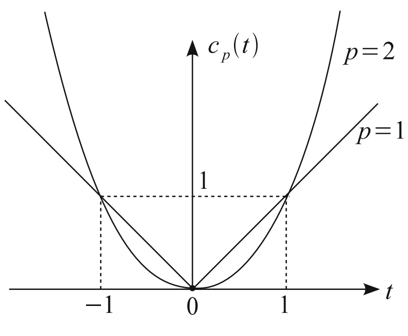

Fig. 7.1 Cost functions c_p

Defining

$$W_0 := \begin{pmatrix} W_{10} & 0 & \dots & 0 \\ 0 & W_{20} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & W_{B0} \end{pmatrix}, \; z := \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_B \end{pmatrix}, \tag{7.21b}$$

we also have

$$\begin{aligned} G_1 = G_1(a, x; F) &= z^T W_0^T W_0 z \\ &= \|W_0 z\|_2^2 = \|z\|_{2,W_0}^2, \end{aligned} \tag{7.21c}$$

where $\|\cdot\|_{2,W_0}$ denotes the *weighted Euclidean norm*

$$\|z\|_{2,W_0} := \|W_0 z\|_2. \tag{7.21d}$$

Using the weighted quadratic cost function (7.20c), the state function $\hat{s}_2^* = \hat{s}_2^*(a, x)$ is replaced by

$$\begin{aligned} \hat{s}_{2,W_0}^*(a, x) &:= \min\left\{\left\|z(F; a, x)\right\|_{2,W_0} : CF = P(a, x)\right\} \\ &= \min\left\{\sqrt{G_1(a, x; F)} : CF = P(a, x)\right\}. \end{aligned} \tag{7.21e}$$

Since

$$\|z\|_{2,W_0} = \|W_0 z\|_2 \leq \|W_0\| \cdot \|z\|$$

with the norm $\|W_0\|$ of the matrix W_0, we find

$$\hat{s}_{2,W_0}^*(a, x) \leq \|W_0\| \hat{s}_2^*(a, x). \tag{7.21f}$$

On the other hand, in case

$$\|W_0 z\|_2 \geq \underline{W}_0 \|z\|_2$$

with a positive constant $\underline{W}_0 > 0$, we have

$$\hat{s}_{2,W_0}^*(a, x) \geq \underline{W}_0 \hat{s}_2^*(a, x) \text{ or } \hat{s}_2^*(a, x) \leq \frac{1}{\underline{W}_0} \hat{s}_{2,W_0}^*(a, x). \tag{7.21g}$$

7.3 Minimum Expected Quadratic Costs

Putting

$$H := \begin{pmatrix} H^{(1)} & & & \\ & H^{(2)} & & \\ & & \ddots & \\ & & & H^{(B)} \end{pmatrix}, F^c := \begin{pmatrix} F_1^c \\ F_2^c \\ \vdots \\ F_B^c \end{pmatrix}, \varrho := \begin{pmatrix} \varrho_1 \\ \varrho_2 \\ \vdots \\ \varrho_B \end{pmatrix}, \tag{7.21h}$$

with (7.8e) we find

$$G_1 = G_1\Big(a(\omega), x; F\Big) = (HF - F^c)^T \varrho_d^{-1} W_0^T W_0 \varrho_d^{-1} (HF - F^c). \tag{7.22a}$$

If

$$F^c = HF^c \text{ with } F^c := \left(\frac{F_i^L + F_i^U}{2}\right)_{i=1,\dots,B} \tag{7.22b}$$

as in the case of no interaction between normal forces and moments, see (7.4a–c), and in the case of symmetric yield stresses

$$\sigma_{yi}^L = -\sigma_{yi}^U, i = 1, \dots, B, \tag{7.22c}$$

we also have

$$G_1\Big(a(\omega), x; F\Big) = (F - F^c)^T H^T \varrho_d^{-1} W_0^T W_0 \varrho_d^{-1} H(F - F^c). \tag{7.22d}$$

Moreover, if (7.22c) holds, then $F^c = 0$ and therefore

$$G_1\Big(a(\omega), x; F\Big) = F^T H^T \varrho_d^{-1} W_0^T W_0 \varrho_d^{-1} HF. \tag{7.22e}$$

For simplification, we assume first in this section that the total cost representation (7.22d) or (7.22e) holds.

According to the equilibrium condition (7.2), the total vector F of generalized forces of the members fulfills

$$CF = P(a(\omega), x).$$

Let $x \in D$ denote a given vector of decision variables, and let be $a = a(\omega)$ a realization of vector $a(\cdot)$ of model parameters. Based on (7.22d) or (7.22e), a cost minimum or "best" internal distribution of the generalized forces

$$F^* = F^*(a(\omega), x)$$

of the structure can be obtained by solving the following optimization problem with quadratic objective function and linear constraints

$$\min_{CF=P(a(\omega),x)} G_1(a(\omega),x;F). \tag{7.23}$$

Solving the related stochastic optimization problem [100]

$$\min_{CF=P(a(\omega),x)\text{ a.s.}} EG_1(a(\omega),x;F), \tag{7.24}$$

for the random configuration $\big(a(\omega),x\big)$ we get the minimum expected (total) quadratic costs

$$\overline{G}_1^* = \overline{G}_1^*(x),\ x \in D, \tag{7.25a}$$

where $\overline{G}_1^*(x)$ may be obtained by interchanging expectation and minimization

$$\overline{G}_1^*(x) = \overline{G_1^*}(x) := E\min\{G_1(a(\omega),x;F) : CF = P(a(\omega),x)\}. \tag{7.25b}$$

The internal minimization problem (7.23)

$$\min\ G_1(a(\omega),x;F) \text{ s.t. } CF = P(a(\omega),x),$$

hence,

$$\min_{CF=P(a(\omega),x)} (F-F^c)^T H^T \varrho_d^{-1} W_0^T W_0 \varrho_d^{-1} H(F-F^c), \tag{7.26}$$

with quadratic objective function and linear constraints with respect to F can be solved by means of Lagrange techniques. We put

$$W = W(a,x) := H^T \varrho_d^{-1} W_0^T W_0 \varrho_d^{-1} H \tag{7.27}$$

and define the Lagrangian of (7.26):

$$L = L(F,\lambda) := (F-F^c)^T W(F-F^c) + \lambda^T(CF - P(a(\omega),x)). \tag{7.28a}$$

Based on the corresponding piecewise linearised yield domain, W describes the plastic capacity of the plane frame with respect to axial forces and bending moments.

The necessary and sufficient optimality conditions for a minimum point (F^*,λ^*) read:

$$0 = \nabla_F L = 2W(F-F^c) + C^T\lambda, \tag{7.28b}$$

$$0 = \nabla_\lambda L \qquad = CF - P. \tag{7.28c}$$

Supposing that W is regular, we get

$$F = F^c - \frac{1}{2} W^{-1} C^T \lambda \tag{7.28d}$$

and

$$P = CF = CF^c - \frac{1}{2} C W^{-1} C^T \lambda, \tag{7.28e}$$

hence,

$$F^* = F^c - \frac{1}{2} W^{-1} C^T \lambda^* = F^c - W^{-1} C^T (C W^{-1} C^T)^{-1} (CF^c - P). \tag{7.28f}$$

Inserting (7.28f) into the objective function $G_1(a(\omega), x; F)$, according to (7.22a) and (7.27) we find

$$\begin{aligned} G_1^* &= G_1^*(a(\omega), x) \\ &= (F^* - F^c)^T W (F^* - F^c) \\ &= \left((CF^c - P)^T (C W^{-1} C^T)^{-1} C W^{-1}\right) W \left(W^{-1} C^T (C W^{-1} C^T)^{-1} (CF^c - P)\right) \\ &= (CF^c - P)^T (C W^{-1} C^T)^{-1} (CF^c - P) \\ &= tr (C W^{-1} C^T)^{-1} (CF^c - P)(CF^c - P)^T, \end{aligned} \tag{7.28g}$$

where "tr" denotes the trace of a matrix. The minimal expected value $\overline{G_1^*}$ is then given by

$$\begin{aligned} \overline{G_1^*}(x) &= E G_1^*\big(a(\omega), x\big) \\ &= E\big(CF^c(a(\omega), x\big) - P\big(a(\omega), x)\big)^T \Big(CW\big(a(\omega), x\big)^{-1} C^T\Big)^{-1} \\ &\quad \times \Big(CF^c\big(a(\omega), x\big) - P\big(a(\omega), x\big)\Big) \\ &= E\ tr\Big(CW\big(a(\omega), x\big)^{-1} C^T\Big)^{-1} \Big(CF^c\big(a(\omega), x\big) - P\big(a(\omega), x\big)\Big) \\ &\quad \times \Big(CF^c\big(a(\omega), x\big) - P\big(a(\omega), x\big)\Big)^T. \end{aligned} \tag{7.29a}$$

If $\sigma_{yi}^L = -\sigma_{yi}^U$, $i = 1, \ldots, B$, then $F^c = 0$ and

$$\begin{aligned} \overline{G_1^*}(x) &= E P(a(\omega), x))^T (C W(a(\omega), x)^{-1} C^T)^{-1} P(a(\omega), x) \\ &= tr E \Big(CW\big(a(\omega), x\big)^{-1} C^T\Big)^{-1} P\big(a(\omega), x\big) P\big(a(\omega), x\big)^T. \end{aligned} \tag{7.29b}$$

Since the vector $P = P(a(\omega), x)$ of external generalized forces and the vector of yield stresses $\sigma^U = \sigma^U(a(\omega), x)$ are stochastically independent, then in case $\sigma_{yi}^L = -\sigma_{yi}^U$, $i = 1, \ldots, B$, we have

$$\begin{aligned}\overline{G_1^*}(x) &= EP(a(\omega), x))^T \overline{U}(x) P(a(\omega), x)\\ &= tr\overline{U}(x) \mathrm{e} P(a(\omega), x) P(a(\omega), x)^T,\end{aligned} \tag{7.29c}$$

where

$$\overline{U}(x) := EK(a(\omega), x)^{-1} \tag{7.29d}$$

with the matrices

$$K(a, x) := CK_0(a, x)C^T \tag{7.30a}$$

and

$$K_0(a, x) := W(a, x)^{-1} = (H^T \varrho_d^{-1} W_0^T W_0 \varrho_d^{-1} H)^{-1}. \tag{7.30b}$$

We compare now, especially in case $F^c = 0$, formula (7.28g) for the costs $G_1^* = G_1^*(a, x)$ with formula

$$\Gamma := u^T P$$

for the *compliance* of an elastic structure, where

$$u := K_{el}^{-1} P$$

is the vector of displacements, and K_{el} denotes the stiffness matrix in case of an elastic structure. Obviously, the cost function $G_1^* = G_1^*(a, x)$ may be interpreted as a *generalized compliance function*, and the $m \times m$ matrix $K = K(a, x)$ can be interpreted as the "*generalized stiffness matrix*" of the underlying plastic mechanical structure. If we suppose that

$$W_{i0} := (w_{il}^0 \delta_{l\lambda})_{l,\lambda=1,\ldots,l_0+3}, \quad i = 1, \ldots, B, \tag{7.30c}$$

are diagonal weight matrices, then, cf. (7.8g),

$$\varrho_d^{-1} W_0^T W_0 \varrho_d^{-1} = \operatorname{diag}\left(\left(\frac{w_{il}^0}{\varrho_{il}}\right)^2\right). \tag{7.30d}$$

If condition (7.22b) and therefore representation (7.22d) or (7.22e) does not hold, then the minimum total costs $G_1^* = G_1^*\Big(a(\omega), x\Big)$ are determined by the more general quadratic program, cf. (7.22a), (7.23), (7.26),

$$\min_{CF=P\left(a(\omega),x\right)} (HF - F^c)^T W_\varrho\Big(a(\omega), x\Big)(HF - F^c), \tag{7.31a}$$

where

$$W_\varrho(a,x) := \varrho_d^{-1}(a,x) W_0^T W_0 \varrho_d^{-1}(a,x). \tag{7.31b}$$

Though also in this case problem (7.31a) can be solved explicitly, the resulting total cost function has a difficult form. In order to apply the previous technique, the vector F^c is approximated—in the least squares sense—by vectors HF with $F \in \mathbb{R}^{3B}$. Hence, we write

$$F^c \approx HF^{c*}, \tag{7.32a}$$

where the $(3B)$-vector F^{c*} is the optimal solution of the optimization problem

$$\min_F \|HF - F^c\|^2. \tag{7.32b}$$

We obtain

$$F^{c*} = F^{c*}(F^c) := (H^T H)^{-1} H^T F^c. \tag{7.32c}$$

The error $e(F^{c*})$ of this approximation reads

$$\begin{aligned} e(F^c) &:= \|HF^{c*} - F^c\| \\ &= \Big\|\Big(H(H^T H)^{-1} H^T - I\Big) F^c\Big\|. \end{aligned} \tag{7.32d}$$

With the vector $F^{c*} = F^{c*}(F^c)$ the total costs $G_1^a = G_1^a\Big(a(\omega), x; F\Big)$ can be approximated now, see (7.22a), by

$$\begin{aligned} G_1^a\Big(a(\omega), x; F\Big) &:= (HF - HF^{c*})^T W_\varrho\Big(a(\omega), x\Big)(HF - HF^{c*}) \\ &= (F - F^{c*})^T H^T W_\varrho\Big(a(\omega), x\Big) H(F - F^{c*}) \\ &= (F - F^{c*})^T W\Big(a(\omega), x\Big)(F - F^{c*}), \end{aligned} \tag{7.33a}$$

where, cf. (7.27),

$$W = W(a, x) := H^T W_\varrho(a, x) H. \tag{7.33b}$$

Obviously, the approximate cost function $G_1^a = G_1^a(a, x; F)$ has the same form as the cost function $G_1 = G_1(a, x; F)$ under the assumption (7.22b), see (7.22d). Hence, the minimum cost function $G_1^{a*} = G_1^{a*}(a, x)$ can be determined by solving, cf. (7.23),

$$\min_{CF=P(a,x)} G_1^a(a, x; F). \tag{7.34a}$$

We get, see (7.28g),

$$G_1^{a*}(a, x) := tr\Big(CW(a, x)^{-1}C^T\Big)^{-1}(CF^{c*} - P)(CF^{c*} - P)^T, \tag{7.34b}$$

where $F^{c*} = F^{c*}(a, x)$ is given here by (7.32c). Taking expectations in (7.34b), we obtain the approximate minimum expected total cost function

$$\overline{G_1^{a*}} = \overline{G_1^{a*}}(x) = EG_1^{a*}\Big(a(\omega), x\Big). \tag{7.34c}$$

7.4 Deterministic Substitute Problems

In order to determine robust optimal designs x^*, appropriate deterministic substitute problems, cf. [100], must be formulated.

7.4.1 Weight (Volume)-Minimization Subject to Expected Cost Constraints

With the expected primary cost function, see (7.1a, b),

$$\overline{G_0}(x) = EG_0(a(\omega), x)$$

and the expected cost function $\overline{G_1^*} = \overline{G_1^*}(x)$ representing the expected total weighted quadratic costs resulting from a violation of the feasibility condition (7.4a, f), we get [46, 50] the optimization problem

$$\min \quad \overline{G_0}(x) \tag{7.35a}$$

$$\text{s.t.} \quad \overline{G_1^*}(x) \le \Gamma_1 \tag{7.35b}$$

$$x \in D, \tag{7.35c}$$

where Γ_1 is a certain upper cost bound. In case (7.1a) we have

$$\overline{G_0}(x) := \sum_{i=1}^{B} \overline{\gamma_{i0}} L_i A_i(x) \tag{7.35d}$$

with $\overline{\gamma_{i0}} := E\gamma_{i0}(\omega)$, and $\overline{G_1^*} = \overline{G_1^*}(x)$ is defined by (7.29a) or (7.29b).

Due to (7.20c) and (7.21a–c), the upper cost bound Γ_1 can be defined by

$$\Gamma_1 := g_1 G_1^{\max}, \tag{7.35e}$$

where $g_1 > 0$ is a certain reliability factor, and $G_1^{\max}$ denotes the maximum of the total cost function $G_1 = G_1(z)$ on the total admissible z-domain $[-1,1]^{(l_0+3)B}$. Hence,

$$\begin{aligned} G_1^{\max} &:= \max_{z \in [-1,1]^{(l_0+3)B}} \sum_{i=1}^{B} \|W_{i0} z_i\|^2 \\ &= \sum_{i=1}^{B} \max_{z_i \in [-1,1]^{(l_0+3)}} \|W_{i0} z_i\|^2 \\ &= \sum_{i=1}^{B} \max_{1 \le j \le 2^{l_0+3}} \|W_{i0} e^{(j)}\|^2, \end{aligned} \tag{7.35f}$$

where $e^{(j)}, j = 1, \ldots, 2^{l_0+3}$, denote the extreme points of the hypercube $[-1,1]^{l_0+3}$.

As shown in the following, for $W_0 = I$ (identity matrix) the expected cost constraint (7.35b) can also be interpreted as a reliability constraint.

According to Theorem 2.1, (7.12) and (7.15a, b), for the probability of survival $p_s = p_s(x)$ of the elastoplastic structure represented by the design vector x we have

$$\begin{aligned} p_s(x) &:= P\Big(s^*\big(a(\omega), x\big) \le 0\Big) \\ &= P\Big(\hat{s}^*\big(a(\omega), x\big) - 1 \le 0\Big) = P\Big(\hat{s}^*\big(a(\omega), x\big) \le 1\Big). \end{aligned} \tag{7.36}$$

Knowing from (7.18a, b) that, in case $W_0 = I$,

$$\frac{1}{\sqrt{B(l_0+3)}} \hat{s}_2^*(a, x) \le \hat{s}^*(a, x) \le \hat{s}_2^*(a, x),$$

we obtain the probability inequalities

$$P\Big(\hat{s}_2^*\big(a(\omega), x\big) \le 1\Big) \le p_s(x) \le P\Big(\hat{s}_2^*(a, x) \le \sqrt{B(l_0+3)}\Big). \tag{7.37a}$$

Due to the first definition of $G_1^* = G_1^*(a,x)$ by (7.18c) and (7.19), related to the case $W_0 = I$, we also have

$$P\Big(G_1^*\big(a(\omega),x\big) \le 1\Big) \le p_s(x) \le P\Big(G_1^*\big(a(\omega),x\big) \le B(l_0+3)\Big). \tag{7.37b}$$

Using now a nonnegative, nondecreasing, measurable function h on $\mathbb{R}_+$, for any $g_1 > 0$ we find [100]

$$P\Big(G_1^*\big(a(\omega),x\big) \le g_1\Big) \ge 1 - \frac{Eh\Big(G_1^*\big(a(\omega),x\big)\Big)}{h(g_1)}. \tag{7.38a}$$

In the case $h(t) = t$ we get the inequality

$$P\Big(G_1^*\big(a(\omega),x\big) \le g_1\Big) \ge 1 - \frac{\overline{G_1^*}(x)}{g_1}, \tag{7.38b}$$

where the expectation $\overline{G_1^*}(x) = EG_1^*\big(a(\omega),x\big)$ is given by (7.29a) or (7.29b). The probabilistic constraint

$$P\Big(G_1^*\big(a(\omega),x\big) \le g_1\Big) \ge \alpha_{\min} \tag{7.39a}$$

for the quadratic mean rate $\hat{s}_2^* = \sqrt{G_1^*(a,x)}$ of minimum possible use of plastic capacity within the plane frame with configuration (a,x) implies $p_s(x) \ge \alpha_{\min}$ for $g_1 = 1$, cf. (7.37b). Hence, due to (7.38b), constraint (7.39a) and therefore $p_s(x) \ge \alpha_{\min}$ can be guaranteed then by the condition

$$\overline{G_1^*}(x) \le g_1(1-\alpha), \tag{7.39b}$$

see (7.35b).

7.4.2 Minimum Expected Total Costs

For a vector $x \in D$ of decision variables and a vector F of internal generalized forces fulfilling the equilibrium condition (7.2), from (7.1a, b) and (7.22a, b) we have the total costs

$$G(a(\omega),x;F) := G_0(a(\omega),x) + G_1(a(\omega),x;F). \tag{7.40a}$$

Here, the weight or scale matrices W_{i0} and the weight or cost factors γ_{i0}, $i = 1,\dots,B$, must be selected such that the dimensions of G_0 and G_1 coincide. For example, if $W_{i0} = I, i = 1,\dots,B$, and $\sqrt{G_1(a,x)}$ is then the quadratic mean rate of use of plastic capacity for a given distribution of generalized forces F, then we may replace γ_{i0} by the relative weight/cost coefficients

$$\gamma_{i0}^{rel} := \frac{\gamma_{i0}}{G_0^{ref}}, \quad i = 1, \ldots, B,$$

with a certain weight or cost reference value G_0^{ref}.

Minimizing now the expected total costs

$$\begin{aligned} \overline{G} = \overline{G}(x) &= EG(a(\omega), x; F(\omega)) \\ &= E(G_0(a(\omega), x) + G_1(a(\omega), x; F(\omega))) \\ &= EG_0(a(\omega), x) + EG_1(a(\omega), x; F(\omega)) \\ &= \overline{G}_0(x) + EG_1(a(\omega), x; F(\omega)) \end{aligned} \tag{7.40b}$$

subject to the equilibrium conditions (7.2) and the remaining condition for the decision variables

$$x \in D, \tag{7.40c}$$

we obtain the stochastic optimization problem

$$\min_{\substack{CF(\omega)=P\left(a(\omega),x\right)\ a.s.\\ x\in D}} E(G_0(a(\omega), x) + G_1(a(\omega), x; F(\omega)). \tag{7.41}$$

Obviously, (7.41) has the following *two-stage structure*:

Step 1: Select $x \in D$ without knowledge of the actual realization $a = a(\omega)$ of the model parameters, but knowing the probability distribution or certain moments of $a(\cdot)$;

Step 2: Determine the best internal distribution of generalized forces $F = F^*(\omega)$ after realization of $a = a(\omega)$.

Therefore, problem (7.41) is equivalent to

$$\min_{x\in D} E\left(G_0(a(\omega), x) + \min_{CF=P(a(\omega),x)} G_1(a(\omega), x; F)\right). \tag{7.42}$$

According to the definitions (7.35d) of $\overline{G_0}$ and (7.25b) of $\overline{G_1^*}$, problem (7.42) can be represented also by

$$\min_{x\in D}\left(\overline{G_0}(x) + \overline{G_1^*}(x)\right). \tag{7.43}$$

7.5 Stochastic Nonlinear Programming

We first suppose that the structure consists of uniform material with a symmetric random yield stress in compression and tension. Hence, we assume next to

$$\sigma_{yi}^U = -\sigma_{yi}^L = \sigma_y^U = \sigma_y^U(\omega), \quad i = 1, \ldots, B, \tag{7.44}$$

with a random yield stress $\sigma_y^U(\omega)$. Due to (7.8e) we have

$$\begin{aligned} \varrho_i\big(a(\omega), x\big) &= A_i(\sigma_{yi}^U, \sigma_{yi}^U \overline{y}_{ic}, \sigma_{yi}^U \overline{y}_{ic}, \sigma_{yi}^U h_4\eta_i, \ldots, \sigma_{yi}^U h_{l_0+3}\eta_i)^T \\ &= \sigma_y^U A_i(1, \overline{y}_{ic}, \overline{y}_{ic}, h_4\eta_i, \ldots, h_{l_0+3}\eta_i)^T := \sigma_y^U \mathring{\varrho}_i(x) \end{aligned} \tag{7.45a}$$

and therefore, see (7.8d),

$$\varrho\big(a(\omega), x\big) = \sigma_y^U(\omega)\, \mathring{\varrho}(x) \tag{7.45b}$$

with $\mathring{\varrho}_i(x) := A_i(1, \overline{y}_{ic}, \overline{y}_{ic}, h_4\eta_i, \ldots, h_{l_0+3}\eta_i)^T$, $\eta_i := \min\left\{\frac{1}{N_{i0}}, \frac{\overline{y}_{ic}}{M_{i0}}\right\}$ and

$$\mathring{\varrho}(x) = \begin{pmatrix} \mathring{\varrho}_1(x) \\ \vdots \\ \mathring{\varrho}_B(x) \end{pmatrix}. \tag{7.45c}$$

According to (7.30a, b), for *fixed weight matrices* $W_{i0}, i = 1, \ldots, B$, we obtain

$$K\big(a(\omega), x\big) = CK_0\big(a(\omega), x\big)C^T \tag{7.46a}$$

with

$$\begin{aligned} K_0\big(a(\omega), x\big) &= (H^T \varrho_d^{-1} W_0^T W_0 \varrho_d^{-1} H)^{-1} \\ &= \sigma_y^U(\omega)^2 \mathring{K}_0(x), \end{aligned} \tag{7.46b}$$

where

$$\mathring{K}_0(x) := \big(H^T \mathring{\varrho}(x)_d^{-1} W_0^T W_0 \mathring{\varrho}(x)_d^{-1} H\big)^{-1}. \tag{7.46c}$$

Now, (7.29d, e), (7.30a, b), and (7.46a–c) yield

$$K\big(a(\omega), x\big) = \sigma_y^U(\omega)^2 C\mathring{K}_0(x)C^T = \sigma_y^U(\omega)^2 \mathring{K}(x) \tag{7.47a}$$

with the deterministic matrix

$$\mathring{K}(x) := C\mathring{K}_0(x)C^T. \tag{7.47b}$$

Moreover, we get

$$U\big(a(\omega), x\big) := K\big(a(\omega), x\big)^{-1} = \big(\sigma_y^U(\omega)^2 \mathring{K}(x)\big)^{-1}$$
$$= \frac{1}{\sigma_y^U(\omega)^2} \mathring{K}(x)^{-1}. \tag{7.47c}$$

Hence, see (7.29d),

$$\overline{U}(x) = EU\big(a(\omega), x\big) = \left(E \frac{1}{\sigma_y^U(\omega)^2}\right) \mathring{K}(x)^{-1}. \tag{7.47d}$$

In case of a *random* weight matrix $W_0 = W_0\Big(a(\omega)\Big)$, for $\overline{U}(x)$ we also obtain a representation of the type (7.47d), provided that (i) the random variables $W_0\Big(a(\omega)\Big)$ and $\sigma_y^U(\omega)$ are stochastically independent and (ii) $\mathring{K}(x)$ is defined by

$$\mathring{K}(x) := \left(E\left(C\mathring{K}_0\Big(W\big(a(\omega)\big), x\Big)C^T\right)^{-1}\right)^{-1}. \tag{7.47e}$$

From (7.29c) we obtain

$$\overline{G_1^*}(x) = EG_1^*\big(a(\omega), x\big)$$
$$= tr\overline{U}(x)EP\big(a(\omega), x\big)P\big(a(\omega), x\big)^T$$
$$= \left(E\frac{1}{\sigma_y^U(\omega)^2}\right) tr\mathring{K}(x)^{-1}EP\big(a(\omega), x\big)P\big(a(\omega), x\big)^T. \tag{7.48}$$

Representing the $m \times m$ matrix

$$B(x) := EP\big(a(\omega), x\big)P\big(a(\omega)x\big)^T$$
$$= \overline{P}(x)\overline{P}(x)^T + \mathrm{cov}\Big(P\big(a(\cdot), x\big)\Big)$$
$$= \big(b_1(x), b_2(x), \ldots, b_m(x)\big) \tag{7.49a}$$

by its columns $b_j(x)$, $j = 1, \ldots, m$, where we still set

$$\overline{P}(x) := EP\big(a(\omega), x\big) \tag{7.49b}$$

$$\mathrm{cov}\Big(P\big(a(\cdot), x\big)\Big) := E\Big(P\big(a(\omega), x\big) - \overline{P}(x)\Big)\Big(P\big(a(\omega), x\big) - \overline{P}(x)\Big)^T, \tag{7.49c}$$

we find

$$Z(x) = (z_1, z_2, \ldots, z_m) := E\left(\frac{1}{\sigma_y^U(\omega)^2}\right)\mathring{K}(x)^{-1}B(x)$$
$$= E\left(\frac{1}{\sigma_y^U(\omega)^2}\right)\left(\mathring{K}(x)^{-1}b_1(x), \mathring{K}(x)^{-1}b_2(x), \ldots, \mathring{K}(x)^{-1}b_m(x)\right). \quad (7.49d)$$

However, (7.49d) is equivalent to the following equations for the columns $z_j, j = 1, \ldots, B$,

$$\mathring{K}(x)z_j = E\left(\frac{1}{\sigma_y^U(\omega)^2}\right)b_j(x), \quad j = 1, \ldots, m. \quad (7.50)$$

With Eq. (7.50) for $z_j, j = 1, \ldots, m$, the expected cost function $\overline{G_1^*}(x)$ can be represented now by

$$\overline{G_1^*}(x) = tr(z_1, z_2, \ldots, z_m). \quad (7.51)$$

Having (7.50), (7.51), the deterministic substitute problems (7.35a–d) and (7.43, b) can be represented as follows:

Theorem 7.2 (Expected Cost Based Optimization (ECBO)) *Suppose that $W_{i0}, i = 1, \ldots, B$, are given fixed weight matrices. Then the expected costbased optimization problem (7.35a–c) can be represented by*

$$\min \overline{G}_0(x) \quad (7.52a)$$

s.t.

$$tr(z_1, z_2, \ldots, z_m) \le \Gamma_1 \quad (7.52b)$$
$$\mathring{K}(x)z_j = E\left(\frac{1}{\sigma_y^U(\omega)^2}\right)b_j(x), \quad j = 1, \ldots, m \quad (7.52c)$$
$$x \in D, \quad (7.52d)$$

where the vectors $b_j = b_j(x), j = 1, \ldots, m$, are given by (7.49a).

Obviously, (7.52a–d) is an ordinary deterministic parameter optimization problem having the additional auxiliary variables $z_j \in \mathbb{R}^m, j = 1, \ldots, m$. In many important cases the external generalized forces $P = P\big(a(\omega)\big)$ does not depend on the design vector x. In this case $b_1, b_2, \ldots, b_m$ are the fixed columns of the matrix $B = EP\big(a(\omega)\big)P\big(a(\omega)\big)^T$ of 2nd order moments of the random vector of external generalized forces $P = P\big(a(\omega)\big)$, see (7.49a–c).

For the second substitute problem we get this result:

Theorem 7.3 (Minimum Expected Costs (MEC)) *Suppose that $W_{0i}, i=1,\dots,B$, are given fixed weight matrices. Then the minimum expected cost problem (7.43) can be represented by*

$$\min \overline{G}_0(x) + tr(z_1, z_2, \dots, z_m) \tag{7.53a}$$

s.t.

$$\mathring{K}(x)z_j = E\left(\frac{1}{\sigma_y^U(\omega)^2}\right) b_j(x), \quad j = 1,\dots,m \tag{7.53b}$$

$$x \in D. \tag{7.53c}$$

Remark 7.2 According to (7.47b) and (7.46c), the matrix $\tilde{K} = \tilde{K}(x)$ is a simple function of the design vector x.

7.5.1 Symmetric, Non Uniform Yield Stresses

If a representation of

$$U(x) = EU\big(a(\omega), x\big) = EK\big(a(\omega), x\big)^{-1} = \beta(\omega)\mathring{K}(x)^{-1},$$

see (7.29d), (7.30a, b), of the type (7.47d) does not hold, then we may apply the approximative procedure described in the following.

First, the probability distribution $P_{a(\cdot)}$ of the random vector $a = a(\omega)$ of model parameters is approximated, as far it concerns the subvector $a_I = a_I(\omega)$ of $a = a(\omega)$ of model parameters arising in the matrix

$$K = K\big(a(\omega), x\big) = K\big(a_I(\omega), x\big),$$

by a discrete distribution

$$\hat{P}_{a_I(\cdot)} := \sum_{s=1}^{N} \alpha_s \varepsilon_{a_I^{(s)}} \tag{7.54}$$

having realizations $a_I^{(s)}$ taken with probabilities $\alpha_s, s = 1,\dots,N$.

Then, the matrix function $U = U(x)$ can be approximated by

$$\hat{U}(x) := \sum_{s=1}^{N} \alpha_s K^{(s)}(x)^{-1}, \tag{7.55a}$$

where

$$K^{(s)}(x) := K(a_I^{(s)}, x) = CK_0(a_I^{(s)}, x)C^T, \tag{7.55b}$$

see (7.30b). Consequently, the expected cost function $\overline{G_1^*} = \overline{G_1^*}(x)$ is approximated by

$$\begin{aligned}\hat{\overline{G_1^*}}(x) &:= tr\hat{U}(x)EP\big(a(\omega), x\big)P\big(a(\omega), x\big)^T \\ &= \sum_{s=1}^{N} \alpha_s tr K^{(s)}(x)^{-1} EP\big(a(\omega), x\big)P\big(a(\omega), x\big)^T. \end{aligned} \tag{7.56}$$

Corresponding to (7.49d), we define now the auxiliary matrix variables

$$\begin{aligned} z^{(s)} &= (z_1^{(s)}, z_2^{(s)}, \ldots, z_m^{(s)}) := K^{(s)}(x)^{-1}B(x) \\ &= \big(K^{(s)}(x)^{-1}b_1(x), K^{(s)}(x)^{-1}b_2(x) \ldots, K^{(s)}(x)^{-1}b_m(x)\big), \end{aligned} \tag{7.57}$$

where $B = B(x)$ is defined again by (7.49a). Thus, for the columns $z_j^{(s)}, j = 1, \ldots, m$, we obtain the conditions

$$K^{(s)}(x)z_j^{(s)} = b_j(x), j = 1, \ldots, m, \tag{7.58}$$

for each $s = 1, \ldots, N$. According to (7.56) and (7.60), the approximate expected cost function $\hat{\overline{G_1^*}} = \hat{\overline{G_1^*}}$ reads

$$\hat{\overline{G_1^*}}(x) = \sum_{s=1}^{N} \alpha_s tr(z_1^{(s)}, z_2^{(s)}, \ldots, z_m^{(s)}), \tag{7.59}$$

where $z_j^{(s)}, j = 1, \ldots, m, s = 1, \ldots, N$, are given by (7.58).

Because of the close relationship between the representations (7.59) and (7.51) for $\hat{\overline{G_1^*}}, \overline{G_1^*}$, approximate mathematical optimization problems result from (7.59) which are similar to (7.52a–d), (7.53a–c), respectively.

7.5.2 Non Symmetric Yield Stresses

In generalization of (7.44), here we suppose

$$\sigma_{yi}^U(\omega) = \gamma_i^U \sigma_y(\omega), \sigma_{yi}^L(\omega) = \gamma_i^L \sigma_y(\omega), \tag{7.60}$$

where $\sigma_y = \sigma_y(\omega) > 0$ is a nonnegative random variable with a given probability distribution, and $\gamma_i^U > 0, \gamma_i^L < 0, i = 1, \ldots B$, denote given, fixed yield coefficients. However, if (7.60) holds, then

$$\frac{\sigma_{yi}^U \pm \sigma_{yi}^L}{2} = \sigma_y \frac{\gamma_i^U \pm \gamma_i^L}{2} \tag{7.61a}$$

and

$$\frac{\eta_i^U \pm \eta_i^L}{2} = \sigma_y \frac{\tilde{\eta}_i^U(x) \pm \tilde{\eta}_i^L(x)}{2}, \tag{7.61b}$$

where, cf (7.4b, c, g, h),

$$\tilde{\eta}_i^\Lambda(x) := \min\left\{\frac{|\gamma_i^\Lambda| A_i(x)}{N_{i0}}, \frac{\gamma_i^U W_{ipl}(x)}{M_{i0}}\right\}, \Lambda = L, U. \tag{7.61c}$$

Corresponding to (7.45a, b), from (7.8f, g) we obtain

$$F_i^c\Big(a(\omega), x\Big) = \sigma_y(\omega) \mathring{F}_i{}^c(x) \tag{7.62a}$$

$$\varrho_i^c\Big(a(\omega), x\Big) = \sigma_y(\omega)\, \mathring{\varrho}_i(x), \tag{7.62b}$$

where the deterministic functions

$$\mathring{F}_i = \mathring{F}_i{}^c(x), \mathring{\varrho}_i = \mathring{\varrho}_i(x) \tag{7.62c}$$

follow from (7.8f), (7.8g), resp., by inserting formula (7.60) and extracting then the random variable $\sigma(\omega)$. Because of (7.62a–c), the generalized stiffness matrix $K = K(a, x)$ can be represented again in the form (7.47a), hence,

$$K\Big(a(\omega), x\Big) = \sigma_y^2(\omega) \mathring{K}(x), \tag{7.63a}$$

where the deterministic matrix function $\mathring{K} = \mathring{K}(x)$ is defined corresponding to (7.47b). Furthermore, according to (7.21h) and (7.62a), for $F^c\Big(a(\omega), x\Big)$ we have

$$F^c\Big(a(\omega), x\Big) = \sigma(\omega) \mathring{F}^c(x), \tag{7.63b}$$

where

$$\mathring{F}^c(x) := \Big(\mathring{F}_1^c(x)^T, \ldots, \mathring{F}_B^c(x)^T\Big)^T. \tag{7.63c}$$

Thus, due to (7.32b, c), for the vector $F^{c*} = F^{c*}\Big(a(\omega), x\Big)$ defined by (7.32a–c) we find

$$F^{c*}\Big(a(\omega), x\Big) = \sigma(\omega) F^{c**}(x) \tag{7.63d}$$

with

$$F^{c**}(x) := (H^T H)^{-1} H^T \mathring{F}^c(x). \tag{7.63e}$$

Inserting now (7.63a, d) into formula (7.34b), for the (approximate) minimum total costs we finally have, cf. (7.47c), (7.48),

$$\begin{aligned} G_1^{a*}\Big(a(\omega), x\Big) &= \frac{1}{\sigma(\omega)^2} tr\mathring{K}(x)^{-1} \Big(\sigma(\omega) C F^{c**}(x) - P\Big(a(\omega), x\Big)\Big) \\ &\qquad \times \Big(\sigma(\omega) C F^{c**}(x) - P\Big(a(\omega), x\Big)\Big)^T \\ &= \frac{1}{\sigma(\omega)^2} tr\mathring{K}(x)^{-1} P\Big(a(\omega), x\Big) P\Big(a(\omega), x\Big)^T \\ &\quad - \frac{1}{\sigma(\omega)} tr\mathring{K}(x)^{-1} \Big(C F^{c**}(x) P\Big(a(\omega), x\Big)^T \\ &\quad + P\Big(a(\omega), x\Big) F^{c**}(x)^T C^T \Big) + tr\mathring{K}(x)^{-1} C F^{c**}(x) F^{c**}(x)^T C^T. \end{aligned} \tag{7.64}$$

The minimum expected cost function $\overline{G_1^{a*}}(x)$ is then given, cf. (7.48), by

$$\begin{aligned} \overline{G_1^{a*}}(x) &= E\left(\frac{1}{\sigma(\omega)^2}\right) tr\mathring{K}(x)^{-1} B(x) \\ &\quad - E\left(\frac{1}{\sigma(\omega)}\right) tr\mathring{K}(x)^{-1} \Big(C F^{c**}(x) \overline{P}(x)^T + \overline{P}(x) F^{c**}(x)^T C^T \Big) \\ &\quad + tr\mathring{K}(x)^{-1} C F^{c**}(x) F^{c**}(x)^T C^T, \end{aligned} \tag{7.65}$$

where $\overline{P} = \overline{P}(x)$, $B = B(x)$ are again given by (7.49a, b). Obviously, also in the present more general case $\overline{G_1^{a*}}$ can be represented by

$$\overline{G_1^{a*}}(x) = trZ(x), \tag{7.66a}$$

where the matrix $Z = Z(x)$ is given by

$$Z(x) := \mathring{K}(x)^{-1}\left(E\left(\frac{1}{\sigma_y(\omega)^2}\right)B(x) - E\left(\frac{1}{\sigma_y(\omega)}\right)\right.$$
$$\left.\times\left(CF^{c**}(x)\overline{P}(x)^T + \overline{P}(x)F^{c**}(x)^T C^T\right) + CF^{c**}(x)F^{c**}(x)^T C^T\right), \tag{7.66b}$$

see (7.51). Hence, due to the close relationship between (7.49d), (7.51) and (7.66a, b), the deterministic substitute problems stated in Sect. 7.4 can be treated as described in Sect. 7.5.

7.6 Reliability Analysis

For the approximate computation of the probability of survival $p_s = p_s(x)$ in Sect. 4.1 a first method was presented based on certain probability inequalities. In the following subsection we suppose that $x = x_0$ is a fixed design vector, and the vector of yield stresses

$$\sigma_y = \begin{pmatrix} (\sigma_{yi}^L)_{i=1,\dots,B} \\ (\sigma_{yi}^U)_{i=1,\dots,B} \end{pmatrix} = \sigma_{y0}$$

is a given deterministic vector of material strength parameters. Moreover, we assume that the weight matrix W_0, cf. (7.20c), (7.21a–c), is fixed. According to (7.8f, g),(7.21h) and (7.30a, b), the vectors $\tilde{F}^c, \tilde{\varrho}$ and the generalized stiffness matrix $K = K(\sigma_{y0}, x_0)$ are given, fixed quantities. Hence, in this case the cost function

$$G_1^*(a, x) = g_1^*(P) := (C\tilde{F}^c - P)^T K^{-1}(C\tilde{F}^c - P), \tag{7.67}$$

see (7.28g), is a quadratic, strictly convex function of the m-vector P of external generalized forces. Hence, the condition $G_1^*(a, x_0) \le g_1$, see (7.37a, b), or

$$g_1^*(P) \le g_1 \tag{7.68a}$$

describes an ellipsoid in the space $\mathbb{R}^m$ of generalized forces.

In case of normal distributed external generalized forces $P = P(\omega)$, the probability

$$\begin{aligned} p_s(x_0; g_1) &:= P\left(G_1^*\Big(\sigma_{y0}, P(\omega), x_0\Big) \le g_1 \right) \\ &= P\left(g_1^*\Big(P(\omega)\Big) \le g_1 \right) \end{aligned} \tag{7.68b}$$

can be determined approximatively by means of linearization

$$g_1^*(P) = g_1^*(P^*) + \nabla_P g_1^*(P^*)^T (P - P^*) + \cdots \tag{7.69}$$

at a so–called design point P^*, see [38, 46, 50]. Since $g_1^* = g_1^*(P)$ is convex, we have

$$g_1^*(P) \ge g_1^*(P^*) + \nabla_P g_1^*(P^*)^T (P - P^*), \; P \in \mathbb{R}^m, \tag{7.70a}$$

and $p_s(x_0; g_1)$ can be approximated *from above* by

$$\begin{aligned} \tilde{p}_s(x_0; g_1) &:= P\left(g_1^*(P^*) + \nabla_P g_1^*(P^*)^T \Big(P(\omega) - P^*\Big) \le g_1 \right) \\ &= P\Big(\nabla_P g_1^*(P^*)^T P(\omega) \le g_1 - g_1^*(P^*) + \nabla_P g_1^*(P^*)^T P^* \Big) \\ &= \Phi\left(\frac{g_1 - g_1^*(P^*) + \nabla_P g_1^*(P^*)^T (P^* - \overline{P})}{\sqrt{\nabla_P g_1^*(P^*)^T \operatorname{cov}\Big(P(\cdot)\Big) \nabla_P g_1^*(P^*)}} \right), \end{aligned} \tag{7.70b}$$

where Φ denotes the distribution function of the $N(0, 1)$-normal distribution, $\operatorname{cov}\Big(P(\cdot)\Big)$ denotes the covariance matrix of $P = P(\omega)$, and the gradient $\nabla_P g_1^*(P^*)$ reads

$$\nabla_P g_1^*(P^*) = -2K^{-1}(C\tilde{F}^c - P^*). \tag{7.70c}$$

Moreover,

$$\overline{P} := EP(\omega) \tag{7.70d}$$

denotes the mean of the external vector of generalized forces $P = P(\omega)$.

In practice, the following two cases are taken into account [38, 46, 50]:

Case 1: Linearization at $P^* := \overline{P}$
Under the above assumptions in this case we have

$$\tilde{p}_s(x_0; g_1) = \Phi\left(\frac{g_1 - g_1^*(\overline{P})}{\sqrt{\nabla_P g_1^*(\overline{P})^T \operatorname{cov}\Big(P(\cdot)\Big)\nabla_P g_1^*(\overline{P})}}\right) \geq p_s(x_0; g_1). \tag{7.71}$$

Case 2: Linearization at a Boundary Point P^* *of* $[g_1^*(P) \leq g_1]$
Here it is $g_1(P^*) = g_1$ and therefore

$$\tilde{p}_s(x_0; g_1) = \Phi\left(\frac{\nabla_P g_1^*(P^*)^T (P^* - \overline{P})}{\sqrt{\nabla_P g_1^*(P^*)^T \operatorname{cov}\Big(P(\cdot)\Big)\nabla_P g_1^*(P^*)}}\right). \tag{7.72}$$

Because of (7.66a), for each boundary point P^* we have again

$$p_s(x_0; g_1) \leq \tilde{p}_s(x_0; g_1). \tag{7.73}$$

Boundary points P^* of the ellipsoid $\Big[g_1^*(P) \leq g_1\Big]$ can be determined by minimising a linear form $c^T P$ on $\Big[g_1^*(P) \leq g_1\Big]$. Thus, we consider [99] the convex minimisation problem

$$\min_{(C\tilde{F}^c - P)^T K^{-1}(C\tilde{F}^c - P) \leq g_1} c^T P, \tag{7.74}$$

where c is a given, fixed m-vector.
By means of Lagrange techniques we obtain this result:

Theorem 7.4 *For each vector* $c \neq 0$ *the unique minimum point of (7.74) reads*

$$P^* = C\tilde{F}^c - \sqrt{\frac{g_1}{c^T K c}} Kc. \tag{7.75a}$$

The gradient of $g_1^* = g_1^*(P)$ at P^* is then given by

$$\nabla_P g_1^*(P^*) = -2\sqrt{\frac{g_1}{c^T K c}} c. \tag{7.75b}$$

Consequently, for the quotient q arising in formula (7.72) we get

$$q = q(c) := \frac{\sqrt{g_1}\sqrt{c^T K c} - c^T(C\tilde{F}^c - \overline{P})}{\sqrt{c^T \operatorname{cov}\Big(P(\cdot)\Big)c}}. \tag{7.76a}$$

Obviously, this function fulfills the equation

$$q(\lambda c) = q(c),\ \lambda > 0, \tag{7.76b}$$

for each m-vector c such that $c^T \operatorname{cov}\Big(P(\cdot)\Big)c \neq 0$.

Since

$$\tilde{p}_s(x_0; g_1) = \Phi(q(c)) \geq p_s(x_0; g_1),\ c \neq 0, \tag{7.77a}$$

see (7.72), (7.73), the best upper bound $\tilde{p}_s(x_0, g_1)$ can be obtained by solving the minimization problem

$$\min_{c \neq 0} q(c). \tag{7.77b}$$

Because of (7.76b), problem (7.77b) is equivalent to the convex optimization problem

$$\min_{c^T \operatorname{cov}\big(P(\cdot)\big)c=1} \sqrt{g_1}\sqrt{c^T K c} - c^T(C\tilde{F}^C - \overline{P}), \tag{7.78}$$

provided that $\operatorname{cov}(P(\cdot))$ is regular. Representing the covariance matrix of $P = P(\omega)$ by

$$\operatorname{cov}\Big(P(\cdot)\Big) = Q^T Q$$

with a regular matrix Q, problem (7.78) can be represented also by

$$\min_{\|w\|=1} \sqrt{g_1}\sqrt{w^T Q^{-1T} K Q^{-1} w} - w^T Q^{-1T}(C\tilde{F}^c - \overline{P}). \tag{7.79}$$

7.7 Numerical Example: 12-Bar Truss

The new approach for optimal design of elasto-plastic mechanical structures under stochastic uncertainty is illustrated now by means of the 12-bar truss according to Fig. 7.2.

Suppose that $L = 1{,}000$ mm, $E = 7{,}200\,\text{N/mm}^2$ is the elasticity modulus, and the yield stresses with respect to tension and compression are given by

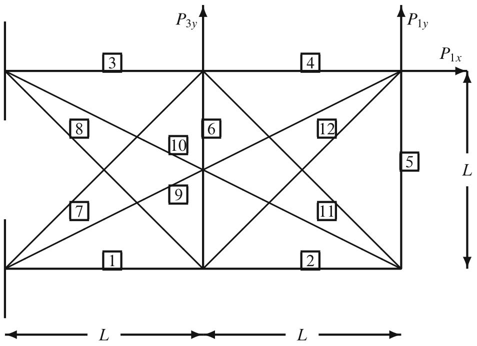

Fig. 7.2 12-Bar truss

$\sigma_y^U = -\sigma_y^L = \sigma_y = 216\,\text{N/mm}^2$. Furthermore, assume that the structure is loaded by the deterministic force components

$$P_{1x} = P_{3y} = 10^5 N$$

and the random force component

$$P_{1y} \cong N(\mu, \sigma^2)$$

having a normal distribution with mean μ and variance σ^2. The standard deviation σ is always 10% of the mean μ.

The numerical results presented in this section have been obtained by Dipl.Math.oec. Simone Zier, Institute for Mathematics and Computer Applications, Federal Armed Forces University, Munich.

The equilibrium matrix C of the 12-bar truss is given by

$$C = \begin{pmatrix} 0 & 0 & 0 & 1.0 & 0 & 0 & 0 & 0 & 0.894427 & 0 & 0 & 0.707107 \\ 0 & 0 & 0 & 0 & 1.0 & 0 & 0 & 0 & 0.447214 & 0 & 0 & 0.707107 \\ 0 & 1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.894427 & 0.707107 & 0 \\ 0 & 0 & 0 & 0 & -1.0 & 0 & 0 & 0 & 0 & -0.447214 & -0.707107 & 0 \\ 0 & 0 & 1.0 & -1.0 & 0 & 0 & 0.707107 & 0 & 0 & 0 & -0.707107 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 & 0.707107 & 0 & 0 & 0 & 0.707107 & 0 \\ 1.0 & -1.0 & 0 & 0 & 0 & 0 & 0 & 0.707107 & 0 & 0 & 0 & -0.707107 \\ 0 & 0 & 0 & 0 & 0 & -1.0 & 0 & -0.707107 & 0 & 0 & 0 & -0.707107 \end{pmatrix}. \tag{7.80}$$

Note that under the above assumptions, condition (2.22b) holds.

Since in the present case of a truss we have $H = I$ ($B \times B$ identity matrix), cf. (2.6f) and (7.21h), the matrix $K_0 = K_0(a, x) = K_0(\sigma_y, x)$, see (2.30b) and (7.46b), is a diagonal matrix represented by

$$K_0(\sigma, x) = \text{diag}\left(\left(\frac{\varrho_i(x)}{w_i^0}\right)^2\right), \tag{7.81a}$$

cf. (7.30d). Here, w_i^0 is the element of the 1×1 weight matrix W_{i0}, and $\varrho_i = \varrho_i(x)$ is defined, cf. (2.9b), by

$$\varrho_i(x) = \frac{F_i^U - F_i^L}{2} = \sigma_y A_i(x), i = 1, \dots, B. \tag{7.81b}$$

Defining

$$\tilde{w}_i = \tilde{w}_i(x) := \left(\frac{\varrho_i(x)}{w_i^0}\right)^2, i = 1, \dots, B, \tag{7.81c}$$

the generalized stiffness matrix $K = K(\sigma_y, x)$, see (2.30a), reads

$$K(\sigma_y, x) = CK_0(\sigma_y, x)C^T$$

$$= \begin{pmatrix}
\tilde{w}_4 + 0.8\tilde{w}_9 + 0.5\tilde{w}_{12} & 0.4\tilde{w}_9 + 0.5\tilde{w}_{12} & 0 & 0 & -\tilde{w}_4 & 0 & -0.5\tilde{w}_{12} & -0.5\tilde{w}_{12} \\
0.4\tilde{w}_9 + 0.5\tilde{w}_{12} & \tilde{w}_5 + 0.2\tilde{w}_9 + 0.5\tilde{w}_{12} & 0 & -\tilde{w}_5 & 0 & 0 & -0.5\tilde{w}_{12} & -0.5\tilde{w}_5 \\
0 & 0 & \tilde{w}_2 + 0.8\tilde{w}_{10} + 0.5\tilde{w}_{11} & 0.4\tilde{w}_{10} - 0.5\tilde{w}_{11} & -0.5\tilde{w}_{11} & 0.5\tilde{w}_{11} & -\tilde{w}_2 & 0 \\
0 & -\tilde{w}_5 & -0.4\tilde{w}_{10} - 0.5\tilde{w}_{11} & \tilde{w}_5 + 0.2\tilde{w}_{10} + 0.5\tilde{w}_{11} & 0.5\tilde{w}_{11} & -0.5\tilde{w}_{11} & 0 & 0 \\
-\tilde{w}_4 & 0 & -0.5\tilde{w}_{11} & 0.5\tilde{w}_{11} & \tilde{w}_3 + \tilde{w}_4 + 0.5\tilde{w}_7 + 0.5\tilde{w}_{11} & 0.5\tilde{w}_7 - 0.5\tilde{w}_{11} & 0 & 0 \\
0 & 0 & 0.5\tilde{w}_{11} & -0.5\tilde{w}_{11} & 0.5\tilde{w}_7 - 0.5\tilde{w}_{11} & \tilde{w}_6 + 0.5\tilde{w}_7 + 0.5\tilde{w}_{11} & 0 & -\tilde{w}_6 \\
-0.5\tilde{w}_{12} & -0.5\tilde{w}_{12} & -\tilde{w}_2 & 0 & 0 & 0 & \tilde{w}_1 + \tilde{w}_2 + 0.5\tilde{w}_8 + 0.5\tilde{w}_{12} & -0.5\tilde{w}_8 + 0.5\tilde{w}_{12} \\
-0.5\tilde{w}_{12} & -0.5\tilde{w}_{12} & 0 & 0 & 0 & -\tilde{w}_6 & -0.5\tilde{w}_8 + 0.5\tilde{w}_{12} & \tilde{w}_6 + 0.5\tilde{w}_8 + 0.5\tilde{w}_{12}
\end{pmatrix}. \tag{7.82}$$

7.7.1 Numerical Results: MEC

In the present case the cost factors γ_{i0} in the primary cost function $G_0(x) = \overline{G_0}(x)$, cf. (4.1a), are defined by

$$\gamma_{i0} := \frac{1}{V_0} = 6,4 \cdot 10^{-4} \left[\frac{1}{mm^3}\right]$$

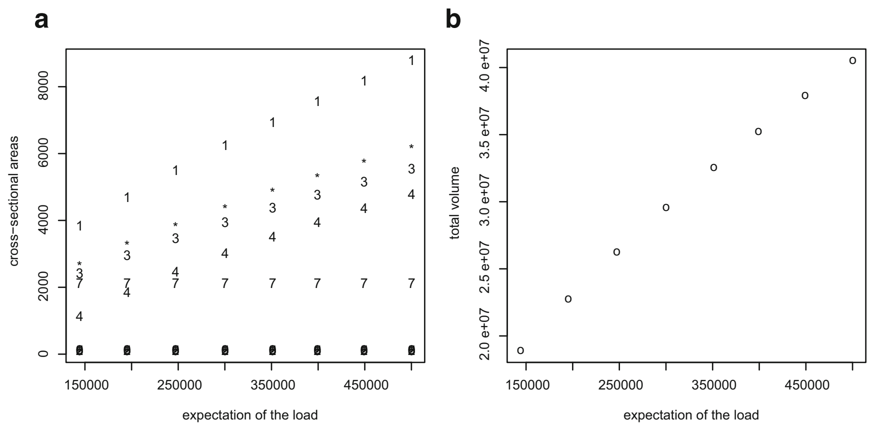

Fig. 7.3 Optimal design using (MEC). (**a**) Optimal cross-sectional areas. (**b**) Total volume

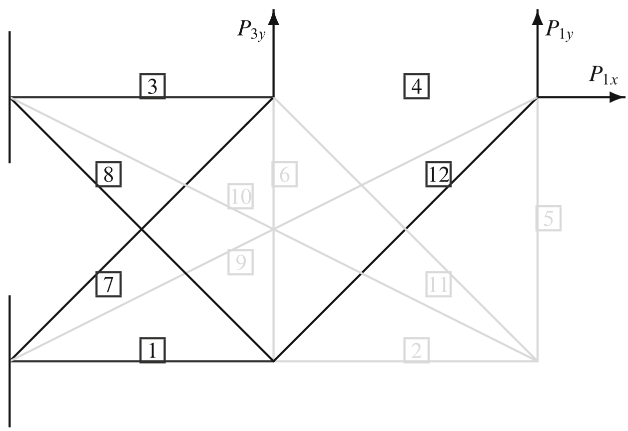

Fig. 7.4 Optimal 6-bar truss using MEC

corresponding to the chosen reference volume $V_0 = 1{,}562.5\,\mathrm{mm}^3$. Thus, the cost function $\overline{G_0}(x)$ and the recourse cost function $\overline{G_1^*}(x)$ are dimensionless, cf. Sect. 4.2. Furthermore, the weight factors in the recourse costs $G_1(x)$ are defined by

$$w_i^0 = 100.$$

In Fig. 7.3a, b the optimal cross-sectional areas $A_i^*, i = 1, \ldots, 12$, and the total volume are shown as functions fo the expectation $\overline{P}_{1y} = EP_{1y}(\omega)$ of the random force component $P_{1y} = P_{1y}(\omega)$. With increasing expected force $\overline{P}_{1y}$, the cross-sectional areas of bar 1,3,4,8,12 are increasing too, while the other remain constant or are near zero. The resulting optimal design of the truss can be seen in Fig. 7.4. Here, bars with cross-sectional areas below $A_{\min} = 100\,\mathrm{mm}^2$ have been deleted.

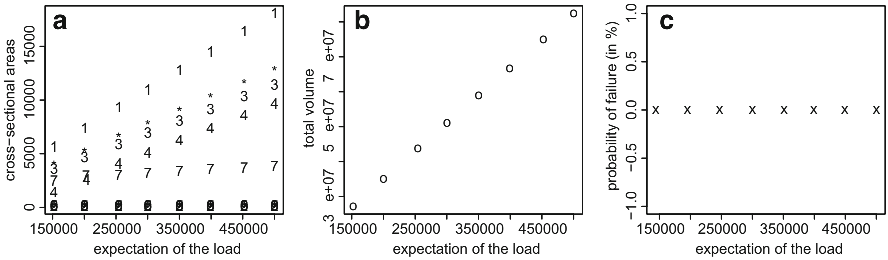

Fig. 7.5 Optimal design using (ECBO). (**a**) Optimal cross-sectional areas. (**b**) Expected minimum volume $\overline{G}_0$. (**c**) Probability of failure

In Figs. 7.3a and 7.5a by the symbol "$*$" the almost equal optimal cross-sectional areas of bar 8 and 12 are marked.

The probability of failure of an (MEC)-optimal truss is always zero showing also the robustness of the optimal 6-bar truss according to Fig. 7.4.

7.7.2 Numerical Results: ECBO

The related numerical results obtained for the expected cost based optimization problem (ECBO) are presented in Fig. 7.5a–c. Here, the optimal cross-sectional areas, the expected minimum volume and the related probability of failure is represented again as functions of the expected form $\overline{P}_{1y}$. The resulting optimal design is the same as in (MEC), where in this case the probability of failure is also zero, which confirms again the robustness of this optimal design.

Chapter 8
Maximum Entropy Techniques

8.1 Uncertainty Functions Based on Decision Problems

8.1.1 Optimal Decisions Based on the Two-Stage Hypothesis Finding (Estimation) and Decision Making Procedure

According to the considerations in the former chapters, in the following we suppose that $v = v(\omega, x)$ denotes the costs or the loss arising in a decision or design problem if the action or design $x \in D$ is taken, and the elementary event $\tilde{\omega} = \omega$ has been realized. Note that $v = v(\omega, x)$ is an abbreviation for $v = v(a(\omega), x)$, where $a(\omega)$ denotes the vector of all random parameters under consideration. As a deterministic substitute for the optimal decision/design problem under stochastic uncertainty

$$\text{minimize } v(\omega, x) \text{ s.t. } x \in D \tag{8.1}$$

we consider, cf. Sect. 1.2, the expectation or mean value minimization problem

$$\text{minimize } v(\lambda, x) \text{ s.t } x \in D, \tag{8.2a}$$

where

$$v(\lambda, x) = v(P_{\tilde{\omega}}, x) := Ev(\tilde{\omega}, x) = \int v(\omega, x)\lambda(d\omega). \tag{8.2b}$$

Here, "E" denotes the expectation operator, and the probability distribution λ on the measurable space $(\Omega, \mathcal{A})$ denotes the true probability distribution $P_{\tilde{\omega}} := \lambda$ of the random element ω. We may assume that λ lies in a certain given set Λ of probability measures on $\mathcal{A}$ (a priori information about λ). In the following we suppose that all integrals, expectations, probabilities, resp., under consideration exist and are finite.

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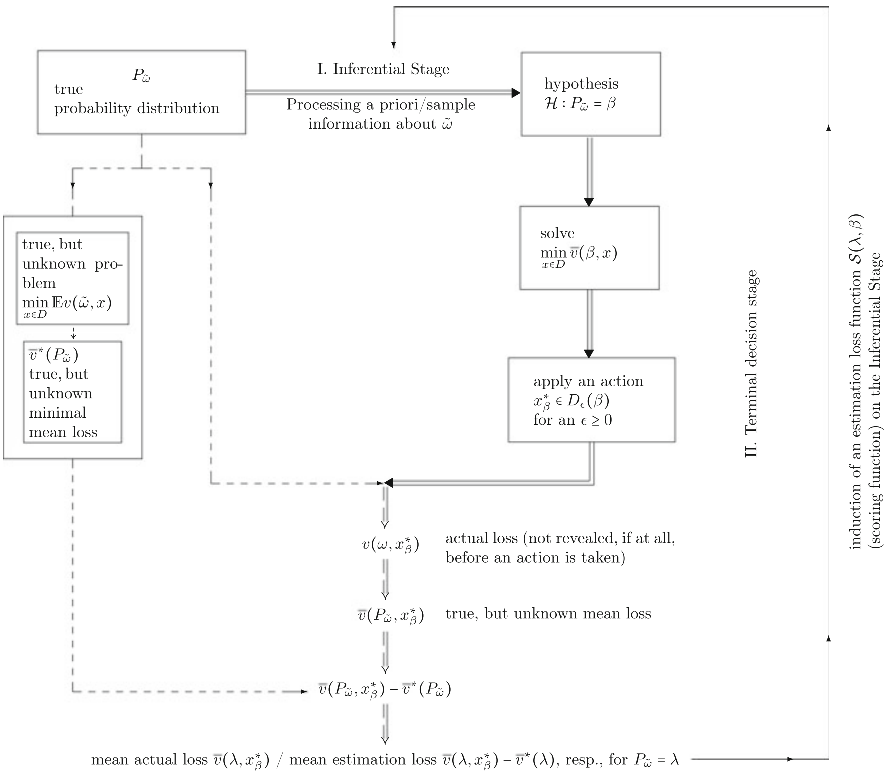

Fig. 8.1 Inference and decision procedure (IDP)

Because in practice also the true probability distribution "$P_{\tilde{\omega}} = \lambda$" of ω is not known in general, one works mostly with the following *two-step Inference and Decision Procedure (IDP)*, according to Fig. 8.1.

Step I: Accept the hypothesis "$P_{\tilde{\omega}} = \beta$". Hence, work with the hypothesis that $\tilde{\omega}$ has the distribution β, where β results from a certain estimation or hypothesis-finding procedure (suitable to (Λ, D, v));

Step II: Instead of (1), solve the approximate optimization problem:

$$\text{minimize } v(\beta, x) \text{ s.t. } x \in D \tag{8.3}$$

and take an ε-optimal decision $x \in D_\varepsilon(\beta)$ with an appropriate bound $\varepsilon > 0$. Here, the set $D_\varepsilon(\beta)$ of ε-optimal decisions is defined by

$$D_\varepsilon(\beta) := \{x \in D : v(\beta, x) \leq v^*(\beta) + \varepsilon\}. \tag{8.4a}$$

Of course,

$$D_0(\beta) := \{x \in D : v(\beta, x) = v^*(\beta)\}. \tag{8.4b}$$

denotes the set of optimal solutions of (8.3).

Note that in (8.4a, b) we use the minimum value of (8.3):

$$v^*(\beta) := inf\{v(\beta, x) : x \in D\}. \tag{8.5}$$

Remark 8.1 Hypothesis-finding in case that there is some a priori information, but **no** sample information $\omega^N = (\omega_1, \ldots, \omega_N)$ of $\tilde{\omega}$:

In the above case a so-called e-estimate β of $P_{\tilde{\omega}}$ can be applied which is defined as follows:

Definition 8.1 Let $e : \Lambda \times \Lambda \to \mathbb{R}$ denote a function on the product $\Lambda \times \Lambda$ of the given set Λ of probability measures—containing the true distribution λ of $\tilde{\omega}$—such that $e(\lambda, \pi)$ can be considered as a measure for the error selecting the distribution π, while $P_{\tilde{\omega}} = \lambda$ is the true distribution. An *e−estimate of* $P_{\tilde{\omega}}$ is then a distribution $\beta \in \Lambda$ such that

$$\sup_{\lambda \in \Lambda} e(\lambda, \beta) = \inf_{\pi \in \Lambda} \left(\sup_{\lambda \in \Lambda} e(\lambda, \pi) \right). \tag{8.6}$$

If $e(\cdot, \cdot)$ is a metric on Λ, then the e-estimate β of $P_{\tilde{\omega}}$ is also called a *"Tchebychev-center"* of Λ.

Though in many cases the approximation, estimation β of λ is computed by standard statistical estimation procedures, the criterion for judging an approximation β of λ should be based on its utility for the decision making process, i.e., one should weight the approximation error according to its influence on decision errors, and the decision errors should be weighted in turn according to the loss caused by an incorrect decision, cf. [113, 162]. A detailed consideration of this concept is given in the following

In order to study first the properties of the above defined 2-step procedure **(I, II)**, resulting from using an estimation/approximation of the unknown or only partly known parameter distribution, we suppose that the set $D_0(\beta)$ of optimal decisions with respect to β, see (8.4b), is non empty. Because $P_{\tilde{\omega}} = \lambda$ is the true distribution, according to the 2-step procedure (I, II) we I), replacing λ by its estimate β, and II) applying then a certain β-optimal decision $x_\beta \in D_0(\beta)$, we have the expected loss $v(\lambda, x_\beta)$. Consequently,

$$H_0(\lambda, \beta) = sup\{v(\lambda, x) : x \in D_0(\beta)\} \tag{8.7}$$

denotes therefore the maximum expected loss if $P_{\tilde{\omega}} = \lambda$ is the true distribution of $\tilde{\omega}$, and the decision maker uses a certain β-optimal decision. Because of $v(\lambda, x) \geq v^*(\lambda), x \in D$, cf. (8.5), we have

$$H_0(\lambda, \beta) \geq v^*(\lambda) \tag{8.8a}$$

and

$$v^*(\lambda) = H_0(\lambda, \lambda) \tag{8.8b}$$

provided that also $D_0(\lambda) \neq \emptyset$. If $D_0(\beta) = \{x^*_\beta\}$, then

$$H_0(\lambda, \beta) = v(\lambda, x^*_\beta). \tag{8.9}$$

In case $D_0(\lambda) \neq \emptyset$, the difference

$$I_0(\lambda, \beta) := H_0(\lambda, \beta) - v^*(\lambda) = H_0(\lambda, \beta) - H_0(\lambda, \lambda), \tag{8.10}$$

is the maximum error relative to the decision making problem (Ω, D, v), if any β-optimal decision is applied, while $P_{\tilde{\omega}} = \lambda$ is the true distribution of $\tilde{\omega}$. Obviously,

$$I_0(\lambda, \beta) \geq 0 \text{ and } I_0(\lambda, \beta) = 0 \text{ if } \beta = \lambda. \tag{8.11}$$

Though, according to the above assumption, problem (8.3) is solvable in principle, due to the complexity of mean value minimization problems, we have to confine in general with a certain ε-optimal solution, hence, with an element of $D_\varepsilon(\beta), \varepsilon > 0$. However, applying any decision x^ε_β of $D_\varepsilon(\beta)$, cf. (8.4a), we have to face the maximum expected loss

$$H_\varepsilon(\lambda, \beta) = sup\{v(\lambda, x) : x \in D_\varepsilon(\beta)\}. \tag{8.12}$$

In order to study the function

$$\varepsilon \to H_\varepsilon(\lambda, \beta), \varepsilon > (=) 0,$$

we introduce still the following notation:

Definition 8.2 Let V denote the **loss set** defined by

$$V := \{v(\cdot, x) : x \in D\}. \tag{8.13a}$$

Moreover, corresponding to $D_\varepsilon(\beta)$, the subset $V_\varepsilon(\beta)$ of V is defined by

$$V_\varepsilon(\beta) := \{v(\cdot, x) : x \in D_\varepsilon(\beta)\}. \tag{8.13b}$$

Based on the loss set V, the functions $H = H_\varepsilon(\lambda, \beta)$ and $v^*(\lambda)$ can be represented also as follows:

$$H_\varepsilon(\lambda, \beta) = sup\{\int v(\omega)\lambda(d\omega) : v \in V_\varepsilon(\beta)\}, \tag{8.13c}$$

$$v^*(\lambda) := inf\{\int v(\omega)\lambda(d\omega) : v \in V\}. \tag{8.13d}$$

Remark 8.2 According to the above assumptions, the loss set V lies in the space $L_1(\Omega, \mathcal{A}, \pi)$ of all π-integrable functions $f = f(\omega$ for each probability distribution $\pi = \lambda, \pi = \beta$ under consideration.

Remark 8.3 Identifying the decision vector $x \in D$ with the related loss function $v = v(\omega, x)$, the set D of decisions can be identified with the related loss set V. Hence, we can consider the loss set V as the *generalized admissible set of decision or design vectors.* On the other hand, the optimal decision problem under stochastic uncertainty can also be described by the set Ω of elementary events and a certain set V of measurable real functions $f = f(\omega)$ on Ω playing the role of loss functions related to the decision vector $x \equiv f(\cdot)$.

We have then the following properties:

Lemma 8.1 *Suppose that $D_\varepsilon(\beta) \neq \emptyset$ for all $\varepsilon > 0$.*

I) $\varepsilon \rightarrow H_\varepsilon(\lambda, \beta)$ *is monotonous increasing on* $(0, +\infty)$;
II) $H_\varepsilon(\lambda, \beta) \geq v^*(\lambda), \varepsilon > 0$;
III) If the loss set V is convex, then $\varepsilon \rightarrow H_\varepsilon(\lambda, \beta)$ is concave;
IV) If $v(\beta, x) \leq v^*(\beta) + \bar{\varepsilon}$ *for all* $x \in D$ *and a fixed* $\bar{\varepsilon} > 0$, *then* $H_\varepsilon(\lambda, \beta) = sup\{v(\lambda, x) : x \in D\}, \varepsilon \geq \bar{\varepsilon}$;
V) The assertions (1)–(4) hold also for $\varepsilon \geq 0$, provided that $D_0(\beta) \neq \emptyset$.

Proof Because of $D_\varepsilon(\beta) \neq \emptyset, \varepsilon > 0$, the maximum expected loss $H_\varepsilon(\lambda, \beta)$ is defined for all $\varepsilon > 0$. I) The monotonicity of $\varepsilon \rightarrow H_\varepsilon(\lambda, \beta)$ follows from $D_\varepsilon(\beta) \subset D_\delta(\beta)$, if $\varepsilon < \delta$. II) The inequality $v(\lambda, x) \geq v^*(\lambda), x \in D$, yields $H_\varepsilon(\lambda, \beta) = \sup\{(\lambda, x) : x \in D_\varepsilon(\beta)\} \geq v^*(\lambda)$. III) Let be $\epsilon_1 > 0, \epsilon_2 > 0, 0 \leq \alpha \leq 1$ and $x_1 \in D_{\varepsilon_1}(\beta), x_2 \in D_{\varepsilon_2}(\beta)$. Because of the convexity of the loss set V, there exists $x_3 \in D$, such that $v(\cdot, x_3) = \alpha v(\cdot, x_1) + (1-\alpha)v(\cdot, x_2)$. This yields then $v(\beta, x_3) = \alpha v(\beta, x_1) + (1-\alpha)v(\beta, x_2) \leq \alpha(v^*(\beta) + \varepsilon_1) + (1-\alpha)(v^*(\beta) + \varepsilon_2) = v^*(\beta) + \bar{\varepsilon}$ with $\bar{\varepsilon} = \alpha\varepsilon_1 + (1-\alpha)\varepsilon_2$. Hence, $x_3 \in D_{\bar{\varepsilon}}(\beta)$ and therefore $H_{\bar{\varepsilon}}(\lambda, \beta) \geq v(\lambda, x_3) = \alpha v(\lambda, x_1) + (1-\alpha)v(\lambda, x_2)$. Since x_1, x_2 were chosen arbitrarily, we get now $H_{\bar{\varepsilon}}(\lambda, \beta) \geq \alpha H_{\varepsilon_1}(\lambda, \beta) + (1-\alpha)H_{\varepsilon_2}(\lambda, \beta)$. The rest of the assertion is clear.

Remark 8.4 According to Lemma 8.1(V) we have $H_\varepsilon(\lambda, \beta) \geq H_0(\lambda, \beta), \varepsilon \geq 0$, provided that $D_0(\beta) \neq \emptyset$.

By the above result the limit "$\lim_{\varepsilon \downarrow 0}$" exists, and we have

$$H(\lambda, \beta) := \lim_{\varepsilon \downarrow 0} H_\varepsilon(\lambda, \beta) = \inf_{\varepsilon > 0} H_\varepsilon(\lambda, \beta). \tag{8.14a}$$

$$H(\lambda, \beta) \geq v^*(\lambda) \tag{8.14b}$$

and

$$H(\lambda, \beta) \geq H_0(\lambda, \beta) \geq v^*(\lambda) \text{ if } D_0(\beta) \neq \emptyset. \tag{8.14c}$$

A detailed study of $H(\lambda, \beta)$ and $I(\lambda, \beta) := H(\lambda, \beta) - v^*(\lambda)$ follows in Sects. 8.1 and 8.2, where we find a close relationship of H, I, resp., with the inaccuracy function of Kerridge [75], the divergence of Kullback [81]. Thus, we use the following notation:

Definition 8.3 The function $H = H(\lambda, \beta)$ is called the *generalized inaccuracy function*, and $I = I(\lambda, \beta) := H(\lambda, \beta) - v^*(\lambda)$ is called the *generalized divergence function*.

8.1.2 Stability/Instability Properties

As shown by the following examples, there are families $(x_\beta^\varepsilon)_{\varepsilon > 0}$ of ε-optimal decisions x_β^ε with respect to β, hence, $x_\beta^\varepsilon \in D_\varepsilon(\beta), \varepsilon > 0$, such that

$$v(\lambda, x_\beta^\varepsilon) \geq H_0(\lambda, \beta) + \delta \text{ for all } 0 < \varepsilon < \varepsilon_0, \tag{8.15}$$

with a fixed constant $\delta > 0$ and for a positive ε_0.

Thus, with a certain distance $\delta > 0$, the expected loss remains—*also for arbitrarily small* accuracy value $\varepsilon > 0$—outside the *error interval* $[v^*(\lambda), H_0(\lambda, \beta)]$, which must be taken into account in any case due to the estimation of (8.2a) by the approximate optimization problem (8.3). However, this indicates a possible *instability* of the 2-step procedure **(I,II)**.

Example 8.1 Let $\Omega := \{\omega_1, \omega_2\}$ with discrete probability distributions $\lambda, \beta \in \mathbb{R}^2_{+,1} := \{\lambda \in \mathbb{R}^2_+ : \lambda_1 + \lambda_2 = 1\}$. Moreover, define the set of decisions, the loss set, resp., by $D \equiv V$, where

$$V = conv\{(1,0)^T, (2,0)^T, (0,2)^T, (0,1)^T\} \backslash conv\{(\frac{1}{2}, \frac{1}{2})^T, (0,1)^T\},$$

where “conv” denotes the convex hull of a set. Selecting $\lambda = (0,1)^T$ and $\beta = (\frac{1}{2}, \frac{1}{2})^T$, we get

$$v^*(\beta) = \frac{1}{2}, H_\varepsilon(\lambda, \beta) = v(\lambda, x_\beta^\varepsilon) = 2(v^*(\beta) + \varepsilon) = 1 + 2\varepsilon$$

with $x_\beta^\varepsilon = (0, 2(v^*(\beta) + \varepsilon))^T = (0, 1 + 2\varepsilon)^T \in D_\varepsilon(\beta)$ for $0 < \varepsilon < \frac{1}{2}$. On the other hand, $D_0(\beta) = conv\{(\frac{1}{2}, \frac{1}{2})^T, (1,0)^T\} \setminus \{(\frac{1}{2}, \frac{1}{2})^T\}$, and therefore $H_0(\lambda, \beta) = \frac{1}{2}$. Hence, (8.15) holds, i.e., $H_\varepsilon(\lambda, \beta) = (\lambda, x_\beta^\varepsilon) = 1 + 2\varepsilon > 1 = H_0(\lambda, \beta) + \delta, \varepsilon > 0$, with $\delta = \frac{1}{2}$.

Remark 8.5 As it turns out later, the instability (8.15) follows from the fact that V is not closed.

Example 8.2 Let $\Omega = \{\omega_1, \omega_2\}$, and suppose that $D \equiv V$ is given by $V = \{(0,0)^T\} \cup \{z \in \mathbb{R}_+^2 : z_1 z_2 \geq 1\}$. Moreover, $\beta = (1,0)^T$ and $\lambda \in \mathbb{R}_{+,1}^2$ with $\lambda_2 > 0$. Then, $v^*(\beta) = 0$ and $D_0(\beta) = \{(0,0)^T\}$. Furthermore, $H_\varepsilon(\lambda, \beta) = +\infty$, and $x_\beta^\varepsilon = (\varepsilon, \frac{1}{\varepsilon})^T \in D_\varepsilon(\beta)$, where $v(\lambda, x_\beta^\varepsilon) = \lambda_1 \varepsilon + \lambda_2 \frac{1}{\varepsilon}, \varepsilon > 0$. Thus, $H_0(\lambda, \beta) = 0$, and also (8.15) holds:

$$v(\lambda, x_\beta^\varepsilon) = \lambda_1 \varepsilon + \lambda_2 \frac{1}{\varepsilon} > H_0(\lambda, \beta) + \delta = \delta$$

for all $0 < \varepsilon < \frac{\lambda_2}{\delta}$ and each (fixed) $\delta > 0$.

Remark 8.6 Here, V is closed, but it is not convex, which is the reason for the instability (8.15) in the present case.

A necessary and sufficient condition excluding the *instability* (8.15) of the two-step procedure (I, II) procedure:

$$(8.1) \equiv (8.2a) \rightsquigarrow (8.3) \rightsquigarrow \text{select an } x_\beta^\varepsilon \in D_\varepsilon(\beta)$$

is given in the following result.

Lemma 8.2 *The* instability *(8.15) of the two-step procedure (I, II) is excluded if and only if $H_\varepsilon(\lambda, \beta) \downarrow H_0(\lambda, \beta)$, $\varepsilon \downarrow 0$, hence, $H(\lambda, \beta) = H_0(\lambda, \beta)$.*

Proof I) Suppose that $H(\lambda, \beta) = H_0(\lambda, \beta)$. Assuming that (8.15) holds, then $H_0(\lambda, \beta) < +\infty$ and therefore $H(\lambda, \beta) \in \mathbb{R}$ as well as $H_0(\lambda, \beta) < H_0(\lambda, \beta) + \delta \leq v(\lambda, x_\beta^\varepsilon) \leq H_\varepsilon(\lambda, \beta), \varepsilon > 0$. However, this is a contradiction to $H_\varepsilon(\lambda, \beta) \downarrow H_0(\lambda, \beta)$ for $\varepsilon \downarrow 0$. Consequently, (8.15) is excluded in this case.

II) Suppose now that the instability (8.15) is excluded. Assuming that $H(\lambda, \beta) > H_0(\lambda, \beta)$, then $H_0(\lambda, \beta) \in \mathbb{R}$, and there is $c \in \mathbb{R}$, such that $H(\lambda, \beta) > c > H_0(\lambda, \beta)$. Because of $H_\varepsilon(\lambda, \beta) \geq H(\lambda, \beta) > c, \varepsilon > 0$, to each $\varepsilon > 0$ there exists an $x_\beta^\varepsilon \in D_\varepsilon(\beta)$ such that $v(\lambda, x_\beta^\varepsilon) > c$. Hence, (8.15) holds with

$\delta := c - H_0(\lambda, \beta)$, which is in contradiction to the assumption. Consequently, we have $H(\lambda, \beta) = H_0(\lambda, \beta)$.

In the following we give now sufficient conditions for the stability condition $H(\lambda, \beta) = H_0(\lambda, \beta)$ or $H_\varepsilon(\lambda, \beta) \downarrow H_0(\lambda, \beta)$ for $\varepsilon \downarrow 0$.

Theorem 8.1 *I) Let $D_\varepsilon(\beta) \neq \emptyset, \varepsilon > 0$, and suppose that there is $\bar{\varepsilon} > 0$ such that $D_{\bar{\varepsilon}}(\beta)$ is compact und $x \to v(\lambda, x), x \to v(\beta, x), x \in D_{\bar{\varepsilon}}(\beta)$ are real valued, continuous functions on $D_{\bar{\varepsilon}}(\beta)$. Then, $v^*(\beta) \in \mathbb{R}, D_0(\beta) \neq \emptyset$, and $H(\lambda, \beta) = H_0(\lambda, \beta)$.*

II) Replacing $x \to v(\lambda, x), x \in D_{\bar{\varepsilon}}(\beta)$, by an arbitrary continuous function $F : D_{\bar{\varepsilon}}(\beta) \to \mathbb{R}$ and assuming that the remaining assumptions of the first part are unchanged, then $sup\{F(x) : x \in D_\varepsilon(\beta)\} \downarrow sup\{F(x) : x \in D_0(\beta)\}$ for $\varepsilon \downarrow 0$.

Proof Obviously, with $F(x) := v(\lambda, x)$ the first assertion follows from the second one. Thus, we have to prove only the second part of Theorem 8.1. We therefore set $F_\varepsilon := sup\{F(x) : x \in D_\varepsilon(\beta)\}$ for $\varepsilon \geq 0$. Corresponding to Lemma 8.1, one can prove that $F_{\varepsilon_1} \leq F_{\varepsilon_2}$, provided that $\varepsilon_1 < \varepsilon_2$. In the first part of the proof we show that $D_0(\beta) \neq \emptyset$, hence, the expression $H_0(\lambda, \beta)$ is defined therefore. Since $D_{\bar{\varepsilon}}(\beta) \neq \emptyset$ and $v(\beta, x) \in \mathbb{R}$ for all $x \in D_{\bar{\varepsilon}}(\beta)$, we get $v^*(\beta) \geq -\bar{\varepsilon} + v(\beta, x) > -\infty$ with some $x \in D_{\bar{\varepsilon}}(\beta)$. Thus, $v^*(\beta) \in \mathbb{R}$. Assuming that $D_0(\beta) = \emptyset$, we would have $\cap_{0<\varepsilon\leq\bar{\varepsilon}} D_\varepsilon(\beta) = D_0(\beta) = \emptyset$. However, according to our assumptions, the set $D_\varepsilon(\beta) = \{x \in D_{\bar{\varepsilon}}(\beta) : v(\beta, x) \leq v^*(\beta) \leq v^*(\beta) + \varepsilon\}, 0 < \varepsilon \leq \bar{\varepsilon}$ is closed for each $0 < \varepsilon \leq \bar{\varepsilon}$. Due to the compactness of $D_{\bar{\varepsilon}}(\beta)$, this yields then $\cap_{i=1}^n D_{\varepsilon_i}(\beta) = \emptyset$ for a finite number of $0 < \varepsilon_i \leq \bar{\varepsilon}, i = 1, 2, \ldots, n$. Defining $\varepsilon_0 = \min_{1\leq i\leq n} \varepsilon_i$, then $\varepsilon_0 > 0$ and $D_{\varepsilon_0}(\beta) = \cap_{i=1}^n D_{\varepsilon_i}(\beta) = \emptyset$, which contradicts to $D_\varepsilon \neq \emptyset, \varepsilon > 0$. Thus, we must have $D_0(\beta) \neq \emptyset$. Since the sets $D_\varepsilon(\beta), 0 \leq \varepsilon \leq \bar{\varepsilon}$ are closed and therefore also compact, we have

$$F_\varepsilon = sup\{F(x) : x \in D_\varepsilon(\beta)\} = max\{F(x) : x \in D_\varepsilon(\beta)\}, 0 \leq \varepsilon \leq \bar{\varepsilon}, \quad (8.16)$$

where, because of the continuity of F, the maximum is taken. In the second part we show that $F_\varepsilon \downarrow F_0$ for $\varepsilon \downarrow 0$. Due to the monotonicity of $\varepsilon \to F_\varepsilon$ and $F_\varepsilon \geq F_0, \varepsilon > 0$, we have $\lim_{\varepsilon\downarrow 0} F_\varepsilon \geq F_0$. Moreover, with (8.16), for $0 \leq \varepsilon \leq \bar{\varepsilon}$ and each $c \in R$ it holds

$$\triangle_c := \{\varepsilon : 0 \leq \varepsilon \leq \bar{\varepsilon}, F_\varepsilon \geq c\} = \{\varepsilon : 0 \leq \varepsilon \leq \bar{\varepsilon}, max\{F(x) : x \in D_\varepsilon(\beta)\} \geq c\} \quad (8.17a)$$

and therefore

$$\triangle_c = \{\varepsilon : 0 \leq \varepsilon \leq \bar{\varepsilon}, \text{ there is } x \in D_{\bar{\varepsilon}}(\beta) \text{ with } v(\beta, x) \leq v^*(\beta) + \varepsilon \text{ and } F(x) \geq c\}. \quad (8.17b)$$

If $\varepsilon^k \to \varepsilon^0, k \to \infty$ is a convergent sequence in $\triangle_c$ for a fixed $c \in \mathbb{R}$,, then, according to (8.17a, b), there are elements $x^k \in D_{\bar{\varepsilon}}(\beta)$, such that

$$v(\beta, x^k) \leq v^*(\beta) + \varepsilon_k \text{ and } F(x^k) \geq c, k = 1, 1, \ldots. \tag{8.18}$$

Since $D_{\bar{\varepsilon}}(\beta)$ is compact, sequence (x^k) has an accumulation point $x^0 \in D_{\bar{\varepsilon}}(\beta)$, and the continuity of $x \to F(x)$ and $x \to v(\beta, x)$ on $D_{\bar{\varepsilon}}(\beta)$ yield the existence of a subsequence (x^{k_j}) of (x^k), such that $F(X^{k_j}) \to F(x^0)$ and $v(\beta, x^{k_j}) \to v(\beta, x^0), j \to \infty$. From 8.18 we get then $v(\beta, x^0) \leq v^*(\beta) + \varepsilon^0$ and $F(x^0) \geq c$. Hence, $F_{\varepsilon^0} \geq c$ and therefore $\varepsilon^0 \in \triangle_c$,, because we have $0 \leq \varepsilon \leq \bar{\varepsilon}$, since $0 \leq \varepsilon^{k_j} \leq \bar{\varepsilon}$. The above considerations yield that $\triangle_c$ is closed for all $c \in R$. Assuming that there is $\tilde{c} \in R$, such that $\lim\limits_{\varepsilon \downarrow 0} F_\varepsilon \geq \tilde{c} > F_0$, we have $F_\varepsilon \geq \tilde{c} > F_0$ for all $\varepsilon > 0$. However, this yields then $\triangle_{\tilde{c}} = \{0 \leq \varepsilon \leq \bar{\varepsilon} : F_\varepsilon \geq \tilde{c} :\} = (0, \bar{\varepsilon}]$, which contradicts to the closedness of $\triangle_c$ for each $c \in R$ shown above. Thus, $\lim\limits_{\varepsilon \downarrow 0} F_\varepsilon = F_0$.

Remark 8.7 According to (8.1–8.3), the second part of the above Theorem 8.1 is used mainly for $F(x) = -v(\lambda, x)$. In this case we have:

$$inf\{v(\lambda, x) : x \in D_\varepsilon(\beta)\} \uparrow inf\{v(\lambda, x) : x \in D_0(\beta)\} \text{ for } \varepsilon \downarrow 0. \tag{8.19}$$

Obviously, the above result can be formulated also with the loss set V in the following way:

Corollary 8.1 *I) Let $V_\varepsilon(\beta) \neq \emptyset, \varepsilon > 0$ and suppose that there is $\bar{\varepsilon} > 0$ such that $V_{\bar{\varepsilon}}(\beta)$ is compact and $f \to \lambda f, f \to \beta f, f \in V_{\bar{\varepsilon}}(\beta)$ are real valued and continuous functions on $V_{\bar{\varepsilon}}(\beta)$. Then $v^*(\beta) \in R, V_0(\beta) \neq \emptyset$, and $H(\lambda, \beta) = H_0(\lambda, \beta)$;*

II) Replacing $f \to \lambda f, f \in V_{\bar{\varepsilon}}(\beta)$ by an arbitrary continuous function $F : V_{\bar{\varepsilon}}(\beta) \to R$, and keeping the remaining assumptions in (I), then $sup\{F(f) : f \in V_\varepsilon(\beta)\} \downarrow sup\{F(f) : f \in V_0(\beta)\}$ for $\varepsilon \downarrow 0$.

Proof The assertion follows immediately from $V_\varepsilon(\beta) = \{v(., x) : x \in D_\varepsilon(\beta)\}$.

8.2 The Generalized Inaccuracy Function $H(\lambda, \beta)$

Let denote $P_{\tilde{\omega}} = \lambda$ the true distribution of $\tilde{\omega}$, and suppose that the hypothesis "$P_{\tilde{\omega}} = \beta$" has been accepted. Moreover, assume that $D_\varepsilon(\beta) \neq \emptyset$ for all $\varepsilon > 0$; This holds if and only if $v^*(\beta) > -\infty$ or $v^*(\beta) = -\infty$ and then $v(\beta, x) = -\infty$ for an $x \in D$. Using a decision $x \in D_\varepsilon(\beta)$, then we have a loss from $\{v(\lambda, x) : x \in D_\varepsilon(\beta)\}$, and

$$H_\varepsilon(\lambda, \beta) = \sup\{v(\lambda, x) : x \in D_\varepsilon(\beta)\},$$

$$h_\varepsilon(\lambda, \beta) = \inf\{v(\lambda, x) : x \in D_\varepsilon(\beta)\}$$

denotes the maximum, minimum, resp., expected loss, if the computation of an ε-optimal decision is based on the hypothesis "$P_{\tilde{\omega}} = \beta$", while $P_{\tilde{\omega}} = \lambda$ is the true probability distribution of $\tilde{\omega}$. Corresponding to Lemma 8.1 on $H_\varepsilon(\lambda, \beta)$, we can show this result:

Lemma 8.3 *Suppose that* $D_\varepsilon(\beta) \neq \emptyset,\ \varepsilon > 0$. *Then,*

I) $\varepsilon \to h_\varepsilon(\lambda, \beta)$, *with* $h_\varepsilon(\lambda, \beta) = \inf\{v(\lambda, x) : x \in D_\varepsilon(\beta)\}$, *is monotonous decreasing on* $(0, +\infty)$*;*

II) $h_\varepsilon(\lambda, \beta) \geq v^*(\lambda)$ *for all* $\varepsilon > 0$*;*

III) $\varepsilon \to h_\varepsilon(\lambda, \beta), \varepsilon > 0$ *is convex, provided that the loss set* V *is convex;*

IV) If $v(\beta, x) \leq v^*(\beta) + \bar{\varepsilon}$ *for all* $x \in D$ *and a fixed* $\bar{\varepsilon} > 0$, *then* $h_\varepsilon(\lambda, \beta) = v^*(\lambda),\ \varepsilon > \bar{\varepsilon}$*;*

v) The assertions (a)–(c) hold also for $\varepsilon \geq 0$, *in case that* $D_o(\beta) \neq \emptyset$.

Lemmas 8.1 and 8.3 yield then this corollary:

Corollary 8.2 *For two numbers* $\varepsilon_1, \varepsilon_2 > 0$ *(*≥ 0*, if* $D_o(\beta) \neq \emptyset, resp.$*) we have*

$$h_{\varepsilon_1}(\lambda, \beta) \leq H_{\varepsilon_2}(\lambda, \beta). \tag{8.20}$$

Proof If $\varepsilon_1 \leq \varepsilon_2$, then $H_{\varepsilon_2}(\lambda, \beta) \geq H_{\varepsilon_1}(\lambda, \beta) \geq h_{\varepsilon_1}(\lambda, \beta)$, and in case $\varepsilon_1 > \varepsilon_2$ it is $h_{\varepsilon_1}(\lambda, \beta) \leq h_{\varepsilon_2}(\lambda, \beta) \leq H_{\varepsilon_2}(\lambda, \beta)$ according to (a) of Lemmas 8.1 and 8.3.

Hence, the limit $\lim\limits_{\varepsilon \downarrow 0} h_\varepsilon(\lambda, \beta) = \sup\limits_{\varepsilon > 0} h_\varepsilon(\lambda, \beta)$, exists, and corresponding to (8.14a) we define

$$h(\lambda, \beta) = \lim_{\varepsilon \downarrow 0} h_\varepsilon(\lambda, \beta) = \sup_{\varepsilon > 0} h_\varepsilon(\lambda, \beta). \tag{8.21}$$

For the functions $h(\lambda, \beta), H(\lambda, \beta)$ defined by (8.14a) and (8.21) the following result holds:

Theorem 8.2 *I) Let* $D_\varepsilon(\beta) \neq \emptyset$ *for* $\varepsilon > 0$. *Then*

$$v^*(\lambda) \leq h(\lambda, \beta) \leq H(\lambda, \beta);$$

II) Let $D_\varepsilon(\lambda) \neq \emptyset$ *for* $\varepsilon > 0$. *Then*

$$v^*(\lambda) = h(\lambda, \lambda) = H(\lambda, \lambda).$$

Proof I) Lemma 8.3 yields $h(\lambda, \beta) = \sup\limits_{\varepsilon > 0} h_\varepsilon(\lambda, \beta) \geq v^*(\lambda)$. Because of (8.20) we have $h_\varepsilon(\lambda, \beta) \leq H_{\bar{\varepsilon}}(\lambda, \beta), \varepsilon > 0$ for each fixed $\bar{\varepsilon} > 0$. From this we obtain $h(\lambda, \beta) = \sup\limits_{\varepsilon > 0} h_\varepsilon(\lambda, \beta) \leq H_{\bar{\varepsilon}}(\lambda, \beta), \bar{\varepsilon} > 0$, hence, $h(\lambda, \beta) \leq \inf\limits_{\varepsilon > 0} H_\varepsilon(\lambda, \beta) = H(\lambda, \beta)$.

II) According to Theorem 8.2.1 and the Definition of $H(\lambda, \lambda)$, we get $v^*(\lambda) \leq H(\lambda, \lambda) \leq H_\varepsilon(\lambda, \lambda) = \sup\{v(\lambda, x) :\ x \in D_\varepsilon(\lambda)\} = \sup\{v(\lambda, x) :\ v(\lambda, x) \leq$

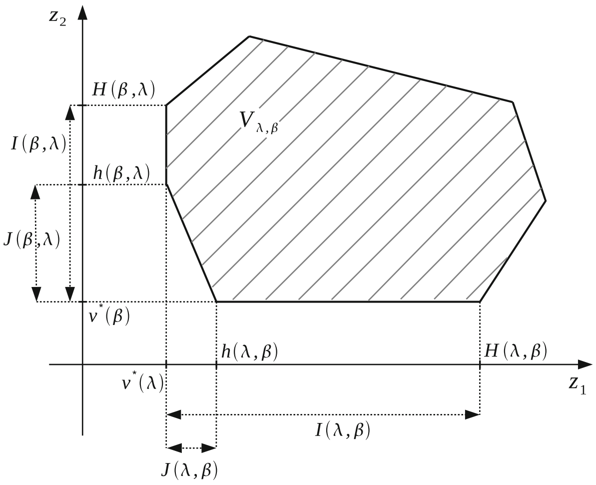

Fig. 8.2 The H, h- and I, J-functions

$v^*(\lambda) + \varepsilon\} \le v^*(\lambda) + \varepsilon$ for each $\varepsilon > 0$. Thus, $H(\lambda, \lambda) = v^*(\lambda)$, and due to the first part we also have $v^*(\lambda) = h(\lambda, \lambda)$.

For the geometrical interpretation of the values $h(\lambda, \beta)$, $H(\lambda, \beta)$ consider now the transformed loss set

$$V_{\lambda,\beta} = \{(\lambda f, \beta f)^T : f \in V\} \quad (\subset \mathbb{R}^2) . \tag{8.22}$$

If the loss set V is convex, then the linear transformation $V_{\lambda,\beta} = T_{\lambda,\beta}(V)$ of V with respect to $T_{\lambda,\beta} : f \to (\lambda f, \beta f)^T$ is again a convex set. Hence, $v^0(\beta) = \inf\{z_2 : z \in V_{\lambda,\beta}\}$, which means that $v^*(\beta)$ can be interpreted as the second coordinate of one of the deepest points of $V_{\lambda,\beta}$. In the same way, $v^*(\lambda)$ is the first coordinate one of the points lying on the left boundary of $V_{\lambda,\beta}$.

According to Fig. 8.2, the values $h(\lambda, \beta)$, $H(\lambda, \beta)$ and also the divergences $I(\lambda, \beta) := H(\lambda, \beta) - H(\lambda, \lambda)$, $J(\lambda, \beta) := h(\lambda, \beta) - h(\lambda, \lambda)$ can be interpreted corresponding to $V_{\lambda,\beta}$ in this way:

$h(\lambda, \beta) :=$ first coordinate of the deepest point of $V_{\lambda,\beta}$ being most left

$H(\lambda, \beta) =$ first coordinate of the deepest point of $V_{\lambda,\beta}$ being most right.

The remaining H, h- and I, J-functions can be interpreted in $V_{\lambda,\beta}$ in the same way.

8.2.1 Special Loss Sets V

In the following we give a justification for the notation "*generalized inaccuracy function*" for $H(\lambda, \beta)$ and $h(\lambda, \beta)$. For this aim, assume next to that $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ contains a finite number of realizations or scenarios. Moreover, suppose that $f : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ is a convex, monotonous decreasing function such that $f(t) \in \mathbb{R}$ for $t > 0$ and $f(0) = \lim_{t \to 0} f(t) = \sup_{t>0} f(t)$. Putting $f(\alpha) \equiv (f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_n))^T$ and $ri\mathbb{R}^n_{+,1} = \{\alpha \in \mathbb{R}^n_{+,1} : \alpha_k > 0, \; k = 1, 2, \ldots, n\}$, let then the loss set

$$V \; := \; C_f,$$

be defined by

$$C_f \; = \; clconv\{f(\alpha) : \; \alpha \in ri\mathbb{R}^n_{+,1}\};. \tag{8.23}$$

Here, "*clconv*" denotes the closed, convex hull of a set. We still put

$$v^*_f(\beta) \; = \; \inf\{\beta^T z : \; z \in C_f\}, \; \beta \in \mathbb{R}^n_{+,1}.$$

Some properties of C_f are stated in the following:

Lemma 8.4 *I) C_f is closed and convex;*

II) From $z \in C_f$ we also have $(z_{\tau(1)}, \ldots, z_{\tau(n)})^T \in C_f$ for each permutation τ of the index set $\{1, 2, \ldots, n\}$;

III) If $f(t) \geq 0, \; 0 \leq t \leq 1$, then $C_f \subset \mathbb{R}^n_+, \; 0 \leq v^*_f(\beta) < +\infty$, and for $\beta_k > 0$ we get $0 \leq z_k \leq (\frac{1/\beta}{k})(v^*_f(\beta) + \varepsilon), \; z \in V_\varepsilon(\beta)$;

IV) If $f(0) \in \mathbb{R}$, then C_f is a compact, convex subset of $\mathbb{R}^n$, it also holds $C_f = conv\{f(\alpha) : \; \alpha \in \mathbb{R}^n_{+,1}\}$, and to each $z \in C_f$ there is an $\alpha \in \mathbb{R}^n_{+,1}$ with $z \geq f(\alpha)$;

V) If $f(0) = +\infty$, then for each $z \in C_f$ there exists an $\alpha \in ri\mathbb{R}^n_{+,1}$ such that $z \geq f(\alpha)$.

Proof I) The first part follows from the definition of C_f.

II) Each $z \in C_f$ has the representation $z = \lim_{\nu \to \infty} z^\nu$ with $z^\nu = \sum_{i=1}^{n} \nu^{\nu i}_\gamma \cdot f(\alpha^{\nu i}), \; \alpha^{\nu i} \in riR^n_{+,1}, \; \gamma^{\nu i} \geq 0, \; i = 1, 2, \ldots, n_\nu, \sum_{i=1}^{n} \nu^{\nu i}_\gamma = 1$. Consequently, with a permutation τ of $1, 2, \ldots, n$, also $(x_{\tau(1)}, \ldots, x_{\tau(n)})^T$ has a representation of this type. Thus, $(x_{\tau(1)}, \ldots, x_{\tau(n)})^T \in C_f$.

III) From $f(t) \geq, \; 0 \leq t \leq 1$ and $0 < \alpha_k < 1$ for $\alpha \in ri\mathbb{R}^n_{+,1}$, we get $f(\alpha) \in \mathbb{R}^n_+$ for $\alpha \in ri\mathbb{R}^n_{+,1}$ and therefore $C_f \subset \mathbb{R}^n_+$. Hence, $v^*_f(\beta) = \inf\{\beta^T z : z \in C_f\} \geq 0$ and $z \geq 0$. Because of $(f(1/n), \ldots, f(1/n))^T \in C_f$ and

$f(1/n) \in \mathbb{R}$, we find $v_f^*(\beta) \leq \sum_{k=1}^{n} \beta_k f(1/n) < +\infty$. In addition, because of $z_k \geq 0,\ k = 1, 2, \ldots, n$ for $z \in C_f$ and with $\beta \geq 0$ we get $z_k \beta_k \leq \beta^T z \leq v^*(\beta) + \varepsilon$ for each $z \in V_\varepsilon(\beta)$, hence, $0 \leq z_k \leq (1/\beta_k)(v_f^*(\beta) + \varepsilon)$, provided that $\beta_k > 0$.

IV) Since $\{f(\alpha) : \alpha \in ri\mathbb{R}^n_{+,1}\} \subset \{f(\alpha) : \alpha \in \mathbb{R}^n_{+,1}\}$ and $\alpha \to f(\alpha) = (f(\alpha_1), \ldots, f(\alpha_n))^T,\ \alpha \geq 0$ is a continuous mapping for real $f(0)$, we find that $\{f(\alpha) : \alpha \in ri\mathbb{R}^n_{+,1}\}$ is bounded as a subset of the compact set $\{f(\alpha) : \alpha \in ri\mathbb{R}^n_{+,1}\}$. Due to [137], Theorem 17.2 we obtain then $C_f = clconv\{f(\alpha) : \alpha \in ri\mathbb{R}^n_{+,1}\} = conv(cl\{f(\alpha) : \alpha \in ri\mathbb{R}^n_{+,1}\}) = conv\{f(\alpha); \alpha \in \mathbb{R}^n_{+,1}\}$; indeed, if $f(\alpha^\nu) \to z,\ \nu \to \infty$ with $\alpha^\nu \in ri\mathbb{R}^n_{+,1}$, then, due to the compactness of $\mathbb{R}^n_{+,1}$ we have a subsequence (α^{ν_j}) of (α^ν), such that $\alpha^{\nu_j} \to \alpha \in \mathbb{R}^n_{+,1},\ j \to \infty$. Because of $f(\alpha^{\nu_j}) \to z,\ j \to \infty$ and the continuity of f, we get then $f(\alpha) = z$, hence, $z \in \{f(\alpha) : \alpha \in \mathbb{R}^n_{+,1}\}$ and therefore, as asserted, $cl\{f(\alpha) : \alpha \in ri\mathbb{R}^n_{+,1}\} = \{f(\alpha) : \alpha \in \mathbb{R}^n_{+,1}\}$. Being the convex hull of a compact set, C_f is also a compact set. For $z \in C_f$ we have $z = \sum_{i=1}^{\nu} \gamma^i f(\alpha^i), \gamma^i \geq 0,\ \alpha^i \in \mathbb{R}^n_{+,1},\ i = 1, 2, \ldots,$ and $\sum_{i=1}^{\nu} \gamma^i = 1$. Thus, $z_k = \sum_{i=1}^{\nu} \gamma^i f(\alpha^i_k) \geq f(\sum_{i=1}^{\nu} \gamma^i \alpha^i_k) = f(\alpha_k)$ with $\alpha = \sum_{i=1}^{\nu} \gamma^i \alpha^i \in \mathbb{R}^n_{+,1}$. Hence, we have therefore found an $\alpha \in \mathbb{R}^n_{+,1}$ such that $z \geq f(\alpha)$.

V) Due to the representation of an element $z \in C_f$ stated in part (II), we have $z_k = \lim_{\nu \to \infty} z^\nu_k$, where $z^\nu_k = \sum_{i=1}^{n_\nu} \gamma^{\nu i} f(\alpha^{\nu i}_k)$ with $\alpha^{\nu i} \in ri\mathbb{R}^n_{+,1}$ and $\gamma^{\nu i} \geq 0,\ \sum_{i=1}^{n_\nu} \gamma^{\nu i} = 1$. As above, for each $\nu = 1, 2, \ldots$ we have the relation $z^\nu \geq f(\alpha^\nu)$, with $\alpha^\nu = \sum_{i=1}^{n} \gamma^{\nu i} \alpha^{\nu i}$. However, the sequence (α^ν) has an accumulation point α in $\mathbb{R}^n_{+,1}$; We show now that $\alpha \in ri\mathbb{R}^n_{+,1}$. Assuming that $\alpha_k = 0$ for an index $1 \leq k \leq n$, with a sequence $\alpha^{\nu_j} \to \alpha, j \to \infty$ we get the relation $\alpha^{\nu_j}_k \to \alpha_k = 0$ and therefore $f(\alpha^{\nu_j}_k) \to$ f(0)= $+\infty,\ j \to \infty$. However, this is not possible, since $z^{\nu_j}_k \geq f(\alpha^{\nu_j}_k)$ and (z^ν) is a convergent sequence. Thus, $\alpha_k > 0$. Furthermore, from $z^{\nu_j}_k \geq f(\alpha^{\nu_j}_k),\ j = 1, 2, \ldots,\ z^{\nu_j}_k \to z_k, \alpha^{\nu_j}_k \to \alpha_k,\ j \to \infty$ we finally obtain $z_k \geq f(\alpha_k)$, hence, $z \geq f(\alpha)$ with an $\alpha \in ri\mathbb{R}^n_{+,1}$.

The above lemma yields now several consequences on $H_\varepsilon(\lambda, \beta),\ h_\varepsilon(\lambda, \beta)$:

Corollary 8.3 *For each $\lambda \in \mathbb{R}^n_{+,1}$ the value $v_f^*(\lambda)$ has the representation*

$$v_f^*(\lambda) = \inf\{\lambda^T f(\alpha) : \alpha \in riR^n_{+,1}\} \tag{8.24}$$

and for $f(0) \in \mathbb{R}$ we may replace $ri\mathbb{R}^n_{+,1}$ also by $\mathbb{R}^n_{+,1}$.

For $H(\lambda, \beta) = H^{(f)}(\lambda, \beta)$ and $h(\lambda, \beta) = h^{(f)}(\lambda, \beta)$, with $V = C_f$, from Lemma 8.4 we get this result:

Corollary 8.4 *I) If $f(0) \in \mathbb{R}$, then $H^{(f)}(\lambda, \beta) = H_0^{(f)}(\lambda, \beta)$ and $h^{(f)}(\lambda, \beta) = h_0^{(f)}(\lambda, \beta) = \inf\{\lambda^T f(\alpha) : \beta^T f(\alpha) = v_f^*(\beta)\}$ for all $\lambda, \beta \in \mathbb{R}^n_{+,1}$. Furthermore, $H_0^{(f)}(\lambda, \beta) = \sup\{\lambda^T f(\alpha) : \beta^T f(\alpha) = v_f^*(\beta)\}$ for all $\lambda \in \mathbb{R}^n_{+,1}$ and $\beta \in ri\mathbb{R}^n_{+,1}$.*

II) If $f(0) = +\infty$ and $f(t) \geq 0,\ 0 \leq t \leq 1$, then $H^{(f)}(\lambda, \beta) = H_0^{(f)}(\lambda, \beta)$, $h^{(f)}(\lambda, \beta) = h_0^{(f)}(\lambda, \beta)$, provided that $\lambda \in \mathbb{R}^n_{+,1}$ and $\beta \in ri\mathbb{R}^n_{+,1}$. If $\lambda, \beta \in \mathbb{R}^n_{+,1}$ are selected such that $\lambda_k > 0,\ \beta_k = 0$ for an $1 \leq k \leq n$, then $H(\lambda, \beta) = +\infty$.

For the case $\beta \in \mathbb{R}^n_{+,1} \backslash ri\mathbb{R}^n_{+,1}$ we obtain this result:

Corollary 8.5 *If $\beta_\kappa = 0$ for an index $l \leq \kappa \leq n$, then*

$$v_f^*(\beta) = \inf\{\sum_{k=\kappa}^{n} \beta_k f(\alpha_k) : \alpha_k > 0,\ k \neq \kappa,\ \sum_{k \neq \kappa}^{n} \alpha_k = 1\}, \tag{8.25}$$

and in the case $f(0) \in \mathbb{R}$ the inequality "$\alpha_k > 0$" may be replaced also by "$\alpha_k \geq 0$". If $f(0) = +\infty$ and in addition f is strictly monotonous decreasing on $[0, 1]$, then $V_0(\beta) = \emptyset$.

Indicating the dependence of the set C_f, cf. (8.23), as well as the values $v_f^*(\beta)$ on the index n (= number of elements of Ω) by means of $C_f^{(n)}$, $v_f^{(n)*}(\beta)$, resp., then $v_f^{(n)*}(\beta) = \inf\{\beta^T z : z \in C_f^{(n)}\}$. Moreover, for a $\beta \in \mathbb{R}^n_{+,1}$ with $\beta_k = 0$, $1 \leq k \leq n$, and using the notation $\hat{\beta} = (\beta_1, \ldots, \beta_{k-1}, \beta_{k+1}, \ldots, \beta_n)^T$, Eqs. (8.24) and (8.25) yield

$$v_f^{(n)*}(\beta) = v_f^{(n-1)*}(\hat{\beta}). \tag{8.26}$$

Extending (8.26), we find the following corollary:

Corollary 8.6 *Suppose that $\lambda, \beta \in \mathbb{R}^n_{+,1}$ with $\lambda_k = \beta_k = 0$ for an index $1 \leq k \leq n$. Moreover, the notation $h_\varepsilon^{(n)}(\lambda, \beta)$, $h^{(n)}(\lambda, \beta)$ indicates the dependence of the functions $h_\varepsilon(\lambda, \beta)$, $h(\lambda, \beta)$ on the dimension n. Then, $h_\varepsilon^{(n)}(\lambda, \beta) = h_\varepsilon^{(n-1)}(\hat{\lambda}, \hat{\beta})$ for $\varepsilon > 0$, as well as $h^{(n)}(\lambda, \beta) = h^{(n-1)}(\hat{\lambda}, \hat{\beta})$, provided that $\hat{\lambda} = (\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_n)'$, and $\hat{\beta}$ is defined in the same way.*

A corresponding result for $H(\lambda, \beta)$ is stated below:

Corollary 8.7 *Consider $\lambda, \beta \in \mathbb{R}^n_{+,1}$ with $\lambda_k = \beta_k = 0$ for a certain $l \leq k \leq n$, and let indicate $H^{(n)}(\lambda, \beta)$ the dependence of $H(\lambda, \beta)$ on n. Then, $H^{(n)}(\lambda, \beta) = H^{(n-1)}(\lambda, \beta)$, provided that λ, β are defined as above, and the implication $\beta_j = 0 \Rightarrow \lambda_j = 0$ holds for $j \neq k$.*

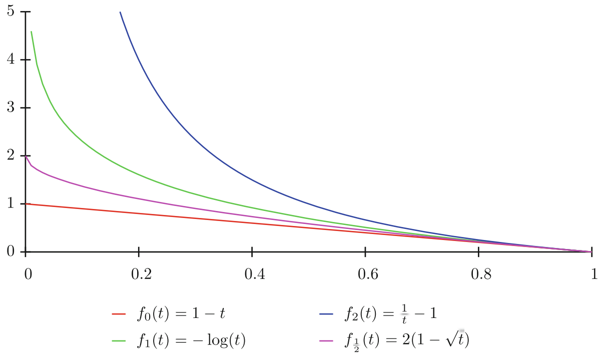

Fig. 8.3 Functions f_b

A relationship between $v_f^*(\lambda)$ and $v_f^*(e),\ e = (1/n, \ldots, 1/n)^T$ is shown in the following:

Corollary 8.8 *For all* $\lambda \in R^n_{+,1}$ *we have* $v_f^*(\lambda) \leq f(1/n) \leq v_f^*(1/n, \ldots, 1/n)$.

Proof Equation (8.24) in Corollary 8.3 yields $v_f^*(\lambda) \leq \lambda^T f(e) = f(1/n)$. Select then an arbitrary $\varepsilon > 0$. According to (8.24) there exists an $\alpha^\varepsilon \in ri\mathbb{R}^n_{+,1}$, such that $v^*(e) + \varepsilon \geq e^T f(\alpha^\varepsilon)$. Hence, $v^*(e) \geq -\varepsilon + \sum_{k=1}^{n} f(\alpha_k^\varepsilon) \geq -\varepsilon + f(\sum_{k=1}^{n} (1/n)\alpha_k^\varepsilon) = -\varepsilon + f(1/n)$. Since $\varepsilon > 0$ was chosen arbitrarily, we have $v_f^*(e) \geq f(1/n)$ and therefore $v_f^*(e) \geq f(1/n) \geq v_f^*(\lambda)$.

Note that the assertion in this corollary can also be found in [3]. Having some properties of $H^{(n)}$ and $h^{(n)}$, we determine now these functions for some important special cases of f. Next to we consider, see Fig. 8.3, the family $(f_b)_{b\geq 0}$, defined by (cf. [3])

$$f_b(t) = \begin{cases} \frac{1}{1-b}(1 - t^{1-b}) & , t \geq 0 \text{ for } b > 0, b \neq 1 \\ -\log t & , t \geq 0 \text{ for } b = 1. \end{cases}$$

The corresponding sets C_{f_b} are shown in Fig. 8.4.

It is easy to see that each f_b is a strictly monotonous decreasing, convex and for $b > 0$ strictly convex function on $[0, +\infty]$, such that $-\infty < f_b(t) < +\infty,\ t > 0$ and

$$\lim_{t\to 0} f_b(t) = \sup_{t>0} f_b(t) = f_b(0) = \begin{cases} \frac{1}{1-b} & \text{for } 0 \leq b \leq 1 \\ +\infty & \text{for } b \geq 1. \end{cases}$$

Moreover, $f_b(t) \geq 0 = f(1)$ for $0 \leq t \leq 1$. Hence, $f = f_b$ fulfills all needed conditions. Next to we want to determine $v_f^*(\lambda)$ and $V_o(\lambda)$, where the dependence on $b \geq$ is denoted by the notations $v_{(b)}^*(\lambda),\ V_{(b)o}(\lambda)$ and $C_{(b)}$.

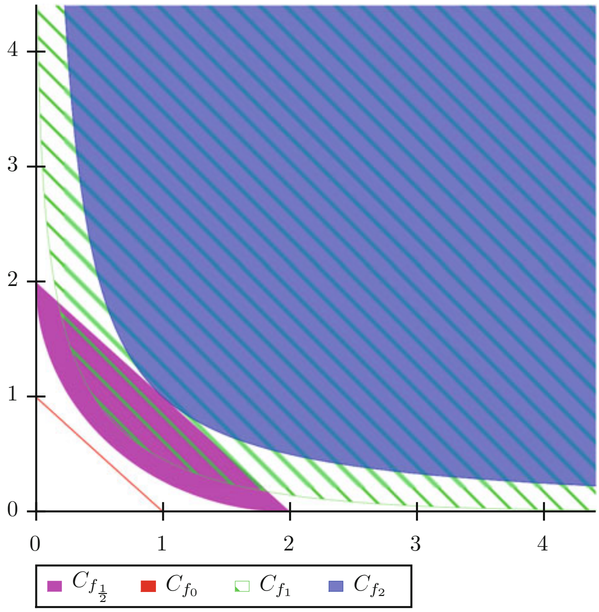

Fig. 8.4 Convex sets C_{f_b}

Theorem 8.3 *I) For each* $\lambda \in \mathbb{R}^n_{+,1}$ *we have*

$$v^*_{(o)}(\lambda) = 1 - \max_{1 \le k \le n} \lambda_k \tag{8.27a}$$

$$v^*_{(b)}(\lambda) = \frac{1}{1-b} \sum_{k=1}^{n} \lambda_k \Big(1 - \Big(\frac{\lambda_k 1/b}{\sum\limits_{k=1}^{n} {\lambda_k}^{1/b}}\Big)^{1-b}\Big) \text{ for } b > 0,\ b \neq 1 \tag{8.27b}$$

$$v^*_{(1)}(\lambda) = \sum_{k=1}^{n} \lambda_k \log(1/\lambda_k) \tag{8.27c}$$

II) If $\lambda_1 > 0, \ldots, \lambda_m > 0, \lambda_{m+1} = \ldots = \lambda_n = 0$*, then for* $b > 0$ *we also have*

$$v^*_{(b)}(\lambda) = \hat{\lambda}^T f_b(\hat{\alpha}^{(b)}) \text{ with } \hat{\lambda} = (\lambda_1, \ldots, \lambda_m)^T \ ,\ \hat{\alpha}^{(b)} = (\alpha^{(b)}_1, \ldots., \alpha^{(b)}_m)^T \tag{8.27d}$$

and

$$\alpha_k^{(b)} = \begin{cases} \lambda_k^{1/b} / \sum\limits_{k=1}^{n} \lambda_k^{1/b} & \textit{for } b > 0,\ b \neq 1 \\ & \qquad\qquad\qquad \textit{and } k = 1, \ldots, m, \\ \lambda_k & \textit{for } b = 1 \end{cases}$$

where $\hat{\alpha}^{(b)}$ is determined uniquely.

III) We have

$$V_{(o)o}(\lambda) = \{f_o(\alpha) : \ \alpha \in R^n_{+,1}, \ \sum_{k=1}^{n} \lambda_k \alpha_k \ = \ \max_{1 \le k \le n} \lambda_k\}, \ \lambda \in \mathbb{R}^n_{+,1}, \tag{8.27e}$$

$$V_{(b)o}(\lambda) = \{f_b(\alpha^{(b)})\} \textit{ with } \alpha^{(b)} = (\alpha_1^{(b)})^T, \lambda \in ri\mathbb{R}^n_{+,1}, \ b > 0 \tag{8.27f}$$

$$V_{(b)o}(\lambda) = \{z : \ z_k \ = \ f_b(\alpha_k^{(b)}), \lambda_k > 0 \textit{ and } z_k \ = \ \frac{1}{1-b}, \lambda_k = 0\} \tag{8.27g}$$

$$= \ \{f_b(\tilde{\alpha}^{(b)})\}, \lambda \in \mathbb{R}^n_{+,1} \backslash ri\mathbb{R}^n_{+,1}, \ 0 < b < 1, \textit{and certain } \tilde{\alpha}^{(b)}$$

$$V_{(b)o}(\lambda) = \emptyset, \ \lambda \in R^n_{+,1} \backslash riR^n_{+,1} \textit{ and } b \geq 1. \tag{8.27h}$$

Proof Consider first the case $b = 0$. From (8.23) we easily find that $C_{(o)} = \textit{clconv}\{f_0(\alpha) : \alpha \in ri\mathbb{R}^n_{+,1}\} = \{(1-\alpha_k)_{k=1,\ldots,n} : \alpha \in \mathbb{R}^n_{+,1}\}$. Because of (8.24) we further have $v^*_{(o)}(\lambda) = \inf\{1 - \lambda^T\alpha : \ \alpha \in \mathbb{R}^n_{+,1}\} = 1 - \sup\{\lambda^T\alpha : \alpha \in \mathbb{R}^n_{+,1}\} = 1 - \max\limits_{1 \le k \le n} \lambda_k$. Hence, $V_{(o)o}(\lambda) = \{z \in C_{(o)} : \lambda^T\alpha = \max\limits_{l \le k \le n} \lambda_k, \ \alpha \in \mathbb{R}^n_{+,1}\}$, which shows the assertion for $b = 0$. Thus, let now $b > 0$ and $\lambda_k > 0, \ k = 1, \ldots, m, \ \lambda_{m+1} = \ldots = \lambda_n = 0$. From (8.24) and (8.26) we get then $v^{(n)*}_{(b)}(\lambda) = v^{(m)*}_{(b)}(\hat{\lambda}) = \inf\{\hat{\lambda}^T f_b(\hat{\alpha}) : \ \hat{\alpha} \in ri\mathbb{R}^m_{+,1}\}$ with $\hat{\lambda} = (\lambda_1, \ldots, \lambda_m)^T$ and $\hat{\alpha} = (\alpha_1, \ldots, \alpha_m)^T$. The Lagrangian of the convex optimization problem

$$\begin{aligned} &\min \ \ \hat{\lambda}^T f_b(\hat{\alpha}) \\ &\text{s.t.} \ \ \hat{\alpha} \in ri\mathbb{R}^m_{+,1} \end{aligned} \tag{8.28}$$

is then given by $L(\hat{\alpha}, u) = \hat{\lambda}^T f_b(\hat{\alpha}) + u(\sum\limits_{k=1}^{m} \alpha_k - 1)$. Moreover, the optimality conditions—without considering the constraints $\alpha_k > 0, \ k = 1, \ldots, m$—read

$$0 \ = \ \frac{\partial L}{\partial u} = \sum_{k=1}^{m} \alpha_k \ - \ 1 \tag{8.29}$$

$$0 \ = \ \frac{\partial L}{\partial \alpha_k} = \lambda_k D f_b(\alpha_k) \ + \ u$$

Inserting $Df_b(t) = -t^{-b},\ t > 0$ for $b > 0, b \neq 1$ and $Df_1(t) = -1/t,\ t > 0$ for $b = 1$ into (8.29), yields

$$\alpha_k = \alpha_k^{(b)} = \begin{cases} \lambda_k^{1/b} / \sum\limits_{k=1}^{m} \lambda_k^{1/b} & \text{for } b > 0,\ b \neq 1 \\ \lambda_k & \text{for } b = 1 \end{cases} \quad \text{and } k = 1, 2, \ldots, m.$$

Since $\alpha_k^{(b)} > 0,\ k = 1, \ldots, m$, also the conditions $\alpha_k > 0,\ k = 1, \ldots, m$, hold. Moreover, $\hat{\alpha}^{(b)} = (\alpha_1^{(b)}, \ldots, \alpha_m^{(b)})^T$ is the unique solution of (8.28), since $\alpha \to \lambda^T f(\alpha)$ is strictly convex on $ri\mathbb{R}^m_{+,1}$. Hence, $v_{(b)}^{(n)*}(\lambda) = v_{(b)}^{(m)*}(\lambda) = \hat{\lambda}^T f_b(\hat{\alpha}^{(b)})$. Using the convention $0 \cdot (+\infty) = 0$, yields the rest of (8.27b). In addition, this proves also part (II). For showing (III) and therefore also (I), let $\lambda \in ri\mathbb{R}^n_{+,1}$ and $z \in V_{(b)o}(\lambda),\ b > 0$. We remember that $f_b(0) = +\infty$ for $b \geq 1$ and $f_b(0) \in R$ for $0 \leq b < 1$. Part (II) yields then $\alpha^{(b)} = (\alpha_1^{(b)}, \ldots, \alpha_n^{(b)})^T \in riR^n_{+,1}$ and $v_{(b)}^{(n)*}(\lambda) = \lambda^T f_b(\alpha^{(b)})$, hence, $f_b(\alpha^{(b)}) \in V_{(b)o}(\lambda)\ (\neq \emptyset)$.

According to Lemma 8.4, for $0 < b < 1$, $b \geq 1$, resp., there is $\alpha \in \mathbb{R}^n_{+,1}$, $\alpha \in ri\mathbb{R}^n_{+,1}$,, resp., such that $z \geq f_b(\alpha)$ (to each $z \in V_{(b)o}(\lambda)$). This immediately yields $v_{(b)}^{(n)*}(\lambda) = \lambda^T z = \lambda^T f_b(\alpha)$, since $f_b(\alpha) \in C_{(b)}$. Thus, $z = f_b(\alpha)$, because of $\lambda_k > 0,\ k = 1, \ldots, n$. Assuming $\alpha \neq \alpha^{(b)}$, for $\tilde{\alpha} = \frac{1}{2}\alpha + \frac{1}{2}\alpha^{(b)}$ we get on the one hand $\tilde{\alpha} \in ri\mathbb{R}^n_{+,1}$, hence, $f_b(\tilde{\alpha}) \in C_{(b)},\ b > 0$, and on the other hand we have $v_{(b)}^{(n)*}(\lambda) \leq \lambda^T f_b(\tilde{\alpha}) < \frac{1}{2}\lambda^T f_b(\alpha) + \frac{1}{2}\lambda^T f_b(\alpha^{(b)}) = v_{(b)}^{(n)*}(\lambda)$, since f_b is strictly convex for $b > 0$. Consequently, $\alpha = \alpha^{(b)}$ and therefore $V_{(b)o}(\lambda) = \{f_b(\alpha^{(b)})\},\ b > 0,\ \lambda \in ri\mathbb{R}^n_{+,1}$. Now consider $\lambda_1 > 0, \ldots, \lambda_m > 0, \lambda_{m+1} = \ldots = \lambda_n = 0$. Again from part (II) we get $v_{(b)}^{(n)*}(\lambda) = v_{(b)}^{(m)*}(\hat{\lambda}) = \lambda^T f_b(\hat{\alpha}^{(b)})$. We put $\tilde{\alpha}^{(b)} = (\alpha_1^{(b)}, \ldots, \alpha_m^{(b)}, 0, \ldots, 0)^T$. Let $0 < b < 1$. Because of $f(0) = 1/(1-b) \in \mathbb{R}$, due to Lemma 8.4 we obtain $C_{(b)} = conv\{f_b(\alpha) :\ \alpha \in \mathbb{R}^n_{+,1}\}$, hence, $f_b(\tilde{\alpha}^{(b)}) \in V_{(b)o}(\lambda)$, since $\tilde{\alpha}^{(b)} \in \mathbb{R}^n_{+,1}$ and $\lambda^T f_b(\tilde{\alpha}^{(b)}) = \hat{\lambda}^T f_b(\hat{\alpha}^{(b)}) = v_{(b)}^{(n)*}(\lambda)$.

Consider now $z \in V_{(b)o}(\lambda)$. According to Lemma 8.4, part V, there is $\alpha \in \mathbb{R}^n_{+,1}$ with $z \geq f_b(\alpha)$. Because of $\hat{\lambda}^T \hat{z} = \lambda^T z = v_{(b)}^{(n)*}(\lambda) = \lambda^T f_b(\alpha) = \hat{\lambda}^T f_b(\hat{\alpha})$ we have $f_b(\alpha) \in V_{(b)o}(\lambda)$ and $\hat{z} = f_b(\hat{\alpha})$, since $\lambda_k > 0,\ k = 1, 2, \ldots, m$. Assuming $\hat{\alpha} \neq \hat{\alpha}^{(b)}$ and considering then under this assumption $\gamma = \frac{1}{2}\alpha + \frac{1}{2}\tilde{\alpha}^{(b)}$, we get $f_b(\gamma) \in C_{(b)}$, since $\gamma \in \mathbb{R}^n_{+,1}$ and $v_{(b)}^{(n)*}(\lambda) \leq \lambda^T f_b(\gamma) = \hat{\lambda}^T f_b(\hat{\gamma}) < \frac{1}{2}\hat{\lambda}^T f_b(\hat{\alpha}) + \frac{1}{2}\hat{\lambda}^T f_b(\hat{\alpha}^{(b)}) = v_{(b)}^{(n)*}(\lambda)$. However, this yields a contradiction, hence, it holds $\hat{\alpha}^{(b)} = \hat{\alpha}$. Obviously, each $\alpha \in \mathbb{R}^n_{+,1}$ fulfilling this equation is contained in $V_{(b)o}(\lambda)$.

Thus, $V_{(b)o}(\lambda) = \{f_b(\alpha) :\ \alpha \in \mathbb{R}^n_{+,1},\ \hat{\alpha} = \hat{\alpha}^{(b)}\} = \{f_b(\alpha^{(b)})\} = \{(f_b(\alpha^{(b)})^T,\ 1/(1-b), \ldots, 1/(1-b))^T\}$, since $\sum\limits_{k=1}^{m} \alpha_k^{(b)} = 1$ and therefore $\alpha_k = 0,\ k = m+1, \ldots, n$. Finally, let $b \geq 1$. Due to $f_b(\hat{\alpha}^{(b)}) =$

$(f_b(\hat{\alpha}^{(b)})^T, +\infty, \dots, +\infty)^T$ and $C_{(b)} \subset \mathbb{R}^n$, we find $f_b(\hat{\alpha}^{(b)}) \notin C_{(b)}$. Suppose that z lies in $V_{(b)o}(\lambda)$. According to Lemma 8.4 ($f(0) = +\infty$) there is then an $\alpha \in ri\mathbb{R}^n_{+,1}$ with $z \geq f(\alpha)$ and therefore $v^{(n)*}_{(b)}(\lambda) = \lambda^T z = \hat{\lambda}^T \hat{z} = \lambda^T f_b(\alpha) = \hat{\lambda}^T f_b(\hat{\alpha})$. Because of $\alpha \in ri\mathbb{R}^n_{+,1}$, it is $\sum\limits_{k=1}^{m} \alpha_k < 1$ and therefore $\hat{\alpha} \neq \hat{\alpha}^{(b)}$. On the other hand we have $\gamma = \frac{1}{2}\alpha + \frac{1}{2}\tilde{\alpha}^{(b)} \in ri\mathbb{R}^n_{+,1}$. This yields $v^{(n)*}_{(b)}(\lambda) \leq \lambda^T f_b(\gamma) = \hat{\lambda}^T f_b(\hat{\gamma}) < \frac{1}{2}\hat{\lambda}' f_b(\hat{\alpha}) + \frac{1}{2}\hat{\lambda}^T f_b(\hat{\alpha})^{(b)}) = v^{(n)*}_{(b)}(\lambda)$, which is again a contradiction. Consequently, $V_{(b)o}(\lambda) = \emptyset$ for $b \geq 1$ and $\lambda \notin ri\mathbb{R}^n_{+,1}$.

Remark 8.8 I) Obviously, $v^*_{(1)}(\lambda) = \sum\limits_{k=1}^{n} \lambda_k \log \frac{1}{\lambda_k}$ is the *(Shannon-) entropy* of the discrete distribution λ.

II) Assume that $\lambda_1 > 0, \dots, \lambda_m > 0,\ \lambda_{m+1} = \dots = \lambda_n = 0$. For $b > 0, b \neq 1$, from (8.27b) we get

$$\begin{aligned} v^*_{(b)}(\lambda) &= \frac{1}{1-b} \sum_{k=1}^{m} \lambda_k (1 - ({\lambda_k}^{1/b} / \sum_{k=1}^{m} {\lambda_k}^{1/b})^{1-b}) \\ &= \frac{1}{1-b} (1 - \sum_{k=1}^{m} {\lambda_k}^{1/b} / (\sum_{k=1}^{m} {\lambda_k}^{1/b})^{1-b}) \\ &= \frac{1}{1-b} (1 - \sum_{k=1}^{m} {\lambda_k}^{1/b})^b) = \frac{1}{1-b} (1 - (\sum_{k=1}^{n} {\lambda_k}^{1/b})^b). \end{aligned} \tag{8.30}$$

Hence,

$$v^*_{(b)}(\lambda) = \frac{1}{1-b} (1 - M_{(1/b)}(\lambda)),$$

provided that—see [56]—the *mean* $M_r(z),\ z \in \mathbb{R}^n_+$ is defined by $M_r(z) = (\sum\limits_{k=1}^{n} {z_k}^r)^{1/r}$. According to [56], where one finds also other properties of M_r, M_r is convex for $r > 1$ and concave for $r < 1$, which follows from the concavity of $v^*(\cdot)$.

Having $V_{(b)o}(\lambda),\ b \geq 0$, also the functions $H^{(b)} = H^{f_b}$ and $h^{(b)} = h^{(f_b)}$ can be determined:

Corollary 8.9 *I) For* $b = 0$ *we have*

$$H^{(o)}(\lambda, \beta) = \sup\{1 - \lambda^T \alpha :\ \alpha \in R^n_{+,1},\ \beta^T \alpha = \max_{1 \leq k \leq n} \beta_k\} \textit{ for all } \lambda, \beta \in \mathbb{R}^n_{+,1},$$

$$h^{(o)}(\lambda, \beta) = \inf\{1 - \lambda^T \alpha :\ \alpha \in R^n_{+,1},\ \beta^T \alpha = \max_{1 \leq k \leq n} \beta_k\} \textit{ for all } \lambda, \beta \in \mathbb{R}^n_{+,1},$$

II) If $0 < b < 1$, *then with* $\alpha^{(b)} = (\alpha_k{}^{1/b} / \sum_{k=1}^{n} \alpha_k{}^{1/b})_{k=1,\dots,n}$, *we get*

$$H^{(b)}(\lambda, \beta) = h^{(b)}(\lambda, \beta) = \sum_{k=1}^{n} \lambda_k f_b(\alpha_k{}^{(b)})$$

$$= \tfrac{1}{1-b}(1 - \sum_{k=1}^{n} \lambda_k (\beta_k{}^{1/b} / \sum_{k=1}^{n} \beta_k{}^{1/b})^{1-b}) \textit{for all} \lambda, \beta \in \mathbb{R}^n_{+,1}.$$

III) If $b = 1$, *then*

$$H^{(1)}(\lambda, \beta) = h^{(1)}(\lambda, \beta)$$
$$= \sum_{k=1}^{n} \lambda_k \log(1/\beta_k),\ \lambda, \beta \in \mathbb{R}^n_{+,1}\ (\textit{with} \log \frac{1}{0} = +\infty).$$

IV) If $b > 1$, *then*

$$H^{(b)}(\lambda, \beta) = h^{(b)}(\lambda, \beta) = \lambda^T f_b(\alpha^{(b)})$$

$$= \sum_{k=1}^{n} \lambda_k \tfrac{1}{1-b}(1 - (\beta_k{}^{1/b} / \sum_{k=1}^{n} \beta_k{}^{1/b})^{1-b}),\ \textit{for all}\ \lambda, \beta \in \mathbb{R}^n_{+,1}.$$

Remark 8.9 Corresponding to Theorem 8.3, we observe that

$$H^{(1)}(\lambda, \beta) \ = \ h^{(1)}(\lambda, \beta) \ = \ \sum_{k=1}^{n} \lambda_k \log(1/\beta_k),\ \lambda, \beta \in \mathbb{R}^n_{+,1}$$

is the Kerridge-Inaccuracy for the hypothesis "$P_\omega = \beta$", while $P_\omega = \lambda$ is the true distribution. However, this justifies the notation *generalized inaccuracy function* for $H(\lambda, \beta)$ and $h(\lambda, \beta)$.

8.2.2 Representation of $H_\varepsilon(\lambda, \beta)$ and $H(\lambda, \beta)$ by Means of Lagrange Duality

In the following we derive a representation of $H_\varepsilon(\lambda, \beta)$ and $H(\lambda, \beta)$ which can be used also to find sufficient conditions for $H(\lambda, \beta) = H_0(\lambda, \beta)$. For this we make the following assumptions on the loss set V, cf. (8.13a, b), of the decision problem (Ω, D, v) and the probability measure λ, β on $\mathcal{A}$:

V is a convex subset of $L_1(\Omega, \mathcal{A}, \lambda) \cap L_1(\Omega, \mathcal{A}, \beta)$, where $-\infty < v^*(\beta) < +\infty$.

Defining the mappings $F\colon V \to \mathbb{R}$ and $g_\varepsilon : L_1(\Omega, \mathcal{A}, \beta) \to \mathbb{R}, \varepsilon \geq 0$ by

$$F(f) = \int f(\omega)\lambda(d\omega), f \in V$$

and

$$g_\varepsilon(f) = \int f(\omega)\beta(d\omega) - (v^*(\beta) + \varepsilon), f \in L_1(\Omega, \mathcal{A}, \beta), \varepsilon \geq 0,$$

F is an affine, real valued functional on V and g_ε an affine, real valued functional on the linear space $X = L_1(\Omega, \mathcal{A}, \beta)$. Moreover, it holds

$$H_\varepsilon(\lambda, \beta) = sup\{F(f) : g_\varepsilon(f) \leq 0, f \in V\},\ \varepsilon \geq 0.$$

Thus, we have to consider the following convex program in space X:

$$\min\ -F(f) \text{ s.t } g_\varepsilon(f) \leq 0, f \in V. \tag{8.31}$$

According to Luenberger [85], Section 8.6, Theorem 1, concerning programs of the type (8.31), we get immediately this result:

Theorem 8.4 *If $H_\varepsilon(\lambda, \beta) \in \mathbb{R}$ for $\varepsilon > 0$, then*

$$\begin{aligned} H_\varepsilon(\lambda, \beta) &= \min_{a \geq 0}(\sup_{x \in D}(v(\lambda, x) - av(\beta, x)) + a(v^*(\beta) + \varepsilon) \qquad (8.32) \\ &= \min_{a \geq 0}(\sup_{x \in D}(v(\lambda, x) - a(v(\beta, x) - v^*(\beta))) + a\varepsilon), \end{aligned}$$

where the minimum in (8.32) is taken in a point $a_\varepsilon \geq 0$.

Proof The dual functional related to (8.31) is defined here by

$$\begin{aligned} \phi_\varepsilon(a) &= \inf_{f \in V}(-F(f) + ag_\varepsilon(f)) = \inf f \in V(-\int f(\omega)\lambda(d\omega) \\ &+ a(\int f(\omega)\beta(d\omega) - v^*(\beta) - \varepsilon)) \\ &= -(\sup_{x \in D}(v(\lambda, x) - av(\beta, x) + a(v^*(\beta) + \varepsilon)). \end{aligned}$$

Due to $v^*(\beta) > -\infty$, we also have $D_{\varepsilon/2}(\beta) \neq \emptyset$. Hence, there is $f_1 \in V$ such that $\beta f_1 \leq v^*(\beta) + \frac{\varepsilon}{2} < v^*(\beta) + \varepsilon$ and therefore $g_\varepsilon(f_1) < 0$ for all $\varepsilon > 0$. According to our assumptions, (8.31) has a finite infimum, hence, all assumptions in the above mentioned theorem of Luenberger are fulfilled. Consequently, *inf* (8.31) $=$ $\max_{a>0} \phi_\varepsilon(a)$,, where the maximum is taken in a point $a_\varepsilon \geq 0$. Thus,

$$\begin{aligned} H_\varepsilon(\lambda,\beta) &= -\mathit{inf}(8.31) = -\max_{a\geq 0}\phi_\varepsilon(a) = \min_{a\geq 0}(-\phi_\varepsilon(a)) \\ &= \min_{a\geq 0}(\sup_{x\in D}(v(\lambda,x) - av(\beta,x) + a(v^*(\beta)+\varepsilon)) \end{aligned}$$

and the maximum is taken in point $a_\varepsilon \geq 0$.

Remark 8.10 Note that for the derivation of (8.32) only the convexity of $V = \{v(.,x) : x \in D\} \subset L_1(\Omega,\mathcal{A},\lambda) \cap L_1(\Omega,\mathcal{A},\beta)$, $D_\varepsilon(\beta) \neq \emptyset$ and the condition $H_\varepsilon(\lambda,\beta) \in \mathbb{R}$ was needed.

For a comparison between $H(\lambda,\beta)$ and $H_0(\lambda,\beta)$ we show the following result:

Theorem 8.5 *Suppose that $H_{\bar\varepsilon}(\lambda,\beta) < +\infty$ for an $\bar\varepsilon > 0$. Then $H(\lambda,\beta)$ has the representation*

$$H(\lambda,\beta) = \inf_{a\in R}(\sup_{x\in D}(v(\lambda,x) - av(\beta,x)) + av^*(\beta)). \tag{8.33}$$

Proof Let h denote the right hand side of (8.33). Then, $h \leq H(\lambda,\beta)$. Put now

$$\delta(a) = \sup_{x\in D}(v(\lambda,x) - a(v(\beta,x) - v^*(\beta))), a \in \mathbb{R}.$$

If $a_1 < a_2$, then $a_1(v(\beta,x) - v^*(\beta) \leq a_2(v(\beta,x) - v^*(\beta)), x \in D$, since $v(\beta,x) \geq v^*(\beta), x \in D$. Hence, $-a_1(v(\beta,x) - v^*(\beta)) \geq -a_2(v(\beta,x) - v^*(\beta)), v(\lambda,x) - a_1(v(\beta,x) - v^*(\beta)) \geq v(\lambda,x) - a_2(v(\beta,x) - v^*(\beta))$ and therefore $\delta(a_1) \geq \delta(a_2)$. Thus, δ is monotonous decreasing. However, this yields $h = \inf_{a\in\mathbb{R}}\delta(a) = \inf_{a\geq 0}\delta(a) = H(\lambda,\beta)$.

Suppose now that $D_0(\beta) \neq \emptyset$ and $H_{\bar\varepsilon}(\lambda,\beta) < +\infty$ for an $\bar\varepsilon > 0$, hence, $H_\varepsilon(\lambda,\beta) \in \mathbb{R}$ for $0 < \varepsilon < \bar\varepsilon$. Because of $H_0(\lambda,\beta) = \mathit{sup}\{F(f) : g_0(f) = 0, f \in V\}$, for the consideration of $H_0(\lambda,\beta)$, in stead of (8.31), we have to consider the optimization problem

$$\min \quad -F(f) \tag{8.34}$$

$$\text{s.t.} \quad g_0(f) = 0, f \in V. \tag{8.35}$$

The dual functional related to this program reads:

$$\phi_0(a) = \inf_{f\in V}(-F(f) + ag_0(f)) = -\sup_{\in D}((v(\lambda,x) - av(\beta,x)) + av^*(\beta)) = -\delta(a).$$

A *Kuhn–Tucker-coefficient* related to (8.34) is, cf. e.g. [137], Section 28, a value a_0, such that

$$\inf(8.34) = \phi_0(a_0) \text{ or } H_0(\lambda, \beta) = -\phi_0(a_0). \tag{8.36}$$

In case that such an a_0 exists, then

$$H_0(\lambda, \beta) = \inf(8.34) = -\phi_0(a_0) = \sup_{x \in D}(v(\lambda, x) - a_0(v(\beta, x) - v^*(\beta))),$$

due to $H_0(\lambda, \beta) \leq H(\lambda, \beta)$ and (8.33), we have

$$H_0(\lambda, \beta) \leq H(\lambda, \beta) = \inf_{a \in \mathbb{R}}(\sup_{x \in D}(v(\lambda, x) - a(v(\beta, x) - v^*(\beta))))$$

$$\leq \sup_{x \in D}(v(\lambda, x) - a_0(v(\beta, x) - v^*(\beta))) = H_0(\lambda, \beta),$$

hence,

$$H(\lambda, \beta) = H_0(\lambda, \beta) = \min_{a \in \mathbb{R}}(\sup_{x \in D}(v(\lambda, x) - a(v(\beta, x) - v^*(\beta)))).$$

Thus, we have this result:

Theorem 8.6 *Let $D_0(\beta) \neq \emptyset$, and assume $H_{\bar{\varepsilon}}(\lambda, \beta) < +\infty$ for a certain $\bar{\varepsilon} > 0$. If the program*

$$\begin{aligned} \max\ & \lambda f \\ \textit{bez.}\ & \beta f = v^*(\beta),\ f \in V \end{aligned}$$

admits a Kuhn–Tucker coefficient, i.e., an $a_0 \in R$, such that

$$sup\{\lambda f : \beta f = v^*(\beta), f \in V\} = \sup\{f \in V\}(\lambda f - a_0(\beta f - v^*(\beta))),$$

then

$$H(\lambda; \beta) = H_0(\lambda, \beta) = \min_{a \in \mathbb{R}}(\sup_{x \in D}(v(\lambda, x) - a(v(\beta, x) - v^*(\beta)))), \tag{8.37}$$

and the minimum in (8.37) is taken at a point $a_0 \in \mathbb{R}$.

8.3 Generalized Divergence and Generalized Minimum Discrimination Information

8.3.1 Generalized Divergence

As in the preceding section, assume that $P_{\tilde{\omega}} = \lambda$ is the true probability distribution of $\tilde{\omega}$ and let denote $P_{\tilde{\omega}} = \beta$ a certain hypothesis on the true distribution λ; For all $\varepsilon > 0$ suppose that $D_\varepsilon \neq \emptyset$. Selecting a decision $x \in D_\varepsilon$, then with respect to the true distribution λ, with respect to the hypothesis β, resp., we have the error

$$e_1 = e_1(\lambda, x) = v(\lambda, x) - v^*(\lambda),$$

$$e_2 = e_2(\lambda, x) = v(\lambda, x) - v^*(\beta),$$

resp., where $e_1(\lambda, x)$, $e_2(\lambda, x)$, are defined only if $v^*(\lambda) \in \mathbb{R}$, $v^*(\beta) \in \mathbb{R}$, respectively.

Evaluating this error still by means of a function γ, then

$$I^e_{\gamma,\varepsilon}(\lambda, \beta) = sup\{\gamma(e(\lambda, x)) : x \in D_\varepsilon(\beta)\}, e = e_1, e_2,$$

$$J^e_{\gamma,\varepsilon}(\lambda, \beta) = inf\{\gamma(e(\lambda, x)) : x \in D_\varepsilon(\beta)\}, e = e_1, e_2$$

denotes the maximum, minimum error relative to γ, for the computation of an ε-optimal decision based on the hypothesis β, while $P_{\tilde{\omega}} = \lambda$ is the true distribution. Hence, as far as the limits under consideration exist, we define the class of generalized divergences by

$$I^e_\gamma(\lambda, \beta) = \lim_{\varepsilon \downarrow 0} I^e_{\gamma,\varepsilon}(\lambda, \beta) = \lim_{\varepsilon \downarrow 0}(\sup_{x \in D_\varepsilon(\beta)} \gamma(e(\lambda, x))), e = e_1, e_2$$

and

$$J^e_\gamma(\lambda, \beta) = \lim_{\varepsilon \downarrow 0} J^e_{\gamma,\varepsilon}(\lambda, \beta) = \lim_{\varepsilon \downarrow 0}(\inf_{x \in D_\varepsilon(\beta)} \gamma(e(\lambda, x))), e = e_1, e_2.$$

We consider now I^e, J^e, e_2 for some special cases of the cost function γ. For this purpose we set

$$I(\lambda, \beta) = H(\lambda, \beta) - H(\lambda, \lambda) = H(\lambda, \beta) - v^*(\lambda), \tag{8.38a}$$

$$J(\lambda, \beta) = h(\lambda, \beta) - h(\lambda, \lambda) = h(\lambda, \beta) - v^*(\lambda). \tag{8.38b}$$

I) If $\gamma(t) = t, t \in \mathbb{R}$, then

$$I_{\gamma,\varepsilon}^{e_1}(\lambda,\beta) = \sup_{x\in D_\varepsilon(\beta)} e_1(\lambda,x) = \sup_{x\in D_\varepsilon(\beta)} (v(\lambda,x) - v^*(\lambda)) = H_\varepsilon(\lambda,\beta) - v^*(\lambda)$$

$$I_{\gamma,\varepsilon}^{e_2}(\lambda,\beta) = \sup_{x\in D_\varepsilon(\beta)} e_2(\lambda,x) = \sup_{x\in D_\varepsilon(\beta)} (v(\lambda,x) - v^*(\beta)) = H_\varepsilon(\lambda,\beta) - v^*(\lambda)$$

$$J_{\gamma,\varepsilon}^{e_1}(\lambda,\beta) = \inf_{x\in D_\varepsilon(\beta)} e_1(\lambda,x) = \inf_{x\in D_\varepsilon(\beta)} (v(\lambda,x) - v^*(\lambda)) = h_\varepsilon(\lambda,\beta) - v^*(\lambda)$$

$$J_{\gamma,\varepsilon}^{e_2}(\lambda,\beta) = \inf_{x\in D_\varepsilon(\beta)} e_2(\lambda,x) = \inf_{x\in D_\varepsilon(\beta)} (v(\lambda,x) - v^*(\beta)) = h_\varepsilon(\lambda,\beta) - v^*(\lambda).$$

Taking the limit $\varepsilon \downarrow 0$, we obtain then

$$I_\gamma^{e_1}(\lambda,\beta) = H(\lambda,\beta) - v^*(\lambda) = I(\lambda,\beta),$$
$$I_\gamma^{e_2}(\lambda,\beta) = H(\lambda,\beta) - v^*(\beta) = I(\lambda,\beta) + (H(\lambda,\lambda) - H(\beta,\beta)),$$
$$J_\gamma^{e_1}(\lambda,\beta) = h(\lambda,\beta) - v^*(\lambda) = J(\lambda,\beta),$$
$$J_\gamma^{e_2}(\lambda,\beta) = h(\lambda,\beta) - v^*(\beta) = J(\lambda,\beta) + (h(\lambda,\lambda) - h(\beta,\beta)).$$

Remark 8.11 Estimations of the variation $H(\lambda,\lambda) - H(\beta,\beta) = h(\lambda,\lambda) - h(\beta,\beta) = v^*(\lambda) - v^*(\beta)$ of the inaccuracy function $\lambda \to v^*(\lambda)$ in the transfer from λ to β can be found e.g. in [124].

II) Let now $\gamma(t) =| t |$. Because of $v(\lambda,x) \geq v^*(\lambda)$ for all $x \in D$ we have

$$\begin{aligned} I_{\gamma,\varepsilon}^{e_1}(\lambda,\beta) &= \sup_{x\in D_\varepsilon(\beta)} | e_1(\lambda,x) | = \sup_{x\in D_\varepsilon(\beta)} (v(\lambda,x) - v^*(\lambda)) \\ &= H_\varepsilon(\lambda,\beta) - v^*(\lambda), \\ J_{\gamma,\varepsilon}^{e_1}(\lambda,\beta) &= \inf_{x\in D_\varepsilon(\beta)} | e_1(\lambda,x) | = \sup_{x\in D_\varepsilon(\beta)} (v(\lambda,x) - v^*(\lambda)) \\ &= h_\varepsilon(\lambda,\beta) - v^*(\lambda). \end{aligned}$$

Thus, also here we get

$$I_\gamma^{e_1}(\lambda,\beta) = H(\lambda,\beta) - v^*(\lambda) = I(\lambda,\beta),$$
$$J_\gamma^{e_1}(\lambda,\beta) = h(\lambda,\beta) - v^*(\lambda) = J(\lambda,\beta).$$

Furthermore, we have

$$\begin{aligned} I_{\gamma,\varepsilon}^{e_2}(\lambda,\beta) &= \sup_{\in D_\varepsilon(\beta)} | e_2(\lambda,x) | \\ &= \max\{| \inf_{x\in D_\varepsilon(\beta)} e_2(\lambda,x) |, | \sup_{x\in D_\varepsilon(\beta)} e_2(\lambda,x) |\} \end{aligned}$$

$$= \max\{| \inf_{x \in D_\varepsilon(\beta)} (v(\lambda, x) - v^*(\beta) |, | \sup_{x \in D_\varepsilon(\beta)} (v(\lambda, x) - v^*(\beta)) |\}$$
$$= \max\{| h_\varepsilon(\lambda, \beta) - v^*(\beta) |, | H_\varepsilon(\lambda, \beta) - v^*(\beta) |\}.$$

Because of the continuity of $(x, y) \to max\{x, y\}, x, y \in R$, by means of the limit $\varepsilon \downarrow 0$ we find

$$I_\gamma^{e_2}(\lambda, \beta) = max\{| h(\lambda, \beta) - v^*(\beta) |, | H(\lambda, \beta) - v^*(\beta) |\} =$$
$$= max\{| h(\lambda, \beta) - h(\beta, \beta) |, | H(\lambda, \beta) - H(\beta, \beta) |\}.$$

In the special case $H(\lambda, \beta) = h(\lambda, \beta)$, we get

$$I_\gamma^{e_2}(\lambda, \beta) =| H(\lambda, \beta) - H(\beta, \beta) | .$$

Especially important are the generalized divergences defined by (2.47a) and (2.47b), hence, $I(\lambda, \beta) = H(\lambda, \beta) - H(\lambda, \lambda)$ and $J(\lambda, \beta) = h(\lambda, \beta) - h(\lambda, \lambda)$. We study now I, J, where—as above—$D_\varepsilon(\beta) \neq \emptyset, \varepsilon > 0$ and $H(\lambda, \lambda) = h(\lambda, \lambda) = v^*(\lambda) \in \mathbb{R}$.

Corollary 8.10 *We have $I(\lambda, \beta) \geq 0, J(\lambda, \beta) \geq 0$ and $I(\lambda, \beta) = J(\lambda, \beta) = 0$ for $\beta = \lambda$. Moreover, $I(\lambda, \beta) \geq J(\lambda, \beta)$ und $I(\lambda, \beta) = J(\lambda, \beta)$ if and only if $H(\lambda, \beta) = h(\lambda, \beta)$.*

Proof According to Theorem 8.2 we have $H(\lambda, \beta) \geq h(\lambda, \beta) \geq v^*(\lambda) = h(\lambda, \lambda) = H(\lambda, \lambda)$, which yields all assertions in the above corollary.

In order to justify the notation *generalized divergence* for the class of functions $I_\gamma^e(\lambda, \beta), J_\gamma^e(\lambda, \beta), e = e_1, e_2$, we consider now the case $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ with the loss set, cf. (8.23),

$$V = C_f, f = f_b, b \geq 0$$

treated in detail in the former section.

Denoting the dependence of the divergences I, J on b by $I^{(b)}, J^{(b)}$, then Corollary 8.9 yields immediately this result:

Corollary 8.11 *For all $\lambda, \beta \in \mathbb{R}^n_{+,1}$ we have*

I)

$$I^{(0)}(\lambda, \beta) = \max_{1 \leq k \leq n} \lambda_k - inf\{\lambda^T \alpha : \alpha \in \mathbb{R}^n_{+,1}, \beta^T \alpha = \max_{1 \leq k \leq n} \beta_k\},$$

$$J^{(0)}(\lambda, \beta) = \max_{1 \leq k \leq n} \lambda_k - sup\{\lambda^T \alpha : \alpha \in \mathbb{R}^n_{+,1}, \beta^T \alpha = \max_{1 \leq k \leq n} \beta_k\};$$

II)

$$I^{(b)}(\lambda,\beta) = J^{(b)}(\lambda,\beta) = \frac{1}{1-b}\sum_{k=1}^{n}\lambda_k((\lambda_k^{1/b})^{-b} - (\beta_k^{1/b}/\sum_{k=1}^{n}\beta_k)^{1-b})$$

for $b > 0, b \neq 1$;

III)

$$I^{(1)}(\lambda,\beta) = J^{(1)}(\lambda,\beta) = \sum_{k=1}^{n}\lambda_k log(\lambda_k/\beta_k).$$

Remark 8.12 Obviously, we see now that $I^{(1)}(\lambda,\beta) = J^{(1)}(\lambda,\beta)$ is the die Kullback-divergence between λ and β, which justifies now the notation *generalized divergence* for I_γ^e, J_γ^e .

According to Corollary 8.10 we have $I(\lambda,\beta) = 0 \implies J(\lambda,\beta) = 0$, and $I(\lambda,\beta) = J(\lambda,\beta) = 0$ for $\beta = \lambda$. However, $I(\lambda,\beta) = 0$ or $J(\lambda,\beta) = 0$ does not imply $\beta = \lambda$ in general.

Example 8.3 Putting $\lambda = (1/n,\ldots,1/n)^T$ in the above Corollary 8.11a, then for all $\beta \in \mathbb{R}^n_{+,1}$ we get

$$\begin{aligned} I^{(0)}(\lambda,\beta) &= 1/n - inf\{\sum_{k=1}^{n}\frac{1}{n}\alpha_k : \alpha \in \mathbb{R}^n_{+,1}, \beta^T\alpha = \max_{1\le k\le n}\beta_k\} \\ &= 1/n - 1/n = 0, \\ J^{(0)}(\lambda,\beta) &= 1/n - sup\{\sum_{k=1}^{n}\frac{1}{n}\alpha_k : \alpha \in \mathbb{R}^n_{+,1}, \beta^T\alpha = \max_{1\le k\le n}\beta_k\} \\ &= 1/n - 1/n = 0. \end{aligned}$$

In this case we have therefore $\{\beta : I^{(0)}(\lambda,\beta) = 0\} = \{\beta : J^{(0)}(\lambda,\beta) = 0\} = \mathbb{R}^n_{+,1}$.

Theorem 8.7 *I) Suppose again* $D_0(\beta) \neq \emptyset$. *If* $I(\lambda,\beta) = 0$, *then* $D_0(\lambda) \neq \emptyset$, $D_0(\beta) \subset D_0(\lambda)$, *and* $H(\lambda,\beta) = H_0(\lambda,\beta) = h_0(\lambda,\beta)$. *If in addition* $D_0(\lambda) = \{x_\lambda\}$, *then* $D_0(\beta) = \{x_\beta\}$ *with* $x_\beta = x_\lambda$.

II) In the case $D_0(\beta) = \{x_\beta\}$, $H(\lambda,\beta) = v(\lambda,x_\beta)$ *for all* $\lambda,\beta \in \Lambda$ *with a subset* Λ *of* $ca_{+,1}(\Omega,\mathcal{A})$, *then* $I(\lambda,\beta) = \lambda(v(.,x_\beta) - v(.,x_\lambda))$ *and* $I(\lambda,\beta) = 0 \iff x_\lambda = x_\beta$, *provided that* $\lambda,\beta \in \Lambda$.

III) Let $D_0(\beta) = \{x_\beta\}$ *and* $H(\lambda,\beta) = v(\lambda,x_\beta)$ *for all* $\lambda,\beta \in \Lambda$. *If* $\hat{x} \in D$ *denotes then a least element of* D *with respect to the order "*$\preceq_\Lambda$*", then* $x_\beta = \hat{x}, \beta \in \Lambda$, $H(\lambda,\beta) = v(\lambda,\hat{x})$ *and* $I(\lambda,\beta) = 0, \lambda,\beta \in \Lambda$.

The representations of $H(\lambda, \beta)$ and $h(\lambda, \beta)$ given in the Theorems 8.4 and 8.5 yield the following representation of I,J:

Corollary 8.12 *I) If $H_{\bar{\varepsilon}}(\lambda, \beta) < +\infty$ for an $\bar{\varepsilon} > 0$ and $v^*(\lambda) \in \mathbb{R}$, then*

$$I(\lambda, \beta) = \inf_{a \geq 0}(\sup_{x \in D}((v(\lambda, x) - v^*(\lambda)) - a(v(\beta, x) - v^*(\beta))));$$

II) If $v^*(\lambda) \in \mathbb{R}$ and $v^*(\beta) \in \mathbb{R}$, then

$$J(\lambda, \beta) = \sup_{a \geq 0}(\sup_{x \in D}((v(\lambda, x) - v^*(\lambda)) - a(v(\beta, x) - v^*(\beta)))).$$

Remark 8.13 Equation $I(\lambda, \beta) = 0$, $J(\lambda, \beta) = 0$. Having $I(\lambda, \beta) = v(\lambda, x_\beta) - v(\lambda, x_\lambda)$, $J(\lambda, \beta) = v(\lambda, x_\beta) - v(\lambda, x_\lambda)$, resp., with two elements $x_\lambda, x_\beta \in D$, such that $D_0(\lambda) = \{x_\lambda\}$, $D_0(\beta) = \{x_\beta\}$, then $I(\lambda, \beta) = 0$, $J(\lambda, \beta) = 0$, resp., provided that $x_\lambda = x_\beta \equiv x^0$, hence, if the true distribution λ as well as the hypothesis β yield the same (unique) optimal decision $x^0 \in D$. See also the following interpretation of I, J.

As can be seen from Corollaries 8.10, 8.11 and Theorem 8.7 the generalized divergences $I(\lambda, \beta), J(\lambda, \beta)$ can be considered as measures for the deviation between the probability measures λ and β relative to the decision problem (Ω, D, v) or to the loss set V.

Based on the meaning of I and J, we introduce the following definition:

Definition 8.4 I) The right-I-ρ-, right-J-ρ-neighborhood of a distribution $\lambda \in ca_{+,1}(\Omega, \mathcal{A})$ with $v^*(\lambda) \in \mathbb{R}$ is the set defined by

$$U_\rho^{I,r}(\lambda)(U_\rho^{J,r}(\lambda), \textit{resp.}) = \{\beta \in ca_{+,1}(\Omega, \mathcal{A}) : D_\varepsilon(\beta) \neq \emptyset,$$
$$\varepsilon > 0, I(\lambda, \beta) < \rho \text{ resp. } J(\lambda, \beta) < \rho\}$$

II) The left-I-ρ-, left-J-ρ-neighborhood of a distribution $\beta \in ca_{+,1}(\Omega, \mathcal{A})$ with $D_\varepsilon(\beta) \neq \emptyset, \varepsilon > 0$ is the set defined by

$$U_\rho^{I,l}(\beta)(U_\rho^{J,l}(\beta), \textit{resp.}) = \{\lambda \in ca_{+,1}(\Omega, \mathcal{A}) : v^*(\lambda) \in \mathbb{R}, I(\lambda, \beta) < \rho$$
$$\text{resp. } J(\lambda, \beta) < \rho\}.$$

Of course, the divergences I,J, yield also the notion of a "convergence":

Definition 8.5 I) A sequence (β^j) in $ca_{+,1}(\Omega, \mathcal{A})$ is called right-I-, right-J-convergent, resp., towards an element $\lambda \in ca_{+,1}(\Omega, \mathcal{A})$, $\beta^j \to^{I,r} \lambda$, $\beta^j \to^{J,r} \lambda$, $j \to \infty$, resp., provided that $D_\varepsilon(\beta^j) \neq \emptyset, \varepsilon > 0$, $j = 1, 2, \ldots, v^*(\lambda) \in \mathbb{R}$ and $I(\lambda, \beta^j) \to 0$, $J(\lambda, \beta^j) \to 0$, $j \to \infty$, respectively.

II) A sequence (λ^k) in $ca_{+,1}(\Omega, \mathcal{A})$ is called left-I- , left-J-convergent, resp., towards an element $\beta \in ca_{+,1}(\Omega, \mathcal{A})$, $\lambda^k \to^{I,l} \beta$, $\lambda^k \to^{J,l} \beta$, $j \to \infty$, resp., provided that $D_\varepsilon(\beta) \neq \emptyset, \varepsilon > 0, v^*(\lambda^k) \in \mathbb{R}, k = 1, 2, \ldots$, and $I(\lambda^k, \beta) \to 0$, $J(\lambda^k, \beta) \to 0, k \to \infty$, respectively.

Remark 8.14 The distinction between the left- and right-convergence, see the above definitions, is necessary, since $I(\lambda, \beta) \neq I(\lambda, \beta), J(\lambda, \beta) \neq J(\lambda, \beta)$ in general, see the following example.

Example 8.4 Consider $\Omega = \omega_1, \ldots, \omega_n)$ and $V = C_{(1/2)}$. According to Corollary 8.11b we have $I^{(1/2)}(\lambda, \beta) = J^{(1/2)}(\lambda, \beta) = 2(\| \lambda \| - \frac{\lambda^T \beta}{\|\beta\|}), \lambda, \beta \in \mathbb{R}^n_{+,1}$, where $\| \cdot \|$ denotes the Euclidean norm. Then, $I^{(1/2)}(\beta, \lambda) = 2(\| \beta \| - \frac{\beta^T \lambda}{\|\lambda\|})$, and for $\lambda = (1, 0, \ldots, 0)^T, \beta = (1/n, \ldots, 1/n)^T$ we get $I^{(1/2)}(\lambda, \beta) = 2(1 - 1/n^{1/2})$ and $I^{(1/2)}(\beta, \lambda) = 2(1/n^{1/2} - 1/n) = (1/n^{1/2})I(\lambda, \beta)$. Hence, we have $I^{(1/2)}(\beta, \lambda) \neq I^{(1/2)}(\lambda, \beta)$ for $n > 1$. We still mention that $H^{(1/2)}(\lambda, \beta) = 2(1 - \frac{\lambda^T \beta}{\|\beta\|})$, hence, $H^{(1/2)}(\beta, \lambda) = 2(1 - \frac{\beta' \lambda}{\|\lambda\|})$. Consequently, $H^{(1/2)}(\lambda, \beta) = H^{(1/2)}(\beta, \lambda)$ holds if and only if $\| \beta \| = \| \lambda \|$. However, this holds not for all $\lambda, \beta \in \mathbb{R}^n$.

For the class of inaccuracy functions $H^{(b)}(\lambda, \beta), h^{(b)}(\lambda, \beta), b \geq 0$ with $H^{(b)}(\lambda, \beta) = h^{(b)}(\lambda, \beta)$ for $b > 0$, given in Corollary 8.9, we have this result:

Corollary 8.13 *Let* $I(\lambda, \beta) = H^{(b)}(\lambda, \beta) - H^{(b)}(\lambda, \lambda), \lambda, \beta \in \mathbb{R}^n_{+,1}, b > 0$. *Then* $U^{I,l}_\rho(\beta)$ *is convex for all* $\rho > 0, \beta \in \mathbb{R}^n_{+,1}$ *and each* $b > 0$; $U^{I,r}_\rho(\lambda)$ *is convex for all* $\rho > 0, \lambda \in \mathbb{R}^n_{+,1}$ *and each* $1/2 \leq b \leq 1$.

Remark 8.15 Generalizations of the Kullback-Divergence

Pure mathematical generalizations of the Kullback-divergence $I^{(l)}(\lambda, \beta) = \int p_\lambda(\omega) log(p_\lambda(\omega)/p_\beta(\omega)) m(d\omega)$ as well as of the related Kerridge-inaccuracy $H^{(l)}(\lambda, \beta) = \int p_\lambda(\omega) log(1/p_\beta(\omega)) m(d\omega)$, where $p_\lambda = \frac{d\lambda}{dm}, p_\beta = \frac{d\beta}{dm}$ and m denotes a measure on $(\Omega, \mathcal{A})$, are suggested by several authors, see e.g.[33, 34, 53]. We mention here, cf. [33], the f-divergence

$$J_f(\lambda, \beta) = \int p_\beta(\omega) f(p_\lambda(\omega)/p_\beta(\omega)) m(d\omega),$$

where f: $\mathbb{R}_+ \to \mathbb{R}$ is a convex function.

In these papers information can be found about the type of *geometry* induced by an f-divergence, i.e., by the related system of J_f-neighbourhoods, see Definition 8.4 on (subsets of) $ca_{+,1}$. For example, in [35] is shown that certain topological properties of $J_f(\lambda, \beta)$ with $f(t) = t log(t)$, hence, $J_f(\lambda, \beta) = I^{(l)}(\lambda, \beta))$, are related to the squared distance $d^2(\lambda, \beta)$ of the Euclidean distance $d(\lambda, \beta) = \| \lambda - \beta \|, \lambda, \beta \in H$ in a Hilbert space.

In the following we show now that also the generalized divergences I, J defined by (8.38a, b) have similar properties as the squared Euclidean distance in a Hilbert space.

8.3.2 *I*-, *J*-Projections

As was shown in the literature, see e.g. [53, 67, 81], in the information-theoretical foundation of statistics, the following minimization problem plays a major role:

$$\begin{aligned} \min \quad & I^{(1)}(\lambda, \beta) \\ \text{s.t.} \quad & \lambda \in C. \end{aligned} \tag{8.39}$$

Here, $I^{(1)}(\lambda, \beta)$ denotes the die Kullback-divergence between a given probability measure $\beta \ll m$(measure on $\mathcal{A}$) and $\lambda \in C$, where C is a certain subset of $\{\lambda \in ca_{+,1}(\Omega, \mathcal{A}) : \lambda \ll m\}$.

In order to find a further relation between the divergences I, J and the squared Euclidean distance in a Hilbert space H, we replace the divergence $I^{(l)}(\lambda, \beta)$ in (8.39) by $\| \lambda - \beta \|^2$, where λ, β, C, resp., are considered as elements, a subset of a Hilbert space H, then we obtain the optimization problem

$$\begin{aligned} \min \quad & \| \lambda - \beta \|^2 \\ \text{s.t.} \quad & \lambda \in C. \end{aligned} \tag{8.40}$$

However, this problem represents the projection of $\beta \in H$ onto the subset $C \subset H$. As is well known, in case of a convex set C, a solution β_0 of (8.40) is then characterized by the condition

$$< \beta_0 - \beta, \lambda - \beta_0 > \ \geq \ 0 \text{ for all } \lambda \in C, \tag{8.41}$$

where $< \lambda, \beta >$ denotes the scalar product in the Hilbert space H. Putting $d^2(\lambda, \beta) = \| \lambda - \beta \|^2$, then (8.41) is equivalent with

$$d^2(\lambda, \beta) \geq d^2(\lambda, \beta_0) + d^2(\beta_0, \beta) \text{ for all } \lambda \in C, \tag{8.42}$$

where in (8.41) and in (8.42) the equality sign (and therefore, of course, the theorem of Pythagoras) holds, if β_0 lies in the relative algebraic interior of C. As was shown in [35], the optimization problem (8.39) can also be interpreted as a (generalized) projection problem, sind a solution β_0 (8.39) can be characterized by the condition

$$I^{(1)}(\lambda, \beta) \geq I^{(1)}(\lambda, \beta_0) + I^{(1)}(\beta_0, \beta) \text{ for all } \lambda \in C \tag{8.43}$$

analogous to (8.42).

We show now that a corresponding result can be obtained also for the minimization problems

$$\begin{aligned} \min \quad & I(\lambda, \beta) \\ \text{s.t.} \quad & \lambda \in C \end{aligned} \tag{8.44}$$

and

$$\begin{aligned} \min \quad & J(\lambda, \beta) \\ \text{s.t.} \quad & \lambda \in C, \end{aligned} \tag{8.45}$$

where $I(\lambda, \beta)$, $J(\lambda, \beta)$ denote the divergences according to (8.38a), (8.38b).

Let denote Λ a convex subset of $ca_{+,1}$, such that $H(\lambda, \beta), h(\lambda, \beta)$ are defined and $H(\lambda, \beta) \in \mathbb{R}, h(\lambda, \beta) \in \mathbb{R}, v^*(\lambda) \in \mathbb{R}$ for all $\lambda, \beta \in \Lambda$. Moreover, let be β a fixed element of Λ and C a subset of Λ. Now, a "projection" of β onto C is defined as follows:

Definition 8.6 A solution β_0 of (8.44), (8.45), resp., is called an I-, an J-projection, resp., of β onto C.

Some properties of I-, J-projections are given in the following:

Theorem 8.8 *Suppose that C is convex, and $\lambda \to H(\lambda, \lambda_0)$, $\lambda \to h(\lambda, \lambda_0)$, resp., is affine linear on Λ for $\lambda_0 = \beta$ and all $\lambda_0 \in C$. Moreover, assume that for all $\lambda, \lambda_0 \in C$ the continuity condition $H(\lambda, \lambda_0 + t(\lambda - \lambda_0)) \to H(\lambda, \lambda_0)$, $h(\lambda, \lambda_0 + t(\lambda - \lambda_0)) \to h(\lambda, \lambda_0)$ holds for $t \downarrow 0$.*

I) A necessary condition for an I-, J-projection β_0, resp., of β onto C is then the condition (analogous to (8.42), (8.43))

$$I(\lambda, \beta) \geq I(\lambda, \beta_0) + I(\beta_0, \beta) \ \text{ for all } \lambda \in C, \tag{8.46}$$

$$J(\lambda, \beta) \geq J(\lambda, \beta_0) + I(\beta_0, \beta) \text{ for all } \lambda \in C, \tag{8.47}$$

resp., where the sign "=" holds, provided that β_0 lies in the relative algebraic interior of C.

II) If $\lim_{t \downarrow 0} \frac{1}{t} I(\beta_0, \beta_0 + t(\lambda - \beta_0)) = \lim_{t \downarrow 0} \frac{1}{t}(H(\beta_0, \beta_0 + t(\lambda - \beta_0)) - H(\beta_0, \beta_0)) = 0$,

$$\lim_{t \downarrow 0} \frac{1}{t} J(\beta_0, \beta_0 + t(\lambda - \beta_0)) = \lim_{t \downarrow 0} \frac{1}{t}(h(\beta_0, \beta_0 + t(\lambda - \beta_0)) - h(\beta_0, \beta_0)) = 0,$$

resp., for all $x \in C$ and a $\beta_o \in C$, then (8.46), (8.47) is also sufficient for an I-, J-projection β_0 of β onto C.

8.3.3 Minimum Discrimination Information

An important reason for the consideration of the I-, J-projections according to Definition 8.6 is the following generalization of the *minimum discrimination information*, a concept that was introduced in [81] for the foundation of methods of statistics. We suppose here the unknown (partly known) probability distribution $P_{\tilde{\omega}} = \lambda$ of $\tilde{\omega}$ lies in a subset Λ of $ca_{+,1}(\Omega, \mathcal{A})$ and satisfies an equation of the type

$$\int T(\omega)\lambda(d\omega) = \theta(\equiv ET(\omega)), \tag{8.48}$$

where $T : \Omega \to \theta$ is a measurable mapping from $(\Omega, \mathcal{A})$ into a further measurable set $(\Theta, \mathcal{B})$, and θ is an element of Θ. The element θ is interpreted as a certain parameter of the distribution P_ω; moreover, it is assumed that estimates $\tilde{\theta} = \tilde{\theta}_N(\omega_1, \ldots, \omega_n)$ are available for θ, where ω_k is a realization of $\tilde{\omega}$.

For a known parameter θ the set

$$C = C(\theta) = \{\lambda \in ca_{+,1}(\Omega, \mathcal{A}) : \lambda \in \Lambda, \int T(\omega)\lambda(d\omega) = \theta\}$$

describes the information available on $P_{\tilde{\omega}} = \lambda$. For a given hypothesis $P_{\tilde{\omega}} = \beta$ the I-projection of β onto $C(\theta)$ describes then the nearest element of $C = C(\theta)$ to β, and

$$\begin{aligned} I(\star, \beta) = I(\star, \beta; \theta) &= \mathit{inf}\big\{I(\lambda, \beta) : \lambda \in C(\theta)\big\} \\ &= \mathit{inf}\Big\{I(\lambda, \beta) : \lambda \in \Lambda, \int T(\omega)\lambda(d\omega) = \theta\Big\} \end{aligned} \tag{8.49}$$

denotes the distance between β and $C(\theta)$ (often identified with θ). Hence, an increasing distance $I(\star, \beta)$ between β and $C(\theta)$ means a decreasing quality of the hypothesis $P_{\tilde{\omega}} = \beta$.

Corresponding to [81] we introduce therefore the following notion:

Definition 8.7 The value $I(\star, \beta) = I(\star, \beta; \theta)$ is called the minimum useful discrimination information—relative to the decision problem (Ω, D, v)—against the (zero-)hypothesis $P_{\tilde{\omega}} = \beta$.

Remark 8.16 Useful Discrimination-Information

The notion useful discrimination information emphasizes the fact that the generalized divergence $I(\lambda, \beta)$ measures the difference between the distributions λ and β relative to a (subsequent) decision problem (Ω, D, v); see also the definition of *economic information measures* used in [53, 88].

We show now some properties of the function $\theta \to I(\star, \beta; \theta)$.

Lemma 8.5 *Let Λ be convex, Θ a linear parameter space and $\lambda \to I(\lambda, \beta)$ convex on Λ. Furthermore, let $\Theta_0 = \{\theta \in \Theta$: there is $\lambda \in \Lambda$, such that $\int T(\omega)\lambda(d\omega) = \theta\}$. Then Θ_0 is convex, and $\theta \to I(\star, \beta; \theta)$ is convex on Θ_0.*

Proof Let $\theta_1, \theta_2 \in \Theta_0$ and $0 < \alpha < 1$. Then there are elements $\lambda_1, \lambda_2 \in \Lambda$, such that $\theta_i = \int T(\omega)\lambda_i(d\omega), i = 1, 2$. This yields $\int T(\omega)(\alpha\lambda_1 + (1-\alpha)\lambda_2)(d\omega) = \alpha \int T(\omega)\lambda_1(d\omega) + (1-\alpha)\int T(\omega)\lambda_2(d\omega)$ and $\alpha\lambda_1 + (1-\alpha)\lambda_2 \in \Lambda$, hence, $\alpha\theta_1 + (1-\alpha)\theta_2 \in \Theta_0$. Furthermore,

$$I(\star, \beta; \alpha\theta_1 + (1-\alpha)\theta_2 \leq I(\alpha\lambda_1 + (1-\alpha)\lambda_2, \beta) \leq \alpha I(\lambda_1, \beta) + (1-\alpha) I(\lambda_2, \beta),$$

which yields the rest of the assertion, since, up to the above conditions, λ_1, λ_2 were arbitrary.

Remark 8.17 Convexity of $I(., \beta)$. Because of the concavity of $\lambda \to v^*(\lambda) = H(\lambda, \lambda)$, the function $\lambda \to I(\lambda, \beta)$ is convex, provided that $H(\lambda, \beta) = H_0(\lambda, \beta) = sup\{\lambda f : f \in V_0(\beta)\}$.

If Θ is a finite dimensional space, then the convexity of $\theta \to I(\star, \beta; \theta)$ on Θ_0 yields the continuity of this function—at least—on the relative interior $ri\Theta_0$ of Θ_0. If the function $\lambda \to I(\star, \beta; \lambda)$ is continuous on a sufficiently large range of definition, then

$$I(\star, \beta; \hat{\theta}_N) \to I(\star, \beta; \theta) = I(\star, \beta) \ \ a.s.,$$

provided that $\hat{\theta}_N \to \theta$ a.s., where $\hat{\theta}_1, \hat{\theta}_2, \ldots$ is a sequence of estimation functions for θ. In this case we interpret then

$$\hat{I}(\star, \beta) = I(\star, \beta; \hat{\theta}_N(\omega_1, \ldots, \omega_N)) \ \ (N = 1, 2, \ldots)$$

as an estimate of $I(\star, \beta)$. For the testing of hypotheses we have then, cf. [81] the following procedure:

Definition 8.8 Reject the (null-) hypothesis $P_{\tilde{\omega}} = \beta$, if $\hat{I}(\star, \beta)$ is significantly large.

For illustration of this test procedure we give still the following example:

Example 8.5 Let $\Omega = \Theta = D = \mathbb{R}$ and $v(\omega, x) = (a(\omega)x - b(\omega))^2$, where $x \in \mathbb{R}$ and $a(\cdot), b(\cdot)$ are square integrable random variables. Then,

$$I(\lambda, \beta) = \overline{a^2}^{\lambda}\left(\overline{ab}^{\lambda}/\overline{a^2}^{\lambda} - \overline{ab}^{\beta}/\overline{a^2}^{\beta}\right)^2,$$

where $\overline{a^2}^{\lambda} = \int a(\omega)^2\lambda(d\omega)$ etc. If now $T(\omega) = a(\omega)b(\omega)$, then $I(\star, \beta) = inf\left\{\overline{a^2}^{\lambda}(\overline{ab}^{\lambda}/\overline{a^2}^{\lambda} - \overline{ab}^{\beta}/\overline{a^2}^{\beta})^2 : \int a(\omega)b(\omega)\lambda(d\omega) = \theta\right\} = \inf_{u>0} u(\theta/u - \overline{ab}^{\beta}/\overline{a^2}^{\beta})^2$, hence, $I(\star, \beta) = 0$, provided that $sign\theta = sign(\overline{ab}^{\beta})$ and $I(\star, \beta) = 4 \mid \theta\overline{ab}^{\beta} \mid$, if $sign\theta = -sign(\overline{ab}^{\beta})$.

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Index

K. Marti, *Stochastic Optimization Methods*, DOI 10.1007/978-3-662-46214-0