The Firefighter Problem: A survey of results, directions and questions

Stephen Finbow
Department of Mathematics, Statistics and Computer Science
St. Francis Xavier University, Antigonish, Nova Scotia, Canada
sfinbow@stfx.ca

Gary MacGillivray *
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia, Canada
gmacgill@math.uvic.ca

1 Introduction

The Firefighter Problem was introduced by Bert Hartnell in 1995 at the 25th Manitoba Conference on Combinatorial Mathematics and Computing [23]. Imagine that, at time 0, a fire breaks out at a vertex of a graph G. At each subsequent time, the firefighter "defends" a vertex of G and then the fire spreads from each "burning" vertex to all of its undefended neighbours. Once a vertex is burning or defended, it remains so from then onwards. The process terminates when the fire can no longer spread. An example is given in Figure 1. In the general form of the problem, more than one fire can break out, and the fire or fires are defended by one or more firefighters.

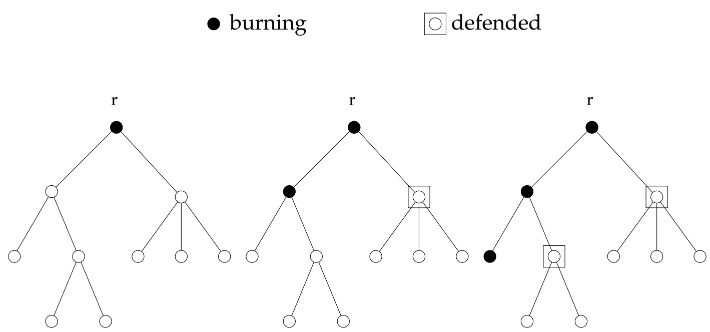
There are a number of different objectives that can be pursued, including:

- Minimize the expected number of vertices burned if the fire breaks out at a random vertex.
- Save the maximum possible number of vertices.
- Put the fire out as quickly as possible. That is, minimize the number of time units until the process ends.
- Determine whether all vertices in a specified collection can be prevented from burning.

*Research supported by the Natural Sciences and Engineering Council of Canada.

- Find the number of firefighters needed to save a particular number of, or fraction of, or subset of the vertices. For infinite graphs, find the smallest number of firefighters that can "contain" the fire in a finite number of steps.

These objectives are sometimes in conflict. For example, it is easy to find trees to illustrate that putting the fire out as quickly as possible does not necessarily lead to saving the maximum possible number of vertices.



(a) time 0 (b) time 1 (c) time 2; terminates

Figure 1: An illustration of the process.

The Firefighter Problem is a deterministic, discrete-time model of the spread of a fire on the vertices of a graph. However, if the graph G represents a social network – the vertices represent groups of people, and the edges represent contact between groups – then the Firefighter Problem can be viewed as a simple model of the spread of a virus or epidemic through a population. According to Devlin and Hartke [13] (also see [22]), some recent epidemiological models attempt to include information about which individuals can come in contact, rather than assuming that the population being studied is well-mixed.

This paper is a survey of directions, results, and open problems that have arisen in this area to date. Definitions, terminology and preliminary observations appear in Section 2. The first topic to be surveyed is algorithms and complexity, in Section 3. These results serve as motivation for many of those in later sections, even though they did not necessarily come first. Infinite and finite grids are the topic of Sections 4 and 5, respectively. Graphs such that if the fire breaks out at random then the expected number of vertices burned is minimized are the subject of Section 6. Related, sometimes earlier, work is briefly mentioned in Section 7. Open problems and possible directions for future research are listed in Section 8.

2 Definitions, terminology, and preliminaries

For a positive integer f, an *f-rooted graph* is an ordered pair (G, F) where G is a graph and $F \subseteq V$ is an f-subset of vertices of G called *roots*. A *rooted graph* is one which is f-rooted for some f. When $F = \{r\}$ we may write (G, r) instead of $(G, \{r\})$.

Let f and d be positive integers. Let (G, F) be an f-rooted graph whose vertices are initially unlabelled. Consider the following deterministic, discrete-time process. At time $t = 0$, the vertices in F are labelled as burning (a set of f fires *breaks out* at these vertices). At each subsequent time $t = 1, 2, \ldots$, a set of d or fewer unlabelled vertices are labelled as defended, and then each unlabelled neighbour of a vertex labelled as burning is labelled as burning. The process terminates when no further vertices can be labelled as burning.

As mentioned, this labelling procedure models the spread, among the vertices of G, of fires that break out at vertices in a set F and are defended by d firefighters. If the process terminates in a finite number of steps, the fires are said to be *contained*. The containment question is of interest for infinite graphs only, but the question of the number of time units needed to contain a fire or fires is of interest for all graphs.

A vertex that is labelled as burning is called *burning*, or sometimes, *burned*. A vertex that is labelled as defended is called *defended*. Since the label of a vertex does not change, once a vertex is burning or defended, it remains so. It is easy to see that, at each time step, the set of burning vertices induces a connected subgraph and, if there are vertices which are neither burning nor defended at the end of the process, then the set of defended vertices is a vertex cut. A vertex is called *saved* if it never burns. The *damage* is the total number of vertices that are burning at the end of the process. For a finite graph with n vertices, the number of vertices saved plus the damage equals n. The notation $MVS(G, F; d)$ stands for the maximum number of vertices that can be saved in G when the fires break out at vertices in F, over all possible ways of defending vertices using d firefighters.

The following propositions summarizes some easy computations.

Proposition 2.1. *1. For $n \geq 2$, $MVS(K_n, r; 1) = 1$.*

2. For $n \geq 3$, $MVS(C_n, r; 1) = n - 2$.

3. For $n \geq 2$, $MVS(P_n, r; 1) = \begin{cases} n-1 & \text{if } r \text{ is a leaf} \\ n-2 & \text{otherwise,} \end{cases}$

where P_n denotes the path on n vertices.

4. For $n \geq 2$, $MVS(Q_n, r; 1) = n$, where Q_n denotes n-cube [29] .

The graph in Figure 2 illustrates that it is not in general advantageous for the firefighter to only defend vertices adjacent to the fire. On the other hand:

Proposition 2.2. [29] *Suppose the fire breaks out at vertex r of the tree T. In an optimum strategy, each vertex defended is adjacent to a burning vertex.*

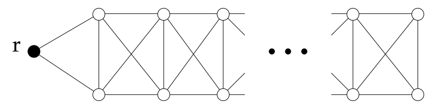


Figure 2: An example which shows it can be best to defend away from the fire.

3 Algorithms and complexity

The decision version of the Firefighter Problem is formally stated below:

FIREFIGHTER
INSTANCE: A rooted graph (G, r) and an integer $k \geq 1$.
QUESTION: Is $MVS(G, r; 1) \geq k$? That is, is there a finite sequence $d_1, d_2, \ldots,$ d_t of vertices of G such that if the fire breaks out at r then,
(i) vertex d_i is neither burning nor defended at time i,
(ii) at time t no undefended vertex is adjacent to a burning vertex, and
(iii) at least k vertices are saved at the end of time t?

This problem is NP-complete for bipartite graphs [29]. Much stronger results, involving a sharp dividing line, hold.

Theorem 3.1. [16] *FIREFIGHTER is NP-complete even if restricted to trees with maximum degree three. The problem is solvable in polynomial time for graphs of maximum degree three, provided the fire starts at a vertex of degree two.*

The NP-completeness proof is a complicated transformation from not-all-equal 3-SAT. In the instance of FIREFIGHTER constructed in the transformation, the fire breaks out a vertex of degree three.

Although the Firefighter Problem has not been investigated for directed graphs, it is worth noting that a rooted tree can be viewed as a digraph in which all arcs are directed "away" from the root. (In the directed version of the Firefighter Problem, the fire can spread only from the tail of an arc to its head.) Thus, FIREFIGHTER is also NP-complete for orientations of trees with maximum in-degree one and maximum out-degree three.

Corollary 3.2. [27] *FIREFIGHTER is NP-complete for cubic graphs.*

Figure 3 shows a slight adaptation of a 0-1 integer program due to Devlin and Hartke [13] that determines MVS$(G, F; d)$ for any f-rooted graph (G, F). Here, T is an upper bound on the number of time units needed to contain the fire, and for $0 \leq t \leq T$ and $x \in V$, the boolean variables $b_{x,t}$ and $d_{x,t}$ are defined by

$$b_{x,t} = \begin{cases} 1 & x \text{ burned before or at time } t \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Min} \sum_{x \in V} b_{x,T}$$

Subject to:

$$\begin{cases}
b_{x,t} \geq b_{x,t-1} & x \in V, 1 \leq t \leq T \\
d_{x,t} \geq d_{x,t-1} & x \in V, 1 \leq t \leq T \\
b_{x,t} + d_{x,t} \geq b_{y,t-1} & x \in V, y \in N(x), 1 \leq t \leq T \\
b_{x,t} + d_{x,t} \leq 1 & x \in V, 1 \leq t \leq T \\
\sum_{y \in N(x)} b_{y,t-1} \geq b_{x,t} & x \in V, 1 \leq t \leq T \\
\sum_{x \in V} (d_{x,t} - d_{x,t-1}) \leq d & 1 \leq t \leq T \\
d_{x,0} = 0 & x \in V \\
b_{x,0} = \begin{cases} 1 & \text{if } x = r_i \text{ for some } i, 1 \leq i \leq f \\ 0 & \text{otherwise} \end{cases} & x \in V \\
b_{x,t}, d_{x,t} \in \{0,1\} & 1 \leq t \leq T
\end{cases}$$

Figure 3: A 0-1 integer program for the Firefighter Problem

$$d_{x,t} = \begin{cases} 1 & x \text{ defended before or at time } t \\ 0 & \text{otherwise.} \end{cases}$$

The first (second) set of constraints assures that a vertex burning (defended) at time $t-1$ is still burning (defended) at time t. The third set of constraints assures that every neighbour of a vertex burning at time $t-1$ is burning or defended at time t. The fourth set of constraints assures that no vertex can be both burning and defended at any time. The fifth set of constraints prevents a vertex with no burning neighbours at time $t-1$ from burning at time t. The sixth set of constraints guarantees that at most d firefighters are used per time step. Finally, the next two sets of constraints initialize the variables corresponding to time $t = 0$.

This integer program can also be used to determine the minimum number of time units needed to contain the fires. If the objective function is replaced by

$$\text{Min} \sum_{1 \leq t \leq T} |V|^t \sum_{x \in V} (b_{x,t} - b_{x,t-1})$$

then the optimum solution is a base-V integer in which the least positive t such that coefficient of V^t equals zero is the required minimum time.

Besides asking whether a certain number of vertices can be saved, a reasonable question is whether one firefighter can save all members of a given subset of vertices. This decision problem is formalized below.

S-FIRE
INSTANCE: A rooted graph (G, r) and a subset $S \subseteq V(G)$.
QUESTION: Can one firefighter save all vertices in S?

Corollary 3.3. [27] *S-FIRE is NP-complete, even if S is the set of leaves of a tree of maximum degree three. The problem is solvable in polynomial time for trees of maximum degree three in which the fire breaks out at a vertex of degree 2.*

The problems FIREFIGHTER and S-FIRE can be brought into a common framework by introducing vertex weights. These can be regarded as the value of each vertex, and the goal is to maximize the value (i.e., weight) of the set of vertices saved. Theorem 3.1 then states that the decision version of this problem, "can a subset of vertices with weight at least the given integer k be saved?", is NP-complete for trees of maximum degree three and all weights equal. Note, however, that the theorem does not imply that the problem is polynomial for all weighted graphs of maximum degree three if the fire breaks out at a vertex of degree two, although this may indeed be true. Given an instance of S-fire, construct a weighted graph by assigning to each vertex in S the weight $n = |V(G)|$, assigning to each non-root vertex not in S the weight 1, and assigning the root weight 0. Set $k = n|S|$. Then, a subset of vertices with weight at least k can be saved if and only if all vertices in S can be saved.

The results above suggest that it is reasonable to focus attention on trees, and on the case where there is one fire and one firefighter. There are three directions that have been pursued. The first of these is approximation algorithms, the second is exact (exponential) algorithms, and the third is special classes of trees for which the problem is solvable in polynomial time.

Since there is a unique path between any two vertices in a tree, once a vertex is defended, that vertex and all of its children are saved. Each vertex v of a rooted tree (T, r) can therefore be regarded as having a weight, $w(v)$, equal to the number of vertices that will be saved if v is defended. Since the root r can not be defended, its weight is defined to be 0.

A natural candidate for an approximation algorithm for the Firefighter Problem on trees is the *greedy algorithm*: at each time step save the vertex of largest weight which is not burning. For the rooted tree (T, r) shown in Figure 4, the greedy algorithm saves five vertices whereas MVS$(T, r; 1)$ equals six. It is easy to generalize this example to show that the solution produced by the greedy algorithm can be arbitrarily far from optimum in absolute terms. On the other hand, the solution obtained from the greedy algorithm can not be far from the optimum solution in relative terms.

Theorem 3.4. [25] *If $Greedy(T, r; 1)$ denotes the number of vertices saved by a single firefighter using the greedy algorithm, then*

$$Greedy(T, r; 1) \geq \frac{1}{2} MVS(T, r; 1).$$

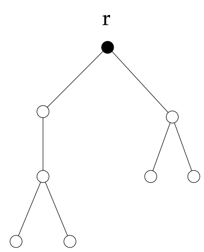


Figure 5: A tree on which the greedy algorithm does not find an optimum solution.

That is, the Greedy Algorithm is a $\frac{1}{2}$-approximation algorithm for the Firefighter Problem on trees.

The ratio $\frac{Greedy(T,r;1)}{MVS(T,r;1)}$ can be seen to be arbitrarily close to $\frac{1}{2}$. Consider the family of rooted graphs (G_k, r) obtained by subdividing one edge of a star on $k \geq 3$ vertices k times and setting r to be the vertex of degree two adjacent to the vertex of degree k. Hence the bound in the theorem is sharp. The same result holds for one fire and any number of firefighters [25].

The "greedy" strategy of defending the vertex of highest degree adjacent to a burning vertex always finds an optimum solution in a caterpillar [29], but not an arbitrary tree. It is unknown whether there a constant $c \in (0, 1]$ such that this algorithm saves at least $c \cdot \mathrm{MVS}(T, r; 1)$ vertices.

It follows from Proposition 2.2 that, in an optimal strategy for the Firefighter Problem on trees, the vertex defended at time i is at distance i from root (i.e., at *level* i). The additional observation that no descendant of a defended vertex is also defended in an optimal solution leads to an 0-1 integer programming formulation of the Firefighter Problem on trees [29]: For each vertex v of T, let d_v be a boolean decision variable such that $d_v = 1$ if and only if vertex v is defended ($d_v = 0$ otherwise). Recall that, for $v \neq r$, the weight $w(v)$ is the number of vertices that will be saved if v is defended (that is, the number of vertices in the subtree rooted at v), and $w(r) = 0$. The objective is to maximize $\sum_{v \in V} d_v w(v)$, subject to constraints that guarantee that at most one vertex is defended at every level, and at most one ancestor of each vertex is defended. The latter condition will be satisfied if and only if exactly one ancestor of each leaf is defended. Such a 0-1 integer program is shown in Figure 5.

The solution to the LP relaxation of this 0-1 integer program can be arbitrarily far from the optimum integral solution in absolute terms. While no bound is known on how far the LP optimum can be from the integral optimum in relative terms, there is strong experimental evidence that suggests such a bound exists [22]: About 1.68 million trees on 100 vertices were randomly generated and both the LP optimum and integral optimum computed. Of these, the LP optimum was strictly greater than the integral optimum less than 6%

$$\text{Max} \sum_{v \in V} d_v w(v)$$

Subject to:

$$\begin{cases} \displaystyle\sum_{\text{level(v) = i}} d_v \leq 1 & \text{for each level } i \\ \displaystyle d_v + \sum_{u \text{ an ancestor of } v} d_u \leq 1 & \text{for every leaf } v \text{ of } T \\ d_v \in \{0, 1\} & \end{cases}$$

Figure 6: A 0-1 integer program for the firefighter problem on a tree [29].

of the time. In these cases, the LP optimum was no more than 1.0634 times the integral optimum. These numbers were improved to 0.7% of the cases and 1.0373 times the integral optimum by adding additional constraints.

Conjecture 3.5. [22] *There is a constant c such that the optimum solution to LP relaxation of the 0-1 integer program in Figure 5 is at most c times the optimum integral solution. That is, linear programming gives a c-approximation algorithm for the Firefighter Problem on trees.*

It is possible to guarantee an integer optimum solution to the relaxation of the above integer linear program by replacing the constraints corresponding to ancestors by different, but non-linear, constraints [29]: For each vertex $v \neq r$ and each descendant w of v, add the constraint $x_v x_w = 0$.

A recursive algorithm that works for all trees has been described by MacGillivray and Wang [29]. Because of Theorem 3.1, it is of interest to find classes of trees for which the Firefighter Problem is solvable in polynomial time. Two such classes of trees have previously been noted: caterpillars and trees of maximum degree three where the root has degree two (these include binary trees).

Another special class of trees, besides binary trees and caterpillars, for which the problem is solvable in polynomial time is the so-called P-trees [29]. For P-trees, the Firefighter Problem can be transformed into a maximum weight independent set problem on a related perfect graph, which can be solved in polynomial time by linear programming [21].

4 Infinite graphs: Containing fires

For infinite graphs, the main issues are whether the fire can be *contained* by the given number of firefighters, how quickly, and finding the minimum number of vertices burned. Research to date has mostly focussed on various infinite grids, and numbers of fires and firefighters.

A useful method for obtaining a lower bound on the number of firefighters needed to contain a fire in an infinite graph was introduced by Fogarty [17], and generalized by Devlin and Hartke [13]. Suppose a fire breaks out at the vertex r of the graph G. Let $V_0 = \{r\}$. For each integer $k \geq 1$, let V_k be the set of vertices at distance k from r, let $B_k \subseteq V_k$ be the subset of these vertices that are burning at time k, and let x_k be the number of vertices in $\cup_{i>k} V_i$ defended at the end of k time units.

Theorem 4.1. [17] *Suppose that a fire breaks out at the vertex r of the graph G. For a given positive integer d, if for every positive integer k, every non-empty subset $A \subseteq V_k$ satisfies $|N(A) \cap V_{k+1}| \geq |A| + d$, then $|B_t| > x_t$ for every t.*

The Hall-like condition in the statement assures that the set of vertices that are at distance t from r and burning at time t (that is, the boundary of the fire) has more neighbours than can be defended. Thus, d firefighters can not contain the fire.

As an example, we apply Theorem 4.1 to the infinite *square grid* with vertex set $\mathbf{Z} \times \mathbf{Z}$ and edge set $\{(u, v)(x, y) : u = x \text{ and } v - y = \pm 1, \text{ or } v = y \text{ and } u - x = \pm 1\}$. For each $k \geq 0$, $A \subseteq V_k$ satisfies $|N(A) \cap V_{k+1}| \geq 2|A| \geq |A| + 1$. Hence one firefighter can not contain the fire.

Now suppose that the square grid is augmented by adding the edges $(u, u)(u+1, u+1)$, and $(u, -u)(u+1, -(u+1))$, where $u \in \mathbf{Z}$. One firefighter can not contain the fire on this new graph because it contains the square grid as a subgraph. But Theorem 4.1 can not be applied: for each $k \geq 1$, there exists $A \subseteq V_k$ with $|N(A) \cap V_{k+1}| = |A|$. Hence the theorem is somehow dependent on the structure of the graph, and in particular on the "shape" of the *k-sphere* V_k. This example was found by S. Lowdon (personal communication, 2007).

The following proposition gives an easy upper bound on the number of firefighters necessary to contain a fire in an infinite grid.

Proposition 4.2. [31] *Let G be an r-regular graph. Then a single fire can be contained by $r - 1$ firefighters in two time units, and the minimum number of vertices burned is two.*

Since an n-dimensional square grid is $2n$-regular, Proposition 4.2 immediately implies an upper bound on the number of firefighters needed to contain the fire.

Corollary 4.3. [31] *In the infinite n-dimensional square grid a single fire can be contained by $2n - 1$ firefighters in two time units, with two vertices burned.*

The fact that a single firefighter can not contain the fire in an infinite 2-dimensional square grid was first proved by Moeller and Wang [31], and later by Fogarty using Theorem 4.1, as above [17]. Moeller and Wang also showed that two firefighters can contain the fire in eight time units, and this is the best possible. This improved an earlier result of Finbow, Hartnell and Schmeisser that 11 time units suffice for two firefighters to contain the fire. Moeller and Wang's strategy results in 18 vertices being burned. Devlin and Hartke [13] used (essentially) the integer program in Figure 3 to show that this is the best possible, even if more than eight time units are used to contain the fire. They also used a modified version of the integer program to give another proof that eight time units are necessary to contain the fire under any strategy.

The following theorem summarizes the situation for infinite square grids.

Theorem 4.4. [13, 31] *The following statements hold with respect to containing a single fire in an infinite n-dimensional square grid.*

1. *If $n = 1$ or $n \geq 3$ then $2n - 1$ firefighters are necessary and sufficient to contain the fire. The minumum number of time units needed to contain the fire is two. The minimum damage when the fire is contained by $2d - 1$ firefighters is two, and this can be achieved in two time units.*

2. *If $n = 2$ then two firefighters are necessary and sufficient to contain the fire. The minimum number of time units needed by two firefighters to contain the fire is eight. The minimum damage when the fire is contained by two firefighters is 18, and this can be achieved in eight time units. Three firefighters can contain the fire in two time units, with two vertices burned.*

The above theorem suggests that, in some sense, the 2-dimensional case is different from the remaining cases. The following theorem provides more evidence that this is true.

Theorem 4.5. *The following statements hold with respect to containing an outbreak of one or more fires in an infinite n-dimensional square grid.*

1. *If $n = 1$, then a single firefighter can contain any finite number of fires.*

2. [17] *If $n = 2$, then two firefighters are necessary and sufficient to contain any finite number of fires.*

3. [13] *If $n \geq 3$, then for any positive integer d there is a finite number of fires that can not be contained by d firefighters.*

With respect to the above theorem and others, note that one way of showing that any finite number of fires can be contained by d firefighters is to argue that if a single fire is allowed to burn for t time units, then d firefighters can still contain the fire.

The problem of containing fires on other infinite grids has been investigated. Here, the *strong grid* has vertex set $\mathbf{Z} \times \mathbf{Z}$ and edge set $\{(u, v)(x, y) : |u - x| \leq 1, |v - y| \leq 1, \text{ and } (u, v) \neq (x, y)\}$. the *triangular grid* is the infinite planar graph whose faces form a tiling of the plane with equilateral triangles, and the *hexagonal grid* is the infinite planar graph whose faces form a tiling of the plane with regular hexagons. The latter two grids are planar duals. The main results follow.

Theorem 4.6. 1. [30] *In the 2-dimensional infinite strong grid, four firefighters are necessary and sufficient to contain any finite number of fires.*

2. [17, 30] *In the 2-dimensional infinite triangular grid, three firefighters are necessary and sufficient to contain any finite number of fires.*

3. [30] *In the 2-dimensional infinite hexagonal grid, two firefighters suffice to contain any finite number of fires.*

Other grid graphs that have been considered include quarter planes (square grid [17] or strong grid [30]), and grids that are finite in one dimension (square grid [17], strong grid [30], or the triangular grid [17] (called the hexagonal grid by Fogarty)).

Variations of the problem that have been considered include determining the number of firefighters needed to contain a fire to a given region [30], and using a periodic sequence of numbers of firefighters at each time step [30, 33]. In the latter case, an interesting result is that the fire can be contained in such a way that the average, over all time units to the end of the process, of the number of firefighters used is smaller than the optimum number of firefighters required to contain the fire or fires. This has been done for the strong grid and triangular grid [30], and for the 2-dimensional square grid [30, 33]. For the 2-dimensional square grid, the average can be as low as $\frac{3}{2}$, and any ratio greater than or equal to $\frac{3}{2}$ can be achieved [33].

Conjecture 4.7. [33] *There is no periodic sequence for which the average, over all time units to the end of the process, of the number of firefighters used to contain the fire is less than* $\frac{3}{2}$.

The fractional version of the Firefighter Problem was first considered by Fogarty [17]. Her point of view was that the firefighters could extinguish some fraction of the fire at a vertex, and then the remaining fraction of the fire is passed on to each non-burning, undefended neighbour of such a vertex. The fire is extinguished when no fraction of it can spread. When $f = d = 1$, a solution to the LP relaxation of the integer program in Figure 3 with can be interpreted in exactly this way. Although Fogarty's work was for the infinite 2-dimensional square grid, the same arguments also apply to other 2-dimensional grids.

Proposition 4.8. *Let (G, r) be a rooted infinite graph, and for $k \geq 0$, let V_k be the set of vertices at distance k from r. If there is a constant c such that $|V_k| \leq ck$, then for $d = 1$ the LP relaxation of the 0-1 integer program in Figure 3 has a finite optimum solution. That is, a single fractional firefighter can contain the fire.*

Proof. Rather than setting the values of the variables in the linear program, we take Fogarty's point of view. Since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, there exists a least integer T such that $\displaystyle\frac{1}{c}\sum_{i=1}^{T} \frac{1}{k} \geq 1$. At times $t = 1, 2, \ldots, T$, assign $\frac{1}{|V_k|} \geq \frac{1}{ck}$ of a firefighter to each vertex in V_k. Hence one firefighter is used at each time step. By definition of T, after at most T time units no fraction of the fire can spread. □

Corollary 4.9. *One fractional firefighter suffices to contain the fire in each of the following infinite grids: the 2-dimensional square grid, the strong grid, the triangular grid and the hexagonal grid.*

P. Dukes and S. Lowdon (personal communication, 2007) have observed that the proof of Proposition 4.8 fails for n-dimensional square grids with $n \geq 3$. They note that the underlying reason is that $|V_k|$ is a quadratic function of k, and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. This provides further evidence that the 2-dimensional case is different.

5 Finite grids

The finite grids that have received attention are square grids of dimension 2 or 3, that is, Cartesian products of two or three paths. Only the case of one fire and one firefighter has been considered.

Exact (exponential) algorithms for solving the Firefighter problem on these grids have been described by Moeller and Wang [31]. The integer program in Figure 3 can also be used. Clearly, the number of vertices burned is a function of the vertex where the fire breaks out. Some bounds and exact answers were given by Moeller and Wang [31] and MacGillivray and Wang [29]. Subsequent to the appearance of the latter paper, the problem for the 2-dimensional square grid when the fire breaks out in the first row was included on the 2005 Asian Pacific Mathematics Olympiad [3].

Suppose the rows and columns of the 2-dimensional square grid $P_n \Box P_n$ are indexed with $1, 2, \ldots, n$ in the natural way. The following strategy is believed to save the maximum number of vertices:

Proposition 5.1. [31] *In $P_n \Box P_n$, when the fire breaks out at (r, c), $1 \leq r \leq c \leq \lceil n/2 \rceil$, if the firefighter defends vertices in the following order: $(r+1, c), (r+1, c+1), (r+2, c-1)$, $(r+2, c+2), (r+3, c-2), (r+3, c+3), \ldots, (r+c, 1), (r+c, 2c), (r+c, 2c+1), \ldots, (r+c, n)$, then $n(n-r) - (c-1)(n-c)$ vertices are saved.*

The following was proved in the case $(r, c) = (1, 1)$ by Moeller and Wang [31].

Proposition 5.2. [29] *The strategy described in Proposition 5.1 saves the maximum number of vertices when $r \in \{1, 2\}$.*

The difference between the number of vertices saved by the strategy in Proposition 5.1 and the best known upper bound on the number of vertices that can be saved becomes arbitrarily large, as a function of n, as the vertex at which the fire starts approaches the middle of the grid. For example, if n is odd and the fire starts at $(\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil)$, then the difference between these two quantities is roughly $\frac{3}{8}n^2$ [29].

Very little is known about the Firefighter Problem in three dimensional square grids. The evidence suggests that relatively few vertices can be saved [13, 31].

Conjecture 5.3. [31] *For any vertex v of $P_n \Box P_n \Box P_n$,*

$$\lim_{n\to\infty} \frac{MVS(P_n \Box P_n \Box P_n, v; 1)}{n^3} = 0.$$

Let G be the subgraph of the 3-dimensional infinite square grid induced by the vertices with non-negative coordinates and at distance at most $3n$ from the $(0,0,0)$. Then G contains $P_n \Box P_n \Box P_n$ as an induced subgraph.

Theorem 5.4. [13] *Let G be the subgraph of the 3-dimensional infinite square grid induced by the vertices with non-negative coordinates and at distance at most $3n$ from the $(0,0,0)$. Then,*

$$\lim_{n\to\infty} \frac{MVS(G, (0,0,0); 1)}{n^3} = 0.$$

6 Minimizing the expected damage and the maximum damage

Two scenarios involving f fires breaking out at a random subset of vertices of a finite graph G and being defended by d firefighters are discussed in this section. In the first of these, the goal is to determine the graphs for which the expected damage is minimized. In the second scenario, the goal is to determine the graphs for which the maximum damage is minimized.

Let (G, F) be an f-rooted graph. Let $mindam(F, d) = n - MVS(G, F; d)$, the minimum damage if the f fires are defended by d firefighters. If each f-subset is equally likely to be the location where the fires start, the *expected damage* is

$$E(G, f; d) = \frac{1}{\binom{|V(G)|}{f}} \sum_{|X|=f} mindam(F, d).$$

Another way of looking at the expected damage is the *surviving rate*, defined by Cai and Wang [7]:

$$\rho(G, f; d) = \frac{1}{\binom{|V(G)|}{f}} \sum_{F\subseteq V} \frac{MVS(G, F; d)}{|V|} = 1 - \frac{1}{|V|} E(G, f; d).$$

This is the average percentage of vertices that can be saved when f fires randomly break out at vertices of G.

For given positive integers f and d, a graph G is called *optimal* if $E(G, f; d)$ is minimum over all graphs with the same number of vertices as G.

Theorem 6.1. [15] *Suppose $f = d = 1$. Then a connected graph G is optimal if and only G is a tree such that*

1. *each vertex has at most two neighbours of degree at least 2, and*

2. *each vertex has at most one neighbour of degree at least 3.*

Thus, when there is one fire and one firefighter, the optimal graphs on n vertices are caterpillars in which any two vertices of degree at least three are at distance at least three. These include paths and stars. It turns out that a star is always an optimal graph when there is a single firefighter, irrespective of the number of fires [15].

Let $B_{k,t}$ denote the tree obtained from a star on $t + k + 1$ vertices by subdividing t of its edges once each.

Theorem 6.2. [11] *Suppose $f = d = 2$. Then a graph G with $n \geq 13$ vertices is optimal if and only if $G \cong B_{n-9,4}$.*

For the complete list of optimal graphs for $n < 13$, see reference [11].

Cai and Wang [7] have determined lower bounds on the surviving rate of trees, outerplanar graphs and Halin graphs. Recall that a graph is *outerplanar* if it has a plane embedding in which all vertices are on the same face, and is a *Halin graph* if it can be formed from an outerplane embedding of a tree with no vertices of degree two by adding a cycle through its leaves in counterclockwise order.

Theorem 6.3. [7]

$$\rho(G,1;1) > \begin{cases} 1-\sqrt{2/n} & \textit{if } G \textit{ is a tree with } n \textit{ vertices} \\ 1/6 & \textit{if } G \textit{ is an outerplanar graph} \\ 3/10 & \textit{if } G \textit{ is a Halin graph with at least 5 vertices.} \end{cases}$$

According to Cai and Wang, the bounds given in the theorem are not tight. Finding improved lower bounds is an open problem, as is extending the results to other classes of graphs. Since trees, outerplanar graphs and Halin graphs are all planar, it is natural to wonder whether there is a constant lower bound for the surviving rate of a planar graph. However, they note that $\lim_{n\to\infty} \rho(K_{2,n}, 1; 1) = 0$, so no such lower bound exists in general. The following conjecture is made.

Conjecture 6.4. [7] *There is a positive constant c such that every nontrivial planar graph G of maximum degree three satisfies $\rho(G, 1; 1) \geq c$.*

For given positive integers f and d, an *mmd-graph* is a graph G such that $\max_{|X|=f} mindam(X, d)$ is minimized over all graphs on the same number of vertices as G. When $f = d = 1$, the mmd-graphs are exactly paths and cycles. This is also true when $f = d = 2$ and the number of vertices is at least 11. In this case there are two further mmd-graphs on 10 vertices [15].

Let S_{2n+1} be the graph obtained by subdividing each edge of a star on $n+1$ vertices exactly once, and let S'_{2n} be the graph obtained by subdividing all but one edge of a star on $n + 1$ vertices exactly once.

Suppose $n = 5q + r$, where $1 \leq r \leq 5$. Let F_n be the tree constructed from $q + 1$ disjoint paths $L_1, L_2, \ldots, L_{q+1}$, such that $L_1, L_2, \ldots, L_q$ have 5 vertices and L_{q+1} has r vertices, by adding edges joining a centre vertex of L_i to a centre vertex of L_{i+1} for $1 \leq i \leq q$.

Theorem 6.5. [15] *Suppose that there are f fires and d firefighters.*

1. *If $n \leq 2f + 1$ then any tree on n vertices is an mmd-graph.*
2. *If $f \leq 3$ and $n \geq 4f$ then a path on n vertices is an mmd-graph.*
3. *If $f \leq 3$ and $2f + 1 < n \leq 4f - 1$ then*
   (a) *if $n = 2m + 1$ then S_n is an mmd-graph, and*
   (b) *if $n = 2m + 2$ and $d \geq \frac{m-f+1}{2}$ then S'_n is an mmd-graph.*
4. *If $f \geq 4$, $n \geq 5f - 4$, and $d \geq f$ then a path on n vertices is an mmd-graph.*
5. *If $f \geq 4$ and $2f + 1 < n < 5f - 4$ then*
   (a) *If $f = 3t + 1$ (so that $6t + 3 < n < 15t + 1$) then*
      i. *if $6t + 3 < n \leq 10t + 5$ then S_n is an mmd-graph when n is odd and S'_n is an mmd-graph when n is even, and*
      ii. *if $10t + 5 < n \leq 15t$ then F_n is an mmd-graph.*
   (b) *If $f = 3t + 2$ (so that $6t + 5 < n < 15t + 6$) then*
      i. *if $6t + 5 < n \leq 10t + 9$ then S_n is an mmd-graph when n is odd and S'_n is an mmd-graph when n is even, and*
      ii. *if $10t + 9 < n \leq 15t + 5$ then F_n is an mmd-graph.*
   (c) *If $f = 3t + 3$ (so that $6t + 7 < n < 15t + 11$) then*
      i. *if $6t + 7 < n \leq 10t + 13$ then S_n is an mmd-graph when n is odd and S'_n is an mmd-graph when n is even, and*
      ii. *if $10t + 13 < n \leq 15t + 10$ then F_n is an mmd-graph.*

7 Related work

In this section some other problems which resemble the Firefighter Problem are briefly mentioned. Our intention is to indicate a few pointers to the literature, and not to undertake any sort of detailed survey of results.

In *Politicians Firefighting* [36] the number of firefighters available varys each time step according to the growth of the fire. Specifically, for each vertex that caught fire at time t and has at least one undefended non-burning neighbour, there is one firefighter available to be placed on such a neighbour in the next time step. This problem is solvable in linear time for trees but NP-hard for planar graphs of maximum degree five. It is fixed parameter tractable for general graphs.

There is a considerable literature on the propagation of rumours, viruses or epidemics in small-world and/or scale-free networks (for a sample see [12, 32, 35, 39, 40]), most of which uses probabilistic methods. There can be (and often is) a threshold for the spreading rate below which an epidemic will die out before reaching the entire population, but scale-free networks do not admit such a threshold [35]. The discrete model considered by Comellas, Mitjana and Peters [9] exhibits similar rates of infection to the probabilistic models just mentioned. In this model, a vertex of a graph that is *infected* remains contagious for $\mathcal{A}$ time units and can infect k of its neighbours in each time unit in which the infection is active. The same authors subsequently [10] determined the time for a virus to spread to all vertices – which they prove is guaranteed to happen for any $k \geq 2$ and $\mathcal{A} \geq 1$ – and then considered the problem of stopping the spread of the virus by installing "firewalls" at some vertices of the graph. The firewalls are installed beforehand (as opposed to at a rate of several per time step), and prevent an infected vertex from spreading the virus to any of its neighbours. Upper and lower bounds on the number of firewalls needed to prevent the virus from infecting all vertices are presented.

The angel problem was introduced by E. R. Berlekamp, Conway, and R. K. Guy in *Winning Ways* [6]. The problem is to decide if a devil eating a square per turn from an infinite checkerboard can strand an angel able to jump up to p squares per turn. When $p = 1$ this can be viewed as the problem of containing a fire (the angel) that can spread to only one neighbouring vertex per time unit. Such an angel can be trapped, but the problem is not solved for other values of p. Bollobas and Leader [4] have proved that, on the 3-dimensional grid, the angel can always escape if p is large enough. They indicate that $p = 50$ is provably large enough, and conjecture that in fact $p = 1$ is large enough.

In the Cops and Robber game, the question is whether k cops can trap a robber who moves along the edges of a graph. Different assumptions lead to different variations of the game. In each of them, the cops play a similar role to the fire (or the devil), in that the are trying to contain the robber (trap the robber on an edge or vertex). One difference from the Firefighter Problem is that the vertices occupied by the cops are not permanently destroyed and may later be occupied by the robber. The case where the robber is arbitrarily fast and invisible is related to searching and sweeping [2, 28]. When the robber is arbitrarily fast and visible, the number k of cops needed to catch the robber is the tree-width of G [37]. (For variations involving width parameters for digraphs or hypergraphs, see [1, 5, 21].) Otherwise it is a typically studied as a perfect information game in which the players move alternately, from vertex to vertex along edges of the graph (or digraph) [8, 26, 34]. The graphs on which one cop has a winning strategy are precisely the retracts of products of paths [34]. Finding a characterization of the graphs on which two (or k) cops have a winning strategy is a long standing open problem.

According to Snatzke [38], the game of Amazons was invented in 1988 by W. Zamkauskas. It is a 2-player game, normally played on a 10×10 chessboard. Each player has a number of Amazons, normally four, which move like a Queen in chess and which also shoot arrows that "move" in the same way as a chess

Queen. Each arrow permanently burns the square on which it lands. A move in the game consists of relocating an Amazon and then shooting an arrow with that same Amazon. Neither Amazons nor arrows can jump over any piece. The last player to be able to move wins. The authors thank R. K. Guy for pointing out this game.

In 1979, Gunther and Hartnell considered the problem of establishing a communication network among members of an underground resistance movement such that the betrayal of members, as a consequence of treachery or subversion by one or more members of the group, is minimized [20]. The members of the movement are modelled by the vertices of a graph. The edges of the graph represent communication links. If a vertex is subverted, then all of its neighbours are betrayed. Given two natural numbers b and n, the problem find connected graphs on n vertices such that the number of betrayals resulting from b subversions is minimum. In particular, for a given graph G and positive integer b, the quantity $K(G, b)$ is the maximum, over all b-subsets $B \subseteq V(G)$, of $|N(B)|$. The quantity $K(n, b)$ is the minimum, over all connected graphs G on n vertices, of $K(G, b)$ (pun intended, no doubt). This problem can be viewed as minimizing the maximum damage that be done by b fires that are never defended but spread for only one time unit.

In 1989, A. Finbow and Hartnell considered the related problem of designing graphs so that the expected damage done by (in the terminology of previous paragraph) the betrayals resulting from b subversions is minimum [14]. This can be viewed as minimizing the expected damage done by b fires that break out at a random set of vertices and are never defended, but spread for only one time unit.

8 Open problems and possible future directions

In this section some open problems and possible directions for future study are listed. We believe that this list contains projects at all levels of difficulty.

1. Study the function $\ell(G, F)$, the minimum number of time units needed to contain f fires that break out at vertices in F, where this quantity is defined to be infinity if the fires can not be contained.

2. Find algorithms and complexity results for the natural extension of S-FIRE in which there are $d \geq 2$ firefighters.

3. Given a weighted graph (G, r) of maximum degree three and in which r has degree two, is there a polynomial-time algorithm that finds the maximum weight subset of vertices that can be saved?

4. Is there a constant $c \in (0, 1]$ such that the "greedy" strategy of defending the vertex of highest degree adjacent to a burning vertex saves at least $c \cdot \mathrm{MVS}(T, r; 1)$ vertices of a rooted tree (T, r)?

5. For a subset $S \subseteq V(G) - \{r\}$ of vertices of a rooted graph (G, r), define $F_{(G,r)}(S)$ to be the minimum number of firefighters needed to save all vertices in S. Find a theorem that gives $F_{(T,r)}(S)$, where S is the set of leaves of the tree T. Under what conditions is this number $\Delta - 1$? More generally, study the S-FIRE number of trees and graphs, especially in the case when $f = 1$.

6. Investigate the Firefighter Problem on trees in the cases where there is more than one fire and more than one firefighter. Determine the the approximation ratio of the greedy algorithm on trees for an arbitrary number f of fires and d of firefighters.

7. Is Conjecture 3.5 true?

8. Find classes of trees for which the Firefighter Problem can be solved in polynomial time. Find a structural characterization of the trees for which the LP relaxation of the 0-1 integer program in Figure 5 gives an optimum solution, or identify non-trivial classes of such trees.

9. B. Alspach suggested a heuristic for trees in which the LP relaxation of the 0-1 integer program is Figure 5 is solved, and then the resulting solution vector is used as a probability distribution for a randomized algorithm for defending the tree. Investigate the performance of such a heuristic experimentally and theoretically.

10. [30] Does one firefighter suffice to contain a fire in the infinite hexagonal grid? What about any finite number of fires?

11. For each of the cases in Theorem 4.6, determine the minimum number of time units needed to contain a single fire, and the minimum number of vertices that must burn over all strategies that use a given number of firefighters.

12. For $n \geq 3$, find the number of fractional firefighters needed to contain a single fire in the n-dimensional square grid and the n-dimensional strong grid.

13. Is Conjecture 4.7 true?

14. Improve the upper bound on the maximum number of vertices that can be saved when a fire in $P_n \Box P_n$ breaks out at (r, c), $1 \leq r \leq c \leq \lceil n/2 \rceil$. Does the strategy in Proposition 5.1 save the maximum number of vertices?

15. Is Conjecture 5.3 true?

16. Investigate optimal graphs (with respect to expected damage) in situations other than $f = d = 1$ and $f = d = 2$. In particular, what about $f = 1$ and $d > f$?

17. Is Conjecture 6.4 true?

18. [7] Determine the minimum d, if it exists, such that there is a positive constant c such that every planar graph G satisfies $\rho(G, 1, d) \geq c$

19. [7] Improve the lower bounds in Theorem 6.3.

20. [7] Is it true that, for outerplanar graphs G, $\lim\limits_{n \to \infty} \rho(G, 1, 1) = 1$? What about for Halin graphs?

21. [7] Is it true that, for every n vertex tree T, $\rho(T, 1, 1) \geq 1 - \Theta(\frac{\log n}{n})$?

22. [7] Prove that it is NP-hard to determine $\rho(T, 1, 1)$ for a given tree T.

23. [7] Determine the approximation ration for the greedy algorithm for the surviving rate $\rho(T, 1, 1)$ of a tree T. Is it $1 - \Theta(\frac{\log n}{n})$?

24. What are the mmd-graphs when $f = d = 2$ and the number of vertices is less than 10?

25. Find a theorem that describes the structure of mmd-graphs and use it to determine whether there are other mmd-graphs besides the ones listed in Theorem 6.5.

26. Directed graphs can be used to situations where a fire can spread from x to y but not from y to x (say because of topography or wind direction). Investigate the Firefighter Problem for (weighted) digraphs.

References

[1] I. Adler, Marshalls, monotone marshalls and hypertree width. *J. Graph Theory* 47 (2004), 275-296.

[2] B. Alspach, D. Dyer, and B. Yang, Sweeping graphs with large clique number. Algorithms and Computation, 902-920, *Lecture Notes in Computer Science* 3341, Springer, Berlin, 2004.

[3] XVII Asian Pacific Mathematics Olympiad (2005). http://www.math.ca/Competitions/APMO/exam/apmo2005.html.

[4] B. Bollobs and I. Leader, The angel and the devil in three dimensions. J. Combin. Theory Ser. A **113** (2006), 176-184.

[5] D. Borowanger, A. Dawar, P. Hunter and S. Kreutzer, DAG-width and parity games. STACS 2006, 524-536, *Lecture Notes in Computer Science* 3884, Springer, Berlin, 2006.

[6] E. R. Berlekamp, J. Conway, and R. K. Guy, *Winning Ways for your mathematical plays.* Academic Press, New York, 1982.

[7] L. Cai and W. Wang, The Surviving Rate of a Graph. Manuscript, 2007.

[8] N. E. Clarke and Emma L. Connon, Cops, robber, and alarms. *Ars Combin.* 81 (2006), 283-296.

[9] F. Comellas, M. Mitjana and J. G. Peters, Broadcasting in Small-World Communication Networks. Proc. 9th Int. Coll. on Structural Information and Communication Complexity (2002), eds. C. Kaklamanis and L. Kirousis, *Proceedings in Informatics*, Carleton Scientific, Waterloo, Canada.

[10] F. Comellas, M. Mitjana and J. G. Peters, Epidemics in Small-World Communication Networks. Manuscript, 2007.

[11] S. Crosby, A. Finbow, B. Hartnell, R. Moussi, K. Patterson and D. Wattar, Designing Fire Resistant Graphs, Manuscript 2005.

[12] Z. Dezso and A-L. Barabasi, Halting viruses in scale-free networks. *Physical Review E* **65** 055103 (2002).

[13] M. Devlin and S. Hartke, Fire containment in grids of dimension three and higher. Manuscript, 2004.

[14] A.S. Finbow and B.L. Hartnell, On Designing a Network to Defend Against Random Attacks on Radius Two, *Networks*, Vol. 19 (1989), 771-792.

[15] S. Finbow, B. Hartnell, Q. Li, and K. Schmeisser, On minimizing the effects of fire or a virus on a network, *JCMCC* 33 (2000), 311-322.

[16] S. Finbow, A. D. King, G. MacGillivray and R. Rizzi, The firefighter problem for graphs of maximum degree three, *Discrete Math.*, to appear.

[17] P. Fogarty, *Catching the fire on grids*, M.Sc. Thesis (2003), Department of Mathematics, University of Vermont, USA.

[18] G. Gottlieb, N. Leone, and F. Sarcello, Robbers, marshals and guards: game theoretical and logical characterizations of hypertree width. *J. Comput. Systems Sci.* 66 (2003), 775-808.

[19] G. Gunther and B. Hartnell, Graphs with isomorphic survivor graphs, *Congressus Numerantium 79*, 69-77, 1990.

[20] G. Gunther and B. L. Hartnell, On M-connected and k-neighbour-connected graphs, *Graph Theory, Combinatorics, and Applications*, 585-596, 1991.

[21] M. Grötschel, L. Lovász,and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*, Springer-Verlag, New York, 1988.

[22] S. G. Hartke, Attempting to narrow the integrality gap for the firefighter problem on trees. *Discrete Methods in Epidemiology*, J.Abello and G.Cormode, eds., DIMACS Series in Discrete Mathematics and Theoretical Computer Science, 70 (2006), 179-185.

[23] B. L. Hartnell, Firefighter! An application of domination. Presentation, *25th Manitoba Conference on Combinatorial Mathematics and Computing*, University of Manitoba in Winnipeg, Canada, 1995.

[24] B. L. Hartnell and W. Kocay, On minimal neighbourhood-connected graphs, *Discrete Math.* 92, 95-105, 1991.

[25] B. Hartnell and Q. Li, Firefighting on trees: How bad is the greedy algorithm?, *Proceedings of the Thirty-first Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 2000)*, Congr. Numer. 145 (2000), 187-192.

[26] G. Hahn and G. MacGillivray, A note on k-cop, l-robber games on graphs. *Discrete Math.* 306 (2006), 2492-2497.

[27] A. King and G. MacGillivray, The firefighter problem for cubic graphs. *Discrete Math,* (to appear).

[28] A. S. LaPaugh, Recontamination does not help to search a graph. *J. ACM* 40 (1993), 224-245.

[29] G. MacGillivray and P. Wang, On the firefighter problem, *JCMCC* 47 (2003), 83-96.

[30] Margaret-Ellen Messinger, *Firefighting on Infinite Grids*, M.Sc. Thesis (2004), Department of Mathematics and Statistics, Dalhousie University, Halifax, Canada.

[31] S. A. Moeller and P. Wang, Fire control on graphs, *JCMCC* 41 (2002), 19 - 34.

[32] M. E. Newman, I. Jensen and R. M. Ziff, Percolation and epidemics in a two-dimensional small world. *Physical Review E* **65** 021904 (2002).

[33] K. L. Ng and P. Raff, Fractional firefighting in the two dimensional grid. DIMACS Technical Report 2005-23, Rutgers University, New Brunswick, NJ, USA.

[34] R. Nowakowski, P. Winkler, Vertex-to-vertex pursuit in a graph *Discrete Math.* **43** (1983) 235-239.

[35] R. Pastor-Satorras and A. Vespignani, Epidemic spreading in scale-free networks. *Phys. Rev. Lett.* 86 (2001), 3200-3203.

[36] A. E. Scott, U. Stege and N. Zeh, *Politician's Firefighting*. Manuscript, 2007.

[37] P. D. Seymour and R. Thomas, Graph searching and a min-max theorem for tree-width. *J. Comb. Theory, Ser. B* 58 (1993), 22-33.

[38] R. G. Snatzke, New results of exhaustive search in the game of Amazons. Theoretical Computer Science, *313* (2004), 499-509.

[39] D. J. Watts and S. Strogatz, Collective dynamics of 'small-world' networks. *Nature*, 393 (1998), 440-442.

[40] D. Zanette and M. Kuperman, Effects of immunization in small-wrld epidemics. *Physica A* 309 (2002), 445-552.