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ABSTRACT

The present paper is the sequel to a previous paper by the author. The main purpose of the present paper is to justify the claim made earlier that the approach to stochastic optimal control previously employed can be extended to the case of noisy observations of the state.

The partial differential equation, for the conditional probability density of the present state given the past history of the noisy observations, is discussed. Using this equation, it is shown that the stochastic optimal control problem can be viewed as a problem in the theory of control of distributed parameter systems. Dynamic programming is then applied to this distributed parameter problem, to obtain a stochastic Hamilton–Jacobi equation in function space.

§ 1. INTRODUCTION

THE present paper is the sequel to a paper previously published in this Journal (Mortensen 1966 b). The basic philosophy of both papers concerning stochastic optimal control problems is the following. Rather than attempt to steer the controlled object from a specified initial state to a specified target state, the proper point of view is to deal only with probability distributions of the state. The evolution in time of the probability distribution of the state is described by a partial differential equation of the Fokker–Planck–Kolmogorov type. The aim of control should be to influence this evolution in time so as to steer from a given initial probability distribution to a given terminal distribution in such a way as to minimize the expected value of the cost of control.

When the problem is formulated in this way, the dynamic equations of the controlled object itself no longer enter directly into the mathematical description of the control problem. Rather, one is now attempting to apply control to the partial differential equation for the probability distribution. Thus, in this formulation, the stochastic optimal control problem becomes a special case of the problem of optimal control of distributed parameter systems.

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Distributed parameter control problems have been studied recently by a number of workers, e.g. Axelband (1965), Balakrishnan (1965), Wang (1964), among others. The main feature of these problems is that the state vector is now infinite dimensional rather than finite dimensional. Stated another way, one now deals with a *state function* (similar to the Schroedinger formulation of quantum mechanics) rather than the usual *state vector*. The optimization problem now requires the use of function space techniques, e.g. the Frechet derivative.

In the previous paper (Mortensen 1966 b) the stochastic optimal control problem was studied for the case when no observations at all of the actual state of the controlled object are available. The probability density of the actual state was taken as the state function, and dynamic programming was used to determine an optimal control input which is a *functional of the state function*. This procedure led to a Hamilton–Jacobi type of equation in which Frechet derivatives occurred instead of the conventional partial derivatives. Also in that paper it was claimed that the same techniques could be extended to apply to the case when noisy observations of the actual state are available. It is the purpose of the present paper to justify that claim.

§ 2. Conditioning on Noisy Observations

During the past several years a number of workers have considered the problem of finding the conditional probability distribution of the current state of a dynamic system perturbed by noise, given the entire past history of noisy observations of the state (Stratonovich 1960, Kashyap 1964, Kushner 1964 a, b, Bucy 1965). Perhaps the first satisfying presentation of the results was given in the note by Bucy.

The main result is a partial differential equation for the conditional probability density, which is similar to the usual Fokker–Planck equation except for the presence of an additional stochastic forcing term involving the noisy observations. When this equation is solved, the conditional probability density for the current state will be a *functional* of the entire past history of the noisy observations.

When the stochastic optimal control problem is treated in the manner described in the Introduction, one seeks the optimal control input as a functional of the conditional probability density. Ultimately, therefore (by composition of functionals), the optimal control input may be expressed as a functional of the entire past history of the noisy observations, which is exactly what is desired. However, by taking the conditional probability density as the *state function*, the optimal control input will depend upon the past history of the observations *only intermediately through this state function*. The state function therefore plays the role of a *sufficient statistic* upon the observations for control purposes.

This type of approach to the problem of optimal control of continuous time, continuous state, non-linear stochastic dynamic systems, when only noisy observations of the state are available, was sketched very briefly by Kushner (1964a). However, Kushner's derivation suffered from some mathematical defects, mainly due to difficulties involving the so-called Ito stochastic calculus (Skorokhod 1965). Although Bucy's note (Bucy 1965) correctly employed the Ito calculus, no application was made to the control problem. In the author's dissertation (Mortensen 1966a) a fully rigorous derivation of the equation of evolution of the conditional probability density is given, along with its application to the control problem.

In the present paper, we will not aim to achieve the standard of rigor which was striven for in the dissertation. Rather, a decidedly engineering approach will be taken. Our programme is to start from the partial differential equation for an auxiliary function related to the conditional density, and to use dynamic programming to derive a Hamilton-Jacobi equation whose solution will yield the optimal control.

§ 3. Problem Formulation

Following the standard probability–theoretic approach to problems involving stochastic processes, we begin by assuming there exists an underlying abstract probability space (Ω, A, P), where Ω is the sample space, A is a σ-algebra of events and P is a probability measure. The sample space variable is denoted by ω.

The dynamics of the object to be controlled are described by the stochastic differential equation:

$$dx(t, \omega) = f(x, u, t)\, dt + dw(t, \omega). \tag{1}$$

Here x is the state, u is the control input, t is time and (t, ω) is a standard Wiener process.

The description of the noisy observations of the state is given by another stochastic differential equation:

$$dy(t, \omega) = h(x, t)\, dt + dv(t, \omega). \tag{2}$$

Here y is the observed output and $v(t, \omega)$ is another standard Wiener process, which we assume is stochastically independent of $w(t, \omega)$. For simplicity of exposition all the variables x, u, y, w, v are assumed to be scalar valued, although all of our results generalize to the vector-valued case.

In order for the rigorous theory to apply, certain restrictions must be imposed on the functions f and h in (1) and (2). Sufficient restrictions are that $f(x, u, t)$ be jointly continuous in x, u and t; once differentiable with respect to each x and u; and uniformly bounded. Also, $h(x, t)$ must be jointly continuous in x and t and uniformly bounded.

Unfortunately, the uniform boundedness restriction is so strong as to exclude the case where f is linear in x and u, and h is linear in x, which is the one case for which the present problem has been solved by other means. The reason for the boundedness condition is a highly technical one involving certain considerations pertaining to integration in function space. Refer to Mortensen (1966a) for details.

We expect that it should be possible to weaken this restriction in the future. Physically, it means that only saturating nonlinearities are admissible. Since in fact practically all physical devices do saturate at a high enough level, this restriction may not be too inconsistent with reality. See Skorokhod (1965) for a thorough discussion of stochastic differential equations and Wiener processes.

For definiteness we assume we are working on a fixed, finite interval of time, $0 \leq t \leq T$.

The result obtained by Bucy, as well as in the author's dissertation, is that the desired conditional probability density $p(x, t, \omega)$ may be expressed as the normalization of an auxiliary function $\Gamma(x, t, \omega)$:

$$p(x, t, \omega) = \frac{\Gamma(x, t, \omega)}{\int_{-\infty}^{\infty} \Gamma(x, t, \omega)\, dx}. \tag{3}$$

This function $\Gamma(x, t, \omega)$ is obtained as the solution to the following stochastic partial differential equation:

$$d\Gamma(x, t, \omega) = \left\{ -\frac{\partial}{\partial x}\left[f(x, u, t)\Gamma\right] + \frac{1}{2}\frac{\partial^2 \Gamma}{\partial x^2}\right\} dt + \Gamma h(x, t)\, dy\,(t, \omega). \tag{4}$$

Although the observations $y(t, \omega)$ will be a known, definite function of time from $t = 0$ up to the present, for control purposes we must use (4) to predict the future evolution of the state function Γ, from the present until the terminal time $t = T$. Of course, the observations yet to be received in the future are unknown, and therefore we must insist on including the dependence upon the sample space variable ω in (4); i.e. the function $\Gamma(x, t, \omega)$ is in fact a random function.

The appropriate initial condition for both Γ and p at time $t = 0$ is:

$$\Gamma(x, 0, \omega) = p(x, 0, \omega) = p_0(x), \tag{5}$$

where $p_0(x)$ is the given *a priori* probability density of the state at time zero.

Incidentally, in the approach suggested by Kushner (1964a), the density $p(x, t, \omega)$ was taken as the state function for control purposes. However, it is clear from (3) that if we know $\Gamma(x, t, \omega)$ then also we know $p(x, t, \omega)$. We prefer to use $\Gamma(x, t, \omega)$ as the state function because eqn. (4) obeyed by it is considerably simpler than the corresponding equation for $p(x, t, \omega)$.

Let $p_T(x)$ be the terminal probability density function toward which it is desired to steer $p(x, t, \omega)$ at the terminal time $t = T$. Let Φ be a penalty functional for missing this goal. By (3) it is possible to express this penalty in terms of p_T and $\Gamma(T)$, as $\Phi(\Gamma(T), p_T)$.

The total cost of control, which we seek to minimize, is assumed given in the form:

$$J = \Phi(\Gamma(T), p_T) + \int_0^T V(\Gamma(t), u, t)\, dt. \tag{6}$$

It is understood here that we are seeking u as a functional of the current state function $\Gamma(t)$ and the current time t, i.e. $u=u(\Gamma(t),t)$. In a deterministic problem, one might seek to minimize a cost functional of the form:

$$J_1=\int_0^T L(x,u,t)\,dt. \tag{7}$$

A stochastic version of the problem would be to minimize the expected value of J_1. From the definition of Γ, this can be accomplished by choosing V in (6) as follows:

$$V(\Gamma,u,t)=\frac{\int_{-\infty}^{\infty} L(x,u,t)\,\Gamma(x,t)\,dx}{\int_{-\infty}^{\infty} \Gamma(x,t)\,dx}. \tag{8}$$

Henceforth we will suppress dependence on the sample space variable ω, for brevity.

§ 4. Dynamic Programming in Function Space

We now proceed to the dynamic programming formalism. From the above discussion, it is clear that at any time prior to the terminal time T, the quantity J in (6) will be a random quantity even if a deterministic control were used, because the future evolution of Γ is unknown. We define the functional $S(\Gamma,t,p_T,T)$ to be the minimum of the conditional expected value of the cost yet to be incurred from the present to the terminal time, given the present state function $\Gamma(x,t)$. Henceforth we will often omit the explicit dependence of Γ upon x:

$$\begin{aligned} S(\Gamma,t,p_T,T)= \min_{\substack{u(\Gamma(\tau),\tau)\\ t\le\tau\le T}} \Bigg[& E\{\Phi(\Gamma(T),p_T)|\Gamma(t)\} \\ & +E\left\{\int_t^T V(\Gamma(\tau),u(\Gamma(\tau),\tau),\tau)\,d\tau|\Gamma(t)\right\}\Bigg]. \end{aligned} \tag{9}$$

All expectation operators are to be interpreted in terms of integrations on the sample space Ω with respect to the probability measure P.

Break the time interval $[t,T]$ into the two sub-intervals $[t,t+\Delta t]$ and $[t+\Delta t,T]$. One may then write:

$$\begin{aligned} S(\Gamma,t,p_T,T) &= \min_{\substack{u\\ t\le\tau\le T}} \Bigg[E\{\Phi|\Gamma(t)\} \\ &\quad +E\left\{\int_t^{t+\Delta} Vd\tau|\Gamma(t)\right\}+E\left\{\int_{t+\Delta t}^{T} Vd\tau|\Gamma(t)\right\}\Bigg] \\ &= \min_{\substack{u\\ t\le\tau\le t+\Delta t}} \Bigg\{ \min_{\substack{u\\ t+\Delta t\le\tau\le T}} \Bigg[E\{\Phi|\Gamma(t)\} \\ &\quad +E\left\{\int_t^{t+\Delta t} Vd\tau|\Gamma(t)\right\}+E\left\{E\left\{\int_{t+\Delta t}^{T} Vd\tau|\Gamma(t+\Delta t)\right\}|\Gamma(t)\right\}\Bigg]\Bigg\}. \end{aligned} \tag{10}$$

Here we have used the properties of iterated minimizations, iterated conditional expectations, and the fact that

$$E\left\{\int_{t+\Delta t}^{T} V\,d\tau \middle| \Gamma(t+\Delta t),\,\Gamma(t)\right\} = E\left\{\int_{t+\Delta t}^{T} V\,d\tau \middle| \Gamma(t+\Delta t)\right\} \tag{11}$$

which follows from the definition of Γ, i.e. $\Gamma(t+\Delta t)$ contains all the information that $\Gamma(t)$ does, and more besides.

Now it is clear that the minimization operation

$$\min_{\substack{u \\ t+\Delta t \leq \tau \leq T}}$$

cannot affect the term $E\left\{\int_{t}^{t+\Delta t} V d\tau \middle| \Gamma(t)\right\}$, so that this term may be brought out from under this operation. Also write:

$$\Gamma(t+\Delta t) = \Gamma(t) + \Delta\Gamma, \tag{12}$$

and observe that

$$S(\Gamma+\Delta\Gamma, t+\Delta t, p_T, T) = \min_{\substack{u \\ t+\Delta t \leq \tau \leq T}} \left[E\{\Phi | \Gamma(t+\Delta t)\} + E\left\{\int_{t+\Delta t}^{T} V d\tau \middle| \Gamma(t+\Delta t)\right\}\right]. \tag{13}$$

Again using iterated conditional expectations,

$$E\{\Phi|\Gamma(t)\} = E\left\{E\{\Phi|\Gamma(t+\Delta t)\}\middle|\Gamma(t)\right\}. \tag{14}$$

Combining these results and applying them to (10), one obtains:

$$S(\Gamma, t, p_T, T) = \min_{\substack{u \\ t \leq \tau \leq t+\Delta t}} \left[E\left\{\int_{t}^{t+\Delta t} V d\tau \middle| \Gamma(t)\right\} + E\{S(\Gamma+\Delta\Gamma, t+\Delta t, p_T, T)|\Gamma(t)\}\right]. \tag{15}$$

Now expand $S(\Gamma+\Delta\Gamma, t+\Delta t, p_T, T)$ in a Taylor series in Frechet derivatives with respect to Γ and conventional partial derivatives with respect to t. For the definition of the Frechet derivative, see Appendix 1 of Mortensen (1966 b):

$$\begin{aligned} S(\Gamma+\Delta\Gamma, t+\Delta t, p_T, T) &= S(\Gamma, t, p_T, T) \\ &+ \frac{\partial S(\Gamma, t, p_T, T)}{\partial t}\Delta t + \int_{-\infty}^{\infty} \frac{\delta S(\Gamma, t, p_T, T)}{\delta \Gamma(x)} \Delta\Gamma(x,t)\,dx \\ &+ \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\delta^2 S(\Gamma, t, p_T, T)}{\delta\Gamma(x_1)\delta\Gamma(x_2)} \Delta\Gamma(x_1,t)\Delta\Gamma(x_2,t)\,dx_1\,dx_2 \\ &+ \dots \end{aligned} \tag{16}$$

In the double integral in (16), x_1 and x_2 are merely dummy variables of integration.

Using eqns. (2)–(4), the definition of Γ, and the properties of the Wiener process, one finds that

$$E\{\Delta\Gamma(x,t)|\Gamma(t)\} = \left\{ -\frac{\partial}{\partial x}[f(x,u,t)\Gamma(x,t)] + \frac{1}{2}\frac{\partial^2\Gamma}{\partial x^2} + \Gamma(x,t)\,h(x,t)\,\hat{h}(t)\right\}\Delta t + \mathrm{O}(\Delta t^2) \tag{17}$$

and

$$E\{\Delta\Gamma(x_1,t)\Delta\Gamma(x_2,t)|\Gamma(t)\} = \Gamma(x_1,t)h(x_1,t)\Gamma(x_2,t)h(x_2,t)\Delta t + \mathrm{O}(\Delta t^2) \tag{18}$$

In (17) we have defined

$$\hat{h}(t) = \frac{\int_{-\infty}^{\infty} h(x,t)\,\Gamma(x,t)\,dx}{\int_{-\infty}^{\infty} \Gamma(x,t)\,dx}\ . \tag{19}$$

In (15), make the approximation:

$$\begin{aligned} &E\left\{\int_t^{t+\Delta t} V(\Gamma(\tau),u(\Gamma(\tau),\tau),\tau)\,d\tau|\Gamma(t)\right\} \\ &= E\{V(\Gamma(t),u(\Gamma(t),t),t)\Delta t + \mathrm{O}(\Delta t^2)|\Gamma(t)\} \\ &= V(\Gamma(t),u(\Gamma(t),t),t)\Delta t + \mathrm{O}(\Delta t^2). \end{aligned} \tag{20}$$

Also, with the understanding that we will eventually let $\Delta t \to 0$, replace

$$\min_{\substack{u(\Gamma(\tau),\tau) \\ t \le \tau \le t+\Delta t}} \quad \text{by} \quad \min_{u(\Gamma(t),t)}\ .$$

Using (16)–(20) in (15), one obtains:

$$\begin{aligned} S(\Gamma,t,p_T,T) = \min_{u(\Gamma(t),t)} \Bigg[& S(\Gamma,t,p_T,T) \\ & + V(\Gamma(t),u(\Gamma(t),t),t)\,\Delta t + \frac{\partial S(\Gamma,t,p_T,T)}{\partial t}\Delta t \\ & + \Delta t \int_{-\infty}^{\infty} \frac{\delta S(\Gamma,t,p_T,T)}{\delta\Gamma(x)}\left\{-\frac{\partial}{\partial x}[f(x,u,t)\Gamma(x,t)] \right. \\ & \left. + \frac{1}{2}\frac{\partial^2\Gamma(x,t)}{\partial x^2} + \Gamma(x,t)\,h(x,t)\hat{h}(t)\right\} dx \\ & + \frac{1}{2}\Delta t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\delta^2 S(\Gamma,t,p_T,T)}{\delta\Gamma(x_1)\delta\Gamma(x_2)}\,\Gamma(x_1,t)h(x_1,t)\Gamma(x_2,t)h(x_2,t)\,dx_1\,dx_2 \\ & + \mathrm{O}(\Delta t^2)\Bigg]. \end{aligned} \tag{21}$$

The higher-order terms indicated by the dots in (16) are all terms whose conditional expectation is $\mathrm{O}(\Delta t^2)$ or higher.

In (21), bring out from under the $\min\limits_{u(\Gamma(t),t)}$ operation those terms which are not affected by it, subtract the common term $S(\Gamma, t, p_T, T)$ from both sides, divide through by Δt, and then pass to the limit $\Delta t \to 0$. The result is the following functional equation of dynamic programming or Hamilton–Jacobi type of equation:

$$0 = \frac{\partial S(\Gamma, t, p_T, T)}{\partial t} + \min_{u(\Gamma(t),t)} \left\{ V(\Gamma(t), u(\Gamma(t), t), t) - \int_{-\infty}^{\infty} \frac{\delta S(\Gamma, t, p_T, T)}{\delta \Gamma(x)} \frac{\partial}{\partial x} [f(x, u(\Gamma(t), t), t) \Gamma(x, t)] \, dx \right\}$$
$$+ \int_{-\infty}^{\infty} \frac{\delta S(\Gamma, t, p_T, T)}{\delta \Gamma(x)} \left[\frac{1}{2} \frac{\partial^2 \Gamma(x, t)}{\partial x^2} + \Gamma(x, t) h(x, t) \hat{h}(t) \right] dx$$
$$+ \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\delta^2 S(\Gamma, t, p_T, T)}{\delta \Gamma(x_1) \delta \Gamma(x_2)} \Gamma(x_1, t) h(x_1, t) \Gamma(x_2, t) h(x_2, t) \, dx_1 \, dx_2. \quad (22)$$

Equation (22) is similar to the Hamilton–Jacobi equation in function space obtained by Mortensen (1966 b) except for the presence of the second-order Frechet derivatives. The relationship between the two is similar to that between the conventional Hamilton–Jacobi equation for deterministic lumped parameter systems and the stochastic Hamilton–Jacobi equation for stochastic lumped parameter systems with perfect observations (e.g. see Wonham 1963). An equation similar to (22) was obtained by Kushner (1964 a) in a much less explicit form.

The question of existence and uniqueness of solutions to the conventional stochastic Hamilton–Jacobi equation (which, of course, only involves conventional partial derivatives rather than Frechet derivatives) has been discussed in a special case by Fleming (1963), using the method of quasi-linearization (Bellman and Kalaba 1965).

Even the theory of linear equations in Frechet derivatives is very rudimentary at the present time, although such equations do occur in quantum electrodynamics (e.g., see Martin and Segal 1964). Note that when the minimization over $u(\Gamma(t), t)$ in (22) is actually carried out, and $u(\Gamma(t), t)$ is eliminated from (22) by expressing it in terms of $S(\Gamma, t, p_T, T)$ and its Frechet derivatives, that (22) will be, in general, nonlinear. However, by generalizing Fleming's approach to cover Frechet differential equations, it should be possible to reduce the problem of solving the non-linear eqn. (22) to that of solving a sequence of linear Frechet differential equation. This would also be a generalization of the method of solving a Hamilton–Jacobi equation in Frechet derivatives by quasi-linearization which was presented by Mortensen (1966 b).

The only case for which we have obtained even a formal solution to (22) is that in which the function $f(x, u, t)$ is linear in x and u, $h(x, t)$ is linear in x, the functional $V(\Gamma, u, t)$ has the form given in (8), and the function $L(x, u, t)$ appearing in (8) is quadratic in x and u. Of course, this solution merely yields the well-known Kalman result for the optimal control (e.g. see Wonham 1963). Even the details of this formal solution were too long to include here, so unfortunately we are unable to provide any examples of the application of (22) at this time.

The main purpose of this paper was to reduce the stochastic optimal control problem to a problem in analysis, i.e. to the problem of solving a differential equation. This has been done, at least formally, and (22) is the result. Even without solving (22), however, it is possible to draw certain conclusions.

§ 5. Conclusions

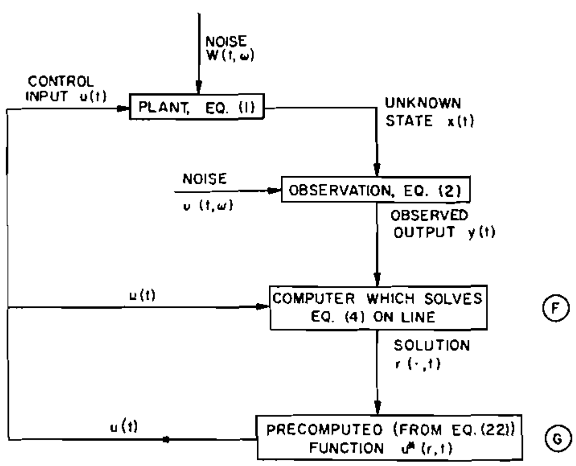
The solution to the control problem thus 'reduces' to finding a functional $S(\Gamma, t, p_T, T)$ which satisfies (22) with the boundary condition:

$$S(\Gamma, T, p_T, T) = \Phi(\Gamma, p_T). \tag{23}$$

This boundary condition follows at once from (9) by letting $t \to T$.

The process of carrying out the $\min\limits_{u(\Gamma(t),t)}$ operation in (22) will yield the optimal control u^* as a functional (i.e. involving an integration with respect to x from $-\infty$ to $+\infty$) of the function $\Gamma(x, t)$. In order to apply this control u^* to the control of the original system (defined by eqns. (1) and (2)), it is then necessary to build a computer which solves eqn. (4) in real time. The function $y(t, \omega)$ in (4) will be the actual observed output $y(t, \omega)$ of (2). The control input u in (4) will be the optimal control u^*. The solution of (4) will be the state function $\Gamma(x, t)$, which is also a functional of the observations $y(\tau)$, $0 \leqslant \tau \leqslant t$. Thus u^* is actually a functional of the past history of the observations, although this dependence occurs intermediately through the state function Γ, as discussed earlier.

A block diagram of the resulting realization of the closed-loop optimal control system is shown in the fig. For the problem considered in this paper,



Closed-loop optimal control system.

all quantities are scalars. However, the function $\Gamma(x,t)$ depends on two independent variables, x and t. It is actually more in keeping with our philosophy of approach to control problems to view this function as an infinite-dimensional vector function $\Gamma(\cdot,t)$ of the one independent variable t, treating x as a continuous index. The fact that $\Gamma(\cdot,t)$ is infinite-dimensional in this sense could lead to certain data transmission problems in practice, of course.

In the case when both the plant and the observations are linear and the function $L(x,u,t)$ in (8) is quadratic in x and u, as mentioned earlier one obtains the well-known Kalman result. The block labelled Ⓕ in the figure reduces to a Kalman–Bucy filter and the block labelled Ⓖ reduces to multiplication by a time-varying gain. Furthermore, the output of Ⓕ is not an infinite-dimensional vector $\Gamma(\cdot,t)$, but rather a scalar $\hat{x}(t)$. This scalar $\hat{x}(t)$ is the conditional expected value of the state of the plant at time t, given the observations $y(\tau)$, $0 \leqslant \tau \leqslant t$. This scalar $\hat{x}(t)$ is a so-called sufficient coordinate for control (see Stratonovich 1962).

It is hoped that, by use of eqns. (4) and (22), it will be possible to find other cases of interest for which the infinite-dimensional vector $\Gamma(\cdot,t)$ in the figure can be replaced by an at most finite number of scalar sufficient coordinates.

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