Bounding Numbers of Equations Required to Describe Compartmental Models on Graphs

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1 Lower Bound

In this section, we aim to prove the following theorem.

Theorem 1 *The minimum number of equations required to describe a compartmental model on a connected graph on n vertices with no cut-vertex sets is the number of equations required to describe a compartmental model on an n-cycle.*

Let G be a connected sub-graph on n vertices with m edges. In particular, G contains no cut-vertices (or bridge edges). We first show that the minimum number of edges for a connected graph is $m = n - 1$.

Result 1 *The minimum number of edges required for a connected graph with n vertices is $n-1$.*

Proof. In a connected graph on n vertices, there exists a unique path from every vertex to every other vertex. We require a minimum of $n - 1$ edges for this to be true, hence the minimum number of edges in a connected graph must be $n - 1$. □

Result 2 *A connected graph on n vertices with $n - 1$ edges is a tree.*

Proof. From 1, we know that the minimum number of edges required to ensure a graph on n vertices is connected is $n - 1$. In the proof of this result, we observed that the removal of any single edge would make such a graph disconnected. It therefore follows that such a graph cannot contain any circuits, so such a graph must be a tree. □

Now, by proving that the minimum number of edges for a connected graph G on n vertices with no cut-vertices is n, we also show the main result (Theorem 1).

Proof. Let T be a tree of order $n \geq 3$. Then, the average degree of T, given by $2(n - 1)/n$, is strictly greater than 1, meaning that the tree must contain at least one cut-vertex since all vertices of degree greater than one in a tree are cut-vertices. Since we know the minimum number of edges for a connected graph on n vertices is $n - 1$ (by Result 1), and all graphs on n vertices with $n - 1$ edges are trees (by Result 2), we know that $n - 1$ edges is too few for a connected sub-graph on n vertices without cut-vertex sets.

Consider, therefore, n edges. From the preceding analysis, we know that a suitable candidate lower-bound-case graph would have to be structured so that the removal of any one edge (or indeed vertex) would result in the graph becoming disconnected. The obvious choice for this is a simple cycle, which is produced from a tree by adding a single vertex. We know that the removal of an edge from a cycle does not result in disconnection of the graph, rather the graph becomes a tree. In particular, there are no cut vertices in a cycle. This means that indeed the minimum number of edges in the graph representing the lower-bound case for the number of equations required for a given compartmental model on the graph is n and in particular this graph is a simple cycle on n vertices. $\square$