A Stochastic Approach to Dubins Feedback Control for Target Tracking

Ross Anderson and Dejan Milutinović

Abstract— A nonlinear system gives rise to many inherent difficulties when designing a feedback control. Motivated by a fixed-speed, fixed-altitude Unmanned Aerial Vehicle (UAV) that tracks an unpredictable target, we seek to control the turning rate of a planar Dubins vehicle. We introduce stochasticity in the problem by assuming the target performs a random walk, which both aides in the computation of a smooth value function and further accounts for all realizations of target kinematics. A Bellman equation based on an approximating Markov chain that is consistent with the stochastic kinematics is used to compute a control policy that minimizes the expected value of a cost function based on a nominal UAV-target distance. Our results indicate how uncertainty in the target motion affects the control law, and simulations illustrate that the control can further be applied to any continuous, smooth trajectory with no need for further computation.

I. INTRODUCTION

The use of Unmanned Aerial Vehicles (UAVs) to track, protect, or provide surveillance of a ground-based vehicle has recently been the focus of much attention and research. In our problem, the UAV is assumed to fly at a constant altitude and with a bounded turning radius. This behavior is modeled by a planar Dubins vehicle [1], which gives a good approximation for feasible UAV trajectories, but this yields a nonlinear system. The Lyapunov stability-based control design is commonly used to develop feedback controllers for problems of this type [2][3][4], but constructing a Lyapunov function may not be a straightforward task.

Alternatively, the control problem may be defined as an infinite horizon optimal control problem, resulting in a Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE) whose solution, the value of the cost-to-go function, prescribes a control that guarantees that the value function will not increase along the trajectory in state space. In this sense, the value function serves as a Lyapunov function that can be constructed computationally. The HJB equation, however, is difficult to solve, and in many scenarios the resulting value function may not be smooth enough to satisfy the HJB equation, or solution characteristics may give rise to shocks. A variety of numerical techniques have been developed, including the use of viscosity solutions [5] and max-plus basis methods [6]. In many cases, a perturbation or approximation is required [7][8]. Artificially adding a small non-degenerate noise to a deterministic problem smooths out

the value function in the area of the shock; this is also a standard PDE technique in related fields [8].

Our goal in this work is to develop a feedback control policy that allows a UAV to maintain a nominal standoff distance from a ground target. We introduce the noise in our problem by assuming that the target motion is a planar random walk [9]. Because of this, we not only provide a sufficiently smooth solution of the HJB equation, but account for wild target kinematics, perhaps that of a target avoiding pursuit. Moreover, any continuous, smooth target trajectory can be considered as a realization of the random walk, although the probability of a particular realization may be very small. Therefore, although strictly speaking our tracking control is optimal in the expected value sense for the random walk, it can be applied to a wider class of continuous and smooth target trajectories.

In many cases it is also easier to characterize a system as a stochastic process than as a low-dimensional chaotic process that could arise, for example, from ensembles of nonlinearly-coupled systems (e.g., UAV targets). We wish to not delve into the philisophical differences between chaos and stochastic processes, but deterministic target motion may mimic Gaussian noise over time scales that are long compared to the characteristic time of the target subsystem, and it is often difficult to distinguish white noise from chaos with a small noise added, such as that which arises from observation error [10].

Our feedback control policy is computed off-line using a Bellman equation discretized as an approximating Markov chain that is spatially and temporally consistent with the stochastic kinematics [11]. This method is well-accepted, but it is rarely used in the robotics community.

Path planning and shortest path problems in time and space for Dubins vehicles have been studied previously in [12], [13], [14], and [15], among others. This type of vehicle has further been studied for the tracking, geolocation, and coverage time of one or more targets by one or more UAVs ([16], [17], [18], [2], [19], [3]). Deterministic variants of this paper's problem that maximize UAV coverage time of a target [16][17] or minimize the geolocation error covariance of target observations [19] use a planning horizon that assumes the target moves in a straight line, which we avoid with the work presented herein.

Stochastic problems in the control of Dubins vehicles typically concentrate on variants of the Traveling Salesperson Problem and other routing problems, in which the target location is unknown or randomly-generated [20][21][22]. Other works examine control methods that direct UAV motion toward the maximum of a scalar field [23][24]. In a previous

work by the authors [25], the effect of penalizing the control and adding a state-dependence in the cost function's time-discounting were investigated using the same techniques. Here we make the case for the inclusion of stochastic motion simplices in robust deterministic robotic control applications and examine how the assumed level of noise intensity of the target will change the control policy. To the best of our knowledge, these works are the first use of stochastic optimal control for the type of problem at hand.

In what follows, we first formulate our problem in Section II for the case of a Brownian target. Section III provides the methodologies for accurate discretization and value iteration to compute the control for this stochastic problem. In Section IV we analyze the connection between the chosen noise intensity and the resulting control law, as well as demonstrate the effectiveness of this approach for both Brownian targets and targets with unknown trajectory. Section V concludes this paper and provides directions for future research.

II. PROBLEM STATEMENT

We consider a UAV flying at a constant altitude in the vicinity of a ground-based target, tasked with maintaining a nominal distance from the target. The target is located at position $\vec{r}_T(t) = [x_T(t), y_T(t)]^T$ at the time point t (see Fig. 1). The UAV, located at position $\vec{r}_A(t) = [x_A(t), y_A(t)]^T$, moves in the direction of its heading angle θ at a constant speed v_A. The turning rate is determined by a non-anticipative [11], bounded control $u(t) \in \mathcal{U} \equiv \{u : |u| \leq u_{\max}\}$, which has to be found.

In our problem formulation, the target motion is unknown. We therefore assume that it is random and described by a 2D stochastic process. Drawing from the field of estimation, the simplest signal that can be used to describe an unknown model suggests that the motion of the target should be described by a 2D Brownian particle:

$$\begin{aligned} dx_T(t) &= \sigma dw_x \\ dy_T(t) &= \sigma dw_y \end{aligned} \quad (1)$$

where dw_x and dw_y are increments of unit intensity Wiener processes along the x and y axes, respectively, which are mutually independent. The level of noise intensity σ determining the target motion is assumed to be in a range that allows the UAV to effectively pursue or maintain pace with the target with high probability. For simplicity, any additional noise due to UAV observation error is also incorporated into the random walk, i.e., the parameter σ. Furthermore, this level of noise intensity is constant over the UAV observation area.

The UAV can be modeled as a planar Dubins vehicle [1]:

$$\begin{aligned} dx_A(t) &= v_A \cos(\theta(t)) \, dt \\ dy_A(t) &= v_A \sin(\theta(t)) \, dt \\ d\theta(t) &= -u(t)dt, \qquad u \in \mathcal{U} \end{aligned} \quad (2)$$

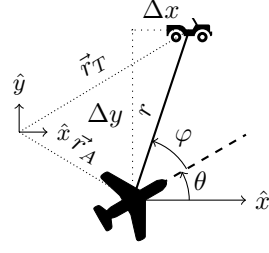where, without loss of generality, we have chosen the sign of $d\theta(t)$ to clarify later results.

Fig. 1. Diagram of a UAV at position $\vec{r}_A$ that is moving at heading angle θ and tracking a randomly-moving target at $\vec{r}_T$ with distance $r = |\vec{r}_A - \vec{r}_T|$ and relative angle φ.

In order for the control to be independent of the heading angle of the UAV or the absolute position of the UAV or target, we relate the problem to relative dynamics based on a time-varying coordinate system aligned with the direction of the UAV velocity. The reduced system state is composed of the distance between the UAV and target $r = |\vec{r}_T - \vec{r}_A|$ and the viewing angle φ between the UAV's direction of motion and the vector from the UAV to the target, as seen in Fig. 1:

$$\begin{aligned} r &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \quad (3) \\ \varphi &= \tan^{-1}\left(\frac{\Delta y}{\Delta x}\right). \quad (4) \end{aligned}$$

The combined UAV-target system (1-4) should maintain the relative distance r at the nominal distance d for all times. To this end we seek to minimize the expectation of an infinite-horizon cost function $W(\cdot)$ with a discounting factor $\beta > 0$ and with zero penalty for control:

$$\begin{aligned} W(r, \varphi, u) &= E_r^u \left\{ \int_0^\infty e^{-\beta t} k\left(r(t)\right) dt \right\} \quad (5) \\ k(r) &= (r - d)^2. \end{aligned}$$

A high value of β places more weight on the instantaneous cost, while β near 0 considers future costs, as well as instantaneous cost.

Next, taking into account the random motion of the target, the evolution of $r = |\vec{r}| = |\vec{r}_T - \vec{r}_A|$ can be derived by first considering the differential components $d(\Delta x)$ and $d(\Delta y)$ of the distance vector $\vec{r}$ along the x and y axes:

$$\begin{aligned} d(\Delta x) &= \sigma dw_x - v_A \cos(\theta) \, dt \\ d(\Delta y) &= \sigma dw_y - v_A \sin(\theta) \, dt \end{aligned} \quad (6)$$

Using Itô's Lemma with (4), the differentials dr and $d\varphi$ can be found as

$$\begin{aligned} dr &= \left(-v_A \cos\varphi + \frac{\sigma^2}{2r}\right) dt \\ &\quad + \sigma \cos(\theta + \varphi) \, dw_x + \sigma \sin(\theta + \varphi) \, dw_y \quad (7) \end{aligned}$$

$$\begin{aligned} d(\theta + \varphi) &= \frac{v_A}{r} \sin\varphi \, dt \\ &\quad - \frac{\sigma}{r} \sin(\theta + \varphi) \, dw_x + \frac{\sigma}{r} \cos(\theta + \varphi) \, dw_y. \end{aligned} \quad (8)$$

Recalling that our coordinate frame is aligned with the direction of UAV motion ($\theta = 0$) and that $d\theta = -udt$ (2), the relative UAV-target kinematics model is

$$
\begin{align}
dr &= \left(-v_A \cos\varphi + \frac{\sigma^2}{2r}\right) dt + \sigma\, dw_0 \tag{9} \\
d\varphi &= \left(\frac{v_A}{r}\sin\varphi + u\right) dt + \frac{\sigma}{r} dw_\perp, \tag{10}
\end{align}
$$

where dw_0 and $dw_\perp$ are mutually independent increments of unit intensity Wiener processes. Since the components of the original 2D random walk model are scaled with the same parameter σ, the noise is invariant under a rotation of the coordinate frame [26]. Note the presence of a positive bias $\sigma^2/2r$ in the relation for $r(t)$, which is a consequence of the random process included in our analysis.

To determine the set of admissible control $\mathcal{U}$, we look to the limit of no noise ($\sigma = 0$), where the distance and angle differentials are described by

$$
\begin{align}
dr &= -v_A \cos\varphi\, dt \\
d\varphi &= \left(\frac{v_A}{r}\sin\varphi + u\right) dt. \tag{11}
\end{align}
$$

With the steady state of (11) in mind, the admissible set is given by $|u| \leq v_A/r_{\min} \equiv u_{\max}$ for the nominal distance $d \in (r_{\min}, r_{\max})$.

III. MARKOV CHAIN APPROXIMATION AND VALUE ITERATION

When discretizing a state space for dynamic programming in stochastic problems, spatial and temporal step sizes should be accurately scaled with respect to the stochastic process. To take this into account, we employ the *Markov chain approximation method* [11] for numerically determining the optimal control policy corresponding to the controlled diffusion process (9-10) and cost function (5).

This well-accepted technique involves the careful construction of a discrete-time and discrete-state approximation in the form of a controlled Markov chain that is "locally-consistent" with the process under control. Likewise, an appropriate approximation to the cost function $W(\cdot)$ is chosen by one of several procedures. Details of the construction for our problem in terms of finite difference methods may be found in [25].

For simplicity and to follow the notation of [11], we will write the discrete version of the current system state as $\mathbf{x} = [r,\ \varphi]^T$. Denote $\mathcal{L}^u$ the differential operator associated with the stochastic process (9-10). It can be shown [11] that a sufficiently smooth $W(\mathbf{x}, u)$ given by (5) satisfies

$$
\mathcal{L}^u W(\mathbf{x}, u) - \beta(\mathbf{x})W(\mathbf{x}, u) + k(\mathbf{x}, u) = 0. \tag{12}
$$

The Bellman equation for the minimum cost $V(\mathbf{x})$ over all control sequences is

$$
\inf_u \left[\mathcal{L}^u V(\mathbf{x}) - \beta(\mathbf{x})V(\mathbf{x}) + k(\mathbf{x}, u)\right] = 0. \tag{13}
$$

It is assumed that if the Dubins vehicle is leaving the computational domain, the value function does not change, since the vehicle is capable of returning to the domain. This results in reflective [11] boundary conditions $(\nabla V(\mathbf{x}))^T \hat{n} = 0$ at the domain boundaries with normals $\hat{n}$.

The transition probabilities for the approximation to the cost function, $p(\mathbf{x}\,|\,\mathbf{y}, u)$, from the state $\mathbf{x}$ to the state $\mathbf{y} \in \mathbb{R}^2$ under the control u appear as coefficients in the finite-difference approximations of the operator $\mathcal{L}^u$ in (12). The relation between the time step $\Delta t(\mathbf{x}, u)$ and spatial grid spacing allows the chain to satisfy the requirement of "local consistency," in the sense that the drift and covariance of the cost function are consistent with the drift and covariance of the cost function associated with original process.

The recursive dynamic programming equation for value iteration on the cost function is then

$$
V(\mathbf{x}) = \min_{u \in \mathcal{U}} \Bigg\{ k(\mathbf{x}, u)\Delta t(\mathbf{x}, u) \tag{14}
$$
$$
+ \sum_{\mathbf{y}} e^{-\beta\Delta t(\mathbf{x}, u)} p(\mathbf{x}\,|\,\mathbf{y}, u) V(\mathbf{y}) \Bigg\}
$$

for $\mathbf{x}$ inside the computational domain. The domain boundary is reflective, and for the states in this boundary, we use ([11], pp. 143), instead of (14):

$$
V(\mathbf{x}) = \sum_{\mathbf{y}} p(\mathbf{x}\,|\,\mathbf{y})V(\mathbf{y}). \tag{15}
$$

Equations (14-15) are used in the standard method of value iteration until the cost converges. From this, we obtain the optimal angular velocity of the Dubins vehicle for any relative distance r and viewing angle φ. Under a sufficient control and reasonable target noise intensity, the Dubins vehicle will remain within a region centered about $r = d$. To this end, we restrict our attention to the semi-periodic computational domain $\mathbf{x} \in (r_{\min}, 2d - r_{\min}) \times [-\pi, \pi)$ discretized into a square grid with spacing $(\Delta r, \Delta\varphi)$. In the examples, the control is obtained by interpolating the current system state to the discretized control $u(r, \varphi)$.

IV. RESULTS

Here we describe the control computed by dynamic programming and provide the results of scenarios that show the effectiveness of the method in maintaining the relative distance $r = d$ from the UAV to the target. In the first example, the UAV tracks a Brownian target as we have previously discussed. Then the same control is applied to the case where the target travels along continuous, smooth trajectories.

In all examples, the UAV travels at speed $v_A = 10$ [m/s]. The nominal distance in all examples is 50 [m], and $r_{\min} = 10$ [m] and $r_{\max} = 90$ [m]. We choose the UAV velocity v_A as 10 [m], the target noise intensity $\sigma = 5$, and discount factor $\beta = 1$.

As there is no penalty for control, the optimal turning rate for the UAV as computed by the methods of Section III is given by a bang-bang controller $u(r, \varphi) \in \{-u_{\max}, u_{\max}\}$, as seen in Fig. 2 for $\sigma = 5$. Based on previous works, this type of controller is not unexpected [1], [27]. We have
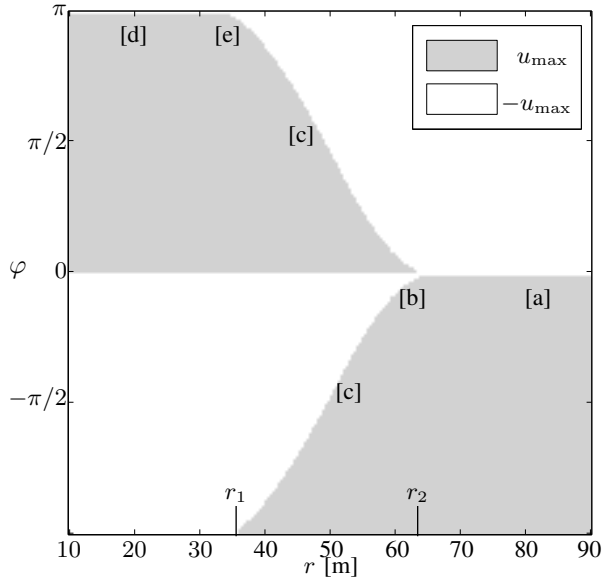
Fig. 2. Optimal control based on distance to the target r and viewing angle φ for $\sigma = 5$. Indicated points are [a] UAV heading toward target at a small angle displacement from a direct line, [b] Start of clockwise rotation about target, [c] Steady states at $\left(d, \pm \cos^{-1}\left(\sigma^2/100v_A\right)\right)$, [d] UAV heading directly away from target at a small angle displacement away from a direct line, [e] Start of counter-clockwise rotation about target. The middle of the open regions direct the UAV to turn left (white) or right (gray) to reach the bounding lines.

Fig. 3. (a) Switching curves from Fig. 2 for various values of σ. (b) Radii r_1 and r_2 at which the UAV enters into turning pattern about target, corresponding to the labeled regions in Fig. 2 for $\sigma = 5$. As the noise intensity of the target increases, the UAV has less information of where the turning circle should be centered and must begin its turn sooner. (c) Coordinate $\varphi^\star$ of switching boundary intersection with $r = d$. As the target noise increases, the UAV expects a bias that tends to increase the target distance, and it reduces its steady-state viewing angle accordingly.

labeled several locations on the policy to help indicate the salient features of the control profile.

The lines that compose the boundaries of the two control regions may be interpreted as an optimal path of the UAV-target system in state space, assuming that the initial system state belongs to the lines. The original sign of $d\theta = -u\,dt$ was chosen so that the path in this state space in some cases reflects the actual path of the UAV. Open regions away from these boundaries have the effect of directing the state back to the lines. If the UAV is in state $(80\ [\text{m}], 0)$, for example, it is far from the target and directed toward it, but with a small angular offset that hints at the future rotation about the target. As it approaches the target, the curvature of switching boundaries gradually directs the UAV into a clockwise circle about the target, beginning at $r = r_2$. This continues until the UAV reaches the steady circle. Likewise, when the UAV is moving directly away from the target, it begins to arc toward the steady state configuration at $r = r_1$ as it follows the curvature near the outer boundaries of the figure. It should be noted that the points $(50, \pm\pi/2)$ are found inside the control regions and not on the switching boundaries due to the bias in $r(t)$ (Fig. 3c).

In simulations, the actual evolution of the state (r, φ) is not smooth due to the random motion of the target. The UAV spends more time in the open regions attempting to return to the lines than it does on the lines; this does not, however, detract from the trajectory of the UAV, as the control is able to compensate for random motion. Since we began this problem with an infinite-horizon cost, the control is highly robust to such deviations, as we will show in simulations below.
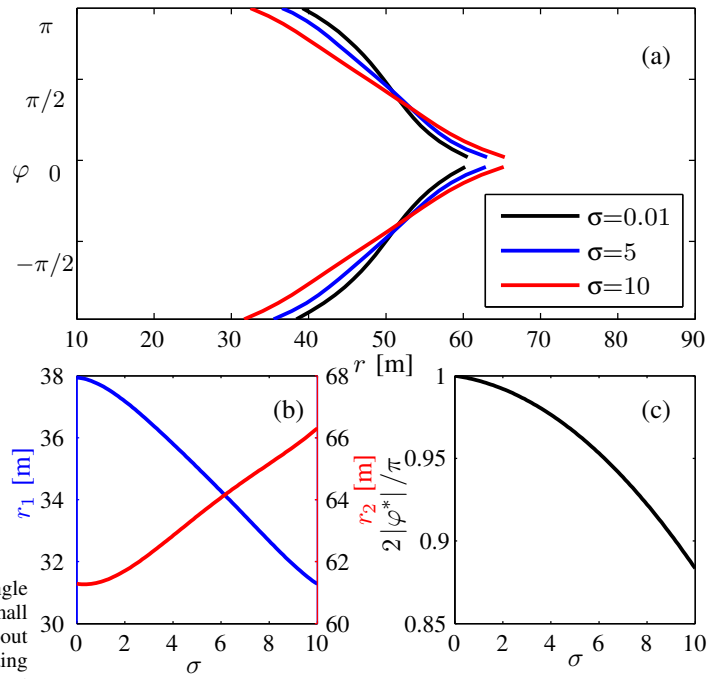
The positioning of r_1 and r_2 determine the state at which the UAV transitions from the act of "avoiding" or "chasing" the target when it is too close or too far, respectively, into the act of maintaining the distance d. With increased noise intensity, the position of the target throughout the duration of the planning horizon is less certain. Consequently, a UAV approaching the nominal standoff distance of a target with higher random motion will begin its turning pattern sooner, as seen by the elongation of the switching boundaries in Fig. 3a and by the locations of r_1 and r_2 in Fig. 3b. Owing to the bias in the mean drift of dr (9), $|r_1 - d| > |r_2 - d|$, e.g., a UAV avoiding the target must expect the bias to "help" with this action, while when chasing, the bias might hinder these efforts. These effects are more prominent with higher noise intensities (Fig. 3b). Finally, recall that as $\sigma \to 0$ (Fig. 3a), the value function converges to the viscosity solution of the deterministic HJB equation [5][28].

A. Brownian Motion Case

For this case, the target noise intensity σ is chosen so that the UAV is capable of keeping pace with the random motion of the target. The total magnitude of the speed v_T of the Brownian target will be distributed as a Rayleigh PDF:

$$v_T \sim f(v) = \frac{v}{\sigma^2} e^{-v^2/2\sigma^2} \qquad (16)$$

with parameter σ. Note that this PDF has mean $\bar{v} = \sigma\sqrt{\pi/2}$. We choose $\sigma = 5$ so that if the UAV is facing the direction
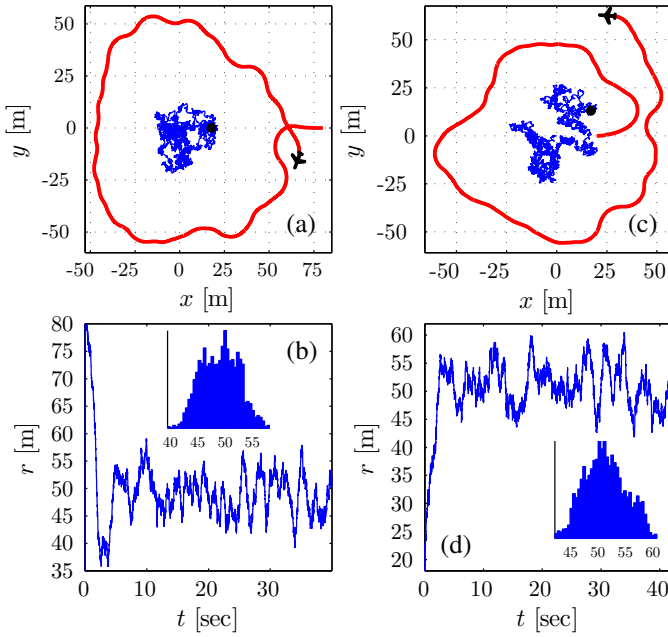
Fig. 4. A UAV (red) tracking a Brownian target (blue). (a) The UAV must approach the target before entering into a clockwise circular pattern ($t_f = 40$) and (b) its associated distance r ($mean(r) = 49.62$, $std(r) = 6.42$). Inset: Distribution of $r(t)$ for $t > 10$. (c) The UAV begins near the target and must first avoid the target before beginning to circle ($t_f = 42$). (d) The associated distance ($mean(r) = 50.23$, $std(r) = 5.23$). Inset: Distribution of $r(t)$ for $t > 10$.
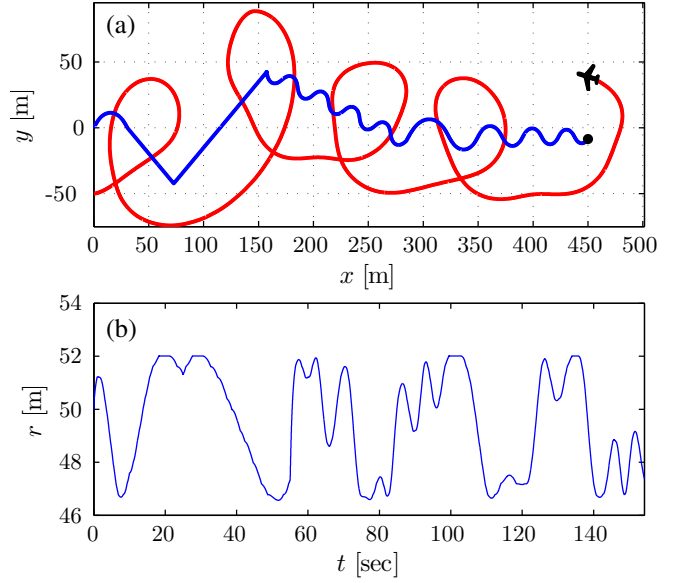

Fig. 5. A UAV (red) tracking a target (blue) moving in a complex sinusoidal path using the control for a Brownian target. (a) The UAV follows the target in eccentric circles whose shape is determined by the current position of the target along its trajectory, $t \in [0, 154.2]$. (b) The distance r ($mean(r) = 49.5$ [m], $std(r) = 3.74$).

of target motion,

$$P_r\{v_T > v_A\} = 1 - P_r\{v_T \le v_A\} < 15\% \qquad (17)$$

We point out that our UAV is a nonholonomic vehicle tracking a random target; the time required to align motion with that of the target is a hindrance that no bounded control can overcome. It is therefore less likely that (17) will hold, and the positive bias in $r(t)$ will augment this effect.

We show implementations of this control when the UAV is initially too far or too close to the target in Fig. 4(a-d), and it is seen that the associated costs remain small.

B. Smooth/Deterministic Trajectory Cases

To emphasize the level of robustness provided by the control, the original assumption that the target position evolves as a 2D random walk is dropped. We first exhibit the response to a target which moves with fixed velocity in arbitrarily-chosen combination of sinusoidal paths:

$$
\begin{aligned}
dx_T(t) &= v_T \cos\theta_T \, dt \\
dy_T(t) &= v_T \sin\theta_T \, dt \\
\theta_T(t) &=
\begin{cases}
\cos(\pi t/10) & t \in [0, 10) \\
-\pi/4 & t \in [10, 25) \\
\pi/4 & t \in [25, 55) \\
\cos(\pi t/5) - \pi/8 & t \in [55, 100) \\
\cos((\pi t/10 - .005(t - 100))\,t) & t \in [100, t_f]
\end{cases}
\end{aligned}
$$

where $v_T = 5\sqrt{2/\pi} \approx 4$ [m/s] is based on the noise intensity for which control has previously been computed. Clearly,

since target motion is no longer random, our control is no longer strictly optimal. It is seen in Fig. 5a, however, that distances remain near the nominal distance d. Note that a constant target speed is not required as long as it is limited, so that the UAV is capable of keeping pace.

Finally, we examine the case of target traveling on a terrain of evenly-spaced hills and valleys in $\hat{x}$. A periodic signal drives damped motion in $\hat{x}$ with frequency ω. The motion in $\hat{y}$ is random. The evolution of a target can be modeled as:

$$
\begin{aligned}
\frac{d^2 x_T}{dt^2} + b\frac{dx_T}{dt} + \sin x_T &= a \sin \omega t \\
dy_T &= \sigma_C \, dw
\end{aligned}
$$

It has been shown that for specific values of parameters ($a = 1.6, b = 0.1, \omega = 0.8$), $x_T(t)$ enters a chaotic regime with a level of noise that is approximately white over a broad frequency range [29]. The distribution of the target position is then Gaussian, with a variance $\sigma_C^2 \approx 4.38^2 t$ that increases linearly with time.

Figure 6 shows the random qualities of the resulting target motion as well as the performance of the Dubins vehicle controller.

V. CONCLUSION AND FUTURE WORK

This paper considers the problem of maintaining a nominal distance between a UAV and a ground-based target with an unknown trajectory. In particular, a model of a Brownian particle is assumed for the target. A Markov chain approximation that is locally-consistent with the system under control is constructed on a discrete state-space, and value iteration on the associated cost function produces a UAV turning rate control to minimize the mean squared distance of the target in excess of a nominal distance.
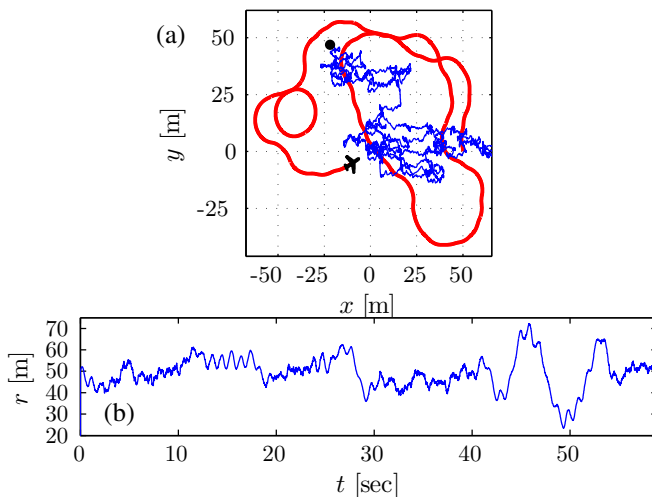
Fig. 6. (a) A UAV (red) tracking a target (blue) that moves in the chaotic regime along the $x-$direction and randomly in the $y-$direction. (b) The distance r ($mean(r) = 49.4$ [m], $std(r) = 7.72$).

The off-line control needs to only be computed once for a given target noise, regardless of trajectory shape or initial conditions, and takes into account kinematic nonlinearities. When approaching the nominal standoff distance from too close or too far a proximity, the UAV flight pattern that initializes revolutions about the target and the revolutions themselves are determined by the amount information known about the unpredictable target motion, even if that motion is smooth and deterministic.

The assumption of random target dynamics coupled with an infinite horizon cost has the fundamental advantage of creating a highly robust control. A variety of trajectories can be tracked using these methods, despite the fact that the bias in the UAV-target distance is no longer present in the case of natural motion, without the need for specialized non-smooth numerical routines.

If a target unexpectedly advances in proximity, the Dubins vehicle must have a relatively large velocity and turning rate to dodge it. This possibility is central to the uninformative property of the problem, but can be avoided if the target motion can be locally predicted. Should the target also be modeled as a Dubins vehicle with a Brownian heading angle, the knowledge of the target's heading angle would provide the UAV with an indicator of its immediate motion and an appropriate response (control) to this prediction.

In future work, it would be interesting to explicitly incorporate observation errors and asymmetry in the UAV viewing angles.

REFERENCES

[1] L. E. Dubins, "On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents," *American Journal of Mathematics*, vol. 79, no. 3, 1957.

[2] E. Lalish, K. A. Morgansen, and T. Tsukamaki, "Oscillatory control for constant-speed unicycle-type vehicles," *Proc. 46th IEEE Conference on Decision and Control*, 2007.

[3] R. A. Wise and R. T. Rysdyk, "UAV coordination for autonomous target tracking," in *Proc. AIAA Guidance, Navigation, and Control Conference*, 2006.

[4] G. Roussos and K. Kyriakopoulos, "Decentralized and prioritized navigation and collision avoidance for multiple mobile robots," *Proc. 10th International Symposium on Distributed Autonomous Robotic Systems*, 2010.

[5] M. Bardi and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*. Birkhauser Boston, 1997.

[6] W. M. McEneaney, "A curse-of-dimensionality-free numerical method for solution of certain HJB PDEs," *SIAM Journal on Control and Optimization*, vol. 46, no. 4, 2007.

[7] F. Hanson, *Techniques in Computational Stochastic Dynamic Programming*, ch. 1, pp. 103–162. Academic Press, 1996.

[8] O. Zienkiewicz, R. Taylor, and P. Nithiarasu, *The Finite Element Method for Fluid Dynamics*, 6th ed. Butterworth-Heinemann, 2005.

[9] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry*, 3rd ed. North Holland, 2007.

[10] V. S. Anishchenko, T. E. Vadivasova, G. a. Okrokvertskhov, and G. I. Strelkova, "Statistical properties of dynamical chaos," *Physics-Uspekhi*, vol. 48, no. 2, Feb. 2005.

[11] H. J. Kushner and P. Dupuis, *Numerical Methods for Stochastic Control Problems in Continuous Time*, 2nd ed. Springer, 2001.

[12] P. K. Agarwal, T. Biedl, S. Lazard, S. Robbins, S. Suri, and S. Whitesides, "Curvature-constrained shortest paths in a convex polygon," in *Proc. 14th Symposium on Computational Geometry*, vol. 31, no. 6. ACM, 1998.

[13] J.-D. Boissonnat, A. Cerezo, and J. Leblond, "Shortest paths of bounded curvature in the plane," *Journal of Intelligent & Robotic Systems*, vol. 11, no. 1-2, Mar. 1994.

[14] H. Chitsaz and S. M. LaValle, "Time-optimal paths for a Dubins airplane," in *Proc. 46th IEEE Conference on Decision and Control*. IEEE, 2008.

[15] J. Lee, R. Huang, A. Vaughn, X. Xiao, J. K. Hedrick, M. Zennaro, and R. Sengupta, "Strategies of path-planning for a UAV to track a ground vehicle," in *Proc. 2nd Symposium on Autonomous Intelligent Networks and Systems*, vol. 2003, 2003.

[16] X. C. Ding, A. Rahmani, and M. Egerstedt, "Optimal multi-UAV convoy protection," *Proc. 2nd International Conference on Robotic Communication and Coordination*, 2009.

[17] X. C. Ding, A. R. Rahmani, and M. Egerstedt, "Multi-UAV convoy protection: an optimal approach to path planning and coordination," *IEEE Transactions on Robotics*, vol. 26, no. 2, Apr. 2010.

[18] D. J. Klein and K. A. Morgansen, "Controlled collective motion for trajectory tracking," *American Control Conference*, 2006.

[19] S. Quintero, F. Papi, D. J. Klein, L. Chisci, and J. P. Hespanha, "Optimal UAV coordination for target tracking using dynamic programming," in *Proc. 49th IEEE Conference on Decision and Control*, 2010.

[20] J. J. Enright, E. Frazzoli, K. Savla, and F. Bullo, "On multiple UAV routing with stochastic targets: performance bounds and algorithms," in *Proc. AIAA Conf. on Guidance, Navigation, and Control*, 2005.

[21] K. Savla, F. Bullo, and E. Frazzoli, "Traveling salesperson problems for a double integrator," *IEEE Transactions on Automatic Control*, vol. 54, no. 4, 2009.

[22] K. Savla, E. Frazzoli, and F. Bullo, "On the dubins traveling salesperson problems: novel approximation algorithms," *Robotics: Science and Systems II*, 2006.

[23] S.-J. Liu and M. Krstić, "Stochastic source seeking for nonholonomic unicycle," *Automatica*, vol. 46, no. 9, Sep. 2010.

[24] A. Mesquita, J. Hespanha, and K. Astrom, "Optimotaxis: A stochastic multi-agent optimization procedure with point measurements," *Hybrid Systems: Computation and Control*, 2008.

[25] R. P. Anderson and D. Milutinović, "Dubins vehicle tracking of a target with unpredictable trajectory," in *Proc. 4th ASME Dynamic Systems and Control Conference*, Arlington, VA, 2011.

[26] C. Gardiner, *Stochastic Methods: A Handbook for the Natural and Social Sciences*, 4th ed. Springer, 2009.

[27] H. J. Sussmann, "The Markov-Dubins problem with angular acceleration control," *Proc. 36th IEEE Conference on Decision and Control*, 1997.

[28] W. Fleming and H. Soner, *Controlled Markov Processes and Viscosity Solutions*, 2nd ed. Springer, 2006.

[29] R. L. Kautz, "Using chaos to generate white noise," *Journal of Applied Physics*, vol. 86, no. 10, 1999.