Advances in Engineering Software 46 (2012) 43–62

Contents lists available at ScienceDirect

Advances in Engineering Software

journal homepage: www.elsevier.com/locate/advengsoft

Optimal control of dynamical systems and structures under stochastic uncertainty Stochastic optimal feedback control

K. Marti

Federal Armed Forces University Munich, Aero-Space Engineering and Technology, 85577 Neubiberg/Munich, Germany

ARTICLE INFO

Article history:
Available online 3 December 2010

Keywords:
Optimal control under stochastic uncertainty
Stochastic optimal control
Robust control
Systems of differential equations with random parameters
Expected total costs
Stochastic optimization methods
Convex approximation
H-minimal control
Two-point boundary value problems with random parameters

ABSTRACT

Consider a dynamic mechanical control systems or structure under stochastic uncertainty, as e.g. the active control of a mechanical structure under stochastic applied dynamic loadings. Optimal controls, being most insensitive with respect to random parameter variations, are determined by finding stochastic optimal controls, i.e., controls minimizing the expected total costs composed of the costs arising along the trajectory, the costs for the control (correction), and possible terminal costs. The problem is modeled in the framework of optimal control under stochastic uncertainty, where the process differential equation depends on certain random parameters having a given probability distribution. Since by computing stochastic optimal controls, random parameter variations are incorporated into the optimal control design, most insensitive or robust controls are obtained.

Based on the stochastic Hamiltonian of the optimal control problem under stochastic uncertainty, the class of "*H*-minimal controls" is determined first by solving a finite-dimensional stochastic program for the minimization of the expected Hamiltonian with respect to the input $u(t)$ at time t.

Having a *H*-minimal control, a two-point boundary value problem with random parameters is formulated for the computation of optimal state-and costate trajectories. Inserting then these trajectories into the *H*-minimal control, stochastic optimal controls are found, or at least stationary controls satisfying the necessary optimality conditions for a stochastic optimal control. Numerical solutions of the two-point boundary value problem are obtained by (i) Discretization of the underlying probability distribution of the random parameters, and (ii) Taylor expansion of the expected total costs and the expected Hamiltonian with respect to the random parameter vector at its expectation. The method is illustrated by the stochastic optimal regulation of a robot.

1. Stochastic control systems – stochastic structural control

Because of the development of high strength materials and more efficient methods in structural analysis and design, larger and more complex mechanical structures, like tall buildings, offshore platforms, have been constructed. Constructions of this type provide only small damping in alleviating vibrations under heavy environmental loads such as strong earthquakes, wind turbulences, water waves, etc. The structures usually are stationary, safe, and stable without external dynamic disturbances, and external dynamic loads are the main sources inducing structural vibrations that should be controlled. Obviously, environmental loads, such as earthquakes, wind, waves, etc., are stochastic processes, having unknown time paths. In order to omit severe structural damages and therefore high compensation (recourse) costs, in recent years active control techniques were developed in structural engineering, see e.g. [24,29,32].

Basically, active control depends on the supply of external energy to counteract the dynamic response of a structure. An active control system consists therefore [9,27,32] of (i) sensors installed at suitable locations of the structure to measure either external excitations or structural response quantities, (ii) devices to process the measured information and to compute necessary control forces based on a given control algorithm, and (iii) actuators to produce the required control forces. Possible technical devices, concepts, actuators to realize active structural controls are e.g. electrohydraulic servomechanisms, passive/active/hybrid/semiactive damping strategies, viscoelastic dampers, tuned mass dampers, aerodynamic appendages, gas pulse generators, gyroscopes, active structural members and joints.

Active structural enhancement consists of the use of active control to modify structural behavior. This enhancement can be used to actively stiffen, or strengthen (against Euler buckling) a given structure. Therefore, actively controlled structures can adaptively modify their stiffness properties to be either stiff or flexible as demanded. E.g., optimal control strategies maximize the critical

E-mail address: kurt.marti@unibw-muenchen.de

0965-9978/$ - see front matter © 2010 Civil-Comp Ltd and Elsevier Ltd. All rights reserved.
doi:10.1016/j.advengsoft.2010.09.008

buckling load using sensors and actuators. The aim is then to actively stabilize the structure to prevent it from collapsing.

An other important practical application are optimal tracking problems and optimal regulator problems under stochastic uncertainty. Here, robust optimal (open-loop) feedback functions can be obtained by minimizing the expected total costs resulting from the tracking error (deviation between the actual trajectory and the given optimal reference trajectory) and the costs of control have to be minimized. Supposing moderate deviations from the reference trajectory, a linear system of differential equations with random parameters results for the tracking error.

While the actual realizations of random parameters and random time dependent parameters are not known at the planning stage, in many practical situations, due to past observations of the process and a priori knowledge about the plant and its working neighborhood, we may assume that the probability distribution or at least the moments of the random variables under consideration are known. The performance of the stochastic dynamic system is evaluated by means of a convex cost function along the trajectory and a convex terminal cost function. The problem is then to determine an open-loop, or an open-loop feedback control law minimizing the expected total costs arising during the run time of the control process.

The basic control system (input–output system) [3,13] is modeled mathematically by a system of first order random differential equations [14,30]:

$$\dot{z}(t) = g(t,\omega,z(t),u(t)), \quad t_0 \leqslant t \leqslant t_f, \quad \omega \in \Omega \tag{1a}$$

$$z(0) = z_0(\omega). \tag{1b}$$

Here, ω is the basic random element taking values in a probability space $(\Omega, \mathcal{A}, P)$, and describing the present random variations of model parameters, such as dynamic parameters, applied load factors, initial values, etc. The plant state vector $z = z(t,\omega)$ is an m-vector involving the displacements and their time derivatives, $z_0(\omega)$ is the random initial state. The plant control $u(t)$ is a deterministic or stochastic n-vector denoting system inputs like external forces or moments. Furthermore, $\dot{z}$ denotes the derivative of z with respect to the time t. We assume that $u = u(t)$ is chosen such that $u(\cdot) \in U$, where U is a suitable linear subspace of the space $PC_0^n[t_0,t_f]$ of piecewise continuous functions $u(\cdot) : [t_0,t_f] \to R^n$ normed by the supremum or maximum norm.

Let Z be a linear space of functions $z(\cdot) : [t_0,t_f] \to R^m$ containing the set $PC_1^m[0,1]$ of possible trajectories of the plant, hence the set of piecewise differentiable functions on the interval $[t_0,t_f]$; the space Z is also normed by the supremum norm. $D(\subset U)$ denotes the convex set of admissible controls $u(\cdot)$. Using the available information up to a certain time t, the problem is then to find a robust optimal control function $u^* = u^*(t)$ minimizing the (conditional) expected costs arising along the trajectory $z = z(t)$ and/or at the terminal state $z_f = z(t_f)$ subject to the plant differential Eq. (1a,b) and the given control and state constraints.

1.1. Random differential equations

Concerning the modeling of solutions of the random differential Eq. (1a,b) there are different possibilities, see e.g. [7,10].

In case of a **discrete** probability distribution of the model parameter, i.e., $\Omega = \{\omega_1, \omega_2, \ldots, \omega_\varrho\}$, $P(\omega = \omega_j) = \alpha_j > 0$, $j = 1,\ldots,\varrho$, $\sum_{j=1}^{\varrho} \alpha_j = 1$, we can *redefine* (1a,b) by

$$\dot{z}(t) = g(t,z(t),u(t)), \quad t_0 \leqslant t \leqslant t_f, \tag{1a$'$}$$

$$z(t_0) = z_0, \tag{1b$'$}$$

where $z(t) := \left(z(t,\omega_j)\right)_{j=1,\ldots,\varrho}$, and with corresponding definitions of g, z_0.

Hence, in this case (1a′,b′) represents again an ordinary system of first order differential equations for the ϱm unknown functions in $z = z(t)$.

Also for the general case we consider a **parametric approach** (solution in the point-wise sense):

Thus, for each random element $\omega \in \Omega$, (1a,b) is interpreted as a system of ordinary 1st order differential equations with initial values $z_0(\omega)$. Hence, we assume that to each deterministic control $u(\cdot) \in U$ and each random element $\omega \in \Omega$ there exists a unique measurable solution

$$z(t) = z_u(t,\omega) = z(t,\omega,u(\cdot)), \tag{2a}$$

denoted for short by

$$z(\cdot) = z_u(\cdot,\omega) = S(\omega,u(\cdot))(\cdot), \tag{2b}$$

$z(\cdot) \in PC_1^m[t_0,t_f]$, of the integral equation

$$z(t) = z_0(\omega) + \int_{t_0}^{t} g(s,\omega,z(s),u(s))ds, \quad t_0 \leqslant t \leqslant t_f. \tag{2c}$$

In order to justify the above assumption, let $\theta = \theta(t,\omega)$ be an r-dimensional stochastic process, such that the sample functions $\theta(\cdot,\omega)$ are continuous with probability one. Furthermore, let

$$\tilde{g} : [t_0,t_f] \times R^r \times R^m \times R^n \to R^m$$

be a continuous function having continuous Jacobians $D_\theta\tilde{q}$, $D_z\tilde{g}$, $D_u\tilde{g}$ with respect to θ, z, u. Now consider the case that the function g of the process differential Eq. (1a,b) is given by

$$g(t,\omega,z,u) := \tilde{g}(t,\theta(t,\omega),z,u), \tag{2d}$$

$(t,\omega,z,u) \in [t_0,t_f] \times \Omega \times R^m \times R^n$. Furthermore, put $U = C^n[t_0,t_f]$, $Z = C_0^m[t_0,t_f]$ and $\Theta = C_0^r[t_0,t_f]$, where $C_0^\nu[t_0,t_f]$ denotes the space of all continuous functions of $[t_0,t_f]$ into R^ν normed by the supremum norm $\|\cdot\|_\infty$. By our assumption we have $\theta(\cdot,\omega) \in \Theta$ a.s. (almost sure). Define

$$\Xi = R^m \times \Theta \times U;$$

Ξ may be considered as the total space of inputs $\xi = (z_0, \theta(\cdot), u(\cdot))$ into the plant, consisting of the random initial state z_0, the random input function $\theta = \theta(t,\omega)$ and the control input $u = u(t)$. Let now the mapping $\tau : \Xi \times Z \to Z$ related to the plant Eq. (1a,b) or (2c) be given by

$$\tau(\xi, z(\cdot))(t) = z(t) - \left(z_0 + \int_{t_0}^{t} \tilde{g}(s,\theta(s),z(s),u(s))ds\right), \tag{2e}$$

$t_0 \leqslant t \leqslant t_f$. Operators of this type are well studied (see e.g. [17]): It is known that τ is continuously Fréchet (F)-differentiable, and that the F-derivative of τ at $(\bar{\xi}, \bar{z}(\cdot))$ is given by

$$\begin{aligned}
&\left(D\tau\left(\bar{\xi},\bar{z}(\cdot)\right)\cdot(\xi,z(\cdot))\right)(t) \\
&\quad = z(t) - \left(z_0 + \int_{t_0}^{t} D_z\tilde{g}\left(s,\bar{\theta}(s),\bar{z}(s),\bar{u}(s)\right)z(s)ds\right. \\
&\qquad + \int_{t_0}^{t} D_\theta\tilde{g}\left(s,\bar{\theta}(s),\bar{z}(s),\bar{u}(s)\right)\theta(s)ds \\
&\qquad \left.+ \int_{t_0}^{t} D_u\tilde{g}\left(s,\bar{\theta}(s),\bar{z}(s),\bar{u}(s)\right)u(s)ds\right), \quad t_0 \leqslant t \leqslant t_f,
\end{aligned} \tag{2f}$$

where $\bar{\xi} = (\bar{z}_0, \bar{\theta}(\cdot), \bar{u}(\cdot))$ and $\xi = (z_0, \theta(\cdot), u(\cdot))$. Especially, for the derivative of τ with respect to $z(\cdot)$ we find

$$\begin{aligned}
&\left(D_z\tau\left(\bar{\xi},\bar{z}(\cdot)\right)\cdot z(\cdot)\right)(t) \\
&\quad = z(t) - \int_{t_0}^{t} D_z\tilde{g}\left(s,\bar{\theta}(s),\bar{z}(s),\bar{u}(s)\right)z(s)ds, \quad t_0 \leqslant t \leqslant t_f.
\end{aligned}$$

The related equation

$$D_z\tau\left(\bar{\xi},\bar{z}(\cdot)\right)\cdot z(\cdot) = y(\cdot), \quad y(\cdot) \in Z,$$

is a linear Volterra integral equation. By our assumptions this equation has a unique solution $z(\cdot) \in Z$, see e.g. [25]. Therefore $D_z\tau(\bar{\xi}, \bar{z}(\cdot))$ is a linear, continuous one-to-one map from Z onto Z. Hence, its inverse exists. Using the implicit function theorem based on the fixed point theorem and the method of successive approximation (Picard-Lindelöf), see e.g. [11], we obtain now the following result:

Lemma 1. *Let $(\bar{\xi}, \bar{z}(\cdot)) \in \Xi \times Z$ be such that $\tau(\bar{\xi}, \bar{z}(\cdot)) = 0$, hence, $\bar{z}(\cdot)$ is the solution of*

$$\dot{z}(t) = \tilde{g}\big(t, \bar{\theta}(t), z(t), \bar{u}(t)\big), \quad t_0 \leqslant t \leqslant t_f, \tag{3a}$$

$$z(t_0) = \bar{z}_0, \tag{3b}$$

where $\bar{\xi} = \big(\bar{z}_0, \bar{\theta}(\cdot), \bar{u}(\cdot)\big)$. Then there is an open neighborhood of $\bar{\xi}$, denoted by $V^0(\bar{\xi})$, such that for each open connected neighborhood $V(\bar{\xi})$ of $\bar{\xi}$ contained in $V^0(\bar{\xi})$ there exists a unique continuous mapping $S : V(\bar{\xi}) \to Z$ such that (a) $S(\bar{\xi}) = \bar{z}(\cdot)$; (b) $\tau(\xi, S(\xi)) = 0$ for each $\xi \in V(\bar{\xi})$, i.e. $z(t) = S(\xi)(t)$, $t_0 \leqslant t \leqslant t_f$, is the solution of

$$z(t) = z_0 + \int_{t_0}^{t} \tilde{g}(s, \theta(s), z(s), u(s))ds, \quad t_0 \leqslant t \leqslant t_f, \tag{3c}$$

where $\xi = (z_0, \theta(\cdot), u(\cdot))$; (c) S is continuously differentiable on $V(\bar{\xi})$, and the partial derivative

$$\zeta(\cdot) = \zeta_{u,h} = D_u S(\xi)h(\cdot), \quad h(\cdot) \in U \tag{3d}$$

of S with respect to u satisfies the integral equation

$$\zeta(t) - \int_{t_0}^{t} D_z\tilde{g}(s, \theta(s), S(\xi)(s), u(s))\zeta(s)ds = \int_{t_0}^{t} D_u\tilde{g}(s, \theta(s), S(\xi)(s), u(s))h(s)ds, \tag{3e}$$

where $t_0 \leqslant t \leqslant t_f$ and $\xi = (z_0, \theta(\cdot), u(\cdot))$.

Note. The random input function $\theta = \theta(t, \omega)$ is not just an additive noise term, but may describe different random variations in the dynamics of the plant, the initial values and the applied loadings.

1.2. Objective function

1.2.1. Control and regulator design by stochastic optimization – robust optimal control

In concrete technical problems, uncertainties arise in the following way, see e.g. [12,23]:

* The initial conditions of a system may not be accurately specified or completely known.
* Technical systems experience disturbances, parametric changes, etc., from their environment or work space.
* Variations of the system model parameters (such as dynamic parameters of the equation of motion), inaccuracy of the system model itself, is a central source of uncertainty.

Many of the actual control design methods are based on some chosen *nominal* set of model parameters. Obviously, controls of this type are not *robust* in general: in a parametric system or structure, a decision variable, as e.g. a control input function $u = u(t)$, a controller or regulator, etc., is called **robust** with respect to a certain set of possible system or model parameters, if it has a satisfactory performance for all parameters under consideration. Thus, robust controls, regulators should be insensitive with respect to parameter variations. Obviously, this is a decision theoretical problem which depends essentially on the amount of information available about the unknown parameters, cf. [23]. Usually, robust control methods are designed to function properly as long as uncertain parameters are within some compact set, as e.g. a multidimensional interval. Robust methods aim to achieve or maintain satisfactory performance, such as controllability, reachability, input–output behavior and/or a certain type of stability (eigenvalues of the related homogeneous system lying in the left half of the complex plane), cf. [1,31], for each parameter vector in the given set of model parameters. Describing the desired property by a scalar criterion, as e.g. by H^∞-functions, cf. [8], also minimax decision rules are applied.

Since the minimax-and/or the *"holds for all parameters"*-criterion are very pessimistic decision criteria, and in many practical cases more detailed information than box-information only is available, also in the optimal control design under uncertainty more recent techniques should be applied. One of these new tools is *Stochastic Optimization* for the case of stochastic uncertainty, which means that the unknown parameters are realizations of certain random variables/vectors having a given (joint) probability distribution. Here, the performance of the controlled system or structure is evaluated first by a certain total cost function. E.g., in regulator optimization problems, using the weighted sum of the costs for the tracking error and the costs for the control corrections, see Section 1.5, one obtains robust feedback controls in terms of stability properties. The different realizations of the random parameter vector with their probability distribution are then incorporated into the optimal control design process by taking expectations with respect to the given a priori/a posteriori information about the plant and its working neighborhood at the time of decision. Thus, determining *stochastic optimal controls*, hence, by minimizing the (conditional) total expected costs, parameter-insensitive, and therefore robust controls are obtained.

Summarizing the above considerations, we get the following *design criterion for robust optimal control:*

The major objective of (feedback) control is to minimize the effects of unknown initial conditions and internal/external influences on system behavior, subject to the constraints of not having a complete representation of the system, cf. [12].

Thus, in case of stochastic uncertainty this can be realized by means of stochastic optimization methods, [23], *hence, by computing stochastic optimal controls as described in the following.*

1.2.2. Finding robust optimal control by means of stochastic optimization methods

According to the above design criterion, robust optimal controls will be obtained by finding **stochastic optimal controls** taking into account stochastic variations of the initial values of the process, dynamic and environmental model parameters. For a deterministic control function $u = u(t)$, $t_0 \leqslant t \leqslant t_f$, the cost criterion $F = F(u(\cdot))$ related to the controlled process $z = z(t, \omega)$ is defined by the conditional expectation

$$F(u(\cdot)) := Ef(\omega, z(\cdot, \omega, u(\cdot)), u(\cdot)). \tag{4a}$$

Here, $E = E(\cdot|\mathcal{A}_{t_0})$, denotes the conditional expectation given the information $\mathcal{A}_{t_0}$ about the control process up to the current starting time point t_0. Moreover, $f = f(\omega, z(\cdot), u(\cdot))$ denote the stochastic total costs arising along the trajectory $z = z(t, \omega)$ and at the terminal point $z_f = z(t_f, \omega)$, cf. [3,30], see also Section 1.4. Hence,

$$f(\omega, z(\cdot), u(\cdot)) := \int_{t_0}^{t_f} L(t, \omega, z(t), u(t))dt + G(t_f, \omega, z(t_f)), \tag{4b}$$

$z(\cdot) \in Z$, $u(\cdot) \in U$. Here,

$$L : [t_0, t_f] \times \Omega \times R^m \times R^n \to R,$$
$$G : [t_0, t_f] \times \Omega \times R^m \to R$$

are given cost functions. We suppose that $L(t, \omega, \cdot, \cdot)$ and $G(t, \omega, \cdot)$ are *convex functions* for each $(t, \omega) \in [t_0, t_f] \times \Omega$, having continuous partial derivatives $\nabla_z L(\cdot,\omega,\cdot,\cdot),\ \nabla_u L(\cdot,\omega,\cdot,\cdot),\ \nabla_z G(\cdot,\omega,\cdot)$. Note that in this case $f = f(\omega, z(\cdot), u(\cdot))$

is a convex function on $Z \times U$ for each $\omega \in \Omega$. Moreover, assume that the expectation $F(u(\cdot))$ exists and is finite for each admissible control $u(\cdot) \in C$.

The aim is now to present stochastic optimization-based methods for the (approximative) solution of the resulting minimum expected cost problems: find deterministic controls $u(\cdot)$ solving

$$\min F(u(\cdot)) \text{ s.t. } u(\cdot) \in D. \tag{5}$$

Problem (5) is of course equivalent $\left(E = E(\cdot|\mathcal{A}_{t_0})\right)$ to

$$\min E\left(\int_{t_0}^{t_f} L(t,\omega,z(t),u(t))dt + G(t_f,\omega,z(t_f))|\mathcal{A}_{t_0}\right) \tag{6a}$$

s.t.

$$\dot z(t) = g(t,\omega,z(t),u(t)), \quad t_0 \leqslant t \leqslant t_f, \quad \text{a.s.} \tag{6b}$$

$$z(t_0,\omega) = z_0(\omega), \quad \text{a.s.} \tag{6c}$$

$$u(\cdot) \in D, \tag{6d}$$

cf. [18,19].

1.3. Control laws

Applying stochastic optimization techniques in order to incorporate information about random model parameters, initial values and applied loadings into the optimal control design parameter-insensitive, hence, robust optimal control process are obtained.

In optimal control of dynamic systems the following types of control laws are considered in practice:

(I) Open-Loop control (OL) Here, the control function $u = u(t)$ is a **deterministic** function depending only on the (a priori) information $\mathcal{A}_{t_0} \subset \mathcal{A}$ about the system, the model parameters, resp., available at the starting time point t_0. Hence, for the optimal selection of optimal (OL) controls

$$u(t) = u\big(t; (t_0, \mathcal{A}_{t_0})\big), t \geqslant t_0, \tag{7a}$$

we get optimal control problems of type (5).

(II) Closed-Loop control (CL) In this case the control function $u = u(t)$ is a **stochastic** function

$$u = u(t,\omega) = u(t,\mathcal{A}_t), t \geqslant t_0 \tag{7b}$$

depending on time t and the total (a posteriori) information $\mathcal{A}_t$ on the system available up to time t. Especially $\mathcal{A}_t$ may contain information about the state $z(t) = z(t,\omega)$ up to time t.

(III) Open-Loop Feedback control (OLF) In this combination of (OL) and (CL) control, cf. [2,16,21,22], at each time point $t \geqslant t_0$ first the optimal open-loop control function for the remaining time interval $s \geqslant t$ is computed, hence,

$$u_{[t,t_f]}(s) = u(s; (t, \mathcal{A}_t)), \quad s \geqslant t. \tag{7c}$$

Then the OLF stochastic optimal control is defined by

$$u(t,\omega) := u(t; (t, \mathcal{A}_t)), \quad t \geqslant t_0, \tag{7d}$$

i.e., the OL control $u_{[t,t_f]}(s),\ s \geqslant t$, for the remaining time interval $[t, t_f]$ is used only for $s = t$. Optimal (OLF) controls are obtained therefore by solving again a control problem of the type (5).

1.4. Stochastic structural control: active control

In structural dynamics, the behavior of the m-vector $z = z(t)$ of displacements with respect to time t is described by a system of second order linear differential equations for (t) having a right hand side being the sum of the stochastic applied load process and the control force depending on a control n-vector function $u(t)$:

$$M\frac{d^2q}{dt^2} + D\frac{dq}{dt} + Kq(t) = f(t,\omega,u(t)) \tag{8a}$$

Hence, the force vector $f = f(t,\omega,u(t))$ on the right hand side of the dynamic equation [9,26–28] is given by the sum

$$f(t,\omega,u) = f_0(t,\omega) + f_a(t,\omega,u) \tag{8b}$$

of the applied load $f_0 = f_0(t,\omega)$ being a vector-valued stochastic process describing e.g. external loads or excitation of the structure caused by earthquakes, wind turbulences, water waves, etc., and the actuator or control force vector $f_a = f_a(t,\omega,u)$. Here, ω denotes the random element lying in a certain probability space (Ω, A, P), used to represent random variations. Furthermore, M, D, K, resp., denotes the $m \times m$ mass, damping and stiffness matrix. In many cases the actuator or control force f_a is linear, i.e.

$$f_a = Cu \tag{8c}$$

with a certain $m \times m$ matrix C.

By introducing appropriate matrices, the linear system of second order differential Eq. (8a) can be represented by a system of first order random differential equations as follows:

$$\frac{dz}{dt} = Az(t) + b(t,\omega,u(t)), \tag{9a}$$

where A and b are given by

$$A := \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix} \tag{9b}$$

with the unit matrix I,

$$b(t,\omega,u) := \begin{pmatrix} 0 \\ M^{-1}f(t,\omega,u) \end{pmatrix}. \tag{9c}$$

Moreover, $z = z(t)$ is the $2n$-state vector defined by

$$z = \begin{pmatrix} q \\ w \end{pmatrix} = \begin{pmatrix} q \\ \frac{dq}{dt} \end{pmatrix} \tag{9d}$$

fulfilling a certain initial condition

$$z(t_0) = \begin{pmatrix} q(t_0) \\ \frac{dq}{dt}(t_0) \end{pmatrix} := \begin{pmatrix} q_0 \\ \dot q_0 \end{pmatrix} \tag{9e}$$

with given or stochastic initial values $q_0 = q_0(\omega),\ \dot q_0 = \dot q(\omega)$.

The performance function in case of active structural control is defined by

$$F(u(.)) = Ef(\omega, z(.,\omega,u(.),u(.))), \tag{10a}$$

where

$$f(\omega,z(.),u(.)) := \int_{t_0}^{t_f} L(t,\omega,z(t),u(t))dt + G(t_f,\omega,z(t_f)) \tag{10b}$$

with

$$L(t,\omega,z,u) := z'Q(\omega)z + u'R(\omega)u, \tag{10c}$$

$$G(t_f,\omega,z) := G(z). \tag{10d}$$

Here, Q and R, resp., are certain positive (semi) definite $n \times n,\ m \times m$ matrices.

While the actual time path of the random applied load is not known at the planning stage, we may assume that the probability distribution or at least the occurring moments of the applied load or other random variables are known. The performance of the stochastic dynamic system is evaluated by means of a convex cost function L along the trajectory and a convex terminal cost function G. The problem is then to determine an open-loop, an open-loop feedback control, or a feedback control law minimizing the expected total costs.

1.5. Feedback control of stochastic dynamic systems

Having a prescribed optimal reference trajectory, the problem is to keep the system on this trajectory as exactly as possible by means of control corrections computed by applying a stochastic optimal feedback control law $\varphi = \varphi(t, \Delta q, \Delta\dot{q})$, hence, a *PD*-controller, where $\Delta q,\ \Delta\dot{q}$ denote the tracking error and its time derivative, see e.g. [21,22].

Taylor expansion of the feedback-function φ at $\Delta q = 0,\ \Delta\dot{q} = 0$ and retaining only linear terms, yields the approximation

$$\varphi(t, \Delta q(t), \Delta\dot{q}(t)) \approx D_{\Delta q}\varphi(t,0,0)\Delta q(t) + D_{\Delta\dot{q}}\varphi(t,0,0)\Delta\dot{q}(t). \tag{11}$$

Linearizing also the dynamic equation of the underlying dynamic system around the reference trajectory, its first and second time derivative as well as around the expectation of the stochastic dynamic parameters, we get

$$\begin{aligned} &Y(t)\Delta p_D(\omega) + K(t)\Delta q(t) + D(t)\Delta\dot{q}(t) + M(t)\Delta\ddot{q}(t) \\ &\quad = D_{\Delta q}\varphi(t,0,0)\Delta q(t) + D_{\Delta\dot{q}}\varphi(t,0,0)\Delta\dot{q}(t), \quad t_0 \leqslant t \leqslant t_f, \end{aligned} \tag{12a}$$

with appropriate deterministic matrices $Y(t),\ K(t),\ D(t),\ M(t)$, see [21,22]. Moreover, $\Delta p_D(\omega)$ denotes the deviation of the vector of dynamic parameters from its expectation. The above equation can be rewritten as

$$\begin{aligned} &M(t)\Delta\ddot{q}(t) + \big(K(t) - D_{\Delta q}\varphi(t,0,0)\big)\Delta q(t) \\ &\quad + \big(D(t) - D_{\Delta\dot{q}}\varphi(t,0,0)\big)\Delta\dot{q}(t) \\ &\quad = -Y(t)\Delta p_D(\omega). \end{aligned} \tag{12b}$$

In many applications M is always regular. Hence, we can define

$$K_p(t) := M(t)^{-1}\big(K(t) - D_{\Delta q}\varphi(t,0,0)\big), \tag{12c}$$

$$K_d(t) := M(t)^{-1}\big(D(t) - D_{\Delta\dot{q}}\varphi(t,0,0)\big). \tag{12d}$$

From (12b–d) we get

$$\Delta\ddot{q}(t) = -K_p(t)\Delta q(t) - K_d(t)\Delta\dot{q}(t) - M(t)^{-1}Y(t)\Delta p_D(\omega). \tag{13a}$$

By introducing the state variable

$$\Delta z(t) := \begin{pmatrix} \Delta q(t) \\ \Delta\dot{q}(t) \end{pmatrix}, \tag{13b}$$

the second order differential Eq. (13a) can be transformed into a system of first order random differential equations, cf. (9a–c),

$$\Delta\dot{z}(t,\omega) = \begin{pmatrix} \Delta\dot{q} \\ -K_p(t)\Delta q(t) - K_d(t)\Delta\dot{q}(t) - M(t)^{-1}Y(t)\Delta p_D(\omega) \end{pmatrix}, \tag{13c}$$

$$= \begin{pmatrix} 0 & I \\ -K_p(t) & -K_d(t) \end{pmatrix}\Delta z(t,\omega) + \begin{pmatrix} 0 \\ -M(t)^{-1}Y(t)\Delta p_D(\omega) \end{pmatrix}. \tag{13d}$$

Note. The same technique can also be applied to *PID*-controllers!

2. Convex approximation

Convex approximations of the underlying control problem are obtained by an **inner linearization** of the control problem, i.e., by the linearization of the random dynamic equation.

We observe first that (5) is in general a nonconvex optimization problem, cf. [17]. Since for convex (deterministic) optimization problems there is a well established theory, we approximate the original problems (5) by a sequence of suitable convex problems. In the following we describe a single step of this procedure. We consider here problem (5) or (6a–d) for deterministic controls $u(\cdot)$ as needed in the computation of optimal (OL), (OLF) controls being most important for practical applications.

Let $v(\cdot) \in C$ be an arbitrary, but fixed admissible initial or reference control and assume (see Lemma 1):

Assumption 2.1. *$S(\omega,\cdot)$ is F-differentiable at $v(\cdot)$ for each $\omega \in \Omega$.*

Denote by $DS(\omega, v(\cdot))$ the *F*-derivative of $S(\omega,\cdot)$ at $v(\cdot)$, and replace now the cost function $F = F(u(\cdot))$ by

$$F_{v(\cdot)}(u(\cdot)) := Ef(\omega, S(\omega, v(\cdot)) + DS(\omega, v(\cdot))(u(\cdot) - v(\cdot)), u(\cdot)) \tag{14}$$

where $u(\cdot) \in U$. Assume that $F_{v(\cdot)}(u(\cdot))$ exists and is finite for all pairs $(u(\cdot), v(\cdot)) \in D \times D$. Then, replace (5), see [18], by

$$\min F_{v(\cdot)}(u(\cdot)) \text{s.t.} u(\cdot) \in D. \tag*{$(5)_{\mathrm{v}(\cdot)}$}$$

Lemma 2. *$(5)_{v(\cdot)}$ is a convex optimization problem.*

Proof. According to Section 1.2, function (4b) is convex. The assertion follows now from the linearity of the *F*-differential of *S*.

Remark 3. Note that the approximate $F_{v(\cdot)}$ of F is obtained from (4a,b) by means of linearization of the input–output map $S = S(\omega, u(\cdot))$ with respect to the control $u(\cdot)$ at $v(\cdot)$, hence, by **inner linearization** of the control problem with respect to the control $u(\cdot)$ at $v(\cdot)$.

Let $\phi'_+(x;y)$ denote the directional derivative at $x \in X$ and in direction $y \in X$ of a convex function $\phi : X \to R$, where X is a linear space. With this definition, from Lemma 2 we may infer that for all $u(\cdot), v(\cdot) \in D$ and $h(\cdot) \in U$ the directional derivative of $F_{v(\cdot)}$ is given by

$$\begin{aligned} F'_{v(\cdot)+}(u(\cdot); h(\cdot)) = Ef'_+(&\omega, S(\omega, v(\cdot)) + DS(\omega, v(\cdot))(u(\cdot) - v(\cdot)), \\ &u(\cdot); DS(\omega, v(\cdot))h(\cdot), h(\cdot)). \end{aligned} \tag{15}$$

A solution $\bar{u}(\cdot) \in D$ of $(6)_{v(\cdot)}$ is then characterized by

$$F'_{v(\cdot)+}(\bar{u}(\cdot); u(\cdot) - \bar{u}(\cdot)) \geqslant 0 \quad \text{for all } u(\cdot) \in D. \tag{16}$$

Definition 4. For each $v(\cdot) \in D$, let $M(v(\cdot))$ be the set of solutions of problem $(5)_{v(\cdot)}$.

Assumption 2.2. *$M(v(\cdot)) \neq \emptyset$ for each $v(\cdot) \in D$.*

The relation between our original problem (5) and the family of its approximates $(5)_{v(\cdot)},\ v(\cdot) \in D$, is shown in the following.

Theorem 2.1. *Suppose that*

$$F'_+(v(\cdot); h(\cdot)) = F'_{v(\cdot)+}(v(\cdot); h(\cdot)) \tag{17}$$

for each $v(\cdot) \in D$ and $h(\cdot) \in D - D$. Then:

(1) *If $\bar{u}(\cdot)$ is an optimal control, then $\bar{u}(\cdot) \in M(\bar{u}(\cdot))$, i.e. $\bar{u}(\cdot)$ is a solution of $(5)_{\bar{u}(\cdot)}$.*
(2) *If (5) is convex, then $\bar{u}(\cdot)$ is an optimal control if and only if $\bar{u}(\cdot) \in M(\bar{u}(\cdot))$.*

Lemma 5.

(a) *If $\bar{u}(\cdot) \notin M(\bar{u}(\cdot))$ for a control $\bar{u}(\cdot) \in D$, then*

$$F_{\bar{u}(\cdot)}(u(\cdot)) < F_{\bar{u}(\cdot)}(\bar{u}(\cdot)) = F(\bar{u}(\cdot)) \quad \text{for each } u(\cdot) \in M(\bar{u}(\cdot))$$

(b) *Let the controls* $u(\cdot),\ v(\cdot) \in D$ *be related such that*

$$F_{u(\cdot)}(v(\cdot)) < F_{u(\cdot)}(u(\cdot)).$$

Provided that (17) *holds for the pair* $(u(\cdot), h(\cdot)), h(\cdot) = v(\cdot) - u(\cdot)$, *then* $h(\cdot)$ *is an admissible direction of decrease for F at* $u(\cdot)$, *i.e. we have* $F(u(\cdot) + \varepsilon h(\cdot)) < F(u(\cdot))$ *and* $u(\cdot) + \varepsilon h(\cdot) \in D$ *on a suitable interval* $0 < \varepsilon < \bar{\varepsilon}$.

Proof.

See [19].

The above results suggest the next definition.

Definition 6. A control $u(\cdot) \in D$ such that $u(\cdot) \in M(u(\cdot))$ is called a **stationary** control.

Under the rather weak assumptions in Theorem 2.1 an optimal control is also stationary, and in the case of a convex problem (5) the two concepts coincide. Hence, stationary controls are candidates for optimal controls. As an appropriate substitute/approximate for an optimal control we may determine therefore stationary controls. Based on the above concepts, stationary controls may be obtained by means of "conditional gradient"-type algorithms, cf. [17,19]. Note that the convex approximation $(5)_{v(\cdot)}$ of problem (5) can also be used to find a certain *improvement* of a given *initial* feasible control $v(\cdot) := u_0(\cdot)$.

3. Computation of $F'_{(\cdot)+}$ and H-minimum controls

Considering the necessary and sufficient optimality condition for the convex approximation the original problem, (approximative) optimal controls may be represented by means of H-minimum controls which are solutions of certain finite-dimensional stochastic optimization problems. Furthermore, having the H-minimum controls, the solution may be obtained by solving the related canonical or Hamiltonian system of 1st order random differential equations, hence, a random two-point boundary value problem.

According to the definitions given in Section 2 of a stationary control and characterization (16) of an optimal solution of $(5)_{v(\cdot)}$, we first have to determined the directional derivative $F'_{v(\cdot)+}$. Based on the justification in Section 1, we assume again that the solution $z(t,\omega) = S(\omega, u(\cdot))(t)$ of (2a,b) is measurable in (t,ω) for each $u(\cdot) \in D,\ u(\cdot) \to S(\omega, u(\cdot))$ is continuously differentiable on D for each $\omega \in \Omega$. Furthermore, we suppose that the F-differential $\zeta_{u,h}(t,\omega) = (D_u S(\omega, u(\cdot))h(\cdot))(t),\ h(\cdot) \in U$, is measurable in (t,ω), and is given according to (3d,e) by the *linear perturbation equation*

$$\dot{\zeta}(t) = A(t,\omega,u(\cdot))\zeta(t) + B(t,\omega,u(\cdot))h(t), \quad t_0 \leqslant t \leqslant t_f, \quad \omega \in \Omega \tag{18a}$$

$$\zeta(t_0) = 0 \tag{18b}$$

with the Jacobians

$$A(t,\omega,u(\cdot)) := D_z g(t,\omega,z_u(t,\omega),u(t)) \tag{18c}$$

$$B(t,\omega,u(\cdot)) := D_u g(t,\omega,z_u(t,\omega),u(t)) \tag{18d}$$

and $z_u(t,\omega)$ defined by (2a), $(t,\omega,u(\cdot)) \in [t_0,t_f] \times \Omega \times U$.

This means that the approximate $(6)_{v(\cdot)}$ of (5) has the following explicit form:

$$\min \quad E\Big(\int_{t_0}^{t_f} L(t,\omega,z_v(t,\omega)+\zeta(t,\omega),u(t))dt + G\big(t_f,\omega,z_v(t_f,\omega)+\zeta(t_f,\omega)\big)\Big), \tag{19a}$$

$$\text{s.t. } \dot{\zeta}(t,\omega) = A(t,\omega,v(\cdot))\zeta(t,\omega) + B(t,\omega,v(\cdot))(u(t)-v(t)) \quad \text{a.s.}, \tag{19b}$$

$$\zeta(t_0,\omega) = 0 \quad \text{a.s.}, \tag{19c}$$

$$u(\cdot) \in D. \tag{19d}$$

With the convexity assumptions in Section 1.1, Lemma 2 yields that (19a–d) is a convex stochastic control problem, with a *linear* plant differential equation.

For the subsequent analysis of the stochastic control problem we need a representation of the directional derivative $F'_{v(\cdot)+}(u(\cdot);h(\cdot))$ by a scalar product of the function $h(\cdot)$ with a function $q = q(t)$ to be determined in the following.

From representation (15) of the directional derivative $F'_{v(\cdot)+}$ of the convex approximate $F_{v(\cdot)}$ of F and definition (4b) of $f = f(\omega, z(\cdot), u(\cdot))$, with (3d) we obtain

$$\begin{aligned} F'_{v(\cdot)+}(u(\cdot);h(\cdot)) = E\Big(\int_{t_0}^{t_f} &(\nabla_z L(t,\omega,z_v(t,\omega) \\ &+ \zeta_{v,u-v}(t,\omega),u(t))^T \zeta_{v,h}(t,\omega) + \nabla_u L(t,\omega,z_v(t,\omega) \\ &+ \zeta_{v,u-v}(t,\omega),u(t))^T h(t)dt\Big) + \nabla_z G\big(t_f,\omega,z_v(t_f,\omega) \\ &+ \zeta_{v,u-v}(t_f,\omega)\big)^T \zeta_{v,h}(t_f,\omega)\Big). \end{aligned} \tag{20}$$

Defining the gradients

$$a(t,\omega,v(\cdot),u(\cdot)) := \nabla_z L\big(t,\omega,z_v(t,\omega)+\zeta_{v,u-v}(t,\omega),u(t)\big) \tag{21a}$$

$$b(t,\omega,v(\cdot),u(\cdot)) := \nabla_u L\big(t,\omega,z_v(t,\omega)+\zeta_{v,u-v}(t,\omega),u(t)\big) \tag{21b}$$

$$c\big(t_f,\omega,v(\cdot),u(\cdot)\big) := \nabla_z G\big(t_f,\omega,z_v(t_f,\omega)+\zeta_{v,u-v}(t_f,\omega)\big), \tag{21c}$$

the directional derivative $F'_{v(\cdot)+}$ can be represented by

$$\begin{aligned} F'_{v(\cdot)+}(u(\cdot);h(\cdot)) = E\Big(\int_{t_0}^{t_f} &\big(a(t,\omega,v(\cdot),u(\cdot))^T \zeta_{v,h}(t,\omega) \\ &+ b(t,\omega,v(\cdot),u(\cdot))^T h(t)\big)dt \\ &+ c\big(t_f,\omega,v(\cdot),u(\cdot)\big)^T \zeta_{v,h}(t_f,\omega)\Big). \end{aligned} \tag{22a}$$

Integrating (18a) over $[t_0,t_f]$, for each ω we get

$$\begin{aligned} \zeta_{v,h}(t_f,\omega) &= \int_{t_0}^{t_f} \dot{\zeta}_{v,h}(t,\omega)dt \\ &= \int_{t_0}^{t_f} (A(t,\omega,v(\cdot))\zeta_{v,h}(t,\omega) + B(t,\omega,v(\cdot))h(t))dt. \end{aligned} \tag{22b}$$

Putting (22b) into (22a), we find

$$F'_{v(\cdot)+}(u(\cdot);h(\cdot)) = E\Big(\int_{t_0}^{t_f} \tilde{a}(t,\omega,v(\cdot),u(\cdot))^T \zeta_{v,h}(t,\omega)dt + \int_{t_0}^{t_f} \tilde{b}(t,\omega,v(\cdot),u(\cdot))^T h(t)dt\Big), \tag{22c}$$

where

$$\tilde{a}(t,\omega,v(\cdot),u(\cdot)) := a(t,\omega,v(\cdot),u(\cdot)) + A(t,\omega,v(\cdot))^T c\big(t_f,\omega,v(\cdot),u(\cdot)\big), \tag{22d}$$

$$\tilde{b}(t,\omega,v(\cdot),u(\cdot)) := b(t,\omega,v(\cdot),u(\cdot)) + B(t,\omega,v(\cdot))^T c\big(t_f,\omega,v(\cdot),u(\cdot)\big). \tag{22e}$$

In order to transform the first integral in (22c) to the form of the second integral in (22c), we introduce the m-vector function

$$\lambda = \lambda_{v,u}(t,\omega)$$

defined by the following integral equation depending on the random parameter ω:

$$\lambda(t) - A(t,\omega,v(\cdot))^T \int_t^{t_f} \lambda(s)ds = \tilde{a}(t,\omega,v(\cdot),u(\cdot)). \tag{23}$$

Under the present assumptions, this *Volterra integral equation* has [19] a unique measurable solution $(t,\omega) \to \lambda_{v,u}(t,\omega)$. By means of (23) we obtain

$$\int_{t_0}^{t_f} \tilde{a}(t,\omega,v(\cdot),u(\cdot))^T \zeta_{v,h}(t,\omega)dt = \int_{t_0}^{t_f} \lambda(s)^T\left(\zeta_{v,h}(s,\omega) - \int_{t_0}^{s} A(t,\omega,v(\cdot))\zeta_{v,h}(t,\omega)dt\right)ds, \tag{24a}$$

where $J = J(s,t)$ is defined by

$$J(s,t) := \begin{cases} 0, & t_0 \leqslant s \leqslant t, \\ 1, & t < s \leqslant t_f. \end{cases}$$

Using now again the perturbation Eq. (18a,b), from (24a) we get

$$\int_{t_0}^{t_f} \tilde{a}(t,\omega,v(\cdot),u(\cdot))^T \zeta_{v,h}(t,\omega)dt = \int_{t_0}^{t_f}\left(B(t,\omega,v(\cdot))^T \int_t^{t_f} \lambda(s)ds\right)^T h(t)dt. \tag{24b}$$

Inserting (24b) into (22c), we have

$$F'_{v(\cdot)+}(u(\cdot);h(\cdot)) = E\left(\int_{t_0}^{t_f}\left(B(t,\omega,v(\cdot))^T\int_t^{t_f}\lambda(s)ds + \tilde{b}(t,\omega,v(\cdot),u(\cdot))\right)\right)^T h(t)dt. \tag{25}$$

By means of (22d), the integral Eq. (23) may be written by

$$\lambda(t) - A(t,\omega,v(\cdot))^T\left(c(t_f,\omega,v(\cdot),u(\cdot)) + \int_t^{t_f}\lambda(s)ds\right) = a(t,\omega,v(\cdot),u(\cdot)). \tag{26}$$

According to (26), we define now the m-vector function

$$y = y_{v,u}(t,\omega) := c(t_f,\omega,v(\cdot),u(\cdot)) + \int_t^{t_f}\lambda_{v,u}(s,\omega)ds. \tag{27}$$

Obviously, $y = y_{v,u}(t,\omega)$ solves the system of differential equations

$$\dot{y}(t) = -A(t,\omega,v(\cdot))^T y(t) - a(t,\omega,v(\cdot),u(\cdot)), \quad t_0 \leqslant t \leqslant t_f, \tag{28a}$$

$$y(t_f) = c(t_f,\omega,v(\cdot),u(\cdot)). \tag{28b}$$

System (28a,b) is called the *adjoint differential equation* related to the perturbation Eq. (18a,b).

Using (27), from (22e) and (25) we obtain now

$$F'_{v(\cdot)+}(u(\cdot);h(\cdot)) = E\left(\int_{t_0}^{t_f}\left(B(t,\omega,v(\cdot))^T y_{v,u}(t,\omega) + b(t,\omega,v(\cdot),u(\cdot))\right)^T h(t)dt\right). \tag{29}$$

Summarizing the above transformations, we have the following result:

Theorem 3.1. *Suppose that the adjoint differential* Eq. (28a,b) *has a measurable solution* $(\omega,t) \to y_{v,u}(t,\omega)$. *Then,*

$$F'_{v(\cdot)+}(u(\cdot);h(\cdot)) = \int_{t_0}^{t_f} E\Big(B(t,\omega,v(\cdot))^T y_{v,u}(t,\omega) + b(t,\omega,v(\cdot),u(\cdot))\Big)^T h(t)dt. \tag{30}$$

For a further discussion of formula (30) for $F'_{v(\cdot)+}(u(\cdot);h(\cdot))$, we introduce the **stochastic Hamiltonian** related to the partly linearized control problem (19a–d) based on a reference control $v(\cdot)$:

$$H_{v(\cdot)}(t,\omega,\zeta,y,u) := L(t,\omega,z_v(t,\omega)+\zeta,u) + y^T(A(t,\omega,v(\cdot))\zeta + B(t,\omega,v(\cdot))(u - v(t))), \tag{31a}$$

$(t,\omega,z,y,u) \in [t_0,t_f]\times\Omega\times \mathrm{IR}^m \times \mathrm{IR}^m \times \mathrm{IR}^n$. Using $H_{v(\cdot)}$, for $F'_{v(\cdot)+}$ we find the representation

$$F'_{v(\cdot)+}(u(\cdot);h(\cdot)) = \int_{t_0}^{t_f} E\nabla_u H_{v(\cdot)}(t,\omega,\zeta_{v,u-v}(t,\omega),y_{v,u}(t,\omega),u(t))^T h(t)dt, \tag{31b}$$

where $\zeta_{v,u-v} = \zeta_{v,u-v}(t,\omega)$ is the solution of the perturbation differential Eq. (19b,c), and $y_{v,u} = y_{v,u}(t,\omega)$ denotes the solution of the adjoint differential Eq. (28a,b).

Let $u^0(\cdot) \in U$ denote a given initial control. By means of (31a,b), the necessary and sufficient condition for a control $u^1(\cdot)$ to be an element of the set $M(u^0(\cdot))$, i.e. a solution of the approximate convex problem $(6)_{u^0(\cdot)}$, reads, see Definition 4 and (16):

$$\int_{t_0}^{t_f} E\nabla_u H_{u^0(\cdot)}(t,\omega,\zeta_{u^0,u^1-u^0}(t,\omega),y_{u^0,u^1}(t,\omega),u^1(t))^T(u(t)-u^1(t))dt \geqslant 0, u(\cdot)\in D. \tag{32}$$

In order to construct now, for a given initial control $u^0(\cdot) \in D$, a control $u^1(\cdot) \in M(u^0(\cdot))$, because of (32) we consider the mean value function

$$\overline{H}_{u^0(\cdot),w}(u(\cdot)) := \int_{t_0}^{t_f} EH_{u^0(\cdot)}(t,\omega,\zeta_{u^0,w-u^0}(t,\omega),y_{u^0,w}(t,\omega),u(t))dt, \tag{33a}$$

where $w(\cdot)$ plays the role of a yet unknown solution $u^{(1)}(\cdot)$ of $(6)_{u^0(\cdot)}$. According to the assumptions in Section 1.2, the function $\overline{H}_{u^0(\cdot),w}$ is convex. Then, assuming that integrals, expectations and derivatives may be interchanged, for a solution $\tilde{u}(\cdot) \in D$ of the convex stochastic optimization problem

$$\min \overline{H}_{u^0(\cdot),w}(u(\cdot)) \quad \text{s.t.} \quad u(\cdot) \in D \tag{33b}$$

in $PC_0^n[t_0,t_f]$ we get, cf. (16), the necessary and sufficient optimality condition

$$\int_{t_0}^{t_f} E\nabla_u H_{u^0(\cdot)}(t,\omega,\zeta_{u^0,w-u^0}(t,\omega),y_{u^0,w}(t,\omega),\tilde{u}(t))^T(u(t)-\tilde{u}(t))dt \geqslant 0 \quad \text{for each } u(\cdot)\in D. \tag{33c}$$

Consider now a control $u^1(\cdot) \in M(u^0(\cdot))$ and put $w(\cdot) := u^1(\cdot)$ in (33a–c). Since the conditions (32) and (33c) coincide for $\tilde{u}(\cdot) = u^{(1)}(\cdot) = w(\cdot)$, we have the following characterization:

Theorem 3.2. *Let $u^0(\cdot) \in D$ be a given initial control. A control $u^{(1)}(\cdot) \in U$ is a solution of $(5)_{u^0(\cdot)}$, i.e., $u^1(\cdot) \in M(u^0(\cdot))$, if and only if $u^1(\cdot)$ is an optimal solution of the convex stochastic optimization problem*

$$\min \overline{H}_{u^0(\cdot),u^1}(u(\cdot)) \quad \text{s.t. } u(\cdot) \in D, \tag{34}$$

In the following we study therefore the convex optimization problem (33a,b). In practice the admissible domain D is given in many cases by

$$D = \{u(\cdot) \in U : u(t) \in D_t, t_0 \leqslant t \leqslant t_f\}, \tag{35}$$

where $D_t \subset \mathrm{IR}^n$ is a given convex subset of IR^n for each time t, $t_0 \leqslant t \leqslant t_f$. Since $\overline{H}_{u^0(\cdot),w}(u(\cdot))$ has an integral form, the minimum value of $\overline{H}_{u^0(\cdot),w}(u(\cdot))$ on D can be obtained – in case (35) – by solving the finite-dimensional stochastic optimization problem

$$\min EH_{u^0(\cdot)}(t,\omega,\zeta_{u^0,w-u^0}(t,\omega),y_{u^0,w}(t,\omega),u) \quad \text{s.t. } u \in D_t \qquad (P)^t_{u^0(\cdot),w}$$

for each $t \in [t_0,t_f]$. Let denote then

$$\widetilde{u^*} = \widetilde{u^*}_{u^0(\cdot)}(t,w(\cdot)), \quad t_0 \leqslant t \leqslant t_f, \tag{36a}$$

a solution of $(P)^t_{u^0(\cdot),w}$ for each $t_0 \leqslant t \leqslant t_f$. Obviously, if function (36a) lies in U and therefore also in D, then

$$\widetilde{u^*}_{u^0(\cdot)}(\cdot,w(\cdot)) \in \arg\min_{u(\cdot)\in D} \overline{H}_{u^0(\cdot),w}(u(\cdot)). \tag{36b}$$

Replacing $\zeta_{u^0,w-u^0}(t,\omega)$ and $y_{u^0,w}(t,\omega)$ by some measurable functions $\zeta:\Omega\to\mathbb{R}^m,\ \eta:\Omega\to\mathbb{R}^m$, the convex stochastic optimization problem $(P)^t_{u^0(\cdot),w}$ takes the following form:

$$\min EH_{u^0(\cdot)}(t,\omega,\zeta(\omega),\eta(\omega),u) \quad \text{s.t. } u\in D_t \qquad (P)^t_{u^0(\cdot),\zeta,\eta}$$

Because of Theorem 3.2, problems $(P)^t_{u^0(\cdot),w}, (P)^t_{u^0(\cdot),\zeta,\eta}$ and (36a,b), we introduce, cf. [14], the following definition:

Definition 3.1. Let $u^0(\cdot)\in D$ be a given initial control. For measurable functions $\zeta(\cdot),\eta(\cdot)$ on $(\Omega,\mathcal{A},P)$, a solution

$$\widetilde{u^*}=\widetilde{u^*}_{u^0(\cdot)}(t,\zeta(\cdot),\eta(\cdot)),\quad t_0\leqslant t\leqslant t_f,$$

of $(P)^t_{u^0(\cdot),\zeta,\eta},\ t_0\leqslant t\leqslant t_f$ is called a $\mathbf{H^0_u}(\cdot)$**-minimum control** of (19a–d).

Remark 7. Obviously, by means of Definition 3.3, for an optimal solution $u^1(\cdot)$ of $(5)_{u^0(\cdot)}$ we have then the "*preliminary optimal control*" $\widetilde{u^*}=\widetilde{u^*}_{u^0(\cdot)}(t,\zeta(\cdot),\eta(\cdot))$ depending on still unknown state, costate variables $\zeta(\cdot),\eta(\cdot)$.

3.1. Canonical (Hamiltonian) system of differential equations

For a given initial control $u^0(\cdot)\in D$, let $\widetilde{u^*}=\widetilde{u^*}_{u^0(\cdot)}(t,\zeta(\cdot),\eta(\cdot))$, $t_0\leqslant t\leqslant t_f$, denote a $H_{u^0(\cdot)}$-minimum control of (19a–d) according to the above definition. Due to (19b,c) and (28a,b) we consider, cf. [14], the following so-called canonical or Hamiltonian system of differential equations, hence, a two-point boundary value problem, with random parameters for the vector functions $(\zeta,y)=(\zeta(t,\omega),y(t,\omega)),\ t_0\leqslant t\leqslant t_f,\ \omega\in\Omega$:

$$\dot\zeta(t,\omega)=A(t,\omega,u^0(\cdot))\zeta(t,\omega)+B(t,\omega,u^0(\cdot))\big(\widetilde{u^*}_{u^0(\cdot)}(t,\zeta(t,\cdot),y(t,\cdot))-u^0(t)\big),\quad t_0\leqslant t\leqslant t_f, \tag{37a}$$

$$\zeta(t_0,\omega)=0\quad \text{a.s.}, \tag{37b}$$

$$\dot y(t,\omega)=-A(t,\omega,u^0(\cdot))^Ty(t,\omega)-\nabla_zL\big(t,\omega,z_{u^0}(t,\omega)+\zeta(t,\omega),\widetilde{u^*}_{u^0(\cdot)}(t,\zeta(t,\cdot),y(t,\cdot))\big),\quad t_0\leqslant t\leqslant t_f, \tag{37c}$$

$$y(t_f,\omega)=\nabla_zG\big(t_f,\omega,z_{u^0}(t_f,\omega)+\zeta(t_f,\omega)\big). \tag{37d}$$

Remark 8. Note that the (deterministic) control $\widetilde{u^*}=\widetilde{u^*}_{u^0(\cdot)}(t,\zeta(t,\cdot),y(t,\cdot))$ depends on the whole random variable $(\zeta(t,\omega),y(t,\omega)),\ \omega\in\Omega$, or the occurring moments. In case of a discrete parameter distribution $\Omega=\{\omega_1,\dots,\omega_\varrho\}$, see Section 1, the control $\widetilde{u^*}_{u^0(\cdot)}$ depends,

$$\widetilde{u^*}_{u^0(\cdot)}=\widetilde{u^*}_{u^0(\cdot)}\big(t,\zeta(t,\omega_1),\dots,\zeta(t,\omega_\varrho),y(t,\omega_1),\dots,y(t,\omega_\varrho)\big)$$

on the 2ϱ unknown functions $\zeta(t,\omega_i),\ y(t,\omega_i),\ i=1,\dots,\varrho$.

Suppose now that $(\zeta^1,y^1)=(\zeta^1(t,\omega),y^1(t,\omega)),\ t_0\leqslant t\leqslant t_f$, $\omega\in(\Omega,\mathcal{A},P)$, is the unique measurable solution of the canonical stochastic system (37a–d), and define

$$u^1(t):=\widetilde{u^*}_{u^0(\cdot)}\big(t,\zeta^1(t,\cdot),y^1(t,\cdot)\big),\quad t_0\leqslant t\leqslant t_f. \tag{38}$$

Assuming that

$$u^1(\cdot)\in U, \tag{39a}$$

due to the definition of a $H_{u^0(\cdot)}$-minimum control we also have

$$u^1(\cdot)\in D. \tag{39b}$$

According to the notation introduced in (3d), (27), resp., and the above assumed uniqueness of the solution (ζ^1,y^1) of (37a–d) we have

$$\zeta^1(t,\omega):=\zeta_{u^0,u^1-u^0}(t,\omega) \tag{40a}$$

$$y^1(t,\omega):=y_{u^0,u^1}(t,\omega) \tag{40b}$$

with the control $u^1(\cdot)$ given by (38).

Due to the above construction, we know that $u^1(t)$ solves $(P)^t_{u^0(\cdot),\zeta,\eta}$ for $\zeta(\omega)=\zeta^1(t,\omega),\ \eta(\omega)=y^1(t,\omega)$. Consequently, $u^1(t)$ solves $(P)^t_{u^0(\cdot),u^1}$ for each $t_0\leqslant t\leqslant t_f$. However, this yields that $u^1(\cdot)$ solves (33b) with $w(\cdot):=u^1(\cdot)$. Hence, control $u^1(\cdot)$, given by (38), is a solution of (34).

Summarizing the above construction, from Theorem 3.2 we obtain this result:

Theorem 3.3. *Suppose that D is given by (35), $M(u^0(\cdot))\neq\emptyset$, and $(P)^t_{u^0(\cdot),\zeta,\eta}$ has an optimal solution for each $t,\ t_0\leqslant t\leqslant t_f$, and measurable functions $\gamma(\cdot),\ \eta(\cdot)$. Moreover, suppose that the canonical system (37a–d) has a unique measurable solution $\big(\zeta^1(t,\omega),y^1(t,\omega)\big),\ t_0\leqslant t\leqslant t_f,\omega\in\Omega$, such that $u^1(\cdot)\in U$, where $u^1(\cdot)$ is defined by (38). Then $u^1(\cdot)$ is a solution of $(6)_{u^0(\cdot)}$.*

4. Stationary controls

Having a method for the construction of improved approximative controls $u^1(\cdot)\in M(u^0(\cdot))$ related to an initial control $u^0(\cdot)\in D$, we consider now the construction of stationary controls of (5), i.e. elements $\bar u(\cdot)\in D$ such that $\bar u(\cdot)\in M(\bar u(\cdot))$, see Definition 6.

Starting again with formula (30), by means of (17), for an element $v(\cdot)\in D$ we have

$$F'_+(v(\cdot);h(\cdot))=F'_{v(\cdot)+}(v(\cdot);h(\cdot))=\int_{t_0}^{t_f}E\Big(B(t,\omega,v(\cdot))^Ty_v(t,\omega)+b(t,\omega,v(\cdot),v(\cdot))\Big)^Th(t)dt, \tag{41a}$$

where

$$y_v(t,\omega):=y_{v,v}(t,\omega) \tag{41b}$$

fulfills, cf. (28a,b), the adjoint differential equation

$$\dot y(t,\omega)=-A(t,\omega,v(\cdot))^Ty(t,\omega)-\nabla_zL(t,\omega,z_v(t,\omega),v(t)), \tag{42a}$$

$$y(t_f,\omega)=\nabla_zG\big(t_f,\omega,z_v(t_f,\omega)\big). \tag{42b}$$

Moreover, cf. (18c,d),

$$A(t,\omega,v(\cdot))=D_zg(t,\omega,z_v(t,\omega),v(t)) \tag{42c}$$

$$B(t,\omega,v(\cdot))=D_ug(t,\omega,z_v(t,\omega),v(t)) \tag{42d}$$

and

$$b(t,\omega,v(\cdot),v(\cdot))=\nabla_uL(t,\omega,z_v(t,\omega),v(t)), \tag{42e}$$

where $z_v=z_v(t,\omega)$ solves the dynamic equation

$$\dot z(t,\omega)=g(t,\omega,z(t,\omega),v(t)),\quad t_0\leqslant t\leqslant t_f, \tag{42f}$$

$$z(t_0,\omega)=z_0(\omega). \tag{42g}$$

Using now the stochastic Hamiltonian

$$H(t,\omega,z,y,u):=L(t,\omega,z,u)+y^Tg(t,\omega,z,u) \tag{43a}$$

related to the basic control problem (16a–d), from (41a,b), (42a–g) we get the representation, cf. (31a,b),

$$F'_+(v(\cdot);h(\cdot))=\int_{t_0}^{t_f}E\nabla_uH(t,\omega,z_v(t,\omega),y_v(t,\omega),v(t))^Th(t)dt. \tag{43b}$$

According to condition (16), a stationary control of (5), hence, an element $\bar{u}(\cdot) \in D$ such that $\bar{u}(\cdot)$ is an optimal solution of $(5)_{\bar{u}(\cdot)}$ is characterized, see (17), by

$$F'_{+}(\bar{u}(\cdot); u(\cdot) - \bar{u}(\cdot)) \geqslant 0 \quad \text{for all } u(\cdot) \in D.$$

Thus, for stationary controls $\bar{u}(\cdot) \in D$ we have the characterization

$$\int_{t_0}^{t_f} E\nabla_u H(t,\omega,z_{\bar{u}}(t,\omega),y_{\bar{u}}(t,\omega),\bar{u}(t))^T(u(t)-\bar{u}(t))dt \geqslant 0, \quad u(\cdot) \in D. \tag{44}$$

Comparing (32) and (44), corresponding to (33a,b) for given $w(\cdot) \in D$ we introduce here the function

$$\overline{H}_w(u(\cdot)) := \int_{t_0}^{t_f} EH(t,\omega,z_w(t,\omega),y_w(t,\omega),u(t))dt, \tag{45a}$$

and we consider the optimization problem

$$\min \overline{H}_w(u(\cdot)) \text{s.t.} u(\cdot) \in D. \tag{45b}$$

Because of the assumptions in Section 1, problem (45b) is (strictly) convex, provided that the process differential Eq. (1a) is affine-linear with respect to u, hence,

$$\dot{z}(t,\omega) = g(t,\omega,z,u) = \hat{g}(t,\omega,z) + \widehat{B}(t,\omega)u \tag{46}$$

with a given vector-, matrix-valued function $\hat{g} = \hat{g}(t,\omega,z)$, $\widehat{B} = \widehat{B}(t,\omega)$.

If differentiation and integration/expectation in (45a) may be interchanged, which is assumed in the following, then (44) is a necessary condition for

$$\bar{u}(\cdot) \in \arg\min_{u(\cdot)\in D} \overline{H}_{\bar{u}}(u(\cdot)), \tag{47}$$

cf. (33a–c) and (34). Corresponding to Theorem 3.2, here we have this result:

Theorem 4.1. *Suppose that a control $\bar{u}(\cdot) \in D$ fulfills (47). Then $\bar{u}(\cdot)$ is a stationary control of (5).*

4.1. Canonical system of differential equations for stationary controls

Assume now again that the feasible domain D is given by (35). In order to solve the optimization problem (45a,b), corresponding to $(P)^t_{u^0(\cdot),\zeta,\eta}$, here we consider the finite-dimensional optimization problem

$$\min EH(t,\omega,\zeta(\omega),\eta(\omega),u) \quad \text{s.t. } u \in D_t \qquad P^t_{\zeta,\eta} \tag{48}$$

for each t, $t_0 \leqslant t \leqslant t_f$.

Corresponding to Definition 3.1, here we have the following notion:

Definition 4.1. For measurable functions $\zeta(\cdot), \eta(\cdot)$ on $(\Omega,\mathcal{A},P)$, let denote

$$\widetilde{u^*} = \widetilde{u^*}(t,\zeta(\cdot),\eta(\cdot)), \quad t_0 \leqslant t \leqslant t_f,$$

a solution of $(P)^t_{\zeta,\eta}$. The function $\widetilde{u^*} = \widetilde{u^*}(t,\zeta(\cdot),\eta(\cdot))$, $t_0 \leqslant t \leqslant t_f$, is called a **H-minimum control** of (6a–d).

For a given H-minimum control $\widetilde{u^*} = \widetilde{u^*}(t,\zeta(\cdot),\eta(\cdot))$ we consider now, see (37a–d), the following canonical (Hamiltonian) two-point boundary value problem with random parameters:

$$\dot{z}(t,\omega) = g\big(t,\omega,z(t,\omega),\widetilde{u^*}(t,z(t,\cdot),y(t,\cdot))\big), \quad t_0 \leqslant t \leqslant t_f, \tag{48a}$$
$$z(t_0,\omega) = z_0(\omega), \tag{48b}$$
$$\begin{aligned}\dot{y}(t,\omega) = &-D_z g\big(t,\omega,z(t,\omega),\widetilde{u^*}(t,z(t,\cdot),y(t,\cdot))\big)^T y(t,\omega)\\ &-\nabla_z L\big(t,\omega,z(t,\omega),\widetilde{u^*}(t,z(t,\cdot),y(t,\cdot))\big), \quad t_0 \leqslant t \leqslant t_f,\end{aligned} \tag{48c}$$
$$y(t_f,\omega) = \nabla_z G\big(t_f,\omega,z(t_f,\omega)\big). \tag{48d}$$

Remark 9. In case of a discrete distribution $\Omega = \{\omega_1,\ldots,\omega_\varrho\}$, $P(\omega=\omega_j)$, $j=1,\ldots,\varrho$, corresponding to Section 3.1, for the H-minimum control we have

$$\widetilde{u^*} = \widetilde{u^*}\big(t,z(t,\omega_1),\ldots,z(t,\omega_\varrho),y(t,\omega_1),\ldots,y(t,\omega_\varrho)\big).$$

Thus, (48a–d) is then an ordinary two-point boundary value problem for the 2ϱ unknown functions $z=z(t,\omega_j)$, $y=y(t,\omega_j)$, $j=1,\ldots,\varrho$.

Let denote $(\bar{z},\bar{y}) = (\bar{z}(t,\omega),\bar{y}(t,\omega))$, $t_0 \leqslant t \leqslant t_f, \omega \in (\Omega,\mathcal{A},P)$, the unique measurable solution of (48a–d) and define.

$$\bar{u}(t) := \widetilde{u^*}(t,\bar{z}(t,\cdot),\bar{y}(t,\cdot)), \quad t_0 \leqslant t \leqslant t_f. \tag{49}$$

Due to (2a) (41b) and (42a,b) we have

$$\bar{z}(t,\omega) = z_{\bar{u}}(t,\omega), \quad t_0 \leqslant t \leqslant t_f, \quad \omega \in (\Omega,\mathcal{A},P) \tag{50a}$$
$$\bar{y}(t,\omega) = y_{\bar{u}}(t,\omega), \quad t_0 \leqslant t \leqslant t_f, \quad \omega \in (\Omega,\mathcal{A},P) \tag{50b}$$

and therefore

$$\bar{u}(t) = \widetilde{u^*}(t,z_{\bar{u}}(t,\cdot),y_{\bar{u}}(t,\cdot)), \quad t_0 \leqslant t \leqslant t_f. \tag{50c}$$

Assuming that

$$\bar{u}(\cdot) \in U(\text{and therefore} \bar{u}(\cdot) \in D), \tag{51}$$

we get this result:

Theorem 4.2. *Suppose that the canonical system (48a–d) has a unique measurable solution $(\bar{z},\bar{y}) = (\bar{z}(t,\omega),\bar{y}(t,\omega))$, and define $\bar{u}(\cdot)$ by (49) with a H-minimum control $u^* = u^*(t,\zeta,\eta)$. If $\bar{u}(\cdot) \in U$, then $\bar{u}(\cdot)$ is a stationary control.*

Proof. According to the construction of $(\bar{z},\bar{y},\bar{u})$, the control $\bar{u}(\cdot) \in D$ minimizes $\overline{H}_{\bar{u}}(u(\cdot))$ on D. Hence,

$$\bar{u}(\cdot) \in \arg\min_{u(\cdot)\in D} \overline{H}_{\bar{u}}(u(\cdot)).$$

Theorem 4.1 yields then that $\bar{u}(\cdot)$ is a stationary control.

5. Stochastic optimization methods

5.1. Stochastic programs

The method developed in Sections 3 and 4 is applied now to the following case:

Let the function g of the process differential Eq. (1a,b) and the cost function L "along the trajectory" be given, cf. (46), (8b,c), (9a–e), (10a–d), by

$$g(t,\omega,z,u) = \hat{g}(t,\omega,z) + \widehat{B}(t,\omega)u, \tag{52a}$$
$$L(t,\omega,z,u) = \widehat{L}(t,\omega,z) + Q(t,\omega,u). \tag{52b}$$

Then, programs $(P)^t_{\zeta,\eta}$, $(P)^t_{u^0(\cdot),\zeta,\eta}$ are equivalent to the *stochastic program* [20,23]

$$\min_{u\in D_t} E\Big(\Big(\widehat{B}(t,\omega)^T\eta(\omega)\Big)^T u + Q(t,\omega,u)\Big). \tag{53}$$

Special cases:

(a) If $B(t,\omega)$ does not depend on ω, then $u^*(t,\zeta(\cdot),\eta(\cdot)) = u^*(t,E\eta(\omega))$;

(b) If $Q(t,\omega,u) = u^TRu$, cf. (10c), with a positive definite matrix R, and $D_t = \mathrm{IR}^n$, then$u^*(t,\zeta(\cdot),\eta(\cdot)) = u^*_{u^0(\cdot)}(t,\zeta(\cdot),\eta(\cdot)) = -\frac{1}{2}R^{-1}E\widehat{B}(t,\omega)^T\eta(\omega)$;

(c) If the cost function $Q = Q(t,\omega,u)$ is strictly convex in u for each $(t,\omega) \in [t_0,t_f]\times\Omega$ a.s., and the admissible sets D_t, $t_0 \leqslant t \leqslant t_f$, are compact, then the program (53) has unique optimal solutions. Hence, the H-minimum control $u^* = u^*(t, \zeta(\cdot), \eta(\cdot))$ is determined uniquely;

5.2. Discrete probability distributions

For a discrete probability distribution $\Omega = \{\omega_1, \ldots, \omega_\varrho\}$, $P(\omega = \omega_j) = \alpha_j,\ j = 1, \ldots, \varrho,\ \sum_{j=1}^{\varrho} \alpha_j = 1$, (53) is reduced to the ordinary parameter optimization problem

$$\min_{u \in D_t} \sum_{j=1}^{\varrho} \alpha_j \left(\widehat{B}(t, \omega_j)^T \eta(\omega_j) \right)^T u + \sum_{j=1}^{\varrho} \alpha_j Q(t, \omega_j, u) \tag{53'}$$

depending on the parameters $\eta(\omega_j)$ and probabilities $\alpha_j,\ j = 1, \ldots, \varrho$.

5.3. Computation of expectations by means of Taylor expansion

Corresponding to the assumptions in Section 1, here we suppose that

$$g(t, \omega, z, u) = \tilde{g}(t, \theta, z, u) \tag{54a}$$

$$z_0(\omega) = \tilde{z}_0(\theta) \tag{54b}$$

$$L(t, \omega, z, u) = \widetilde{L}(t, \theta, z, u) \tag{54c}$$

$$G(t, \omega, z) = \widetilde{G}(t, \theta, z) \tag{54d}$$

with a stationary random r-vector

$$\theta = \theta(\omega), \quad \omega \in (\Omega, \mathcal{A}, P), \tag{54e}$$

of random model parameters and initial values and sufficiently smooth functions $\tilde{g},\ \tilde{z}_0,\ \widetilde{L},\ \widetilde{G}$ of the indicated variables. For simplification of notation we omit "~" and write again $g(t, \theta, z, u)$ instead of $g(t, \omega, z, u)$, etc.

Since the approximate problem (19a–d) has the same basic structure as the original problem (6a–d), it is sufficient to describe the procedure for problem (6a–d). Again, for simplification, the conditional expectation $E(\ldots|\mathcal{A}_{t_0})$ given the information $\mathcal{A}_{t_0}$ up to the considered starting time t_0 is denoted by "E". Thus, let denote

$$\overline{\theta} := E\theta(\omega) = E\big(\theta(\omega)|\mathcal{A}_{t_0}\big) \tag{55a}$$

the conditional expectation of the random vector $\theta(\omega)$ given the information $\mathcal{A}_{t_0}$ at time point t_0. Taking into account the properties of the solution

$$z(t, \theta) = z(t, \theta, u(\cdot)) = S(z_0(\theta), \theta, u(\cdot))(t), \quad t \geqslant t_0, \tag{55b}$$

of the dynamic Eq. (3b–d), see Lemma 1, and due to properties of the Hamiltonian 2-point boundary value problem (48a–d), the expectations arising in the objective function (6a) can be computed approximatively by means of Taylor expansion with respect to θ at $\overline{\theta}$.

According to (54a–e) we have-*omitting for simplicity the argument* $u(\cdot)$ -the parametric representation

$$z(t, \omega) = z(t, \theta(\omega)), \quad y = y(t, \theta(\omega)). \tag{56}$$

We consider then the Taylor expansions

$$z(t, \omega) = z(t, \theta(\omega)) = z(t, \overline{\theta}) + z_\theta(t, \overline{\theta})\big(\theta(\omega) - \overline{\theta}\big) + \cdots, \tag{57a}$$

$$y(t, \omega) = z(t, \theta(\omega)) = y(t, \overline{\theta}) + y_\theta(t, \overline{\theta})\big(\theta(\omega) - \overline{\theta}\big) + \cdots. \tag{57b}$$

Deleting second and higher order terms, we have to determine the functions

$$z = z(t, \overline{\theta}), y = y(t, \overline{\theta}), \quad t_0 \leqslant t \leqslant t_f, \tag{58a}$$

$$z_\theta = z_\theta(t, \overline{\theta}), y_\theta = y_\theta(t, \overline{\theta}), \quad t_0 \leqslant t \leqslant t_f. \tag{58b}$$

Using this expansion, the stochastic optimization problem $(P)_{\zeta,\eta}^t$ for finding a H-minimum control can be approximated by

$$\min \quad EH\big(t, \theta(\omega), z(t, \overline{\theta}) + z_\theta(t, \overline{\theta})(\theta(\omega) - \overline{\theta}), y(t, \overline{\theta}) \tag{59a}$$

$$+ y_\theta(t, \overline{\theta})(\theta(\omega) - \overline{\theta}), u\big) \tag{59b}$$

$$\text{s.t.} \quad u \in D_t. \tag{59c}$$

Thus, the resulting approximate H-minimum control

$$\widetilde{u^*} = \widetilde{u^*}\big(t, z(t, \overline{\theta}), z_\theta(t, \overline{\theta}), y(t, \overline{\theta}), y_\theta(t, \overline{\theta})\big) \tag{60}$$

depends on the unknown vector and matrix functions stated in (58a,b).

Describing the random parameter variations be means of a certain random parameter vector $\theta = \theta(\omega)$, a two-point boundary value problem for the unknown functions stated in (58a,b) can be obtained then from the parametric Hamiltonian system of differential Eqs. (48a–d) by

- putting $\theta = \overline{\theta}$ and by
- differentiation with respect to θ and putting then $\theta = \overline{\theta}$.

For simplification and due to the main application of the present analysis to feedback control of mechanical systems and structures with linear dynamics and quadratic costs along the trajectory as described in Sections 1.4 and 1.5, the unknown control function $u(t) := V(t)$ is the matrix $V(t) = (K_P(t), K_d(t))$ of regulator parameter functions, see (12c,d).

According to the results obtained in Section 4, for the computation of an optimal regulator parameter matrix function $u^* = u^*(t) = V^*(t)$ we have the following result:

Theorem 5.1. *Suppose that the random parameter variations of the structural control problem (6a–d) are described by means of a random parameter vector $\theta = \theta(\omega)$. Assuming the existence of the derivatives under consideration, for $z = z(t, \overline{\theta}),\ y = y(t, \overline{\theta})$ and $z_\theta = z_\theta(t, \overline{\theta}),\ y_\theta = y_\theta(t, \overline{\theta})$ we have the two-point boundary value problems*

$$\dot{z}(t, \overline{\theta}) = A(\widetilde{u^*})z(t, \overline{\theta}) + b(t, \overline{\theta}), \quad t_0 \leqslant t \leqslant t_f, \tag{61a}$$

$$z(t_0, \overline{\theta}) = z_0(\overline{\theta}), \tag{61b}$$

$$\dot{y}(t, \overline{\theta}) = -A(\widetilde{u^*})^T y(t, \overline{\theta}) - 2\big(Q + \widetilde{u^*}^T R(t)\widetilde{u^*}\big)z(t, \overline{\theta}), \tag{61c}$$

$$y(t_f, \overline{\theta}) = 0, \tag{61d}$$

$$\dot{z}_\theta(t, \overline{\theta}) = A(\widetilde{u^*})z_\theta(t, \overline{\theta}) + b_\theta(t, \overline{\theta}), \tag{61e}$$

$$z_\theta(t_0, \overline{\theta}) = z_{0\theta}(\overline{\theta})\dot{y}_\theta(t, \overline{\theta}) = -A(\widetilde{u^*})^T y_\theta(t, \overline{\theta}), \tag{61f}$$

$$-2\big(Q + \widetilde{u^*}^T R(t)\widetilde{u^*}\big)z_\theta(t, \overline{\theta}), \tag{61g}$$

$$y_\theta(t_f, \overline{\theta}) = 0, \tag{61h}$$

where the H-minimum control $\widetilde{u^*} = \widetilde{u^*}\big(t, z(t, \overline{\theta}), z_\theta(t, \overline{\theta}), y(t, \overline{\theta}), y_\theta(t, \overline{\theta})\big)$ *is a solution of (59a,b). An optimal regulator parameter matrix function* $u^* = u^*(t)$ *is then given by*

$$u^*(t) = \widetilde{u^*}\big(t, z^*(t, \overline{\theta}), z_\theta^*(t, \overline{\theta}), y^*(t, \overline{\theta}), y_\theta^*(t, \overline{\theta})\big) \tag{62}$$

with a solution $(z^*(t, \overline{\theta}), z_\theta^*(t, \overline{\theta}), y^*(t, \overline{\theta}), y_\theta^*(t, \overline{\theta}))$ *of (61a–h).*

6. Numerical example

The above described method for finding robust optimal regulators is applied now, see Section 1.5, to the stochastic optimal feedback control of a manipulator (Manutec r3) with three rotational links (three degrees of freedom) and a normal distributed random payload. The system of second order linear differential equations for the tracking error were stated already in Section 1.5, and we use here also a quadratic cost function, as described by (10a–d) in Section 1.4; in the present case there are zero terminal costs. According to the linear-quadratic structure of the regulator problem, an approximate robust optimal feedback control can be obtained now by determining a H-minimum control $\widetilde{u^*}$ and the solution of the related 2-point boundary value problem (61a–h).

The numerical results presented in this section have been obtained by *Dipl.Math. Michael Schacher, Institute for Mathematics and Computer Applications, Federal Armed Forces University Munich*.

In the present case we have, see (12b–d), to determine the regulator parameters contained in the following matrix $V(t)$. Hence, for the unknown function $u = u(t)$ we have

$$u(t) := V(t) \tag{63a}$$

with

$$V(t) := (K_p(t), K_d(t)), \tag{63b}$$

where

$$K_p(t) := M(t)^{-1}\left(K(t) - D_{\Delta q}\varphi(t,0,0)\right), \tag{63c}$$
$$K_d(t) := M(t)^{-1}\left(D(t) - D_{\Delta \dot{q}}\varphi(t,0,0)\right). \tag{63d}$$

Note that in the present application instead of the arguments q, $\dot{q}$, z, $\ddot{q}$, θ we use the standard variables Δq, $\Delta\dot{q}$, Δz, $\Delta\ddot{q}$, Δp_D in regulator theory, see Section 1.5.

In order to find now a H-minimum control

$$\widetilde{V^*} = \widetilde{V^*}\left(t, \Delta z(t,\overline{\theta}), \Delta z_\theta(t,\overline{\theta}), y(t,\overline{\theta}), y_\theta(t,\overline{\theta})\right) \tag{64}$$

we have to solve the minimization problem

$$\min EH(t, \Delta p_D, \Delta z, y, V), \qquad (65a) s.t.$$
$$\text{s.t. } V \in D_t. \tag{65b}$$

Here, the Hamilton function reads

$$H(t, \Delta p_D, \Delta z, y, V) := \Delta z^T \widetilde{Q}(t, V(t))\Delta z + y^T(A(V(t))\Delta z + b(t, \Delta p_D)), \tag{66a}$$

where

$$A(V(t)) := \begin{pmatrix} 0 & I \\ -K_p(t) & -K_d(t) \end{pmatrix}, \tag{66b}$$

$$b(t, \Delta p_D) := \begin{pmatrix} 0 \\ -M(t)^{-1}Y(t) \end{pmatrix}\Delta p_D(\omega), \tag{66c}$$

$$\widetilde{Q}(t, V(t)) := Q + B(t, V(t))^T R B(t, V(t)), \tag{66d}$$
$$B(t, V(t)) := (K(t), D(t)) - M(t)V(t). \tag{66e}$$

The expected Hamiltonian is now approximated by means of Taylor approximation at $\overline{\theta} = E\Delta p_D(\omega) = 0$ as described by (57a,b), (58a,b) and (59a,b).

However, if the vector oft dynamic parameters p_D is known, hence, if $\Delta p_D = 0$, then we have

$$\Delta z(t, 0) = 0. \tag{67a}$$

Moreover, since also $y(t_f, 0) = 0$, the Taylor expansion of $\Delta z(t, \Delta p_D(\omega)), y(t, \Delta p_D(\omega))$ is reduced, cf. (58a,b), therefore to

$$\Delta z(t, \Delta p_D(\omega)) \approx \Delta z(t, 0) + \Delta z_{\Delta p_D}(t, 0)\Delta p_D(\omega)$$
$$= \Delta z_{\Delta p_D}(t, 0)\Delta p_D(\omega), \tag{67b}$$

$$y(t, \Delta p_D(\omega)) \approx y(t, 0) + y_{\Delta p_D}(t, 0)\Delta p_D(\omega)$$
$$= y_{\Delta p_D}(t, 0)\Delta p_D(\omega). \tag{67c}$$

Assuming that the payload mass is a scalar stochastic variable, the approximated expected Hamiltonian EH is given by, cf. (66a),

$$EH(t, \Delta p_D(\omega), \Delta z, y, V)$$
$$= \sigma_{p_D}^2 \Big(\Delta z_{\Delta p_D}(t,0)^T \widetilde{Q}(t, V(t))\Delta z_{\Delta p_D}(t,0)$$
$$+ y_{\Delta p_D}(t,0)^T A(V(t))\Delta z_{\Delta p_D}(t,0) + y_{\Delta p_D}(t,0)^T \begin{pmatrix} 0 \\ -M(t)^{-1}Y(t) \end{pmatrix}\Big), \tag{68}$$

where $\sigma_{p_D}^2$ denotes the variance of the scalar payload. Depending on the type of the feasible domain D_t, the minimization of (68) subject to the constraint

$$V \in D_t \tag{69}$$

can be done analytically. In the present case we work with the lower bound 0.5 for the regulator parameters contained in V. At each time t and for given $(\Delta z_{\Delta p_D}(t,0), y_{\Delta p_D}(t,0))$, the minimization of the approximate expected Hamiltonian H is carried out numerically by using the BFGS-B solver.

Assuming zero terminal costs, hence $G = 0$, because of (67a–c), in the present case the 2-point boundary value problem (61a–h) is reduced to

$$\Delta\dot{z}_{\Delta p_D}(t,0) = A(\widetilde{V^*})\Delta z_{\Delta p_D}(t,0) + b_{\Delta p_D}(t,0), \quad t_0 \leqslant t \leqslant t_f, \tag{70a}$$

$$\dot{y}_{\Delta p_D}(t,0) = -A(\widetilde{V^*})^T y_{\Delta p_D}(t,0) - 2\widetilde{Q}(t, \widetilde{V^*})\Delta z_{\Delta p_D}(t,0), \quad t_0 \leqslant t \leqslant t_f, \tag{70b}$$

$$\Delta z_{\Delta p_D}(t_0, 0) = 0, \tag{70c}$$

$$y_{\Delta p_D}(t_f, 0) = 0, \tag{70d}$$

where $\widetilde{V^*}$ is obtained-as described above-from the numerical minimization of the approximate expected Hamiltonian $EH(t, \Delta p_D(\omega), \Delta z, y, V)$ with respect to V, cf. (68).

For the above described 2-point boundary value problem we have then the following twelve state variables x_i, $i = 1, \ldots, 12$:

$$x_1(t) = \frac{\partial \Delta q_1}{\partial \Delta p_D}(t,0), \qquad x_7(t) = \frac{\partial y_1}{\partial \Delta p_D}(t,0),$$
$$x_2(t) = \frac{\partial \Delta q_2}{\partial \Delta p_D}(t,0), \qquad x_8(t) = \frac{\partial y_2}{\partial \Delta p_D}(t,0),$$
$$x_3(t) = \frac{\partial \Delta q_3}{\partial \Delta p_D}(t,0), \qquad x_9(t) = \frac{\partial y_3}{\partial \Delta p_D}(t,0),$$
$$x_4(t) = \frac{\partial \Delta \dot{q}_1}{\partial \Delta p_D}(t,0), \qquad x_{10}(t) = \frac{\partial y_4}{\partial \Delta p_D}(t,0),$$
$$x_5(t) = \frac{\partial \Delta \dot{q}_2}{\partial \Delta p_D}(t,0), \qquad x_{11}(t) = \frac{\partial y_5}{\partial \Delta p_D}(t,0),$$
$$x_6(t) = \frac{\partial \Delta \dot{q}_3}{\partial \Delta p_D}(t,0), \qquad x_{12}(t) = \frac{\partial y_6}{\partial \Delta p_D}(t,0).$$

Moreover, at the starting time point $t_0 = 0$ we have the initial conditions

$$x_1(0) = x_2(0) = \cdots = x_6(0) = 0,$$

and the terminal conditions read:

$$x_7(t_f) = x_8(t_f) = \cdots = x_{12}(t_f) = 0.$$

For simplification, we suppose now that the matrix $V(t)$ of regulator parameter functions has diagonal sub matrices $K_p(t), K_d(t)$, cf. (63b,c), hence,

$$v_1(t) := [K_p]_{11}(t) \quad v_2(t) := [K_p]_{22}(t) \quad v_3(t) := [K_p]_{33}(t), \tag{71}$$
$$v_4(t) := [K_d]_{11}(t) \quad v_5(t) := [K_d]_{22}(t) \quad v_6(t) := [K_d]_{33}(t). \tag{72}$$

In addition, we define

$$c_{ij} = [\widetilde{Q}(t, V(t))]_{ij}. \tag{73}$$

Note that the these diagonal elements of $K_p(t)$, $K_d(t)$ are determined by minimizing the approximate expected Hamiltonian (68) with respect to the above mentioned elements $v = v_i(t)$, $i = 1, \ldots, 6$.

Differential Eq. (70a) can then be represented by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -v_1 & 0 & 0 & -v_4 & 0 & 0 \\ 0 & -v_2 & 0 & 0 & -v_5 & 0 \\ 0 & 0 & -v_3 & 0 & 0 & -v_6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -[(M(t))^{-1}\ Y(t)]_1 \\ -[(M(t))^{-1}\ Y(t)]_2 \\ -[(M(t))^{-1}\ Y(t)]_3, \end{pmatrix} \tag{74}$$

and (70b) reads:

$$\begin{pmatrix} \dot{x}_7 \\ \dot{x}_8 \\ \dot{x}_9 \\ \dot{x}_{10} \\ \dot{x}_{11} \\ \dot{x}_{12} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & v_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & v_3 \\ -1 & 0 & 0 & v_4 & 0 & 0 \\ 0 & -1 & 0 & 0 & v_5 & 0 \\ 0 & 0 & -1 & 0 & 0 & v_6 \end{pmatrix} \begin{pmatrix} x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \end{pmatrix} - 2 \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}. \tag{75}$$

Having representation (74) and (75), the 2-point boundary value problem (70a–c) is solved numerically by means of the collocation solver COLNEW [6]. The robust optimal parameters of the feedback law $\varphi(t,\cdot,\cdot)$ are then obtained, cf. (63b–d), from

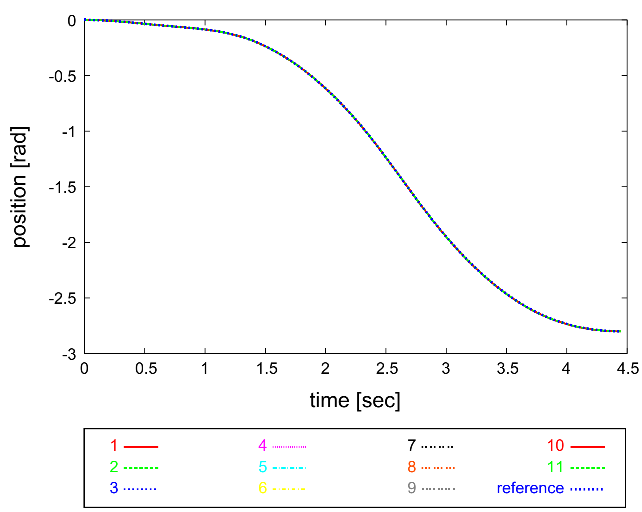

Fig. 1. $q_1(t)$, $t_0 \leqslant t \leqslant t_f$.

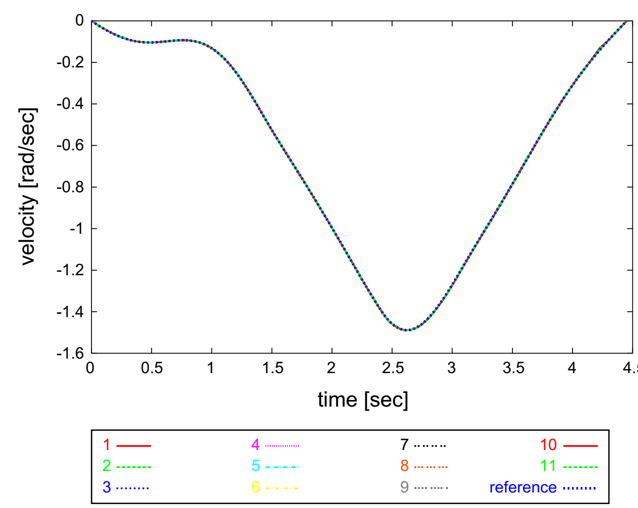

Fig. 3. $\dot{q}_1(t)$, $t_0 \leqslant t \leqslant t_f$.

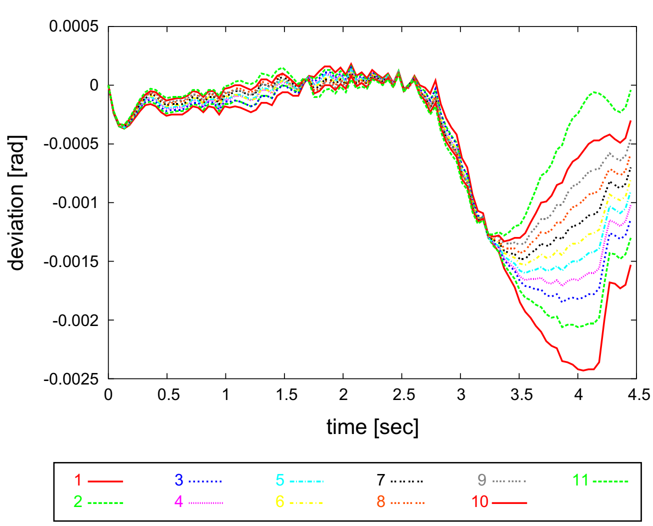

Fig. 2. $\Delta q_1(t)$, $t_0 \leqslant t \leqslant t_f$.

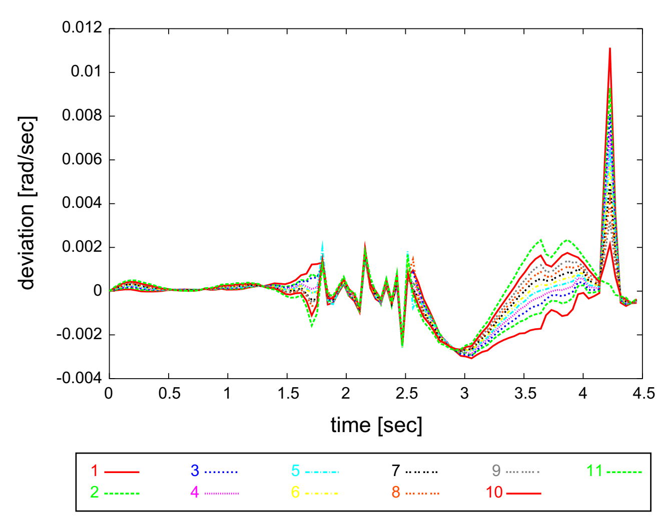

Fig. 4. $\Delta \dot{q}_1(t)$, $t_0 \leqslant t \leqslant t_f$.

$$D_{\Delta q}\varphi(t,0,0) = K(t) - M(t)K_p^*(t), \tag{76a}$$
$$D_{\Delta \dot{q}}\varphi(t,0,0) = D(t) - M(t)K_d^*(t), \tag{76b}$$

where $V^*(t) = (K_p^*(t), K_d^*(t))$ results from the insertion of the solution of (70a–d) into the approximate H-minimum control $\widetilde{V}^*$ obtained from minimizing the approximate expected Hamiltonian (68) subject to $V(t)$.

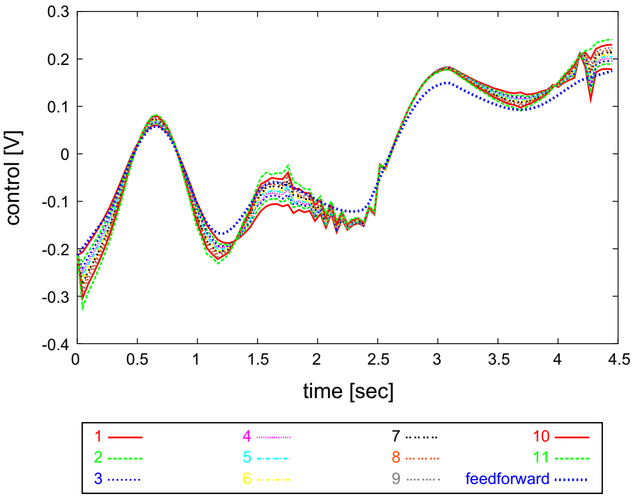

Fig. 5. $u_1(t),\ t_0 \leqslant t \leqslant t_f$.

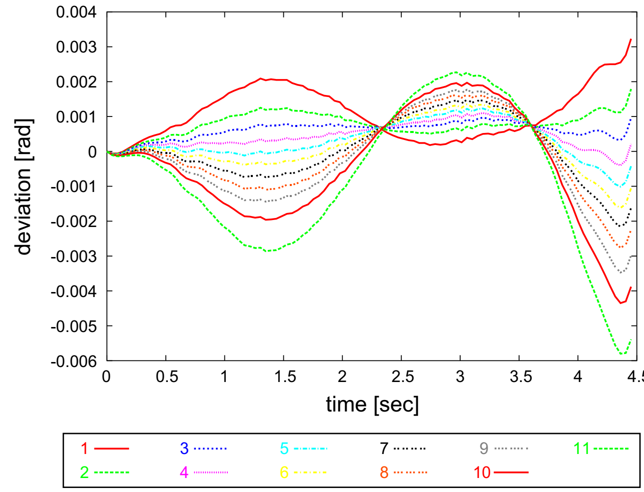

Fig. 8. $\Delta q_2(t),\ t_0 \leqslant t \leqslant t_f$.

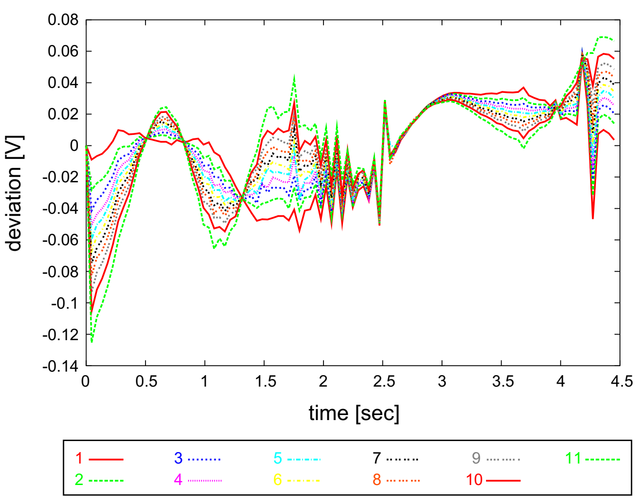

Fig. 6. $\varphi_1(t, \Delta q(t), \Delta\dot{q}(t)),\ t_0 \leqslant t \leqslant t_f$.

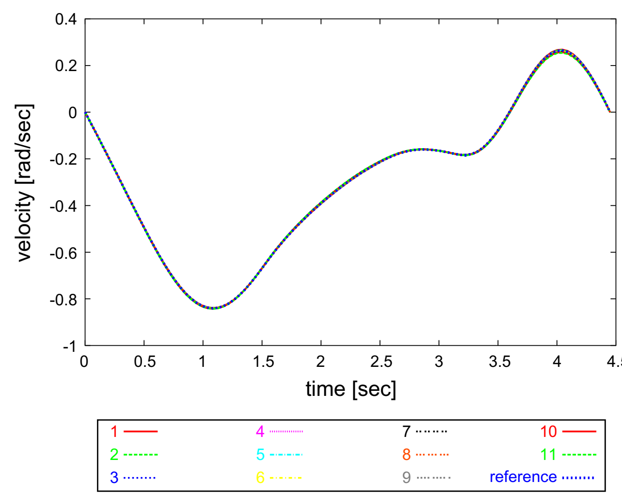

Fig. 9. $\dot{q}_2(t),\ t_0 \leqslant t \leqslant t_f$.

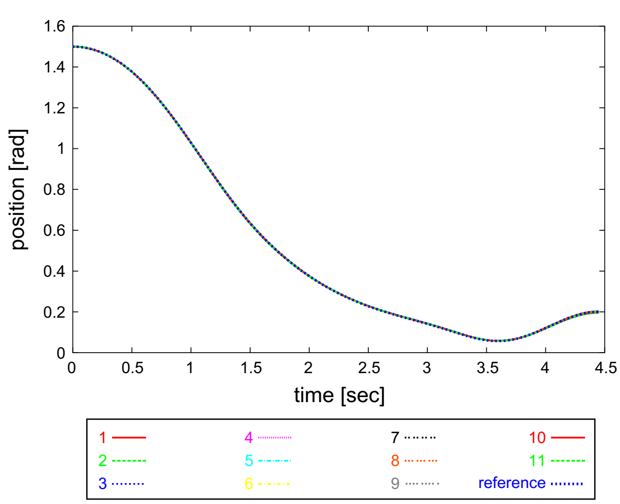

Fig. 7. $q_2(t),\ t_0 \leqslant t \leqslant t_f$.

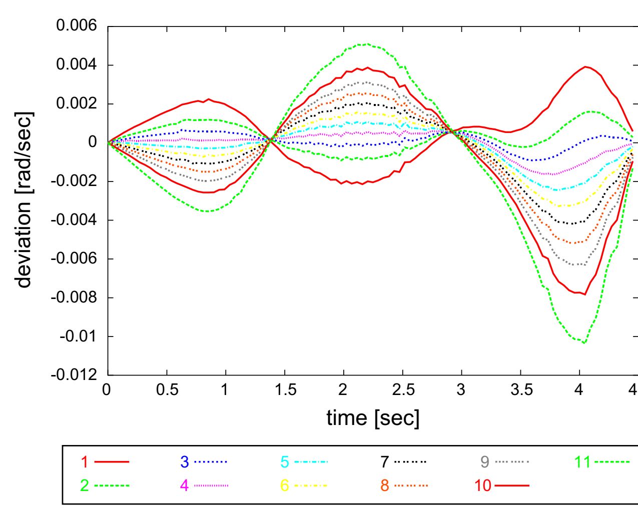

Fig. 10. $\Delta\dot{q}_2(t),\ t_0 \leqslant t \leqslant t_f$.

In order to evaluate the tracking properties of the robust optimal feedback control law obtained by the above described method, we apply the obtained robust optimal regulator to 11 actual trajectories of the manipulator generated by 11 realizations of the random payload mass assumed to be N(0,5)-normal distributed. We assume that these realizations lie in confidence interval related

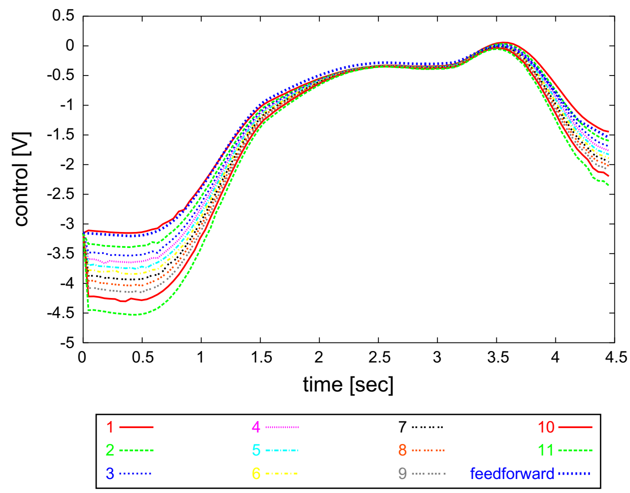

Fig. 11. $u_2(t),\ t_0 \leqslant t \leqslant t_f$.

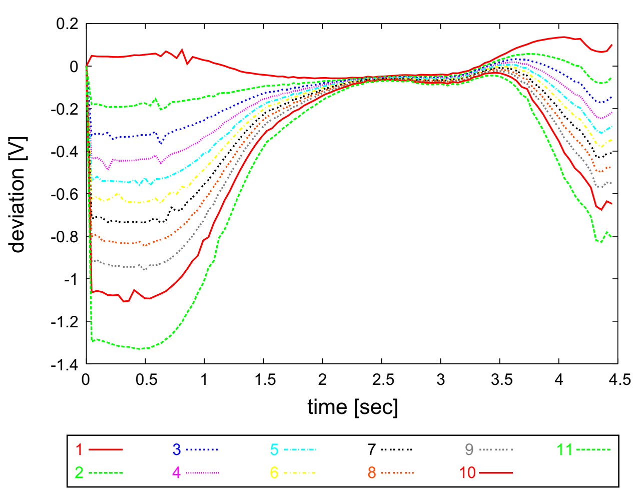

Fig. 12. $\varphi_2(t, \Delta q(t), \Delta\dot{q}(t)),\ t_0 \leqslant t \leqslant t_f$.

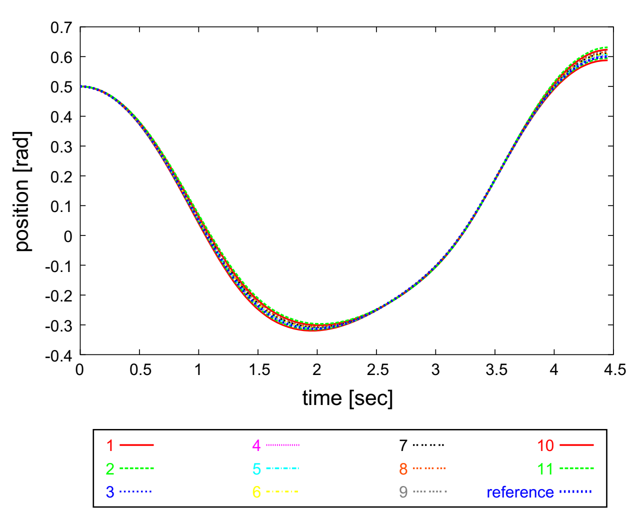

Fig. 13. $q_3(t),\ t_0 \leqslant t \leqslant t_f$.

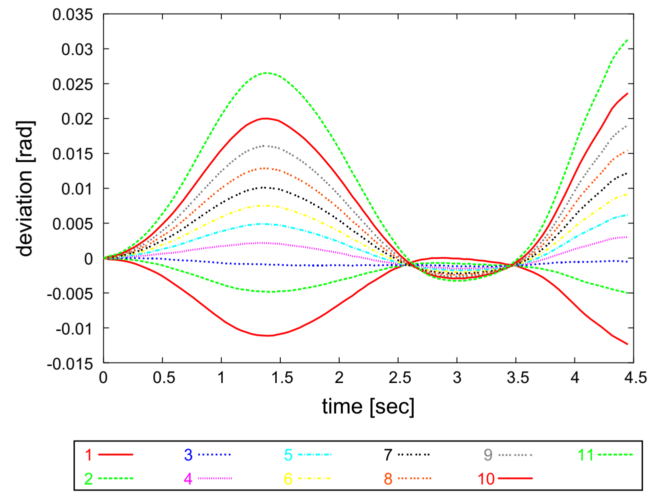

Fig. 14. $\Delta q_3(t),\ t_0 \leqslant t \leqslant t_f$.

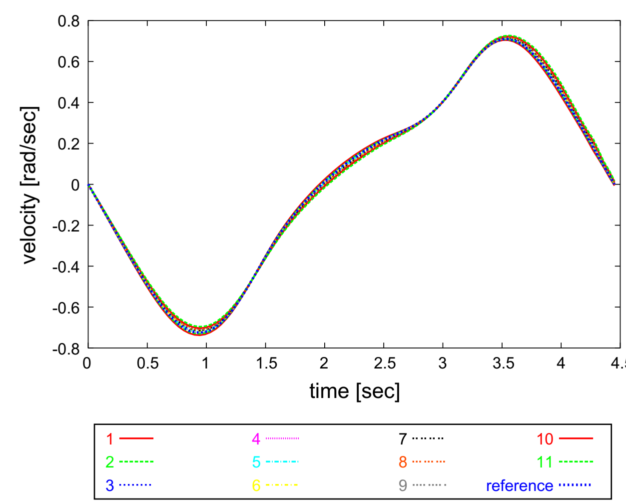

Fig. 15. $\dot{q}_3(t),\ t_0 \leqslant t \leqslant t_f$.

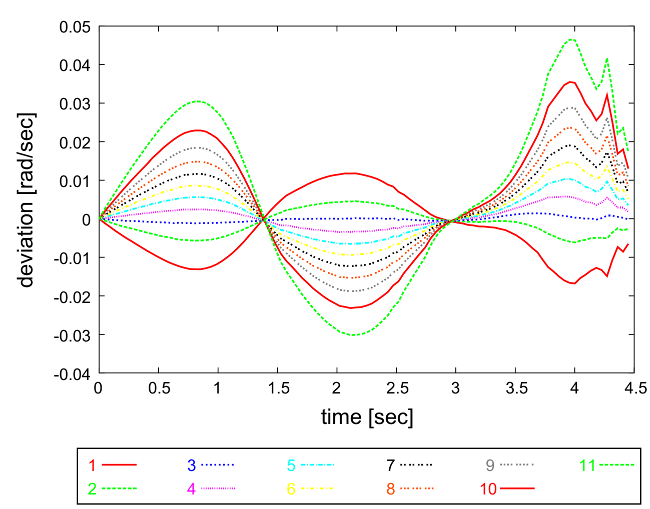

Fig. 16. $\Delta\dot{q}_3(t),\ t_0 \leqslant t \leqslant t_f$.

to the confidence number α. Events having a probability smaller than $\frac{1-\alpha}{2}$, larger than $\frac{1+\alpha}{2}$, resp. are not taken into account. For $\alpha = 0.9$, the remaining range is partitioned then by equidistant points. The resulting realizations are given in the following table:

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7	Case 8	Case 9	Case 10	Case 11
Prob. x	0.05	0.14	0.23	0.32	0.41	0.5	0.59	0.68	0.77	0.86	0.95
$p_D = \Phi^{-1}(x)$	1.7755	4.5985	6.3055	7.6615	8.8625	10	11.1375	12.3385	13.6945	15.4015	18.2245

The optimal feed forward control and the reference trajectory for a point-to-point trajectory optimization problem from the initial point

$$\begin{pmatrix} 0.0 \\ 1.5 \\ 0.5 \end{pmatrix} \text{ to the terminal point } \begin{pmatrix} -2.8 \\ 0.2 \\ 0.6 \end{pmatrix},$$

were computed by means of the stochastic roboter optimization software (*OSTP*) [5]. The trajectories were computed by *SIMPACK* [15], a multi-body simulation software package.

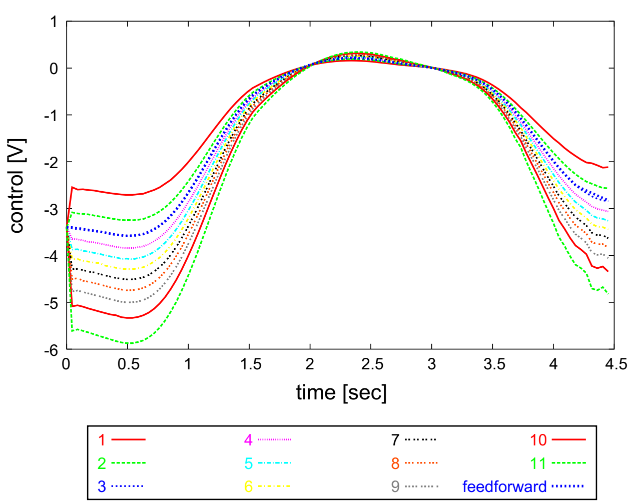

Fig. 17. $u_3(t),\ t_0 \leqslant t \leqslant t_f$.

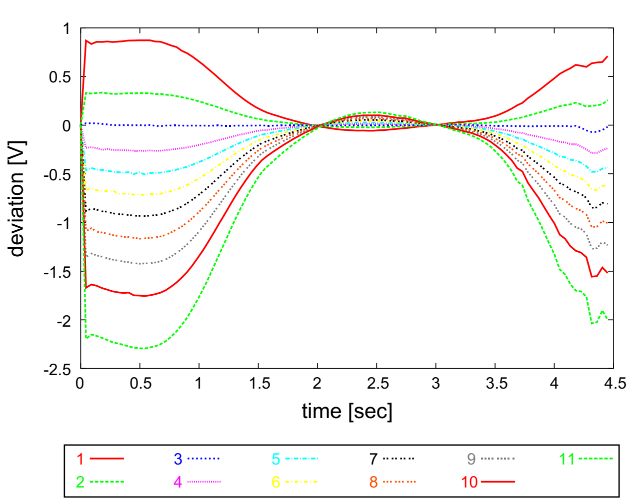

Fig. 18. $\varphi_3(t, \Delta q(t), \Delta \dot{q}(t)),\ t_0 \leqslant t \leqslant t_f$.

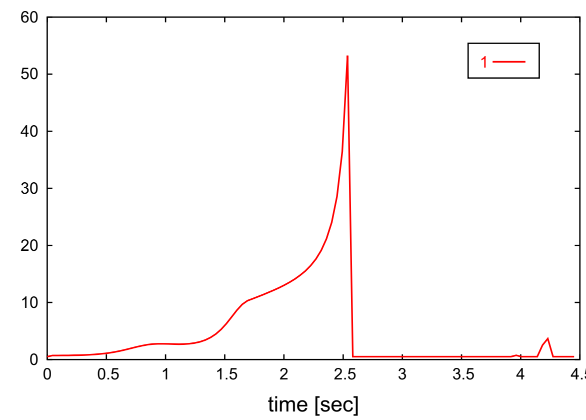

Fig. 19. $[K_p]_{11},\ t_0 \leqslant t \leqslant t_f$.

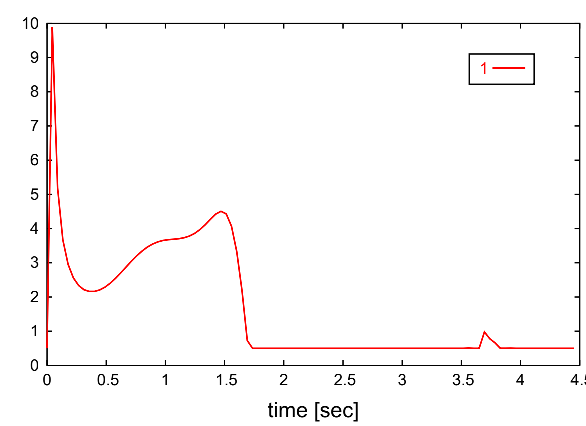

Fig. 20. $[K_d]_{11},\ t_0 \leqslant t \leqslant t_f$.

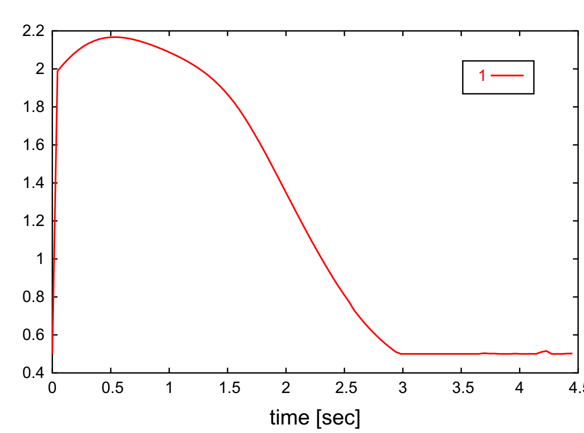

Fig. 21. $[K_p]_{22},\ t_0 \leqslant t \leqslant t_f$.

In the following Figs. 1–18 the simulation results for all three links are presented; also the basic reference trajectories and feed forward controls can be seen. In the figures the time development of the configuration coordinates q_i, velocities $\dot{q}_i$, their deviations $\Delta q_i, \Delta\dot{q}_i$ from the corresponding reference values, the total manipulator control input u_i and the robust optimal feedback φ_i are given. Moreover, Figs. 19–24 present the behavior of the basic robust optimal regulator parameter functions

$$K^*_{p,11} = K^*_{p,11}(t), \ldots, K^*_{p,33} = K^*_{p,33}(t), \quad K^*_{d,11} = K^*_{d,11}(t), \ldots, K^*_{d,33} = K^*_{d,33}(t). \tag{76c}$$

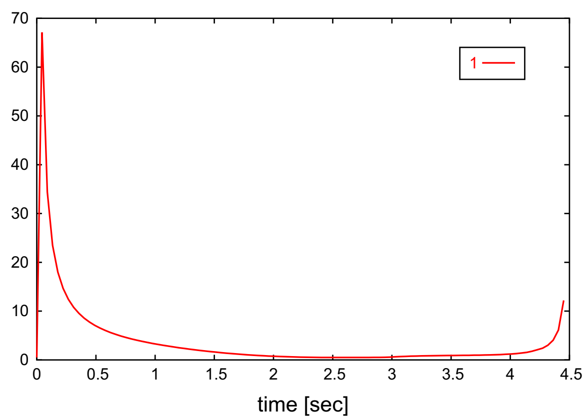

Fig. 22. $[K_d]_{22},\ t_0 \leqslant t \leqslant t_f$.

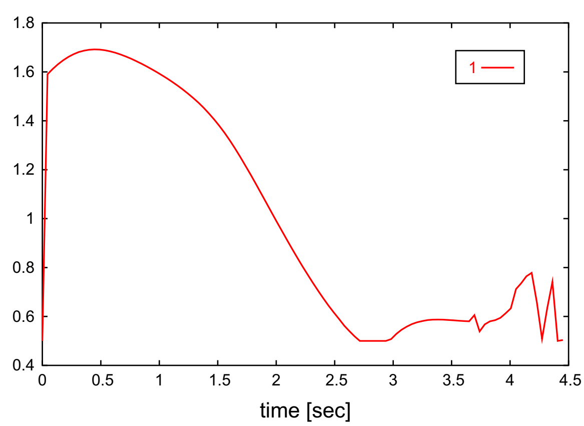

Fig. 23. $[K_p]_{33},\ t_0 \leqslant t \leqslant t_f$.

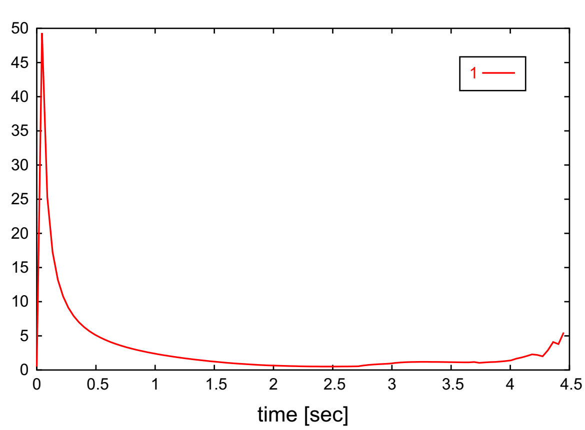

Fig. 24. $[K_d]_{33},\ t_0 \leqslant t \leqslant t_f$.

For comparison, in the next series of Figs. 25–42 the behavior of a standard PD-controller with regulator parameters taken from the literature [4,33] is presented, where the same realizations of the

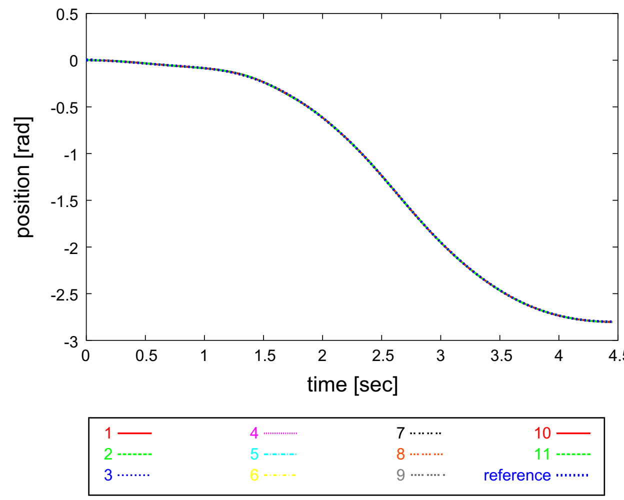

Fig. 25. $q_{1,s}(t),\ t_0 \leqslant t \leqslant t_f$.

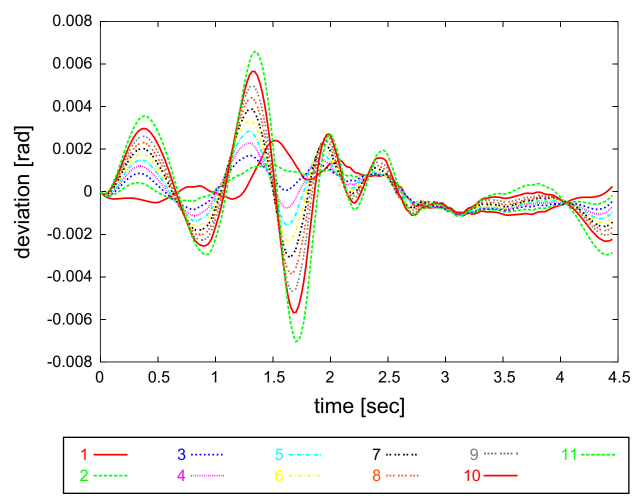

Fig. 26. $\Delta q_{1,s}(t),\ t_0 \leqslant t \leqslant t_f$.

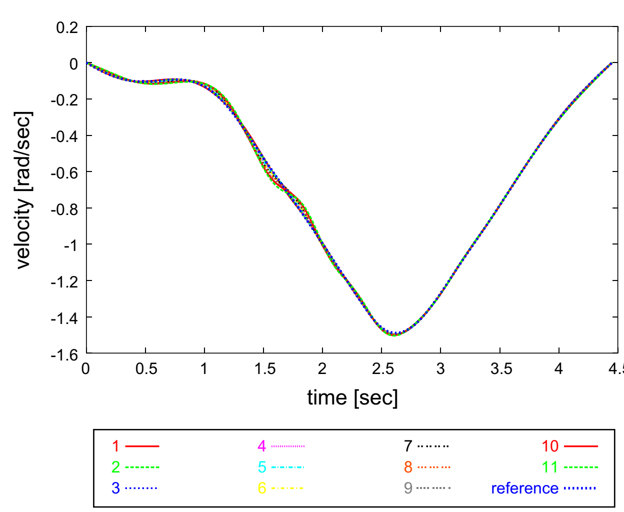

Fig. 27. $\dot{q}_{1,s}(t),\ t_0 \leqslant t \leqslant t_f$.

payload and also the same remaining assumptions are used as for the robust optimal regulator. Here, the subscript "s" denotes the results corresponding to the standard PD-controller.

Obviously, the tracking of the given reference trajectory in configuration space of the manipulator provided by the robust optimal regulation based on the stochastic optimization method is

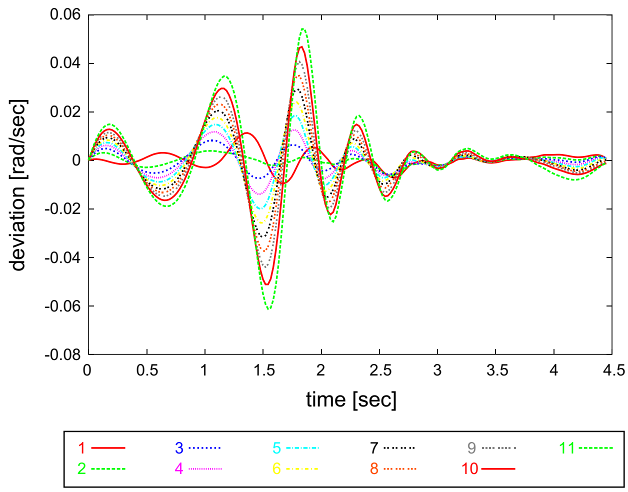

Fig. 28. $\Delta\dot{q}_{1,s}(t),\ t_0 \leqslant t \leqslant t_f$.

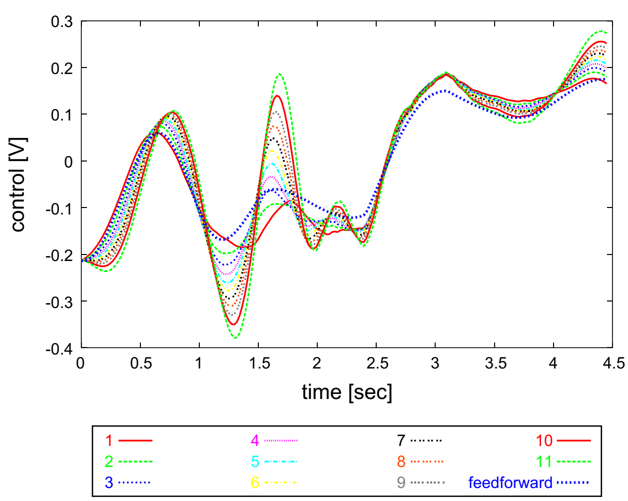

Fig. 29. $u_{1,s}(t),\ t_0 \leqslant t \leqslant t_f$.

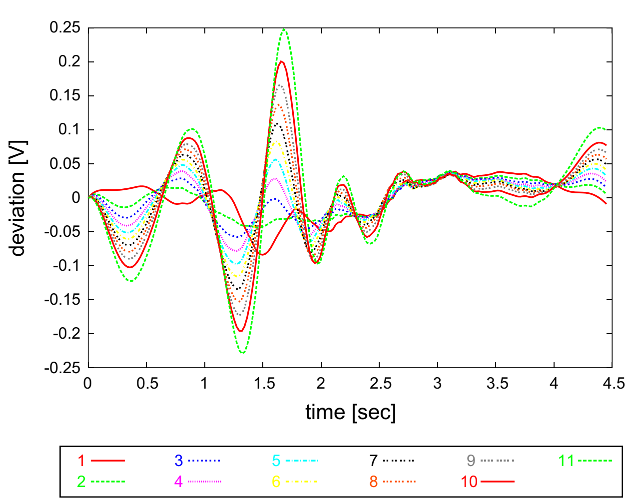

Fig. 30. $\varphi_{1,s}(t, \Delta q(t), \Delta\dot{q}(t)),\ t_0 \leqslant t \leqslant t_f$.

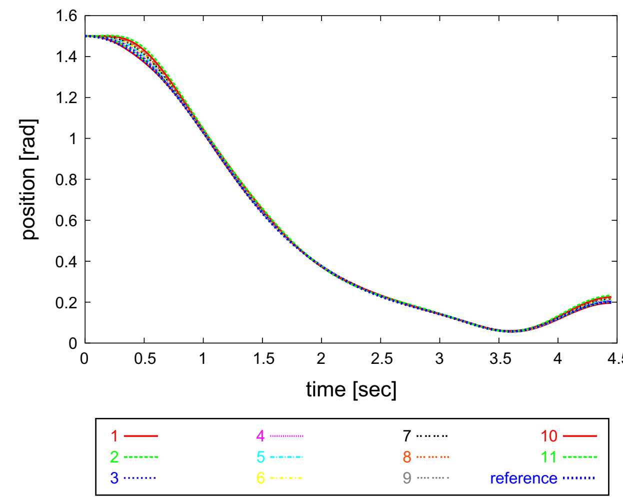

Fig. 31. $q_{2,s}(t),\ t_0 \leqslant t \leqslant t_f$.

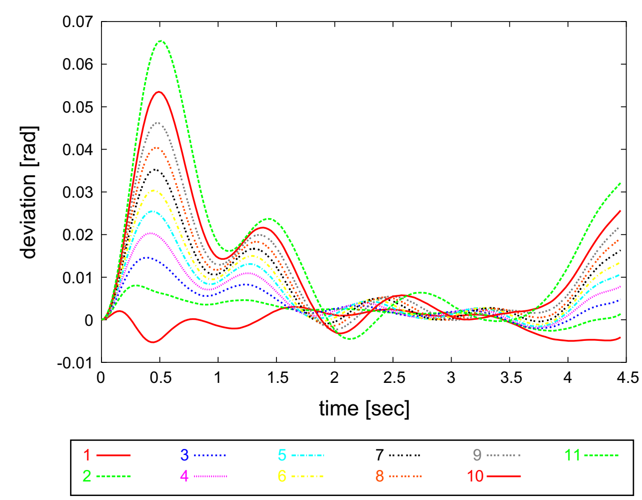

Fig. 32. $\Delta q_{2,s}(t),\ t_0 \leqslant t \leqslant t_f$.

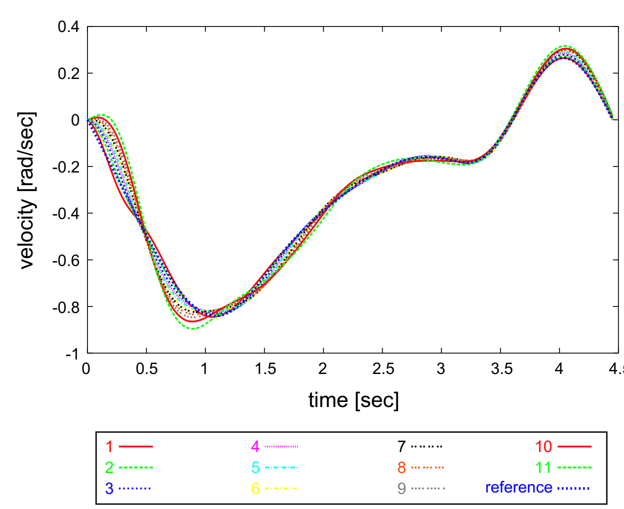

Fig. 33. $\dot{q}_{2,s}(t),\ t_0 \leqslant t \leqslant t_f$.

- much faster
- more exact
- more stable

than the tracking by the standard PD-regulator. From Figs. 19–24 it can be seen that, in contrast to the standard *PD*-regulator, the robust optimal regulator parameter functions K_p^*, K_d^* tend to zero towards the end of the control period.

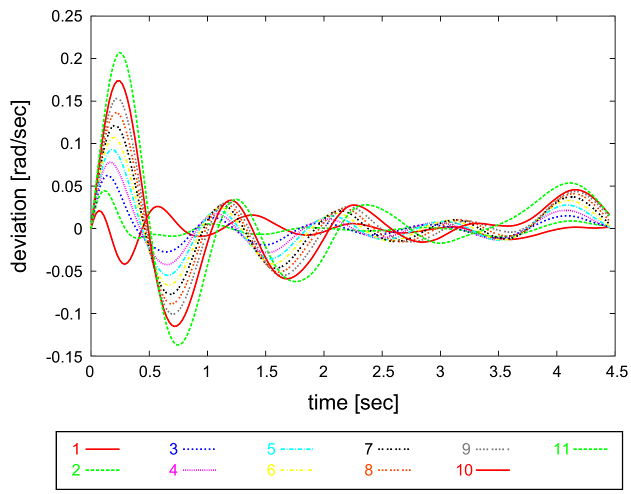

Fig. 34. $\Delta\dot{q}_{2,s}(t),\ t_0 \leqslant t \leqslant t_f$.

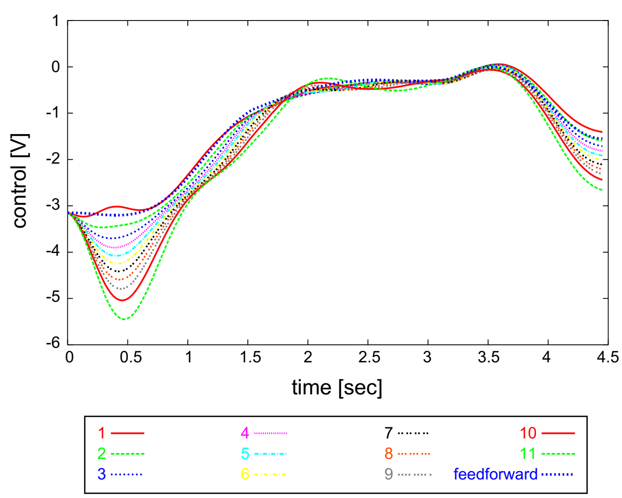

Fig. 35. $u_{2,s}(t),\ t_0 \leqslant t \leqslant t_f$.

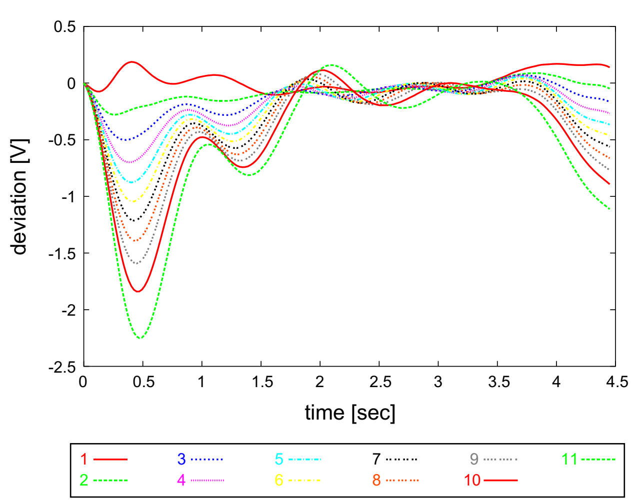

Fig. 36. $\varphi_{2,s}(t, \Delta q(t), \Delta\dot{q}(t)),\ t_0 \leqslant t \leqslant t_f$.

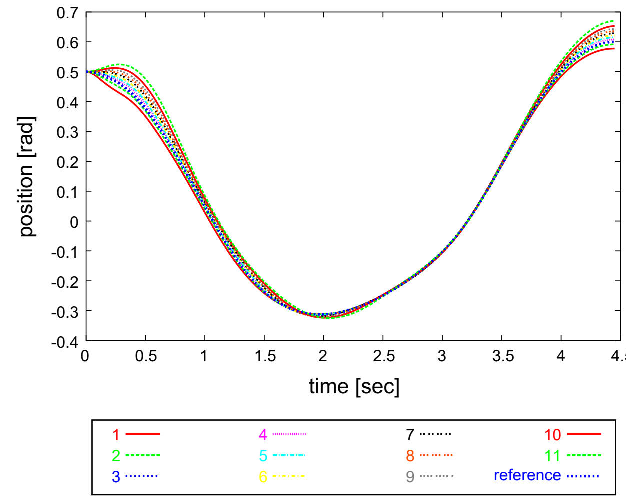

Fig. 37. $q_{3,s}(t),\ t_0 \leqslant t \leqslant t_f$.

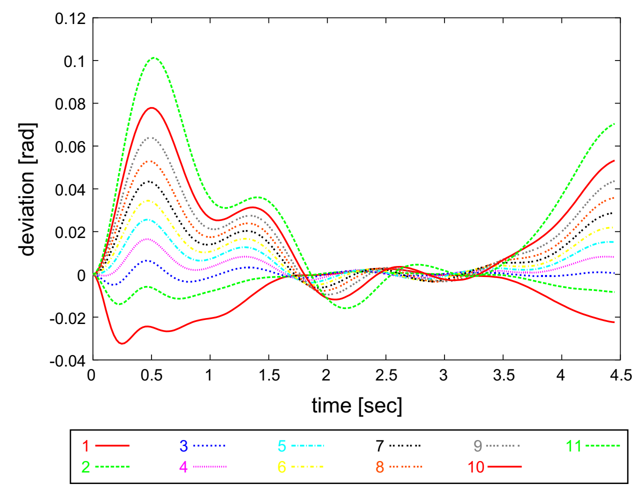

Fig. 38. $\Delta q_{3,s}(t),\ t_0 \leqslant t \leqslant t_f$.

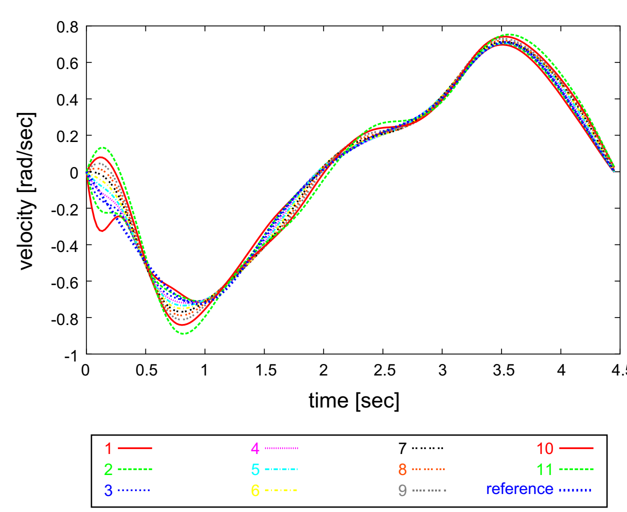

Fig. 39. $\dot{q}_{3,s}(t),\ t_0 \leqslant t \leqslant t_f$.

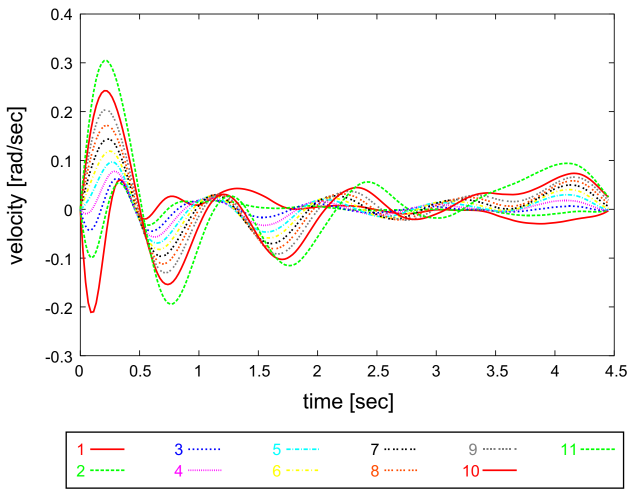

Fig. 40. $\Delta\dot{q}_{3,s}(t),\ t_0 \leqslant t \leqslant t_f$.

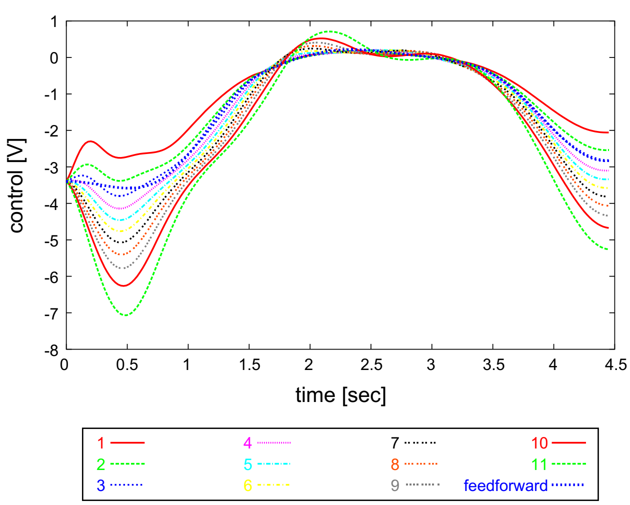

Fig. 41. $u_{3,s}(t),\ t_0 \leqslant t \leqslant t_f$.

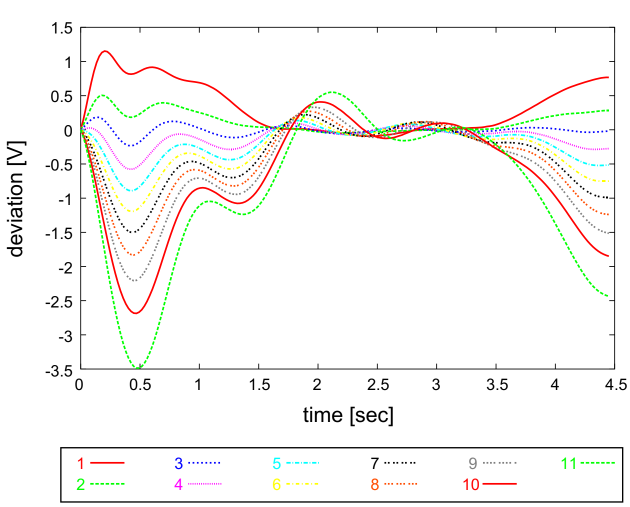

Fig. 42. $\varphi_{3,s}(t, \Delta q(t), \Delta\dot{q}(t)),\ t_0 \leqslant t \leqslant t_f$.

7. Conclusions

Classical stochastic optimal control problems are based on process differential equations described by stochastic differential equations having an additive white noise term. However, for many practical dynamic structures/systems the modeling of the process dynamic equation by a system of ordinary differential equations with random dynamic parameters, random initial values and random external loadings, payloads, etc., is more appropriate. The problem is then to determine robust optimal controls, hence, feed forward/feedback controls u^* minimizing the expected total costs arising along the trajectory and at the terminal state. Using stochastic optimization methods, the mathematical basis is available now for the construction of robust optimal controls by means of the following procedure:

- Finding a *H*-minimum control $\widetilde{u^*}$.
- Solving the related *Hamiltonian or canonical* 2-point boundary value problem with random dynamic parameters, random initial values and random applied dynamic loadings/excitations.
- Inserting the solution of the 2-point boundary value problem, i.e. the state-and costate-trajectory, into the *H*-minimum control $\widetilde{u^*}$ to get an optimal, or at least a stationary control u^*.

Here, for finding a *H*-minimum control, a finite-dimensional stochastic programming problem has been derived. The solution of the related 2-point boundary value problem with random parameters can be obtained then by two different methods: (a) discretization of the underlying probability distribution of the random parameters, and (b) Taylor expansion of the Hamilton function with respect to the vector of model parameters at its mean. In both cases a standard 2-point boundary value problem results, which can be solved numerically by an appropriate software package. The functioning of the method was shown by the computation of a robust optimal feedback controller for a manipulator with three degrees of freedom. The obtained numerical results of the robust optimal regulator are compared with the corresponding results of the standard regulator from the literature.

The present method can be extended also to the construction of robust optimal *PID*-regulators.

References

[1] Ackermann J. Robust Regelung. Berlin–Heidelberg–New York: Springer-Verlag; 1993.
[2] Aoki M. Optimization of stochastic systems – topics in discrete-time systems. New York–London: Academic Press; 1967.
[3] Aström KJ. Introduction to stochastic control theory. New York: Academic Press; 1970.
[4] Aurnhammer A. Optimale Stochastische Trajektorienplanung und Regelung von Industrierobotern. Fortschrittberichte VDI, Reihe 8, Mess-, Steuerungs-und Regelungstechnik, vol. 1032. Düsseldorf: VDI-Verlag GmbH; 2004.
[5] Aurnhammer A, Marti K, Schacher M. OSTP – Fortran-program for optimal stochastic trajectory planning for robots, version 0.5. Neubiberg/Munich: Federal Armed Forces University Munich, Institute for Mathematics and Computer Applications; 2009.
[6] Bader G, Ascher U. A new basis implementation for a mixed order boundary value ODE solver. Siam J Sci Stat Comput 1987;8:483–500.
[7] Barucha-Reid AT. Random integral equations. New York–London: Academic Press; 1972.
[8] Basar T, Bernhard P. H_∞-optimal control and related minimax design problems: a dynamic approach. Boston: Birkhäuser; 1991.
[9] Block C. Aktive Minderung personeninduzierter Schwingungen an weit gespannten Strukturen im Bauwesen. Fortschrittberichte VDI, Reihe 11, Schwingungstechnik, vol. 336. Düsseldorf: VDI-Verlag GmbH; 2008.
[10] Bunke H. Gewöhnliche Differentialgleichungen mit zufälligen Parametern. Berlin: Akademie-Verlag; 1972.
[11] Diedonné J. Foundations of modern analysis. New York: Academic Press; 1969.
[12] Dullerud GE, Paganini F. A course in robust control theory – a convex approach. New York: Springer-Verlag; 2000.
[13] Howard RA. Dynamic probabilistic systems. New York: Wiley; 1971.
[14] Kalman RE, Falb PL, Arbib MA. Topics in mathematical system theory. New York: McGraw-Hill Book Company; 1969.

[15] Kick W, Möller G, Rulka W. SIMPACK – ein Programm zur Simulation von Mehrkörpersystemen. Düsseldorf: VDI-Verlag; 1989. Nr. 1087.
[16] Ku R, Athans M. On the adaptive control of linear systems using the open-loop-feedback-optimal approach. IEEE Trans Automat Contr 1973;AC-18:489–93.
[17] Luenberger DG. Optimization by vector space methods. New York–London: Wiley; 1969.
[18] Marti K. Convex approximations of stochastic optimization problems. Methods Oper Res 1975;20:66–76.
[19] Marti K. Approximationen stochastischer Optimierungsprobleme. Königstein/Ts.: Hain; 1979.
[20] Marti K. Approximationen der Entscheidungsprobleme mit linearer Ergebnisfunktion und positiv homogener subadditiver Verlustfunktion. Zeitschrift für Wahrscheinlichkeitstheorie verwandte Gebiete 1975;31:203–33.
[21] Marti K. Stochastic optimization methods in robust adaptive control of robots. In: Grötschel M et al., editors. Online optimization of large scale systems. Berlin–Heidelberg–New York: Springer-Verlag; 2001. p. 545–77.
[22] Marti K. Adaptive optimal stochastic trajectory planning and control (AOSTPC) for robots. In: Marti K, Ermoliev Y, Pflug G, editors. Dynamic stochastic optimization. Berlin–Heidelberg: Springer-Verlag; 2004. p. 155–206.
[23] Marti K. Stochastic optimization problems. 2nd ed. Berlin–Heidelberg: Springer-Verlag; 2008.
[24] Nagarajaiah S, Narasimhan S. Optimal control of structures. In: Arora JS, editor. Optimization of structural and mechanical systems. New Jersey: World Scientific; 2007. p. 221–44.
[25] Smirnov WI. Lehrgang der Höheren Mathematik. Berlin: Deutscher Verlag der Wissenschaft; 1966.
[26] Soong TT. Active structural control in civil engineering. Eng Struct 1988;10:74–84.
[27] Soong TT. Active structural control: theory and practice. New York: Longman Scientific and Technical, J. Wiley; 1990.
[28] Soong TT, Costantinou MC. Passive and active structural vibration control in civil engineering. CISM courses and lectures, vol. 345. Wien–New York: Springer-Verlag; 1994.
[29] Spencer BF, Nagarajaiah S. State of the art of structural control. J Struct Eng, ASCE 2003;129(7):845–56.
[30] Stengel RF. Stochastic optimal control – theory and application. New York: J. Wiley; 1986.
[31] Stengel RF, Ray LR. Stochastic robustness of linear time-invariant control systems. IEEE Trans Automat Contr 1991;AC-36(1):82–7.
[32] Yang JN, Soong TT. Recent advance in active control of civil engineering structures. Probabilist Eng Mech 1988;3(4):179–88.
[33] Türk S. Dynamische Robotemodelle am Beispiel des Manutec R3, Technischer Bericht 88-16. Oberpfaffenhofen: DFVLR, Institut für Dynamik der Flugsysteme; 1988.